Applied Signal Processing
Lecture 7

Tomas McKelvey

Department of Electrical Engineering
Chalmers University of Technology

Agenda

- Repetition
- IIR filter design
- Bilinear design method
- Example
- Comparison IIR and FIR filter

Filter design

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

In DTFT domain

$$Y(\omega) = H(\omega)X(\omega)$$

Filter $H(\omega)$ shape the signal.

FIR Filter Design

- FIR - Finite impulse response - filters $h(n)$
- Easy to design and analyze
- Fast filtering algorithms with FFT

Convolution

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

Frequency function

$$H(\omega) = \sum_{n=0}^{M-1} h(n)e^{j\omega\Delta tn}$$

The filter *shape* the DTFT of the output:

$$Y(\omega) = H(\omega)X(\omega)$$

Window design method

Desired frequency function $H_D(\omega)$.

Derive impulse response

$$h_I(n) = \frac{1}{\omega_s} \int_0^{\omega_s} H_D(\omega) e^{j\omega \Delta t n} \, d\omega$$

- Infinte long. $\Rightarrow$ Truncate the length (windowing) with window function $w(n)$
  - $h_T(n) = w(n) h_I(n), \quad n = 0, \pm 1, \ldots, \pm (M-1)/2$
- Non-causal. $\Rightarrow$ Shift it to make it causal
  - $h(n) = h_T(n - (M-1)/2), \quad n = 0, \ldots, M-1)$

This introduces a delay of $(M-1)/2$ samples.

IIR Motivations

Discret time IIR filter structures is given by

$$y(n) = -\sum_{k=1}^{n_a} a(k)y(n-k) + \sum_{k=0}^{n_b} b(k)x(n-k)$$

with frequency function (DTFT of impulse response)

$$H(\omega) = \frac{\sum_{k=0}^{n_b} b(k)z^{-j\omega\Delta tk}}{1 + \sum_{k=0}^{n_a} a(k)z^{-j\omega\Delta tk}}$$

- A rational function has different flexibility compared to a polynomial function (FIR filter)
- Frequency function is non-linear with regard to the parameters $a(k)$. More difficult to design filters.

CT design techniques

Several continuous time design techniques exist for analog IIR rational filters.

$$H(\Omega) = \frac{\sum_{k=0}^{n_b} b(k)(j\Omega)^k}{1 + \sum_{k=0}^{n_a} a(k)(j\Omega)^k}$$

- Analytical expressions on $b(k)$ and $a(k)$ given design specs
- Examples
  - Butterworth filter (maximally flat passband)
  - Chebyshev filter (small transition band)
  - Elliptic filter (equiripple filter)
  - and more ...

How can we use these analog filter designs to obtain a DT IIR filter of rational type?

Motivating example

Assume we have an analog filter design such that $H(\Omega)$ satisfy some design specs, e.g. filter order and pass band edge frequency Ω_p. Assume the frequency function is (where $a = 1/\omega_p$)

$$H(\Omega) = \frac{1}{1 + aj\Omega}$$

Consider the identity

$$\begin{aligned} j\Omega &\triangleq 2j\tan(\omega\Delta t/2) = 2j\frac{\sin(\omega\Delta t/2)}{\cos(\omega\Delta t/2)} \\ &= \frac{2(e^{j\omega\Delta t/2} - e^{-j\omega\Delta t/2})}{e^{j\omega\Delta t/2} + e^{-j\omega\Delta t/2}} = \frac{2(1 - e^{-j\omega\Delta t})}{1 + e^{-j\omega\Delta t}} \end{aligned}$$

Insert in frequency function above

$$H(\Omega) = \frac{1}{1 + aj\Omega} = \frac{1}{1 + a\frac{2(1-e^{-j\omega\Delta t})}{1+e^{-j\omega\Delta t}}} = \frac{1 + e^{-j\omega\Delta t}}{1 + 2a + (1 - 2a)e^{-j\omega\Delta t}} = H(\omega)$$

Which we recognize as the DT frequency function for an IIR filter of order 1.

