ESS101- Modeling and Simulation
Lecture 7

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Physical modelling

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Physical modeling. Three phases method

- Structuring
  - Divide into subsystems
  - Inputs, outputs, internal variables?

  → Graph or block diagram

- Relationships
  - Conservation laws
  - Constitutive relations

  → differential equations and algebraic relationships

- Form state-space model
  - Choose state variables
  - Rearrange the equations

  →

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

System identification

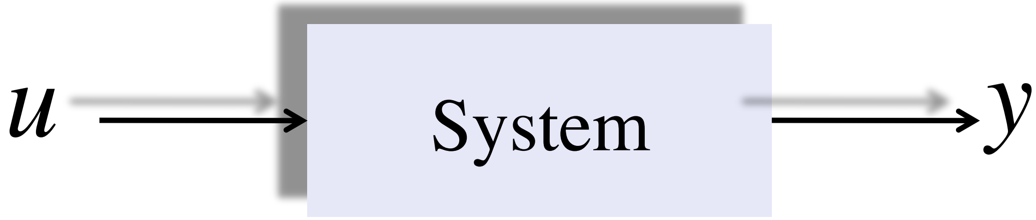
Limitations in physical modeling:

✏ Constitutive relationships may be unknown

✏ Physical parameters may be either unknown or highly uncertain

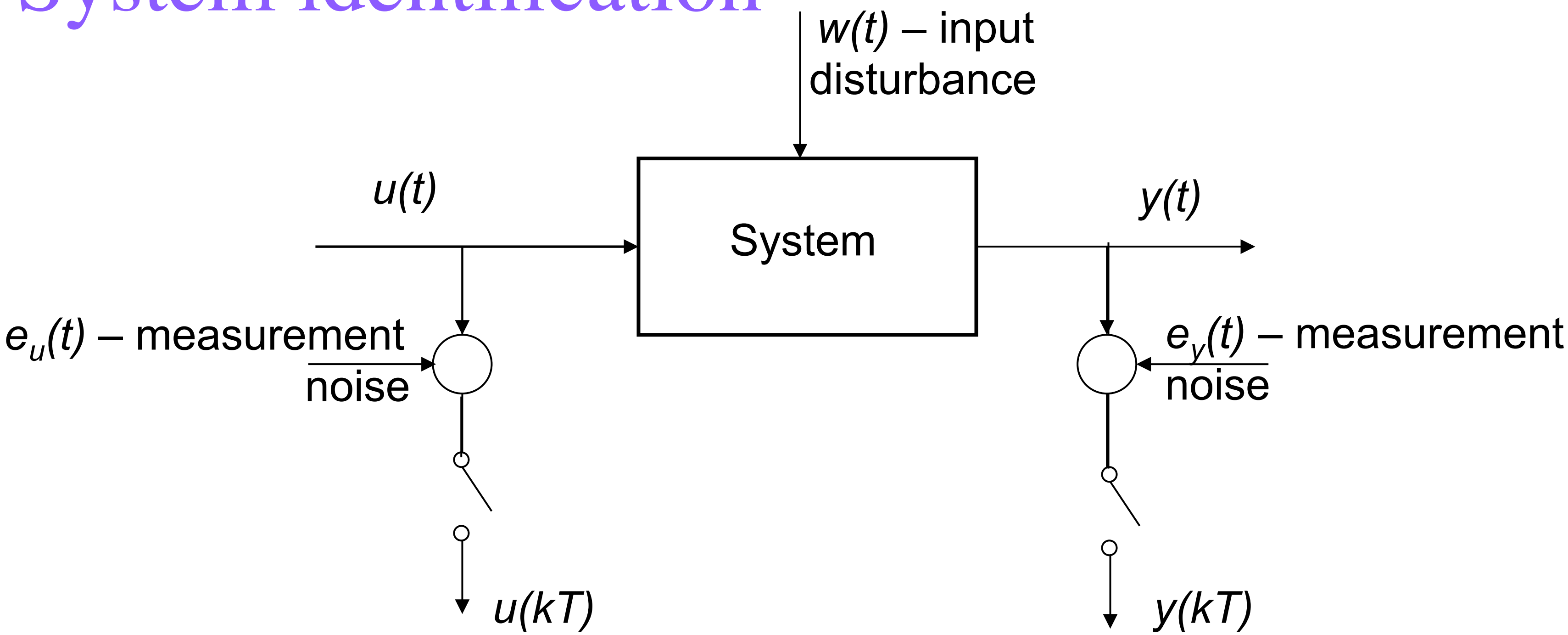
✏ The problem is too complex

Basic idea in SysId



Collect measurements of u and y and find a model for the System fitting the collected data

System identification



- Model derived from input and output measurements
- In practice, measurements are noisy (e_u, e_y) and system is affected by disturbances
- ***Goal.*** Model the effects of disturbances

Today (Chapter 3)

- ☞ Disturbance modeling
  - ☞ Deterministic models in time domain
  - ☞ Stochastic models in time domain
  - ☞ Deterministic models in the frequency domain
  - ☞ Stochastic models in the frequency domain

Time domain deterministic models

IDEA: to describe the disturbance signal as the output of a dynamical model

$$\dot{x}_w(t) = f\left(x_w(t), u_w(t)\right)$$
$$w(t) = g\left(x_w(t), u_w(t)\right)$$

or

$$W(s) = G_w(s)U_w(s)$$

if f and g are linear

where typical choices for u_w are pulse, pulse train or sinusoids. In discrete time

$$w(k+n) + d_1 w(k+n-1) + \cdots + d_n w(k) =$$
$$c_0 u_w(k+n) + c_1 u_w(k+n-1) + \cdots + c_n u_w(k)$$

