ESS101- Modeling and Simulation
Lecture 6

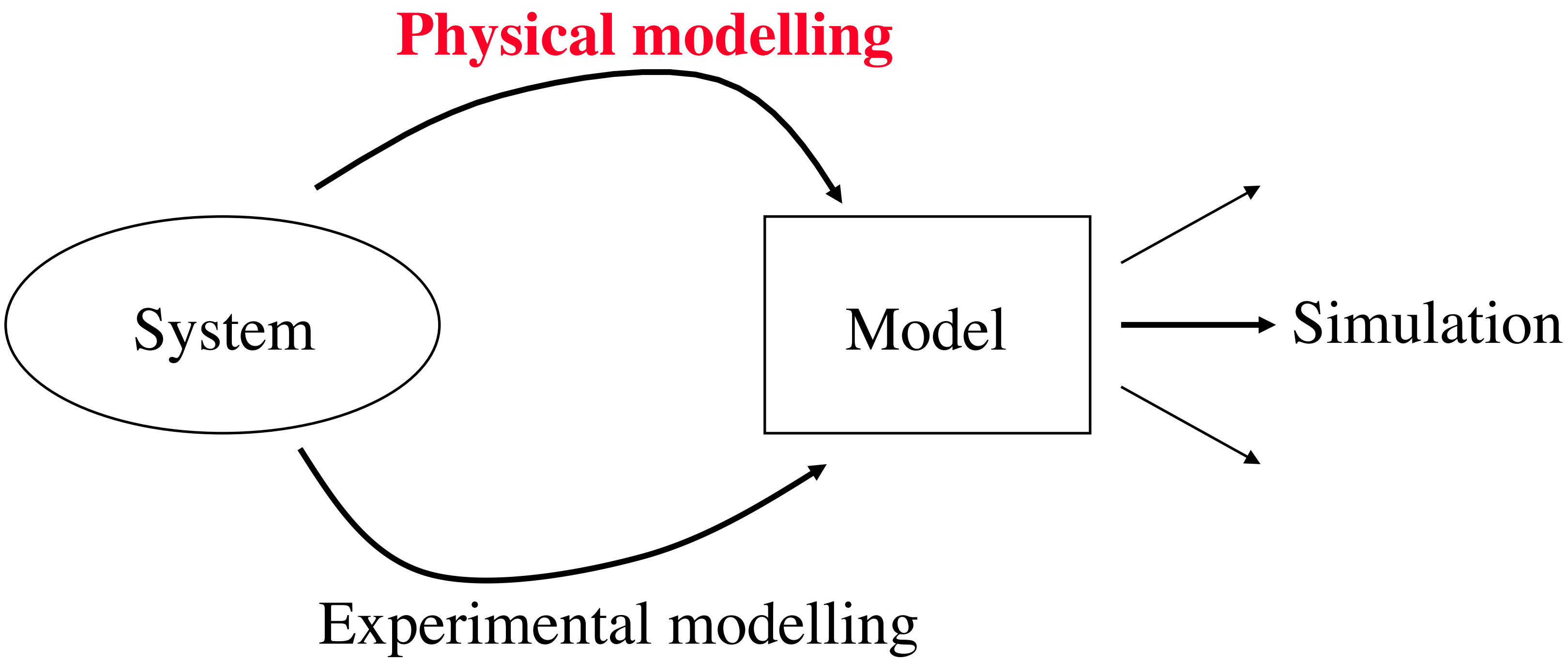
Paolo Falcone

Department of Signals and Systems

Chalmers University of Technology

Göteborg, Sweden

Today (Chapter 4, 5)

