Applied Signal Processing
Lecture 2

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Agenda

- Finalize continuous time signals and systems
- Discrete time signals and systems
- Sampling

Admin stuff...

- On Thursday 11:30-12:30 Pick up DPS-kit
  - Bring signed agreement
- Sign up for project groups (4 students per group)
- Sign up for tutorial group

Complex exponential input

Assume $x(t) = e^{j\omega_0 t}$, the system output is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{j\omega_0(t-\tau)}\,d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{-j\omega_0\tau}\,d\tau e^{j\omega_0 t} = H(\omega_0)e^{j\omega_0 t} \end{aligned}$$

We can define

Amplitude function:	$A(\omega) \triangleq \lvert H(\omega)\rvert$
Phase function:	$\phi(\omega) \triangleq \angle H(\omega)$

which gives

$$y(t) = A(\omega_0)e^{j(\omega_0 t + \phi(\omega_0))}$$

Complex exponential input, alternative way

Assume $x(t) = e^{j\omega_0 t}$, then $X(\omega) = \mathrm{FT}[x(t)] = 2\pi\delta(\omega - \omega_0)$ and the system output is

$$
\begin{aligned}
Y(\omega) &= H(\omega)X(\omega) = H(\omega)2\pi\delta(\omega - \omega_0) \quad \Rightarrow \\
y(t) &= \mathrm{FT}^{-1}[Y(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)2\pi\delta(\omega - \omega_0)e^{j\omega t}\, d\omega = \\
&= H(\omega_0)e^{j\omega_0 t}
\end{aligned}
$$

which gives

$$y(t) = A(\omega_0)e^{j(\omega_0 t + \phi(\omega_0))}$$

A discrete time (DT) signal is an collection of values $x(n)$ where $n = 0, \pm 1, \pm 2, \ldots$ is the sample index.

Often (bot not necessary) $x(n)$ is the result of sampling a continuous time (CT) signal $x_c(t)$

$$x(n) \triangleq x_c(n\Delta t), \quad n = 0, \pm 1, \pm 2, \ldots$$

where Δt is the sampling period.

sampling period	Δt	[s]
sampling frequency:	$f_s \triangleq \frac{1}{\Delta t}$	[Hz=1/s]
sampling frequency:	$\omega_s \triangleq \frac{2\pi}{\Delta t}$	[rad/s]
Nyquist frequency:	$\frac{f_s}{2}$	[Hz]

Dimensionless normalized frequencies f/f_s (or ω/ω_s).

Discrete Time Fourier Transform (DTFT)

The DTFT is

$$X(\omega) = \text{DTFT}[x(n)] \triangleq \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n \Delta t}$$

ω has unit radians per second.

For integer k we note that $X(\omega + k\omega_s) = X(\omega)$ since $e^{jkn\omega_s \Delta t} = e^{jkn2\pi} = 1$, hence DTFT is a periodic function.

The Inverse DTFT is

$$x(n) = \text{DTFT}^{-1}[X(\omega)] = \frac{1}{\omega_s} \int_0^{\omega_s} X(\omega) e^{j\omega n \Delta t} \, d\omega$$

Discrete time convolution

For DT signals convolution (filtering) is defined as

$$y(n) \triangleq \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The DTFT of $y(n)$ is simply

$$Y(\omega) = H(\omega)X(\omega)$$

Proof: Since $H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k\Delta t}$ we have

$$y(n) = \frac{1}{\omega_s}\int_0^{\omega_s} \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k\Delta t}X(\omega)e^{j\omega n\Delta t}\, d\omega =$$

$$= \sum_{k=-\infty}^{\infty} h(k)\frac{1}{\omega_s}\int_0^{\omega_s} X(\omega)e^{j\omega(n-k)\Delta t}\, d\omega =$$

$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

	Time dom.	Fourier dom.
Delay	$x(n-k)$	$e^{-j\Delta t\omega k}X(\omega)$
Modulation	$e^{j\omega_0\Delta tn}x(n)$	$X(\omega-\omega_0)$
Constant	1	$\omega_s\tilde{\delta}(\omega)$
Kronecker delta	δ_n	1
Conv.	$\sum_{k=-\infty}^{\infty} h(k)x(n-k)$	$H(\omega)X(\omega)$
Freq. conv.	$x(n)w(n)$	$\frac{1}{\omega_s}\int_0^{\omega_s} X(\lambda)W(\omega-\lambda)\,d\lambda$

where $\tilde{\delta}(\omega) \triangleq \sum_{k=-\infty}^{\infty} \delta(\omega + k\omega_s)$

The Kronecker delta function (DT delta function) is

$$\delta_n \triangleq \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

DT Convolution is linear filtering

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$x(n)$ - input,
$y(n)$ - output,
$h(n)$ - impulse response

