Problem Sheet 4

Last modified November 4, 2019

Note: You may work in groups of two on the homework problems. Next week, for Monday **you are required to hand in problem 1**. For Wednesday **you are required to hand in problem 3**.

Please submit your results via Canvas (one submission per group, state all group members in the comments box). State the number of the problem sheet and the name of each group member on the top of the first page. You may submit a scanned handwritten solution. If using a smartphone for scanning, ensure that it is properly readable (resolution, lighting, angle).

Problems for Monday, December 2

Problem 1

The code $\mathcal{C}$ is defined as

$$\mathcal{C} = \left\{ \begin{array}{c} (0000000), \\ (1101001), \\ (1011010), \\ (0110011), \\ (0111100), \\ (1010101), \\ (1100110), \\ (0001111) \end{array} \right\}.$$

Questions

1. How can you show that this code is a linear block code? What are the code parameters (k, n, d_{min})?
2. Find a $k \times n$ generator matrix. Find a systematic generator matrix of the form $\mathbf{G}_{\mathsf{s}} = [\mathbf{P}_{3\times 4}\, \mathbf{I}_3]$. Find a $(n-k) \times n$ parity check matrix $\mathbf{H}$. Verify that $\mathbf{G}_{\mathsf{s}}\mathbf{H}^T = \mathbf{0}$.
3. What are the error correction and error detection capabilities of this code over the binary symmetric channel with cross-over probability $p < 0.5$?
4. Determine the decoding table for transmission over the binary symmetric channel with cross-over probability $p < 0.5$. Motivate the choices you make. (You may also use MATLAB to construct the decoding table.)
5. Suppose you receive $\bar{\mathbf{y}} = (1111111)$. What is the output of the ML decoder?
6. Assume the binary symmetric channel with cross-over probability $p \in [0, 1]$. Should you modify your decoder when $p > 0.5$, for example $p = 0.9999$? If so, how?

Problem 2

Consider a $(4, 3)$ Parity Check Code transmitted over a BSC with crossover probability p.

1. Determine the generator matrix $\mathbf{G}$ and the parity check matrix $\mathbf{H}$ for this code.
2. What is the minimum Hamming distance d_{min} and the rate R_{c} of this code?

3. What are the error detection and error correction capabilites of this code?

4. Assume that a codeword from the $(4, 3)$ Parity Check Code is transmitted. When an error is detected in the receiver, the word is retransmitted. If there is an error in the retransmission, the word is *not* retransmitted again. Find the word error probability of this system and compare it to an uncoded system for $p = 0.1$.

Problems for Wednesday, December 4

Problem 3

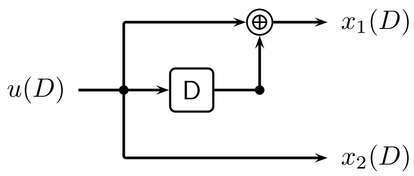

Figure 1: Block diagram of the convolutional encoder.

Consider the convolutional encoder shown in Figure 1. The encoder is assumed to be initialized to the all-zero state. The output bits are transmitted over a binary symmetric channel with crossover probability $\epsilon = 0.01$.

1. Draw three full sections of the Trellis diagram. Make sure that all state transitions are clearly labeled with the corresponding input and output bits (use the notation u/x_1x_2, where u corresponds to the input bit and x_1 and x_2 to the two output bits of a particular transition).

2. What is the free distance of the code?

3. Assume that three information bits are encoded *without zero-termination of the encoder* (that is, the encoder is truncated). The observed bit sequence after the channel is given by $\bar{\boldsymbol{y}} = (000111)$. Find the ML codeword and the corresponding estimate of the information bits by using the Viterbi algorithm. It is important that you clearly show all involved branch metrics, state metrics, and survivor paths.

4. Verify the correctness of your answer for the Viterbi algorithm by using an exhaustive brute-force approach to the ML decoding problem. That is, enumerate all possible codewords and calculate the distance between them and the received word.

