ESS101- Modeling and Simulation
Lectures 2-5

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Today (Murray and Åstrom)

☞ State space models

☞ Linearity and linearization

☞ Differential equations

☞ Matrix exponential

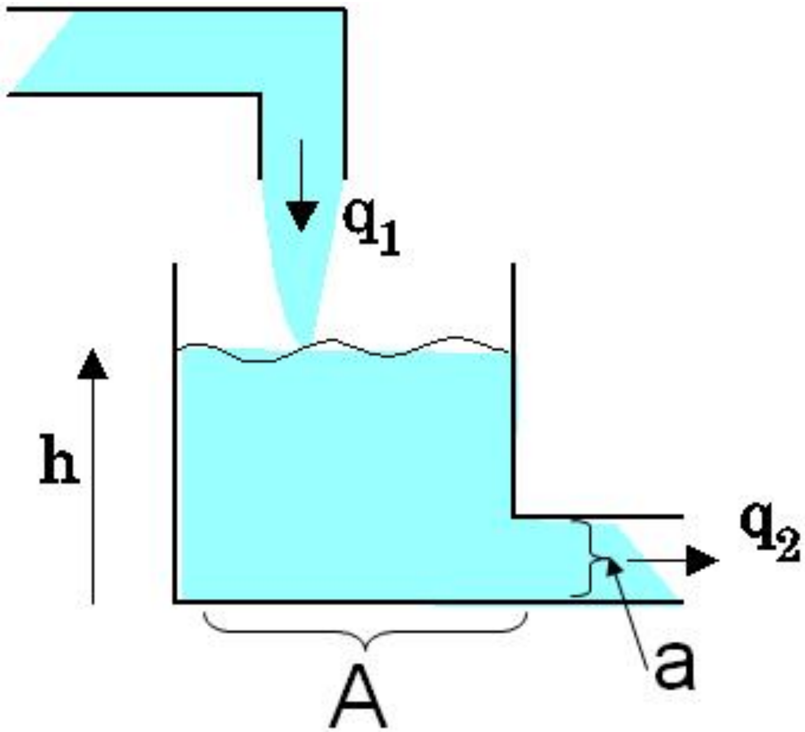
☞ Initial condition response

☞ Linear algebra

☞ Input-output response

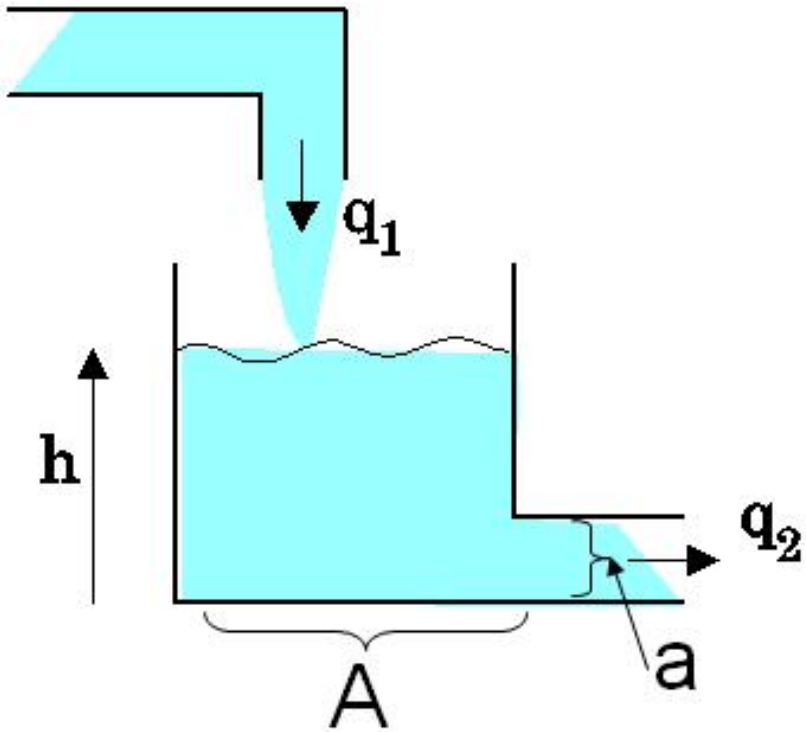
☞ Stability

Example. A tank model



Objective. Model the outlet flow

Example. A tank model



Physics

$$v = a\sqrt{2hg}$$ Torricelli's law

$$q_2 = av$$ Outlet flow

$$\frac{d}{dt}Ah = q_1 - q_2$$ Mass balance

Example. A tank outlet flow control

The resulting model:

$$\dot{h} = -\frac{a}{A}\sqrt{2hg} + \frac{q_1}{A}$$
$$q_2 = a\sqrt{2hg}$$

is in the form

$$\dot{x} = f(x,u)$$
$$y = g(x)$$

where:

$x = h$ state

$u = q_1$ input

$y = q_2$ output

State space models

A ***dynamical*** model in a state-space form is

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

where

- $x(t) = (x_1(t), \ldots, x_n(t))$ is the state vector
- $u(t)$ is the input (scalar or vector)
- $y(t)$ is the output (scalar or vector)

Note. For physical systems the state variables account for storage of mass, momentum and energy.

State space models. A bit of terminology

Consider the state-space model

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

where

- $f : R^n \times R^m \to R^n$, $h : R^n \times R^m \to R^p$ are called ***state update*** and ***output functions*** or ***mappings***. They are usually assumed ***smooth*** and in general are ***nonlinear***.
- n is the ***order*** of the system
- the system is said ***time-invariant*** since f and h do ***not*** depend on the time

Linearity and linearization

Consider the system

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$
$$x(0) = x_0$$

Denote by $y(t, x_0, u)$ the output response to u starting from x_0

The system is linear if

$$y(t, ax_{0,1} + bx_{0,2}, au_1 + bu_2) = ay(t, x_{0,1}, u_1) + by(t, x_{0,2}, u_2)$$

Principle of superposition

A special case. Linear systems

In case f and h are linear in the state and the input vectors

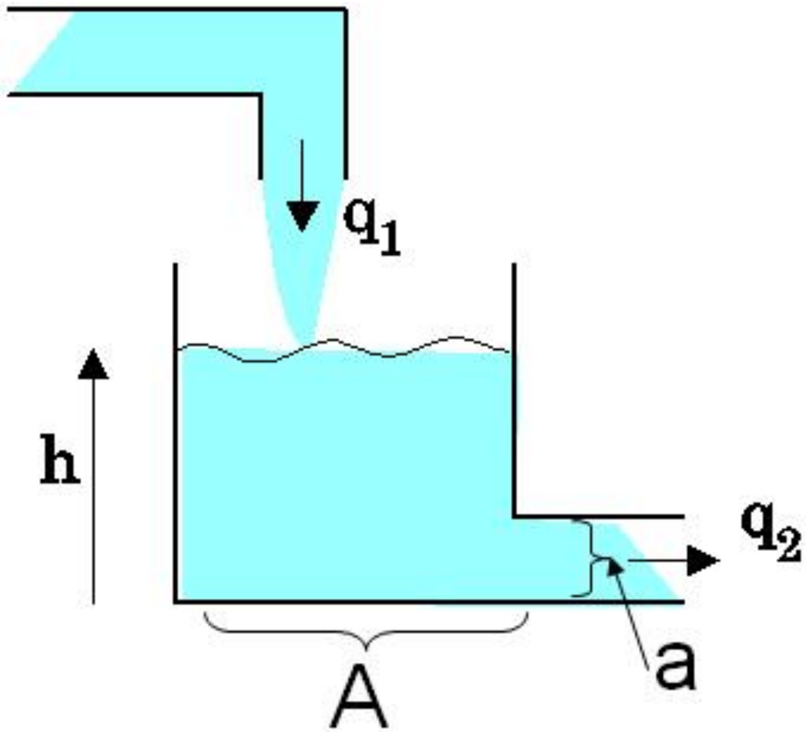
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where

- A, B, C are the ***dynamics***, ***control*** and ***sensor*** or ***output matrices***, respectively, D is called ***direct term***,
- the system is said ***time-invariant*** if A, B, C, D are constant matrices,

it can be shown that the system is linear (the superposition principle holds)

Example. The tank model



$$\dot{h} = -\frac{a}{A}\sqrt{2hg} + \frac{q_1}{A}$$

$$q_2 = a\sqrt{2hg}$$

Nonlinear

Solution of differential equations

Consider the differential equation

$$\dot{x} = f(x,u)$$

Solving the differential equation is the problem of finding a function $\hat{x}(t)$ such that

$$\dot{\hat{x}}(t) = f(\hat{x}(t),u(t)),\ \forall t \geq 0$$

and for a given function $u(t)$

In an ***initial value problem*** $x(0)$ is assigned

Note. Terms like *evolution*, *behavior* always refer to the solution of the ODE

Solution of differential equations

Solving a *nonlinear* differential equation *in general* can be difficult. It may *not* have a solution for *all t* or a unique solution.

