Chapter 4

Analog Measurand: Time Dependant Characteristics

4.1 Introduction

A parameter common to all of measurement is time: All measurands have time-related characteristics. As time progresses, the magnitude of the Measurand either changes or does not change. The time variation of any change is often fully as important as is any particular amplitude.

In this chapter we will discuss those quantities necessary to define and describe various time-related characteristics of measurands. As in Chapter 1, we classify time-related measurands as either

1. Static—constant in time
2. Dynamic—varying in time
   a. Steady-state periodic
   b. Nonrepetitive or transient
      i. Single-pulse or aperiodic
      ii. Continuing or random

4.2 Periodic signals

4.2.1 Simple Harmonic Relations

A function is said to be a simple harmonic function of a variable when its second derivative is proportional to the function but of opposite sign. More often than not, the independent variable is time t, although any two variables may be related harmonically. As an example of a simple harmonic signal is:

$$f(t)=A \sin\omega t$$

Where
A = the amplitude,
ω = frequency (circular frequency) rad/s

4.2.2 Fourier series expansion for Periodic Functions

In this section we develop the Fourier series expansion of periodic functions and discuss how closely they approximate the functions. We also indicate ho symmetrical properties of the function may be taken advantage of in order reduce the amount of mathematical manipulation involved in determining Fourier series.

Periodic functions:

A function $f(t)$ is said to be periodic if its image values are repeated at regular intervals in its domain. Thus the graph of a periodic function can be divided it 'vertical strips' that are replicas of each other, as illustrated in Figure 4.1. The interval between two successive replicas is called the ***period*** of the function. therefore say that a function $f(t)$ is periodic with period T if, for all its domain value

$$f(t + mT) = f(t)$$

for any integer m.
To provide a measure of the number of repetitions per unit of t, we de the frequency of a periodic function to be the reciprocal of its period, so that

$$\text{frequency} = \frac{1}{\text{period}} = \frac{1}{T}$$

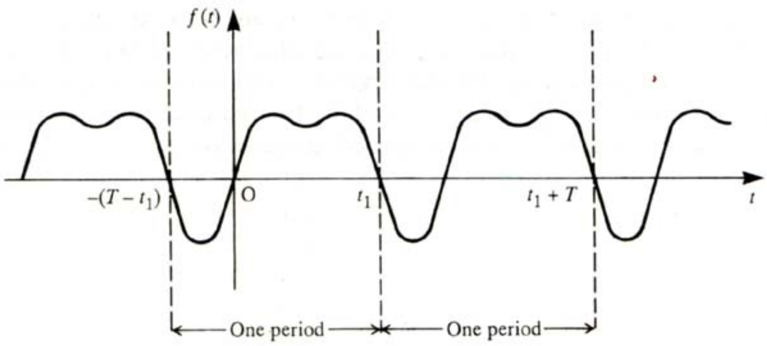

Figure 4.1 An example of a periodic function

The term circular frequency, ω is also used in engineering, and is defined by:

$$\omega = \text{circular frequency} = 2\pi \text{ x frequency} = 2\pi/T$$

and is measured in radians per second. It is common to drop the term 'circular' and refer to this simply as the frequency when the context is clear.
'Fourier's theorem': This theorem states that a periodic function that satisfies certain conditions can be expressed as the sum of a number of sine functions of different amplitudes, phases and periods. That is, if *f(t)* is a periodic function with period T then

$$f(t) = A_0 + A_1 \sin(\omega t + \phi_1) + A_2 \sin(2\omega t + \phi_2) + \dots + A_n \sin(n\omega t + \phi_n) + \dots \quad (4.2.1)$$

where the A's and φ 's are constants and ω = 2π /T is the frequency of *f(t)*. The term

$$A_1 \sin(\omega t + \varphi_1))$$

is called the first harmonic or the fundamental mode, and it has the same frequency ω as the parent function *f(t)*. The term

$$A_n \sin(n\omega t + \varphi_n)$$

is called the nth harmonic, and it has frequency $n\omega$, which is n times that of the fundamental. A_n denotes the amplitude of the nth harmonic and φ_n is its phase angle, measuring the lag or lead of the nth harmonic with reference to a pure sine wave of the same frequency.
Since

$$A_n \sin(n\omega t + \phi_n) \equiv (A_n \cos\phi_n)\sin n\omega t + (A_n \sin\phi_n)\cos n\omega t$$
$$\equiv b_n \sin \omega t + a_n \cos n\omega t$$

where

$$b_n = A_n \cos\phi_n, \qquad a_n = A_n \sin\phi_n \tag{4.2.2}$$

the expansion (4.2.1) may be written as

$$f(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \tag{4.2.3}$$

where $a_0 = 2A_0$ (we shall see later that taking the first term as $1/2\ a_0$ rather than a_0 is a convenience that enables us to make a_0 fit a general result). The expansion (4.2.3) is called the Fourier series expansion of the function *f(t)*, and the *a's* and *b's* are called the Fourier coefficients. In mechanical and electrical engineering it is common practice to refer to a_n and b_n respectively as the in-phase and phase quadrature components of the nth harmonic, this terminology arising from the use of the phasor notation $e^{jn\omega t} = \cos n\omega t + j \sin n\omega t$. Clearly, (4.2.1) is an alternative representation of the Fourier series with the amplitude and phase of the nth harmonic being determined from (4.2) as

$$A_n = \sqrt{(a_n^2 + b_n^2)}, \qquad \phi_n = \tan^{-1}\left(\frac{a_n}{b_n}\right)$$

with care being taken over choice of quadrant.

The Fourier coefficients:

Before proceeding to evaluate the Fourier coefficients, we first recall the following integrals, in which ω =circular frequency = 2π x frequency = $2\pi/T$
:

$$\int_d^{d+T} \cos n\omega t \,\mathrm{d}t = \begin{cases} 0 & (n \neq 0) \\ T & (n = 0) \end{cases} \tag{4.2.4}$$

$$\int_d^{d+T} \sin n\omega t \,\mathrm{d}t = 0 \quad (\text{all } n) \tag{4.2.5}$$

$$\int_d^{d+T} \sin m\omega t \sin n\omega t \,\mathrm{d}t = \begin{cases} 0 & (m \neq n) \\ \tfrac{1}{2}T & (m = n \neq 0) \end{cases} \tag{4.2.6}$$

$$\int_d^{d+T} \cos m\omega t \cos n\omega t \,\mathrm{d}t = \begin{cases} 0 & (m \neq n) \\ \tfrac{1}{2}T & (m = n \neq 0) \end{cases} \tag{4.2.7}$$

$$\int_d^{d+T} \cos m\omega t \sin n\omega t \,\mathrm{d}t = 0 \ (\text{all } m \text{ and } n) \tag{4.2.8}$$

The results (4.2.4)—(4.2.8) constitute the ***orthogonality relations*** for *sine* and *cosine* functions, and show that the set of functions

$$\{1, \cos \omega t, \cos 2\omega t, \ldots, \cos n\omega t, \sin \omega t, \sin 2\omega t, \ldots, \sin n\omega t\}$$

is an orthogonal set of functions on the interval $d \le t \le d + T$. The choice of d is arbitrary in these results, it only being necessary to integrate over a period of duration T.

Integrating the series (4.2.3) with respect to t over the period $t = d$ to $t = d + T$', and using (4.2.4) and (4.2.5), we find that each term on the right-hand side is zero except for the term involving a_0 that is, we have

$$\begin{aligned}\int_d^{d+T} f(t)\,\mathrm{d}t &= \tfrac{1}{2}a_0 \int_d^{d+T} \mathrm{d}t + \sum_{n=1}^{\infty} \left(a_n \int_d^{d+T} \cos n\omega t\,\mathrm{d}t + b_n \int_d^{d+T} \sin n\omega t\,\mathrm{d}t \right) \\ &= \tfrac{1}{2}a_0(T) + \sum_{n=1}^{\infty} [a_n(0) + b_n(0)] \\ &= \tfrac{1}{2}Ta_0\end{aligned}$$

Thus

$$\tfrac{1}{2}a_0 = \frac{1}{T}\int_d^{d+T} f(t)\,\mathrm{d}t$$

and we can see that the constant term a_0 in the Fourier series expansion represents the mean value of the function $f(t)$ over one period. For an electrical signal it represents the bias level or DC (direct current) component. Hence

$$a_0 = \frac{2}{T}\int_d^{d+T} f(t)\,\mathrm{d}t \tag{4.2.9}$$

To obtain this result, we have assumed that term-by-term integration of the series (4.2.3) is permissible. This is indeed so because of the convergence properties of the series — its validity is discussed in detail in more advanced texts.

