SSY125 Digital Communications

Department of Signals and Systems

Exam Date: January 14, 2017, 14:00-18:00
Location: HA, HB, HC

Teaching Staff Alexandre Graell i Amat, Arni Alfredsson, Andreas Buchberger

Material Allowed material is

- Chalmers-approved calculator.
- L. Råde, B. Westergren. Beta, Mathematics Handbook, any edition.
- A dictionary.

Grading A correct, clear and well-motivated solution gives a maximum of 15 points.

An erroneous answer, unclear, incomplete or badly motivated solutions give point reductions down to a minimum of 0 points. Answers in any other language than English are ignored.

Solutions The solutions are made available as soon as possible on the course web page.

Review The grading review will be on January 30, 2017, 9:00-10:00, and on February 15, 2017, 13:00-14:00, in room 6414 (Blue Room) in the EDIT-building.

Grades The final grade on the course will be decided by the project (maximum score 30), quizzes (maximum score 3), tutorial grade (maximum score 7), and final exam (maximum score 60). The sum of all scores will decide the grade according to the following table.

Total Score	0–39	40–59	60–79	≥ 80
Grade	Fail	3	4	5

PLEASE NOTE THAT THE PROBLEMS ARE NOT NECESSARILY ORDERED IN DIFFICULTY.

Good luck!

Problem 1 - Source Coding and Channel Capacity [15 points]

Part I

Consider a discrete memoryless source whose output X takes values on the alphabet $\mathcal{X} = \{x_1, x_2\}$ with probabilities $P_X(x_1) = 0.3$ and $P_X(x_2) = 0.7$.

1. [*3 pt*] Use the Huffman coding algorithm to encode over three consecutive symbols of this source.
2. [*2 pt*] What is the efficiency of the code?
3. [*1 pt*] Show that the code satisfies Kraft's inequality.

Part II

Consider a channel whose input and output are random variables X and Y that take values on $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Y} = \{y_1, y_2\}$, respectively. The channel is described by the conditional probability distribution

$$
\begin{aligned}
P_{Y|X}(y_1|x_1) &= 1 \\
P_{Y|X}(y_1|x_2) &= \varepsilon \\
P_{Y|X}(y_2|x_1) &= 0 \\
P_{Y|X}(y_2|x_2) &= 1 - \varepsilon
\end{aligned}
$$

where $\varepsilon \in [0, 1]$. The probabilities of the input symbols are $P_X(x_1) = P_X(x_2) = 1/2$.

1. [*2 pt*] Compute the distributions $P_{X,Y}(x, y)$ and $P_Y(y)$.
2. [*5 pt*] Find numerical values of the mutual information $I(Y; X)$ in bits, for $\varepsilon = 0$, $\varepsilon = 1/2$, and $\varepsilon = 1$.
3. [*1 pt*] Are X and Y independent for any of these three values of ε? Explain why.
4. [*1 pt*] Does the capacity of this channel depend on $P_X(x_1)$ and $P_X(x_2)$? Justify your answer.

Problem 2 - Signal Constellations and Detection [15 points]

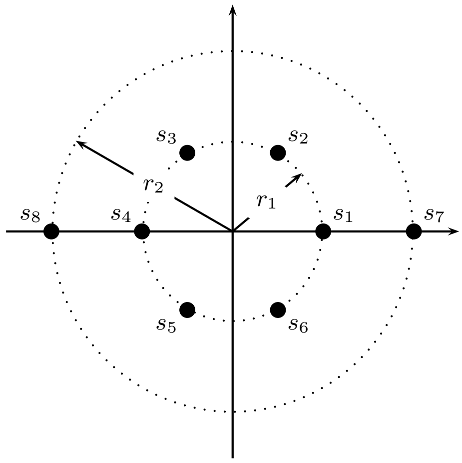

Figure 1: Complex signal constellation Ω.

Consider the complex signal constellation Ω shown in Figure 1, where $r_2 > r_1$. Assume that the received signal is given as $r = s + n$, where s is a point selected from Ω, and n is a realization of a zero-mean, complex Gaussian random variable N with $\mathbb{E}[|N|^2] = N_0$. All symbols in the constellation are assumed to be equally likely to be transmitted.

Questions

1. [*2 pt*] Determine the relation between r_1 and r_2 such that the minimum distance of Ω is maximized. *Hint:* Let $x, y \in [0, 1]$. Then the function $\min(x, y)$ is maximized when $x = y$.

2. [*1 pt*] For the remainder of this problem, use the relation between r_1 and r_2 that you found previously. If you could not compute the previous part, assume the relation $r_2 = 2r_1$. Find r_1 and r_2 such that Ω has unit energy $\mathbb{E}[|S|^2] = 1$.

3. [*1 pt*] Carefully draw the maximum likelihood decision regions for Ω.

4. [*3 pt*] Find the nearest neighbor approximation for the symbol error probability $P_s(e)$ for Ω. The final expressions for the approximation should only be a function of N_0.

5. [*2 pt*] Compare Ω with 8-PSK in terms of spectral efficiency and $P_s(e)$ at high SNR.

6. [*1 pt*] Suppose that all the points in Ω are rotated by an arbitrary phase ϕ to obtain a new constellation Ω_{R}. Are the power efficiencies of Ω and Ω_{R} equal or different? Explain why.

