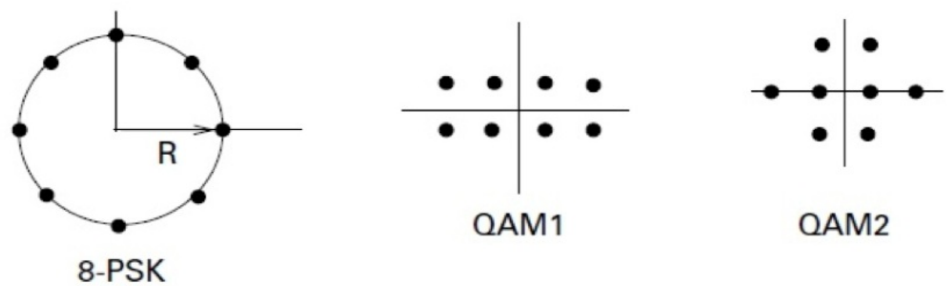
# TUTORIAL 6

*Question 1:*

The 8-ary signal constellations are shown in Fig.1.

Fig. 1. Signal Constellations for Question 1




(a) Express $R$ and $d_{\min}^{(2)}$ in terms of $d_{\min}^{(1)}$ so that all three constellations have the same $E_b$.

(b) For a given $E_b/N_0$, which constellation do you expect to have the smallest bit error probability over a high SNR AWGN channel?

(c) For each constellation, determine whether you can label signal points using three bits so that the label for nearest neighbours differs by at most one bit. If so, find such labeling. If not, say why not and find some "good" labeling.

(d) For the labelings found in part (c), compute nearest neighbours approximations for the average bit error probability as a function of $E_b/N_0$ for each constellation. Evaluate these approximations for $E_b/N_0 = 15$ dB.

*Solution:*

(a) For 8-PSK, the symbol energy $E_s = R^2$. From Fig.1,

$$\begin{aligned} E_s(\text{QAM1}) &= \text{I}-\text{channel avg energy} + \text{Q}-\text{channel avg energy} \\ &= \frac{4}{8}\left[(d_1/2)^2 + (3d_1/2)^2\right] + (d_1/2)^2 = \frac{3}{2}d_1^2 \\ E_s(\text{QAM2}) &= \text{I}-\text{channel avg energy} + \text{Q}-\text{channel avg energy} \end{aligned}$$

$$= \left[\frac{6}{8}(d_2/2)^2 + \frac{2}{8}(3d_2/2)^2\right] + \left[\frac{4}{8}d_2^2 + \frac{4}{8}\cdot 0\right] = \frac{5}{4}d_2^2 \quad (1)$$

where $d_1 = d_{\min}^{(1)}$ and $d_2 = d_{\min}^{(2)}$. Since the number of constellation points is the same for each constellation, equal energy per bit corresponds to equal symbol energy, and, from Eq.(1), this occurs when

$$R = \sqrt{3/2}d_1,\ d_2 = \sqrt{6/5}d_1 \quad (2)$$

(b) From Eq.(2), for the same $E_b$, using the sine law,

$$d_{\min}^{\text{8PSK}} = \frac{R\sin(\pi/4)}{\sin(3\pi/8)} = \frac{\sqrt{3/2}\sin(\pi/4)}{\sin(3\pi/8)} = 0.937d_1 < d_1 < d_2.$$

Thus, for high SNR, since the power efficiency of each constellation is $\eta_p = d_{\min}^2/E_b$ and $P_e \approx \bar{N}_{d_{\min}} Q\left(\sqrt{\frac{\eta_p E_b}{2N_0}}\right)$, where $\bar{N}_{d_{\min}}$ is the average number of nearest neighbours, it is expected that

$$P_e(\text{8QPSK}) > P_e(\text{QAM1}) > P_e(\text{QAM2}), \quad (3)$$

since $\bar{N}_{d_{\min}}$ is negligible at high SNR.

(c) Each symbol is assigned 3 bits. Since 8-PSK and QAM1 are regular constellations with at most 3 nearest neighbors per point, we expect to be able to Gray code. However, QAM2 has some points with 4 nearest neighbors, so we definitely cannot Gray code it. We can, however, try to minimize the number of bit changes between neighbors. Fig.2 shows Gray codes for 8-PSK and QAM1. The labeling for QAM2 is arbitrarily chosen to be such that points with 3 or fewer nearest neighbors are Gray coded.

