Introduction to Communication Engineering SSY121, Lecture # 13

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Announcements!

- Exercise E_5 on Wed Oct 23, 10:00–11:45 – Open questions about problems from the homeworks, old exams, etc.
- Noon Fri Oct. 18: Test Report, Software, and Time Report 6.
- Noon Mon Oct. 21: Experience report deadline. Its contents will not be graded (but the delivery will)
- Project demonstrations on Fri Oct 18 and Mon Oct 21, at 17:00 – 20:00. Signup list is in Canvas.
- Exam: Wed Oct. 30 at 08:30-12:30 (in 14 days!)
- One or two articles will be uploaded to Canvas. Please remember to read them together with the other material when preparing for the exam!
- Course evaluation opens after the course. Everyone are encouraged to answer!

Ericsson visit!

- Wed Oct. 23, 13:15–15:30: Experience workshop (voluntary) at Ericsson, Lindholmen
- Transportation is **not** arranged (bus 16 or 55 from Chalmers)
- Sign up as either "Attending" or "Not attending" in Canvas (list under People) no later than Sun Oct 20!
- Preliminary agenda:
  - 13.15 Short Introduction (Fredrik)
  - 13:25 Way of working (WoW) Ericsson Research Radio (Jingya Li)
  - 13:55 Coffee break
  - 14.15 WoW Packet Core (Martin Trygg)
  - 14.45 Project introduction and Lessons learned (Fredrik)
  - 15.05 Demo (students)
  - 15.30 End

Some words about the exam

- Understanding is more important than memorizing
- Explain the line of reasoning clearly
  - Good solution with minor error $\rightarrow$ almost full points
  - Correct answer without clear solution $\rightarrow$ 0 points
  - Obviously unreasonable answer $\rightarrow$ 0 points
- We will visit the exam after about 1 and 3 hours
- We will be happy to clarify the questions if needed, but please do not ask about your solutions
- Problems are given in random order and the points for each question are clearly marked
- The course contents are defined in the Course Memo: The exam is based on the project, the lectures, the book (pages listed in course memo), the articles, Erik Svenske's slides (Lecture 3), and the document "Working in projects."

Some words about the exam

- Read Chalmers' examination rules (link in the Course Memo)
- Allowed aids:
  - Mathematics Handbook (Beta)
  - A simple "Chalmers-approved calculator"
  - Dictionary (see examination rules)
  - Do not bring the formula sheet – it will be handed out
- Write clearly and only in English. If we do not understand it, we do not give points for it
- Do not solve two problems on the same sheet of paper (Subproblems may share the same paper)

Part I

Lecture 1

Linear and Time Invariant (LTI) System

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- The impulse response of the LTI system is given by $h(t)$
- In the time domain, $y(t) = x(t) * h(t)$
- In the frequency domain, $Y(f) = X(f)H(f)$

The Sampling Theorem

Let $x(t)$ be a signal with Fourier transform $X(f)$ such that $x(t)$ is band-limited, i.e., $X(f) = 0$ for $|f| \geq B$. If the signal $x(t)$ is sampled at uniformly spaced time instants using a sampling frequency $f_\mathrm{s} = 1/T_\mathrm{s}$, $x(t)$ can be completely recovered if $f_\mathrm{s} \geq 2B$.

And how?

By interpolating, i.e.,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_\mathrm{s}) \mathrm{sinc}\left(\frac{t - nT_\mathrm{s}}{T_\mathrm{s}}\right),$$

where $\mathrm{sinc}\,(x)$ is the normalized sinc function

$$\mathrm{sinc}\,(x) = \frac{\sin \pi x}{\pi x}.$$

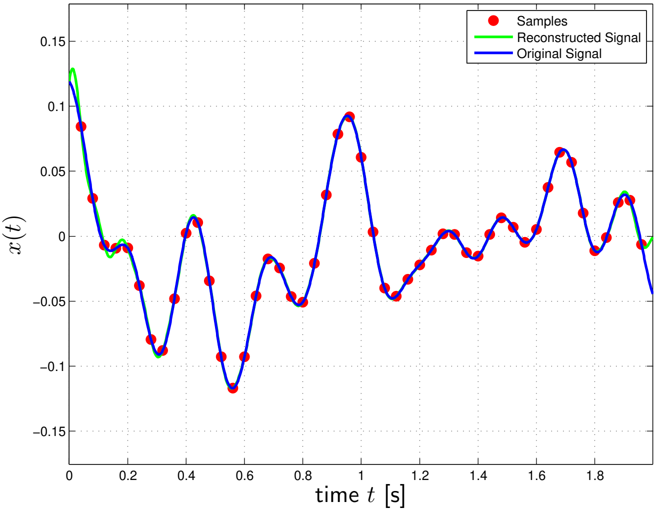


Figure: The band-limited signal $x(t)$ is reconstructed using samples. The BW of $x(t)$ is $B \approx 10$ Hz and $f_s = 25$ sample/s.

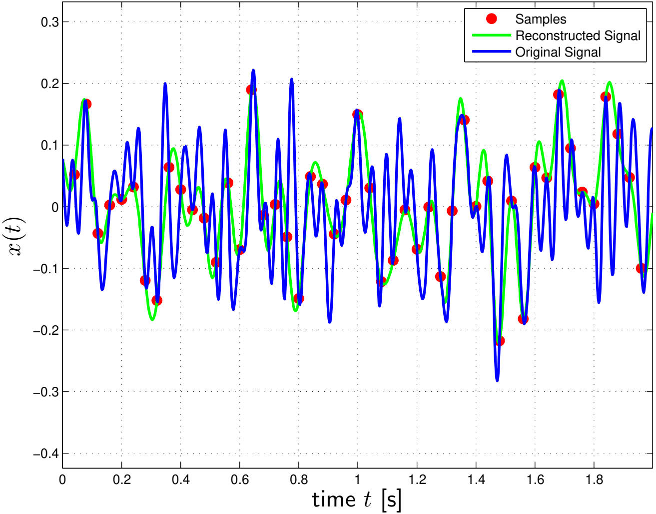
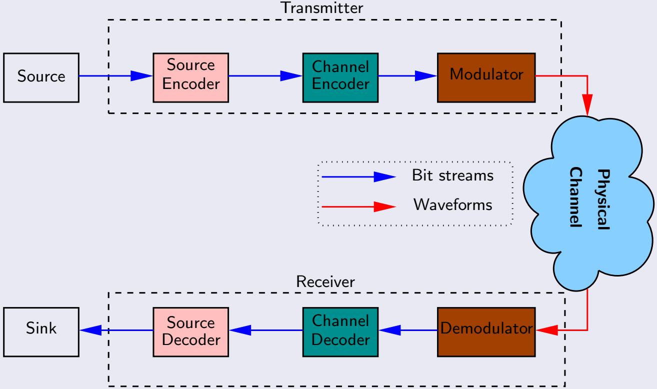


Figure: The band-limited signal $x(t)$ is reconstructed using samples. The BW of $x(t)$ is $B \approx 100$ Hz and $f_s = 25$ sample/s.

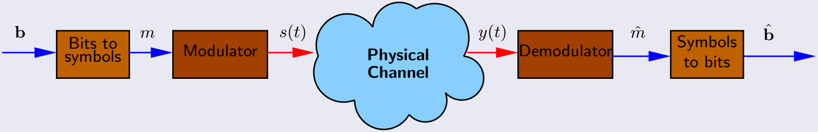
Shannon's Communication Model



Part II

Lecture 2

The General Diagram



- The length-l binary codeword $\mathbf{b} = [b_1, \ldots, b_l]$ has to be transmitted
- Each of the possible $M = 2^l$ codewords is mapped to a message (symbol) $m \in \{m_1, \ldots, m_M\}$
- The modulator sends a continuous-time baseband signal $s(t)$ through the physical channel each T_s [s], where $s(t)$ is selected from the set of signal alternatives $\mathcal{S} = \{s_1(t), \ldots, s_M(t)\}$.
- The channel introduces some distortion such that $y(t) \neq s(t)$

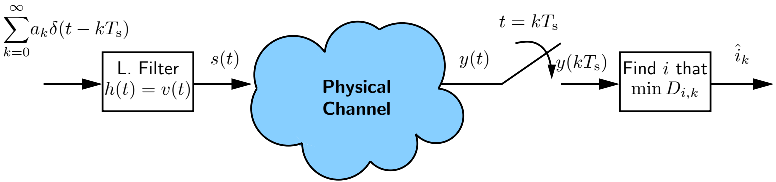
The General Problem

Using only the channel observation $y(t)$ during one symbol period T_s, *guess* what the transmitted message m was.

