Introduction to Communication Engineering
SSY121, Lecture # 10

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Project Quiz, Wed Oct 16, at 10:00-10:20 (first part of the last lecture)

10 questions (10 points) is answered individually. The quiz is 20 minutes long, and it is composed of 10 multiple choice questions which evaluate the understanding of the scientific base of the project.

Written exam Oct 30 at 08:30-12:30

Sign up for the exam before Oct 10!

Mid-course meeting tomorrow Wed Oct 2!

- Elia Saquand (ERASMUS) saquand@student.chalmers.se
- Yuling Zhang (MPCOM) yulingz@student.chalmers.se
- Isac Olofsson (MPSYS) isaco@student.chalmers.se

Part I

Review: Performance Analysis

ML and MAP detectors for the AWGN channel

For the AWGN channel, the conditional PDF is

$$f_{\mathsf{Y}|\mathsf{S}}(\mathsf{y}|\mathsf{s}_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{||\mathsf{y}-\mathsf{s}_i||^2}{N_0}\right),$$

and therefore,

$$\hat{i}_{\mathsf{ML}} = \arg\max_i \{\mathbb{P}[\mathsf{Y}=\mathsf{y}|\mathsf{S}=\mathsf{s}_i]\} \equiv \min_i \{||\mathsf{y}-\mathsf{s}_i||\}$$

$$\hat{i}_{\mathsf{MAP}} = \arg\max_i \{\mathbb{P}[\mathsf{S}=\mathsf{s}_i|\mathsf{Y}=\mathsf{y}]\} \equiv \min_i \left\{||\mathsf{y}-\mathsf{s}_i||^2 - N_0 \log \mathbb{P}[\mathsf{S}=\mathsf{s}_i]\right\}$$

PEP for an AWGN channel in N dimensions

$$\mathrm{PEP}^{(i,j)} = \mathbb{P}\left[||\mathsf{Y} - \mathsf{s}_j|| < ||\mathsf{Y} - \mathsf{s}_i|| \middle| \mathsf{S} = \mathsf{s}_i \right] = \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}} \right),$$

where $N_0/2$ the variance of the noise, and $D_{i,j}^2$ is the squared Euclidean distance between s_i and s_j, i.e., $D_{i,j}^2 = \|\mathsf{s}_j - \mathsf{s}_i\|^2$.

Symbol Error Probability (equally likely symbols)

$$\begin{aligned} P_{\mathrm{e}} &= \frac{1}{M} \sum_{i=1}^{M} \sum_{j \neq i} \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_j \middle| \mathsf{S} = \mathsf{s}_i \right] \\ &\leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \neq i} \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}} \right) \approx \frac{2K}{M} \cdot \mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}} \right), \end{aligned}$$

where $D_{\min} = \min_{i \neq j}\{D_{i,j}\}$ is the *minimum distance* of the constellation, and K is the number of signal pairs at minimum distance.

Symbol Error Probability (equally likely symbols)

$$P_{\mathrm{e}} = \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathbb{P}\left[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right]$$

$$\leq \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \approx \frac{2K}{M}\cdot\mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

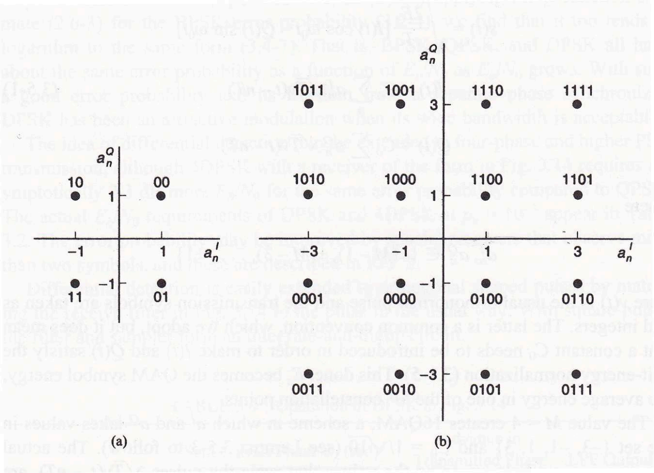
Bit Error Probability (equally likely symbols)

$$B_{\mathrm{e}} = \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\frac{H_{i,j}}{m}\mathbb{P}\left[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right]$$

$$\leq \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\frac{H_{i,j}}{m}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \approx \frac{2H_{\min}}{Mm}\cdot\mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right),$$

where $H_{i,j}$ is the Hamming distance (the number of different bits) between the labels of symbols s_i and s_j, and $H_{\min}$ is the total number of bits differing between signal pairs at minimum distance.