(d) For Gray coded 8-PSK and QAM1, a symbol error due to decoding to a nearest neighbor causes only 1 out of the 3 bits to be in error. Hence, using the nearest neighbors approximation, $P[\text{bit error}] \approx \frac{1}{3}P[\text{symbol error}]$. On the other hand, $P[\text{symbol error}] \approx \bar{N}_{d_{\min}} Q\left(\frac{d_{\min}}{2\sigma}\right)$. For 8-PSK, $d_{\min} = R\frac{\sin(\pi/4)}{\sin(3\pi/8)}$ and $E_s = 3E_b = R^2$. Plugging in $\sigma^2 = N_0/2$ and $\bar{N}_{d_{\min}} = 2$, we obtain

$$P[\text{bit error}]_{\text{8PSK}} \approx \frac{2}{3}Q\left(\sqrt{\frac{3\left(1-\frac{1}{\sqrt{2}}\right)E_b}{N_0}}\right) \quad (4)$$

For QAM1, $\bar{N}_{d_{\min}} = 5/2$ and $E_s = 3E_b = 3d_1^2/2$, so that

$$P[\text{bit error}]_{\text{QAM1}} \approx \frac{5}{6}Q\left(\sqrt{\frac{E_b}{N_0}}\right) \quad (5)$$

For QAM2, we need to make nearest neighbors approximation specifically for bit error probability. Let $N(b)$ total number of bit changes due to decoding into nearest neighbors when symbol $b$ is sent. For the labeling given, these are specified by Table I. Let $\bar{N}_{\text{bit}} = \frac{1}{8}\Sigma_b N(b) = 11/4$ denote

Fig. 2. Bit Mapping for Question 1

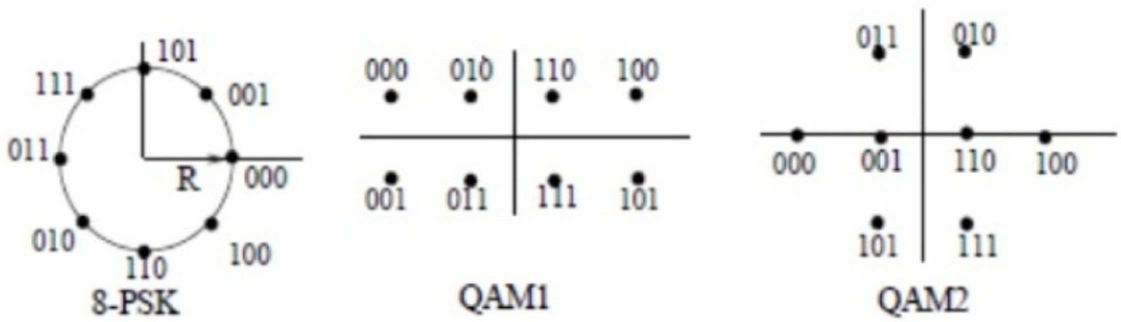



TABLE I

NUMBER OF BIT CHANGES DUE TO DECODING INTO NEAREST NEIGHBORS FOR EACH SYMBOL OF QAM2

| $b$ | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| $N(b)$ | 1 | 6 | 2 | 2 | 1 | 2 | 6 | 2 |

the average number of bits wrong due to decoding into nearest neighbors. Since each signal point is labeled by 3 bits, the nearest neighbors approximation for the bit error probability is now given by

$$P[\text{bit error}]_{\text{QAM2}} \approx \frac{1}{3}\bar{N}_{\text{bit}} Q\left(\frac{d_{\min}}{2\sigma}\right) = \frac{11}{12} Q\left(\sqrt{\frac{6E_b}{5N_0}}\right) \tag{6}$$

(We can reduce the factor from 11/12 to 5/6 using an alternative labeling.)

While we have been careful about the factors multiplying the $Q$ function, these are insignificant at high SNR compared to the argument of the Q function, and are often ignored in practice. For $E_b/N_0 = 15$ dB, from Eqs.(4), (5) and (6), the preceding approximations give the values shown in Table II. The ordering of error probabilities is as predicted in Eq.(3).

TABLE II

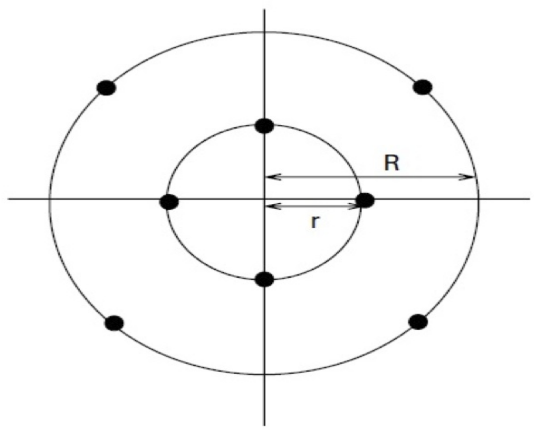
BIT ERROR PROBABILITIES AT $E_b/N_0$ OF 15 DB

| Constellation | 8PSK | QAM1 | QAM2 |
|---|---|---|---|
| $P$[bit error] | $4.5 \times 10^{-8}$ | $7.8 \times 10^{-9}$ | $3.3 \times 10^{-10}$ |

*Question 2:*

Consider the signal constellation shown Fig.3, which consists of two QPSK constellations of different radii, offset from each other by $\pi/4$. The constellation is to be used to communicate over a passband AWGN channel.

Fig. 3. Constellation for Question 2




(a) Carefully redraw the constellation for $r = 1$ and $R = \sqrt{2}$. Sketch the ML decision regions.

