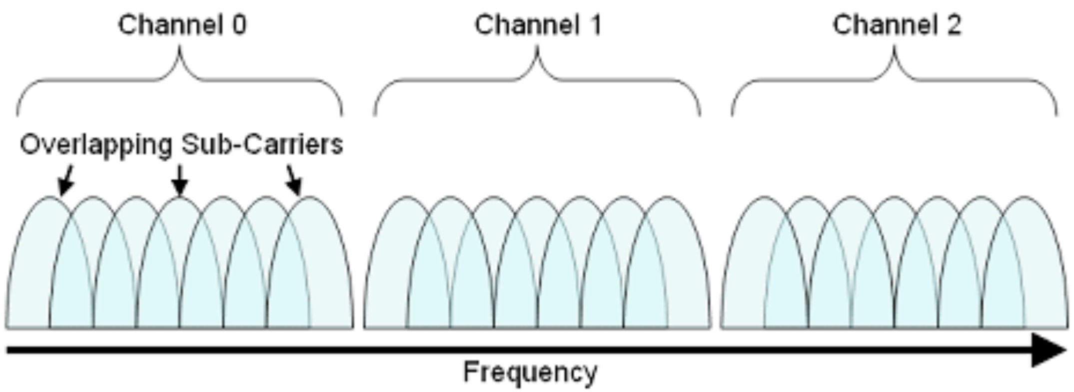
Wireless Communications SSY135 – Lecture 8

Henk Wymeersch
Department of Electrical Engineering
Chalmers University of Technology
http://tinyurl.com/hwymeers
email: henkw@chalmers.se



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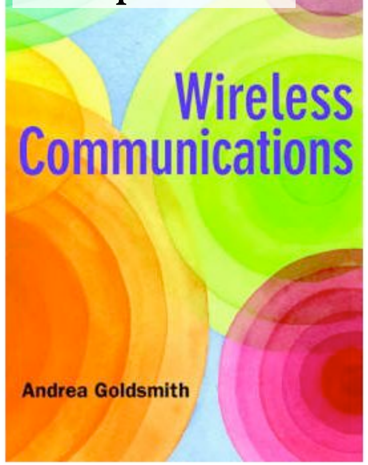
Topics for today

- Lecture learning outcomes
- Single carrier vs multicarrier modulation
- Basics: circular convolution, Toeplitz and circulant matrices, DFT and IDFT matrices, diagonalization of circulant matrices, cyclic prefix
- OFDM transmitter and receiver

Chapter 12



Suggested reading:
- Every section from Chapter 12, except 12.4.2

Today's learning outcomes

At the end of this lecture, you must be able to

- Motivate the need for multicarrier modulation
- Describe main benefits and drawbacks of analog multicarrier modulation
- Compute the DFT of a vector
- Diagonalize a circulant matrix
- Design the length of the cyclic prefix and the OFDM symbol for a given data rate, coherence time, and delay spread
- Determine the overhead to the cyclic prefix
- Describe the main drawbacks of OFDM
- Interpret the transmitted spectrum

Last lecture: adaptation

- Received power is random (over time, over diversity branches, over parallel channels), so performance is random
- **Rate adaptation:** reduce constellation size for lower received power to maintain fixed SER or BER
- **Power adaptation:** increase transmit power to maintain a fixed received power
- **Rate and power adaptation:** "waterfilling"

$$\begin{aligned} \text{maximize}_{\mathbf{x}} \quad & \sum_{k=1}^{L} a_k \log_2(1 + b_k x_k) \\ \text{s.t.} \quad & \sum_{k=1}^{L} a_k x_k = P \\ & x_k \geq 0, k = 1, \ldots, L \end{aligned}$$

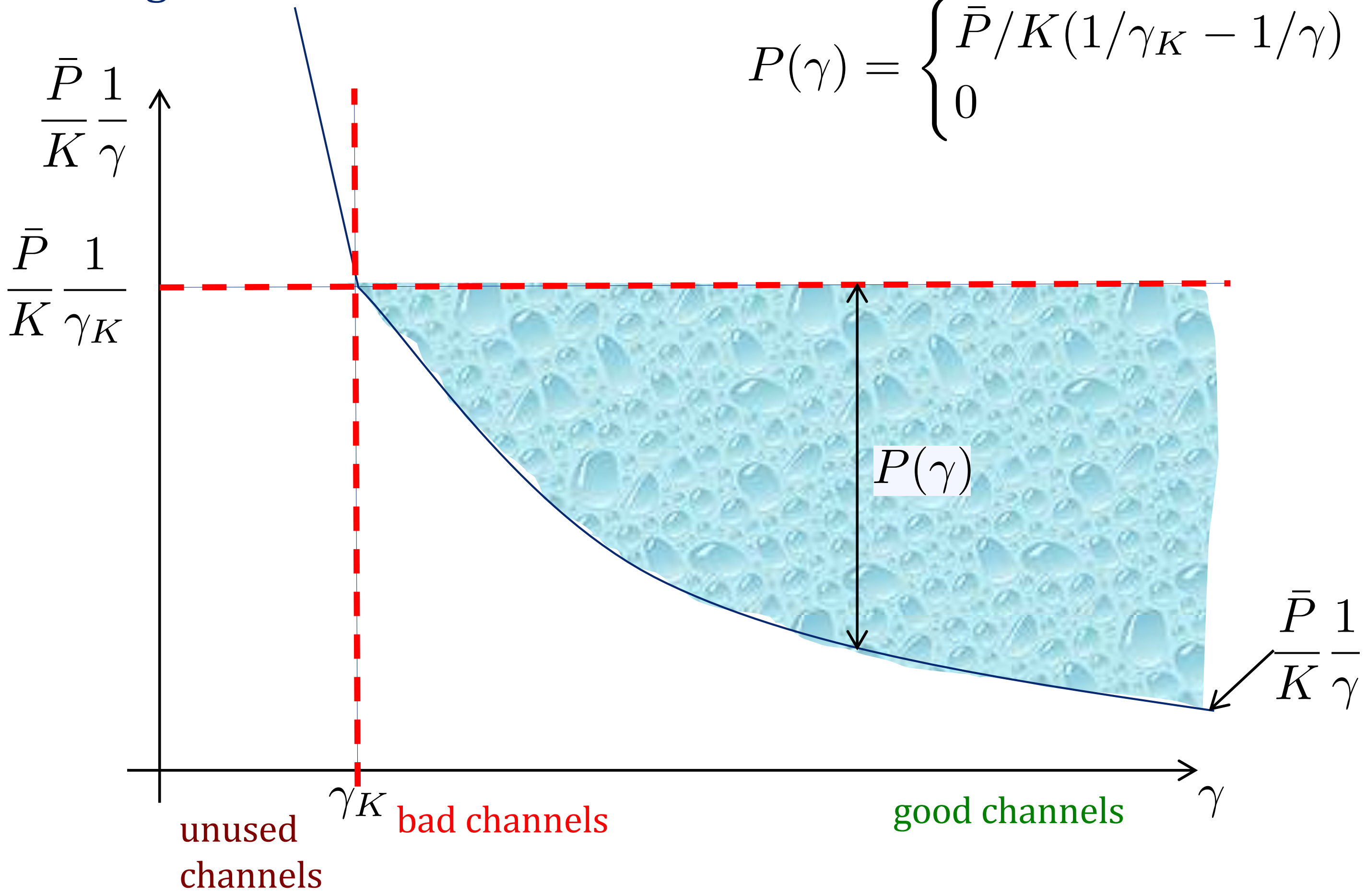
$$\Longrightarrow \quad x_k = \max\left(0, \boxed{\frac{1}{b_c}} - \frac{1}{b_k}\right)$$

