Digital Communications
SSY125, Lecture 7

Analysis of Linear Modulations
(Chapter 6)

Alexandre Graell i Amat
alexandre.graell@chalmers.se
https://sites.google.com/site/agraellamat

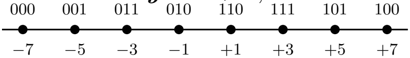
November 18, 2019

CHALMERS

Analysis of Linear Modulations

AWGN channel,

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{n},$$



- $x_i \in \mathcal{X} = \{\mathsf{X}_1, \mathsf{X}_2, \ldots, \mathsf{X}_M\} \subset \mathbb{C}$, with average energy per symbol $\mathsf{E_s}$.
- $|\mathcal{X}| = M \longrightarrow m = \log M$ bits per symbol.
- $N_i \sim \mathcal{CN}(0, \mathsf{N_0}/2)$.
- The energy per information bit is

$$\mathsf{E_b} = \frac{\mathsf{E_s}}{\log M}.$$

Optimum Decoding Rule

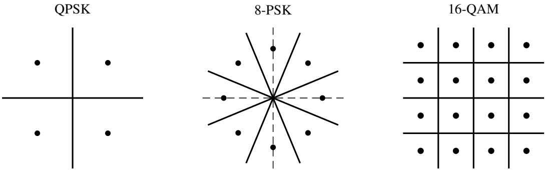
- Based on $\boldsymbol{y}$, infer $\boldsymbol{u}$ optimally $\longrightarrow$ Based on $\boldsymbol{y}$ infer $\boldsymbol{x}$ optimally.
- Criterion: minimizing the probability of error $\rightarrow$ maximum a posteriori (MAP) decoding rule,

$$\hat{\boldsymbol{x}}_{\mathsf{MAP}} = \arg\max_{\boldsymbol{x}} p(\boldsymbol{x}|\boldsymbol{y}).$$

- If all sequences $\boldsymbol{x}$ are equiprobable, the MAP decoding rule boils down to the maximum likelihood (ML) decoding rule,

$$\hat{\boldsymbol{x}}_{\mathsf{ML}} = \arg\max_{\boldsymbol{x}} p(\boldsymbol{y}|\boldsymbol{x}),$$

Optimum Decoding Rule



- If the channel is memoryless: symbol-by-symbol decoder that makes an independent decision on each received symbol y_i,

$$\hat{x} = \arg\max_{x \in \mathcal{X}} p(y|x).$$

- For transmission over the Gaussian channel (and equiprobable symbols),

$$\begin{aligned}\hat{x} &= \arg\max_{x \in \mathcal{X}} p(y|x) \\ &= \arg\min_{x \in \mathcal{X}} ||y - x||^2,\end{aligned}$$

i.e., selecting among all possible constellation symbols $x \in \mathcal{X}$ the one at minimum Euclidean distance to the received value y.

Performance of Modulation Schemes

The evaluation of a modulation scheme is based on three parameters:

- The error probability (symbol error probability or bit error probability);
- The E_b/N_0 required to achieve such probability of error, and
- The spectral efficiency $R = \log M$.

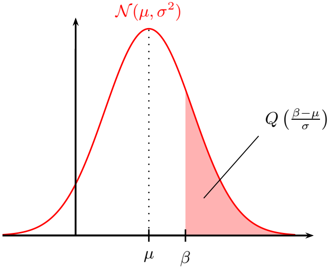
The Union Bound

Given a number of events $E_1, \ldots, E_N$

$$\mathsf{Pr}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mathsf{Pr}(E_i),$$

where $E_i \cup E_j$ stands for the union of the events E_i and E_j.

The Q-function



For $Z \sim \mathcal{N}(\mu,\sigma^2)$,

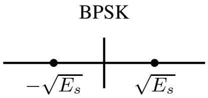
$$\Pr(Z > \beta) = \mathsf{Q}\left(\frac{\beta-\mu}{\sigma}\right),$$

where $\mathsf{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\tau^2/2\right) \mathrm{d}\tau$ is the tail probability of the standard normal distribution.

