Exercise 2

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1. In time $t = 0$, we have $X_0 = B_1 - (a + b0)B_{\frac{1}{1-0}} = B_1 - aB_1 = (1 - a)B_1$. But for X_t to be a Brownian Motion, there must exist some constant $y \in \mathbb{R}$ (specifically $y = 0$), such that a.s. $X_0 = y$, but $(1 - a)B_1$ is a random variable, not a constant. Unless $a = 1$, and then a.s. $X_0 = B_1 - 1B_1 = 0$.

   Using ths identities $\mathbb{E}B_t^2 = t$ and $\mathbb{E}B_tB_s = \min\{t, s\}$, we calculate

   For $t < s$,

   $$\mathbb{E}(X_tX_s) = \mathbb{E}[(B_1 - (a + bt)B_{\frac{1}{1-t}})(B_1 - (a + bs)B_{\frac{1}{1-s}}) =$$

   $$= \mathbb{E}[B_1^2] - \mathbb{E}[(a+bt)B_{\frac{1}{1-t}}B_1] - \mathbb{E}[(a+bs)B_{\frac{1}{1-s}}B_1] + \mathbb{E}[(a+bt)(a+bs)B_{\frac{1}{1-t}}B_{\frac{1}{1-s}}] =$$

   $$= \mathbb{E}[B_1^2] - (a+bt)\mathbb{E}[B_{\frac{1}{1-t}}B_1] - (a+bs)\mathbb{E}[B_{\frac{1}{1-s}}B_1] + (a+bt)(a+bs)\mathbb{E}[B_{\frac{1}{1-t}}B_{\frac{1}{1-s}}]$$

   $0 < t < s < 1 \Rightarrow 0 < 1 - s < 1 - t < 1 \Rightarrow 1 < \frac{1}{1-t} < \frac{1}{1-s}$, then

   $$= 1 - (a + bt)1 - (a + bs)1 + (a + bt)(a + bs)\frac{1}{1 - t} =$$

   $$= 1 - a - bt - a - bs + \frac{a^2 + abt + abs + b^2ts}{1 - t} =$$

   $$= \frac{1 - t - 2a + 2at - bt + bt^2 - bs + bst + a^2 + abt + abs + b^2ts}{1 - t}$$

   But if X_t and X_s are Brownian Motion random variables, then $\mathbb{E}[X_tX_s] = \min\{t, s\} = t$

   $$\Rightarrow 1 - t - 2a + 2at - bt + bt^2 - bs + bst + a^2 + abt + abs + b^2ts = t - t^2$$

   We collect all the constant expressions (expressions with only a or 1) to obtain $a^2 - 2a + 1 = 0 \Rightarrow a = 1$, which we already saw. So we have $-t + 2t - bt + bt^2 - bs + bst + bt + bs + b^2ts - t + t^2 = 0 \Rightarrow bt^2 + bst + b^2ts + t^2 = 0$. We write the equations $bt^2 + t^2 = 0$ and $b^2ts + bts = 0$ to obtain $b = -1$. So $X_t = B_1 - (1 - t)B_{\frac{1}{1-t}}$. We already saw that for $a = 1$, a.s. $X_0 = 0$, and since it is a linear combination of B_t, it is also a.s continuous. $\mathbb{E}[X_{t+s} - X_t] = \mathbb{E}[B_1 - (1 - (t + s))B_{\frac{1}{1-t-s}} - [B_1 - (1 - t)B_{\frac{1}{1-t}}]] = \mathbb{E}[(1 - t)B_{\frac{1}{1-t}}] - \mathbb{E}[(1 - (t + s)B_{\frac{1}{1-t-s}}] = (1 - t)0 - (1 - (t + s))0 = 0$.

$var(X_{t+s} - X_t) = \mathbb{E}[(X_{t+s} - X_t)^2] - \mathbb{E}[X_{t+s} - X_t]^2 = \mathbb{E}[(X_{t+s} - X_t)]^2] = \mathbb{E}[[(1-t)B_{\frac{1}{1-t}} - (1-(t+s))B_{\frac{1}{1-t-s}}]^2] = (1-t)^2\frac{1}{1-t} - 2(1-t)(1-t-s)\frac{1}{1-t} + (1-t-s)^2\frac{1}{1-t-s} = 1-t-2(1-t-s)+1-t-s = 1-t-2+2t+2s+1-t-s = s$, so $X_{t+s} - X_t \sim \mathcal{N}(0, s)$.

