A self- stabilizing algorithm for maximal matching in link-register model in $O(n\Delta^3)$ moves

Johanne Cohen¹, Georges Manoussakis^{*1}, Laurence Pilard² and Devan Sohier²

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<sup>1</sup>LRI, CNRS, Université Paris Sud, Université Paris-Saclay, France,

{ johanne.cohen,georges.manoussakis}@lri.fr

<sup>2</sup>LI-PaRAD, Université Versailles-St. Quentin, Université Paris-Saclay, France,

{ laurence.pilard, devan.sohier}@uvsq.fr
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Abstract

In the matching problem, each node maintains a pointer to one of its neighbor or to null, and a maximal matching is computed when each node points either to a neighbor that itself points to it (they are then called married), or to null, in which case no neighbor can also point to null. This paper presents a self-stabilizing distributed algorithm to compute a maximal matching in the link-register model under read/write atomicity, with complexity $O(n\Delta^3)$ moves under the adversarial distributed daemon, where Δ is the maximum degree of the graph.

Keywords: Self-stabilization, Maximal Matching.

1 Introduction

The matching problem consists in building disjoint pairs of adjacent nodes. The matching is maximal if no new pair can be built, *i.e.*, if among any two adjacent nodes, at least one of them is part of a pair. This problem has a wide range of applications in networking and parallel computing, such as the implementation of load balancing.

All nodes participating in the distributed matching algorithm maintain a variable, called the pointer, that can take any neighbor as value, or the special value null; a node pointing to null is single, and two nodes pointing one toward the other are called married. The set of pairs of married nodes constitutes the matching.

The matching problem definition is thus local; however, the building process needs to take long range phenomena into account, to avoid cycles in which node pointers form a cycle of size greater than 2. To break such symmetries, we suppose that all nodes have a unique identifier.

2 State of the art

The matching problem has recently received much attention, both in graph theory and in distributed computing.

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For instance, in graph theory, some (almost) linear time approximation sequential algorithm for the maximum weighted matching problem (the maximum matching being the largest matching in terms of the weight of the edges) have recently been studied [6, 13].

Self-stabilizing algorithms for computing maximal matching have been designed in various models (anonymous network [1] or not [14], weighted or unweighted, see [7] for a survey). For an unweighted graph, Hsu and Huang [9] gave the first self-stabilizing algorithm and proved a bound of $O(n^3)$ on the number of moves under a sequential adversarial daemon. Hedetniemi et al. [8] completed the complexity analysis proving a O(m) move complexity. Manne et al. [11] gave a self-stabilizing algorithm that converges in O(m) moves under a distributed adversarial daemon. Cohen et al. [2] extend this result and propose a randomized self-stabilizing algorithm for computing a maximal matching in an anonymous network. The complexity is $O(n^2)$ moves with high probability, under the adversarial distributed daemon.

All these algorithms work in a state model, in which nodes can directly access the variables of adjacent nodes. This model was introduced by Dijkstra in [3]. However, this model fails to capture the aynchronous phenomena that happen in many real-life distributed systems, which led [4] to introduce the link-register model. In this model, communications are abstracted by registers in which nodes can write and read values. Atomicity conditions define the granularity of the algorithm. A variety of atomicity conditions exist (see [10] for a survey and some results on the strength of the different atomicities).

Read/write atomic registers are registers associated to each (directed) link, in which one of the nodes can write, and the other read; each read or write operation on the register is atomic (meaning that it cannot be interrupted), but a read in the register can happen arbitrarily long after the previous write, forcing the reading process to act based on outdated values (as opposed to what happens in Dijkstra-type state model). Read/write atomic registers can be implemented over message-passing models (at a large cost however), as in [12] for instance. This kind of atomicity is the strongest, meaning that algorithm written under this model also solve the problem under the other classical models.

The possible occurrence of faults in the execution of the algorithm is taken into account with the paradigm of self-stabilisation, as defined in [3]. A fault (or a sequence of faults) can lead the system to an arbitrary configuration of the processes and registers, starting from which the execution (seen as a completely new execution starting from this arbitrary configuration) must eventually resume a correct behavior in finite time. A complete study of this notion can be found in [5].

We propose the first distributed self-stabilizing algorithm solving the matching problem in a link-register model with read/write atomicity. The algorithm is presented under the form of guarded rules (usual for state model algorithms, but as far as we are aware of, never used before for link-register algorithms). This allows to underscore the granularity of the model, each configuration being the result of the application of an arbitrary subset of guarded rules to the previous configuration.

The algorithm works as follows: the lowest id node of a pair proposes to its neighbor, that accept (or not) the proposition; then the marriage is confirmed in three steps. This scheme can be interrupted at any point, either because of another marriage being concluded, or because of a faulty initialization. Once the marriage is reached, the married nodes do not take any more move, and the algorithm is eventually silent. At worst, the algorithm has to take $O(n\Delta^3)$ moves before reaching a maximal matching.

3 Model

The system consists in a set of processors V and a set $E \subset V \times V$ of links. We suppose that communications are bidirectional, so that $(u, v) \in E \Rightarrow (v, u) \in E$. Considering two adjacent processors u and v (i.e., $(u, v) \in E$), there exists a register r_{uv} in which u is the only process allowed to write, and that v can read.

The set of neighbors of a process u is denoted by N(u) and is the set of all processes adjacent to u, and Δ is the maximum degree of G.

All nodes have the same variables; if var is a variable, var_u denotes the instance of this variable on process u. Each node u has a unique id id_u ; in the following, for the sake of simplicity, we do not distinguish between u and id_u .

In the matching problem, each node u maintains a variable $p_u \in N(u) \cup \{null\}$ indicating the neighbor it is married or attempting to marry.

A configuration describes the situation of the algorithm at a point in its execution: in the link-register model, it is the vector of the values of all variables and registers. In particular, a configuration solves the maximal matching if it is such that $\forall u, (p_u \neq null \Rightarrow p_{p_u} = u) \land (p_u = null \Rightarrow \forall v \in N(u), p_v \neq null)$. The first part of this specification means that if a node points to one of its neighbor, this neighbor points to it; the second one implies maximality: if a node points to no other node, none of its neighbors is in the same situation, since they could marry and create a larger matching.

The algorithm is presented under the form of a set of guarded rules. A guarded rule consists in a guard which is a predicate on the values of the variables of a node, and an atomic action that can be executed if the guard is true. To respect read/write atomicity, if a guard refers to the value of a neighbor's register (which implies the reading of this register), the associated action cannot write in a register. In particular, we decided to introduce the *Write* guarded rule, that writes the adequate value in a register; other actions never write in any register. Thus, all actions consist either in readings in neighbor's registers and taking local actions, or in writing in its own register, which respects the read/write atomicity. A guarded rule is activable in a configuration if its guard is true.

Link-register algorithms are generally presented as an infinite loop of readings, local actions and writings for each node, and the chosen atomicity allows to decide the points in the algorithm at which the execution of a node can be suspended to let another node take over. We chose to show more explicitly the points at which a node suspends action, and thus the result in terms of configuration of each of its atomic actions: the next configuration in an execution is obtained by applying one or several actions of guarded rules whose guards are true.

A daemon is a predicate on the executions. We consider only the most powerful one: the adversarial distributed daemon that, at each step, picks a nonempty subset of activable rules (at most one for each node) and executes them. The next configuration is then the result of the concurrent application of the actions of all these rules (as only one rule can be selected for each node, and that no action can modify the variables of other nodes, this concurrent application yields unambiguous result).

The moment when a process u writes in register r_{uv} is the time starting from which the written value r_{uv} is available to v, thus, the writing is analogous to a message reception by v in a message-passing model. A node u reads in its register r_{uv} in all guards. This allows it to check that the writing register of a node has reached its correct value. This can be paralleled with an acknowledgement.

An execution is then an alternate sequence of configurations and actions $\mathcal{E} = (C_0, A_0, \dots, C_i, A_i, \dots)$, such that:

• C_i is a configuration;

- A_i is a nonempty set of couples (R, u) such that R is activable for node u in configuration C_i , and no two of these couples concern the same node;
- C_{i+1} is obtained from C_i by executing all actions in A_i .

We then write $C_i \mapsto_{\mathcal{E}} C_{i+1}$, or $C_i \mapsto_{i+1} C_{i+1}$ if the execution is clear from the context, $C_i \mapsto^* C_j$ if $i \leq j$, and $C_i \mapsto^+ C_j$ if i < j; in this case, we say for any action $(R, u) \in A_k$ with $i \leq k \leq j$ that there has been a transition (R, u) between C_i and C_j .

An algorithm is self-stabilizing for a given specification, if there exists a sub-set \mathcal{L} of configurations such that every execution starting from a configuration in \mathcal{L} verifies the specification (correctness), and every execution, starting from any configuration, eventually reaches a configuration in \mathcal{L} (convergence). A configuration is stable if no process is activable in the configuration. The algorithm presented here, is silent: its maximal executions are finite and end in a stable configuration. We call legitimate a stable configuration verifying the maximal matching specification.

4 Algorithm

The presented algorithm is based on the algorithm by Manne *et al* [11] written under the state model. A marriage is contracted in two phases: (1) *the selection* of the edge to add to the matching and (2) *the confirmation* or the *lock* of the edge in the matching in three steps.

Variables description: Each node u has two local variables. Variable p_u is the identifier of the node u points to: nodes u and v are said to be married to each other if and only if u and v are neighbors, p_u points to v, and p_u points to u. We also use a variable m_u indicating the progress of u's marriage. $m_u \in \{0, 1, 2\}$.

