ESS101- Modeling and Simulation
Lecture 15

Paolo Falcone

Department of Signals and Systems

Chalmers University of Technology

Göteborg, Sweden

Today (Chapters 4, 5)

- Linear Multistep Methods
  - Trapezoidal rule
  - The Adams-Bashforth method
- Two-step methods
  - Consistency
- Convergence and Zero-stability

Basics on simulation (IVP)

We assume the mathematical model is in the form

$$\dot{x}(t) = f(t, x(t), u(t))$$

By assuming that *u(t)* is a known function of the time

$$\dot{x}(t) = f(t, x(t))$$

The problem of simulating the model $\dot{x} = f(t, x(t))$ can be formulated as follows

Given the initial condition $x(0)$, finding a sequence of points $x_1, x_2, \ldots x_{tf}$ approximating the solution x at the time instants $0<t_1<t_2<\ldots<t_f$, i.e., the $x(t_1), x(t_2), \ldots, x(t_f)$

FE and TS(p) methods

For FE

$$x_{n+1} = x_n + h\dot{x}_n$$

$$E_n = O(h)$$

For TS(p)

$$x_{n+1} = x_n + \sum_{i=1}^{p} \frac{1}{i!} h^i x_n^{(i)}$$

$$E_n = O(h^p)$$

But, TS(p) methods require high order derivative of the rhs

Impratical for larger ODE and when an analytical expression of $\dot{x}$ is not available

Idea. Achieve accuracy of TS(p) methods by exploiting *previous solutions rather than higher order derivatives*

Trapezoidal rule

Idea. Achieve accuracy of TS(p) methods by exploiting *previous solutions rather than higher order derivatives*

Calculate the Taylor expansion

$$\dot{x}(t+h) = \dot{x}(t) + h\ddot{x}(t) + O(h^2)$$

Substitute $h\ddot{x}(t) = \dot{x}(t+h) - \dot{x}(t) + O(h^2)$ in

$$x(t+h) = x(t) + h\dot{x}(t) + \frac{1}{2!}h^2\ddot{x}(t) + O(h^3)$$

$$\underline{x_{n+1}} = x_n + h\dot{x}_n + \frac{1}{2}h\left(\dot{x}_{n+1} - \dot{x}_n\right)$$

$$\underline{= x_n + \frac{1}{2}h\left(f_{n+1} + f_n\right)}$$

Trapezoidal rule

Note that in $x_{n+1} = x_n + \frac{1}{2}h\left(f_{n+1} + f_n\right)$

We need to calculate $f_{n+1} = f\left(t_{n+1}, x_{n+1}\right)$

That is, x_{n+1} is on both left and right hand sides

We may not have an explicit expression of x_{n+1} as function of the previous samples ***(implicit method)***

Implicit and explicit methods

In general, an algorithm for solving a differential equation (***or ODE solver***) can be expressed as

$$x_{n+1} = G(t, x_{n-k+1}, x_{n-k+2}, \ldots, x_n, x_{n+1})$$

Implicit method

k-steps method

$$x_{n+1} = G(t, x_{n-k+1}, x_{n-k+2}, \ldots, x_n, x_{n+1})$$

This equation needs to be solved to find x_{n+1}

Explicit method

$$x_{n+1} = G(t, x_{n-k+1}, x_{n-k+2}, \ldots, x_n)$$

2-step Adam-Bashforth method (AB(2))

Instead of

$$\dot{x}(t+h) = \dot{x}(t) + h\ddot{x}(t) + O(h^2)$$

calculate the Taylor expansion as

$$\dot{x}(t-h) = \dot{x}(t) - h\ddot{x}(t) + O(h^2)$$

The resulting iteration is

$$\underline{x_{n+1}} = x_n + h\dot{x}_n + \frac{1}{2}h\left(\dot{x}_n - \dot{x}_{n-1}\right)$$

$$\underline{= x_n + \frac{1}{2}h\left(3f_n - f_{n-1}\right)}$$

Example. Comparison of AB(2) and TS(2)

Simulate the system

$$\dot{x}(t) = (1-2t)x(t), \quad x(0) = 1$$

over the time interval $0 \le t \le 0.9$

Recall the ***exact solution*** $x(t) = e^{\frac{1}{4}-\left(\frac{1}{2}-t\right)^2}$