or

$$G_w(z) = \frac{c_0 z^n + c_1 z^{n-1} + \cdots + c_n}{z^n + d_1 z^{n-1} + \cdots + d_n}$$

Examples in the textbook

Time domain stochastic models

Time domain stochastic models

✔ We describe a signal through a stochastic model when we are not able to predict it. E.g., the wind gust velocity.

✔ Although such a signal cannot be predicted, a guess can be made on its expected value, based on its *stochastic characterizations*

A signal *x(t)* can be viewed as a *Random Variable* (RV). As such, *x(t)* can be characterized by means of its *Cumulative Distribution Function* (CDF)

$$F_x(x,t) = \Pr\{x(t) \leq x\}$$

Time domain stochastic models

CDF can be interpreted as

$$F_x(x,t) = \lim_{N \to \infty} \frac{N(x,t)}{N}$$

where *N(x,t)* is the number of times *x(t)* is below the threshold *x* over *N* realizations ("tosses")

For the RV *x(t)*, we also define the *Probability Density Function* (PDF) as

$$f_x(x,t) = \frac{\partial F_x(x,t)}{\partial x}$$

Its integral, over a range of x, is the probability the value of the RV falls in that range

Time domain stochastic models

A RV can be more compactly described by mean of the *expected value* or *mean*

$$m_x(t) = E[x(t)] = \int_{-\infty}^{+\infty} x f_x(x,t)\,dx$$

✓ the *covariance*

$$R_x(t,s) = E\Big[\big(x(t) - m_x(t)\big)\big(x(s) - m_x(s)\big)\Big]$$

✓ the *cross covariance*

$$R_{xw}(t) = E\Big[\big(x(t) - m_x(t)\big)\big(w(t) - m_w(t)\big)\Big]$$

zero for ***independent*** random variables

✓ the *variance*

$$V_x(t) = R_x(t,t)$$

Stationary signals

A signal *x(t)* is *stationary* if $m_x(t) = m_x$

the *covariance* $R_x(t,s) = R_x(t-s) = R_x(\tau)$

the *cross covariance* $R_{xw}(t) = R_{xw}$

the *variance* $V_x(t) = V_x$

That is, the signal x(t) is stationary if its stochastic characterizations do not depend on the time

