ESS101- Modeling and Simulation
Lecture 16

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Today (Chapters 6, 7)

- ☞ Motivating examples
- ☞ Absolute stability of Linear Multistep Methods (scalar case)
- ☞ A-Stability
- ☞ Systems of ODEs
- ☞ Stiff systems

Absolute stability. Motivation

Convergence (consistency and zero-stability) guarantees that

$$x_n \rightarrow x(t^*), \text{ as } n \rightarrow \infty \text{ and } h \rightarrow 0, \text{ with } t_n = t_0 + nh = t^*$$

The approximations calculated by a LMM can be arbitrarily close to the real solution, if h is chosen arbitrarily small

What happen to x_n when h is *not* arbitrarily small?

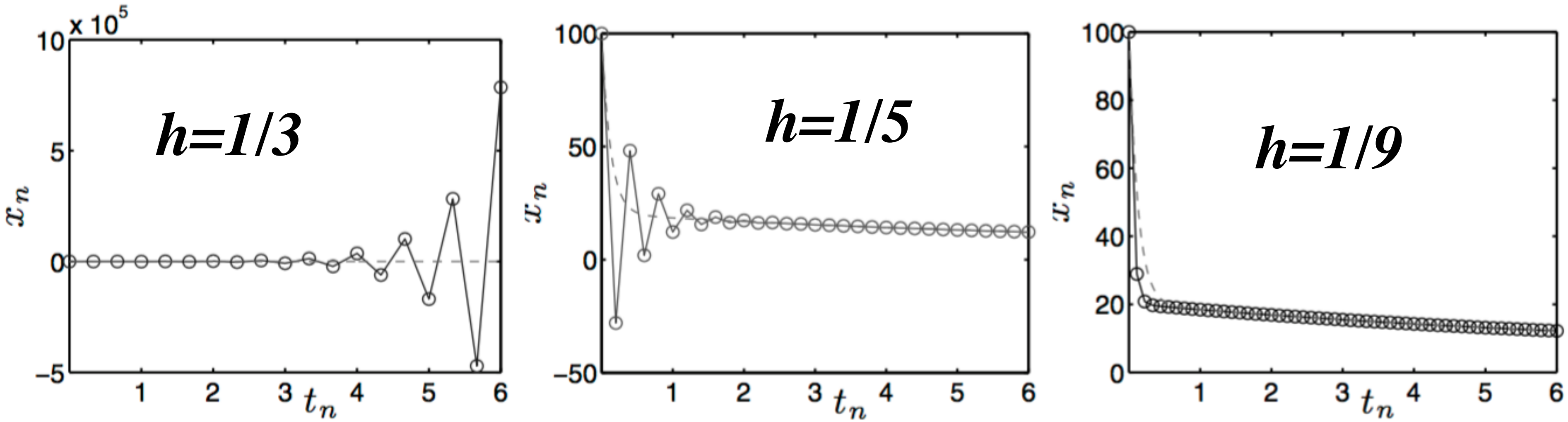
Example

Consider the IVP

