Each problem is restated and followed by my solution to the problem, indicated by the "$\hookrightarrow$" symbol. For multi-part problems, my solutions are placed directly following the statement of the relevant part and preceeding the statement of the next part.

1. (Problem 3.1-1) Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Using the basic definition of Θ-notation, prove that $f(n) + g(n) = \Theta(\max(f(n), g(n)))$.

   $\hookrightarrow$ By hypothesis, there exists a constant n_0 such that for all $n \geq n_0$, we have $f(n) \geq 0$ and $g(n) \geq 0$. From this it follows that $f(n) \leq f(n) + g(n)$ and $g(n) \leq f(n) + g(n)$ for all $n \geq n_0$, which implies that for all $n \geq n_0$,

   $$\max(f(n), g(n)) \leq f(n) + g(n)$$

   Suppose that on some interval $I \subseteq [n_0, \infty)$, we have $f(n) \geq g(n)$ so that $\max(f(n), g(n)) = f(n)$ and

   $$f(n) + g(n) \leq 2f(n) = 2\max(f(n), g(n))$$

   This argument works regardless of whether $f(n) \geq g(n)$ or vice-versa, so the inequality $f(n) + g(n) \leq 2\max(f(n), g(n))$ holds for all $n \geq n_0$.
   Let $c_1 = 1$ and $c_2 = 2$. We cannot specify an exact value of n_0 since are keeping $f(n)$ and $g(n)$ general, but by hypothesis we know that an appropriate value n_0 exists. Then

   $$0 \leq c_1 \max(f(n), g(n)) \leq f(n) + g(n) \leq c_2 \max(f(n), g(n))$$

   for all $n \geq n_0$. Therefore $f(n) + g(n) = \Theta(\max(f(n), g(n)))$, as required.

2. (Problem 3.1-3) Explain why the statement "The running time of algorithm A is at least $O(n^2)$" is meaningless.

   $\hookrightarrow$ By saying that the running time $f(n)$ of algorithm A is at least $O(n^2)$, we mean that there is a function $h(n) = O(n^2)$ such that $h(n) \leq f(n)$ for all sufficiently large values of n. However, the definition of big-O is such that the class $O(n^2)$ contains functions that grow at a rate commensurate with n^2 as well as all functions that grow at a rate strictly less than that of n^2. This means that the function $h(n)$ can grow arbitrarily slowly, and, in fact, could even be a constant function, so long as that constant is non-negative. This lack of restriction on $h(n)$ makes the inequality $h(n) \leq f(n)$, and thus the statement at hand, meaningless.

3. (Problem 3.1-4) Determine whether the following statements are true or false.

   a. $2^{n+1} = O(2^n)$

      $\hookrightarrow$ (True.) Notice that $2^{n+1} = 2 \cdot 2^n$. Let $c = 2$ and $n_0 = 1$. Then $0 \leq 2^{n+1} \leq c \cdot 2^n$ for all $n \geq n_0$. Hence $2^{n+1} = O(2^n)$.

   b. $2^{2n} = O(2^n)$

      $\hookrightarrow$ (False.) Suppose that this statement is true and there are positive constants c and n_0 such that $0 \leq 2^{2n} \leq c \cdot 2^n$ for all $n \geq n_0$. Then we can divide the inequality by 2^n and obtain $2^n \leq c$ which implies that $n \leq \lg c$. But n can take on any value in $\mathbb{R}^+$ (*e.g.* $n = 1 + \lg c$), so we have derived a contradiction. Hence the supposition is false and no such constants exist. Therefore $2^{2n} \neq O(2^n)$.

4. (Problem 3-2) Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1$, $\varepsilon \geq 0$, and $c > 1$ are constants. Place 'yes' or 'no' in each of the empty cells below, and justify your answers.

