ESS101- Modeling and Simulation
Lecture 14

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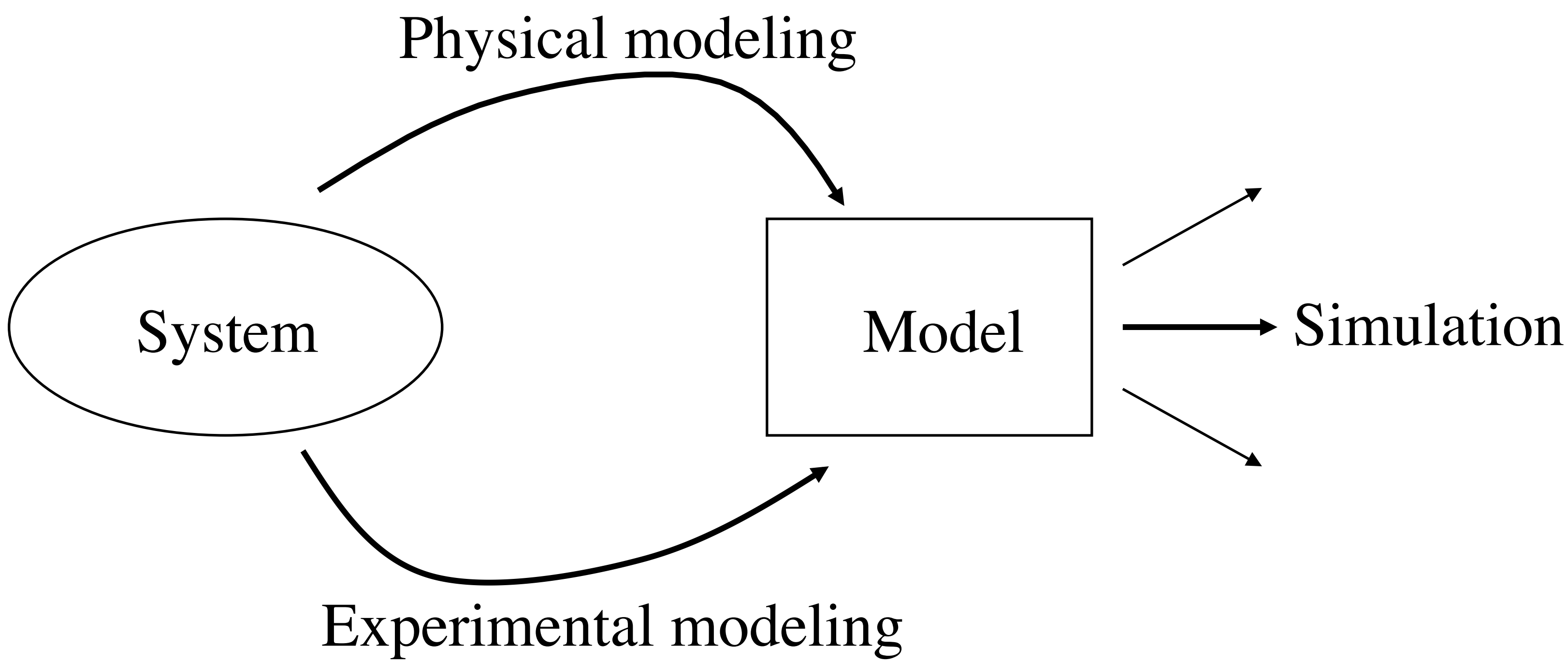
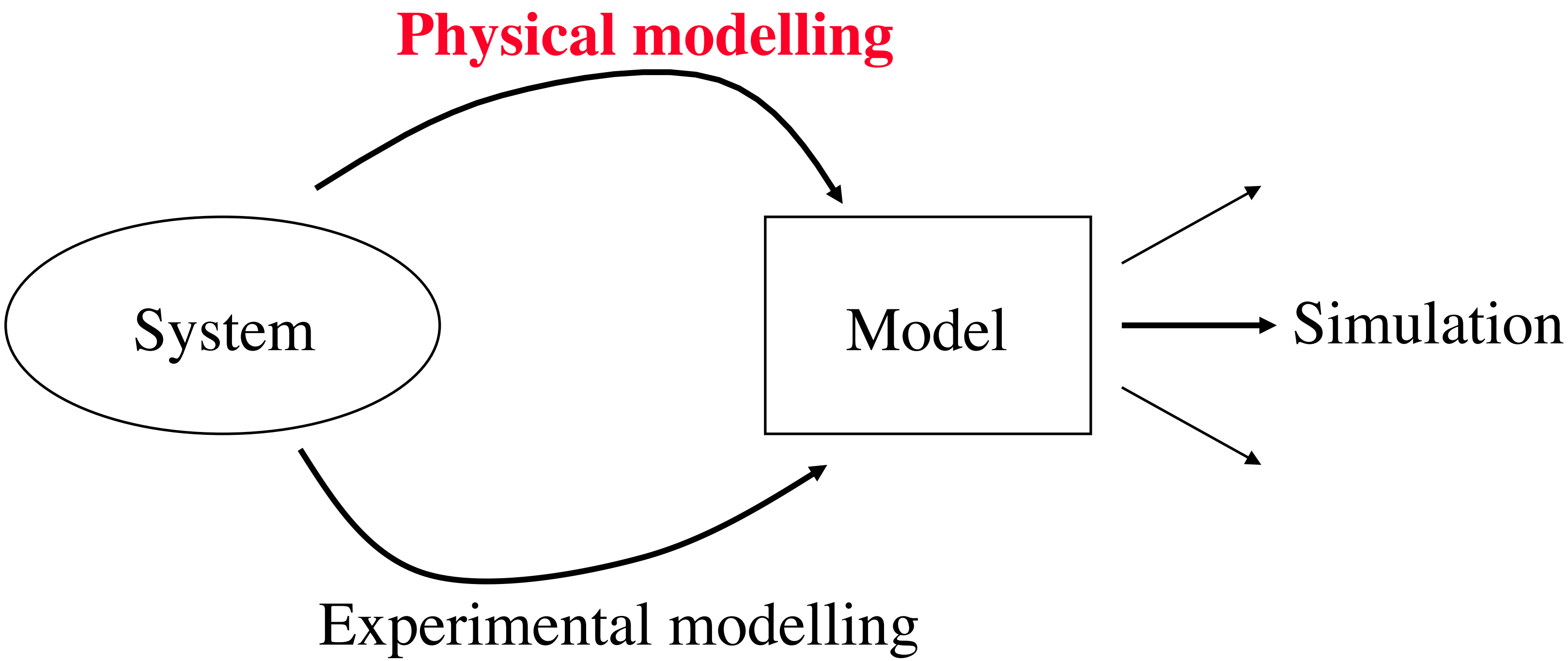
Today (Chapter 2, 3)