$$\dot{x}(t) = -8x(t) - 40\left(3e^{-t/8} - 1\right), \quad x(0) = 100$$

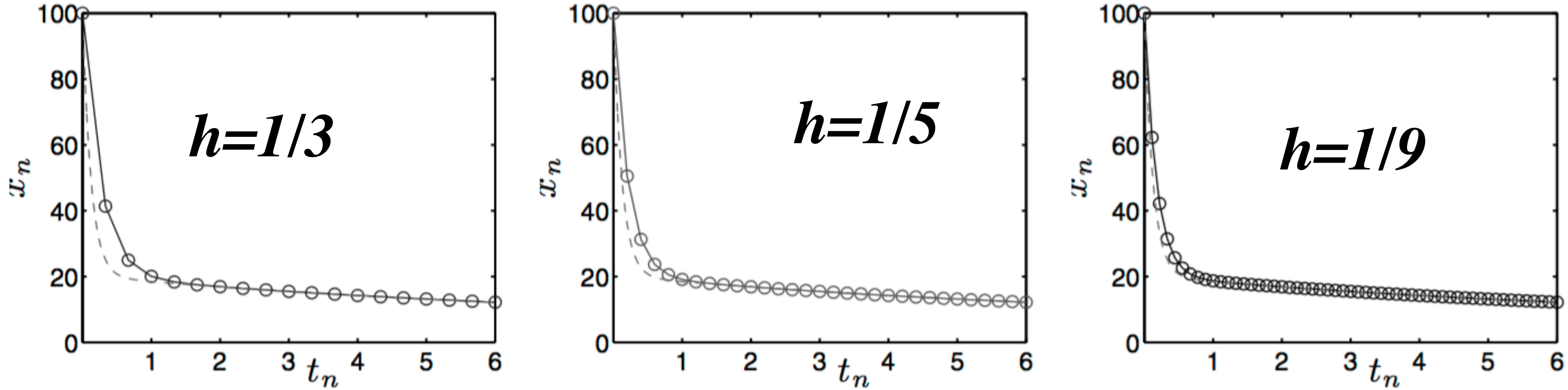
The exact solution is $x(t) = \dfrac{1675}{21} e^{-8} - \dfrac{320}{21} e^{-t/8} + 5$

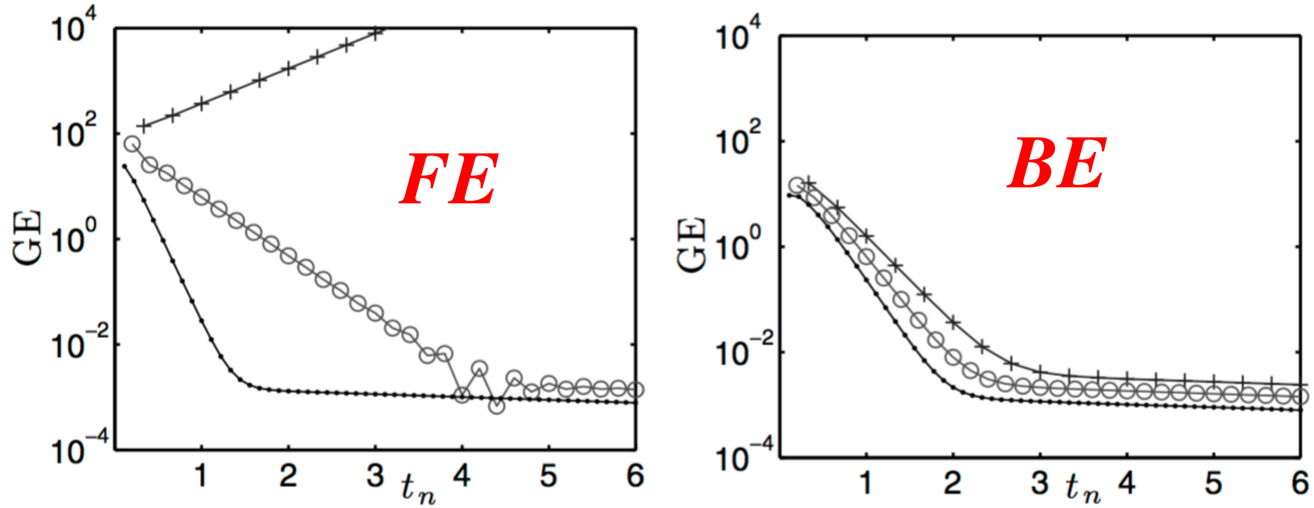
The ***FE method*** results in $x_{n+1} = \left(1 - 8h\right) x_n + h\left(120e^{-t_n/8} + 40\right)$



Example

The ***BE method*** results in $x_{n+1} = x_n - 8hx_{n+1} + h\left(120e^{-t_{n+1}/8} + 40\right)$





$+:\ h = 1/3$

$\circ:\ h = 1/5$

$\bullet:\ h = 1/9$

Example

FE method

$$x_{n+1} = (1-8h)x_n + h\left(120e^{-t_n/8} + 40\right)$$

BE method

$$x_{n+1} = \frac{1}{1+8h}\left[x_n + h\left(120e^{-t_{n+1}/8} + 40\right)\right]$$

Where the instability is coming from?

