Problem 3, sheet 3

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Problem 3

Part I (uncoded)

Transmitted code $c = (c_1, c_2)$

Received code $= \hat{c} = (\hat{c_1}, \hat{c_2})$

Given, $y = x + n$, where Y is received symbol and x is the transmitted Symbol What is transmitted is well-known.

where $n \sim N(0, \sigma^2)$

$n \sim N(0, N_0)$

QPSK, 2 bits per symbol

$M = 4$

Optimum Decoding rule

for Baye's law

$$P(x|y)\,P(y) = P(y|x)\,P(x) \quad (1)$$

$$P(x|y) = \frac{P(y|x)\,P(x)}{P(y)}$$

Let received symbol be represented as $\hat{X}$

To maximize $\hat{x}$, for a corresponding y

$$\hat{X} = \arg\max_x P(x|y) = \arg\max_x \frac{P(y|x)\,P(x)}{P(y)}$$

Since $P(y)$ is independent of x, we can do away with $P(y)$

$$\hat{X} = \arg\max_x P(x|y) = \arg\max_x P(y|x)\,P(x)$$

for QPSK, equiprobable symbols, $P(x) = \frac{1}{M} = \frac{1}{4}$

$$\hat{X} = \arg\max_x P(x|y) = \arg\max_x P(y|x)$$

Since we are transmitting over a Gaussian channel

$$\hat{x} = \arg\max_x P(x|y) = \arg\max_x \left(\frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{\|y-x\|^2}{2\sigma^2}}\right)$$

$$\hat{X} = \arg\max_{x} \left(\ln \frac{1}{\sqrt{2\pi}\sigma} - \frac{\|y-x\|^2}{2\sigma^2} \right)$$

$$\hat{X} = \arg\max_{x} \left(- \frac{\|y-x\|^2}{2\sigma^2} \right)$$

$$\hat{X} = \arg\max_{x} \left(-\|y-x\|^2 \right)$$

$$\hat{X} = \arg\min_{X} \left(\|y-x\|^2 \right)$$

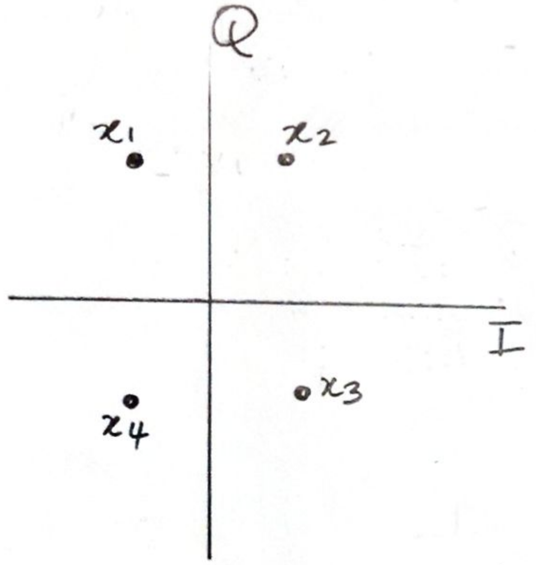
for AWGN Channel, based on Maximum-Likelihood

$$\hat{X} = \arg\min_{x} \left(\|y-x\|^2 \right) = \arg\min_{x} d_E(x, y)$$

where d_E represents the Euclidean distance between y and x on the signal constellation, for $x \in \mathcal{X}$

Therefore optimal rule is based on the Euclidean distance between the symbols in the constellation

X



QPSK Constellation

$x = \{x_1, x_2, x_3, x_4\}$

There are four (4) symbols in the constellatio

From eqn(1), we can therefore conclude that

$$P_{X|Y}(\hat{x}|Y) \geq P_{X|Y}(X|Y), \text{ for all } x \in \mathcal{X}$$

$$P_{Y|X}(y|\hat{x})\, P_X(\hat{x}) \geq P_{Y|X}(Y|X)\, P_X(x), \text{ for all } x \in \mathcal{X}$$

Taken that $\hat{x}$ belongs to $X = (x_1, x_2, x_3, x_4)$

$$\hat{x} = \begin{cases} x_1, & \text{if } y_I < 0 \text{ and } Y_Q > 0 \\ x_2, & \text{if } y_I > 0 \text{ and } Y_Q > 0 \\ x_3, & \text{if } y_I > 0 \text{ and } Y_Q < 0 \\ x_4 & \text{if } y_I < 0 \text{ and } Y_Q < 0 \end{cases}$$

