Applied Signal Processing
Lecture 6

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Agenda

- Frequency Analysis - DTFT of a window of data
- Window functions
- FIR filters
  - Specifications
  - Linear phase
  - The Window design method
  - FIR-LS design method
  - FIR-PM design method
- Design demo

DTFT results

Multiplication of two time domain signals

$$y(n) = x(n)w(n)$$
$$\Leftrightarrow$$
$$Y(\omega) = \frac{1}{\omega_s} \int_0^{\omega_s} X(\lambda) W(\omega - \lambda)\, d\lambda$$

Frequency domain convolution of the DTFTs

We observe a signal $x(n)$ in an observation window of length N.

We have the sequence $x(0), x(1),\ldots,x(N-1)$ avaialble for analysis

$$\hat{x}(n) \triangleq r_N(n)x(n), \quad \forall n$$

where

$$r_N(n) = \begin{cases} 1 & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

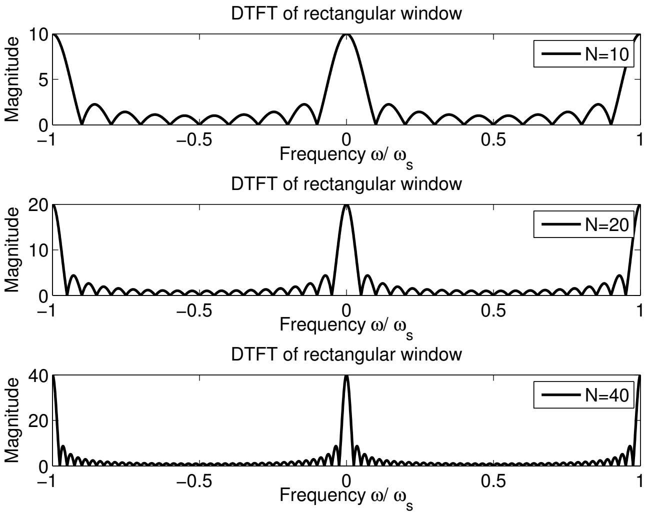
$$\hat{x}(n) = x(n)r_N(n) \quad \Leftrightarrow \quad \hat{X}(\omega) = \frac{1}{\omega_s}\int_0^{\omega_s} R_N(\lambda)X(\omega-\lambda)\,d\lambda$$

where

$$R_N(\omega) = \sum_{n=-\infty}^{\infty} r_N(n)e^{-j\omega\Delta tn} = \sum_{n=0}^{N-1} e^{-j\omega\Delta tn} =$$

$$= \frac{1-e^{-j\omega\Delta tN}}{1-e^{-j\omega\Delta t}} = e^{-j\frac{N-1}{2}\omega\Delta t}\frac{\sin(\frac{N\omega\Delta t}{2})}{\sin(\frac{\omega\Delta t}{2})}$$

The rectangular window $R_N(\omega)$



Observations

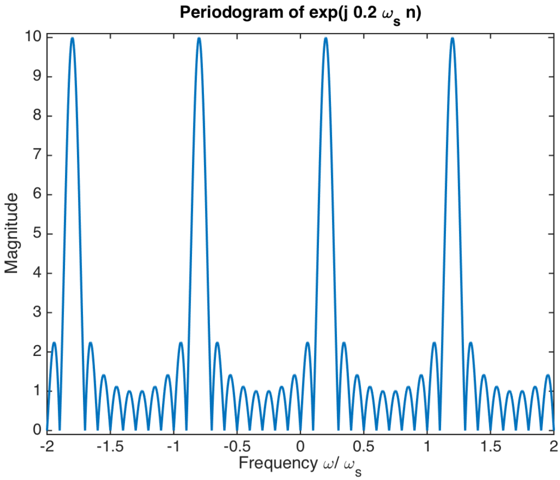
- $R_N(\omega) = 0$ for $\omega = \pm\omega_s/N, \pm 2\omega_s/N$
- Main lobe:
  - High and narrow is good
  - Width $\sim 1/N$
- Side lobes: Low levels are good
- As $N \to \infty$ then $R_N(\omega) \approx \omega_s \tilde{\delta}(\omega)$

Example

Consider a discrete time signal $x(n) \triangleq e^{j\omega_0 n}$ with $\Delta t = 1$

Then $X(\omega) = 2\pi\tilde{\delta}(\omega - \omega_0)$

$$\begin{aligned}\hat{X}(\omega) &= \frac{1}{2\pi}\int_0^{2\pi} X(\omega-\lambda)R_N(\lambda)\,d\lambda = \\ &= \frac{1}{2\pi}\int_0^{2\pi} 2\pi\tilde{\delta}(\omega-\omega_0-\lambda)R_N(\lambda)\,d\lambda = R_N(\omega-\omega_0)\end{aligned}$$