Example. Consider the system

$$\dot{x} = x^2, \;\; x(0) = 1$$

The function $x(t) = \dfrac{1}{1-t}$ is a solution of the equation but is

defined for $0 \le t < 1$

Existence and ***uniqueness*** can be guaranteed by requiring that the rhs is Lipschitz

$$\dot{x} = f(x) \qquad\qquad \|f(x) - f(y)\| < c\|x - y\|, \;\; \forall x, y$$

Linearity and linearization

Consider the system

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

Assume $(x_e = 0,\ u_e = 0)$ is an ***equilibrium point***

$$0 = f(x(t), u(t))$$

Observe that if $(x_e \neq 0, u_e \neq 0)$ a change of variables can be done as follows

$$\tilde{x} = x - x_e,\ \ \tilde{u} = u - u_e,\ \ \tilde{y} = y - y_e,$$

Linearity and linearization

By expanding the rhs in Taylor series

$$\dot{x}(t) \approx \overbrace{f\left(x_e, u_e\right)}^{=0} + \overbrace{\left.\frac{\partial f\left(x,u\right)}{\partial x}\right|_{\substack{x=x_e,\\ u=u_e}}}^{A} \overbrace{\left(x - x_e\right)}^{\tilde{x}} + \overbrace{\left.\frac{\partial f\left(x,u\right)}{\partial u}\right|_{\substack{x=x_e,\\ u=u_e}}}^{B} \overbrace{\left(u - u_e\right)}^{\tilde{u}}$$

$$y(t) \approx \underbrace{h\left(x_e, u_e\right)}_{y_e} + \underbrace{\left.\frac{\partial h\left(x,u\right)}{\partial x}\right|_{\substack{x=x_e,\\ u=u_e}}}_{C} \left(x - x_e\right) + \underbrace{\left.\frac{\partial h\left(x,u\right)}{\partial u}\right|_{\substack{x=x_e,\\ u=u_e}}}_{D} \left(u - u_e\right)$$

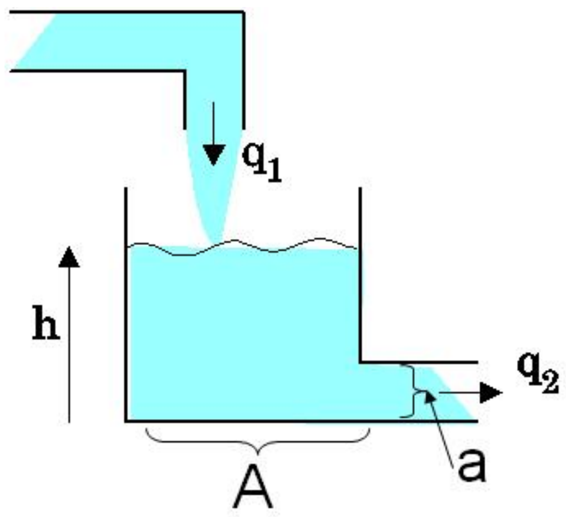
The corresponding linear system is

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$$

$$\tilde{y} = C\tilde{x} + D\tilde{u}$$

Valid only for small deviations from x_e, u_e

Example. The tank model



1. Find an equilibrium point

$$0 = \dot{h} = -\frac{a}{A}\sqrt{2h_e g} + \frac{q_1}{A} \Rightarrow h_e = \frac{u_e^2}{2ga^2}$$

2. Calculate the matrices

$$\dot{h} = -\frac{a}{A}\sqrt{2hg} + \frac{q_1}{A}$$

$$q_2 = a\sqrt{2hg}$$

$$A_{lin} = \frac{\partial}{\partial h}\left(-\frac{a}{A}\sqrt{2hg} + \frac{q_1}{A}\right)\Bigg|_{h_e, u_e} = -\frac{a^2 g}{Au_e}$$

$$B_{lin} = \frac{\partial}{\partial u}\left(-\frac{a}{A}\sqrt{2hg} + \frac{q_1}{A}\right)\Bigg|_{h_e, u_e} = \frac{1}{A}$$

$$C_{lin} = \frac{\partial}{\partial h}\left(a\sqrt{2hg}\right)\Bigg|_{h_e, u_e} = \frac{a^2 g}{u_e}$$

$$D_{lin} = \frac{\partial}{\partial u}\left(-a\sqrt{2hg}\right)\Bigg|_{h_e, u_e} = 0$$

Initial condition response

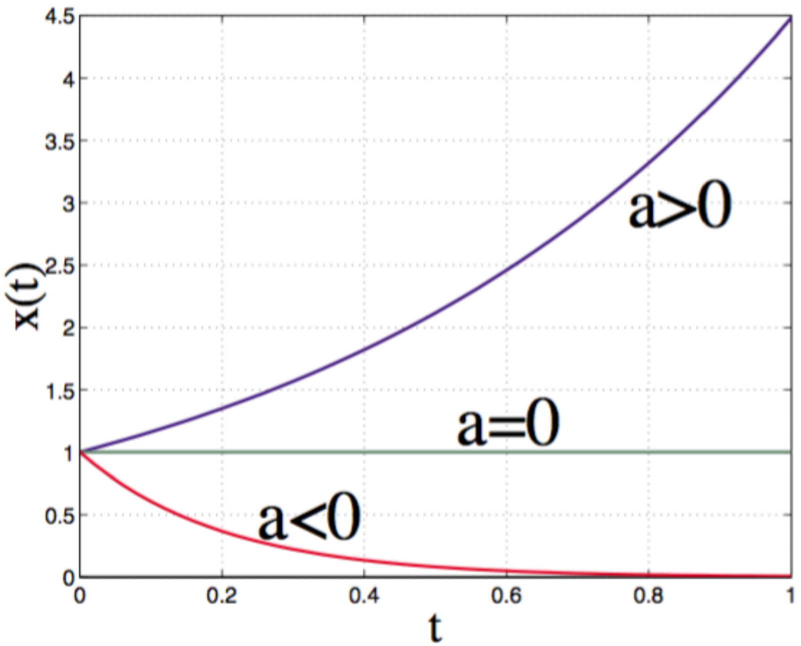
Consider the system

$$\dot{x} = Ax$$

For a “scalar” version

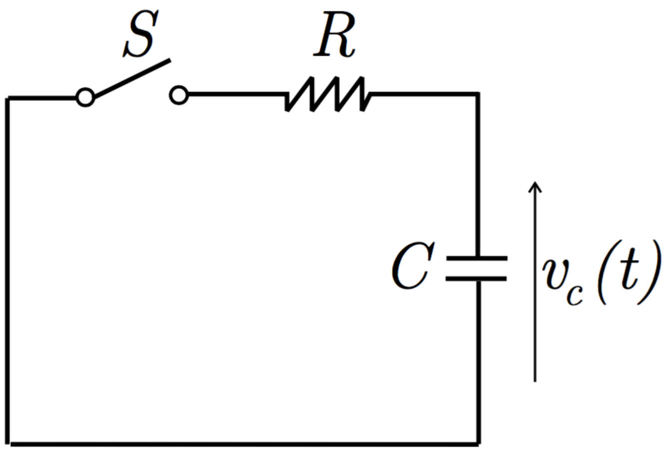
$$\dot{x} = ax$$

we know the solution is *unique* and $x(t) = e^{at}x(0)$



Initial condition response. Example

Consider the system

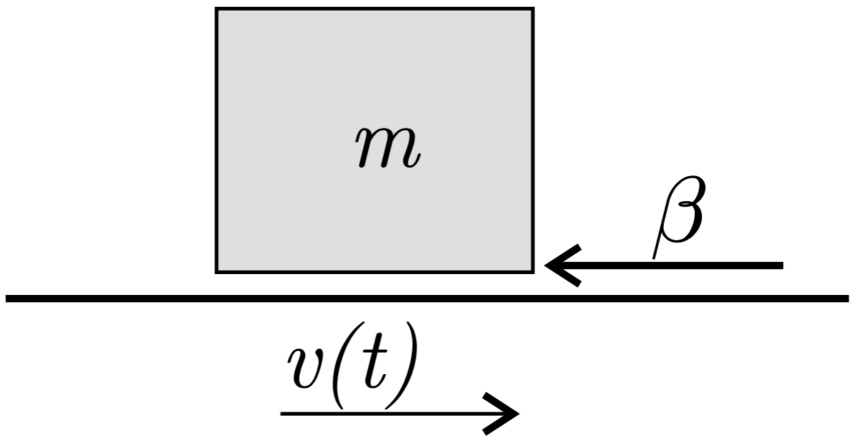


$$v(t) + RC\dot{v}_c(t) = 0 \qquad \text{(Voltage Kirchoff law)}$$

$$v_c(t) = e^{-t/RC} v_c(0)$$

Initial condition response. Example

Consider the system



$$m\dot{v}(t) = -\beta v(t)$$ (Newton's law)

$$v(t) = e^{-t\beta/m} v(0)$$

Response to input signal. The convolution equation

Theorem. The solution of the linear differential equation

$$\dot{x} = ax + bu$$

$$y = cx + du$$

is

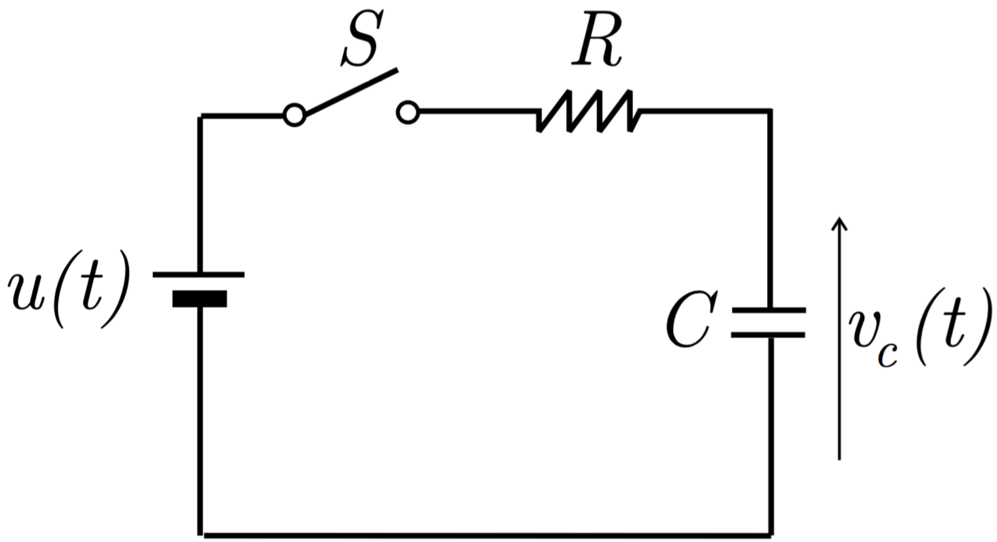
$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)} bu(\tau)\,d\tau$$

