Appendix B. The two SLPs based on MILP models in (Fernandez-Viagas et al., 2019)

Among the four MILP models mentioned in (Fernandez-Viagas et al., 2019), P4 model integrates the sequencing and scheduling rules of P2 model and P3 model, its solution space is the smallest, and it has the highest solving efficiency. However, it is noteworthy that P4 model is not an equivalent model of the two-stage hybrid flow shop. Without loss of generality, we will use P1 and P4 models for comparison purposes. To facilitate further discussion, the SAA method is applied to construct the SLP version of P1 and P4, i.e., SAA-P1 and SAA-P4.

(SAA-P1)

$$\min \sum_{u=1}^{\mathcal{N}} C_{max}$$

$$\text{s.t.}\quad Y^u_{1,1,j} = 1 \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$\sum_{l=1}^{m} Y^u_{2,l,j} = 1 \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C_{s,j} - C^u_{s-1,j} \geq p^u_{s,j} \quad s = 1, 2; j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{1,j} \geq C^u_{1,k} + p^u_{1,j} - M(1 - x^u_{1,j,k})$$
$$j = 1, \ldots, n-1; k = j+1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{2,j} \geq C^u_{2,k} + p^u_{2,j} - M(3 - X^u_{2,j,k} - Y^u_{2,l,j} - Y^u_{2,l,k})$$
$$l = 1, \ldots, m; j = 1, \ldots, n-1; k = j+1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{1,k} \geq C^u_{1,j} + p^u_{1,k} - MX^u_{1,j,k}$$
$$j = 1, \ldots, n-1; k = j+1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{2,k} \geq C^u_{2,j} + p^u_{2,k} - MX^u_{2,j,k} - M(2 - Y^u_{2,l,j} - Y^u_{2,l,k})$$
$$l = 1, \ldots, m; j = 1, \ldots, n-1; k = j+1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{max} \geq C^u_{2,j} \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{1,j}, C^u_{2,j} \geq 0 \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{0,j} = 0 \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|.$$

(SAA-P4)

$$min \sum_{u=1}^{|\mathcal{N}|} C_{max}$$

s.t. $Y^u_{1,1,j} = 1 \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$

$$\sum_{l=1}^{m} Y^u_{2,l,j} = 1 \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{s,j} = C^u_{s-1,j} + p^u_{s,j} + h^u_{s,j} \quad s = 1, 2; j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{1,j} = C^u_{1,k} + p^u_{1,j} - M(1 - X^u_{1,j,k}) + h^{'u}_{1,1,j,k}$$

$$j = 1, 2, \ldots, n; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; u = 1, 2, \ldots, |\mathcal{N}|$$

$$C^u_{2,j} = C^u_{2,k} + p^u_{2,j} - M(3 - X^u_{2,j,k} - Y^u_{2,l,j} - Y^u_{2,l,k}) + h^{'u}_{2,l,j,k}$$

$$l = 1, \ldots, m; j = 1, \ldots, n; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; u = 1, \ldots, |\mathcal{N}|$$

$$X^u_{s,j,k} + X^u_{s,k,j} = 1$$

$$s = 1, 2; j = 1, \ldots, n; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; u = 1, \ldots, |\mathcal{N}|$$

$$h^u_{s,j} \le M \cdot V^u_{s,j} \quad s = 1, 2; j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$h^{'u}_{1,1,j,k} + M(1 - X^u_{1,j,k}) \le M^{'}(1 - V^{'u}_{1,1,j,k})$$

$$j = 1, \ldots, n; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; u = 1, \ldots, |\mathcal{N}|$$

$$h^{'u}_{2,l,j,k} + M(3 - X^u_{2,j,k} - Y^u_{2,l,j} - Y^u_{2,l,k}) \le M^{'}(1 - V^{'u}_{1,1,j,k})$$

$$j = 1, \ldots, n; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; l = 1, \ldots, m; u = 1, \ldots, |\mathcal{N}|$$

$$\sum_{k=1, k \ne j}^{n} V^{'u}_{1,1,j,k} \ge 1 - M(1 - V^u_{1,j}) \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$\sum_{l=1}^{m} \sum_{k=1, k \ne j}^{n} V^{'u}_{2,l,j,k} \ge 1 - M(1 - V^u_{2,j})$$

$$j = 1, \ldots, n; l = 1, \ldots, m; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{s,j} - p^u_{s,j} + M(1 - X^u_{s,j,k}) \ge C^u_{s,k} - p^u_{s,k}$$

$$s = 1, 2; j = 1, \ldots, n; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{s-1,j} + M(1 - X^u_{s,j,k}) \ge C^u_{s-1,k}$$

$$s = 1, 2; j = 1, \ldots, n; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{max} \geq C^u_{2,j} \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$S^u_{1,1,j} + M(2 - X^u_{1,j,k} - Y^u_{1,1,k}) = C^u_{1,k} + h''^u_{1,1,j,k}$$

$$j = 1, \ldots, n; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; u = 1, \ldots, |\mathcal{N}|$$

$$S^u_{2,l,j} + M(2 - X^u_{2,j,k} - Y^u_{2,l,k}) = C^u_{2,k} + h''^u_{2,l,j,k}$$

$$j = 1, \ldots, n; l = 1, \ldots, m; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; u = 1, \ldots, |\mathcal{N}|$$

$$h''^u_{1,1,j,k} \leq M' V''^u_{1,1,j,k}$$

$$j = 1, \ldots, n; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; u = 1, \ldots, |\mathcal{N}|$$

