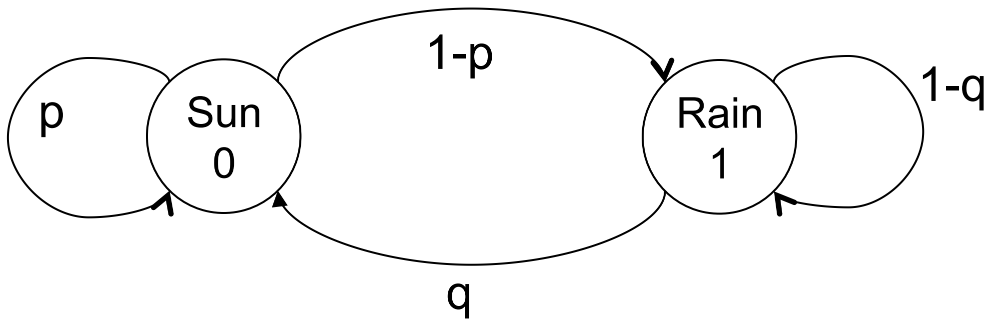
Markov Chains
(Part 5)

Estimating Probabilities and Absorbing States

Weather Example: How to estimate *p* and *q*?

- State Transition Diagram



- Probability Transition Matrix

$$
\begin{array}{lcc}
 & & \begin{array}{cc} 0 & 1 \end{array} \\
Sunny & 0 & \multirow{2}{*}{$\begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix}$} \\
Rainy & 1 &
\end{array}
$$

Weather Example: Estimation from Data

- Estimate transition probabilities from data

  Weather data for 1 month (R – rainy, S – sunny)

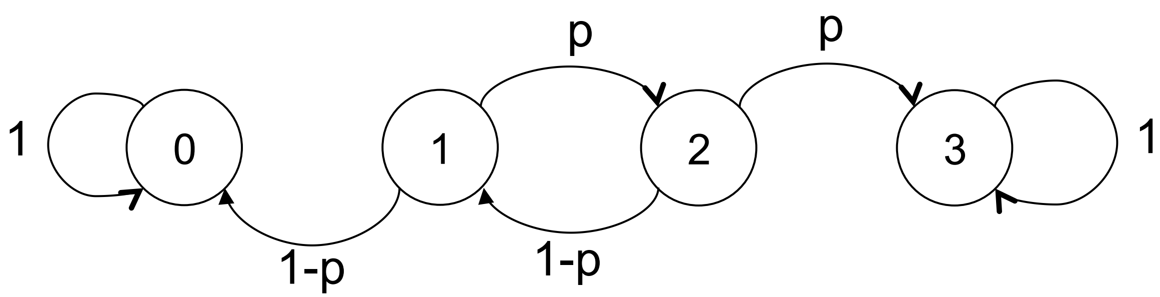
Sun	M	T	W	Th	F	Sat
R	R	R	S	S	R	R
R	S	S	S	S	R	R
R	S	S	R	R	R	R
S	S	R	S	R	R	R

$$
\begin{array}{lc c c c}
 & & \textit{Sunny} & \textit{Rainy} & \text{Row sum} \\
\textit{Sunny} & 0 & \left[\begin{array}{cc} 6 & 5 \\ 5 & 11 \end{array}\right. & & 11 \\
\textit{Rainy} & 1 & & \left.\right] & 16
\end{array}
$$

$$
\begin{array}{lc}
 & \begin{array}{cc} \textit{Sunny} & \textit{Rainy} \\ 0 & 1 \end{array} \\
P = \begin{array}{lc} \textit{Sunny} & 0 \\ \textit{Rainy} & 1 \end{array} & \begin{bmatrix} 6/11 & 5/11 \\ 5/16 & 11/16 \end{bmatrix}
\end{array}
$$

Gambler's Ruin Example: How to estimate *p*?