Example (16-QAM from (Anderson))

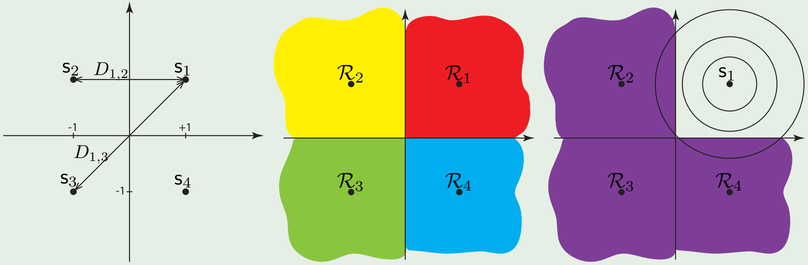


In (b), what is $K =$ $H_{1,11} =$ $H_{\min} =$ $H_{\min} =$ (Gray)?

Part II

Review: Performance Analysis

Example (Exact SEP for QPSK)

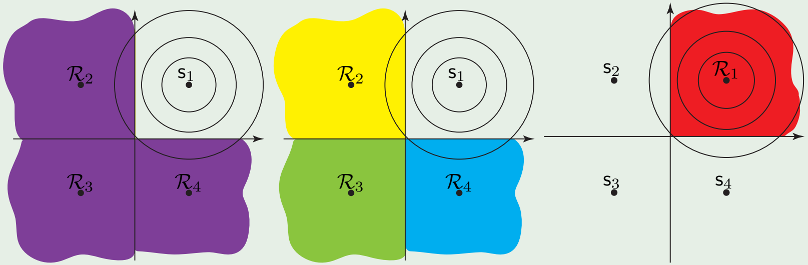


$$E_{\mathrm{s}} = 2, \quad D_{1,2}^2 = D_{1,4}^2 = 4 = 2E_{\mathrm{s}}, \quad \text{and } D_{1,3}^2 = 8 = 4E_{\mathrm{s}}$$

$$P_{\mathrm{e}} = \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_j | \mathsf{S} = \mathsf{s}_i\right] = \frac{1}{M}\sum_{i=1}^{M}\mathbb{P}\left[\mathsf{Y} \notin \mathcal{R}_i | \mathsf{S} = \mathsf{s}_i\right]$$

$$= \mathbb{P}\left[\mathsf{Y} \notin \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right] \qquad \text{(since QPSK is symmetric)}$$

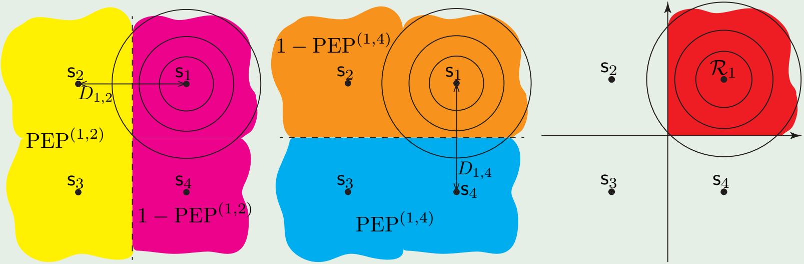
Example (Exact SEP for QPSK)



- What is $P_e = \mathbb{P}\left[\mathsf{Y} \notin \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right]$?

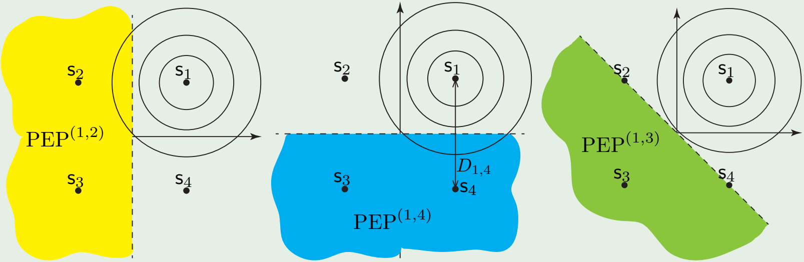
$$P_e = \mathbb{P}\left[\mathsf{Y} \notin \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right] = \sum_{j=2}^{4} \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_j | \mathsf{S} = \mathsf{s}_1\right]$$
$$= 1 - \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right]$$

Example (Exact SEP for QPSK)