- Mathematical models. Summary
- Initial Value Problem (IVP) statement
- Euler method
  - Global and local error
  - Accuracy
  - Convergence
- Taylor series method
  - Accuracy
  - Convergence

Overview of the course (*from first lecture*)

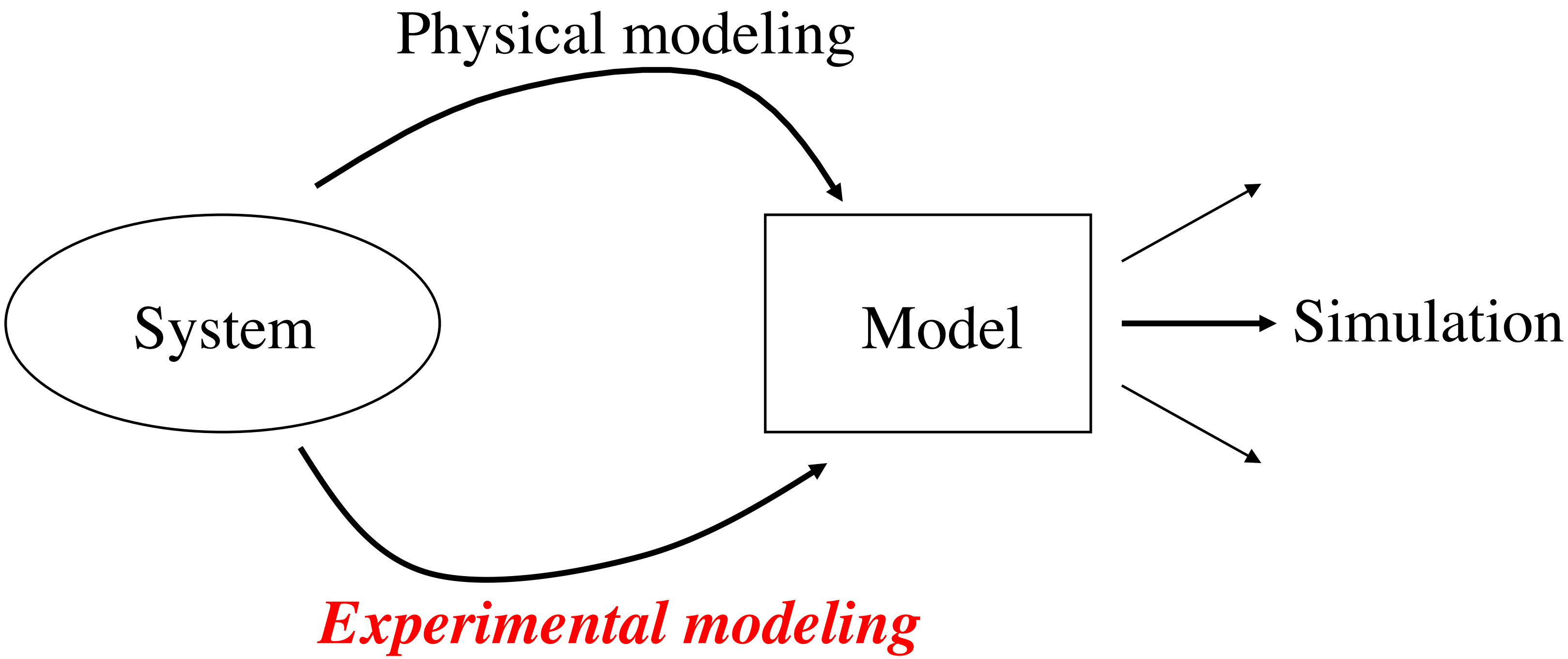


Physical modeling

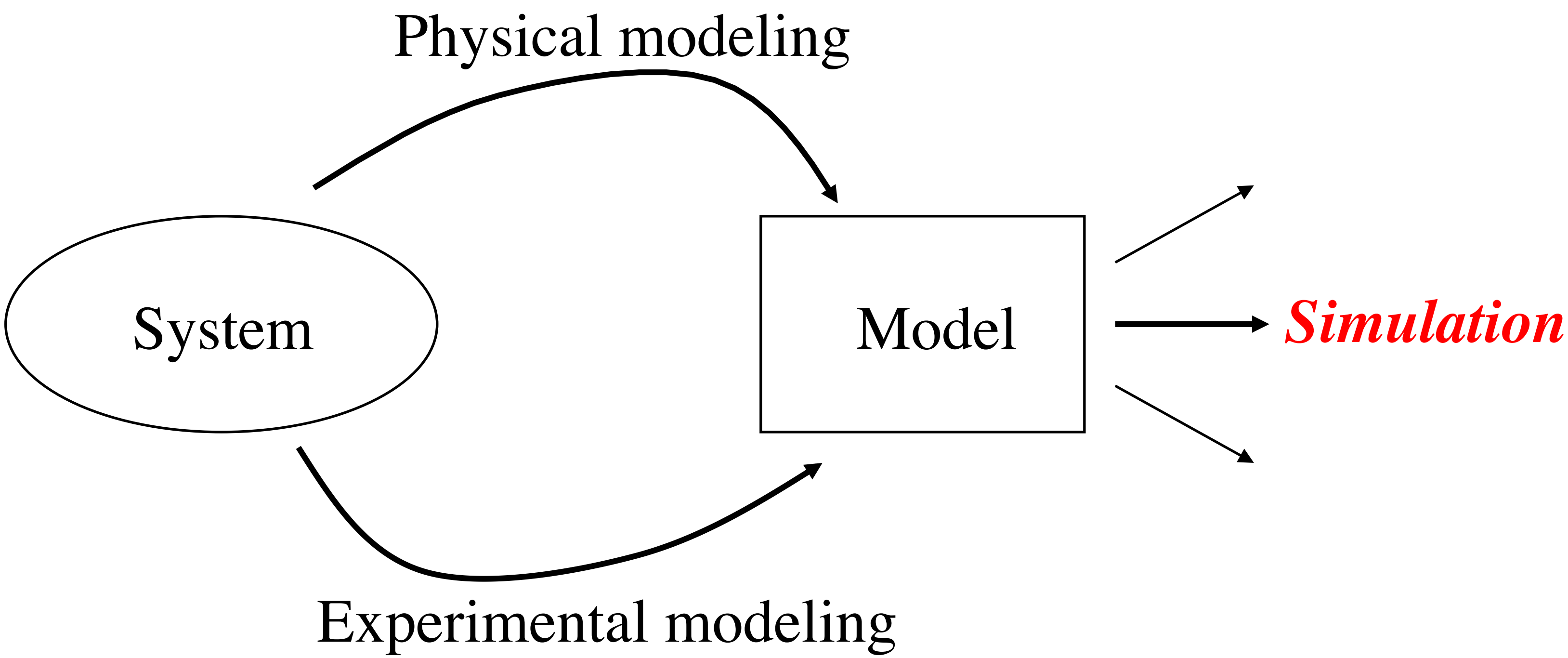


- ✓ Physical modeling
- ✓ Three phases method

System identification



In summary…



Regardless the path we take, we end up in a mathematical model in the form $\dot{x}(t) = f(t, x(t), u(t))$

Basics on simulation (IVP)

We assume the mathematical model is in the form

$$\dot{x}(t) = f(t, x(t), u(t))$$

By assuming that $u(t)$ is a known function of the time