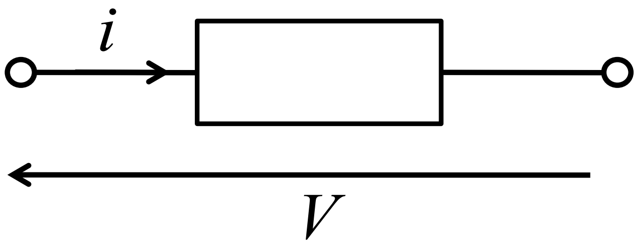


- ✓ Physical modeling
- ✓ Three phases method

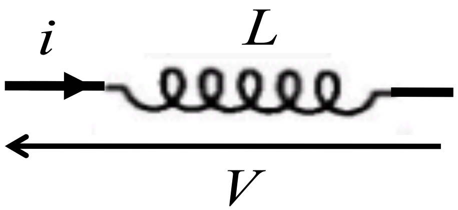
Three phases method

- Structuring
  - Divide into subsystems
  - Inputs, outputs, internal variables
- Relationships
  - Conservation laws
  - Constitutive relations
- Form state-space model
  - Choose state variables
  - Form $\dot{x} = \ldots$

Electrical systems



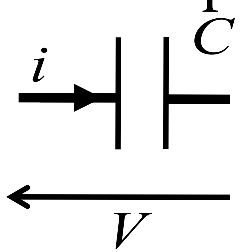
Ideal inductor



$$V(t) = L\frac{d}{dt}i(t)$$

$$i(t) = \frac{1}{L}\int_0^t V(s)ds$$

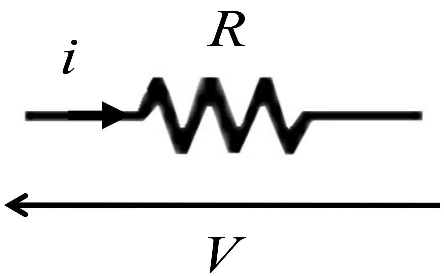
Ideal capacitor



$$i(t) = C\frac{d}{dt}V(t)$$

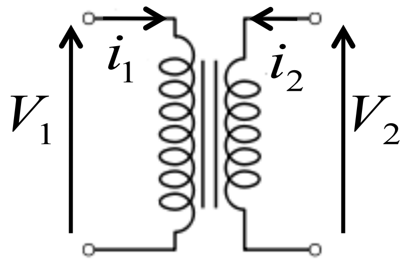
$$V(t) = \frac{1}{C}\int_0^t i(s)ds$$

Ideal resistor



$$V(t) = Ri(t)$$

Ideal transformers



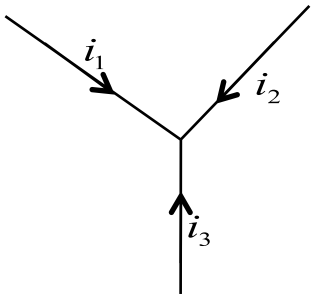
$$V_1 i_1 = V_2 i_2$$

$$V_1 = kV_2$$

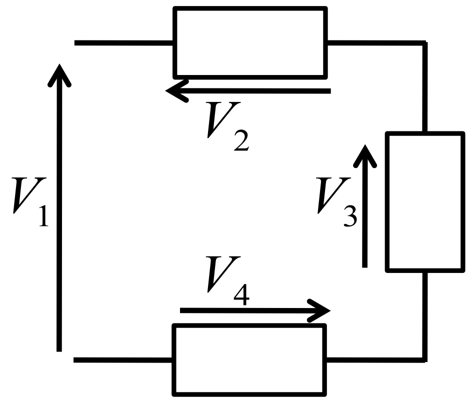
$$i_1 = \frac{1}{k}i_2$$

Electrical systems

Kirchhoff's laws



$$\sum_k i_k = 0$$

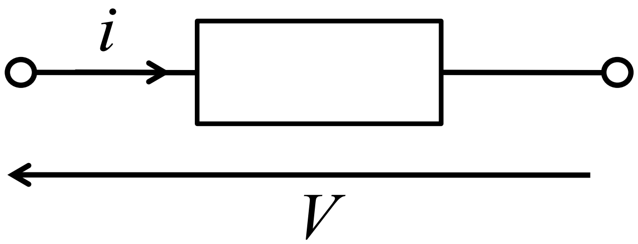


