ESS101- Modeling and Simulation
Lecture 20

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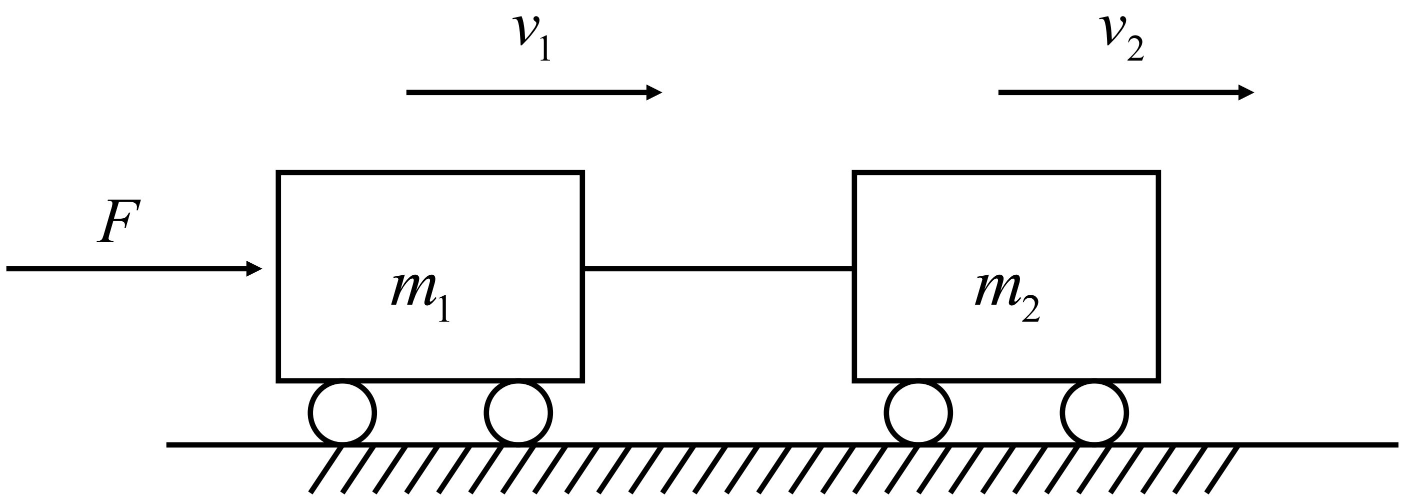
Today (**Handouts**)

- ☞ Differential Algebraic Equations (DAE)
  - ☞ Examples
  - ☞ DAE
  - ☞ Linear DAE
  - ☞ Kronecker normal form
  - ☞ Index
  - ☞ Index 1 and 2 DAE
  - ☞ Reducing the differentiation index

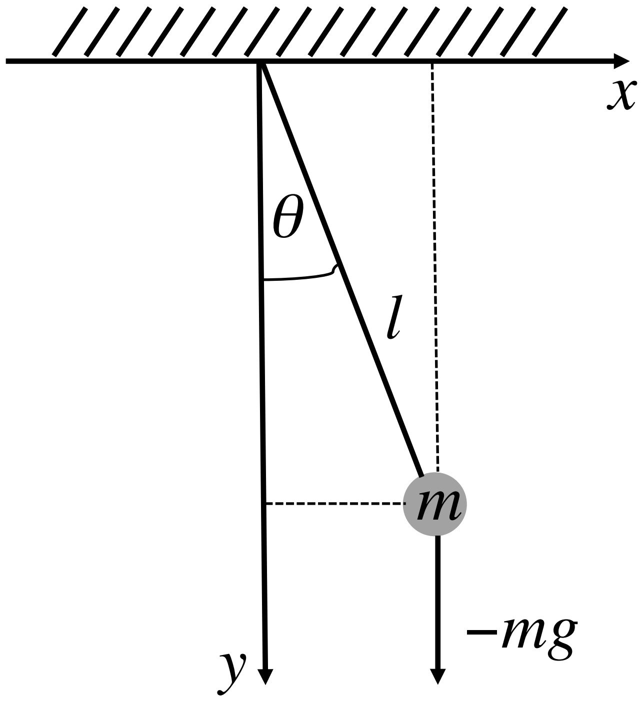
Mechanical example #1



$$\left.\begin{aligned} m_1 \frac{dv_1}{dt} &= F_{m_1} \\ m_2 \frac{dv_2}{dt} &= F_{m_2} \end{aligned}\right\} \qquad \dot{x} = f(x,u) \quad \text{Differential equations}$$