$$\sum_{k=1}^{L} a_k x_k = P$$

Statistical water-filling: interpretation

- Idea: taking advantage of good channel conditions by using more power, not using channel below cutoff value

$$P(\gamma) = \begin{cases} \bar{P}/K(1/\gamma_K - 1/\gamma) & \gamma \geq \gamma_K \\ 0 & \gamma < \gamma_K \end{cases}$$



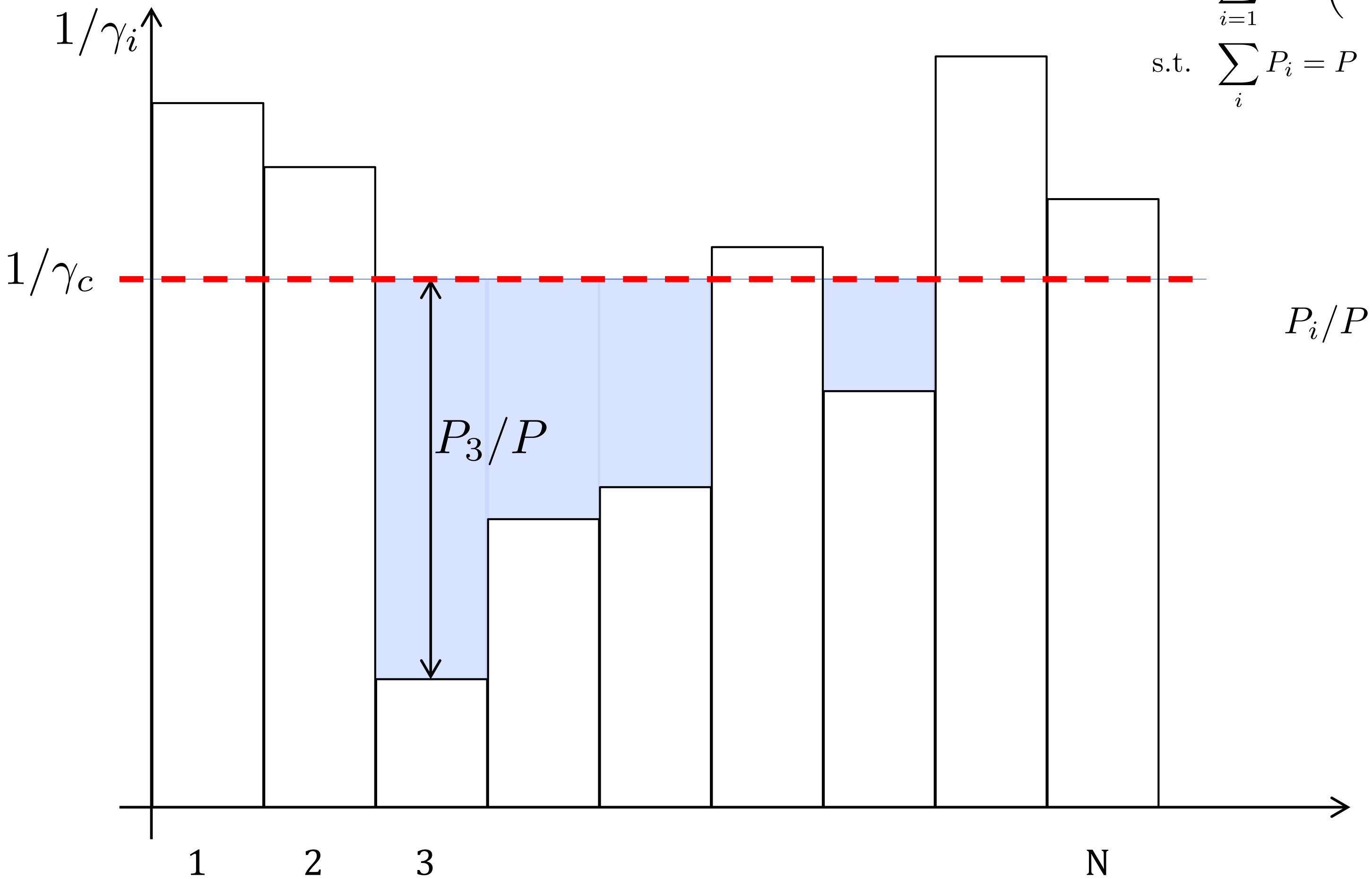
Deterministic water-filling / adaptive loading: interpretation

- Idea: taking advantage of good channel conditions by using more power, not using channel below cutoff value

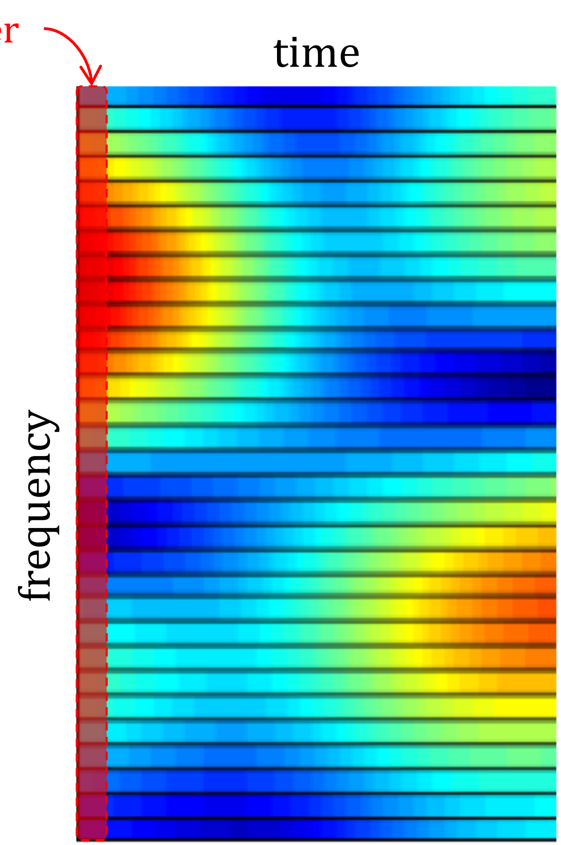
$$\text{maximize}_{\mathbf{P}} \quad \sum_{i=1}^{N} \log_2\left(1+\frac{\alpha_i^2 P_i}{N_0 B}\right) = \sum_{i=1}^{N} \log_2\left(1+\frac{\gamma_i P_i}{P}\right)$$

