The Evolution of Insurance Pricing

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1. Introduction

The growing sophistication of insurance pricing, particularly for property-casualty insurance and reinsurance risk, has created a proliferation of approaches used in practice. Even within firms, pricing methodologies can vary from line to line, ranging from simplistic expected loss ratio targets to sophisticated return on capital models and even more sophisticated probability transform methods.

This paper begins with an overview of these varied approaches, using a progression from the least sophisticated to the most sophisticated to reinforce the link between these various methods. Limitations of the popular return on capital pricing model will be discussed to motivate exploration of some of the increasingly attractive alternatives.

A unified pricing framework will be presented to highlight the similarities of these alternative pricing methods. Their relationship to well known financial pricing models, including CAPM and Black-Scholes, will be explored as well. The paper will also suggest how each of these models can be calibrated and used in insurance applications.

2. Expected Loss Ratio Targets

The simplest approach to pricing follows an elementary notion that insurers collect premiums up front and then must pay overhead expenses, claims and claim adjusting expenses. Therefore, if for every dollar of premium collected 30% is paid in overhead expenses then a break-even price would require that no more than 70% of the premium is paid on claims and claim expenses (hereafter combined and referred to as "claims").

Pricing under this simple approach is simply a matter of setting premium equal to *expected* claims divided by 70%, where 70% is the *target loss ratio* based on the 30% overhead expense ratio.

$$\text{Premium} = \frac{\text{Expected Claims}}{1 - \text{Expense Ratio}}$$

It should be noted that actuaries have always recognized the uncertainty associated with the claim payments they will make for a given policy (or book of policies) and so the expected value of the claims is used here. When using this rather crude approach to pricing, several alternative approaches might be used to estimate these expected claims:

- Subjective Estimates of Claim Costs – Some actuaries are comfortable judgmentally estimating the claims to be paid during the accounting period and use this directly in the pricing formula. Attention must be paid to whether the estimate reflects the mean (probability weighted average value), median (value that has 50% chance of being too high or too low) or mode (most likely value) of the possible values.
- Historical Estimate of Expected Claims – While this sounds quite antiquated, in many lines of insurance it is still common to estimate expected losses using simple averages of historical losses for similar policies. These so-called *burning costs* were, until recently, used extensively to price catastrophe insurance coverages.
- Distributional Assumptions – Various approaches might be used to make distributional assumptions about total claims, ranging from simple ranges of possible values to specific models of claim frequency and claim severity. Regardless of the approach, the resulting distribution of losses is then used to estimate the expected value (the mean of the distribution) of the claim payments for use in this simple pricing model (see below for methods that consider other characteristics of the distribution).

- Simulation Approaches – When the distributional assumptions become more complex and include interdependencies among different distribution assumptions, it is often easier to derive the claim distribution using simulation techniques.

  A good example is the use of simulation models that are now prevalent for the pricing of property catastrophe risks. These models generate a wide (probability weighted) range of possible catastrophe events, overlay these with estimates of property damage that might be result from the events and then apply estimates of a given portfolio's exposure to these losses based on their policy terms. While such a model may be difficult to describe analytical using specific formulas, the simulation approach allows one to derive the resulting claim distribution nonetheless.

  One commonly overlooked fact is that there is no conceptual difference between simulation models and analytical results based on distribution assumptions. The choice between the two approaches rests solely on mathematical convenience. Producing the resulting claim distributions based on simulations does not produce any more valid or appropriate estimates of the mean (or other characteristics) of the claim distribution. In fact, with insufficient numbers of iterations or insufficient care taken to produce the results, simulation results may be less reliable than analytical results.

The discussion of the various approaches to estimating the expected claim amount highlights an important issue related to pricing models. The distributional output from either analytical or simulation-based models may produce valuable *additional* information to the pricing actuary besides just the expected value of a claim, such as a measure of the potential variability of the outcome or the potential magnitude of an extreme event. However, the resulting premium calculations do not necessarily contain any allowance for these characteristics unless the premium formulas explicitly reflect them. Therefore, in this crude pricing model discussed so far in which we simply load the expected claims for overhead expenses, there is no explicit provision for *risk* merely because distributions were assumed or simulations were used as the basis for estimating the expected value.

Three obvious refinements of the simple loss ratio model include the following:

<u>Profit Loads</u> – Although this was not addressed in the previous discussion, it is quite common to incorporate a target profit load into the expense component in the previous formula. At some point in the evolution of actuarial pricing methods 5% was a common profit load for some lines, though a combination of *ad hoc* or more sophisticated methods are now also used.

<u>Discounted Claims</u> – Another refinement is to reflect the discounted value of the claims rather than the nominal value of the claims. Assuming (for now) that this discounting is done using risk-free rates such as US Treasury rates, this recognizes the economic reality that when writing certain long-tailed liability coverages it is reasonable to charge the policyholder for the expected discounted value of the claims.

<u>Pure Premium Presentation</u> – For simplicity, it is often easier to focus on the *pure premium* without the loading for overhead expenses and express this as:

$$\text{Pure Premium} = \text{Expected Present Value Claims}$$

In the discussions that follow, the focus will generally be on the Pure Premium and the obvious adjustments to reflect overhead expenses will be ignored for convenience.

3. Risk Loaded Premiums

The simple model discussed so far may still be used in many short-tailed, personal lines coverages in which large numbers of relatively homogeneous insureds produce claim results that year after year do not substantially deviate from their expected values. However, in many other lines of business, particularly reinsurance, substantial variation from the expected claim amounts can occur and this potential variation must be reflected in the price.

Pricing models respond to this reality by incorporating explicit *risk loads* into the premium. The premiums are therefore higher when the potential deviations from the expected value are higher. Although many practitioners combine the risk load as defined here with the traditional profit load discussed earlier, it is best to consider this a *risk load* when it is based on a measure of risk and as a *profit load* when it is not risk-based.

Conceptually then, pure premium calculations can be depicted as:

$$\text{Pure Premium} = \text{Expected Present Value Claims} + \text{Risk Load}$$

In practice there are a wide variety of methods to estimate risk loads. Some of the most widely used risk load methods can be summarized as follows:

<u>Variance Based Risk Load</u>

One method of determining risk loads is to use a multiple of the variance of the claim amount distribution, reflecting the common-sense notion that policies with greater potential deviation from the expected claim amounts should contain a larger risk load. The resulting pricing formula is then given as follows:

$$\text{Pure Premium} = \text{Expected Present Value Claims} + \lambda * \text{Variance(Claims)}$$

In this calculation, the lambda variable (λ) represents the multiple to be applied to the variance of the claim distribution and is often assumed to be constant across businesses. The Insurance Services Office (ISO) used this methodology at one point to derive their published pure premium estimates.

One feature of this approach is that it reflects an adjustment to the pure premium that depends solely on the risk characteristics of the individual policy and can therefore be applied to all companies regardless of their existing portfolio. Whether this is an advantage or a disadvantage depends on one's perspective.

<u>Standard Deviation Based Risk Load</u>

As a closely related alternative, some actuaries prefer to use the claim distribution's standard deviation as opposed to its variance[1]. Using the standard deviation as the basis for the risk load results in the following alternative formula:

$$\text{Pure Premium} = \text{Expected Present Value Claims} + \lambda * \text{Std.Dev(Claims)}$$

This approach, like the variance based approach, also relies solely on the characteristic of the claim distribution and can therefore be applied consistently, using a constant λ parameter.

<u>Covariance Based Risk Load</u>

Rodney Kreps has proposed a pricing methodology that does not rely solely on the claim distribution. Instead, Kreps' method reflects the *marginal* impact of the new policy on the total portfolio's risk (and hence the total surplus requirement).

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[1] The variance and the standard deviation based risk loads seem to focus on the wrong measure of risk. They focus on the probable range of claim outcomes based on the assumed claim distribution and the assumed parameters of this distribution, a quantity commonly referred to as the *process risk*. While the process risk might be of concern to the insured and may impact his or her willingness to pay a given premium, an insurer writing many similar policies is less exposed to the process risk, in the aggregate. More significant sources of risk for the insurer are the risk that wrong distributional form is used (*model risk*) or the risk that the parameters of the distribution have been incorrectly estimated (*parameter risk*), since these could potentially impact many or all such policies. See Feldblum for further discussion of this point.

Kreps' fundamental result relies on a particular manner of determining the lambda parameter in the previously discussed standard deviation based risk load formula, where Kreps used the term *reluctance*, denoted $\Re$, to define his constant:

$$\text{Pure Premium} = \text{Expected Present Value Claims} + \Re * \text{Std.Dev(Claims)}$$

Although the formulas Kreps derived for his reluctance parameter involve rather complex notation, they are related to items such as the yield on the investment portfolio (y), the target solvency probability (z), the correlation of the policy with the existing portfolio (C), the standard deviations of the existing policies (S) and the standard deviation of the new policies (σ).

The resulting formula for the reluctance parameter, $\Re$, in the case where the standard deviation of the new policies is substantially smaller than the standard deviation for the existing policies is approximated as follows:

$$\Re = \frac{yz}{1+y}\left[C + \frac{\sigma}{2S}\right]$$

An interesting observation made by Bault is that when the new policy is much smaller than the existing portfolio (as in the case Kreps presents) and is also uncorrelated with the existing portfolio, the reluctance term is approximately proportional to the standard deviation of the individual claim distribution. This makes the risk load proportional to the variance of the new claim distribution. At the other extreme, if the new policy is much smaller than the existing portfolio and perfectly correlated with the existing policies, the reluctance term is approximately a constant and therefore the risk load is proportional to the standard deviation of the new claim distribution.

All other cases fall between these extremes. Bault therefore argues that Kreps's risk load can actually be written as a function of the covariance between the new policy's claim distribution and the existing portfolio's claim distribution.

Kreps' resulting formula for the pure premium can be depicted approximately as follows:

$$\begin{aligned}\text{Pure Premium} &= \text{Expected Present Value Claims} + \Re\sigma \\ &= \text{Expected Present Value Claims} + \lambda * \text{Covariance}(l, L)\end{aligned}$$

where, l represents the claims for the new policy and L represents the claims for the existing policies.

The Kreps approach is more theoretically grounded than either the variance or standard deviation methods, both of which are relatively *ad hoc*. While both of those methods can be seen as special cases of Kreps' formulation, neither provides any guidance regarding the appropriate constant to be applied to the risk measure. The Kreps approach shows that this constant is a function of the investment yield, the solvency probability, the correlation coefficient between the existing and new policies and the standard deviations of the existing and new policies.

Note though that measuring covariances of new policies with existing policies is not trivial. This is one reason why many actuaries continue to use either the variance or standard deviation formulations as opposed to the covariance formulation. Either of these risk measures can be calculated using solely the modeling assumptions required for the specific risk, rather than an estimate of its impact and covariance with the entire portfolio. But given the close relationship among all three approaches, it is sufficient to recognize these as nearly identical approaches, at least conceptually.

4. Discounted Cash Flow Models Using Risk-Adjusted Discount Rates

An alternative to the Kreps methodology (or the closely related variance and standard deviation based approaches) some actuaries prefer an approach in which the pure premium is determined as simply the discounted value of the expected claims. However, the key element of this discounting is the fact that the discount rate is no longer assumed to be the risk free rate. Instead, the discount rate is *risk-adjusted* in a manner that results in a discount rate <u>lower</u> than the risk free rate[2]. This has the desired effect of increasing the pure premium (relative to the risk-free present value of the expected claims) without the use of an explicit risk load as a separate term.

The formulas used can be written as:

$$\text{Pure Premium} = \frac{\text{Expected Nominal Claims}}{1 + \text{Risk Adjusted Rate}}$$

$$\text{Risk Adjusted Rate} = \text{Risk Free Rate - Risk Adjustment}$$

This approach is closely tied to the method of valuing risky cash flows often presented in introductory finance textbooks. As a result, a variety of pricing models have been derived that appear more sophisticated than those discussed above (see, for example, the Myers-Cohn and Fairley insurance pricing models).