Each node u has a four bit register r_{uv} for each of its neighbors v. The first two bits $r_{uv}.p$ can take the value Idle if u points to null (ie $p_u = null$), You if it points to v, and Other if it points to a node $\neq v$. The last two bits $r_{uv}.m$ can be 0, 1 or 2, indicating the progress of u's marriage.

Algorithm description: The Seduction and Marriage rules implement the selection of an edge for the matching: they set the p variable of a node to a candidate for a marriage. First, the node with the smallest id in a pair executes the Seduction rule, to which the node with the highest id can respond by executing the Marriage rule. This asymetrical process avoids situations with nodes trying to seduce neighbors in a cycle, that could be reproduced by an adversarial daemon.

Under read/write atomicity, an offset is possible between the value of the local variables of a node (p, m) and the value of its registers. In order to avoid infinite executions during which the distributed daemon lets a node u attempt to marry a neighbor v just at the same time when v abandons its attempt to marry u, and then conversely at the next step, it is necessary to design a mechanism locking progressively a marriage. We achieve that with variable m, which takes values in $\{0,1,2\}$: except for faulty initialization, $m_u = 0$ means that u did not start locking any marriage, and $m_u \geq 1$ means that u has a neighbor v such that $p_u = v \wedge p_v = u$. If $m_u = 1$ then the marriage lock is in progress and if $m_u = 2$ then the lock is done. m is incremented in the execution of rule Increase.

The Reset rule ensures that local variables p and m for a node u have consistent values. It is executed when predicate PRabandonment(u) or PRreset(u) is true. PRabandonment(u) means that u's marriage process should be restarted if u is trying to marry a node v < u that

is not seducing it, or if u is trying to seduce a node that is already married (at least, when the registers of v indicate that). This last case can happen when v is responding to several proposals at the same time, while the first one is provoked by "bad" initializations. PRreset(u) indicates a discrepancy between the steps taken in the locking mechanism by the two processes involved in it. This is due to "bad" initializations.

Predicates and functions of the algorithm:

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\begin{aligned} Correct\_register\_value(u,a) &\equiv \text{ if } p_u = null \text{ then return } (Idle,0) \\ &= \text{ else if } p_u = a \text{ then return } (You,m_u) \\ &= \text{ else return } (Other,m_u) \end{aligned} PRabandonment(u) &\equiv \left[ p_u \neq null \land (r_{p_u}.p \neq You \land (u > p_u \lor m_u \neq 0)) \\ \lor (r_{p_u}u = (Other,2) \land u < p_u) \right] \end{aligned} PRreset(u) &\equiv (p_u \neq null) \land (r_{p_u}.p = You) \land (\\ (m_u = 0 \land r_{p_u}u.m = 2) \\ \lor (m_u = 0 \land r_{p_u}u.m = 2) \\ \lor (m_u = 2 \land r_{p_u}u.m = 0) \\ \lor (m_u = 1 \land r_{p_u}u.m = 1 \land u > p_u) \\ \lor (m_u = 1 \land r_{p_u}u.m = 2 \land u < p_u) \\ \lor (m_u = 1 \land r_{p_u}u.m = 2 \land u < p_u) \\ \lor (m_u = 2 \land r_{p_u}u.m = 1 \land u > p_u)) \end{aligned}
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Rules for each node u:

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\forall \mathbf{a} \in \mathbf{N}(\mathbf{u}), \\ \mathbf{Write}(\mathbf{a}) \quad :: \quad r_{ua} \neq Correct\_register\_value(u, a) \\ \quad \rightarrow r_{ua} := Correct\_register\_value(u, a) \\ \mathbf{Seduction}(\mathbf{a}) :: \quad p_u = null \ \land \ r_{ua} = Correct\_register\_value(u, a) \\ \quad \wedge \ r_{au} = (Idle, 0) \land (u < a) \quad \rightarrow (p_u, m_u) := (a, 0) \\ \mathbf{Marriage}(\mathbf{a}) \quad :: \quad p_u = null \ \land \ r_{ua} = Correct\_register\_value(u, a) \\ \quad \wedge \ r_{au} = (You, 0) \land (u > a) \quad \rightarrow (p_u, m_u) := (a, 0) \\ \mathbf{Increase} \quad :: \quad p_u \neq null \land r_{up_u} = Correct\_register\_value(u, p_u) \\ \quad \wedge \ (r_{p_uu}.p = You) \land (\\ \quad (m_u = 0) \land [(u < p_u \land r_{p_uu}.m = 1) \lor (u > p_u \land r_{p_uu}.m = 0)] \\ \quad \vee \ (m_u = 1) \land [(u < p_u \land r_{p_uu}.m = 1) \lor (u > p_u \land r_{p_uu}.m = 2)] \\ \quad ) \rightarrow \\ \quad m_u := m_u + 1 \\ \mathbf{Reset} :: \quad p_u \neq null \land r_{up_u} = Correct\_register\_value(u, p_u) \\ \quad \wedge \ (PRabandonment(u) \lor PRreset(u)) \quad \rightarrow (p_u, m_u) := (null, 0) \\ \end{cases}
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About the rules: A node u has $3 \deg(u) + 2$ rules: one rule Write, Seduction and Marriage for each neighbor, plus one rule Increase and one Reset, that are not associated to a link. Given a neighbor v of u, u can be activable only for rule Marriage(v) or Seduction(v), because the first one necessitates that u > v and the second that u < v.

Definition 1 (v-Increase/v-Reset, v-rule and R(-) rule) Let u and v be two nodes. We say that node u is activable in the configuration C for a v-Increase (resp. v-Reset) rule if, in C, u is activable for an Increase (resp. Reset) rule and $p_u = v$. We say that node u is eligible in the configuration C for a v-rule if u is eligible for one of the following rule: $\{Write(v), Seduction(v), Marriage(v), v-Increase, v-Reset\}$ in C. Finally, let R be any rule

among Write, Marriage and Seduction. We say that u is eligible for a R(-) rule, if there exists a neighbor of u, say a, such that u is eligible for a R(a) rule.

Observation 1 Let u be a node and C be a configuration. In C, we have:

- 1. if $p_u = null$ then:
 - u is not eligible for Increase nor Reset;
 - $\forall v \in N(u) : u \text{ is eligible for at most one rule among the set of rules } \{Write(v), Marriage(v), Seduction(v)\};$
- 2. if $p_u \neq null$ then:
 - u is not eligible for Marriage(-) nor Seduction(-);
 - u is eligible for at most one rule among the set of rules $\{Write(p_u), Increase, Reset\}$; moreover, if this rule is an Increase (resp. Reset), this is necessarily a p_u -Increase (resp. p_u -Reset);
 - and $\forall x \in N(u) \setminus \{p_u\}$: among all the x-rules, u can only be eligible for Write(x);

Execution examples: Below is an execution of the algorithm under the adversarial distributed daemon. Figure 1a shows the initial state of the execution. Node identifiers are indicated inside the circles. Black arrows show the content of the local variable p and the absence of arrow means that p = null. s:Write(t) means that node s executes the Write(t) rule.

Consider an initial configuration (Figure 1a in which variable and register values as follows: $(p_s, m_s) = (p_t, m_t) = (null, 0), r_{st} = (You, 2)$ and $r_{ts} = (Idle, 0)$. Thus, initially, nodes s and t are not matched. Since the local variables of s are not consistent with register r_{st} , node s executes Write(t) in order to set $r_{st} = (Idle, 0)$ (Figure 1b). In this execution, we take s < t.

Now, since s and t are not matched they can start a selection process in order to marry. The node with the smallest identifier, s, starts the process and thus, since $r_{ts} = (Idle, 0)$, s executes a Seduction(t) rule in order to set p_s to t. s is then eligible to execute a series of Write(v) rules to update its registers. Consider an execution it executes at some point Write(t) (Figure 1c).

Once register r_{st} is updated, node t answers to the proposition of s by executing a Marriage(s) rule, setting $p_t = s$. It is then eligible to execute a series of Write(v) rules to update its registers. As long as t does not update its r_{ts} register, the process of locking the marriage cannot start since s needs $r_{ts}.p = You$ in order to start increasing its m variable. So assume t updates its r_{ts} register with a Write(s) (Figure 1d).

From this point, both nodes point towards each other. The locking process of the marriage starts from this point. First, the node with the highest identifier, that is t, sets its m variable to 1 with a s-Increase and then updates its registers (see Figure 1e). Then node s executes a t-Increase and sets $m_s = 1$ followed by a Write(t) rule to update its register (Figure 1f). It executes yet another t-Increase rule to set $m_s = 2$ (Figure 1g). This execution of two consecutive t-Increase by s guarantees that t has correct register values. Finally, after registers have been updated accordingly, t executes a last s-Increase rule to set $m_t = 2$. At this point the matching of s and t is locked.

One might wonder why the Increase rule is not alternately executed by s and t. Indeed, s executes two consecutive t-Increase to set its m variable from 0 to 1 and then to 2, while t does not change its m value. Actually, the algorithm does not converge if nodes would perform the Increase rule alternately. These two consecutive Increase are a key point on the lock process of a marriage.

Figure 1: A typical execution of the algorithm. The absence of arrow means that the p-variable is equal to null.

5 Proof

5.1 State of an edge

We first focus on an edge (s,t) of the matching $\mathcal{M} = \{(a,b) \in E : p_a = b \land p_b = a\}$ built by our algorithm when s < t. In particular, we focus on values of local variables and registers of this edge in some chosen configurations.