$$\text{s.t.} \quad \sum_i P_i = P$$

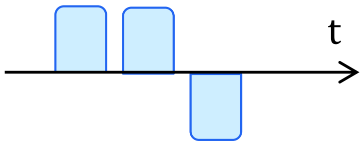
$$P_i/P = \max(1/\gamma_c - 1/\gamma_i, 0)$$

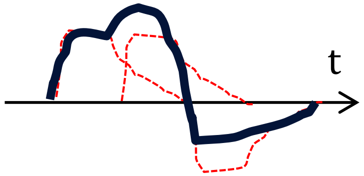


Today: Higher data rates



- Increasing the data rate
  - Increase bits per symbol: limited by SNR
  - Reduce duration per symbol: limited by ISI
- When data rate is close to coherence bandwidth
  - Frequency-selective channel and ISI





- Receiver: equalization

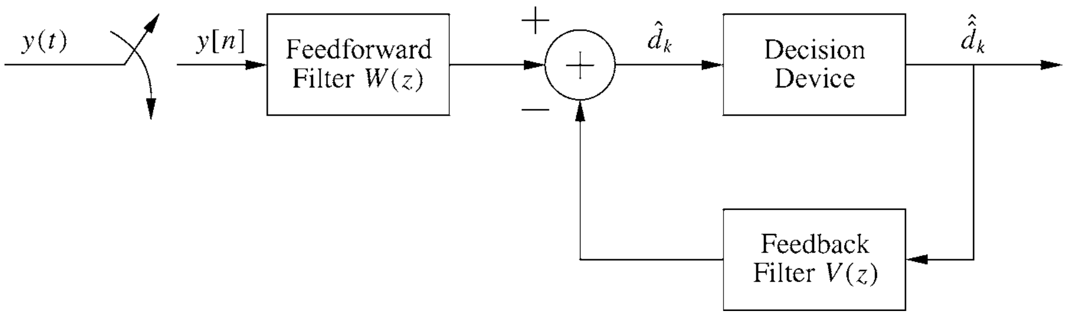


Figure 11.5: Decision-feedback equalizer structure.

- Linear (poor performance), nonlinear (high complexity or error bursts)
- Complexity per symbol for L-tap channel: $\mathcal{O}(L)$, $\mathcal{O}(L^2)$, $\mathcal{O}(L^3)$, $\mathcal{O}(2^L)$

Analog multicarrier

- Split spectrum into parallel non-overlapping channels
- Each channel is now flat and low-rate

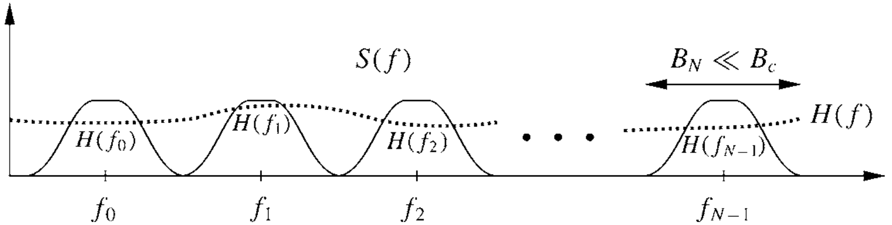
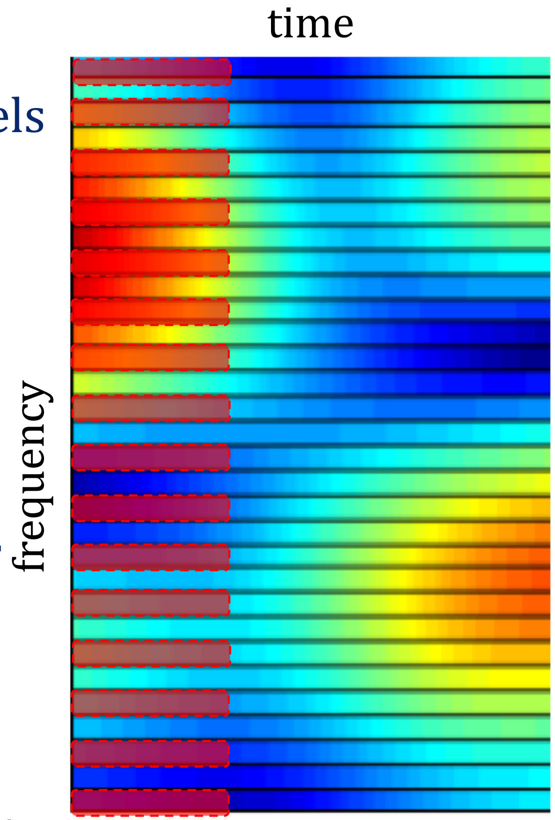


Figure 12.1: Multicarrier transmitter.