- $\mathsf{Q}(-x) = 1 - \mathsf{Q}(x)$.
- For $0 < a \ll b < c$,

$$\mathsf{Q}(a) + \mathsf{Q}(b) + \mathsf{Q}(c) \approx \mathsf{Q}(a).$$

Symbol Error Probability of BPSK



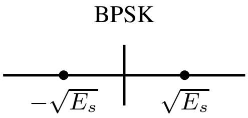
- $\mathcal{X} = \{\mathsf{X}_1, \mathsf{X}_2\} \subset \mathbb{R}$, where $\mathsf{X}_1 = -\sqrt{\mathsf{E_s}}$ and $\mathsf{X}_2 = \sqrt{\mathsf{E_s}}$.
- The received signal is

$$y = x + n,$$

where $n \sim \mathcal{N}(0, \sigma^2)$.
- For equiprobable symbols, $\hat{x} = \mathsf{X}_2$ if $y > 0$ and $\hat{x} = \mathsf{X}_1$ otherwise.

$$\begin{aligned}
P_{\mathsf{s}}^{\mathsf{BPSK}} &= \sum_{x \in \mathcal{X}} \mathsf{Pr}(\hat{x} \neq x | x) P(x) \\
&= \mathsf{Pr}(\mathsf{X}_2 | \mathsf{X}_1) P(\mathsf{X}_1) + \mathsf{Pr}(\mathsf{X}_1 | \mathsf{X}_2) P(\mathsf{X}_2) \\
&= \frac{1}{2} \mathsf{Pr}(\mathsf{X}_2 | \mathsf{X}_1) + \frac{1}{2} \mathsf{Pr}(\mathsf{X}_1 | \mathsf{X}_2),
\end{aligned}$$

Symbol Error Probability of BPSK



If $X = \mathsf{X}_1$, $Y \sim \mathcal{N}(-\sqrt{\mathsf{E_s}}, \sigma^2)$, hence

$$\begin{aligned}
\mathsf{Pr}(\hat{x} = \mathsf{X}_2 | x = \mathsf{X}_1\,) &= \mathsf{Pr}(Y > 0 | X = \mathsf{X}_1\,) \\
&= \mathsf{Pr}(Y > 0)\Big|_{Y \sim \mathcal{N}(-\sqrt{\mathsf{E_s}}, \sigma^2)} \\
&= \mathsf{Q}\left(\frac{\sqrt{\mathsf{E_s}}}{\sigma}\right) = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_s}}{\mathsf{N_0}}}\right) = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right).
\end{aligned}$$

Due to symmetry, $\mathsf{Pr}(\hat{x} = \mathsf{X}_1 | x = \mathsf{X}_2\,) = p(\hat{x} = \mathsf{X}_2 | x = \mathsf{X}_1\,)$, and

$$P_{\mathsf{s}}^{\mathsf{BPSK}} = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_s}}{\mathsf{N_0}}}\right) = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right).$$

Symbol Error Probability and Euclidean Distance

For BPSK, $d_{\mathsf{E}}(\mathsf{X}_1, \mathsf{X}_2) = 2\sqrt{\mathsf{E_s}}$, hence

$$P_{\mathsf{s}}^{\mathsf{BPSK}} = \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_1, \mathsf{X}_2)}{2\mathsf{N}_0}}\right).$$

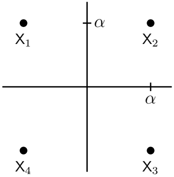
In the general case...

