Applied Signal Processing
Lecture 4

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Agenda

- Discrete Fourier Transform (DFT)
- Inverse Discrete Fourier Transform (IDFT)
- DFT a linear invertible operator
- Filtering using DFT
- Equalization

DFT of a length N sequence $\{x(n)\}_{n=0}^{N-1}$ is defined as

$$\text{DFT:} \quad X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

- index k corresponds to a frequency $\omega_k = \frac{2\pi k}{N\Delta t} = \frac{k}{N}\omega_s$
- $X(k)$ is $N-$periodic, i.e. $X(k) = X(k+N)$.

Relation between DTFT and DFT

Assume $x(n) = 0$ for $n < 0$ and $n > N - 1$.

For DTFT we have

$$X(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega \Delta t n} = \sum_{n=0}^{N-1} h(n) e^{-j\omega \Delta t n}$$

If we set $\omega = \omega_k = \frac{2\pi k}{N \Delta t}$ we obtain

$$X(\omega_k) = \sum_{n=0}^{N-1} h(n) e^{-j\frac{2\pi k}{N\Delta t}\Delta t n} = \sum_{n=0}^{N-1} h(n) e^{-j\frac{2\pi kn}{N}} = X(k)$$

i.e. the DFT of $x(n)$.

DFT is hence equidistant samples of the function $X(\omega)$ at ω_k.

Inverse Discrete Fourier Transform

IDFT of a length N sequence $\{X(k)\}_{k=0}^{N-1}$ is defined as

$$\text{IDFT:} \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}}$$

- index k corresponds to a frequency $\omega_k = \frac{2\pi k}{N\Delta t} = \frac{k}{N}\omega_s$
- $x(n)$ is $N-$periodic, i.e. $x(n) = x(n+N)$.

$$
\begin{aligned}
\mathrm{IDFT}[X(k)] &= \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi km}{N}}\right) e^{j\frac{2\pi kn}{N}} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} x(m) \underbrace{\sum_{k=0}^{N-1} e^{j\frac{2\pi k(n-m)}{N}}}_{\begin{cases} N, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}} \\
&= x(n)
\end{aligned}
$$

DFT is a linear invertible operator

We have

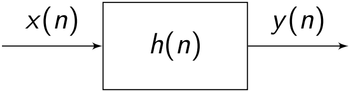
$$\mathbf{Y} \triangleq \begin{bmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(N-1) \end{bmatrix} = \mathbf{F} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \triangleq \mathbf{F}\mathbf{x}$$

where

$$\mathbf{F} = \begin{bmatrix} z_0 & z_0^1 & \cdots & z_0^{N-1} \\ z_1 & z_1^1 & \cdots & z_1^{N-1} \\ \vdots & \vdots & & \vdots \\ z_{N-1} & z_{N-1}^1 & \cdots & z_{N-1}^{N-1} \end{bmatrix}$$

and $z_k = e^{-j2\pi k/N}$. The inverse is

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{Y} = \frac{1}{N}\bar{\mathbf{F}}^T\mathbf{Y} \quad \text{Easy to calculate!}$$



$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

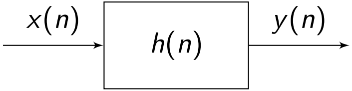
Filtering: How to determine $y(n)$ given $h(n)$ and $x(n)$?

Equalization: How to determine $x(n)$ given $h(n)$ and $y(n)$?

System Identification: How to determine $h(n)$ given $x(n)$ and $y(n)$?

Note that Equalization and System Identification are similar problems! Why?

System response to a periodic input



Assume $h(k)$ is zero for $k < 0$ and $k > N-1$. We have

$$y(n) = \sum_{k=0}^{N-1} h(k)x(n-k)$$

If $x(n) = x(n+N)$ then we get $y(n) = y(n+N)$.