Based on the condition, that for QPSK Symbol Error rate,

$$P_{SER} = P(\hat{x} \neq x_1 | x = x_1)$$
$$= P(Y_I > 0 \cup Y_Q < 0 | X_1), \quad \text{Hint: } P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(Y_I > 0 | X_1) + P(Y_Q < 0 | X_1) - P(Y_I > 0 | X_1) P(Y_Q < 0 | X_1)$$

Since we assume the symbols are normally distributed and considering AWGN channel, $N \sim (0, \sigma^2)$. $\sigma^2 = N_0 \Rightarrow \sigma = \sqrt{N_0}$

Average Symbol Energy $E_s = 2E_b$

Taking $\alpha = \|y - x\|^2 \Rightarrow$

$\alpha = \sqrt{E_s/2} \Rightarrow E_s = 2\alpha^2, \quad \alpha = \sqrt{\frac{E_s}{2}} = \sqrt{E_b}$

$E_b = \alpha^2$

Therefore, $Y_I \sim \mathcal{N}(-\alpha, \sigma^2)$ and $Y_Q \sim \mathcal{N}(\alpha, \sigma^2)$

So, $P(Y_I > 0 | X_1) = P(Y_I > 0) | Y_I \sim \mathcal{N}(-\alpha \sigma^2)$

$$= Q\left(\frac{\alpha}{\sigma}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

For QPSK, by Symmetry, $P(Y_Q < 0 | X_1) = P(Y_I > 0 | X_1)$

$P_{SER} = P(Y_I > 0 | X_1) + P(Y_Q < 0 | X_1) - P(Y_I > 0 | X_1) P(Y_Q < 0 | X_1)$

$$P_{SER} = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left[Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2$$

The Symbol Error Probability for QPSK

$$= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left[Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2$$

Bit Error Probability for QPSK, for the first and second bit

BER for an M-ary Constellation is generally given as

$$P_e = \frac{1}{M} \sum_{i=1}^{M} (\text{Probability for ith bit in Error})$$

Let P_1 and P_2 represent probabilities for bit 1 and 2. $M=4$ for QPSK

for gray labelling

$P_1 = P(eb_1)$,

$$P_1 = \frac{1}{4} \sum_{x \in X} P(eb_1 | x)$$

$$= \frac{1}{4} \left[P(eb_1 | x_1) + P(eb_1 | x_2) + P(eb_1 | x_3) + P(eb_1 | x_4) \right]$$

$$= \frac{1}{4} \left[(P(b_1(\hat{x}) = 0 | x_1) + P(b_1(\hat{x}) = 1 | x_2) + P(b_1(\hat{x}) = 1 | x_3) + Pr(b_1(\hat{x}) = 0 | x_4) \right)$$

for QPSK , $P(x) = 1/M = \frac{1}{4}$

Therefore $P(b_1(\hat{x}) = 0 | x_1) = P(Y_1 > 0 | x_1)$

$$= P(Y_1 > 0) | Y_1 \sim N(-\alpha, \sigma^2)$$

$$= Q\left(\sqrt{\frac{E_s}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \longrightarrow \text{1st Bit}$$

For the Second bit, $P(b_1(\hat{x}) = 1 | x_2) = P(Y_1 < 0 | x_2)$

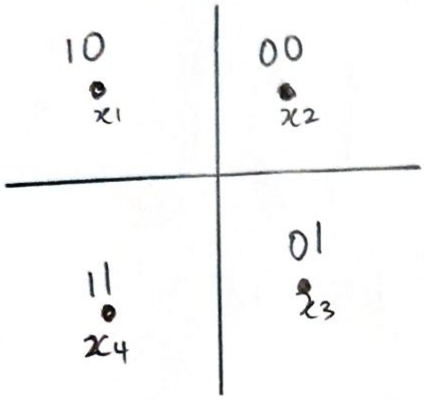
$$= P(Y_1 < 0) | Y_1 \sim N(\alpha, \sigma^2)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \longrightarrow \text{Second bit}$$

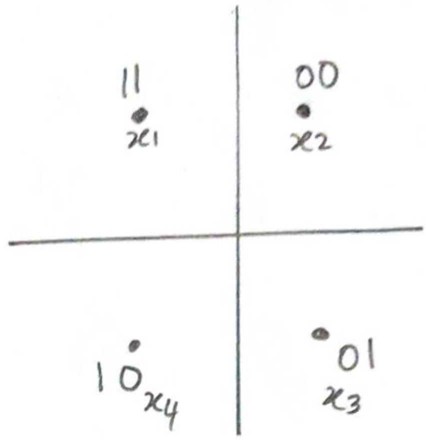
Average Bit Error rate for QPSK, with gray labelling, will be

$$P_{QPSK-Gray} = \frac{1}{2}(P_1 + P_2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

The average error probability for bits 1 and 2 is the Same as the Bit error rate for either of the bits



Gray Labelling



Lexiographical Labelling

Taken that

$$x_1 = \alpha(-1+j), \quad x_2 = \alpha(1+j), \quad x_3 = \alpha(1-j), \quad x_4 = \alpha(-1-j)$$

Average Symbol Energy is the same for either constellations.

