Applied Signal Processing
Lecture 8

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Agenda

- Multirate Processing
  - Up-sampling
  - Interpolation
  - Down-sampling
  - Decimation
  - Rate changes
  - Modulation - Demodulation

Motivations

- Rate changes to match different standards
- Oversampling
  - Use sample-rate higher than required Nyquist-rate at signal acquisition
  - Signal reconstruction at a rate higher than Nyquist-rate
- Handle signals in frequency divided channels
- Efficient processing of band pass signals by using downsampling

Notation

Use the delay operator z^{-1} to denote filtering:

$$H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$$

$$\Rightarrow$$

$$y(n) = H(z)x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

since $z^{-k}x(n) = x(n-k)$.

Up-sampling

$$x(n) \longrightarrow \boxed{\Uparrow_L} \longrightarrow x_u(n)$$

We define *up-sampling* with factor L

$$x_u(n) \triangleq \begin{cases} x(n/L), & n = 0, \pm, L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

The signal x_u have sampling frequency $\omega_{s_u} = L\omega_s$ and sampling interval $\Delta t_u = \Delta t / L$ where ω_s and Δt correspond to the signal $x(n)$.

We use the short-form notation

$$x_u(n) = \Uparrow_L [x(n)]$$

$$
\begin{aligned}
X_u(\omega) &= \sum_{n=-\infty}^{\infty} x_u(n) e^{-j\omega n \Delta t_u} = [m = n/L \text{ and } \Delta t_u = \Delta t/L] \\
&= \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega L m \Delta t/L} = [n = m] \\
&= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n \Delta t} = X(\omega)
\end{aligned}
$$

where ω is un-normalized, i.e. rad/s.

DTFT up-sampling

$$X_u(\omega) = X(\omega)$$

The DTFT is *unchanged* after up-sampling. The new sampling frequency is L times higher.

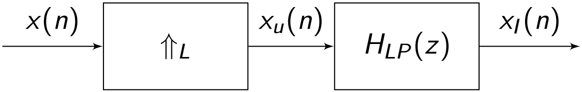
DTFT properties

Since $X(\omega)$ is periodic with period ω_s the identity

$$X_u(\omega) = X(\omega)$$

imply that $x_u(\omega)$ will have $L-1$ extra images of the original DTFT from ω_s to $L\omega_s = \omega_{s_u}$.

Interpolation



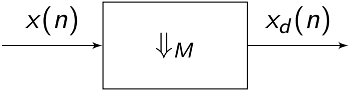
The combined operation of up-sampling and LP filtering is known as *interpolation*.

The LP-filter will remove the extra images introduced by interpolation. To get the desired behaviour the LP filter should (ideally) have the specifications.

$$H(\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \frac{\omega_s}{2L} \\ 0, & \text{otherwise} \end{cases}$$

The result $X_I(n)$ will be a signal which resembles the shape of $x(n)$ but will have a factor L more samples.

Down-sampling



We define *down-sampling* with factor M as

$$x_d(n) \triangleq \Downarrow_M[x(n)] \triangleq x(Mn), \quad n = 0, \pm 1, \pm 2, \ldots$$

The signal x_d have sampling frequency $\omega_{s_d} = \omega_s / M$ and sampling interval $\Delta t_d = M\Delta t$ where ω_s and Δt correspond to the signal $x(n)$.

We use the short-form notation

$$x_u(n) = \Downarrow_M[x(n)]$$

Down-sampling in the DTFT domain

Clearly the down-sampling is an operation which potentially reduce/distort the information in the signal. In the DTFT domain we have

DTFT for down-sampling

$$X_d(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\omega + \frac{\omega_s k}{M}\right).$$

where ω us un-normalized, i.e. rad/s.

Consider expressing the down-sampled signal in terms of the inverse DTFT of the original signal:

$$
\begin{aligned}
x_d(n) = x(nM) &= \frac{1}{\omega_s}\int_0^{\omega_s} X(\omega)e^{j\omega nM\Delta t}\,d\omega = [\Delta t_d = M\Delta t] \\
&= \frac{1}{\omega_s}\sum_{k=0}^{M-1}\int_{\omega_s k/M}^{\omega_s(k+1)/M} X(\omega)e^{j\omega n\Delta t_d}\,d\omega = [\omega = \lambda + \omega_s k/M] \\
&= \frac{1}{\omega_s/M}\int_0^{\omega_s/M} \underbrace{\frac{1}{M}\sum_{k=0}^{M-1} X\left(\lambda + \frac{\omega_s k}{M}\right)}_{X_d(\lambda)} e^{j\lambda n\Delta t_d}\,d\lambda.
\end{aligned}
$$