$$v_1 = v_2 \Big\} \qquad h(x) = 0 \quad \text{Algebraic constraints}$$

Mechanical example #2



Apply the Euler-Lagrange's method

$$T(\dot{x},\dot{y}) = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) \quad \text{(Kinetic energy)}$$

$$U(x,y) = mg(l - y) \quad \text{(Potential energy)}$$

$$x^2 + y^2 - l^2 = h(x,y) = 0 \quad \text{(Constraints)}$$

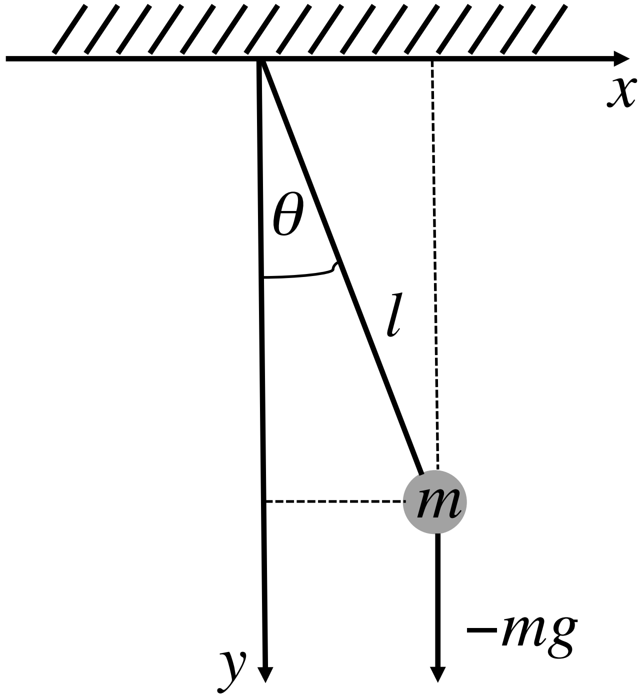
$$L(q,\dot{q}) = T(\dot{q}) - U(q) - \lambda h(q), \quad q = \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix}$$

Lagrangian

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1,2,3$$

Euler equation

Mechanical example #2



Euler's equation

$$m\ddot{x} + 2\lambda x = 0$$

$$m\ddot{y} - mg + 2\lambda y = 0$$

$$h(x,y) = 0$$

Introduce $z = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}$

$$\left.\begin{aligned} \dot{z}_1 &= z_3 \\ \dot{z}_2 &= z_4 \\ \dot{z}_3 &= -\frac{2}{m}\lambda z_1 \\ \dot{z}_4 &= -\frac{2}{m}\lambda z_2 + g \end{aligned}\right\} \text{Differential equations}$$

$$0 = z_1^2 + z_2^2 - l^2$$

Algebraic constraints

DAEs

- Conservation laws
- Constitutive equations
- Design and physical constraints

can lead to a set of Differential and Algebraic Equations (DAEs).

$$F\left(\dot{z}, z, t\right) = 0$$

DAEs

DAE can be found in different forms

$$F(\dot{y}, y, t) = 0 \qquad \textit{Fully implicit}$$

$$A\dot{y} + f(y,t) = 0 \qquad \textit{Linear implicit}$$

$$\begin{aligned} \dot{x} &= f(x,z,t) \\ 0 &= g(x,z,t) \end{aligned} \qquad \textit{Semi-explicit}$$

x *differential variables* $\qquad y^T = \begin{bmatrix} x^T & z^T \end{bmatrix}$

z *algebraic variables*

Solving DAEs

- *How do we solve DAEs?*
- *Can we extends ODE solvers to solve DAEs as well?*

Differentiation index

Definition. The DAE

$$F(\dot{y}, y, t) = 0$$

has differentiation index m, if m is the minimal number of differentiations of F that is necessary in order to solve for the derivative $\dot{y}$

Differentiation index. DAE with index 1

A DAE in the form

$$\dot{x} = f(x, y)$$
$$0 = g(x, y)$$

with g not singular in a neighborhood of the solution has index 1.