Definition 2 Let (s,t) be an edge with s < t. We say that in a configuration C, the edge (s,t) is in state (You, α, β) if $(p_s = t \land p_t = s) \land (m_s = \alpha \land m_t = \beta)$

If an edge (s,t) is in state (You, α, β) , then this edge belongs to the matching. Unfortunately, due to some "bad" initialization for instance, this edge can be removed from the matching at some point of the execution. In the following, we characterize an edge that belongs to the matching and that will forever remain in it.

A correct state corresponds to the situations appearing in the Figure 1 starting from step (d). Starting from a configuration where edge (s,t) is in a correct state, the two nodes, one after the other, execute Increase and Write rules. A link is in an updated correct state when all registers of the edge are updated (and so exactly one node among s and t is eligible for an Increase), while it is toUpdate when the register of one of the two nodes is not up to date (and so exactly one node among s and t is eligible for a Write).

Definition 3 (Updated correct state) Consider an edge (s,t) with s < t in state (You, α, β) in a configuration C. This link is in an updated correct state if

$$(r_{st} = (You, \alpha) \land r_{ts} = (You, \beta)) \land (\alpha, \beta) \in \{(0, 0), (0, 1), (1, 1), (2, 1), (2, 2)\}$$

Definition 4 (toUpdate correct state) Let (s,t) be an edge with s < t in the state (You, α, β) in a configuration C. This state is said to be a toUpdate correct state if

$$[(\alpha, \beta) \in \{(0, 1), (2, 2)\} \land (r_{st} = (You, \alpha) \land r_{ts} = (You, \beta - 1))]$$

$$\lor [(\alpha, \beta) \in \{(1, 1), (2, 1)\} \land (r_{st} = (You, \alpha - 1) \land r_{ts} = (You, \beta))]$$

Definition 5 (Correct state) Let (s,t) be an edge with s < t in the state (You, α, β) in a configuration C. This state is said to be correct if the state is an updated or a to Update correct state.

All four previous definitions deal with an edge in which the first node has a smaller identifier than the second node. In the following, we will write (s,t) to denote such an edge. This constraint is due to the fact that nodes execute their *Increase* rule one after the other in a specific order, and a link in state (You, 0, 1) can be correct while a link in state (You, 1, 0) never is. When we do not make any assumption on which node has the smallest identifier in an edge, we use the notation (u, v).

We now state that a node in an edge in a correct state is only activable for Increase and Write and that an edge in a correct state will forever remain in a correct state.

Lemma 1 Let (s,t) be an edge with s < t. Let C be a configuration. If (s,t) is in a correct state (You, α, β) in C then neither s nor t is eligible for Seduction(-), Marriage(-) or Reset in C:

Proof: Neither s nor t are eligible for a Seduction(-) or a Marriage(-) rule in C, since $p_s \neq null$ and $p_t \neq null$ in C. Since $p_s = t$, then s is not eligible for a x-Reset if $x \neq t$. We now study the case of a t-Reset. PRabandonment(s) is False since $r_{ts}.p = You$. In C, we have: $(m_s, r_{ts}.m) \in \{(0,0), (0,1), (1,1), (2,1), (2,2)\}$ and $r_{ts}.p = You$ and s < t. So, PRreset(s), does not hold in C. Thus, s is not eligible for Reset in C. Since $p_t = s$, then t is not eligible for a x-Reset if $x \neq s$. We now study the case of a s-Reset. PRabandonment(t) is False since $r_{st}.p = You$. In C, we have: $(m_s, r_{ts}.m) \in \{(0,0), (1,0), (1,1), (1,2), (2,2)\}$ and $r_{st}.p = You$ and s < t. So, PRreset(t), does not hold in C. Thus, t is not eligible for Reset in C. \Box

In fact, our algorithm is designed in such a manner that, once an edge is in a correct state, it remains in a correct state forever.

Lemma 2 Let (s,t) be an edge with s < t. Let C be a configuration. If (s,t) is in a correct state (You, α, β) in C then

- (s,t) forever remains in a correct state;
- if neither s nor t is eligible for any rule in C, then (s,t) is in the updated correct state (You, 2, 2) in C.

Thus, we are now going to perform a case study: we check for all possible correct state in C, which rules s (resp. t) is eligible for among the rules t-Increase and Write(t) (resp. s-Increase and Write(s)). e call these rules the relevant rules since these are the only one that can change the state of (s,t). For all possible transitions $C \mapsto C'$ that contains at least one of these rules, we prove that (s,t) is in a correct state in C'. The two following tables present this case study.

The α and β values in C are given in column 1. Column 2 gives the values of the quadruplet $(m_s, m_t, r_{st}.m, r_{ts}.m)$, according to the values of α and β . Columns 3 and 4 give rules s and t are eligible for.

Observe that at each line, there is at most one node among s and t that is eligible for a relevant rule in C. If this node does not perform any rule in the transition starting in C, then the state of edge (s,t) remains constant and the proof is done. Otherwise, we obtain the considered configuration in the tables, called D. Thus, in column 5 and 6, we give the state that is reached after s or t performs its rule. Observe that the last line of table 1 does not contain any value in the last two columns because there is no such a D configuration since neither s nor t is eligible for a relevant rule.

In the following table, we assume (s,t) is in an updated correct state in C: (toUpdateCS) means toUpdate correct state)

	in	\overline{C}	in D		
(α, β)	$(m_s, m_t, r_{st}, r_{ts})$	relevant rules eligibility		(m m r r r)	state of (s,t)
		for s	for t	$(m_s, m_t, r_{st}, r_{ts})$	state or (s,t)
(0,0)	(0,0,0,0)	Ø	s-Increase	(0,1,0,0)	toUpdateCS $(You, 0, 1)$
(0,1)	(0,1,0,1)	t-Increase	Ø	(1, 1, 0, 1)	toUpdateCS $(You, 1, 1)$
(1,1)	(1, 1, 1, 1)	t-Increase	Ø	(2,1,1,1)	toUpdateCS $(You, 2, 1)$
(2,1)	(2,1,2,1)	Ø	s-Increase	(2, 2, 2, 1)	toUpdateCS $(You, 2, 2)$
(2,2)	(2, 2, 2, 2)	Ø	Ø	-	-

In table 2, we assume (s,t) is in a toUpdate correct state in C: (updatedCS means updated correct state)

	in	C	in D		
(α, β)	$(m_s, m_t, r_{st}, r_{ts})$	relevant rules eligibility		(m m, r, r,	state of (s,t)
		for s	for t	$(m_s, m_t, r_{st}, r_{ts})$	state of (s,t)
(0,1)	(0,1,0,0)	Ø	Write(s)	(0,1,0,1)	updatedCS $(You, 0, 1)$
(1,1)	(1, 1, 0, 1)	Write(t)	Ø	(1, 1, 1, 1)	updatedCS $(You, 1, 1)$
(2,1)	(2,1,1,1)	Write(t)	Ø	(2,1,2,1)	updatedCS $(You, 2, 1)$
(2,2)	(2, 2, 2, 1)	Ø	Write(s)	(2, 2, 2, 2)	updatedCS $(You, 2, 2)$

Corollary 1 Let (s,t) be an edge with s < t. Let C be a configuration. If (s,t) is in the correct state (You, α, β) in C then neither s nor t is eligible for Seduction(-), Marriage(-) or Reset from C.

5.2 Correctness Proof

Definition 6 A configuration is called stable if no node can execute a rule in this configuration.

In particular, a configuration is stable iff all guards are false.

We now show that if our algorithm reaches a stable configuration then p-values define a maximal matching. The matching built by the algorithm is $\mathcal{M} = \{(u, v) \in E : p_u = v \land p_v = u\}$.

9

Lemma 3 Let u be a node. In any stable configuration

$$\forall v \in N(u) : r_{uv} = Correct_register_value(u, v)$$

Proof: Let C be a stable configuration. If $\exists v \in N(u) : r_{uv} \neq Correct_register_value(u, v)$ in C, then u is eligible for a Write(v) rule and C is not stable.

Lemma 4 Let u be a node. In any stable configuration:

$$p_u = null \Rightarrow \forall v \in N(u) : p_v \not\in \{null, u\}$$

Proof: Let C be a stable configuration where $p_u = null$. Consider a neighbor v. After Lemma 3, $r_{uv} = (Idle, 0)$.

If $p_v = null$ then we can assume without loss of generality that u < v. Then, according to Lemma 3, $r_{vu} = (Idle, 0)$. Thus u is eligible for a Seduction(v) rule and C is not stable.

If $p_v = u$ and u < v then PRabandonment(v) holds since $r_{uv} \neq You$ and then v is eligible for a Reset rule and C is not stable. Finally, if $p_v = u$ and v < u then, according to Lemma 3, $r_{vu} = (You, m_v)$. Then either $m_v = 0$ and so u is eligible for a Marriage(v) rule, or $m_v \neq 0$ and so PRabandonment(v) holds and then v is eligible for a Reset rule. In both cases, C is not stable.

Lemma 5 Let (s,t) be an edge with s < t. In any stable configuration, if $p_s = t$ and $p_t = s$ then edge (s,t) is in the updated correct state (You, 2, 2).

Proof: Consider the state (You, α, β) of edge (s, t) in a stable configuration C. Observe that if an edge (s, t) is in a correct state, then edge (s, t) is in the updated correct state (You, 2, 2) from Lemma 2, point 2.