- Requires suitable pulse shape and complex receiver

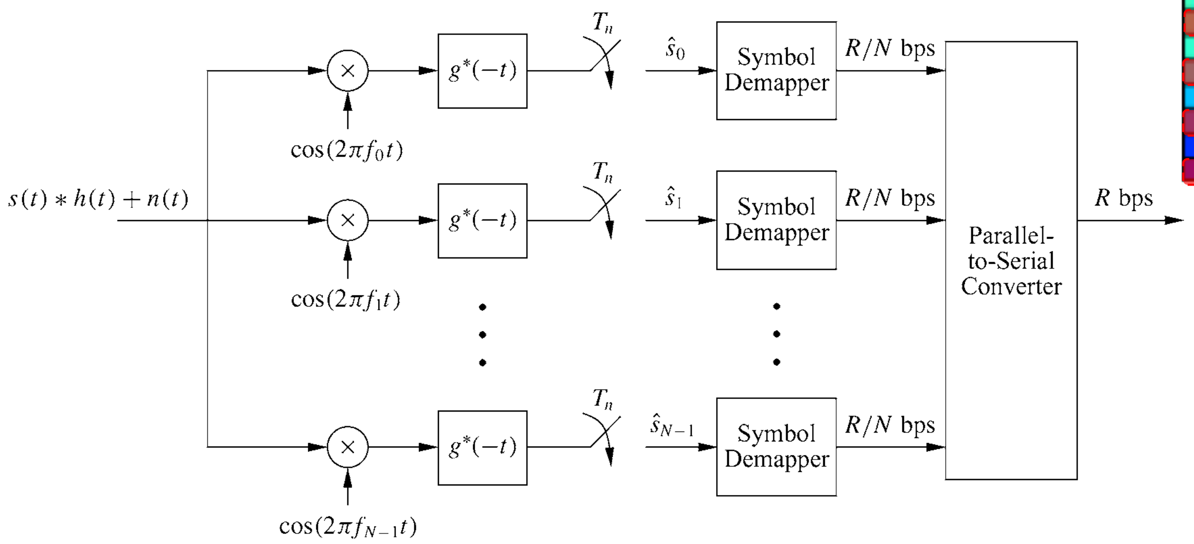


Figure 12.4: Multicarrier receiver for overlapping subcarriers.

Digital implementation of multicarrier modulation

- 1966: Chang (Bell Labs)
- 1971: Weinstein, Ebert: FFT and cyclic prefix
- Main ideas
  - parallel sub-channels
  - fully digital, efficient implementation
  - cyclic prefix to avoid ISI
- Mathematical components

Circular convolution

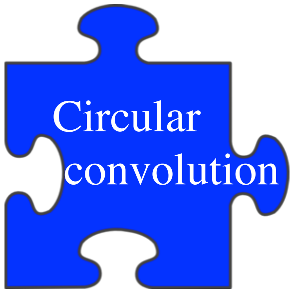
- Convolution with discrete-time channel
  - input: $x[n]$
  - channel: $h[n], \quad n = 0, \dots, L$
  - output:

$$y[n] = \sum_{k=0}^{L} h[k]x[n-k] = h[0]x[n] + h[1]x[n-1] + \ldots + h[L]x[n-L]$$
$$= h[n] \star x[n]$$

- Circular convolution

  - periodic input (period N): $x[n] = x[n+N]$
  - channel: $h[n], \quad n = 0, \dots, L$
  - output is also periodic with period N:

$$y[n] = \sum_{k=0}^{L} h[k]x[n-k] = \sum_{k=0}^{L} h[k]x[n-k+N]$$
$$= h[n] \circledast x[n]$$



`cconv(x,h,N)` in matlab

Convolution in matrix form

Given

- Channel [1 -0.5] at delays [0 1]
- Input [1 1 2] at times [0 1 2] (preceded and succeeded by zeros)

Task

- Write down the input/output relationship in matrix form.
- Write down the input/output relationship in matrix form when the input is extended with period 4.

Discrete Fourier transform

- Fourier transform of finite sequence of samples $\mathbf{x} = [x[0], x[1], x[2], \ldots, x[N-1]]^T$
- N x N DFT unitary matrix **Q**:

$$Q_{m,n} = \frac{1}{\sqrt{N}} W^{mn} \quad \text{for } m, n \in \{0, 1, \ldots, N-1\}$$

$$W = \exp(-j2\pi/N)$$

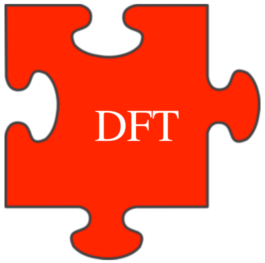
N should be power of 2

- DFT and IDFT (*N log N* implementation = FFT/IFFT): unitary transform

$$X[i] = \text{DFT}\{x[n]\}$$

$$x[i] = \text{IDFT}\{X[n]\}$$

$$\mathbf{X} = \mathbf{Q}\mathbf{x} \qquad \mathbf{x} = \mathbf{Q}^H \mathbf{X}$$



- Note: slightly different notation from MATLAB

```
X = 1/sqrt(N)*fft(x);
x = sqrt(N)*ifft(x);
```

- Note: bold capital letters will be used for DFT and for matrices. Be careful!

Toeplitz and circulant matrices

Convolution: Toeplitz matrix

$$\mathbf{x} = [0\ 1\ 1\ 3\ 0]^T,\ (k = 0, \ldots, 4)$$

$$\mathbf{h} = [1\ \ -1]^T,\ (k = 0, 1)$$

$$\mathbf{y} = [0\ 1\ 0\ 2\ \ -2\ 0]^T,\ (k = 0, \ldots, 5)$$

$$\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$$

$$= \mathbf{H}\mathbf{x}$$

Circular Convolution: circulant matrix

$$\mathbf{x} = [0\ 1\ 1\ 3]^T,\ \text{period} = 4$$

$$\mathbf{h} = [1\ \ -1]^T,\ (k = 0, 1)$$

$$\mathbf{y} = [-3\ 1\ 0\ 2]^T,\ \text{period} = 4$$

$$\mathbf{y} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \mathbf{H}_c\mathbf{x}$$

Properties:

- Circulant matrices are square
- (I)DFT columns are always eigenvectors of circulant matrices

Diagonalization property of DFT

- Continuous time: convolution in time domain becomes point-wise multiplication in frequency domain

$$y(t) = \int_{-\infty}^{+\infty} c(\tau)u(t-\tau)\mathrm{d}\tau$$

$$Y(f) = C(f)U(f)$$

- Discrete time: circular convolution in time domain becomes point-wise multiplication in frequency domain. Due to eigenvector relationship: $\mathbf{H}\mathbf{q}_n = \lambda_n \mathbf{q}_n$

$$\mathbf{H}\mathbf{Q}^H = \mathbf{Q}^H \boldsymbol{\Lambda}$$

$$\mathbf{Q}\mathbf{H}\mathbf{Q}^H = \boldsymbol{\Lambda} = \sqrt{N}\mathrm{diag}[H_0, \ldots, H_{N-1}]$$

$$\mathbf{y} = \underbrace{\mathbf{H}}_{\text{circulant}} \mathbf{x}$$

$$\mathbf{Q}\mathbf{y} = \mathbf{Q}\mathbf{H}\mathbf{x}$$

$$\mathbf{Y} = \underbrace{\mathbf{Q}\mathbf{H}\mathbf{Q}^H}_{\text{diagonal matrix}} \mathbf{X}$$

