Applied Signal Processing
Lecture 9

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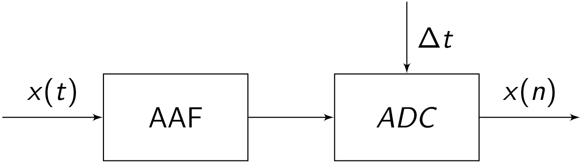
Agenda

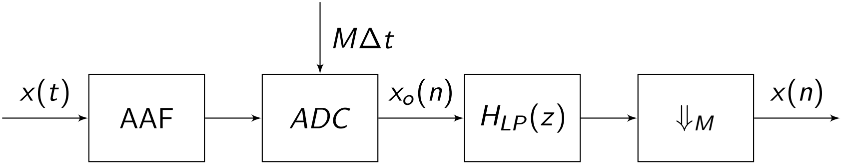
- Multirate Processing
  - Oversampling ADC
  - Oversampling DAC (ZOH)
- Statistical Signal Processing
  - Random Variables / Random Processes
  - Auto-correlation / Power spectral density (spectrum)
  - Cross-correlation / Cross-spectrum

Oversampled data acquisition

A classical data sampling frontend



An oversampled data sampling frontend



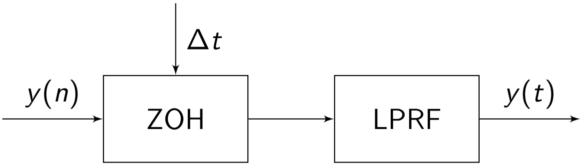
If M large enough the analog anti-aliasing filter (AAF) is not needed!

Properties

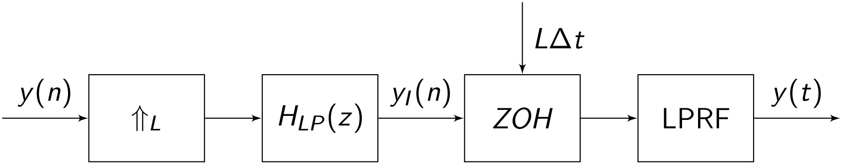
- The anti-aliasing filtering can be made in the sampled domain by a digital LP filter
- Only signals with power at frequencies above $M\omega_s/2$ will contribute to the alias distortion in the initial ADC step.

Oversampled signal reconstruction

A classical reconstruction backend



An oversampled signal reconstruction backend



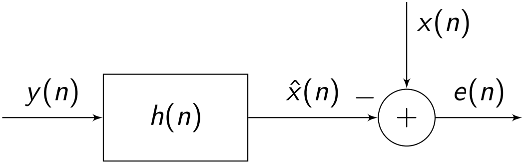
If M large enough the analog reconstruction filter (LPRF) is not needed!

Properties

- The ZOH reconstruction has two non-ideal behaviors
  - Magnitude change in the passband
  - The sinc function create high-frequency reconstruction distortion
- Both problem are mitigated by oversampling
  - The passband will be closer to magnitude 1 in the band $|\omega| \leq \frac{\omega_s}{2}$
  - The first high frequency distortion from the sinc function is moved from just above $\omega_s/2$ to around $L\ \omega_s$

Motivating example

Consider the generic filtering problem



We want to find filters h which make e small.

$$\min_{\mathbf{h}} V(\mathbf{h}) = \min_{\mathbf{h}} \sum_n \left(x(n) - \sum_k h(k) y(n-k) \right)^2$$

$V(\mathbf{h})$ contain terms with factors $\sum_n (x(n)y(n-k))$, $\sum_n y(n)y(n-k)$ and $\sum_n x^2(n)$ known as *sample correlations*.

The best (optimal) filter will depend on these correlations and not on the individual signals $y(n)$ and $x(n)$.

Modeling signals as *stochastic processes* is a viable alternative.

Random variables

A *random variabel* x take on a value upon a *random event*.

For a random variabel x, the *cumulative distribution function* $F_x(x_0)$ give the probability of the event that the random variable x take a value less than x_0.

$$P(x < x_0) = F_x(x_0)$$

The probability of an outcome of the random variable x in the interval between x_l and x_h is

$$P(x_l < x < x_h) = F_x(x_h) - F_x(x_l) = \int_{x_l}^{x_h} p_x(x)\, dx$$

where $p_x(x)$ is the *probability density function* where

$$\frac{d}{dx} F_x(x) = p_x(x)$$

From above it follows that $p_x(x) \geq 0$ and $\lim_{l \to \infty} P(-l < x < l) = 1$ so the area under $p_x(x)$ is equal to one.