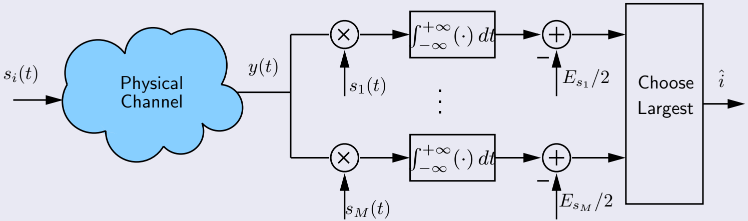
The Sampling Rx

The Sampling Rx simply samples $y(t)$ at time instants $t = kT_\mathrm{s}$ for $k = 0, 1, 2, \ldots$, each time computing the M distances $D_{i,k}$ and selecting the i that gives the minimum $D_{i,k}$.

$$\hat{i}_k = \arg \min_{i \in \{1,\ldots,M\}} D_{i,k}, \quad \text{where} \quad D_{i,k} = |y(kT_\mathrm{s}) - s_i(0)|.$$



The Correlator Rx



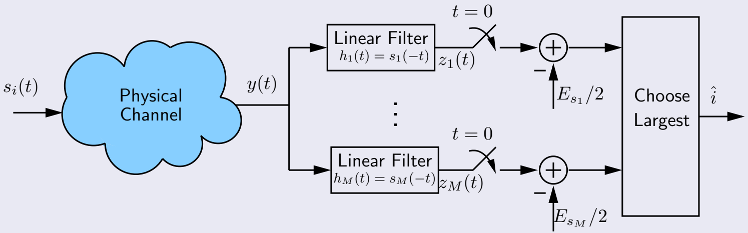
Some Comments

- If all the signal alternatives have the same energy ($E_{s_1} = E_{s_2} = \ldots$), the problem is reduced to

$$\hat{i} = \arg\max_i \left\{ \int_{-\infty}^{\infty} y(t) s_i(t)\, dt \right\}$$

- Since the "heart" of the processing are correlation integrals, this Rx is called the **correlator Rx** (CR)

The Matched Filter Rx



Some Comments

- Since the filters are *matched* to the signal alternatives, it is called the **matched filter Rx** (MFR)
- Both the **correlator Rx** (CR) and MFR can be implemented using *linear filters* and are therefore called **linear Rx** (LR)
- In terms of performance, the CR and the MFR are equivalent
- However, the performance of the CR and MFR is different than the performance of **Sampling Rx**

Part III

Lecture 3

Lecture by Erik Svenske

- Development of projects
- Working in teams
- Project phases
- Project elements

Part IV

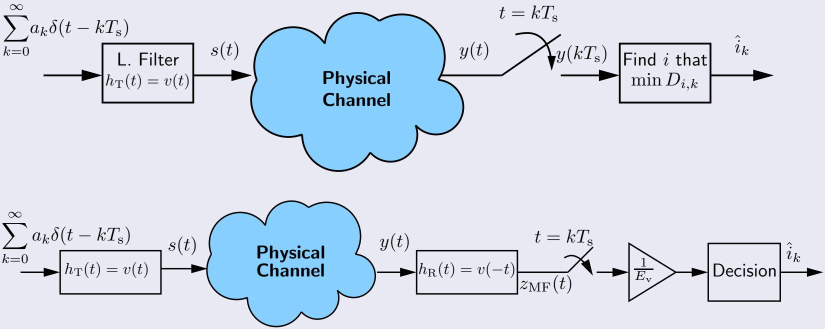
Lecture 4

The Transmitted Signal is a Sequence of M-ary Pulses

$$s(t) = \sum_{k=0}^{\infty} a_k v(t - kT_{\mathrm{s}}),$$

where $a_k \in \mathcal{A}$ is the amplitude transmitted at the kth time instant.

Sampling Rx (SR) vs. Matched Filter Rx (MFR)



Definition (Pulses in Time Domain)

$$\text{Nyquist Pulse for SR: } v(nT_\mathrm{s}) = 0$$

$$T_\mathrm{s}\text{-orthogonal Pulse for MFR: } \int_{-\infty}^{\infty} v(t)v(t-nT_\mathrm{s})\,dt = 0$$

if $n = \pm 1, \pm 2, \pm 3, \ldots$

Definition (Pulses in Frequency Domain)

If $v(t)$ is symmetric around $t = 0$

$$\text{Nyquist Pulse for SR: } \sum_{n=-\infty}^{\infty} V\left(f - \frac{n}{T_\mathrm{s}}\right) = T_\mathrm{s} v(0)$$

$$T_\mathrm{s}\text{-orthogonal Pulse for MFR: } \sum_{n=-\infty}^{\infty} \left| V\left(f - \frac{n}{T_\mathrm{s}}\right)\right|^2 = T_\mathrm{s} E_v.$$

The signal and its vectorial representation

The signal alternatives $s_i(t)$ $i = 1, 2, \ldots, M$ can be represented by the vectors $\mathsf{s}_i = [\mathsf{s}_{i,1}, \ldots, \mathsf{s}_{i,N}] \in \mathbb{R}^N$

$$s_i(t) = \sum_{n=1}^{N} \mathsf{s}_{i,n} \phi_n(t),$$

$$\mathsf{s}_{i,n} = \int_{-\infty}^{\infty} s_i(t) \phi_n(t)\, dt,$$

where $\phi_n(t)$ is an orthonormal basis

$$\int_{-\infty}^{\infty} \phi_n(t) \phi_m(t)\, dt = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Distance Measures

- The *energy* of a signal $s_i(t)$ is

$$E_{s_i} = \|s_i(t)\|^2 = \int_{-\infty}^{\infty} s_i^2(t)\,dt = \|\mathsf{s}_i\|^2 = \mathsf{s}_i \cdot \mathsf{s}_i^{\mathsf{T}} = \sum_{n=1}^{N} \mathsf{s}_{i,n}^2$$

- The *length* of a signal $s_i(t)$ is

$$\sqrt{E_{s_i}} = \|s_i(t)\| = \sqrt{\int_{-\infty}^{\infty} s_i^2(t)\,dt} = \|\mathsf{s}_i\| = \sqrt{\mathsf{s}_i \cdot \mathsf{s}_i^{\mathsf{T}}}$$

- The *correlation* between $s_i(t)$ and $s_j(t)$ is

$$\langle s_i(t), s_j(t) \rangle = \int_{-\infty}^{\infty} s_i(t) s_j(t)\,dt = \mathsf{s}_i \cdot \mathsf{s}_j^{\mathsf{T}} = \sum_{n=1}^{N} \mathsf{s}_{i,n} \mathsf{s}_{j,n}$$

Distance Measures (cont.)