$\hookrightarrow$ The justifications for each answer follow.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ε	yes	yes	no	no	no
b.	n^k	c^n	yes	yes	no	no	no
c.	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d.	2^n	$2^{n/2}$	no	no	yes	yes	no
e.	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f.	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

a. We determine the asymptotic relationship between $\lg^k n$ and n^ε analyzing the the limit of their ratio as $n \to \infty$. We assume first that $k \in \mathbb{Z}$, and then prove from this case that the same results hold for $k \in \mathbb{R}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{(\lg n)^k}{n^\varepsilon} &= \lim_{n\to\infty} \frac{\frac{1}{n}k(\lg n)^{k-1}}{\varepsilon n^{\varepsilon-1}} = \frac{k}{\varepsilon}\lim_{n\to\infty}\frac{(\lg n)^{k-1}}{n^\varepsilon} && \text{by L'Hopital's rule} \\
&= \frac{k!}{\varepsilon^k}\lim_{n\to\infty}\frac{1}{n^\varepsilon} = 0 && \text{after } k \text{ applications of L'H.}
\end{aligned}$$

Since the limit is 0, we conclude that $\lg^k n = o(n^\varepsilon)$, which tells us also that $\lg^k n = O(n^\varepsilon)$ and that the other relations are not satisfied. Generalizing to the case $k \in \mathbb{R}$, we remember that $1 \leq k \leq \lceil k \rceil$, which implies that $\lg^k n \leq \lg^{\lceil k \rceil}(n)$. But we just showed (since $\lg^{\lceil k \rceil}(n) = o(n^\varepsilon)$) that for all constants $c > 0$ there exists a constant $n_0 > 0$ such that $0 \leq \lg^{\lceil k \rceil} n < cn^\varepsilon$ for all $n \geq n_0$. Hence, under the same conditions for c and n_0, we have $0 \leq \lg^k n < cn^\varepsilon$, which means by definition that $\lg^k n = o(n^\varepsilon)$. The other relations follow by subset arguments.

b. Suppose for now that $k \in \mathbb{Z}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^k}{c^n} &= \frac{k}{\ln c}\lim_{n\to\infty}\frac{n^{k-1}}{c^n} && \text{by L'Hopital's rule} \\
&= \frac{k!}{\ln^k c}\lim_{n\to\infty}\frac{1}{c^n} = 0 && \text{after } k \text{ applications of L'H.}
\end{aligned}$$

Hence $A = o(B)$ and $A = O(B)$, but $A \neq \omega(B)$, $A \neq \Omega(B)$, and $A \neq \Theta(B)$, for the case when $k \in \mathbb{Z}$. To generalize this to $k \in \mathbb{R}$, we unwrap the definition of little-o notation for $n^{\lceil k \rceil} = o(c^n)$: for every constant $c_1 > 0$ there exists a constant n_0 such that $0 \leq n^{\lceil k \rceil} < c_1 c^n$ for all $n \geq n_0$. Because the inequality $1 \leq k \leq \lceil k \rceil$ holds, we have that $n^k \leq n^{\lceil k \rceil}$ and thus $0 \leq n^k < c_1 c^n$ for all $n \geq n_0$. In other words, $n^k = o(c^n)$ and the other relations follow by subset arguments.

c. The functions in question are $A(n) = n^{1/2}$ and $B(n) = n^{\sin n}$. Since they are both functions defined by taking some power of n, we can compare them by comparing their powers. The power of A is the constant function $1/2$, and the power of B is the sine function $\sin n$, which has range $[-1, 1]$. Because $\sin n$ periodically satisfies $\sin n > 1/2$ and $\sin n < 1/2$ for different values of n, the value of the function $B(n)$ is periodically greater than the function $A(n)$, and the opposite is true as well. For this reason, the functions are incomparable and none of the asymptotic relationships at hand exist between them.

d. The limit of the ratio of the functions is

$$\lim_{n\to\infty} \frac{2^n}{2^{n/2}} = \lim_{n\to\infty} 2^{n/2} = \infty.$$

Therefore $A = \omega(B)$, which implies that $A = \Omega(B)$, $A \neq o(B)$, $A \neq O(B)$, and $A \neq \Theta(B)$.

