ESS101- Modeling and Simulation
Lecture 8-9

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Today (Chapter 8)

- ☞ System identification
- ☞ Nonparametric identification
  - ☞ Correlation analysis
    - ☞ Matlab example
  - ☞ Transient analysis
  - ☞ Frequency analysis
  - ☞ Fourier analysis
    - ☞ Matlab example
  - ☞ Spectral analysis

System identification

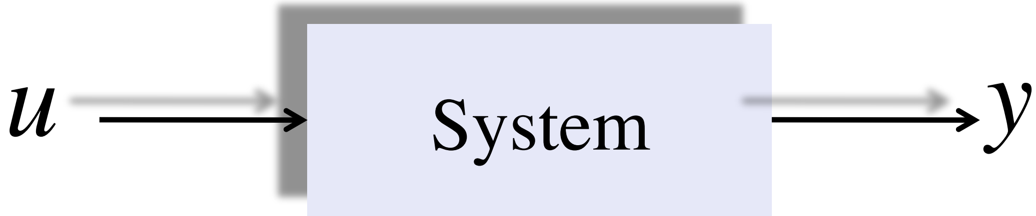
Limitations in physical modeling:

✏ Constitutive relationships may be unknown

✏ Physical parameters may be either unknown or highly uncertain

✏ The problem is too complex

Basic idea in SysId



Collect measurements of u and y and find a model for the System fitting the collected data

System identification

1. The system model is known from physical modeling, but some parameters are unknown or uncertain ("transparent" or "gray box" modeling)

2. Limited or no knowledge about the system ("black box" modeling)
   - ✔ Linear nonparametric (***today***)
   - ✔ Linear parametric (***next***)
   - ✘ Nonlinear (System Identification Course. Prof. Jonas Sjöberg (Signals and Systems))

System identification

- ☞ Parametric identification
  - ✔ *A priori* defined model structure

- ☞ Nonparametric identification
  - ✔ Linear system is assumed
  - ✔ A choice of model structure is not needed

Parametric system identification

Problem statement. Given a system in the form

$$\dot{x} = f\left(x,u,\theta\right)$$

$$y = g\left(x,u,\theta\right)$$

find the vector θ based on measurements of $\underline{u}$ and y.

Linear parametric identification. The function f is linear. Possible forms

$$\begin{aligned}\dot{x} &= A(\theta)x + B(\theta)u \\ y &= C(\theta)x + D(\theta)u\end{aligned} \qquad \textbf{\textit{or}} \qquad Y(s) = G(s,\theta)U(s)$$

Recall that….

Recall that $$y(t) = \underbrace{Ce^{At}x(0)}_{\text{initial condition response}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}_{\text{input response}}$$

Consider a *pulse*

$$\delta(t) = \lim_{\varepsilon \to 0} p_\varepsilon(t), \quad p_\varepsilon(t) = \begin{cases} 0, & t < 0 \\ 1/\varepsilon, & 0 \le t < \varepsilon \\ 0, & t \ge \varepsilon \end{cases}$$

The response to a pulse from a zero initial condition is

$$h(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau = Ce^{A(t)}B$$

Hence

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{initial condition response}} + \underbrace{\overbrace{\int_0^t h(t-\tau)u(\tau)d\tau}^{\text{convolution of the impulse response and the input signal}} + Du(t)}_{\text{input response}}$$

Linear nonparametric time-domain models

The response in the time domain of a ***linear*** system is

$$y(t) = \int_0^t h(\tau)u(t-\tau)\,d\tau$$

where h is called ***impulse response***.

If u is an impulse

$$u(t) = \delta(t) \Rightarrow \begin{cases} u(t) = 1, \; t = 0 \\ u(t) = 0, \; t \neq 0 \end{cases}$$

$$y(t) = h(t)$$

Calculating the impulse response from output samples

Correlation analysis

The response in the time domain of a ***linear*** system is

$$y(t) = \int_0^t h(\tau)u(t-\tau)\,d\tau$$

In ***discrete time***

$$y(k) = \sum_{i=1}^{k} g_i u(k-i) + v(k)$$

where v is measurement noise.

Assume u is white noise, i.e., $u(t) \in N(0,\lambda)$. Hence

$$R_u(\tau) = \begin{cases} \lambda, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}, \qquad R_{uv} = 0$$

Correlation analysis

Calculate the cross covariance

$$\underline{R_{yu}(\tau)} = E\left[y(k)u(k-\tau)\right]$$

$$= E\left[\left(\sum_{i=1}^{k} g_i u(k-i) + v(k)\right)u(k-\tau)\right]$$

$$= \sum_{i=1}^{k} g_i \underbrace{E\left[u(k-i)u(k-\tau)\right]}_{\substack{=0, \quad i\neq\tau \\ =\lambda, \quad i=\tau}} + \underbrace{E\left[v(k)u(k-\tau)\right]}_{=0}$$

$$\underline{= \lambda g_\tau}$$

Idea. Estimating the cross covariance from data and solve for g_τ

Correlation analysis. Estimation of the cross covariance

Estimate the cross covariance from a data set of N samples of input and output

$$\hat{R}_{yu}^{N}(\tau) = \frac{1}{N}\sum_{i=1}^{N} y(i)u(i-\tau)$$

Calculate an ***approximation over N samples*** of the impulse response

$$\hat{g}_{\tau}^{N} = \frac{1}{\lambda}\hat{R}_{yu}^{N}(\tau)$$

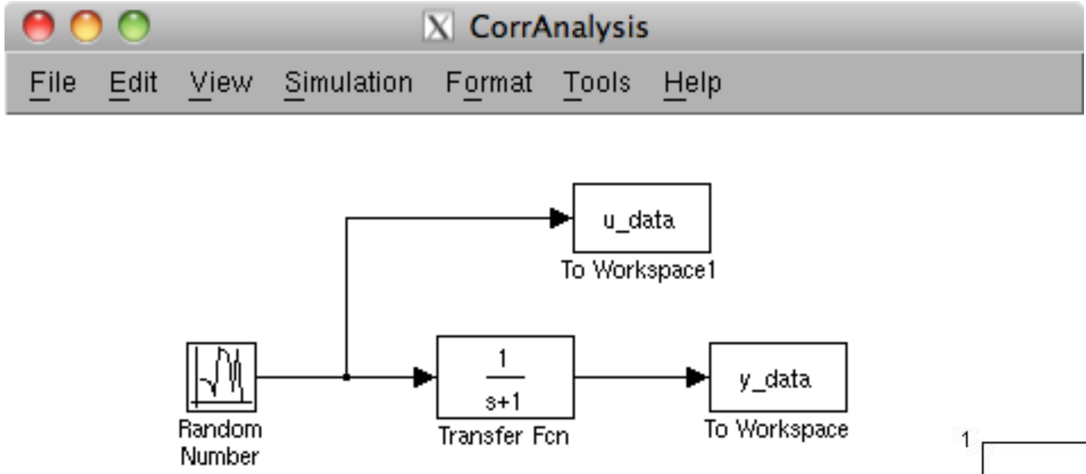
Q: What if *u* is NOT white noise?

