Exercise 2

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1. In time $t = 0$, we have $X_0 = B_1 - (a + b0)B_{\frac{1}{1-0}} = B_1 - aB_1 = (1 - a)B_1$. But for X_t to be a Brownian Motion, there must exist some constant $y \in \mathbb{R}$ (specifically $y = 0$), such that a.s. $X_0 = y$, but $(1 - a)B_1$ is a random variable, not a constant. Unless $a = 1$, and then a.s. $X_0 = B_1 - 1B_1 = 0$.

   Using ths identities $\mathbb{E}B_t^2 = t$ and $\mathbb{E}B_tB_s = \min\{t, s\}$, we calculate

   For $t < s$,

   $$\mathbb{E}(X_tX_s) = \mathbb{E}[(B_1 - (a + bt)B_{\frac{1}{1-t}})(B_1 - (a + bs)B_{\frac{1}{1-s}}) =$$

   $$= \mathbb{E}[B_1^2] - \mathbb{E}[(a+bt)B_{\frac{1}{1-t}}B_1] - \mathbb{E}[(a+bs)B_{\frac{1}{1-s}}B_1] + \mathbb{E}[(a+bt)(a+bs)B_{\frac{1}{1-t}}B_{\frac{1}{1-s}}] =$$

   $$= \mathbb{E}[B_1^2] - (a+bt)\mathbb{E}[B_{\frac{1}{1-t}}B_1] - (a+bs)\mathbb{E}[B_{\frac{1}{1-s}}B_1] + (a+bt)(a+bs)\mathbb{E}[B_{\frac{1}{1-t}}B_{\frac{1}{1-s}}]$$

   $0 < t < s < 1 \Rightarrow 0 < 1 - s < 1 - t < 1 \Rightarrow 1 < \frac{1}{1-t} < \frac{1}{1-s}$, then

   $$= 1 - (a + bt)1 - (a + bs)1 + (a + bt)(a + bs)\frac{1}{1-t} =$$

   $$= 1 - a - bt - a - bs + \frac{a^2 + abt + abs + b^2ts}{1-t} =$$

   $$= \frac{1 - t - 2a + 2at - bt + bt^2 - bs + bst + a^2 + abt + abs + b^2ts}{1-t}$$

   But if X_t and X_s are Brownian Motion random variables, then $\mathbb{E}[X_tX_s] = \min\{t, s\} = t$

   $$\Rightarrow 1 - t - 2a + 2at - bt + bt^2 - bs + bst + a^2 + abt + abs + b^2ts = t - t^2$$

   We collect all the constant expressions (expressions with only a or 1) to obtain $a^2 - 2a + 1 = 0 \Rightarrow a = 1$, which we already saw. So we have $-t + 2t - bt + bt^2 - bs + bst + bt + bs + b^2ts - t + t^2 = 0 \Rightarrow bt^2 + bst + b^2ts + t^2 = 0$. We write the equations $bt^2 + t^2 = 0$ and $b^2ts + bts = 0$ to obtain $b = -1$. So $X_t = B_1 - (1 - t)B_{\frac{1}{1-t}}$. We already saw that for $a = 1$, a.s. $X_0 = 0$, and since it is a linear combination of B_t, it is also a.s continuous. $\mathbb{E}[X_{t+s} - X_t] = \mathbb{E}[B_1 - (1 - (t + s))B_{\frac{1}{1-t-s}} - [B_1 - (1 - t)B_{\frac{1}{1-t}}]] = \mathbb{E}[(1 - t)B_{\frac{1}{1-t}}] - \mathbb{E}[(1 - (t + s)B_{\frac{1}{1-t-s}}] = (1 - t)0 - (1 - (t + s))0 = 0$.

$var(X_{t+s} - X_t) = \mathbb{E}[(X_{t+s} - X_t)^2] - \mathbb{E}[X_{t+s} - X_t]^2 = \mathbb{E}[(X_{t+s} - X_t)]^2] = \mathbb{E}[[(1-t)B_{\frac{1}{1-t}} - (1-(t+s))B_{\frac{1}{1-t-s}}]^2] = (1-t)^2\frac{1}{1-t} - 2(1-t)(1-t-s)\frac{1}{1-t} + (1-t-s)^2\frac{1}{1-t-s} = 1-t-2(1-t-s)+1-t-s = 1-t-2+2t+2s+1-t-s = s$, so $X_{t+s} - X_t \sim \mathcal{N}(0, s)$.

