Applied Signal Processing SSY130
Tutorial 1
Solutions

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Problems

1 Denote the signal after the first summation with $w(n)$.

By following the diagram backwards from $w(n)$ we obtain the equation

$$w(n) = x(n) - d_1 w(n-1)$$

Using the DTFT on both sides and solve for $W(\omega)$ results in

$$W(\omega) = \frac{1}{1 + d_1 e^{-j\omega}} X(\omega) \tag{1}$$

From the diagram we have the output $y(n) = d_1 w(n) + w(n-1)$ which yields in the frequency domain

$$Y(\omega) = (d_1 + e^{-j\omega}) W(\omega)$$

Inserting $W(\omega)$ from (1) yields

$$Y(\omega) = (d_1 + e^{-j\omega}) \frac{1}{1 + d_1 e^{-j\omega}} X(\omega)$$

which we can rearrange to

$$(1 + d_1 e^{-j\omega})Y(\omega) = (d_1 + e^{-j\omega})X\omega)$$

Finally we obtain the time domain relation using the IDTFT as

$$y(n) = -d_1 y(n-1) + d_1 x(n) + x(n-1)$$

2

$$y(n+N) = \sum_{k=0}^{\infty} h(k)x(n+N-k) = [\text{since } x \text{ is N-periodic}] = \sum_{k=0}^{\infty} h(k)x(n-k)$$

which shows that also $y(n)$ is $N-$periodic.

3

$$\begin{aligned} x_1(n) * x_2(n) &= \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k) \\ &[\text{Change variables } k' = n-k] \\ &= \sum_{k'=-\infty}^{\infty} x_1(k'-n)x_2(k') = \\ &= \sum_{k'=-\infty}^{\infty} x_2(k')x_1(k'-n) = x_2(n) * x_1(n) \end{aligned}$$

4 Consider the signal $u(n) - u(n-1)$. Clearly this is the Kronecker delta function δ_n. Since $s(n)$ is the filter response from $u(n)$ then by linearity $s(n) - s(n-1)$ is the response from the input δ_n. Consequently, the impulse response is $h(n) = s(n) - s(n-1)$ and the output for an arbitrary input $x(n)$ is

$$y(n) = \sum_{k=-\infty}^{\infty} (s(k) - s(k-1))x(n-k) \tag{2}$$

5 Assume $x(n) = \delta_n$, the Kronecker delta function. Clearly for $n < 0$, $y(n) = 0$. For $n = 0$ we have $y(0) = 1$ and hence $y(n) = \alpha y(n-1) = \alpha^n y(0) = \alpha^n$ We conclude that the impulse response of the system is $h(n) = \alpha^n$.

If $|\alpha| \leq 1$ the system is bounded input bounded output stable since if $|x(n)| < c_x$ we have

$$|y(n)| = \left| \sum_{k=0}^{\infty} h(k)x(n-k) \right| \leq \sum_{k=0}^{\infty} |h(k)|c_x = \frac{1}{1-|\alpha|}c_x < \infty$$

6 Consider

$$\begin{aligned}\cos n\omega &= \mathrm{Re}(e^{j\omega n}) = \mathrm{Re}(e^{j\omega(n-1)}e^{j\omega})\\ &= \mathrm{Re}((\cos(n-1)\omega + j\sin(n-1)\omega)(\cos\omega + j\sin\omega)\\ &= \cos(n-1)\cos\omega - \sin(n-1)\omega\sin\omega\end{aligned}$$

similarly

$$\begin{aligned}\sin n\omega &= \mathrm{Im}(e^{j\omega n}) = \mathrm{Im}(e^{j\omega(n-1)}e^{j\omega})\\ &= \mathrm{Im}((\cos(n-1)\omega + j\sin(n-1)\omega)(\cos\omega + j\sin\omega)\\ &= \cos(n-1)\sin\omega + \sin(n-1)\omega\cos\omega\end{aligned}$$

Now we directly see that $y_r(n) = \cos n\omega$ and $y_i(n) = \sin n\omega$ are the desired solutions which also satisfy the given initial conditions.

7 Causal means that $h(n) = 0$ for $n < 0$. b) and d) are causal.

8 a) Linearity:

$$\sum_{n=-\infty}^{\infty}(\alpha g(n)+\beta x(n))e^{-j\omega n\Delta t} = \sum_{n=-\infty}^{\infty}\alpha g(n)e^{-j\omega n\Delta t}+\sum_{n=-\infty}^{\infty}\beta x(n)e^{-j\omega n\Delta t} = \alpha G(\omega)+\beta G(\omega)$$

Time shifting:

$$\sum_{n=-\infty}^{\infty} g(n-n_0)e^{-j\omega n\Delta t} = [s = n-n_0] = \sum_{s=-\infty}^{\infty} g(s)e^{-j\omega(s+n_0)\Delta t} = e^{-j\omega n_0\Delta t}\sum_{s=-\infty}^{\infty} g(s)e^{-j\omega s\Delta t}$$

Frequency shifting/modulation:

$$\sum_{n=-\infty}^{\infty} e^{j\omega_0 n\Delta t}g(n)e^{-j\omega n\Delta t} = \sum_{n=-\infty}^{\infty} g(n)e^{-j(\omega-\omega_0)n\Delta t} = G(\omega-\omega_0)$$

Convolution:

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} g(k)x(n-k)e^{-j\omega n\Delta t} &= [s = n-k] = \sum_{s=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} g(k)x(s)e^{-j\omega(s+k)\Delta t}\\ &= \sum_{s=-\infty}^{\infty} x(s)e^{-j\omega s\Delta t}\sum_{k=-\infty}^{\infty} g(k)e^{-j\omega k\Delta t} = X(\omega)G(\omega)\end{aligned}$$