Correlation analysis. Summary

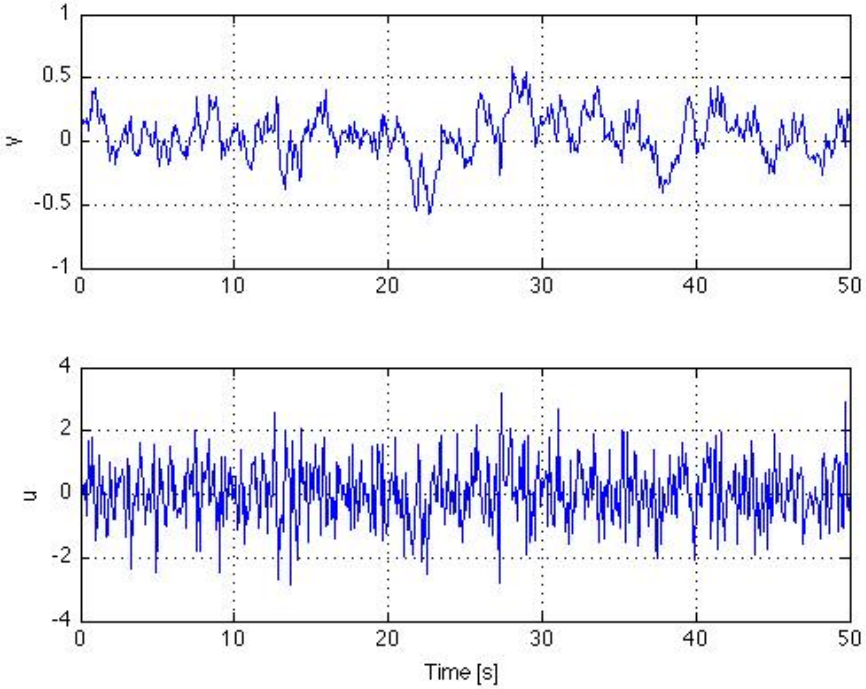
✓ Simple
✓ Immediate result
✓ Disturbances compensated by long measurement series

✗ The form of the results
✗ Results depend on initial conditions (might not be 0)

Correlation analysis. Example



The input is a white noise with variance 1



Correlation analysis. Example

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```
% Example of estimation of the impulse response via correlation analysis
%
% Input and output data are generated by feeding the system
%          1
% G(s)=-----, with white noise with variance 1
%         s+1
%

% Simulating the model to obtain data
sim('CorrAnalysis')

% Plotting the input and output data
figure,subplot(2,1,1),plot(tout,y_data),ylabel('y'), grid
subplot(2,1,2),plot(tout,u_data),ylabel('u'), xlabel('Time [s]'),grid

% Packing the collected data into the structure used by the command
% 'impulse'
data=iddata(y_data,u_data,1e-1);

% Plotting the impulse response from the input output data
figure, impulse(data,5);

