Introduction to Communication Engineering
SSY121, Lecture # 5

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Information

Project Groups

51 MSc students and 1 PhD student signed up for the project: 12 groups of 4 MSc students, 1 group of 3 MSc students, and 1 group with 1 PhD student.

Deadline for Common Values

The deadline for Common Values is tomorrow Thu Sept 9 at noon. The deadline for Time Report is Every Friday at noon! Template soon posted!

Request for Proposal

The request for proposal (RFP) and the MATLAB functions needed for the project will be released tomorrow Thu Sept 9 at noon.

Student Representatives

- Sara Akbari (MPBME) saraakb@student.chalmers.se
- Andreas Benjaminsson (MPEPO) andbenj@student.chalmers.se
- Chitra Suresh Hebbar (MPCOM) chitra@student.chalmers.se
- Julia Ohlslöf (MPBME) juliaoh@student.chalmers.se

Part I

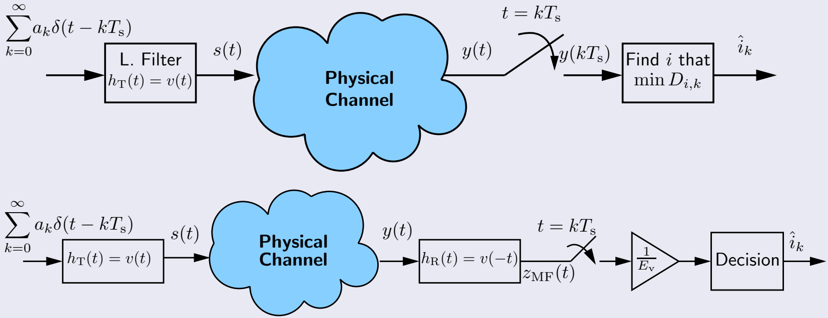
A Short Summary of Last Lecture

The Transmitted Signal is a Sequence of M-ary Pulses

$$s(t) = \sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s}),$$

where $a_k \in \mathcal{A}$ is the amplitude transmitted at the kth time instant.

Sampling Rx (SR) vs. Matched Filter Rx (MFR)



Definition (Pulses in Time Domain)

$$\text{Nyquist Pulse for SR: } v(nT_\mathrm{s}) = 0$$

$$\text{Orthogonal Pulse for MFR: } \int_{-\infty}^{\infty} v(t)v(t - nT_\mathrm{s})\, dt = 0$$

if $n = \pm 1, \pm 2, \pm 3, \ldots$

Definition (Pulses in Frequency Domain)

If $v(t)$ is symmetric around $t = 0$

$$\text{Nyquist Pulse for SR: } \sum_{n=-\infty}^{\infty} V\left(f - \frac{n}{T_\mathrm{s}}\right) = T_\mathrm{s} v(0)$$

$$\text{Orthogonal Pulse for MFR: } \sum_{n=-\infty}^{\infty} \left| V\left(f - \frac{n}{T_\mathrm{s}}\right)\right|^2 = T_\mathrm{s} E_v.$$

The signal and its vectorial representation

The signal alternatives $s_i(t)$ $i = 1, 2, \ldots, M$ can be represented by the vectors $\mathsf{s}_i = [\mathsf{s}_{i,1}, \ldots, \mathsf{s}_{i,N}] \in \mathbb{R}^N$

$$s_i(t) = \sum_{n=1}^{N} \mathsf{s}_{i,n} \phi_n(t),$$

$$\mathsf{s}_{i,n} = \int_{-\infty}^{\infty} s_i(t) \phi_n(t)\, dt,$$

where $\phi_n(t)$ is an orthonormal basis

$$\int_{-\infty}^{\infty} \phi_n(t) \phi_m(t)\, dt = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Distance Measures

- The *energy* of a signal $s_i(t)$ is

$$E_{s_i} = \|s_i(t)\|^2 = \int_{-\infty}^{\infty} s_i^2(t)\,dt = \|\mathsf{s}_i\|^2 = \mathsf{s}_i \cdot \mathsf{s}_i^{\mathsf{T}} = \sum_{n=1}^{N} \mathsf{s}_{i,n}^2$$

