Introduction to Communication Engineering
SSY121, Lecture # 3

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September 6, 2021

Information

Project Groups

50 students signed up for the project. If you change your mind and decide not to take the course, please let me know as soon as possible!

Deadline for Common Values

Thu Sept 9 at noon! Remember the file name and format!

Student Representatives

As student representative, you are expected to represent yourself and the other students of the course and help us better understand our students' expectations and how they experience our courses.

- TBD
- ...

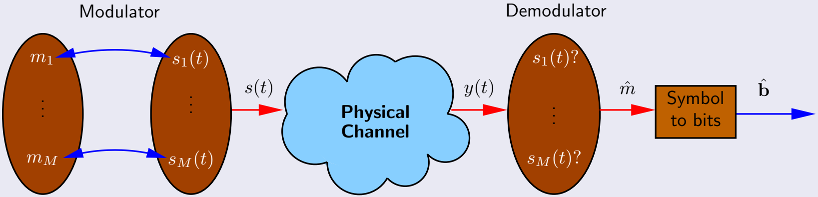
Outline

1. A Short Summary of Last Week
   - The General Problem
   - The CR and the MFR for One Pulse
   - The CR and the MFR for a Train of Pulses
2. Orthogonality Criterion
   - Motivation
   - Definition and Consequences
3. Signal Space
   - Distance Measures for Signals
   - The Gram-Schmidt Orthonormalization
4. Vector Representation of Signals
   - Definitions
   - Theorem of Irrelevance
   - Minimum Distance vs. Correlation
   - Decision Regions
5. Summary

Part I

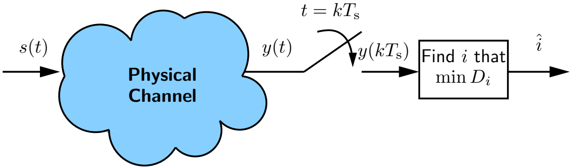
The Linear Receiver and Orthogonal Pulses

The General Problem

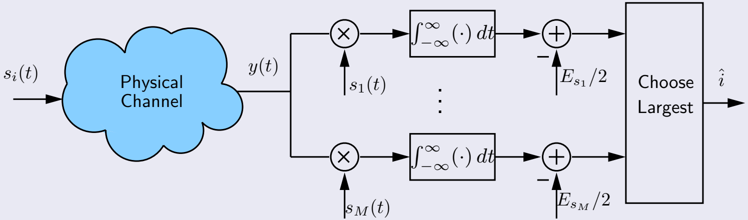


The Sampling Receiver

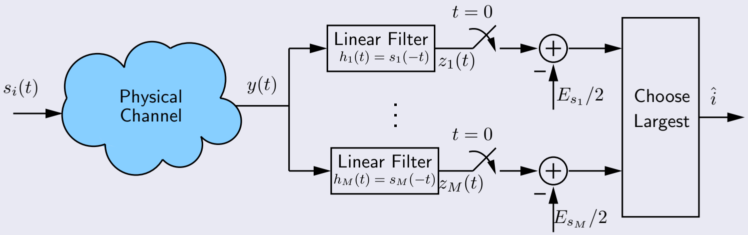
Sample the received signal $y(t)$ at $t = 0, T_\mathrm{s}, 2T_\mathrm{s}, \ldots$, and for each sample, choose the signal alternative that is the closest to $y(kT_\mathrm{s})$, i.e., find the $i \in \{1, \ldots, M\}$ that minimizes $D_i = |y(kT_\mathrm{s}) - s_i(0)|$



The Correlator Receiver (one pulse)



The Matched Filter Receiver (one pulse)

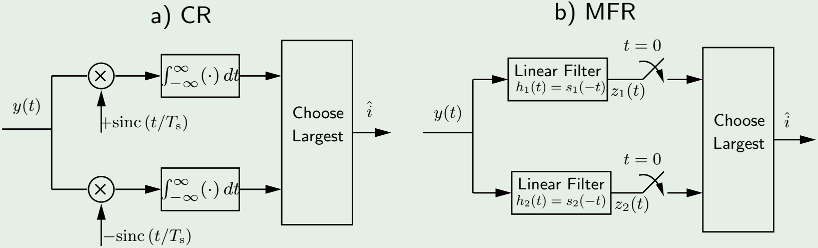


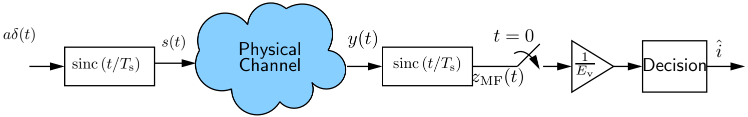
Example (Binary transmission with a sinc pulse)

