Drake Pitts · CMPS 101 · hw3

Each problem is restated and followed by my solution to the problem, indicated by the "$\hookrightarrow$" symbol. For multi-part problems, my solutions are placed directly following the statement of the relevant part and preceeding the statement of the next part.

1. Prove that for all $n \geq 1$: $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Do this twice:

   a. Using form IIa of the induction step.

   $\hookrightarrow$ **Proof.** For all $n \in \mathbb{Z}^+$, let $P(n)$ be the above proposition.

   I. BASE Notice that $\displaystyle\sum_{i=1}^{1} i^3 = 1^3 = 1 = 1^2 = \frac{1^2 \cdot 2^2}{2^2} = \left(\frac{1 \cdot 2}{2}\right)^2$, so $P(1)$ is true.

   IIa. INDUCTION Let $n \geq 1$ be arbitrarily chosen. Assume for this particular n that $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. We must show that $\displaystyle\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$. In the following algebraic manipulations, we start with the left-hand side and derive the right-hand side.

$$
\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= (n+1)^3 + \sum_{i=1}^{n} i^3 && \text{from the definition of summation} \\
&= (n+1)^3 + \left(\frac{n(n+1)}{2}\right)^2 && \text{by the induction hypothesis} \\
&= \frac{4(n+1)^3}{4} + \frac{n^2(n+1)^2}{4} \\
&= \frac{(n+1)^2[4(n+1)+n^2]}{4} = \frac{n^2(n^2+4n+4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}
$$

   So we have shown that $P(n) \to P(n+1)$.

   By the First Principle of Mathematical Induction, it follows that $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ holds for all positive integers n.

   b. Using form IIb of the induction step.

   $\hookrightarrow$ **Proof.** For all $n \in \mathbb{Z}^+$, let $P(n)$ be the above proposition.

   I. BASE Notice that $\displaystyle\sum_{i=1}^{1} i^3 = 1^3 = 1 = 1^2 = \frac{1^2 \cdot 2^2}{2^2} = \left(\frac{1 \cdot 2}{2}\right)^2$, so $P(1)$ is true.

   IIb. INDUCTION Let $n > 1$ be arbitrarily chosen. Assume for this particular n that $\displaystyle\sum_{i=1}^{n-1} i^3 =$

$\left(\frac{(n-1)n}{2}\right)^2$. We must show that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. In the following algebraic manipulations, we start with the left-hand side and derive the right-hand side.

$$
\begin{aligned}
\sum_{i=1}^{n} i^3 &= n^3 + \sum_{i=1}^{n-1} i^3 && \text{from the definition of summation} \\
&= n^3 + \left(\frac{(n-1)n}{2}\right)^2 && \text{by the induction hypothesis} \\
&= \frac{4n^3}{4} + \frac{(n-1)^2 n^2}{4} \\
&= \frac{n^2[4n + (n-1)^2]}{4} = \frac{n^2(n^2+2n+1)}{4} \\
&= \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}
$$

So we have shown that $P(n-1) \to P(n)$.

By the First Principle of Mathematical Induction, it follows that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ holds for all positive integers n.

2. Define $S(n)$ for $n \in \mathbb{Z}^+$ by the recurrence:

$$
S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}
$$

Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

$\hookrightarrow$ **Proof.** For all $n \in \mathbb{Z}^+$, let $P(n)$ be the proposition "$S(n) \geq \lg(n)$".

I. BASE We must show that $S(1) \geq \lg(1)$. This inequality simply asserts that $0 \geq 0$, which is true because $0 = 0$. Hence $P(1)$ is true.

IId. INDUCTION Let $n > 1$ be arbitrarily chosen and assume for all k in the range $1 \leq k < n$ that $S(k) \geq \lg(k)$. We must show that $S(n) \geq \lg(n)$. So

$$
\begin{aligned}
S(n) &= S\left(\lceil n/2 \rceil\right) + 1 && \text{by the definition of } S(n) \\
&\geq \lg(\lceil n/2 \rceil) + 1 && \text{by the IH with } k = \lceil n/2 \rceil \\
&\geq \lg(n/2) + 1 && \text{since } \lg x \text{ is increasing} \\
&= \lg(n) - \lg(2) + 1 \\
&= \lg(n)
\end{aligned}
$$

Hence $S(n) \geq \lg(n)$, so $[P(1) \wedge \cdots \wedge P(n-1)] \to P(n)$.

By the Second Principle of Mathematical Induction, we have $S(n) \leq \lg(n)$ for all $n \geq 1$. Because $S(n) \geq 0$, we conclude that $S(n) = \Omega(\log n)$.