- The *length* of a signal $s_i(t)$ is

$$\sqrt{E_{s_i}} = \|s_i(t)\| = \sqrt{\int_{-\infty}^{\infty} s_i^2(t)\,dt} = \|\mathsf{s}_i\| = \sqrt{\mathsf{s}_i \cdot \mathsf{s}_i^{\mathsf{T}}}$$

- The *correlation* between $s_i(t)$ and $s_j(t)$ is

$$\langle s_i(t), s_j(t) \rangle = \int_{-\infty}^{\infty} s_i(t) s_j(t)\,dt = \mathsf{s}_i \cdot \mathsf{s}_j^{\mathsf{T}} = \sum_{n=1}^{N} \mathsf{s}_{i,n}\mathsf{s}_{j,n}$$

Distance Measures (cont.)

- The *distance* between $s_i(t)$ and $s_j(t)$ is

$$\|s_i(t) - s_j(t)\| = \sqrt{\int_{-\infty}^{\infty} [s_i(t) - s_j(t)]^2 \, dt}$$

$$= \|\mathsf{s}_i - \mathsf{s}_j\| = \sqrt{(\mathsf{s}_i - \mathsf{s}_j) \cdot (\mathsf{s}_i - \mathsf{s}_j)^\mathsf{T}}$$

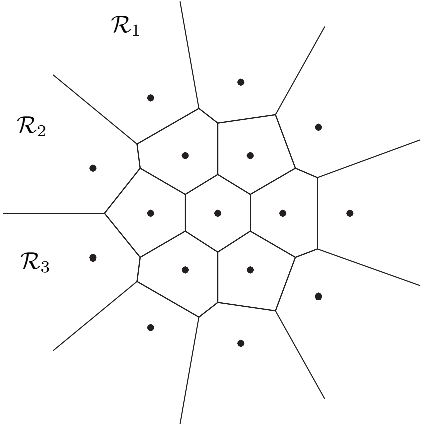
- The *angle* between $s_i(t)$ and $s_j(t)$ is

$$\cos\alpha = \frac{\langle s_i(t), s_j(t) \rangle}{\|s_i(t)\| \cdot \|s_j(t)\|} = \frac{\mathsf{s}_i \cdot \mathsf{s}_j^\mathsf{T}}{\|\mathsf{s}_i\| \cdot \|\mathsf{s}_j\|}$$

Note that $\cos\alpha = 0$ ($\alpha = \pi/2$) $\Rightarrow$ orthogonality.

The problem we are trying to solve

$$\min_i \left\{ \int_{-\infty}^{\infty} [y(t) - s_i(t)]^2 \, dt \right\} \equiv \boxed{\min_i \left\{ \|\mathsf{y} - \mathsf{s}_i\| \right\}}$$

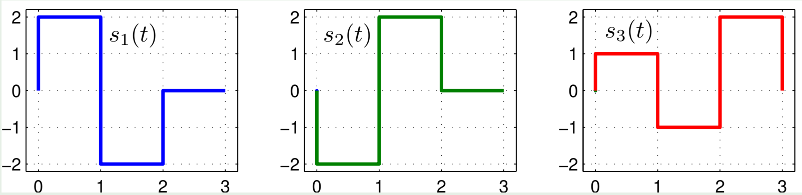


Part II

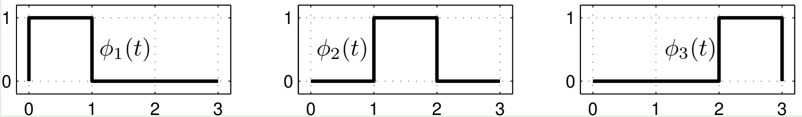
The Gram-Schmidt Process

Example (Finding $\phi_n(t)$)

$$s_i(t) = \sum_{n=1}^{N} \mathsf{s}_{i,n}\phi_n(t), \quad \mathsf{s}_{i,n} = \int_{-\infty}^{\infty} s_i(t)\phi_n(t)\,dt,$$

