Digital Communications
SSY125, Lectures 5 and 6

Communication over a Noisy Channel
(Chapters 4 and 5)

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digital data, voice
movie, audio
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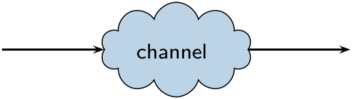

satellite link,
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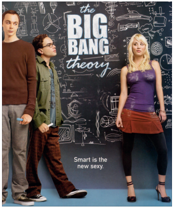

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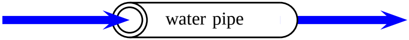
In These Lectures...

- How much information can we transmit reliably over an unreliable channel?
- How do we achieve this in practice?

Shannon's Seminal Contribution

There is a fundamental limit, i.e., a highest rate, at which information can be transmitted reliably over the channel: channel capacity.

Model of a Noisy Communication Channel



Discrete Memoryless Channel

- The input and output of the channel, $\boldsymbol{x} = (x_1, x_2, \ldots,)$ and $\boldsymbol{y} = (y_1, y_2, \ldots,)$, are discrete, i.e., $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$.
- Channel is defined by $P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})$ with (memoryless)

$$P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) = \prod_i P_{Y|X}(y_i|x_i). \quad \boxtimes$$

- Channel input: $X \in \mathcal{X}$.
- Channel output: $Y \in \mathcal{Y}$.
- Entirely specified by the conditional PMF $P_{Y|X}(y|x)$.

System Entropies

- Both the channel input and the channel output are random variables $\longrightarrow$ The information content of a random variable, its uncertainty, is given by the entropy!

Average amount of uncertainty on the input of the channel X:

$$\mathsf{H}(X) = \sum_{x \in \mathcal{X}} P(x) \log \frac{1}{P(x)}.$$

We have shown...

$$0 \leq \mathsf{H}(X) \leq \log |\mathcal{X}|.$$

Average amount of uncertainty on the channel output Y:

$$\mathsf{H}(Y) = \sum_{y \in \mathcal{Y}} P(y) \log \frac{1}{P(y)}.$$

System Entropies

Definition (Conditional Entropy of X given $Y = y$)

The conditional entropy of X given the even $Y = y$ is

$$\mathsf{H}(X|Y=y) \triangleq \sum_{x \in X} P(x|y) \log \frac{1}{P(x|y)}$$

Definition (Conditional Entropy)

The conditional entropy of X given Y is

$$\begin{aligned}
\mathsf{H}(X|Y) &= \sum_{y \in \mathcal{Y}} P(y) \mathsf{H}(X|Y=y) \\
&= \sum_{y \in \mathcal{Y}} P(y) \sum_{x \in \mathcal{X}} P(x|y) \log \frac{1}{P(x|y)} \\
&= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P(y) P(x|y) \log \frac{1}{P(x|y)} \\
&= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P(x,y) \log \frac{1}{P(x|y)}.
\end{aligned}$$

System Entropies

Theorem (Conditioning reduces entropy)

For any two random variables X and Y,

$$\mathsf{H}(X|Y) \leq \mathsf{H}(X). \qquad \boxtimes$$

with equality if and only if X and Y are statistically independent.

System Entropies

Definition (Joint Entropy)

The joint entropy of X and Y is defined as

$$\mathsf{H}(X,Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P(x,y) \log \frac{1}{P(x,y)}.$$

It gives the uncertainty of the overall system.

Lemma

The joint entropy of two statistically independent random variables X and Y is

$$\boxed{\mathsf{H}(X,Y) = \mathsf{H}(X) + \mathsf{H}(Y).}$$

System Entropies

For any *useful* communication channel, the output Y and the input X are not independent.

Lemma (Chain Rule)

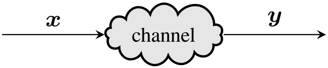
The joint entropy of two random variables X and Y is given by

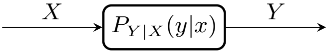
$$\mathsf{H}(X, Y) = \mathsf{H}(X) + \mathsf{H}(Y|X).$$

Since $P(x, y) = P(y, x)$, we can also write

$$\mathsf{H}(X, Y) = \mathsf{H}(Y) + \mathsf{H}(X|Y).$$

Information Conveyed by the Channel





- How much information about the transmitted random variable X we gain from Y?