The iteration is

$$x_{n+2} = x_{n+1} + \frac{1}{2}h\left(3f_{n+1} - f_n\right)$$

$$= x_{n+1} + \frac{1}{2}h\left[3\left(1-2t_{n+1}\right)x_{n+1} - \left(1-2t_n\right)x_n\right]$$

Example. Comparison of AB(2) and TS(2)

For $n=0$

$$x_2 = \left[1 + \frac{3}{2}h(1-2h)\right]x_1 - \frac{1}{2}h$$

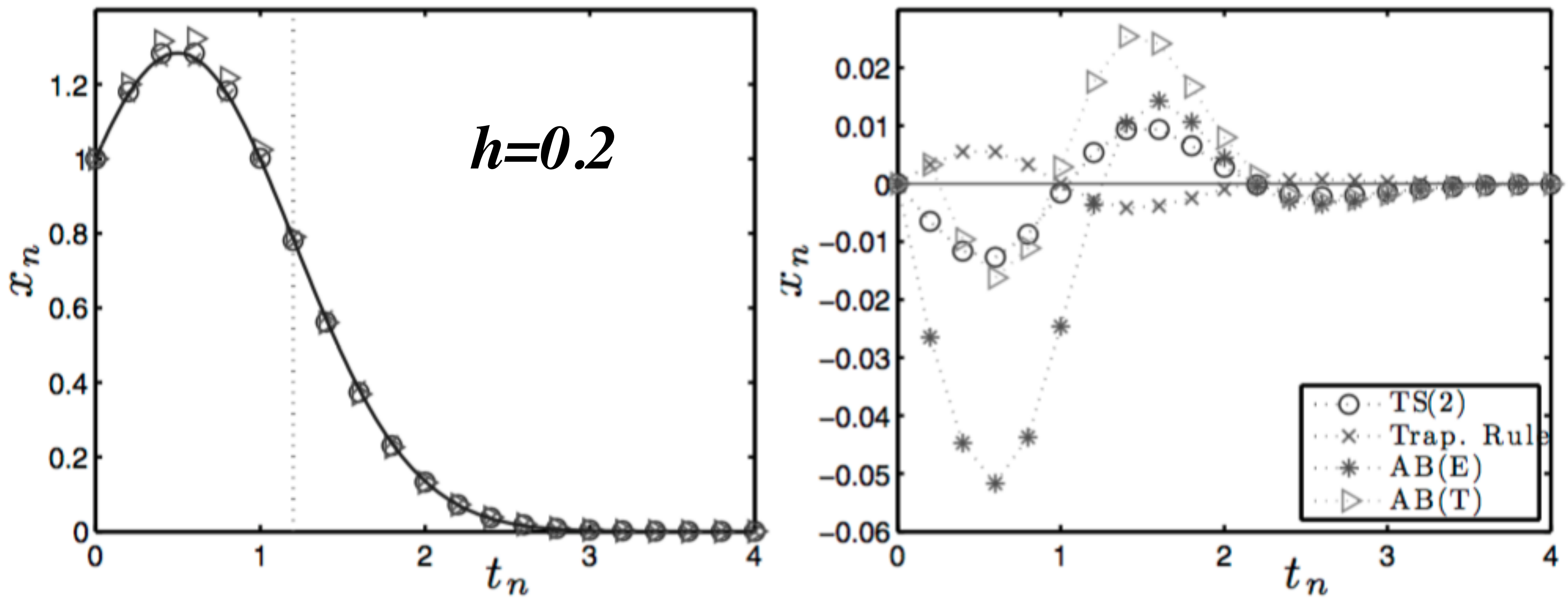
How do we calculate x_1?

1. $x_1 = (1+h)x_0$ Euler ***(ABE)***
2. $x_1 = x_0 + \frac{1}{2}h\left[(1-2t_1)x_1 + (1-2t_0)x_0\right] \Rightarrow$

$$\Rightarrow x_1 = \frac{1 + \frac{1}{2}(1-2t_0)}{1 - \frac{1}{2}(1-2t_1)}x_0$$

Trapezoidal rule ***(ABT)***

Example. Comparison of AB(2) and TS(2)