The signal alternatives are such that

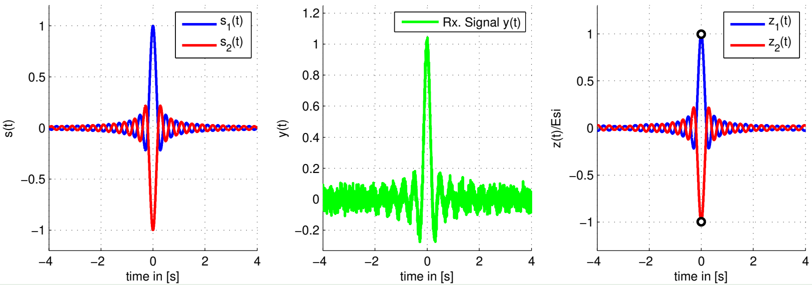
$$b = 0 \Rightarrow m = m_1 \Rightarrow s(t) = s_1(t) = +\operatorname{sinc}(t/T_\mathrm{s})$$
$$b = 1 \Rightarrow m = m_2 \Rightarrow s(t) = s_2(t) = -\operatorname{sinc}(t/T_\mathrm{s})$$

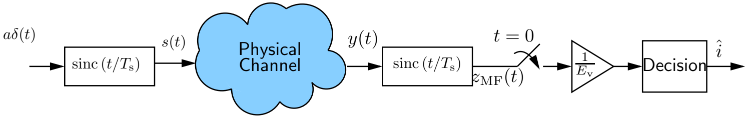
The energies $E_{s_1} = E_{s_2}$, and therefore, the CR/MFR becomes:



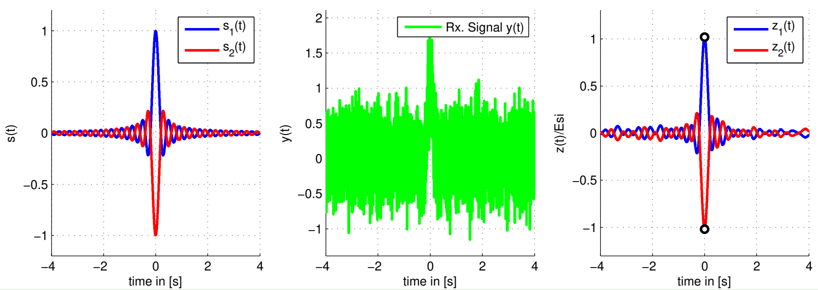


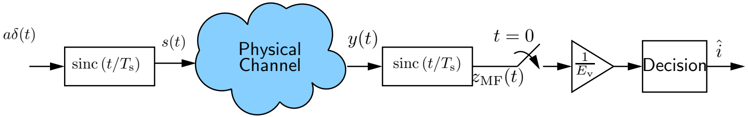
Example (One sinc pulse + a "Good" channel + MFR)



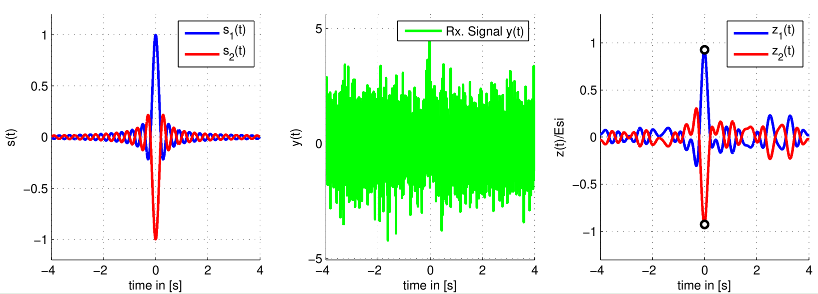


Example (One sinc pulse + a “Bad” channel + MFR)





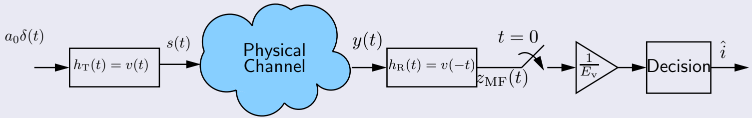
Example (One sinc pulse + a "Really Bad" channel + MFR)



Amazed with the MFR?