(b) For $r = 1$ and $R = \sqrt{2}$, find an intelligent union bound for the conditional error probability, given that a signal point from the inner circle is sent, as a function of $E_b/N_0$.

(c) How would you choose the parameters $r$ and $R$ so as to optimize the power efficiency of the constellation (at high SNR)?

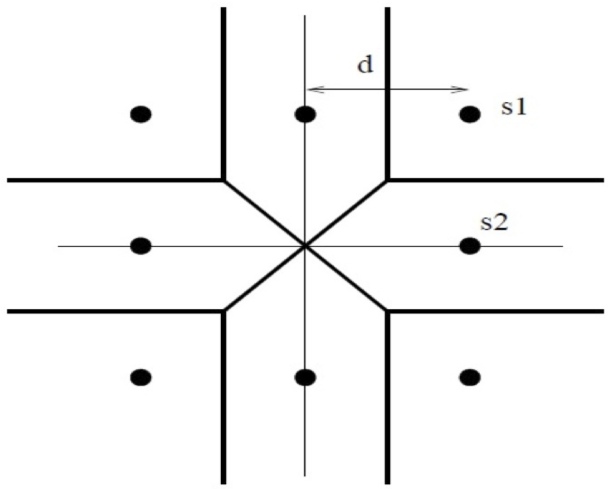
*Solution:*

(a) For $R/r = \sqrt{2}$. the constellation takes the rectangular shape shown in Fig.4.

(b) We compute the intelligent union bound on the error probability conditioned on two typical signal points shown in Fig.4.

$$P(e|s1) \leq 2Q\left(\frac{d}{2\sigma}\right) = 2Q\left(\sqrt{\frac{d^2}{E_b}}\sqrt{\frac{E_b}{2N_0}}\right). \tag{7}$$

Fig. 4. ML Decision Regions for Question 2a




We now compute $\eta = \frac{d^2}{E_b}$ to express the result in terms of $E_b/N_0$. The energy per symbol is

$$E_s = 2 \times \mathrm{I-channel\ energy} = 2\left(\frac{6}{8} \times d^2 + \frac{2}{8} \times 0\right) = 3d^2/2. \tag{8}$$

Since $E_b = E_s/\log_2 8$, from Eq.(8), we have $\eta = d^2/E_b = 2$, and, from Eq.(7), this yields

$$P(e|s1) \leq 2Q\left(\sqrt{E_b/N_0}\right). \tag{9}$$

Similarly,

$$P(e|s2) \leq 2Q\left(\frac{d}{2\sigma}\right) + 2Q\left(\frac{\sqrt{2}d}{2\sigma}\right) = 2Q\left(\sqrt{E_b/N_0}\right) + 2Q\left(\sqrt{2E_b/N_0}\right) \tag{10}$$

Thus, from Eqs.(9) and (10), the average error probability is given by

$$P_e = \frac{1}{2}(P(e|s1) + P(e|s2)) \leq 2Q\left(\sqrt{E_b/N_0}\right) + Q\left(\sqrt{2E_b/N_0}\right). \tag{11}$$

(c) We wish to design the parameter $x = R/r \geq 1$ to optimize the power efficiency, which is given by

$$\eta = \min(d_1^2, d_2^2)/E_b \tag{12}$$

where $d_1$ and $d_2$ shown in Fig.5 are given by

$$d_1^2 = 2r^2$$

$$d_2^2 \quad = \quad \left(R/\sqrt{2}\right)^2 + \left(R/\sqrt{2} - r\right)^2 = R^2 + r^2 - \sqrt{2}Rr = r^2(1 + x^2 - \sqrt{2}x). \tag{13}$$

The energy per symbol is $E_s = (r^2 + R^2)/2$, so that

$$E_b = E_s/log_2 8 = (r^2 + R^2)/6 = r^2(1 + x^2)/6. \tag{14}$$

From Eqs.(13) and (14), it is easy to check the following: $\eta_1 = d_1^2/E_b$ decreases with $x$, and $\eta_2 = d_2^2/E_b$ increases with $x$ (for $x \geq 1$). Furthermore, at $x = 1$, $\eta_1 > \eta_2$. This shows that the optimal $x \geq 1$ corresponds to $\eta_1 = \eta_2$, i.e. $d_1^2 = d_2^2$. That is,

$$2r^2 = r^2(1 + x^2 - \sqrt{2}x)$$

or

$$x^2 - \sqrt{2}x - 1 = 0.$$

The solution to this in the valid range $x \geq 1$ is given by

$$x = R/r = \frac{\sqrt{2} + \sqrt{6}}{2} \approx 1.93, \tag{15}$$

and, from Eq.(15), the corresponding power efficiency is given by

$$\eta = 2(3 - \sqrt{3}) \approx 2.54. \tag{16}$$

which is about 1 dB better than the power efficiency of $\eta = 2$ for $x = \sqrt{2} \approx 1.41$ in part (a).

Fig. 5. Signal Space Geometry for Question 2c