%                                                         1
% Plotting the impulse response og the system G(s)=-----
%                                                        s+1
hold on, impulse(tf(1,[1 1]),'r',5), grid
```

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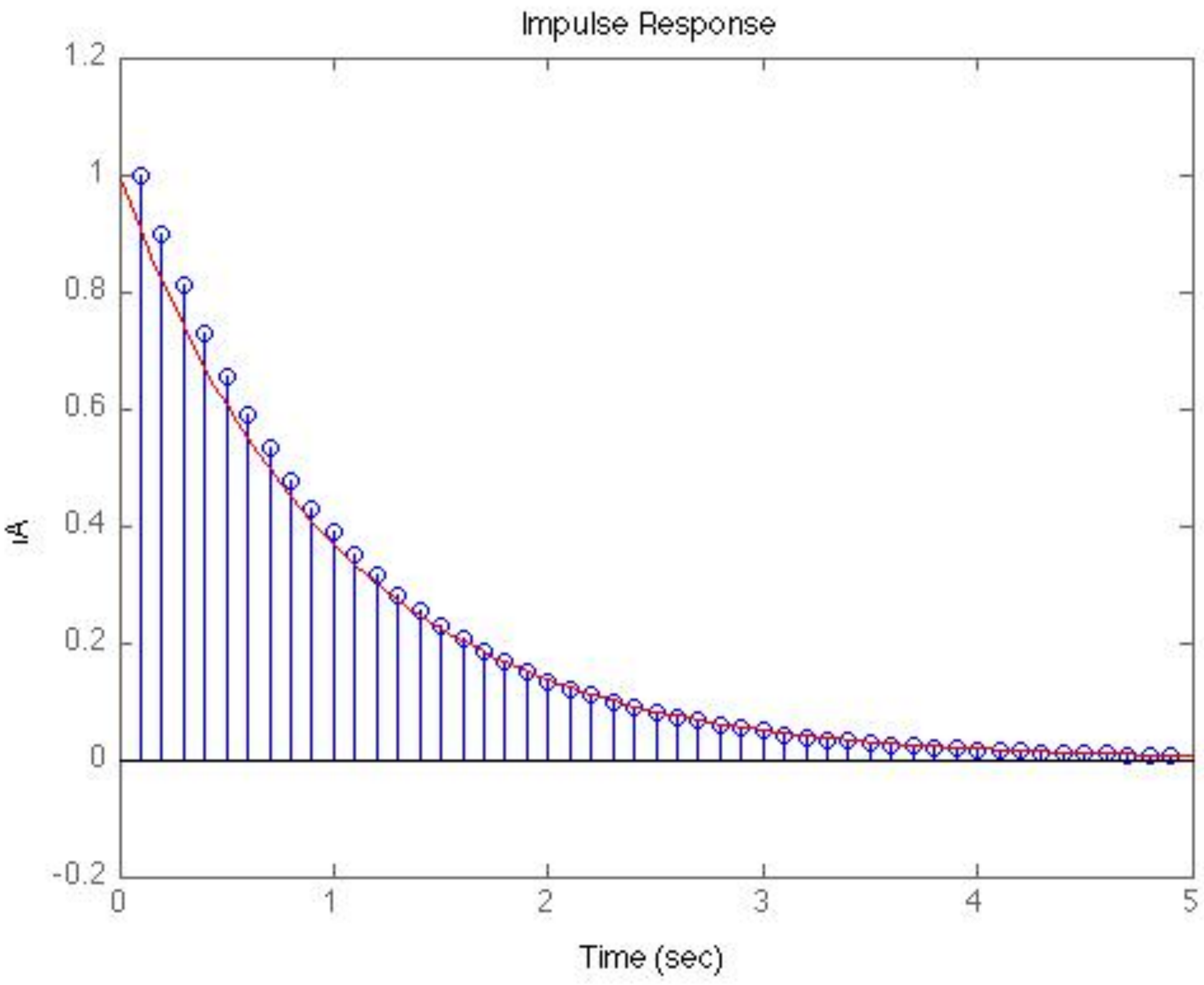
Correlation analysis. Example



Transient analysis

Idea. Give a *step input* and analyze the step response to determine

☛ the model order (e.g., 1^{st} or 2^{nd} order)

☛ time constants

☛ DC gain

In summary

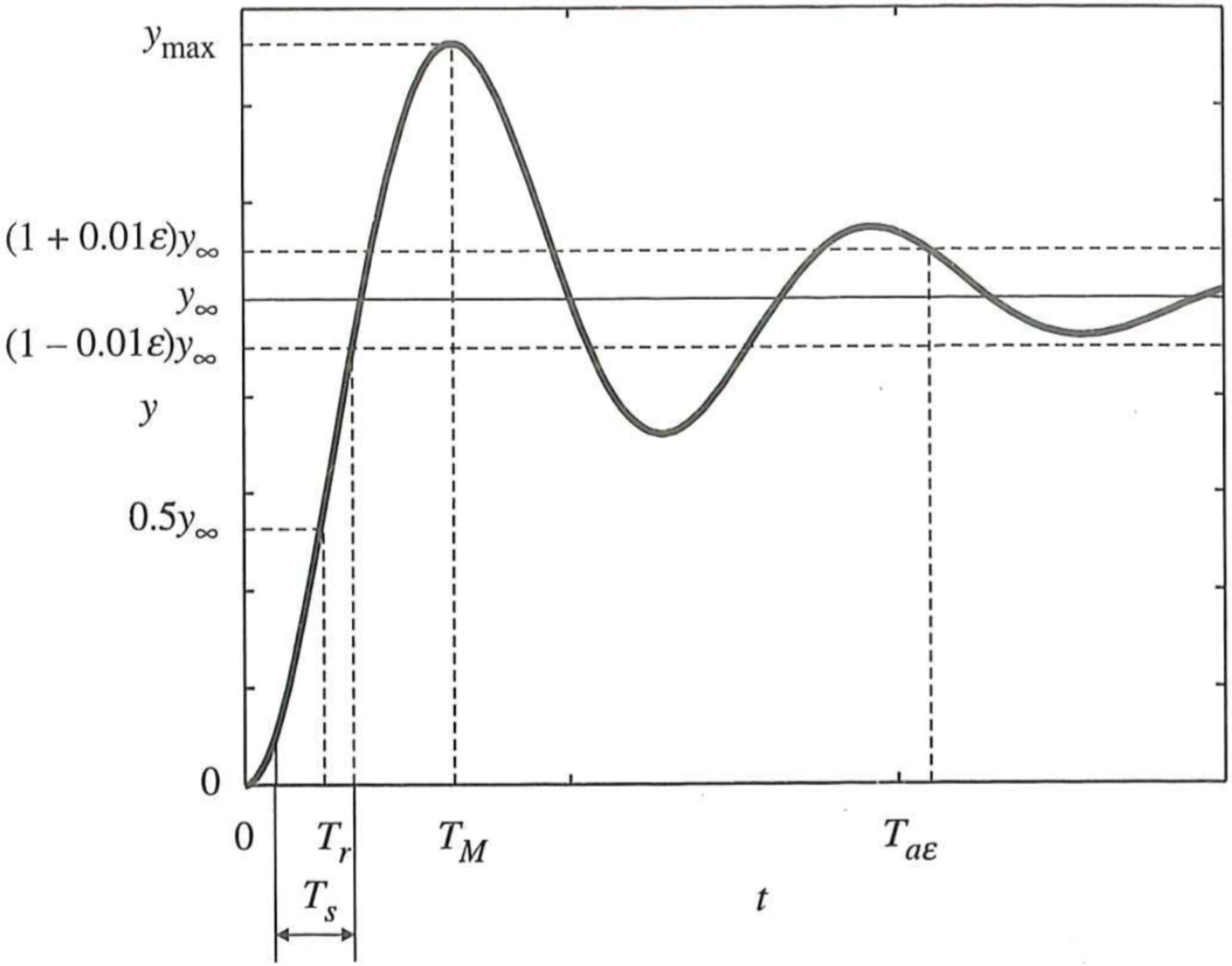
✔ fast

✔ results ready to use (DC gain, time constants)

✘ results depend on initial conditions (might not be 0)