Nonetheless, there are some important weaknesses of this approach, including the following:

- Discount vs. Risk Adjustment – Despite the manner in which textbooks often present the valuation risky cash flows, it is not always appropriate to combine the risk adjustment and the discounting into a single step.

  Leigh Halliwell has presented an excellent example of the dangers using an example similar to the following scenario. Suppose an insurer wrote an insurance policy in which a coin is flipped in 1 year and they either pay a claim of $100 if heads appeared or they pay a claim of $0 if tails appeared. What would be the value today of this policy, assuming risk free interest rates are 5%? In other words, what should the insurer charge someone in order to assume this risk?

  The expected cash flow is $50. If that amount were discounted at the risk free rate, the pure premium would be $50/1.05 = $47.62. However, since this is a *risky* cash flow, the insurer might prefer to use a lower discount rate to reflect a risk load. Suppose that 2% is an appropriate discount rate, which is 3% lower than the risk free rate. In that case, the pure premium would be $50/1.02 = $49.02.

  This $49.02 pure premium reflects the risk free value of the potential claim payment equal to $47.62 and a risk load of $1.40. The use of a risk-adjusted discount rate (2% in this case) is therefore conceptually no different than the use of an explicit risk load.

  The problem with this method becomes clear when we change the situation slightly to reflect an instantaneous flip of the coin and an immediate payment. Here, the expected value is the same and the *risk* is the same as well. But there is no discounting for the time value of money, so there is no basis from which to reflect the risk adjustment in the discount rate. Given the infinitesimally small time until payment, there is no meaningful risk adjustment one could apply to the risk free discount rate to provide a risk-adjusted premium.

  What worked well in a simple case with one year to expiration doesn't work at all when the time dimension disappears. This suggests that it may be best to reflect risk adjustments explicitly in the cash flows rather than in the discount rate.

- Basis for Risk Adjusted Discount Rate – Some are undaunted by the previous argument in favor of separating the risk load from the discount rate. This leads to the next challenge – deriving a methodology for determining the proper risk adjusted discount rate.

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[2] See Butsic for a detailed discussion.

Some authors advocate the use of the Capital Asset Pricing Model (CAPM) as the basis for risk-adjusted discount rates. But while the CAPM is commonly used to estimate *risk-adjusted* rates of return in a variety of applications, its notion of "risk" is often very different than practitioners recognize. Briefly stated, CAPM (and many other similar models, such as Arbitrage Pricing Theory and the Fama-French 3-Factor Model) makes a critical distinction between systematic risk and non-systematic risk.

Systematic risk is the risk that cannot be diversified away by spreading a fixed investment amount over multiple imperfectly correlated investment opportunities. When a particular investment (or risk-taking activity) is viewed in the context of its systematic risk only, the return models commonly used in financial theory textbooks simply argue that the expected return from a risky activity is related, often linearly, to the activity's measure of systematic risk. Although investments may also contain other diversifiable risks, these risks will not be priced in a competitive financial market as no single investor would worry about its influence on their total portfolio risk. By definition, the risk can be diversified away and so nobody needs to reflect this risk in their pricing.

The CAPM measures systematic risk as the covariance[3] of a particular investment's return with the "market portfolio" of all risky assets, which is a theoretical construct that is often proxied using the S&P 500, or a similarly diversified portfolio of equities.

This definition seems to suggest that to use CAPM for insurance, one could estimate the covariance of the *policy's* return with the market portfolio. However, this leads to numerous errors in logic, including failed attempts to use reported accounting results to measure CAPM accounting betas. It is also troubling that the CAPM focuses on *rate of return* because this is so poorly defined in an insurance pricing context, a topic we will discussed in the next section.

But another way to describe the CAPM is that it uses the market portfolio as an index of good and bad times. Investors are willing to pay more than the expected value for an asset whose payoffs are high in bad times and low in good times[4]. Such an asset, because it costs more, will have a low expected rate of return. Because this asset has high payoffs in bad times (when the market is down), and low payoffs in good times (when the market index is up), the covariance with the market index will be low. This will lead to a low "CAPM beta" and the CAPM result that low beta assets have low expected returns is obtained.

With this formulation, one can ask whether the "market portfolio" is really the right index of good/bad times for an insurer to use when pricing its policies. Feldblum argues that it is not the appropriate index and instead proposes that the index be based solely on insurance industry profit margins. His argument is based on tax and regulatory restrictions on the asset portfolios of insurers, but his conclusions are consistent with an argument that a new index must be created in order to formulate an "Insurance CAPM".

- Systematic vs. Non-Systematic Risk – The standard CAPM argument is that only systematic risk matters for the pricing of financial risk. Other sources of risk and variability can be diversified away by investors and so these do not affect the market price. But this argument is often counterintuitive to practitioners and is increasingly being challenged by academics. Authors including Froot & Stein, Froot, Perold, Stulz and Schimko have argued that when applied in these settings, the return model must

[3] Technically, the covariance is scaled by the variance of the market portfolio. The scaled covariance is then referred to as the CAPM Beta.

[4] Further details of this interpretation of CAPM are provided later in this document.

reflect sources of risk that might otherwise be considered non-systematic. In essence, their argument resembles Feldblum's argument that the index used should not be solely the market portfolio.

An interesting observation regarding the use of risk-adjusted discount rates to derive the pure premium is that this is not fundamentally different than the variance, standard deviation or covariance based risk load methods discussed in Section 3. However, the *rate of return* focus often obscures this fact. For instance, the version of CAPM that practitioners are most often familiar with is:

$$E(r_i) = r_f + \beta_i \left[E(r_m) - r_f\right]$$

where $\beta_i = Cov(r_i, r_m)/\sigma_m^2$.

This formula indicates that the expected rate of return is related to the risk free rate plus a risk premium that is determined as the product of the beta and the equity risk premium. However, an equivalent expression for CAPM can be specified in terms of the price, *p*¸ of a risky cash flow, *x*, rather than its expected return, as follows[5]:

$$p = \frac{E(x_i)}{1+r_f} - \lambda Cov(x_i, r_m)$$

This states that the under the CAPM, the risk-adjusted value of a cash flow is the risk free present value of the expected cash flow less a risk-adjustment equal to a constant, λ, times the covariance of the cash flow and the rate of return on the market portfolio.

Recall that the Kreps risk load formula could be written in terms of the covariance of the policy in question with the firm's overall cash flows. This shows that the two formulas are essentially the same. The difference exists solely because the CAPM measures the risk in terms of the covariance with the market returns (and other CAPM assumptions result in a specific definition of the λ parameter) while the Kreps formula measures risk in terms of the covariance with the firm's existing portfolio:

$$\text{Kreps Pure Premium} = \text{Expected Present Value Claims} + \lambda * \text{Covariance}(l, L)$$

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[5] To derive this alternative form of CAPM, note that if the cash flow is x_i and the price is p, then the rate of return is simply $r_i = x_i / p - 1$. From this, it is easy to derive the following alternative definition of beta:

$$\beta_i = Cov(r_i, r_m)/\sigma_m^2 = Cov(x_i, r_m)/(p\sigma_m^2).$$

The formula for p can be then be written as the expected value of x_i discounted at the risk-adjusted rate:

$$p = \frac{E(x_i)}{\text{Risk-Adjusted Discount Rate}} = \frac{E(x_i)}{1 + r_f + \dfrac{Cov(x_i, r_m)}{p\sigma_m^2}\left[E(r_m) - r_f\right]}$$

When this equation is solved for p, it simplifies to:

$$p = \frac{E(x_i) - \dfrac{\left[E(r_m) - r_f\right]}{\sigma_m^2} Cov(x_i, r_m)}{1 + r_f}$$

$$p = \frac{E(x_i)}{1+r_f} - \lambda Cov(x_i, r_m)$$

Notice that the second to last equation is sometimes referred to as the "certainty equivalent" form of CAPM. It reflects a risk-adjusted cash flow in the numerator, which is then discounted at the risk free rate (as if the cash flow were certain and not risky).

5. Return on Capital Models

One criticism of the discounted cash flow models, which rely on risk-adjusted discount rates, is the inability of these methods to explicitly reflect the amount of capital required to support the policies. As a result, it is difficult to tie the results back to the total return required by the firm's shareholders.

This criticism leads some actuaries to favor models that attempt to attribute (or "allocate") capital to particular lines of business or policies. Then, the price needed to achieve a target return on this capital can be solved for in a relatively straightforward manner.

Background

It is useful to begin this discussion with a more complete description of how this approach is typically applied. This will make it easier to focus the later discussion on a critical weakness of this method.

- Cash Flow Estimates – Return on capital models are often, but not always, applied in a cash flow context in which specific cash flows are estimated. This is not a unique element of these models, since all of the methods discussed so far require the estimation of the expected amount and timing of the underlying policy cash flows.

  However, a unique aspect of most return on capital models is the explicit modeling of the full distribution of cash flows. Unlike the other models that rely solely on the expected value of the cash flows, return on capital models require explicit recognition of either the cash flow variability or estimates of "tail" outcomes representing a worst case result[6].

  It was noted earlier that simulations are often performed to estimate expected values in many of the pricing models already discussed. Here, these simulation models become more valuable because of their ability to more readily quantify the variability and tail outcomes.

- Capital Attribution – After estimating the distribution of cash flows, some standard must be adopted to determine the amount of capital needed to support the policy. A variety of methods have been developed, ranging from the simplistic use of premium to surplus all the way to sophisticated option pricing frameworks. Some examples of these approaches include:

  - Premium to Surplus Ratios – Early return on capital models defined the capital required to support a policy as simple ratios of premium to surplus. This approach was tied to early regulatory models of capital requirements and often relied on judgment to vary the ratios by line of business.

  - Risk-Based Capital – In the early 1990's, the NAIC and various credit rating agencies developed risk-based capital models that added sophistication to the process of attributing more capital to businesses with more variable cash flows. While these models emphasized aggregate capital requirements for the total firm, most were readily adapted to attribute capital requirements to individual lines of business or policies.

  - Probability of Ruin Models – Actuaries also use probability of ruin frameworks that set capital requirements such that the probability of "ruin" is at or below some threshold level, usually a value such as 1.00% or lower. These models parallel more sophisticated Value at Risk models used in banking, though interestingly in order to make such models practical, significant simplifications are often required.

[6] Technically the true "worst case" is rarely, if ever, used for this purpose. The values used generally reflect a "very bad case" defined as being something like the 99th percentile of the distribution.

- Marginal Capital Models – A problem with many of the capital models, such as those that are based on probability of ruin, is that the lack of perfect correlation across policies or across businesses means that the capital requirements for the firm are less than the sum of the stand-alone capital requirements for all of the respective sub-categories. As a result, models have to be used carefully to ensure that a policy being evaluated is assigned only its *marginal* contribution to the firmwide capital requirement.

  Several approaches have been proposed to attribute the firmwide capital requirement back to individual sub-categories, including the Myers-Read method and the RMK algorithm.

  The Myers-Read method is based on the Expected Policyholder Deficit risk measure[7] and results in additive capital requirements, though it assumes that the shape of the loss distributions are unchanged when the volume is increased or decreased, which is unrealistic for most lines of insurance[8].

  The RMK algorithm, which was derived separately based on "Conditional Risk Charges" of Ruhm and Mango and "Riskiness Leverage Models" of Kreps, is particularly easy to apply in a simulation setting and is flexible enough to be used with a variety of risk measures. The RMK algorithm provides a mechanism to attribute the total capital required to each sub-category based on its contribution to the scenarios used to measure the aggregate risk.

A variety of other approaches exist, but they differ from those discussed above in ways that are not material for the present purposes. At this stage it is sufficient to note that various methods can be used to determine the "capital" needed to support a given policy or line of business. This is typically measured on an aggregate basis for the whole firm and then attributed to each policy or line.

An important detail to note is that a portion of the "capital" required will come from the policyholder in the form of insurance premiums. Therefore the return on capital measure must be net of this contribution. In practice, the circularity caused by the fact that the required capital depends on the premium charged, which is itself a function of the required capital, is often ignored.

