Formula sheet, SSY121

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This sheet is an allowed aid at written exams in SSY121, Introduction to Communication Engineering, at Chalmers in 2016. It will be handed out with the exam problems. Students may not bring their own copy.

Decibels

$$\left(\frac{E_1}{E_2}\right)_{\mathrm{dB}} = 10\log_{10}\frac{E_1}{E_2}$$

Energies E_s and E_b

$$E_\mathrm{s} = \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S} = \mathsf{s}_i\right] ||\mathsf{s}_i||^2$$

$$E_\mathrm{s} = E_\mathrm{b}\log_2 M$$

Normalized minimum distance

$$d_{\min} = \frac{D_{\min}}{\sqrt{2E_\mathrm{b}}}$$

Nyquist criterion

- In time domain
$$v(nT_\mathrm{s}) = 0, \quad n = \pm1, \pm2, \ldots$$
- In frequency domain
$$\sum_{n=-\infty}^{\infty} \Re\left\{V\left(f - \frac{n}{T_\mathrm{s}}\right)\right\} = T_\mathrm{s}v(0)$$
$$\sum_{n=-\infty}^{\infty} \Im\left\{V\left(f - \frac{n}{T_\mathrm{s}}\right)\right\} = 0,$$
where $T_\mathrm{s}v(0)$ is a real constant.
- If the $v(t)$ is symmetric respect to zero, the definition in frequency domain is
$$\sum_{n=-\infty}^{\infty} V\left(f - \frac{n}{T_\mathrm{s}}\right) = T_\mathrm{s}v(0)$$

T_s-orthogonality

- In time domain
$$\int_{-\infty}^{\infty} v(t)v(t-nT_\mathrm{s})dt = 0, \quad n = \pm1, \pm2, \ldots$$
- In frequency domain
$$\sum_{n=-\infty}^{\infty} \left|V\left(f - \frac{n}{T_\mathrm{s}}\right)\right|^2 = T_\mathrm{s}E_v$$

Sinc, Raised-cosine, and Root raised-cosine pulses

$$v_{\mathrm{sinc}}(t) = \mathrm{sinc}(t/T_\mathrm{p}) = \frac{\sin(\pi t/T_\mathrm{p})}{\pi t/T_\mathrm{p}}$$

$$V_{\mathrm{sinc}}(f) = \begin{cases} T_\mathrm{p}, & |f| < \frac{1}{2T_\mathrm{p}} \\ 0, & |f| \geq \frac{1}{2T_\mathrm{p}} \end{cases}$$

$$v_{\mathrm{RC}}(t) = \mathrm{sinc}\left(\frac{t}{T_\mathrm{p}}\right)\frac{\cos\left(\frac{\pi\alpha t}{T_\mathrm{p}}\right)}{1-\left(\frac{2\alpha t}{T_\mathrm{p}}\right)^2}$$

$$V_{\mathrm{RC}}(f) = \begin{cases} T_\mathrm{p}, & |f| < f_1 \\ \frac{T_\mathrm{p}}{2}\left(1+\cos\left[\frac{\pi T_\mathrm{p}}{\alpha}\left(|f| - \frac{1-\alpha}{2T_\mathrm{p}}\right)\right]\right), & f_1 \leq |f| < f_2 \\ 0, & |f| \geq f_2, \end{cases}$$

where $f_1 = \dfrac{1-\alpha}{2T_\mathrm{p}}$ and $f_2 = \dfrac{1+\alpha}{2T_\mathrm{p}}$

$$v_{\mathrm{RRC}}(t) = \sqrt{T_\mathrm{p}}\frac{\sin\left(\frac{(1-\alpha)\pi t}{T_\mathrm{p}}\right) + \frac{4\alpha t}{T_\mathrm{p}}\cos\left(\frac{(1+\alpha)\pi t}{T_\mathrm{p}}\right)}{\pi t\left(1-\left(\frac{4\alpha t}{T_\mathrm{p}}\right)^2\right)}$$

$$V_{\mathrm{RRC}}(f) = \sqrt{V_{\mathrm{RC}}(f)}$$

Correlation receiver

$$\min_i\left\{\int_{-\infty}^{\infty}[y(t) - s_i(t)]^2\,dt\right\}$$
$$\max_i\left\{\int_{-\infty}^{\infty}y(t)s_i(t)\,dt - \frac{E_{s_i}}{2}\right\},$$

where $E_{s_i} = \int_{-\infty}^{\infty} s_i^2(t)\,dt$.

PAM (baseband)

$$s(t) = \sum_{k=0}^{\infty} a_k v(t-kT_\mathrm{s}),$$

where $a_k \in \{\pm(M-1), \pm(M-3), \ldots, \pm1\}$.

