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Problem 1

Given $X\{X_i, P_i\}$ and $Y\{Y_i, q_i\}$

Then, $P_i = q_i$

To calculate average information content,

$$H(x) = \sum_{i=1}^{M} P_i \log_2\left(\frac{1}{P_i}\right)$$

$$H(y) = \sum_{i=1}^{M} q_i \log_2\left(\frac{1}{q_i}\right)$$

Since $q_i = P_i$

Therefore; $H(x) = H(y)$

$$\sum_{i=1}^{M} P_i \log_2\left(\frac{1}{P_i}\right) = \sum_{i=1}^{M} q_i \log_2\left(\frac{1}{q_i}\right)$$

$$\sum_{i=1}^{M} P_i \log_2\left(\frac{1}{P_i}\right) - \sum_{i=1}^{M} q_i \log_2\left(\frac{1}{q_i}\right) = 0$$

$$\sum_{i=1}^{M}\left(P_i \log_2\left(\frac{1}{P_i}\right) - q_i \log_2\left(\frac{1}{q_i}\right)\right) = 0$$

where $H(x) = H(y) = 0$, if $P_i = q_i = 0$

OR

$H(x) = H(y) = \infty$, or otherwise > 0

We know that $P_i \log\left(\frac{1}{P_i}\right) \begin{cases} 0, \text{ if } P_i = 1 \text{ or } P_i = 0 \\ > 0, \text{ otherwise} \end{cases}$

$$\sum_{i=1}^{M}\left(P_i \frac{\ln(1/P_i)}{\ln 2} - q_i \frac{\ln(1/q_i)}{\ln 2}\right) = 0$$

$$\frac{1}{\ln 2}\sum_{i=1}^{M}\left(P_i \ln(1/P_i) - q_i \ln\left(\frac{1}{q_i}\right)\right) = 0$$

note: $\ln x \leq x - 1$

Continuation

$$\sum_{i=1}^{M} P_i \log_2\left(\frac{1}{P_i}\right) = \sum_{i=1}^{M} q_i \log_2\left(\frac{1}{q_i}\right)$$

$$\frac{1}{\ln 2} \sum_{i=1}^{M} P_i \ln\left(\frac{1}{P_i}\right) = \frac{1}{\ln 2} \sum_{i=1}^{M} q_i \ln\left(\frac{1}{q_i}\right)$$

$$\Rightarrow P_i \ln\left(\frac{1}{P_i}\right) = q_i \ln\left(\frac{1}{q_i}\right)$$

We know that $\ln x \leq x-1$

$$P_i \ln\left(\frac{1}{P_i}\right) \leq P_i\left(\frac{1}{P_i}\right) - P_i$$

and

$$q_i \ln\left(\frac{1}{q_i}\right) \leq q_i\left(\frac{1}{q_i}\right) - q_i$$

Since we are given that $q_i = P_i \Rightarrow q_i - P_i = 0$

$$P_i \ln\left(\frac{1}{P_i}\right) \leq P_i\left(\frac{1}{P_i}\right) - P_i + q_i - P_i$$

$$\ln\left(\frac{1}{P_i}\right) \leq \left(\frac{1}{P_i}\right) - 1 + \frac{q_i}{P_i} - 1$$

$$\ln\left(\frac{1}{P_i}\right) \leq \frac{1}{P_i} + \frac{q_i}{P_i} - 2$$

$$\ln\left(\frac{1}{P_i}\right) \leq \left(\frac{q_i+1}{P_i}\right) - 2$$

$$2 + \ln\left(\frac{1}{P_i}\right) \leq \frac{q_i+1}{P_i}$$

$$2P_i + P_i \ln\left(\frac{1}{P_i}\right) \leq q_i + 1$$

$$2P_i + P_i \ln\left(\frac{1}{P_i}\right) - 1 \leq q_i \quad \text{---} \ (*)$$

So if $q_i = P_i$, as stated before

Equation above can only hold if and only if $H(y) \geq H(x)$, where q_i belongs to $H(y)$

It is both greater and equal to.