SSY125 Digital Communications

Department of Electrical Engineering

Exam Date: January 13, 2018, 14:00-18:00
Location: HA, HB, HC

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Material Allowed material is

- Chalmers-approved calculator.
- L. Råde, B. Westergren. Beta, Mathematics Handbook, any edition.
- A dictionary.

Grading A correct, clear and well-motivated solution gives a maximum of 15 points.

An erroneous answer, unclear, incomplete or badly motivated solutions give point reductions down to a minimum of 0 points. Answers in any other language than English are ignored.

Solutions The solutions are made available as soon as possible on the course web page.

Review The grading review will be on January 30, 2018, 9:00-10:00, and on February 15, 2018, 13:00-14:00, in room 6337 (office of the TAs) in the EDIT-building.

Grades The final grade on the course will be decided by the project (maximum score 30), quizzes (maximum score 3), tutorial grade (maximum score 7), and final exam (maximum score 60). The sum of all scores will decide the grade according to the following table.

Total Score	0–39	40–59	60–79	≥ 80
Grade	Fail	3	4	5

PLEASE NOTE THAT THE PROBLEMS ARE NOT NECESSARILY ORDERED IN DIFFICULTY.

Good luck!

Problem 1 - Source Coding and Channel Capacity [15 points]

Part I

Let X and Y be two independent integer-valued random variables. Let X be uniformly distributed over $\{0, 2, 4, 6, 8, 10, 12, 14\}$, and let $\Pr(Y = k) = 2^{-k}$, where $k = 1, 2, 3, \ldots$ The following expression may be useful:

$$\sum_{i=0}^{\infty} i a^i = \frac{a}{(1-a)^2}, \quad |a| < 1.$$

1. [*3 pt*] Find the entropies $\mathsf{H}(X|Y)$, $\mathsf{H}(Y|X)$, and $\mathsf{H}(X,Y)$, in bits.

Part II

Consider a source with 5 symbols which have probabilities $P = (\frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{2}{15}, \frac{2}{15})$.

1. [*1 pt*] What is the source entropy?
2. [*2 pt*] Apply the Huffman coding algorithm to this source.
3. [*1 pt*] What is the efficiency of the code?
4. [*2 pt*] Argue that this code is also optimal for a source with probabilities $P' = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

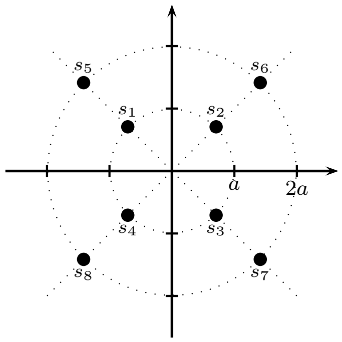
Part III

Consider a channel whose input is a random variable X which takes values on $\mathcal{X} = \{0, 1\}$ with probabilities $P_X(0) = P_X(1) = 0.5$. The channel output is a random variable Y which takes values on $\mathcal{Y} = \{0, 1, 2\}$. The channel is defined by the conditional distribution $P_{Y|X}(y|x)$,

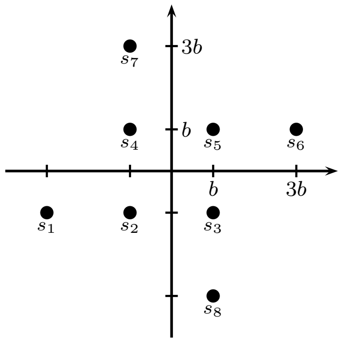
$$\begin{aligned} P_{Y|X}(0|0) &= P_{Y|X}(1|1) = 1 - \varepsilon \\ P_{Y|X}(1|0) &= P_{Y|X}(0|1) = \varepsilon \\ P_{Y|X}(2|0) &= P_{Y|X}(2|1) = 0. \end{aligned}$$

