Introduction to Communication Engineering
SSY121, Lecture # 2

Fredrik Brännström
Fredrik.Brannstrom@chalmers.se

Communication Systems Group
Department of Electrical Engineering
Chalmers University of Technology
Göteborg, Sweden

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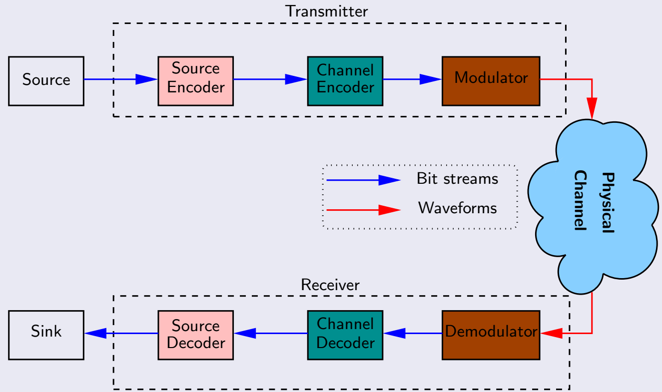
Outline

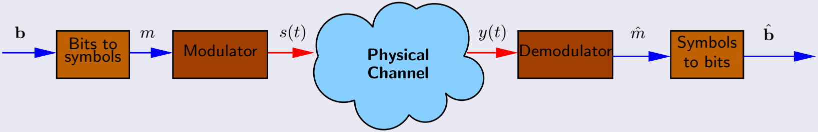
Part I

The Sampling Rx and Nyquist Pulses

Shannon's Communication Model



The General Diagram

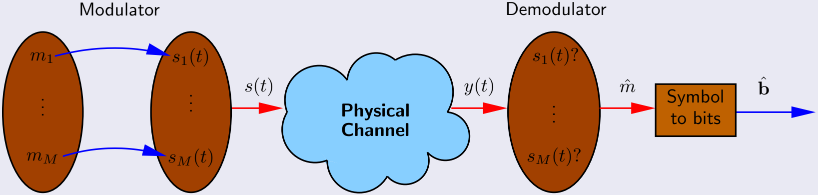


- The length-l binary codeword $\mathbf{b} = [b_1, \ldots, b_l]$ has to be transmitted
- Each of the possible $M = 2^l$ codewords is mapped to a message (symbol) $m \in \{m_1, \ldots, m_M\}$
- The modulator sends a continuous-time baseband signal $s(t)$ through the physical channel each T_s [s], where $s(t)$ is selected from the set of signal alternatives $\mathcal{S} = \{s_1(t), \ldots, s_M(t)\}$.
- The channel introduces some distortion such that $y(t) \neq s(t)$

The General Problem

Using only the channel observation $y(t)$ during one symbol period T_s, *guess* what the transmitted message m was.

The General Problem



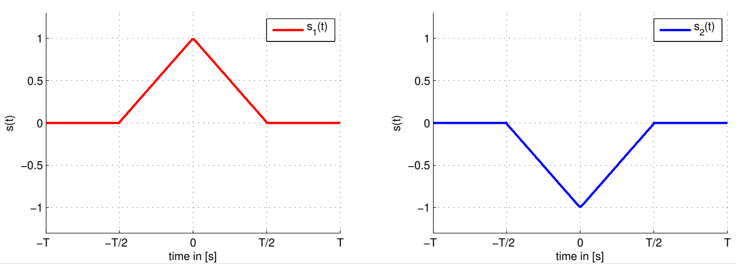
- A symbol is what is transmitted in each time slot (every T_s [s])
- T_s is the symbol duration and T_b is the bit duration, where $lT_\mathrm{b} = T_\mathrm{s}$ ($T_\mathrm{b} = T_\mathrm{s}$ for binary transmission since $l = 1$)
- The symbol rate (baud rate) is the number of symbols per second $R_\mathrm{s} = 1/T_\mathrm{s}$
- The bit rate is the number of bits per second $R_\mathrm{b} = 1/T_\mathrm{b} = l/T_\mathrm{s} = lR_\mathrm{s}$