- The MFR seems to work perfectly!
- Even if the channel is extremely bad, the results are very reliable!
- Moreover, it is not very complex to implement (linear filters, or "conv" if implemented digitally)
- But...
  - How does it look for a train of pulses?
  - How fast can we transmit?
  - Do we have the ISI problem (like for the Sampling Rx)?

The MFR for one M-ary Pulse



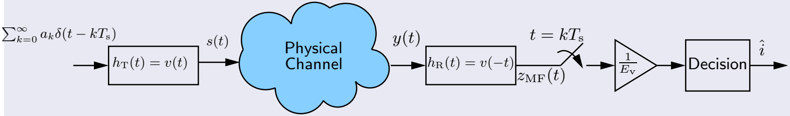
A Sequence of M-ary Pulses

The transmitted signal is

$$s(t) = \sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s})$$
$$= a_0 v(t) + a_1 v(t - T_\mathrm{s}) + a_2 v(t - 2T_\mathrm{s}) + \dots,$$

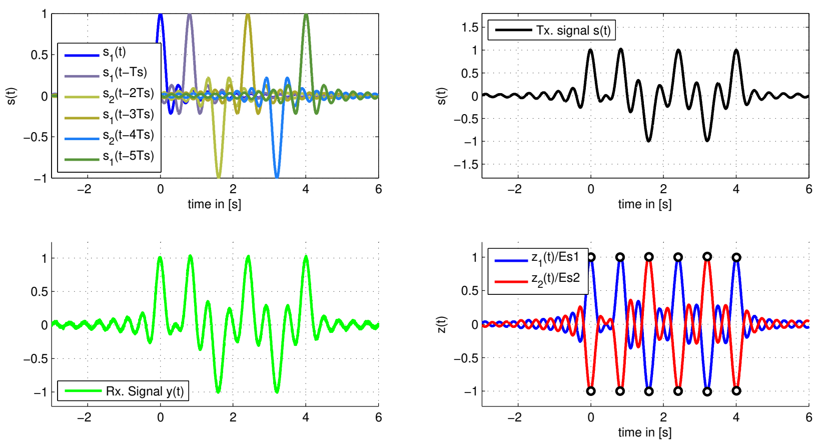
where $a_k \in \mathcal{A}$ represent the amplitude of the symbol transmitted at the kth time instant.

The MFR for This Case?



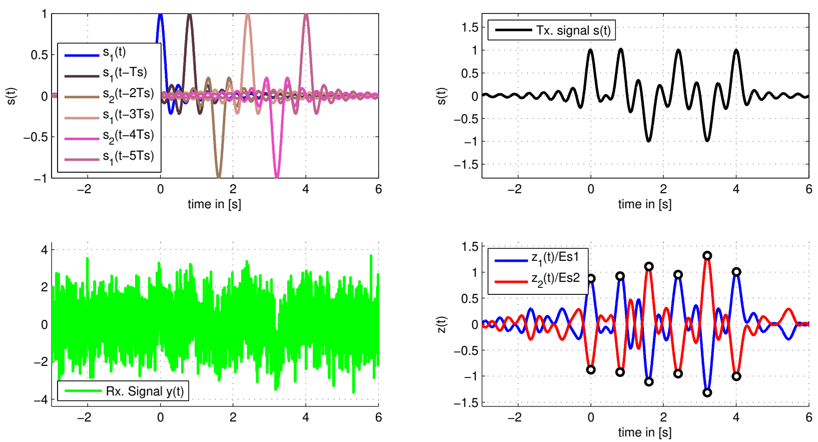
Example (A slow train of sinc pulses and a good channel)

Assume that the sequence $\mathbf{b} = [0, 0, 1, 0, 1, 0]$ needs to be transmitted and binary transmission using pulses $\mathrm{sinc}\,(t/T)$ are used. Assume $T_\mathrm{s} = 4T$.



