SSY125 Digital Communications

Department of Electrical Engineering

Exam Date: January 19, 2019, 14:00-18:00
Location: HA, HB, HC

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Material Allowed material is

- Chalmers-approved calculator.
- L. Råde, B. Westergren. Beta, Mathematics Handbook, any edition.
- A dictionary.

Grading A correct, clear and well-motivated solution gives a maximum of 15 points.

An erroneous answer, unclear, incomplete or badly motivated solutions give point reductions down to a minimum of 0 points. Answers in any other language than English are ignored.

Solutions The solutions are made available as soon as possible on the course web page.

Review The grading review will be on February 4, 2019, 10:00-11:00, and on February 11, 2019, 13:00-14:00, in room 6337 (office of the TAs) in the EDIT-building.

Grades The final grade on the course will be decided by the project (maximum score 30), quizzes (maximum score 3), tutorial grade (maximum score 7), and final exam (maximum score 60). The sum of all scores will decide the grade according to the following table.

Total Score	0–39	40–59	60–79	≥ 80
Grade	Fail	3	4	5

PLEASE NOTE THAT THE PROBLEMS ARE NOT NECESSARILY ORDERED IN DIFFICULTY.

Good luck!

Problem 1 - Channel Capacity [15 points]

Part I [9 points]

Consider a channel whose input is a random variable X that takes values on $\mathcal{X} = \{0, 1\}$ with probabilities $P_X(0) = P_X(1) = 1/2$. The channel output is a random variable Y that takes values on $\mathcal{Y} = \{0, \varepsilon, 1\}$. The channel is defined by the conditional distribution $P_{Y|X}(y|x)$ given by

	$X = 0$	$X = 1$
$Y = 0$	$1 - \beta - \alpha$	β
$Y = \varepsilon$	α	α
$Y = 1$	β	$1 - \beta - \alpha$

where $\alpha = \beta = 1/2$.

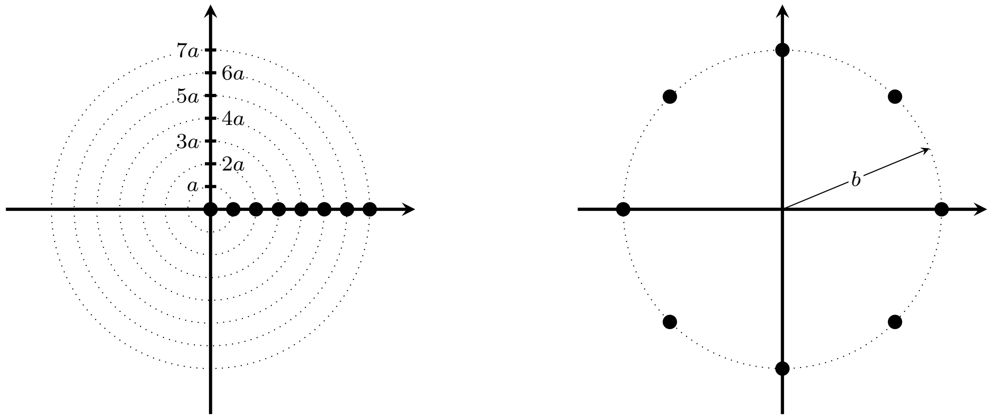
1. [*2 pt*] Compute the following probability distributions: $P_{X,Y}(x, y)$ and $P_Y(y)$.
2. [*2 pt*] Compute the following entropies in bits: $\mathsf{H}(X)$, $\mathsf{H}(Y)$, $\mathsf{H}(Y|X)$, and $\mathsf{H}(X, Y)$.
3. [*2 pt*] Compute the mutual information of this channel, $\mathsf{I}(X;Y)$, in bits. In general, how does the channel capacity relate to $\mathsf{I}(X;Y)$?
4. [*3 pt*] Assume that you can now change α and β, but the input distribution is still fixed at $P_X(0) = P_X(1) = 0.5$. What values of α and β maximize $\mathsf{I}(X;Y)$?

Part 2 [6 points]

Consider the discrete-time complex additive white Gaussian noise (AWGN) channel. The capacity of this channel is given by $C = \log_2(1 + \mathsf{E_s}/\mathsf{N_0})$, where $\mathsf{E_s}$ is the average symbol energy and $\mathsf{N_0}$ is the noise power spectral density.