Example (Binary transmission with triangular pulse)

In binary transmission $l = 1$, $M = 2$, $b \in \{0, 1\}$ and therefore $b = 0 \Rightarrow m = m_1 \Rightarrow s_1(t)$, and $b = 1 \Rightarrow m = m_2 \Rightarrow s_2(t)$. Moreover, we could use only one basic pulse such that

$$s(t) = \begin{cases} s_1(t) = +1 \cdot v(t) & \text{if } m = m_1 \\ s_2(t) = -1 \cdot v(t) & \text{if } m = m_2 \end{cases},$$

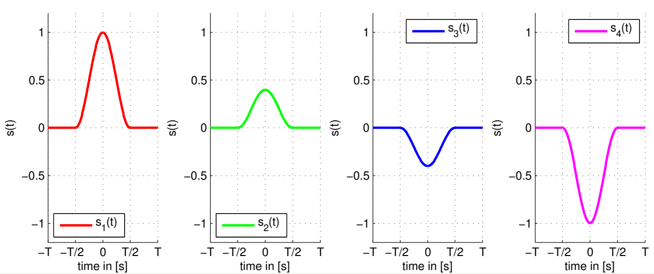
where $v(t)$ is a triangular pulse.



Example (4-ary transmission with arbitrary pulse)

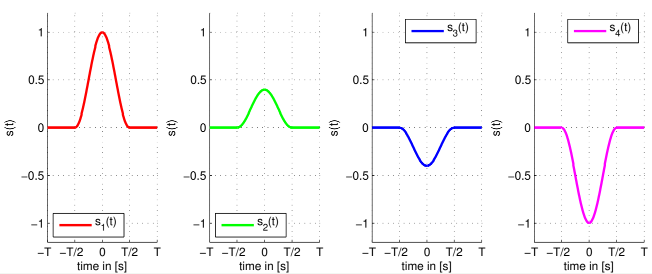
In this case $l = 2$, $M = 4$, $\mathbf{b} \in \{00, 01, 10, 11\}$ and $s_i(t) = a_i \cdot v(t)$ where $a_i \in \{+1, +0.4, -0.4, -1\}$, i.e.,

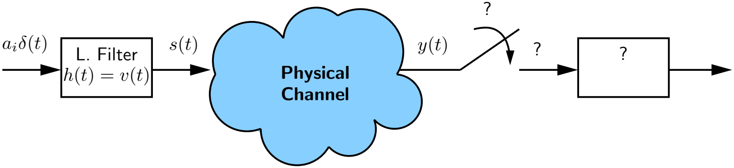
$\mathbf{b}$	m	$s(t)$
00	m_1	$s_1(t) = +1.0 \cdot v(t)$
01	m_2	$s_2(t) = +0.4 \cdot v(t)$
10	m_3	$s_3(t) = -0.4 \cdot v(t)$
11	m_4	$s_4(t) = -1.0 \cdot v(t)$



Example (4-ary transmission with arbitrary pulse)

Transmitted signal $s(t) = a_i \cdot v(t)$ where $a_i \in \{+1, +0.4, -0.4, -1\}$





What can we do with in the receiver?

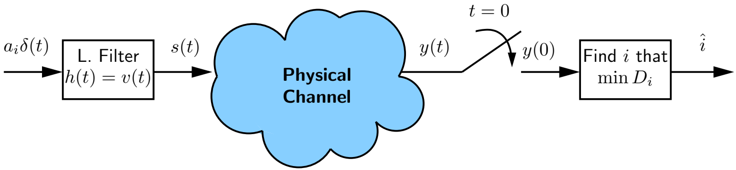
You are given a box which contains:

- a linear filter (with an impulse response you can decide)
- a “bag” with elementary signals (steps, impulses, sinusoids, ...)

