Introduction to Communication Engineering
SSY121, Lecture # 8

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Outline

1. Summary of Synchronization
   - Carrier Frequency synchronization
   - Carrier Phase synchronization
   - Symbol synchronization
   - Frame synchronization
2. A Random Model for Communications and AWGN
   - Additive noise channels
   - Gaussian random variables
   - The AWGN channel
3. ML and MAP Detection in AWGN Channels
   - ML detection
   - MAP detection

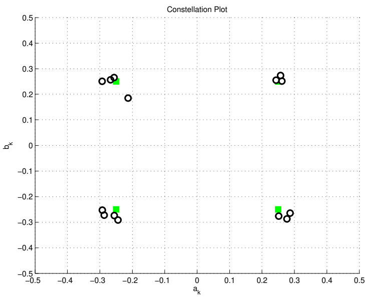
Part I

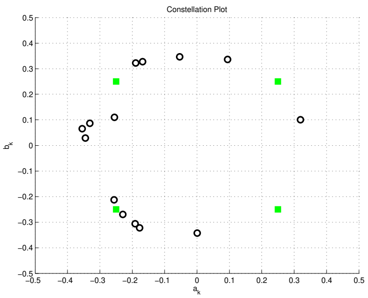
Summary of Synchronization

Synchronization in Rx

- Carrier frequency synchronization:
  - The local oscillators in Rx is not exactly f_c but $f_c + \Delta$
- Carrier phase synchronization:
  - The phase of the local oscillators in Rx is different than in Tx, i.e.,
  - $\cos(w_c t)$ and $\sin(w_c t)$ are $\cos(w_c t + \theta)$ and $\sin(w_c t + \theta)$
- Symbol synchronization:
  - The output of the MFs are taken at the wrong instant, e.g.,
  - $t = 0.0T_\mathrm{s}, 1.1T_\mathrm{s}, 2.2T_\mathrm{s}, \ldots$ wrong sampling frequency
  - $t = 0.1T_\mathrm{s}, 1.1T_\mathrm{s}, 2.1T_\mathrm{s}, \ldots$ wrong timing
- Frame synchronization:
  - Locate where does the block of information bits start, i.e., where the beginning of the "sentence" starts.

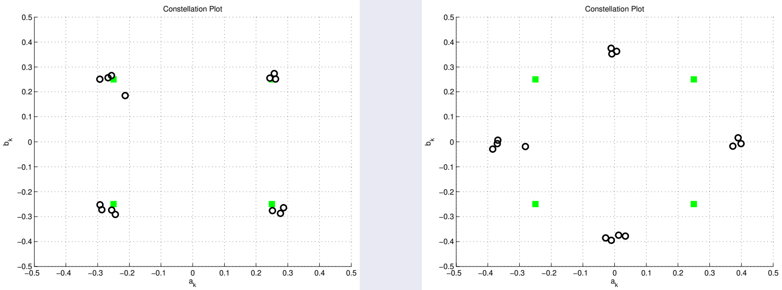
Constellation Plot for QPSK, good channel, and $\Delta = 0.1 f_c$





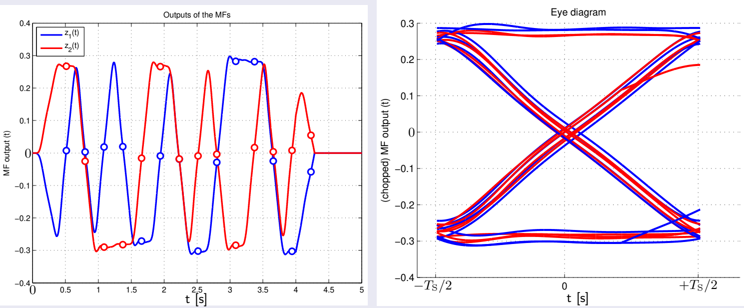
Wrong frequency $\Rightarrow$ a "rotating" constellation