$$\dot{x}(t) = f(t, x(t))$$

The problem of simulating the model $\dot{x} = f(t, x(t))$ can be formulated as follows

Given the initial condition $x(0)$, finding a sequence of points $x_1, x_2, \ldots x_{tf}$ approximating the solution x at the time instants $0 < t_1 < t_2 < \ldots < t_f$, i.e., the $x(t_1), x(t_2), \ldots, x(t_f)$

Example. Euler method for the IVP

Simulate the system

$$\dot{x}(t) = (1-2t)x(t), \quad x(0) = 1$$

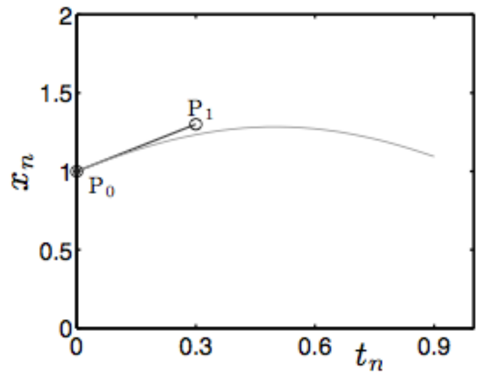
over the time interval $0 \le t \le 0.9$

In this simple case we know the ***exact solution***

$$x(t) = e^{\frac{1}{4}-\left(\frac{1}{2}-t\right)^2}$$

At time ***t=0*** $x(0)=1$ and $\dot{x}(0)=1$. Hence, the tangent to the solution x is $x(t)=1+t$