Expectation, Mean value and Variance

Expectation of a function of the random variabel x, $f(x)$ is an operation defined as

$$\mathbf{E}\{f(x)\} \triangleq \int_{-\infty}^{\infty} f(x) p_x(x)\, dx$$

The mean value of a random variable x is obtained when $f(x) = x$,

$$m_x \triangleq \mathbf{E}\{x\} \triangleq \int_{-\infty}^{\infty} x p_x(x)\, dx$$

The *variance* σ_x^2 is

$$\sigma_x^2 = \mathbf{E}(x - m_x)^2 = \mathbf{E}\, x^2 - 2m_x \mathbf{E}\, x + m_x^2 = \mathbf{E}\, x^2 - m_x^2.$$

If the stochastic variable is zero mean $(m_x = \mathbf{E}\, x = 0)$ then the variance is $\mathbf{E}\, x^2$.

The *standard deviation* $\sigma_x = \sqrt{\sigma_x^2}$.

A multidimensional pdf provide the information on how several variables are statistically related.

Say x and y are two variables with a joint pdf $p_{x,y}(x,y)$. Then

$$P(x_l < x < x_h, y_l < y < y_h) = \int_{y_l}^{y_h} \int_{x_l}^{x_h} p_{x,y}(x,y)\, dxdy.$$

Expectation for two variables are defined as

$$\mathbf{E}\{f(x,y)\} \triangleq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) p_{x,y}(x,y)\, dxdy.$$

If the joint pdf can be factored as $p_{x,y}(x,y) = p_x(x)p_y(y)$, the two variables are called *independent* and

$$\begin{aligned}\mathbf{E}\{xy\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyp_{x,y}(x,y)\, dxdy = \int_{-\infty}^{\infty} xp_x(x)\, dx \int_{-\infty}^{\infty} yp_y(y)\, dy. \\ &= \mathbf{E}\{x\}\, \mathbf{E}\{y\}\end{aligned}$$

An enumbered sequence of random variables $x(n)$, $n = 0, \pm 1, \pm 2, \ldots$ is a *stochastic process*.

A signal can thus be regarded as a *realization* of a stochastic process.

The joint pdf:s $p_{x(n),x(n+k)}$ describe how the variables in the sequence is pairwise related.

Here we only deal with processes which are *stationary*

$$p_{x(n),x(n+k)} = p_{x(0),x(k)} \quad \text{for all } \; n$$

For a stationary and *ergodic* process

$$\begin{aligned} \mathbf{E}\, f(x(n), x(n+k)) &= \mathbf{E}\, f(x(0), x(k)) \\ &= \lim_{M \to \infty} \frac{1}{2M+1} \sum_{n=-M}^{M} f(x(n), x(n+k)) \end{aligned}$$

Auto-correlation and Power spectral density

The *auto-correlation function* is defined as

$$\phi_{xx}(n) \triangleq \mathbf{E}\{x(0)x(n)\}$$

and (trivially) satisfy $\phi_{xx}(-n) = \phi_{xx}(n)$.

If $m_x = 0$ then $\phi_{xx}(0)$ is the variance of $x(n)$.

The *Power spectral density* (spectrum) is

$$S_{xx}(\omega) \triangleq \sum_{n=-\infty}^{\infty} \phi_{xx}(n)e^{-j\omega\Delta tn}.$$

$S_{xx}(\omega)$ is a real-valued and non-negative function. We have

$$\sigma_x^2 = \phi_{xx}(0) = \frac{1}{\omega_s}\int_0^{\omega_s} S_{xx}(\omega)\,d\omega$$

$S_{xx}(\omega)$ is the signal power distribution over frequencies.

The *cross-correlation function* between two signals is defined as

$$\phi_{xy}(n) \triangleq \mathbf{E}\{x(0)y(n)\}$$

and satisfy $\phi_{xy}(n) = \phi_{yx}(-n)$.

The *Cross-spectral density* (cross-spectrum) is

$$S_{xy}(\omega) \triangleq \sum_{n=-\infty}^{\infty} \phi_{xy}(n)e^{-j\omega\Delta tn}.$$

If $\phi_{xy}(n)$ is real valued $S_{xy}(\omega) = S^*_{xy}(-\omega) = S^*_{yx}(\omega)$.

The Periodogram

The *periodogram* $P_x(\omega)$ of a signal $x(n)$, observed for $n = 0, 1, \ldots, N-1$ is given by

$$P_x(\omega) = |\hat{X}(\omega)|^2 = \hat{X}(\omega)\hat{X}^*(\omega)$$

where $\hat{X}(\omega)$ is the DTFT of $\hat{x}(n) = r_N(n)x(n)$, the truncated (windowed) signal.