Equivalently...

How much uncertainty remains on the knowledge of X after receiving Y $\longrightarrow$ The conditional entropy of X given Y, $\mathsf{H}(X|Y)$!

- How much information have we conveyed over the channel?
  - *Before* receiving Y: Uncertainty on X is $\mathsf{H}(X)$
  - *After* receiving Y: Uncertainty on X is $\mathsf{H}(X|Y)$
  - Thus, the information conveyed is: $\mathsf{H}(X) - \mathsf{H}(X|Y)$!

Information Conveyed by the Channel

Definition (Mutual Information)

The mutual information between X and Y is

$$\mathsf{I}(X;Y) \triangleq \mathsf{H}(X) - \mathsf{H}(X|Y).$$

- The mutual information measures the average reduction in uncertainty about X that results from learning the value of Y.

By simple manipulation of the entropies,

$$\mathsf{I}(X;Y) = \sum_{x \in X} \sum_{y \in Y} P(x,y) \log \frac{P(x,y)}{P(x)P(y)}.$$

Properties

P1 $\mathsf{I}(X;Y) \geq 0$,

P2 $\mathsf{I}(X;Y) = 0$ if and only if X and Y are independent, $\boxtimes$

P3 $\mathsf{I}(X;Y) = \mathsf{I}(Y;X)$,

P4 $\mathsf{I}(X;Y) \leq \min(\mathsf{H}(X), \mathsf{H}(Y))$.

The Channel Capacity

- How much information can we transmit over the cannel?
  - No control over the channel.
  - However, control over $P_X(x)$.
- The mutual information $\mathsf{I}(X;Y)$ depends on $P_X(x)$!.

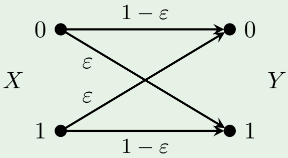
Definition (Channel Capacity)

For a given channel, the channel capacity is defined to be the maximum of the mutual information, maximized over all possible input distributions $P_X(x)$,

$$\mathsf{C} \triangleq \max_{P_X} \mathsf{I}(X;Y). \quad \boxtimes$$

Mutual Information and Channel Capacity

Running Example: The Binary Symmetric Channel



- Input: $X \in \mathcal{X} = \{0, 1\}$.
- Output: $Y \in \mathcal{Y} = \{0, 1\}$.
- Channel defined by the transition probabilities

$$P(0|0) = P(1|1) = 1 - \varepsilon$$
$$P(1|0) = P(0|1) = \varepsilon.$$

Mutual Information and Channel Capacity

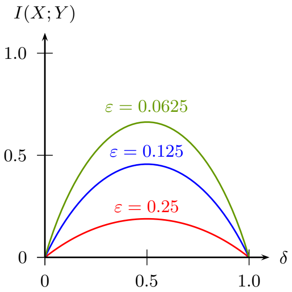
Running Example: The Binary Symmetric Channel

- We compute first the mutual information,

$$
\begin{aligned}
\mathsf{I}(X;Y) &= \mathsf{H}(Y) - \mathsf{H}(Y|X) \\
&= \mathsf{H}(Y) - \sum_{x \in \mathcal{X}} P(x)\mathsf{H}(Y|X=x) \\
&= \mathsf{H}(Y) - (P(0)\mathsf{H}(Y|X=0) + P(1)\mathsf{H}(Y|X=1)) \\
&= \mathsf{H}(Y) - \underbrace{(P(0)+P(1))}_{=1}\mathsf{H}_\mathsf{b}(\varepsilon) \\
&= \mathsf{H}(Y) - \mathsf{H}_\mathsf{b}(\varepsilon) \\
&\leq 1 - \mathsf{H}_\mathsf{b}(\varepsilon).
\end{aligned}
$$

$\mathsf{I}(X;Y)$ depends on P_Y, thus on P_X!

