ctmc3

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August 2025

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$dX_t = 2X_t dB_t$, $X_0 = 3$.

We divide both sides of the equation by $X_t \Rightarrow \frac{dX_t}{X_t} = 2dB_t$.

We set $Y_y := \ln X_t$, and calculate partial derivatives,

$\frac{\partial Y_n}{\partial t}$ does not exist, because Y_t depends only on X_t.

$\frac{\partial Y_t}{\partial X_t} = \frac{\partial \ln X_t}{\partial X_t} = \frac{1}{X_t}$.

$\frac{\partial^2 Y_t}{\partial {X_t}^2} = \frac{\partial^2 \ln X_t}{\partial {X_t}^2} = -\frac{1}{X_t^2}$.

By Ito, we get,

$$dY_t = d\ln X_t = \frac{1}{X_t}dX_t + \frac{1}{2} \cdot -\frac{1}{X_t^2}(dX_t)^2 = \frac{dX_t}{X_t} - \frac{1}{2X_t^2}(2X_t dB_t)^2 =$$

$$= \frac{dX_t}{X_t} - \frac{4X_t^2(dB_t)^2}{2X_t^2} = \frac{dX_t}{X_t} - 2(dB_t)^2.$$

We use the identity $(dB_t)^2 = dt$, and the equation $\frac{dX_t}{X_t} = 2dB_t$, and we get,

$dY_t = d\ln X_t = \frac{dX_t}{X_t} - 2dt = 2dB_t - 2dt$.

We integrate both sides, and add the constant $Y_0 = \ln X_0$,

$$Y_t = 2B_t - 2t + Y_0 \Rightarrow \ln X_t - \ln X_0 = 2B_t - 2t \Rightarrow$$

$$\Rightarrow \ln \frac{X_t}{X_0} = 2B_t - 2t \Rightarrow \frac{X_t}{X_0} = e^{2B_t - 2t} \Rightarrow X_t = X_0 e^{2Bt-2t} = 3e^{2B_t - 2t}.$$

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$X_t = \cos(t + B_t)$.

We reconstruct the equation using Ito.

Denote $F(t, B_t) = \cos(t + B_t)$.

We calculate partial derivatives,

$\frac{\partial F}{\partial t} = -\sin(t + B_t)$.
$\frac{\partial F}{\partial B_t} = -\sin(t + B_t)$.
$\frac{\partial^2 F}{\partial {B_t}^2} = -\cos(t + B_t)$.

Setting $F(t, B_t) = \cos(t, B_t) = X_t$, we get,

$$dX_t = dF(t, B_t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial B_t}dB_t + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial {B_t}^2}(dB_t)^2 =$$

$$= -\sin(t + B_t)dt - \sin(t + B_t)dB_t + \frac{1}{2} \cdot -\cos(t + B_t)(dB_t)^2 =$$

$$= [-\sin(t + B_t) - \frac{1}{2}\cos(t + B_t)]dt - \sin(t + B_t)dB_t.$$

But $\cos(t + B_t) = X_t \Rightarrow -\sin(t + Bt) = -\sqrt{1 - \cos^2(t + Bt)} = -\sqrt{1 - X_t^2}$.

Hence, $dX_t = [-\sqrt{1 - X_t^2} - \frac{1}{2}X_t]dt - \sqrt{1 - X_t^2}dB_t$.

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