Problem Sheet 3

Last modified November 4, 2019

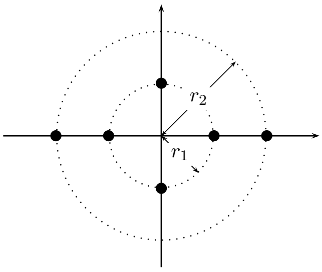
Note: You may work in groups of two on the homework problems. Next week, for Monday **you are required to hand in problem 1**. For Wednesday **you are required to hand in problem 3**.

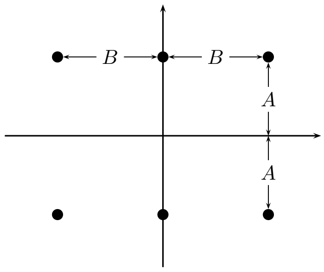
Please submit your results via Canvas (one submission per group, state all group members in the comments box). State the number of the problem sheet and the name of each group member on the top of the first page. You may submit a scanned handwritten solution. If using a smartphone for scanning, ensure that it is properly readable (resolution, lighting, angle).

Problems for Monday, November 25

Problem 1

Consider the two constellations (a) (left) and (b) (right), both with $M = 6$ points, shown below. Assume all constellation points to be equally likely to be transmitted.





1. For constellation (a), determine the relation between r_1 and r_2 so that the average energy per symbol is 1, i.e., $\mathbb{E}[|s|^2] = 1$.
2. For constellation (b), determine the relation between A and B so that the average energy per symbol is 1, i.e., $\mathbb{E}[|s|^2] = 1$.
3. Draw the ML decision regions for constellation (a).
4. Draw the ML decision regions for constellation (b).
5. Using $r_1 = \sqrt{1/2}$ and $r_2 = \sqrt{2}$, give the expression of the nearest neighbor approximation for the symbol error probability for constellation (a) (in terms of $\mathsf{E_s}/\mathsf{N_0}$).
6. Using $A = \sqrt{1/3}$ and $B = 1$, give the expression of the nearest neighbor approximation for the symbol error probability for constellation (b) (in terms of $\mathsf{E_s}/\mathsf{N_0}$).
7. Which of the two constellations is more power efficient? Motivate your answer.

Problem 2

Consider a constellation with four points $s_1 = A$, $s_2 = Ae^{j\phi_1}$, $s_3 = Ae^{j\phi_2}$, and $s_4 = 0$, where $A \in \mathbb{R}$, $A > 0$ and $0 < \phi_1 < \phi_2 < 2\pi$. We assume the observation model

$$r = s + n,$$

where n is a realization of a zero-mean, complex Gaussian random variable N with $\mathbb{E}[|N|^2] = \mathsf{N}_0$, and $s \in \{s_1, s_2, s_3, s_4\} = \Omega$. Notice that the union bound for non-equiprobable symbols is given by

$$P_{\mathsf{s}}^{(4)} \leq \sum_{s \in \Omega} P(s) \sum_{s' \in \Omega \setminus \{s\}} \Pr(\hat{s} = s' \mid s),$$

where

$$\Pr(\hat{s} = s' \mid s) = \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(s, s')}{2\mathsf{N}_0}}\right).$$

Questions

1. Assuming $P(s) = 1/4,\ \forall s \in \Omega$, what is A such that $\mathbb{E}[|S|^2] = \mathsf{E}_\mathsf{s}$?
2. Choose any value for ϕ_1 and $\phi_2 \neq \phi_1$ you like and draw the decision regions for the MAP detector when $P(s) = 1/4,\ \forall s \in \Omega$.
3. Choose any value for ϕ_1 and $\phi_2 \neq \phi_1$ you like and draw the decision regions when $P(s_1) = P(s_2) = P(s_3) = 1/3$ and $P(s_4) = 0$.
4. Determine the Euclidean distance between any two points as a function of ϕ_1 and ϕ_2.
5. Determine values for ϕ_1 and ϕ_2 to maximize the average Euclidean distance.
6. Assume equiprobable symbols and let $A = 2\sqrt{\mathsf{E}_\mathsf{s}/3}$, $\phi_1 = \pi/2$, and $\phi_2 = \pi$. Determine an approximation of the symbol error rate. Which terms dominate?