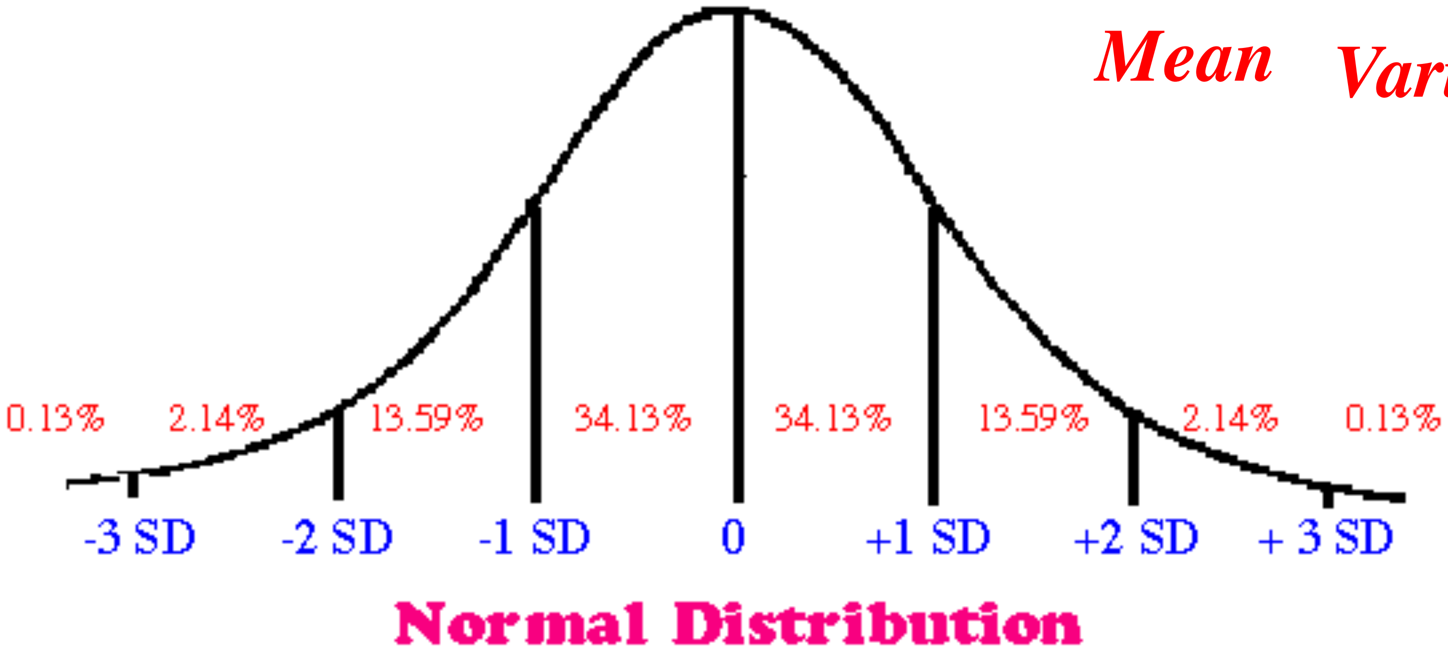
Special signals. White noise

A *white noise e(t)* is a sequence of *independent*, *normally distributed* stochastic variables

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\lambda = \sigma^2$$

$$e(t) \in N(\underbrace{0}_{\text{Mean}},\underbrace{\lambda}_{\text{Variance}})$$



White noise

White Noise. Used to describe disturbances and noises in engineering systems. A sequence of independent, normally distributed random variables

$$e(t) \in N(0,\lambda) \Rightarrow \begin{cases} m_e(t) = m_e = 0 \\ R_e(t,s) = R_e(t-s) = R_e(\tau) = \lambda\delta(\tau) \end{cases}$$

Colored Noise. Obtained from white noise by filtering it through linear systems.

Stochastic processes. Exercise.

$$e(t) \in N(0,1)$$

$$e(t) \longrightarrow \boxed{H(z) = \frac{1}{1-0.9z^{-1}}} \longrightarrow w(t)$$

Compute mean and covariance of $w(t)$.