- The *distance* between $s_i(t)$ and $s_j(t)$ is

$$\|s_i(t) - s_j(t)\| = \sqrt{\int_{-\infty}^{\infty} [s_i(t) - s_j(t)]^2 \, dt}$$
$$= \|\mathsf{s}_i - \mathsf{s}_j\| = \sqrt{(\mathsf{s}_i - \mathsf{s}_j) \cdot (\mathsf{s}_i - \mathsf{s}_j)^\mathsf{T}}$$

- The *angle* between $s_i(t)$ and $s_j(t)$ is

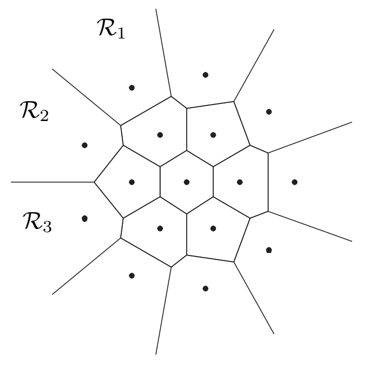
$$\cos\alpha = \frac{\langle s_i(t), s_j(t) \rangle}{\|s_i(t)\| \cdot \|s_j(t)\|} = \frac{\mathsf{s}_i \cdot \mathsf{s}_j^\mathsf{T}}{\|\mathsf{s}_i\| \cdot \|\mathsf{s}_j\|}$$

Note that $\cos\alpha = 0$ $(\alpha = \pi/2) \Rightarrow$ orthogonality.

Decision Regions

For a given received vector y_1, we should choose s_i instead of s_j if $\|\mathsf{y}_1 - \mathsf{s}_i\| \leq \|\mathsf{y}_1 - \mathsf{s}_j\|$. One can plot decision regions $\mathcal{R}_1, \ldots, \mathcal{R}_M$ by drawing boundaries halfway between all pair of signal vectors.

Example (Graphically in 2D)



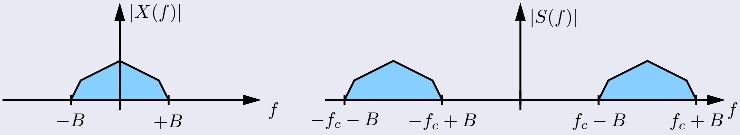
Part V

Lecture 5

What do we do then?

- In FM, the signals are not transmitted in baseband, but instead using a carrier
- This is simply done by multiplying the baseband signal $x(t)$ by a sinusoid of frequency f_c: $s(t) = x(t)\cos(2\pi f_c t)$
- What is the spectrum of such a signal?

$$S(f) = \mathcal{F}\{x(t)\cos(2\pi f_c t)\} = \mathcal{F}\{x(t)\} * \mathcal{F}\{\cos(2\pi f_c t)\}$$
$$= X(f) * \frac{1}{2}\Big[\delta(f-f_c)+\delta(f+f_c)\Big] = \frac{1}{2}\Big[X(f-f_c)+X(f+f_c)\Big]$$



- Other reasons for doing this
  - Makes best use of the channel
  - Allows us to assign different users to different frequencies

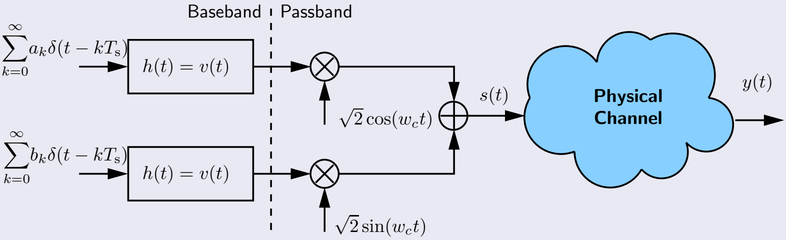
Passband signal for 2D modulations

- The passband signal for the 2D constellations can be written as

$$s(t) = \sqrt{2} \sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s}) \cos(w_c t) + b_k v(t - kT_\mathrm{s}) \sin(w_c t)$$

- Every point $(a_k, b_k) \in \mathbb{R}^2$ can be represented using an amplitude $A_k = \sqrt{a_k^2 + b_k^2}$ and an angle $\psi_k = \arctan(b_k / a_k)$
- The 1D signals can be obtained by using $b_k = 0$.

2D Tx



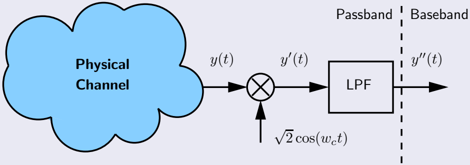
Summary of constellations

$a_k \in$	$b_k \in$	Name(s)	$\psi_k \in$
$\{0, +A\}$	0	OOK	0
$\{-A, +A\}$	0	BPSK (2-PAM)	$\{0, \pi\}$
$\{-3A, -A, +A, +3A\}$	0	4-PAM	$\{0, \pi\}$
$\{-A, +A\}$	$\{-A, +A\}$	QPSK (4-QAM)	$\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$
$\{-3A, -A, +A, +3A\}$	$\{-3A, -A, +A, +3A\}$	16-QAM	–
$\cos(\psi_k)$	$\sin(\psi_k)$	M-PSK	$\{\frac{2\pi i}{M}\}$ for $i = 0, \ldots, M-1$

Part VI

Lecture 6

1D Rx



Why?

If the channel is good, we can assume $y(t) \approx s(t)$, and thus

$$
\begin{aligned}
y'(t) &= \sqrt{2}y(t)\cos(w_c t) \approx \sqrt{2}s(t)\cos(w_c t) \\
&= 2\sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s})\cos^2(w_c t) \\
&= \sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s})\left[1 + \cos(2w_c t)\right] \\
&= \sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s}) + {\color{red}\sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s})\cos(2w_c t)}
\end{aligned}
$$

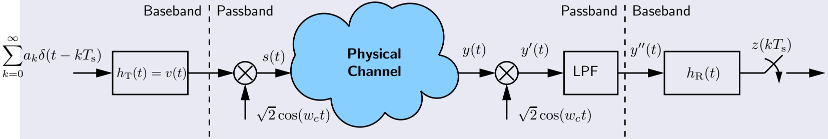
After the LPF

After the LPF, we obtain

$$y''(t) = \sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s}),$$

which is an expression we have seen before, haven't we?

1D Tx and Rx



The selection of $h_\mathrm{R}(t)$?

How do we choose the impulse response of the Rx filter?

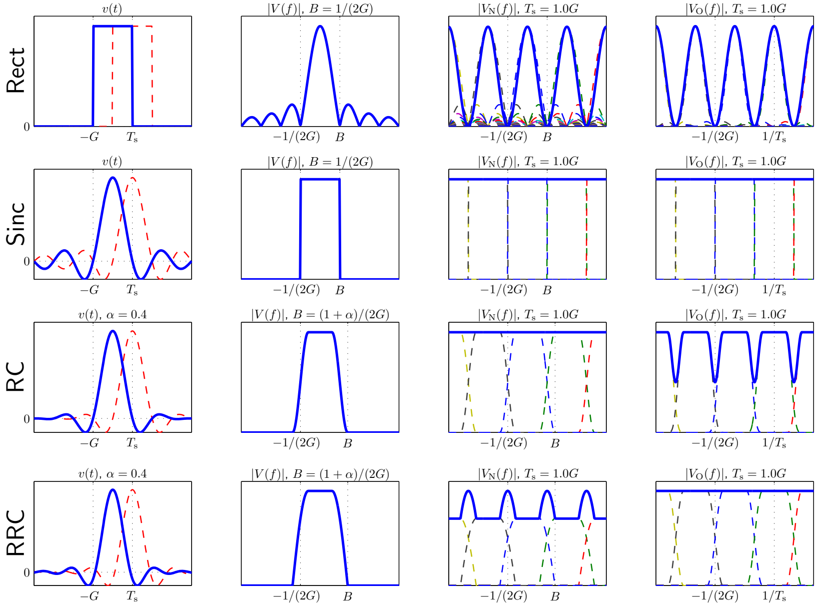
What is the relationship between B, f_c, f_{samp}, R_{s}, T_{s}, and T_{samp}?