The Channel Capacity



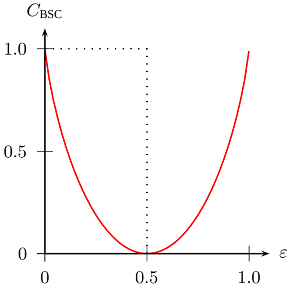
Running Example: The Binary Symmetric Channel

$$\mathsf{I}(X;Y) = \mathsf{H}(Y) - \mathsf{H}(Y|X) \leq 1 - \mathsf{H}_\mathsf{b}(\varepsilon).$$

- Let $P(X=0) = \delta$ and $P(X=1) = 1-\delta$.
- The mutual information is maximized for $\delta = \frac{1}{2}$ (equally likely symbols)
- The maximum of the mutual information is the channel capacity,

$$\mathsf{C}_\mathsf{BSC} = 1 - \mathsf{H}_\mathsf{b}(\varepsilon) \ \text{ bits.}$$

The Channel Capacity



Running Example: The Binary Symmetric Channel

$$\mathsf{C}_{\mathsf{BSC}} = 1 - \mathsf{H}_{\mathsf{b}}(\varepsilon) \quad \text{bits.}$$

- It is maximum $(1$ bit$)$ for $\varepsilon = 0$ and $\varepsilon = 1$.
- It is zero for $\varepsilon = \frac{1}{2} \longrightarrow$ The channel is useless!

The Channel Coding Theorem

Definition (Channel Capacity)

For a given channel, the channel capacity is defined to be the maximum of mutual information, maximized over all possible input distributions $P(x)$,

$$\mathsf{C} \triangleq \max_{P_X(x)} \mathsf{I}(X;Y).$$

- The channel capacity is a measure of the information conveyed by a channel, but... What is its operational meaning?

The Channel Coding Theorem

The channel capacity is the maximum transmission rate at which we can communicate reliably over the channel!

Theorem (Shannon's Channel Coding Theorem)

For a discrete-time channel, it is possible to transmit information with an arbitrarily small probability of error if the communication rate R is below the channel capacity, i.e., $R < \mathsf{C}$. More precisely, for any $R \leq \mathsf{C}$, there exist a sequence of coding schemes of length N with average error probability $P_{\mathsf{e}}^{(N)}$ that tends to zero as $N \to \infty$, i.e., $P_{\mathsf{e}}^{(N)} \to 0$ as $N \to \infty$.
Conversely, any sequence of coding schemes with vanishing error probability must have $R \leq C$. Hence, the probability of error for transmission above capacity is bounded above zero.

Running Example: The Binary Symmetric Channel

For $\varepsilon = 0.25$, $\mathsf{C}_{\mathsf{BSC}} = 1 - \mathsf{H}_{\mathsf{b}}(\varepsilon) = 1 - \mathsf{H}_{\mathsf{b}}(0.25) = 0.1887 \longrightarrow$ Reliable communication is possible as long as we transmit at a rate < 0.1887 bits per channel use.

Communication Over the AWGN Channel

The AWGN Channel

Additive White Gaussian Noise (AWGN) Channel

Continuous-time, complex AWGN channel,

$$y(t) = x(t) + n(t),$$

where $x(t)$ is bandlimited with bandwidth W and has signal power P, the symbol interval is $T = 1/W$, and $n(t)$ is complex AWGN with PSD N_0 and

$$\mathsf{SNR} \triangleq \frac{P}{\mathsf{N}_0 W}.$$

Discrete-time AWGN channel,

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{n},$$

where $\boldsymbol{x} = (x_1, x_2, \ldots)$ is the transmitted sequence of constellation symbols with average energy per symbol E_s, and $\boldsymbol{n}$ is a sequence of i.i.d. Gaussian noise random variables with zero mean and variance $\sigma^2 = \mathsf{N}_0/2$ per real dimension.

The AWGN Channel

- Input of the channel x takes values on $\mathcal{X} = \{\mathsf{X}_1, \mathsf{X}_2, \ldots, \mathsf{X}_M\} \subset \mathbb{C}$ according to $P_X(x)$.
- Channel law (conditional PDF):

$$p_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{||y-x||^2}{2\sigma^2}}.$$

- Signal-to-noise ratio:

$$\mathsf{SNR} \triangleq \frac{\mathsf{E_s}}{\mathsf{N_0}} = \frac{\mathsf{E_s}}{2\sigma^2}.$$