We now prove by contradiction that this is the only possible case. Assume that the edge (s,t) is not in a correct state. First observe that from Lemma 3, we have $r_{st} = Correct_register_value(s,t)$, and node s (resp. t) is not eligible for a Write(t) (resp. Write(s)) rule. This implies that if (s,t) is in the state (You,α,β) , then $r_{st} = (You,\alpha)$ and $r_{ts} = (You,\beta)$. Thus, according to Definition 3, the only remaining possibilities for (α,β) are (2,0), (0,2), (1,0) and (1,2). In all of these cases, s is activable for a Reset rule which contradicts the fact that C is a stable configuration.

Lemma 6 Let (u,v) be an edge. In any stable configuration, $p_u = v$ if and only if $p_v = u$.

Proof:

Assume, by contradiction, that $p_u = v$ and $p_v \neq u$. By Lemma 4, $p_v \neq null$, thus $\exists v_1 \in N(v) : p_v = v_1$ with $v_1 \neq u$.

If u > v, after Lemma 3, $r_{vu}.p = Other$ (i.e., $\neq You$). So the predicate PRabandonment(u) holds and node u is eligible for a Reset rule. Thus the configuration is not stable, which is impossible. If u < v and $m_v = 2$ then Lemma 3 implies that $r_{vu} = (Other, 2)$. Then the predicate PRabandonment(u) holds and u is eligible for a Reset rule. Thus the configuration is not stable neither.

Thus $(p_u = v \land p_v \neq u) \Rightarrow (u < v \land m_v \in \{0,1\} \land \exists v_1 \in N(v) \setminus \{u\} : p_v = v_1).$

Suppose by contradiction $p_{v_1} = v$. From Lemma 5, the edge (v_1, v) is in updated correct state (You, 2, 2). This implies that $m_v = 2$ which contradicts the fact $m_v \in \{0, 1\}$. So, $p_{v_1} \neq v$. Using the same argument for edge (u, v), we can deduce: $v < v_1 \land m_{v_1} \in \{0, 1\} \land \exists v_2 \in N(v_1) \setminus \{v\} : p_{v_1} = v_2$). Now we can continue the construction in the same way. We construct a path $(u, v, v_1, v_2, \ldots, v_r, \ldots)$ where $\forall i \geq 1 : p_{v_i} = v_{i+1} \land v_i < v_{i+1} \land m_{v_i} \in \{0, 1\}$. Since the

number of nodes is finite, there exists a node v_{y_1} that appears at least twice in the path. Thus, this path contains the cycle $(v_{y_1}, v_{y_2}, \dots, v_r, v_{y_1})$ and by construction, we have $v_r < v_{y_1}$ and $v_{y_1} < v_r$. This gives the contradiction.

From these Lemmas, we deduce:

Theorem 1 In any stable configuration, the set of edges $\mathcal{M} = \{(u, v) \in E : p_u = v \land p_v = u\}$ is a maximal matching.

Proof: Let C be a stable configuration. By definition, the constructed set of edge \mathcal{M} is a matching. From Lemma 6, any node u such that $p_u = v$ is in \mathcal{M} . Lemma 4 implies that in C no edge (u, v) is such that $p_u = \bot$ and $p_u = \bot$. So, \mathcal{M} is maximal.

5.3 Convergence Proof

The three following lemmas put in relation the number of moves of all rules except the Write rule.

5.4 Relationship between the all rules except the Write rule

Lemma 7 Let (s,t) be an edge with s < t. Let \mathcal{E} be an execution containing two transitions $C_0 \mapsto C_1 \mapsto^* D_0 \mapsto D_1$ during which t executes a Marriage(s) rule. Then s executes a Seduction(t) rule between C_1 and D_0 .

Proof: From the Marriage(s) rule, we have $r_{st} = (You, 0)$ in C_0 and D_0 . Moreover, according to Lemma 11, t executes a Reset rule between C_1 and D_0 . Let $C_4 \mapsto C_5$ be the transition where it does for the first time. Then $p_t = s$ from C_1 to C_4 .

In the first step of the proof, we will prove that there exists two configurations γ and γ' between C_0 and D_0 and with $\gamma \mapsto^+ \gamma'$, such that $(p_s, m_s) \neq (t, 0)$ in γ and $(p_s, m_s) = (t, 0)$ in γ' . Then, we will prove that s must execute a Seduction(t) rule between γ and γ' . For the first step of the proof, we study two cases according to the value of r_{st} in C_4 .

First, assume that $r_{st} \neq (You, 0)$ in C_4 . So, there exists two transitions $C_2 \mapsto C_3$ and $C_6 \mapsto C_7$, with $C_0 \mapsto^* C_2 \mapsto^+ C_4$ and $C_4 \mapsto^* C_6 \mapsto^+ D_0$ such that: (i) in $C_2 \mapsto C_3$, s executes Write(t) to set r_{st} to a couple $\neq (You, 0)$ and so $(p_s, m_s) \neq (t, 0)$ in C_2 ; and (ii) in $C_6 \mapsto C_7$, s executes Write(t) to set r_{st} to (You, 0) and so $(p_s, m_s) = (t, 0)$ in C_6 . We then have $\gamma = C_2$ and $\gamma' = C_6$.

Second, if $r_{st} = (You, 0)$ in C_4 , then PRabandonment(t) is false and according to the Reset rule and in particular to the PRreset(t) predicate, $m_t = 2$ in C_4 (it is the only possible case as s < t and $r_{st}.m = 0$). Since $m_t \neq 2$ in C_1 , there exists a transition $C_2 \mapsto C_3$ between C_1 and C_4 where t executes an Increase rule in order to write 2 in its m-variable. Since $p_t = s$ from C_1 to C_4 , then $p_t = s$ in C_2 and then, according to the Increase rule, in C_2 $r_{st} = (You, 2)$. Since $r_{st} = (You, 0)$ in C_0 and C_4 , then there exists two transitions $B_0 \mapsto B_1$ and $B_2 \mapsto B_3$, with $C_0 \mapsto^* B_0 \mapsto^+ C_2$ and $C_2 \mapsto^* B_1 \mapsto^+ C_4$ such that: (i) in $B_0 \mapsto B_1$, s executes Write(t) to set r_{st} to (You, 2) and so $(p_s, m_s) = (t, 2)$ in B_0 ; and (ii) in $B_2 \mapsto B_3$, s executes Write(t) to set r_{st} to (You, 0) and so $(p_s, m_s) = (t, 0)$ in B_2 . We then have $\gamma = B_0$ and $\gamma' = B_2$.

We now prove the second step of the proof, that is s must execute a Seduction(t) rule between γ and γ' . If $p_s \neq t$ in γ , and since s < t, then s must execute a Seduction(t) rule between γ and γ' in order to set t in its p-variable. Otherwise, we have $p_s = t \land m_s \neq 0$ in γ . Let us assume that s does not perform any Seduction(t) rule between γ and γ' . Thus, the only two rules to write 0 in its m-variable are Marriage(-) and Reset. Since s < t, s cannot

execute a Marriage(t) rule, thus after writing 0 in its m variable, $p_s \neq t$ and we go back to the case 1, leading to the conclusion that s must perform a Seduction(t) rule in order to write t in its p-variable. Finally, in any cases, s must execute a Seduction(t) rule between γ and γ' . \square

Lemma 8 Let u be a node. Let \mathcal{E} be an execution where u executes at least two Reset moves. Let $C_0 \mapsto C_1 \mapsto^* C_2 \mapsto C_3$ be two transitions corresponding to two consecutive Reset rule executed by u. Then u executes a rule in $\{Seduction(-), Mariage(-)\}$ once between C_1 and C_2 .

Proof: According to the Reset rule, $p_u = null$ in C_1 and $p_u \neq null$ in C_2 . so, u has to execute a rule between C_1 and C_2 to set a neighbor identifier in its p-variable. There are only two rules doing that: the Seduction and the Marriage rules. Thus, u executes such a rule at least once between C_1 and C_2 . Now, assume that node u executes such a rule more than once between C_1 and C_2 . Then, from Lemma 11, u executes a Reset rule between C_1 and C_2 . This contradicts the fact that node u does not execute any Reset rule between C_1 and C_2 . Thus, the lemma holds.

Lemma 9 Let u be a node. Let \mathcal{E} be an execution where u executes at least three Increase moves. Let $C_0 \mapsto C_1$, and $C_2 \mapsto C_3$ and $C_4 \mapsto C_5$ be three transitions corresponding to three consecutive Increase rules executed by u. Then u executes a Reset rule once between C_0 and C_5 .

Proof: We now prove this lemma, by contradiction. Let us assume node u does not executes an Reset rule a between C_0 and C_5 .

According to the *Increase* rule, $m_u \in \{1,2\}$ in C_3 and C_5 . According to Lemma 11, if $m_u = 1$ in C_3 (resp. C_5) then u executes a *Reset* rule between C_1 and C_2 (resp. C_3 and C_4). This contradicts the fact that node u does not execute any *Reset* rule between C_1 and C_2 . Thus, $m_u = 2$ in C_3 and C_5 . And so $m_u = 2$ in C_3 and $m_u = 1$ in C_4 .

There is only one way to decrease the value of an m variable: to write 0. Thus u has to execute an Reset rule between C_3 and C_4 in order to decrease the value m_u . However this contradicts the fact that node u does not execute any Reset rule between C_0 and C_5 .

These lemmas bound the number of Marriage (Lemma 7), Reset (Lemma 8) and Increase (Lemma 9) in function of the number of Seduction. Then an upper bound on the number of Write follows since one modification of the local variables of u leads u to execute at most deg(u) Write. So, in the following, we present the sketch of the proof leading to an upper bound on the number of Seduction. In the Lemma below, we state that the number of Seduction(t) by s is strongly connected to the number of times that t writes 2 in m_t .