- Holding Period of Capital – Once the capital required for the new policy is determined, care must be taken to determine how and when this capital will be released. Will it be held until final policy maturity? Can it be expected to be released as the risks associated with the policy decline? Will the release be dependent on the experience? Does the release of capital depend upon the *future* marginal contribution of the remaining policy risk to the overall firm risk? What happens if losses on the policy exceed the capital "committed"?

  These turn out to be extremely challenging questions, which we will explore a bit later on in this discussion.

- Return Targets – Return targets are used to convert the capital requirement for the policy and its holding period to profit loads. The basis for these return targets is quite often arbitrary, for reasons that will be made clear later in this discussion.

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[7] The method can be more accurately described as being based on the value of the firm's insolvency put option. However, under certain conditions, the expected policyholder deficit can be assumed to be identical to the value of the insolvency put.

[8] See Mildenhall's discussion for details.

Example

A very simple example of this approach can be shown here.

Suppose an insurer were to write a one-year policy with an expected claim payment of \$100. The standard deviation of the claim amount is \$25. If the claim amount is normally distributed, then one could argue that a "worst case" loss (at the 99th percentile level) is \$158.16.

Assuming the risk free rate is 4%, the discounted pure premium is \$100/1.04 = \$96.15, which will grow to \$100 by the time the claim payment must be made. In a worst case scenario, where the claim payment is in fact \$158.16, the insurer will need an additional \$58.16 of assets. This would have to be funded today with \$58.16/1.04 = \$55.92, assuming that this is the only policy written and that the policyholder will allow the firm to be capitalized such that there is a 1% probability the full claim payment cannot be made. In other words, there is a 1% probability that the claim will exceed \$158.16, in which case there will be insufficient funds to pay the full claim.

In this case, the holding period for the required capital is merely one year. To determine the pure premium for the policy, a rate of return is assumed to be required on the \$55.92 of capital contributed by the firm's shareholders and the dollar value of the required return (the risk load) is added to the (risk free) discounted value of the expected claim. Because the expected claims are covered in the pure premium, the expected return on capital is simply the risk load divided by the capital commitment.

Suppose the return target is 20% and that the risk load charged to the policyholder will be invested to earn the risk free rate of 4%. The resulting risk load is then 20%*\$55.92/1.04 = \$10.75.

The risk-adjusted pure premium is then \$96.15 + \$10.75 = \$106.90.

Of course, if there were multiple holding periods the risk load would be adjusted accordingly to ensure that the target rate of return on capital is earned in *each* period for which the risk capital is held.

Implementation of Return on Capital Models

This pricing method is seemingly more sophisticated than the other methods, which might explain its widespread acceptance among practitioners. More importantly though, the methodology is quite intuitive and results in risk loaded prices for insurance coverages that explicitly reflects relative risk, however broadly or narrowly "risk" is defined.

However, there are some subtle issues that must be addressed by practitioners implementing this method:

- Target Rate of Return – Where should the target return come from? Presumably the methods mentioned earlier to determine the risk-adjusted discount rate could be used, but that will introduce the weaknesses of that method into this model.

  In addition, care is needed to ensure that the arbitrary choices made to calculate and attribute the capital to each policy or line of business also are taken into account when determining the target rate of return. For instance, suppose that instead of the calculations discussed above the 99.9% quantile had been chosen as the "worst-case" outcome. Then, the capital required would have increased to \$74.29, which is \$18.37 more capital than before.

  Should this additional \$18.37 of capital still have to earn a 20% return? Is it equally exposed to the same risk that the \$55.92 of capital is exposed to? Notice that the \$74.29 of capital is sufficient to pay all claims 99% of the time. Therefore, there is only a 1% chance of dipping into any portion of the \$18.37 of additional capital and only a .1% chance of requiring the entire \$18.37 of additional capital to pay claims. Clearly this additional capital is less "at risk" than the other capital.

This shows that the target rate of return, however determined, would need to take into account the varying amounts of overall average return requirements, depending upon the methods and assumptions used to determine the required capital.

- Leverage – Assuming that the insurer only needs some arbitrary level of allocated capital to write a policy essentially leverages the return on capital calculation by dividing the expected profit by a smaller amount of capital than is contractually at risk.

  When *real* leverage is possible through non-recourse borrowing, it is appropriate to adjust the *hurdle rate* to account for the additional risk to the equity investor. However, in an insurance pricing context, the leverage is different. Insurers write many policies and therefore *all* of their capital stands behind each policy. In the event that the claim exceeds the capital *allocated* to the policy, the leverage effect doesn't really exist. Since the return on the allocated capital metric arbitrarily ignores the worst outcomes (and many not-so-bad ones too) without the ability to literally walk away from paying those claims, the leveraged ratio loses its usefulness.

For these and other reasons, numerous authors have concluded that return on capital models are inappropriate for the task of pricing insurance. A sample of some of these authors' comments includes the following:

- Butsic[9] – "*The role of capital in pricing cannot be as the denominator in a return on equity calculation…Although ROE as a profit measurement may be useful at the level of the entire insurance company, we conclude here that the ROE is an inadequate and potentially misleading profitability measure at the line of business level and below. For determining fair prices for lines, policies and individual layers of insurance, a present value method is more appropriate*."
- Kreps[10] – "*While the author does not particularly advocate allocating capital to do pricing, this class of models does allow pricing at the individual policy clause level, if so desired.*"
- Venter[11] – "*But there is no theory to suggest that equalizing the return on this capital – or that from any other risk-measure's allocation - would produce appropriate by-line pricing…This could be a problem with the entire enterprise of allocating capital by a logical but arbitrary measure then pricing to equalize return on that capital*."

6. Economic Value Added

Many of the problems with return on capital models stem from the fact that they attempt to use a rate of return framework for something that is inherently not conducive to the calculation of rates of return. Individual insurance policies do not literally require that capital be allocated in any real sense and the capital attributed to the policy does not in any way limit the claims that could be paid. As such, return on capital is an intuitive but perhaps inappropriate concept to apply to insurance.

The concept of allocated capital can be salvaged, to some degree, by treating the allocated capital merely as a mechanism to allocate the cost of risk and the frictional costs of holding capital to business units or policies.

This approach merely reflects an additional source of costs in a standard discounted cash flow model. The cost is based on the *marginal* capital allocated and a charge for the costs of this capital. A host of issues arise in quantifying this cost and practice varies. Some argue that

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[9] Butsic, Robert, "Capital Allocation for Property-Liability Insurers: A Catastrophe Reinsurance Application".

[10] Kreps, Rodney, "Riskiness Leverage Models".

[11] Venter, Gary, "Discussion of 'Capital Allocation for Insurance Companies' by Myers and Read".

the only quantifiable cost, at least on a risk-adjusted basis, of holding capital stems from the double taxation of investment income. Others prefer to incorporate a systematic risk load in this cost as well.

Mango argues that what really should be allocated is the value of each business unit's (or each policy's) ability to call upon the total firmwide resources when needed. Sherris argues that what should be allocated is the business unit's contribution to the overall value of the firm's *insolvency put option* – their ability to default on some or all of their policyholder obligations.

7. Probability Transform Methods

A more recent innovation in insurance pricing, and I believe the most promising, has two important elements. The first is to abandon the so-called rate of return paradigm and focus instead on the fair *value*, **in dollars**, of an insurance policy. The second is to borrow the logic of modern option pricing and avoid the use of the discount rate as the basis for risk-adjusting the present values of cash flows.

Background

Although option pricing theory will not be reviewed in detail in this section[12], it is worthwhile to consider a few key highlights of this theory.

Option pricing and insurance pricing share numerous commonalities. Prior to the breakthroughs of Black, Scholes and Merton in the early 1970's, economists struggled with the proper pricing of options for essentially the same reason that actuaries have struggled with the pricing of insurance. The development of CAPM in the early 1960's put many financial economists on a path in which they saw risk-adjusted rates of return as the most elegant way to price securities. Unfortunately, the leveraged nature of options made the standard CAPM inappropriate and a lot of effort was spent trying to determine the proper risk-adjusted rate of return for an option.

Black, Scholes and Merton's insight was to develop an approach to pricing that did not require the use of a risk-adjusted rate of return to discount the expected option payoffs. Instead, arbitrage arguments were used to value options relative to the prices of the underlying asset and risk free bonds. Later, Cox, Ross & Rubenstein showed that the Black-Scholes option pricing formula could be derived as the limiting case of a simple binomial model. Their simplified approach emphasized the insight that option prices can be derived using transformed, or risk-adjusted, probabilities.

Prior to this development, securities were priced by discounting the expected cash flows at risk-adjusted discount rates. Now, options and other complex securities are priced by adjusting the probability distribution to calculate *risk-adjusted* expected cash flows analogous to what some refer to as "certainty equivalent" cash flows (see discussion in the next section). Then, these risk-adjusted cash flows are discounted using risk-free interest rates.

It is the use of risk-free interest rates in the last step that has resulted in the use of the term *risk-neutral* pricing to describe this method. This is unnecessarily confusing terminology, since it is often misinterpreted as suggesting that risk is ignored or that risky cash flows can be valued using risk-free discount rates. Discounting can done using risk-free rates only because the risk adjustment has been incorporated into the probability distribution used to calculate the expected payoffs[13]. The resulting price is in fact a risk-adjusted price, equivalent in virtually every respect to our risk-adjusted pure premium.

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[12] See Hull, *Options, Futures and Other Derivative Securities*, for a thorough summary of modern option pricing theory.

[13] Leigh Halliwell has suggested that the resulting adjusted probabilities are more accurately referred to as "risk neutralized" rather than "risk neutral".

It is also worth noting that despite the volumes of research that has been conducted on the theory of option pricing in the past 30 years, the rate of return (i.e. return on capital) framework has not been resurrected. Option pricing models are specified on a dollar value basis (i.e. in units of dollars) and return on capital measures rarely, if ever, mentioned in option pricing textbooks or by option traders[14].

Application to Insurance

The reason this risk-neutral approach works so well for traded options is that it essentially prices the option relative to the prices of the underlying asset and risk-free bonds. The method does not ignore the risk of the option. Instead, it merely allows the price of the underlying asset to incorporate the risk adjustment and then values the option in relation to that risk-adjusted price.

When the same logic of pricing with probability transforms is applied to cash flows that are contingent upon the value of underlying assets for which risk-adjusted market prices do not exist, there is not a unique solution as in the case of options on marketable securities in liquid markets. Insurance is unfortunately one of these applications where such a price rarely exists.

Transform Methodology Example

Despite the lack of a unique solution, the conceptual foundation of the method can still be applied. There are multiple theoretically valid probability transforms and one is forced to choose among them based on empirical and practical considerations.

In essence, the probability transform merely has to re-weight the outcomes so that their contribution to the risk-adjusted expected value is either greater or smaller than their true probabilities suggest. Consider the following highly simplified, and completely hypothetical, example which will serve to simply demonstrate the mechanics.

Suppose there are four equally likely outcomes with the following probabilities and assume that the risk free rate is 4%. Also assume that a separate analysis has suggested that the proper risk-adjusted discount rate is 15%. Using the methods discussed previously, the risk-adjusted value of this set of cash flows can be found to equal 217, as shown below:

Cash Flow (CF)	Probability (P)	CF * P
100	0.25	25
200	0.25	50
300	0.25	75
400	0.25	100
	Expected Cash Flow	250
	Risk-Adjusted Discount Rate	15%
	Risk-Adjusted Value	217

The value of 217 was obtained by discounting the expected value of 250 by the risk-adjusted discount rate of 15%, obtaining 250/1.15 = 217.

A mathematically equivalent approach would risk-adjust the probabilities, rather than risk-adjust the discount rate, and then discount the expected cash flows at the risk free rate. This approach to the same example shown above would proceed as follows:

[14] In certain instances, return on risk capital, return on regulatory capital or return on required margins may be used as *constraints* to select from among a set of otherwise fairly priced options. But this practice should be distinguished from the use of those metrics to actually derive the prices of these instruments.

