Solutions Stochastic Processes and Simulation II, May 18, 2017

Problem 1: Poisson Processes

Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process on $(0, \infty)$ with rate λ. Let $\{S_i, i = 1, 2, \cdots\}$ be the points of the Poisson Process, such that $S_1 \leq S_2 \leq S_3 \leq \cdots$.

a) Provide the distribution of $S_2 - S_1$. (4p)

Solution: The difference between two subsequent points in a homogeneous Poisson process is exponentially distributed; in this case with expectation $1/\lambda$.

b) For $x > 0$, compute $\mathbb{P}[S_2 - S_1 \leq x | N(1) = 2]$. (4p)

Solution: By $N(1) = 2$ we know that there are exactly two points in the interval $[0, 1]$. By the order statistic property the positions of the two points (not taking into account their order) are independent Uniforms on $[0, 1]$ (Say U_1 and U_2). The distance between those two uniforms can be computed as follows. (In the computation s and t are the positions of the uniforms).

$$\mathbb{P}[S_2 - S_1 \leq x | N(1) = 2] = \mathbb{P}[|U_2 - U_1| \leq x] = \int_0^1 \int_0^1 \mathbb{1}(|s - t| \leq x) ds dt$$

$$= \int_0^1 \int_0^t \mathbb{1}(|s - t| \leq x) ds dt + \int_0^1 \int_t^1 \mathbb{1}(|s - t| \leq x) ds dt$$

$$= \int_0^1 \int_0^t \mathbb{1}(|s-t| \leq x) ds dt + \int_0^1 \int_0^s \mathbb{1}(|s-t| \leq x) dt ds = 2 \int_0^1 \int_0^t \mathbb{1}(|s-t| \leq x) ds dt$$

$$= 2 \left(\int_0^x \int_0^t ds dt + \int_x^1 \int_{t-x}^t ds dt \right) = 2 \left(\int_0^x t dt + \int_x^1 x dt \right) = x^2 + 2x(1-x).$$

c) Assume that at the points of the Poisson process $\{N(t), t \geq 0\}$ (as defined above) a busload of passangers arrives at a restaurant. The number of passangers in each bus is independent and identically distributed with a geometric distribution with parameter p. That is, if k is a strictly positive integer, then the number of passengers in a bus is k with probability $p(1-p)^{k-1}$ and the mean of the number of passengers in a bus is $1/p$. Let $N^*(t)$ be the number of passengers that arrived at the restaurant up to t. What sort of process is $\{N^*(t), t \geq 0\}$? Compute the mean of $N^*(t)$. (4p)

Solution: The process is a compound Poisson Process with $\mathbb{E}[N^*(t)] = \lambda t/p$ by Wald's equation.

Problem 2: Renewal Theory

Let $\{N(t), t \geq 0\}$ be a renewal process with inter-arrival time distribution function

$$F(x) = \mathbb{P}(X \leq x) = (1-p) + p(1-e^{-\lambda x}) = 1 - pe^{-\lambda x} \qquad \text{for } x \geq 0.$$

Note that one might write

$$F(x) = \mathbb{P}(X = 0) + \mathbb{P}(X > 0)F_1(x),$$

where $F_1(x) = 1 - e^{-\lambda x}$ is the distribution function of an exponentially distributed random variable. That is, the interarrival time X is 0 with probability $(1-p)$ and conditioned on not being 0 the interarrival time is exponentially distributed with expectation $1/\lambda$.

a) Provide the distribution of $N(0)$. (4p)

Solution: For the interested reader: note that the process is a compound Poisson Process with extra weight $N(0)$ at time 0, where $\mathbb{P}(N(0) \geq k)$ for $k \geq 0$ is the probability that the first k arrivals where all with interarrival time 0. So, $\mathbb{P}(N(0) \geq k) = (1-p)^k$ and $\mathbb{P}(N(0) = k) = p(1-p)^k$.

b) Show that $m(t) = \mathbb{E}[N(t)] = \frac{1-p}{p} + \lambda t \frac{1}{p}$. (4p)

Hint: One way (most straightforward, perhaps not the easiest) is to deduce the renewal equation, which in this case reads:

$$m(t) = F(t) + \mathbb{P}(X = 0)m(t) + \int_0^t m(t-x)\mathbb{P}(X > 0)f_1(x)dx. \qquad \text{for } t \geq 0$$

If you use this equation, you have to justify the equation.