1. [*2 pt*] What is the entropy of the source, $\mathsf{H}(X)$, the probability distribution of the output, $P_Y(y)$, and the entropy of the output, $\mathsf{H}(Y)$?
2. [*2 pt*] What is the joint probability distribution for the source and the output, $P_{X,Y}(x, y)$, and what is the joint entropy, $\mathsf{H}(X, Y)$?
3. [*1 pt*] Compute the mutual information of this channel, $\mathsf{I}(X; Y)$, as a function of ε.
4. [*0.5 pt*] For which values of ε is the mutual information maximal? What is the mutual information in this case?
5. [*0.5 pt*] For which values of ε is the mutual information minimal? What is the mutual information in this case?

Problem 2 - Signal Constellations and Detection [15 points]



(a) Ω_1



(b) Ω_2

Figure 1: Two different complex signal constellations

Consider the two complex signal constellation Ω_1 and Ω_2 shown in Figure 1 (a) and (b). Assume that the received signal is given as $r = s + n$, where s is a point selected from one of the constellations, and n is a realization of a zero-mean, complex Gaussian random variable N with $\mathbb{E}[|N|^2] = \mathsf{N_0}$. All symbols in both constellations are assumed to be equally likely to be transmitted.

Questions

1. [*1 pt*] Determine a and b, such that both constellations Ω_1 and Ω_2 have unit energy $\mathbb{E}[|S|^2] = 1$.
2. [*2 pt*] Carefully draw the maximum likelihood (ML) decision regions for both constellations.
3. [*3 pt*] Find the nearest neighbor approximation for the symbol error probability P_{s} for both constellations. The final expressions for the approximation should only be a function of $\mathsf{N_0}$.
4. [*1 pt*] How do the constellations compare in terms of symbol error probability at high signal-to-noise ratio?
5. [*2 pt*] Rotate all the points in the outer circle of Ω_1 by $\pi/4$ to obtain a new constellation Ω_3. Are the power efficiencies of Ω_1 and Ω_3 equal or different? Explain why.
6. [*2 pt*] Is it possible to find a Gray mapping for either Ω_1 or Ω_2? If so, show the bit mapping. If not, explain why.
7. [*2 pt*] Suppose that the *a priori* probabilities of selecting the points are given by $P(s_1) = P(s_2) = P(s_3) = 1/3$, and $P(s_j) = 0$ for $j \in \{4, 5, 6, 7, 8\}$. Carefully draw the maximum *a posteriori* decision regions for both constellations.
8. [*2 pt*] Let $\mathcal{X}$ be a 16-QAM constellation, and assume that $s \in \mathcal{X}$. Show that the ML decision rule for s can be written as
$$\hat{s}_{\mathrm{ML}} = \underset{s \in \mathcal{X}}{\operatorname{argmax}}\, 2|r||s| \cos(\angle r - \angle s) - |s|^2,$$
where $\angle z$ denotes the angle of a complex number z.

Problem 3 - Linear Block Codes and LDPC Codes [15 points]

Part I

Consider a code $\mathcal{C}_1$ defined by the Tanner graph in Fig. 2.

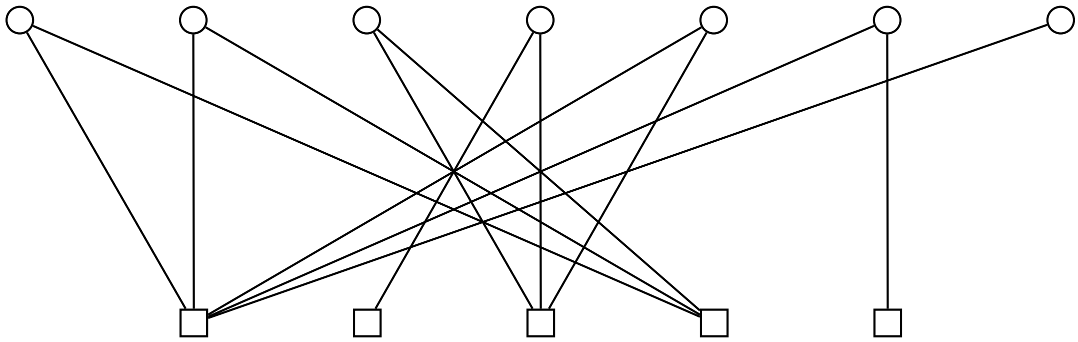


Figure 2: Tanner graph of code $\mathcal{C}_1$.