Mutual Information and Channel Capacity

- For a given P_X, the amount of information that can be conveyed over the channel is given by the mutual information

$$\mathsf{I}(X;Y) = \iint p(x,y) \log \frac{p(x,y)}{p(x)p(y)} dxdy,$$

- If X is a discrete RV,

$$\mathsf{I}(X;Y) = \sum_{x \in X} \int P(x)p(y|x) \log \frac{p(x,y)}{P(x)p(y)} dy,$$

- The ultimate limit at which we can transmit reliably is given by the channel capacity,

$$\mathsf{C} \triangleq \max_{p_X} \mathsf{I}(X;Y).$$

Capacity of the AWGN channel

Capacity of the Continuous-Time AWGN Channel

The channel capacity of the continuous-time AWGN channel of bandwidth W is

$$\mathsf{C}_{\mathsf{AWGN-C}} = W \log\left(1 + \mathsf{SNR}\right) \quad [\text{bits/s}],$$

where the $\mathsf{SNR} \triangleq \frac{P}{\mathsf{N}_0 W}$.
The channel capacity is achieved by a Gaussian input distribution, i.e., X is Gaussian-distributed.

The capacity depends on only two parameters, the channel bandwidth W and the SNR.

Capacity of the AWGN channel

Capacity of the Discrete-Time AWGN Channel

The channel capacity of the discrete-time real AWGN channel with average energy per symbol $\mathsf{E_s}$ and noise variance $\sigma^2 = \mathsf{N_0}/2$ is

$$\mathsf{C}_{\mathsf{AWGN-D}} = \frac{1}{2}\log\left(1 + \frac{\mathsf{E_s}}{\sigma^2}\right) \text{ [bits/channel use] or [bits/symbol]},$$

and is achieved by a Gaussian input distribution, i.e., $X \sim \mathcal{N}(0, \mathsf{E_s})$.

Capacity of the Discrete-Time AWGN Channel

The channel capacity of the discrete-time complex AWGN channel with average energy per symbol $\mathsf{E_s}$ and noise variance $\sigma^2 = \mathsf{N_0}/2$ per dimension is

$$\mathsf{C}_{\mathsf{AWGN-D}} = \log\left(1 + \frac{\mathsf{E_s}}{2\sigma^2}\right) \text{ [bits/channel use] or [bits/symbol]},$$

and is achieved by a Gaussian input distribution, i.e., $X \sim \mathcal{CN}(0, \mathsf{E_s})$.

The Channel Coding Theorem for the AWGN Channel

Theorem (Channel Coding Theorem, Discrete-Time Channel)

All rates R below $\mathsf{C}_{\mathsf{AWGN-D}}$ are achievable, i.e., for every $R < \mathsf{C}_{\mathsf{AWGN-D}}$ there exists a sequence of coding schemes with vanishing error probability $P_{\mathsf{e}}^{(N)} \to 0$ as the block length $N \to \infty$. Conversely, any sequence of coding schemes of rate R and block length N with error probability $P_e^N \to 0$ must have a rate $R < \mathsf{C}_{\mathsf{AWGN-D}}$.

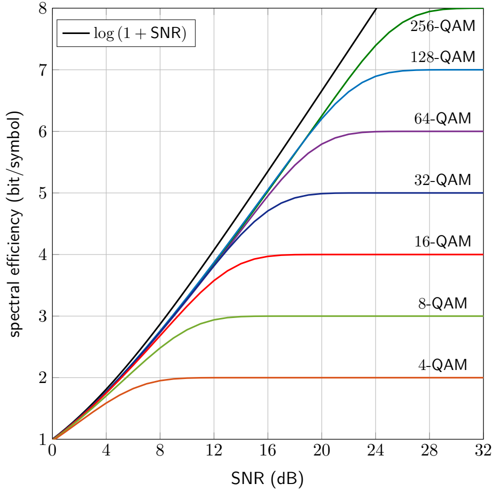
What the Theorem Tells Us: Achievable Rates

Arbitrarily reliable transmission can be achieved on the discrete-time channel at any rate $R < \mathsf{C}_{\mathsf{AWGN-D}}$ if codes of large length are used. $\longrightarrow$ Rate R is achievable.

For the Continuous-Time Channel...