By setting $t_1=h=0.3$ $\quad x_1 = 1.3 \approx x(t_1)$



Example

In summary

Step n=0

$$t_0 = 0,$$
$$x_0 = 1,$$
$$\dot{x}_0 = 1$$

Step n=1

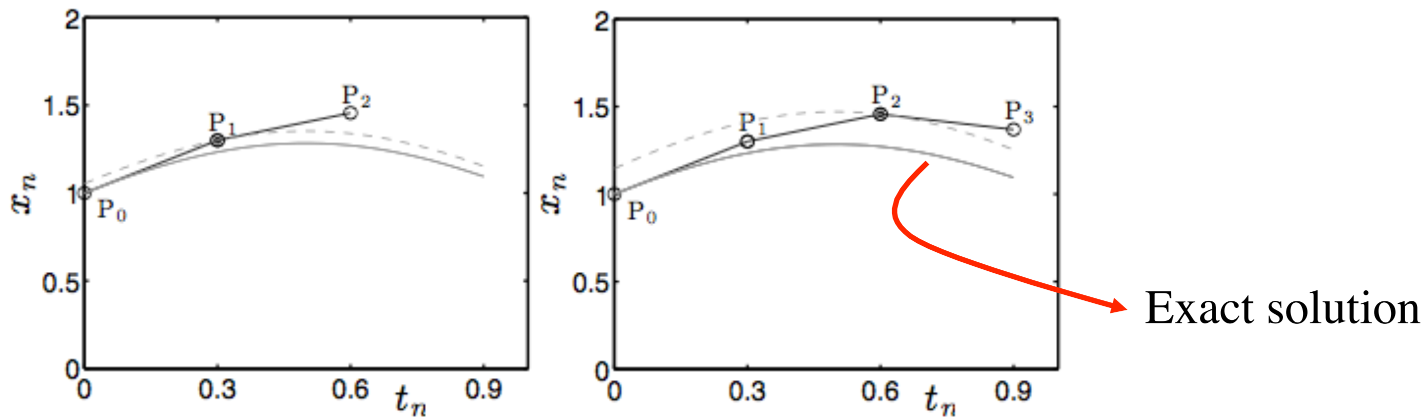
$$t_1 = 0.3 = t_0 + h,$$
$$x_1 = x_0 + h\dot{x}_0 = 1.3,$$
$$\dot{x}_1 = (1 - 2t_1)x_1 = 0.52$$

Step n=2

$$t_2 = 0.6 = t_1 + h,$$
$$x_2 = x_1 + h\dot{x}_1 = 1.456,$$
$$\dot{x}_2 = (1 - 2t_2)x_2 = -0.2912$$

Step n=3

$$t_3 = 0.9 = t_2 + h,$$
$$x_3 = x_2 + h\dot{x}_2 = 1.3686,$$
$$\dot{x}_3 = (1 - 2t_3)x_3 = -1.0949$$



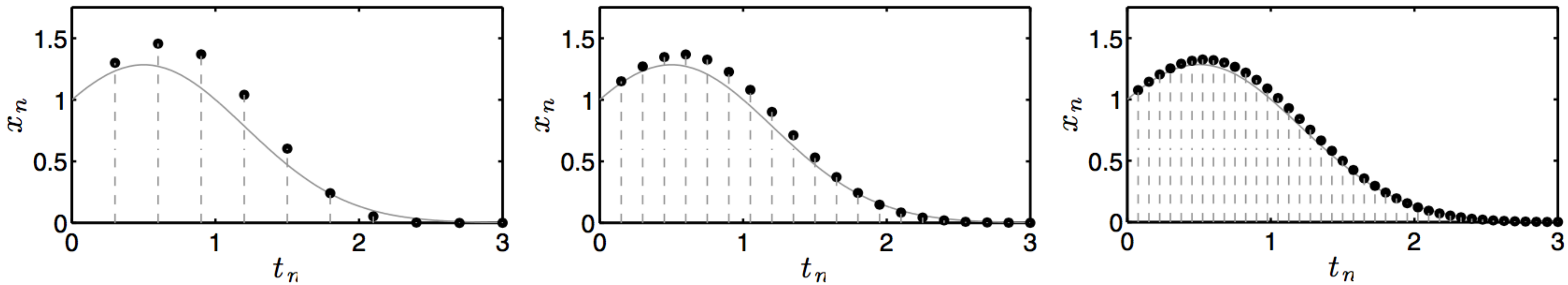
Example

In general ***(Euler method)***

$$t_{n+1} = t_n + h,$$

$$x_{n+1} = x_n + h\dot{x}_n$$

What happen if we reduce the step size h?