- Different notation

$$y[n] = x[n] \circledast h[n]$$

$$Y[i] = \sqrt{N} \times X[i] \times H[i]$$

- DFT diagonalizes all circulant matrices

DFT example

Given

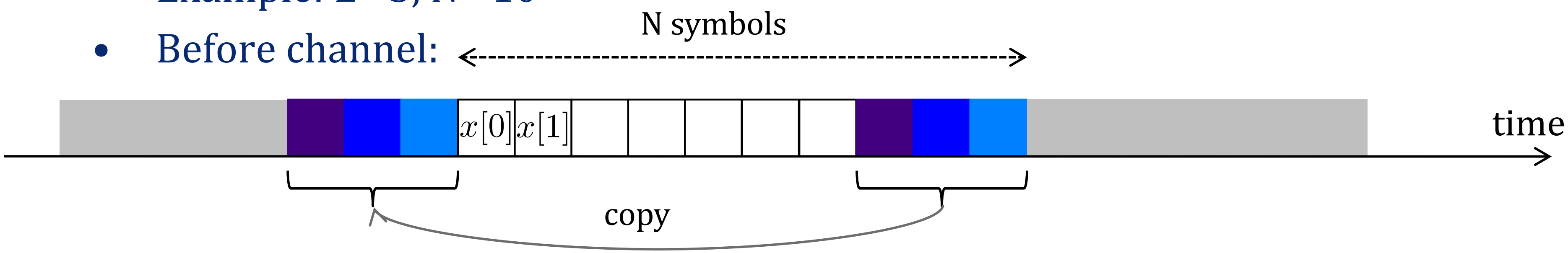
- Channel [1 -0.5] at delays [0 1]
- Input [1 1 2] at times [0 1 2] (preceded and succeeded by zeros)
- period = 4

Task

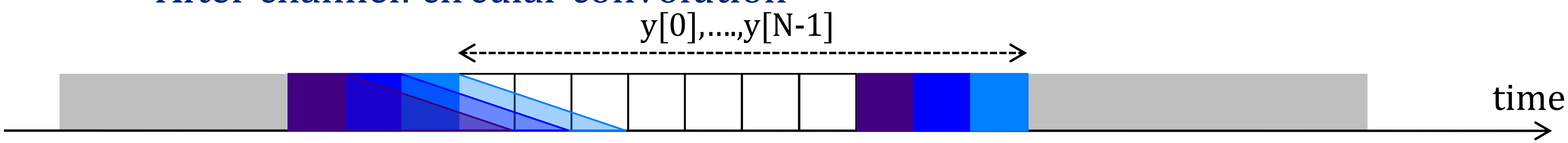
- write down the periodic input output relationship
- What is the DFT matrix?
- Determine DFT of x, y, h

Creating circular convolution: the cyclic prefix

- Channel of length L+1, input sequence of length N>L: cyclic prefix of length L
- Example: L=3, N=10
- Before channel:



- After channel: circular convolution



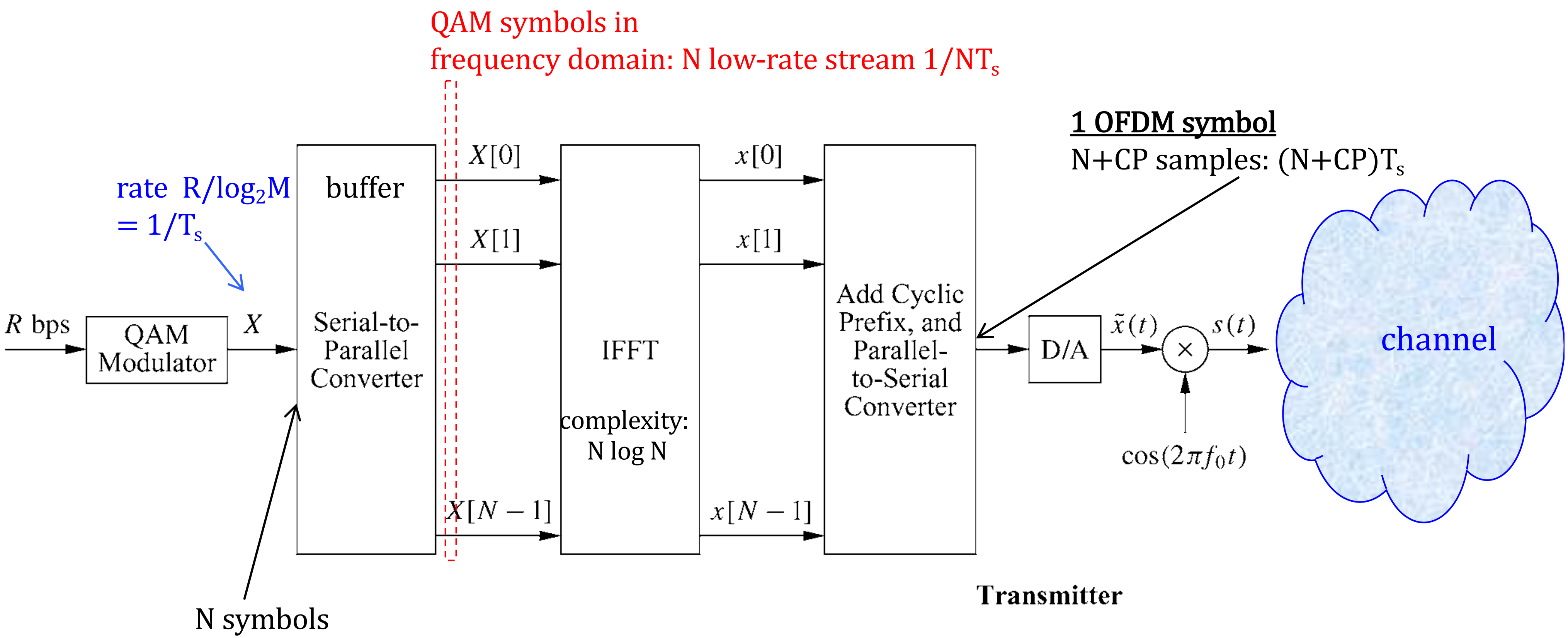
$$y[0] = h[0]x[0] + h[1]x[-1] + h[2]x[-2] + h[3]x[-3] = h[0]x[0] + h[1]x[N-1] + h[2]x[N-2] + h[3]x[N-3]$$