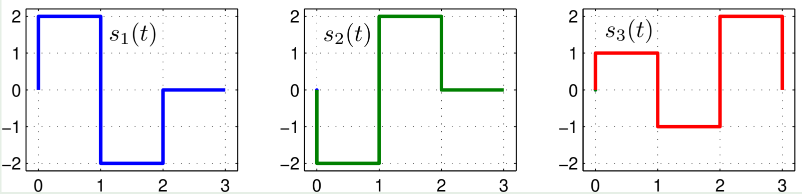


- One set of orthonormal basis functions, $\phi_n(t)$, are:



- What are the vectors s_1, s_2, and s_3?
- $\mathsf{s}_1 = [2, -2, 0]$, $\mathsf{s}_2 = [-2, 2, 0]$, and $\mathsf{s}_3 = [1, -1, 2]$

Example (The Gram-Schmidt Process (Anderson p. 47))

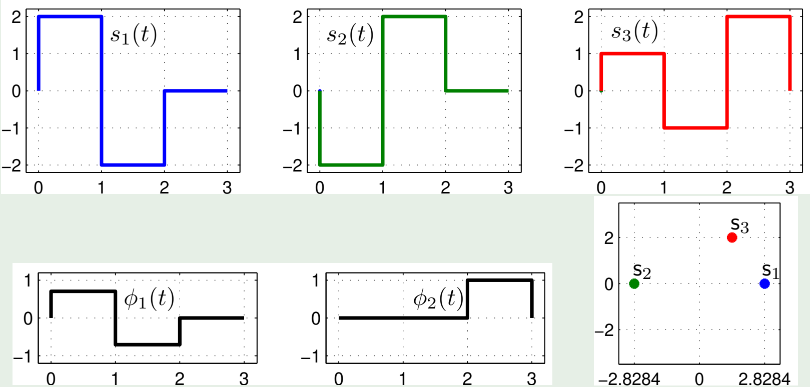


$$\Theta_j(t) = s_j(t) - \sum_{k=1}^{j-1} \langle s_j(t), \phi_k(t) \rangle \phi_k(t)$$

$$\phi_j(t) = \frac{\Theta_j(t)}{\sqrt{E_j}}, \quad \text{where} \quad E_j = \langle \Theta_j(t), \Theta_j(t) \rangle$$

- $\Theta_1(t) = s_1(t)$, $E_1 = 8$, $\phi_1(t) = s_1(t)/\sqrt{8}$, $s_1(t) = \sqrt{8}\phi_1(t)$
- $\Theta_2(t) = s_2(t) - \langle s_2(t), \phi_1(t) \rangle \phi_1(t) = 0$, $s_2(t) = -s_1(t) = -\sqrt{8}\phi_1(t)$
- $\Theta_3(t) = s_3(t) - \langle s_3(t), \phi_1(t) \rangle \phi_1(t) = s_3(t) - \sqrt{2}\phi_1(t)$, $E_3 = 4$, $\phi_2(t) = (s_3(t) - \sqrt{2}\phi_1(t))/2$, $s_3(t) = \sqrt{2}\phi_1(t) + 2\phi_2(t)$

Example (The Gram-Schmidt Process (Anderson p. 47))



$$s_1(t) = +\sqrt{8}\phi_1(t), \qquad \mathsf{s}_1 = [\sqrt{8}, 0]$$

$$s_2(t) = -\sqrt{8}\phi_1(t), \qquad \mathsf{s}_2 = [-\sqrt{8}, 0]$$

$$s_3(t) = +\sqrt{2}\phi_1(t) + 2\phi_2(t), \qquad \mathsf{s}_3 = [\sqrt{2}, 2]$$

Part III

Passband Transmission

In baseband

- For a baseband pulse $v(t)$ with Fourier transform $V(f)$, the BW of the transmitted signal is about $\frac{1}{2T_\mathrm{s}}$, where $R_\mathrm{s} = 1/T_\mathrm{s}$ is the symbol rate
- Best case scenario, we use pulses $v(t) = \mathrm{sinc}\,(t/T_\mathrm{s})$ and the BW is exactly $R_\mathrm{s}/2$
- Such a signal can be sent for example over a telephone line
- If radio waves are used, the wavelengths are very big, and therefore, antennas are huge
- How does an FM receiver work then?