10^3GE@ t=1.2

h	TS(2)	Trap.	ABE	ABT
0.2	5.4	−2.8	−3.6	17.6
0.1	1.4	−0.71	−0.66	4.0
Ratio	3.90	4.00	5.49	4.40

$$E_n \propto h^2$$

Two-step methods. Consistency

k	p	Method	Name
1	1	$x_{n+1} - x_n = hf_n$	Euler
1	1	$x_{n+1} - x_n = hf_{n+1}$	Backward Euler
1	2	$x_{n+1} - x_n = \frac{1}{2}h(f_{n+1} + f_n)$	trapezoidal
2	2	$x_{n+2} - x_{n+1} = \frac{1}{2}h(3f_{n+1} - f_n)$	two-step Adams–Bashforth
2	2	$x_{n+2} - x_{n+1} = \frac{1}{12}h(5f_{n+2} + 8f_{n+1} - f_n)$	two-step Adams–Moulton
2	4	$x_{n+2} - x_n = \frac{1}{3}h(f_{n+2} + 4f_{n+1} + f_n)$	Simpson's rule
2	3	$x_{n+2} + 4x_{n+1} - 5x_n = h(4f_{n+1} + 2f_n)$	Dahlquist (see Example 4.11)

More in general

$$x_{n+2} + \alpha_1 x_{n+1} + \alpha_0 x_n = h\left(\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n\right)$$

that is ***consistent*** and of ***order p*** if

$$x(t+2h) + \alpha_1 x(t+h) + \alpha_0 x(t) - h\left[\beta_2 \dot{x}(t+2h) + \beta_1 \dot{x}(t+h) + \beta_0 \dot{x}(t)\right] = O(h^{p+1})$$

Two-step methods. Consistency

Definition (Consistency). A two-step LMM method is ***consistent*** if a *p>0* exists such that

$$x(t+2h)+\alpha_1 x(t+h)+\alpha_0 x(t)-h\left[\beta_2 \dot{x}(t+2h)+\beta_1 \dot{x}(t+h)+\beta_0 \dot{x}(t)\right]=O(h^{p+1})$$

Otherwise, it is ***inconsistent***.

Example. The Euler method

The *n*-th iteration $x_{n+1}=x_n+hf_n$ stems from

$$x(t+h)=x(t)+h\dot{x}(t)+O(h^2)\Rightarrow$$

$$\Rightarrow x(t+h)-x(t)-h\dot{x}(t)=O(h^2)$$

Hence, the method is consistent of order *p=1*

Example. Consistency

Determine the consistency order of the integration method (last in the table)

$$x_{n+2} + 4x_{n+1} - 5x_n = h\left(4f_{n+1} + 2f_n\right)$$

We have to find the p in

$$x(t+2h) + 4x(t+h) - 5x(t) - h\left[4\dot{x}(t+h) + 2\dot{x}(t)\right] = O(h^p)$$

Let's expand in Taylor series the terms on the lhs

$$x(t+2h) = x(t) + 2h\dot{x} + 2h^2\ddot{x}(t) + \frac{4}{3}h^3\dddot{x}(t) + \frac{2}{3}h^4x^{(4)}(t) + O(h^5)$$

$$x(t+h) = x(t) + h\dot{x} + \frac{1}{2}h^2\ddot{x}(t) + \frac{1}{6}h^3\dddot{x}(t) + \frac{1}{24}h^4x^{(4)}(t) + O(h^5)$$

Example. Consistency

$$\dot{x}(t+h) = \dot{x}(t) + h\ddot{x}(t) + \frac{1}{2}h^2\dddot{x}(t) + \frac{1}{6}h^3 x^{(4)}(t) + O(h^4)$$

By summing up the and collecting the terms

$$x(t+2h) + 4x(t+h) - 5x(t) - h\left[4\dot{x}(t+h) + 2\dot{x}(t)\right] =$$

$$\left(1+4-5\right)x(t) + h\left[2+4-\left(4+2\right)\right]\dot{x}(t)$$

$$+h^2\left[2+2-4\right]\ddot{x}(t) + h^3\left[\frac{4}{3} + 4\times\frac{1}{6} - 4\times\frac{1}{2}\right]\dddot{x}(t)$$

$$+h^4\left[\frac{2}{3} + 4\times\frac{1}{24} - 4\times\frac{1}{6}\right]x^{(4)}(t)$$

Example. Consistency

After simplification

$$x(t+2h)+4x(t+h)-5x(t)-h\left[4\dot{x}(t+h)+2\dot{x}(t)\right]=$$
$$\frac{1}{6}h^4x^{(4)}(t)+O(h^5)$$