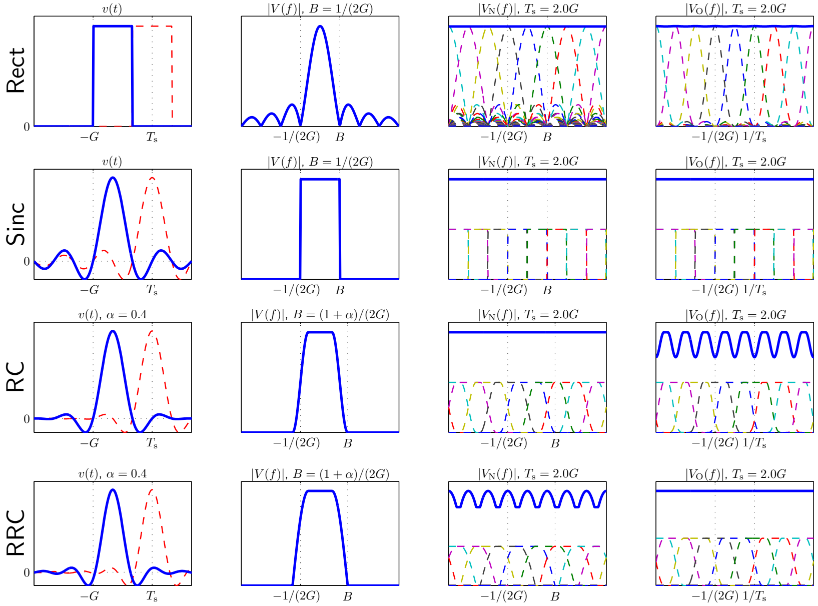
- Define the baseband bandwidth B [Hz] of $V(f) = \mathcal{F}\{v(t)\}$ to be the lowest frequency $f > 0$, where $|V(f)| = 0$ (L_5).
- The passband bandwidth is then $2B$ [Hz] (L_5).
- The symbol rate is always $R_{\text{s}} = 1/T_{\text{s}}$ [symbols/s], where T_{s} [s] is the time between symbols (or samples at the MF output) (L_2).
- The received signal should be sampled every T_{samp} [s] with sampling frequency $f_{\text{samp}} = 1/T_{\text{samp}} > 2(f_c + B)$ [Hz], where f_c [Hz] is the carrier frequency (The Sampling Theorem in L_1).
- For Nyquist pulses (L_2 and L_4):

$$V_{\text{N}}(f) = \sum_{n=-\infty}^{\infty} V\left(f - \frac{n}{T_{\text{s}}}\right) = T_{\text{s}} v(0)$$

- For T_{s}-orthogonal pulses (L_4):

$$V_{\text{O}}(f) = \sum_{n=-\infty}^{\infty} \left|V\left(f - \frac{n}{T_{\text{s}}}\right)\right|^2 = T_{\text{s}} E_v$$


Rect
Sinc
RC
RRC
$v(t)$
$v(t)$, $\alpha = 0.4$
$|V(f)|$, $B = 1/(2G)$
$|V(f)|$, $B = (1+\alpha)/(2G)$
$|V_{\mathrm{N}}(f)|$, $T_{\mathrm{s}} = 1.0G$
$|V_{\mathrm{O}}(f)|$, $T_{\mathrm{s}} = 1.0G$
0
$-G$
T_{s}
$-1/(2G)$
B
$1/T_{\mathrm{s}}$


Rect
Sinc
RC
RRC
$v(t)$
$|V(f)|$, $B = 1/(2G)$
$|V_\mathrm{N}(f)|$, $T_\mathrm{s} = 2.0G$
$|V_\mathrm{O}(f)|$, $T_\mathrm{s} = 2.0G$
$v(t)$, $\alpha = 0.4$
$|V(f)|$, $B = (1+\alpha)/(2G)$
0
$-G$
T_s
$-1/(2G)$
B
$1/T_\mathrm{s}$

Summary and Conclusions

Pulse	Nyquist	T_s-orthogonal	B
Rect	$T_\mathrm{s} > G$	$T_\mathrm{s} \geq 2G$	$1/(2G)$
Sinc	$T_\mathrm{s} = kG$	$T_\mathrm{s} = kG$	$1/(2G)$
RC	$T_\mathrm{s} = kG$	–	$(1+\alpha)/(2G)$
RRC	–	$T_\mathrm{s} = kG$	$(1+\alpha)/(2G)$

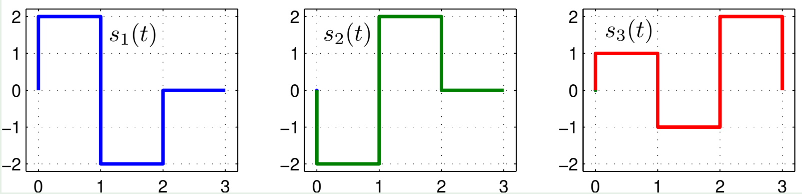
where $k = 1, 2, 3, \ldots$

Part VII

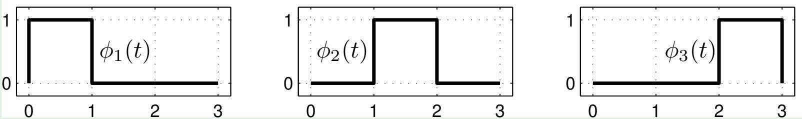
Lecture 7

Example (Finding $\phi_n(t)$)

$$s_i(t) = \sum_{n=1}^{N} \mathsf{s}_{i,n}\phi_n(t), \quad \mathsf{s}_{i,n} = \int_{-\infty}^{\infty} s_i(t)\phi_n(t)\,dt,$$

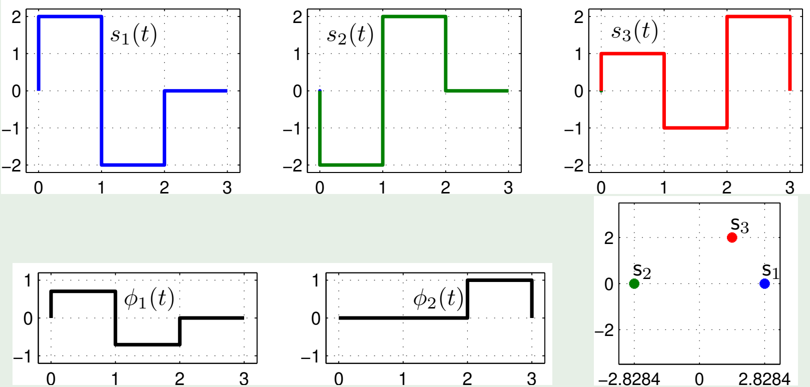


- One set of orthonormal basis functions, $\phi_n(t)$, are:



- What are the vectors s_1, s_2, and s_3?
- $\mathsf{s}_1 = [2, -2, 0]$, $\mathsf{s}_2 = [-2, 2, 0]$, and $\mathsf{s}_3 = [1, -1, 2]$

Example (The Gram-Schmidt Process (Anderson p. 47))



$$s_1(t) = +\sqrt{8}\phi_1(t), \qquad \mathrm{s}_1 = [\sqrt{8}, 0]$$
$$s_2(t) = -\sqrt{8}\phi_1(t), \qquad \mathrm{s}_2 = [-\sqrt{8}, 0]$$
$$s_3(t) = +\sqrt{2}\phi_1(t) + 2\phi_2(t), \qquad \mathrm{s}_3 = [\sqrt{2}, 2]$$