2. Code can be found here: https://github.com/HaimL76/ctmc2.git

Figure 1: Brownian Motion 50 different simulations

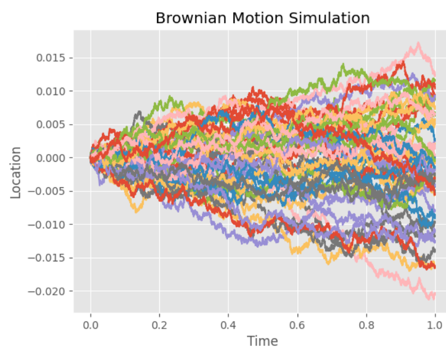
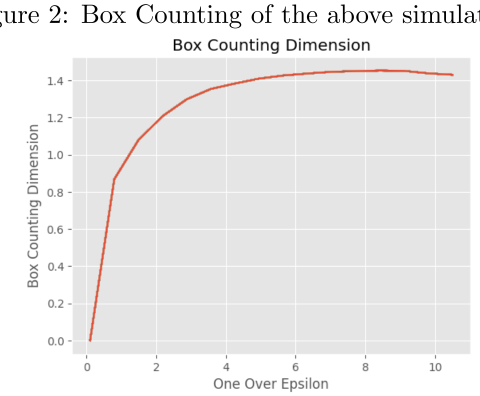


Figure 2: Box Counting of the above simulations



We see that even though the 50 simulations go in different paths, the tendency of their box counting dimension around 1.4 is very similar.

3. Code can be found here: https://github.com/HaimL76/fbm.git We see that for the three different values of the Hurst parameter, the MSD stabilizes around the same value almost from the beginning of the time interval. Looking at the different plots, we see that the higher the Hurst parameter is, we get more moderate fluctuations around each path. The plot for $H = 0.25$ has the shape of half an ellipse, with a significantly broad range of values near the beginning of the time interval, and different paths are

strongly blended into other paths. For $H = 0.75$, the plot has a triangular shape, and each path is fluctuating closely enough to what can be described as a straight line. For $H = 0.5$, where the process is identified with a standard Brownian Motion, we get a plot that is quite in the middle between the above two plots, and should resemble more the plot of the standard Brownian Motion in 2.

Figure 3: Fractal Brownian Motion, Hurst=0.25

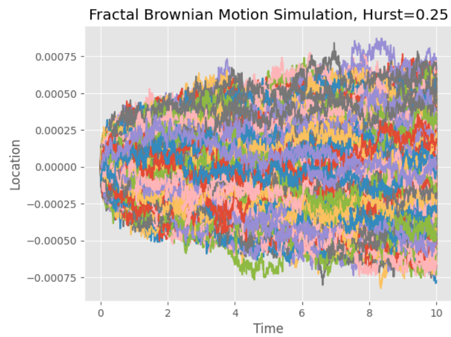


Figure 4: Fractal Brownian Motion MSD, Hurst=0.25

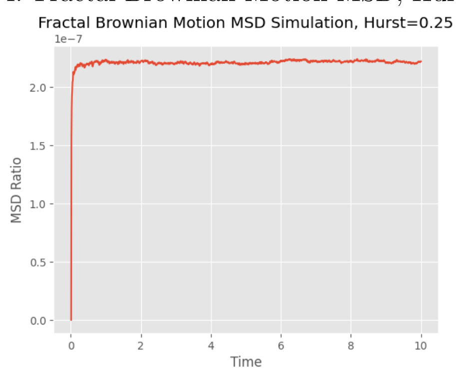


Figure 5: Fractal Brownian Motion, Hurst=0.5

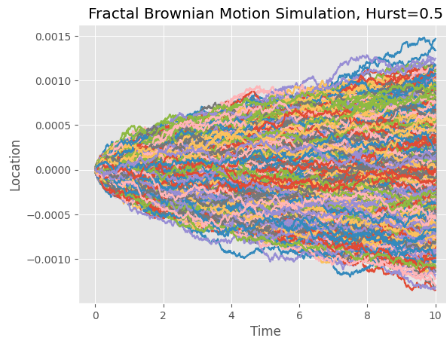


Figure 6: Fractal Brownian Motion MSD, Hurst=0.5

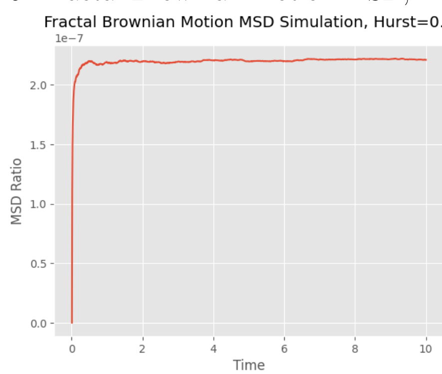


Figure 7: Fractal Brownian Motion, Hurst=0.75

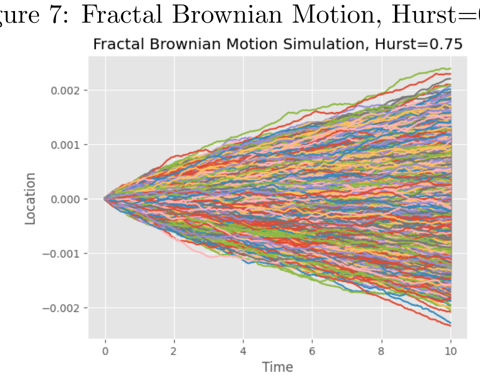


Figure 8: Fractal Brownian Motion MSD, Hurst=0.75