Cash Flow (CF)	Risk-Adjusted Probability (P)	CF * P
100	0.33	33
200	0.26	53
300	0.23	68
400	0.18	73
	Risk-Adjusted Expected Cash Flow	226
	Risk-Free Discount Rate	4%
	Risk-Adjusted Value	217

What this example demonstrates is that risk-adjusted values do not require the use of risk-adjusted rates of return. Instead, they can use "expected" cash flows discounted at the risk free rate, so long as the expected value calculation uses *transformed* probabilities that give more weight to bad outcomes and less weight to good outcomes. This alternative approach is appealing for the reasons noted earlier in the brief discussion of Halliwell's arguments against risk-adjusted discount rates.

Of course, that simplified example did not address the appropriate method for risk-adjusting or transforming the probability. In recent years, there has been significant research aimed at addressing this question. Some methods have been suggested and tested against market prices for insurance coverages, with interesting results[15]. For instance, the Wang Transform[16] has been shown to approximate market prices for catastrophe bonds. Venter, Barnett and Owen tested various transforms and found that the minimum entropy martingale transform fit fairly well to catastrophe reinsurance pricing data.

8. Utility Theory

A closely related approach to the probability transform method has a more direct basis in Utility Theory. The concept of utility underlies many of the methods discussed so far though they are rarely described in this fashion. For clarity, the pricing approaches that are commonly classified as utility approaches are those that risk-adjust the *cash flows* rather than either the probabilities or the discount rate.

The example shown in the previous section could have been done as follows:

Original Cash Flow (CF)	Risk-Adjusted Cash Flow (CF)	Probability (P)	CF * P
100	132	0.25	33
200	210	0.25	53
300	271	0.25	68
400	291	0.25	73
		Risk-Adjusted Expected Cash Flow	226
		Risk-Free Discount Rate	4%
		Risk-Adjusted Value	217

Notice that the cash flows were risk-adjusted in this example and the resulting expected value was discounted at the risk free rate. This produced the same results as were obtained when using a risk-adjusted discount rate or using transformed probabilities[17].

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[15] See Venter for an early discussion of this concept, which he referred to as exponential tilting. A thorough treatment can also be found in the "Premium Principles" chapter of the *Encyclopedia of Actuarial Science*.

[16] The Wang Transform actually transforms the survival function, $G(x) = 1 - F(x)$, where $F(x)$ is the cumulative distribution function for a random cash flow, x. His transformed survival function is given by the formula, $G^*(x) = \Phi[\Phi^{-1}(G(x)) + \lambda]$, where $\Phi(\ )$ is the standard normal cumulative distribution function.

[17] For more details on this approach applied to pricing insurance, see Longley-Cook.

Mathematically, these approaches can be considered to be equivalent. However, the choice of the methods used to derive the risk-adjusted discount rate, the transformed probabilities or the utility-adjusted cash flows will affect the final results. It is unlikely that these three methods, independently applied, would actually produce the same result.

It is this challenge of implementation that has resulted in less explicit use of the probability transform or utility theory approaches in practice. However, as the next section will demonstrate, all financial pricing models are closely tied to both of these methods.

9. Stochastic Discount Factors and the Fundamental Pricing Formula

Introduction

Many practitioners have been reluctant to adopt the Probability Transform or Utility Theory approaches because of the difficulty calibrating these methods to match market prices. While they accept the theoretical basis for the methods, they somehow appear less "practical" than the CAPM or the Black-Scholes formula. But as has been suggested earlier, these models are not directly applicable to many insurance applications. What is needed is a unifying framework that can tie these various methods together.

In fact, *all* financial pricing models (CAPM and Black-Scholes included) can be characterized in a rather elegant fashion using what are known as *stochastic discount factors*. This approach will be explored in detail here with the intent to accomplish three related goals:

- Demonstrate that well-known financial pricing models such as CAPM and Black-Scholes are essentially special cases of a more general pricing formula that can be applied to *any* risky cash flow.
- Show that many of the insurance pricing models discussed so far, including those that rely on risk-adjusted discount rates, probability transforms or utility theory, are also special cases of this more general pricing formula.
- Map out the path of future research needed to apply this more general pricing formula to insurance pricing.

A Fundamental Pricing Formula

All commonly used financial pricing models, including not only the CAPM and the Black-Scholes option pricing formula but also the Arbitrage Pricing Theory (APT), the Fama-French 3 Factor Model and various term structure models such as the Cox, Ingersoll and Ross (CIR) model, can be written in an identical fashion as follows:

$$p = E[m_i x_i]$$

where p is the price (value) of the asset, x_i represents the (risky) payoffs in state i, m_i is the *stochastic discount factor*[18] in state i and $E[]$ is the standard expected value operator. In other words, all financial pricing models are merely calculations of weighted expected values.

Before demonstrating that the common financial pricing models can be characterized by this pricing formula, I will explore the stochastic discount factor, m_i, in more detail. In this discussion, I will draw heavily from Cochrane. The interested reader is strongly encouraged to refer to this source for substantially more theoretical support for, and extensions of, this framework.

[18] The terminology used here is based on Cochrane. The same concepts are often discussed by different authors using the terms *state price densities*, *pricing kernels* or *deflators*. These are all equivalent to the stochastic discount factor terminology used here.

The Stochastic Discount Factor

This section will explore the properties of the stochastic discount factor in more detail, beginning with the discrete case.

Discrete Case

The goal of any pricing formula is to determine the value of a stream of uncertain cash flows. In other words, the goal is to express the value p_t at time t of a risky cash flow x_{t+1} at time $t+1$. Note that x_{t+1} is a random variable and represents the payoff, **not** the return or the profit, as is commonly used n the derivation of most pricing models. I will begin with a discrete version of the fundamental pricing formula in which all of the cash flows occur in one period and where the probability distribution of the cash flows is discrete. That is to say, the payoff x_{t+1} can take on values of either $x^{(1)}$, $x^{(2)}$, ... $x^{(n)}$ with probabilities $\pi^{(1)}$, $\pi^{(2)}$, ... $\pi^{(n)}$.

To find the value of this random payoff, I assume that investors derive their pleasure from consumption and that their ultimate goal is to maximize the *utility* of that consumption. Utility in this context is a numerical representation of the pleasure that is gained from consumption. It is important to note that investors' utility depends on both current and future consumption, so both current and future consumption will be reflected within the utility function. The use of current and future time periods helps to motivate the investor to make investments now with future payoffs, thereby decreasing current consumption but potentially increasing future consumption. To keep this discussion simple, consumption is limited to just one future period, though the extension to multiple future time periods can certainly be done.

I express utility in both the current and future periods as follows:

$$U(c_t, c_{t+1}) = u(c_t) + \beta E_t[u(c_{t+1})]$$

This format for the utility function reflects the fact that total utility, U, over both the current and future periods is equal to the utility of current consumption, $u(c_t)$, and the (discounted) expected utility of next period's consumption, $\beta E_t[u(c_{t+1})]$. Notice that next period's consumption is a random variable that will depend on the payoff x_{t+1}. In addition, the expected value of this utility is adjusted by a *subjective discount factor*, β. The subjective discount factor represents a (downward) adjustment due to the delay in consumption until time period $t+1$.

Following elementary microeconomic arguments, an investor considering the purchase of a small amount, ξ, of a risky cash flow at a price p_t can be assumed to seek to maximize his or her utility. The utility maximization problem is further subject to the constraints that consumption in the current period is equal to his or her original consumption level less the price of the asset and consumption in the next period is equal to consumption in the absence of the risky asset's payoff plus the (random) asset payoff.

The first order condition for the amount of this asset to purchase is such that the marginal cost and marginal benefit are equal. Denoting the change in utility by $u'(c_t)$, the following are obtained:

$$\text{Marginal Cost} = p_t u'(c_t)$$
$$\text{Marginal Benefit} = E_t[\beta u'(c_{t+1}) x_{t+1}]$$

Setting the marginal cost equal to the marginal benefit,

$$p_t u'(c_t) = E_t[\beta u'(c_{t+1}) x_{t+1}]$$

This formulation indicates that the loss in utility from paying p_t today is equal to the expected, discounted gain in utility from the extra payoff at time $t+1$.

This expression can be rewritten as:

$$p_t = E_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}]$$

Note that in this expression, $u'(c_t)$ and β are both known values at time t and only the expression $u'(c_{t+1})$ is a random variable at time t. Defining m_{t+1}, the stochastic discount factor, as:

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

The **fundamental pricing formula** can be written as:

$$p_t = E_t[m_{t+1} x_{t+1}]$$

Since the time subscripts are generally understood (p_t is the price today, $E_t[]$ is the expected value conditional on information known at time t, and both m_{t+1} and x_{t+1} reflect random variables with realization at time $t+1$), I can drop the subscripts and express this formula as:

$$p = E[mx]$$

Notice that this derivation did not require any assumptions about the distribution of the payoffs or the functional form of the utility curve. It merely assumed that investors care about current and future consumption and that they maximize their total utility of this consumption.

Notice also that no mention has yet been made of a market portfolio, normally distributed returns, risk-neutrality or any of the common starting points for derivations of well-known financial pricing models. Later, I will demonstrate that these assumptions used for well-known financial pricing models are only introduced in order to be able to extend the fundamental pricing model to make *equilibrium* statements about market prices. That is to say, in order to go from a model of how any one individual would value a stream of cash flows to a model of what the equilibrium market price for those cash flows will be, it is often necessary to make more restrictive assumptions about how all participants in the market behave. But those restrictions are not needed if we simply desire a model for how an individual investor or firm ought to price risky cash flows.

Continuous Case

For convenience, I would prefer to limit my discussion of the fundamental pricing formula to the discrete case. This greatly simplifies the notation and hopefully will make the key insights more accessible to readers unfamiliar with continuous time finance and stochastic calculus.

However, there are continuous time extensions of this framework that will be needed when this approach is used to derive the Black-Scholes option pricing formula. I will defer discussion of the continuous time version of the stochastic discount factor until that section of the paper[19].

Utility Functions

A detailed discussion of alternative functional forms of the utility function is beyond the scope of this paper. The interested reader should refer to Gerber and Pafumi for alternative functional forms of utility functions and a discussion of the underlying theory. Since it will be necessary to use one or more utility functions in the discussion that follows, a brief discussion of representative utility functions is included in Appendix D.

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[19] See Chapter 4 of Cochrane for additional details.

Models for the Stochastic Discount Factor (m)

To use this fundamental pricing model to price cash flows, all that is needed is a utility function, parameters for the utility function, a value for β and probability distributions for consumption and the payoffs, x_t. This is much easier said than done. In addition, the model actually works poorly in practice due to the difficulty of measuring consumption and the marginal utility parameters accurately.

To use the fundamental pricing formula, it will therefore be necessary to find proxies for consumption (and the marginal utility of consumption). With judiciously chosen functional forms for the stochastic discount factor and appropriate proxies for consumption, rather elegant and useful formulas can be derived. As was stated in the Introduction, **all well-known asset pricing models, including CAPM and the Black-Scholes option pricing model, represent characterizations of the stochastic discount factor using proxies for consumption**.

Implications of the Fundamental Pricing Formula

Before turning to these well-known financial pricing models, it is useful to consider some implications from the general pricing formula presented.

Rate of Return

Many well-known financial pricing models are often characterized in terms of the rate of return, $R = x_{t+1}/p_t$. From the fundamental pricing formula described above, I can divide both sides by the current price and see the following relationship:

$$p_t = E[mx_{t+1}]$$
$$1 = E[m\frac{x_{t+1}}{p_t}]$$
$$1 = E[mR]$$

Risk Free Rate

Consider a risk-free asset with a known payoff equal to R_f and a known price, which I assume equals 1.0 for convenience (we could rescale both values if desired, but here R_f represents the *gross* return and is a value such as 1.05). Using the fundamental pricing formula and noting that by definition the risk free payoff is a constant and uncorrelated with the stochastic discount factor, I can derive an important expression for the *risk-free rate*,

$$1 = E[mR_f] = E[m]R_f$$

which can also be written as:

$$E[m] = \frac{1}{R_f}$$

Risk-Adjusted Values

Notice that the fundamental pricing formula $p = E[mx]$ reflects **both** a risk-adjustment and a discount for the time value of money. It is sometimes useful to separate these two adjustments. Begin by noting that both m and x are random variables.

