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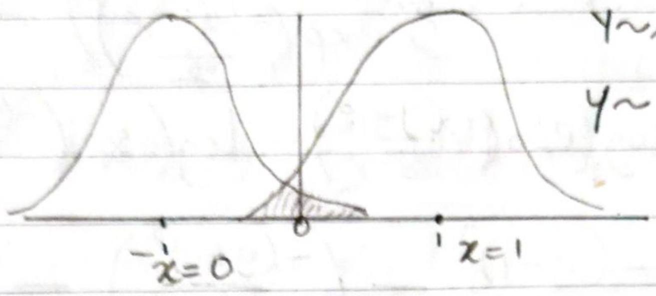
Problem 3



$Y \sim N(1,4)$, for $X=1$

$Y \sim N(-1,1)$ for $X=0$

Gaussian Distribution is given for $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

for $x=1$; $\mu=1, \sigma^2=4$

for $x=0$; $\mu=-1, \sigma^2=1$

$$P(y|x=1) = \frac{1}{\sqrt{2\pi(4)}} \exp\left(\frac{-(y-1)^2}{2(4)}\right) = \frac{1}{\sqrt{8\pi}} \exp\left(\frac{-(y-1)^2}{8}\right)$$

$$P(y|x=0) = \frac{1}{\sqrt{2\pi(1)}} \exp\left(\frac{-(y-(-1))^2}{2(1)}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(y+1)^2}{2}\right)$$

For optimal rule, we compare the log likelihood ratio to a threshold.

$$L(y) = \frac{P(y|x=1)}{P(y|x=0)}$$, if $L=1$, they are neutral

$$L(y) = \frac{1}{\sqrt{8\pi}} \exp\left(\frac{-(y-1)^2}{8}\right) \Big/ \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(y+1)^2}{2}\right)$$

$$L(y) = \left(\frac{\cancel{\sqrt{2\pi}}}{2\cancel{\sqrt{2\pi}}}\right) \cdot \exp\left(\frac{-(y-1)^2}{8}\right) \cdot \frac{1}{\exp\left(\frac{-(y+1)^2}{2}\right)}$$

$$L(y) = \frac{\exp\left(\frac{-(y-1)^2}{8}\right)}{2 \cdot \exp\left(\frac{-(y+1)^2}{2}\right)}$$

Taking Logarithm of both sides

$$\log L(y) = \log\left(\frac{\exp\left(\frac{-(y-1)^2}{8}\right)}{2\exp\left(\frac{-(y+1)^2}{2}\right)}\right)$$

$$= \log\left(\exp\left(\frac{-(y-1)^2}{8}\right)\right) - \log\left(\exp\left(\frac{-(y+1)^2}{2}\right)\right) - \log 2$$

$$= -\frac{(y-1)^2}{8} - \left(\frac{-(y+1)^2}{2}\right) - \log 2$$

$$= -\frac{(y-1)^2}{8} + \frac{(y+1)^2}{2} - \log 2$$

$$= (-1)(y^2-2y+1) + (4)(y^2+2y+1) - \log 2$$

$$= -y^2+2y-1+4y^2+8y+4 - \log 2$$

$$= 3y^2+10y+3 - \log 2$$

comparing this with a quadratic function ay^2+by, with a threshold, this forms the desired equation

Therefore, $a = 3$, $b = 10$, and

Threshold $= 3 - \log 2$

2) Specify the rule explicitly, for $P_x(0) = \frac{1}{3}$

$$\log L(y) = -\frac{(y-1)^2}{8} + \frac{(y+1)^2}{2} - \log 2$$

$P_x(0) = \frac{1}{3}$ and $P_x(1) = \frac{2}{3}$

because $P_x(0) + P_x(1) = 1$

Here, $\log L(y) = \log\frac{P_x(0)}{P_x(1)} = \frac{\log(\frac{1}{3})}{\log(\frac{2}{3})}$

$$= \log 1 - \cancel{\log 3} - \log 2 + \cancel{\log 3}$$

and $\log 1 = 0$

$\log L(y) = -\log 2$

Derived, $\log L(y) = 3y^2 + 10y + 3 - \log 2$

$$L(y) = \frac{P(y|x=1)}{P(y|x=0)} \underset{x=0}{\overset{x=1}{\gtrless}} \frac{P_x(0)}{P_x(1)}$$

If $L(y) = 1$, $\log 1 = 0$

$$-\frac{(y-1)^2}{8} + \frac{(y+1)^2}{2} - \log 2 = 0$$

$$0 = 3y^2 + 10y + 3 - 8\log 2$$

$$3y^2 + 10y = 8\log 2 - 3$$

We can then simply say

$$3y^2 + 10y \underset{x=0}{\overset{x=1}{\gtrless}} 8\log 2 - 3$$

Part II

Given $r = \alpha s + n$, $\alpha \in R$ is a real constant unknown to the receiver,

$n \sim N(0, \sigma^2)$

By comparism, α is the amplitude of the signal is s is known

Then, $r \sim N(\alpha, \sigma^2)$, for S being a positive value, $+1$

$r \sim N(-\alpha, \sigma^2)$ for S being negative value, -1

$$P(r|\alpha, s=1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{-(y-\alpha)^2}{2\sigma^2}$$

$$P(r|\alpha, s=-1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{-(y+\alpha)^2}{2\sigma^2}$$