Lemma 10 Let (s,t) be an edge with s < t. Let \mathcal{E} be an execution containing three transitions $D_0 \mapsto D_1 \mapsto^* D_2 \mapsto D_3 \mapsto^* D_4 \mapsto D_5$ where s executes a Seduction(t) rule. Then there exists a transition $D \mapsto D'$ between D_2 and D_4 where t executes a Write(s) rule and with in D: $p_t \neq null$ and $m_t = 2$.

From the previous Lemma, we know that t has to write 2 in its m-variable for s to reset a previous Seduction(t). The next Theorem gives conditions where the value 2 in a m-variable corresponds to a locked marriage and thus yields to a situation where s cannot seduce t anymore.

Theorem 2 Let (u,v) be an edge. Let \mathcal{E} be an execution. If \mathcal{E} contains two transitions $A_0 \mapsto A_1 \mapsto^* A_2 \mapsto A_3 \mapsto^* A_4$ such that :

• in $A_0 \mapsto A_1$, v executes a u-rule;

- in $A_2 \mapsto A_3$, u executes a Reset rule;
- and in A_4 , $(p_u, m_u) = (v, 2)$;

then the edge (min(u, v), max(u, v)) is in a correct state in A_4 .

5.5 Number of moves for the Seduction Rule

Theorem 3 Let (s,t) be an edge with s < t. The number of Seduction(t) rules executed by node s is in $O(\Delta)$.

Proof: By contradiction. Let \mathcal{E} be an execution where s executes a Seduction(t) rule at least $2\Delta + 4$ times. We are going to show that such an execution is not possible since after the $(2\Delta + 3)^{th}$ Seduction(t) execution, s cannot perform any other Seduction(t) rule.

For $0 \le i \le (2\Delta + 3)$, let $A_i \mapsto B_i$ be the transition where s executes its i^{th} Seduction(t) rule in \mathcal{E} .

According to Lemma 19, between each couple of configurations (B_j, A_{j+1}) where $1 \le j \le (2\Delta + 1)$, there exists a transition $C_j \mapsto D_j$ where t executes a Write(s) rule and with $p_t \ne null$ and $m_t = 2$ in C_j .

Since t has at most Δ neighbors and since there are $2\Delta + 1$ such transitions $C_j \mapsto D_j$, then there exists at least one neighbor of t that appears at least 3 times in p_t among these C_j . More formally, $\exists x \in N(t) : \exists$ distinct $a, b, c \in [1, ..., 2\Delta + 1] :: p_t = x$ in C_a , C_b and C_c . Let us assume w.l.o.g that a < b < c.

First, let us prove that $x \neq s$. By contradiction. The we consider the three transitions $A_0 \mapsto B_0$, $A_a \mapsto B_a$ and $A_{a+1} \mapsto B_{a+1}$. Observe that the execution starting in configuration A_a reaches all the assumptions made in Lemmas 15. Indeed, before A_a , both s executes a trule (Seduction(t)) and t executes a Reset by Lemma 12. Thus, when $(p_t, m_t) = (s, 2)$ in configuration C_a , the edge (s, t) is in a correct state. Thus, by Corollary 1, from this configuration s cannot execute any Seduction, which contradict the fact that it does in transition $A_{b+1} \mapsto B_{b+1}$. So $x \neq s$.

According to Lemma 18, between C_a and C_b , x executes a Write(t) rule and between C_b and C_c , t executes a Reset rule. Finally, in C_c , we have $(p_t, m_t) = (x, 2)$. Thus by Theorem 2, the edge (min(t, x), max(t, x)) is in a correct state in C_c . From Corollary 1, from this configuration t cannot execute any Reset. However, since s executes two Seduction(t) rules in $A_{2\Delta+2} \mapsto B_{2\Delta+2}$ and $A_{2\Delta+3} \mapsto B_{2\Delta+3}$ and by Lemma 12, t executes a Reset between $B_{2\Delta+2}$ and $A_{2\Delta+3}$ which leads the contradiction

Lemma 11 Let u be a node. Let \mathcal{E} be an execution containing two transitions $C_0 \mapsto C_1$ and $C_2 \mapsto C_3$ with $C_1 \mapsto^* C_2$ where u executes a rule.

- 1. If u executes an Increase rule during these two transitions and if $m_u = 1$ in C_3 , then u executes a Reset rule between C_1 and C_2 .
- 2. If u executes a Seduction(-) or a Mariage(-) rule during these two transitions, then u executes a Reset rule between C_1 and C_2 .

Proof: We start by proving the first point. According to the *Increase* rule, $p_u \neq null$ in C_0 and u writes 1 or 2 in m_u during the transition $C_0 \mapsto C_1$. So $m_u \neq 0$ and $p_u \neq null$ in C_1 . Since $m_u = 1$ in C_3 , then $m_u = 0$ in C_2 . Thus either u sets its m variable to 0 executing a Reset rule between C_1 and C_2 and the proof is done, or u executes a Marriage(-) or a Seduction(-) rule, let say in transition $C \mapsto C'$. Then we have $p_u = null$ in C. Since $p_u \neq null$ in C_1 , then u must execute a Reset rule between C_1 and C.

We now prove the second point. According to both rules Seduction and Marriage, $p_u \neq \bot$ in C_1 and $p_u = \bot$ in C_2 . Thus u must execute a Reset rule between C_1 and C_2 in order to set p_u to \bot .

Lemma 12 Let (s,t) be an edge with s < t. Let \mathcal{E} be an execution containing two transitions $C_0 \mapsto C_1$ and $D_0 \mapsto D_1$ with $C_1 \mapsto^* D_0$ where s executes a Seduction(t) rule. We have: both s and t execute Reset between C_1 and D_0 .

Proof: From the Seduction(t) rule, we have $r_{ts} = (Idle, 0)$ in C_0 and D_0 . Moreover, according to Lemma 11, s executes a Reset rule between C_1 and D_0 . Let $C_4 \mapsto C_5$ be the transition where it does for the first time.

We now study two cases: in C_4 , r_{ts} is either equal or different from (Idle, 0).

If it is different then there exists two transitions $C_2 \mapsto C_3$ and $C_6 \mapsto C_7$, with $C_0 \mapsto^* C_2 \mapsto^+ C_4$ and $C_4 \mapsto^* C_6 \mapsto^+ D_0$ such that: (i) in $C_2 \mapsto C_3$, t executes Write(s) to set r_{ts} to a couple $\neq (Idle, 0)$ and so $p_t \neq null$ in C_2 ; and (ii) in $C_6 \mapsto C_7$, t executes Write(s) to set r_{ts} to (Idle, 0) and so $p_t = null$ in C_6 . Thus, there exists a transition between C_3 and C_6 where t executes a Reset move.

Now, if $r_{ts} = (Idle, 0)$ in C_4 . Then, PRreset(s) is false and according to the Reset rule and in particular to the PRabandonment(s) predicate, $m_s \neq 0$ in C_4 . Moreover, from the Seduction(t) rule, $m_s = 0$ in C_1 . Thus there exists a transition $C_2 \mapsto C_3$ between C_1 and C_4 where s executes an Increase rule. Since s executes its first Reset from C_1 in $C_4 \mapsto C_5$, then $p_s = t$ from C_1 to C_4 , and so $p_s = t$ in C_2 . According to the Increase rule, in C_2 $r_{ts} = (You, 1)$. Since $r_{ts} = (Idle, 0)$ in C_0 and C_4 , then there exists two transitions $B_0 \mapsto B_1$ and $B_2 \mapsto B_3$, with $C_0 \mapsto^* B_0 \mapsto^+ C_2$ and $C_2 \mapsto^* B_2 \mapsto^+ C_4$ such that: (i) in $B_0 \mapsto B_1$, t executes Write(s) to set r_{ts} to (You, 1) and so $(p_t, m_t) \neq (s, 1)$ in B_0 ; and (ii) in $B_2 \mapsto B_3$, t executes Write(s) to set r_{ts} to (Idle, 0) and so $(p_t, m_t) = (\bot, 0)$ in B_2 . Thus, there exists a transition between B_1 and B_2 where t executes a Reset move.

Lemma 13 Let (s,t) be an edge with s < t. Let \mathcal{E} be an execution. If \mathcal{E} contains two configurations L_0 and L_1 with $L_0 \mapsto^* L_1$ and such that:

- s executed at least one t-rule before L_0 ;
- $r_{st} = (You, 0) \text{ in } L_0;$
- $(p_s, m_s) = (t, 2)$ in L_1 ;

then, s executes a t-Increase to set $m_s = 2$ between L_0 and L_1 .

Proof: In L_0 there are two cases concerning the value of p_s : either $p_s = t$ or not. We consider the first case where $p_s = t$. Let $D_0 \mapsto D_1$ be the last transition before L_0 in which s executes a t-rule. Thus we have $r_{st} = (You, 0)$ and $p_s = t$ in D_1 .

According to the t-rule s executes in $D_0 \mapsto D_1$, we can deduce its (p_s, m_s) values in D_1 .

- Write(t): $(p_s, m_s) = (t, 0)$ in D_1 ;
- t-Reset: $(p_s, m_s) = (null, 0)$ in D_1 . This is not possible since s cannot execute any t-rule between D_1 and L_0 and since $p_s = t$ in L_0 ;
- Marriage(t): not possible since t > s.