$$y[1] = h[0]x[1] + h[1]x[0] + h[2]x[N-1] + h[3]x[N-2]$$

- After DFT: point-wise multiplication

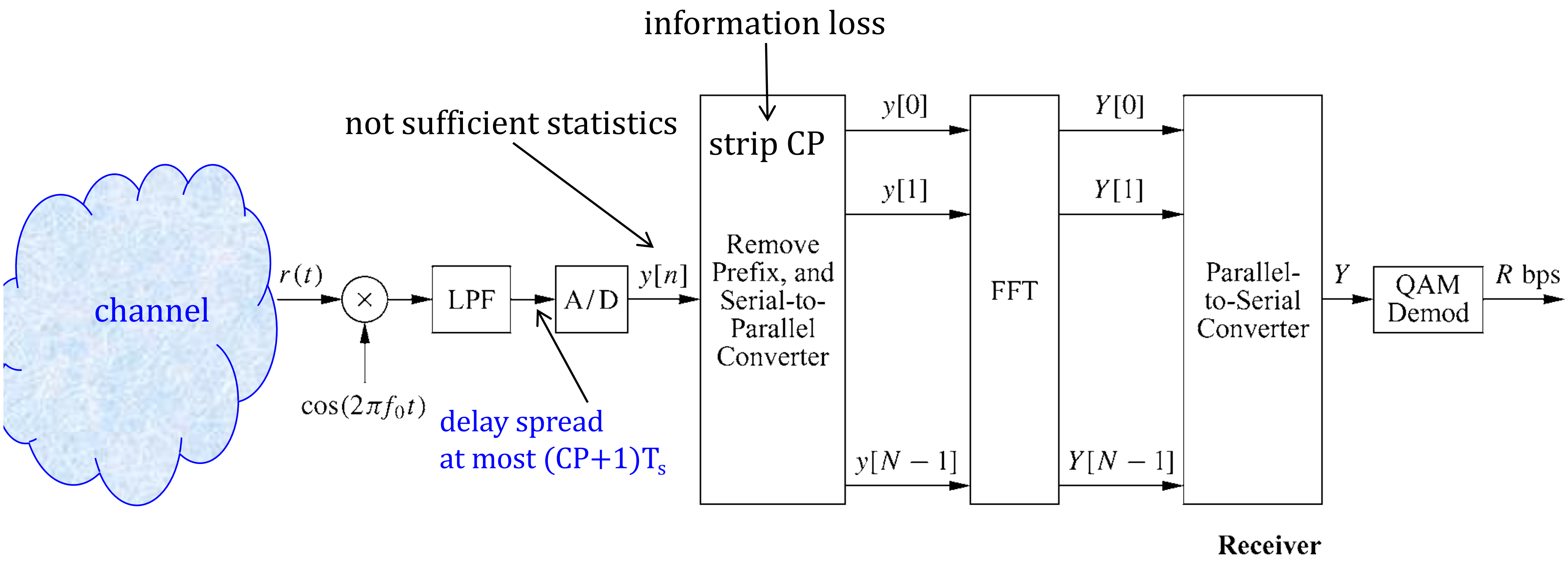
$$y[n] = h[n] \circledast x[n] \longrightarrow \boxed{\text{length N DFT}} \longrightarrow Y[i] = \sqrt{N} \times X[i] \times H[i]$$

OFDM transmitter



- N should be power of 2 for low complexity FFT
- CP overhead based on the channel delay spread
  - Reduction in rate by CP/(N+CP)
  - Additional power
  - Trade-off between robustness and rate loss

OFDM receiver



- Channel must remain constant during each OFDM symbol
- In the absence of noise $Y[i] = \sqrt{N} \times X[i] \times H[i]$

OFDM design

Given

- Maximum delay spread 10 us
- Signal bandwidth 2 MHz
- 16 QAM transmission
- Channel coherence time 20 ms

Task

- How much ISI would a single carrier system experience? What is the data rate (bps)? What is the spectral efficiency (bps/Hz)?
- Design an OFDM system that can operate over this channel. What is the data rate? What is the spectral efficiency?
- What would change if the coherence time was only 2 ms?

OFDM: end to end operation (part 1/3)

- Stream of QAM symbols $\hat{x}[k]$ grouped into block of length N (note notation change)
- OFDM symbol m
  1. Input $\hat{\mathbf{x}}^{(m)} = [\hat{x}[mN]\ \hat{x}[mN+1] \ldots \hat{x}[(m+1)N-1]]^T$
  2. After FFT $\mathbf{x}^{(m)} = \mathbf{Q}^H \hat{\mathbf{x}}^{(m)}$
  3. Add CP: $x^{(m)}_{-k} = x^{(m)}_{N-k}, \quad k = 1, \ldots, L$
  4. Generate baseband signal

$$s^{(m)}(t) = \sum_{k=-L}^{N-1} x^{(m)}[k] p(t - kT_s)$$

- Overall signal

$$x(t) = \sum_{m=-\infty}^{+\infty} s^{(m)}(t - m(N+L)T_s) = \sum_{m=-\infty}^{+\infty} \sum_{k=-L}^{N-1} x^{(m)}[k] p(t - kT_s - m(N+L)T_s)$$

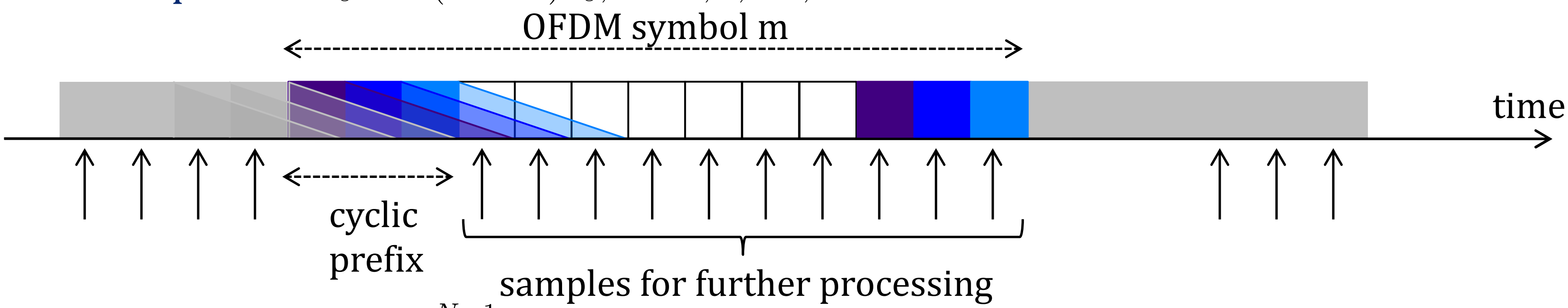
- At receiver, after physical channel and filtering

$$y(t) = \sum_{m=-\infty}^{+\infty} \sum_{k=-L}^{N-1} x^{(m)}[k] \underbrace{\boxed{h(t - kT_s - m(N+L)T_s)}}_{\text{end-to-end channel}} + \underbrace{\boxed{n(t)}}_{\text{band-limited noise}}$$