To obtain the Fourier coefficient a_n (n # 0), we multiply (4.2.3) throughout by $\cos m\omega t$ and integrate with respect to t over the period $t = d$ to $t = d + 7$ giving

$$\begin{aligned}\int_d^{d+T} f(t)\cos m\omega t\,\mathrm{d}t = \tfrac{1}{2}a_0 \int_d^{d+T} \cos m\omega t\,\mathrm{d}t &+ \sum_{n=1}^{\infty} a_n \int_d^{d+T} \cos n\omega t \cos m\omega t\,\mathrm{d}t \\ &+ \sum_{n=1}^{\infty} b_n \int_d^{d+T} \cos m\omega t \sin n\omega t\,\mathrm{d}t\end{aligned}$$

Assuming term-by-term integration to be possible, and using (4.2.4), (4.2.7) and (4.2.8), we find that, when m ≠ 0, the only non-zero integral on the right-hand side is the one that occurs in the first summation when $n = m$. That is, we have

$$\int_d^{d+T} f(t)\cos m\omega t \, dt = a_m \int_d^{d+T} \cos m\omega t \cos m\omega t \, dt = \tfrac{1}{2}a_m T$$

giving

$$a_m = \frac{2}{T}\int_d^{d+T} f(t)\cos m\omega t \, dt$$

which, on replacing m by n, gives

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos n\omega t \, dt \tag{4.2.10}$$

The value of a_0 given in (4.2.9) may be obtained by taking $n = 0$ in (4.2.10), so that we may write

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos n\omega t \, dt \quad (n = 0, 1, 2, \ldots) \tag{4.2.11}$$

This explains why the constant term in the Fourier series expansion was taken as $1/2a_0$ and not a_0, since this ensures compatibility of the results (4.2.9) and (4.2.10). Although a_0 and a_n satisfy the same formula, it is usually safer to work them out separately.

Finally, to obtain the Fourier coefficients b_n, we multiply (4.2.3) throughout by *sin nω t* and integrate with respect to t over the period $t = d$ to $t = d + T$ giving

$$\int_d^{d+T} f(t)\sin m\omega t \, dt = \tfrac{1}{2}a_0 \int_d^{d+T} \sin m\omega t \, dt + \sum_{n=1}^{\infty}\left(a_n \int_d^{d+T} \sin m\omega t \cos n\omega t \, dt + b_n \int_t^{d+T} \sin m\omega t \sin n\omega t \, dt\right)$$

Assuming term-by-term integration to be possible, and using (4.2.5), (4.2.6) and (4.2.8), we find that the only non-zero integral on the right-hand side is the one that occurs in the second summation when $m = n$. That is, we have

$$\int_d^{d+T} f(t)\sin m\omega t \, dt = b_m \int_d^{d+T} \sin m\omega t \sin m\omega t \, dt = \tfrac{1}{2}b_m T$$

giving, on replacing m by n,

$$b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin n\omega t \, dt \quad (n = 1, 2, 3, \ldots) \tag{4.2.12}$$

The equations (4.2.11) and (4.2.12) giving the Fourier coefficients are known as Euler's formulae.

Summary

In summary, we have shown that if a periodic function f(t) of period $T = 2\pi/\omega$ can be expanded as a Fourier series then that series is given by

$$f(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

where the coefficients are given by the Euler formulae

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos n\omega t \, dt \quad (n = 0, 1, 2, \ldots)$$

$$b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin n\omega t \, dt \quad (n = 1, 2, 3, \ldots)$$

The limits of integration in Euler's formulae may be specified over any period, so that the choice of d is arbitrary, and may be made in such a way as to help in the calculation of a_n and b_n. In practice, it is common to specify $f(t)$ over either the period $-1/2\ T < t < 1/2\ T$ or the period $0 < t < T$ leading respectively to the limit of integration being $-1/2\ T$ and $1/2\ T$ (that is, $d = -1/2\ T$) or 0 and T (that is, $d = 0$)

EXAMPLE 4.1 Obtain the Fourier series expansion of the periodic function $f(t)$ of period 2π defined by

$$f(t) = t \quad (0 < t < 2\pi)$$
$$f(t) = f(t + 2\pi)$$

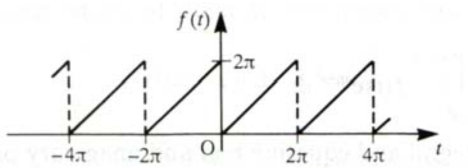

Figure 4.2 Sawtooth wave of Example 4.1.

Solution

A sketch of the function $f(t)$ over the interval — $4\pi < t < 4\pi$ is shown in Figure 4.2. Since the function is periodic we only need to sketch it over one period, the pattern being repeated for other periods. Using (4.2.15) to evaluate the Fourier coefficients a_0 and a gives

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\,\mathrm{d}t = \frac{1}{\pi}\int_0^{2\pi} t\,\mathrm{d}t$$

$$= \frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{2\pi} = 2\pi$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,\mathrm{d}t \quad (n = 1, 2, \ldots)$$

$$= \frac{1}{\pi}\int_0^{2\pi} t\cos nt\,\mathrm{d}t$$

which, on integration by parts, gives

$$a_n = \frac{1}{\pi}\left[t\frac{\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{2\pi}{n}\sin 2n\pi + \frac{1}{n^2}\cos 2n\pi - \frac{\cos 0}{n^2}\right)$$

$$= 0$$

since $\sin 2n\pi = 0$ and $\cos 2n\pi = \cos 0 = 1$. Note the need to work out a_0 separately from a_n in this case. The formula for b_n gives

$$b_n = \frac{1}{n}\int_0^{2\pi} f(t)\sin nt\,\mathrm{d}t \quad (n = 1, 2, \ldots)$$

$$= \frac{1}{\pi}\int_0^{2\pi} t\sin nt\,\mathrm{d}t$$

which, on integration by parts, gives

$$b_n = \frac{1}{\pi}\left[-\frac{t}{n}\cos nt + \frac{\sin nt}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(-\frac{2\pi}{n}\cos 2n\pi\right) \quad \text{(since } \sin 2n\pi = \sin 0 = 0\text{)}$$

$$= -\frac{2}{n} \quad \text{(since } \cos 2n\pi = 1\text{)}$$

Hence the Fourier series expansion of $f(t)$ is

$$f(t) = \pi - \sum_{n=1}^{\infty}\frac{2}{n}\sin nt$$

or, in expanded form,

$$f(t) = \pi - 2\left(\sin t + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \ldots + \frac{\sin nt}{n} + \ldots\right)$$

EXAMPLE 4.2 A periodic function *f(t)* with period 2π is defined within the period $-\pi < t < \pi$ by

$$f(t) = \begin{cases} -1 & (-\pi < t < 0) \\ 1 & (0 < t < \pi) \end{cases}$$

Find its Fourier series expansion.

Solution:

A sketch of the function *f(t)* over the interval $-\pi < t < \pi$ is shown in Figure 4.3. Clearly *f(t)* is an odd function of *t*, so that its Fourier series expansion consists of sine terms only. Taking $T = 2\pi$, that is $\omega = 1$, the Fourier series expansion is given by

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt \, dt \quad (n = 1, 2, 3, \ldots)$$

$$= \frac{2}{\pi}\int_0^{\pi} 1 \sin nt \, dt = \frac{2}{\pi}\left[-\frac{1}{n}\cos nt\right]_0^{\pi}$$

$$= \frac{2}{n\pi}(1 - \cos n\pi) = \frac{2}{n\pi}[1 - (-1)^n]$$

$$= \begin{cases} 4/n\pi & (\text{odd } n) \\ 0 & (\text{even } n) \end{cases}$$

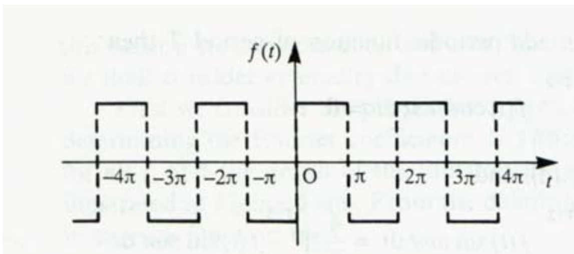

Figure 4.3 Square wave of Example 4.2.