The Sampling Rx

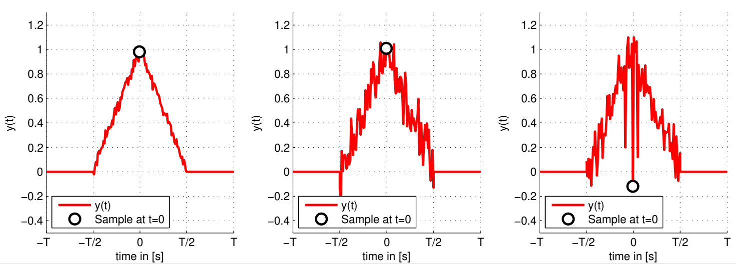
Sample the received signal $y(t)$ at $t = 0$ and choose the signal alternative that is the closest to the sample, i.e.,

$$\hat{i} = \arg \min_{i \in \{1,\ldots,M\}} D_i, \quad \text{where} \quad D_i = |y(0) - s_i(0)|.$$



Example (Sampling Rx for binary transmission with triangular pulse)

Assume that the transmitted message is $m = m_1 \Rightarrow s(t) = s_1(t)$, i.e., a positive triangular pulse. The received signal for three different channels is



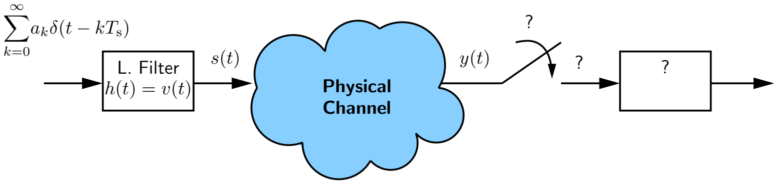
D_1	D_2	$\hat{i}$
$\|+0.98-1\| = 0.02$	$\|+0.98+1\| = 1.98$	1
$\|+1.01-1\| = 0.01$	$\|+1.01+1\| = 2.01$	1
$\|-0.12-1\| = 1.12$	$\|-0.12+1\| = 0.88$	2

A Sequence of M-ary Pulses

Assume that a basic pulse $v(t)$ is used for transmission. The transmitted signal is a linear superposition of time-shifted versions of $v(t)$, i.e.,

$$s(t) = \sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s}),$$

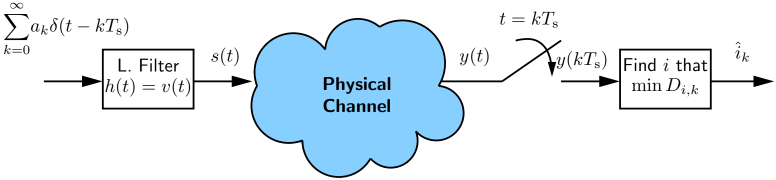
where $a_k \in \mathcal{A}$ where $|\mathcal{A}| = M$.



The Sampling Rx

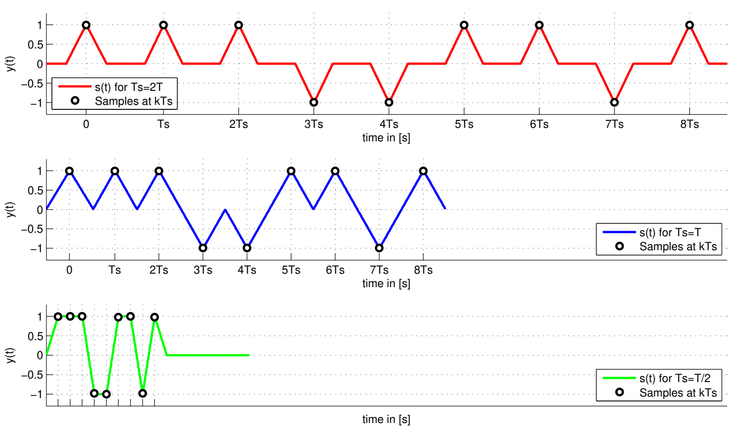
The Sampling Rx simply samples $y(t)$ at time instants $t = kT_\mathrm{s}$ for $k = 0, 1, 2, \ldots$, each time computing the M distances $D_{i,k}$ and selecting the i that gives the minimum $D_{i,k}$.

