Introduction to Communication Engineering SSY121, Lecture # 9

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Information

Computer lab

The computer lab (room 5225) has updated their hardware, so our headphones can no longer be used. Instead, please use the rooms ES61, ES62, and ES63 on floor 6 in Linsen in the EDIT house. The rooms will not be booked, but they should be available after 17:00.

Sign up for exam

The exam in this course is as you know on Wed Oct 30 at 08:30-12:30. Please remember to sign up for the exam. Chalmers' rule is that if you have not registered for the exam, you will not be allowed to take the exam. The last day to register is Oct 10!

Outline

1. Review and Some Definitions
   - The AWGN channel
   - Review of the vector channel model for the AWGN channel
   - Some Definitions
2. Pairwise error probability
3. Error probabilities
   - The average error probability
   - The union bound and a high SNR approximation
   - Bit Error probability and Gray code

Part I

Review and Some Definitions

A very nice property of the AWGN

If $n(t)$ is a white Gaussian random process with PSD $N_0/2$, then the random variables n_i, the projections of $n(t)$ onto **any** set of orthonormal basis functions, are i.i.d. Gaussian random variables with zero mean and variance $N_0/2$. (Theorem 2.5-1 in [Anderson])

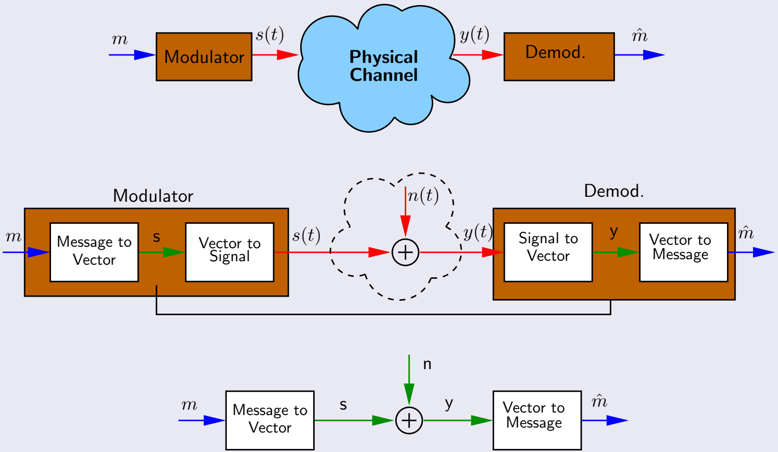
The continuous-time AWGN channel in signal space

- If $n(t)$ is AWGN with PSD $N_0/2$, the continuous time channel $y(t) = s(t) + n(t)$, can be replaced by a vectorial channel model

$$\mathsf{y} = \mathsf{s} + \mathsf{n},$$

  where $\mathsf{s} = [\mathsf{s}_1, \ldots, \mathsf{s}_N]$ is the transmitted symbol, $\mathsf{n} = [\mathsf{n}_1, \ldots, \mathsf{n}_N]$ is the noise, and $\mathsf{y} = [\mathsf{y}_1, \ldots, \mathsf{y}_N]$ is the received signal.
- The components of the noise vector n_i for $i = 1, \ldots, N$ are i.i.d. zero-mean Gaussian random variables with variance $N_0/2$.

The vector channel model for the AWGN channel



Definitions (see also L_2)

- M: Number of symbols
- l: Number of bits per symbol
- R_s: Symbol rate (T_s: symbol period)
- R_b: Bit rate (T_b: bit period). $R_\mathrm{b} = lR_\mathrm{s}$
- E_s: Average symbol energy:

$$E_\mathrm{s} = \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S} = \mathsf{s}_i\right] \|\mathsf{s}_i\|^2$$

- E_b: Average bit energy. $E_\mathrm{b} = E_\mathrm{s}/l$
- Signal to noise-ratio: SNR$=E_\mathrm{s}/N_0$
- Decibel: logarithmic measure of the ratio between two power or energies. For example $\mathrm{SNR}_\mathrm{dB} = 10\log_{10}(E_\mathrm{s}/N_0)$

Example (what can you do after this lecture?)

Consider equally likely 4-PAM transmission where $\mathsf{s} \in \{-3A, -A, A, 3A\}$.