3. Let $T(n)$ be defined by the recurrence formula:

$$T(n) = \begin{cases} 0 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

Show that $\forall n \geq 1 : T(n) \leq \frac{4}{3}n^2$, and hence $T(n) = O(n^2)$.

$\hookrightarrow$ **Proof.** For all $n \in \mathbb{Z}^+$, let $P(n)$ be the proposition "$T(n) \leq \frac{4}{3}n^2$".

I. BASE Since $T(1) = 0 \leq \frac{4}{3} = \frac{4}{3}(1)^2$, we know that $P(1)$ is true.

IId. INDUCTION Let $n > 1$ be chosen arbitrarily and assume for all k in the range $1 \leq k < n$ that $T(k) \leq \frac{4}{3}k^2$. We must show that $T(n) \leq \frac{4}{3}n^2$. We have that

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + n^2 && \text{from the definition of } T(n) \\ &\leq \frac{4}{3}(\lfloor n/2 \rfloor)^2 + n^2 && \text{by the IH with } \lceil n/2 \rceil \\ &\leq \frac{4}{3}(n/2)^2 + n^2 && \text{since } x^2 \text{ is increasing} \\ &= \frac{1}{3}n^2 + n^2 = \frac{4}{3}n^2 \end{aligned}$$

We have now shown that $T(n) \leq \frac{4}{3}n^2$, which means that $[P(1) \wedge \cdots \wedge P(n-1)] \to P(n)$.

From the Second PMI, we have shown that $T(n) \leq \frac{4}{3}n^2$ for all integers $n \geq 1$. It follows that $T(n) = O(n^2)$.

4. Let $T(n)$ be defined by the recurrence formula:

$$T(n) = \begin{cases} 2 & n = 1, 2 \\ 9\,T(\lfloor n/3 \rfloor) + 1 & n \geq 3 \end{cases}$$

Show that $\forall n \geq 1 : T(n) \leq 3n^2 - 1$, and hence $T(n) = O(n^2)$.

$\hookrightarrow$ **Proof.** For all $n \in \mathbb{Z}^+$, let $P(n)$ be the proposition "$T(n) \leq 3n^2 - 1$".

I. BASE Notice that

$$\begin{aligned} T(1) &= 2 \leq 2 = 3 - 1 = 3(1)^1 - 1 && \text{and} \\ T(2) &= 2 \leq 11 = 12 - 1 = 3(4) - 1 = 3(2)^2 - 1 \end{aligned}$$

Hence $P(1)$ and $P(2)$ both hold.

IId. INDUCTION Let $n > 2$ be arbitrarily chosen. Assume for all k in the range $1 \leq k < n$ that $T(k) \leq 3n^2 - 1$. In particular, we have assumed that $T(\lfloor n/3 \rfloor) \leq 3\lfloor n/3 \rfloor^2 - 1$. We must show that $T(n) \leq 3n^2 - 1$.

$$\begin{aligned} T(n) &= 9\,T(\lfloor n/3 \rfloor) + 1 && \text{by the definition of T(n)} \\ &\leq 9\left(3\lfloor n/3 \rfloor^2 - 1\right) + 1 && \text{by the IH with } k = \lfloor n/3 \rfloor \\ &= 27\lfloor n/3 \rfloor^2 - 8 \\ &\leq 27(n/3)^2 - 8 && \text{since } 27x^2 - 8 \text{ is increasing} \\ &= 3n^2 - 8 \\ &\leq 3n^2 - 1 && \text{since } -8 \leq -1 \end{aligned}$$

So we have shown that $[P(1) \wedge \cdots \wedge P(n-1)] \to P(n)$.

By the Second PMI, we conclude that $T(n) \leq 3n^2 - 1$ for all positive integers n, which implies that $T(n) = O(n^2)$.

5. Let $g(n)$ be an asymptotically non-negative function. Prove that $o(g(n)) \cap \Omega(g(n)) = \emptyset$.

   $\hookrightarrow$ Suppose that there exists a function $f(n) = o(g(n)) \cap \Omega(g(n))$. That is, suppose that for all positive constants c_1 there exist positive constants c_2 and n_0 such that the following inequalities hold for all $n \geq n_0$:

   $$0 \leq f(n) < c_1 g(n) \qquad \text{and} \qquad 0 \leq c_2 g(n) \leq f(n)$$

   Combining these, we obtain the chain of inequalities

   $$0 \leq c_2 g(n) \leq f(n) < c_1 g(n)$$

   However, in the case $c_1 = c_2$, these inequalities cannot both be satisfied: $c_2 g(n) \leq f(n)$ cannot be true at the same time as $c_2 g(n) > f(n)$. From this contradiction, we reject our supposition that such a function exists. Thus $o(g(n)) \cap \Omega(g(n)) = \emptyset$.