5. Explain if, how and why your answer to question 3 changes if the crossover probability increases to $\epsilon = 0.1$?

Problem 4

Consider a communication system that encodes independent and equally likely information bits with a convolutional encoder with generator matrix $(1 + D^2, 1 + D + D^2, 1 + D^2)$. The encoder is assumed to be terminated to the all-zero state after encoding the information bits. The coded bits are transmitted over an AWGN channel with a linear modulation using BPSK. Suppose the hard decision on the coded bit sequence is $y = (1, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1)$.

1. What is the rate of the code? What is the memory and number of states? How many information bits have been transmitted?

2. What is the ML codeword and the corresponding information bits?
3. Assuming that the decoded codeword in part 2 is correct, what is the number of bit errors that have been introduced by the channel?

Extra Problems

Problem 5

Consider a linear block code with the generator matrix

$$\boldsymbol{G} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Questions

1. Express $\boldsymbol{G}$ in its systematic form. Also determine a parity check matrix $\boldsymbol{H}$ for the code.
2. Find all codewords of the code and compute the weight spectrum A_d, $d = 1, \ldots, n$, of the code, where A_d is the number of codewords with a particular weight d.
3. What are the code parameters (n, k, d_{min})? What is the code rate?
4. What are the error correction and error detection capabilities of this code assuming a binary symmetric channel channel with cross-over probability $\epsilon < 0.5$?
5. Suppose you receive $\bar{\boldsymbol{y}} = (1111111)$ and $\epsilon < 0.5$. Determine the ML codeword that was transmitted and the corresponding information bits.
6. Now suppose you receive $\bar{\boldsymbol{y}} = (1111111)$ and $\epsilon > 0.5$. Determine again the ML codeword and the corresponding information bits.
7. Suppose you receive $\bar{\boldsymbol{y}} = (1111111)$. Perform error *detection* using the parity check matrix $\boldsymbol{H}$. In general, under which conditions on the error pattern can we *certainly* not detect an error? Explain.

Problem 6

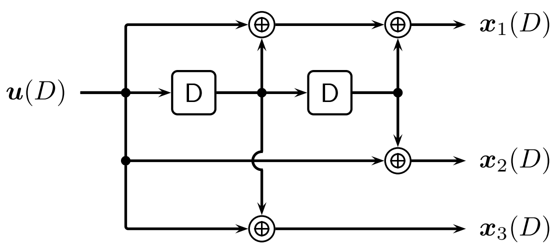

Figure 2: Convolutional encoder

Consider the convolutional encoder shown in Figure 2. The output bits are assumed to be transmitted over a binary symmetric channel with crossover probability $\epsilon < 0.5$.

1. Draw one full section of the trellis diagram. Make sure that all state transitions are clearly labeled with the corresponding input and output bits.

2. Assume now that the encoder is initialized to the all-zero state. Three information bits are encoded (without zero termination) and the resulting coded bits are transmitted over the channel. The observed bit vector after the channel is given by $\bar{\boldsymbol{y}} = (100011110)$. Find the ML codeword and the corresponding estimate of the information bits by using the Viterbi algorithm. It is important that you clearly show all involved branch metrics, state metrics, and survivor paths.

3. Verify your answer by performing a brute-force approach to the ML decoding problem.

4. The convolutional encoder shown in Figure 2 has rate 1/3. One can obtain a higher rate by omitting one of the three output streams $\boldsymbol{x}_1(D)$, $\boldsymbol{x}_2(D)$, or $\boldsymbol{x}_3(D)$ (also known as puncturing). Which output stream should be omitted in order to maximize the free distance of the resulting rate-1/2 convolutional code?

Problem 7

Assume that an (n, k) linear block code has minimum Hamming distance d_{min}.

1. Prove that every set of $d_{\mathsf{min}} - 1$ or fewer columns of the parity-check matrix of the code, $\boldsymbol{H}$, is linearly independent.

2. Prove that there exists at least one set of d_{min} columns of $\boldsymbol{H}$ that is linearly dependent.