Frequency convolution:

$$\begin{aligned}&\frac{1}{\omega_s}\int_0^{\omega_s}\frac{1}{\omega_s}\int_0^{\omega_s} G(\theta)X(\omega-\theta)\,d\theta e^{j\omega n\Delta t}\,d\omega = [\lambda = \omega-\theta]\\ &= \frac{1}{\omega_s}\int_0^{\omega_s}\frac{1}{\omega_s}\int_{0-\theta}^{\omega_s-\theta} G(\theta)X(\lambda)e^{j(\lambda+\theta)n\Delta t}\,d\theta\,d\lambda =\\ &\frac{1}{\omega_s}\int_0^{\omega_s} G(\theta)e^{j(\theta)n\Delta t}\,d\theta\frac{1}{\omega_s}\int_0^{\omega_s} X(\lambda)e^{j\lambda n\Delta t}\,d\lambda = g(n)x(n)\end{aligned}$$

b) First relation

$$\sum_{n=-\infty}^{\infty} x(-n)e^{-j\omega n\Delta t} = [s=-n] = \sum_{s=-\infty}^{\infty} x(s)e^{-j(-\omega)s\Delta t} = X(-\omega)$$

Second relation:

$$\sum_{n=-\infty}^{\infty} x^*(-n)e^{-j\omega n\Delta t} = [s=-n] = \sum_{s=-\infty}^{\infty} x^*(s)e^{j\omega s\Delta t} = \left(\sum_{s=-\infty}^{\infty} x(s)e^{-j\omega s\Delta t}\right)^* = X^*(\omega)$$

Third relation; Combining First and Second relation yields $\text{DTFT}[x^*(n)] = X^*(-\omega)$

Since $\operatorname{Re} x(n) = \frac{1}{2}(x(n) + x^*(n))$ the result follows.

Fourth relation:

Since $\operatorname{Im} x(n) = \frac{1}{2j}(x(n) - x^*(n))$ the result again follows from $\text{DTFT}[x^*(n)] = X^*(-\omega)$.

c) When $x(n)$ is real valued $x^*(n) = x(n)$ which implies that $X(\omega) = X^*(-\omega)$ since $\text{DTFT}[x^*(n)] = X^*(-\omega)$. The magnitude is thus symmetric around zero: $|X(\omega)| = |X(-\omega)|$ and the phase function is anti-symmetric $\arg X(\omega) = -\arg X(-\omega)$ and obviously $X(0)$ is real valued.

9 We obtain the impulse response of the filter by deriving the IDTFT of the frequency function $H_{\text{LP}}(\omega)$.

$$h_{\text{LP}}(n) = \frac{1}{\omega_s}\int_{-\omega_c}^{\omega_c} e^{j\omega n\Delta t}\,d\omega = \frac{1}{\omega_s}\left[\frac{e^{j\omega n\Delta t}}{jn\Delta t}\right]_{-\omega_c}^{\omega_c} = \frac{1}{\omega_s}\frac{e^{j\omega_c n\Delta t} - e^{-j\omega_c n\Delta t}}{jn\Delta t} = \frac{2\omega_c}{\omega_s}\frac{\sin(\omega_c n\Delta t)}{\omega_c n\Delta t} \quad \text{for } n \neq 0$$

and $h_{\text{LP}}(0) = \frac{2\omega_c}{\omega_s}$. The impulse response is non-causal and infinitely long.

10 We assume w.l.o.g. $\Delta t = 1$ The DTFT of $x(n)$ is

$$X(\omega) = \text{DTFT}[\frac{1}{2}(e^{j\pi n/4} + e^{-j\pi n/4})] = \frac{2\pi}{2}\left(\tilde{\delta}(\omega - \pi/4) + \tilde{\delta}(\omega + \pi/4)\right) \qquad (3)$$

The frequency function of the system is obtained by calculating the DTFT of the left and right hands of equation (3)

$$Y(\omega) = \alpha e^{-j\omega}Y(\omega) + X(\omega)$$

which after simplification results in

$$Y(\omega) = \frac{1}{1-\alpha e^{-j\omega}} X(\omega)$$

Hence the frequency function is

$$H(\omega) = \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{1-\alpha\cos\omega + j\alpha\sin\omega} = a e^{j\theta}$$

Where the magnitude is $a = 1/\sqrt{1+\alpha^2 - 2\alpha\cos\omega}$ and angle is $\theta = -\tan^{-1}\frac{\alpha\sin\omega}{1-\alpha\cos\omega}$

Since $Y(\omega) = H(\omega)X(\omega)$ we get

$$Y(\omega) = \frac{2\pi}{2}\Big(H(\omega)\tilde{\delta}(\omega - \pi/4) + H(\omega)\tilde{\delta}(\omega + \pi/4)\Big)$$

and using the IDTFT

$$x(n) = \frac{1}{2}\Big(H(\pi/4)e^{j\pi n/4} + H(-\pi/4)e^{-j\pi n/4}\Big)$$

Inserting the magnitude and phase for H results in

$$x(n) = \frac{1}{2}(a e^{j(\pi n/4+\theta)} + a e^{-j(\pi n/4+\theta)}) = a\cos(\pi n/4 + \theta)$$

where

$$a = 1/\sqrt{1+\alpha^2 - 2\alpha\cos(\pi/4)} = 1/\sqrt{1+\alpha^2-\sqrt{2}\alpha}$$

and

$$\theta = -\tan^{-1}\frac{\alpha\sin(\pi/4)}{1-\alpha\cos(\pi/4)} = -\tan^{-1}\frac{\alpha}{\sqrt{2}-\alpha}$$