Since the last expression is the inverse DTFT of x_d we have shown that

$$
X_d(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(\omega + \frac{\omega_s k}{M}\right).
$$

By inspection of the result

$$X_d(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(\omega + \frac{\omega_s k}{M}\right).$$

we can conclude the following theorem:

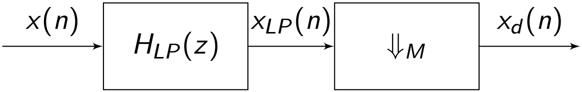
Nyquist theorem for down-sampling

If $X(\omega) = 0$, for $|\omega| \geq |\frac{\omega_s}{2M}$,

then $X(\omega) = M\,X_d(\omega)$, for $|\omega| < |\frac{\omega_s}{2M}$ where $\omega_{s_d} = \omega_s/M$

If the signal $x(n)$ is sufficiently band-limited, the down-sampling does not cause aliasing. Conversely if $X(\omega)$ is non-zero for a frequency band larger than ω_{s_d}, aliasing will occur.

Decimation



The combined operation of LP-filtering and down-sampling is known as *decimation*.

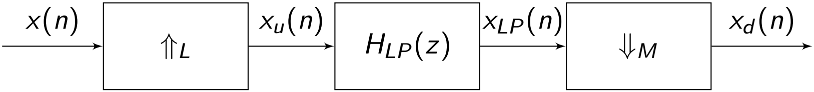
The LP-filter will bandlimit the signal to avoid aliasing. To get the desired behaviour the LP filter should (ideally) have the specification.

$$H_{LP}(\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \frac{\omega_s}{2M} \\ 0, & \text{otherwise} \end{cases}$$

The result $x_d(n)$ will be a signal which resembles the shape of $x_{LP}(n)$ but will have a factor M less samples.

Rate change

The interpolation and decimation methods can be combined to perform a fractional rate change. Assume we want to create a signal with the new rate $\omega_s' = \frac{L}{M}\omega_s$.



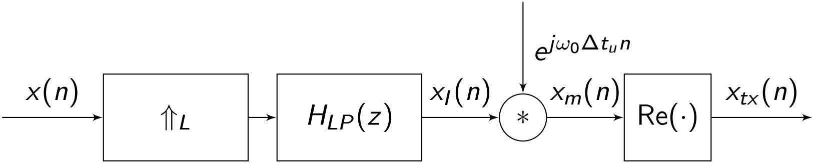
To avoid aliasing in the down-sampling stage the filter should satisfy

$$H_{LP}(\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \frac{\omega_s'}{2} \\ 0, & \text{otherwise} \end{cases}$$

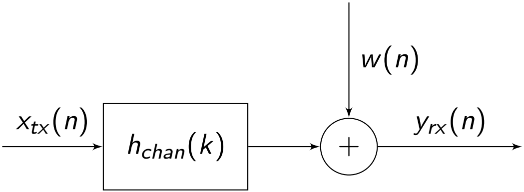
Can we shift the order of the up-sampler and down-sampler?

Interpolation and modulation

On the transmitter side:



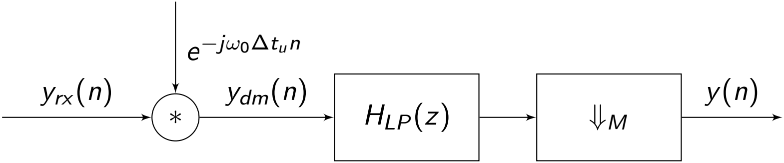
- $x(n)$ complex base band signal
- $x_I(n)$ interpolated base band signal
- $x_m(n)$ modulated complex signal with center frequency ω_0
- $x_{tx}(n)$ real valued modulated signal.
- $\Delta t_u = L\Delta t$



- $x_{tx}(n)$ transmitted real valued signal
- $y_{rx}(n)$ received real valued signal
- $w(n)$ noise signal

$$Y_{rx}(\omega) = H_{chan}(\omega)X_{tx}(\omega) + W(\omega)$$

On the receiver side:



- $y_{rx}(n)$ received real valued signal
- $y_{dm}(n)$ demodulated complex signal
- $y(n)$ complex decimated base band signal
- Here $M = L$ to match the transmitter.