Problems for Wednesday, November 27

Problem 3

Part I (uncoded)

We want to transmit two bits $\boldsymbol{c} = (c_1, c_2)$ by selecting a point x from a QPSK constellation according to the employed binary mapping (either Gray or lexicographical). The channel model is $y = x + n$, where n is a realization of complex Gaussian random variable N, with zero mean and $\mathbb{E}[|N|^2] = \sigma^2 = \mathsf{N}_0$, and $\mathbb{E}[|X|^2] = \mathsf{E}_s$. The receiver first detects the symbol $\hat{x}$ and then outputs the binary mapping corresponding to $\hat{x}$, which we will denote by $\hat{\boldsymbol{c}} = (\hat{c}_1, \hat{c}_2)$.

- Assuming QPSK with a Gray mapping, what is the optimal symbol detection rule for $\hat{x}$? What is the symbol error probability? What is the bit error probability for the first and second bit? What is the average bit error probability? *Hint:* All probabilities will involve Q-functions with the same argument. You may therefore use the abbreviation $\mathsf{Q}(x) = \mathsf{Q}$.
- Repeat all questions for QPSK and the lexicographical mapping. Note that we are looking for exact error probabilities in all cases, not approximations.

Part II (coded)

In this part, we will use a channel code over the above discrete-input, discrete-output channel from $\boldsymbol{c} = (c_1, c_2)$ to $\hat{\boldsymbol{c}} = (\hat{c}_1, \hat{c}_2)$. The code $\mathcal{C}$ consists of the four codewords

$$\mathcal{C} = \left\{ \begin{array}{c} \boldsymbol{c_1} = (000000), \\ \boldsymbol{c_2} = (010101), \\ \boldsymbol{c_3} = (101010), \\ \boldsymbol{c_4} = (111111) \end{array} \right\}.$$

- What is the length n and the dimension k of the code? Is this a linear code? What is the minimum distance? How many QPSK symbols have to be transmitted to convey one codeword?
- Assuming QPSK with the Gray mapping and high SNR: If the receiver outputs (100001), what is the maximum likelihood codeword?
- (Bonus Question, Not Mandatory) Assuming QPSK with the lexicographical mapping and high SNR: If the receiver outputs (100001), what is the maximum likelihood codeword?

Problem 4

Consider QPSK with a lexicographical bit mapping together with the observation model $y = x + n$, where n is a realization of a complex Gaussian random variable with $\mathbb{E}[|N|^2]$. Find expressions for the bit-wise log-likelihood ratios (LLRs), for $i = 1, 2$,

$$L_i = \log \frac{p(y|b_i = 0)}{p(y|b_i = 1)}.$$

Extra Problems

Problem 5

Assuming each symbol is equally likely, derive the following expressions for the average symbol error probability for 4-PAM and 16-QAM:

$$\begin{aligned} P_{\mathsf{s}}^{\mathsf{4PAM}} &= \frac{3}{2}\mathsf{Q}\left(\sqrt{\frac{4\mathsf{E_b}}{5\mathsf{N_0}}}\right) \\ P_{\mathsf{s}}^{\mathsf{16QAM}} &= 3\mathsf{Q}\left(\sqrt{\frac{4\mathsf{E_b}}{5\mathsf{N_0}}}\right) - \frac{9}{4}\mathsf{Q}^2\left(\sqrt{\frac{4\mathsf{E_b}}{5\mathsf{N_0}}}\right) \end{aligned}$$

(Assume 4-PAM with equally spaced levels symmetric about the origin, and rectangular 16-QAM equivalent to two 4-PAM constellations independently modulating the I and Q components.)

Problem 6

Consider all rectangular constellations of the form

$$\{a + jb, a - jb, -a + jb, -a - jb\},$$

where $a, b \in \mathbb{R}$. Which of these constellations whose second moment is one has the largest minimum distance?