- Compute A such that $E_\mathrm{s} = 1$
- Compute the relation between E_s and E_b
- Draw the constellation and the decision regions
- Compute the SNR in dB if $N_0 = 0.5$
- Compute the conditional PDF $f_{\mathsf{Y}|\mathsf{S}}(\mathsf{y}|\mathsf{S} = -3A)$ for SNR$= 3$ dB, for SNR$= -3$ dB, and for SNR$= 10$ dB
- Compute the probability of mistaking the transmitted symbol $\mathsf{s} = -3A$ by the symbol $-A$, A, and $+3A$ for the three SNR values

Pairwise error probability (PEP) definition

The PEP is simply the probability of mistaking the symbol s_i by s_j, i.e.,

$$\mathrm{PEP}^{(i,j)} = \mathbb{P}\left[||\mathsf{Y} - \mathsf{s}_j|| \leq ||\mathsf{Y} - \mathsf{s}_i|| \middle| \mathsf{S} = \mathsf{s}_i\right]$$

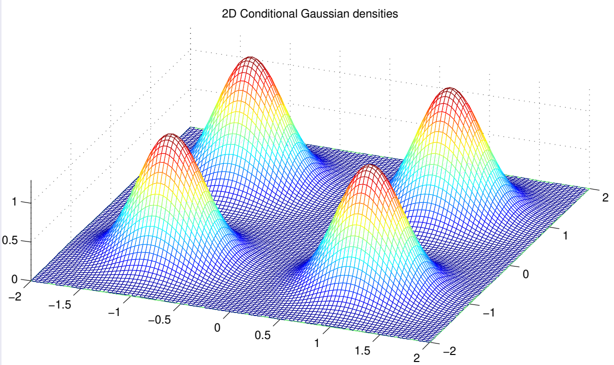
PEP in 1D for an AWGN channel

$$\begin{aligned}
\mathrm{PEP}^{(i,j)} &= \mathbb{P}\left[||\mathsf{Y} - \mathsf{s}_j|| < ||\mathsf{Y} - \mathsf{s}_i|| \middle| \mathsf{S} = \mathsf{s}_i\right] \\
&= \int_{y_0}^{\infty} f_{\mathsf{Y}|\mathsf{S}}(\mathsf{y}|\mathsf{s}_i) d\mathsf{y} = \mathrm{Q}\left(\frac{y_0 - \mathsf{s}_i}{\sigma}\right) \\
&= \mathrm{Q}\left(\frac{\mathsf{s}_i + \mathsf{s}_j - 2\mathsf{s}_i}{2\sqrt{N_0/2}}\right) = \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right),
\end{aligned}$$

where $y_0 = (\mathsf{s}_i + \mathsf{s}_j)/2$ is halfway between the signals, $\sigma^2 = N_0/2$ the variance of the noise, and $D_{i,j}^2$ is the squared Euclidean distance between s_i and s_j, i.e., $D_{i,j}^2 = \|\mathsf{s}_j - \mathsf{s}_i\|^2$ (see also L_7).

PEP in 2D: Still the same (Why?)

$$\mathrm{PEP}^{(i,j)} = \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right).$$



PEP in N dimensions

Still the same! (Why?)

$$\mathrm{PEP}^{(i,j)} = \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right).$$

What we need to know?

- The squared Euclidean distance between the two points $D_{i,j}^2$
- The noise variance $N_0/2$

The "border" (decision region)

- In 1D, the "border" is a point
- In 2D, the "border" is a line
- In 3D, the "border" is a plane
- In 4D, ...

A general error probability formula

The error probability when s_i is transmitted is

$$\mathbb{P}\left[\text{error}|\mathsf{S}=\mathsf{s}_i\right] = \mathbb{P}\left[\mathsf{Y} \notin \mathcal{R}_i|\mathsf{S}=\mathsf{s}_i\right] = \sum_{j\neq i} \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right]$$

The **exact** average error probability P_e is

$$\begin{aligned} P_\mathrm{e} &= \sum_{i=1}^{M} \mathbb{P}\left[\text{error}|\mathsf{S}=\mathsf{s}_i\right] \mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right] \\ &= \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right] \sum_{j\neq i} \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right] \end{aligned}$$

And what is the problem with this?

In general, we do not know how to compute $\mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right]$ ☹

An exception to the rule...