$$\hat{i}_k = \arg \min_{i \in \{1,\ldots,M\}} D_{i,k}, \quad \text{where} \quad D_{i,k} = |y(kT_\mathrm{s}) - s_i(0)|.$$



Example (Sampling Rx for binary transmission with triangular pulses)

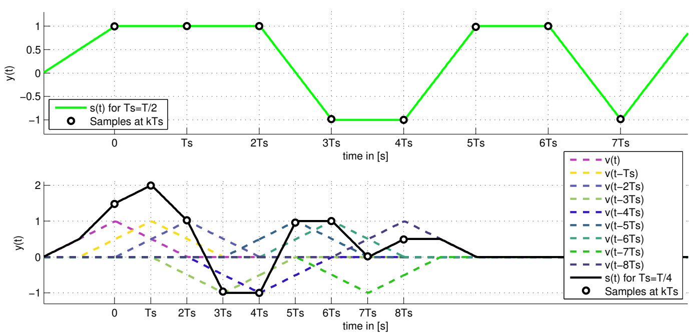
The sequence $\mathbf{b} = [1, 1, 1, 0, 0, 1, 1, 0, 1]$ is transmitted in a perfect channel using a triangular pulse of width T. The symbol rate $R_s = 1/T_s$ is increased...



Increasing and increasing R_s forever?

In the previous example we increased R_s for a fixed BW (fixed T) and the Sampling Rx gave good results. What happens if we keep increasing it?

Example (Previous example with $T_\mathrm{s} = T/4$)



Intersymbol Interference

Intersymbol interference (ISI) appears if R_{s} is too high, i.e., when detecting the kth symbol, interference from other adjacent symbols appear.

Definition (Nyquist pulses)

If the pulse is chosen such that $v(kT_{\mathrm{s}}) = 0$ if $k = \pm 1, \pm 2, \pm 3, \ldots$ and $v(0) \neq 0$, the pulse is called "Nyquist pulse".

Consequences

- $y(kT_{\mathrm{s}})$ depends only on the kth transmitted symbol (since the shifted versions of $v(t)$ coming from other symbols are zero at that time instant)
- No ISI for the Sampling Rx!
- In a perfect channel $y(kT_{\mathrm{s}}) = a_k v(0) \Rightarrow D_k = 0$

Poisson's Sum Formula

$$\sum_{n=-\infty}^{\infty} V\left(f - \frac{n}{T_\mathrm{s}}\right) = T_\mathrm{s} \sum_{m=-\infty}^{\infty} v(mT_\mathrm{s}) \mathrm{e}^{-\jmath 2\pi f m T_\mathrm{s}}$$

Definition (Nyquist pulses in the frequency domain)

A pulse $v(t)$ is called a Nyquist pulse if and only if its Fourier transform $V(f)$ fulfills the following two conditions

$$\sum_{n=-\infty}^{\infty} \Re\left\{V\left(f - \frac{n}{T_\mathrm{s}}\right)\right\} = T_\mathrm{s} v(0)$$

$$\sum_{n=-\infty}^{\infty} \Im\left\{V\left(f - \frac{n}{T_\mathrm{s}}\right)\right\} = 0,$$

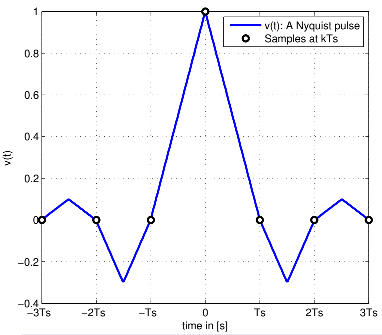
where $T_\mathrm{s} v(0)$ is a real constant.