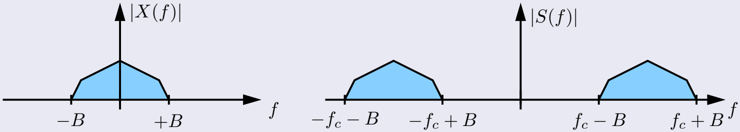
Example (Binary baseband transmission at $R_\mathrm{b} = 1$ kbps)

$R_\mathrm{b} = R_\mathrm{s} = 10^3\ \mathrm{symb/s}$, and therefore $T_\mathrm{s} = 10^{-3}\ \mathrm{s}$. The wavelength $\lambda = c/f$ assuming $f = 1/(2T_\mathrm{s})$ and $c = 3 \cdot 10^8\ \mathrm{m/s}$ (speed of light) is $\lambda \approx 6T_\mathrm{s}10^8\ \mathrm{m}$. The wavelength is then $\lambda \approx 600\ \mathrm{km}$, which means a HUGE antenna. If the frequency is $100\ \mathrm{MHz}$ (FM), $\lambda \approx 3\ \mathrm{m} \Rightarrow$ antennas of $\lambda/2$ or $\lambda/4$ are feasible.

What do we do then?

- In FM, the signals are not transmitted in baseband, but instead using a carrier
- This is simply done by multiplying the baseband signal $x(t)$ by a sinusoid of frequency f_c: $s(t) = x(t)\cos(2\pi f_c t)$
- What is the spectrum of such a signal?

$$\begin{aligned} S(f) &= \mathcal{F}\{x(t)\cos(2\pi f_c t)\} = \mathcal{F}\{x(t)\} * \mathcal{F}\{\cos(2\pi f_c t)\} \\ &= X(f) * \frac{1}{2}\Big[\delta(f-f_c)+\delta(f+f_c)\Big] = \frac{1}{2}\Big[X(f-f_c)+X(f+f_c)\Big] \end{aligned}$$



- Other reasons for doing this
  - Makes best use of the channel
  - Allows us to assign different users to different frequencies

A very common $\mathcal{P}$

- A very common $\mathcal{P} = \{\phi_1(t), \phi_2(t)\}$ is

$$\phi_1(t) = \sqrt{2}v(t)\cos(w_c t)$$
$$\phi_2(t) = \sqrt{2}v(t)\sin(w_c t)$$

- $v(t)$ is a unit-energy baseband pulse
- $f_c = \frac{w_c}{2\pi}$ is the carrier frequency

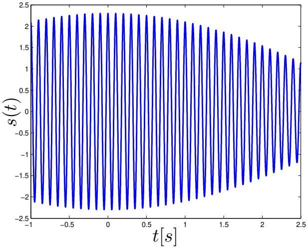
Is $\mathcal{P}$ really an orthonormal basis?