- Consider $\mathsf{X}_1 \in \mathbb{C}$ and $\mathsf{X}_2 \in \mathbb{C}$ with $d_{\mathsf{E}}(\mathsf{X}_1, \mathsf{X}_2) = ||\mathsf{X}_1 - \mathsf{X}_2||$. Assume $x \in \mathcal{X} = \{\mathsf{X}_1, \mathsf{X}_2, \ldots, \}$ is transmitted and we receive $y = x + n$.
- Let $\tilde{y}$ be the projection of y onto the straight line between X_1 and X_2. Then,

$$\begin{aligned}\mathsf{Pr}(\hat{x} = \mathsf{X}_2 | x = \mathsf{X}_1) &= \mathsf{Pr}\left(\tilde{Y} > \frac{d_{\mathsf{E}}(\mathsf{X}_1, \mathsf{X}_2)}{2}\right)\Bigg|_{\tilde{Y} \sim \mathcal{N}(0, \sigma^2)} \\ &= \mathsf{Q}\left(\frac{d_{\mathsf{E}}(\mathsf{X}_1, \mathsf{X}_2)}{2\sigma}\right) = \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_1, \mathsf{X}_2)}{2\mathsf{N}_0}}\right).\end{aligned}$$

- $\mathsf{Pr}(\hat{x} = \mathsf{X}_2 | x = \mathsf{X}_1)$ depends on $d_{\mathsf{E}}(\mathsf{X}_1, \mathsf{X}_2)$ $\Rightarrow$ Construct constellations with high distance between constellation points!

Symbol Error Probability of 4-QAM

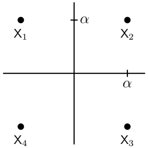


- $\mathcal{X} = \{\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3, \mathsf{X}_4\}$ with $\alpha = \sqrt{\mathsf{E}_\mathsf{s}/2}$ such that the average energy per symbol is E_s.
- The ML decision is

$$\hat{x} = \begin{cases} \mathsf{X}_1 & \text{if } y_\mathsf{I} < 0 \text{ and } y_\mathsf{Q} > 0 \\ \mathsf{X}_2 & \text{if } y_\mathsf{I} > 0 \text{ and } y_\mathsf{Q} > 0 \\ \mathsf{X}_3 & \text{if } y_\mathsf{I} > 0 \text{ and } y_\mathsf{Q} < 0 \\ \mathsf{X}_4 & \text{if } y_\mathsf{I} < 0 \text{ and } y_\mathsf{Q} < 0 \end{cases},$$

where y_I and y_Q are the imaginary and quadrature components of y.

Analysis of Linear Modulations



$$
\begin{aligned}
P_{\mathsf{s}}^{\mathsf{4QAM}} &= \mathsf{Pr}(\hat{x} \neq \mathsf{X}_1 | x = \mathsf{X}_1) = \mathsf{Pr}(Y_{\mathsf{I}} > 0 \cup Y_{\mathsf{Q}} < 0 | \mathsf{X}_1) \\
&= \mathsf{Pr}(Y_{\mathsf{I}} > 0 | \mathsf{X}_1) + \mathsf{Pr}(Y_{\mathsf{Q}} < 0 | \mathsf{X}_1) - \mathsf{Pr}(Y_{\mathsf{I}} > 0 \cap Y_{\mathsf{Q}} < 0 | \mathsf{X}_1) \\
&= \mathsf{Pr}(Y_{\mathsf{I}} > 0 | \mathsf{X}_1) + \mathsf{Pr}(Y_{\mathsf{Q}} < 0 | \mathsf{X}_1) - \mathsf{Pr}(Y_{\mathsf{I}} > 0 | \mathsf{X}_1)\mathsf{Pr}(Y_{\mathsf{Q}} < 0 | \mathsf{X}_1),
\end{aligned}
$$

If X_1 is transmitted, $Y_{\mathsf{I}} \sim \mathcal{N}(-\alpha, \sigma^2)$ and $Y_{\mathsf{Q}} \sim \mathcal{N}(\alpha, \sigma^2)$. Thus,

$$
\mathsf{Pr}(Y_{\mathsf{I}} > 0 | \mathsf{X}_1) = \mathsf{Pr}(Y_{\mathsf{I}} > 0)\Big|_{Y_{\mathsf{I}} \sim \mathcal{N}(-\alpha, \sigma^2)} = \mathsf{Q}\left(\frac{\alpha}{\sigma}\right) = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right),
$$