- Probability *p* of winning on any turn
- State Transition Diagram



- Probability Transition Matrix

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

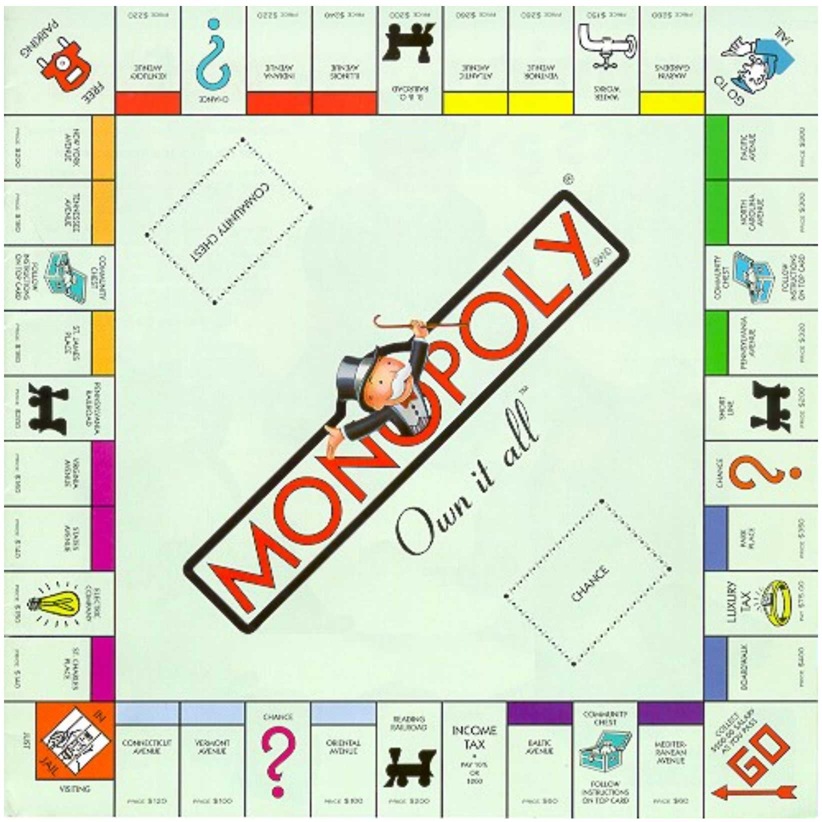
Gambler's Ruin Example: How to estimate *p*?

- Here, *p* can be estimated by analyzing the game.
- For example, a roll of the dice can be analyzed directly.
- A poker game may need more data, and *p* may depend on who is at the table, and what the stakes are.

- Probability Transition Matrix

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

Monopoly Example



- You roll a pair of dice to advance around the board
- If you land on the "Go To Jail" square, you must stay in jail until you roll doubles or have spent three turns in jail
- Let X_t be the location of your token on the Monopoly board after t dice rolls
  - Can a Markov chain be used to model this game?
  - If not, how could we transform the problem such that we can model the game with a Markov chain?

<ins>… more to come …</ins>

Absorbing States

- Recall a state i is an absorbing state if p_{ii}=1
- Example: Gambler's ruin has two absorbing states

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Absorbing States

- If p_{kk}=1 (that is, once the chain visits state *k*, it remains there forever), then we may want to know:

  the probability of absorption, denoted f_{ik}
- These probabilities are important because they provide answers to questions such as
  - If I start out with a dollar, what is the probability that I will go broke in the gambling game? What is the probability I get $3?
- Three observations:
  - f_{kk}=1
  - If *i* is recurrent, and *i* is different from *k*, then f_{ik}=0
  - We are most interested in f_{ik} when *i* is a transient state and *k* is an absorbing state

Linear Equations for Absorption Probabilities

- As before, we condition on the first transition of the Markov chain to get

$$f_{ik} = \sum_{j=0}^{M} p_{ij} f_{jk}, \quad \text{for } i = 0,1,\ldots,M$$

with f_{kk}=1 for absorbing state *k*,
and f_{jk}=0 when *j* is a recurrent state

Another Gambling Example

- Two players A and B, each having \$2, agree to keep betting \$1 at a time until one of them goes broke. The probability that A wins a bet is 1/3. So B wins a bet with probability 2/3. We model the evolution of the number of dollars that A has as a Markov chain. Note that A can have 0,1,2,3, or 4 dollars. The transition probability matrix is given by

