Practicality of complex numbers

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This tutorial aims at explaining how we may transmit both the in-phase and the quadrature-phase information using a real signal. We will denote the baseband signal by $x(t)$ and the passband signal by $s(t)$.

1. Transmitter Side

From lecture 4, you know that when up-converting the baseband signal to a passband signal, a very common way to do so is by multiplying the in-phase and quadrature-phase components by $\sqrt{2}\cos(2\pi f_\mathrm{c} t)$ and $\sqrt{2}\sin(2\pi f_\mathrm{c} t)$, respectively, and then add. This will result in the signal

$$s(t) = \sum_{k=0}^{N-1} a_k v(t - kT_\mathrm{s})\,\sqrt{2}\cos(2\pi f_\mathrm{c} t) + b_k v(t - kT_\mathrm{s})\,\sqrt{2}\sin(2\pi f_\mathrm{c} t) \tag{1}$$

where N is the number of symbols to be transmitted. Now one might ask oneself, **why do we bother using imaginary numbers when we are only transmitting a real signal?** It all comes down to practicality. Indeed, using complex numbers simplifies notation and implementation.

We note that we can represent the signal of a 2-dimensional constellation by a complex signal, $x(t)$, given as

$$x(t) = \sum_{k=0}^{N-1} a_k v(t - kT_\mathrm{s}) + i b_k v(t - kT_\mathrm{s}) \tag{2}$$

where a_k and b_k are the in-phase and quadrature-phase components of the kth symbol, respectively. In the uploaded solution of computer exercise 2, you see that we obtain the passband signal as

$$\begin{aligned}
s(t) &= \Re\Big(x(t)\,\sqrt{2}e^{-i2\pi f_\mathrm{c}}\Big) && (3)\\
&= \sqrt{2}\Re\left(\left(\sum_{k=0}^{N-1} a_k v(t - kT_\mathrm{s}) + i b_k v(t - kT_\mathrm{s})\right)\left(\cos(2\pi f_\mathrm{c} t) - i\sin(2\pi f_\mathrm{c} t)\right)\right) && (4)\\
&= \sqrt{2}\sum_{k=0}^{N-1} \Re((a_k v(t - kT_\mathrm{s})\cos(2\pi f_\mathrm{c} t) + b_k v(t - kT_\mathrm{s})\sin(2\pi f_\mathrm{c} t)) - i(a_k v(t - kT_\mathrm{s})\sin(2\pi f_\mathrm{c} t) - b_k v(t - kT_\mathrm{s})\cos(2\pi f_\mathrm{c} t))) \\
&= \sqrt{2}\sum_{k=0}^{N-1} a_k v(t - kT_\mathrm{s})\cos(2\pi f_\mathrm{c} t) + b_k v(t - kT_\mathrm{s})\sin(2\pi f_\mathrm{c} t) && (5)
\end{aligned}$$

which is the same expression as in (1). Hence, instead of carrying out two parallel streams of computations, we may use only one but now, all the computations are complex.

2. Receiver Side

So how do we again turn our received signal into baseband? Recall that we want to multiply our received signal by a cosine and a sine and then perform low-pass filtering. It turns out that this can also be done very efficiently with complex numbers! Let's pretend that the received obtained is noise-less, hence we receive $s(t)$. Now, we perform a similar operation at the receiver side as in the transmitter side as

$$
\begin{aligned}
\hat{x}(t) &= s(t)\,\sqrt{2}e^{i2\pi f_{\mathrm{c}}t} && (6)\\
&= \left[\sqrt{2}\sum_{k=0}^{N-1} a_k v(t-kT_{\mathrm{s}})\cos(2\pi f_{\mathrm{c}}t) + b_k v(t-kT_{\mathrm{s}})\sin(2\pi f_{\mathrm{c}}t)\right]\left[\sqrt{2}\cos(2\pi f_{\mathrm{c}}t) + i\sqrt{2}\sin(2\pi f_{\mathrm{c}}t)\right] && (7)\\
&= 2\sum_{k=0}^{N-1} a_k v(t-kT_{\mathrm{s}})\cos^2(2\pi f_{\mathrm{c}}t) + b_k v(t-kT_{\mathrm{s}})\sin(2\pi f_{\mathrm{c}}t)\cos(2\pi f_{\mathrm{c}}t)\\
&\quad +2i\left(\sum_{k=0}^{N-1} a_k v(t-kT_{\mathrm{s}})\cos(2\pi f_{\mathrm{c}}t)\sin(2\pi f_{\mathrm{c}}t) + b_k v(t-kT_{\mathrm{s}})\sin^2(2\pi f_{\mathrm{c}}t)\right) && (8)\\
&= 2\sum_{k=0}^{N-1} a_k v(t-kT_{\mathrm{s}})\,\frac{1}{2}(1+\cos(4\pi f_{\mathrm{c}}t)) + b_k v(t-kT_{\mathrm{s}})\,\frac{1}{2}\sin(4\pi f_{\mathrm{c}}t)\\
&\quad +2i\left(\sum_{k=0}^{N-1} a_k v(t-kT_{\mathrm{s}})\,\frac{1}{2}\sin(4\pi f_{\mathrm{c}}t) + b_k v(t-kT_{\mathrm{s}})\,\frac{1}{2}(1-\cos(4\pi f_{\mathrm{c}}t))\right) && (9)
\end{aligned}
$$

where in the final step, we made use of trigonometric identities for $\cos^2(\cdot)$, $\sin^2(\cdot)$ and $\cos(\cdot)\sin(\cdot)$. We note that $\hat{x}(t)$ now consists of signals with frequencies centered around frequencies $f=0$ and around $f=2f_{\mathrm{c}}$. If we apply a low-pass filter to get rid of the $2f_{\mathrm{c}}$-component, we finally arrive at

$$
\hat{x}(t) = \sum_{k=0}^{N-1} a_k v(t-kT_{\mathrm{s}}) + i b_k v(t-kT_{\mathrm{s}})\,. \qquad (10)
$$

3. Conclusion

So just to summarize. Initially we wanted to transmit our symbols using two orthogonal signals (cosine and a sine) at the same frequency. As shown in this document, this can be very efficiently done by exploiting complex numbers to represent two orthogonal signals. The advantage is that instead of processing two signals separately, we only have to consider one complex signal.

Note that the signal that goes over the channel is REAL, we cannot transmit complex signals!