A particular case

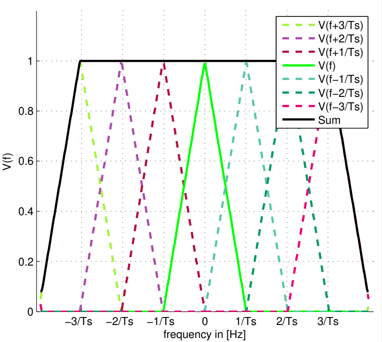
If the pulse is symmetric respect to zero, the previous definition is simplified to

$$\sum_{n=-\infty}^{\infty} V\left(f - \frac{n}{T_\mathrm{s}}\right) = T_\mathrm{s} v(0)$$

A Nyquist Pulse in t-domain



A Nyquist Pulse in f-domain



Part II

The Correlator/Matched Filter Rx

Symbol Rate BW Tradeoff

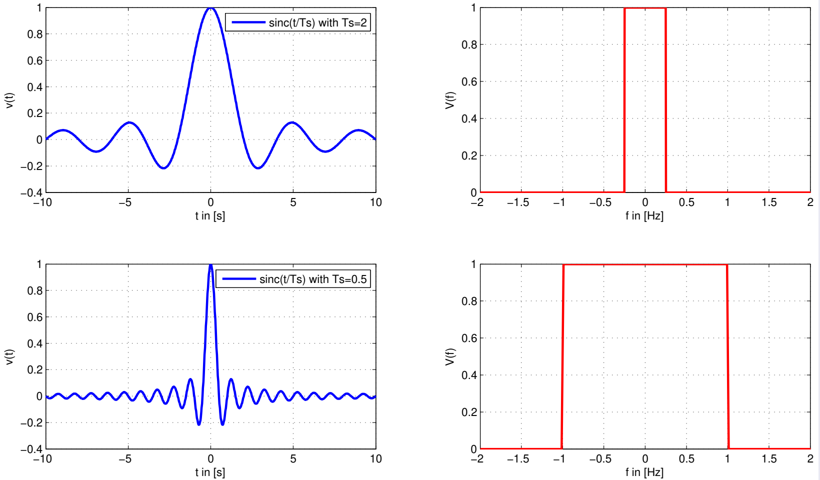
- We know that for a given R_s, ISI-free transmission is indeed possible
- We also know that if R_s increases (T_s decreases), the BW also increases
- We are interested in maximizing R_s using the least possible BW
- The interesting question is then: Is there a fundamental limit on the BW for ISI-free transmission?

The answer is Yes, there is a fundamental lower limit

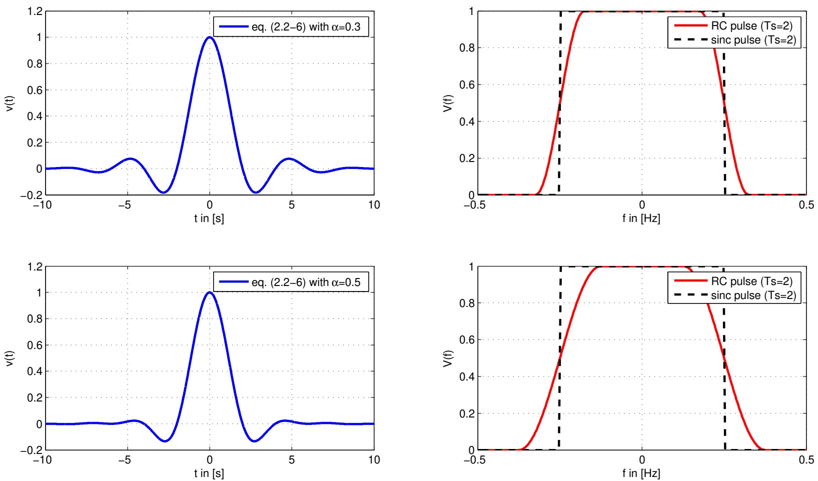
For a given R_s, ISI-free transmission is possible only for BW$\geq \frac{1}{2T_\mathrm{s}}$, with equality if and only if $v(t) = \operatorname{sinc}(t/T_\mathrm{s})$. In other words

