Solution Sheet 5

Last modified November 18, 2019

Problem 1

To design a recursive systematic code (RSC), we need to transform the given generator matrix $\boldsymbol{G} = \left(1 + D + D^2, 1 + D^2\right)$. By dividing the whole matrix by the first element, we get

$$\boldsymbol{G}_\mathsf{s} = \left(1, \frac{1 + D^2}{1 + D + D^2}\right)$$

which is systematic and recursive. This code can then be represented by the block diagram in Fig. 1. For an

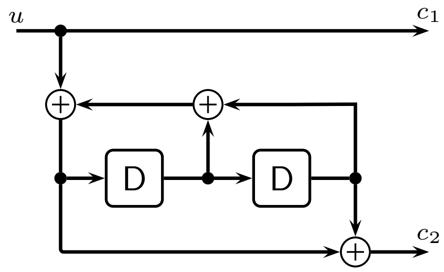


Figure 1: Block diagram

input of $\boldsymbol{u} = [10\,00\,\ldots]$, we can draw the corresponding state transitions in the trellis diagram (Fig. 2). We know that a section in the Trellis diagram is time invariant, and can therefore conclude that the encoder will move between states '01' and '10' forever, since the input will not change. This results in the output sequence

$$\boldsymbol{c} = [11\,01\,01\,00\,01\,\ldots]\,.$$

Since the output contains recurring '1's, the length of this sequence (i.e., the position of the last '1') is infinity.

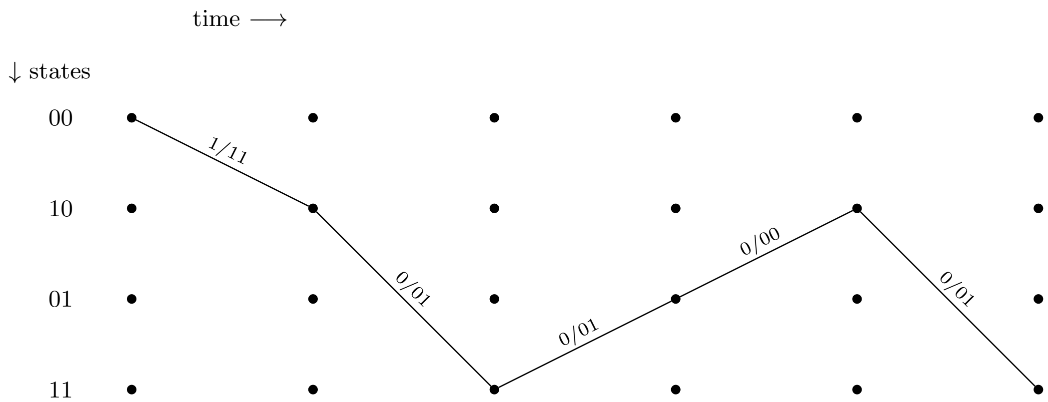


Figure 2: State transitions for the input sequence $\boldsymbol{u} = [10\,00\,\ldots]$.

With the same line of arguing, we can compute the output sequence for an input of $\boldsymbol{u} = [10\,01\,00\,\ldots]$ (Fig. 3). This results in the output sequence

$$\boldsymbol{c} = [11\,01\,01\,11\,00\,00\ldots]\,.$$

Since the last time the ouput contains a '1' is the transition at the forth timestep, the length is finite ($l = 9$). For the input sequence $\boldsymbol{u} = [10\,00\,01\,00\,\ldots]$ (Fig. 4), we get the output sequence

$$\boldsymbol{c} = [11\,01\,01\,00\,01\,10\,01\,00\,01\ldots]\,.$$

Since the output contains recurring '1's, the length of this sequence is infinity.

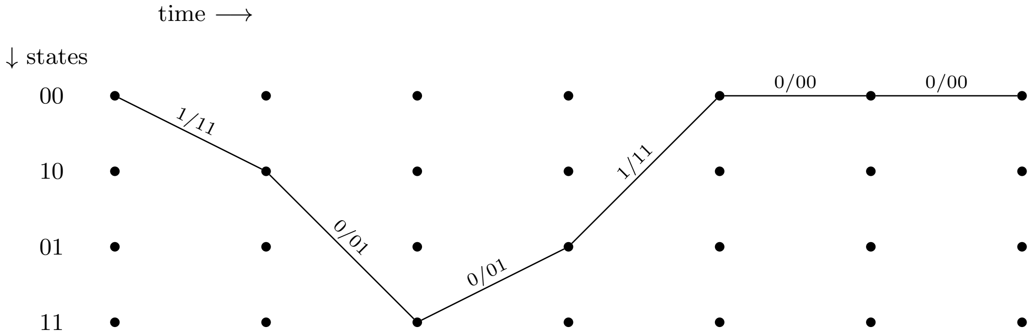


Figure 3: State transitions for the input sequence $\boldsymbol{u} = [10\,01\,00\,00\,\ldots]$.