It follows that

$$y(t) = \underbrace{ce^{at}x(0)}_{\text{initial condition response}} + \underbrace{\int_0^t ce^{a(t-\tau)} bu(\tau)\,d\tau + du(t)}_{\text{input response}}$$

Response to input signal. Example

Consider the system

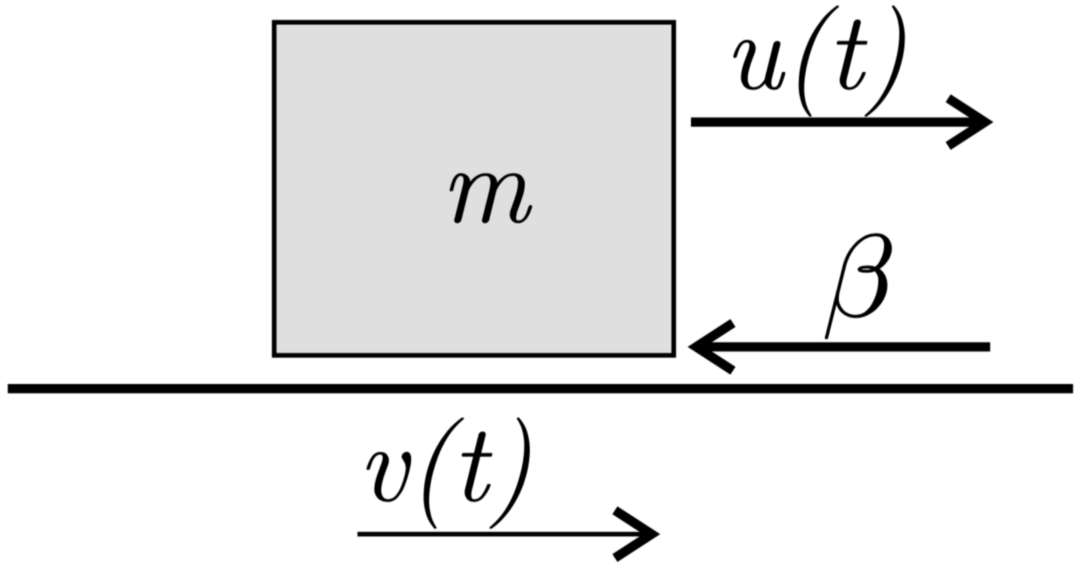


$$v(t) + RC\dot{v}_c(t) = u(t) \quad \text{(Voltage Kirchoff law)}$$

$$v_c(t) = e^{-t/RC} v_c(0) + \int_0^t e^{-(t-\tau)/RC} \frac{1}{RC} u(\tau) d\tau$$

Response to input signal. Example

Consider the system



$$m\dot{v}(t) = -\beta v(t) + u(t) \qquad \text{(Newton's law)}$$

$$v(t) = e^{-t\beta/m}v(0) + \int_0^t e^{-\frac{\beta}{m}(t-\tau)}\frac{1}{m}u(\tau)d\tau$$

Systems of differential equations

Consider the system of differential equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u$$

$$x_1(0) = x_{10},\ x_2(0) = x_{20},\ \cdots, x_n(0) = x_{n0}$$

In a matrix form

$$\dot{x} = Ax + Bu$$

$$x(0) = x_0$$

Linear differential equations and linear systems

Similarly to the scalar case the state response will be

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

How do we calculate e^{At}?

Matrix exponential

Define $$e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}X^k$$

where

$$X^0 = I,\ X^1 = X,\ X^2 = XX$$

By replacing $X \to At$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

By differentiating both sides wrt t

$$\frac{d}{dt}e^{At} = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{3!}A^4t^3 + \cdots = A\sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k = Ae^{At}$$

Initial condition response

Proposition. The solution of the homogenous system of differential equations $\dot{x} = Ax$ is $x(t) = e^{At}x(0)$

Example. Double integrator

$$\ddot{y} = u$$

Rewrite as

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$A^2 = 0 \qquad\qquad e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Recall some linear algebra

Consider the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Its ***characteristic equation*** is $\det(sI - A) = 0$

Its ***characteristic polynomial*** is $P(s) = \det(sI - A)$

Its ***eigenvalues*** $\lambda_1, \ldots, \lambda_n$ are solutions of the characteristic equations $\det(\lambda_i I - A) = 0$

Its ***eigenvectors*** $v_1, \ldots, v_n$ are vectors such that

$$Av_i = \lambda_i v_i$$

Eigenvalues and eigenvectors. Example

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

The characteristic equation is

$$\det(sI - A) = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{vmatrix} = s^3 + 6s^2 + 11s + 6$$

The corresponding roots (eigenvalues) are

$$\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3$$

Eigenvalues and eigenvectors. Example

The eigenvector v_1 corresponding to, e.g., $\lambda_1 = -1$

$$Av_1 = \lambda_1 v_1$$

is found as solution of $\left(A - I\lambda_1\right)v_1 = 0$

$$\begin{bmatrix} -\lambda_1 & 1 & 0 \\ 0 & -\lambda_1 & 1 \\ -6 & -11 & -\lambda_1 - 6 \end{bmatrix} v_1 = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -6 & -11 & -5 \end{bmatrix} v_1 = 0$$

$$\text{null}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -6 & -11 & -5 \end{bmatrix} = \begin{bmatrix} -0.57 \\ -0.57 \\ -0.57 \end{bmatrix}$$

Eigenvalues and eigenvectors. Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(sI - A) = \begin{vmatrix} s-1 & -3 \\ 5 & s-2 \end{vmatrix} = s^2 - 3s + 17$$

The corresponding roots (eigenvalues) are

$$\lambda_1 = \frac{3}{2} + j\frac{\sqrt{59}}{2}, \ \lambda_2 = \frac{3}{2} - j\frac{\sqrt{59}}{2}$$

Initial condition response

Note that if all the eigenvalues of A are distinct, an invertible matrix $T=[v_1,v_2,\dots.v_n]$ exists such that TAT^{-1} is diagonal and with the eigenvalues of A on the main diagonal.

Consider the following change of coordinates

$$z = Tx$$

In the new coordinates the system equations become

$$\dot{z} = T\dot{x} = TAx = TAT^{-1}z$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} z$$

Initial condition response

Proposition. The solution of the homogenous system of differential equations $\dot{x} = Ax$ is $x(t) = e^{At}x(0)$

Hence, the response in the new coordinates system is

$$z(t) = e^{TAT^{-1}t}z(0) = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{\lambda_n t} \end{bmatrix} z(0)$$

$$x(t) = T^{-1} \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{\lambda_n t} \end{bmatrix} Tx(0)$$

Example

For the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Eigenvalues and eigenvectors are $\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3$

$$T = \begin{bmatrix} -0.5774 & 0.2182 & -0.1048 \\ -0.5774 & -0.4364 & 0.3145 \\ -0.5774 & 0.8729 & -0.9435 \end{bmatrix} \qquad A = T\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}T^{-1}$$

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>>[T,Λ]=eig(A)
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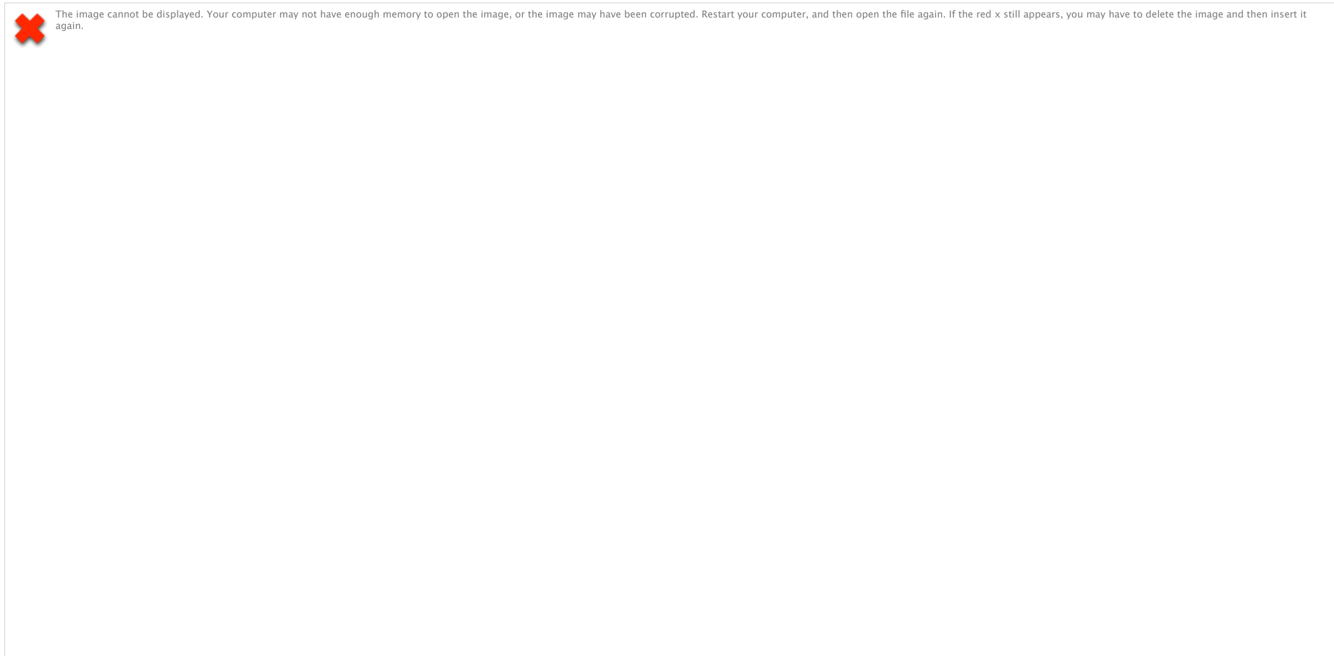
Example

$$T^{-1} = \begin{bmatrix} -5.1962 & -4.3301 & -0.8660 \\ -13.7477 & -18.3303 & -4.5826 \\ -9.5394 & -14.3091 & -4.7697 \end{bmatrix} \qquad e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix} T^{-1}$$



Linear differential equations and linear systems

Consider the following linear differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = u$$

relating the input u and output y variables of a dynamical system.