Solution: The easiest way is to observe that the process described is a compound Poisson Process with extra weight $N(0)$ at time 0. Here $N(0)$ is a shifted geometric distribution with expectation $(1/p) - 1$ and the weights at an arrival have expectation $1/p$.

It is also possible to use the renewal equation, with X_1 the time of the first arrival

$$\begin{aligned}
m(t) &= \mathbb{E}[N(t)] = \mathbb{E}[\mathbb{E}[N(t)|X_1]] \\
&= \mathbb{P}(X_1 = 0)\mathbb{E}[N(t)|X_1 = 0] + \int_0^\infty f_1(x)\mathbb{P}(X_1 > 0)\mathbb{E}[N(t)|X_1 = x]dx \\
&= \mathbb{P}(X_1 = 0)(1 + m(t)) + \mathbb{P}(X_1 > 0)\int_0^t f_1(x)(1 + m(t-x))dx \\
&= \mathbb{P}(X \leq t) + \mathbb{P}(X_1 = 0)m(t) + \mathbb{P}(X_1 > 0)\int_0^t f_1(x)m(t-x)dx,
\end{aligned}$$

as desired. Note that we can rewrite

$$m(t) = F(t) + \mathbb{P}(X = 0)m(t) + \int_0^t m(t-x)\mathbb{P}(X > 0)f_1(x)dx$$

as

$$m(t)\mathbb{P}(X > 0) = F(t) + \int_0^t m(t-x)\mathbb{P}(X > 0)f_1(x)dx$$

and filling $m(t) = \frac{1-p}{p} + \lambda t \frac{1}{p}$ in gives on the left hand side $1 - p + \lambda t$, while the right hand side is given by

$$\begin{aligned}
1 - pe^{-\lambda t} &+ \int_0^t \lambda e^{-\lambda x}(1 - p + \lambda(t-x))dx \\
&= 1 - pe^{-\lambda t} + (1 - p + \lambda t)(1 - e^{-\lambda t}) - \int_0^t x\lambda^2 e^{-\lambda x}dx \\
&= 1 - pe^{-\lambda t} + (1 - p + \lambda t)(1 - e^{-\lambda t}) + \int_0^t (-\lambda e^{-\lambda x})\lambda x dx \\
&= 1 - pe^{-\lambda t} + (1 - p + \lambda t)(1 - e^{-\lambda t}) + \left([e^{-\lambda x}\lambda x]_{x=0}^t - \int_0^t e^{-\lambda x}\lambda dx\right) \\
&= 1 - pe^{-\lambda t} + (1 - p + \lambda t)(1 - e^{-\lambda t}) + \lambda t e^{-\lambda t} - (1 - e^{-\lambda t}) = 1 - p + \lambda t,
\end{aligned}$$

as desired. We have used integration by parts between the third and fourth line.

c) Let $Y(t)$ be the time until the next arrival at time t. So, $Y(t) = S_{N(t)+1} - t$, where S_i is the time of the i-th arrival. Provide $\lim_{t\to\infty} \mathbb{E}[Y(t)]$. (4p)

Solution: See example 7.19 (page 433) in book. Or realize the relation between the renewal process and a compound Poisson process, in which the waiting time to the next event is always $1/\lambda$.

Problem 3: Queueing Theory

Consider an $M/G/1$ queue in which customers arrive according to a Poisson Process with rate λ, and each customer needs exactly one time unit of service. (So, the workloads are not random).

a) Provide the condition that λ should satisfy in order for the number of customers in the queue not to go to infinity. (2p)

Solution: $\lambda < 1$, because the long-run departure rate during busy periods should exceed the long-run arrival rate.