Using the definition of covariance:

$$Cov(m, x) = E(mx) - E(m)E(x)$$

Substituting for $p = E[mx]$ and $E[m] = \frac{1}{R_f}$,

$$p = E(m)E(x) + Cov(m,x)$$
$$p = \frac{E(x)}{R_f} + Cov(m,x)$$

This last expression can be interpreted as follows. The risk-adjusted price of any asset is equal to its risk-free expected value, $\frac{E(x)}{R_f}$, plus a *risk load* that depends on the covariance of its payoffs with the stochastic discount factor, $Cov(m,x)$.

Two observations regarding this formulation are important to highlight:

- Source of the Risk Adjustment – Using the previous formulation, by substituting the formula for the stochastic discount factor, *m*, the asset pricing formula can be rewritten as:

  $$p = \frac{E(x)}{R_f} + \frac{Cov[\beta\, u'(c_{t+1}), x]}{u'(c_t)}$$

  This version of the asset pricing formula helps to highlight why the price of a risky asset (or liability) would have either a positive or negative risk load.

  The risk load for an asset will be negative if the asset's cash flow covaries positively with consumption. That is to say, if you buy an asset whose payoffs are high when you are already feeling wealthy (high consumption) and low when you are already feeling poor (low consumption) your consumption will be more volatile and hence you will require a low price to induce you to buy it.

  On the other hand, if an asset's cash flow covaries negatively with consumption the risk load will be positive. An insurance policy, priced from the insured's perspective, is an example of this case. The insured is willing to pay more than the discounted expected value for the policy precisely because the payoffs covary negatively with their consumption in the absence of the policy.

  This explanation also highlights what the stochastic discount factor, *m*, actually represents. The stochastic discount factor can be thought of as an index of good times and bad times. I am willing to pay more for an asset whose payoffs are high in "bad times" and low in "good times"; I am willing to pay less for an asset whose payoffs are high in "good times" and low in "bad times". Finding an appropriate stochastic discount factor boils down to defining what is meant by "good times" and "bad times".

- Link to Insurance Pricing Models – As shown earlier, many insurance pricing models can be written in the form:

  Premium = Risk-free Expected Claim Costs + Risk Load

  The presentation of the asset pricing model shown above makes it clear that the risk load is appropriately a function of the covariance between the payoff and the stochastic discount factor.

  The common notion that risk loads are required because of the *variability* of the cash flows is certainly incorrect. Risk loads based on a multiple of either variance or standard deviation may not correctly price for risk unless the chosen multiple of these risk measures approximates the covariance with the stochastic discount factor.

The Fundamental Pricing Formula and Common Asset Pricing Models

In this section, I will demonstrate that the fundamental pricing formula,

$$p = E[mx]$$

can be used to derive two commonly used financial pricing models, the CAPM and the Black-Scholes formula, by merely selecting an appropriate functional form for the stochastic discount factor *m*.

There are advantages to this particular formulation, as opposed to the more commonly known formulations. First, it emphasizes the fact that $p = E[mx]$ is explicitly rooted in assumptions about utility. Second, it shows that the price of any asset reflects both the riskiness of the cash flows and the time value of those cash flows and that these two effects can be separated.

Highlighting this link to utility theory makes it easier to determine when a particular pricing formula may or may not be appropriate for a given situation. In other words, reference to a specific assumption about the stochastic discount factor that underlies a model such as CAPM or Black-Scholes helps to determine whether those particular stochastic discount factors are appropriate for a given application. This, in turn, helps determine whether CAPM, Black-Scholes, or some other asset pricing model is appropriate for applications such as the pricing of insurance risks.

This formulation is also appealing for a reason of particular importance to the pricing of insurance risks. Similar to option pricing formulas, and unlike the standard CAPM formulation, the fundamental pricing formula presented here specifies the <u>price</u> or <u>value</u> of a risky cash flow in units of *dollars* rather than in terms of the *rate of return*. Of course, it is trivial in many contexts to flip from one characterization to another and many introductory finance textbooks present CAPM in both a rate of return framework and a price framework. The emphasis on price nonetheless avoids a variety complications that are inevitably introduced when trying to assess rates of return for risky cash flows that don't require an initial investment (e.g. futures, forwards and swaps) or that contain implicit leverage (e.g. options and insurance).

<u>Common Asset Pricing Models</u>

Below I will demonstrate that various financial pricing models can be expressed as $p = E[mx]$. I will focus on CAPM and Black-Scholes for convenience, but the points raised here are applicable to other pricing models as well.

Based on the CAPM, the value of a risky cash flow is given by the following formula:

$$V = \frac{E(CF_1)}{1 + E(r_i)}$$

where the discount rate, $E(r_i)$ is given by the CAPM formula:

$$E(r_i) = r_f + \beta\left[E(r_m) - r_f\right]$$

In contrast, the Black-Scholes option pricing formula indicates that the formula for the value of a call option, which also has an uncertain cash flow at maturity, is given by the formula:

$$\text{Call Value} = S\,\mathrm{N}(d_1) - Ke^{-rT}\,\mathrm{N}(d_2)$$

where the following parameter definitions are used:

S = current stock price

K = option exercise price

r = continuously compounded risk free rate of interest

σ = stock price volatility

T = maturity (in years) of the option

$d_1 = \dfrac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$ and

$d_2 = d_1 - \sigma\sqrt{T}$

Interestingly, the only difference between these seemingly unrelated formulas is that they use different definitions of the stochastic discount factor, *m*. What motivates the different choices of the stochastic discount factor *m* are the characteristics of the cash flows in each of the three models and the assumptions that can be made with regard to prices in equilibrium.

However, in all cases these models will reflect the equilibrium prices in a market with many buyers and sellers. None of these specific models for *m* necessarily reflect the value to any one individual investor unless the choice of *m* is appropriate for that investor and for the particular cash flows being valued.

For those readers unfamiliar with CAPM or the Black-Scholes formula, the standard derivations are presented in Appendix A and Appendix B, respectively. Here, each will be derived using the fundamental pricing formula.

Derivation of CAPM Using Fundamental Pricing Formula

Recall from above the basic relationship between the current price and the uncertain payoffs.

$$p_i = E(mx_i) = \frac{E(x_i)}{R_f} + Cov(m, x_i)$$

After dividing both sides of the expression by the current price, this equation can be written in terms of the total return, $R = x_{t+1}/p_t$, as follows:

$$1 = \frac{E(R)}{R_f} + Cov(m, R)$$

Rearranging terms, this can be written as an expression in terms of the expected return:

$$E(R) = R_f - R_f Cov(m, R)$$

To derive a more specific formula for the expected return, I simply need to determine an appropriate stochastic discount factor, m. If that stochastic discount factor is a linear function of the return on the market, R_m, then:

$$m = a - bR_m$$

And,

$$\begin{aligned} E(R) &= R_f - R_f Cov(m, R) \\ &= R_f - R_f Cov(a - bR_m, R) \\ &= R_f + \lambda Cov(R_m, R) \end{aligned}$$

In the above expression, λ is an unknown constant. However, assuming that the above expression must hold for all assets, including the market portfolio, implies that:

$$E(R_m) = R_f + \lambda Cov(R_m, R_m) = R_f + \lambda \sigma_m^2$$

This equation for the expected return on the market portfolio can be used to solve for the unknown parameter,

$$\lambda = \frac{E(R_m) - R_f}{\sigma_m^2}$$

Plugging this value for λ into the previous equation results in the standard CAPM, as shown below:

$$
\begin{aligned}
E(R) &= R_f + \lambda Cov(R_m, R) \\
&= R_f + \frac{E(R_m) - R_f}{\sigma_m^2} Cov(R_m, R) \\
&= R_f + [E(R_m) - R_f] \frac{Cov(R_m, R)}{\sigma_m^2} \\
&= R_f + \beta [E(R_m) - R_f]
\end{aligned}
$$

This shows that, starting with the fundamental pricing formula and merely assuming a linear relationship between the stochastic discount factor and the return on the market portfolio results in the standard CAPM. All that remains is to justify this linear relationship. Under what conditions is it appropriate to assume that the stochastic discount factor, m, is a linear function of the return on the market portfolio?

Stochastic Discount Factor as Linear Function of Return on Market Portfolio

There are a number of alternative assumptions that can be used to produce a stochastic discount factor that is a linear function of the return on the market portfolio, R_m. This section will show that assuming a two-period investor with quadratic utility and no labor income results in just such a formula for the stochastic discount factor.

Two-Period Investor with Quadratic Utility

The full expression for the stochastic discount factors is:

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

Now assume that an investor will live for only two periods and his only source of wealth is his investment, net of his current consumption, in the stock market. Because he lives for only two periods, his total consumption will equal his consumption this period plus whatever wealth he has next period, inclusive of any investment returns generated on the wealth not consumed in the current period. His current period consumption will be c_t and his consumption next period depends on the rate of return earned on his investment. Denoting his current wealth, W_t, and the return on the stock market, R_m, then his future consumption will be $R_m(W_t - c_t)$.

If his utility function is assumed to be quadratic,

$$
\begin{aligned}
u(c_t) &= -\frac{1}{2}(c_t - c^*)^2 \\
u'(c_t) &= -(c_t - c^*) \\
u'(c_{t+1}) &= -(c_{t+1} - c^*)
\end{aligned}
$$

Plugging in for $c_{t+1} = R_m(W_t - c_t)$ and simplifying:

$$
\begin{aligned}
m_{t+1} &= \beta \frac{u'(c_{t+1})}{u'(c_t)} \\
&= \beta \frac{R_m(W_t - c_t) - c^*}{c_t - c^*} \\
&= -\beta \frac{c^*}{c_t - c^*} + \beta \frac{(W_t - c_t)}{c_t - c^*} R_m \\
&= a - bR_m
\end{aligned}
$$

Alternative Assumptions

While the derivations are not shown here, Cochrane shows that three alternative assumptions can be used to also derive a stochastic discount factor as a linear function of the market return. The three alternative assumptions are:

1. Power Utility Function and Normally Distributed Returns, in a two period model
2. Quadratic Utility and *i.i.d*. Returns, over an infinite horizon
3. Log Utility Function

Black-Scholes Option Pricing Formula

This section will demonstrate that a formula for a European call option can also be derived using the fundamental pricing model and a specific assumption regarding the stochastic discount factor.

The Black-Scholes model will require the use of a continuous-time version of the stochastic discount factor, which was mentioned only briefly earlier. For this reason, a *discrete time* derivation of a call option pricing formula under simplifying assumptions will be shown first and then a derivation of the same formula will be shown using the fundamental pricing model. The continuous time derivation of the Black-Scholes formula follows the same steps, so that version is presented in the Appendix (in slightly less detail, with some of the more complex mathematics ignored).

A simple one-period European call option on a stock will be used for this derivation, though the methodology applies to any derivative security. The call option will give the owner the right to buy one share of stock for a price of K at maturity. The maturity date will be denoted as time period T, the stock price will have a known price today of S_0 and the risk free rate of interest will be denoted using R_f, which again reflects the accumulated value of \$1 rather than a rate of return (i.e. it takes on a value such as 1.05 as opposed to a rate of 5%). In the continuous time version, r is used to denote the continuously compounded risk free rate.

Option Pricing with Discrete Model

Most textbooks begin the discussion of the pricing of call options with a simple, discrete case where the stock price can only go up or down to two values, S_u or S_d, during each time period. Under this assumption, the payoffs for the call option will depend on the final stock price and will be either:

$C_u = Max(S_u - K, 0)$ if the stock price rises to S_u,

or

$C_d = Max(S_d - K, 0)$ if the stock price falls to S_d.

For a concrete example, assume that the current stock price is $S_0 = 20$, that the two possible values are either $S_u = 30$ or $S_d = 10$, that the exercise price is $K = 20$ and that $R_f = 1.05$. In this case, the option payoffs will either be 10 if the stock price goes up to 30 or 0 if the stock price goes down to 10.

