Tutorial 3 Solutions

1 a) $E[x(n)] = 0$, $\phi_{xx}(0) = \sigma_w^2 + 2^2\sigma_v^2$

$\phi_{xx}(\pm 1) = 2^2 \phi_{vv}(1) = 4 \cdot 0{,}5 = 2$

$\phi_{xx}(\ell) = 0 \quad |\ell| > 1$

b) $E[x(n)] = 0 \quad \phi_{xx}(0) = (1^2 + 3^2)\sigma_w^2 + \sigma_v^2 = 10\sigma_w^2 + \sigma_v^2$

$\phi_{xx}(\pm 1) = 3 \cdot 1\, \sigma_w^2 + \phi_{vv}(1) = 3\sigma_w^2 + 0{,}5$

2 a) $\phi_{xx}(\ell) = E[x(n)\,x(n+\ell)] = E[x(n+\ell)\,x(n)]$

$= [n' = n+\ell] = E[x(n')\,x(n'-\ell)] = \phi_{xx}(-\ell)$

b) $S_{xx}(\omega) = \sum \phi_{xx}(\ell)e^{-j\omega\ell} = \sum \phi_{xx}(-\ell)e^{-j\omega\ell} =$

$[\ell = -\ell'] = \sum \phi_{xx}(\ell')\underbrace{e^{j\omega\ell'}}_{\left(e^{-j\omega\ell'}\right)^*} = S_{xx}(\omega)^*$

3 a) $\phi_{xy}(\ell) = E[x(n)\,y(n+\ell)] = E\left[\sum_k h(k)\,x(n+\ell-k)\,x(n)\right]$

$= \sum_k h(k)\,\phi_{xx}(\ell-k) = [x \text{ is white noise}] =$

$= h(\ell)\sigma_x^2$

b) $\phi_{yy}(\ell) = E[y(n)\,y(n+\ell)] = E\left[\sum_k h(k)\,x(n-k)\,y(n+\ell)\right]$

$= \sum_k h(k)\,\phi_{xy}(k+\ell) = [k' = -k]$

$= \sum_{k'} h(-k')\,\phi_{xy}(\ell-k') = [\text{from a)}]$

$= \sum_k h(-k)\,h(\ell-k)\sigma_x^2$

c) $S_{yy}(\omega) = DTFT[\phi_{yy}(\ell)] = [\text{convolution}] = \cancel{DTFT[h(-k)]}\cdot$

$= DTFT[h(-k)] \cdot DTFT[h(k)]\sigma_x^2 = |H(\omega)|^2\sigma_x^2$

(2)

4 a) $$\frac{\partial}{\partial h(m)} e^2(n) = \frac{\partial}{\partial h(m)} \left(x(n) - \sum_k h(k) y(n-k)\right)^2$$

$$= -2\left(x(n) - \sum h(k) y(n-k)\right) y(n-m)$$

$$= -2\left(x(n) - \underbrace{\cancel{\quad}}_{\underline{y}(n)^T \underline{h}}\right) y(n-m)$$

b) $$\nabla_{\underline{h}} e^2(n) = -2\left(x(n) - \underbrace{\cancel{\quad}}_{\underline{y}(n)^T\underline{h}}\right) \begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-M+1) \end{bmatrix} = -2\underline{y}\left(x(n) - \underbrace{\cancel{\quad}}_{\underline{y}(n)^T\underline{h}}\right)$$

c) $$L(\underline{h}) = \sum_{n=0}^{N-1} e^2(n) \Rightarrow \nabla_{\underline{h}} L(\underline{h}) = \sum_{n=0}^{N-1} \nabla_{\underline{h}} e^2(n)$$

$$= -2 \sum_{n=0}^{N-1} \underline{y}(n)\left(x(n) - \underline{y}^T(n)\,\underline{h}\right) =$$

$$= -2 \sum_{n=0}^{N-1} \underline{y}(n)\, x(n) + 2 \sum_{n=0}^{N-1} \underline{y}(n)\,\underline{h}$$

Result follows by setting $\nabla_{\underline{h}} L(\underline{h}) = 0$

d) $$\nabla_h E[e^2] = E\left[\nabla_{\underline{h}} e^2(n)\right] = E\left[-2\underline{y}(n)\left(x(n) - \underline{y}(n)^T \underline{h}\right)\right]$$

$$= -2\left(\underline{\Phi}_{yx} - \Phi_{yy}\underline{h}\right)$$

e) Optimum when $\nabla_{\underline{h}} E[e^2] = 0$

$$\Rightarrow \Phi_{yx} - \Phi_{yy}\underline{h} = 0 \Rightarrow h_{opt} = \Phi_{yy}^{-1}\Phi_{yx}$$