Watch out the covariance!!

Exercise

a) $u(t) - 0.9u(t-1) = e(t), \; e(t) \in N(0,1)$

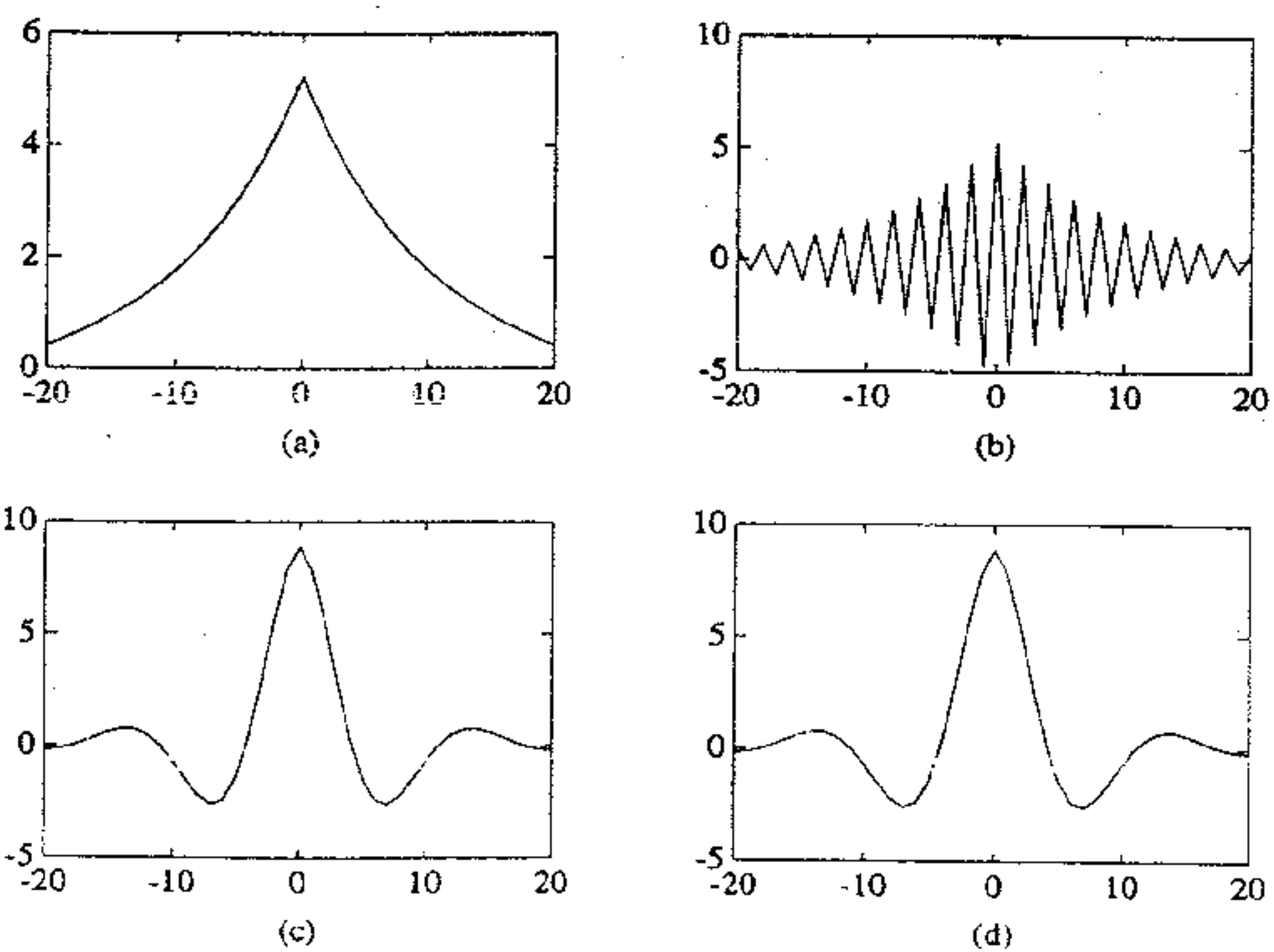
b) $u(t) + 0.9u(t-1) = e(t), \; e(t) \in N(0,1)$

c) $u(t) - 0.5u(t-1) + 0.7u(t-2) = e(t) + 0.5e(t-1), \; e(t) \in N(0,1)$

d) same as in c), but with

$$e(t) = \begin{cases} 0 & \text{w.p. } 0.98 \\ \pm\sqrt{50} & \text{w.p. } 0.01 \end{cases}$$