Basically $P_{SER}^{QPSK} = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left[Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2$

where $E_s = 2E_b$

for QPSK, $P(\hat{x} = x_j \mid x = x_i) = Q\left(\sqrt{\frac{d_E^2(x_i, x_j)}{2N_0}}\right) = Q\left(\sqrt{\frac{d^2_{min}}{2N_0}}\right)$

$$\frac{d^2_{min}}{2N_0} = \frac{2E_b}{N_0} \Rightarrow d^2_{min} = 4E_b = 2E_s$$

$$d_{min} = \sqrt{2E_s} = 2\sqrt{E_b}$$

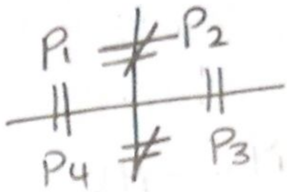
This will be the same for gray labelling and lexiographical labelling

$$SER_{Gray}^{QPSK} = SER_{Lexiographical}^{QPSK}$$

$$= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left[Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2$$

$$= 2Q\left(\sqrt{\frac{d_E^2(x_i, x_j)}{2N_0}}\right) - \left[Q\sqrt{\frac{d_E^2(x_i, x_j)}{2N_0}}\right]^2$$

For Bit Error rate, QPSK, Lexiographical mapping, we can see that $P_1 \neq P_2$



$$P_1 = P_4$$
$$P_2 = P_3$$

for If $P_1 = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

$$P_2 = (P_1 + P_4)(1 - P_1) = (P_1 + P_4)(1 - P_4)$$

$$P_1 = P_4$$

$$P_2 = 2P_1(1 - P_1) = 2P_1 - 2P_1^2$$

$$P_2 = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)$$

$$P^{QPSK}_{Average} = \frac{P_1 + P_2}{2} = \frac{1}{2}\left(Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)\right)$$

$$= \frac{P_1 + (2P_1 - 2P_1^2)}{2} = \frac{3P_1 - 2P_1^2}{2} = \frac{3}{2}P_1 - P_1^2$$

$$\Rightarrow P^{QPSK}_{average} = \frac{3}{2}Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left(Q\sqrt{\frac{2E_b}{N_0}}\right)^2$$

BER for Gray Labelling $\neq$ BER for Lexiographical Labelling

Part II (coded)

1 Given

$$C = \begin{cases} C_1 = (000000) \\ C_2 = (010101) \\ C_3 = (101010) \\ C_4 = (111111) \end{cases}$$

$C(n, k) = ?$

n = length of codewords in bits

$\Rightarrow n = 6$

k = Information block length

2^k = Number of code words and there are 4 code words. (C_1, C_2, C_3, C_4)

$2^k = 4$, $\Rightarrow k = 2$

Linear codes must satisfy two conditions

i) Minimum Hamming distance = Minimum Hamming weight

$\min(d_H) = \min(W_H)$

ii) Addition of two code words gives another code word.

Here, $C_1 + C_2 = C_2$, $C_1 + C_3 = C_3$, $C_1 + C_4 = C_4$, $C_2 + C_3 = C_4$

Minmum Hamming distance

$C_1, C_2 = 3$; $C_1, C_3 = 3$; $C_1, C_4 = 6$; $C_2, C_3 = 6$

Minimum Hamming weight

$C_1 = 0$, $C_2 = 3$, $C_3 = 3$, $C_4 = 6$

The first condition of Linearity is not satisfied, therefore it is not a Linear code.

QPSK is 2bit per symbol and these code words have 6 bits

We will need 3 QPSK symbols for one Codeword

② Maximum Likelihood detection, we need to look into the minimum hamming distance

Receiver outputs $\hat{c} = 100001$

$\hat{c}, c_1 = 2$
$\hat{c}, c_2 = 3$
$\hat{c}, c_3 = 3$
$\hat{c}, c_4 = 4$

$\Rightarrow \min^{Hd}(c, \hat{c}) = c_1$

c_1 compared with $\hat{c}$ offered the minimum Hamming distance

Bonus Question

Since the codeword mapping remains the same irrespective of either gray or lexiographical mapping, the decision region will not change but the bit error rate will change.