1. More iterations necessary (higher computational complexity)
2. More accurate solution (closer to the exact)

Accuracy

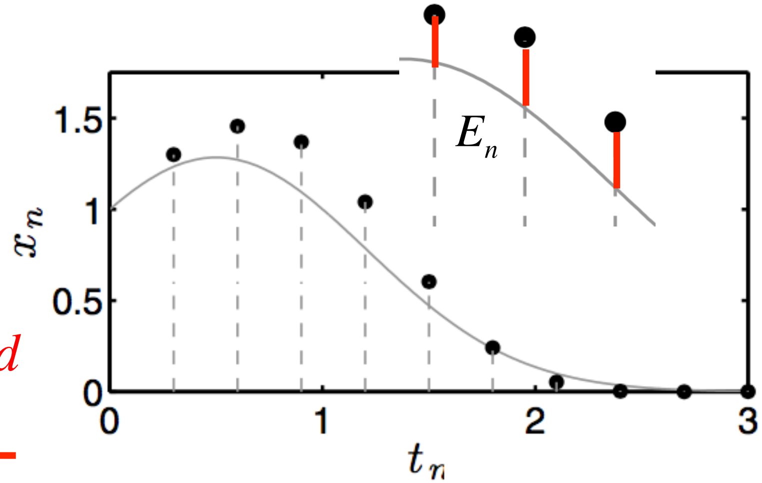
The ***accuracy*** of an ODE solver is given by the

Global Error (GE)

$$E_n = x(t_n) - x_n$$

Real value

Approximated value



Local Error (LE)

$$e_n = x(t_n) - z_n$$

with

$$z_n = G(t, x(t_{n-k}), x(t_{n-k+1}), \ldots, z_n)$$

Example

In general, for the Euler method

$$x(t_n) - x_n \propto h$$

What does this mean?

h	x_n	Global errors (GEs)	GE/h
0.3	$x_3 = 1.3686$	$x(0.9) - x_3 = -0.2745$	-0.91
0.15	$x_6 = 1.2267$	$x(0.9) - x_6 = -0.1325$	-0.89
0.075	$x_{12} = 1.1591$	$x(0.9) - x_{12} = -0.0649$	-0.86
Exact	$x(0.9) = 1.0942$		

$$E_n \cong 0.9h$$

If we require at least three decimal digit exacts

$$|E_n| < 10^{-4} \qquad \Rightarrow h < 10^{-5} \qquad \Rightarrow n = 0.9/h \approx 1620 \text{ steps}$$

Landau notation

We will make extensive use of the following notation

$$O(h^p) \qquad \text{(big-O or Landau symbols)}$$

The notation

$$z = O(h^p) \qquad \text{(reads as "}z\text{ is of order }p\text{")}$$

means $\exists\, C > 0,\ h_0 > 0 : \ |z| \le C h^p, \ \ 0 < h < h_0$

***Example*.** McLaurin expansion of e^h

$$e^h = 1 + h + \frac{1}{2!}h^2 + \frac{1}{3!}h^3 + \cdots + \frac{1}{n!}h^n + \cdots$$

$$e^h = 1 + O(h)$$

$$e^h = 1 + h + O(h^2) \qquad e^h = 1 + h + \frac{1}{2!}h^2 + O(h^3)$$

Convergence

Definition (Convergence). A numerical method converges to the solution *x(t)* of a IVP at $t_n = t^*$ if

$$|E_n| \to 0, \text{ as } h \to 0$$

The method converges with a *p*-th order rate if

$$E_n = O(h^p), \text{ for some } p > 0$$

It can be shown that (see pp. 27-28)

Theorem (Convergence of Euler method). Euler's method, applied to the IVP

$$\dot{x}(t) = \lambda x(t) + g(t), \ 0 < t \le t_f, \ x(0) = 1$$

where *g* is continuously differentiable, converges and the GE is *O(h)*

Example 2.5

Calculate an approximate solution at time *t=0.2* of the problem

$$\ddot{x}(t)+x(t)=t,\ x(0)=1,\ \dot{x}(0)=2$$

with the FE method and a step length *h=0.2*.

Rewrite the system as $\begin{cases} \dot{x}^1 = x^2, \\ \dot{x}^2 = t - x^1, \\ x^1(0)=1,\ x^2(0)=2 \end{cases}$

The FE iteration becomes $\begin{cases} x^1_{n+1} = x^1_n + hx^2_n, \\ x^2_{n+1} = x^2_n + ht_n - hx^1_n, \\ t_{n+1} = t_n + h, \\ x^1_0 = 1,\ x^2_0 = 2 \end{cases}$

Taylor series method

In Euler method, the solution *x(t)* of

$$\dot{x}(t) = f(t, x(t))$$

has been derived from the Taylor series

$$x(t+h) = x(t) + h \underbrace{\dot{x}(t)}_{f(t,x(t))} + \frac{1}{2!} h^2 \ddot{x}(t) + O(h^3)$$

by neglecting the last two terms

$$x(t+h) \approx x(t) + h \underbrace{\dot{x}(t)}_{f(t,x(t))}$$

Why don't we include one more term? Does it get better?