- $||\phi_1(t)|| = 1?$ $(\sqrt{E_{\phi_1}} = 1)$
- $||\phi_2(t)|| = 1?$ $(\sqrt{E_{\phi_2}} = 1)$
- $\langle\phi_1(t), \phi_2(t)\rangle = 0?$

A useful integral

Consider a slow varying signal (band limited) signal $x(t)$ and the integral

$$s(t) = \int_{-\infty}^{\infty} x(t)\cos(w_c t)\,dt = ?$$



A useful integral

- If the BW of $x(t)$, f_x, is less than the carrier frequency, $f_x < f_c$, we can prove using Parseval's theorem that:

$$\int_{-\infty}^{\infty} x(t)\cos(w_c t)\,dt = \int_{-\infty}^{\infty} X(f)\mathcal{F}\{\cos(w_c t)\}^*\,df$$

$$= \int_{-\infty}^{\infty} X(f)\frac{1}{2}\Big[\delta(f-f_c)+\delta(f+f_c)\Big]\,df = \frac{1}{2}\Big[X(f_c)+X(-f_c)\Big] = 0$$

- HW: prove that $\int_{-\infty}^{\infty} x(t)\sin(w_c t)\,dt = 0$ if $f_x < f_c$!

Energy

- Consider now the following integral

$$E_{\phi_1} = \int_{-\infty}^{\infty} \phi_1^2(t)\, dt = 2\int_{-\infty}^{\infty} v^2(t)\cos^2(w_c t)\, dt$$
$$= \int_{-\infty}^{\infty} v^2(t)[1+\cos(2w_c t)]\, dt = 1 + \int_{-\infty}^{\infty} v^2(t)\cos(2w_c t)\, dt$$

- If the BW of $v(t)$ is f_v then the BW of $v^2(t)$ is $2f_v$, since $\mathcal{F}\{v^2(t)\} = V(f) * V(f)$, hence $\int_{-\infty}^{\infty} v^2(t)\cos(2w_c t)\, dt = 0$.

Correlation

$$\langle \phi_1(t), \phi_2(t) \rangle = 2\int_{-\infty}^{\infty} v^2(t)\cos(w_c t)\sin(w_c t)\, dt = \int_{-\infty}^{\infty} v^2(t)\sin(2w_c t)\, dt$$

A summary of what we know

$E_{\phi_1} = 1$, $E_{\phi_2} = 1$, and $\langle \phi_1(t), \phi_2(t) \rangle = 0$ means an orthonormal basis.

ISI-free transmission

- The transmitted signal will be of the form

$$s(t) = \sum_{k=0}^{\infty} a_k \phi_1(t - kT_s) + b_k \phi_2(t - kT_s)$$

- For ISI-free transmission, we need to have that for $n = \pm 1, \pm 2, \ldots$

$$\int_{-\infty}^{\infty} \phi_1(t)\phi_1(t - nT_s) = \int_{-\infty}^{\infty} \phi_2(t)\phi_2(t - nT_s) = 0$$

$$\int_{-\infty}^{\infty} \phi_1(t)\phi_2(t - nT_s) = \int_{-\infty}^{\infty} \phi_2(t)\phi_1(t - nT_s) = 0$$

- The previous equalities are valid when the baseband pulse $v(t)$ is T_s-orthogonal. HW: prove it!

What is important to remember?

The functions $\phi_1(t)$ and $\phi_2(t)$ form an orthonormal basis, and they are also T_s-orthogonal (when the baseband pulse $v(t)$ is T_s-orthogonal).

Part IV

1D and 2D Modulations

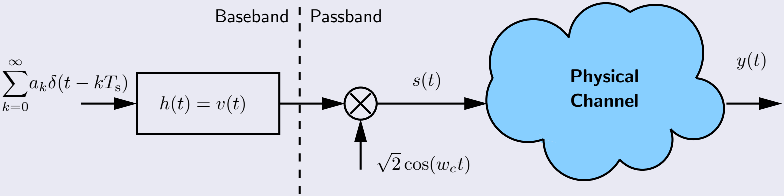
Assumptions

- A square pulse of duration T_s and amplitude $1/\sqrt{T_\mathrm{s}}$.
- The two basis functions are
  - $\phi_1(t) = \sqrt{2}v(t)\cos(w_c t)$
  - $\phi_2(t) = \sqrt{2}v(t)\sin(w_c t)$

1D Constellations

- Only one dimension is used ($N = 1$)
- This makes the Tx/Rx simple to implement