2. Code can be found here: https://github.com/HaimL76/ctmc2.git

Figure 1: Brownian Motion 35 different simulations

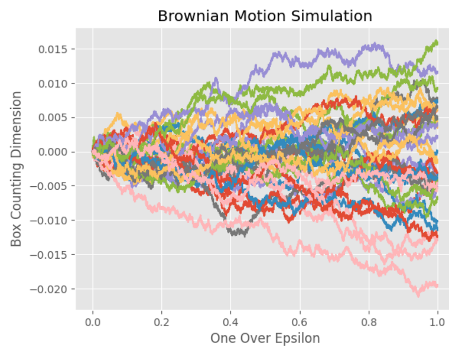
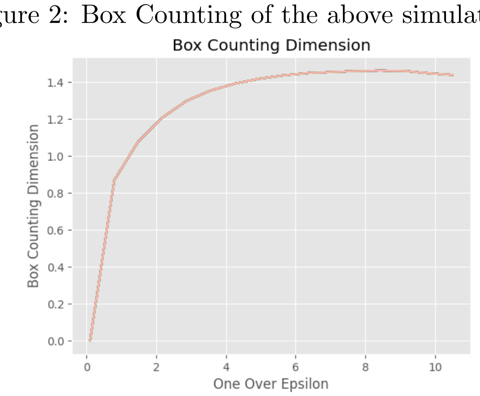


Figure 2: Box Counting of the above simulations



We see that even though the 35 simulations go in different paths, the tendency of their box counting dimension around 1.4 is very similar.

3. Code can be found here: https://github.com/HaimL76/fbm.git

Figure 3: Fractal Brownian Motion, Hurst=0.25

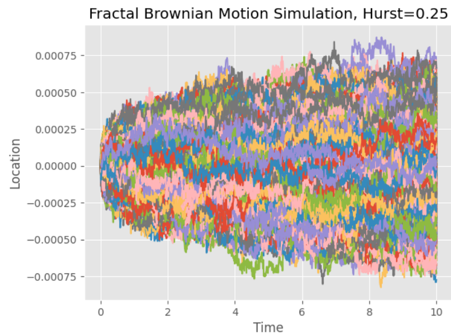


Figure 4: Fractal Brownian Motion MSD, Hurst=0.25

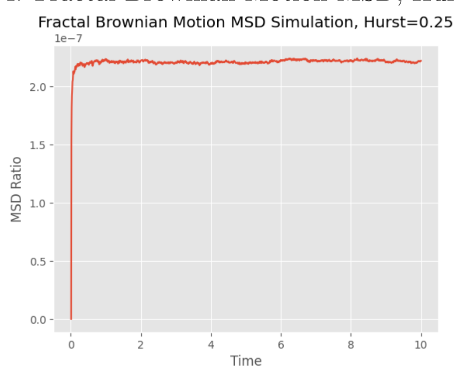


Figure 5: Fractal Brownian Motion, Hurst=0.5

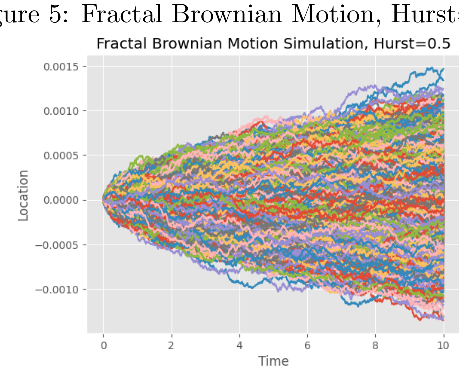


Figure 6: Fractal Brownian Motion MSD, Hurst=0.5

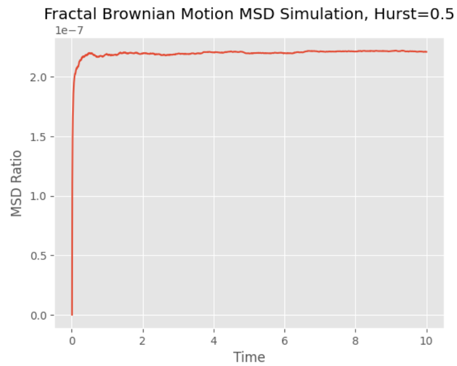


Figure 7: Fractal Brownian Motion, Hurst=0.75

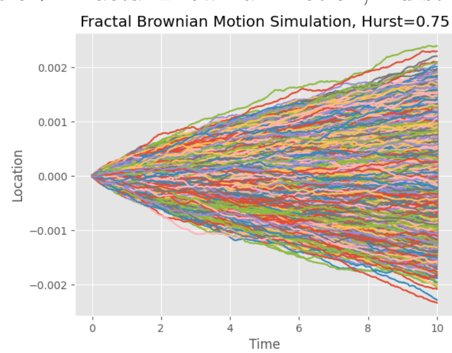


Figure 8: Fractal Brownian Motion MSD, Hurst=0.75