Applied Signal Processing
Lecture 3

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Agenda

- Examples of sampling
- Reconstruction
  - Ideal
  - Zero-order-hold
- Sampling - DT Filtering - Reconstruction
- Reconstruction - CT Filtering - Sampling

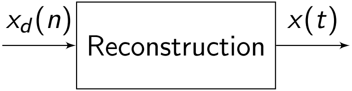
Admin stuff...

- Form Project Groups
  - Group match-making during the break today
- Project 1 material on Canvas
- Hand-in 1 material on Canvas

FT and DTFT after sampling

$$X_d(\omega) = X_c(\omega) = \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} X(\omega + k\omega_s)$$

Remember $X_d(\omega) = X_d(\omega + k\omega_s)$. It is periodic!

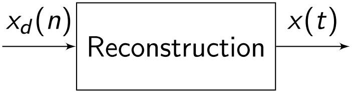


The process to generate a continuous time signal from a sampled signal

$$\{x_d(n)\}_{n=-\infty}^{\infty} \rightarrow x(t)$$

Issues to consider

- Desired relation between $x_d(n)$ and $x(t)$
- Complexity
- Causality



Desired relation between $x_d(n)$ and $x(t)$?

A good start

$$X(\omega) \approx \begin{cases} \Delta t \, X_d(\omega), & -\frac{\omega_s}{2} \leq \omega \leq \frac{\omega_s}{2} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Ideal Reconstruction

Assume that a CT signal $x(t)$ is *band-limited* such that $X(\omega) = 0$ for all $|\omega| > \omega_s/2$.

Then the continuous signal $x(t)$ can be perfectly recovered from the discrete time samples $x_d(n) = x(n\Delta t)$.

The reconstruction is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x_d(n) \left(\frac{\sin(\frac{\omega_s}{2}(t - \Delta tn))}{\frac{\omega_s}{2}(t - \Delta tn)} \right).$$

- $x(n\Delta t) = x_d(n)$
- Clearly non-causal
- Limited practical use without further approximations

Reconstruction as a filtering

Insert the identity

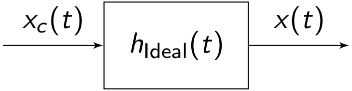
$$x_d(n) = \int_{-\infty}^{\infty} x_d(n)\delta(\Delta t n - \tau)\, d\tau \tag{2}$$

into the ideal reconstruction we get

$$x(t) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x_d(n)\delta(\Delta t n - \tau)\, d\tau \left(\frac{\sin(\frac{\omega_s}{2}(t - \Delta t n))}{\frac{\omega_s}{2}(t - \Delta t n)} \right) \tag{3}$$

By shifting the order of the sum and integration and noting that the integrand is zero whenever $\tau \neq \Delta t n$ we obtain

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} \underbrace{\sum_{n=-\infty}^{\infty} x_d(n)\delta(\Delta t n - \tau)}_{\triangleq x_c(\tau)} \underbrace{\left(\frac{\sin(\frac{\omega_s}{2}(t - \tau))}{\frac{\omega_s}{2}(t - \tau)} \right)}_{\triangleq h_{\text{Ideal}}(t-\tau)} d\tau \\ &= \int_{-\infty}^{\infty} h_{\text{Ideal}}(\tau) x_c(t - \tau)\, d\tau \end{aligned} \tag{4}$$



Where

$$h_{\text{Ideal}}(t) = \frac{\sin(\frac{\omega_s}{2}t)}{\frac{\omega_s}{2}t}$$

The frequency function of $h_{\text{Ideal}}(t)$ is

$$H_{\text{Ideal}}(\omega) = \begin{cases} \Delta t & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

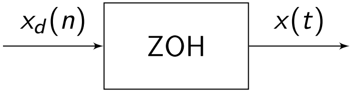
which of course is expected.