$$\sum_k V_k = 0$$

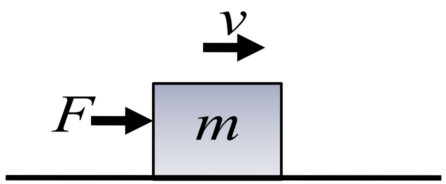
Energy storage

$$E_L(t) = \frac{1}{2} L i^2(t), \qquad E_C(t) = \frac{1}{2} C V^2(t)$$

Mechanical systems



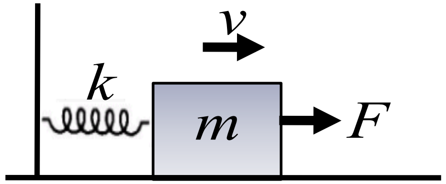
Mass



$$F(t) = m\frac{d}{dt}v(t)$$

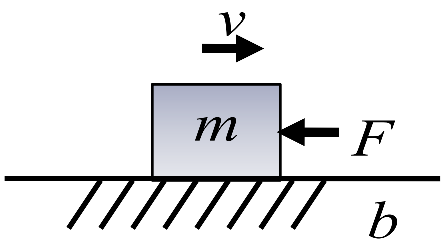
$$v(t) = \frac{1}{m}\int_0^t F(s)ds$$

Spring element



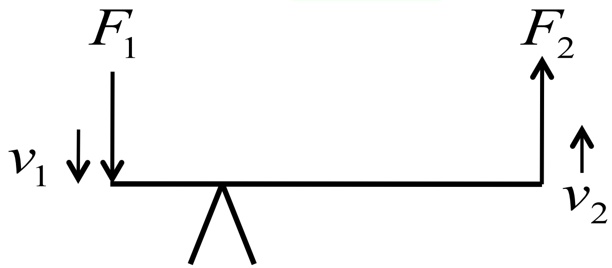
$$F(t) = k\int_0^t v(s)ds$$

Friction element



$$F(t) = -bv(t)$$

Lever



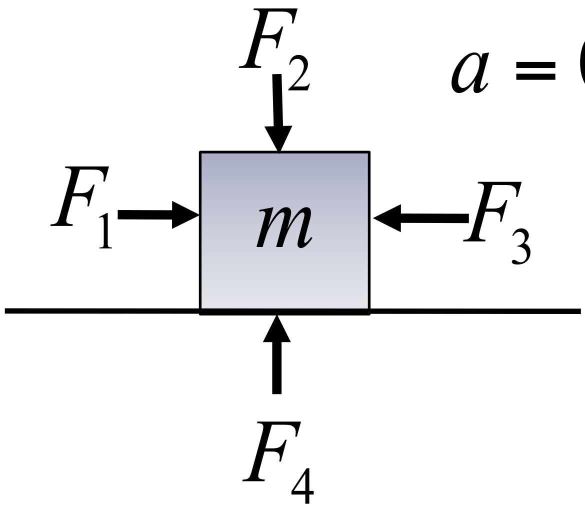
$$F_1 v_1 = F_2 v_2$$

$$F_1 = kF_2$$

$$v_1 = \frac{1}{k}v_2$$

Mechanical systems

Newton's law



$$a = 0$$

$$\sum_k F_k = 0$$

Energy storage

$$E_K(t) = \frac{1}{2} m v^2(t), \qquad E_s(t) = \frac{1}{2k} F^2(t)$$

Flow, thermal systems

Similar analogies exist for rotating mechanical systems, flow and thermal systems.

See pages 112-121 of the textbook

Phase 1: structuring the problem

Intuition of how the system works is necessary

Q: What the model is meant for?