Example. Order-two Taylor series method (TS(2))

Simulate the system

$$\dot{x}(t) = (1-2t)x(t), \quad x(0) = 1$$

over the time interval $0 \le t \le 0.9$

Recall the ***exact solution*** $x(t) = e^{\frac{1}{4}-\left(\frac{1}{2}-t\right)^2}$

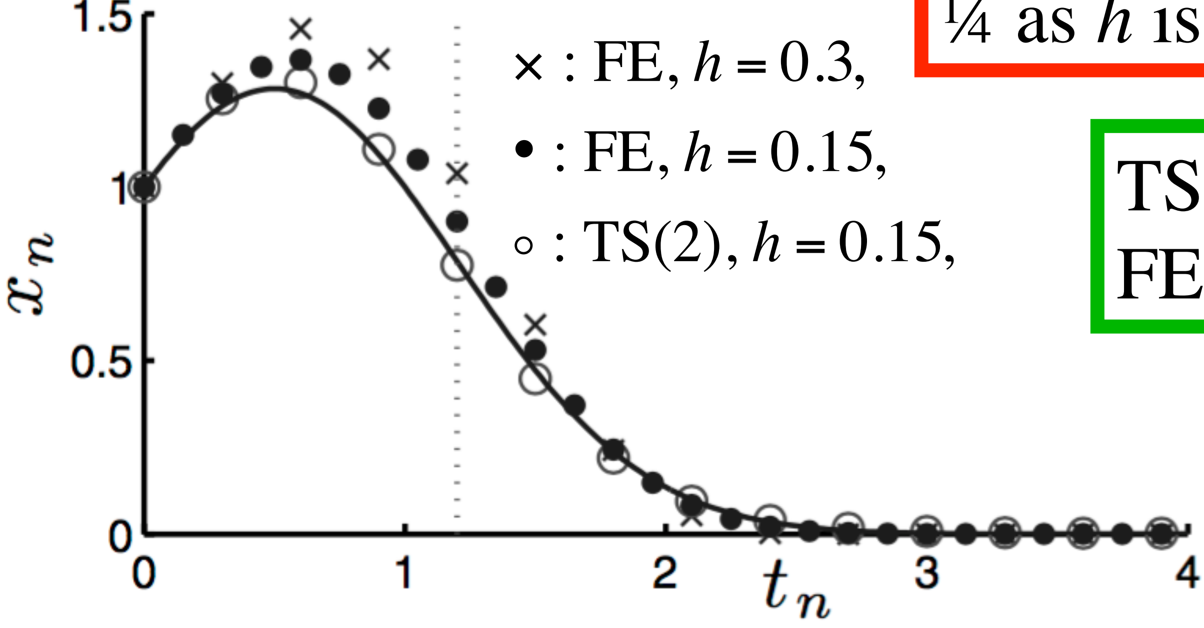
Differentiate the expression of $\dot{x}(t)$ to calculate $\ddot{x}(t)$

$$\ddot{x}(t) = \left[\left(1-2t\right)^2 - 2\right]x(t)$$

The TS(2) iteration results in

$$\underbrace{x_{n+1}}_{\approx x(t_n+h)} = x_n + h\underbrace{\left(1-2t_n\right)x_n}_{\dot{x}(t_n)} + \frac{1}{2}h^2\underbrace{\left[\left(1-2t_n\right)^2 - 2\right]}_{\ddot{x}(t_n)}x_n$$

Example. Order-two Taylor series method (TS(2))



GE for TS(2) reduced of ¼ as h is halved $\Rightarrow \underbrace{GE}_{x(t_n)-x_n} \propto h^2$