- $D_{1,2}^2 = D_{1,4}^2 = 2E_\mathrm{s}$
- $\mathrm{PEP}^{(1,2)} = \mathrm{Q}\left(\sqrt{\frac{D_{1,2}^2}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) = \mathrm{PEP}^{(1,4)}$
- $\mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right] = (1-\mathrm{PEP}^{(1,2)})(1-\mathrm{PEP}^{(1,4)})$
  $= \left(1 - \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)\right)^2 = 1 + \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)^2 - 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)$
- $P_\mathrm{e} = 1 - \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right] = 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) - \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)^2$

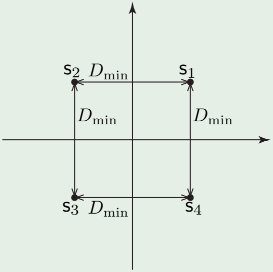
Example (Upper Bound of SEP for QPSK)



- $D_{1,2}^2 = D_{1,4}^2 = 2E_\mathrm{s}$, and $D_{1,3}^2 = 4E_\mathrm{s}$
- $\mathrm{PEP}^{(1,2)} = \mathrm{PEP}^{(1,4)} = \mathrm{Q}\left(\sqrt{\frac{D_{1,2}^2}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)$
- $\mathrm{PEP}^{(1,3)} = \mathrm{Q}\left(\sqrt{\frac{D_{1,3}^2}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{2E_\mathrm{s}}{N_0}}\right)$

$$P_\mathrm{e} \le \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathrm{PEP}^{(i,j)} = \sum_{j=2}^{4}\mathrm{PEP}^{(1,j)} = 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) + \mathrm{Q}\left(\sqrt{\frac{2E_\mathrm{s}}{N_0}}\right)$$

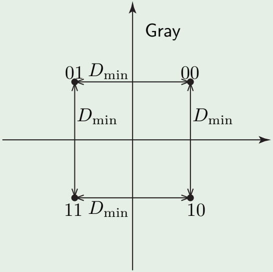
Example (Approximation of SEP for QPSK)

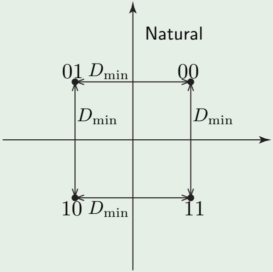


- $D_{\min}^2 = D_{1,2}^2 = D_{2,3}^2 = D_{3,4}^2 = D_{1,4}^2 = 2E_\mathrm{s}$
- Number of signal pairs at minimum distance: $K = 4$

$$P_\mathrm{e} \approx \frac{2K}{M} \cdot \mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) = 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)$$

Example (Approximation of BEP for QPSK)

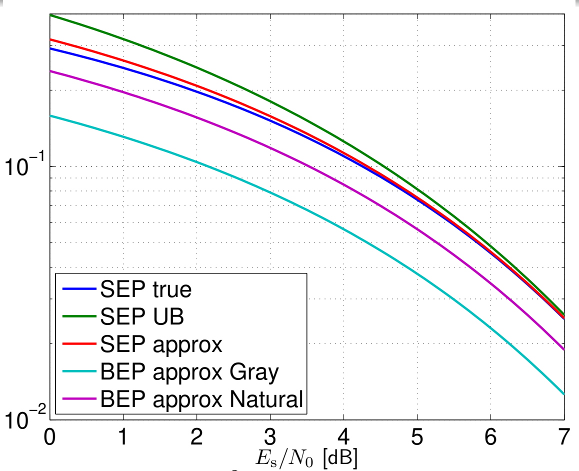




- $E_s = 2E_b, \quad D_{\min}^2 = 2E_s = 4E_b$
- Total number of bits differing between signal pairs at minimum distance: $H_{\min} = 4$ (Gray) $\quad H_{\min} = 6$ (Natural)

$$B_e \approx \frac{2H_{\min}}{Mm} \cdot \mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{E_s}{N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad \text{(Gray)}$$

$$B_e \approx \frac{3}{2}\mathrm{Q}\left(\sqrt{\frac{E_s}{N_0}}\right) = \frac{3}{2}\mathrm{Q}\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad \text{(Natural)}$$



$$P_\mathrm{e} = 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) - \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)^2 \leq 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) + \mathrm{Q}\left(\sqrt{\frac{2E_\mathrm{s}}{N_0}}\right) \approx 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)$$

$$B_\mathrm{e} \approx \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) \quad \text{(Gray)}, \qquad B_\mathrm{e} \approx \frac{3}{2}\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) \quad \text{(Natural)}$$