PAM (passband)

$$s(t) = \sum_{k=1}^{\infty} a_k v(t-kT_\mathrm{s})\sqrt{2}\cos w_c t,$$

where $a_k \in \{\pm(M-1), \pm(M-3), \ldots, \pm1\}$

2D Modulations

$$s(t) = \sum_{k=0}^{\infty} a_k v(t-kT_\mathrm{s})\sqrt{2}\cos(w_c t) - \sum_{k=0}^{\infty} b_k v(t-kT_\mathrm{s})\sqrt{2}\sin(w_c t)$$

M-PSK

$$s(t)=\sum_{k=0}^{\infty} v(t-kT_\mathrm{s})\sqrt{2}\cos\left(w_c t+\frac{2i\pi}{M}\right),$$

where $i=0,1,\ldots,M-1$.

M-FSK

$$s_i(t)=\cos\left(2\pi\left[f_c+\frac{h}{2T_\mathrm{s}}i\right]t\right),$$

where $i=\pm(M-1),\pm(M-3),\ldots,\pm1$.

Link budget

$$P_\mathrm{R}=P_\mathrm{T}G_\mathrm{T}G_\mathrm{R}\left(\frac{\lambda}{4\pi d}\right)^2,$$

where $c=\lambda f=3\cdot10^8\mathrm{m/s}$.

Parabolic dish antenna

$$G_\mathrm{Par}=\frac{4\pi A}{\lambda^2}$$

1D Gaussian PDF (i.i.d.)

$$f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

ND Gaussian PDF with variance σ^2 (i.i.d.)

$$f_\mathsf{Z}(\mathsf{z})=\frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left(-\frac{||\mathsf{z}-\boldsymbol{\mu}||^2}{2\sigma^2}\right)$$

Bayes' rule

$$f_{\mathsf{X}|\mathsf{Y}}(\mathsf{x}|\mathsf{y})=\frac{f_\mathsf{X}(\mathsf{x})}{f_\mathsf{Y}(\mathsf{y})}f_{\mathsf{Y}|\mathsf{X}}(\mathsf{y}|\mathsf{x})$$

Additive white Gaussian noise (AWGN)

The following formulas are for the AWGN channel $\mathsf{Y}=\mathsf{S}+\mathsf{Z}$, where S is the transmitted symbol and Z is Gaussian noise.

Maximum likelihood (ML) detection

$$\max_i\{\mathbb{P}[\mathsf{Y}=\mathsf{y}|\mathsf{S}=\mathsf{s}_i]\}$$

Maximum a posteriori (MAP) detection

$$\max_i\{\mathbb{P}[\mathsf{S}=\mathsf{s}_i|\mathsf{Y}=\mathsf{y}]\}\equiv\max_i\{\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\,\mathbb{P}[\mathsf{Y}=\mathsf{y}|\mathsf{S}=\mathsf{s}_i]\}$$

Pairwise error probability (PEP)

$$\mathrm{PEP}^{(i,j)}=\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right),$$

where $D_{i,j}^2=\|\mathsf{s}_j-\mathsf{s}_i\|^2$.

Symbol error probability (SEP) (exact)

$$P_\mathrm{e}=\sum_{i=1}^{M}\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\sum_{j\neq i}\mathbb{P}[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i]$$

SEP (union bound)

$$\begin{aligned}P_\mathrm{e}&\le\sum_{i=1}^{M}\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\sum_{j\neq i}\mathrm{PEP}^{(i,j)}\\&=\sum_{i=1}^{M}\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\sum_{j\neq i}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)\end{aligned}$$

SEP (High-SNR approximation for equally likely symbols)

$$P_\mathrm{e}\approx\frac{2K}{M}\cdot\mathrm{Q}\left(\sqrt{\frac{D_\mathrm{min}^2}{2N_0}}\right),$$

where K is the number of distinct signal pairs with distance $D_\mathrm{min}=\min_{i\neq j}\{D_{i,j}\}$ and M is the constellation size.

Bit error probability (BEP) (exact)

$$B_\mathrm{e}=\sum_{i=1}^{M}\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\sum_{j\neq i}\frac{H_{i,j}}{m}\mathbb{P}[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i],$$

where $H_{i,j}$ is the Hamming distance (the number of different bits) between the labels of symbols s_i and s_j.

BEP (union bound)

$$B_\mathrm{e}\le\sum_{i=1}^{M}\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\sum_{j\neq i}\frac{H_{i,j}}{m}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

BEP (High-SNR approximation for equally likely symbols)

$$B_\mathrm{e}\approx\frac{2H_\mathrm{min}}{Mm}\cdot\mathrm{Q}\left(\sqrt{\frac{D_\mathrm{min}^2}{2N_0}}\right),$$

where H_min is the *total* number of bits differing between signal pairs at minimum distance and $m=\log_2(M)$ is the number of bits per symbol.

Q-function

See also tables in Mathematics Handbook, where $\mathrm{Q}(x)=1-\Phi(x)$

$\mathrm{Q}(x)$	x
10^{-1}	1.2816
10^{-2}	2.3263
10^{-3}	3.0902
10^{-4}	3.7190
10^{-5}	4.2649
10^{-6}	4.7534
10^{-7}	5.1993
10^{-8}	5.6120
10^{-9}	5.9978
10^{-10}	6.3613
10^{-11}	6.7060
10^{-12}	7.0345