Constellation Plot for QPSK, good channel, and $\theta = \pi/4$

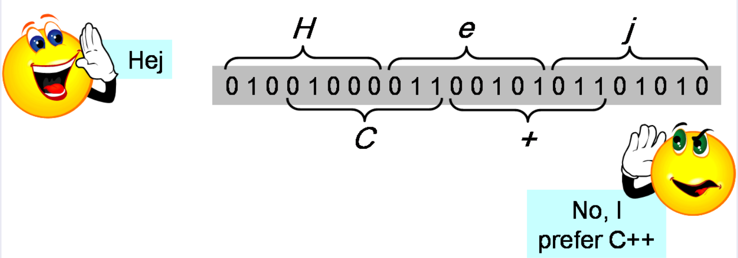


Wrong phase $\Rightarrow$ a "rotated" constellation

When the sampling instant is incorrect... $t_k = kT_\mathrm{s} + 0.5T_\mathrm{s}$



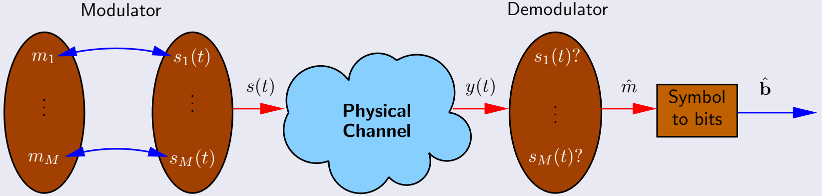
When the frame synchronization is incorrect...



Part II

A Random Model for Communications and AWGN

The general problem



The problem

- Given the observation $y(t)$, *guess* what the transmitted message was
- The channel will modify the transmitted signal $s(t)$ in a random fashion
- If there is no randomness, our guesses will be always correct $\rightarrow$ problem solved $\rightarrow$ no need to study communication theory
- We would like to minimize the number of errors:

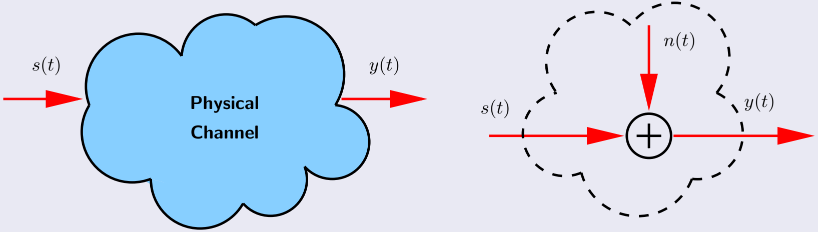
$$\min \{\mathbb{P}[\hat{m} \neq m]\},$$

or equivalently, $\max \{\mathbb{P}[\hat{m} = m]\}$

Additive white Gaussian Noise

- We need a model for the randomness of the channel
- A very common one is the additive white Gaussian noise (AWGN):
  - It represents accurately "thermal noise," generated by movement of electrons at the receiver
  - It represents some cases of "atmospheric noise," caused by the weather
  - The sum of many small independent contributions is Gaussian ("central limit theorem")
  - It allows a simple analysis
- Are there other channel models?
  - Non-Gaussian noise
  - Poisson channels
  - Fading channels (multiplicative interference)
  - Noise from other users

Additive noise



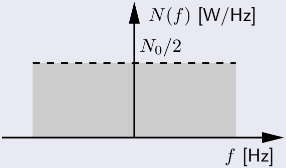
- It is additive noise because

$$y(t) = s(t) + n(t)$$

- The added noise $n(t)$ does not depend on $s(t)$

White Noise

- If the noise has the following properties
  - $\mathbb{E}[n(t)] = 0$
  - $\mathbb{E}[n(t_1)n(t_2)] = \frac{N_0}{2}\delta(t_1 - t_2)$ (uncorrelated for $t_1 \neq t_2$)
- The result is a constant ("white") spectrum



- Theoretically, the power of the noise is $\int_{-\infty}^{\infty} N(f)\,df = \infty$ [W] ...
- When observed over any finite BW, the noise has finite power (no explosions!)
- If the noise is flat over the frequencies of interest, it will still be white for our purposes

1D Gaussian random variables

- Gaussian distribution is equivalent to normal distribution
- The mean value is μ and the variance is σ^2
- We use $\mathcal{N}(\mu, \sigma^2)$ to denote this random variable
- The probability density function (PDF) of $\mathcal{N}(\mu, \sigma^2)$ is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- The normalized Gaussian distribution is obtained when $\mu = 0$ and $\sigma^2 = 1$
- The probability density function of $\mathcal{N}(0, 1)$ is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Normalized Gaussian distribution