Let $V(t)$ be the remaining workload in the system at time t. That is, $V(t)$ is the number of customers in the queue plus the remaining time until the customer in service leaves. Let W_Q be the expected time a customer spends in the queue.

b*) Argue that $\frac{1}{t}\int_0^t V(s)ds \to \lambda(W_Q + 1/2)$ as $t \to \infty$. (4p)

Solution: Assume that every customer pays at a rate which equals its remaining service time. So every customer pays the average time it is in the queue plus 1/2 (The average remaining service time, while in service). Over a long time the number of customers that has arrived is about lambda per time unit.
The rate at which the system earns at time s is V(s) because every customer in the queue pays 1, and every customer in service pays equal to its remaining service time. This gives the desired result.

c) Argue that $\frac{1}{t}\int_0^t V(s)ds \to W_Q$ as $t \to \infty$. (3p)

Solution: By PASTA, a newly arrived customer sees on average $\lim_t \frac{1}{t}\int_0^t V(s)ds$ of work in front of him, which is equal to the time he or she has to stay in the queue (with expectation W_Q).

d) What is the asymptotic fraction of time that there are no customers in the system? (3p)
Hint: this question can be solved without part b) and c).

Solution: Let $N(t)$ be the number of arrivals up to time t, which each bring in 1 time unit of work load. The total time the server has been working up to time t is equal to $N(t)-V(t)$. Since the system does not explode, neither does $V(t)$. So the fraction of time the server has been busy is $(N(t)-V(t))/t$. If t is large $V(t)/t$ converges to 0 and $N(t)/t$ converges to the expected number of arrivals per time unit, which is λ. So, the fraction of time the system is idle converges to $1-\lambda$.

Problem 4: Brownian Motion and Stationary Processes

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion.

a) Provide the definition of a standard Brownian motion. (4p)

Solution: A *standard Brownian motion* $\{B(t); t \geq 0\}$ is a stochastic process in continuous time and continuous state space $\mathbb{R}$ which satisfies:
- $B(0) = 0$
- B has independent increments, i.e. $B(t_3) - B(t_2)$ and $B(t_1) - B(t_0)$ are independent if $0 \leq t_0 < t_1 \leq t_2 < t_3$
- For $0 \leq s < t$, $B(t) - B(s) \sim \mathcal{N}(0, t-s)$
- B is almost surely continuous

Let $\mu \leq 0$ and $\sigma > 0$ be constants. Define $\{Y(t), t \geq 0\}$ through

$$Y(t) = e^{\sigma B(t) + \mu t} \qquad \text{for all } t \geq 0.$$

b) Compute the mean and the variance of $Y(t)$. (4p)

Solution:

$$\begin{aligned}
\mathbb{E}[e^{\sigma B(t)+\mu t}] &= e^{\mu t}\mathbb{E}[e^{\sigma B(t)}] = e^{\mu t}\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{\sigma x} e^{-x^2/(2t)} dx \\
= e^{\mu t}\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-(x^2-2\sigma t x)/(2t)} dx &= e^{\mu t}\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-[(x-\sigma t)^2-(\sigma t)^2]/(2t)} dx \\
&= e^{\mu t + \sigma^2 t/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi t}} e^{-(x-\sigma t)^2/(2t)} dx = e^{\mu t+\sigma^2 t/2}.
\end{aligned}$$

The last equality is because the integrand is the density of a normal distribution with mean σt and variance t.