For the last two rules, since $r_{st} = (You, 0)$ in D_1 and since s does not perform a Write(t) in $D_0 \mapsto D_1$, then $r_{st} = (You, 0)$ in D_0 . Moreover, according to the Seduction(t) and the t-Increase rules, $r_{st} = Correct_register_value(s, t)$ in D_0 if s executes one of these two rules in $D_0 \mapsto D_1$. Thus, $(p_s, m_s) = (t, 0)$ in D_0 .

- Seduction(t): not possible since we should have $p_s = null$ in D_0 ;
- t-Increase: $(p_s, m_s) = (t, 1)$ in D_1 .

Thus $(p_s, m_s) \in \{(t, 0), (t, 1)\}$ in D_1 . Observe now that s cannot execute a x-Increase rule for any node x, between D_1 and L_0 : by construction it cannot execute it for node t. Also, it cannot execute it for any other node, since $p_s = t$ in D_1 and it cannot be modified between D_1 and L_0 (because s cannot execute any t-rule between these two configurations). Thus s is not eligible for an Increase rule between D_1 and L_0 . We obtain $m_s \in \{0,1\}$ in L_0 and $(p_s, m_s) = (t, 2)$ in L_1 . Thus s must execute a t-Increase to set $m_s = 2$ between L_0 and L_1 and the proof is done.

We now study the second case where $p_s \neq t$ in L_0 . Since in L_1 , $p_s = t$ by assumption, then s must execute a Seduction(t) rule in some transition $C_0 \mapsto C_1$ between L_0 and L_1 and so $m_s = 0$ in C_1 . Since $(p_s, m_s) = (t, 2)$ in L_1 , then s must execute a t-Increase to set $m_s = 2$ between C_1 and L_1 . Finally, the proof is done because $L_0 \mapsto^+ C_1 \mapsto^+ L_1$.

Lemma 14 Let (s,t) be an edge with s < t. Let \mathcal{E} be an execution. If \mathcal{E} contains two configurations L_0 and L_1 with $L_0 \mapsto^* L_1$ and such that:

- t executed at least one s-rule before L_0 ;
- $r_{ts} = (Idle, 0) in L_0;$
- $(p_t, m_t) = (s, 1)$ in L_1 ;

then, t executes a s-Increase to set $m_s = 1$ between L_0 and L_1 .

Proof: In L_0 there are two cases concerning the value of p_t : either $p_t = s$ or not. We consider the first case where $p_t = s$. Let $D_0 \mapsto D_1$ be the last transition before \mathcal{E} in which t executes a s-rule. Thus we have $r_{ts} = (Idle, 0)$ and $p_t = s$ in D_1 .

According to the s-rule t executes in $D_0 \mapsto D_1$, we can deduce its m_t value in D_1 :

- Write(u): not possible since this would imply $p_t = s$ in D_0 and then $r_{ts}.p = You$ in D_1 . But $r_{ts} = (Idle, 0)$ in D_1 .
- Reset(s): not possible since this would imply $p_t = \bot$ in D_1 .
- Increase(s): not possible since this would implies in D_0 : $\neg PRwriting(s)$ and $p_s = t$ and $r_{ts} = (Idle, 0)$ (since only m_s would be modified). And these three conditions are not compatible.
- Seduction(s): t cannot perform this rule since s < t.
- $Marriage: (p_t, m_t) = (s, 0) \text{ in } D_1.$

Thus $(p_t, m_t) = (s, 0)$ and $r_{ts} = Idle, 0$ in D_1 . Since this is the last s-rule t executes before L_0 then observe that p_t cannot be modified. Thus between D_1 and L_0 t cannot execute Seduction, Marriage or Reset rules. Since t is not eligible for Write(s) then r_{ts} remains equal to (Idle, 0) until L_0 . Observe also that since $p_t = s$ between D_1 and L_0 , t cannot execute Increase(x) for

any node x. Thus $m_t = 0$ remains True between these configurations. This implies that $m_t = 0$ in L_0 . We obtain that $m_t = 0$ in L_0 and since $(p_t, m_t) = (s, 1)$ in L_1 then t must execute an Increase(s) writing $m_t := 1$ between L_0 and L_1 .

We now study the second case where $p_t \neq s$ in L_0 . Since in L_1 , $p_t = s$ by assumption, then t must execute a Marriage(s) rule in some transition $C_0 \mapsto C_1$ between L_0 and L_1 and so $m_t = 0$ in C_1 . Since $(p_t, m_t) = (s, 1)$ in L_1 , then t must execute a s-Increase to set $m_t := 1$ between C_1 and L_1 . Finally, the proof is done because $L_0 \mapsto^+ C_1 \mapsto^+ L_1$.

Lemma 15 Let (s,t) be an edge with s < t. Let \mathcal{E} be an execution. If \mathcal{E} contains two transitions $A_0 \mapsto A_1$, $A_2 \mapsto A_3$ and a configuration A_4 , with $A_1 \mapsto^* A_2$ and $A_3 \mapsto^* A_4$ and such that :

- in $A_0 \mapsto A_1$, s executes a t-rule;
- in $A_2 \mapsto A_3$, t executes a Reset rule;
- and in A_4 , $(p_t, m_t) = (s, 2)$;

then the edge (s,t) is in a correct state in A_4 .

Proof: Let $C_0 \mapsto C_1$ be the last Reset executed by t between A_2 and A_4 (we can have $C_0 = A_2$). In C_1 , $p_t = null$ and in A_4 , $p_t = s$, with t > s. Thus t must execute a Marriage(s) rule between C_1 and A_4 to set $p_t = s$

Let $C_2 \mapsto C_3$ be the last Marriage(-) rule executed by t between C_1 and A_4 . Since t does not perform any Reset from C_3 to A_4 by construction, then p_t remains contant from C_3 to A_4 . $p_t = s$ in A_4 , thus $p_t = s$ in C_3 and so t performs a Marriage(s) rule in $C_2 \mapsto C_3$.

Observe that between C_3 and A_4 , we have by construction: t does not perform any Reset nor Marriage and $p_t = s$. Thus, t cannot perform any Seduction rule neither. So, the only rule t can perform between C_3 and A_4 are Write(-) and s-Increase (since $p_t = s$). So the value of m_t can only change by a +1 incrementation between C_3 and A_4 . In C_3 , $m_t = 0$ and in A_4 , $m_t = 2$. Thus, beside the Write rule, t executed exactly two s-Increase between C_3 and A_4 .

Let $C_4 \mapsto C_5$ and $C_6 \mapsto C_7$ be these two s-Increase executed by t, with $C_5 \mapsto^* C_6$. In $C_4 \mapsto C_5$, t sets $m_t = 1$ and in $C_6 \mapsto C_7$, t sets $m_t = 2$. So in C_4 , $m_t = 0$ and in C_6 , $m_t = 1$. According to the Increase rule, we have: in C_4 , $r_{st} = (You, 0)$ and in C_6 , $r_{st} = (You, 2)$. Thus, s performs a Write(t) rule between C_4 and C_6 to set $r_{st} = (You, 2)$.

Let $D_0 \mapsto D_1$ be this transition. We thus have $(p_s, m_s) = (t, 2)$ in D_0 with $C_4 \mapsto^* D_0$.

From Lemma 13 – by setting $C_4 = L_0$ and $D_0 = L_1$ and considering that s executed a t-rule in $A_0 \mapsto A_1$ – there exists a transition $F_0 \mapsto F_1$ between C_4 and D_0 where s executes a t-Increase rule to set $m_s = 2$.

We are now going to prove that the edge (s,t) is in the updated correct state (You, 1, 1) in F_0 .

Since s executes a t-Increase in $F_0 \mapsto F_1$ with $m_s = 2$ in F_1 , then in F_0 we have: $(p_s, m_s) = (t, 1)$ and, according to the Increase rule, $r_{ts} = (You, 1)$. We also have $r_{st} = (You, 1)$ otherwise s would have executed a Write(t) instead of an Increase in $F_0 \mapsto F_1$.

In $C_4 \mapsto C_5$, t executes a s-Increase setting $m_t = 1$, so $(p_t, m_t) = (s, 0)$ in C_4 and $(p_t, m_t) = (s, 1)$ in C_5 . Moreover, t can only execute some Write(-) rules between C_5 and C_6 . Thus in C_4 , $(p_t, m_t) = (s, 0)$ and from C_5 to C_6 , $(p_t, m_t) = (s, 1)$.

We have: $C_4 \mapsto^* F_0 \mapsto^* D_0 \mapsto^+ C_6$ and we know that $r_{st} = (You, 0)$ in C_4 and $r_{st} = (You, 1)$ in F_0 . So $C_4 \neq F_0$ and so $C_5 \mapsto^* F_0 \mapsto^* D_0 \mapsto^+ C_6$. So $(p_t, m_t) = (s, 1)$ in F_0 .

We finally obtain for F_0 : $(p_s, m_s) = (t, 1)$, $(p_t, m_t) = (s, 1)$, $r_{st} = (You, 1)$ and $r_{ts} = (You, 1)$. Thus the edge (s, t) is in the updated correct state (You, 1, 1) in F_0 . As $F_0 \mapsto^+ C_6 \mapsto^+ A_4$ and according to lemma 2, the edge (s, t) is in a correct state in A_4 .

Lemma 16 Let (s,t) be an edge with s < t. Let \mathcal{E} be an execution. If \mathcal{E} contains two transitions $A_0 \mapsto A_1$, $A_2 \mapsto A_3$ and a configuration A_4 , with $A_1 \mapsto^* A_2$ and $A_3 \mapsto^* A_4$ and such that :

- in $A_0 \mapsto A_1$, t executes a s-rule;
- in $A_2 \mapsto A_3$, s executes a Reset rule;
- and in A_4 , $(p_s, m_s) = (t, 1)$;

then the edge (s,t) is in a correct state in A_4 .