Example

Consider the IVP

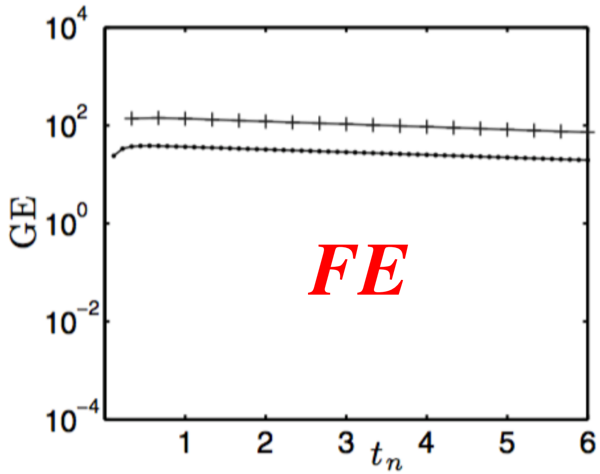
$$\dot{x}(t) = -\frac{1}{8}\left(x(t) - 5 - 5025e^{-8t}\right), \quad x(0) = 100$$

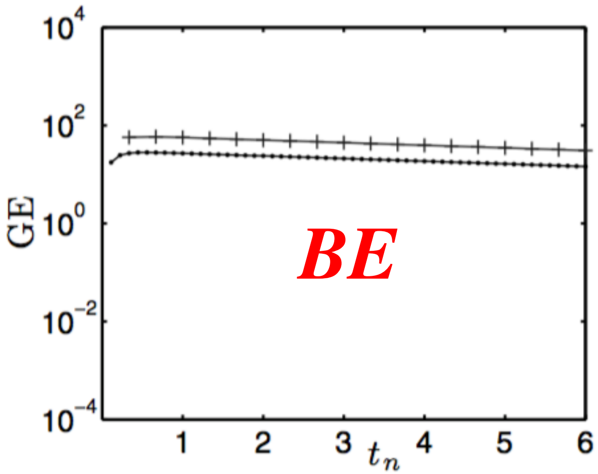
FE method

$$x_{n+1} = \left(1 - \frac{1}{8}h\right)x_n - h\left(5 - 5025e^{-8t_n}\right)$$

BE method

$$x_{n+1} = \frac{1}{1 + h/8}\left[x_n - h\left(5 - 5025e^{-8t_{n+1}}\right)\right]$$





$+: \; h = 1/3$

$\bullet: \; h = 1/9$

Example. Conclusions

- FE is unstable for large h. BE is not
- Instability depends on the system dynamics
- The GE decays faster for smaller h
- Small h for stability may generate over-accurate results

Absolute stability

Stability depends on both h and the system dynamics

Hence we start by studying ***absolute stability*** of a convergent LMM for the *scalar model problem*

$$\dot{x}(t) = \lambda x(t), \quad \mathrm{Re}(\lambda) < 0$$

Unlike convergence, here h is fixed

Absolute stability

Definition (Absolute stability). A LMM method is ***absolutely stable*** if its solution x_n to the problem

$$\dot{x}(t) = \lambda x(t), \quad \mathrm{Re}(\lambda) < 0$$

converges to zero as $n \to \infty$

That is the LMM resembles the asymptotic behavior of our system

Absolute stability criterion

Consider a 2-step LMM method

$$x_{n+2} + \alpha_1 x_{n+1} + \alpha_0 x_n = h\left(\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n\right)$$