This is an n-th order system that can be converted into a state space form as follows.

Define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d^{n-1} y / dt^{n-1} \\ d^{n-2} y / dt^{n-2} \\ \vdots \\ dy/dt \\ y \end{bmatrix}$$

Linear differential equations and linear systems

The state space equation then becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} x$$

Linear differential equations and linear systems

Consider the following linear differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = 0$$

Define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ dy/dt \\ \vdots \\ d^{n-2}y/dt^{n-2} \\ d^{n-1}y/dt^{n-1} \end{bmatrix}$$

Linear differential equations and linear systems

The state space equation then becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Example

Consider the following linear differential equation

$$\ddot{y} + 2\dot{y} + 5y = 0$$

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned} \quad \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -5x_1 - 2x_2 \end{cases}, \qquad x(0) = \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix}$$

Linear differential equations and linear systems

Consider the following linear differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y =$$

$$b_{n-1}\frac{d^{n-1} u}{dt^n} + b_{n-2}\frac{d^{n-2} u}{dt^{n-2}} + \cdots + b_1 \dot{u} + b_0 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} x$$

Example

Consider the following linear differential equation

$$\ddot{y} - 2\dot{y} + y = u$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Example

Consider the following linear differential equation

$$\dddot{y} + 5\ddot{y} + 3\dot{y} + 6y = 3u + 4\dot{u}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -3 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} x$$

Input/Output response. The convolution equation

Recall that

$$y(t) = \underbrace{Ce^{At}x(0)}_{\substack{\text{initial condition}\\ \text{response}}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}_{\text{input response}}$$

Let's calculate the response to some input signal

The convolution equation

Consider a *pulse*

$$\delta(t) = \lim_{\varepsilon \to 0} p_\varepsilon(t), \quad p_\varepsilon(t) = \begin{cases} 0, & t < 0 \\ 1/\varepsilon, & 0 \le t < \varepsilon \\ 0, & t \ge \varepsilon \end{cases}$$

The response to a pulse from a zero initial condition is

$$h(t) = \int_0^t Ce^{A(t-\tau)} B\delta(\tau)d\tau = Ce^{A(t)}B$$

Hence

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{initial condition response}} + \underbrace{\overbrace{\int_0^t h(t-\tau)u(\tau)d\tau}^{\text{convolution of the impulse response and the input signal}} + Du(t)}_{\text{input response}}$$

Example. Step response

Use the convolution equation to calculate the output response to a step signal

$$u(t) = \begin{cases} 1, & \mathrm{t} \geq 0 \\ 0, & t < 0 \end{cases}$$

assuming *x(0)=0*

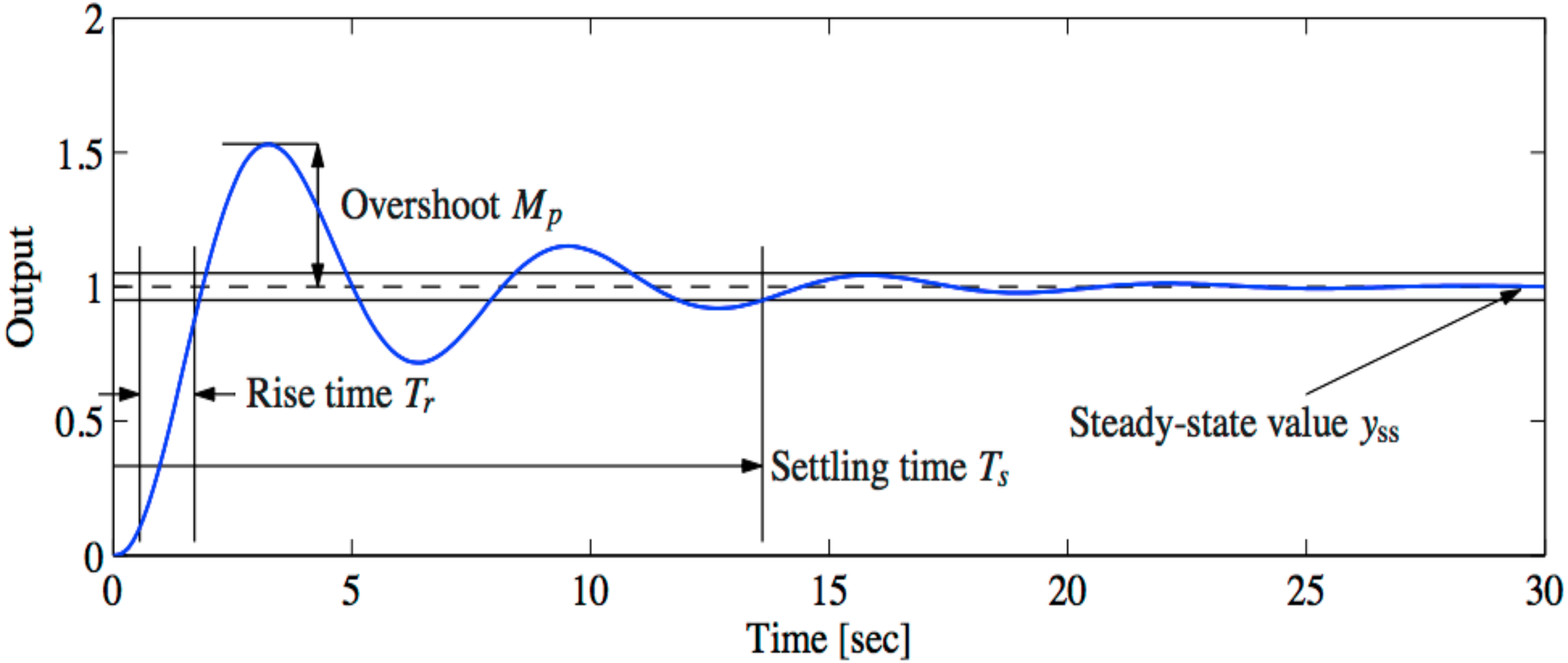
$$y(t) = Ce^{At}\overbrace{x(0)}^{0} + \int_0^t Ce^{A(t-\tau)}B\overbrace{u(\tau)}^{1}d\tau + D\overbrace{u(t)}^{1}$$

$$= \int_0^t Ce^{A(t-\tau)}Bd\tau + D =_{|\sigma = t-\tau} \int_0^t Ce^{A\sigma}Bd\sigma + D$$

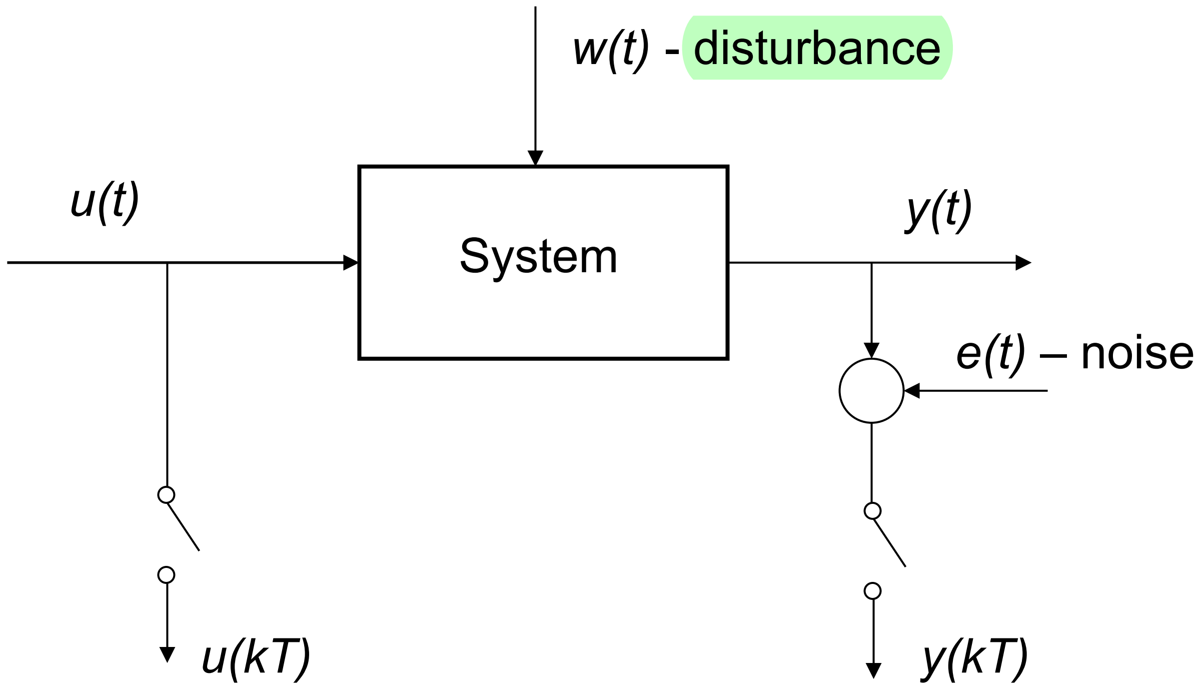
$$= C\left(A^{-1}e^{A\sigma}B\right)\Big|_0^t + D$$

$$= \underbrace{CA^{-1}e^{At}B}_{\text{transient}} - \underbrace{CA^{-1}B + D}_{\text{steady state}}$$