Based on the answer, break down the overall system into simpler subsystems

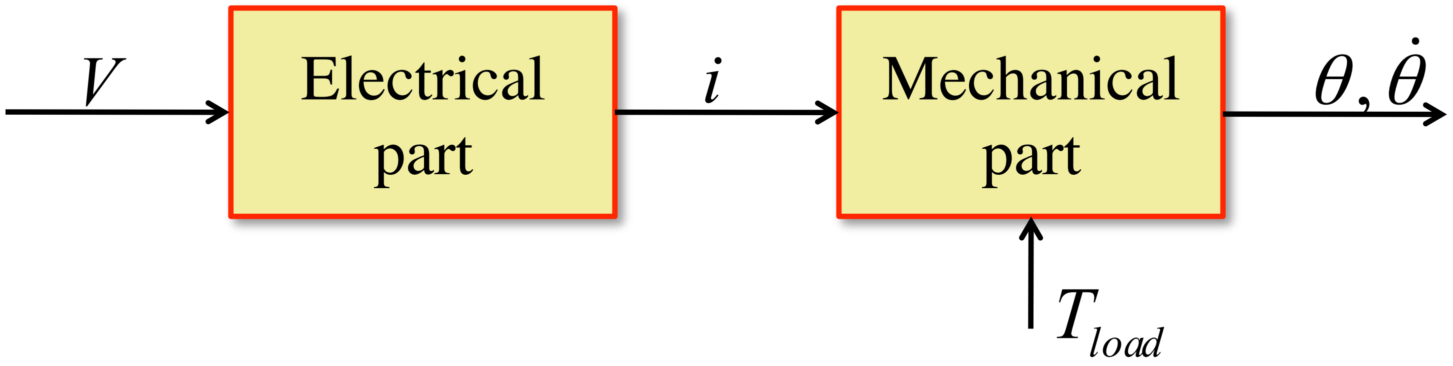
Hints:

1. Identify the manipulated variables (u), disturbances (d) and output variables (y)
2. Identify cause and effect relationships from u and d to y
3. Identify the variables important to describe how the system evolves
4. Identify the constant (slowly varying) and varying variables
5. Identify the static and dynamic relationships

Phase 1: structuring the problem. Example. DC motor

Remarks:

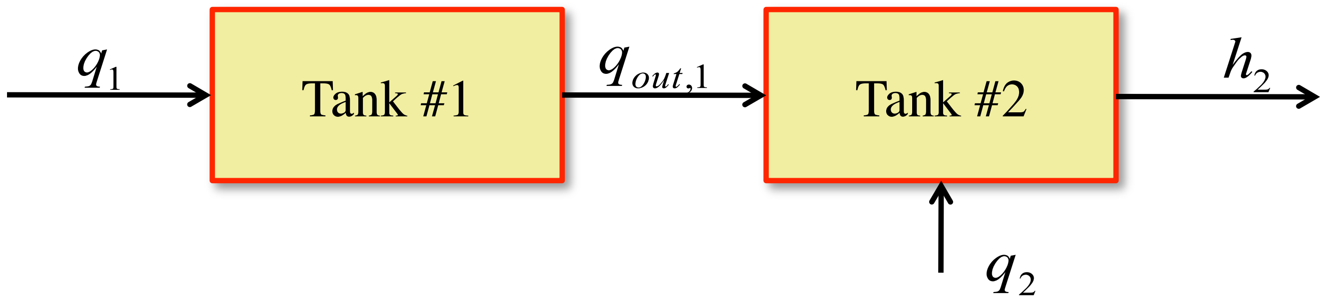
1. We manipulate the input voltage (u) to vary the angular position and velocity (y), while the rotor is subject to external torque (d)
2. The input voltage increase the current in the windings (electric part). The current generates a torque rotating the rotor (mechanical part).
3. Input voltage, current, rotor speed
4. Parameters in electric and mechanical part can be considered constants
5. Electrical part much faster than mechanical part



Phase 1: structuring the problem. Example. Coupled tanks

Remarks:

1. We manipulate the input flow to the first tank (u) to vary the level of the first tank (internal variable). The level of the first tank affects the input flow to the second tanks. An input flow to the second tank (d) affects the level of the second tank
2. The level of the first flow determines the input flow to the second tank (first tank). The output flow from the first tank together with the second input flow (d) and the output flow determines the level in the second tank (second tank).
3. Manipulated flow, tanks levels, input flow to the second tank
4. Parameters in both tanks can be considered constant
5. Depending on the tank geometry one tank might have faster filling-empty dynamics