Thus the Fourier series expansion of $f(t)$ is

$$f(t) = \frac{4}{\pi}\left(\sin t + \tfrac{1}{3}\sin 3t + \tfrac{1}{5}\sin 5t + \ldots\right)$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{2n-1}$$

4.2.3Typical periodic Waveforms

A number of frequently used special waveforms may be written as infinite trigonometric series. Several of these are shown in Fig. 4.4. Table 4.1 lists the corresponding equations.

Both the square(see example 4.2) wave and the sawtooth wave (see example 4.1) are useful in checking the response of dynamic measuring systems. In addition, the skewed sawtooth form, Fig. 4.4(c), is of the form required for the voltage-time relation necessary for driving the horizontal sweep of a cathode-ray oscilloscope. All these forms may be obtained as voltage-time relations from electronic signal generators.

For each case shown in Fig. 4.4, all the terms in the infinite series are necessary if the precise waveform indicated is to be obtained. Of course, with increasing harmonic order, their effect on the whole sum becomes smaller and smaller.

As an example, consider the square wave shown in Fig. 4.4(a). The complete series includes all the terms indicated in the relation

$$y = \frac{4A}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right).$$

By plotting only the first three terms, which include the fifth harmonic, the waveform shown in Fig. 4.5(a) is obtained. Figure 4.5(b) shows the results of plotting terms through and including the ninth harmonic, and Fig. 4.5(c) shows the form for the terms including the fifteenth harmonic. As more and more terms are added, the waveform gradually approaches the square wave, which results from the infinite series

Table 4.1 Fourier expansion of the functions given in Figure 4.4

$$y = \frac{4A}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}\sin(2n-1)\omega t\right]$$

$$y = \frac{2A}{\pi}\left(\sin \omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right) = \frac{2A}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin n\omega t\right]$$

$$y = \frac{2A}{\pi}\left(\sin \omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \frac{1}{4}\sin 4\omega t + \cdots\right) = \frac{2A}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\sin n\omega t\right]$$

$$y = \frac{A}{2} - \frac{4A}{(\pi)^2}\left(\cos \omega t + \frac{1}{(3)^2}\cos 3\omega t + \frac{1}{(5)^2}\cos 5\omega t + \cdots\right) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)^2}\cos(2n-1)\omega t\right]$$

$$y = \frac{8A}{(\pi)^2}\left(\sin \omega t - \frac{1}{(3)^2}\sin 3\omega t + \frac{1}{(5)^2}\sin 5\omega t - \cdots\right) = \frac{8A}{(\pi)^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)\omega t\right]$$

$$y = -\frac{8A}{(\pi)^2}\left(\cos \omega t + \frac{1}{(3)^2}\cos 3\omega t + \frac{1}{(5)^2}\cos 5\omega t + \cdots\right) = \frac{8A}{(\pi)^2}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)^2}\cos n\omega t\right]$$

$$y = \frac{2A}{\alpha(\pi-\alpha)}\left(\sin\alpha\sin\omega t + \frac{1}{(2)^2}\sin 2\alpha\sin 2\omega t + \frac{1}{(3)^2}\sin 3\alpha\sin 3\omega t + \cdots\right) = \frac{2A}{\alpha(\pi-\alpha)}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\sin n\alpha\sin n\omega t\right]$$

tions does not necessarily represent the harmonic order.

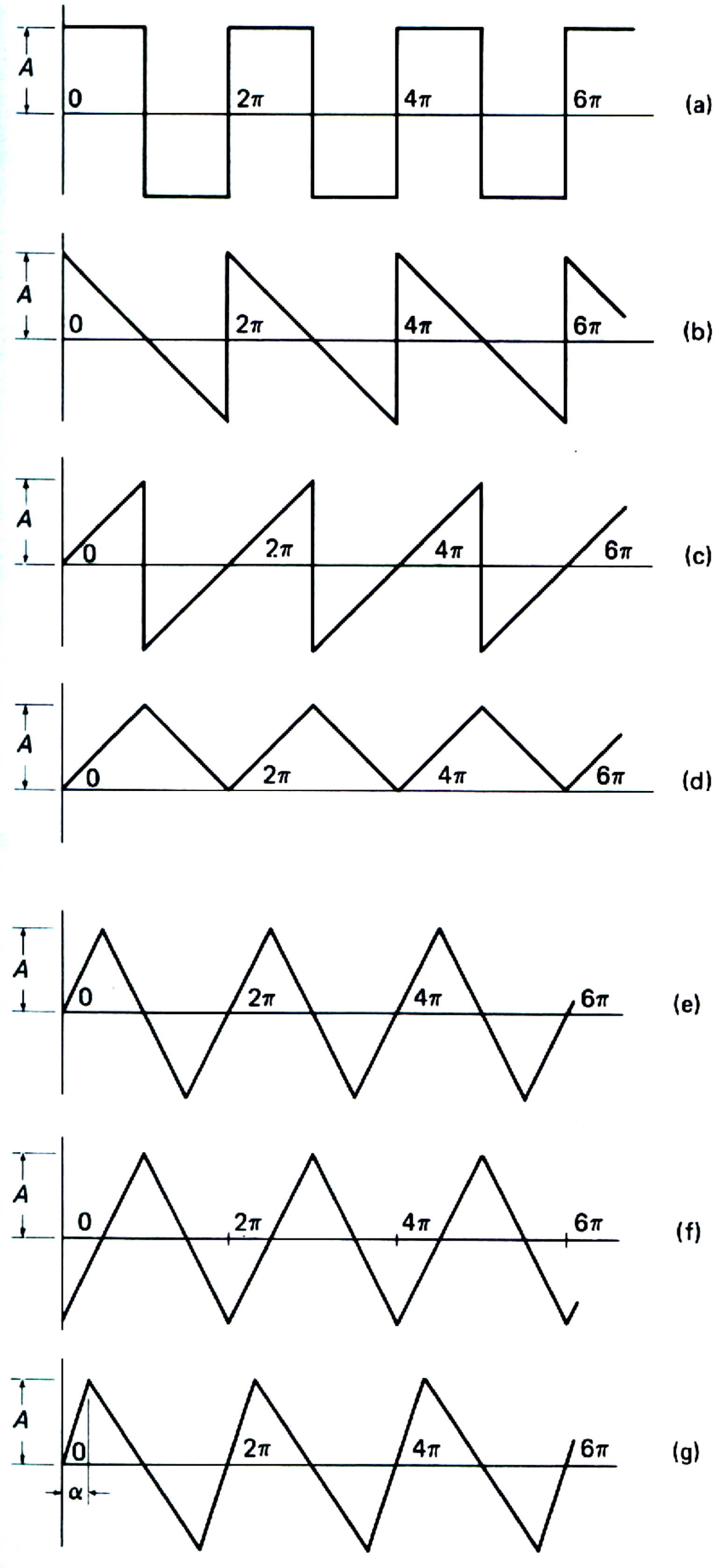

Figure 4.4 Various special waveforms of harmonic nature. In each case, the ordinate is y and the abscissa is *ωt*.