1. [*1 pt*] What is the capacity-achieving input distribution for this channel?
2. [*2 pt*] Assume that $\mathsf{E_s} = 10^{-3}$ W·s and $\mathsf{N_0} = 10^{-6}$ W/Hz. Is reliable communication possible at an information rate of $R = 10$ bits/symbol? What is the lowest signal-to-noise ratio (SNR), $\mathsf{E_s}/\mathsf{N_0}$, that enables reliable communication at $R = 10$ bits/symbol?
3. [*3 pt*] Show that the SNR per information bit, $\mathsf{E_b}/\mathsf{N_0}$, must be greater than $\ln 2$ for reliable communication to be possible through the considered channel. What is the value of R at the minimum required $\mathsf{E_b}/\mathsf{N_0}$?

Problem 2 - Signal Constellations and Maximum Likelihood [15 points]



(a) Ω_1 (b) Ω_2

Figure 1: Constellations of two modulation formats.

Consider the two constellations Ω_1 and Ω_2 shown in Figures 1 (a) and (b). Assume that the received signal is given as $r = s + n$, where s is a point selected from one of the constellations and n is a realization of a zero-mean, complex Gaussian random variable N with $\mathbb{E}[|N|^2] = \mathsf{N}_0$. All constellation points in Ω_1 and Ω_2 are assumed equally likely to be transmitted.

Questions

1. [*1 pt*] Determine a and b, such that both constellations Ω_1 and Ω_2 have average symbol energy $\mathbb{E}[|S|^2] = \mathsf{E_s}$.
2. [*2 pt*] Find a Gray mapping for both constellations.
3. [*3 pt*] Find an upper bound for the symbol error probability, P_s, for Ω_2. The final expressions for the bound should only be a function of $\mathsf{E_s}/\mathsf{N}_0$. *Hint: The distance between two points on the unit circle with relative angle* ϕ *is* $2\sin(\phi/2)$. What happens to P_s as $\mathsf{N}_0 \to 0$?
4. [*2 pt*] Can the constellation points in Ω_1 be shifted on the real axis such that the average symbol energy of Ω_1 decreases? If no, explain why. If yes, find the shift that minimizes the average symbol energy.
5. [*2 pt*] Suppose now that the received signal has an unknown phase offset, i.e., $r = se^{j\theta} + n$, where j is the imaginary unit. Assuming that s is known, find the maximum-likelihood estimator of θ, which should only be a function of r and s.
6. [*2 pt*] Consider two symbol-detection strategies for transmission of a point s selected from Ω_1, with the received signal given by $r = se^{j\theta} + n$, where θ is uniformly distributed in $[0, 2\pi)$. The first strategy computes
$$\hat{s} = \underset{s\in\Omega_1}{\operatorname{argmin}}\, |s - r|$$
whereas the second strategy computes
$$\hat{s} = \underset{s\in\Omega_1}{\operatorname{argmin}}\, |s - |r||.$$
Which strategy will perform better in terms of P_s when $\mathsf{N}_0 \to 0$? Explain why.
7. [*3 pt*] Assuming equiprobable constellation points, show that the exact symbol error probability of an 8PAM constellation is given by
$$P_\mathsf{s}^{\mathsf{8PAM}} = AQ\left(\sqrt{B\frac{\mathsf{E_b}}{\mathsf{N}_0}}\right),$$
and determine the values of A and B. Using this fact, show that the exact symbol error probability of a 64QAM constellation is
$$P_\mathsf{s}^{\mathsf{64QAM}} = 2AQ\left(\sqrt{B\frac{\mathsf{E_b}}{\mathsf{N}_0}}\right) - A^2Q^2\left(\sqrt{B\frac{\mathsf{E_b}}{\mathsf{N}_0}}\right),$$
assuming that the complex AWGN affects the in-phase and quadrature components of the signal independently.

Problem 3 - Linear Block Codes and LDPC Codes [15 points]

Part I [4 points]

Consider a code $\mathcal{C}_1$ defined by the parity check matrix.

$$\boldsymbol{G}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

1. [*1 pt*] Draw the Tanner graph of $\mathcal{C}_1$.
2. [*1 pt*] Find the girth and highlight it in the graph.
3. [*2 pt*] Determine the variable node degree distribution $\Lambda(x)$ and the check node degree distribution $P(x)$. Is the LDPC code regular or irregular? Justify your answer!

Note: Even though $\mathcal{C}_1$ has small code length and is not sparse, consider it an LDPC code.

