Exercise 2

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1. In time $t = 0$, we have $X_0 = B_1 - (a + b0)B_{\frac{1}{1-0}} = B_1 - aB_1 = (1 - a)B_1$. But for X_t to be a Brownian Motion, there must exist some constant $y \in \mathbb{R}$ (specifically $y = 0$), such that a.s. $X_0 = y$, but $(1 - a)B_1$ is a random variable, not a constant. Unless $a = 1$, and then a.s. $X_0 = B_1 - 1B_1 = 0$.

   Using ths identities $\mathbb{E}B_t^2 = t$ and $\mathbb{E}B_tB_s = \min\{t, s\}$, we calculate

   For $t < s$,

$$\mathbb{E}(X_tX_s) = \mathbb{E}[(B_1 - (a + bt)B_{\frac{1}{1-t}})(B_1 - (a + bs)B_{\frac{1}{1-s}}) =$$

$$= \mathbb{E}[B_1^2] - \mathbb{E}[(a+bt)B_{\frac{1}{1-t}}B_1] - \mathbb{E}[(a+bs)B_{\frac{1}{1-s}}B_1] + \mathbb{E}[(a+bt)(a+bs)B_{\frac{1}{1-t}}B_{\frac{1}{1-s}}] =$$

$$= \mathbb{E}[B_1^2] - (a+bt)\mathbb{E}[B_{\frac{1}{1-t}}B_1] - (a+bs)\mathbb{E}[B_{\frac{1}{1-s}}B_1] + (a+bt)(a+bs)\mathbb{E}[B_{\frac{1}{1-t}}B_{\frac{1}{1-s}}]$$

   $0 < t < s < 1 \Rightarrow 0 < 1 - s < 1 - t < 1 \Rightarrow 1 < \frac{1}{1-t} < \frac{1}{1-s}$, then

$$= 1 - (a + bt)1 - (a + bs)1 + (a + bt)(a + bs)\frac{1}{1 - t} =$$

$$= 1 - a - bt - a - bs + \frac{a^2 + abt + abs + b^2ts}{1 - t} =$$

$$= \frac{1 - t - 2a + 2at - bt + bt^2 - bs + bst + a^2 + abt + abs + b^2ts}{1 - t}$$

   But if X_t and X_s are Brownian Motion random variables, then $\mathbb{E}[X_tX_s] = \min\{t, s\} = t$

$$\Rightarrow 1 - t - 2a + 2at - bt + bt^2 - bs + bst + a^2 + abt + abs + b^2ts = t - t^2$$

   We collect all the constant expressions (expressions with only a or 1) to obtain $a^2 - 2a + 1 = 0 \Rightarrow a = 1$, which we already saw. So we have $-t + 2t - bt + bt^2 - bs + bst + bt + bs + b^2ts - t + t^2 = 0 \Rightarrow bt^2 + bst + b^2ts + t^2 = 0$. We write the equations $bt^2 + t^2 = 0$ and $b^2ts + bts = 0$ to obtain $b = -1$. So $X_t = B_1 - (1 - t)B_{\frac{1}{1-t}}$. We already saw that for $a = 1$, a.s. $X_0 = 0$, and since it is a linear combination of B_t, it is also a.s continuous. $\mathbb{E}[X_{t+s} - X_t] = \mathbb{E}[B_1 - (1 - (t + s))B_{\frac{1}{1-t-s}} - [B_1 - (1 - t)B_{\frac{1}{1-t}}]] = \mathbb{E}[(1 - t)B_{\frac{1}{1-t}}] - \mathbb{E}[(1 - (t + s)B_{\frac{1}{1-t-s}}] = (1 - t)0 - (1 - (t + s))0 = 0$.

$var(X_{t+s}-X_t) = \mathbb{E}[(X_{t+s}-X_t)^2] - \mathbb{E}[X_{t+s}-X_t]^2 = \mathbb{E}[(X_{t+s}-X_t)]^2] = \mathbb{E}[[(1-t)B_{\frac{1}{1-t}} - (1-(t+s))B_{\frac{1}{1-t-s}}]^2] = (1-t)^2\frac{1}{1-t} - 2(1-t)(1-t-s)\frac{1}{1-t} + (1-t-s)^2\frac{1}{1-t-s} = 1-t-2(1-t-s)+1-t-s = 1-t-2+2t+2s+1-t-s = s$, so $X_{t+s}-X_t \sim \mathcal{N}(0,s)$.

2. Code can be found here: https://github.com/HaimL76/ctmc2.git

Figure 1: Brownian Motion 8 different simulations

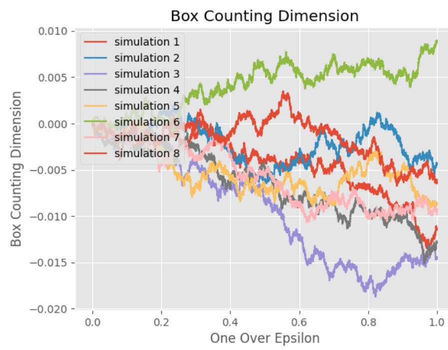
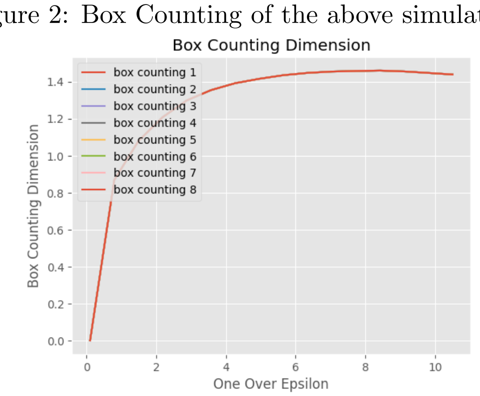


Figure 2: Box Counting of the above simulations



We see that even though the 8 simulations go in different paths, the tendency of their box counting dimension around 1.4 is very similar.