Problem Sheet 1

Last modified November 4, 2019

Note: You may work in groups of two on the homework problems. Next week, for Monday **you are required to hand in problem 1**. For Wednesday **you are required to hand in problem 3**.

Please submit your results via Canvas (one submission per group, state all group members in the comments box). State the number of the problem sheet and the name of each group member on the top of the first page. You may submit a scanned handwritten solution. If using a smartphone for scanning, ensure that it is properly readable (resolution, lighting, angle).

Problems for Monday, November 11

Problem 1

Suppose we have a random variable X that takes values on the alphabet $\mathcal{X} = \{x_0, x_1, \dots, x_M\}$ with probabilities $p_0, p_1, \dots, p_M$. Suppose we have another random variable Y that takes values on the alphabet $\mathcal{Y} = \{y_0, y_1, \dots, y_M\}$ with probabilities $q_0, q_1, \dots, q_M$, where

$$q_i = p_i,$$

for $i = 0, 1, \dots, j-2, j+1, \dots, M$, with $0 < j < M$, and

$$q_j = q_{j-1} = \frac{p_j + p_{j-1}}{2}.$$

The entropies of X and Y are denoted by $\mathsf{H}(X)$ and $\mathsf{H}(Y)$, respectively. How are $\mathsf{H}(X)$ and $\mathsf{H}(Y)$ related (greater, equal, less)? Prove your answer mathematically.

Problem 2

For the analysis of data transmission systems, it is usually assumed that the length of the transmitted messages are geometrically distributed. If N is the number of bits in a message, then it holds that $P_N(n) = p(1-p)^{n-1}$ for $n = 1, 2, \dots$ and $0 \leq p \leq 1$.

1. What is the expected value of the random variable N?
2. Calculate the uncertainty $\mathsf{H}(N)$ about the length N of the message subject to p and simplify this expression with the help of the expected value in the previous part and the binary entropy function $\mathsf{H}_\mathsf{b}(p) = -p\log_2(p) - (1-p)\log_2(1-p)$.

Problems for Wednesday, November 13

Problem 3

For source coding, only uniquely decodable codes are used in practice. They have the property that each finite sequence with the corresponding symbol alphabet can be uniquely mapped to a sequence of codewords.

1. Explain why a prefix-free code is always uniquely decodable.
2. For each of the following codes, determine if it is uniquely decodable and prefix-free. If the code is not uniquely decodable, give an example of a sequence for which more than one valid interpretation as a sequence of codewords exists.
   a) $\{00, 10, 01, 11\}$
   b) $\{0, 10, 11\}$
   c) $\{0, 01, 11\}$
   d) $\{1, 101\}$
   e) $\{0, 1, 01\}$
3. For the following codeword lengths, does there exist a prefix-free code? If no, explain why not. If yes, construct a code.
   a) $(1, 2, 3, 4, 5, 5)$
   b) $(1, 2, 3, 4, 4, 5)$
   c) $(2, 2, 2, 3, 4)$

Problem 4

Consider the following random experiment involving four different urns. A ball is drawn from urn A, which contains six balls labeled with B, three balls with C, and another three balls with D. The ball determines from which urn a second ball has to be drawn. Urn B contains five red and five white balls, urn C four red and six white balls, and urn D two red and eight white balls.

1. Assuming that the second ball is red, what is the probability that the first ball was labeled B?
2. Are these two events independent?
3. Consider the two events "the first ball is labeled with C" and "the second ball is red". Are these events independent?

Extra Problems

Problem 5

1. Given the random variable X with $P_X(0) = 0.1$, $P_X(1) = 0.2$, and $P_X(2) = 0.7$. Calculate the mean values $\mathbb{E}_X[X]$, $\mathbb{E}_X[P_X(X)]$, and $\mathbb{E}_X[-\log_2(P_X(X))]$. What happens to those quantities if we change the "alphabet" to $P_X(1) = 0.1$, $P_X(2) = 0.2$, and $P_X(3) = 0.7$?
2. Prove that $\mathbb{E}_Y[\mathbb{E}_X[X|Y]] = \mathbb{E}_X[X]$. *Hint:* Consider $\mathbb{E}_X[X|Y]$ as a function of the random variable Y.

Problem 6

A manipulated coin is tossed three times. The probabilities of the events “heads” and “tails” are 1/3 and 2/3, respectively.

1. Let X be a random variable that counts how often “heads” was tossed in the three independent tosses. Calculate the probability distribution function $P_X(x)$.

2. Let Y be a random variable that counts how often “heads” was tossed in the first of the three tosses. Specify the joint probability distribution function $P_{X,Y}(x,y)$ in the form of a table.

3. Calculate the covariance between X and Y. Are X and Y correlated?

4. Now, assume that the coin is tossed as many times as desired. The random variable N denotes the number of “tails” until the first occurrence of “heads”. Specify a formula for the probability $\Pr(N \leq n)$.

Problem 7

Let the random variables X and Z be independent with $X \sim \mathcal{N}(0,1)$ and Z taking on the values ± 1 equiprobably. Let $Y = ZX$ denote their product.

1. Find the density of Y.

2. Are X and Y correlated?

3. Compute $\Pr(|X| \geq 1)$ and $\Pr(|Y| \geq 1)$.

4. Compute the probability that both $|X|$ and $|Y|$ exceed 1.

5. Are X and Y independent?