7. [*2 pt*] Suppose that the received signal is given as $r = se^{j\theta} + n$, where $\theta > 0$, and assume that the symbols are detected using maximum likelihood as if $\theta = 0$. For which values of θ does the symbol error probability $P_s(e)$ for Ω go to zero as the SNR goes to infinity, i.e., in the limit $N_0 \to 0$?

8. [*3 pt*] Let $\mathcal{X}$ be a $2M$-PAM constellation, for any integer $M > 1$, and assume that $r = s + n$, where $s \in \mathcal{X}$. Prove or disprove that the maximum likelihood decision rule for s can be written as

$$\hat{s}_{\mathrm{ML}} = \underset{s \in \mathcal{X}}{\operatorname{argmax}}\, \Re\{rs\},$$

where $\Re\{z\}$ denotes the real part of a complex number z.

Problem 3 - Linear Block Codes and LDPC Codes [15 points]

Part I

Consider a code $\mathcal{C}$ defined by the parity-check matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Note: Even though $\mathbf{H}$ is not sparse, consider $\mathcal{C}$ to be an LDPC code.

1. [*2 pt*] Draw the Tanner graph for $\mathbf{H}$. Find the girth and highlight it in the graph.
2. [*1 pt*] Determine the variable node degree distribution $\Lambda(x)$ and the check node degree distribution $P(x)$. Is the LDPC code regular or irregular? Justify your answer!
3. [*3 pt*] For the code $\mathcal{C}$, is it possible to find a different parity check matrix $\tilde{\mathbf{H}}$ whose Tanner graph has a larger girth than the Tanner graph corresponding to $\mathbf{H}$? If yes, find $\tilde{\mathbf{H}}$ and the corresponding Tanner graph. If no, justify why not.

Part II

Consider a system that takes an input of 4 bits, duplicates the first 2 and appends them to the sequence and calculates a parity check bit. *Example:* $\boldsymbol{u} = [1, 0, 0, 1]$ then $\boldsymbol{x} = [1, 0, 0, 1, 1, 0, 1]$.

1. [*2 pt*] Find the generator matrix $\mathbf{G}_s$ and parity check matrix $\mathbf{H}_s$ in systematic form.
2. [*2 pt*] What are the code parameters $(N, K, d_{\min})$? What is the code rate? How many errors can this code correct and detect?
3. [*2 pt*] Assuming transmission over the binary symmetric channel with crossover probability $p < 0.5$, generate a syndrome table based on $\mathbf{H}_s$. (It is required to generate the complete table for *all* possible syndromes.)
4. [*3 pt*] Suppose you transmit data over an AWGN channel using BPSK according to the model

$$r_i = (2x_i - 1) + n_i,$$

where $x_i \in \mathbb{B}$ is the transmitted bit and $n_i \sim \mathcal{N}(0, \sigma^2)$. Assume now that you receive

$$\boldsymbol{r} = \begin{bmatrix} -0.4623 & 2.8339 & -3.2588 & 1.8622 & 1.3188 & 0.3077 & 0.5664 \end{bmatrix}.$$

What is the maximum likelihood codeword based on the syndrome table and parity-check matrix?

Note: If you could not find $\mathbf{G}_s$ and $\mathbf{H}_s$ in II.1, you may use

$$\mathbf{G}_s^\star = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}. \qquad \mathbf{H}_s^\star = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

for II.2-II.4.

Problem 4 - Viterbi Algorithm [15 points]

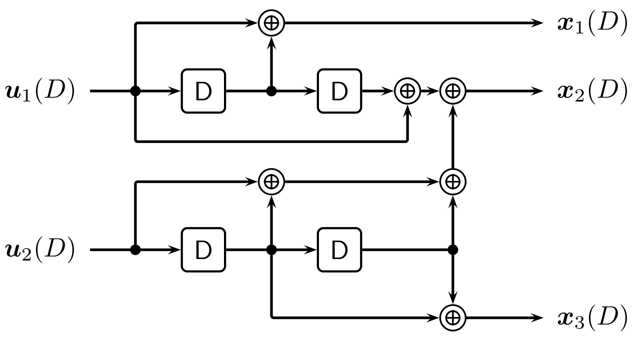

Figure 2: Encoder $\mathcal{E}_1$

1. [*2 pt*] Find the generator matrix $\mathbf{G}$ for encoder $\mathcal{E}_1$ in Fig. 2.
2. [*2 pt*] Transform the encoder $\mathcal{E}_1$ in Fig. 2 into recursive systematic form $\mathcal{E}_{\text{RSC}}$.
3. [*5 pt*] Now assume that the second input is set to $\boldsymbol{u}_2(D) = \mathbf{0}$. Draw one full section of the Trellis diagram of $\mathcal{E}_1$. Only display possible transitions and reachable states. Make sure that all state transitions are clearly labeled with the corresponding input and output bits.
4. [*6 pt*] Assume that the encoder $\mathcal{E}_1$ is initialized to the all-zero state. Again, the second input is set to $\boldsymbol{u}_2(D) = \mathbf{0}$. Eight information bits are encoded (without zero termination). The bits are transmitted over a BSC with $p = 0.2$ and the received observation is given by $\boldsymbol{r} = (100, 110, 000, 010)$. Find the maximum likelihood estimate of the information bits by using the Viterbi algorithm.

For all parts of this problem, it is important that you clearly show all involved branch metrics, state metrics, and survivor paths.