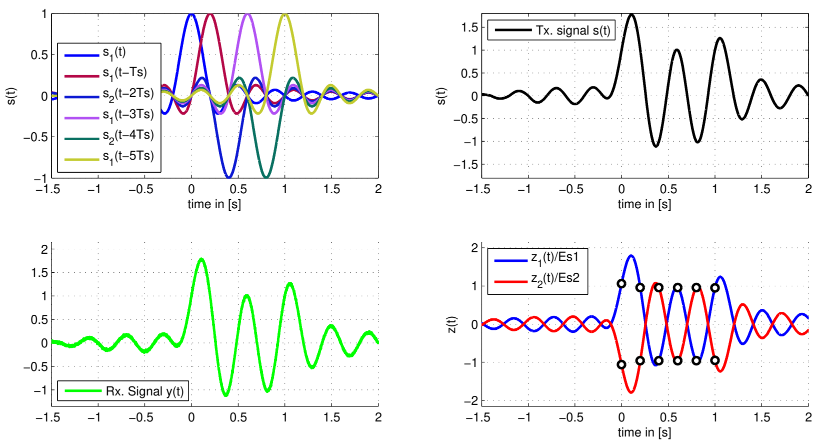
Example (A slow train of sinc pulses and a bad channel)

Assume that the sequence $\mathbf{b} = [0, 0, 1, 0, 1, 0]$ needs to be transmitted and binary transmission using pulses $\mathrm{sinc}\,(t/T)$ are used. Assume $T_\mathrm{s} = 4T$.



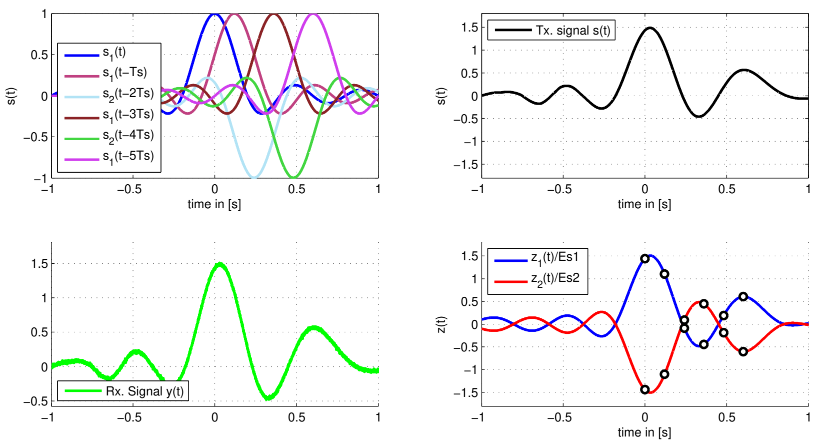
Example (A fast train of sinc pulses and a good channel)

Assume that the sequence $\mathbf{b} = [0, 0, 1, 0, 1, 0]$ needs to be transmitted and binary transmission using pulses $\text{sinc}\,(t/T)$ are used. Now assume a transmission 4 times faster than before, i.e., $T_\text{s} = T$.



Example (A very fast train of sinc pulses and a good channel)

Assume that the sequence $\mathbf{b} = [0, 0, 1, 0, 1, 0]$ needs to be transmitted and binary transmission using pulses $\text{sinc}\,(t/T)$ are used. Now assume a transmission **even faster** than before, i.e., $T_s = 0.6T$.



A Train of M-ary Pulses and the MFR

For simplicity, assume a train of M-ary pulses based on the basic pulse $v(t)$. The transmitted signal is

$$s(t) = \sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s}) = a_0 v(t) + a_1 v(t - T_\mathrm{s}) + a_2 v(t - 2T_\mathrm{s}) + \ldots,$$

where $a_k \in \mathcal{A}$ is the amplitude of the kth transmitted symbol.