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$\Leftrightarrow$$

$$Y(\omega) = H(\omega)X(\omega)$$

Causal system if	$h(n) = 0$ for $n < 0$
Anti-causal if	$h(n) = 0$ for $n > 0$
Non-causal	otherwise

Linear and Time Invariant (LTI):
If inputs $x_1(n)$ and $x_2(n)$ yields outputs $y_1(n)$ and $y_2(n)$, input $\alpha x_1(n-n_1) + \beta x_2(n-n_2)$ yields output $\alpha y_1(n-n_1) + \beta y_2(n-n_2)$

Parseval and unit delay operator

For DT signals with finite energy, Parseval's relation is

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{\omega_s} \int_0^{\omega_s} |X(\omega)|^2 \, d\omega$$

Convenient to use the delay operator z^{-1} :

$$z^{-1}x(n) \triangleq x(n-1).$$

Complex exponential input

Assume $x(n) = e^{j\omega_0 \Delta t n}$, the system output is

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
&= \sum_{k=-\infty}^{\infty} h(k)e^{j\omega_0\Delta t(n-k)} \\
&\left(\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega_0\Delta t k}\right) e^{j\omega_0\Delta t n} \\
&= H(\omega_0)e^{j\omega_0\Delta t n}
\end{aligned}
$$

As before

$$
\begin{array}{ll}
\text{Amplitude function:} & A(\omega) \triangleq |H(\omega)| \\
\text{Phase function:} & \phi(\omega) \triangleq \angle H(\omega)
\end{array}
$$

which gives

$$
y(t) = A(\omega_0)e^{j(\omega_0 t + \phi(\omega_0))}
$$

A general difference equation

We can define the output as the following difference equation

$$y(n) \triangleq -\sum_{k=1}^{n_a} a_k y(n-k) + \sum_{k=0}^{n_b} b_k x(n-k)$$

Assuming stability (the recursion could make it unstable) take the DTFT of both sides

$$Y(\omega) = -\sum_{k=1}^{n_a} a_k e^{-j\omega\Delta tk} Y(\omega) + \sum_{k=0}^{n_b} b_k e^{-j\omega\Delta tk} X(\omega)$$

which imply

$$Y(\omega) = \frac{\sum_{k=0}^{n_b} b_k e^{-j\omega\Delta tk}}{1 + \sum_{k=1}^{n_a} a_k e^{-j\omega\Delta tk}} X(\omega) = H(\omega)X(\omega)$$

We aim to describe the following operations using the CT Fourier Transform:

- Sampling
- DT filtering
- Reconstruction

Need a CT signal model which can represent DT signals x_d and y_d.

The sampled DT signal is

$$x_d(n) = x(n\Delta t)$$

Consider the CT signal model of $x_d(n)$:

$$x_c(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)x(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)x_d(n)$$

The FT of this CT model signal is

$$X_c(\omega) = \int_{-\infty}^{\infty} x_c(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) x_d(n) e^{-j\omega t}\, dt$$

$$= \ldots = \sum_{n=-\infty}^{\infty} x_d(n) e^{-j\omega n \Delta t}$$

which we recognize as the DTFT of $x_d(n)$. In summary we have the identity

FT and DTFT relation

$$\mathrm{FT}[x_c(t)] = \mathrm{DTFT}[x_d(n)]$$
$$X_c(\omega) = X_d(\omega)$$

To complete the picture we need to express $X_c(\omega)$ (and thus also $X_d(\omega)$) in terms of $X(\omega)$.

Start by expressing $x(t)$ exactly at the sampling time instances using the inverse FT:

$$
\begin{aligned}
x_d(n) = x(n\Delta t) &= \int_{-\infty}^{\infty} X(\omega) e^{j\omega n \Delta t} \, \frac{d\omega}{2\pi} \\
&= \sum_{k=-\infty}^{\infty} \int_{k\omega_s}^{(k+1)\omega_s} X(\omega) e^{j\omega n \Delta t} \, \frac{d\omega}{2\pi} \\
&= [\text{Variable change: } \omega = \omega' + k\omega_s] \\
&= \sum_{k=-\infty}^{\infty} \int_{0}^{\omega_s} X(\omega' + k\omega_s) e^{j(\omega' + k\omega_s) n \Delta t} \, \frac{d\omega'}{2\pi} \\
&= \frac{1}{\omega_s} \int_{0}^{\omega_s} \underbrace{\frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} X(\omega' + k\omega_s)}_{X_d(\omega')} \, e^{j\omega' n \Delta t} \, d\omega'
\end{aligned}
$$

last expression is the inverse DTFT of the signal $x_d(n)$.

FT and DTFT after Sampling

We have shown

FT and DTFT after sampling

$$X_d(\omega) = X_c(\omega) = \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} X(\omega + k\omega_s)$$