e. Because $A = n^{\lg c}$, we know that

$$\lg A = \lg\left(n^{\lg c}\right) = \lg c \cdot \lg n = \lg c^{\lg n}.$$

Hence $A = c^{\lg n} = B$. Therefore $A = \Theta(B)$.

f. Stirling's formula tells us that the factorial function $n!$ has the following asymptotic tight bound:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + (1 + \Theta(1/n)))$$

Taking the binary logarithm of both sides gives us

$$\begin{aligned} \lg n! &= \lg \sqrt{2\pi n} + \lg \left(\frac{n}{e}\right)^n + \lg(1 + (1 + \Theta(1/n))) \\ &= \frac{1}{2}\lg(2\pi) + \frac{1}{2}\lg n + n\lg n - n\lg e + \lg(1 + (1 + \Theta(1/n))) \\ &= n\lg n + o(n\lg n) \end{aligned}$$

By a theorem presented in class, $n! = \Theta(n \lg n)$.

5. (Problem 3-4) Let $f(n)$ and $g(n)$ be asymptotically positive functions (*i.e.* $f(n) > 0$ and $g(n) > 0$ for sufficiently large n). Prove or disprove the following statements.

c. Assume $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n. Then $f(n) = O(g(n))$ implies that $\lg(f(n)) = O(\lg(g(n)))$.
$\hookrightarrow$ By hypothesis, there exist constants $c > 0$ and $n_0 > 0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$. Taking the binary logarithm of each expression in this inequality chain gives

$$1 \leq \lg(f(n)) \leq \lg(cg(n)) = \lg(g(n)) + \lg c.$$

Let $c_1 = |\lg c|$. Since $\lg(g(n)) \geq 1$ by supposition, we have $c_1 \lg(g(n)) \geq \lg(g(n)) + \lg c$. This implies that $0 \leq \lg(f(n)) \leq c_1 \lg(g(n))$ for all $n \geq n_0$. Therefore $\lg(f(n)) = O(\lg(g(n)))$.

d. $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$.
$\hookrightarrow$ Consider the functions $f(n) = 3n$ and $g(n) = n$. Then $f(n) = O(g(n))$, but $2^{f(n)} \neq O(g(n))$.

e. $f(n) = O((f(n))^2)$.
$\hookrightarrow$ Suppose the function $f(n) = 1/n$. When we square it, we obtain $(f(n))^2 = 1/n^2$. The the limit of their ratio is

$$\lim_{n\to\infty} \frac{1/n}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n} = \infty.$$

By the limit theorem for little-ω, we have $f(n) = \omega((f(n))^2)$, so $f(n) \neq O((f(n))^2)$.

h. $f(n) + o(f(n)) = \Theta(f(n))$.

$\hookrightarrow$ Let $h(n) = o(f(n))$. Then the ratio

$$\frac{f(n) + h(n)}{f(n)} = 1 + \frac{h(n)}{f(n)} \longrightarrow 1$$

as $n \to \infty$. By the limit theorem for big-Θ, $f(n) + h(n) = \Theta(f(n))$ (because $0 < 1 < \infty$). Since $h(n)$ is an arbitrary function in the class $o(f(n))$, we have $f(n) + o(f(n)) = \Theta(f(n))$, as required.

6. Let $f(n) = \Theta(n)$. Prove that $\sum_{i=1}^{n} f(i) = \Theta(n^2)$.

$\hookrightarrow$ Because $f(n)$ is a linear function, we know that it satisfies the linearity additivity property

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Applying this property to the summation in question gives

$$\sum_{i=1}^{n} f(i) = f\left(\sum_{i=1}^{n} i\right) = f\left(\frac{n(n+1)}{2}\right) = \Theta(n^2).$$

7. Use Stirling's formula to prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

$\hookrightarrow$ From Stirling's formula, we can express the binomial coefficient $\binom{2n}{n}$ as the following:

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}\left(1 + \Theta(\frac{1}{n})\right)}{2\pi n\left(\frac{n}{e}\right)^{2n}\left(1 + \Theta\left(\frac{1}{n}\right)\right)^2} = \frac{4^n\left(1 + \Theta\left(\frac{1}{n}\right)\right)}{\sqrt{\pi n}\left(1 + \Theta\left(\frac{1}{n}\right)\right)^2}$$

Taking the limit of the ratio of this representation for this binomial coefficient with the function $4^n/\sqrt{n}$ yields

$$\lim_{n\to\infty} \frac{4^n\left(1 + \Theta\left(\frac{1}{n}\right)\right)}{\sqrt{\pi n}\left(1 + \Theta\left(\frac{1}{n}\right)\right)^2} \cdot \frac{\sqrt{n}}{4^n} = \lim_{n\to\infty} \frac{\left(1 + \Theta\left(\frac{1}{n}\right)\right)}{\sqrt{\pi}\left(1 + \Theta\left(\frac{1}{n}\right)\right)^2} = \frac{1}{\sqrt{\pi}}$$

which satisfies the inequalities $0 < 1/\sqrt{\pi} < \infty$. By the limit theorem for big-Θ, we have $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$, as required.

8. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Show that

$$\sum_{i=1}^{n} f(i) = \Theta(nf(n)).$$

$\hookrightarrow$ Showing the asymptotic upper-bound relation is quick. We have

$$\sum_{i=1}^{n} f(i) \leq \sum_{i=1}^{n} f(n) = nf(n) = \Theta(nf(n))$$

Hence $\sum_{i=1}^{n} f(i) = O(nf(n))$. Showing the asymptotic lower-bound relation is somewhat less quick. We have

$$\begin{aligned}
\sum_{i=1}^{n} f(i) \geq \sum_{i=\lceil n/2 \rceil}^{n} f(i) \geq \sum_{i=\lceil n/2 \rceil}^{n} f(\lceil n/2 \rceil) \\
&= (n - \lceil n/2 \rceil + 1) f(\lceil n/2 \rceil) \\
&= (\lfloor n/2 \rfloor + 1) f(\lceil n/2 \rceil) \\
&> (n/2 - 1 + 1) f(n/2) \\
&= \frac{1}{2} n f(n/2) = \frac{1}{2} n \Theta(f(n)) = \Theta(nf(n)).
\end{aligned}$$

Thus $\sum_{i=1}^{n} f(i) = \Omega(nf(n))$ and therefore $\sum_{i=1}^{n} f(i) = \Theta(nf(n))$, as desired.

9. Use the result of problem 4 above to give an alternate proof of $\log(n!) = \Theta(n \log n)$ that does not use Stirling's formula.

$\hookrightarrow$ The factorial function is defined by the product $n! = 1 \cdot 2 \cdots (n-1) \cdot n$. By applying the logarithm (of arbitrary base) to this function, we obtain

$$\log(n!) = \log 1 + \log 2 + \cdots + \log n = \sum_{i=1}^{n} \log i$$

We now consider the upper bound of this function. Because $\log n \geq 0$ for $n \geq 1$, we know that

$$\log(n!) = \sum_{i=1}^{n} \log i \leq \sum_{i=1}^{n} \log n = n \log n$$

so that $\log(n!) = O(n \log n)$. The lower bound is proven in the following:

$$\begin{aligned}
\log(n!) &= \sum_{i=1}^{n} \log i \geq \sum_{i=\lceil n/2 \rceil}^{n} \log i \geq \sum_{i=\lceil n/2 \rceil}^{n} \log \lceil n/2 \rceil \\
&= (n - \lceil n/2 \rceil + 1) \log \lceil n/2 \rceil \\
&= (\lfloor n/2 \rfloor + 1) \log \lceil n/2 \rceil \\
&> (n/2 - 1 + 1) \log(n/2) \\
&= \frac{1}{2} n \log(n) - \frac{1}{2} \log \frac{1}{2} = \Theta(n \log n)
\end{aligned}$$

Thus $\log(n!) = \Omega(n \log n)$. It follows that $\log(n!) = \Theta(n \log n)$.