Covariance



6
4
2
0
-20
-10
0
10
20
(a)
10
5
0
-5
-20
-10
0
10
20
(b)
10
5
0
-5
-20
-10
0
10
20
(c)
10
5
0
-5
-20
-10
0
10
20
(d)

Frequency domain models

(Density) Spectrum

$\Phi_w(\omega)$ describes the frequency content of the signal *w(t)*

$$\int_{\omega_1}^{\omega_2} \Phi_w(\omega)d\omega$$

is a measure of the signal energy in the frequency interval $\omega \in [\omega_1, \omega_2]$

Spectrum. Continuous deterministic signals

Consider the signal $w(t), -\infty < t < \infty$ with $\int_{-\infty}^{\infty} |w(t)| dt < \infty$

The spectrum is defined as:

$$\Phi_w(\omega) = |W(\omega)|^2$$

where

$$W(\omega) = \int_{-\infty}^{\infty} w(t) e^{-i\omega t} dt \qquad \text{(Fourier transform)}$$

Recall that

$$w(t) = \sum_{n=-\infty}^{\infty} a_n e^{-in\omega t} \qquad \text{with} \qquad a_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} w(t) e^{-i \overbrace{\frac{2\pi}{T}}^{\omega} nt} dt$$

Spectrum. Sampled deterministic signals

Consider the signal $w(k), -\infty < t < \infty$ with $\sum_{k=-\infty}^{\infty} |w(kT)| < \infty$

The spectrum is defined as:

$$\Phi_w^T(\omega) = \left|W^T(\omega)\right|^2$$

where

$$W^T(\omega) = T \sum_{k=-\infty}^{\infty} w(k) e^{-i\omega kT} \qquad \text{(Fourier transform)}$$

Spectrum. Sampled stochastic processes

The spectrum for a sampled stochastic signal *w(k)* is defined as

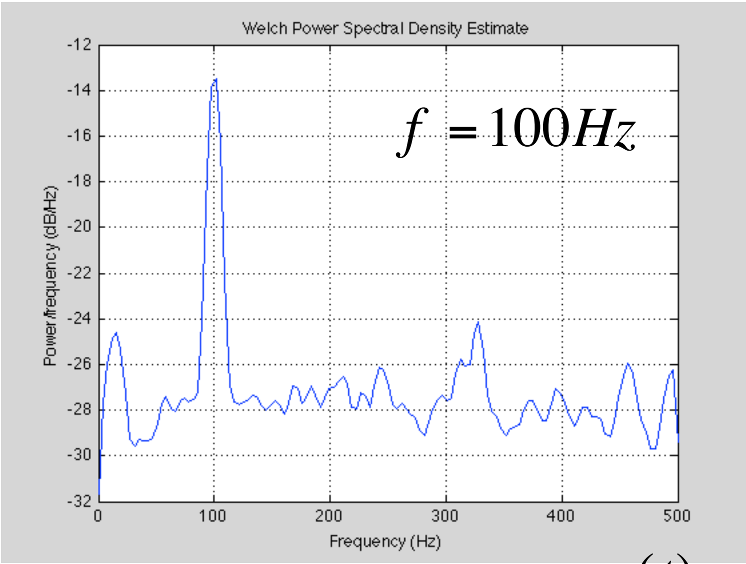
$$\Phi_w^T(\omega) = T \sum_{k=-\infty}^{k=\infty} R_w(kT) e^{-i\omega kT}$$