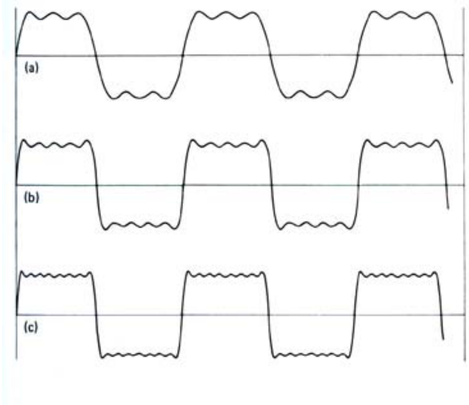

Figure 4.5 Plot of square-wave function: (a) plot of first three terms only (includes the fifth harmonic), (b) plot of the first five terms (includes the ninth harmonic), (c) plot of the first eight terms (includes the fifteenth harmonic)

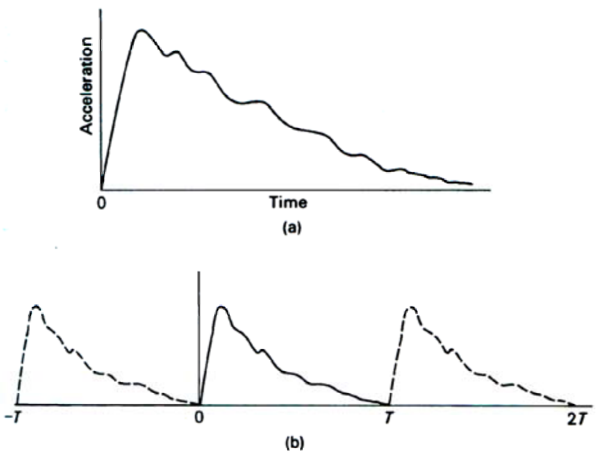

Figure 4.6 (a) Acceleration-time relationship resulting from shock-test,
(b) considering the nonrepeating function as one real cycle of a periodic relationship