$\frac{1}{6}h^4x^{(4)}(t)$ is dominating and $O(h^5)$ can be neglected

$\frac{1}{6}h^4x^{(4)}(t)=O(h^4)$ and the method is of order 3

Two-step methods. Consistency conditions

Theorem (consistency conditions for two-step LMMs).
Given the two-step LMM

$$x_{n+2} + \alpha_1 x_{n+1} + \alpha_0 x_n = h\left(\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n\right)$$

define the two polynomials

$$\rho(r) = r^2 + \alpha_1 r + \alpha_0$$
$$\sigma(r) = \beta_2 r^2 + \beta_1 r + \beta_0$$

The method is consistent if and only if

$$\rho(1) = 0,\ \rho'(1) = \sigma(1)$$

Two-step methods. Necessary condition for convergence

Theorem (necessary condition for convergence). A convergent LMM is consistent.

A consistent LMM is not guaranteed to be convergent

Example

Solve the IVP $\dot{x}(t) = -x(t),\ x(0) = 1$ with the LMM

$$x_{n+2} + 4x_{n+1} - 5x_n = h\left(4f_{n+1} + 2f_n\right)$$

with *h=0.1, 0.01, 0.001*

The LMM can be rewritten as

$$x_{n+2} + 4x_{n+1} - 5x_n = -h\left(4x_{n+1} + 2x_n\right)$$

and rearranged as

$$x_{n+2} = -4(1+h)x_{n+1} + \left(5 - 2h\right)x_n$$

Example

$h = 0.1$		$h = 0.01$		$h = 0.001$	
$x_7 =$	0.544	$x_{13} =$	0.938	$x_{19} =$	1.070
$x_8 =$	0.199	$x_{14} =$	0.567	$x_{20} =$	0.535
$x_9 =$	1.735	$x_{15} =$	2.384	$x_{21} =$	3.205
$x_{10} =$	-6.677	$x_{16} =$	-6.810	$x_{22} =$	-10.159
$x_{11} =$	37.706	$x_{17} =$	39.382	$x_{23} =$	56.697
$x_{12} =$	-197.958	$x_{18} =$	-193.017	$x_{24} =$	-277.788

No matter how small is h, the solution oscillates

k-step methods

The two-step LMM

$$x_{n+2} + \alpha_1 x_{n+1} + \alpha_0 x_n = h\left(\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n\right)$$

can be generalized to *k*-step LMM

$$x_{n+k} + \alpha_{k-1} x_{n+k-1} + \cdots + \alpha_0 x_n = h\left(\beta_k f_{n+k} + \beta_{k-1} f_{n+k-1} + \cdots + \beta_0 f_n\right)$$

Given the starting values $x_0, x_1, \ldots, x_{k-1}$

Any method is required to *converge* as $h \to 0$

That is, the approximated solution should be equal to the actual as $h \to 0$

Convergence is required also for $x_1, \ldots, x_{k-1}$, since they are calculated with a numerical method as well

k-step methods. Consistency

In order to have convergence, we must have consistency

Define the polynomials

$$\rho(r) = \sum_{i=0}^{k} \alpha_i r^i,\ \alpha_k = 1 \text{ and } \sigma(r) = \sum_{i=0}^{k} \beta_i r^i$$

The consistency conditions become

$$\sum_{i=0}^{k} \alpha_i = 0,\ \alpha_k = 1, \text{ and } \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k} \beta_i$$

Example

Solve the IVP $\dot{x}(t) = 0,\ x(0) = 1$ with the LMM

$$x_{n+2} + 4x_{n+1} - 5x_n = h\left(4f_{n+1} + 2f_n\right)$$

Use $x_1 = 1 + h$

The LMM becomes $x_{n+2} + 4x_{n+1} - 5x_n = 0$

Note that the polynomial $\rho(r) = r^2 + 4r - 5$ has two roots $r=1$ and $r=5$

The solution of the equation is $x_n = A + B(-5)^n$ where A, B are determined from the initial condition and x_1

x_n diverges while it should be constant and equal to 1

Convergence and Zero-stability

Definition (Zero-stability). A LMM is zero stable if all the roots of

$$\rho(r) = \sum_{i=0}^{k} \alpha_i r^i, \ \alpha_k = 1$$

satisfy the condition $|r| \leq 1$ ***(root condition)***

and the ones satisfying $|r| = 1$ have multiplicity 1

Theorem (necessary and sufficient condition for convergence). A LMM is convergent if and only if is consistent and zero-stable