Step response



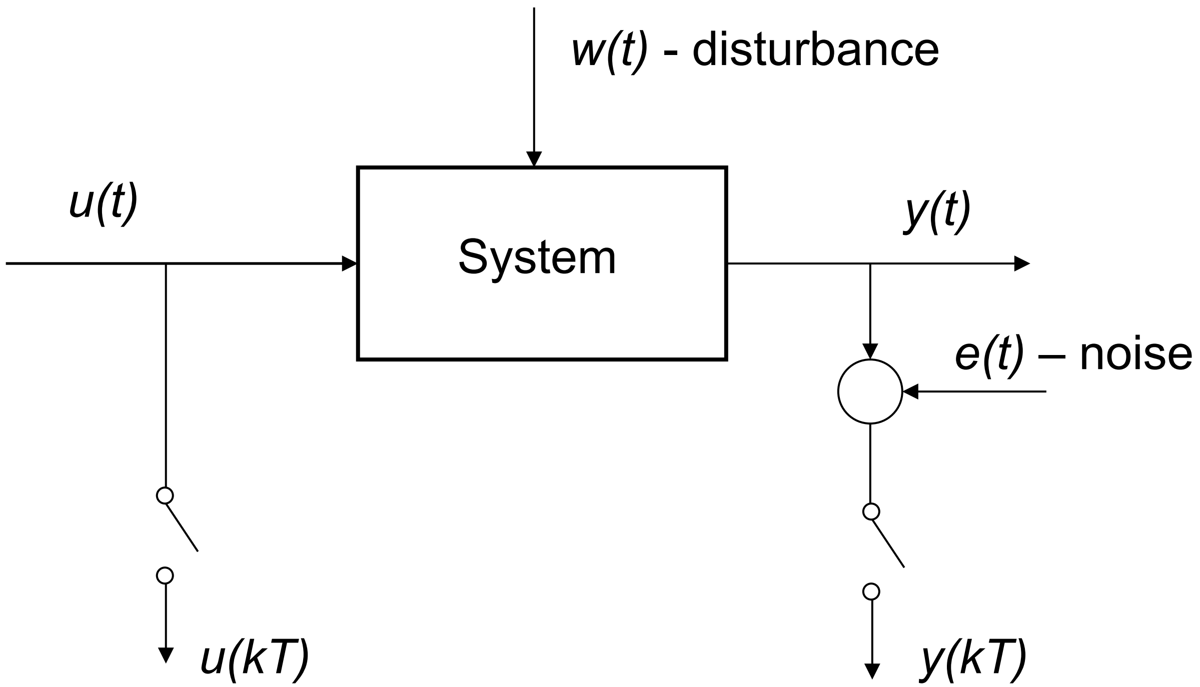
Discrete time linear systems

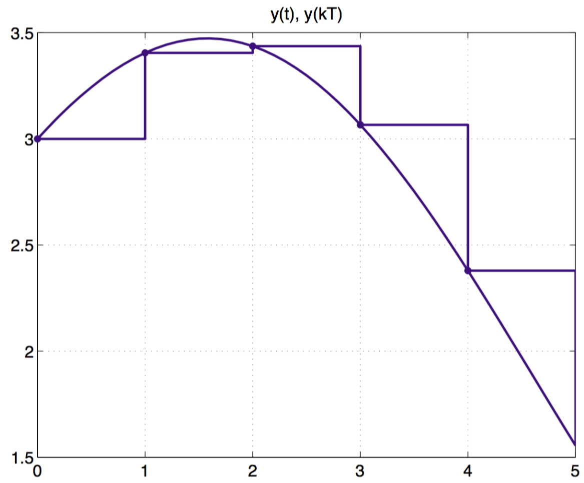


- Signals in discrete time
- Describing the system behavior in discrete time

Discrete time linear systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x((k+1)T) &= \tilde{A}x(kT) + \tilde{B}u(kT) \\ y(kT) &= Cx(kT) + Du(kT) \end{aligned}$$





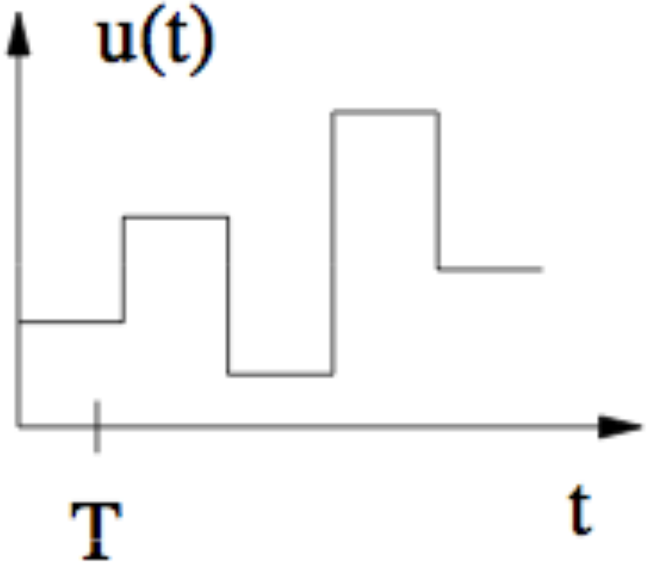
Piecewise constant signals (zoh)

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

has solution

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-t_0-\tau)}Bu(\tau)d\tau$$

Piecewise constant signals



for $t = (k+1)T,\ t_0 = kT$ and

$$u(\tau) = \tilde{u}(k) \text{ for } kT \le \tau \le (k+1)T$$

$$\tilde{x}(k+1) = e^{AT}\tilde{x}(k) + \int_0^T e^{A(T-\tau)}d\tau\, B\tilde{u}(k)$$

Piecewise constant signals

$$\tilde{x}(k+1) = e^{AT}\tilde{x}(k) + \int_0^T e^{A(T-\tau)} d\tau \, B\tilde{u}(k)$$

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}\tilde{u}(k)$$
$$y(k) = C\tilde{x}(k) + D\tilde{u}(k)$$

with:

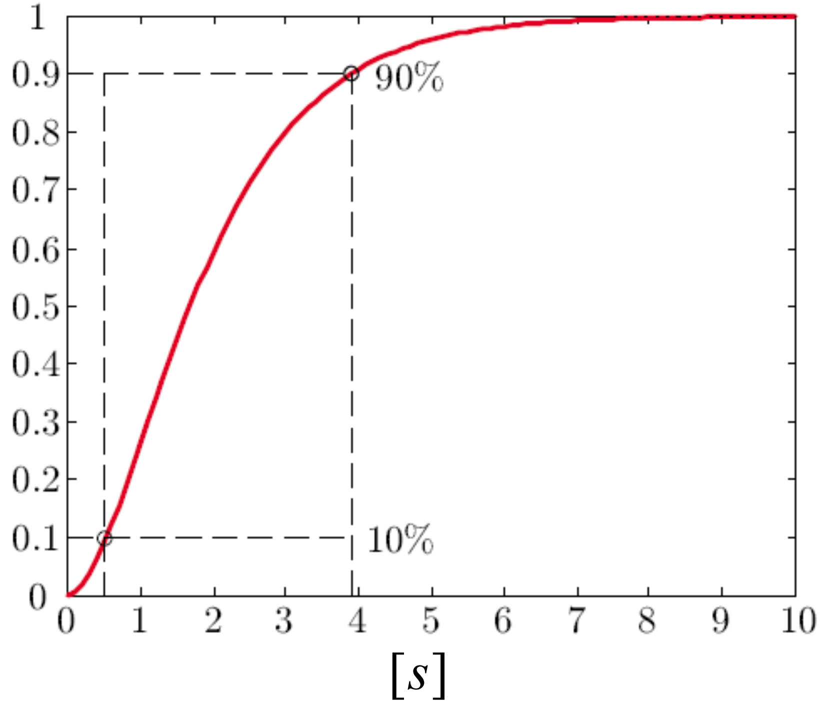
$$\tilde{A} = e^{AT} \text{ and } \tilde{B} = \int_0^T e^{A(T-\tau)} d\tau \, B$$

NOTE: the system $\left(\tilde{A}, \tilde{B}, C, D\right)$ ***exactly*** describes the system $\left(A, B, C, D\right)$ at $t=kT$ if the input is piecewise constant

Sampling time selection

Q: how to select the sampling time?

A: for linear systems $T = \frac{1}{10} t_r$ with a unitary step



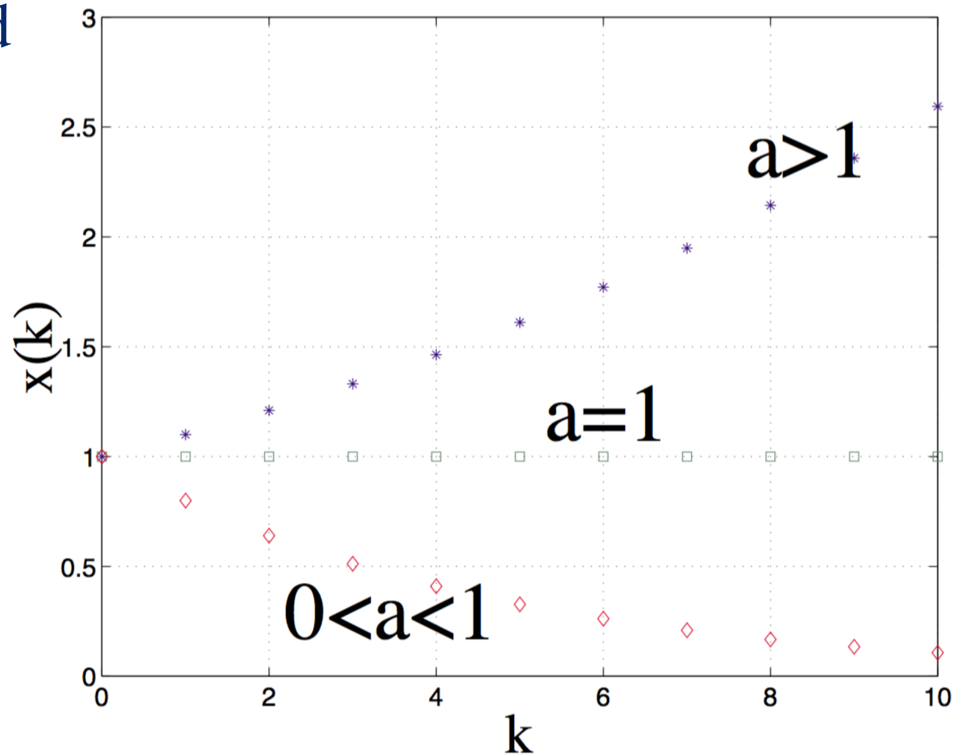
Discrete time systems

Consider the system

$$x(k+1) = ax(k),$$
$$x(0) = x_0$$

The solution exists, is unique and

$$x(k) = a^k x_0$$



Discrete time systems

Consider the system with inputs

$$x(k+1) = ax(k) + bu(k),$$
$$x(0) = x_0$$

The solution exists, is unique and $x(k) = a^k x_0 + \sum_{i=0}^{k-1} a^i bu(k-1-i)$