Examples 4.1 and 4.2 illustrate the analytical calculation of the equations for Fig. 4.4(a) and (b).

```
%MEG 381  ;
%Fourier Series;
%Chapter 4;Figure 4.9 a;
%Table 4.1 Eqn 4.9(a)
%--------------------------------------------------
%Dr. Mostafa S/. Habib
%------------------------------------------------------
A=1;N=3 ;w=10;f=w/2/pi;T=1/f
t=0:.001:3*T;
y=0;
for n=1:N;  %Loop to add harmonics n=1
```

```
        %to n=N for each time step 1
         y = y+1/(2*n-1)*sin((2*n-1)*w*t); % 4.9 a
        % y = y+1./(n).*sin((n).*w.*t); % 4.9 b
end
y=4*A/pi*y;
figure;
plot(t,y);grid;
xlabel('time, t sec');
ylabel('y')
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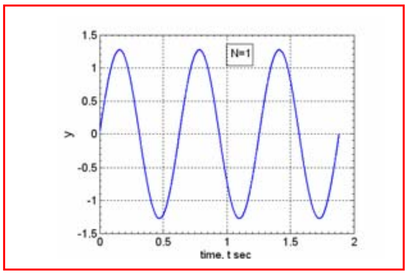
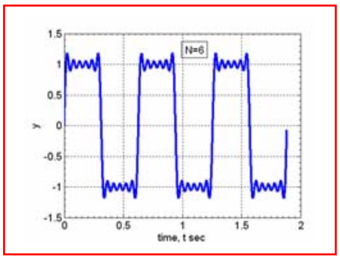
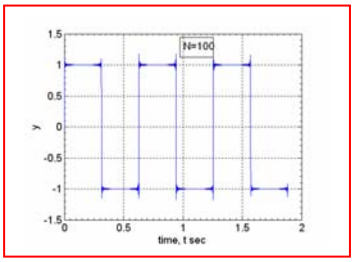






4.3 Nonperiodic or Transient Waveforms

In the foregoing examples of special waveforms, various combinations of harmonic components were used. In each case the result was a periodic relation repeating indefinitely in every detail. Many mechanical inputs are not repetitive—for example, consider the acceleration-time relation resulting from an impact test [Fig. 4.6(a)]. Although such a relation is transient, it may be thought of as one cycle of a periodic relation in which all other cycles are fictitious [Fig. 4.6(b)]. On this basis, nonperiodic functions may be analyzed in exactly the same manner as periodic functions. If the nonperiodic waveform is sampled for a time period T, then the fundamental frequency of the fictitious periodic wave is $f = 1/T$ (cyclic) or $\omega = 2\pi/T$ (circular frequency).

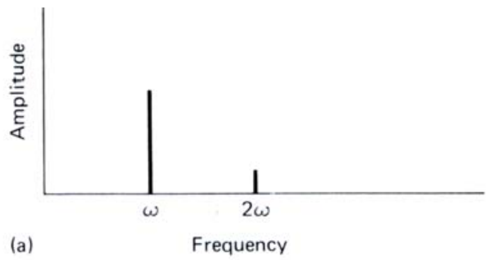
4.4 Frequency Spectrum

Figures 4.1 through 4.6 are plotted using time as the independent variable. This is the most common and familiar form. The waveform is displayed as it would appear on the face of an ordinary *cathode-ray oscilloscope* or on the paper of a strip-chart recorder. A second type of plot is the *frequency spectrum*, in which frequency is the independent variable and the amplitude of each frequency component is displayed as the ordinate. For example, the frequency spectra for the signals:

$$f(t)=10\sin\omega t + 2\sin 2\omega t \quad \text{and} \quad f(t)=10\sin\omega t + 8\sin 2\omega t$$

are shown in Fig. 4.7, respectively and the frequency spectrum for the square wave is shown in Fig. 14.9.
The frequency spectrum is useful because it allows us to identify at a glance the frequencies present in a signal. For example, if the waveform results from a vibration test of a structure, we could use the frequency spectrum to identify the structure's natural frequencies.
The application of frequency spectrum plots has increased greatly since the development of the spectrum analyzer and fast Fourier transform. The spectrum analyzer is an electronic device that displays the frequency spectrum on a cathode-ray

screen. The fast Fourier transform (FFT) is a computer algorithm that calculates the frequency spectrum from computer- acquired data.



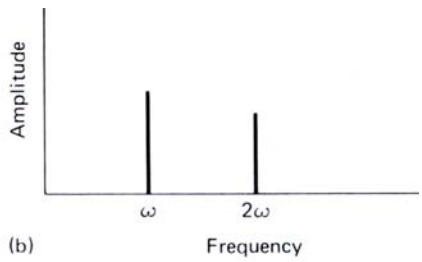
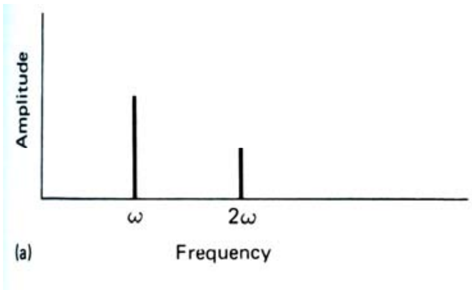


Figure 4.7 (a) Frequency spectrum corresponding to $f(t)=10 \sin\omega t + 2 \sin 2\omega t$,
(b) Frequency spectrum corresponding to $f(t)=10 \sin\omega t + 8 \sin 2\omega t$



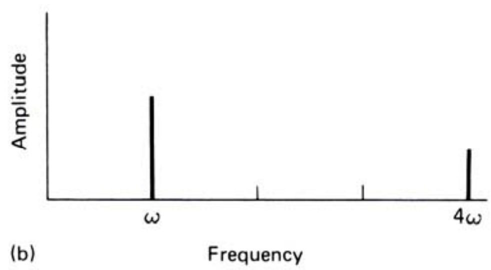


Figure 4.8 (a) Frequency spectrum corresponding to $f(t)=10 \sin\omega t + 5 \sin 2\omega t$
(b) Frequency spectrum corresponding to $f(t)=10 \sin\omega t + 5 \sin 4\omega t$

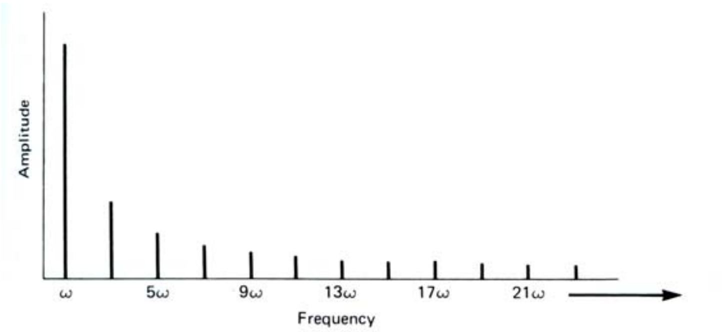


Figure 4.9 Frequency spectrum for the square wave shown in Fig. 4.4(a)

4.5 Discrete Fourier Analysis

In the preceding sections, we saw how known combinations of waves could be summed to produce more complex waveforms. In an experiment, the task is reversed: We measure the complex waveform and seek to determine which frequencies are present in it! The process of determining the frequency spectrum of a known waveform is called *harmonic analysis* (*spectral analysis*), or *Fourier analysis*.

Fourier analysis is a branch of classical mathematics on which entire textbooks have been written (see section 4.2). The basic equations are derived in section 4.2, and more detailed discussions are available in the Suggested Readings for this chapter.

The key to harmonic analysis is that the harmonic coefficients in Eq. (4.2.11) and (4.2.12) are integrals of the waveform *f(t)*:

$$A_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} y(t)\cos(\omega nt)\,dt \qquad n = 0, 1, 2, \ldots, \tag{4.2.11}$$

$$B_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} y(t)\sin(\omega nt)\,dt \qquad n = 1, 2, \ldots. \tag{4.2.12}$$

These relations are reciprocal to Eq. (4.2.3). When the harmonic coefficients are already known, Eq. (4.2.3) can be summed to obtain f*(t)*. Conversely, when f(t) is known (as from an experiment), the integrals can be evaluated to determine the harmonic coefficients.
Experimentally, the waveform is usually measured only for a finite time period *T*. It turns out to be more convenient to write the integrals in terms of this time period, rather than the fundamental circular frequency, ω , that is $\omega = 2\pi/T$, the integrals are just

$$A_n = \frac{2}{T}\int_0^{T} y(t)\cos\left(\frac{2\pi}{T}nt\right)dt \qquad n = 0, 1, 2, \ldots,$$

$$B_n = \frac{2}{T}\int_0^{T} y(t)\sin\left(\frac{2\pi}{T}nt\right)dt \qquad n = 1, 2, \ldots.$$

Practical harmonic analysis usually falls into one of the following four categories:
1. The waveform f(t) is known mathematical function. In this case, the integrals (4.2.11) and (4.2.12) can be evaluated analytically. These calculations are illustrated through examples.

2. The waveform *f(t)* is an analog signal from a transducer. In this case, the waveform may be processed with an electronic spectrum analyzer to obtain the signal's spectrum.