$$h''^u_{2,l,j,k} \leq M' V''^u_{2,l,j,k}$$

$$j = 1, \ldots, n; l = 1, \ldots, m; k = \{1, \ldots, j-1\} \cup \{j+1, n\}; u = 1, \ldots, |\mathcal{N}|$$

$$\sum_{k=1, k \neq j}^{n} V''^u_{1,1,j,k} = 1 \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$\sum_{k=1, k \neq j}^{n} V''^u_{2,l,j,k} = 1$$

$$j = 1, \ldots, n; l = 1, \ldots, m; u = 1, \ldots, |\mathcal{N}|$$

$$S^u_{2,l,j} \leq S^u_{2,w,j} + M(1 - Y^u_{2,l,j})$$

$$j = 1, \ldots, n; l = 1, \ldots, m; w = 1, \ldots, m; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{s,j} \geq 0 \quad s = 1, 2; j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$h^u_{s,j}, h'^u_{s,j}, h''^u_{s,j} \geq 0 \quad s = 1, 2; j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$V^u_{s,j}, V'^u_{s,j}, V''^u_{s,j} \in \{0, 1\} \quad s = 1, 2; j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|$$

$$C^u_{0,j} = 0 \quad j = 1, \ldots, n; u = 1, \ldots, |\mathcal{N}|.$$

To test the performance of the MILPs listed above, the following comparative study is conducted. With the sample size $N = 100$, the SAA model in this paper is denoted as SAA, which is compared with SAA-P1 and SAA-P4 in terms of the CPU time and the worst-case out-of-sample performance. The experimental results are shown in Table B.1,

Table B.1: Performance comparison of different SLP models with $\lambda_1 = \lambda_2 = 0$

m	n	DRO	SAA		SAA-P1		SAA-P4	
		CPU time	CPU time	Worst-case	CPU Time	Worst-case	CPU Time	Worst-case
3	3	0.04	1.27	263.22	1.19	263.22	1.15	263.22
	4	0.10	14.87	407.84	14.63	407.84	10.46	407.85
	5	0.87	173.86	419.58	146.84	419.58	113.30	419.59
	6	5.62	> 1200	480.60	> 1200	480.60	> 1200	480.60
	7	88.21	> 1200	699.60	> 1200	699.60	> 1200	699.60
	8	146.56	> 1200	585.89	> 1200	585.89	> 1200	585.94
4	3	0.03	1.77	315.37	1.76	315.37	1.61	315.38
	4	0.15	20.90	375.86	19.75	375.86	19.22	375.86
	5	0.32	79.11	460.37	78.76	460.37	75.80	460.37
	6	2.03	> 1200	700.99	> 1200	700.99	> 1200	700.99
	7	20.36	> 1200	785.99	> 1200	785.99	> 1200	785.99
	8	209.39	> 1200	840.13	> 1200	840.13	> 1200	840.14
5	3	0.03	3.49	239.52	3.49	239.52	3.41	239.53
	4	0.19	36.13	280.40	36.10	280.40	31.01	280.42
	5	0.63	168.76	473.00	158.06	473.00	135.53	473.00
	6	1.20	813.25	593.19	803.26	593.19	802.24	593.27
	7	4.82	> 1200	633.43	> 1200	633.43	> 1200	633.46
	8	16.02	> 1200	765.17	> 1200	765.17	> 1200	765.34
6	3	0.05	2.65	416.38	2.52	416.38	2.43	416.38
	4	0.35	34.46	496.31	29.26	496.31	27.68	496.32
	5	0.88	162.78	426.98	154.08	426.98	127.62	427.23
	6	8.87	> 1200	521.03	1075.28	521.03	982.56	521.04
	7	9.17	> 1200	606.12	> 1200	606.12	> 1200	606.55
	8	14.57	> 1200	720.83	> 1200	720.83	> 1200	721.32

The underlined items in Table B.1 represent that the solution of the SAA-P4 model in the worst-case scenario is inferior to those of the SAA and SAA-P1 models. When the solving time exceeds 1,200 seconds, the solution is the best possible, rather than the optimal solution. Based on the above table, it is observed that the SAA-P4 model demonstrates the highest computational efficiency. Compared with the SAA model, the SAA-P4 demonstrates an advantage ranging from 1.35% to 34.83%. This advantage is mainly attributed to the effective compression of the solution space by the P4 model, which significantly accelerates the solving speed. Regarding the SAA-P1 model, its solving speed is faster than the SAA model in all but some cases, with a cap of 16% gap. Overall, the CPU times of the three SLP models are quite close, with the SAA-P4 holding a slight edge over the others. Interestingly, it is discovered that the worst-case outcomes obtained by the SAA are consistent with the SAA-P1 model, while the solutions of the SAA-P4 model are not as good. One possible explanation for that is because the solution space of the SAA-P4 model is relatively small and is not an equivalent reformulation of the

DRO model, resulting in the fact that its solutions are not optimal in most of the cases. Nevertheless, this worst-case performance gap is also considerably small. For instance, when $m - n = 6 - 5$, the worst-case solutions of the SAA and SAA-P1 models are both 426.98, and the SAA-P4 model is 427.23, with a gap of only 0.033%.

References

Fernandez-Viagas, V., Perez-Gonzalez, P., Framinan, J.M., 2019. Efficiency of the solution representations for the hybrid flow shop scheduling problem with makespan objective. Computers & Operations Research 109, 77–88.