SSY130 - Applied Signal Processing
Hand in Problem 2
Qixun Qu (901001-5551)
December 12, 2016

1. Discrete Time Model

The state vector and state equations in continuous formations are shown as Equation 1.1 and 1.2.

$$\text{s(t)} = \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ y(t) \\ \dot{y}(t) \end{bmatrix} \quad (1.1)$$

$$\begin{aligned} \dot{s_1}(t) &= \dot{x}(t) = s_2(t) \\ \dot{s_2}(t) &= \ddot{x}(t) = w_x(t) \\ \dot{s_3}(t) &= \dot{y}(t) = s_4(t) \\ \dot{s_4}(t) &= \ddot{y}(t) = w_y(t) \end{aligned} \quad (1.2)$$

The finite-difference approximation is shown as Equation 1.3, in which *T* is the sampling time.

$$\dot{x}(t)|_{t=kT} \approx \frac{x(kT+T) - x(kT)}{T} \quad (1.3)$$

After which, the discrete-time state equations can be obtained as Equation 1.4.

$$\begin{aligned} s_1(k+1) &= s_1(k) + T\dot{s_1}(k) = s_1(k) + Ts_2(k) \\ s_2(k+1) &= s_2(k) + T\dot{s_2}(k) = s_2(k) + Tw_x(k) \\ s_3(k+1) &= s_3(k) + T\dot{s_3}(k) = s_3(k) + Ts_4(k) \\ s_4(k+1) &= s_4(k) + T\dot{s_4}(k) = s_4(k) + Tw_y(k) \end{aligned} \quad (1.4)$$

Thus, the discrete-time state-space model can be derived as Equation 1.5. *A* is the system matrix, *w(k)* is the process noise matrix.

$$s(k+1) = As(k) + w(k)$$

$$A = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad w(k) = \begin{bmatrix} 0 \\ Tw_x(k) \\ 0 \\ Tw_y(k) \end{bmatrix} \quad (1.5)$$

The measurement equation is shown as below. *C* is the measurement matrix, *v(k)* is the measured noise matrix.

$$\text{z(k)} = Cs(k) + v(k)$$

$$\text{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad v(k) = \begin{bmatrix} v_x(k) \\ v_y(k) \end{bmatrix} \quad (1.6)$$

According to *w(k)* and *v(k)*, covariance matrix *Q* of process noise and covariance matrix *R* of measurement noise can be calculated as below.

$$\begin{aligned} Q &= q \cdot E\{w(k) \cdot w(k)^T\} \\ &= q \cdot E\left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & T^2w_x^2(k) & 0 & T^2w_x(k)w_y(k) \\ 0 & 0 & 0 & 0 \\ 0 & T^2w_x(k)w_y(k) & 0 & T^2w_y^2(k) \end{bmatrix}\right\} \end{aligned} \quad (1.7)$$

$$\begin{aligned} R &= r \cdot E\{v(k) \cdot v(k)^T\} \\ &= r \cdot E\left\{\begin{bmatrix} v_x^2(k) & v_x(k)v_y(k) \\ v_x(k)v_y(k) & v_y^2(k) \end{bmatrix}\right\} \end{aligned} \quad (1.8)$$

In Equation 1.7 and 1.8, $w_x(k)$, $w_y(k)$, $v_x(k)$ and $v_y(k)$ are zero mean white noise that is generated by function *randn*. Thus, $w_x(k)$ and $w_y(k)$ are uncorrelated, $v_x(k)$ and $v_y(k)$ are uncorrelated. Covariance of each noise is 1. *q* and *r* are factors of each covariance matrix. Equations can be simplified as below.

