SSY125 Digital Communications

Department of Signals and Systems

Exam Date: January 11 2016, 14:00-18:00
Location: HA, HB, HC

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Material Allowed material is

- Chalmers-approved calculator.
- L. Råde, B. Westergren. Beta, Mathematics Handbook, any edition.
- A dictionary.

Grading A correct, clear and well-motivated solution gives a maximum of 15 points.

An erroneous answer, unclear, incomplete or badly motivated solutions give point reductions down to a minimum of 0 points. Answers in any other language than English are ignored.

Solutions The solutions are made available as soon as possible on the course web page.

Review The grading review will be on February 1, 2016, 10AM-11AM, and on February 9, 2016, 15PM-16PM, in room 6414 (Blue Room) in the ED-building.

Grades The final grade on the course will be decided by the project (maximum score 30), quizzes (maximum score 3), tutorial grade (maximum score 7), and final exam (maximum score 60). The sum of all scores will decide the grade according to the following table.

Total Score	0–39	40–59	60–79	≥ 80
Grade	Fail	3	4	5

PLEASE NOTE THAT THE PROBLEMS ARE NOT NECESSARILY ORDERED IN DIFFICULTY.

Good luck!

Problem 1 - Entropy, Source Coding, and Mutual Information [15 points]

Part I

Let X and Y be two independent integer-valued random variables. Let X be uniformly distributed over $\{1, 2, \dots, 16\}$, and let $\mathrm{P}(Y = k) = 2^{-k}$, where $k = 1, 2, 3, \dots$ The following expression may be useful:

$$\sum_{i=0}^{\infty} i a^i = \frac{a}{(1-a)^2}, \quad |a| < 1.$$

1. [*3 pt*] Find the entropies $H(X)$, $H(Y)$, and $H(X, Y)$, in bits.

Part II

Consider a source with 5 symbols which have probabilities $P = \{0.3, 0.3, 0.2, 0.1, 0.1\}$.

1. [*1 pt*] What is the source entropy?
2. [*2 pt*] Apply the Huffman coding algorithm to this source.
3. [*1 pt*] What is the efficiency of the code?
4. [*2 pt*] Let $l_1, l_2, \dots, l_5$ be the codeword lengths of the code and consider an alternative probability distribution $P' = \{p'_1, p'_2, \dots, p'_5\}$ for the 5 symbols. Determine $p'_1, p'_2, \dots, p'_5$ such that the expected codeword length $\bar{L}$ for the code,
$$\bar{L} = \sum_{i=1}^{5} p'_i l_i,$$
is equal to the entropy of the random variable defined by the distribution P'. Use the codeword lengths that you found in question 2.

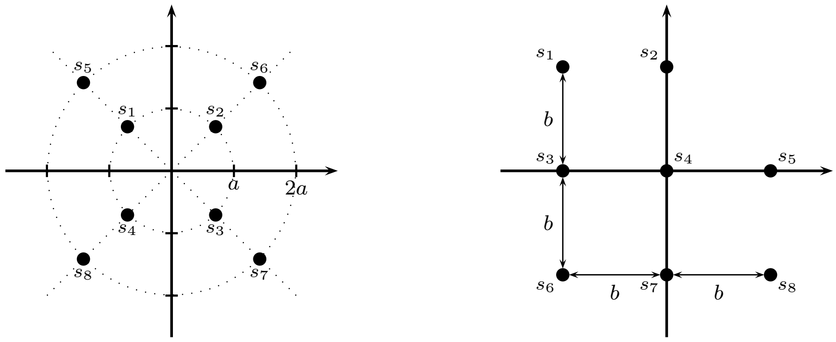
Part III

Consider a channel whose input is a random variable X which takes values on $\mathcal{X} = \{0, 1\}$ with probabilities $P(X = 0) = P(X = 1) = 0.5$. The channel output is a random variable Y which takes values on $\mathcal{Y} = \{0, 2\}$. The channel is defined by the conditional distribution $P(y|x)$,

