Notes on Perfectly Matched Layers (PMLs)

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4.2 An example: PML for 2d scalar waves

$$\frac{\partial^2 u}{\partial t^2} = b\nabla \cdot (a\nabla u) \tag{1}$$

$$\Longrightarrow \begin{cases} \frac{\partial^2 u}{\partial t} = b\nabla \cdot \overrightarrow{v} \\ \frac{\partial \overrightarrow{v}}{\partial t} = a\nabla u \end{cases} \tag{2}$$

Let's talk about 2-D in complex domian , so $u(\tilde{x}, \tilde{y}, t)$.

$$\overrightarrow{v} = (v_{\tilde{x}}, v_{\tilde{y}}) \tag{3}$$

$$\tilde{x} = x + if(x) = x + i\frac{\sigma_x}{\omega} \tag{4}$$

$$\tilde{y} = y + if(y) = y + i\frac{\sigma_y}{\omega} \tag{5}$$

Note: Perform the Fourier Transform with respect to time t on $u, v_{\tilde{x}}, v_{\tilde{y}}$.

$$\frac{\partial \tilde{x}}{\partial x} = \left(1 + i\frac{\sigma_x}{\omega}\right)\frac{\partial}{\partial x} \tag{6}$$

$$\frac{\partial \tilde{y}}{\partial y} = \left(1 + i\frac{\sigma_y}{\omega}\right)\frac{\partial}{\partial y} \tag{7}$$

$$\begin{aligned} u &\xrightarrow{\mathcal{F}} \tilde{u}, \quad \frac{\partial u}{\partial t} \xrightarrow{\mathcal{F}} i\omega\tilde{u} \\ v_{\tilde{x}} &\xrightarrow{\mathcal{F}} \tilde{v}_{\tilde{x}}, \quad \frac{\partial v_x}{\partial t} \xrightarrow{\mathcal{F}} i\omega\tilde{v}_{\tilde{x}} \\ v_{\tilde{y}} &\xrightarrow{\mathcal{F}} \tilde{v}_{\tilde{y}}, \quad \frac{\partial v_y}{\partial t} \xrightarrow{\mathcal{F}} i\omega\tilde{v}_{\tilde{y}} \\ \varphi &\xrightarrow{\mathcal{F}} \tilde{\varphi}, \quad \frac{\partial \varphi}{\partial t} \xrightarrow{\mathcal{F}} i\omega\tilde{\varphi} \end{aligned} \tag{8}$$

$$\frac{\partial u}{\partial t} = b\left(\frac{\partial v_{\tilde{x}}}{\partial \tilde{x}} + \frac{\partial v_{\tilde{y}}}{\partial \tilde{y}}\right) \xrightarrow{\mathcal{F}} i\omega\tilde{u}(\tilde{x},\tilde{y},\omega) = b\left(\frac{\partial \tilde{v}_{\tilde{x}}}{\partial \tilde{x}} + \frac{\partial \tilde{v}_{y}}{\partial \tilde{y}}\right) \tag{9}$$

$$i\omega\tilde{u} = b\left(\frac{\partial \tilde{v}_{\tilde{x}}}{\partial \tilde{x}} + \frac{\partial \tilde{v}_{\tilde{y}}}{\partial \tilde{y}}\right) \quad \text{with} \quad \partial\tilde{x} = \left(1 + \frac{i\sigma_x}{\omega}\right)\partial x \quad \partial\tilde{y} = \left(1 + \frac{i\sigma_y}{\omega}\right)\partial y \tag{10}$$

$$i\omega\tilde{u}\left(1 + \frac{\sigma_x}{\omega}\right)\left(1 + \frac{i\sigma_y}{\omega}\right) = b\left(\left(1 + \frac{i\sigma_y}{\omega}\right)\frac{\partial v_{\tilde{x}}}{\partial x} + \left(1 + \frac{i\sigma_x}{\omega}\right)\frac{\partial v_{\tilde{y}}}{\partial y}\right) \tag{11}$$

$$\left(i\omega - \sigma_x - \sigma_y\right)\tilde{u} = b\left(\frac{\partial \tilde{v}_{\tilde{x}}}{\partial x} + \frac{\partial \tilde{V}_{\tilde{y}}}{\partial y}\right) + \frac{ib}{\omega}\left(\sigma_y\frac{\partial \tilde{V}_{\tilde{x}}}{\partial x} + \sigma_x\frac{\partial \tilde{V}_{\tilde{y}}}{\partial y}\right) + \frac{1}{\omega}\sigma_x\sigma_y\tilde{u} \tag{12}$$