For binary transmission ($M = 2$), we do know how to compute the exact P_e, since $\mathbb{P}[\mathsf{Y} \in \mathcal{R}_j | \mathsf{S} = \mathsf{s}_i] = \mathrm{PEP}^{(1,2)}$.

For the particular case of $M = 2$ and equally likely symbols

For binary modulation ($M = 2$), the exact average error probability P_e can be written in terms of the PEP

$$\begin{aligned} P_\mathrm{e} &= \mathbb{P}[\mathsf{S} = \mathsf{s}_1] \, \mathrm{PEP}^{(1,2)} + \mathbb{P}[\mathsf{S} = \mathsf{s}_2] \, \mathrm{PEP}^{(2,1)} \\ &= \frac{1}{2} \mathrm{Q}\left(\sqrt{\frac{D_{1,2}^2}{2N_0}}\right) + \frac{1}{2} \mathrm{Q}\left(\sqrt{\frac{D_{2,1}^2}{2N_0}}\right) \\ &= \mathrm{Q}\left(\sqrt{\frac{D_{1,2}^2}{2N_0}}\right) \end{aligned}$$

The union bound (UB)

A good **approximation** of the average error probability P_{e} is

$$P_{\mathrm{e}} = \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right] \sum_{j\neq i} \mathbb{P}\left[\mathsf{Y}\in\mathcal{R}_j | \mathsf{S}=\mathsf{s}_i\right]$$

$$\leq \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right] \sum_{j\neq i} \mathrm{PEP}^{(i,j)} = \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right] \sum_{j\neq i} \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

The UB for equally likely symbols

$$P_{\mathrm{e}} \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j\neq i} \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

The previous approximation...

- It is actually **not** an approximation (it is exact) for $M=2$

Can we simplify it even more?

- The Q-functions decrease very fast when the argument increases
- For large arguments (high SNR), one of the Q-functions will dominate

Definition (High-SNR approximation)

For sufficiently high SNR, the average error probability can be approximated by

$$P_{\mathrm{e}} = \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S} = \mathsf{s}_i\right] \sum_{j \neq i} \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_j | \mathsf{S} = \mathsf{s}_i\right]$$

$$\leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \neq i} \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \approx \frac{2K}{M} \cdot \mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right),$$

where $D_{\min} = \min_{i \neq j}\{D_{i,j}\}$ is the *minimum distance* of the constellation, and K is the number of signal pairs at minimum distance (section 2.6 in [Andersson])

Example (Error probability of OOK and BPSK)

Which modulation scheme is better in terms of error probability, OOK or BPSK?

- Normalize both constellations to unit energy
- For a given N_0, find the PEP
- Plot both of them using Matlab

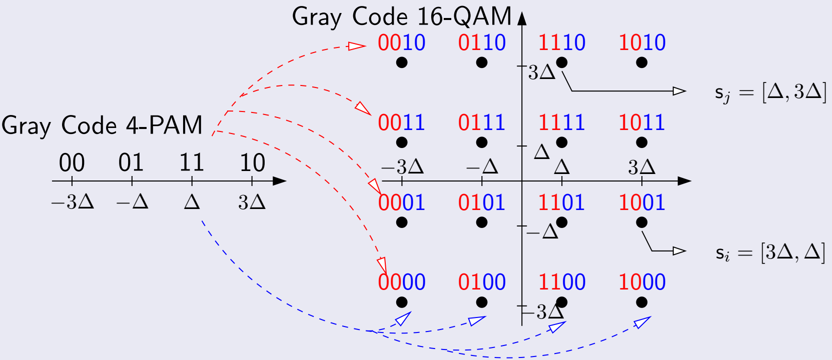
Homeworks!

Compute and plot error probabilities (bounds/approximations) for OOK, BPSK, 4-PAM, 8-PAM, QPSK, 16-QAM, 8-PSK, etc.

Symbol error probability and bit error probability

- The relation between symbol error probability and bit error probability is in general not simple
- It depends on the binary labeling
- The "best" we could do is to have one bit-error for each symbol error
- For the high-SNR approximation, we would like to use a binary labeling such that symbols at $D_{\min}$ differ in only one bit
- How: Gray Code

Gray code for 4-PAM and 64-QAM



Today's Summary

- Vector channel model for the AWGN channel.
- Pairwise error probability (PEP), exact average error probability, union bound, and high-SNR approximation.
- Symbol error probability vs. bit error probability and Gray Code.