By symmetry, $\mathsf{Pr}(Y_{\mathsf{Q}} < 0 | \mathsf{X}_1) = \mathsf{Pr}(Y_{\mathsf{I}} > 0 | \mathsf{X}_1)$, and

$$
P_{\mathsf{s}}^{\mathsf{4QAM}} = 2\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right) - \left(\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)\right)^2.
$$

Upper Bound on the Symbol Error Probability

For general constellations, the exact symbol error probability P_{s} is hard to compute $\longrightarrow$ Use union bound to compute an upper bound

$$
\begin{aligned}
P_{\mathsf{s}}^{(M)} &= \sum_{i=1}^{M} \mathsf{Pr}(\hat{x} \neq \mathsf{X}_i | x = \mathsf{X}_i) P(\mathsf{X}_i) \\
&= \frac{1}{M} \sum_{i=1}^{M} \mathsf{Pr}(\hat{x} \neq \mathsf{X}_i | x = \mathsf{X}_i) \\
&= \frac{1}{M} \sum_{i=1}^{M} \mathsf{Pr}\left(\bigcup_{j \neq i} \hat{x} = \mathsf{X}_j | x = \mathsf{X}_i \right) \\
&\leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \neq i} \mathsf{Pr}(\hat{x} = \mathsf{X}_j | x = \mathsf{X}_i).
\end{aligned}
$$

Upper Bound on the Symbol Error Probability

Using

$$\Pr(\hat{x} = \mathsf{X}_j | x = \mathsf{X}_i) = \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_i, \mathsf{X}_j)}{2\mathsf{N}_0}}\right).$$

the symbol error probability of an M-ary constellation can be upperbounded as

$$P_{\mathsf{s}}^{(M)} \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \neq i} \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_i, \mathsf{X}_j)}{2\mathsf{N}_0}}\right).$$

Upper bound on P_s for 4-QAM

Example: 4-QAM

$$P_\mathsf{s}^{4\mathsf{QAM}} \leq \frac{1}{4}\sum_{i=1}^{4}\sum_{j\neq i}\mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_i,\mathsf{X}_j)}{2\mathsf{N}_0}}\right) = \sum_{j=2}^{4}\mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_j)}{2\mathsf{N}_0}}\right)$$
$$= \mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_2)}{2\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_3)}{2\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_4)}{2\mathsf{N}_0}}\right).$$

Using $d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_2) = d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_4) = ||\alpha(-1+j)-\alpha(1+j)||^2 = 4\alpha^2 = 2\mathsf{E}_\mathsf{s}$ and $d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_3) = ||\alpha(-1+j)-\alpha(1-j)||^2 = 8\alpha^2 = 4\mathsf{E}_\mathsf{s}$,

$$P_\mathsf{s}^{4\mathsf{QAM}} \leq 2\mathsf{Q}\left(\sqrt{\frac{\mathsf{E}_\mathsf{s}}{\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E}_\mathsf{s}}{\mathsf{N}_0}}\right)$$
$$= 2\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E}_\mathsf{b}}{\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{4\mathsf{E}_\mathsf{b}}{\mathsf{N}_0}}\right).$$

Nearest Neighbor Approximation

- For large M, even the union bound may be difficult to compute $\longrightarrow$ Derive an approximation.

$$P_{\mathsf{s}}^{(M)} \leq \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_i,\mathsf{X}_j)}{2\mathsf{N}_0}}\right).$$

- The dominant terms are the ones with smaller $d_{\mathsf{E}}(\mathsf{X}_i,\mathsf{X}_j)$
- Let

$$d_{\mathsf{E,min}}(\mathsf{X}_i) = \min_{\mathsf{X}_j\neq\mathsf{X}_i} d_{\mathsf{E}}(\mathsf{X}_i,\mathsf{X}_j)$$

and $A_{\mathsf{min}}(\mathsf{X}_i)$ the number of constellation points at distance $d_{\mathsf{E,min}}(\mathsf{X}_i)$ from X_i (nearest neighbors of X_i). Then,

$$P_{\mathsf{s}}^{(M)} \approx \frac{1}{M}\sum_{i=1}^{M} A_{\min}(\mathsf{X}_i)\mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E,min}}^2(\mathsf{X}_i)}{2\mathsf{N}_0}}\right).$$

