Bonus Assignment 2 –Parameter Estimation & Bootstrap

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Problem A

A histogram of dataset ‘whales’ is made as shown in Figure 1, from which it can be seen that the distribution of this dataset looks quite like the Gamma distribution.

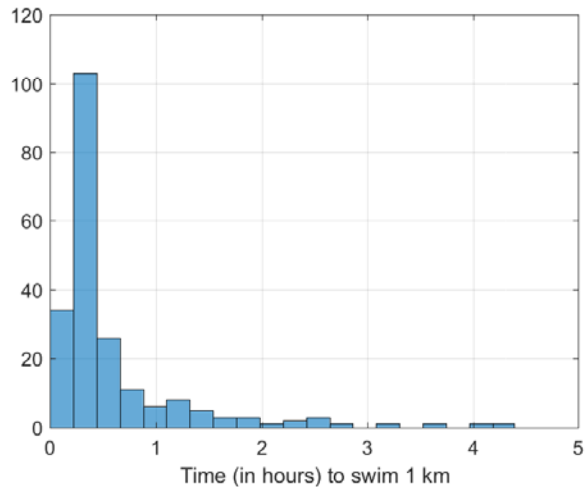

Figure 1. The Histogram of Dataset ‘whales’

Problem B

In this problem, two parameters of Gamma distribution are estimated by the method of moments. To fit the distribution, sample mean $\bar{x}$ and sample square mean $\overline{x^2}$ need to be computed at first. Two estimated parameters $\hat{\alpha}$ (shape parameter) and $\hat{\lambda}$ (scale parameter, not rate parameter) can be solved by Equation 1 and 2 as follows. Results of estimation are shown in Equation 3 and 4.

$$E(X) = \alpha\lambda \quad \text{Eq. 1} \qquad E(X^2) = \alpha(\alpha+1)\lambda^2 \quad \text{Eq. 2}$$

$$\hat{\alpha} = \frac{\bar{x}^2}{\overline{x^2} - \bar{x}^2} = \mathbf{0.7992} \quad \text{Eq. 3} \qquad \hat{\lambda} = \frac{\bar{x}}{\hat{\alpha}} = \mathbf{0.7583} \quad \text{Eq. 4}$$

Problem C

In this problem, parameters α and λ of Gamma distribution will be estimated by the method of maximum likelihood. The likelihood function is shown as Equation 5.

$$L(\alpha,\lambda) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)} \lambda^{-\alpha} x_i^{\alpha-1} e^{-\lambda^{-1} x_i} = \frac{\lambda^{-n\alpha}}{\Gamma^n(\alpha)} (x_1 \dots x_n)^{\alpha-1} e^{-\lambda^{-1}(x_1+\cdots+x_n)} \quad \text{Eq. 5}$$

Let $t_1 = x_1 + \cdots + x_n$ and $t_2 = x_1 \dots x_n$, Equation 5 is converted to Equation 6.

$$L(\alpha,\lambda) = \frac{\lambda^{-n\alpha}}{\Gamma^n(\alpha)} t_2^{\alpha-1} e^{-\lambda^{-1}t_1} \quad \text{Eq. 6}$$

After which, the log-likelihood function is obtained as Equation 7.

$$log[L(\alpha,\lambda)] = -n\alpha\, log(\lambda) - nlog\big(\Gamma(\alpha)\big) + (\alpha-1)\, log(t_2) - \lambda^{-1}t_1 \quad \text{Eq. 7}$$

To maximize the log-likelihood function, partial derivatives of the function with respect to α and λ should be generated first. The solutions of two derivatives that equal to 0 are the estimation of two parameters.

$$\frac{\partial}{\partial\alpha} log[L(\alpha,\lambda)] = -nlog(\lambda) - n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + log(t_2) = 0 \quad \text{Eq. 8}$$

$$\frac{\partial}{\partial\lambda} log[L(\alpha,\lambda)] = -\frac{n\alpha}{\lambda} + \frac{t_1}{\lambda^2} = 0 \quad \text{Eq. 9}$$

These two equations are needed to be solved as shown below.