3. Alternatively, the analog waveform may be recorded by a digital computer. The computer will *store* f*(t)* only at a series of *discrete points* in time. Integrals (4.2.11) and (4.2.12) are replaced by sums and evaluated, as discussed in the next section.

4. The waveform is known graphically, for instance, from a strip-chart recorder or the screen of an oscilloscope. In this case, *f(t)* may be read from the graph at a discrete series of points, and the integrals may again be evaluated as sums.

4.5.1 The Discrete Fourier Transform (DFT)
The case when *f(t)* (or say *y(t)*) is known only at discrete points in time is very important in practice because of the wide use of computers and microprocessors for recording signals. Normally, a computer will read and store signal input at time intervals of Δt (Fig. 4.10). The computer records a total of *N* points over the time period $T = N\,\Delta t$. Therefore, in the computer's memory, the analog signal *y(t)* has been reduced to a series of points measured at times $t = \Delta t,\ 2\,\Delta t, \ldots, N\,\Delta t$, specifically,

y(Δt), y(2 Δt), . . . , y(N Δt). We can write this series more compactly as $y(t_r)$ by setting $t_r = r\,\Delta t$ *for* $r = 1, 2, \ldots, N$.
To perform a Fourier analysis of a discrete time signal like this, the integrals in Eqs. (4.2.11) and (4.2.12) must be replaced by sums. Likewise, the continuous time *t* is replaced by the discrete time $t_r = r\,\Delta t$, and the period *T* is replaced by $N\,\Delta t$. Making these substitutions in Eq. (4.2.11), we get

$$A_n = \frac{2}{N\,\Delta t}\sum_{r=1}^{N} y(t_r)\cos\left(\frac{2\pi}{N\,\Delta t}nr\,\Delta t\right)\Delta t$$
$$= \frac{2}{N}\sum_{r=1}^{N} y(r\,\Delta t)\cos\left(\frac{2\pi rn}{N}\right).$$

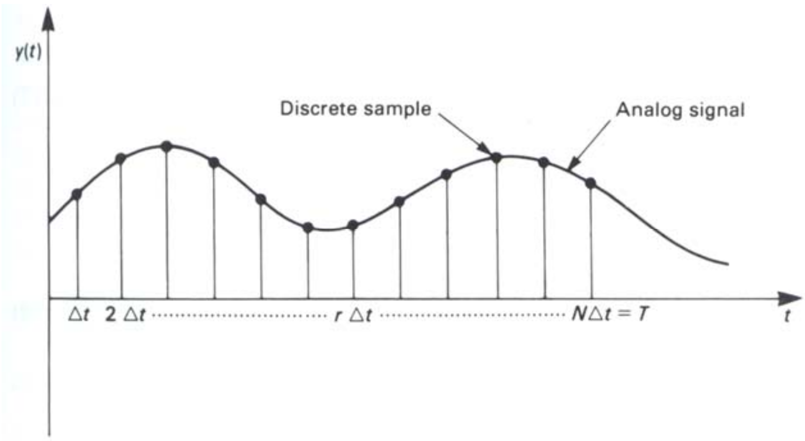


Figure 4.10 Discrete sampling of a continuous analog signal. The value of the signal is recorded at intervals Δt apart for a period *T*.

Thus, the harmonic coefficients of a discretely sampled waveform are

$$A_n = \frac{2}{N}\sum_{r=1}^{N} y(r\,\Delta t)\cos\left(\frac{2\pi rn}{N}\right) \qquad n = 0, 1, \ldots, \frac{N}{2} \tag{4.5.1}$$

$$B_n = \frac{2}{N}\sum_{r=1}^{N} y(r\,\Delta t)\sin\left(\frac{2\pi rn}{N}\right) \qquad n = 1, 2, \ldots, \frac{N}{2} - 1 \tag{4.5.2}$$

for *N* an even number. The corresponding expression for the discrete waveform $y(t_r)$

$$y(t_r) = \frac{A_0}{2} + \sum_{n=1}^{N/2-1}\left[A_n\cos\left(\frac{2\pi rn}{N}\right) + B_n\sin\left(\frac{2\pi rn}{N}\right)\right] + \frac{A_{N/2}}{2}\cos(\pi r). \tag{4.5.3}$$

Equations (4.5.1) and (4.5.2) are called the discrete Fourier transform (DFT) of y(tr). Equation (4.5.3) is called the discrete Fourier series.
In practice, the discrete sample is taken by an analog-to-digital converter connected to a computer or a microprocessor-driven electronic spectrum analyzer. The computer or

microprocessor evaluates the sums, Eqs. (4.5.1) and (4.5.2), often by using the fast Fourier transform algorithm. (The fast Fourier transform, or FFT, algorithm is special factorization of these sums that applies when N is a power of 2 ($N = 2^m$). The number of calculations normally required to evaluate these sums is proportional to N^2 when the FFT algorithm is used, the number is proportional to N $\log_2 N$. Thus, the FFT requires less computer work when N is large.) The result, which approximates the spectrum of the original analog signal, is then displayed.
Like the ordinary Fourier series [Eq. (4.2.3)], the discrete Fourier series expresses y(t) as a sum of frequency components, since

$$\frac{2\pi rn}{N} = 2\pi\left(\frac{n}{N\,\Delta t}\right)r\,\Delta t = 2\pi(n\,\Delta f)t_r$$

for a fundamental cyclic frequency

$$\Delta f \equiv \frac{1}{N\,\Delta t} \tag{4.5.4}$$

and harmonic orders *n = 1,. . . , N/2*. In other words,

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{N/2-1}[A_n\cos(2\pi n\,\Delta f\,t) + B_n\sin(2\pi n\,\Delta f\,t)] + \frac{A_{N/2}}{2}\cos\left(2\pi\frac{N\,\Delta f}{2}t\right). \tag{4.5.5}$$

Note that the DFT yields only harmonic components up to *n = N/2*, whereas the ordinary Fourier series [Eq. (4.2.3)] may have an infinite number of frequency components. This very important fact is a consequence of the discrete sampling process itself.

4.5.2 Frequencies in Discretely Sampled Signals: *Aliasing and Frequency Resolution*

When an analog waveform is recorded by discrete sampling, some care is needed to ensure that the waveform is accurately recorded. The two sampling parameters that we can control are the sample rate, $f_s = 1/\Delta t$, which is the frequency with which samples are recorded, and the number of points recorded, *N*. Typically, the software controlling the data-acquisition computer will request values off, and N as input.
Figure 4.11 shows two examples of sampling a particular waveform. In Fig. 4.11(a), the sample rate is low (At is large), and as a result the high frequencies of the original waveform are not well resolved by the discrete samples—the signal seen in the discrete sample (the dashed curve) does not show the sharp peaks of the original waveform. The total time period of sampling is also fairly short (N is small), and thus the low frequencies of the signal are missed as well; it isn't clear how often the signal repeats itself. In Fig. 4.11(b), the sample rate and the number of points are each increased, improving the resolution of both high and low frequencies. These figures illustrate the importance of sample rate and total sampling time period in determining how well a discrete sample represents the original waveform.
What is the minimum sample rate needed to resolve a particular frequency? Consider the cases shown in Fig. 4.12, where a signal of frequency f is sampled at increasing