Output is also periodic with same period!

$$x(n) \longrightarrow \boxed{\;h(n)\;} \xrightarrow{\;y(n)\;}$$

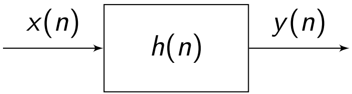
The convolution can be expressed as

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} = \mathbf{H}_c \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

where the *circulant* matrix $\mathbf{H}_c$ has the structure

$$\mathbf{H}_c = \begin{bmatrix} h(0) & h(N-1) & h(N-2) & \cdots & h(1) \\ h(1) & h(0) & h(N-1) & \cdots & h(2) \\ h(2) & h(1) & h(0) & \cdots & h(3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h(N-1) & h(N-2) & h(N-3) & \cdots & h(0) \end{bmatrix}.$$

System response to a periodic input



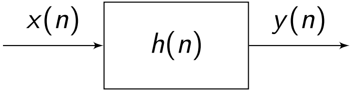
Assume for all n, $x(n) = x(n+N)$. From before we can then express $x(n)$ as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\omega_k \Delta t n}$$

where $\omega_k = \frac{2\pi k}{N\Delta t}$. The output is then

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(\omega_k) X(k) e^{j\omega_k \Delta t n}$$

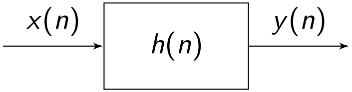
where $H(\omega_k) = \mathrm{DTFT}[h(n)]\big|_{\omega=\omega_k}$



Periodic case

1. Calculate $X(k) = \text{DFT}[x(n)]$ for one period of $x(n)$
2. Determine DFT of output $Y(k) = H(\omega_k)X(k)$
3. Calculate one period of output $y(n) = \text{IDFT}[Y(k)]$.

Filtering finite signals



Assume

$$
\begin{aligned}
x(n) &= 0, \quad n < 0 \text{ and } n > N-1, \quad \text{i.e. length } N \\
h(n) &= 0, \quad n < 0 \text{ and } n > M-1, \quad \text{i.e. length } M
\end{aligned}
$$

The output $y(n)$ is given by

$$
y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)
$$

which imply

$$
y(n) = 0, \quad n < 0 \text{ and } n > N+M-2 \quad \text{i.e. length } N+M-1 \triangleq P
$$

We can express the filtering as a linear operator $\mathbf{H}$

$$\mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(P-1) \end{bmatrix} = \mathbf{H} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \mathbf{Hx}$$

where the matrix $\mathbf{H}$ has the structure

$$\mathbf{H} = \begin{bmatrix} h(0) & 0 & 0 & \cdots & 0 \\ h(1) & h(0) & 0 & \cdots & 0 \\ \vdots & & & & \\ h(M-1) & h(M-2) & h(M-3) & \cdots & h(0) \\ 0 & h(M-1) & h(M-2) & \cdots & h(1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h(0) \end{bmatrix}.$$

We notice for a signal $y(n)$ of length P

$$Y(\omega) = \text{DTFT}[y(n] = \sum_{n=-\infty}^{\infty} y(n)e^{-j\omega\Delta tn} = \sum_{n=0}^{P-1} y(n)e^{-j\omega\Delta tn}$$

which if we set $\omega = \omega_k = \frac{2\pi k}{P\Delta t}$ we get

$$Y(\omega_k) = \sum_{n=0}^{P-1} y(n)e^{-j\frac{2\pi kn}{P}} = Y(k)$$

Hence, if we know $Y(\omega_k)$ for $k = 0, 1, \dots, P-1$ we can derive $y(n)$ from the IDFT. From DTFT theory we know

$$Y(\omega_k) = H(\omega_k)X(\omega_k)$$

so we need to obtain $H(\omega_k)$ and $X(\omega_k)$.