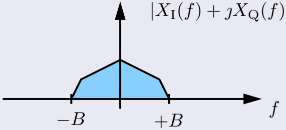
2D Passband Tx

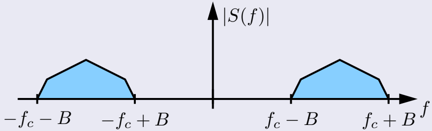
- Transmitted signal (time domain):

$$s(t) = x_{\mathrm{I}}(t)\cos(2\pi f_c t) + x_{\mathrm{Q}}(t)\sin(2\pi f_c t)$$

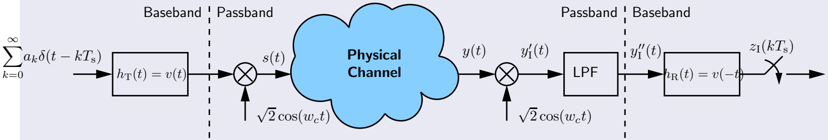
- Transmitted signal (frequency domain):

$$\begin{aligned}
S(f) &= \mathcal{F}\{x_{\mathrm{I}}(t)\cos(2\pi f_c t)\} + \mathcal{F}\{x_{\mathrm{Q}}(t)\sin(2\pi f_c t)\} \\
&= \mathcal{F}\{x_{\mathrm{I}}(t)\} * \mathcal{F}\{\cos(2\pi f_c t)\} + \mathcal{F}\{x_{\mathrm{Q}}(t)\} * \mathcal{F}\{\sin(2\pi f_c t)\} \\
&= X_{\mathrm{I}}(f) * \frac{1}{2}\Big[\delta(f+f_c)+\delta(f-f_c)\Big] + X_{\mathrm{Q}}(f) * \frac{\jmath}{2}\Big[\delta(f+f_c)-\delta(f-f_c)\Big] \\
&= \frac{1}{2}\Big[X_{\mathrm{I}}(f+f_c)+X_{\mathrm{I}}(f-f_c) + \jmath X_{\mathrm{Q}}(f+f_c) - \jmath X_{\mathrm{Q}}(f-f_c)\Big]
\end{aligned}$$





2D Passband Rx



If the channel is good, we can assume $y(t) \approx s(t)$, and thus

$$y_I'(t) \approx \left[\sqrt{2}\sum_{k=0}^{\infty} a_k v(t-kT_s)\cos(w_c t) + b_k v(t-kT_s)\sin(w_c t)\right]\sqrt{2}\cos(w_c t)$$

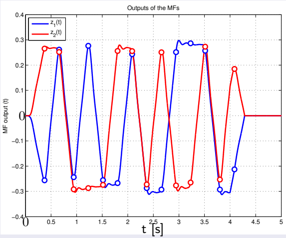
$$= \sum_{k=0}^{\infty} a_k v(t-kT_s) + \sum_{k=0}^{\infty} v(t-kT_s)(a_k\cos(2w_c t) + b_k\sin(2w_c t))$$

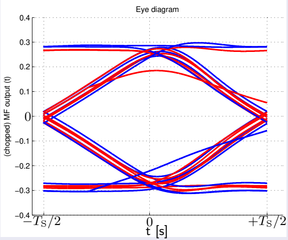
$$y_Q'(t) \approx \left[\sqrt{2}\sum_{k=0}^{\infty} a_k v(t-kT_s)\cos(w_c t) + b_k v(t-kT_s)\sin(w_c t)\right]\sqrt{2}\sin(w_c t)$$

$$= \sum_{k=0}^{\infty} b_k v(t-kT_s) + \sum_{k=0}^{\infty} v(t-kT_s)(a_k\sin(2w_c t) - b_k\sin(2w_c t))$$

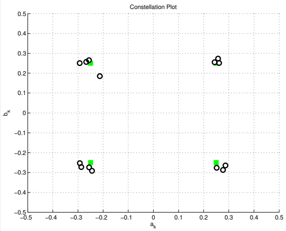
The LPF removes the red terms at frequency $2w_c$!

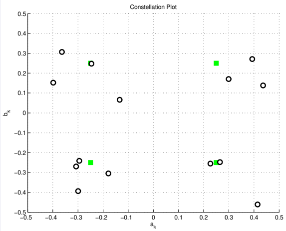
MF's outputs and Eye Diagram





Constellation Plot (Good and Bad Channel)

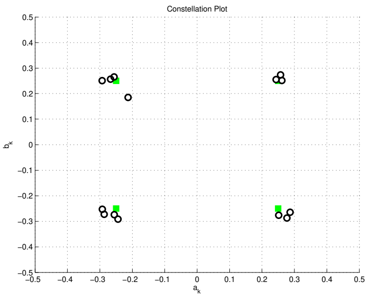


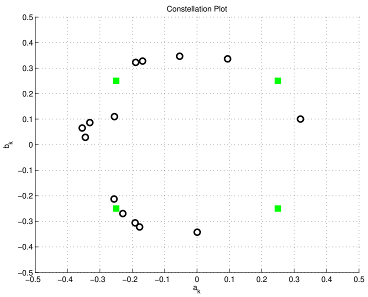


Synchronization in Rx

- Carrier frequency synchronization:
  - The local oscillators in Rx is not exactly f_c but $f_c + \Delta$
- Carrier phase synchronization:
  - The phase of the local oscillators in Rx is different than in Tx, i.e.,
  - $\cos(w_c t)$ and $\sin(w_c t)$ are $\cos(w_c t + \theta)$ and $\sin(w_c t + \theta)$
- Symbol synchronization:
  - The output of the MFs are taken at the wrong instant, e.g.,
  - $t = 0.0T_\mathrm{s}, 1.1T_\mathrm{s}, 2.2T_\mathrm{s}, \ldots$ wrong sampling frequency
  - $t = 0.1T_\mathrm{s}, 1.1T_\mathrm{s}, 2.1T_\mathrm{s}, \ldots$ wrong timing
- Frame synchronization:
  - Locate where does the block of information bits start, i.e., where the beginning of the "sentence" starts.

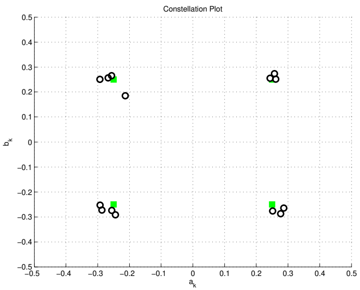
Constellation Plot for QPSK, good channel, and $\Delta = 0.1 f_c$

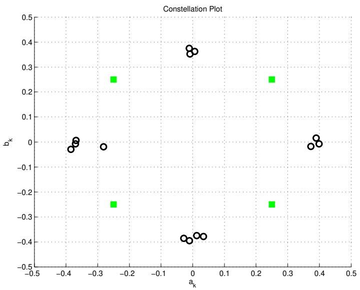




Wrong frequency $\Rightarrow$ a “rotating” constellation

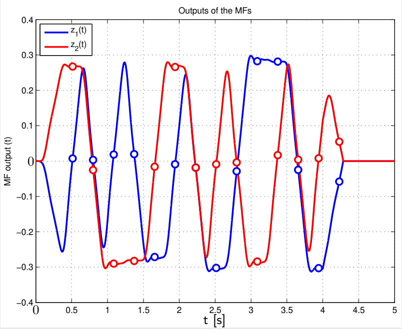
Constellation Plot for QPSK, good channel, and $\theta = \pi/4$

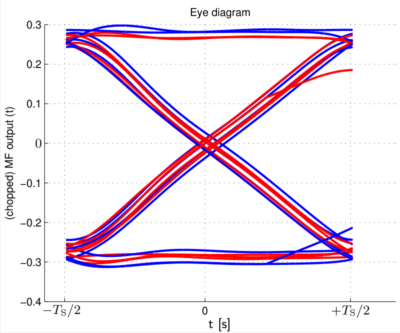




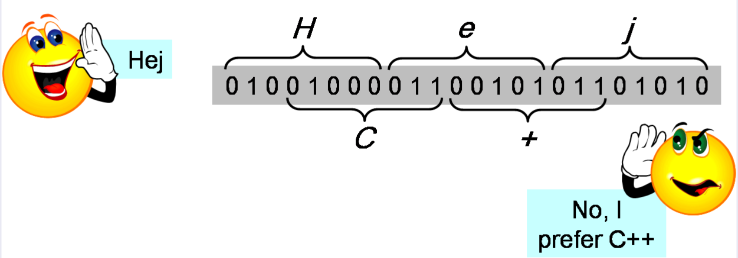
Wrong phase $\Rightarrow$ a “rotated” constellation

When the sampling instant is incorrect... $t_k = kT_s + 0.5T_s$





When the frame synchronization is incorrect...