Reliable transmission can be achieved on the continuous-time channel at any bit rate R_{b} [bits/second] ($R_{\mathsf{b}} = R/T$) such that $R_{\mathsf{b}} < \mathsf{C}_{\mathsf{AWGN-C}}$.

Channel Capacity and Mutual Information



- There is a gap between the mutual information curves and the capacity.
- QAM constellations are not capacity-achieving!

Power Efficiency and Energy per Information Bit

- To compare coded communication systems we consider the ratio

$$\frac{\mathsf{E_b}}{\mathsf{N_0}},$$

where $\mathsf{E_b}$ is the energy per information bit. $\mathsf{E_b}/\mathsf{N_0}$ is referred to as the power efficiency.

- For a coded system that encodes sequences $\boldsymbol{u}$ of K bits onto sequences $\boldsymbol{x}$ of N symbols with average energy per symbol $\mathsf{E_s}$. Then,

$$\mathsf{E_b} = \frac{\mathsf{E_s}}{R}. \qquad \boxtimes$$

- $\mathsf{SNR} = \frac{\mathsf{E_s}}{\mathsf{N_0}} = R\frac{\mathsf{E_b}}{\mathsf{N_0}}$.

Power Efficiency and Bandwidth Fundamental Tradeoff

Communication is reliable if

$$\begin{aligned} R &< \mathsf{C}_{\mathsf{AWGN-D}} \\ &= \log\left(1 + \frac{\mathsf{E_s}}{2\sigma^2}\right) \\ &= \log\left(1 + R\frac{\mathsf{E_b}}{2\sigma^2}\right) \\ &= \log\left(1 + R\frac{\mathsf{E_b}}{\mathsf{N_0}}\right). \end{aligned}$$

For reliable transmission, the minimum $\mathsf{E_b}/\mathsf{N_0}$ to support a rate R is

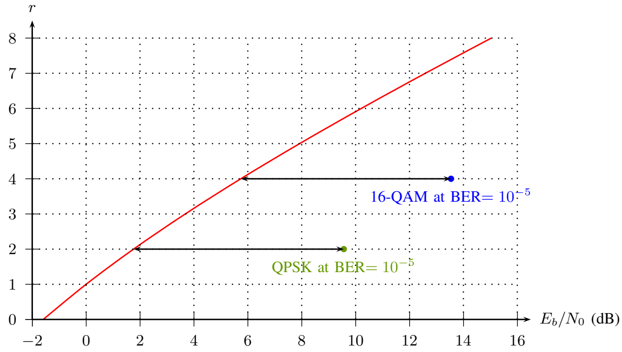
$$\boxed{\frac{\mathsf{E_b}}{\mathsf{N_0}} > \frac{2^R - 1}{R}.}$$

Using, $\lim_{R\to 0} \frac{2^R-1}{R} = \ln 2$,

$$\mathsf{E_b}/\mathsf{N_0} > \ln 2 = -1.59 \text{ dB},$$

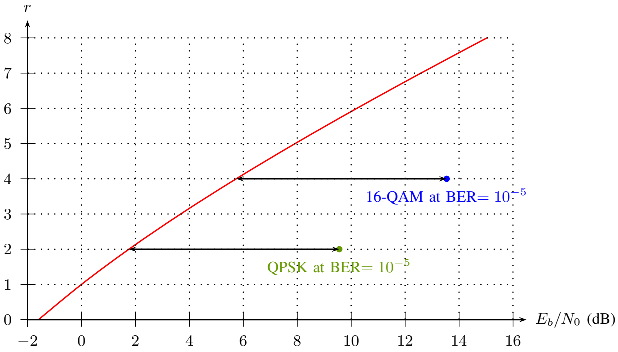
i.e., it is not possible to transmit reliably over the AWGN channel at $\mathsf{E_b}/\mathsf{N_0}$ smaller than -1.59 dB, even when we let $R \to 0$!.

Power Efficiency and Bandwidth Fundamental Tradeoff



- All rates R below the red curve are achievable.
- For a given R, all values of $\mathsf{E_b/N_0}$ in the right of the red curve are achievable.
- R is usually referred to as spectral efficiency, since determines how efficiently we use the spectrum ($R = R_\mathsf{b}T = R_\mathsf{b}/W$).
- Uncoded transmission performs far away from the theoretical limit.