The output of the MFR

If the channel is very good ($y(t) \approx s(t)$) the output of the MFR $z(t)$ is the convolution of $s(t)$ and the filter $h_\mathrm{R}(t) = v(-t)$:

$$z(t) = s(t) * v(-t) = \sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s}) * v(-t)$$

$$= \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} a_k v(\tau - kT_\mathrm{s}) v(\tau - t)\, d\tau = \sum_{k=0}^{\infty} a_k \int_{-\infty}^{\infty} v(\tau - kT_\mathrm{s}) v(\tau - t)\, d\tau$$

A Train of M-ary Pulses and the MFR

We would like to study the samples of $z(t)$ at $t = 0, T_\mathrm{s}, 2T_\mathrm{s}, \ldots$, where

$$z(t) = \sum_{k=0}^{\infty} a_k \int_{-\infty}^{\infty} v(\tau - kT_\mathrm{s}) v(\tau - t)\, d\tau$$

For example, take the 3rd sample, i.e., $t = 2T_\mathrm{s}$

$$z(t = 2T_\mathrm{s}) = a_0 \int_{-\infty}^{\infty} v(\tau) v(\tau - 2T_\mathrm{s})\, d\tau + a_1 \int_{-\infty}^{\infty} v(\tau - T_\mathrm{s}) v(\tau - 2T_\mathrm{s})\, d\tau + \\ a_2 \int_{-\infty}^{\infty} [v(\tau - 2T_\mathrm{s})]^2\, d\tau + a_3 \int_{-\infty}^{\infty} v(\tau - 3T_\mathrm{s}) v(\tau - 2T_\mathrm{s})\, d\tau + \ldots$$

And use the change of variables $p = \tau - 2T_\mathrm{s}$ to obtain

$$z(t = 2T_\mathrm{s}) = a_0 \int_{-\infty}^{\infty} v(p + 2T_\mathrm{s}) v(p)\, dp + a_1 \int_{-\infty}^{\infty} v(p + T_\mathrm{s}) v(p)\, dp + \\ a_2 E_v + a_3 \int_{-\infty}^{\infty} v(p - T_\mathrm{s}) v(p)\, dp + \ldots$$

Definition (Orthogonal Pulses in Time Domain)

A pulse $v(t)$ is orthogonal ("T_s-orthogonal" in the book) if

$$\int_{-\infty}^{\infty} v(t)v(t-nT_\mathrm{s})\,dt = 0$$

for $n = \pm 1, \pm 2, \pm 3, \ldots$

Definition (Orthogonal Pulses in Frequency Domain)

A pulse $v(t)$ is orthogonal if (proof on p. 27 in the book)

$$\sum_{n=-\infty}^{\infty} \left| V\left(f - \frac{n}{T_\mathrm{s}}\right)\right|^2 = T_\mathrm{s} E_v.$$

Consequences

- Orthogonal pulses imply zero-ISI for the CR/MFR
- The output of MF at $t = kT_\mathrm{s}$ is $z(t = kT_\mathrm{s}) \approx a_k E_v$ if $y(t) \approx s(t)$.

Definition (Pulses in Time Domain)

$$\text{Nyquist Pulse: } v(nT_\mathrm{s}) = 0$$

$$\text{Orthogonal Pulse: } \int_{-\infty}^{\infty} v(t)v(t - nT_\mathrm{s})\,dt = 0$$

if $n = \pm 1, \pm 2, \pm 3, \ldots$

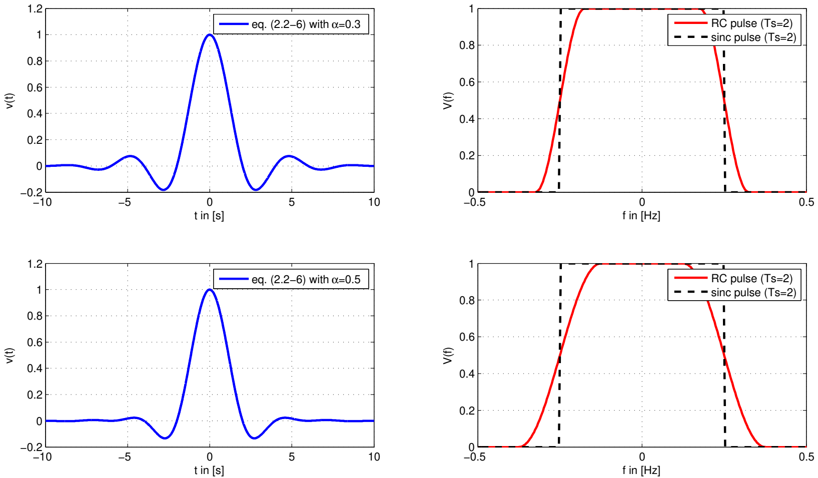
Definition (Pulses in Frequency Domain)