Phase 2: basic equations formulation

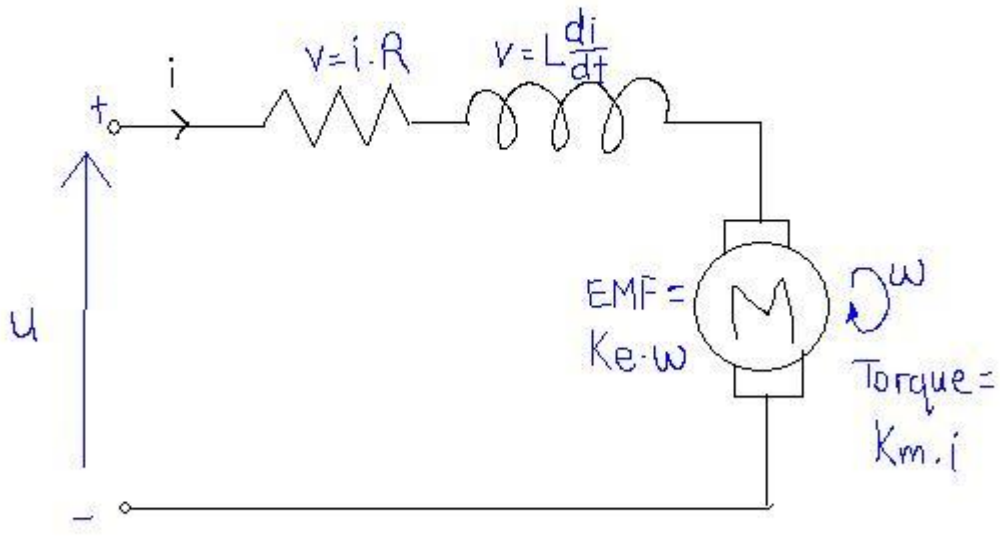
For each subsystems use basic physical equations and principles to derive cause-effect relationship

Simplifying assumptions are always needed

Hints:

1. Simplest assumption: No assumption at all. Complex model. Model simplification. Hard to interpret the results.
2. Start with a simple model. Complicate it where needed. Drop important part. Unexpected behavior in some operating point.
3. Conservation laws.
   - ✓ Power in – Power out = stored energy per time unit
   - ✓ Input flow rate – Input flow rate = stored volume per time unit
   - ✓ Input mass flow rate - Output mass flow rate = stored mass per time unit
4. Constitutive relationships. Current-voltage relationships in inductors, capacitors