Periodogram based on $N = 10$ samples of the signal

$$x(n) = e^{j0.2\omega_s \Delta t n}$$

Window functions

We are not restricted to the rectangular window

$$\hat{x}(n) = w(n)x(n)$$

Window	-3dB bw [Hz]	PSLL [dB]	SLRO [dB/oct.]
Rectangular	$\frac{0.89}{N\Delta t}$	-13	-6
Hanning	$\frac{1.4}{N\Delta t}$	-32	-18
Hamming	$\frac{1.3}{N\Delta t}$	-43	-6
Dolph-Chebyshev	$\frac{1.44}{N\Delta t}$	-60	0

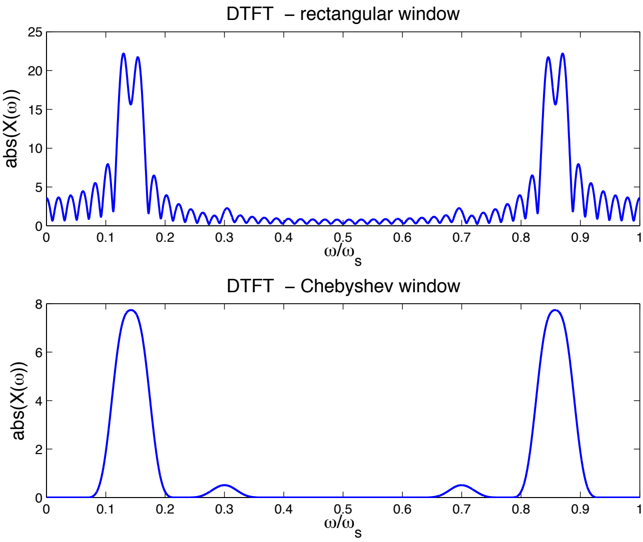
BW - Band width
PSLL - Peak side-lobe level
SLRO - Side lobe roll off

Example

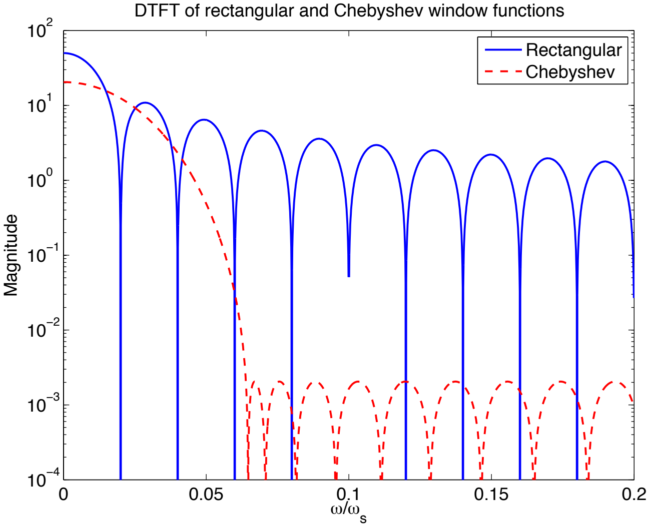
$$x(n) = \sin(2\pi f_0 n) + \sin(2\pi f_1 n) + 0.05 \sin(2\pi f_2 n)$$

where $f_0 = 0.135$, $f_1 = 0.150$ and $f_2 = 0.3$. The number of samples are $N = 50$

1. Rectangular window
2. Chebyshev window



Frequency analysis using DTFT.



DTFT of the rectangular and Chebyshev window of length 50

Zero padding

To calculate the DTFT for arbitrary frequencies ω is straight-forward. It is just to implement the definition

$$\hat{X}(\omega) = \sum_{n=0}^{N-1} x(n) e^{-j\omega \Delta t n}$$

However, using an equidistantly spaced frequency grid the calculations can be made more efficiently by employing the FFT algorithm.

Assume the signal is of length N

We want to calculate the DTFT at frequncies $\omega_k = \omega_s k/N_Z$, $k = 0, \ldots, N_Z - 1$ where $N_Z > N$.

We get the desired result if we calculate the FFT (i.e. DFT) for the extended signal

$$\hat{x}_z(n) = \begin{cases} \hat{x}(n), & n = 0, \ldots, N-1 \\ 0 & n = N, \ldots N_Z - 1 \end{cases}$$

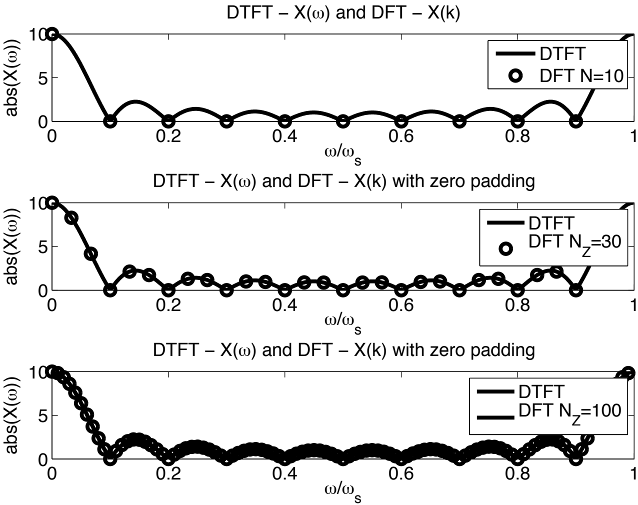


Illustration of zero padding.