For our system $\dot{x}(t) = \lambda x(t)$, it becomes

$$x_{n+2} + \alpha_1 x_{n+1} + \alpha_0 x_n = \lambda h\left(\beta_2 x_{n+2} + \beta_1 x_{n+1} + \beta_0 x_n\right)$$

After rearranging

$$\left(1 - \lambda h \beta_2\right) x_{n+2} + \left(\alpha_1 - \lambda h \beta_1\right) x_{n+1} + \left(\alpha_0 - \lambda h \beta_0\right) x_n = 0$$

This is a dynamical system in DT whose solution is

$$x_n = a r_1^n + b r_2^n$$

Absolute stability criterion

r_1 and r_2 are solutions of the ***characteristic (or stability) polynomial***

$$\underbrace{\left(1-\lambda h\beta_2\right)r^2+\left(\alpha_1-\lambda h\beta_1\right)r+\left(\alpha_0-\lambda h\beta_0\right)}_{\rho(r)-\lambda h\sigma(r)}=0$$

In order to have stability the roots r of $\rho(r)-\lambda h\sigma(r)=0$ must be $|r|<1$

Lemma (necessary and sufficient condition for absolute stability). A LMM is absolutely stable for a given λh *if and only if* the roots of the stability polynomial are strictly within the unitary circle

Absolute stability region

Definition (Absolute stability region). The region of absolute stability for a two-step LMM method is the set of complex values λh that make the method absolutely stable

Example. Consider the Euler method

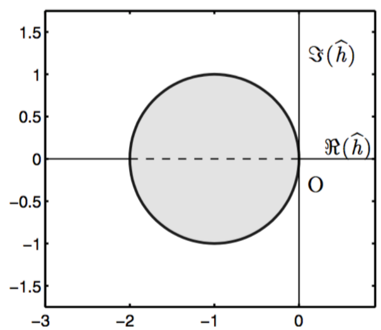
$$x_{n+1} = x_n + hf_n$$

Applied to the system $\dot{x}(t) = \lambda x(t), \quad \mathrm{Re}(\lambda) < 0$

$$x_{n+1} = x_n + h\lambda x_n \qquad \Rightarrow \rho(r) - \lambda h \sigma(r) = r - 1 - \lambda h = 0$$

The stability criterion gives

$$|1 + \lambda h| < 1$$



Example

What the largest h with $\lambda = -3 + 4j$?

$$|1 + \lambda h| < 1 \quad \Rightarrow |1 - 3h + 4jh| < 1 \quad \Rightarrow |1 - 3h + 4jh|^2 < 1$$

$$\Rightarrow -8h + 25h^2 < 0 \Rightarrow h < 8/25$$

Example. Find the stability region of the trapezoidal rule

$$x_{n+1} - x_n = \frac{1}{2}h\left(f_{n+1} + f_n\right)$$

The stability polynomial is $\left(1 - \frac{1}{2}h\lambda\right)r - \left(1 + \frac{1}{2}h\lambda\right)$

$$\left|\frac{1 + h\lambda/2}{1 - h\lambda/2}\right| < 1 \quad \text{satisfied} \quad \forall \lambda : \quad \mathrm{Re}(\lambda) < 0$$

A-Stability

Definition (A-Stability). A method is *A-stable* if its stability region is the left half-plane of

Theorem (Dahlquist's Second Barrier Theorem).

1. There's no A-stable explicit LMM
2. An A-stable LMM cannot have $p>2$

Systems of ODEs

The stability results obtained so far can be easily extended to systems of ODEs

Consider the system of ODEs

$$\dot{x}(t) = Ax(t), \quad \mathrm{Re}(\lambda_i) < 0, \quad \forall i$$

Assuming the system has n distinct eigenvalues a coordinate transformation $x = Vz$ exists such that

$$\dot{z}(t) = \Lambda z(t), \quad \Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Systems of ODEs

The resulting system of ODEs consists of n decoupled *scalar* equations that can be studied with the results presented so far

For each equation, the roots r_i of the corresponding stability polynomial must be $|r_i|<1$

Example

Consider the second order system

$$\dot{x}(t) = \begin{bmatrix} -11 & 100 \\ 1 & -11 \end{bmatrix} x(t),$$

with eigenvalues λ_1=-1, λ_2=-21.