Phase 2: basic equations formulation. Example. DC motor



Kirchhoff's laws

$$u = V_R + V_L + V_{EMF}$$

$$i_R = i_L = i$$

Newton's law

$$J\dot{\omega} = T_M + T_F + T_L$$

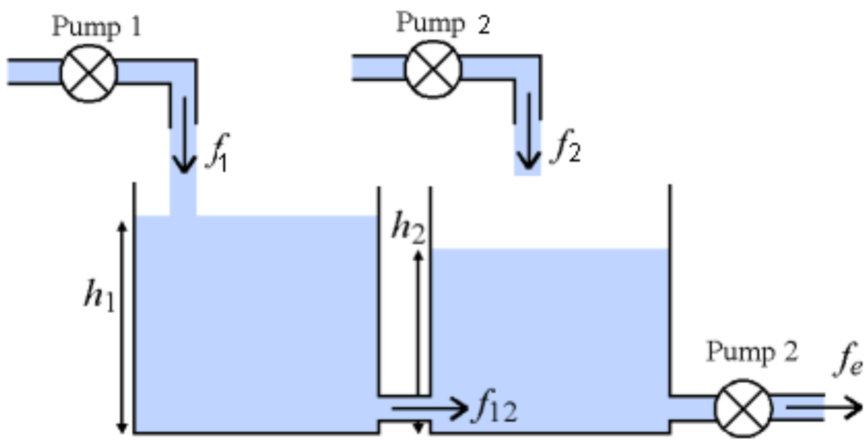
Constitutive relationships

$$V_R = Ri_R \qquad T_M = k_m i$$

$$V_L = L\frac{di_L}{dt} \qquad T_F = b_f \omega$$

$$V_{EMF} = K_e \omega$$

Phase 2: basic equations formulation. Example. Coupled tanks



Input flow rate balancing

$$\dot{V}_1 = f_1 - f_{12}$$

$$\dot{V}_2 = f_2 + f_{12} - f_e$$

Constitutive relationships

$$V_1 = A_1 h_1$$

$$V_2 = A_2 h_2$$

$$f_{12} = a_1 \sqrt{2g(h_1 - h_2)}$$

$$f_e = a_2(u)\sqrt{2gh_2}$$

Phase 3: deriving a model representation

Hints:

1. Choose a set of state variables
   - ✓ The state variables represent the memory of the system (energy).
2. Express the time derivative of each state variable as function of the state and input variables
   - ✓ Additional state variable can be added
3. Express the output variables of each state variable as function of the state and input variables

State variables. Examples

- Position of point mass
- Velocity of point mass
- Charge of capacitor
- Current through inductor
- Temperature
- Tank level

Phase 3: deriving a model representation. Example. DC motor

Defining state, input, output and disturbance vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1 = i, x_2 = \omega \qquad u = V \qquad y = \omega \qquad d = T_L$$

State space representation

$$\dot{x} = Ax + Bu + Ed \qquad A = \begin{bmatrix} -\frac{R}{L} & -\frac{k_e}{L} \\ \frac{k_m}{J} & -\frac{b}{J} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ -\frac{1}{J} \end{bmatrix}$$

$$y = Cx + Du$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad D = 0$$

Input-output representation

$$Y(s) = \left[C(sI - A)^{-1}B + D\right]U(s)$$

Phase 3: deriving a model representation. Example. Coupled tanks

HOMEWORK

Study the example of three phases physical modeling in Chapter 4

Simplified models

- ☞ Neglecting small effects
- ☞ Separating time constants
- ☞ Aggregating state variables

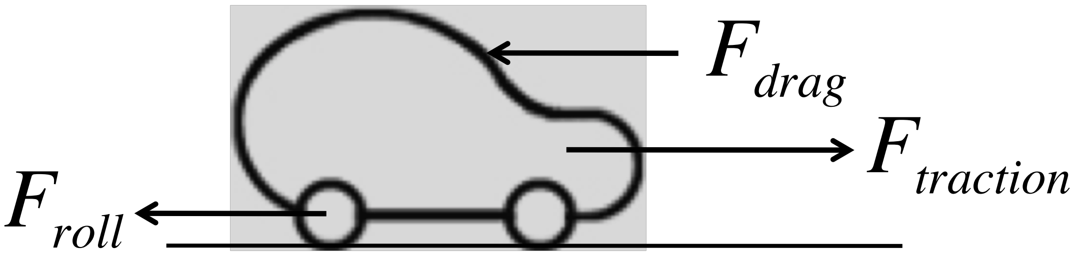
Small effects are neglected

Make assumptions:

- No mass
- No friction
- Incompressible
- Linear
- Perfect mixing
- Ideal gas

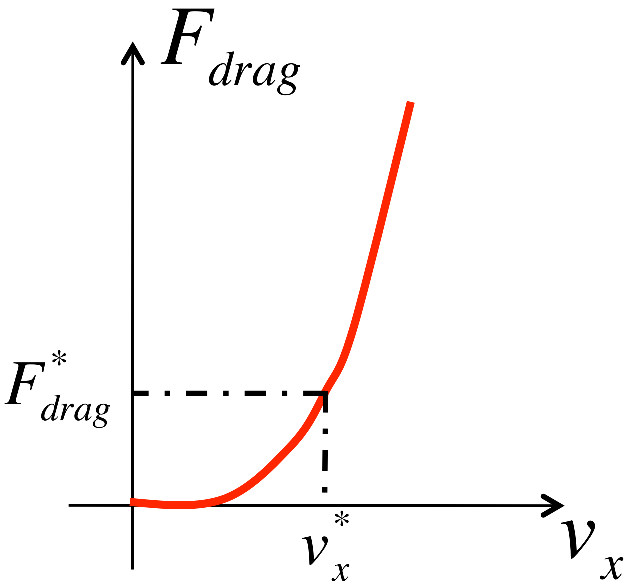
Make balanced approximations

Example. The longitudinal vehicle dynamics



with $\begin{cases} F_{roll} = k \cdot m \\ F_{drag} = \dfrac{\rho}{2} C_x v_x^2 \end{cases}$

$$\boxed{m\ddot{x} = F_{traction} - F_{roll} - F_{drag}}$$



Acceleration

$$v_x \text{ small } \Rightarrow F_{traction} >> F_{drag}$$

$$m\ddot{x} = F_{traction} - F_{roll}$$

Steady state, high speed

$$v_x \text{ high } \Rightarrow F_{traction} \approx F_{drag}$$

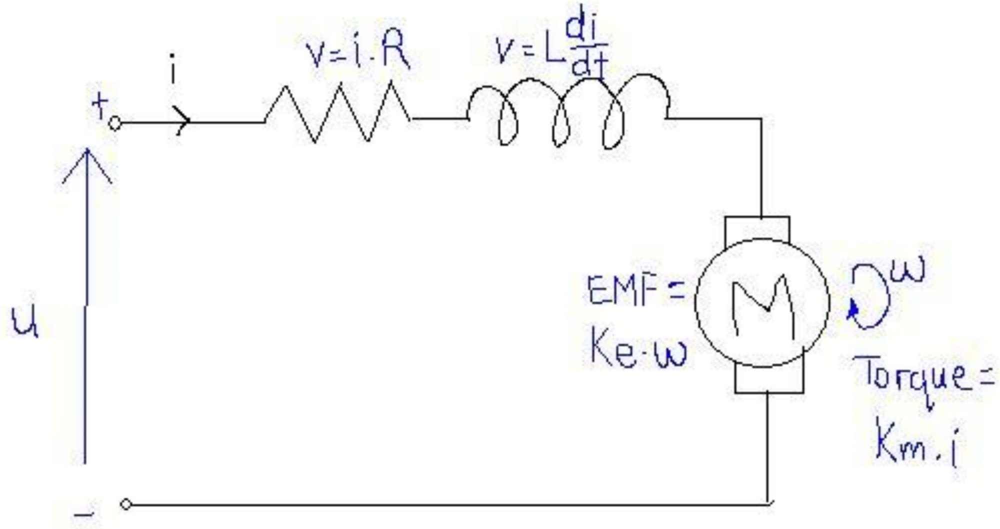
$$m\ddot{x} = F_{traction} - F_{roll} - F_{drag}$$

Separation of time constants

- Subsystem whose dynamics are considerable *faster* are approximated with *static relationships*

- Subsystems whose dynamics are appreciable *slower* are approximated as *constants*