1D Tx



On-off keying (OOK)

$$s_1(t) = 0 \qquad \mathsf{s}_1 = [0, 0]$$

$$s_2(t) = A\phi_1(t) \qquad \mathsf{s}_2 = [A, 0]$$

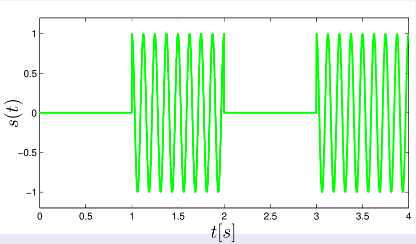
Energies

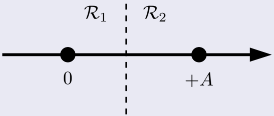
$$E_{s_1} = \|\mathsf{s}_1\|^2 = 0$$

$$E_{s_2} = \|\mathsf{s}_2\|^2 = A^2$$

$$\overline{E} = \frac{1}{2}E_{s_1} + \frac{1}{2}E_{s_2} = \frac{A^2}{2}$$

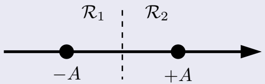
OOK signal using square pulses





Binary phase shift keying (BPSK)

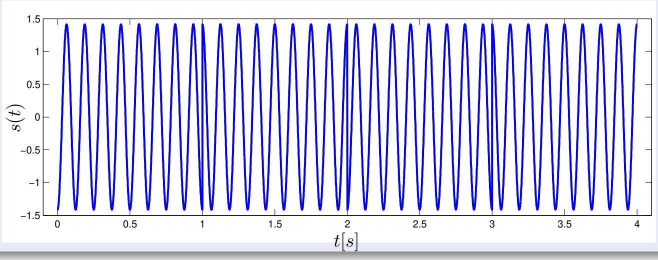
$$s_1(t) = -A\phi_1(t) \qquad \mathsf{s}_1 = [-A, 0]$$
$$s_2(t) = +A\phi_1(t) \qquad \mathsf{s}_2 = [+A, 0]$$



Energies

$$E_{s_1} = \|\mathsf{s}_1\|^2 = A^2$$
$$E_{s_2} = \|\mathsf{s}_2\|^2 = A^2$$
$$\overline{E} = \frac{1}{2}E_{s_1} + \frac{1}{2}E_{s_2} = A^2$$

BPSK signal using square pulses



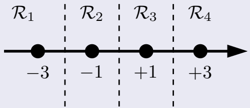
M-ary pulse amplitude modulation (M-PAM)

$$s_1(t) = -(M-1)A\phi_1(t), \quad \mathsf{s}_1 = [-(M-1)A, 0]$$

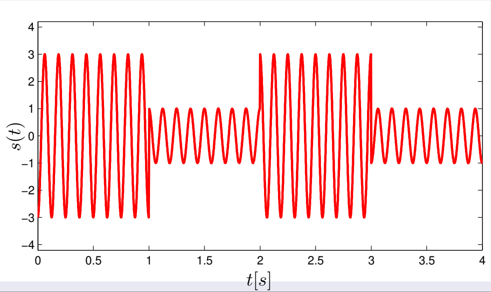
$$\vdots \qquad\qquad \vdots$$

$$s_M(t) = +(M-1)A\phi_1(t), \quad \mathsf{s}_M = [+(M-1)A, 0]$$

4-PAM



4-PAM signal using square pulses



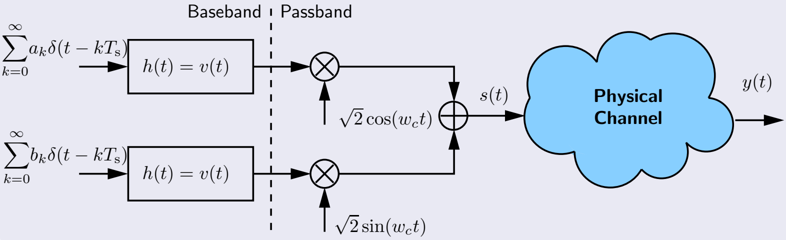
Passband signal for 2D modulations

- The passband signal for the 2D constellations can be written as

$$s(t) = \sqrt{2}\sum_{k=0}^{\infty} a_k v(t - kT_{\mathrm{s}}) \cos(w_c t) + b_k v(t - kT_{\mathrm{s}}) \sin(w_c t)$$