- Many different pulses offer ISI-free transmission at expenses of BW$> \frac{1}{2T_\mathrm{s}}$
- Only one pulse offers ISI-free transmission for BW$= \frac{1}{2T_\mathrm{s}}$, the $\operatorname{sinc}(t/T_\mathrm{s})$ pulse
- ISI-free transmission is **not** possible for BW$< \frac{1}{2T_\mathrm{s}}$

The $\operatorname{sinc}(t/T_s)$ pulse and its spectrum



The Raised-Cosine (RC) pulse and its spectrum



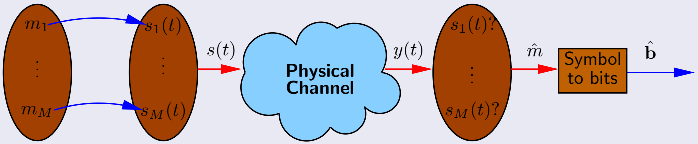
Properties of the Sampling Rx

- Very simple to implement
- Good results for good channel conditions
- Sensitive to bad channel conditions

What do we do if the channel is very noisy?

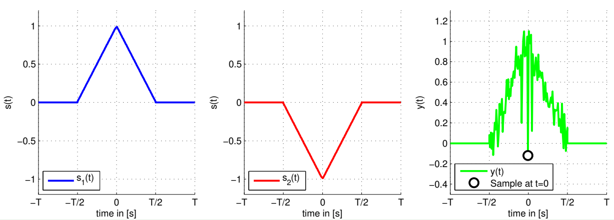
We use a better (but more complex) Rx: **The linear Rx**

The system we are analyzing...



Example (From the previous Example)

A binary message is transmitted using a triangular pulse. At the Rx, and based on $y(t)$, we need to guess which message was transmitted.

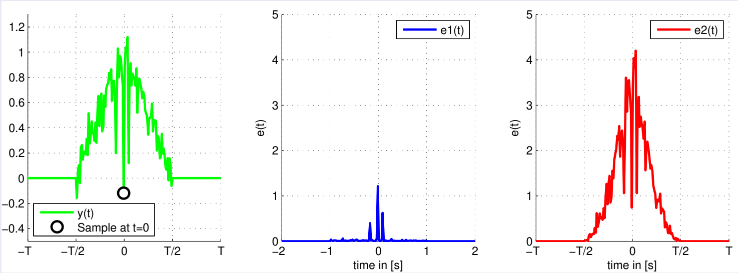


Definition (The error energy)

The error between the received signal $y(t)$ and the ith signal alternative is defined as $e_i(t) = |y(t) - s_i(t)|$, and therefore, one can define the error energy as follows

$$E_{e_i} = \int_{-\infty}^{\infty} [y(t) - s_i(t)]^2 \, dt.$$

For the previous example



Minimizing the error energy

- The error energies can be used for decision ⇒ The problem we need to solve is

$$\boxed{\hat{i} = \arg\min_i \left\{ \int_{-\infty}^{\infty} [y(t) - s_i(t)]^2 \, dt \right\}}$$

- It can be shown that

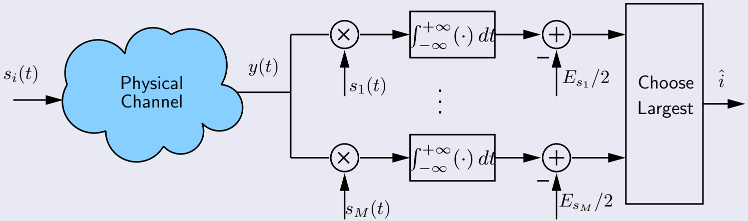
$$\min_i \left\{ \int_{-\infty}^{\infty} [y(t) - s_i(t)]^2 \, dt \right\} \equiv \max_i \left\{ \int_{-\infty}^{\infty} y(t) s_i(t) \, dt - \frac{E_{s_i}}{2} \right\},$$

where E_{s_i} is the energy of the ith signal alternative,
$E_{s_i} = \int_{-\infty}^{\infty} s_i^2(t) \, dt$.