Nearest Neighbor Approximation

- Further approximate P_{s} considering only the minimum Euclidean distance,

$$d_{\mathsf{E,min}} = \min_{\mathsf{X}_i} d_{\mathsf{E,min}}(\mathsf{X}_i).$$

Then,

$$\begin{aligned}
P_{\mathsf{s}}^{(M)} &\approx \frac{1}{M}\sum_{i=1}^{M} A_{\min}(\mathsf{X}_i)\mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}(\mathsf{X}_i)}{2\mathsf{N}_0}}\right)\\
&\leq \frac{1}{M}\sum_{i=1}^{M} A_{\mathsf{min}}(\mathsf{X}_i)\mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N}_0}}\right)\\
&= \left(\frac{1}{M}\sum_{i=1}^{M} A_{\mathsf{min}}(\mathsf{X}_i)\right)\mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N}_0}}\right) = \bar{A}_{\mathsf{min}}\mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N}_0}}\right),
\end{aligned}$$

where $\bar{A}_{\mathsf{min}} = \frac{1}{M}\sum_{i=1}^{M} A_{\mathsf{min}}(\mathsf{X}_i)$ is the average number of nearest neighbors.

- Requires only knowledge of $\bar{A}_{\mathsf{min}}$ and $d_{\mathsf{E,min}}$!

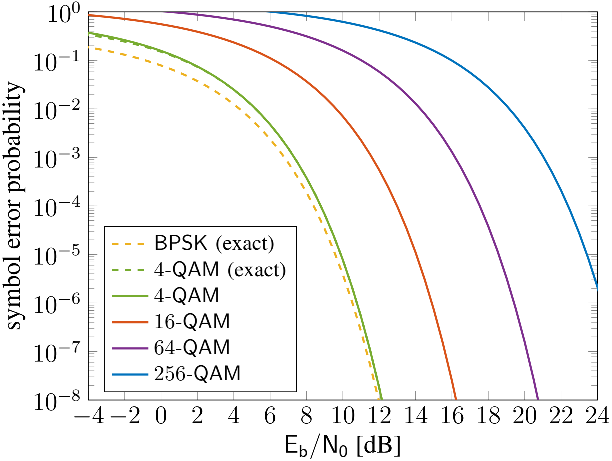
Nearest Neighbor Approximation

Nearest Neighbor Approximation for squared M-QAM

For squared M-QAM constellations, there are 4 constellation points with 2 neighbors, $4(\sqrt{M}-2)$ points with 3 neighbors and the remaining points have 4 neighbors, hence $\bar{A}_{\mathsf{min}} = 4 - 4/\sqrt{M}$. Furthermore, $d_{\mathsf{E,min}} = \sqrt{\frac{6\mathsf{E_s}}{M-1}}$. Hence,

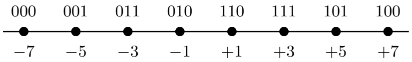
$$
\begin{aligned}
P_{\mathsf{s}}^{M\mathsf{QAM}} &\approx \left(4 - \frac{4}{\sqrt{M}}\right)\mathsf{Q}\left(\sqrt{\frac{3\mathsf{E_s}}{(M-1)\mathsf{N_0}}}\right)\\
&= \left(4 - \frac{4}{\sqrt{M}}\right)\mathsf{Q}\left(\sqrt{\frac{3\mathsf{E_b}\log M}{(M-1)\mathsf{N_0}}}\right).
\end{aligned}
$$

Nearest Neighbor Approximation of P_s of QAM



- The power efficiency decreases with M.
- The spectral efficiency increases with M.