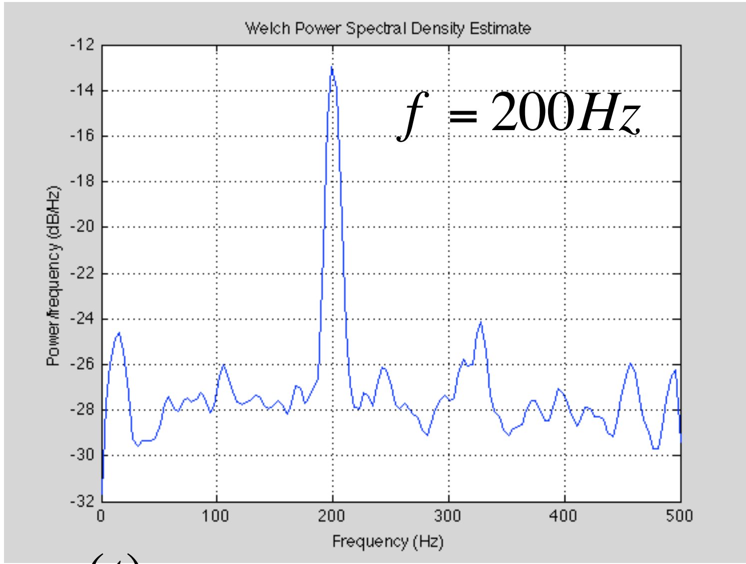
The *cross spectrum* of two signals *u* and *y* is defined as

$$\Phi_{yu}^T(\omega) = T \sum_{k=-\infty}^{k=\infty} R_{yu}(kT) e^{-i\omega kT}$$

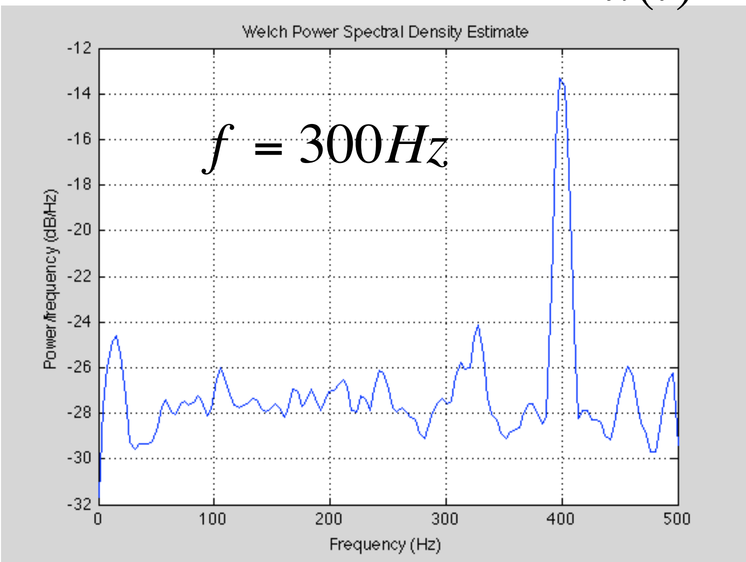
and provide information their joint variation