Nyquist sampling and reconstruction theorem

Nyquist

If $|X(\omega)| = 0$ for all $|\omega| \geq \omega_s/2$ and $x_d(n) = x(n\Delta t)$. then

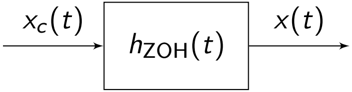
- $X_d(\omega) = \frac{1}{\Delta t}X(\omega). \quad \forall|\omega| < \omega_s/2$
- $x(t)$ can be exactly reconstructed from $x_d(n)$.



Zero-order-hold (ZOH) reconstruction is defined as

$$x(t) \triangleq x_d(n), \quad n\Delta t \leq t < (n+1)\Delta t.$$

An equivalent mathematical CT model is



$$h_{\text{ZOH}}(t) = \begin{cases} 1 & 0 \leq t < \Delta t \\ 0 & \text{otherwise} \end{cases}$$

ZOH frequency function

The impulse response of the ZOH filter has the frequency function

$$H_{\mathrm{ZOH}}(\omega) = FT[h_{\mathrm{ZOH}}(t)] = \Delta t e^{-j\frac{\omega\Delta t}{2}} \frac{\sin(\frac{\omega\Delta t}{2})}{\frac{\omega\Delta t}{2}}$$

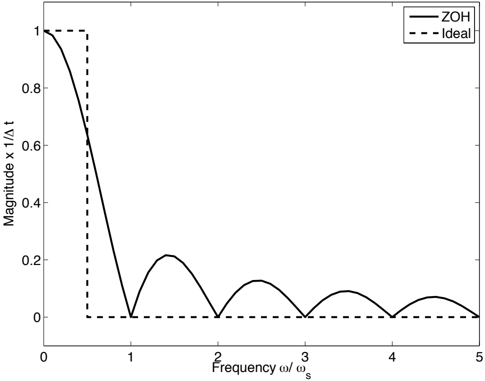
$$= \Delta t e^{-j\pi\frac{\omega}{\omega_s}} \frac{\sin(\pi\frac{\omega}{\omega_s})}{\pi\frac{\omega}{\omega_s}}.$$

We notice that since $\lim_{x\to 0}\sin(x)/x = 1$ we obtain at zero frequency

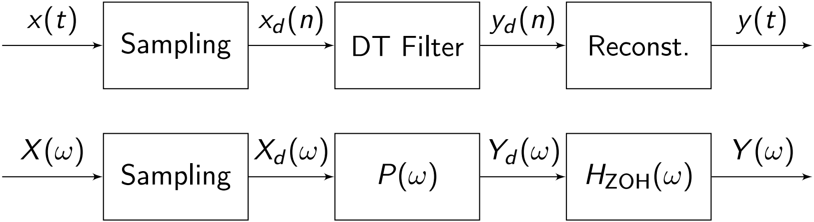
$$H_{\mathrm{ZOH}}(0) = \Delta t$$

and the frequency function is zero for all $\omega = k\omega_s$, $k = \pm 1, \pm 2, \ldots$.

Ideal and ZOH frequency functions



Full Chain CT-DT-CT

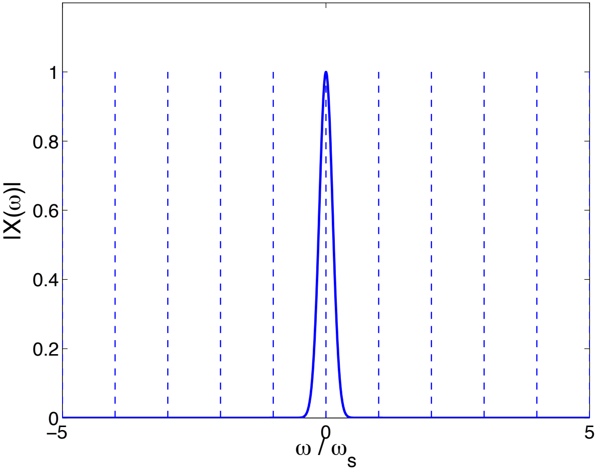


$$Y(\omega) = H_{\text{ZOH}}(\omega) P(\omega) \underbrace{\left[\frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} X(\omega + \omega_s k) \right]}_{X_c(\omega) = X_d(\omega)}$$