- The PDF of $\mathcal{N}(0,1)$ is

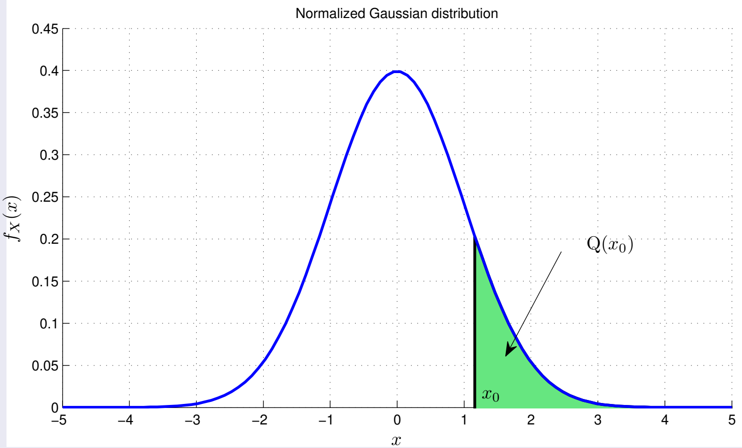
$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right)$$

- The probability of X being larger than x_0 is

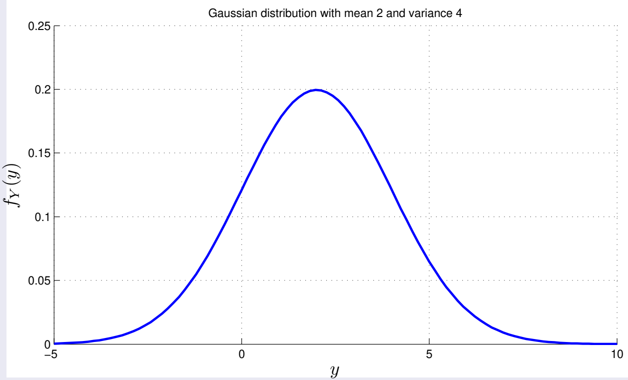
$$\mathbb{P}\left[X > x_0\right] = \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = \mathrm{Q}(x_0)$$

- $\mathrm{Q}(0) = 1/2$
- $\mathrm{Q}(-x_0) = 1 - \mathrm{Q}(x_0)$
- $1 - \mathrm{Q}(x_0)$ is tabulated in Mathematics Handbook
- $\mathrm{Q}(x_0)$ is "*qfunc*(x_0)" in Matlab

Normalized Gaussian distribution



Another Gaussian distribution ($\mu = 2$ and $\sigma^2 = 4$)



The probability of $Y \sim \mathcal{N}(\mu, \sigma^2)$ being larger than y_0 is

$$\mathbb{P}\left[Y > y_0\right] = \mathrm{Q}\left(\frac{y_0 - \mu}{\sigma}\right)$$

2D Gaussian random variables

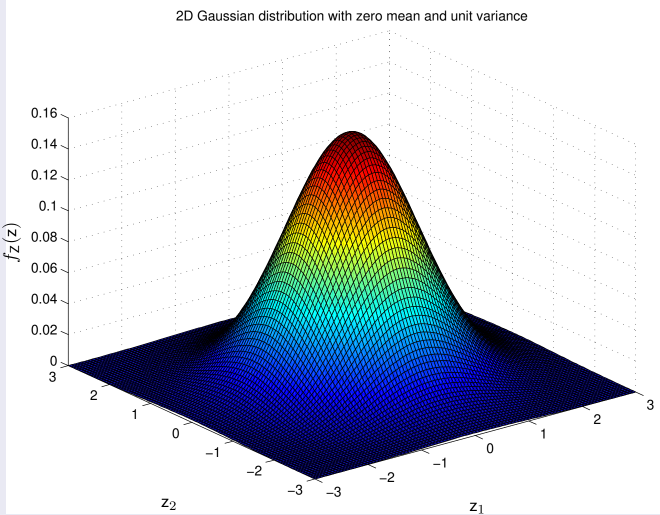
- N independent Gaussian random variables with the same variance σ^2 have a PDF given by:

$$f_{\mathsf{Z}}(\mathsf{z}) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{||\mathsf{z}-\boldsymbol{\mu}||^2}{2\sigma^2}\right)$$