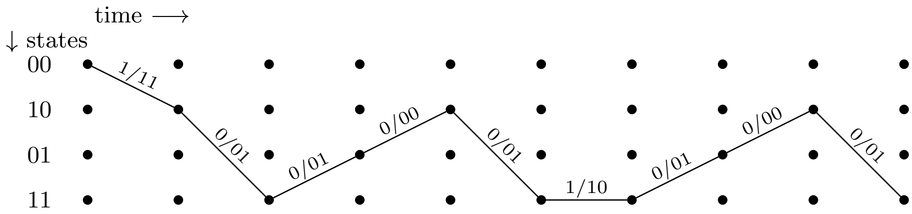


Figure 4: State transitions for the input sequence $\boldsymbol{u} = [10\,00\,01\,00\,00\,\ldots]$.

Problem 2

1. The block diagram is depicted in Fig. 5

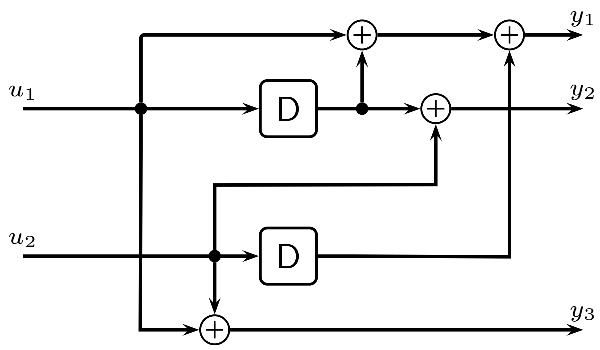
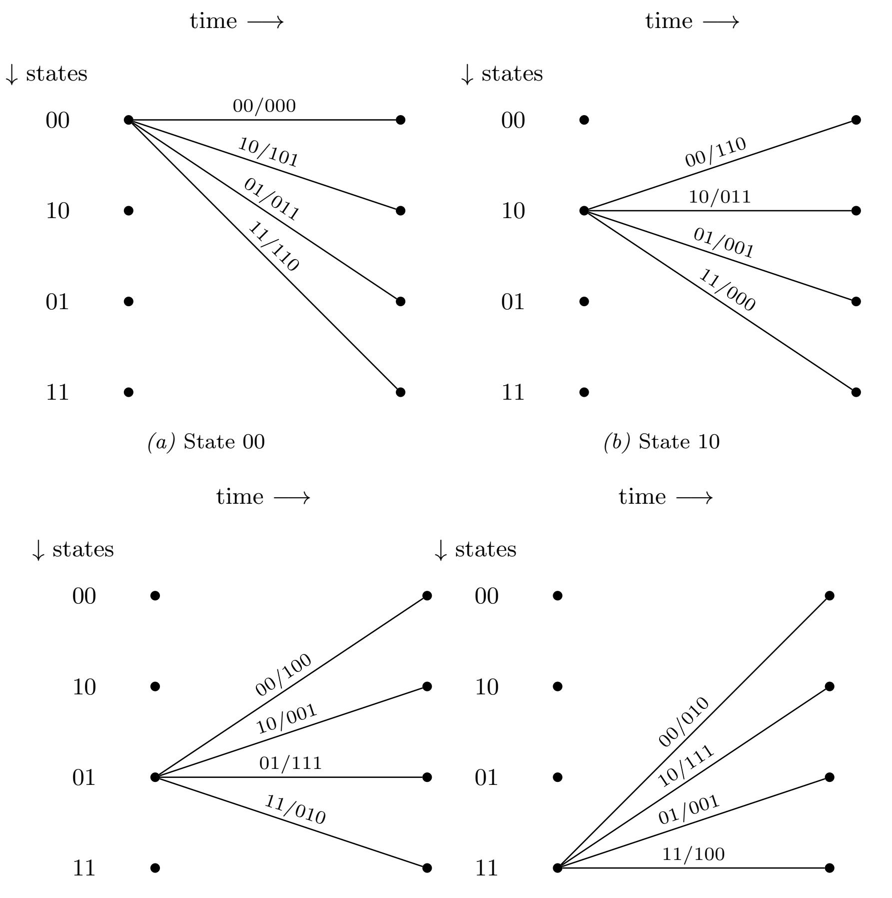


Figure 5: Block diagram

2. The Trellis diagramis depicted in Fig. 6
3. From the Trellis diagram, we can see that the minimum weight of a path diverging from the all-zero state and remerging into it is 3, e.g., $00-01-00$ or $00-11-00$. Therefore, $d_{\mathsf{free}} = 3$.