$$P(0|0) = P(2|1) = 1 - \varepsilon$$
$$P(2|0) = P(0|1) = \varepsilon.$$

1. [*2 pt*] What is the entropy of the source, $H(X)$, the probability distribution of the output, $P(y)$, and the entropy of the output, $H(Y)$?
2. [*2 pt*] What is the joint probability distribution for the source and the output, $P(x, y)$, and what is the joint entropy, $H(X, Y)$?
3. [*1 pt*] Use $H(X)$, $H(Y)$, and $H(X, Y)$ to compute the mutual information of this channel, $I(X; Y)$, as a function of ε.
4. [*0.5 pt*] What is the maximum mutual information of the channel in bits?
5. [*0.5 pt*] For what value of ε is the mutual information minimal? What is the mutual information in this case?

Problem 2 - Signal Constellations and Detection [15 points]



(a) Ω_1 (b) Ω_2

Figure 1: Two different complex signal constellations

Consider the two complex signal constellation Ω_1 and Ω_2 shown in Figure 1 (a) and (b). Assume that the received signal is given as $r = s + n$, where s is a point selected from one of the constellations, and n is a realization of a zero-mean, complex Gaussian random variable N with $\mathbb{E}[|N|^2] = N_0$. All symbols in both constellations are assumed to be equally likely to be transmitted.

Questions

1. [*1 pt*] Determine a and b, such that both constellations Ω_1 and Ω_2 have unit energy $\mathbb{E}[|S|^2] = 1$.
2. [*2 pt*] Carefully draw the maximum likelihood decision regions for both constellations.
3. [*3 pt*] Find the nearest neighbor approximation for the symbol error probability $P_s(e)$ for both constellations. The final expressions for the approximation should only be a function of N_0.
4. [*1 pt*] How do the constellations compare in terms of symbol error probability at high signal-to-noise ratio?
5. [*2 pt*] Rotate all the points in the outer circle of Ω_1 by $\pi/4$ to obtain a new constellation Ω_3. Are the power efficiencies of Ω_1 and Ω_3 equal or different? Explain why.
6. [*2 pt*] Is it possible to find a Gray mapping for either Ω_1 or Ω_2? If so, show the bit mapping. If not, explain why.
7. [*2 pt*] Suppose now that the received signal is given as $r = se^{j\theta} + n$, where θ is phase noise. The phase noise θ is uniformly distributed between $-\pi/6$ and $\pi/6$. Assume that symbols are detected by finding the constellation point that is closest to r in terms of Euclidean distance. What happens to the symbol error probability for Ω_1 and 8-PSK as the signal-to-noise ratio goes to infinity, i.e., in the limit $N_0 \to 0$?
8. [*2 pt*] Assume again that the received signal is given as $r = s + n$. Let $\mathcal{X}$ be an M-PSK constellation, for any integer $M \geq 2$, and assume that $s \in \mathcal{X}$. Show that the maximum likelihood decision rule for s can be written as
$$\hat{s}_{\text{ML}} = \underset{s \in \mathcal{X}}{\operatorname{argmax}} \cos(\arg\{r\} - \arg\{s\}),$$
where $\arg\{z\}$ denotes the argument of a complex number z.

Problem 3 - Linear Block Codes [15 points]

Consider a linear block code $\mathcal{C}$ defined by the parity-check matrix

$$\boldsymbol{H} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Questions

1. [*2 pt*] Determine a systematic parity-check of the form $\boldsymbol{H}_s = (\boldsymbol{I}_3, \boldsymbol{P})$ and the corresponding systematic generator matrix $\boldsymbol{G}_s$, where $\boldsymbol{I}_3$ denotes the identity matrix of size 3.

2. [*2 pt*] Compute the weight spectrum A_d, $d = 1, \ldots, N$, of the code, where A_d is the number of codewords with a particular weight d.

3. [*1 pt*] What are the code parameters $(N, K, d_{\min})$? What is the code rate?