Limitations

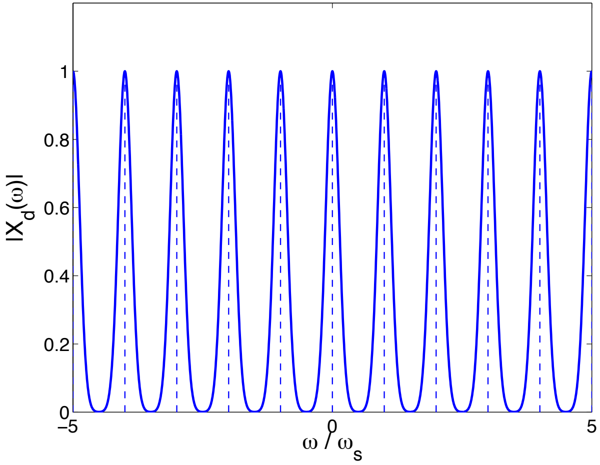
- Sampling frequency limited
  - High frequency signal components will cause sampling distorition (aliasing)
  - Anti-Aliasing filter
- Non-ideal reconstruction
  - ZOH reconstruction will produce signal energy above the Nyquist frequency
  - Analog (CT) Reconstruction filter can mitigate this
  - ZOH reconstruction is also non-ideal below the Nyquist frequency
  - Can be compensated in the DT processing step.