- Every point $(a_k, b_k) \in \mathbb{R}^2$ can be represented using an amplitude $A_k = \sqrt{a_k^2 + b_k^2}$ and an angle $\psi_k = \arctan(b_k/a_k)$
- The 1D signals can be obtained by using $b_k = 0$.

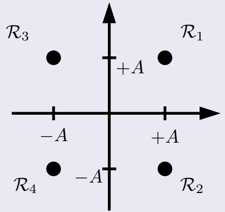
2D Tx



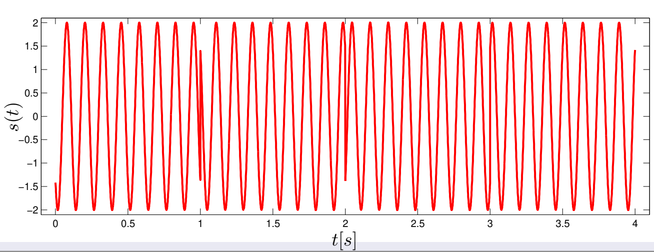
Quaternary phase shift keying (QPSK)

$$
\begin{aligned}
s_1(t) &= +A\phi_1(t)+A\phi_2(t), & \mathsf{s}_1 &= [+A,+A] \\
s_2(t) &= +A\phi_1(t)-A\phi_2(t), & \mathsf{s}_2 &= [+A,-A] \\
s_3(t) &= -A\phi_1(t)+A\phi_2(t), & \mathsf{s}_3 &= [-A,+A] \\
s_4(t) &= -A\phi_1(t)-A\phi_2(t), & \mathsf{s}_4 &= [-A,-A]
\end{aligned}
$$

Constellation



QPSK signal using square pulses



Quadrature amplitude modulation (M-QAM)

The signal alternatives are

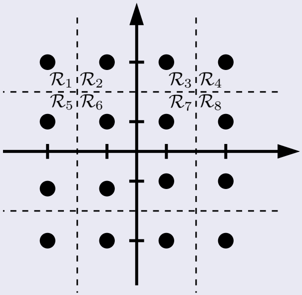
$$s_i(t) = a_k \phi_1(t) + b_k \phi_2(t)$$

with $i = 1, 2, \ldots, M$, $a_k, b_k \in \mathcal{A}$, and

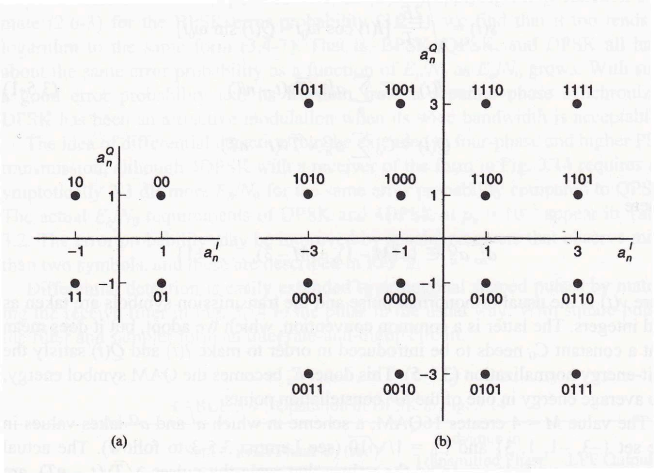
$$\mathcal{A} = \{-(M-1), \ldots, -1, +1, \ldots, (M-1)\}.$$

The constellation points are $\mathsf{s}_{i,1} \in \mathcal{A}$ and $\mathsf{s}_{i,2} \in \mathcal{A}$.