- Channel: $h(t) = 0 \quad t < 0 \quad \text{and} \quad t > LT_s$

OFDM: end to end operation (part 2/3)

- Sample at $k'T_s + m(N+L)T_s, k = 0, 1, \ldots, N-1$



- Samples $y^{(m)}[k'] = \sum_{k=-L}^{N-1} x^{(m)}[k]h(k'T_s - kT_s) + n^{(m)}[k']$

$$\mathbb{E}\{n^{(m)}[k']n^{*(m')}[k'']\} = N_0\delta_{k'-k''}\delta_{m-m'}$$

$$= \sum_{k=-L}^{N-1} x^{(m)}[k]h[k'-k] + \boxed{n^{(m)}[k']}$$

- In vector notation $\mathbf{y}^{(m)} = \boxed{\mathbf{H}}\mathbf{x}^{(m)} + \mathbf{n}^{(m)}$ the N x N circulant matrix
- Channel matrix circulant by design

OFDM: end to end operation (part 3/3)

- Apply the DFT to $\mathbf{y}^{(m)} = \mathbf{H}\mathbf{x}^{(m)} + \mathbf{n}^{(m)}$ so

$$\begin{aligned}\hat{\mathbf{y}}^{(m)} &= \mathbf{Q}\mathbf{y}^{(m)} = \mathbf{Q}\mathbf{H}\mathbf{x}^{(m)} + \mathbf{Q}\mathbf{n}^{(m)} \\ &= \mathbf{Q}\mathbf{H}\mathbf{Q}^{H}\hat{\mathbf{x}}^{(m)} + \mathbf{Q}\mathbf{n}^{(m)}\end{aligned}$$

- Noise statistics are unchanged!
- For subcarrier k, OFDM symbol m: $\hat{y}_k^{(m)} = \hat{h}_k \times \hat{x}_k^{(m)} + \hat{n}_k^{(m)}$
- Maximum likelihood detection

$$\hat{x}_{\mathrm{ML},k}^{(m)} = \arg\max_{x} p(\hat{y}_k^{(m)} | \hat{x}_k^{(m)} = x)$$

OFDM properties

Task

- Given a certain propagation channel, how would you design the receiver front end filter?
- Prove that the noise is white after DFT
- What happens if the data symbols are all equal?

OFDM design

Given

- Maximum delay spread 10 us
- Signal bandwidth 2 MHz, no roll-off
- 16 QAM transmission
- Channel coherence time 20 ms

Task

- How much ISI would a single carrier system experience? What is the data rate (bps)? What is the spectral efficiency (bps/Hz)?
- Design an OFDM system that can operate over this channel. What is the data rate? What is the spectral efficiency?
- What would change if the coherence time was only 2 ms?

Solution

Single carrier

- The symbol slot is 0.5 us, so ISI would stretch over 20 symbols
- The data rate is 8 Mbit/s and the spectral efficiency 4/bits/s/Hz

OFDM for coherence time of 20 ms

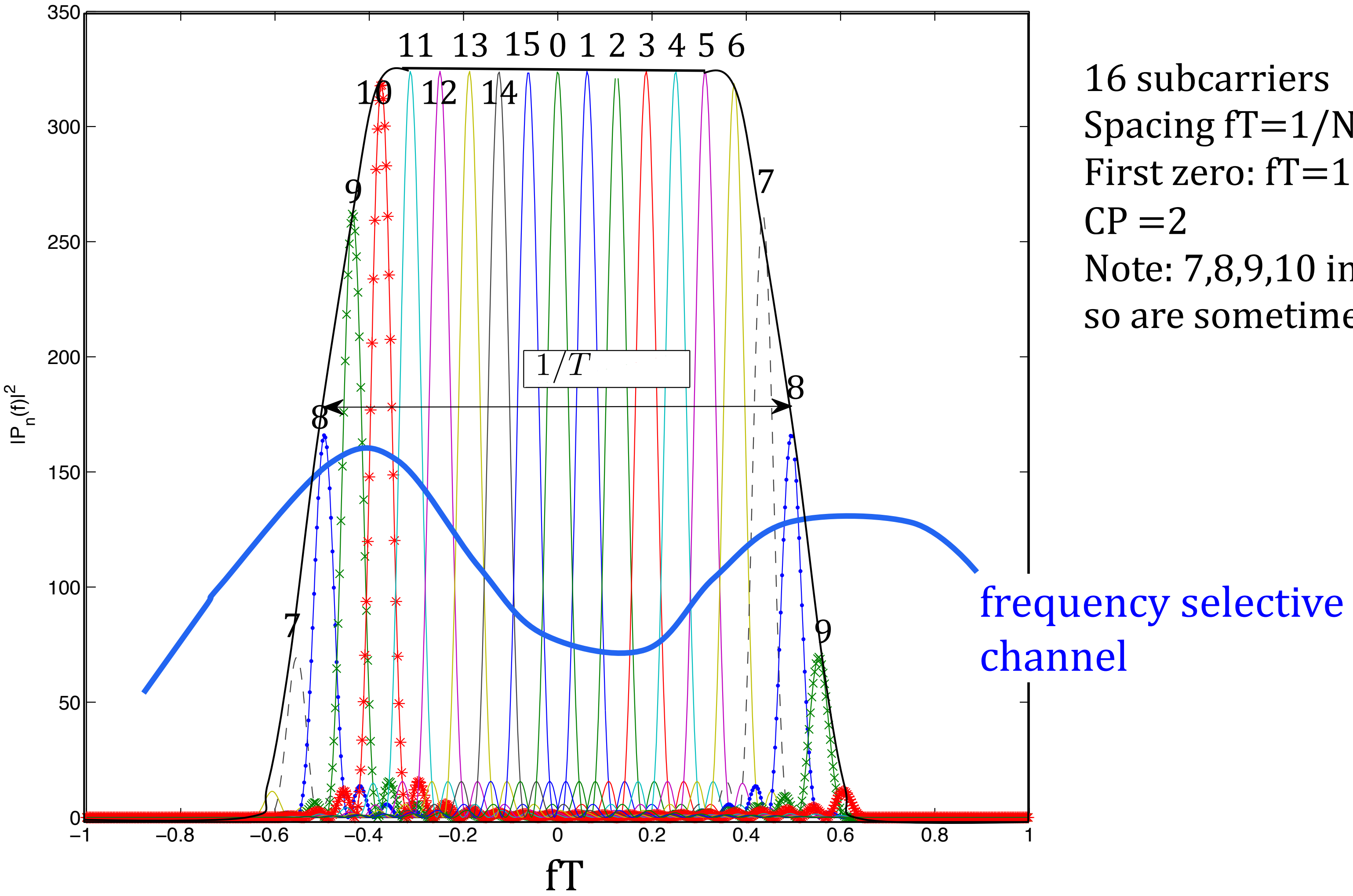
- We can design an OFDM system with L=20 symbols slots (=10 us)
- To determine N, we need the channel to remain constant over (N+L)*0.5us, so we want (N+L)*0.5us < 2 ms (coherence time/10). We find that N should be below 4000. The largest power of 2 below 4000 is 2048. So we set L=20, N=2048.
- The rate is 2048x4 bits / (2068 x 0.5us) = 7.92 Mbit/s, corresponding to a spectral efficiency of 3.96 bits/s/Hz