Similarly, in the n^{th} order case

$$x(k+1) = Ax(k) + Bu(k),$$
$$x(0) = x_0$$

the solution exists, is unique and $x(k) = A^k x_0 + \sum_{i=0}^{k-1} A^i Bu(k-1-i)$

where $A^k = T^{-1}\begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} T$

Example

Consider the system

$$x_1(k+1) = \frac{1}{2}x_1(k) + \frac{1}{2}x_2(k),$$
$$x_2(k+1) = x_2(k) + u(k),$$
$$x_1(0) = -1, \quad x_2(0) = 1$$

The resulting matrices are

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = 1$$

The response is

$$x(k) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \sum_{i=0}^{k-1} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^i \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k-1-i)$$

Example

Let's calculate $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^k$

The eigenvalues are

$$\lambda_1 = 0.5,\ \lambda_2 = 1$$

The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

Hence $A^k = [\ v_1 \quad v_2\ ]\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}[\ v_1 \quad v_2\ ]^{-1}$

$$= \begin{bmatrix} 1 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{bmatrix}^{-1} = \begin{bmatrix} 1 - \frac{1}{2^{k-1}} \\ 1 \end{bmatrix}$$

Stability

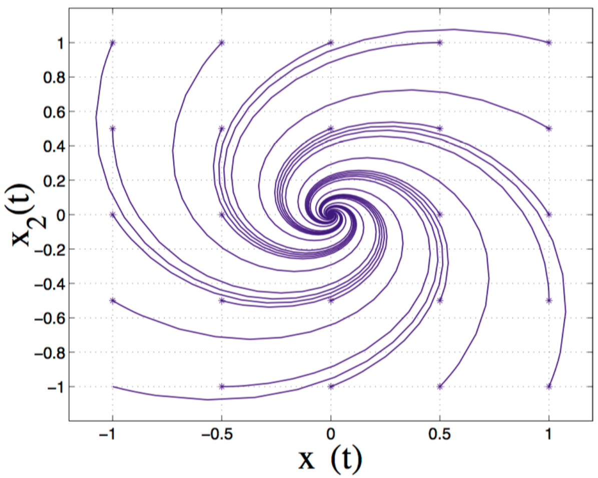
Consider the system

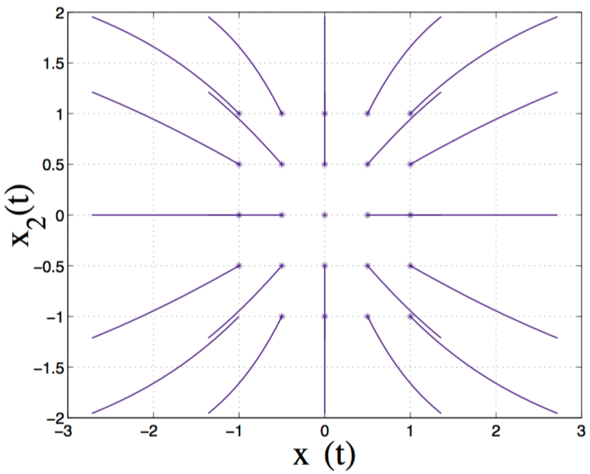
$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

Assume $\left(x_e,\ u_e\right)$ is an ***equilibrium point***

Definition (Stability). The equilibrium $\left(x_e,\ u_e\right)$ is stable if

$$\forall \varepsilon > 0,\ \ \exists \delta > 0 :\ \ \left\| x_0 - x_e \right\| < \delta \ \ \Rightarrow \ \ \left\| x(t, x(0), u_e) - x_e \right\| < \varepsilon$$





Stability of continuous time systems

Consider the system with inputs

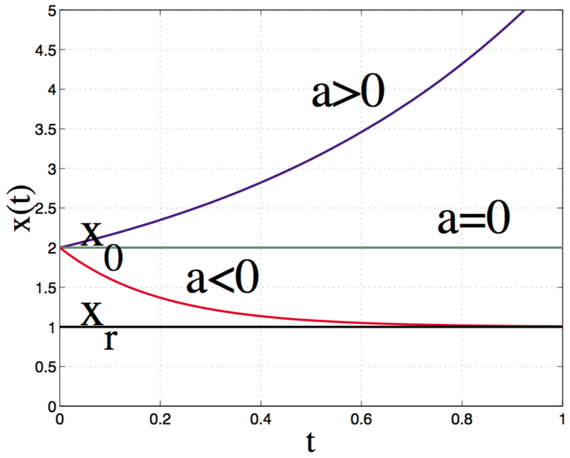
$$\dot{x}(t) = ax(t) + bu(t),$$
$$y(t) = cx(t)$$

At the equilibrium $x_e = -\frac{b}{a}u_e$ Does the system remain close to x_e?

Define the error variable $e = x - x_e$

$$\dot{e}(t) = \dot{x}(t) = ax(t) + bu_e = \Big|_{u_e = -\frac{a}{b}x_e} ae(t)$$

We found that $e(t) = e^{at}e(0)$



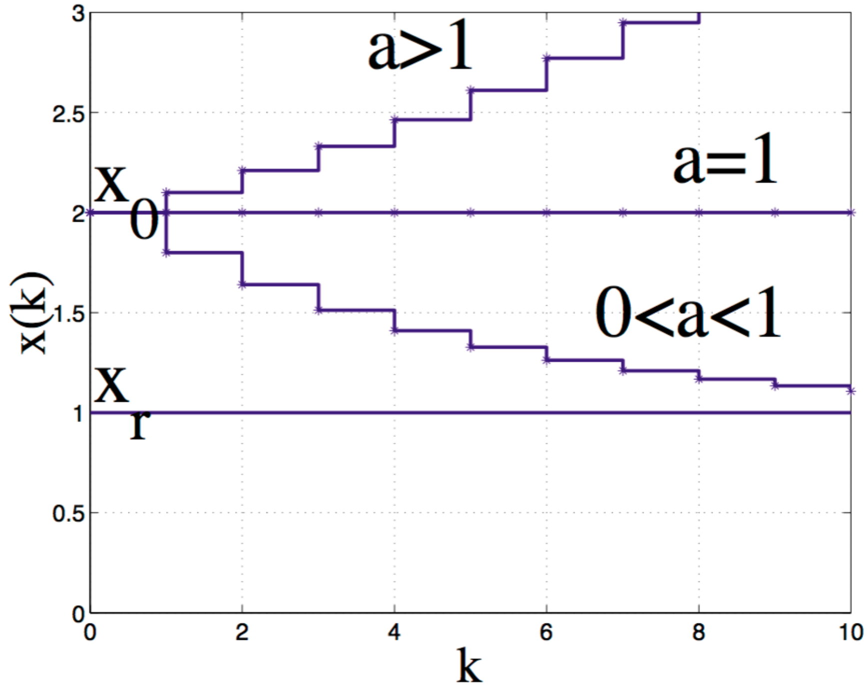
For higher order systems we look at the real parts of the eigenvalues

Stability of discrete time systems

Consider the system with inputs

$$x(k+1) = ax(k) + bu(k),$$
$$x(0) = x_0$$

Similarly to the continuous time case, it can be shown that



Example

Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} x(t), \quad x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

The roots of $|\lambda I - A| = \lambda^2 + 3s + 2$ are $\lambda_1 = -1,\ \lambda_2 = -2$

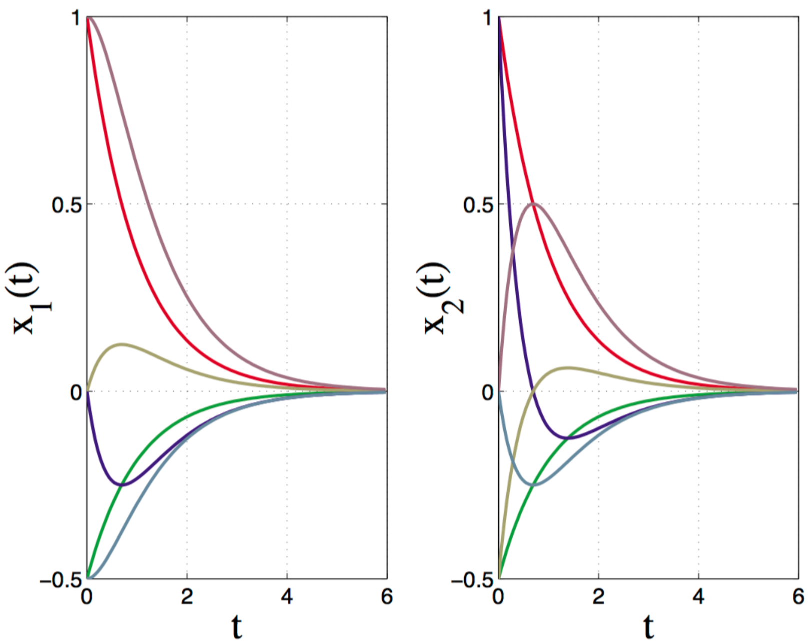
The eigenvectors are

$$T = \begin{bmatrix} 0.71 & 0.45 \\ 0.71 & 0.89 \end{bmatrix}$$

Example

The response is $x(t) = T\begin{bmatrix} e^{-t} & \\ & e^{-2t} \end{bmatrix} T^{-1}x(0)$

$$= \begin{bmatrix} x_{10}\left(2e^{-t} - e^{-2t}\right) + x_{20}\left(-e^{-t} + e^{-2t}\right) \\ x_{10}\left(2e^{-t} - 2e^{-2t}\right) + x_{20}\left(-e^{-t} + 2e^{-2t}\right) \end{bmatrix}$$



