ctmc3

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$dX_t = 2X_t dB_t$, $X_0 = 3$.

We divide both sides of the equation by $X_t \Rightarrow \frac{dX_t}{X_t} = 2dB_t$.

We set $Y_y := \ln X_t$, and calculate partial derivatives,

$\frac{\partial Y_n}{\partial t} = \frac{\partial \ln X_t}{\partial t} = 0$.

$\frac{\partial Y_t}{\partial X_t} = \frac{\partial \ln X_t}{\partial X_t} = \frac{1}{X_t}$.

$\frac{\partial^2 Y_t}{\partial {X_t}^2} = \frac{\partial^2 \ln X_t}{\partial {X_t}^2} = -\frac{1}{X_t^2}$.

By Ito, we get,

$$dY_t = d\ln X_t = \frac{1}{X_t}dX_t + \frac{1}{2}\cdot -\frac{1}{X_t^2}(dX_t)^2 = \frac{dX_t}{X_t} - \frac{1}{2X_t^2}(2X_t dB_t)^2 =$$

$$= \frac{dX_t}{X_t} - \frac{4X_t^2(dB_t)^2}{2X_t^2} = \frac{dX_t}{X_t} - 2(dB_t)^2.$$

We use the identity $(dB_t)^2 = dt$, and the equation $\frac{dX_t}{X_t} = 2dB_t$, and we get,

$dY_t = d\ln X_t = \frac{dX_t}{X_t} - 2dt = 2dB_t - 2dt$.

We integrate both sides, and add the constant $Y_0 = \ln X_0$,

$$Y_t = 2B_t - 2t + Y_0 \Rightarrow \ln X_t - \ln X_0 = 2B_t - 2t \Rightarrow$$

$$\Rightarrow \ln\frac{X_t}{X_0} = 2B_t - 2t \Rightarrow \frac{X_t}{X_0} = e^{2B_t - 2t} \Rightarrow X_t = X_0 e^{2Bt-2t} = 3e^{2B_t-2t}.$$

2

$X_t = \cos(t + B_t)$.

We reconstruct the equation using Ito.

Denote $F(t, B_t) = \cos(t + B_t)$.

We calculate partial derivatives,

$\frac{\partial F}{\partial t} = -\sin(t + B_t)$.

$\frac{\partial F}{\partial B_t} = -\sin(t + B_t)$.

$\frac{\partial^2 F}{\partial {B_t}^2} = -\cos(t + B_t)$.

Setting $F(t, B_t) = \cos(t, B_t) = X_t$, we get,

$$dX_t = dF(t, B_t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial B_t}dB_t + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial {B_t}^2}(dB_t)^2 =$$

$$= -\sin(t + B_t)dt - \sin(t + B_t)dB_t + \frac{1}{2} \cdot -\cos(t + B_t)(dB_t)^2 =$$

$$= [-\sin(t + B_t) - \frac{1}{2}\cos(t + B_t)]dt - \sin(t + B_t)dB_t.$$

But $\cos(t + B_t) = X_t \Rightarrow -\sin(t + Bt) = -\sqrt{1 - \cos^2(t + Bt)} = -\sqrt{1 - X_t^2}$.

Hence, $dX_t = [-\sqrt{1 - X_t^2} - \frac{1}{2}X_t]dt - \sqrt{1 - X_t^2}dB_t$.

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$dX_t = (\alpha - \mu X_t)dt + \sigma dW_t$, $\alpha, \mu, \sigma > 0$, $X_0 = \frac{\alpha}{\mu}$.

We bring the equation to the form $dX_t = -\mu(X_t - \tilde{\alpha})dt + \sigma dW_t$,

where $\tilde{\alpha} = \frac{\alpha}{\mu}$.

Set $Y_t := (X_t - \tilde{\alpha})e^{\mu t}$.

We calculate partial derivatives,

$\frac{\partial Y_t}{\partial t} = \mu(X_t - \tilde{\alpha})e^{\mu t}$.

$\frac{\partial Y_t}{\partial X_t} = dX_t e^{\mu t}$.

$\frac{\partial^2 Y_t}{\partial {X_t}^2} = \frac{\partial e^{\mu t}}{\partial {X_t}^2}dX_t = 0$.

Then by Ito, we get,

$dY_t = \mu(X_t - \tilde{\alpha})e^{-\mu t}dt + e^{-\mu t}dX_t$.

But $dX_t = -\mu(X_t - \tilde{\alpha})dt + \sigma dW_t \Rightarrow$

$$\Rightarrow dY_t = \mu(X_t - \tilde{\alpha})e^{\mu t}dt + e^{\mu t}(-\mu(X_t - \tilde{\alpha})dt + \sigma dW_t) =$$

$$= \sigma e^{\mu t}dW_t \Rightarrow Y_t = \int_0^t \sigma e^{\mu s}dW_s + Y_0 = \sigma \int_0^t e^{\mu s}dW_s + (X_0 - \tilde{\alpha})e^{\mu \cdot 0}.$$

But

$$X_0 = \frac{\alpha}{\mu} = \tilde{\alpha} \Rightarrow Y_t = \sigma \int_0^t e^{\mu s}dW_s \Rightarrow .$$

But

$$X_t = Y_t e^{-\mu t} + \tilde{\alpha} = e^{-\mu t}\sigma \int_0^t e^{\mu s}dW_s + \tilde{\alpha}.$$

We calculate the expectation,

$$\mathbb{E}[X_t] = \mathbb{E}[e^{-\mu t}\sigma \int_0^t e^{\mu s}dW_s + \tilde{\alpha}].$$

But expectation is linear, so we get,

$$\mathbb{E}[X_t] = \mathbb{E}[e^{-\mu t}\sigma \int_0^t e^{\mu s}dW_s] + \mathbb{E}[\tilde{\alpha}] = \sigma\mathbb{E}[\int_0^t e^{\mu(s-t)}dW_s] + \tilde{\alpha}.$$

But the expectation of a Brownian motion is zero, so we get,

$\mathbb{E}[X_t] = \tilde{\alpha} = \frac{\alpha}{\mu}$.

This means that if we start at the mean value, then the expectation remains the same, which means that this process has the behavior of a martingale.

$$Var(X_t) = \mathbb{E}[(X_t - \mathbb{E}[X_t])^2] = \mathbb{E}[(\sigma \int_0^t e^{\mu(s-t)} + \tilde{\alpha} - \tilde{\alpha})^2] =$$

$$= \mathbb{E}[(\sigma \int_0^t e^{\mu(s-t)})^2].$$

But by Ito isometry, we get,

$$Var(X_t) = \sigma^2 \int_0^t e^{2\mu(s-t)}ds.$$

But this is a standard integral, with no stochastic element, so we get,

$$Var(X_t) = \frac{\sigma^2}{2\mu}[e^{2\mu(s-t)}]_0^t = \frac{\sigma^2}{2\mu}[e^{2\mu \cdot t - t} - e^{-2\mu t}] = \frac{\sigma}{2\mu}[1 - e^{-2\mu t}].$$