$$-\mathrm{nlog}\left(\frac{\bar{x}}{\hat{a}}\right) - n\frac{\Gamma'(\hat{a})}{\Gamma(\hat{a})} + log(t_2) = 0 \quad \text{Eq. 10} \qquad \hat{\lambda} = \frac{\bar{x}}{\hat{a}} \quad \text{Eq. 11}$$

To solve two equations, using parameters estimated by the method of moments as initial values. Then, the estimations will be generated as:

$$\hat{\alpha} = \mathbf{1.5954} \qquad \hat{\lambda} = \mathbf{0.3798}$$

Problem D

Two Gamma distributions fitted by the method of moments and the method of maximum likelihood are plotted in Figure 2, the histogram of dataset is also displayed. From the figure, it is observed that the distribution estimated by the method of maximum likelihood is more reasonable than the method of moments does.

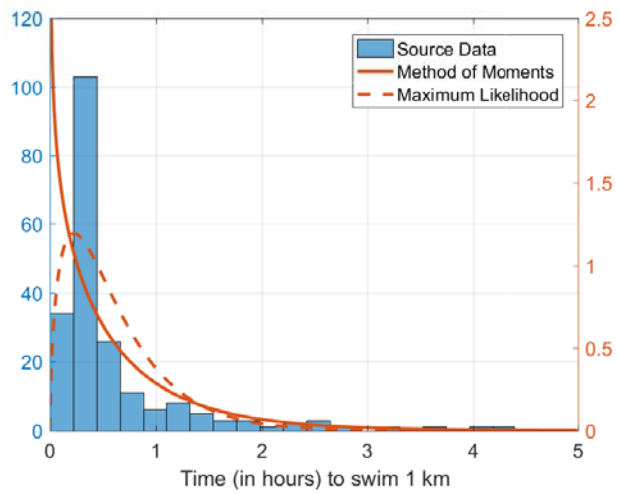

Figure 2. Two Estimated Gamma Distribution and Histogram of Dataset ‘whales’

Problem E

In this section, the standard errors of two parameters estimated by the method of moments is going to be computed by applying the bootstrap. First of all, 1000 samples of size 210 from Gamma(0.7992, 0.7583) are generated. The Gamma distribution has been fitted in problem B by the method of moments.

After which, the estimated $\hat{\alpha}$ and $\hat{\lambda}$ from each sample can be computed by the method of moments, resulting in 1000 $\hat{\alpha}$ and $\hat{\lambda}$ shown as below. Figure 3 displays their distributions, which look like normal distribution for all two parameters.

$$\hat{\alpha}_{mm} = [\hat{\alpha}_1 \; \hat{\alpha}_2 \; \dots \; \hat{\alpha}_{1000}] \qquad \hat{\lambda}_{mm} = [\hat{\lambda}_1 \; \hat{\lambda}_2 \; \dots \; \hat{\lambda}_{1000}]$$

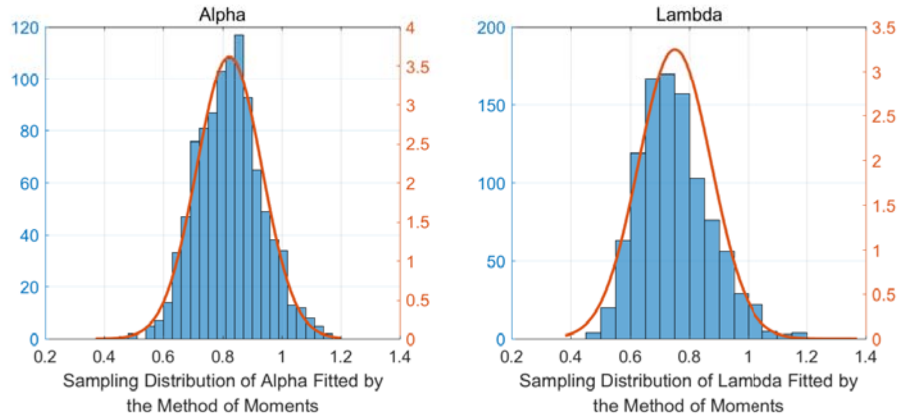

Figure 3. Sampling Distribution of Alpha and Lambda
Fitted by the Method of Moments