✘ sensitive to disturbances

Step responses of 1st and 2nd order systems

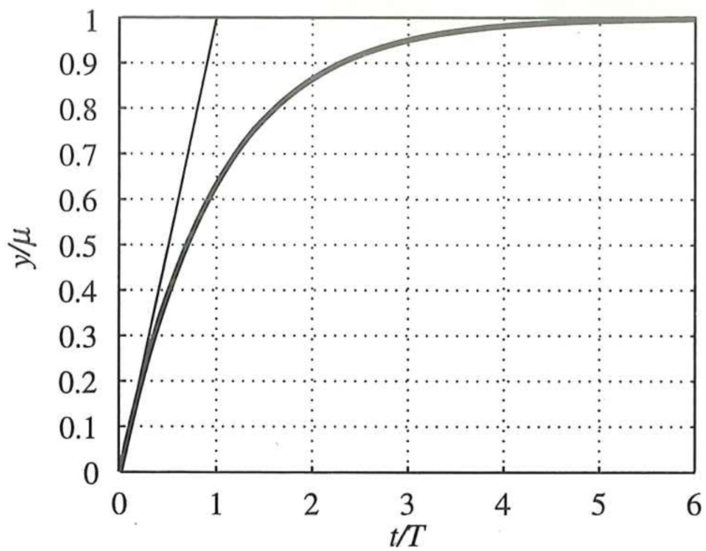


Step responses of 1st and 2nd order systems

Consider the system $G(s) = \dfrac{\mu}{1+sT}$

By the initial value theorem and the Laplace of time derivatives

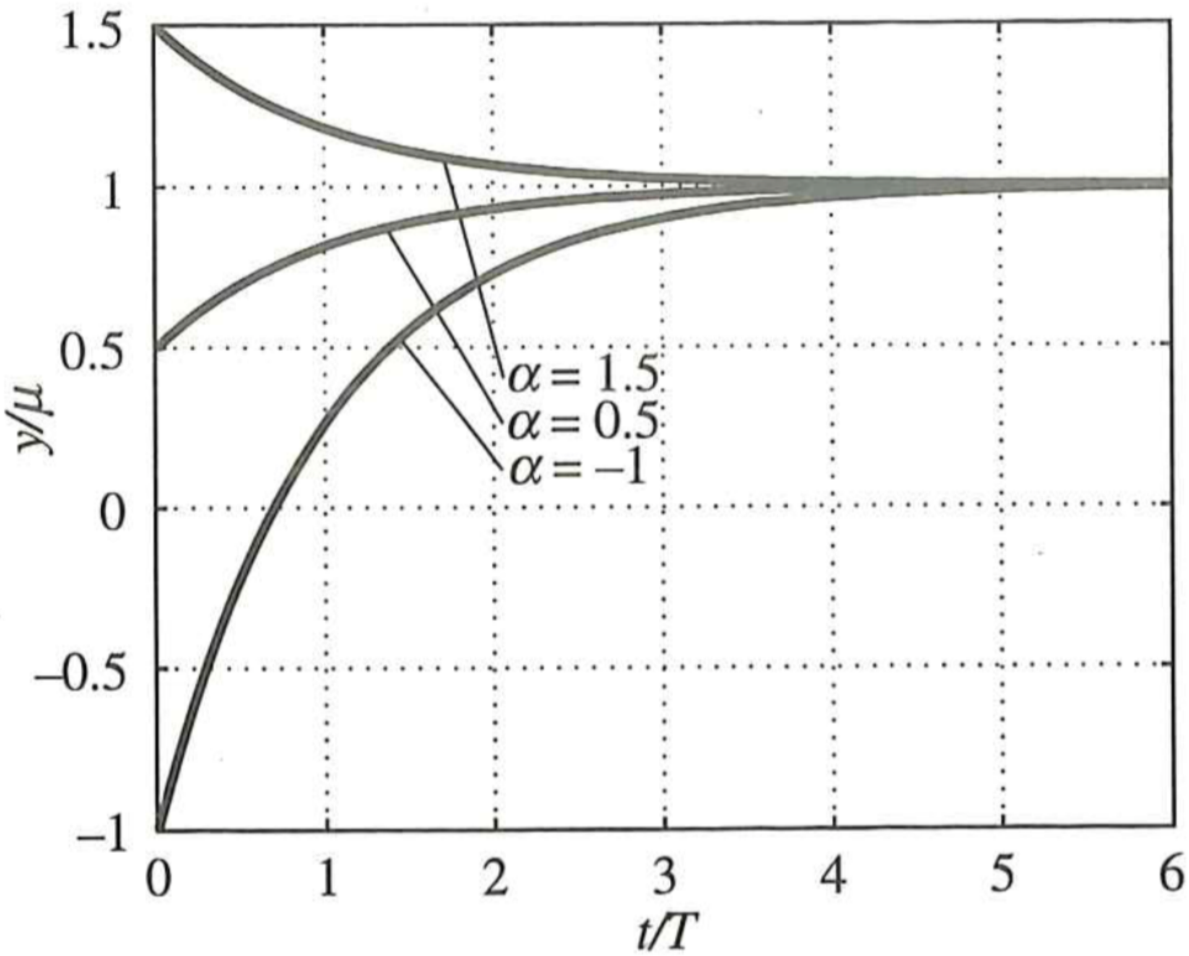
$$\frac{dy}{dt}(0) = \lim_{s\to\infty} \overbrace{s\, \underbrace{s \overbrace{\frac{\mu}{1+sT}\underbrace{\frac{1}{s}}_{\text{Laplace transform of the step}}}^{\text{Step response}}}_{\text{Derivative in time domain}}}^{\text{Initial value theorem}} = \frac{\mu}{T}$$



y_∞	T_s	T_r	T_{a5}	T_{a1}
μ	$\cong 2.2T$	$\cong 0.7T$	$\cong 3T$	$\cong 4.6T$