Example

Consider the system

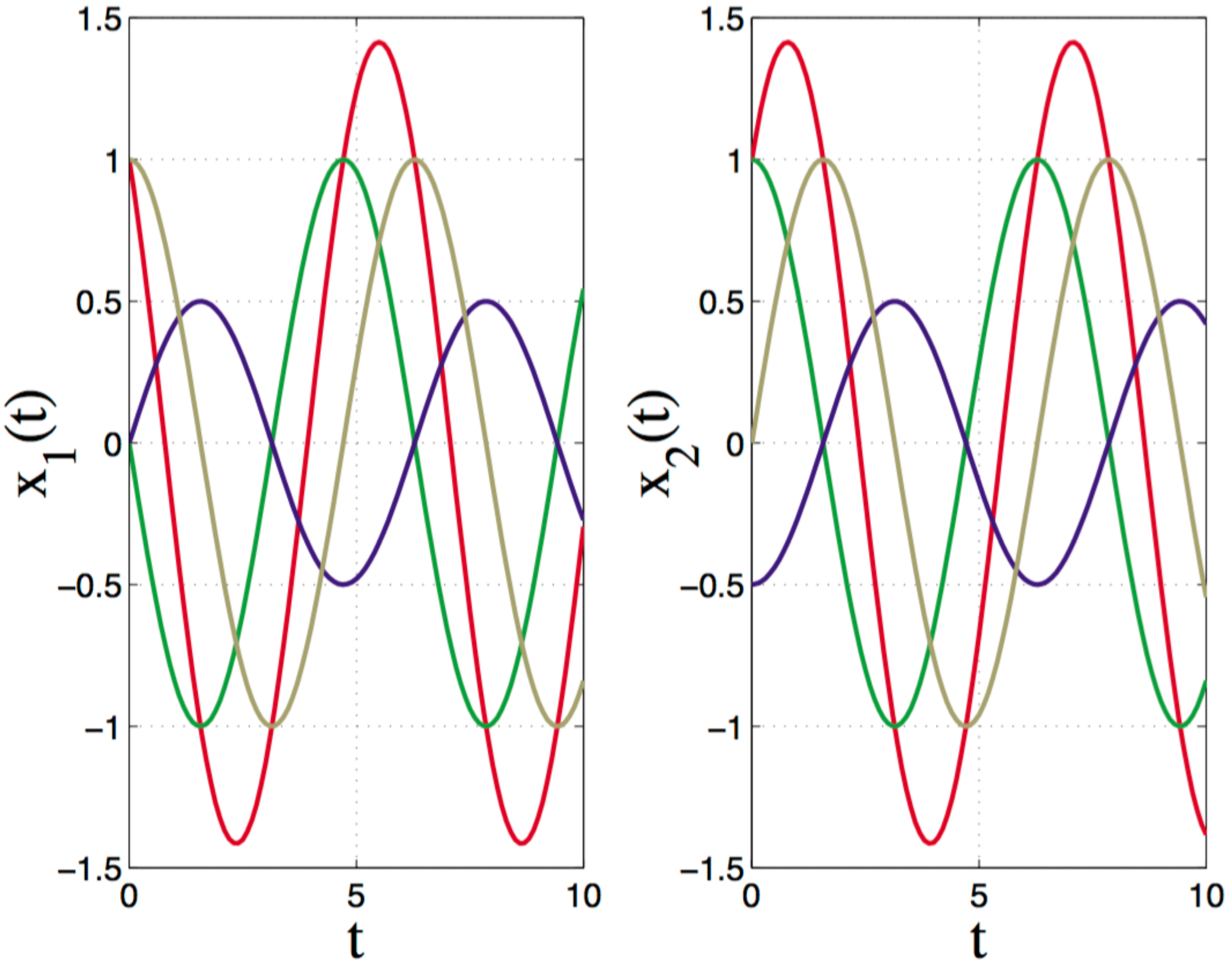
$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(t), \quad x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

The eigenvalues are $\lambda_1 = j,\ \lambda_2 = -j$

The response is $x(t) = T^{-1} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} T x(0)$

$$= \begin{bmatrix} x_{10}\cos t - x_{20}\sin t \\ x_{10}\sin t + x_{20}\cos t \end{bmatrix}$$

Example



Laplace transform

Consider the function $f(t)$. Its Laplace transform is a complex function of a complex variable

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = L[f]$$

Example. Impulse

$$f(t) = \delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases} \qquad \int_0^\infty \delta(t) = 1 \qquad \Rightarrow L[\delta] = 1$$

Example. Step

$$f(t) = \mathbf{1}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \qquad \Rightarrow L[\mathbf{1}] = \frac{1}{s}$$

Laplace transform. Properties

Linearity.

$$L[af_1 + bf_2] = aL[f_1] + bL[f_2]$$

Time shifting

$$L[f(t-\tau)] = e^{-s\tau}L[f(t)] \qquad L[\mathbf{1}(t-2)] = \frac{e^{-2s}}{s}$$

Frequency shifting

$$L[e^{at}f(t)] = F(s-a) \qquad f(t) = \cos(\omega t)\mathbf{1}(t)$$

Recall that $\cos(\omega t) = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2}$

$$L[f(t)] = \frac{s}{s^2 + \omega^2}$$

Laplace transform. Properties

Derivative in time domain

$$L\left[\frac{d}{dt}f(t)\right] = sL[f] + f(0^+)$$

Derivative in frequency domain

$$L\left[tf(t)\right] = -\frac{d}{ds}L[f]$$

Initial value theorem

$$f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

Final value theorem

$$f(+\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Transfer functions in continuous time

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$
$$y(t) = Cx(t) + Du(t)$$

Be *X(s)=L[x(t)], U(s)=L[u(t)], Y(s)=L[y(t)]*

$$sX(s) - x_0 = AX(s) + BU(s),$$
$$Y(s) = CX(s) + DU(s)$$

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s),$$
$$Y(s) = \underbrace{C(sI - A)^{-1}x_0}_{\text{Response to initial condition}} + \underbrace{[C(sI - A)^{-1}B + D]U(s)}_{\text{Input response}}$$

Transfer functions in continuous time

For $x_0=0$

$$G(s)=\frac{Y(s)}{U(s)}=C(sI-A)^{-1}B+D$$ ***Transfer function***

Note that

$$\begin{aligned} \mathrm{Y}(s) &= [C(sI-A)^{-1}B+D]U(s) \\ &= \Big|_{u(t)=\delta(t)} \underbrace{C(sI-A)^{-1}B+D}_{G(s)} \end{aligned}$$

The transfer function is the Laplace transform of the impulse response

Example

Consider the system

$$\dot{x}(t) = \begin{bmatrix} -10 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 2 & 2 \end{bmatrix} x(t)$$

Recall that $A^{-1} = \dfrac{\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}}{|A|}$

$$G(s) = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} s+10 & -1 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{2s+22}{s^2+11s+10}$$

Example

Consider the following linear differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y =$$

$$b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \dot{u} + b_0 u$$

Set $y(0) = \dot{y}(0) = \cdots = y^{(n-1)}(0) = 0$

By the time derivative property

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Example. $\ddot{y}(t) + 2\dot{y}(t) + y(t) = \dot{u}(t) + u(t)$

$$G(s) = \frac{s+1}{s^2 + 2s + 1}$$

Example

Integrator

$$\dot{x} = u$$

$$y = x$$

$$G(s) = \frac{1}{s}$$

Double integrator

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = u(t)$$

$$y(t) = x_1(t)$$

$$G(s) = \frac{1}{s^2}$$

Oscillator

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -x_1(t) + u(t)$$

$$y(t) = x_1(t)$$

$$G(s) = \frac{1}{s^2 + 1}$$

Inverse Laplace transform

Consider the transfer function

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Find the poles $p_1, \ldots, p_n$ of $G(s)$ and break it down into

$$G(s) = \frac{\alpha_1}{s - p_1} + \cdots + \frac{\alpha_n}{s - p_n}$$

Find the *residuals* $\alpha_1, \ldots, \alpha_n$ as

$$\alpha_i = \lim_{s \to p_i} (s - p_i) G(s)$$

Inverse transform each term as by using the time shifting property

$$g(t) = \alpha_1 e^{p_1 t} + \cdots + \alpha_n e^{p_n t}$$

Z-transform

Consider the function *f(k)*. Its Z-transform is a complex function of a complex variable

$$F(z) = \sum_{k=0}^{\infty} z^{-k} f(k) = Z[f]$$

Example. Impulse

$$f(k) = \delta(k) = \begin{cases} 0, & \mathrm{k} \neq 0 \\ 1, & t = 0 \end{cases} \qquad \Rightarrow Z[\delta] = 1$$

Example. Step

$$f(k) = \mathbf{1}(k) = \begin{cases} 0, & \mathrm{k} < 0 \\ 1, & \mathrm{k} \geq 0 \end{cases} \qquad \Rightarrow Z[\mathbf{1}] = \frac{z}{z-1}$$