Power Efficiency and Bandwidth Fundamental Tradeoff



A fundamental tradeoff between power and bandwidth

- The required $\mathsf{E_b/N_0}$ increases with increasing R, while increasing R decreases the required bandwidth to support the same information rate R_b.
- Decreasing R requires less $\mathsf{E_b/N_0}$, but higher bandwidth to support the same R_b.

Power-Limited and Bandwidth-Limited Regimes

Fundamental tradeoff in terms of SNR:

$$\mathsf{SNR} > 2^{R} - 1.$$

Power-Limited and Bandwidth-Limited Channels

Ideal band-limited AWGN channels may be classified as bandwidth-limited ($\mathsf{SNR} \gg 1$) or power-limited ($\mathsf{SNR} \ll 1$) according to whether they permit transmission at high spectral efficiencies or not.

Power-Limited Regime

Power-Limited Regime, SNR$\ll 1$ (While W Can Grow Very Large)

We can approximate channel capacity as

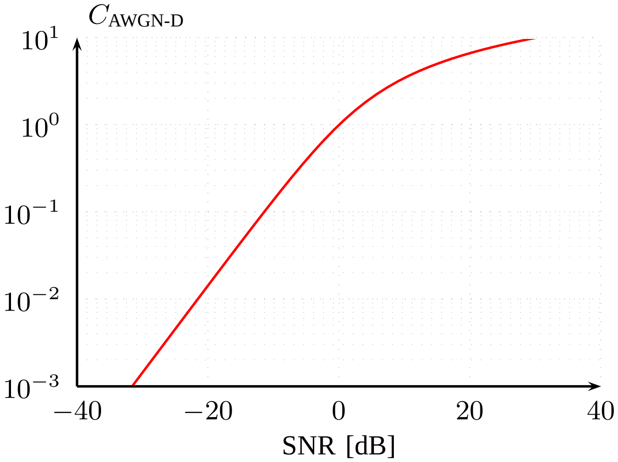
$$\mathsf{C}_{\mathsf{AWGN-D}} = \log(1 + \mathsf{SNR}) \approx \frac{1}{\ln 2}\mathsf{SNR},$$

i.e., $\mathsf{C}_{\mathsf{AWGN-D}}$ increases linearly with SNR.

In the Power-Limited Regime

- Doubling the SNR doubles the capacity.
- We commonly use BPSK to conserve power, at the expense of bandwidth efficiency.

Capacity Curve



- For low SNR the gain is linear.

Bandwidth-Limited Regime

Bandwidth-Limited Regime, SNR$\gg 1$

When the SNR is large, we have

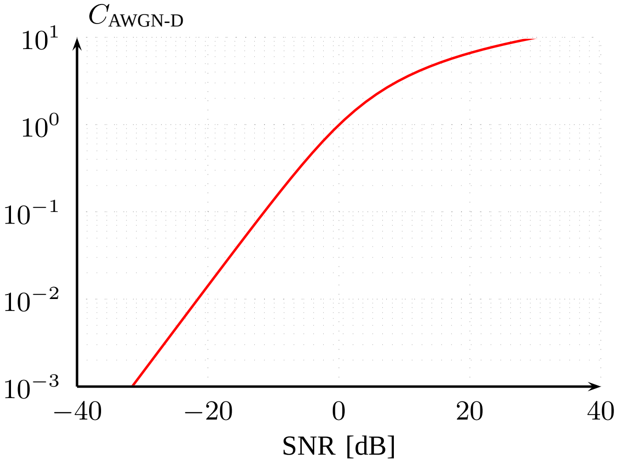
$$\begin{aligned} \mathsf{C}_{\mathsf{AWGN-D}} &= \log(1+\mathsf{SNR}) \\ &\approx \log \mathsf{SNR}, \end{aligned}$$

i.e., $\mathsf{C}_{\mathsf{AWGN-D}}$ increases logarithmically with SNR.

In the Bandwidth-Limited Regime

- Doubling the SNR (every additional 3 dB in SNR) yields an increase in achievable spectral efficiency of only 1 (bit/s)/Hz.
- We commonly use high-order modulation at the expense of power.

Capacity Curve



- For low SNR the gain is linear.
- For high SNR the gain is only logarithmic.