Exercise 2

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In time $t=0$, we have $X_0 = B_1 - (a+b0)B_{\frac{1}{1-0}} = B_1 - aB_1 = (1-a)B_1$. But for X_t to be a Brownian Motion, there must exist some constant $y \in \mathbb{R}$ (specifically $y=0$), such that a.s. $X_0 = y$, but $(1-a)B_1$ is a random variable, not a constant. Unless $a=1$, and then a.s. $X_0 = B_1 - 1B_1 = 0$.

Using ths identities $\mathbb{E}B_t^2 = t$ and $\mathbb{E}B_tB_s = \min\{t,s\}$, we calculate

For $t<s$,

$$\mathbb{E}(X_tX_s) = \mathbb{E}[(B_1-(a+bt)B_{\frac{1}{1-t}})(B_1-(a+bs)B_{\frac{1}{1-s}}) =$$

$$= \mathbb{E}[B_1^2] - \mathbb{E}[(a+bt)B_{\frac{1}{1-t}}B_1] - \mathbb{E}[(a+bs)B_{\frac{1}{1-s}}B_1] + \mathbb{E}[(a+bt)(a+bs)B_{\frac{1}{1-t}}B_{\frac{1}{1-s}}] =$$

$$= \mathbb{E}[B_1^2] - (a+bt)\mathbb{E}[B_{\frac{1}{1-t}}B_1] - (a+bs)\mathbb{E}[B_{\frac{1}{1-s}}B_1] + (a+bt)(a+bs)\mathbb{E}[B_{\frac{1}{1-t}}B_{\frac{1}{1-s}}]$$

$0<t<s<1 \Rightarrow 0<1-s<1-t<1 \Rightarrow 1<\frac{1}{1-t}<\frac{1}{1-s}$, then

$$= 1-(a+bt)1-(a+bs)1+(a+bt)(a+bs)\frac{1}{1-t} =$$

$$= 1-a-bt-a-bs+\frac{a^2+abt+abs+b^2ts}{1-t} =$$

$$= \frac{1-t-2a+2at-bt+bt^2-bs+bst+a^2+abt+abs+b^2ts}{1-t}$$

But if X_t and X_s are Brownian Motion random variables, then $\mathbb{E}[X_tX_s] = \min\{t,s\} = t$

$$\Rightarrow 1-t-2a+2at-bt+bt^2-bs+bst+a^2+abt+abs+b^2ts = t-t^2$$

We collect all the constant expressions (expressions with only a or 1) to obtain $a^2-2a+1=0 \Rightarrow a=1$, which we already saw. So we have $-t+2t-bt+bt^2-bs+bst+bt+bs+b^2ts-t+t^2=0 \Rightarrow bt^2+bst+b^2ts+t^2=0$. We write the equations $bt^2+t^2=0$ and $b^2ts+bts=0$ to obtain $b=-1$. So $X_t = B_1-(1-t)B_{\frac{1}{1-t}}$. We already saw that for $a=1$, a.s. $X_0=0$, and since it is a linear combination of B_t, it is also a.s continuous. $\mathbb{E}[X_{t+s}-X_t] = \mathbb{E}[B_1-(1-(t+s))B_{\frac{1}{1-t-s}} - [B_1-(1-t)B_{\frac{1}{1-t}}]] = \mathbb{E}[(1-t)B\frac{1}{1-t}] - \mathbb{E}[(1-(t+s)B_{\frac{1}{1-t-s}}] = (1-t)0-(1-(t+s))0 = 0$.

$var(X_{t+s}-X_t) = \mathbb{E}[(X_{t+s}-X_t)^2] - \mathbb{E}[X_{t+s}-X_t]^2 = \mathbb{E}[(X_{t+s}-X_t)]^2] = \mathbb{E}[[(1-t)B_{\frac{1}{1-t}} - (1-(t+s))B_{\frac{1}{1-t-s}}]^2] = (1-t)^2\frac{1}{1-t} - 2(1-t)(1-t-s)\frac{1}{1-t} + (1-t-s)^2\frac{1}{1-t-s} = 1-t-2(1-t-s)+1-t-s = 1-t-2+2t+2s+1-t-s = s$, so $X_{t+s}-X_t \sim \mathcal{N}(0,s)$.