1. [*1 pt*] Find the girth and highlight it in the graph.
2. [*1 pt*] Find the parity check matrix $\boldsymbol{H}$ of the code $\mathcal{C}_1$.
3. [*1 pt*] Determine the variable node degree distribution $\Lambda(x)$ and the check node degree distribution $P(x)$. Is the LDPC code regular or irregular? Justify your answer!

Note: Even though $\mathcal{C}_1$ has small code length and is not sparse, consider it an LDPC code.

Part II

Consider the code $\mathcal{C}_2$,

$$\mathcal{C}_2 = \left\{ \begin{array}{c} (00000000) \\ (10011001) \\ (01001110) \\ (11010111) \\ (00110011) \\ (10101010) \\ (01111101) \\ (11100100) \end{array} \right\}.$$

1. [*3 pt*] Is this code a linear block code? *Justify your answer!* Find the generator matrix $\boldsymbol{G}_\mathrm{s}$ and parity check matrix $\boldsymbol{H}_\mathrm{s}$ in systematic form.
2. [*2 pt*] What are the error correction and error detection capabilities of this code over the binary symmetric channel with cross-over probability $p = 0.3$?
3. [*4 pt*] Complete the partial decoding table for syndrome-based decoding for a BSC with $p = 0.3$ below. Motivate all choices you make. How many entries would the full decoding table have?

error pattern	syndrome	error pattern	syndrome
	00100		01000
	10000		10011
	01110		11001
	11000		11011

4. [*2 pt*] You received $\bar{\boldsymbol{y}} = (11111111)$ after transmission over a BSC with $p = 0.3$. Perform syndrome-based decoding. What was the transmitted codeword?
5. [*1 pt*] You received $\bar{\boldsymbol{y}} = (11111111)$ after transmission over a BSC with $p = 0.7$. What was the transmitted codeword?

Problem 4 - Viterbi Algorithm [15 points]

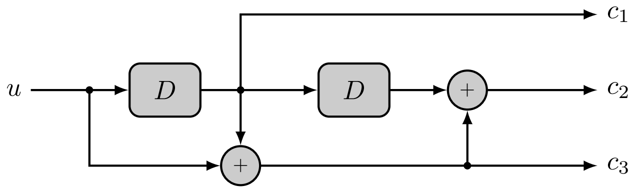


Figure 3: Encoder $\mathcal{E}_1$

1. [*2 pt*] Find the generator matrix $\boldsymbol{G}$ for encoder $\mathcal{E}_1$ in Fig. 3.
2. [*3 pt*] Transform the encoder $\mathcal{E}_1$ in Fig. 3 into recursive systematic form $\mathcal{E}_{\text{RSC}}$. Show the corresponding block diagram and generator matrix.
3. [*4 pt*] Draw one full section of the Trellis diagram of $\mathcal{E}_1$. Only display possible transitions and reachable states. Make sure that all state transitions are clearly labeled with the corresponding input and output bits.
4. [*6 pt*] Assume that the encoder $\mathcal{E}_1$ is initialized to the all-zero state and zero-termination. The bits are transmitted over a BSC with $p = 0.6$ and the received observation is given by $\bar{\boldsymbol{y}} = (100, 110, 000, 010)$. Find the maximum likelihood estimate of the information bits by using the Viterbi algorithm.

For all parts of this problem, it is important that you clearly show all involved branch metrics, state metrics, and survivor paths.