If $v(t)$ is symmetric around $t = 0$

$$\text{Nyquist Pulse: } \sum_{n=-\infty}^{\infty} V\left(f - \frac{n}{T_\mathrm{s}}\right) = T_\mathrm{s} v(0)$$

$$\text{Orthogonal Pulse: } \sum_{n=-\infty}^{\infty} \left|V\left(f - \frac{n}{T_\mathrm{s}}\right)\right|^2 = T_\mathrm{s} E_v.$$

Is the Raised-Cosine (RC) Pulse Orthogonal?



Some HWs

- The root-raised cosine (RRC) pulse is presented and studied in the book. Why is it so important? Is it a Nyquist pulse? Is it an orthogonal pulse?
- What is the connection between RRC pulses and raised-cosine (RC) pulses? How are their spectra connected?
- What happens if an MFR is used with RC pulses?
- ...

Part II

Signal Space and Vector Representation of Signals

Distance Measures

- The *energy* of a signal $s_i(t)$ is

$$E_{s_i} = \int_{-\infty}^{\infty} s_i^2(t)\,dt$$

- The *length* of a signal $s_i(t)$ is

$$\|s_i(t)\| = \sqrt{\int_{-\infty}^{\infty} s_i^2(t)\,dt} = \sqrt{E_{s_i}}$$

- The *correlation* between $s_i(t)$ and $s_j(t)$ is

$$\langle s_i(t), s_j(t)\rangle = \int_{-\infty}^{\infty} s_i(t)s_j(t)\,dt$$

Distance Measures

- The *distance* between $s_i(t)$ and $s_j(t)$ is

$$\|s_i(t) - s_j(t)\| = \sqrt{\int_{-\infty}^{\infty} [s_i(t) - s_j(t)]^2\,dt}$$

- The *angle* between $s_i(t)$ and $s_j(t)$ is

$$\cos\alpha = \frac{\langle s_i(t), s_j(t)\rangle}{\|s_i(t)\| \cdot \|s_j(t)\|}$$

Note that $\cos\alpha = 0$ $(\alpha = \pi/2) \Rightarrow$ orthogonality.

The Signal Alternatives

The signal set alternatives are simply a set of finite-energy signals $\mathcal{S} = \{s_1(t), \ldots, s_M(t)\}$.

The Orthonormal Basis $\mathcal{P}$

Each of the elements of $\mathcal{S}$ can be always represented as a linear combination of $N \leq M$ orthonormal basis $\mathcal{P} = \{\phi_1(t), \ldots, \phi_N(t)\}$, i.e.,

$$s_i(t) = \sum_{n=1}^{N} \mathsf{s}_{i,n} \phi_n(t),$$

where $\mathsf{s}_{i,n} \in \mathbb{R}$, and where

$$\int_{-\infty}^{\infty} \phi_n(t)\phi_m(t)\, dt = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

The Gram-Schmidt Orthonormalization

- The *Gram-Schmidt Orthonormalization procedure* is a step-by-step procedure that finds such set of orthonormal basis $\mathcal{P}$
- The procedure is described in the book (Sec. 2.5.2)

Vectorial Representation of Signals

Once the basis functions have been found, each signal in $\mathcal{S}$ can be represented by the N components, i.e.,

$$\begin{aligned} s_1(t) &\equiv \mathsf{s}_1 = [\mathsf{s}_{1,1}, \ldots, \mathsf{s}_{1,N}] \\ s_2(t) &\equiv \mathsf{s}_2 = [\mathsf{s}_{2,1}, \ldots, \mathsf{s}_{2,N}] \\ &\vdots \\ s_M(t) &\equiv \mathsf{s}_M = [\mathsf{s}_{M,1}, \ldots, \mathsf{s}_{M,N}] \end{aligned}$$

i.e., each signal can be represented using a point in an N-dimensional Euclidean space.