The standard deviation of the sampling distribution of $\hat{\alpha}$ can be regarded as standard error here as shown in Equation 12, in which B is the number of samples that is 1000 in this case. The standard error of $\hat{\lambda}$ can be calculated in the same way.

$$\bar{\alpha} = \frac{1}{B}\sum_{i=1}^{B}\hat{\alpha}_i \;, \quad s_{\hat{\alpha}}^2 = \frac{1}{B-1}\sum_{i=1}^{B}(\hat{\alpha}_i - \bar{\alpha})^2 \qquad \text{Eq. 12}$$

The results of the standard errors of two parameters are shown as follows.

$$s_{\hat{\alpha}} = \mathbf{0.1103} \qquad s_{\hat{\lambda}} = \mathbf{0.1228}$$

Problem F

The solution of this problem is quite same as the solution in problem E, but the 1000 samples are generated from the distribution of Gamma(1.5954, 0.3798) that is estimated by the method maximum likelihood. Then, estimate $\hat{\alpha}$ and $\hat{\lambda}$ from each sample by the method of maximum likelihood. Figure 4 shows the plot of each parameter's histogram and the distribution. The results of the standard errors of two parameters are shown as follows.

$$s_{\hat{\alpha}} = \mathbf{0.2040} \qquad s_{\hat{\lambda}} = \mathbf{0.0522}$$

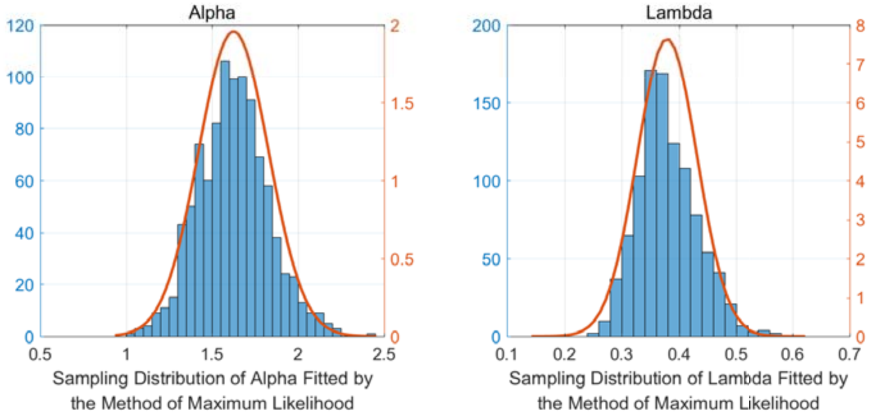

Figure 4. Sampling Distribution of Alpha and Lambda
Fitted by the Method of Maximum Likelihood

Compared the result with the previous one, it is observed that parameters estimated by the maximum likelihood method has a larger standard error in shape parameter but a smaller standard error in scale parameter.

Problem G

This problem is to find an approximate 95% confidence intervals (CI) for $\hat{\alpha}$ and $\hat{\lambda}$ that estimated by the method of maximum likelihood in problem F. First, obtain 2.5 and 97.5 percentile of 1000 $\hat{\alpha}$ and $\hat{\lambda}$ of all samples. The CI of $\hat{\alpha}$ and $\hat{\lambda}$ can be calculated as below, in which c_1 and c_2 are 2.5 and 97.5 percentile of all $\hat{\alpha}$, c_1' and c_2' are 2.5 and 97.5 percentile of all $\hat{\lambda}$. Results are given as follows.

CI for $\hat{\alpha}$: $(2\hat{\alpha} - c_2, 2\hat{\alpha} - c_1)$ CI for $\hat{\lambda}$: $(2\hat{\lambda} - c_2', 2\hat{\lambda} - c_1')$ Eq. 13

CI for $\hat{\alpha}$: **(1.1312, 1.9496)** CI for $\hat{\lambda}$: **(0.2694, 0.4684)**

2017/02/14