Example. Exp

$$f(k) = a^k \mathbf{1}(k) \qquad \Rightarrow Z[f] = \frac{z}{z-a}$$

Z-transform. Properties

Linearity

$$Z[af_1 + bf_2] = aZ[f_1] + bZ[f_2]$$

Forward time shifting

$$Z[f(k+1)] = zZ[f] - zf(0) \qquad Z[a^{k+1}\mathbf{1}(k)] = z\frac{z}{z-a} - z$$

Backward time shifting

$$Z[f(k-1)] = z^{-1}Z[f] \qquad Z[\mathbf{1}(k-1)] = \frac{z}{z(z-1)}$$

Derivative in freq. domain

$$Z[kf(k)] = -z\frac{d}{dz}Z[f] \qquad Z[k\mathbf{1}(k)] = \frac{z}{(z-1)^2}$$

Z-transform. Properties

Initial value theorem $f(0) = \lim_{z \to \infty} F(z)$

$$Z[\mathbf{1}(k) - k\mathbf{1}(k)] = \frac{z}{z-1} - \frac{z}{\left(z-1\right)^2} \qquad f(0) = \lim_{z \to \infty} F(z) = 1$$

Final value theorem

$$f(+\infty) = \lim_{t \to \infty} f(t) = \lim_{z \to 1} \left(z-1\right) F(z)$$

$$Z[\mathbf{1}(k) + (-0.7)^k \mathbf{1}(k)] = \frac{z}{z-1} + \frac{z}{z+0.7}$$

$$f(+\infty) = \lim_{z \to 1} \left(z-1\right) F(z) = 1$$

Transfer functions in discrete time

Consider the system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0$$
$$y(k) = Cx(k) + Du(k)$$

Be *X(z)=Z[x(k)], U(z)=Z[u(k)], Y(z)=Z[y(k)]*

$$zX(z) - zx_0 = AX(z) + BU(z),$$
$$Y(z) = CX(z) + DU(z)$$

$$X(z) = z(zI - A)^{-1}x_0 + (zI - A)^{-1}BU(z),$$
$$Y(z) = \underbrace{zC(zI - A)^{-1}x_0}_{\text{Response to initial condition}} + \underbrace{[C(zI - A)^{-1}B + D]U(z)}_{\text{Input response}}$$

Transfer functions in discrete time

For $x_0=0$

$$G(z)=\frac{Y(z)}{U(z)}=C(zI-A)^{-1}B+D$$

Transfer function

Note that

$$\mathrm{Y}(z)=[C(zI-A)^{-1}B+D]U(z)$$
$$=\Big|_{u(k)=\delta(k)}\underbrace{C(zI-A)^{-1}B+D}_{G(z)}$$

The transfer function is the Z-transform of the impulse response

Example

Consider the system

$$x(k+1)=\begin{bmatrix} 0.5 & 1 \\ 0 & -0.5 \end{bmatrix}x(k)+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k),\quad x(0)=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y(k)=\begin{bmatrix} 1 & -1 \end{bmatrix}x(k)$$

Recall that $A^{-1}=\dfrac{\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}}{|A|}$

$$G(z)=\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} z-0.5 & -1 \\ 0 & z+0.5 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$=\frac{-z+1.5}{z^2-0.25}$$

Example

Consider the following linear differences equation

$$a_n y(k-n) + a_{n-1} y(k-n+1) + \cdots + y =$$
$$b_m u(k-m) + \cdots + b_1 u(k-1)$$

By the backward time shifting property

$$G(z) = \frac{b_m z^{-m} + b_{m-1} z^{-m+1} + \cdots + b_1 z^{-1}}{a_n z^{-n} + a_{n-1} z^{-n+1} + \cdots + a_1 z^{-1} + 1}$$

Example. $3y(k-2) + 2y(k-1) + y(k) = 2u(k-1)$

$$G(z) = \frac{2z^{-1}}{3z^{-2} + 2z^{-1} + 1}$$

Example

Integrator

$$x(k+1) = x(k) + u(k)$$
$$y(k) = x(k)$$

$$G(z) = \frac{1}{z-1}$$

Double integrator

$$x_1(k+1) = x_1(k) + x_2(k)$$
$$x_2(k+1) = x_2(k) + u(k)$$
$$y(k) = x_1(k)$$

$$G(z) = \frac{1}{z^2 - 2z + 1}$$

Oscillator

$$x_1(k+1) = x_1(k) - x_2(k) + u(k)$$
$$x_2(k+1) = x_1(k)$$
$$y(t) = \frac{1}{2}x_1(t) + \frac{1}{2}x_2(t)$$

$$G(z) = \frac{0.5z + 0.5}{z^2 - z + 1}$$

Frequency response

Calculate the *steady-state* output response to the signal

$$u(t) = \cos \omega t$$

Observe that (*Euler formula*)

$$\cos \omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

We can calculate the response to e^{st} and then, by linearity, derive the response to the cosine

The convolution equation with $u(t)=e^{st}$ gives

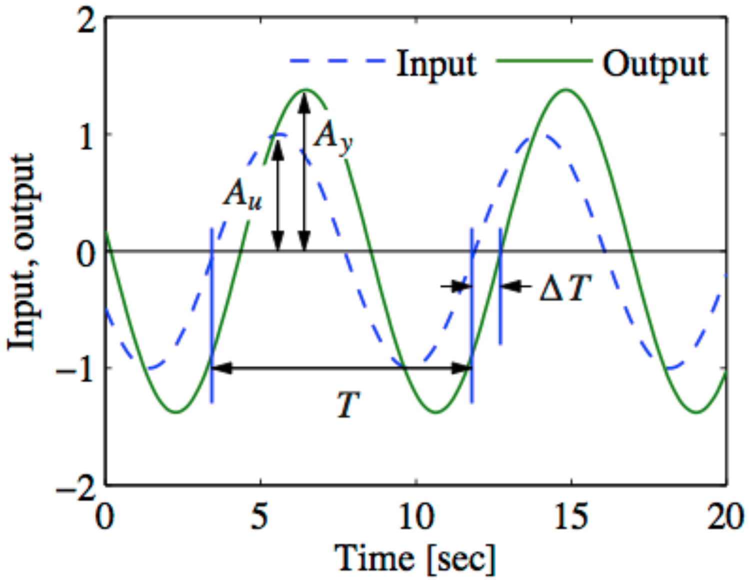
$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

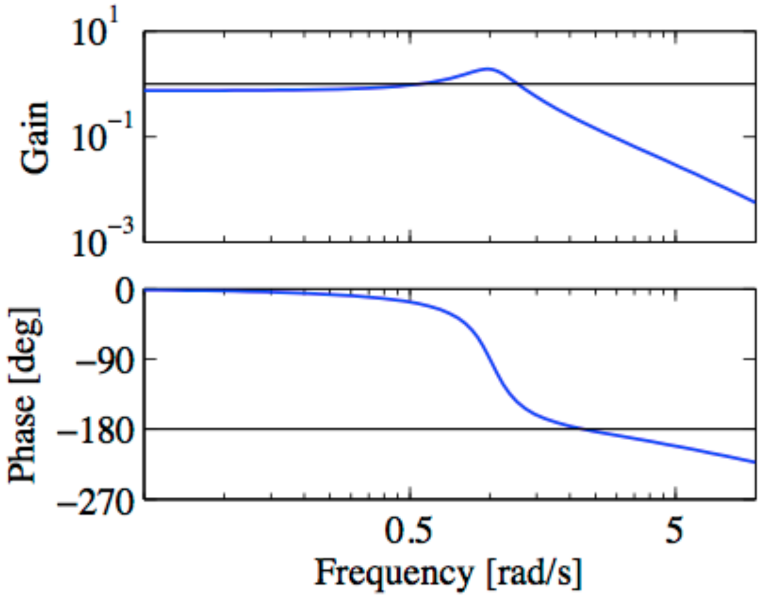
Frequency response

The *steady-state* output response can be rewritten as

$$y_{ss}(t) = \left(C\left(sI - A\right)^{-1} B + D\right)e^{st}$$
$$= Me^{i\theta}e^{st}$$

Hence, the input signal is amplified by M and shifted by θ





Frequency response

At $\omega=0$, the frequency response become

$$M_0 = -CA^{-1}B + D$$

This is the ratio between the amplitudes of the steady output and the constant input

Note that, the zero frequency gain (*DC gain*) makes sense only if *A* does not have eigenvalues in 0

Example

Calculate the response of the system

$$G(s) = \frac{10}{(1+0.1s)^2}$$

to the signal $u(t) = 5\sin 10t$

The poles are $p_1=p_2=-10$. Hence the system is stable

$$y(t) = 5\left|G(j\omega)\right|\sin\left(10t + \angle G(j\omega)\right)$$

Calculate $\left|G(10j)\right|, \ \angle G(10j)$

$$G(10j) = -5j \ \Rightarrow \ \left|G(10j)\right| = 5, \ \angle G(10j) = -\frac{\pi}{2}$$

$$y(t) = 25\sin\left(10t - \frac{\pi}{2}\right)$$

Frequency response for discrete time systems

Calculate the response of the system

$$G(z) = \frac{1.5 - z}{z(z - 0.8)}$$

to the signal $u(k) = 20 \sin \frac{\pi}{6} k$

The poles are $p_1 = 0.8$, $p_2 = 0$. Hence the system is stable

$$y(k) = 20 \left| G(e^{\frac{\pi}{6} j}) \right| \sin\left(\frac{\pi}{6} k + \angle G(e^{\frac{\pi}{6} j}) \right)$$

Literature

Further reading from Murray and Åstrom is recommended

- Chapter 2. Section 2.2, pp. 34-37. Exercises
- Chapter 3.
- Chapter 4. Sections 4.1, 4.2 pp. 100-102, 4.3 pp. 104-111
- Chapter 5. Sections 5.1, 5.2, 5.3, 5.4