Part VIII

Lecture 8

1D Gaussian random variables

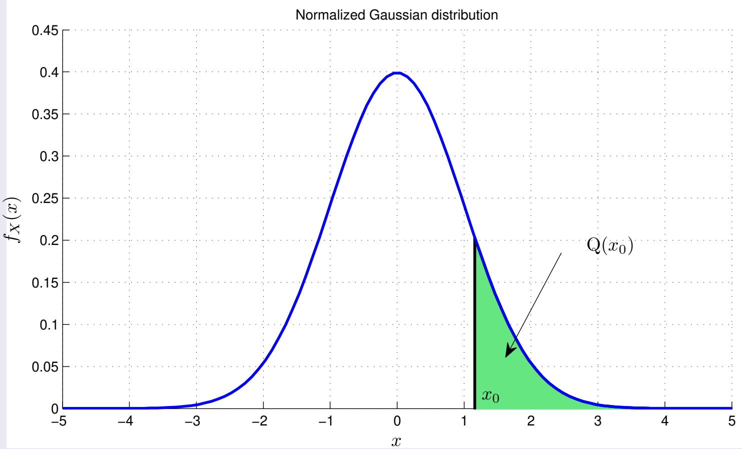
- Gaussian distribution is equivalent to normal distribution
- The mean value is μ and the variance is σ^2
- We use $\mathcal{N}(\mu, \sigma^2)$ to denote this random variable
- The probability density function (PDF) of $\mathcal{N}(\mu, \sigma^2)$ is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- The normalized Gaussian distribution is obtained when $\mu = 0$ and $\sigma^2 = 1$
- The probability density function of $\mathcal{N}(0, 1)$ is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Normalized Gaussian distribution



ML detection

In summary, minimizing the error probability is equivalent to maximizing the conditional probabilities (or PDFs)

$$\max_i \{\mathbb{P}[\mathsf{Y} = \mathsf{y} | \mathsf{S} = \mathsf{s}_i]\},$$

which is called **maximum likelihood** (ML) detection.

ML for the AWGN channel

For the AWGN channel, the conditional PDF is

$$f_{\mathsf{Y}|\mathsf{S}}(\mathsf{y}|\mathsf{s}_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{||\mathsf{y} - \mathsf{s}_i||^2}{N_0}\right),$$

and therefore,

$$\max_i \{\mathbb{P}[\mathsf{Y} = \mathsf{y} | \mathsf{S} = \mathsf{s}_i]\} \equiv \min_i \{||\mathsf{y} - \mathsf{s}_i||\}$$

Maximum a posteriori detection

- For ML detection, we assumed that all the symbols are equally likely
- If they are not, the problem is

$$\max_i \{\mathbb{P}[\mathsf{S} = \mathsf{s}_i | \mathsf{Y} = \mathsf{y}]\} \equiv \max_i \{\mathbb{P}[\mathsf{S} = \mathsf{s}_i] \, \mathbb{P}[\mathsf{Y} = \mathsf{y} | \mathsf{S} = \mathsf{s}_i]\}$$

- This is called **maximum a posteriori** (MAP) detection.

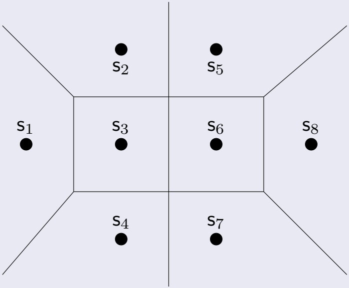
MAP detection for the AWGN channel

- If the channel is AWGN the MAP detection is

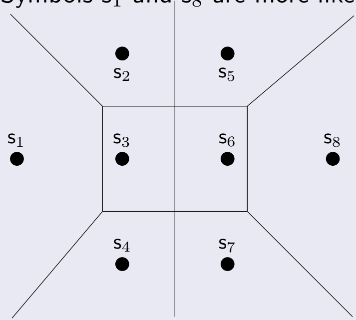
$$\begin{aligned}
&\max_i \{\mathbb{P}[\mathsf{S} = \mathsf{s}_i | \mathsf{Y} = \mathsf{y}]\} \\
&\quad \equiv \max_i \left\{ \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{||\mathsf{y} - \mathsf{s}_i||^2}{N_0} \right) \right\} \\
&\quad \equiv \max_i \left\{ \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \exp\left(-\frac{||\mathsf{y} - \mathsf{s}_i||^2}{N_0} \right) \right\} \\
&\quad \equiv \min_i \left\{ ||\mathsf{y} - \mathsf{s}_i||^2 - N_0 \log \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \right\}
\end{aligned}$$

OTTO constellation

Equally likely symbols



Symbols s_1 and s_8 are more likely



Part IX

Lecture 9

Pairwise error probability (PEP) definition

The PEP is simply the probability of mistaking the symbol s_i by s_j, i.e.,

$$\mathrm{PEP}^{(i,j)} = \mathbb{P}\left[||\mathsf{Y} - \mathsf{s}_j|| \leq ||\mathsf{Y} - \mathsf{s}_i|| \middle| \mathsf{S} = \mathsf{s}_i \right]$$

PEP in 1D for an AWGN channel

$$\begin{aligned}
\mathrm{PEP}^{(i,j)} &= \mathbb{P}\left[||\mathsf{Y} - \mathsf{s}_j|| < ||\mathsf{Y} - \mathsf{s}_i|| \middle| \mathsf{S} = \mathsf{s}_i \right] \\
&= \int_{y_0}^{\infty} f_{\mathsf{Y}|\mathsf{S}}(\mathsf{y}|\mathsf{s}_i) dy = \mathrm{Q}\left(\frac{y_0 - \mathsf{s}_i}{\sigma} \right) \\
&= \mathrm{Q}\left(\frac{\mathsf{s}_i + \mathsf{s}_j - 2\mathsf{s}_i}{2\sqrt{N_0/2}} \right) = \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}} \right),
\end{aligned}$$

where $y_0 = (\mathsf{s}_i + \mathsf{s}_j)/2$ is halfway between the signals, $\sigma^2 = N_0/2$ the variance of the noise, and $D_{i,j}^2$ is the squared Euclidean distance between s_i and s_j, i.e., $D_{i,j}^2 = \|\mathsf{s}_j - \mathsf{s}_i\|^2$ (see also L_7).

The union bound (UB)

A good **approximation** of the average error probability P_e is

$$P_\mathrm{e} = \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right] \sum_{j\neq i} \mathbb{P}\left[\mathsf{Y}\in\mathcal{R}_j | \mathsf{S}=\mathsf{s}_i\right]$$

$$\leq \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right] \sum_{j\neq i} \mathrm{PEP}^{(i,j)} = \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right] \sum_{j\neq i} \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

The UB for equally likely symbols

$$P_\mathrm{e} \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j\neq i} \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

The previous approximation...

- It is actually **not** an approximation (it is exact) for $M=2$

Can we simplify it even more?