(a) State 00 *(b)* State 10

(c) State 01 *(d)* State 11

Figure 6: Note that the Trellis diagram has been split up into four subdiagrams for a better readability. To form a full stage of the Trellis diagram, all four subdiagrams must be superimposed.

Problem 3

The given parity check matrix has four rows and six colums which corresponds to four check nodes (CN) and six variable nodes (VN). The resulting Tanner graph is depicted in Fig. 7. It has one cycle of length four, three cycles of length six and two cycles of length eight. Some examples are depicted in Fig. 8 to 13. The girth of the graph is defined by the length of the shortest cycle. For this paticular case, the girth is 4.

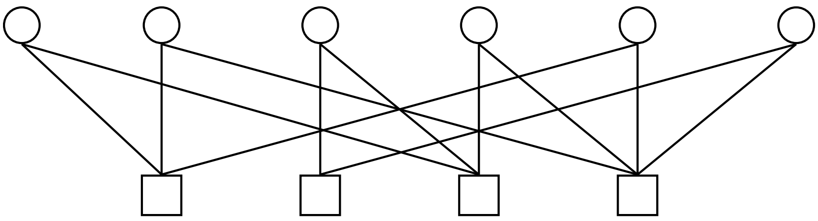


Figure 7: Full Tanner graph.

From Fig. 7, we can easily derive the degree distributions. For the CNs we get

$$d_{\mathrm{c},1} = 3 \qquad d_{\mathrm{c},2} = 2 \qquad d_{\mathrm{c},3} = 3 \qquad d_{\mathrm{c},4} = 4$$

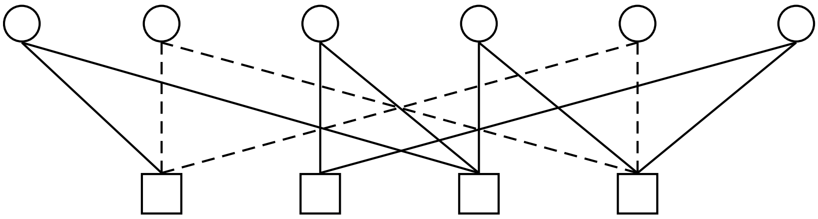


Figure 8: Tanner graph with one cycle of length four (dashed).

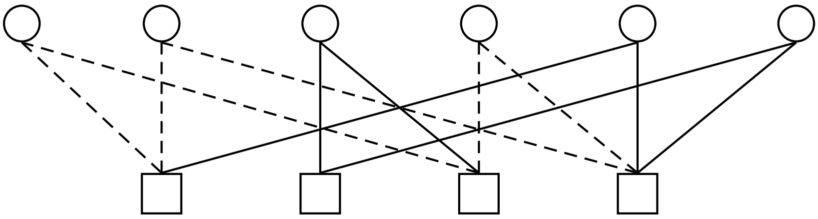


Figure 9: Cycle of of length six (dashed).

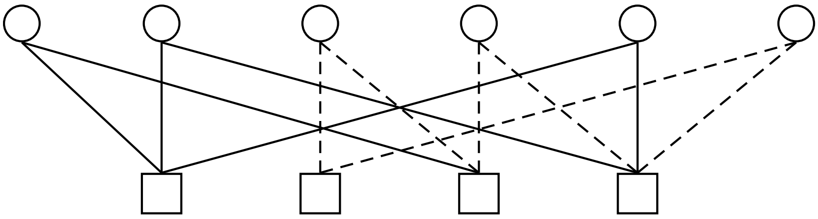


Figure 10: Cycle of of length six (dashed).

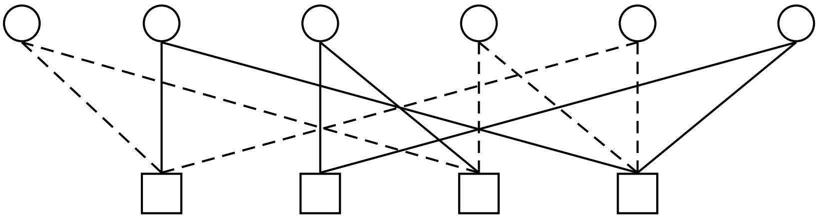


Figure 11: Cycle of of length six (dashed).