Bit Error Probability of Linear Modulations



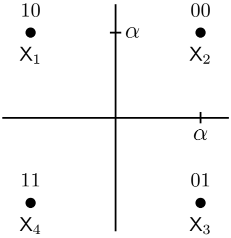
- The bit error probability depends on the binary labeling, i.e., how tuples of $m = \log M$ bits are mapped to the constellation symbols.
- Let

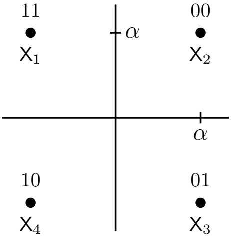
$$\mathsf{L}(x) = (b_1(x), \ldots, b_m(x))$$

  be the m-bit labeling associated to constellation symbol $x \in \mathcal{X}$.
- Example: $\mathsf{L}(+1) = (1, 1, 0)$, $\mathsf{L}(+5) = (1, 0, 1)$.
- The bit error probability P_b of an M-ary constellation is given by

$$P_\mathsf{b} = \frac{1}{m} \sum_{i=1}^{m} \mathsf{Pr}(i\text{th bit in error}) = \frac{1}{m} \sum_{i=1}^{m} p_i.$$

Bit Error Probability of 4-QAM with Gray and Lexicographic Labeling





Gray Labeling

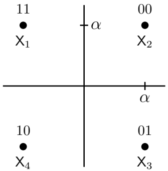
- e_{b_i}: event that bit i is decoded in error.

$$
\begin{aligned}
p_1 = \Pr(\mathsf{e}_{b_1}) &= \sum_{x \in \mathcal{X}} \Pr(\mathsf{e}_{b_1}|x)P(x) = \frac{1}{4}\sum_{x \in \mathcal{X}} \Pr(\mathsf{e}_{b_1}|x) \\
&= \frac{1}{4}(\Pr(\mathsf{e}_{b_1}|\mathsf{X}_1) + \Pr(\mathsf{e}_{b_1}|\mathsf{X}_2) + \Pr(\mathsf{e}_{b_1}|\mathsf{X}_3) + \Pr(\mathsf{e}_{b_1}|\mathsf{X}_4)) \\
&= \frac{1}{4}(\Pr(b_1(\hat{x}) = 0|\mathsf{X}_1) + \Pr(b_1(\hat{x}) = 1|\mathsf{X}_2) \\
&\quad + \Pr(b_1(\hat{x}) = 1|\mathsf{X}_3) + \Pr(b_1(\hat{x}) = 0|\mathsf{X}_4)),
\end{aligned}
$$

- All terms are $\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$, thus $p_1 = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$.
- By symmetry, $p_2 = p_1$, hence,

$$
P_{\mathsf{b}}^{\mathsf{4QAM-Gray}} = \frac{1}{2}(p_1 + p_2) = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right).
$$

Lexicographic Labeling



$$p_1 = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$$

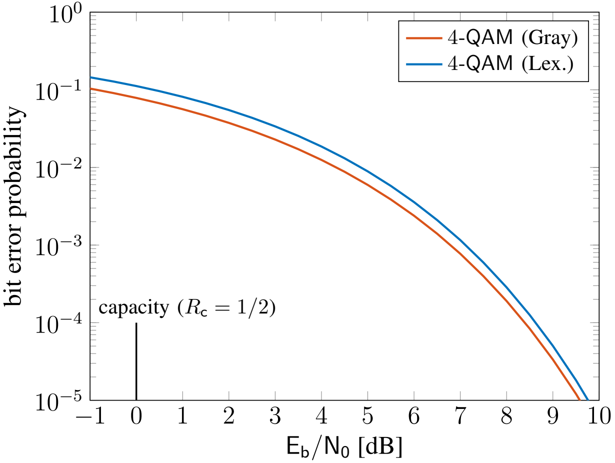
$$p_2 = 2\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)\left(1 - \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)\right).$$

Thus,

$$P_{\mathsf{b}}^{\mathsf{4QAM-Lex}} = \frac{1}{2}(p_1 + p_2) = \frac{3}{2}\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right) - \left(\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)\right)^2.$$

- Due to the lack of symmetry, $p_1 \neq p_2$.

Bit Error Probability for 4-QAM



Bit Error Probability of M-ary Constellations

- For general M-ary constellations and arbitrary labelings, the computation of P_b is cumbersome $\longrightarrow$ Nearest neighbor approximation.