We can interpret $P_x(\omega)$ as DTFT of the signal resulting from convolving $\hat{x}(n)$ with $\hat{x}(-n)$.

$$p_x(n) = \sum_{k=0}^{N-1} x(k)x(k-n), \quad n = 0, \pm 1, \ldots \pm (N-1).$$

Expected value of periodogram

Clearly

$$\mathbf{E}\, p_x(n) = \underbrace{(N - |n|)}_{w_{\text{tri}}(n)} \phi_{xx}(n) = w_{\text{tri}}(n)\phi_{xx}(n), \quad n = 0, \pm 1, \ldots, \pm(N-1)$$

So we obtain

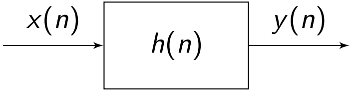
$$\begin{aligned} \mathbf{E}\, P_x(\omega) &= \sum_{k=-(N-1)}^{N-1} w_{\text{tri}}(n)\phi_{xx}(n)e^{-j\omega\Delta tn} \\ &= \frac{1}{\omega_s}\int_0^{\omega_s} W_{\text{tri}}(\lambda)S_{xx}(\omega-\lambda)\,d\lambda. \end{aligned}$$

It can be established

$$\lim_{N\to\infty}\frac{1}{N}\mathbf{E}\, P_x(\omega) = S_{xx}(\omega)$$

The periodogram is an asymptotically unbiased estimate of the spectrum (but the asymptotic variance does not vanish with N).

Filtered Stochastic processes



Input $x(n)$ is a stochastic process. Output $y(n)$ is also a stochastic process.

$$
\begin{aligned}
\phi_{xy}(n) = \mathbf{E}\, x(0)y(n) &= \mathbf{E} \sum_{k=-\infty}^{\infty} h(k)x(0)x(n-k) \\
&= \sum_{k=-\infty}^{\infty} h(k)\phi_{xx}(n-k)
\end{aligned}
$$

This imply

$$S_{xy}(\omega) = H(\omega)S_{xx}(\omega).$$

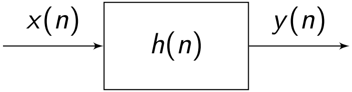
Furthermore

$$\phi_{yy}(n) = \mathbf{E}\, y(0)y(n) = \mathbf{E} \sum_{k=-\infty}^{\infty} h(k)y(0)x(n-k)$$
$$= \sum_{k=-\infty}^{\infty} h(k)\phi_{yx}(n-k)$$

Which yields

$$S_{yy}(\omega) = H(\omega)S_{yx}(\omega) = H(\omega)S^*_{xy}(\omega)$$
$$= H(\omega)H^*(\omega)S_{xx}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

Filtered stochastic process



- Signal:

$$y(n) = \sum_k h(k)x(n-k) \quad \Leftrightarrow \quad Y(\omega)X(\omega)$$

- Cross-correlation:

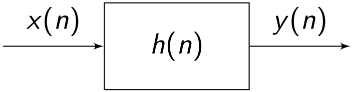
$$\phi_{xy}(n) = \sum_k h(k)\phi_x x(n-k) \quad \Leftrightarrow \quad S_{xy}(\omega) = H(\omega)S_{xx}(\omega)$$

- Output auto-correlation

$$\phi_{yy}(n) = \sum_k h(k)\phi_{yx}(n-k) \quad \Leftrightarrow \quad S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

Filtered white noise

Assume $\phi_{xx}(n) = \delta_n \sigma^2$ Then $S_{xx}(\omega) = \sigma^2$



We have

- Cross-correlation

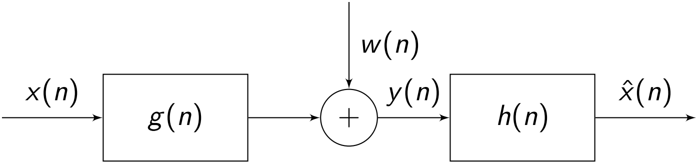
$$\phi_{xy}(n) = h(n)\sigma^2 \quad \Leftrightarrow \quad S_{xy}(\omega) = H(\omega)\sigma^2$$

- Output auto-correlation

$$\phi_{yy}(n) = \sum_k h(k)h(k-n)\sigma^2 \quad \Leftrightarrow \quad S_{yy}(\omega) = |H(\omega)|^2\sigma^2$$

The output spectrum become colored by the filter!

Inverse filtering



A naive approach

$$H(\omega) = \frac{1}{G(\omega)} \quad \Rightarrow \quad \hat{X}(\omega) = X(\omega) + \frac{1}{G(\omega)} W(\omega)$$

Issues:

- $\frac{1}{G(\omega)}$ might not be stable and causal
- If $|G(\omega)| << 1$ then $|H(\omega)W(\omega)| >> 1$ The noise will dominate.

Wiener filtering will provide a systematic approach how to design $H(\omega)$.