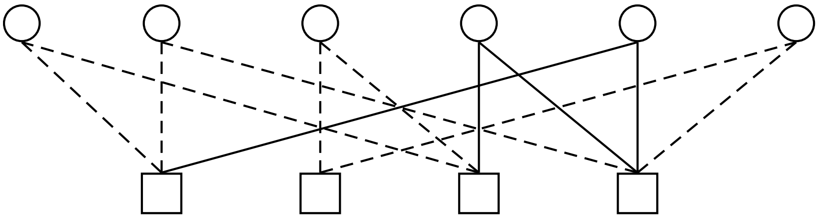


Figure 12: Cycle of length eight (dashed).

which we can write in polynomial form

$$P(x) = \frac{1}{4}\,x^2 + \frac{1}{2}\,x^3 + \frac{1}{4}\,x^4.$$

For the VNs we get

$$d_{\mathrm{v},1} = 2 \qquad d_{\mathrm{v},2} = 2 \qquad d_{\mathrm{v},3} = 2 \qquad d_{\mathrm{v},4} = 2 \qquad d_{\mathrm{v},5} = 2 \qquad d_{\mathrm{v},6} = 2$$

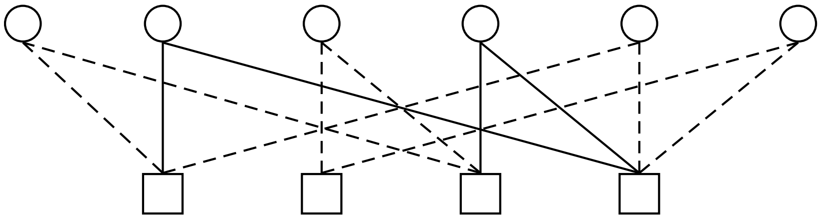


Figure 13: Cycle of length eight (dashed).

which is equivalent to

$$\Lambda(x) = x^2.$$

Since the degree distribution is not constant, it is an irregular LDPC code.

Problem 4

If we look at the Tanner graph, we can identify VNs that are only connected by one edge and therefore can not be part of a cycle. These are the VNs 1, 4 6 and 8. Looking at the remaining VNs, we can see that the only possible cycle is the sequence VN2-CN1-VN3-CN3-VN2. Since we only have one cycle, the length of it is also the girth of this Tanner graph, which in this case is 4.
By looking at the connected edges of each node, we get the degree distribution for the VNs

$$\Lambda(x) = \frac{1}{2}\,x + \frac{1}{2}\,x^2$$

and for the CNs

$$P(x) = x^3.$$

This LDPC code is irregular since the degree for each VN is not the same.

Problem 5

Part I

The entropy of X is

$$\mathsf{H}(X) = \sum_{i=1}^{16} p_i \log_2\left(\frac{1}{p_i}\right) = \log_2(16) = 4 \text{ bits}.$$

The entropy of Y is

$$\begin{aligned}
\mathsf{H}(Y) &= \sum_{k=1}^{\infty} 2^{-k}\log_2(2^k) \\
&= \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k \\
&= \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k \\
&= \frac{0.5}{(1-0.5)^2} \\
&= 2 \text{ bits}.
\end{aligned}$$

Since X and Y are independent, the joint entropy of X and Y is

$$\mathsf{H}(X,Y) = \mathsf{H}(X) + \mathsf{H}(Y) = 2 + 4 = 6 \text{ bits}.$$

Part II

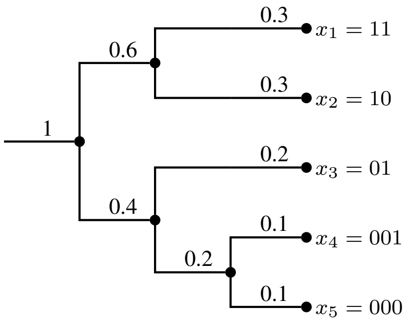
1. The source entropy is

$$\mathsf{H}(X) = \sum_{i=1}^{5} p_i \log_2\left(\frac{1}{p_i}\right) = 2.171 \text{ bits.}$$

2. One possibility is

symbol	codeword	probability
x_1	11	0.3
x_2	10	0.3
x_3	01	0.2
x_4	001	0.1
x_5	000	0.1

where the corresponding tree is depicted below.