Constellation



4-QAM and 16-QAM, Gray and non-Gray (from [Anderson])



M-PSK constellations

- M-PSK is simply a straightforward extension of BPSK/QPSK to more phases
- If M is the number of constellation points,

$$s(t) = \sum_{k=0}^{\infty} v(t - kT_{\mathrm{s}}) \cos\left(w_c t + \frac{2i\pi}{M}\right),$$

  where $i = 0, 1, \ldots, M-1$
- The M-PSK is equivalent to BPSK $(M = 2)$ and QPSK $(M = 4)$
- Note that for $M = 4$, $\psi_k \in \{0, \pi/2, \pi, 3\pi/2\}$, which does not change *anything* compared to $\psi_k \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$
- M-PSK constellations have *constant-energy*

8-PSK constellation (from [Anderson])

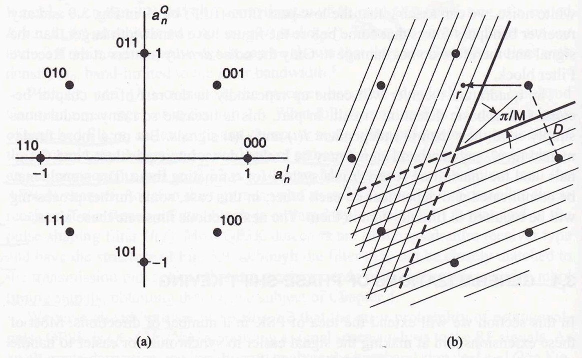


Figure 3.10 **(a)** Transmission symbols for 8PSK; labels show corresponding data bits. **(b)** Calculation of symbol error probability for MPSK for 8PSK case. Minimum distance is D; circle radius is $\sqrt{E_s}$.

Summary of constellations

$a_k \in$	$b_k \in$	Name(s)	$\psi_k \in$
$\{0, +A\}$	0	OOK	0
$\{-A, +A\}$	0	BPSK (2-PAM)	$\{0, \pi\}$
$\{-3A, -A, +A, +3A\}$	0	4-PAM	$\{0, \pi\}$
$\{-A, +A\}$	$\{-A, +A\}$	QPSK (4-QAM)	$\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$
$\{-3A, -A, +A, +3A\}$	$\{-3A, -A, +A, +3A\}$	16-QAM	–
$\cos(\psi_k)$	$\sin(\psi_k)$	M-PSK	$\{\frac{2\pi i}{M}\}$ for $i = 0, \ldots, M-1$

Other constellations (from [Anderson])

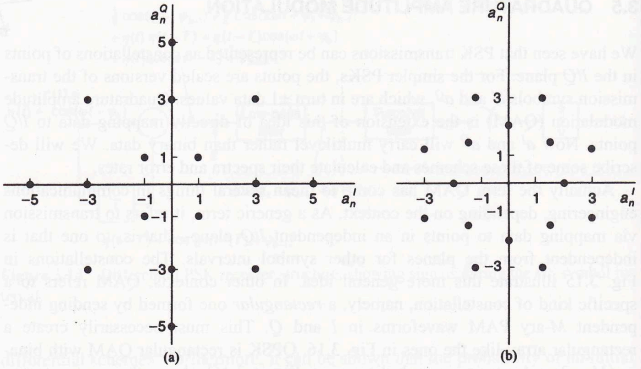


Figure 3.15 Two nonrectangular QAM constellations: **(a)** V.29 modem standard; **(b)** double-circle constellation (outer radius 3.16; inner radius 2).

Today's Summary

- SR vs. MFR (L_3)
- Time vs. Frequency Domain (L_3)
- Signals in Vectorial Representation (L_3)
- Distance Measure and Decision Regions (L_3)
- Gram-Schmidt Process
- Baseband and Passband Transmission
- A Very Common $\mathcal{P}$
- 1D and 2D Constellations
- Binary Labeling