Step responses of 1st and 2nd order systems

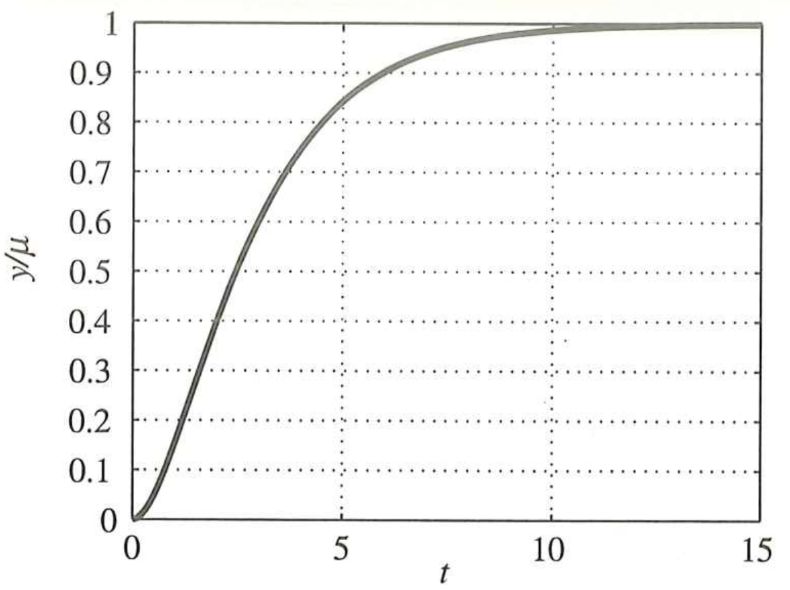
Consider the system $G(s) = \mu \dfrac{1 + s\tau}{1 + sT}$



Step responses of 1st and 2nd order systems

Consider the system $G(s) = \dfrac{\mu}{(1+sT)^2}$

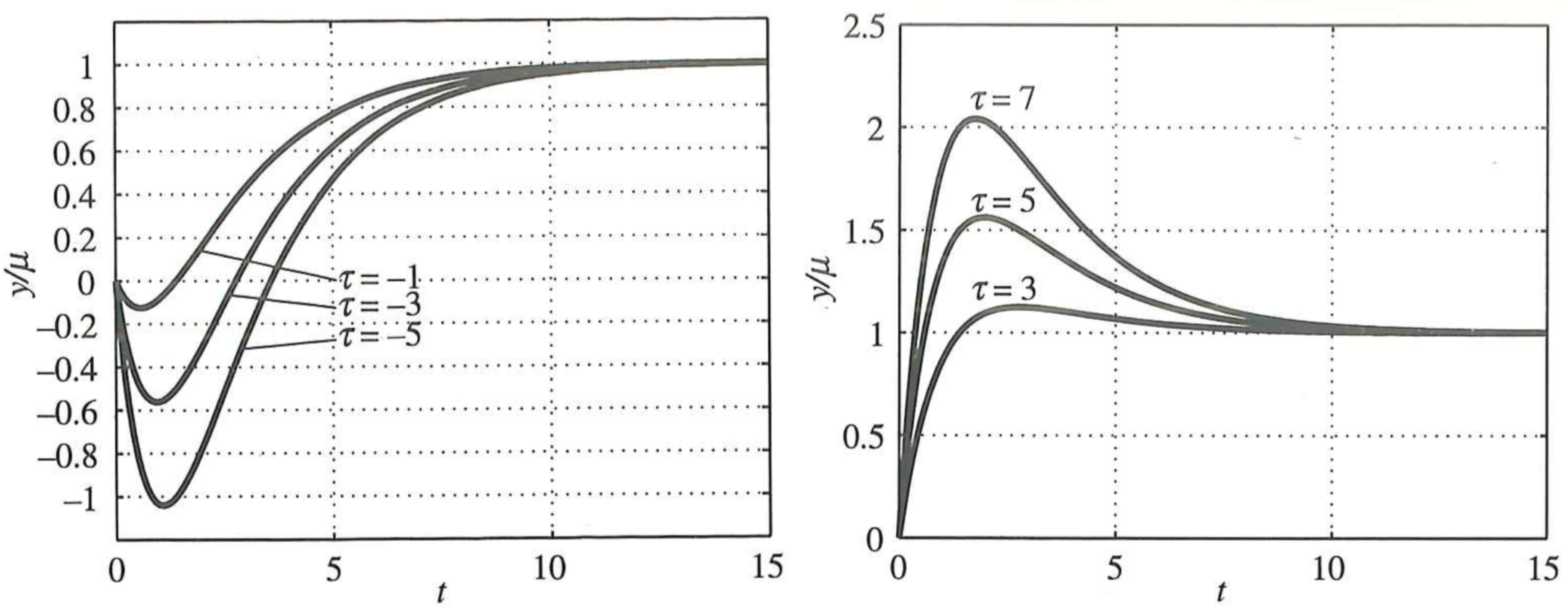
The step response is flat in 0 $\dfrac{dy}{dt}(0) = \lim_{s\to\infty} s^2 \dfrac{\mu}{(1+sT)^2}\dfrac{1}{s} = 0$



y_∞	T_s	T_r	T_{a5}	T_{a1}
μ	$\approxeq 3.36T$	$\approxeq 1.68T$	$\approxeq 4.74T$	$\approxeq 6.64T$

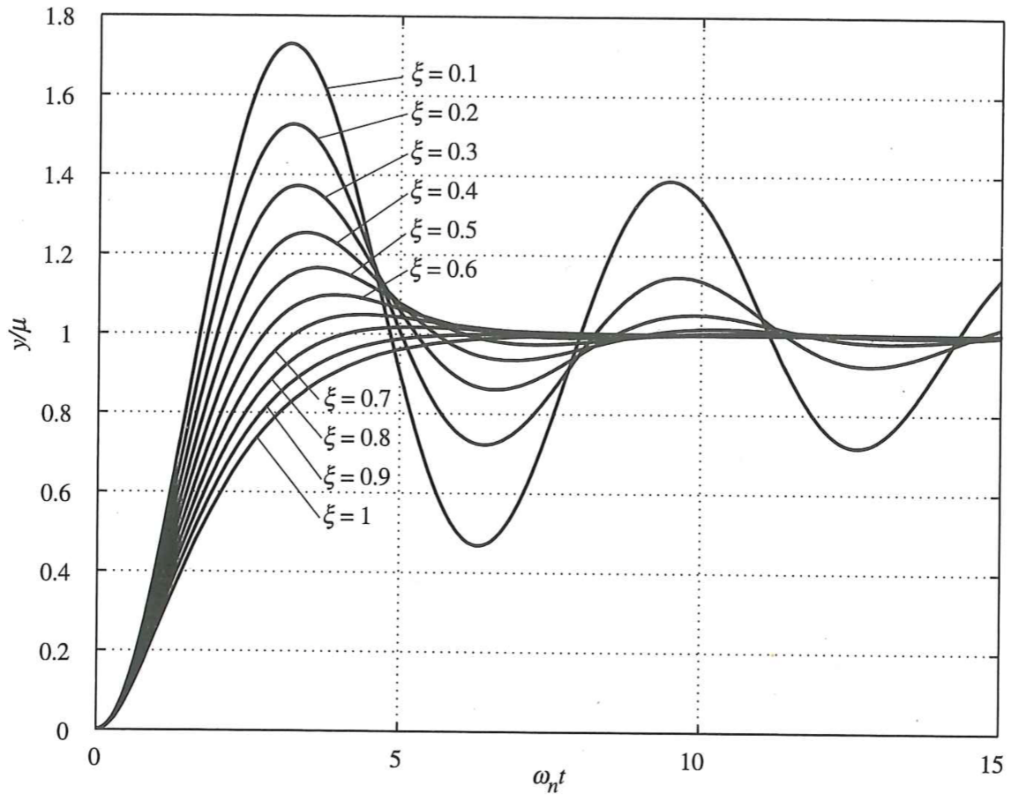
Step responses of 1st and 2nd order systems

Consider the system $G(s) = \mu \dfrac{1+s\tau}{(1+sT_1)(1+sT_2)}$