FIR Filter Design

- FIR - Finite impulse response - filters $h(n)$
- Easy to design and analyze
- Fast filtering algorithms with FFT

Convolution

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

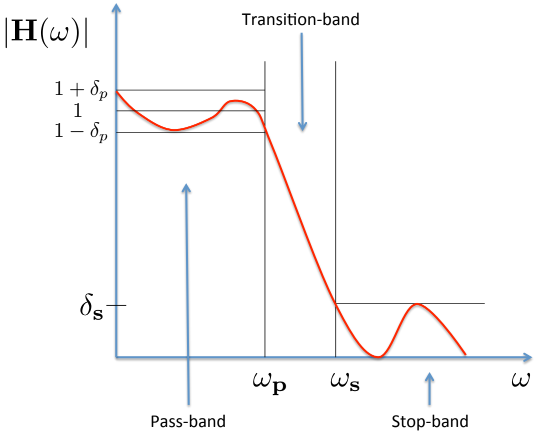
Frequency function

$$H(\omega) = \sum_{n=0}^{M-1} h(n)e^{j\omega\Delta tn}$$

The filter *shape* the DTFT of the output:

$$Y(\omega) = H(\omega)X(\omega)$$

Objective of the filtering are to *enhance* or *suppress* certain bands of the frequency content of the signal.



$\delta_\mathbf{p}$	Pass-band ripple (dB)	$\omega_\mathbf{p}$	Pass-band edge freq.
$\delta_\mathbf{s}$	Stop-band attenuation (dB)	$\omega_\mathbf{s}$	Stop-band edge freq.

Typical filter types

- Low-pass filter (LP)
- High-pass filter (HP)
- Band-bass filter (BP)
- Band-stop filter (BS)

Ideal LP filter

Frequency function

$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \omega_s/2. \end{cases}$$

Impulse response

$$h_{LP}(n) = \begin{cases} \frac{2\omega_c}{\omega_s} & n = 0 \\ \frac{2\omega_c}{\omega_s} \frac{\sin \omega_c \Delta t n}{\omega_c \Delta t n} & n \neq 0 \end{cases}$$

- Infinte long. $\Rightarrow$ Truncate the length (windowing)
  - $h_T(n) = w(n) h_{LP}(n), \quad n = 0, \pm 1, \ldots, \pm (M-1)/2$
- Non-causal. $\Rightarrow$ Shift it to make it causal (introduces a delay)
  - $h(n) = h_T(n - (M-1)/2), \quad n = 0, \ldots, M-1)$

Linear phase FIR filters

Non-causal truncated filters

- Odd filters $h_T(-n) = -h_T(n)$
- Even filters $h_T(-n) = h_T(n)$

Have a zero (or constant π) phase function

Shifted filters $h(n) = h_T(n - (M-1)/2), \quad n = 0, \dots, M-1)$

have linear phase $-\omega \Delta t (M-1)/2$.

Linear phase filters act as a delay in the passband and preserves pulse shapes.

Analysis of the filter

Truncation yieleds

$$H_T(\omega) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H_{LP}(\lambda) W(\omega - \lambda)\, d\lambda$$

The delay introduces a phase shift and the final result is

$$H(\omega) = e^{-j\omega\Delta t \frac{M-1}{2}} H_T(\omega) = e^{-j\omega\Delta t \frac{M-1}{2}} \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H_{LP}(\lambda) W(\omega - \lambda)\, d\lambda$$

The choice of window function will influence the stop-band attenuation and the width of the transition band.

Window function properties

Window's name	Trans band	Peak $20\log_{10}\delta_s$
Rectangular	1/M	-21 dB
Hamming	3.3/M	-53 dB
Blackman	5.5/M	-74 dB

Optimal FIR design methods

- FIR-LS:
  - Minimizes the sum of the squared frequency function deviation from the spec.
- FIR-PM:
  - Minimizes the maximum frequency deviation from the spec.

FIR-LS and FIR-PM

FIR-LS

$$\min_{\mathbf{h}} \sum_{k=1}^{N_{\text{spec}}} W_K |H_D(\omega_k) - H(\omega_k, h)|^2$$

FIR-PM

$$\min_{\mathbf{h}} \max_{\omega} |H_D(\omega) - H(\omega, h)|^2.$$

Efficent algoritrhms exist to solve both problems.