The signal and its vectorial representation

If we have the signal alternatives $s_i(t)$ $i = 1, 2, \ldots, M$ represented by the vectors $\mathsf{s}_i = [\mathsf{s}_{i,1}, \ldots, \mathsf{s}_{i,N}]$, we could convert...

From vectors to signals

$$s_i(t) = \sum_{n=1}^{N} \mathsf{s}_{i,n} \phi_n(t)$$

From signals to vectors

$$\mathsf{s}_{i,n} = \int_{-\infty}^{\infty} s_i(t) \phi_n(t) \, dt$$

And we could also "re-define" the distance measures as

- The *energy* of a signal $s_i(t)$:

$$E_{s_i} = \int_{-\infty}^{\infty} s_i^2(t)\,dt = \|\mathsf{s}_i\|^2 = \mathsf{s}_i \cdot \mathsf{s}_i^\mathsf{T} = \sum_{n=1}^{N} \mathsf{s}_{i,n}^2$$

- The *length* of a signal $s_i(t)$: $\|s_i(t)\| = \|\mathsf{s}_i\| = \sqrt{\mathsf{s}_i \cdot \mathsf{s}_i^\mathsf{T}}$
- The *correlation* between $s_i(t)$ and $s_j(t)$:

$$\left\langle s_i(t), s_j(t)\right\rangle = \mathsf{s}_i \cdot \mathsf{s}_j^\mathsf{T} = \sum_{n=1}^{N} \mathsf{s}_{i,n}\mathsf{s}_{j,n}$$

- The *distance* between $s_i(t)$ and $s_j(t)$ is: $\|s_i(t) - s_j(t)\| = \|\mathsf{s}_i - \mathsf{s}_j\|$
- The *angle* between $s_i(t)$ and $s_j(t)$ is:

$$\cos\alpha = \frac{\mathsf{s}_i \cdot \mathsf{s}_j^\mathsf{T}}{\|\mathsf{s}_i\| \cdot \|\mathsf{s}_j\|}$$

Are the previous results true?

The proof of $E_{s_i} = \|\mathsf{s}_i\|^2$ can be found in Viterbi-Omura Book on p. 50:

$$E_{s_i} = \int_{-\infty}^{\infty} s_i^2(t)\,dt = \int_{-\infty}^{\infty} \left[\sum_{n=1}^{N} \mathsf{s}_{i,n}\phi_n(t)\right]^2 dt$$

$$= \sum_{n=1}^{N}\sum_{j=1}^{N} \mathsf{s}_{i,n}\mathsf{s}_{i,j} \int_{-\infty}^{\infty} \phi_n(t)\phi_j(t)\,dt = \sum_{n=1}^{N} \mathsf{s}_{i,n}^2 = \|\mathsf{s}_i\|^2$$

HW: Prove the previous equalities!

Why do we care about vectorial representation of signals?

- Because it is easier to work with points in Euclidean space instead of signals in time
- The "problems" get a geometrical interpretation
- We could use it to design receivers or signal alternatives
- We could analyze the performance of the system

Splitting the received signal in two

Assume we have an orthonormal basis $\mathcal{P} = \{\phi_1(t), \ldots, \phi_N(t)\}$ that span the signal alternatives $\mathcal{S} = \{s_1(t), \ldots, s_M(t)\}$, and the ith signal $s_i(t)$ is represented by the N-dimensional vector $\mathsf{s}_i = [\mathsf{s}_{i,1}, \ldots, \mathsf{s}_{i,N}]$.
The received signal $y(t)$ can also be represented using a larger orthonormal basis $\mathcal{P}' = \{\phi_1(t), \ldots, \phi_N(t), \phi_{N+1}(t), \ldots, \phi_{N'}(t)\}$ $(N' \geq N)$ as

$$
\begin{aligned}
y(t) &= \sum_{n=1}^{N} \mathsf{y}_{1,n}\phi_n(t) + \sum_{n=N+1}^{N'} \mathsf{y}_{2,n}\phi_n(t) \\
&= y_1(t) + y_2(t).
\end{aligned}
$$

Remarks

- $y_1(t)$ lies in the signal space of $\mathcal{S}$. Why?
- $y_2(t)$ is orthogonal to the signal space of $\mathcal{S}$. Why?