$$Q = q \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & T^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T^2 \end{bmatrix} \quad (1.9)$$

$$R = r \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (1.10)$$

2. Estimated Results

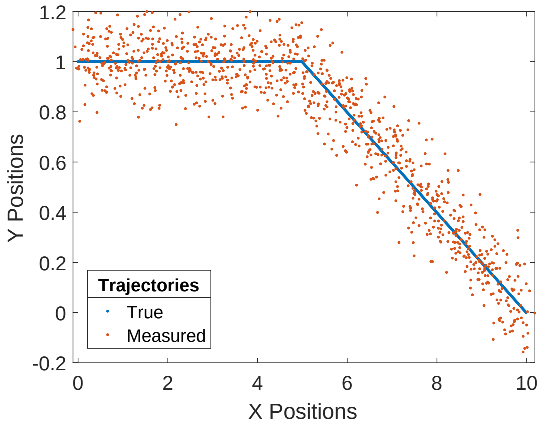

Figure 2.1 True and Measured Trajectories

Figure 2.1 shows the true trajectory and the measured trajectory. In the following filtering process, initial state vector is a vector full of zeros, and initial state covariance matrix $P_0 = 10^6 \times \boldsymbol{I}$. Figure 2.2, 2.3 and 2.4 show different estimations of various combination of q and r in case of the same T (0.01). Figure 2.5 shows the result with T of 0.1. What is important is the ratio between q and r. Since increasing measurement noise covariance matrix R with a factor has the equal effect of descending process noise covariance matrix Q with the same factor. For example, when T is 0.01, estimation obtained with q=0.1 and r=0.1 is same as the result of q=1 and r=1.

3. Explanation

In this assignment, tracking a target in *xy*-plane is focused on. There are two kinds of descriptions of target's moving. The first one is that the present state of target (position and velocity) can be obtained by making use of previous state, which is called "prediction". Equation 1.5 shows the method to predict moving state of target. The other one is that the *xy*-position of target has been measured by a camera-based sensor. Kalman filter is able to determine the weight of difference between prediction and measurement to correct estimation. The weight is well known as Kalman gain.

$$\hat{x}_k^+ = \hat{x}_k + K(z_k - C\hat{x}_k) \quad (3.1)$$

$$K = \frac{P_k C^T}{C P_k C^T + R} \quad (3.2)$$

$$P_{k+1} = A P_k (1 - KC) A^T + Q \quad (3.3)$$

In Equation 3.1, $\hat{x}_k^+$ is the estimation result, $\hat{x}_k$ is prediction, z_k is measured value and K is Kalman gain that is shown in Equation 3.2. Equations 3.3 shows the update of state covariance. Equation 3.2 indicates the proportion of k^{th} pridiction's minimum mean square error in whole error (prediction error and measurement error). If K is close to zero, which means measurement noise covariance matrix R is large (or process noise covariance matrix Q is small), the estimation result is close to prediction value, because the error of measured value is larger than prediction. As shown in Figure 2.3, estimation has little noise but slow convergence rate. If K approximates 1 (process noise covariance matrix Q is large or measurement noise covariance matrix R is small), the estimation result is close to measured value. As shown in Figure 2.4, estimation is able to converge quickly but is not very stable with much noise.