rates. In (a), when the waveform is sampled at a frequency of f, = f, the discretely sampled signal appears to be constant! No frequency is seen. In (b), the waveform is sampled at a higher rate, between *f* and *2f*; the discrete signal now appears to be a wave, but it has a frequency lower than f. In (c), the waveform is sampled at a rate *f* = *2f*, and the discrete sample appears to be a wave of the correct frequency, *f*. Unfortunately, if the sampling begins a quarter-cycle later at this same rate [(d)], then the signal again appears to be constant. Only when the sample rate is increased above *2f*, as in (e), do we always obtain the correct signal frequency with the discrete sample.

The highest frequency resolved at a given sampling frequency is determined by the Nyquist frequency,

$$f_{\mathrm{Nyq}} = \frac{f_s}{2}. \tag{4.5.6}$$

Signals with frequencies lower than $f_{Nyq} = f_s /2$ are accurately sampled. Signals with frequencies greater than f_{Nyq} are not accurately sampled; the frequencies above the Nyquist frequency incorrectly appear as lower frequencies in the discrete sample. The phenomenon of a discretely sampled signal taking on a different frequency, as in Fig. 4.12(b), is called aliasing. Aliasing occurs whenever the Nyquist frequency falls below the signal frequency. Furthermore, the phase ambiguity shown by Fig. 4.12(c) and (d) prohibits sampling at the Nyquist frequency itself. To prevent these problems, the sampling frequency should always be chosen to be more than twice a signal's highest frequency.

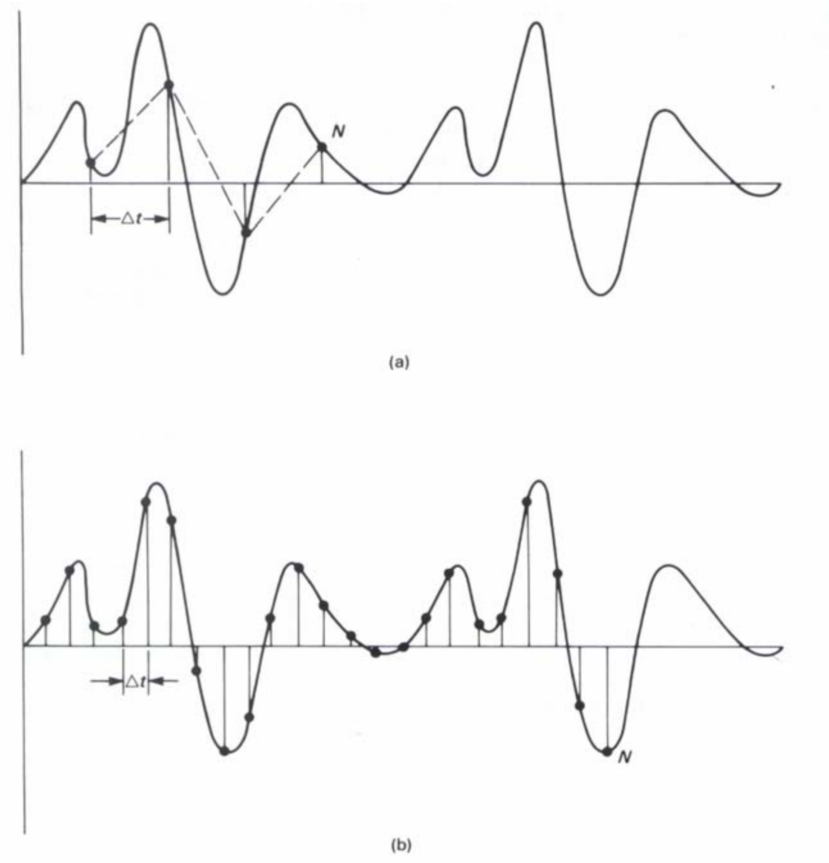


Figure 4.11 Effect of sample rate, f_s and number of samples taken:
(a) undersampled: both sample rate and number of points are too low,
(b) resolution of waveform is improved by raising the sample rate and number of points

The Nyquist frequency tells us how to resolve correctly the highest frequencies of a signal. It also tells us why the DFT contains only a finite umber of frequency components: Frequencies higher than the Nyquist are too fast to be resolved with samples taken at the given sample rate.
In a similar fashion, we can determine the lowest nonzero frequency in the DFT and justify the discrete spacing of frequencies. If a waveform is to be resolved by discrete sampling, one or more *full periods* of that waveform must be present in the discrete sampling period, as shown in Fig. 4.13(a). Since the discrete sampling period, as shown for one wave in Fig. 4.13a. Since the period of sampling is $T = N\,\Delta t = N/f_s$, the frequency of this wave is:

$$f_{\text{lowest}} = \frac{1}{T} = \frac{1}{N\,\Delta t} = \frac{f_s}{N} \equiv \Delta f \qquad (4.5.7)$$

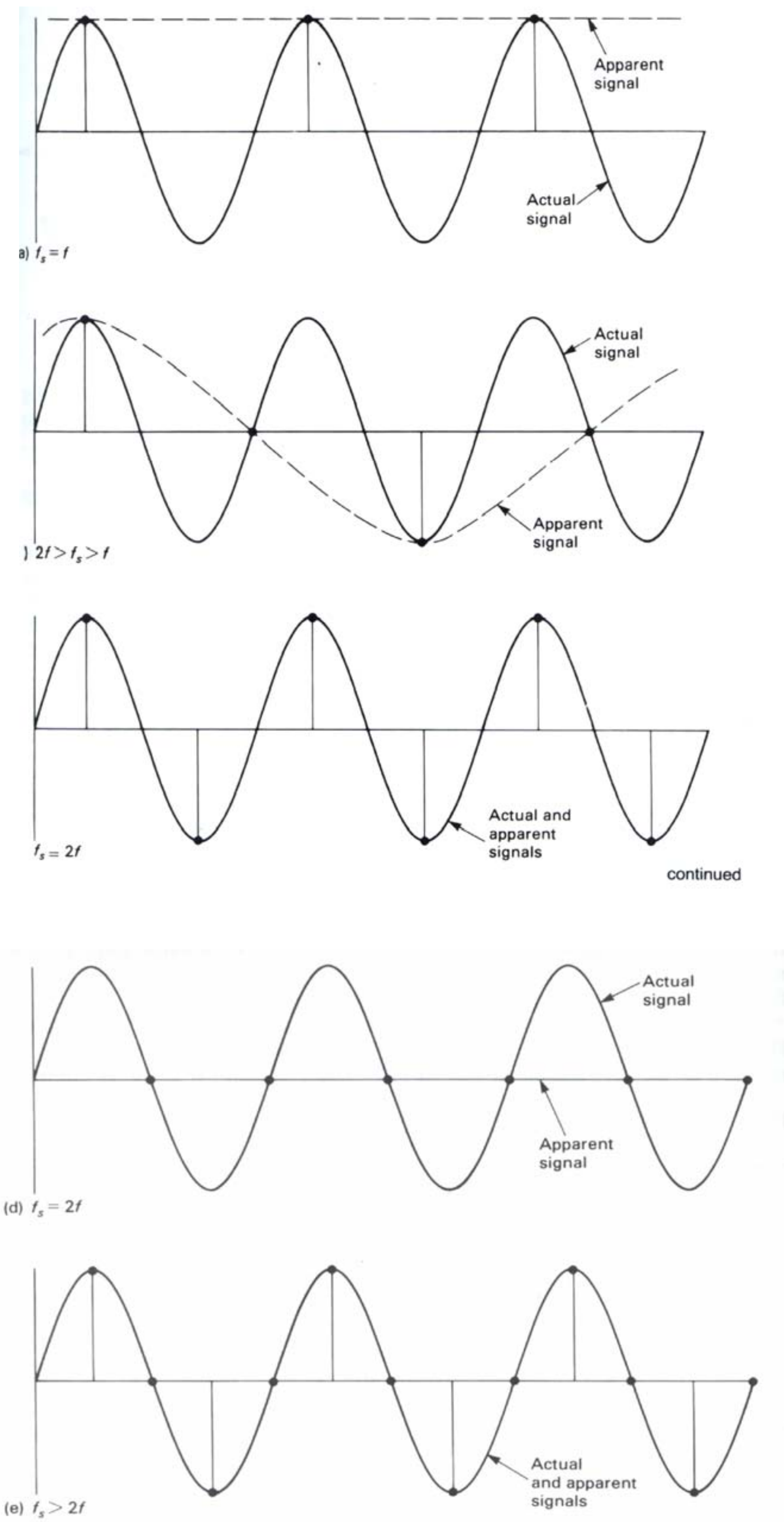


Figure 4.12 Effect of varying the sample rate, f_s on the apparent signal obtained by discrete sampling

Thus the fundamental frequency of the DFT, Δf, is also that of the lowest-frequency full wave that fits within the sampling period. No lower frequency (other than $f = 0$) is resolved.

The next-lowest frequency is that for which two full waves fit in the sampling period [Fig. 4.13(b)]. Since two periods of the wave equal the sampling period, the wave's frequency is

$$f_2 = \frac{2}{T} = 2\frac{f_s}{N} = 2\,\Delta f.$$

We can continue adding full waves to show that the only frequencies resolved by the DFT are

$$0,\ \Delta f,\ 2\,\Delta f,\ \ldots,\ n\,\Delta f,\ \ldots,\ \frac{N}{2}\Delta f = f_{\mathrm{Nyq}}.$$

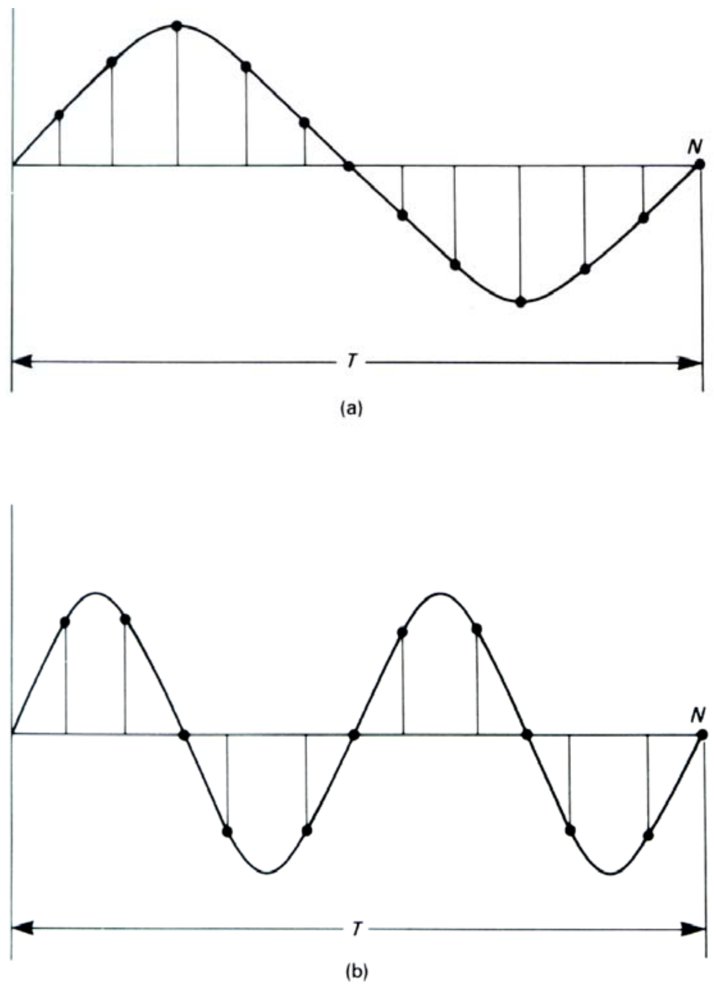


Figure 4.13 Resolving low frequencies:
(a) one full wave in the sampling period,
(b) two full waves in the sampling period

Note that although the Nyquist frequency itself is present in the DFT, it may not correctly represent the underlying signal, owing to phase ambiguity.
The frequencies of the DFT are spaced in increments of Δf, and thus Δf is sometimes called the *frequency resolution* of the DFT. If an analog signal contains a frequency f_o that lies between two resolved frequencies, say $n\Delta f < f_o < (n + 1)\,\Delta f$, then this frequency component will *"leak"* to the adjacent frequencies of the DFT. The adjacent frequencies can each show some contribution from f_o. As a result, each frequency component observed in the DFT has an uncertainty in frequency of approximately $\pm\Delta f/2$ (95%) relative to the frequencies actually present in the original analog signal. We can reduce leakage and sharpen the peaks in the frequency spectrum by decreasing Δf.
This discussion leads us to the following steps for accurate discrete sampling:

1. First, estimate the highest frequency in the signal and choose the Nyquist frequency to be greater than it. In other words, make the sample rate, greater than twice the highest frequency in the signal.
2. If limitations in the sample rate force you to pick a Nyquist frequency less than the highest frequency in the signal, then use a low-pass filter to block frequencies greater than the Nyquist frequency while sampling.
3. After the sample rate is chosen, estimate the lowest frequency in the signal or estimate the frequency resolution needed to accurately resolve the frequency components in the signal. Then choose the number of points in the sample, N, to yield the desired $\Delta f = f_s/N$ at the previously determined value of the sampling frequency, f.

4.6 Amplitudes of Waveforms

The magnitude of a waveform can be described in several ways. The simplest waveform is a sine or cosine wave:

$$V(t) = Va \sin 2\pi f t.$$

The amplitude of this waveform is Va. The peak-to-peak amplitude is 2Va.
On the other hand, we may want a time-average value of this wave. If we simply average it over one period, however, we obtain an uninformative result:

$$\bar{V} = \frac{1}{T}\int_0^T V_a \sin 2\pi f t\, dt = -\frac{V_a}{2\pi}(\cos 2\pi - \cos 0) = 0.$$

The net area beneath one period of a sine wave is zero. Thus it is more useful to work with a root-mean-square (rms) value:

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T V^2(t)\, dt} = \sqrt{\frac{1}{T}\int_0^T V_a^2 \sin^2 2\pi f t\, dt} = \cdots = \frac{V_a}{\sqrt{2}}.$$