Assuming that the probabilities of these two payoffs were known, one could theoretically determine the expected option payoff. However, the non-linear nature of the payoffs makes it difficult to determine the appropriate risk-adjustment. As a result, it is difficult to value these payoffs and an alternative approach is needed.

Consider a portfolio, Π, that consists of a short position in one call option and a long position in 0.5 shares[20] of stock. The short call option has identical payoffs as before but with the

[20] The 0.5 shares used here are based upon the *option delta*, which reflects the change in option values over the change in the stock values. Later, when we allow the stock price to change continuously, the option delta will be defined as the partial derivative of the option with respect to the stock price.

opposite sign, since it is an option that was sold rather purchased. This portfolio would have the following alternative payoffs:

Portfolio Payoffs at Option Maturity

	$S_u = 30$	$S_d = 10$
Short Call	-10	0
Long 1/2 Share	15	5
Total Portfolio Value	5	5

Since the payoffs are fixed at 5 regardless of the outcome, the total portfolio value has to simply be the risk-free present value of 5, or $\Pi = 5/R_f = 5/1.05 = 4.762$. Since the one-half share of stock itself is worth 10, the call option itself must be worth:

$$C = .5S_0 - \text{PV}(5) = 10 - 4.762 = 5.238.$$

Option Pricing with Discrete Version of Fundamental Pricing Model

Notice that because it was possible to combine the call option with a certain number of shares (0.5 in this case) and achieve a total portfolio payoff that was fixed, it was easy to determine the value of the call option as a function of the current stock price and the risk-free interest rate. In the context of the fundamental pricing formula, this is equivalent to finding a stochastic discount factor, m, that correctly values both the stock and the risk free asset. **Any stochastic discount factor that correctly values the stock and the risk free bond simultaneously will also correctly value the option payoffs.**

Sticking with the simplifying assumption that there are only two possible states of the world, the probability of the stock pricing going up can be depicted as π_u and the probability of the stock price going down can be depicted as π_d. The formulas for the value of the stock and the value of a risk free payoff can then be written as follows in terms of the fundamental pricing formula:

$$S = E[mx] = \pi_u m_u S_u + \pi_d m_d S_d$$

$$1 = E[mR_f] = R_f(\pi_u m_u + \pi_d m_d)$$

Any values for m_u and m_d that correctly value the stock and the risk free asset will also correctly value the option. Values for m_u and m_d, can be found by first noting that there are two equations but four unknowns, since the probabilities are unknown as well. However, the parameters π_u and π_d appear only with the m_u and m_d terms, so there will not actually be any need to solve for these values.

The second equation can be used to write m_d as a function of m_u:

$$\pi_d m_d = \frac{1}{R_f} - \pi_u m_u$$

$$m_d = \frac{1}{\pi_d}\left[\frac{1}{R_f} - \pi_u m_u\right]$$

Plugging this in to the equation for the stock price and solving for m_u results in the following:

$$S = \pi_u m_u S_u + \left(\frac{1}{R_f} - \pi_u m_u \right) S_d$$

$$S = \pi_u m_u (S_u - S_d) + \frac{S_d}{R_f}$$

$$\pi_u m_u = \frac{S - \frac{S_d}{R_f}}{S_u - S_d} = \frac{R_f S - S_d}{R_f (S_u - S_d)}$$

$$m_u = \frac{R_f S - S_d}{\pi_u R_f (S_u - S_d)}$$

Given these formulas for m_u and m_d, the known quantities from the previous example can be used to calculate numeric values for these quantities. Since it is actually the products $\pi_u m_u$ and $\pi_d m_d$, that are needed, the results are:

$$\pi_u m_u = \frac{R_f S - S_d}{R_f (S_u - S_d)} = \frac{1.05(20) - 10}{1.05(30 - 10)} = \frac{11}{21} = 0.5238$$

$$\pi_d m_d = \frac{1}{R_f} - \pi_u m_u = \frac{1}{1.05} - .5238 = .4285$$

Finally, the value of the call option can be found using the fundamental pricing formula and these values:

$$\begin{aligned} C &= E[mx] \\ &= \pi_u m_u C_u + \pi_d m_d C_d \\ &= .5238\,Max(30 - 20{,}0) - .4285\,Max(10 - 20{,}0) \\ &= .5238(10) - .4285(0) \\ &= 5.238 \end{aligned}$$

Notice that the value is identical to the value determined earlier. This confirms the fact that the value of an option can be determined using the fundamental pricing formula and a stochastic discount factor that correctly values both the stock and the risk free bond.

Link to Risk Neutral Pricing – Many finance textbooks discuss the pricing of options using the so-called *risk neutral* pricing model. In this model, *risk-neutral probabilities* are derived by assuming that the expected return for the stock is equal to the risk free rate of interest. The option value is then determined as the risk-free present value of the expected call payoffs, where the expectation is taken with respect to these risk-neutral probabilities (denoted π^*) as opposed to the *realistic* probabilities (π). The formula, using consistent notation as before, is as follows:

$$C = \frac{1}{R_f} \left[\pi^* C_u + (1 - \pi^*) C_d \right]$$

Comparing the risk-neutral formula with the fundamental pricing formula, it is clear that the risk-neutral probabilities are really just a combination of the *realistic probabilities* (π), the *stochastic discount factor* (m) and the *risk free discount factor* (R_f):

$$\frac{1}{R_f} \pi^* = \pi_u m_u \Rightarrow \pi^* = R_f \pi_u m_u$$

<u>*Option Pricing in Continuous Time*</u>

Although the logic used in the discrete case is identical to the logic underlying the famous Black-Scholes option pricing formula, the resulting formulas are not the same. The difference is due to the simplifying assumption that the stock price can take on only one of two possible values at the end of the period. If instead the stock price can change continuously, the math will be more complex but the results will be similar.

A complete derivation of the Black-Scholes formula is beyond the scope of this paper. The mathematical details are presented in a simplified fashion in Appendix B. A less mathematically complex derivation using the insight of the *risk neutral method* discussed at the end of the previous section is shown in Appendix C for those readers unfamiliar with the stochastic calculus techniques used in Appendix B.

Option Pricing in Continuous Time Using the Fundamental Pricing Formula

As previously mentioned, the derivation of the Black-Scholes option formula requires the continuous time version of the fundamental pricing formula, which is somewhat more complicated than the discrete version. However, note that the Black-Scholes continuous time derivation follows the same underlying logic as in the discrete case. Since in the discrete case it was easy to show that the fundamental pricing formula could be used, this should also be true in the continuous time case as well.

Appendix D derives a formula for European call option by determining a formula for the stochastic discount factor, m, that correctly prices both the stock and the risk free bond. The resulting formula for m can then be used to price the payoffs for a European call option, producing a formula that is identical to the Black-Scholes option pricing formula.

Summary of Derivations

In this section, common asset pricing models such as the CAPM and the Black-Scholes formula were shown to be special cases of the fundamental pricing formula:

$$p = E[mx]$$

To derive the CAPM, a proxy for the aggregate marginal utility growth rate was used. When the stochastic discount function was written as a linear function of the return on the market portfolio, the fundamental pricing formula simplified to the standard CAPM. A variety of alternative assumptions can be used to justify the use of a linear function of the market portfolio return as a proxy for marginal utility. To determine whether CAPM is appropriate to an application such as insurance policy pricing, one must therefore assess whether these assumptions are valid. If not, then the same fundamental pricing formula can be used as a starting point, but a more appropriate proxy for marginal utility would have to be used.

Similarly, to derive the Black-Scholes formula, *any* stochastic discount factor that correctly prices both the underlying stock and the risk-free bond can also be used to correctly price an option. In fact, any cash flow that can be replicated using the stock and risk free bond can be valued using the same stochastic discount factor. However, cash flows that cannot be replicated risklessly with these two assets *cannot* be valued using this stochastic discount factor. Instead, the fundamental pricing formula can still be used, but again an appropriate proxy for the marginal utility would be needed.

Link Between Fundamental Pricing Formula and Insurance Pricing Models

The fundamental pricing formula described here has been written as:

$$p = E[mx]$$

However, this can also be written, in the discrete case, in terms of the sum of the probability-weighted terms $m_i x_i$, where x_i is the risky payoff in state i and m_i is the stochastic discount factor for state i. The stochastic discount factor m_i recall contains both a component for the time value of money, denoted as β, and a term related to the marginal utility of consumption. The marginal utility of consumption term incorporates a risk-adjustment into the pricing formula, causing the cash flows to be valued higher than their risk-free discounted expected value if their payoffs cause consumption to increase at what would otherwise be a “bad” time.

When written in this alternative fashion, using π_i for the probability of state *i*, the formula is:

$$\begin{aligned} p &= E[mx] \\ &= \sum_i \pi_i m_i x_i \\ &= \sum_i \pi_i \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right] x_i \end{aligned}$$

Now consider three special cases of this formula.

1. Risk-Adjusted Discount Rate Method – If the terms related to marginal utility, $\frac{u'(c_{t+1})}{u'(c_t)}$, are interpreted as a multiplicative risk-adjustment to the otherwise risk-free discount rate for the time value of money, β, then the product can be thought of as a risk-adjusted discount rate. If this term is time-dependent but not state dependent, then the above expression simplifies to calculating the expected payoffs and discounting them at a risk-adjusted discount rate.

$$\begin{aligned} p &= \sum_i \pi_i \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right] x_i \\ &= \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right] \sum_i \pi_i x_i \end{aligned}$$

Risk-Adjusted Discount Rate

2. Probability Transform Method – As an alternative, assume the discount rate term, β, is constant and equal to the risk free rate. If the terms related to marginal utility are multiplied by the probability π_i, then the product can be thought of as a transformed or risk-neutral probability. This expression then simplifies to the risk-free present value of the expected cash flows, using the transformed rather than the realistic probability.

$$\begin{aligned} p &= \sum_i \pi_i \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right] x_i \\ &= \beta \sum_i \left[\pi_i \frac{u'(c_{t+1})}{u'(c_t)} \right] x_i \end{aligned}$$

Risk-Adjusted Probability

3. Utility Theory Method – Again assume the discount rate term β is constant and equal to the risk free rate. However, if now the marginal utility terms are multiplied by the cash flow, x_i, then the expression simplifies to calculating the discounted value of the expected *risk-adjusted* cash flows. Since this risk-adjustment for the cash flows involves the marginal utility of consumption, this is essentially a measure of the utility of those cash flows and is hence identical to the utility theory based insurance pricing model.

$$\begin{aligned} p &= \sum_i \pi_i \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right] x_i \\ &= \beta \sum_i \pi_i \left[\frac{u'(c_{t+1})}{u'(c_t)} x_i \right] \end{aligned}$$

Risk-Adjusted Cash Flow

As these examples show, not only are commonly used financial pricing models such as CAPM and Black-Scholes special cases of the fundamental pricing formula, so too are the various insurance pricing models discussed earlier. Notice too that these insurance pricing models are fundamentally quite similar. All define the price as the product of four terms. They differ only with respect to how these terms are combined and interpreted.

10. Applying the Fundamental Pricing Formula to Insurance Pricing

Using the fundamental pricing formula requires an appropriate proxy for marginal utility of consumption, since utility functions themselves are impossible to observe. The CAPM assumes that a linear function of the market return is an appropriate proxy while the Black-Scholes models assumes that this proxy can simply be inferred from the marketable securities used to replicate the risky cash flows.

Neither of theses two proxies seems appropriate to price insurance. To begin, it is not clear how “consumption” for an insurance firm should even be defined. When dealing with an individual investor it may be reasonable to assume that their consumption is based entirely on the financial performance of their investments. But when pricing an individual policy, what matters is how the policy’s cash flows will affect the shareholders’ consumption. It is doubtful that the market portfolio is a good proxy for this, since it will also depend upon factors such as the other policies’ cash flows, how claims may have impacted future pricing, etc.

Second, inferring a stochastic discount factor from other marketable securities that can be used to replicate specific cash flows, so-called relative pricing or market consistent pricing, is reasonable when such prices exist and when the cash flows can be replicated. Most insurance cash flows, excluding of course certain embedded options in life insurance liabilities, cannot be replicated with marketable securities.