We want $H(\omega_k)$ and $X(\omega_k)$ for $\omega_k = \frac{2\pi k}{P\Delta t}$.

$$H(\omega_k) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\frac{2\pi k}{P\Delta t}\Delta t n} = \sum_{n=0}^{M-1} h(n)e^{-j\frac{2\pi k}{P}n}$$

$$= \sum_{n=0}^{P-1} h_{zp}(n)e^{-j\frac{2\pi k}{P}n} = H_{zp}(k)$$

where

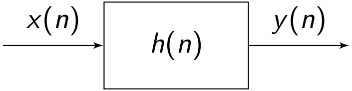
$$h_{zp}(n) = \begin{cases} h(n), & 0 \le n \le M-1 \\ 0, & M \le n \le P-1 \end{cases}$$

is the zero padded impulse response.
Similarly we get

$$X(\omega_k) = X_{zp}(k)$$

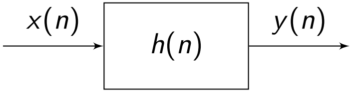
which yields $Y(k) = H_{zp}(k)X_{zp}(k)$ for $k = 0, 1, \ldots, P-1$

Filtering finite signals with DFT



1. Set $P = N + M - 1$
2. Calculate $X_{zp}(k)$ and $H_{zp}(k)$ using DFT.
3. Set $Y(k) = X_{zp}(k)H_{zp}(k)$, for $k = 0, 1, \ldots, P-1$.
4. Calculate $y(n) = \text{IDFT}[Y(k)]$

Filtering, Equalization and System Identification



$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Filtering: How to determine $y(n)$ given $h(n)$ and $x(n)$?

Equalization: How to determine $x(n)$ given $h(n)$ and $y(n)$?

System Identification: How to determine $h(n)$ given $x(n)$ and $y(n)$?

Note that Equalization and System Identification are similar problems! Why?

Time domain equalization

For the *periodic case*:

$$\mathbf{x} = \mathbf{H}_c^{-1}\mathbf{y}$$

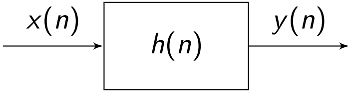
For the *finite signals case*:

$$\mathbf{x} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y}$$

Issues:

- High computational complexity to invert matrices
- Matrices to be inverted could be ill-conditioned

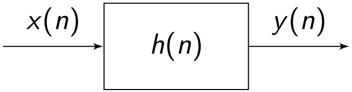
Equalization periodic case



How to determine $x(n)$ given $h(n)$ and $y(n)$?

For periodic case:

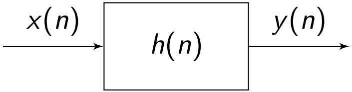
1. Calculate $Y(k) = \mathrm{DFT}[Y(n)]$ for one period of $Y(n)$
2. Determine DFT of input $X(k) = Y(k)/H(\omega_k)$
3. Calculate one period of input $x(n) = \mathrm{IDFT}[X(k)]$.



How to determine $x(n)$ given $h(n)$ and $y(n)$?

For finite signals case:

1. Calculate $Y(k) = \text{DFT}[Y(n)]$ for the length P signal
2. Calculate $H_{zp}(k) = \text{DFT}[h_{zp}(n)]$.
3. Determine DFT of input $X_{zp}(k) = Y(k)/H_{zp}(k)$
4. Calculate the finite input $x_{zp}(n) = \text{IDFT}[X_{zp}(k)]$.



Assume $x(n) = x(n + N)$.

If $h(n)$ has finite impulse length M how long must the input signal $x(n)$ be turned on to make $y(0), y(1), \ldots, y(N-1)$ be the correct values for the assumed periodic signal $y(n)$?

$$y(0) = \sum_{k=0}^{M-1} h(k)x(-k)$$

Enough to start $x(n)$ at sample $x(-M+1)$.

This is know as adding a *cyclic-prefix*.

Making finite signals look periodic

Using the cyclic-prefix technique make it possible to perform equalization with a DFT of length N instead of a DFT of length $N + M - 1$.