Proof: Let $C_0 \mapsto C_1$ be the last Reset executed by s between A_2 and A_4 (we can have $C_0 = A_2$). In C_1 , $p_s = null$ and in A_4 , $p_s = t$, with s < t. Thus s must execute a Seduction(t) rule between C_1 and A_4 to set $p_s = t$

Let $C_2 \mapsto C_3$ be the last Seduction(-) rule executed by s between C_1 and A_4 . Since s does not perform any Reset from C_3 to A_4 by construction, then p_s remains contant from C_3 to A_4 . $p_s = t$ in A_4 , thus $p_s = t$ in C_3 and so s performs a Seduction(t) rule in $C_2 \mapsto C_3$.

Observe that between C_3 and A_4 , we have by construction: s does not perform any Reset nor Seduction and $p_s = t$. Thus, s cannot perform any Marriage rule neither. So, the only rule s can perform between C_3 and A_4 are Write(-) and t-Increase (since $p_s = t$). So the value of m_s can only change by a +1 incrementation between C_3 and A_4 . In C_3 , $m_s = 0$ and in A_4 , $m_s = 1$. Thus, beside the Write rule, s executed exactly one t-Increase between C_3 and A_4 .

Let $C_4 \mapsto C_5$ be this t-Increase executed by s. In $C_4 \mapsto C_5$, s sets $m_s = 1$. So, in C_4 , $(p_s, m_s) = (t, 0)$ and then $r_{ts} = (You, 1)$. Moreover, according to the Seduction(t) rule, in C_2 , $r_{ts} = (Idle, 0)$. So there exists a transition $D_0 \mapsto D_1$ between C_2 and C_4 where t executes a Write(s) rule to set $r_{ts} = (You, 1)$. Thus, in D_0 , $(p_t, m_t) = (s, 1)$.

From Lemma 14 – by setting $C_2 = L_0$ and $D_0 = L_1$ and considering that t executed a s-rule in $A_0 \mapsto A_1$ – there exists a transition $F_0 \mapsto F_1$ between C_2 and D_0 where t executes a s-Increase rule to set $m_t = 1$.

We are now going to prove that the edge (s,t) is in the updated correct state (You, 0, 0) in F_0 .

Since t executes a s-Increase in $F_0 \mapsto F_1$ with $m_t = 1$ in F_1 , then in F_0 we have: $(p_t, m_t) = (s, 0)$ and, according to the Increase rule, $r_{st} = (You, 0)$. We also have $r_{ts} = (You, 0)$ otherwise t would have perform a Write(s) instead of an s-Increase in $F_0 \mapsto F_1$.

Recall, that $C_2 \mapsto^* F_0 \mapsto^+ D_0$. We already know that $r_{ts} = (Idle, 0)$ in C_2 , so $C_3 \mapsto^* F_0 \mapsto^+ D_0$.

Moreover, we know that between C_3 and C_4 , s does not perform any rule but some Write(-) rule. So (p_s, m_s) remains constant between C_3 and C_4 . s executes a Seduction(t) rule in $C_2 \mapsto C_3$, so $(p_s, m_s) = (t, 0)$ in C_3 . Moreover, $C_3 \mapsto^* D_0 \mapsto^+ C_4$, so $(p_s, m_s) = (t, 0)$ in D_0 .

We finally obtain for F_0 : $(p_t, m_t) = (s, 0)$, $(p_s, m_s) = (t, 0)$, $r_{ts} = (You, 0)$ and $r_{st} = (You, 0)$. Thus the edge (s, t) is in the updated correct state (You, 0, 0) in F_0 . As $F_0 \mapsto^+ C_4 \mapsto^+ A_4$ and according to lemma 2, the edge (s, t) is in a correct state in A_4 .

Theorem 4 lem:mvaut2 Let (u,v) be an edge. Let \mathcal{E} be an execution. If \mathcal{E} contains two transitions $A_0 \mapsto A_1$, $A_2 \mapsto A_3$ and a configuration A_4 , with $A_1 \mapsto^* A_2$ and $A_3 \mapsto^* A_4$ and such that:

- in $A_0 \mapsto A_1$, v executes a u-rule;
- in $A_2 \mapsto A_3$, u executes a Reset rule;

• and in A_4 , $(p_u, m_u) = (v, 2)$;

then the edge (min(u, v), max(u, v)) is in a correct state in A_4 .

Proof: If u > v, we conclude immediately using Lemma 15. Thus assume that u < v.

Let $C_0 \mapsto C_1$ be the last Reset executed by u between A_2 and A_4 (we can have $C_0 = A_2$). In C_1 , $p_u = null$ and in A_4 , $p_u = v$, with u < v. Thus u must execute a Seduction(v) rule between C_1 and A_4 to set $p_u = v$

Let $C_2 \mapsto C_3$ be the last Seduction(-) rule executed by u between C_1 and A_4 . Since u does not perform any Reset from C_3 to A_4 by construction, then p_u remains constant from C_3 to A_4 . Since $p_u = v$ in A_4 , then $p_u = v$ in C_3 and so u performs a Seduction(v) rule in $C_2 \mapsto C_3$.

Observe that between C_3 and A_4 , we have by construction: u does not perform any Reset nor Seduction and $p_u = v$. Thus, u cannot perform any Marriage rule neither. So, the only rule u can perform between C_3 and A_4 are Write(-) and v-Increase (since $p_u = v$). So the value of m_u can only change by a +1 incrementation between C_3 and A_4 . In C_3 , $m_u = 0$ and in A_4 , $m_u = 2$. Thus, beside the Write rule, u executed exactly two v-Increase between C_3 and A_4 . Let $C_4 \mapsto C_5$ be the transition in which u executes the first such v-Increase rule with $C_4 \geq C_3$. By definition of this rule, we have that in C_5 , $(p_u, m_u) = (v, 1)$ holds. We now apply Lemma 16. Observe that C_5 is after transition $A_0 \mapsto A_1$ in which v executes a v-rule and after v-1 and v-1

Lemma 17 Let (s,t) be an edge with s < t. Let \mathcal{E} be an execution containing the two following transitions:

- $C_0 \mapsto C_1$ where s executes a Seduction(t) rule;
- $C_2 \mapsto C_3$ where s executes a t-Reset rule;
- with $C_1 \mapsto^* C_2$ and with s does not execute any Reset between C_1 and C_2 .

Then, t executes a Write(s) rule between C_0 and C_2 .

Proof: Since s executes a Seduction(t) rule in $C_0 \mapsto C_1$, then $r_{ts} = (Idle, 0)$ in C_0 . Since s executes a t-Reset rule in $C_2 \mapsto C_3$, then $r_{ts} = (Idle, 0)$ in C_0 . Either $r_{ts} \neq (Idle, 0)$ in C_2 and so the proof is done, or $r_{ts} = (Idle, 0)$ in C_2 .

In the second case, $m_s = 0$ in C_1 (by the Seduction rule) and $m_s \neq 0$ in C_2 . So, s executes an Increase rule between C_1 and C_2 . Let $C_3 \mapsto C_4$ be the transition where s does so for the first time. In C_1 , $p_s = t$ and $m_s = 0$, by the Seduction rule. Moreover, by assumption, s does not execute any Reset from C_1 to C_2 . Thus, s can only perform some Write(-) rule or a t-Increase rule from C_1 to C_2 . And so, in C_3 , $p_s = t$ and $m_s = 0$ (since $C_3 \mapsto C_4$ is the first Increase from s). s performs then $m_s := 1$ in $C_3 \mapsto C_4$ and so $r_{ts} = (You, 1)$ in C_3 . Since $r_{ts} = (Idle, 0)$ in C_0 , the proof is done.

Lemma 18 Let (s,t) be an edge. Let \mathcal{E} be an execution containing two transitions $A_0 \mapsto A_1$ and $A_2 \mapsto A_3$ with $A_1 \mapsto^* A_2$ where t executes a write(s) rule. If we have $(p_t, m_t) = (v, 2)$ with $v \neq s$ in A_0 and A_2 , then t executes a Reset rule and v executes a Write(t) rule between A_0 and A_2 .

Proof:

Since t executes a Write(s) rule in $A_0 \mapsto A_1$ and $A_2 \mapsto A_3$, then $r_{ts} = (Other, 2)$ in A_1 and A_3 . By definition of Write(s) rule, we also have $r_{ts} \neq (Other, 2)$ in A_2 . Thus t must execute a Write(s) rule between A_1 and A_2 to modify r_{ts} . Let $F \mapsto F'$ be the last transition before A_2 in which t executes such a rule. Thus in F, either $(p_t, m_t) = (v, m)$ with m < 2, or $p_t \neq v$.

First assume that $(p_t, m_t) = (v, m)$ with m < 2. Thus, $m_t = 2$ in A_0 and $m_t < 2$ in F. There is only one way to decrease the value of an m variable: to write 0 by executing a Reset rule. On the second case, $p_t \neq v$ in F and $p_t = v$ in A_0 . The only way to change the value of a non-null p variable is that t executes a Reset rule between A_0 and F. Then in both cases, t executes a Reset rule between A_0 and A_2 .

Let $C_0 \mapsto C_1$ be the last Reset executed by t between A_1 and A_2 (we can have $C_0 = A_1$). In C_1 , $m_t = 0$ and in A_2 , $m_t = 2$. Thus t must execute an Increase rule between C_1 and A_2 to set $m_t = 2$.