Third, note that models such as CAPM or Black-Scholes are primarily concerned with *equilibrium prices in complete markets*. That is to say, they are not intended to assess the value of a given stream of cash flows to any particular individual. Rather, they reflect what the value would reasonably be in a “market” with many buyers and sellers. Because of the illiquid nature of most insurance liabilities, insurance pricing is more often concerned with the private valuation that a given individual may place on a stream of risky cash flows. This value could differ significantly from firm to firm, given their unique “utility” functions and, more importantly, how their “consumption” will be affected by the cash flows.

The challenge for actuaries is to determine a set of stochastic discount factors that appropriately capture the realities of insurance cash flows and that can be used consistently to value these cash flows. The resulting models will share a common foundation with models such as CAPM and Black-Scholes, but are unlikely to literally use the standard textbook forms of those models.

The papers by Wang and Venter, Barnett and Owen that attempted to calibrate probability transforms to match market prices of catastrophe risk, which were mentioned in Section 7, demonstrate one approach that could be used to calibrate a stochastic discount factor.

The RMK Conditional Risk Charge

A related approach that relies on the same principles underlying the fundamental pricing formula presented in Section 9 is contained in Ruhm, Mango and Kreps’ “Applications of the RMK Conditional Risk Charge Algorithm”. In that paper, the authors use stochastic discount factors and the fundamental pricing formula, albeit using different terminology and rationale, to determine the risk charge (i.e. risk load) for insurance cash flows.

In their application though, they were not concerned directly with the calibration of the risk charge to derive a pure premium, as the discussion here has focused on. Instead, they used their algorithm to derive *relative* risk charges for different risks within a portfolio of risks and then used those relative risk charges to allocate, or more accurately stated to *attribute*, a total portfolio risk measure to each component of the portfolio.

To make their results more directly applicable to pricing risky insurance cash flows, one would need to establish a rationale for using any particular portfolio risk measure and converting this to a dollar charge for the aggregate portfolio risk.

In the RMK presentation, they discuss two different derivations of their risk charge, one using Kreps' Riskiness Leverage Ratios and one using Ruhm and Mango's Conditional Risk Charge. In either case, the resulting risk charge is as follows:

$$\text{Risk Charge}: R(X_i) = E[X_i \cdot RMK]$$

where X_i is the (random) cash flow for a particular risk within a portfolio *P*, *RMK* is a weighting or leverage variable that depends only on the total portfolio result and E[*] is the expected value function estimated with respect to the true probability distribution.

If the risk-adjusted value, *p*, is defined as the risk-free expected value of *X* plus a risk load, then:

$$\begin{aligned} p_i &= \frac{E[X_i]}{R_f} + E[X_i \cdot RMK] \\ &= E[X_i \cdot (\frac{1}{R_f} + RMK)] \\ &= E[X_i \cdot RMK^*] \end{aligned}$$

where R_f represents one plus the risk free rate and RMK^* is a weighting variable that takes into account both the discounting for the time value of money and the risk adjustment.

Depending on whether the Kreps or Ruhm-Mango derivation is used, the RMK^* functional form will differ but the results are identical. Kreps defines RMK^* in terms of a "riskiness leverage function"; Ruhm and Mango relate RMK^* to a utility function that weights the portfolio outcomes using multiples of the true probabilities. These are both in the spirit of the probability transform methods and the utility theory methods discussed earlier.

In each case, the formula suggests that the value of any risky cash flow is given by a simple expected value calculation. However, when calculating this expected value, either the probabilities or the cash flows are adjusted by a factor that is perhaps different for each possible state of the world. This is clearly identical to the fundamental pricing formula discussed in Section 9.

11. Conclusion

In this paper, various insurance pricing models were presented to provide an incomplete, but perhaps insightful view of how these models have evolved from the most simplistic to increasingly sophisticated methods.

The link between many of these models, particularly those that use risk-adjusted discount rates (either directly in the discounting of cash flows or as a basis for determining a charge for allocated capital), those that use probability transforms and those that use utility theory was emphasized. In addition, each of these models were shown to be special cases of a general pricing methodology, described as a *fundamental pricing formula*, which relies on a stochastic discount factor.

Commonly used financial pricing models, such as CAPM and Black-Scholes, were shown to also be special cases of this general pricing methodology, albeit under fairly specific and perhaps restrictive assumptions. Because these specific assumptions are not applicable to most risky insurance cash flows, the limits of those models for pricing insurance were highlighted.

Nonetheless, the fundamental pricing formula does suggest that a unified and coherent approach to pricing insurance is likely to be found through further research into appropriate probability transforms or utility functions that can be calibrated to market prices for insurance. Papers by Wang; Venter, Barnett and Owen; and Ruhm, Mango and Kreps offer insight into how this research should progress and the form that the results may ultimately take.

Appendix A – Derivation of the Capital Asset Pricing Model[21]

The Capital Asset Pricing Model (CAPM) is commonly expressed in textbooks in terms of a rate of return, as follows:

$$E(r_i) = r_f + \beta\left[E(r_m) - r_f\right]$$

where, $E(r_i)$ is the expected rate of return for a given stock, r_f is the risk free rate, β is the stock's *beta* and $E(r_m)$ is the expected return on the *market portfolio*.

The standard CAPM derivation usually begins by assuming that investors make portfolio choices in the following manner. First, they evaluate all possible combinations of *risky* asset to determine an optimal risky portfolio. When selecting this optimal risky portfolio, they seek to maximize their return and minimize their risk, using the variance of returns as their measure of risk.

Next, they combine this risky portfolio with either borrowing at the risk free rate to increase (leverage) their overall risk and return or with lending at the risk free rate to decrease their overall risk and return. Their final portfolio composition will depend upon their attitudes towards risk, expressed in terms of their *utility function*, in order to maximize their expected utility.

If *all* investors have the same information regarding the expected returns and standard deviations of returns for all risky assets, and they all follow this same procedure, then all investors will wind up selecting the same risky asset portfolio. However, if they each have their own unique attitudes regarding risk, they may select different combinations of this risky portfolio and the risk free asset.

In equilibrium, each investor's risky asset portfolio will have to mirror the *market portfolio*. In other words, if everyone chooses the same portfolio of risky assets, this portfolio has to contain risky assets in the same proportion as these assets exist in the total market portfolio. Similarly, the aggregate borrowing and lending will have to be equal.

To determine what this implies about the expected returns and risk of an individual security, consider what would happen if an investor wanted to increase the risk in their portfolio. The investor would have two ways to accomplish this – he could either buy a small amount more of a particular risky asset or he could maintain the same risky asset portfolio (the market portfolio) and use more leverage (borrow) to increase the risk.

An investor considering buying a small amount more of a risky security will have to evaluate the *marginal* contribution this will make to his already optimal portfolio's risk and his current expected risk premium over the risk free rate.

Suppose their portfolio contains *y* percent invested in the market portfolio and (1- *y*) invested in the risk free asset. Further assume that w_i reflects the weight on each of the risky assets in the market portfolio. Their total portfolio expected returns and variance would be given by the following equations:

1. Expected Return: $E(r_p) = y\left[\sum w_i E(r_i)\right] + (1-y)r_f$
2. Expected Risk Premium: $E(r_p) - r_f = y\left[\sum w_i E(r_i) - r_f\right]$
3. Variance: $\sigma_p^2 = y^2\left[\sum w_i^2\sigma_i^2 + \sum_i\sum_{j\neq i} w_i w_j \sigma_{ij}\right]$

From these equations, the marginal risk premium and the marginal risk can be determined by taking partial derivatives with respect to w_i.

The marginal risk premium is $yw_i[E(r_i) - r_f]$.

[21] The derivation here follows the presentation in Bodie, Kane & Marcus

The marginal risk is $2y^2 w_i Cov(r_i, r_m)$.

Taking the ratio of these two marginal amounts, a reward-to-risk ratio for any particular stock can be defined as:

$$\frac{\text{Contribution to Risk Premium}}{\text{Contribution to Variance}} = \frac{yw_i[E(r_i) - r_f]}{2y^2 w_i Cov(r_i, r_m)} = \frac{E(r_i) - r_f}{2yCov(r_i, r_m)}$$

The alternative approach for the investor to increase risk is to simply increase the proportion, *y*, invested in the market portfolio. This would produce a marginal risk premium equal to $E(r_i) - r_f$ and marginal risk equal to $2y\sigma_m^2$.

This would produce a reward to risk ratio of:

$$\frac{\text{Contribution to Risk Premium}}{\text{Contribution to Variance}} = \frac{E(r_m) - r_f}{2y\sigma_m^2}$$

Given these two choices of either investing a small amount in a particular stock or investing in the overall market, investors have two possible values of their expected rewards per unit of risk. Unless these two ratios are equal, investors will have a preference for one over the other and market forces will cause the prices to adjust, the expected returns to change and the difference to disappear. In equilibrium, the reward-to-risk ratios will be equal.

This will lead to the following equality in equilibrium:

$$\frac{E(r_m) - r_f}{2y\sigma_m^2} = \frac{E(r_i) - r_f}{2yCov(i, m)}$$

Rearranging terms, this can be written as,

$$E(r_i) = r_f + \frac{Cov(i, m)}{\sigma_m^2}\left[E(r_m) - r_f\right]$$

Defining beta as $\beta = \dfrac{Cov(i, m)}{\sigma_m^2}$ I can rewrite this as:

$$E(r_i) = r_f + \beta\left[E(r_m) - r_f\right]$$

This is the standard CAPM. Note that its derivation required specific assumptions regarding the manner by which investors select their preferred portfolio based on their utility functions and level of risk aversion. But these parameters don't appear in the final model because it is assumed that all investors perform the same analysis, giving rise to the notion that they all own the market portfolio (in some proportion with risk free borrowing or lending). In equilibrium, all potential investments must have the same marginal reward-to-risk ratio as the market portfolio and CAPM results.

Appendix B – Derivation of Black-Scholes

This Appendix will present a derivation of the Black-Scholes formula to price a European call option on a stock that does not pay dividends.

Assume that Π represents the value of a portfolio that contains a short position in the option and delta shares of stock. Given a particular formula, *f*, for the value of a call option, the goal is to determine how the value of this formula changes as time passes and the stock price changes. This can be derived using a second order Taylor expansion and taking care to note that terms of higher order than *dt* can safely be ignored[22].

If the stock price follows the stochastic process $dS = \mu S dt + \sigma S dz$, then Ito's Lemma indicates that the derivative security follows the process:

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial T} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S}\sigma S\, dz\,.$$

A portfolio containing a short position in the derivative and $\frac{\partial f}{\partial S}$ shares of the stock would today be worth,

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

By plugging in the formulas for dS and df , the change in value of the total portfolio over a small period of time, dt, would follow the following process:

$$d\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) dt$$

Notice that the dz term drops out of the combined portfolio equation, making the change in portfolio value riskless. As a result, this can be set equal to the initial portfolio value multiplied by the risk free rate and the length of the time period. After simplifying the resulting expression the following relationship is obtained:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf$$

This equation is known as the Black-Scholes-Merton differential equation. It indicates that the formula, *f*, for the value of a call option must be related to the risk free rate, the stock price, the stock volatility and the partial derivatives with respect to *S* and *t*. Along with appropriate boundary conditions, such as the value of the option at maturity, an appropriate formula for *f* that meets this criterion and these boundary conditions can be derived. Although the details are not shown here, one solution to this equation is the Black-Scholes call option formula:

$$f = S\,N(d_1) - Ke^{-rT} N(d_2)$$

Where,

S = current stock price,

K = exercise price,

$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

[22] Ito's Lemma can be used to determine the change in the value of the option whose value is a function of the stochastic process for the stock price.

Appendix C – Alternative Derivation of the Black-Scholes Formula

The previous Appendix B derived the Black-Scholes formula using stochastic calculus. However, the risk-neutral method discussed in the text can also be used to derive an identical formula by assuming that the stock's return is equal to the risk free rate.

The original assumption that the stock price follows the process $dS = \mu S\,dT + \sigma S \varepsilon \sqrt{dT}$ results in a final stock price with an expected return[23] equal to μT.