Step responses of 1st and 2nd order systems

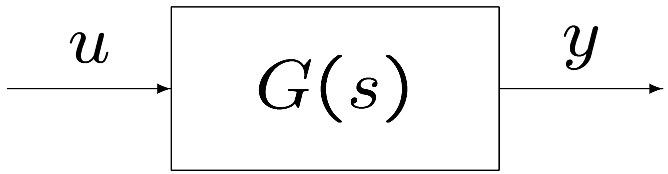
Consider the system $G(s) = \dfrac{\mu\omega^2}{s^2 + 2\omega\zeta s + \omega^2}$



y_∞	$S\%$	T_M	$T_{a\varepsilon}$
μ	$100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$	$\dfrac{\pi}{\omega\sqrt{1-\zeta^2}}$	$-\dfrac{1}{\zeta\omega}\ln 0.01\varepsilon$

Frequency analysis

Basic idea for frequency analysis:



$$u = u_o sin(\omega_o t)$$

$$\Rightarrow$$

$$y = |G(j\omega_o)| u_o sin(\omega_o t + arg(G(j\omega_o)))$$

Frequency analysis

✓ Simple
✓ Bode diagram
✓ No model structure assumed (other than linearity)

✗ Magnitude and phase vs. frequency table
✗ Simulation in time domain not straightforward

Fourier analysis

Recall that

$$Y(\omega) = G(i\omega)U(\omega)$$

with $Y(\omega)$, $G(i\omega)$, $U(\omega)$ Fourier transforms of the output, the impulse response and the input

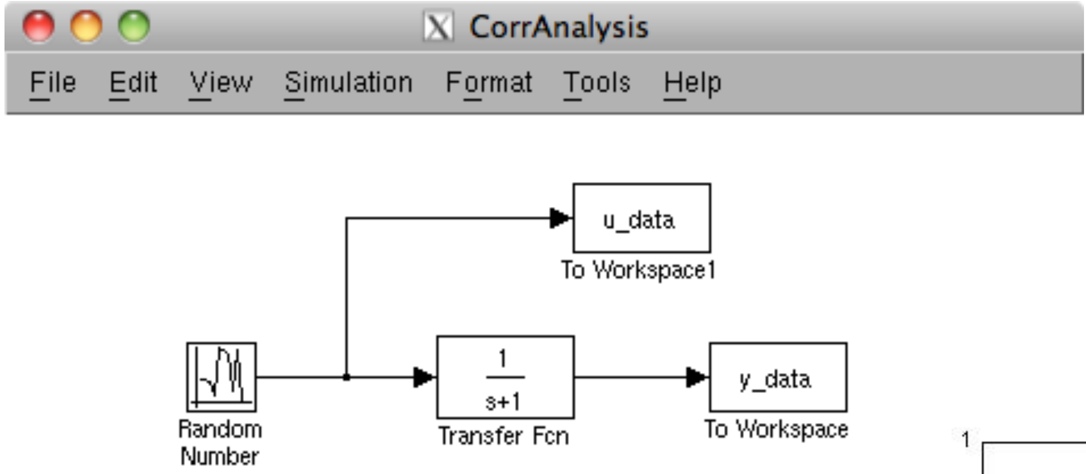
For sampled signals

$$Y_S(\omega) = T\sum_{k=1}^{N} y(kT)e^{-i\omega kT}, \quad U_S(\omega) = T\sum_{k=1}^{N} u(kT)e^{-i\omega kT}$$

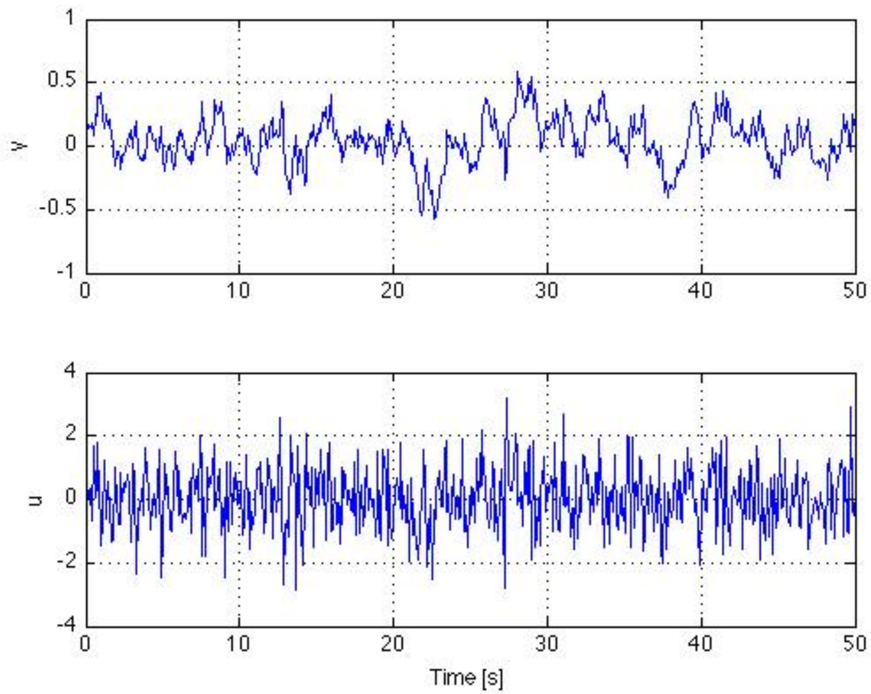
Estimate $G(i\omega)$ as

$$\hat{G}_S(i\omega) = \frac{Y_S(\omega)}{U_S(\omega)}$$

Fourier analysis. Example



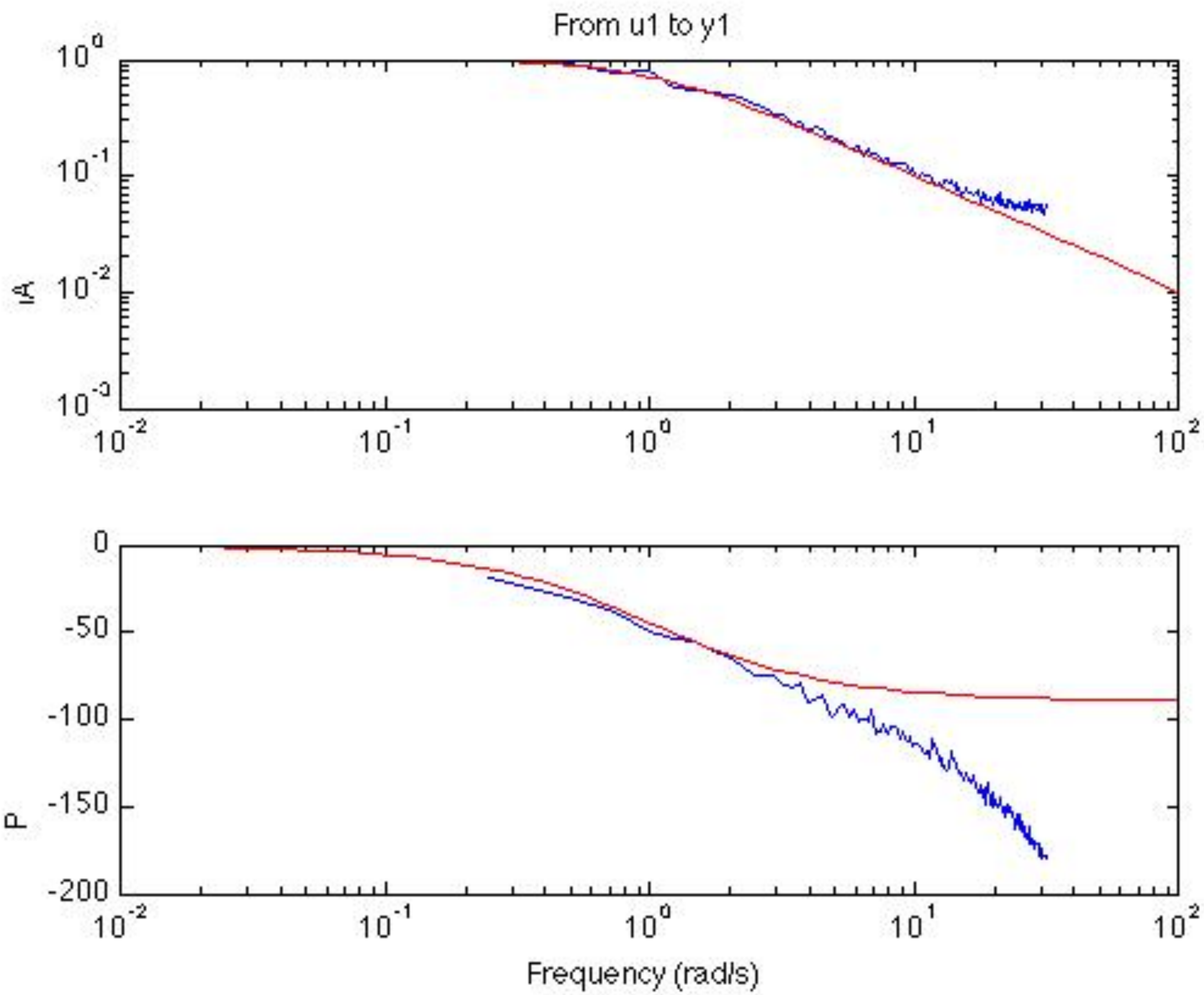
The input is a white noise with variance 1



Fourier analysis. Example