Let $C_5 \mapsto C_6$ be the last Increase rule executed by t between C_1 and A_2 . Since t does not perform any Reset from C_6 to A_3 by construction, then p_t remains contant from C_6 to A_3 . $p_t = v$ in A_3 , thus $p_t = v$ in C_5 and so t performs a v-Increase rule in $C_5 \mapsto C_6$.

First we assume that v < t. Let $C_2 \mapsto C_3$ be the first Marriage(-) rule executed by t between C_1 and C_5 . Since t does not perform any Reset from C_3 to C_5 by construction, then p_t remains contant from C_3 to C_5 . $p_t = v$ in C_5 , thus $p_t = v$ in C_3 and so t performs a Marriage(v) rule in $C_2 \mapsto C_3$. Thus in C_2 , $r_{vt} = (You, 0)$, by the Marriage(v) rule. In C_5 , $r_{vt} = (You, 2)$, by the v-Increase rule. Thus, v has performed at least once Write(t) between C_2 and C_5 and the lemma holds.

Second we assume that t < v. Let $C_2 \mapsto C_3$ be the first Seduction(-) rule executed by t between C_1 and C_5 . Since t does not perform any Reset from C_3 to C_5 by construction, then p_t remains contant from C_3 to C_5 . $p_t = v$ in C_5 , thus $p_t = v$ in C_3 and so t performs a Seduction(v) rule in $C_2 \mapsto C_3$. Thus in C_2 , $r_{vt} = (Idle, 0)$, by the Seduction(v) rule, and in C_5 , $r_{vt} = (You, 1)$, by the v-Increase rule. Thus, v has performed at least once Write(t) between C_2 and C_5 and this concludes the proof.

Lemma 19 Let (s,t) be an edge with s < t. Let \mathcal{E} be an execution containing three transitions $D_0 \mapsto D_1$, $D_2 \mapsto D_3$ and $D_4 \mapsto D_5$ with $D_1 \mapsto^* D_2$ and $D_3 \mapsto^* D_4$ and where s executes a Seduction(t) rule. Then there exists a transition $D \mapsto D'$ between D_2 and D_4 where t executes a Write(s) rule and with in D: $p_t \neq null$ and $m_t = 2$.

Proof: According to Lemma 11.(2), s executes a Reset between D_1 and D_2 . Let $C_0 \mapsto C_1$ be the transition where s does so for the first time. So s executes a t-Reset rule in $C_0 \mapsto C_1$. And then according to Lemma 17, t executes a Write(s) rule between D_1 and C_0 .

Observe now that the execution starting in configuration D_2 reaches all the assumptions made in Lemmas 16. Indeed, before D_2 , t executes a s-rule and then s executes a Reset rule.

According to Lemma 12, s executes a Reset rule between D_3 and D_4 . Let $C_2 \mapsto C_3$ be the transition where s does so for the first time. So s executes a t-Reset in $C_2 \mapsto C_3$. In C_2 , either $m_s > 0$ or $m_s = 0$.

If $m_s > 0$ in C_2 and since $m_s = 0$ in D_3 by the Seduction rule, then s performs $m_s := 1$ between D_3 and C_2 , let say in transition $B \mapsto B'$. In this transition, s executes an Increase rule. Recall that $p_s = t$ in D_3 since s executes a Seduction(t) rule, and that s does not execute any Reset between D_3 and C_2 . Thus $p_s = t$ in B and so $(p_s, m_s) = (t, 1)$ in B'. By Lemma 16, the edge (s, t) is in a correct state in B'. Thus, by Corollary 1, from this configuration, s cannot execute any Reset, which contradict the fact that it does in $C_2 \mapsto C_3$. Then $m_s = 0$ in C_2 .

Since $(p_s, m_s) = (t, 0)$ in C_2 with s < t then, according to the Reset rule, $r_{ts}.m = 2$ in C_2 . Moreover, since $r_{ts}.m = 0$ in D_2 by the Seduction(t) rule, then t executes a Write(s) rule

between D_2 and C_2 . This is the transition $D \mapsto D'$ and we have $m_t = 2$ and $p_t \neq null$ in D. This conclude the proof.

5.6 Main Result

Using Lemmas 7, 8, 9, we can conclude on the time complexity of the algorithm.

Theorem 5 The algorithm stabilizes in $O(n\Delta^3)$ moves.

Proof: Let (s,t) be an edge with s < t. By Theorem 3, node s can execute the Seduction(t)rule $\mathcal{O}(\Delta)$ times. By Lemma 7, between two executions of the Marriage(s) rule by node t, node s must execute a Seduction(t) rule. This implies that t can execute $\mathcal{O}(\Delta)$ Marriage(s) rules. In total, a node can execute $\mathcal{O}(\Delta)$ Seduction(-) and Marriage(-) rules per neighbor, which gives a $\mathcal{O}(\Delta^2)$ total number of these rules, per node. Let now u be a node. By Lemma 8, between two executions of the Reset rule by node u, it must execute a Marriage(-) or Seduction(-) rule. Since we proved that it can do so $\mathcal{O}(\Delta^2)$ times, then it can execute the Reset rule $\mathcal{O}(\Delta^2)$ times as well. Now by Lemma 9, between three executions of the *Increase* rule, node u must execute the Reset rule. As a consequence, u can execute the Increase rule $\mathcal{O}(\Delta^2)$ times. Altogether, a node can execute at most $\mathcal{O}(\Delta^2)$ Seduction(-), Marriage(-), Increase and Reset rules. Let's call such rules high level rules. Each time it executes such a rule, a node may execute a Write(-) rule $\mathcal{O}(\Delta)$ times to update all its registers. If it does not execute any high level rule, a node can execute at most $\mathcal{O}(\Delta)$ Write rules. Thus in total a node can do $\mathcal{O}(\Delta^3)$ Write rules. Finally, nodes can, in total, execute $\mathcal{O}(n\Delta^2)$ high level rules and $\mathcal{O}(n\Delta^3)$ Write rules which gives a $\mathcal{O}(n\Delta^3)$ bound on the total number of moves.

6 Conclusion

In this paper, we propose the first algorithm to solve the matching problem in link-register model. This algorithm is self-stabilizing, and takes at worst $O(n\Delta^3)$ moves before converging from the worst possible initialization, with the worst possible scheduling of communications. This is to be compared with similar solutions in state model, that converge in O(m) moves. Indeed, asynchronous communications allow executions with a node taking steps in the algorithm ignoring the actual state of its neighbors. Moreover, to discard outdated values of the register, the matching process between two nodes requires a number of steps, to ensure that eventually, the two nodes agree regarding their marriage and will no longer take any move.

References

- [1] Y. Asada and M. Inoue. An efficient silent self-stabilizing algorithm for 1-maximal matching in anonymous networks. In *WALCOM: Algorithms and Computation 9th International Workshop*, pages 187–198. Springer International Publishing, 2015.
- [2] Johanne Cohen, Jonas Lefevre, Khaled Maâmra, Laurence Pilard, and Devan Sohier. A self-stabilizing algorithm for maximal matching in anonymous networks. *Parallel Processing Letters*, 26(04):1650016, 2016.
- [3] Edsger W. Dijkstra. Self-stabilizing systems in spite of distributed control. *Commun. ACM*, 17(11):643–644, 1974.
- [4] S. Dolev, A. Israeli, and S. Moran. Self-stabilization of dynamic systems assuming only read/write atomicity. dc, 7:3–16, 1993.

- [5] Shlomi Dolev. Self-Stabilization. MIT Press, 2000.
- [6] D. E. Drake and S. Hougardy. A simple approximation algorithm for the weighted matching problem. *Inf. Process. Lett.*, 85(4):211–213, 2003.
- [7] N. Guellati and H. Kheddouci. A survey on self-stabilizing algorithms for independence, domination, coloring, and matching in graphs. *J. Parallel Distrib. Comput.*, 70(4):406–415, 2010.
- [8] S. T. Hedetniemi, D. Pokrass Jacobs, and P. K. Srimani. Maximal matching stabilizes in time o(m). *Inf. Process. Lett.*, 80(5):221–223, 2001.
- [9] S.-C. Hsu and S.-T. Huang. A self-stabilizing algorithm for maximal matching. *Inf. Process. Lett.*, 43(2):77–81, 1992.
- [10] Colette Johnen and Lisa Higham. Fault-tolerant implementations of atomic registers by safe registers in networks. In *Proceedings of the Twenty-Seventh Annual ACM Symposium on Principles of Distributed Computing, PODC 2008, Toronto, Canada, August 18-21, 2008*, page 449, 2008.
- [11] F. Manne, M. Mjelde, L. Pilard, and S. Tixeuil. A new self-stabilizing maximal matching algorithm. *Theoretical Computer Science (TCS)*, 410(14):1336–1345, 2009.
- [12] Achour Mostéfaoui, Matoula Petrolia, Michel Raynal, and Claude Jard. Atomic read/write memory in signature-free byzantine asynchronous message-passing systems. *Theory Comput. Syst.*, 60(4):677–694, 2017.
- [13] R. Preis. Linear time 1/2-approximation algorithm for maximum weighted matching in general graphs. In 16th Annual Symposium on Theoretical Aspects of Computer Science (STACS), Lecture Notes in Computer Science, pages 259–269. Springer, 1999.
- [14] V. Turau and B. Hauck. A new analysis of a self-stabilizing maximum weight matching algorithm with approximation ratio 2. *Theoretical Computer Science (TCS)*, 412(40):5527–5540, 2011.