Example (approximation of fast dynamics). The electric DC motor



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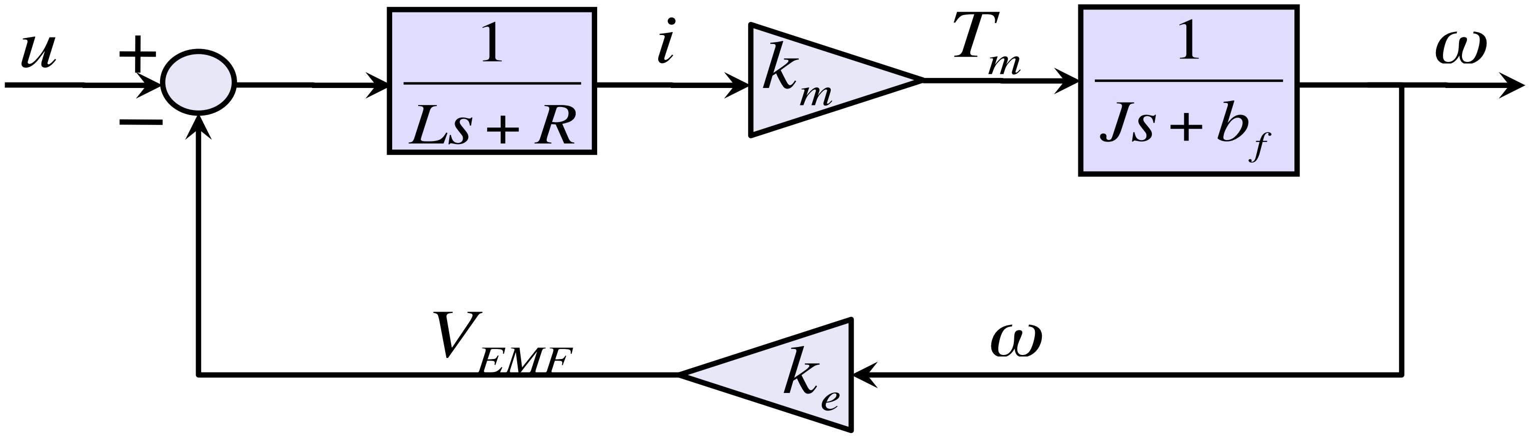
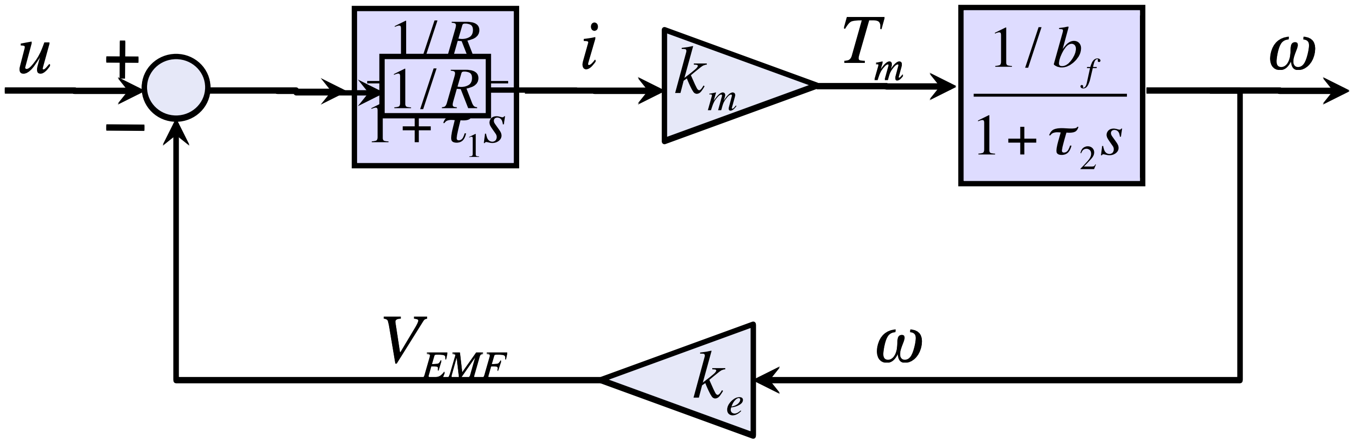
$$V_{EMF} = K_e \omega$$

Example (approximation of fast dynamics). The electric DC motor

Parameter	Value	Dimension
k_e	2.39	$V/kRPM$
L	0.03	μH
k_T	2.28	$N*cm/Amp$
J	0.37	$Kg*cm^2$
R	0.640	$Ohms$
b	0.2	$N*cm/kRPM$

Example (approximation of fast dynamics). The electric DC motor





$$\tau_1 = L/R \approx 0.04$$

$$\tau_2 = J/b_f \approx 1.85$$

$$\tau_2 >> \tau_1$$