- The problem we need to solve is then

$$\boxed{\hat{i} = \arg\max_i \left\{ \int_{-\infty}^{\infty} y(t) s_i(t) \, dt - \frac{E_{s_i}}{2} \right\}}$$

The Correlator Rx



Some Comments

- If all the signal alternatives have the same energy ($E_{s_1} = E_{s_2} = \ldots$), the problem is reduced to

$$\hat{i} = \arg\max_i \left\{ \int_{-\infty}^{\infty} y(t) s_i(t)\, dt \right\}$$

- Since the "heart" of the processing are correlation integrals, this Rx is called the **correlator Rx** (CR)

The Correlation Integral

- It can be proved that

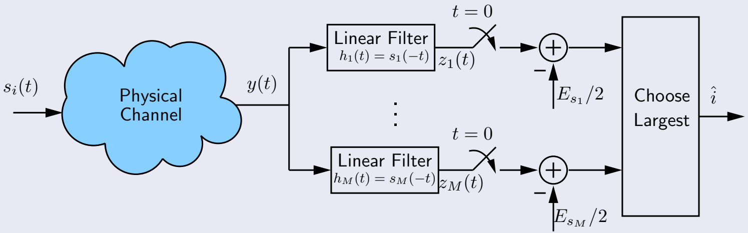
$$\int_{-\infty}^{\infty} y(t)s_i(t)\,dt = y(t) * h_i(t)\Big|_{t=0},$$

  where $h_i(t) = s_i(-t)$, or in frequency domain $H(f) = S^*(f)$.
- Proof?
- The previous result implies that the correlation integrals in the previous block diagram can be replaced by linear filters and sampling at $t = 0$.

The Causality Problem

For a linear filter to be realizable, it must be causal, i.e., if $s_i(t) \neq 0$ for $-T_1 \leq t \leq T_2$ $(T_1, T_2 > 0)$, the filter with impulse response $h_i(t) = s_i(-t)$ is not causal (not realizable). To have a causal filter the impulse response must be shifted T_2 [s], i.e., $h_i(t) = s_i(-t + T_2)$. The corresponding samples will be delayed in T_2 [s].

The Matched Filter Rx



Some Comments

- Since the filters are *matched* to the signal alternatives, it is called the **matched filter Rx** (MFR)
- Both the **correlator Rx** (CR) and MFR can be implemented using *linear filters* and are therefore called **linear Rx** (LR)
- In terms of performance, the CR and the MFR are equivalent
- However, the performance of the CR and MFR is different than the performance of **Sampling Rx**

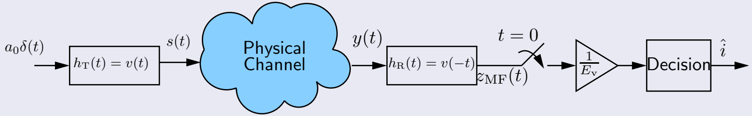
One M-ary Pulse

The transmitted signal is

$$s(t) = a_0 v(t)$$

where $a_0 \in \mathcal{A}$ is the amplitude of the transmitted symbol, and E_v is the energy of the pulse $v(t)$.

The MF Rx for this case?



Example (MF Rx for 4-ary transmission)

Assume $\mathcal{A} = \{-3, -1, 1, 3\}$ and a channel which is perfect.
Why do we need the gain $1/E_\mathrm{v}$?

A sequence of M-ary pulses

The transmitted signal is

$$s(t) = \sum_{k=0}^{\infty} a_k v(t - kT_\mathrm{s})$$
$$= a_0 v(t) + a_1 v(t - T_\mathrm{s}) + a_2 v(t - 2T_\mathrm{s}) + \dots,$$

where $a_k \in \mathcal{A}$ represent the amplitude of the symbol transmitted at the kth time instant.

The MF Rx for this case?

HW for L_3!!

Today's Summary

- Sampling Rx (one pulse and a train of pulses)
- Nyquist pulses. No ISI for Sampling Rx
- Rate and BW tradeoff
- Sinc pulses and RC pulses
- MFR/CR (one pulse and a train of pulses)
- Scaling by $1/E_{\text{v}}$