Introduce the auxiliary variable φ,let:

$$i\omega\tilde{\varphi} = b\left(\sigma_y\frac{\partial \tilde{v}_{\tilde{x}}}{\partial x} + \sigma_x\frac{\partial \tilde{v}_{\tilde{y}}}{\partial y}\right) + \sigma_x\sigma_y\tilde{U} \tag{13}$$

$$i\omega\tilde{u} = b\left(\frac{\partial \tilde{v}_{\tilde{x}}}{\partial \tilde{x}} + \frac{\partial \tilde{v}_{\tilde{y}}}{\partial \tilde{y}}\right) + (i\omega\tilde{\varphi})\frac{i}{\omega} + (\sigma_x + \sigma_y)\tilde{u} \tag{14}$$

$$i\omega\tilde{u} = b\left(\frac{\partial \tilde{v}_{\tilde{x}}}{\partial x} + \frac{\partial \tilde{v}_{\tilde{y}}}{\partial y}\right) - \tilde{\varphi} + (\sigma_x + \sigma_y)\tilde{u} \tag{15}$$

$$\mathcal{F}^{-1} \quad \Rightarrow \quad \frac{\partial u}{\partial t} = b\left(\frac{\partial v_{\tilde{x}}}{\partial x} + \frac{\partial v_{\tilde{y}}}{\partial y}\right) - \varphi + (\sigma_x + \sigma_y)u \tag{16}$$

$$\frac{\partial \varphi}{\partial t} = b\left(\sigma_y\frac{\partial v_{\tilde{x}}}{\partial x} + \sigma_x\frac{\partial v_{\tilde{y}}}{\tilde{y}}\right) + \sigma_x\sigma_y u \tag{17}$$

$$\frac{\partial v_{\tilde{x}}}{\partial t} = a\frac{\partial u}{\partial \tilde{x}} \quad \xrightarrow{\mathcal{F}} i\omega\widetilde{v}_{\tilde{x}} = a\frac{\partial \widetilde{u}}{\partial \tilde{x}} \tag{18}$$

$$\frac{\partial v_{\tilde{y}}}{\partial t} = a\frac{\partial u}{\partial \tilde{y}} \quad \xrightarrow{\mathcal{F}} i\omega\widetilde{v}_{\tilde{y}} = a\frac{\partial \widetilde{u}}{\partial \tilde{y}} \tag{19}$$

$$\Rightarrow \quad i\omega\widetilde{v}_{\tilde{x}} = a\frac{1}{1 + i\omega\sigma_x}\frac{\partial \widetilde{u}}{\partial x} \tag{20}$$

$$i\omega\widetilde{v}_{\tilde{y}} = a\frac{1}{1 + i\omega\sigma_y}\frac{\partial \widetilde{u}}{\partial y} \tag{21}$$

$$\mathcal{F}^{-1} \quad \Longrightarrow \quad \frac{\partial v_{\tilde{x}}}{\partial t} = a\frac{\partial u}{\partial x} + \sigma_x v_{\tilde{x}} \tag{22}$$

$$\frac{\partial v_{\tilde{y}}}{\partial t} = a\frac{\partial u}{\partial y} + \sigma_y v_{\tilde{y}} \tag{23}$$

Finally, we obtain: (22), (23), (28), and (29), as follows:

$$\frac{\partial u}{\partial t} = b\left(\frac{\partial v_{\tilde{x}}}{\partial x} + \frac{\partial v_{\tilde{y}}}{\partial y}\right) - \varphi + (\sigma_x + \sigma_y)u \tag{22}$$

$$\frac{\partial \varphi}{\partial t} = b\left(\sigma_y\frac{\partial v_{\tilde{x}}}{\partial x} + \sigma_x\frac{\partial v_{\tilde{y}}}{\partial y}\right) + \sigma_x\sigma_y u \tag{23}$$

$$\frac{\partial v_{\tilde{x}}}{\partial t} = a\frac{\partial u}{\partial x} + \sigma_x v_{\tilde{x}} \tag{28}$$

$$\frac{\partial v_{\tilde{y}}}{\partial t} = a\frac{\partial u}{\partial y} + \sigma_y v_{\tilde{y}} \tag{29}$$