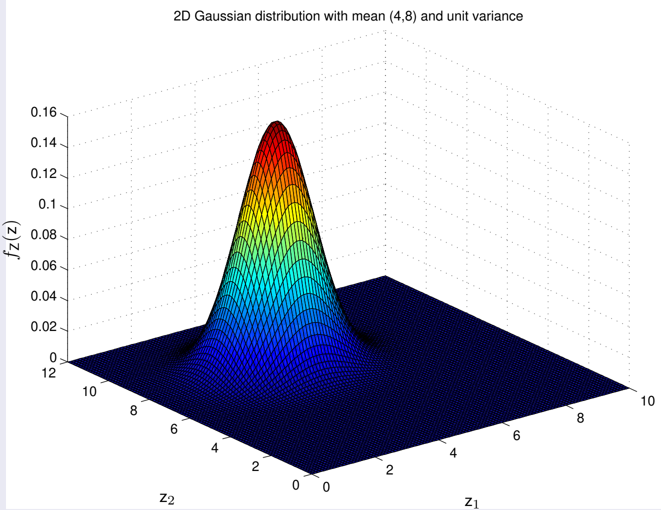
- For the 2D case, the normalized Gaussian distribution is $\mathsf{Z} = [X_1, X_2]$, $\boldsymbol{\mu} = [0, 0]$, $\sigma^2 = 1$ and the PDF is

$$f_{\mathsf{Z}}(\mathsf{z}) = \frac{1}{2\pi} \exp\left(-\frac{||\mathsf{z}||^2}{2}\right)$$

Normalized 2D Gaussian distribution



Not-normalized 2D Gaussian distribution



Definition (The AWGN channel)

If the noise is *additive*, *white*, and also *Gaussian*, we talk about an AWGN channel: $y(t) = s(t) + n(t)$

The AWGN channel in vector space

- Or in vectorial notation (L_4)

$$\mathsf{y} = \mathsf{s} + \mathsf{n},$$

  where $\mathsf{s} = [\mathsf{s}_1, \ldots, \mathsf{s}_N]$

- The noise can be also represented using the basis functions $\phi_1(t), \ldots, \phi_N(t)$ plus other basis functions $\phi_{N+1}(t), \ldots, \phi_{N'}(t)$, i.e.,

$$n(t) = \sum_{i=1}^{N} \mathsf{n}_i \phi_i(t) + \sum_{i=N+1}^{N'} \mathsf{n}_i \phi_i(t)$$

- More than N basis functions are needed
- Only the N that span the signals $s_i(t)$ are relevant for detection (L_4)

The AWGN channel

- The continuous time channel

$$y(t) = s(t) + n(t)$$

can then be replaced by a vectorial model

$$\mathsf{y} = \mathsf{s} + \mathsf{n},$$

where $\mathsf{s} = [\mathsf{s}_1, \ldots, \mathsf{s}_N]$, $\mathsf{n} = [\mathsf{n}_1, \ldots, \mathsf{n}_N]$, $\mathsf{y} = [\mathsf{y}_1, \ldots, \mathsf{y}_N]$

- Each component of the received vector is $\mathsf{y}_i = \mathsf{s}_i + \mathsf{n}_i$

- If $n(t)$ is AWGN with PSD $N_0/2$, what is the distribution of n?
  - $\mathbb{E}[\mathsf{n_i}] = 0$ for all i
  - $\text{var}[\mathsf{n_i}] = N_0/2$ for all i
  - $\mathsf{n_i}$ and $\mathsf{n_k}$ are independent for $i \neq k$
- Note that
  - The proof is given in Sec. 2.5.3 of [Anderson]
  - The previous result is valid for any basis functions
  - PSD and variance are different entities, but have the same value

Summarizing...

In summary, the vectorial channel model is $\mathsf{y} = \mathsf{s} + \mathsf{n}$, where

- s is the transmitted symbol (a point in an N-dimensional Euclidean space)
- n is the noise (N independent and identically distributed (i.i.d.) $\mathcal{N}(0, N_0/2)$ components) independent of s
- y is the received symbol (a point in an N-dimensional Euclidean space which is a "translated" version of s)
- The PDF of the 2D noise is (zero mean and variance $N_0/2$)

$$f_{\mathsf{N}}(\mathsf{n}) = \frac{1}{\pi N_0} \exp\left(-\frac{||\mathsf{n}||^2}{N_0}\right)$$