Part II [11 points]

Consider the code $\mathcal{C}_2$,

$$\mathcal{C}_2 = \left\{ \begin{array}{l} (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0), \\ (1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0), \\ (1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1), \\ (0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1) \end{array} \right\}.$$

1. [*3 pt*] Is this code a linear block code? *Justify your answer!* Find the generator matrix $\boldsymbol{G}_{\mathsf{s}}$ and parity check matrix $\boldsymbol{H}_{\mathsf{s}}$ in systematic form.
2. [*2 pt*] What are the error correction and error detection capabilities of this code over the binary symmetric channel with cross-over probability $p = 0.3$?
3. [*6 pt*] To transmit one codeword $\boldsymbol{c} \in \mathcal{C}_2$, for each two bits (c_1, c_2), we select a point x from the QPSK constellation as shown below in Fig. 2. The channel model is $y = x + n$, where n is a realization of complex Gaussian random variable N, with zero mean and $\mathbb{E}[|N|^2] = \sigma^2 = \mathsf{N}_0$, and $\mathbb{E}[|X|^2] = \mathsf{E}_{\mathsf{s}}$. The receiver first detects the symbol $\hat{x}$ and then outputs the binary mapping corresponding to $\hat{x}$, which we will denote by $\bar{\boldsymbol{y}} = (\bar{y}_1, \bar{y}_2)$. Under the assumption of high SNR, the receiver outputs $\bar{\boldsymbol{y}} = (1\,0\,1\,0\,1\,0)$. What is the ML codeword?

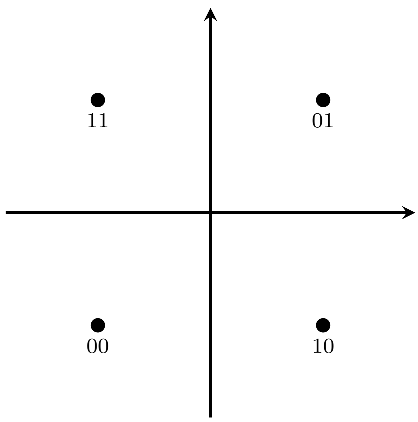


Figure 2: QPSK constellation.

Problem 4 - Convolutional Codes and the Viterbi Algorithm [15 points]

Part I [11 points]

Consider the encoder $\mathcal{E}_{\mathsf{RSC}}$ with generator matrix

$$\boldsymbol{G} = \begin{pmatrix} 1 & \frac{1+\mathsf{D}+\mathsf{D}^2}{1+\mathsf{D}^2} & \frac{1+\mathsf{D}}{1+\mathsf{D}^2} \end{pmatrix}.$$

1. [*1 pt*] Draw the block diagram of the encoder $\mathcal{E}_{\mathsf{RSC}}$.
2. [*2 pt*] Transform the encoder $\mathcal{E}_{\mathsf{RSC}}$ into feed-forward form $\mathcal{E}_1$. Show the corresponding block diagram and generator matrix.
3. [*3 pt*] Draw one full section of the Trellis diagram of $\mathcal{E}_{\mathsf{RSC}}$. Only display possible transitions and reachable states. Make sure that all state transitions are clearly labeled with the corresponding input and output bits.
4. [*5 pt*] Assume that the encoder $\mathcal{E}_{\mathsf{RSC}}$ is initialized to the all-zero state and zero-termination. The bits are transmitted over an AWGN channel using BPSK where bit 0 is mapped to 1 and bit 1 is mapped to -1. The $\mathsf{E_b}/\mathsf{N_0} = 4$ dB and the received observation is given by

   $$\boldsymbol{y} = \begin{pmatrix} 0.72 & -0.06 & 0.71 & -0.20 & -0.12 & 1.48 & 1.40 & 1.41 & 0.67 & -1.20 & 0.71 & 1.63 \end{pmatrix}.$$

   Find the maximum likelihood estimate of the information bits by using the Viterbi algorithm and hard decision decoding.

Part II [4 points]

Consider a linear block code with parity-check matrix $\boldsymbol{H}$ with columns $\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_n$. For each valid codeword, $\boldsymbol{c} = (c_1, c_2, \ldots c_n)$, the parity-check equation $c_1\boldsymbol{h}_1 + c_2\boldsymbol{h}_2 + \ldots + c_n\boldsymbol{h}_n = \boldsymbol{0}$ must be fullfilled. We then define the result of the partial parity-check equation as

$$S_\ell = \sum_{i=1}^{\ell} c_i \boldsymbol{h}_i.$$

We can then create a trellis representation of the linear block code by using S_ℓ as the state at the ℓ-th section of the trellis.

Considering now the parity-check matrix

$$\boldsymbol{H} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

1. [*1 pt*] How many trellis sections do you need?
2. [*1 pt*] How is the trellis terminated and why (zero-termination, truncated, ...)?
3. [*2 pt*] Draw paths of all possible codewords of the above code.

For all parts of this problem, it is important that you clearly show all involved branch metrics, state metrics, and survivor paths.