Proof.

By differentiating the second equation wrt the time

$$0 = g_x f + g_y \dot{y} \Rightarrow \dot{y} = -g_y^{-1} g_x f$$

Differentiation index. DAE with index 2

A DAE in the form

$$\dot{x} = f(x, y)$$
$$0 = g(x)$$

with $(g_{xy} f + g_x f_y)$ not singular in a neighborhood of the solution has index 2

Proof.

By differentiating twice the second equation wrt the time

$$0 = g_x f$$

$$0 = (g_{xx} f + g_x f_x) f + (g_{xy} f + g_x f_y)\dot{y} \Rightarrow \dot{y} = -(g_{xy} f + g_x f_y)^{-1} (g_{xx} f + g_x f_x) f$$

Initial conditions

Consider the DAE with index 2

$$\dot{x} = f(x,y)$$

$$\dot{y} = -\left(g_{xy} f + g_x f_y\right)^{-1} \left(g_{xx} f + g_x f_x\right) f$$

The initial condition have to satisfy the set of equations

$$0 = g_x f$$

$$0 = g(x)$$

What's the meaning of the index?

- ☣ (Numerical) Differentiation is required.
- ☣ The higher the index, the more differentiations are required
- ☣ The higher the index, the more initial conditions on the input have to be set

The index is a measure of the complexity of the DAE.

In summary

1. DAE with differentiation index equal to ***0*** is a ODE. ***A standard ODE solver can be used***

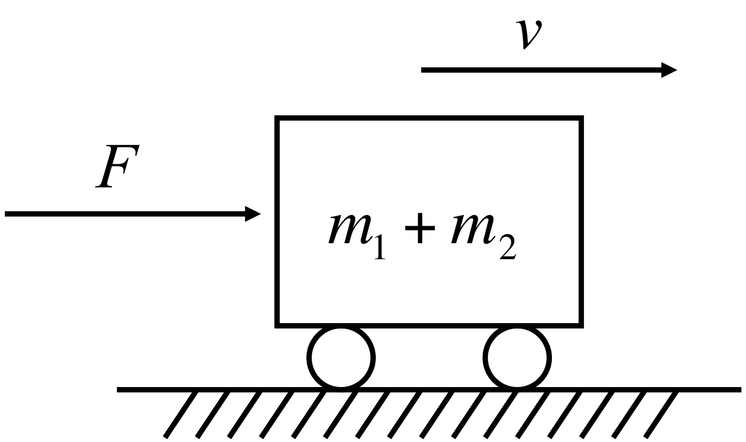
2. DAE with differentiation index equal to ***1*** can be transformed in ODE. ***A standard ODE solver can be used***

