ctmc3

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$dX_t = 2X_t dB_t,\ X_0 = 3$

Following a similar example by Oksendal, we solve by Ito and by Stratonovich.

This is a private case of the general form $dX_t = \mu X_t dt + \sigma X_t dB_t$, where $\mu = 0$ and $\sigma = 2$.

X_t is not identically zero, so we divide both sides of the equation by X_t,

$\frac{dX_t}{X_t} = 2dB_t \Rightarrow \int_0^t \frac{dX_s}{X_s} = \int_0^t 2dB_s = 2\int_0^t dB_s = 2B_t$.

To calculate the integral we use Ito formula for the function $Y_t = \ln X_t$,

$dY_t = d\ln X_t = \frac{1}{X_t} \cdot dX_t + \frac{1}{2}(-\frac{1}{X_t^2})(dX_t)^2 = \frac{dX_t}{X_t} - \frac{1}{2X_t^2} \cdot 4X_t^2 dt = \frac{dX_t}{X_t} - 2dt \Rightarrow \frac{dX_t}{X_t} = dY_t + 2dt \Rightarrow Y_t = 2B_t - 2t + Y_0 \Rightarrow \ln X_t - \ln X_0 = 2B_t - 2t \Rightarrow \ln \frac{X_t}{X_0} = 2B_t - 2t \Rightarrow \frac{X_t}{X_0} = e^{2B_t - 2t} \Rightarrow X_t = 3e^{2B_t - 2t}$

By Stratonovich,

$dX_t = \mu X_t dt + \sigma X_t \circ dB_t$, where $\mu = 0$ and $\sigma = 2$, and the solution is $X_t = 3e^{2B_t}$

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$X_t = \cos(t + B_t)$

Denote $f(t, B_t) := \cos(t + B_t)$. We calculate partial derivatives,

$\frac{\partial f}{\partial t} = -\sin(t + B_t)$

$\frac{\partial f}{\partial B_t} = -\sin(t + B_t)$

$\frac{\partial f^2}{\partial^2 B_t} = -\cos(t + B_t)$

So by Ito,

$dX_t = df(t, B_t) = [\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial f^2}{\partial^2 B_t}]dt + \frac{\partial f}{\partial B_t} dB_t = [-\sin(t + B_t) - \frac{1}{2}\cos(t + B_t)]dt - \sin(t + B_t)dB_t$