TS(2) $E_n \approx 0.14h^2$
FE $E_n \approx -0.77h$

For accuracy of **0.01**

TS(2) 5 steps
FE 92 steps

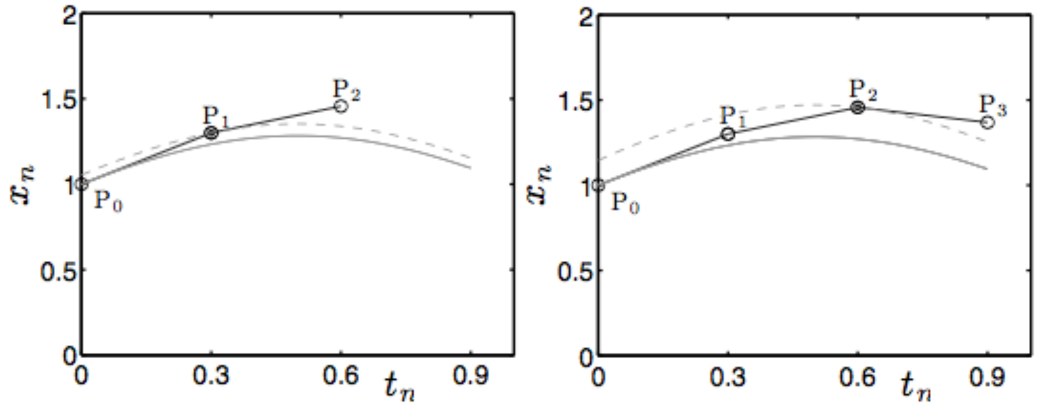
$x(1.2) = 0.7866$

	Solutions at $t = 1.2$		GEs at $t = 1.2$		
h	Euler: TS(1)	TS(2)	Euler: TS(1)	TS(2)	GE for TS(2)/h^2
0.30	1.0402	0.7748	−0.2535	0.0118	0.131
0.15	0.9014	0.7836	−0.1148	0.0031	0.138

Remark on TS(2)

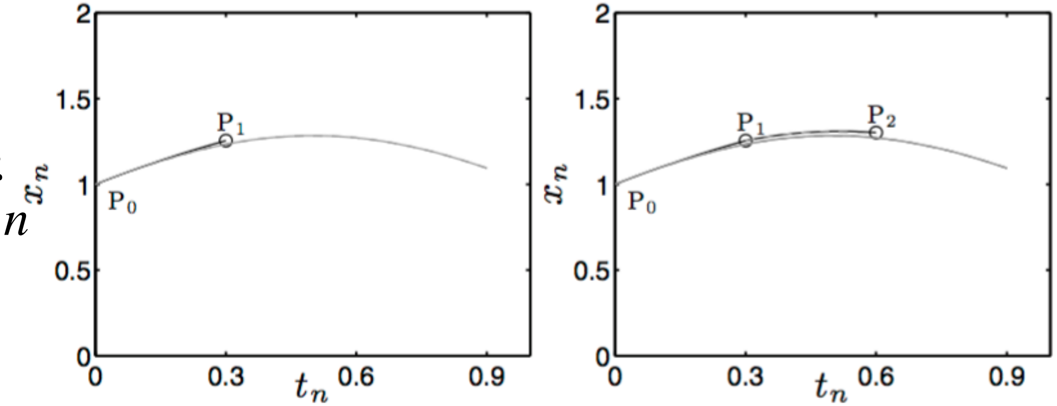
For FE we found that

$$x_{n+1} = x_n + h\dot{x}_n$$



For TS(2) we found that

$$x_{n+1} = x_n + h\dot{x}_n + \frac{1}{2}h^2\ddot{x}_n$$



Convergence of TS(p)

TS(2) can be extended to any *p*th-order TS method (TS(p))

Theorem (Convergence of TS(p)). The TS(p) method, applied to the IVP

$$\dot{x}(t) = \lambda x(t) + g(t),\ 0 < t \le t_f,\ x(0) = 1$$

where g is ***p times*** continuously differentiable, converges and the GE is $O(h^p)$

Conclusion

For FE

$$x_{n+1} = x_n + h\dot{x}_n$$

$$E_n = O(h)$$

For TS(p)

$$x_{n+1} = x_n + \sum_{i=1}^{p} \frac{1}{i!} h^p x_n^{(p)}$$

$$E_n = O(h^p)$$

But, TS(p) methods require high order derivative of the rhs