States	\$0	\$1	\$2	\$3	\$4
\$0	1	0	0	0	0
\$1	2/3	0	1/3	0	0
\$2	0	2/3	0	1/3	0
\$3	0	0	2/3	0	1/3
\$4	0	0	0	0	1

Another Gambling Example

- Find the probability that A will lose her \$2. That is, find f_{20}.
- Use, f_{kk}=1 for absorbing *k*, and f_{jk}=0 for recurrent state *j*

$$
\begin{aligned}
f_{20} &= \sum_{j=0}^{4} p_{2j} f_{j0} = p_{20} f_{00} + p_{21} f_{10} + p_{22} f_{20} + p_{23} f_{30} + p_{24} f_{40} \\
&= (0)(1) + (2/3) f_{10} + (0) f_{20} + (1/3) f_{30} + (0)(0) \\
&= (2/3) f_{10} + (1/3) f_{30}
\end{aligned}
$$

$$
\begin{aligned}
f_{10} &= \sum_{j=0}^{4} p_{1j} f_{j0} = p_{10} f_{00} + p_{11} f_{10} + p_{12} f_{20} + p_{13} f_{30} + p_{14} f_{40} \\
&= (2/3)(1) + (0) f_{10} + (1/3) f_{20} + (0) f_{30} + (0)(0) \\
&= (2/3) + (1/3) f_{20}
\end{aligned}
$$

$$
\begin{aligned}
f_{30} &= \sum_{j=0}^{4} p_{3j} f_{j0} = p_{30} f_{00} + p_{31} f_{10} + p_{32} f_{20} + p_{33} f_{30} + p_{34} f_{40} \\
&= (0)(1) + (0) f_{10} + (2/3) f_{20} + (0) f_{30} + (1/3)(0) \\
&= (2/3) f_{20}
\end{aligned}
$$

Another Gambling Example

- Solve three equations, with three unknowns

$$f_{20} = (2/3)f_{10} + (1/3)f_{30}$$

$$f_{10} = (2/3) + (1/3)f_{20}$$

$$f_{30} = (2/3)f_{20}$$

- Substitute, to get $f_{20} = (2/3)\big((2/3) + (1/3)f_{20}\big) + (1/3)\big((2/3)f_{20}\big)$
$= (4/9) + (2/9)f_{20} + (2/9)f_{20}$

- Solving, yields $f_{20} = 4/5, \text{ and } f_{24} = 1 - f_{20} = 1/5$

$$f_{10} = 14/15, \text{ and } f_{14} = 1/15$$

$$f_{30} = 8/15, \text{ and } f_{34} = 7/15$$

Credit Evaluation Example

- Every month, credit accounts are checked to determine the state of each customer :
  - State 0: Fully paid
  - State 1: Payment on account is 1 month (1-30 days) overdue
  - State 2: Payment on account is 2 months (31-60 days) overdue
  - State 3: Account is written off as bad debt
- The transition probability matrix is given as

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.7 & 0.2 & 0.1 & 0 \\ 0.5 & 0.1 & 0.2 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Credit Evaluation Example

- What is the probability an account 1 month overdue eventually gets fully paid? Becomes a bad debt?

$$
\begin{aligned}
f_{00} &= 1 \\
f_{10} &= 0.7 f_{00} + 0.2 f_{10} + 0.1 f_{20} \\
f_{20} &= 0.5 f_{00} + 0.1 f_{10} + 0.2 f_{20} + 0.2 f_{30} \\
f_{30} &= 0
\end{aligned}
$$

- Solving, yields
$$
\begin{aligned}
f_{00} &= 1 \\
f_{10} &= 0.968, \text{ and } f_{13} = 0.032 \\
f_{20} &= 0.746, \text{ and } f_{23} = 0.254 \\
f_{30} &= 0
\end{aligned}
$$

- The probability an account 1 month overdue eventually gets fully paid is $f_{10} = 0.968$
- Becomes a bad debt? $f_{13} = 0.032$