Example: Band limited signal



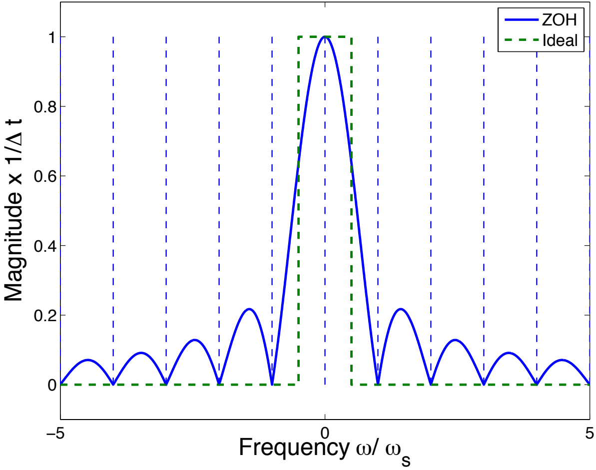
$X(\omega)$

Example: Band limited signal



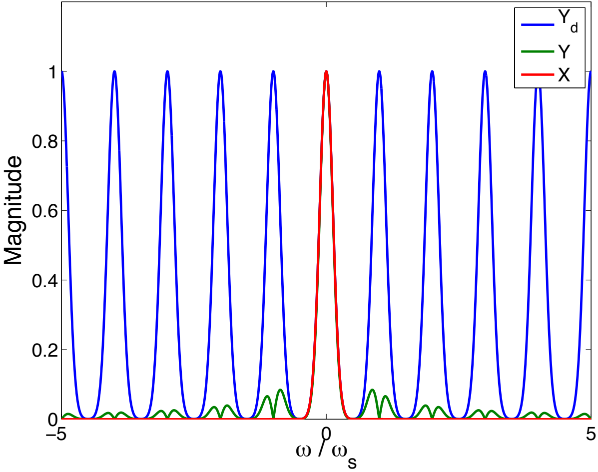
$$X_d(\omega) = Y_d(\omega)$$

Example: Band limited signal



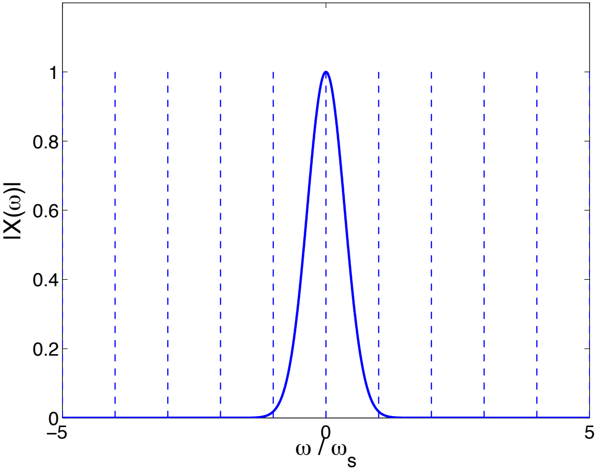
$H_{\mathrm{ZOH}}(\omega)$

Example: Band limited signal



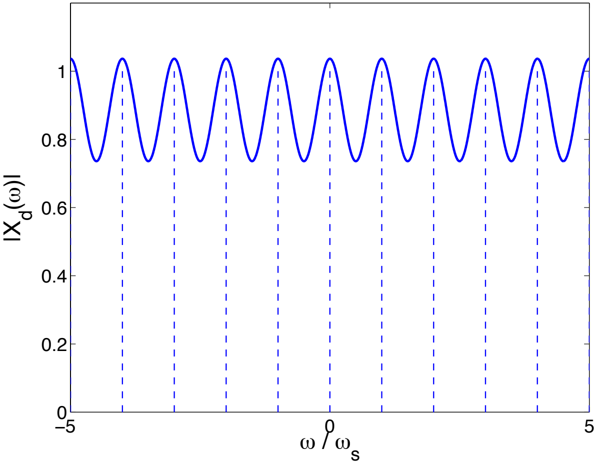
$$Y(\omega), \; Y_d(\omega)\, X(\omega)$$

Example: Non Band limited signal



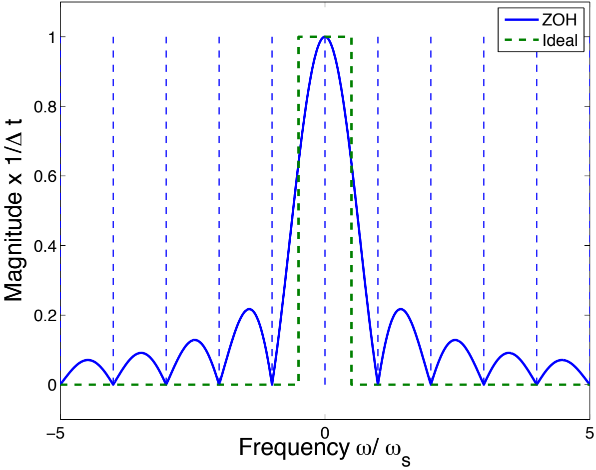
$X(\omega)$

Example: Non Band limited signal



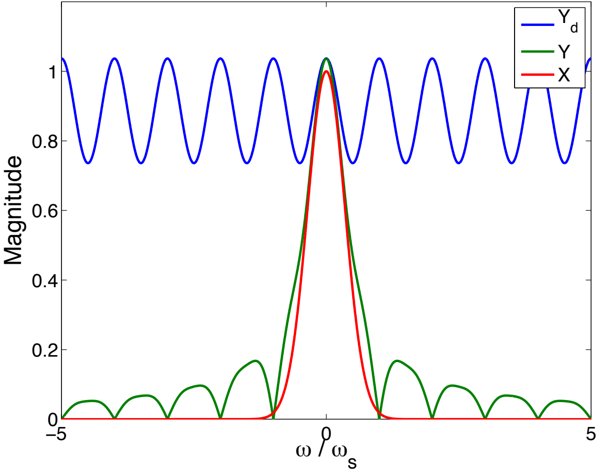
$$X_d(\omega) = Y_d(\omega)$$

Example: Non Band limited signal



$H_{\text{ZOH}}(\omega)$

Example: Non Band limited signal



$$Y(\omega),\ \ Y_d(\omega)\, X(\omega)$$