4. [*4 pt*] Assuming transmission over the binary symmetric channel with crossover probability $p < 0.5$, generate *two* syndrome tables based on both $\boldsymbol{H}$ and $\boldsymbol{H}_s$ from part 2. (It is required to generate the complete tables for *all* possible syndromes.)

5. [*2 pt*] Suppose that you receive $\overline{r} = (011000)$. What is the ML codeword based on *both* syndrome tables and parity-check matrices?

6. [*2 pt*] Determine the dual code $\mathcal{C}_\perp$ of the code $\mathcal{C}$ and list all codewords in $\mathcal{C}_\perp$. The dual code is defined as the set of all binary words of length N that are orthogonal to every codeword in $\mathcal{C}$, i.e., $\mathcal{C}_\perp = \{\tilde{\boldsymbol{x}} \in \{0,1\}^N : \langle \tilde{\boldsymbol{x}}, \boldsymbol{x} \rangle = 0 \text{ for all } \boldsymbol{x} \in \mathcal{C}\}$ where $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{N} x_i y_i \bmod 2$ denotes the inner product of two binary vectors.

7. [*2 pt*] Compare the two codes $\mathcal{C}$ and $\mathcal{C}_\perp$ in terms of their weight spectrum (as defined in part 2) and error-correction and detection capabilities. Explain the result.

Problem 4 - Viterbi Algorithm [15 points]

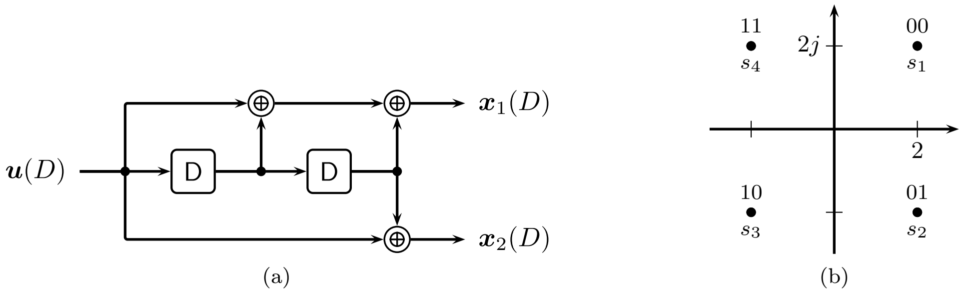


Figure 2: Rate-1/2 convolutional encoder in (a) and QPSK with lexicographical labeling in (b).

Consider the convolutional encoder shown in Figure 2 (a).

1. [*2 pt*] Draw one full section of the trellis diagram. Make sure that all state transitions are clearly labeled with the corresponding input and output bits.

2. [*4 pt*] Assume that the encoder is initialized to the all-zero state. Three information bits are encoded (without zero termination). The resulting coded bits are transmitted over the binary symmetric channel with crossover probability $p < 0.5$. The observed bit vector after the channel is given by $\bar{\boldsymbol{r}} = (111010)$. Find the ML codeword and the corresponding estimate of the information bits by using the Viterbi algorithm.

3. [*5 pt*] Assume that the encoder is initialized to the all-zero state. Three information bits are encoded (without zero termination) and the resulting coded bits are then mapped to a QPSK constellation $\mathcal{X} = \{2+2j, 2-2j, -2-2j, -2+2j\}$ with a lexicographical labeling as shown in Fig. 2 (b). The symbols are transmitted over the complex AWGN channel and the received observation is given by $\boldsymbol{r} = (2+j, -1, 1-j)$. Find the ML estimate of the information bits by using the Viterbi algorithm.

4. [*4 pt*] Assume now that you have intercepted the transmission in part 2 and you have access to the observation $\bar{\boldsymbol{r}} = (111010)$. However, the transmission was not intended for you, and you do not know how the encoder state was initialized. Your best guess is that all initial states were equally likely. With this assumption, find again the ML codeword and the new estimate of the information bits by using the Viterbi algorithm. (The assumed channel is again the binary symmetric channel with crossover probability $p < 0.5$).

For all parts of this problem, it is important that you clearly show all involved branch metrics, state metrics, and survivor paths.