- The Q-functions decrease very fast when the argument increases
- For large arguments (high SNR), one of the Q-functions will dominate

Definition (High-SNR approximation)

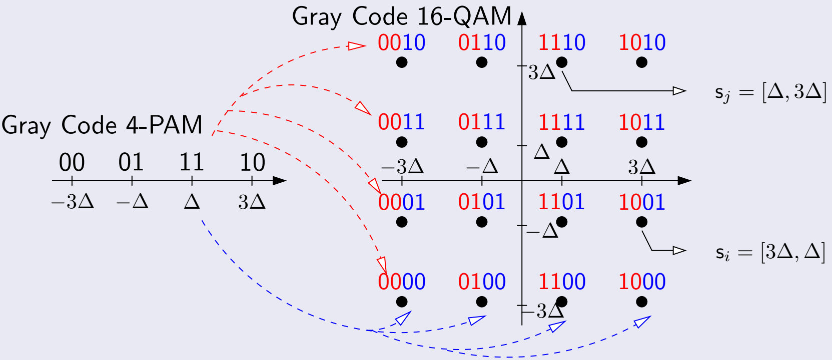
For sufficiently high SNR, the average error probability can be approximated by

$$P_{\mathrm{e}} = \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S} = \mathsf{s}_i\right] \sum_{j \neq i} \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_j | \mathsf{S} = \mathsf{s}_i\right]$$

$$\leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \neq i} \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \approx \frac{2K}{M} \cdot \mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right),$$

where $D_{\min} = \min_{i \neq j}\{D_{i,j}\}$ is the *minimum distance* of the constellation, and K is the number of signal pairs at minimum distance (section 2.6 in [Andersson])

Gray code for 4-PAM and 64-QAM



Part X

Lecture 10

The linear channel

- The channel is represented using a linear filter
- Noise is added at the receiver, but not by the channel
- The output of the channel is

$$r(t) = s(t) * h(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau,$$

or in the frequency domain $R(f) = S(f) \cdot H(f)$.

Different types of channels

- Ideal channel: $h(t) = \delta(t)$
- Multipath channel: $h(t) = k_0\delta(t) + k_1\delta(t-\tau_1) + k_2\delta(t-\tau_2) + \dots$
- Fading channel: $h(t) = \beta(t)$
- Guided channels: wire pair, wave guide, coaxial, cable, and optical fibers

Link Budget

The received power P_{R} [W] is

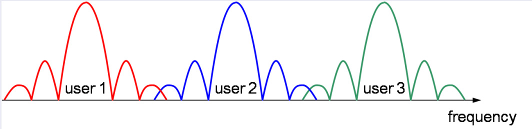
$$P_{\mathrm{R}} = P_{\mathrm{T}} G_{\mathrm{T}} G_{\mathrm{R}} \left(\frac{\lambda}{4\pi d} \right)^2,$$

where P_{T} [W] is the transmitted power by an isotropic antenna, G_{T} and G_{R} are the gains in TX and RX, $\lambda = \frac{3\cdot 10^8}{f_c}$ is the wavelength in [m], f_c is the carrier frequency [Hz], and d is the distance [m].

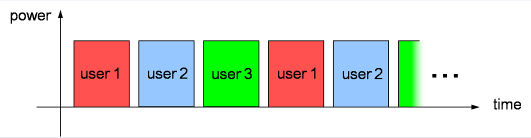
Transmission Impairments

- Additive white Gaussian noise (AWGN)
- Cochannel interference (CCI)
- Adjacent channel interference (ACI)
- Intersymbol interference (ISI)
- Nonlinearities, e.g., PA with clipping

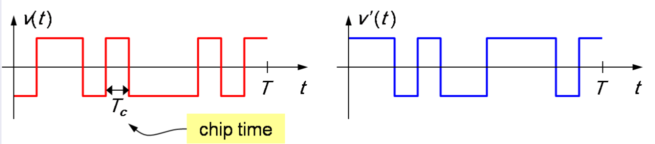
Frequency-Division Multiple Access (FDMA)



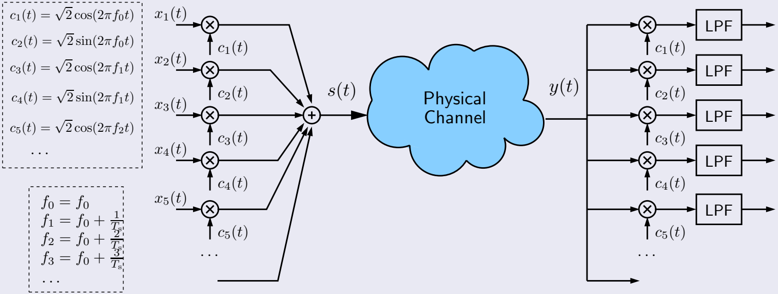
Time-Division Multiple Access (TDMA)



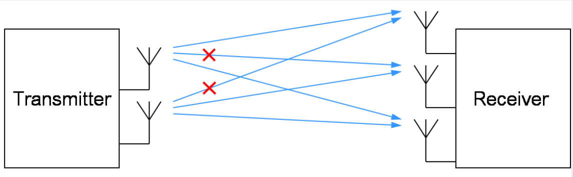
Code-Division Multiple Access (CDMA)



Orthogonal Frequency-Division Multiplexing (OFDM) Tx and Rx



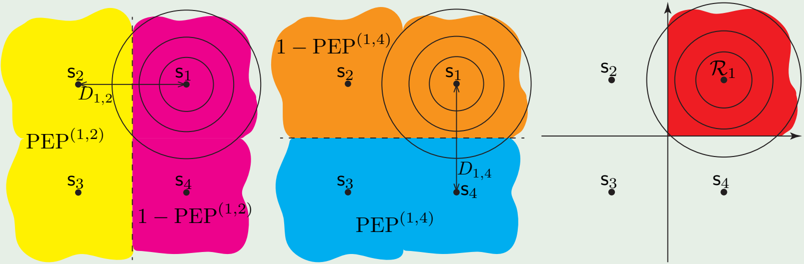
Multiple-Input Multiple-Output (MIMO)



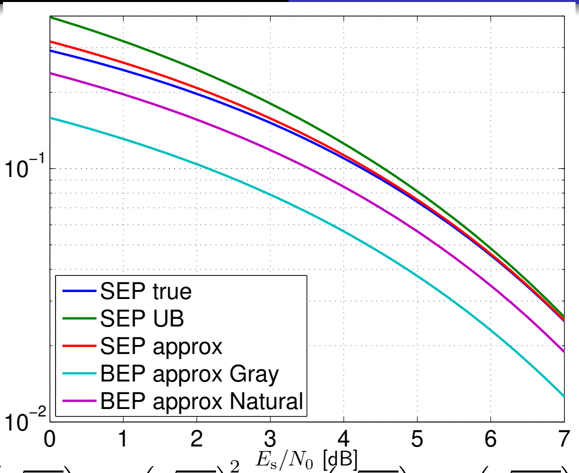
Part XI

Lecture 11

Example (Exact SEP for QPSK)



- $D_{1,2}^2 = D_{1,4}^2 = 2E_\mathrm{s}$
- $\mathrm{PEP}^{(1,2)} = \mathrm{Q}\left(\sqrt{\frac{D_{1,2}^2}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) = \mathrm{PEP}^{(1,4)}$
- $\mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right] = (1-\mathrm{PEP}^{(1,2)})(1-\mathrm{PEP}^{(1,4)})$
  $= \left(1 - \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)\right)^2 = 1 + \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)^2 - 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)$
- $P_\mathrm{e} = 1 - \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right] = 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) - \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)^2$