Bilinear transformation

The idea generalizes:

- Take any arbitrary rational CT frequency function $H(\Omega)$.
- Make the substitution

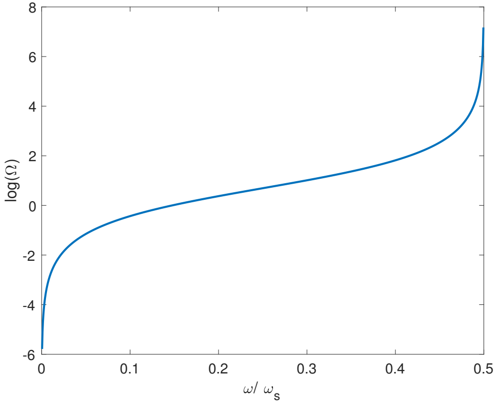
$$j\Omega \to \frac{2(1 - e^{-j\omega\Delta t})}{1 + e^{-j\omega\Delta t}}$$

- This yields $H(\omega)$ which is a rational function in the variable $e^{-j\omega\Delta t} \Rightarrow H(\omega)$ is a DT frequency function.

Operation known as bilinear transformation

- The DT and CT rational functions have the same order n_a
- If the CT filter is stable so is the DT filter
- By construction we have

$$H(\Omega) = H(\omega) \text{ for all } \Omega = 2\tan(\omega\Delta t/2)$$



Design Method (bilinear)

1. Decide filterype and DT filter specifications, e.g. ω_p, $|H(\omega_p)| = \alpha$
2. Convert the specifications to CT domain
   - For frequencies
     $$\Omega_p \triangleq 2\tan(\omega_p/2)$$
   - For amplitudes (and phase)
     $$H(\Omega_p) \triangleq H(\omega_p)$$
   - Design CT filter
   - Substitute
     $$j\Omega \to \frac{2(1-e^{-j\omega\Delta t})}{1+e^{-j\omega\Delta t}}$$

     to yield a DT filter

Comparison IIR and FIR

FIR filter of order $M = 11$. FIR-PM design method

IIR filter of order $n_a = n_b = 5$. Bilinear method with Butterworth CT design method.

Complexity is equal between the two filters. For each time -domain implementation 11 multiplications required for each time sample.

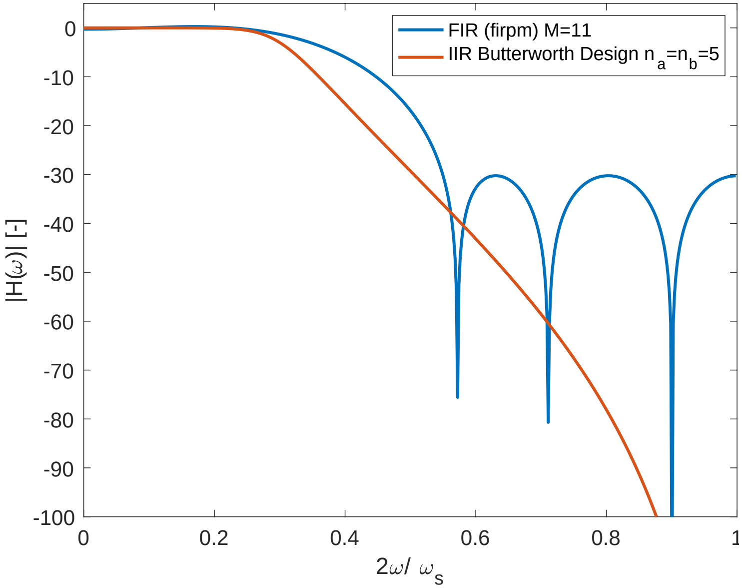
```
%% firPM design
f = [0 0.25 0.55  1]; aa = [1 1 0 0];
b_pm = firpm(10,f,aa);
[hfir,w] = freqz(b_pm,1,512);


%% IIR filter design
[b,a] = butter(5,0.3);
[hiir,w] = freqz(b,a,512);

pp=plot(w/pi,20*log10(abs([hfir hiir])))
fixplot(pp)

legend('FIR (firpm) M=11','IIR Butterworth Design n_a=n_b=5')
axis([0 1 -100 5])
```

Result



FIR (firpm) M=11
IIR Butterworth Design n_a=n_b=5
|H(ω)| [-]
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
0
0.2
0.4
0.6
0.8
1
2ω/ ω_s

IIR FIR comparison

Property	IIR	FIR
Flexible design techniques	-	+
Complexity v.s. stop-band attenuation	+	-
Complexity v.s. with of transition region	+	-
Linear phase	-	+
Robustness to filter coefficient truncation	-	+
Guaranteed stability	-	+