For more complex waveforms, the frequency spectrum provides a complete description of the amplitude of each individual frequency component in the signal. However, the spectrum can be cumbersome to use, and a single time-average value is often more convenient to work with. For this reason, the rms amplitude is generally applied to complex waveforms as well. Meters that measure the rms value of a waveform are discussed later

NB: The materials in this chapter are combined from the text book and Math references.

Suggested Readings

Churchill, R. V., and J. W. Brown. Fourier Series and Boundary Value Problems. 3rd ed. New York: McGraw-Hill, 1978.
Greenberg, M. D. Foundations of Applied Mathematics. Englewood Cliffs, N.J.: Prentice Hall, 1978.
Oppenheim, A. V., A. S. Willsky, and I. T. Young. Signals and Systems. Englewood Cliffs, N.J.: Prentice Hall, 1983.

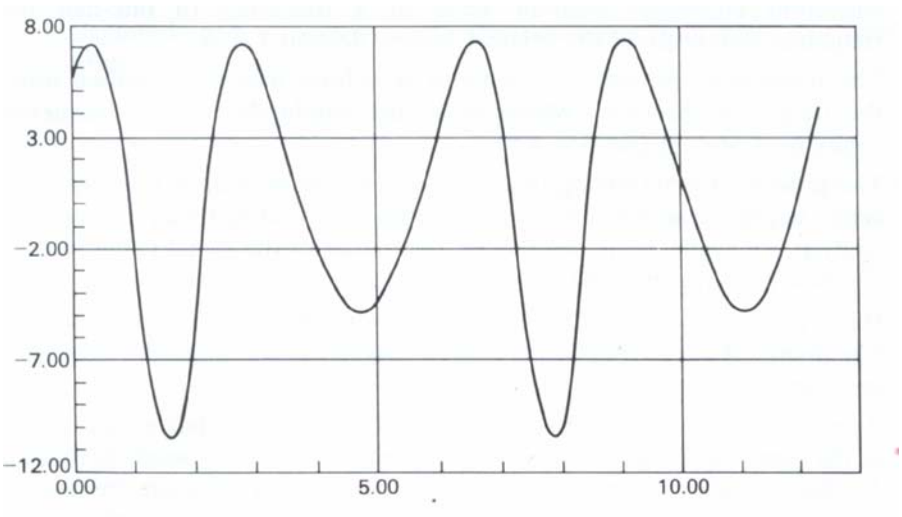
Problems

4.1 The following expression represents the displacement of a point as a function of time:

$$y(t) = 100 + 95 \sin 15t + 55 \cos 15t.$$

a. What is the fundamental frequency in hertz?
b. Rewrite the equation in terms of cosines only.
4.2 Rewrite each of the following expressions in the form of Eq. (4.6).
a. $y = 3.2\cos(0.2t - 0.3) + \sin(0.2t + 0.4)$ b. $y = 12 \sin(t - 0.4)$

Figure 4.14 Oscilloscope trace for Problem 4.8



4.3 Construct a frequency spectrum for Fig. 4.9(a).

4.4 Construct a frequency spectrum for Fig. 4.9(c).

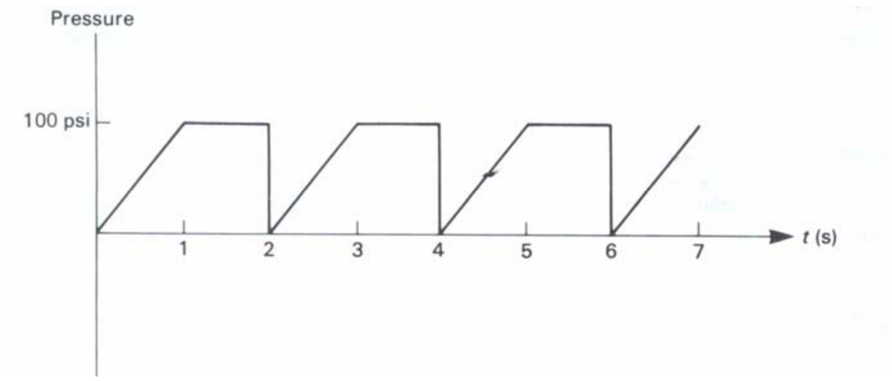
4.5 Construct a frequency spectrum for Fig. 4.9(e).

4.6 Construct a frequency spectrum for Fig. 4.9(f).

4.7 Construct a frequency spectrum for Fig. 4.9(g).

4.8 Figure 4.14 represents a trace from an oscilloscope where the ordinate is in vol and the abscissa is in milliseconds. Determine its discrete Fourier transform ai sketch its frequency spectrum. Use a sampling frequency of 2000 Hz. Check I periodicity in choosing your data window.

Figure 4.15 Pressure-time record for Problem 4.9



4.9 Consider a pressure-time record as shown in Fig. 4.15. Determine its frequency spectrum.

4.10 If the signal of Problem 4.9 were to be sampled digitally for its discrete Fourier transform, what sampling frequency would you recommend?

4.11 Solve Problem 4.8 using a sampling frequency of 10 Hz and compare your results with those of Problem 4.8.

4.1.2 Using the data files in Table 4.4, if t is in milliseconds and fQ) is in volts, determine the discrete Fourier transform for each set of digital data.
a. Use f1(t).
b. Use f2(t).
c. Use f3(t).
d. Use f4(t).
e. Use f5(t).
f. Use f6(t).

4.13 A 500 Hz sine wave is sampled at a frequency of 4096 Hz. A total of 2048 points are taken.
a. What is the Nyquist frequency?
b. What is the frequency resolution?
c. The student making the measurement suspects that the sampled waveform contains several harmonics of 500 Hz. Which of these can be accurately measured? What happens to the others?

4.14 A 150 Hz cosine wave is sampled at a rate of 200 Hz.
a. Draw the wave and show the temporal locations at which it is measured.
b. What apparent frequency is measured?
c. Describe the relation of the measured frequency to aliasing. Give a numerical justification for your answer.

4.15 a. Suppose that a 500 Hz sinusoidal signal is sampled at 750 Hz. Draw the discrete time signal found and determine the apparent frequency of the signal.
b. If a 200 Hz component were present in the signal of part a, would it be detected? Explain.
c. If a 375 Hz component were present in the signal of part a, would it be detected? Explain.

4.16 A engineer is studying the vibrational spectrum of a large diesel engine. Her modeling estimates suggest that a strong resonance is likely at 250 Hz, and that weaker frequeicies of up to 2000 Hz may be excited also. She has placed an accelerometer on the machine to measure the vibration spectrum. She samples the accelerometer output voltage using her computer's analog-to-digital converter board.

a. What is the minimum sample rate she should use?
b. To reliably test her model of the machine's vibration, she must resolve the peak resonant frequency to ±1 Hz. How can she achieve this level of resolution?

4.17 A temperature measuring circuit responds fully to frequencies below 8.3 kHz; above this frequency, the circuit attenuates the signal. This circuit is to be used to measure a temperature signal with an unknown frequency spectrum. Accuracy of ±1 Hz is desired in the frequency components. If no frequency components above 8.3 kHz are present in the circuit's output, what sample rate and number of samples should be used?

Table 4.2 Data for Problem 4.12

t	$f_1(t)$	$f_2(t)$	$f_3(t)$	$f_4(t)$	$f_5(t)$	$f_6(t)$
0	5.4	3.76	10.2	4	10.1	−9.4
1	4.74	3.88	10	2.58	9.34	−6.8
2	3.01	4.19	9.83	0.99	8.32	−3.5
3	0.8	4.54	9.57	1.54	7.05	−0.2
4	−1.2	4.67	9.26	4.4	5.57	2.52
5	−2.4	4.46	8.92	8.02	3.91	4.11
6	−2.7	4.06	8.56	10.7	2.14	4.58
7	−2.4	3.73	8.21	11.6	0.3	4.17
8	−1.8	3.6	7.89	11.2	−1.6	3.3
9	−1.6	3.55	7.62	10	−3.4	2.47
10	−1.8	3.35	7.43	8.52	−5.1	2.03
11	−2.4	2.89	7.34	7.19	−6.6	2.11
12	−2.7	2.36	7.36	6.66	−7.9	2.57
13	−2.4	2.01	7.5	7.3	−9	3.05
14	−1.2	1.98	7.74	8.45	−9.9	3.11
15	0.8	2.14	8.07	8.46	−10	2.36
16	3.01	2.24	8.48	5.85	−11	0.64
17	4.74	2.16	8.94	0.9	−11	−1.9
18	5.4	2.06	9.41	−4	−10	−4.9
19	4.74	2.16	9.87	−6	−9.3	−7.6
20	3.01	2.57	10.3	−4.5	−8.3	−9.3
21	0.8	3.13	10.6	−1.5	−7.1	−9.5
22	−1.2	3.56	10.9	−0.9	−5.6	−7.9
23	−2.4	3.73	11.1	−4.6	−3.9	−4.5
24	−2.7	3.76	11.2	−11	−2.1	0.06
25	−2.4	3.88	11.3	−15	−0.3	4.93
26	−1.8	4.19	11.2	−15	1.55	9.15
27	−1.6	4.54	11.2	−10	3.35	11.8
28	−1.8	4.67	11.1	−5.1	5.05	12.4
29	−2.4	4.46	10.9	−3.7	6.59	10.7
30	−2.7	4.06	10.8	−6.7	7.94	7.1
31	−2.4	3.73	10.7	−11	9.04	2.29
32	−1.2	3.6	10.6	−12	9.87	−2.7
33	0.8	3.55	10.5	−8.5	10.4	−7
34	3.01	3.35	10.4	−2.4	10.6	−9.7
35	4.74	2.89	10.3	2.57	10.5	−11
36	5.4	2.36	10.2	4	10.1	−9.4

References

1. Beckwith, T. G., and R. D. Marangoni. Mechanical Measurements. 4th ed. Reading, Mass.: Addison-Wesley, 1990.
2. Oppenheim, A. V., A. S. Willsky, and I. T. Young. Signals and Systems. Englewood Cliffs, N.J.: Prentice Hall, 1983.