The risk neutral method is therefore equivalent to the following *alternative* assumption for the stock price process, where the drift term μ is replaced by the risk free rate r:

$$dS = r S\,dT + \sigma S \varepsilon \sqrt{dT}$$

The call option value is then simply the risk-free present value of the expected call option payoff. Since the option payoff is $max(S_T - K, 0)$, then the call option is worth:

$$\text{Call} = e^{-rT}\, E[max(S_T - K, 0)]$$

The solution to this involves a fairly complicated integral. Actuaries familiar with the limited expected value and excess mean formulas for lognormally distributed random variables should already be familiar with these formulas. These are used to determine the expected value of a claim payment in excess of a deductible for lognormally distributed ground-up claims. If L is the loss random variable, lognormally distributed with parameters α and β, and D is the deductible, then E[max(L - D, 0)] is the expected excess losses.

$$E[max(L - D, 0)] \quad = e^{\alpha+\beta^2/2}\left\{1-\Phi\left[\frac{\ln(D)-\alpha-\beta^2}{\beta}\right]\right\} - D\left\{1-\Phi\left[\frac{\ln(D)-\alpha}{\beta}\right]\right\}$$

Instead of using L and D in this formula, simply use S_T and K and the appropriate parameters for the distribution for S_T. Given the assumed process for the stock price, S_T is lognormal with parameters $ln(S)+(\mu-.5\sigma^2)T$ and $\sigma\sqrt{T}$.

Under the *risk neutral* assumption, the process is adjusted by setting $\mu = r$. Then, S_T is lognormal with parameters $\alpha = ln(S)+(r-.5\sigma^2)T$ and $\beta = \sigma\sqrt{T}$.

Plugging into the excess mean formula and simplifying the algebra produces the following:

$$\begin{aligned}
EM &= e^{\alpha+\beta^2/2}\left\{1-\Phi\left[\frac{\ln(D)-\alpha-\beta^2}{\beta}\right]\right\} - D\left\{1-\Phi\left[\frac{\ln(D)-\alpha}{\beta}\right]\right\} \\
&= e^{ln(S_0)+(r-.5\sigma^2)T+.5\sigma^2T}\left\{1-\Phi\left[\frac{\ln(K)-ln(S)-(r-.5\sigma^2)T-\sigma^2T}{\sigma\sqrt{T}}\right]\right\} - K\left\{1-\Phi\left[\frac{\ln(K)-ln(S)-(r-.5\sigma^2)T}{\sigma\sqrt{T}}\right]\right\} \\
&= e^{ln(S_0)+rT}\left\{1-\Phi\left[\frac{\ln(K)-ln(S)-rT-.5\sigma^2T}{\sigma\sqrt{T}}\right]\right\} - K\left\{1-\Phi\left[\frac{\ln(K)-ln(S)-(r-.5\sigma^2)T}{\sigma\sqrt{T}}\right]\right\}
\end{aligned}$$

Notice that $N(-x) = 1-N(x)$, so the signs within the N( ) terms can be reversed. In addition, $\exp[\ln(S_0) + rT] = Se^{rT}$ and $-ln(K) + ln(S) = ln(S/K)$.

Making these two changes allows the above equation to be written as follows:

[23] This is the expected continuously compounded return, not the price of the stock S_T. Under this assumption about the stochastic process for the stock price, the price at time T is lognormally distributed with parameters equal to $ln(S_0) + (\mu - .5\sigma^2)T$ and $\sigma T^{1/2}$.

$$EM = S_0 e^{rT} \Phi\left[\frac{ln\,(\mathrm{S/K}) + (r + .5\sigma^2)T}{\sigma\sqrt{T}}\right] - K\,\Phi\left[\frac{ln\,(\mathrm{S/K}) + (r - .5\sigma^2)T}{\sigma\sqrt{T}}\right]$$

EM $= Se^{rT} N(d_1) - K\, N(d_2)$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The formula for the excess mean, *EM*, only reflects the expected value of the option payoffs at maturity. The present value for the option is then found by multiplying by e^{-rT}:

$$\begin{aligned} \text{Call} &= e^{-rT}\, E[max(S_T - K, 0)] \\ &= e^{-rT}\, [Se^{rT} N(d_1) - K\, N(d_2)] \\ &= S\, N(d_1) - Ke^{-rT} N(d_2) \end{aligned}$$

This is the Black-Scholes formula. The key step in the derivation is the ability to replace the realistic probability distribution for the stock price with the risk neutral probability distribution.

Appendix D – Two Useful Utility Functions

Quadratic Utility

This functional form assumes that utility is for a given level of consumption follows:

$$u(c_t) = -\frac{1}{2}(c_t - c^*)^2$$
$$u'(c_t) = -(c_t - c^*)$$

Notice that this utility function implies that marginal utility is a linear function of consumption.

Power Utility

This functional form assumes that utility is for a given level of consumption follows:

$$u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}$$
$$u'(c_t) = c_t^{-\gamma}$$

Notice that this utility function implies that marginal utility is an exponential function of consumption.

Appendix E – Derivation of Black-Scholes Using the Fundamental Pricing Formula[24]

To begin, note that by definition the stochastic discount factor, *m*, is such that the current price of the stock can be found as the expected value of the stochastic discount factor times the future stock price. Assume that the future stock price is governed by the following stochastic process:

$$dS = \mu S\,dt + \sigma S \varepsilon \sqrt{dt}$$

or equivalently,

$$\frac{dS}{S} = \mu dt + \sigma S \varepsilon \sqrt{dt}\ .$$

It should be intuitive that the functional form of the stochastic discount factor might also have to be described in a similar fashion. Using the notation Λ to refer to the continuous time stochastic discount factor, then the continuous time version of the stochastic discount factor that correctly prices the stock and the risk free bond can be written as[25]:

$$\frac{d\Lambda}{\Lambda} = -r\,dt - \frac{\mu - r}{\sigma}\varepsilon\sqrt{dt}$$

To see why this is the case, consider the value of the stock price today according to the fundamental pricing formula:

$$S_0 = E\left[\frac{\Lambda_T}{\Lambda_0} S_T\right] = \int \frac{\Lambda_T}{\Lambda_0} S_T df(\Lambda_T, S_T)$$

Calculating the value of this integral is easy once a few simplifications are noted. The integral shown is taken with respect to the *joint* distribution of Λ_T and S_T. However, since both of these quantities follow geometric Brownian motion, both of these values are lognormally distributed and their values at time *T* can be written as stochastic functions of a normally distributed error term, ε (see Footnote 23 for details):

$$S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\varepsilon}$$

$$\Lambda_T = \Lambda_0 e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon}$$

Now, the integral can be written directly as a function of the normal distribution, as follows:

$$\int \frac{\Lambda_T}{\Lambda_0} S_T df(\Lambda_T, S_T) = \int e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon} S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\varepsilon} f(\varepsilon)d\varepsilon$$

Plugging in the normal density function for $f(\varepsilon)$, removing the S_0 term from the integral and simplifying all of the terms in the exponential[26], this expression simplifies to:

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[24] This derivation is based on the presentation in Cochrane.

[25] This form of the stochastic discount function is not unique. Any random noise term of the form $\sigma_w w\sqrt{dt}$, where *w* is a standard normal random variable that is uncorrelated with the other terms, could be added to this equation without impacting the ability of the stochastic discount function to correctly price the stock. But since the call payoff only depends on S_T and S_T does not depend on this noise term, the noise term cannot affect the value of the call option. For simplicity then, the noise term can be ignored here.

[26] This final step of simplifying the expressions in the exponent is unlikely to be intuitive. The interested reader can expand the final expression to confirm that this is in fact identical to the previous formula.

$$S_0 \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}\left[\varepsilon - \left(\sigma - \left(\frac{\mu - r}{\sigma}\right)\sqrt{T}\right)\right]^2} d\varepsilon$$

Notice that this expression is merely the current stock price multiplied by an integral. However, the integral is identical to the form of a normal density function with a mean equal to $\sigma - \left(\frac{\mu - r}{\sigma}\right)\sqrt{T}$ and a standard deviation equal to 1.0. Since the integral from negative infinity to positive infinity for a normal density function is equal to 1.0, the entire expression is simply equal to the current stock price, S_0.

A similar set of steps can be used to confirm that this stochastic discount factor also correctly prices a risk-free bond with a fixed payoff of $1.

$$\begin{aligned}
\int \frac{\Lambda_T}{\Lambda_0}(1) df(\Lambda_T) &= \int e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon} f(\varepsilon) d\varepsilon \\
&= e^{-rT} \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}\left[\varepsilon = \left(\frac{\mu - r}{\sigma}\right)\sqrt{T}\right]^2} d\varepsilon \\
&= e^{-rT}
\end{aligned}$$

Now, it should be trivial to see that this stochastic discount factor also correctly values the option and in fact results in the Black-Scholes formula for the value of a European call option. The steps are similar to those shown above, with the only exception being that the integrals are not taken from negative infinity to positive infinity. Instead, the integrals range from the strike price to infinity.

A brief sketch of these steps is as follows:

$$\begin{aligned}
C_0 &= E\left[\frac{\Lambda_T}{\Lambda_0} \max(S_T - K, 0)\right] \\
&= \int_{S_T = K}^{\infty} \frac{\Lambda_T}{\Lambda_0}(S_T - K) df(\Lambda_T, S_T) \\
&= \int_{S_T = K}^{\infty} \frac{\Lambda_T}{\Lambda_0} S_T df(\Lambda_T, S_T) - K \int_{S_T = K}^{\infty} \frac{\Lambda_T}{\Lambda_0} df(\Lambda_T) \\
&= S_0 \frac{1}{\sqrt{2\pi}} \int_{S_T = K}^{\infty} e^{-\frac{1}{2}\left[\varepsilon - \left(\sigma - \left(\frac{\mu - r}{\sigma}\right)\sqrt{T}\right)\right]^2} d\varepsilon - Ke^{-rT} \frac{1}{\sqrt{2\pi}} \int_{S_T = K}^{\infty} e^{-\frac{1}{2}\left[\varepsilon - \left(\frac{\mu - r}{\sigma}\right)\sqrt{T}\right]^2} d\varepsilon \\
&= S_0 N(d_1) - Ke^{-rT} N(d_2)
\end{aligned}$$

Notice that in this derivation, I left the lower bounds of the integrals in terms of S_T until the very last step, and then I left out a lot of the details behind this final step. To see what was left out, notice that setting $S_T = K$ as the lower bound for the integral is equivalent to the following expression for ε:

$$S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\varepsilon} = K \Rightarrow \varepsilon = \frac{\ln(K) - \ln(S_0) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

The final step then is merely to recall that the integrals are equivalent to normal distributions with means equal to $\sigma - \left(\frac{\mu - r}{\sigma}\right)\sqrt{T}$ and $-\left(\frac{\mu - r}{\sigma}\right)\sqrt{T}$, respectively and standard deviations equal to 1.0. Therefore, the integrals can both be written in the form of the standard normal distribution function, N(x) evaluated at the quantity equal to the mean less the lower bound[27].

In the case of the first integral in the above expression, this is:

$$N\left(\left(\sigma-\left(\frac{\mu-r}{\sigma}\right)\sqrt{T}\right)-\frac{\ln(K)-\ln(S_0)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}-\right)$$

$$=N\left(\frac{\ln(S_0/K)+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$=N(d_1)$$

And in the case of the second integral, this is:

$$N\left(\left(-\left(\frac{\mu-r}{\sigma}\right)\sqrt{T}\right)-\frac{\ln(K)-\ln(S_0)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}-\right)$$

$$=N\left(\frac{\ln(S_0/K)+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$=N\left(d_1-\sigma\sqrt{T}\right)$$

$$=N(d_2)$$

Finally, plugging those expressions in to the second to last line of the derivation produces the Black-Scholes formula for the value of a European call option.

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[27] The integral when μ represents the mean and 1.0 is the standard deviation can be evaluated as follows:

$$\frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-\frac{1}{2}\left(\frac{x-\mu}{1.0}\right)^2}dx=1-N(a-\mu)=N(\mu-a)$$

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