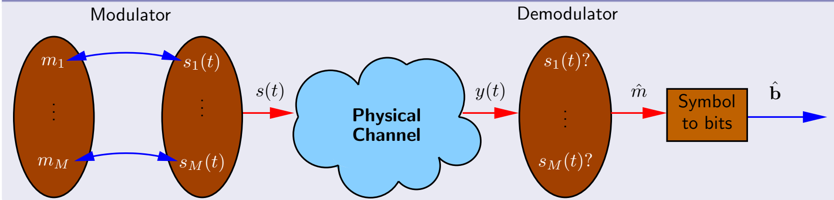
- For a given transmitted $\mathsf{s} = \mathsf{s}_i$, the received signal y has a similar distribution as n, but centered at s_i

$$f_{\mathsf{Y}}(\mathsf{y}|\mathsf{s}_i) = \frac{1}{\pi N_0} \exp\left(-\frac{||\mathsf{y} - \mathsf{s}_i||^2}{N_0}\right)$$

Part III

ML and MAP Detection in AWGN Channels

The general problem



The fundamental problem

- At the Tx, one of the possible M symbols was transmitted ($\mathsf{S} = \mathsf{s}$).
- Given the observation of the random variable $\mathsf{Y} = \mathsf{y}$, the receiver must *guess* what the transmitted message was.
- We would like to minimize the number of errors,i.e.,

$$\min \left\{ \mathbb{P}\left[\hat{\mathsf{s}} \neq \mathsf{s} \right] \right\}$$

- Equivalently, choose i ($\hat{\mathsf{s}} = \mathsf{s}_i$) which maximizes

$$\max_i \left\{ \mathbb{P}\left[\mathsf{S} = \mathsf{s}_i | \mathsf{Y} = \mathsf{y} \right] \right\}$$

Bayes' rule

Bayes' rule tells us how to reverse conditional probabilities or PDFs:

$$f_{\mathsf{X}|\mathsf{Y}}(\mathsf{x}|\mathsf{y}) = \frac{f_{\mathsf{X}}(\mathsf{x})}{f_{\mathsf{Y}}(\mathsf{y})} f_{\mathsf{Y}|\mathsf{X}}(\mathsf{y}|\mathsf{x}),$$

using that the joint PDF can be factorized as

$$f_{\mathsf{X},\mathsf{Y}}(\mathsf{x},\mathsf{y}) = f_{\mathsf{X}|\mathsf{Y}}(\mathsf{x}|\mathsf{y}) f_{\mathsf{Y}}(\mathsf{y}) = f_{\mathsf{Y}|\mathsf{X}}(\mathsf{y}|\mathsf{x}) f_{\mathsf{X}}(\mathsf{x})$$

The fundamental problem

Choose i ($\hat{\mathsf{s}} = \mathsf{s}_i$) which maximizes the conditional probabilities

$$\begin{aligned}
\max_i \{\mathbb{P}[\mathsf{S} = \mathsf{s}_i|\mathsf{Y} = \mathsf{y}]\} &\equiv \max_i \left\{ \frac{\mathbb{P}[\mathsf{S} = \mathsf{s}_i]}{\mathbb{P}[\mathsf{Y} = \mathsf{y}]} \mathbb{P}[\mathsf{Y} = \mathsf{y}|\mathsf{S} = \mathsf{s}_i] \right\} \\
&\equiv \max_i \{\mathbb{P}[\mathsf{S} = \mathsf{s}_i]\, \mathbb{P}[\mathsf{Y} = \mathsf{y}|\mathsf{S} = \mathsf{s}_i]\} \\
&\equiv \max_i \{\mathbb{P}[\mathsf{Y} = \mathsf{y}|\mathsf{S} = \mathsf{s}_i]\},
\end{aligned}$$

where in the last step we assume that the symbols are transmitted with equal probability, i.e., $\mathbb{P}[\mathsf{S} = \mathsf{s}_i] = \frac{1}{M}$

ML detection

In summary, minimizing the error probability is equivalent to maximizing the conditional probabilities (or PDFs)

$$\max_i \{\mathbb{P}[\mathsf{Y}=\mathsf{y}|\mathsf{S}=\mathsf{s}_i]\},$$

which is called **maximum likelihood** (ML) detection.