What is orthogonal to what?

If the previous remarks are true we have that

- $\langle y_2(t), \phi_n(t) \rangle = 0$ for $n = 1, 2, \ldots, N$
- $\langle y_2(t), y_1(t) \rangle = 0$
- $\langle y_2(t), s_i(t) \rangle = 0$ for $i = 1, 2, \ldots, M$

What do we want to maximize?

The correlation between $y(t)$ and $s_i(t)$, i.e.,

$$\langle y(t), s_i(t) \rangle = \langle y_1(t), s_i(t) \rangle + \langle y_2(t), s_i(t) \rangle$$

Definition (Theorem of Irrelevance)

Only the part of $y(t)$ that lies in the signal space of $\mathcal{S}$ matters for detection, i.e., if the received signal is split in two parts, $y(t) = y_1(t) + y_2(t)$, $y_1(t)$ is important for detection, and $y_2(t)$ is *irrelevant* for detection.

The problem we are trying to solve

In L_2 we decided we wanted to minimize the error energy... remember?

- The problem we need to solve is

$$\min_i \left\{ \int_{-\infty}^{\infty} [y(t) - s_i(t)]^2 \, dt \right\}$$

- Or equivalently

$$\max_i \left\{ \int_{-\infty}^{\infty} y(t) s_i(t) \, dt - \frac{E_{s_i}}{2} \right\} \equiv \max_i \left\{ \langle y(t), s_i(t) \rangle - \frac{E_{s_i}}{2} \right\}$$

- Or in terms of vectors:

$$\max_i \left\{ \mathsf{y} \cdot \mathsf{s}_i^{\mathsf{T}} - \frac{\|\mathsf{s}_i\|^2}{2} \right\} \equiv \min_i \left\{ \|\mathsf{y} - \mathsf{s}_i\|^2 \right\} \equiv \boxed{\min_i \left\{ \|\mathsf{y} - \mathsf{s}_i\| \right\}}$$

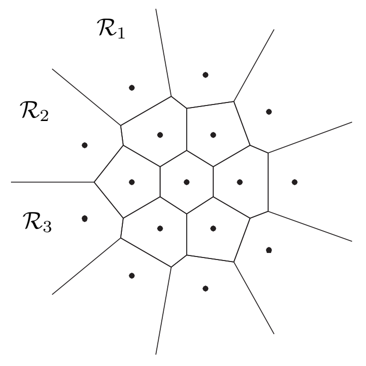
Correlation vs. Distance

Maximizing correlation is equivalent to minimizing Euclidean distance!

Decision Regions

For a given received vector y_1, we should choose s_i instead of s_j if $\|\mathsf{y}_1 - \mathsf{s}_i\| \leq \|\mathsf{y}_1 - \mathsf{s}_j\|$. One can plot decision regions $\mathcal{R}_1, \ldots, \mathcal{R}_M$ by drawing boundaries halfway between all pair of signal vectors.

Example (Graphically in 2D)



Today's Summary

- The transmitter is a linear filter $h_{\mathrm{T}}(t)$ with a train of pulses as input $\sum_{k=0}^{\infty} a_k \delta(t - kT_{\mathrm{s}})$.
- The Sampling Rx samples the signal at $t = kT_{\mathrm{s}}$, for $k = 0, 1, 2, \ldots$ and has zero-ISI if $h_{\mathrm{T}}(t)$ is a Nyquist pulse.
- The Matched Filter Rx samples the output of the receive filter $h_{\mathrm{R}}(t) = h_{\mathrm{T}}(-t)$ at $t = kT_{\mathrm{s}}$, for $k = 0, 1, 2, \ldots$ and has zero-ISI if $h_{\mathrm{T}}(t)$ is T_{s}-orthogonal.
- Any finite-energy signal set $\mathcal{S} = \{s_1(t), \ldots, s_M(t)\}$ can be represented by a set of N-dimensional vectors $\{\mathsf{s}_1, \ldots, \mathsf{s}_M\}$ based on $N \leq M$ orthonormal basis $\mathcal{P} = \{\phi_1(t), \ldots, \phi_N(t)\}$.
- Minimizing the error energy is equivalent to minimizing the Euclidean distance.