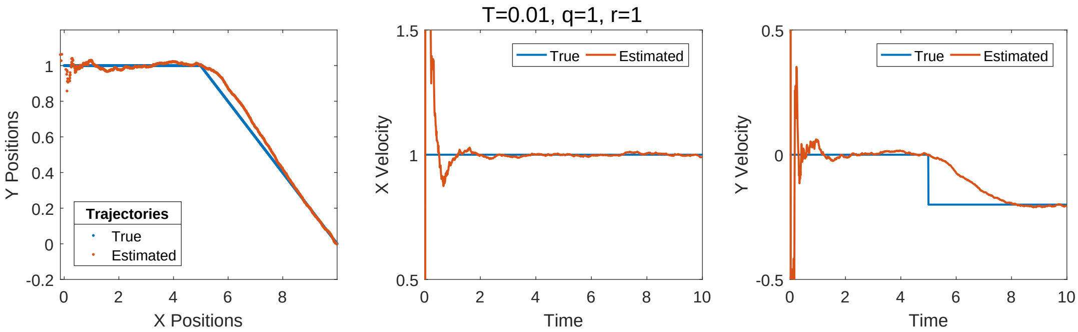

Figure 2.2 Estimated Result in case of T=0.01, q=1, r=1

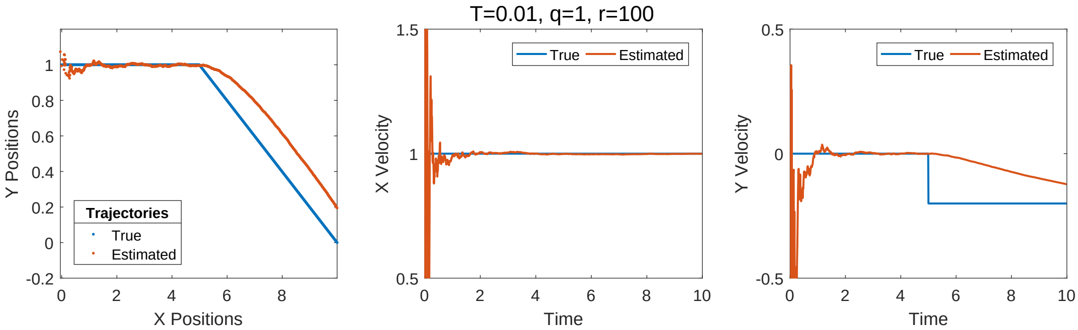

Figure 2.3 Estimated Result in case of T=0.01, q=1, r=100

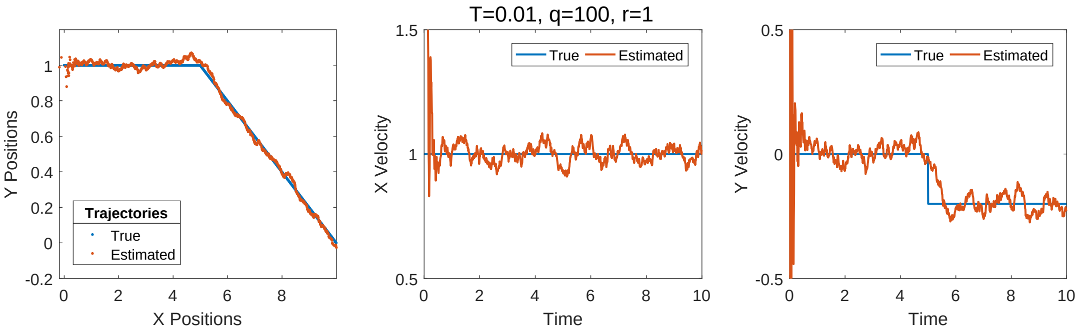

Figure 2.4 Estimated Result in case of T=0.01, q=100, r=1

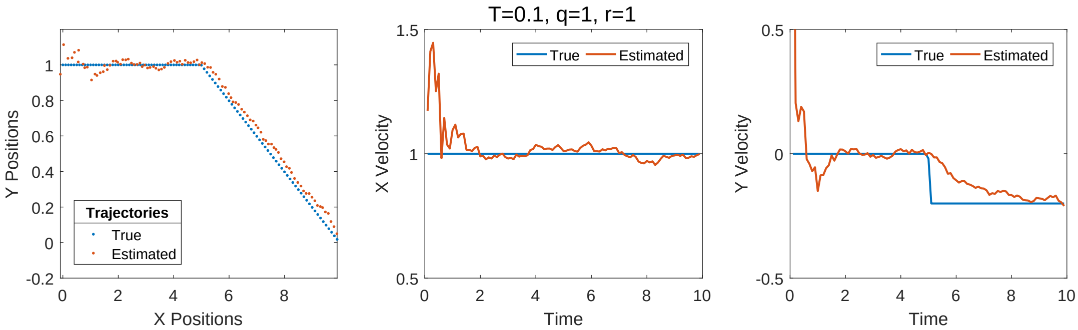

Figure 2.5 Estimated Result in case of T=0.1, q=1, r=1

Sampling time T determines the covariance level of process noise matrix Q in this case as shown in Equation 1.9. The larger the T is, more error prediction value introduces. Thus, Kalman gain is closer to 1 than 0, taking measured value into more account to do the estimation. Comparing Figure 2.5 with Figure 2.2, it is observed that there are more noise in estimation result when T=0.1. On the contrary, if T decreases, prediction value will take a greater part in estimation result since prediction has little error.

The initial state covariance matrix P_0 influences the convergence rate of estimation. In general, P_0 is large to make sure that initial estimations converge quickly. Because initial Kalman gain is close to 1 if P_0 is large, causing the initial estimations are close to measured values which are near the true values.