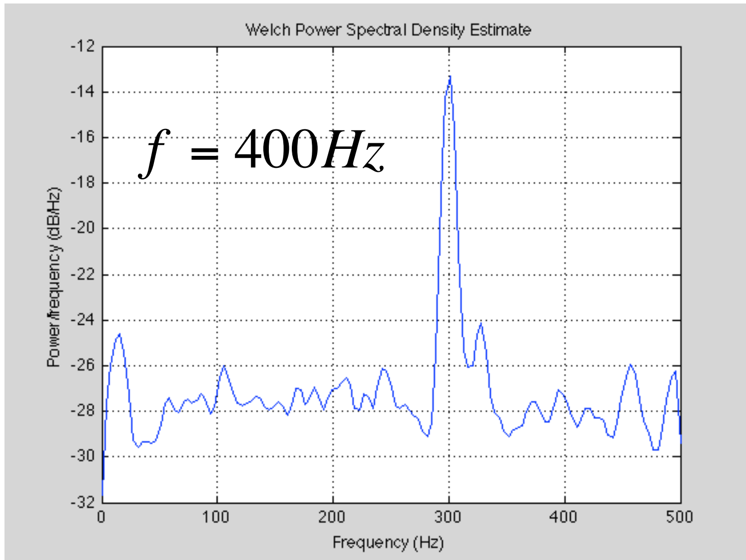
Examples





$$x(t) = \cos(2\pi f t) + n(t)$$





Example

Let's calculate the spectrum of a ***white noise w(t)*** with variance λ

Use the definition of spectrum for stochastic signals

$$\Phi_w^T(\omega) = T\sum_{k=-\infty}^{k=\infty} R_w(kT)e^{-i\omega kT}$$

Recall that $R_e(t) = \lambda\delta(t)$

Hence

$$\Phi_w^T(\omega) = \lambda T$$

Spectra and linear systems

Consider the sampled signal *y(k)* obtained as output of a linear system

$$y(k) = G(q)u(k) + v(k)$$

where *u(k)* and *v(k)* are uncorrelated

The spectra $\Phi_y^T(\omega)$ and $\Phi_{yu}^T(\omega)$ are given by

$$\Phi_y^T(\omega) = |G(i\omega)|^2 \Phi_u^T(\omega) + \Phi_v^T(\omega)$$

$$\Phi_{yu}^T(\omega) = G(i\omega)\Phi_u^T(\omega)$$

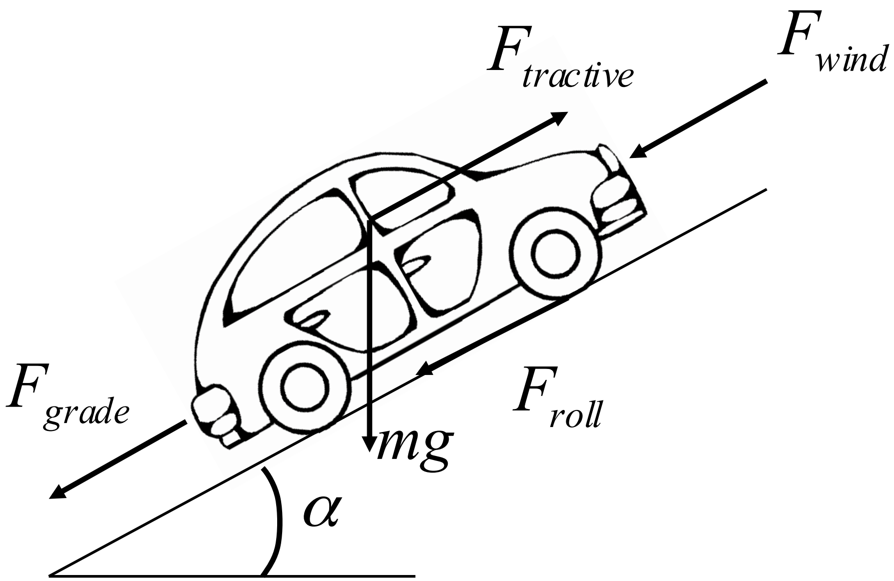
Spectra and linear systems. Example

Let *w(t)* be a stationary stochastic process. *w(t)* is generated from the time discrete relation

$$w(t) - 0.4w(t-1) = e(t) - 0.5e(t-1)$$

where *e(t)* is white noise with variance λ. Determine the spectrum of *w(t)*

Example. Vehicle Dynamics



$$m\dot{v} = F_{tractive} - F_{roll} - F_{wind} - F_{grade}$$

with

$$F_{roll} = kmg\,\mathrm{sign}(v)$$

$$F_{wind} = \frac{1}{2}\rho C_x A\left(v - v_{wind}\right)^2$$

$$F_{grade} = mg\sin\alpha$$

The road grade might be available (e.g., through GPS and maps).

Wind velocity is not

Road slope spectra