```
% Example of estimation of the impulse response via correlation analysis
%
% Input and output data are generated by feeding the system
%         1
% G(s)=-----, with white noise with variance 1
%        s+1
%

% Simulating the model to obtain data
sim('CorrAnalysis')

% Plotting the input and output data
figure,subplot(2,1,1),plot(tout,y_data),ylabel('y'), grid
subplot(2,1,2),plot(tout,u_data),ylabel('u'), xlabel('Time [s]'),grid

% Packing the collected data into the structure used by the command
% 'impulse'
data=iddata(y_data,u_data,1e-1);

% Estimating the frequency response from data
g=etfe(data);

%                                                   1
% Plotting the frequency response of the system G(s)=-----
%                                                  s+1
figure, bode(g,tf(1,[1 1]),'r'), grid
```

Fourier analysis. Example



Spectral analysis

Consider the (continuous time) signal *y(t)* obtained as output of a linear system

$$y(t) = G(p)u(t) + v(t)$$

where *u(t)* and *v(t)* are uncorrelated

The spectra $\Phi_y^T(\omega)$ and $\Phi_{yu}^T(\omega)$ are given by

$$\Phi_y^T(\omega) = |G(i\omega)|^2 \Phi_u^T(\omega) + \Phi_v^T(\omega)$$

$$\Phi_{yu}^T(\omega) = G(i\omega)\Phi_u^T(\omega) \quad \Rightarrow \quad G(i\omega) = \frac{\Phi_{yu}^T(\omega)}{\Phi_u^T(\omega)}$$

Spectral analysis

Q: How do we calculate the spectrum based on a data set of <u>N</u> samples?

Recall that the spectrum for a sampled stochastic signal *w(k)* is defined as

$$\Phi_w^T(\omega) = T\sum_{k=-\infty}^{k=\infty} R_w(kT)e^{-i\omega kT}$$

while the *cross spectrum* of two signals *u* and *y* is defined as

$$\Phi_{yu}^T(\omega) = T\sum_{k=-\infty}^{k=\infty} R_{yu}(kT)e^{-i\omega kT}$$

Estimation of signals spectra. Periodogram

The spectrum has been defined as ***square of the magnitude of the Fourier transform of the signal***

$$\Phi_w^T(\omega) = \left|W^T(\omega)\right|^2 \qquad W^T(\omega) = T\sum_{k=-\infty}^{\infty} w(k)e^{-i\omega kT}$$

Given N samples of w, approximate the spectrum as

$$\hat{\Phi}_{w,N}^T(\omega) = T\left|W_N^T(\omega)\right|^2,$$

$$W_N^T(\omega) = \sum_{k=1}^{N} w(k)e^{-i\omega kT}$$

Periodogram

Properties of the periodogram

Assume w is a stochastic process. Hence the ***periodogram is a stochastic process*** too.

$$E\left[\hat{\Phi}^T_{w,N}(\omega)\right] = \Phi_w(\omega) + R_N^{(1)}$$

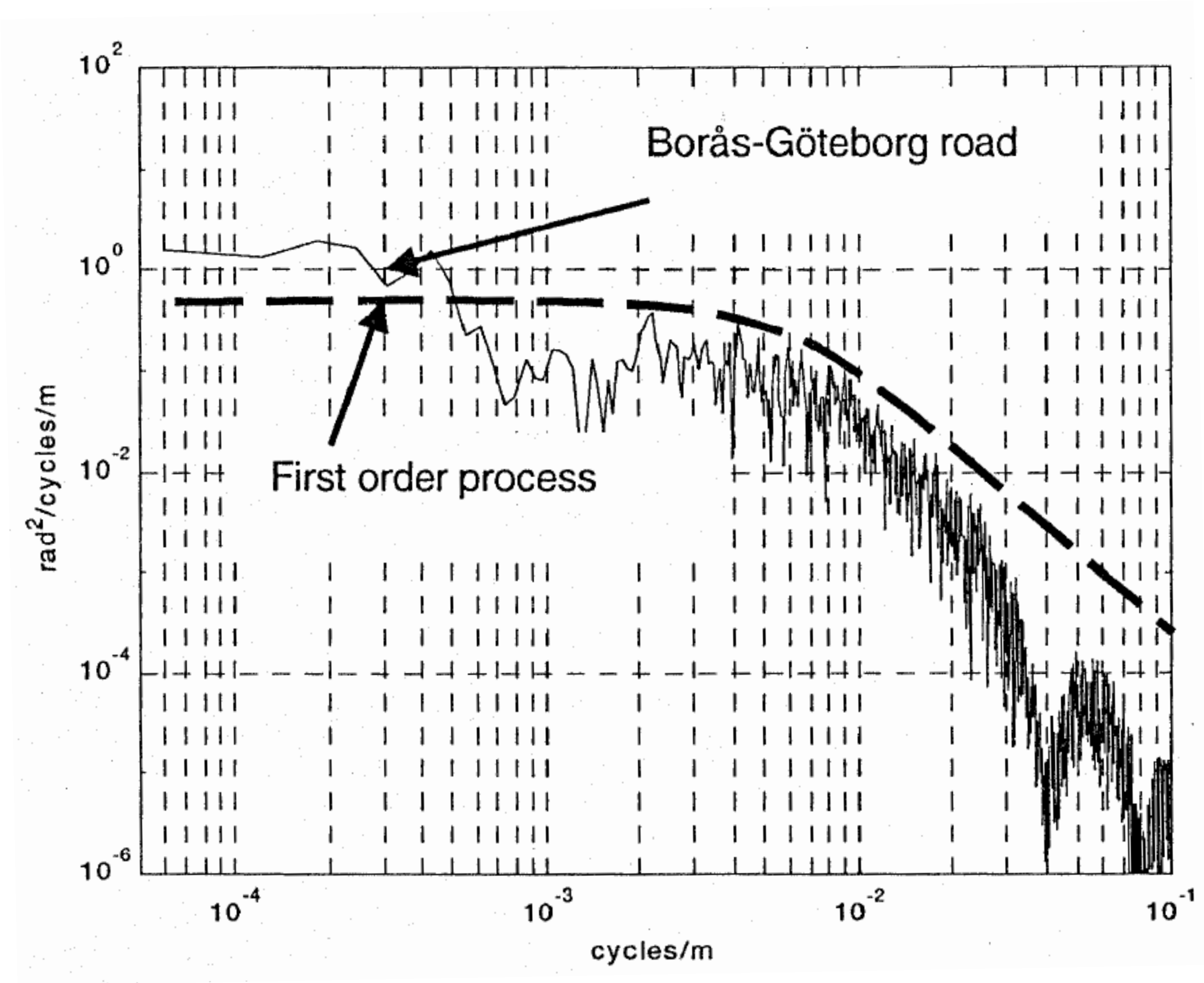
$$E\left[\left(\hat{\Phi}^T_{w,N}(\omega) - \Phi_w(\omega)\right)^2\right] = \Phi_w^2(\omega) + R_N^{(2)}$$

$$E\left[\left(\hat{\Phi}^T_{w,N}(\omega_1) - \Phi_w(\omega_1)\right)\left(\hat{\Phi}^T_{w,N}(\omega_2) - \Phi_w(\omega_2)\right)\right] = R_N^{(3)}$$

$$R_N^{(i)} \rightarrow 0,\ N \rightarrow \infty,\ i = 1,2,3,..$$

The periodogram is fluctuating.

Road slope spectra



Averaging the periodogram

Welch's method

IDEA:

1. Split the signal into R parts of length M
2. For each part i compute the periodogram $\Phi_M^{(i)}(\omega)$
3. Estimate the spectrum as the average of the periodograms

$$\hat{\Phi}_N(\omega) = \frac{1}{R}\sum_{k=1}^{R}\Phi_M^{(k)}(\omega)$$

Smoothing the periodogram

Blackman-Tuckey's (BT) method

$$E\left(\hat{\Phi}_N(\omega_1) - \Phi_u(\omega_1)\right)\left(\hat{\Phi}_N(\omega_2) - \Phi_u(\omega_2)\right) \to 0, \text{ as } N \to \infty$$

The periodgrams are uncorrelated for neighboring frequencies

IDEA: Averaging the periodograms of a number of neighboring frequencies

$$\hat{\Phi}_N(\omega) = \int_{-\pi}^{\pi} W_\gamma(\omega - \xi)\hat{\hat{\Phi}}_N(\xi)d\xi \qquad \int_{-\pi}^{\pi} W_\gamma(\omega)d\omega = 1$$

Smoothing the periodogram

Blackman-Tuckey's (BT) method

Rectangular window

$$W_\gamma(\xi) = \begin{cases} \gamma, \text{if } |\xi| < \dfrac{1}{2\gamma} \\ 0, \text{otherwise} \end{cases}$$

In the time domain:

$$\hat{\Phi}_N(\omega) = \sum_{k=-\lambda}^{\lambda} w_\lambda(k) \hat{R}_u^N(k) e^{-i\omega k}$$ with

$w_\lambda(k)$ Inverse Fourier transform of $W_\gamma(\xi)$

$$\hat{R}_u^N(k) = \frac{1}{N} \sum_{t=1}^{N} u(t+k)u(t)$$

Spectral analysis. Summary

fast
✔ used and well known method.
✔ general, no assumption on the model structure

✘ input an measurement noise must noise must be uncorrelated
✘ results is a frequency response. Not suitable for simulation purposes