ML for the AWGN channel

For the AWGN channel, the conditional PDF is

$$f_{\mathsf{Y}|\mathsf{S}}(\mathsf{y}|\mathsf{s}_i)=\frac{1}{(\pi N_0)^{N/2}}\exp\left(-\frac{||\mathsf{y}-\mathsf{s}_i||^2}{N_0}\right),$$

and therefore,

$$\max_i \{\mathbb{P}[\mathsf{Y}=\mathsf{y}|\mathsf{S}=\mathsf{s}_i]\} \equiv \min_i \{||\mathsf{y}-\mathsf{s}_i||\}$$

Does it look familiar?

Does the previous expression look familiar? See L_4

The Matched filter again

The matched filter (correlator) receiver is the optimal choice in two cases:

- Minimizing the error energy (L_2)
- Minimizing the error probability for AWGN channels (ML detection, previous slide)

Maximum a posteriori detection

- For ML detection, we assumed that all the symbols are equally likely
- If they are not, the problem is

$$\max_i \{\mathbb{P}[\mathsf{S} = \mathsf{s}_i | \mathsf{Y} = \mathsf{y}]\} \equiv \max_i \{\mathbb{P}[\mathsf{S} = \mathsf{s}_i]\, \mathbb{P}[\mathsf{Y} = \mathsf{y} | \mathsf{S} = \mathsf{s}_i]\}$$

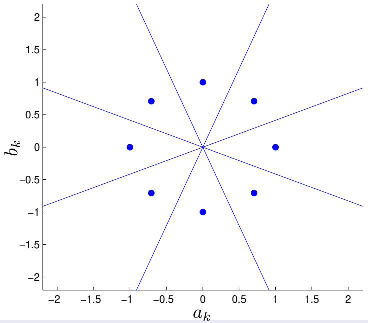
- This is called **maximum a posteriori** (MAP) detection.

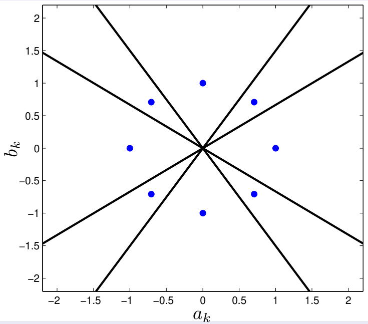
MAP detection for the AWGN channel

- If the channel is AWGN the MAP detection is

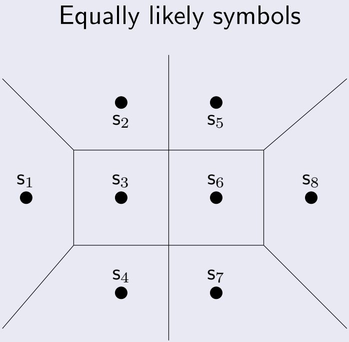
$$\begin{aligned} &\max_i \{\mathbb{P}[\mathsf{S} = \mathsf{s}_i | \mathsf{Y} = \mathsf{y}]\} \\ &\equiv \max_i \left\{ \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{||\mathsf{y} - \mathsf{s}_i||^2}{N_0} \right) \right\} \\ &\equiv \max_i \left\{ \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \exp\left(-\frac{||\mathsf{y} - \mathsf{s}_i||^2}{N_0} \right) \right\} \\ &\equiv \min_i \left\{ ||\mathsf{y} - \mathsf{s}_i||^2 - N_0 \log \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \right\} \end{aligned}$$

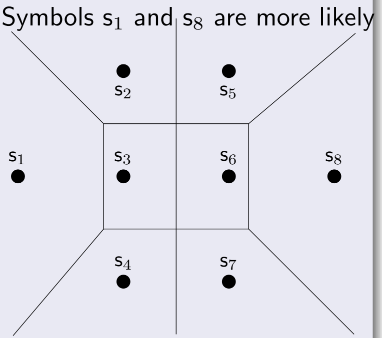
8-PSK with equally likely (left) and non equally likely symbols (right)





OTTO constellation





Today's Summary

- Short summary of AWGN
- The MFR minimizes both the error energy and the error probability in AWGN channels
- Minimizing the error probability assuming the transmitted symbols are equally likely is called **maximum likelihood (ML)** detection
- Minimizing the error probability knowing the transmitted symbol probabilities is called **maximum a posteriori (MAP)** detection
- The decision regions for ML and MAP are different if the transmitted symbols are not equally likely