3. The average codeword length is

$$\bar{L} = 2 \cdot 0.3 \cdot 2 + 0.2 \cdot 2 + 2 \cdot 0.1 \cdot 3 = 2.2 \text{ bits.}$$

Therefore, the efficiency is

$$\eta = \frac{\mathsf{H}(P)}{\bar{L}} = \frac{2.171}{2.2} = 98.7\%.$$

4. The average codeword length is equal to the entropy of P' if and only if the codeword lengths satisfy

$$\ell_i = \log_2\left(\frac{1}{p'_i}\right) \quad \forall i.$$

Solving for p'_i gives

$$p'_i = \frac{1}{2^{\ell_i}} = 2^{-\ell_i}.$$

Using the codeword lengths from question 2, P' becomes

$$P' = (2^{-2}, 2^{-2}, 2^{-2}, 2^{-3}, 2^{-3}) = (0.25, 0.25, 0.25, 0.125, 0.125).$$

Part III

1. The entropy of the source is

$$\mathsf{H}(X) = -0.5 \log_2(0.5) - 0.5 \log_2(0.5) = 1.$$

The probability distribution of the output is

$$\begin{aligned}P(Y=0) &= P(Y=0|X=0)P(X=0) + P(Y=0|X=1)P(X=1)\\ &= 0.5(1-\varepsilon) + 0.5\varepsilon\\ &= 0.5,\\ P(Y=2) &= P(Y=2|X=0)P(X=0) + P(Y=2|X=1)P(X=1)\\ &= 0.5\varepsilon + 0.5(1-\varepsilon)\\ &= 0.5.\end{aligned}$$

Therefore, the entropy of the output distribution is

$$\mathsf{H}(Y) = -P(Y=0)\log_2(P(Y=0)) - P(Y=2)\log_2(P(Y=2)) = 1.$$

2. The product rule gives

$$P(x,y) = P(y|x)P(x).$$

Therefore, the joint probability distribution of the source and the output is

$$\begin{aligned}P(X=0,Y=0) &= P(Y=0|X=0)P(X=0) = 0.5(1-\varepsilon)\\ P(X=1,Y=0) &= P(Y=0|X=1)P(X=1) = 0.5\varepsilon\\ P(X=0,Y=2) &= P(Y=2|X=0)P(X=0) = 0.5\varepsilon\\ P(X=1,Y=2) &= P(Y=2|X=1)P(X=1) = 0.5(1-\varepsilon).\end{aligned}$$

The joint entropy is then computed as

$$\begin{aligned}\mathsf{H}(X,Y) = &-P(X=0,Y=0)\log_2(P(X=0,Y=0)) - P(X=0,Y=2)\log_2(P(X=0,Y=2))\\ &- P(X=1,Y=0)\log_2(P(X=1,Y=0)) - P(X=1,Y=2)\log_2(P(X=1,Y=2))\\ &= -0.5(1-\varepsilon)\log_2(0.5(1-\varepsilon)) - 0.5\varepsilon\log_2(0.5\varepsilon) - 0.5\varepsilon\log_2(0.5\varepsilon) - 0.5(1-\varepsilon)\log_2(0.5(1-\varepsilon))\\ &= -(1-\varepsilon)\log_2(1-\varepsilon) - (1-\varepsilon)\log_2(0.5) - \varepsilon\log_2(\varepsilon) - \varepsilon\log_2(0.5)\\ &= 1 + \mathsf{H_b}(\varepsilon).\end{aligned}$$

3. The joint entropy, $\mathsf{H}(X,Y)$, can be rewritten as

$$\mathsf{H}(X,Y) = \mathsf{H}(X|Y) + \mathsf{H}(X).$$

Therefore, the mutual information can be computed, using $\mathsf{H}(X)$, $\mathsf{H}(Y)$, and $\mathsf{H}(X,Y)$, as

$$\begin{aligned}\mathsf{I}(X;Y) &= \mathsf{H}(Y) - \mathsf{H}(Y|X)\\ &= \mathsf{H}X) + \mathsf{H}(Y) - \mathsf{H}(X,Y)\\ &= 1 + 1 - 1 - \mathsf{H_b}(\varepsilon)\\ &= 1 - \mathsf{H_b}(\varepsilon).\end{aligned}$$

4. For $\varepsilon = 0$ or $\varepsilon = 1$, $\mathsf{H_b}(\varepsilon) = 0$. In that case, $\mathsf{I}(X;Y)$ is maximal and the mutual information is 1 bit.

5. For $\varepsilon = 0.5$, $\mathsf{H_b}(\varepsilon) = 1$. In that case, $\mathsf{I}(X;Y)$ is minimal and the mutual information is 0.