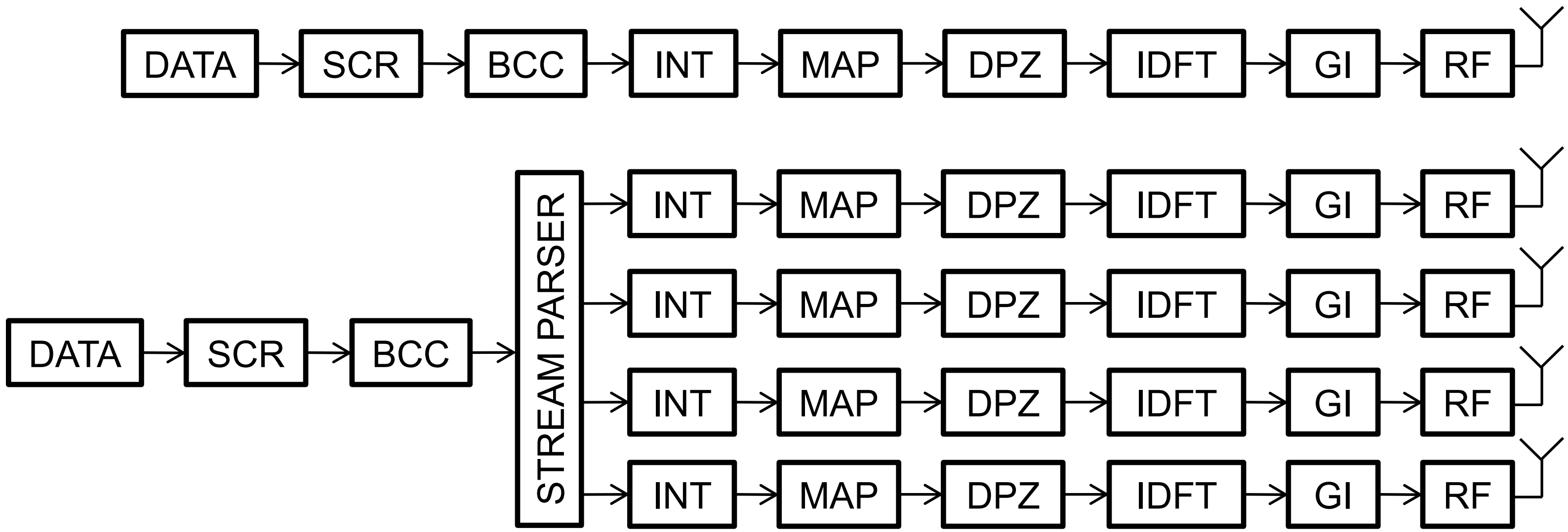
$$P_e = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q\left(\sqrt{\frac{E_s}{N_0}}\right)^2 \leq 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) + Q\left(\sqrt{\frac{2E_s}{N_0}}\right) \approx 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$B_e \approx Q\left(\sqrt{\frac{E_s}{N_0}}\right) \quad \text{(Gray)}, \qquad B_e \approx \frac{3}{2}Q\left(\sqrt{\frac{E_s}{N_0}}\right) \quad \text{(Natural)}$$

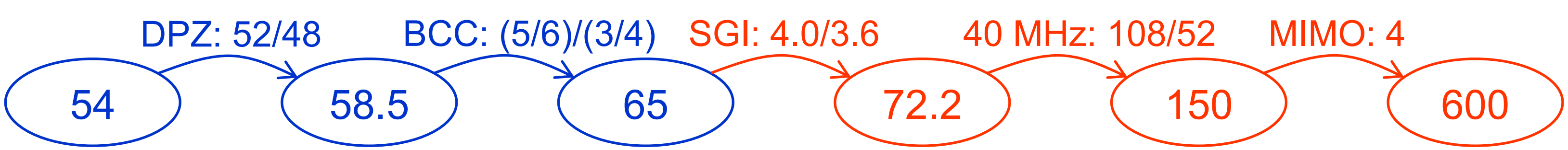
Part XII

Lecture 12

802.11n: Performance?



- The data rate is increased from 54 to 600 Mbit/s.
- Mandatory (DPZ and BCC) and optional (SGI, 40 MHz, MIMO) features.



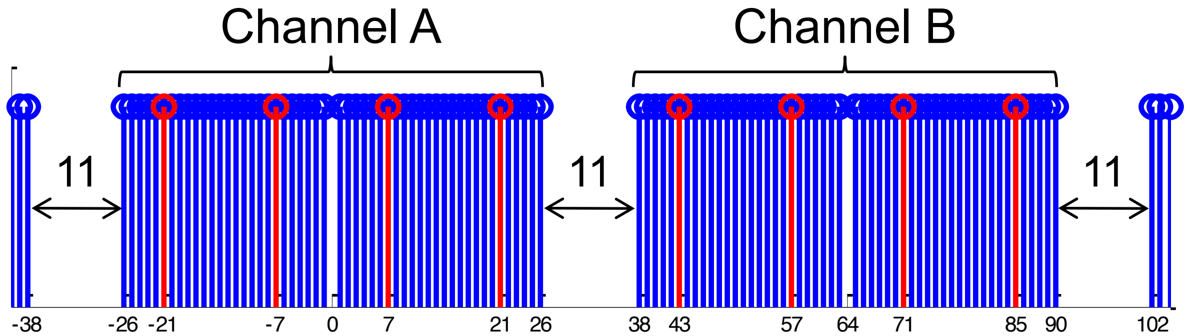
- What about the performance? Can it be improved? Suggestions?

802.11n: DPZ Allocation, 40 MHz

802.11a/g 20 MHz

48 D, 4 P, 12 Z

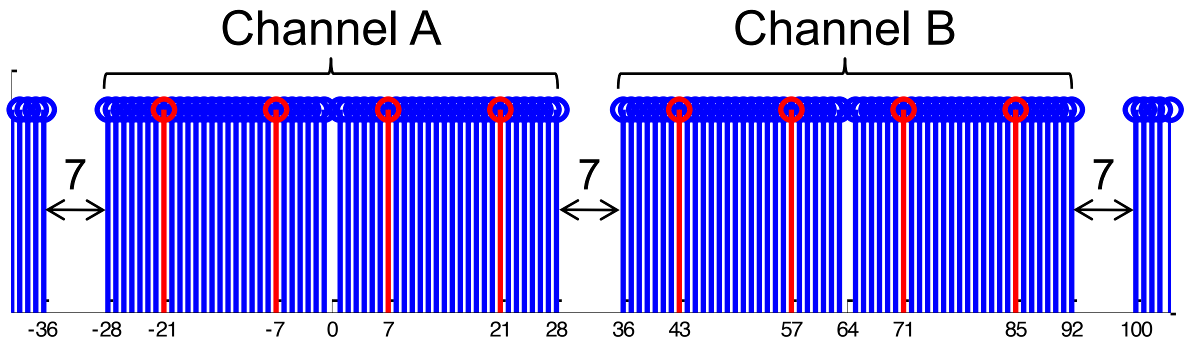
54 Mbit/s



802.11n 20 MHz

52 D, 4 P, 8 Z

72.2 Mbit/s



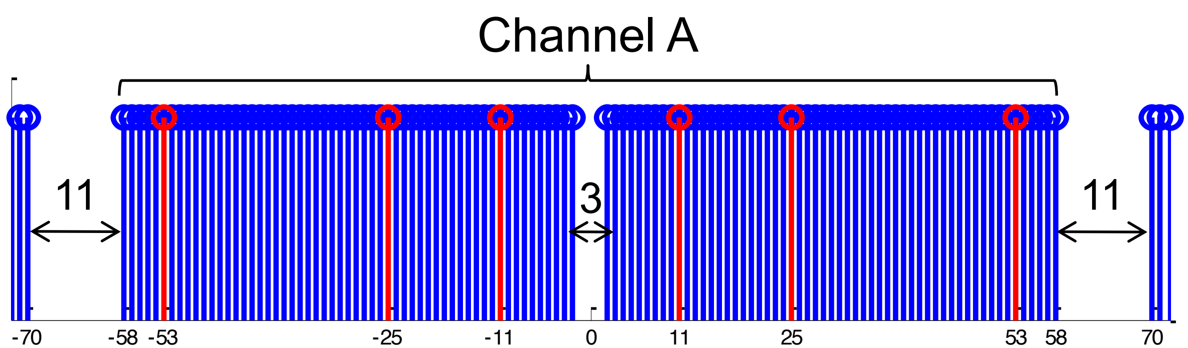
802.11n 40 MHz

108 D, 6 P, 14 Z

72.2·108/52

= 150 Mbit/s

128-point IDFT



Good Luck!