If we use the Euler method

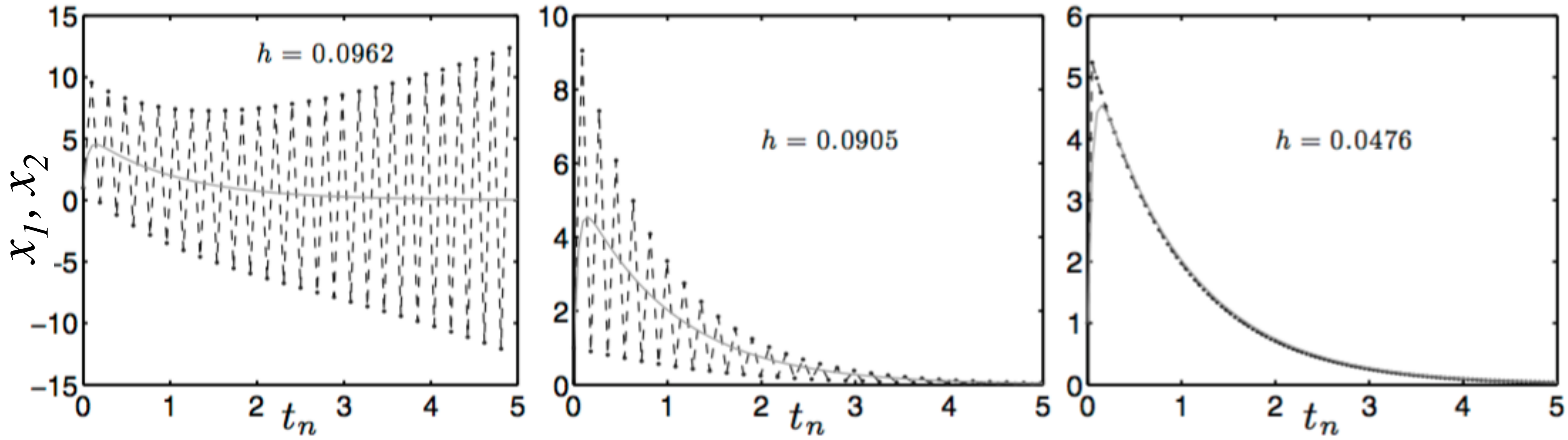
$$x_{1,n+1} = x_{1,n} + h\left(-11x_{1,n} + 100x_{2,n}\right),$$

$$x_{2,n+1} = x_{2,n} + h\left(x_{1,n} - 11x_{2,n}\right),$$

For Euler method we found that the stability region is

$$\left|1 + \lambda_i h\right| < 1 \Rightarrow \begin{cases} \lambda_i h < 0 \\ -2 < \lambda_i h \end{cases} \qquad 0 < h < \frac{2}{21} \approx 0.0952$$

Example



Smaller stepsize is required because of λ_2. Fulfilling the stability requirement lead to over-accurate solution and unnecessary complexity

Stiff systems

Linear systems

$$\dot{x}(t) = Ax(t), \quad \mathrm{Re}(\lambda_i) < 0, \quad \forall i$$

may have eigenvalue with the real part very much different in absolute value. I.e., high condition number

$$\left| \frac{\max_i \mathrm{Re}(\lambda_i)}{\min_i \mathrm{Re}(\lambda_i)} \right|$$

Stiff systems

In order to avoid over accurate solutions due to the absolute stability requirements set by the fastest eigenvalue, A-stable *(implicit)* methods should be used