3. DAE with differentiation index equal to **2** can be transformed in ODE. ***Special ODE solvers are necessary***

4. DAEs with higher differentiation index lack numerical solver achieving high accuracy

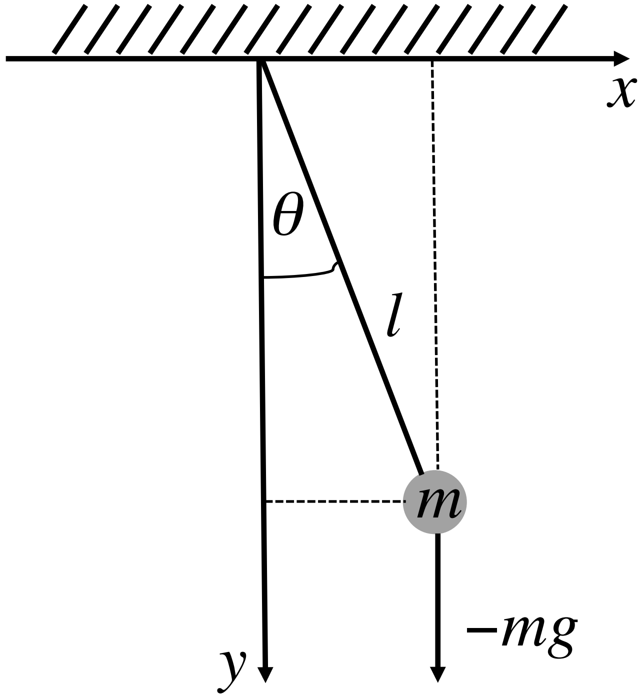
Alternatives? Example #1

In some case algebraic constraints can be avoided



$$(m_1 + m_2)\frac{dv}{dt} = F$$

Alternatives? Example #2



Introduce $z = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = -\frac{g \sin z_1}{l^2}$$

Alternatives?

In more complicated cases, where how to remove the algebraic constraints is not straightforward,

1. Calculate the differentiation index

2. If the differentiation index is higher than 2, ***problem!!***

3. Otherwise, use available ODE solvers

Is any general approach to calculate the index of a DAE?

Linear DAEs

Consider the DAE

$$F(\dot{y}, y, t) = 0$$

linearize around 0 (vector)

$$-E\frac{dy}{dt} = Ay + u(t) \qquad E = \frac{dF}{d\dot{y}} \qquad A = \frac{dF}{dy}$$

if E is *not* singular

$$\dot{y} = -E^{-1}Ay + -E^{-1}u(t)$$

Standard ODE

Linear DAEs

Consider the matrix

$$\lambda E - A$$

with λ scalar, called *matrix pencil*.

Definition. The matrix pencil λE-A is *singular* if det(λE-A)=0, for all λ. It is *regular* otherwise

Petzold (1982) defined the difficulty of solving a DAEs system in terms of nilpotency (degree of singularity) of matrix pencil

Linear DAEs. Kronecker theorem

Theorem. Be (E, A) a regular matrix pencil. Two matrices U and V exist such that

$$UEV = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} \qquad UAV = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}$$

where N=diag(N_1,N_2,…,N_k) with N_i a Jordan block to the eigenvalue 0, i.e.,

$$N_i = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

Kronecker normal form

Based on the previous results a change of variables exists such that the DAE can be rewritten as:

$$\begin{bmatrix} I & O \\ O & N \end{bmatrix} \begin{bmatrix} \dot{t}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} C & O \\ O & I \end{bmatrix} \begin{bmatrix} t(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} a(t) \\ b(t) \end{bmatrix} \quad \text{with} \quad y = V \begin{bmatrix} t \\ v \end{bmatrix},\ Uu = \begin{bmatrix} a \\ b \end{bmatrix}$$

Then:

$$\dot{t}(t) = Ct(t) + a(t)$$

$$N\dot{v}(t) = v(t) + b(t)$$

Kronecker normal form

Consider the second equation

$$N\dot{v}(t) = v(t) + b(t)$$

Assume $N^2=0$

$$N^2\ddot{v}(t) = N\dot{v}(t) + N\dot{b}(t)$$

$$\Downarrow$$

$$0 = v(t) + b(t) + N\dot{b}(t)$$

In general

$$v(t) = \sum_{i=0}^{\mu-1}(-N)^i b^{(i+1)}(t) \qquad t(t) = e^{Ct}t(0) + \int_0^t e^{C(t-s)}a(s)ds$$

where μ is the nilpotency index defined as

$$\mu \in N^+ : N^{\mu-1} \neq O \text{ and } N^{\mu} = O$$

Result

A linear DAE forming a regular matrix pencil have differentiation index μ if the nilpotency index is μ