Applied Signal Processing SSY130
Tutorial 3

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Problems

1. The stochastic signals $w(n)$ and $v(n)$ have zero mean and the autocorrelation functions

$$\phi_{ww}(l) = \mathbf{E}[w(n)w(n+l)] = \begin{cases} \sigma_w^2 & l = 0 \\ 0 & l \neq 0 \end{cases}, \quad \phi_{vv}(l) = \mathbf{E}[v(n)v(n+l)] = \begin{cases} \sigma_v^2 & l = 0 \\ 0.5 & l = \pm 1 \\ 0 & |l| > 1 \end{cases}$$

   The signals are also uncorrelated which means that

$$\mathbf{E}[w(n)v(l)] = 0 \quad \forall n, l$$

   Derive the mean value and the autocorrelation function for the following signals derived from $v(n)$ and $w(n)$

   (a) $x(n) = w(n) + 2v(n)$
   (b) $x(n) = w(n) + 3w(n-1) + v(n)$

2. Define $\phi_{xx}(l) \triangleq \mathbf{E}[x(n)x(n+l)]$ and $S_{xx}(\omega) \triangleq \sum_{l=-\infty}^{\infty} \phi_{xx}(l)e^{-j\omega l}$.

   (a) Show that $\phi_{xx}(l) = \phi_{xx}(-l)$, i.e. it is a symmetric function.
   (b) Show that $S_{xx}(\omega)$ is a real valued function. (Hint: Use the identity $e^{j\omega} = \cos\omega + j\sin\omega$)

3. Let $x(n)$ be a zero mean white noise signal with variance σ_x^2 and let $y(n)$ be the output when $x(n)$ is filtered through a filter with impulse response $h(k)$. $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$. Show the following results:

   (a) $\phi_{xy}(l) \triangleq \mathbf{E}[x(n)y(n+l)] = h(l)\sigma_x^2$
   (b) $\phi_{yy}(l) \triangleq \mathbf{E}[y(n)y(n+l)] = \sum_{k=-\infty}^{\infty} h(-k)h(l-k)\sigma_x^2$
   (c) $S_{yy}(\omega) \triangleq \sum_{l=-\infty}^{\infty} = \phi_{yy}(l)e^{-j\omega l} = |H(\omega)|^2\sigma_x^2$.
   (Hint: If DTFT$[h(k)] = H(\omega)$ then DTFT$[h(-k)] = H(\omega)^*$)

4. *Wiener filtering - Optimal filter*
Consider the standard Wiener filter setup

$$\hat{x}(n) = \sum_{k=0}^{M-1} h(k)y(n-k) = \mathbf{y}^T(n)\mathbf{h}$$
$$e(n) = x(n) - \hat{x}(n) = x(n) - \mathbf{y}^T(n)\mathbf{h}$$

where

$$\mathbf{h}^T = \begin{bmatrix} h(0) & h(1) & \dots & h(M-1) \end{bmatrix}$$
$$\mathbf{y}^T(n) = \begin{bmatrix} y(n) & y(n-1) & \dots y(n-M+1) \end{bmatrix}.$$

The optimal solution $\mathbf{h}_{\text{opt}}$ is the filter coefficients (vector) which minimize the quadratic least-squares criterion

$$L(\mathbf{h}) = \sum_{n=0}^{N-1} e^2(n)$$

This is a purely signal sample based criterion. If we further assume that $y(n)$ and and $x(n)$ are random signals (stochastic processes) we obtain the optimal Wiener filter which is the filter (vector) which minimizes the variance of the error. i.e. $\mathbf{E}\, e^2(n)$ (LMMSE solution). We will now derive the solution in a few steps.

(a) Show $\frac{\partial}{\partial h(m)} e^2(n) = -2y(n-m)(x(n) - \mathbf{h}^T\mathbf{y})$

(b) Let

$$\bigtriangledown_{\mathbf{h}} e^2(n) \triangleq \begin{bmatrix} \frac{\partial}{\partial h(0)} e^2(n) \\ \frac{\partial}{\partial h(1)} e^2(n) \\ \vdots \\ \frac{\partial}{\partial h(M-1)} e^2(n) \end{bmatrix}$$

Show that $\bigtriangledown_{\mathbf{h}} e^2(n) = -2\mathbf{y}(n)\left(x(n) - \mathbf{y}^T(n)\mathbf{h}\right)$

(c) A quadratic positive function has its minimum when the gradient of the function is zero, i.e. $\bigtriangledown_{\mathbf{h}} L(\mathbf{h}) = 0$. Show that the optimal solution (assuming the inverse exists) is

$$\mathbf{h}_{\text{opt}} = \left[\sum_{n=0}^{N-1} \mathbf{y}(n)\mathbf{y}^T(n)\right]^{-1} \sum_{n=0}^{N-1} \mathbf{y}(n)x(n)$$

(d) Show that $\bigtriangledown_{\mathbf{h}} \mathbf{E}[e^2(n)] = -2(\Phi_{yx} - \Phi_{yy}\mathbf{h})$ where $\Phi_{yx} \triangleq \mathbf{E}[\mathbf{y}(n)x(n)]$ and $\Phi_{yy} \triangleq \mathbf{E}[\mathbf{y}(n)\mathbf{y}^T(n)]$

(e) Show that $\min_{\mathbf{h}} \mathbf{E}[e^2(n)]$ has the solution $\mathbf{h}_{\text{opt}} = \Phi_{yy}^{-1}\Phi_{yx}$ (the Wiener solution)