For the variance compute (multiplying μ and σ by 2 in the above comp.)

$$\mathbb{E}[Y(t)^2] = \mathbb{E}[e^{2\sigma B(t)+2\mu t}] = e^{2\mu t + 2\sigma^2 t}.$$

So $Var[Y(t)] = \mathbb{E}[Y(t)^2] - (\mathbb{E}[Y(t)])^2 = e^{\sigma^2 t}$.

c) For this subquestion, assume that $\mu = 0$. Provide the distribution function of the first time $Y(t)$ is larger or equal than 2. That is, let

$$T = \inf\{t \geq 0; Y(t) \geq 2\}$$

and find $F(t) = \mathbb{P}(T \leq t)$. (4p)

Solution: Note $T = \inf\{t \geq 0; Y(t) \geq 2\} = \inf\{t \geq 0; B(t) \geq \log[2]/\sigma\}$. By $\mathbb{P}(T \leq t) = \mathbb{P}(\sup_{0\leq s\leq t} B(s) \geq \log[2]/\sigma)$ and the reflection principle we obtain $\mathbb{P}(T \leq t) = 2\mathbb{P}(B(t) > \log[2]/\sigma) = \mathbb{P}(\mathcal{N}(0,t) > \log 2/\sigma)$.

Problem 5: Simulation

Consider a non-homogeneous Poisson Process on $[0, \infty)$ with density

$$\lambda(x) = 1 + \frac{1}{1+x}.$$

Assume that we are only able to generate realisations of uniform random variables and Poisson distributed random variables. Throughout the question all random variables generated are independent.

a) Let $T > 0$ and create the above non-homogeneous Poisson process on the interval $[0, T)$ as follows.

- Generate N, a Poisson distributed random variable with expection $2T$.
- Generate N uniform random variables on $[0, 1]$ and denote those random variables by $U_1, U_2, \cdots, U_N$.
- Generate N uniform random variables on $[0, 1]$ and denote those random variables by $U'_1, U'_2, \cdots, U'_N$.
- If $2U'_i < 1 + 1/(1 + TU_i)$ then place a point at point TU_i. If $2U'_i > 1 + 1/(1 + TU_i)$, then ignore U_i further.

Argue that this procedure indeed generates the points of non-homogeneous Poisson Process on $[0, T)$ with density $\lambda(x) = 1 + 1/(1 + x)$. (4p)

Solution: First create a homogeneous Poisson process with intensity 2 on $[0, T)$. The number of points is by the definition of a homogeneous Poisson Process distributed as a Poisson $(2T)$ random variable. And if the number of points is known, you can use the order statistic property to place the points on the interval $[0, T)$ as independent uniforms.
Then use rejection sampling, by maintaining a point at x with probability $\lambda(x)/2$, independently of other points.

b) How can we generate a realisation of the same non-homogeneous Poisson Process using two Poisson distributed random variables and a number of uniformly distributed random variables, where this number of uniformly distributed random variables has lower expectation than the corresponding number in part a) ? (3p)

Solution: Create independently green and red points each according to a homogeneous Poisson process with intensity 1 (as above). Keep all green points (so there is no need for generating extra uniforms for rejection sampling on the green points) and use rejection sampling with accepting probability $1/(1+x)$ on the red points.

c*) A method to find the first point of this inhomogeneous Poisson Process is the following. Simulate two independent Uniform random variables on $[0,1]$ (say U_1 and U_2). Set $X_1 = -\log[U_1]$ and $X_2 = \frac{U_2}{1-U_2}$ and let the first point of the non-homogeneous Poisson Process be $\min(X_1, X_2)$. Show that $\min(X_1, X_2)$ is indeed the first point of the non-homogeneous Poisson Process. (5p)

Hint: Show that X_1 is distributed as the first point in a Poisson Process with intensity $\lambda_1(x) = 1$ and X_2 is distributed as the first point in a non-homogeneous Poisson Process with intensity $\lambda_2(x) = 1/(1+x)$. Argue that the minimum of those two random variables is distributed as the first point in the original non-homogeneous Poisson Process.

Solution:

Use the idea of part b with red and green points. The first green point arrives after an exponential distributed time with expectation 1 (Which is distributed as $-\log[U_1)$. The first red point is obtained using Example 11.13 (page 677) with $a = 1$. The first point in the combination of the two Poisson Processes is the minimum of the first red point and the first green point.