OFDM for coherence time of 2 ms

- When the coherence time is reduced by a factor of 10, we must set N < 400, so we set N= 256.
- The rate is 7.42 Mbit/s and the spectral efficiency 3.7 bits/s/Hz

OFDM spectrum for non-square pulse

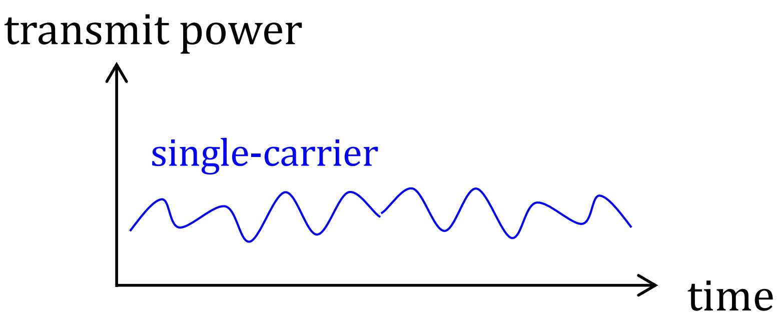
- Transmitted signal = sum of N signals

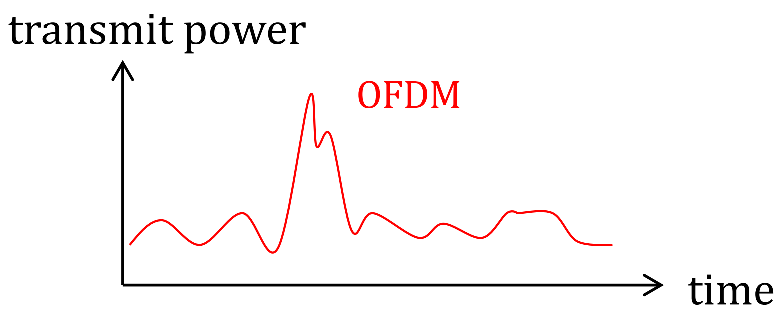


16 subcarriers
Spacing fT=1/N
First zero: fT=1/(N+CP)
CP =2
Note: 7,8,9,10 in roll-off zone so are sometimes set to zero

OFDM design and properties

- Design of OFDM systems for given T_s
  - Cyclic prefix must exceed maximum delay spread: make CP long. Overhead: CP/(CP+N), so make N long
  - Channel must remain static: make N short: $(N+CP)T_s << T_c$
- Easy to combine with adaptive loading using waterfilling
- OFDM challenges
  1. Sensitive to synchronization errors (inter-carrier interference)
  2. High peak to average power ratio $\mathrm{PAPR} = \frac{\max_n |x[n]|^2}{\mathbb{E}\{|x[n]|^2\}}$





- PAPR example
  - constant signal gives large value of 1st time domain value
  - periodic signal gives large value on one of the time domain values

Today's learning outcomes

At the end of this lecture, you must be able to

- Motivate the need for multicarrier modulation
- Describe main benefits and drawbacks of analog multicarrier modulation
- Compute the DFT of a vector
- Diagonalize a circulant matrix
- Design the length of the cyclic prefix and the OFDM symbol for a given data rate, coherence time, and delay spread
- Determine the overhead to the cyclic prefix
- Describe the main drawbacks of OFDM
- Interpret the transmitted spectrum

Convolution in matrix form

Given

- Channel [1 -0.5] at delays [0 1]
- Input [1 1 2] at times [0 1 2] (preceded and succeeded by zeros)

Task

- Write down the input/output relationship in matrix form.
- Write down the input/output relationship in matrix form when the input is extended with period 4.

Solution

- The relation **y=Hx** is given by

$$\begin{bmatrix} 1 \\ 0.5 \\ 1.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.5 & 1 \\ 0 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Note that the channel [1 -0.5] is visible in the columns of **H** and (time-reversed) in the rows of **H**.

- In the periodic case, we have a periodic input and a 4 x 4 channel matrix

$$\begin{bmatrix} 1 \\ 0.5 \\ 1.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ -0.5 & 1 & 0 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

- The periodic H matrix is easier to see if you use as input 1 1 2 3 1 1 2 3 1 1 2 3…

DFT

Given

- Channel [1 -0.5] at delays [0 1]
- Input [1 1 2] at times [0 1 2] (preceded and succeeded by zeros)
- period = 4

Task

- write down the periodic input output relationship
- What is the DFT matrix?
- determine DFT of x, y, h

Solution

- The 4x4 DFT matrix is $\mathbf{Q} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$

```
x=[1 1 2 0];
X=fft(x)/2
y=[1 0.5 1.5 -1];
Y=fft(y)/2
```

- The DFT of x is [2 0.5*(1-j) 1 -0.5*(1+i)]^T
- The DFT of y is [1 0.25*(-1-3j) 3/2 0.25*(-1+3*j)]^T
- The DFT of h is [0.25 0.25*(2+j) ¾ 0.25*(2-j)]^T
- Note that
  - DFT(y)=2*DFT(h).*DFT(x)
  - Q * H_c *Q' = 2*DFT(h), where H_c is the 4 x 4 circulant matrix $\begin{bmatrix} 1 & 0 & 0 & -0.5 \\ -0.5 & 1 & 0 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 0 & -0.5 & 1 \end{bmatrix}$

OFDM design

Given

- Maximum delay spread 10 us
- Signal bandwidth 2 MHz, no roll-off
- 16 QAM transmission
- Channel coherence time 20 ms

Task

- How much ISI would a single carrier system experience? What is the data rate (bps)? What is the spectral efficiency (bps/Hz)?
- Design an OFDM system that can operate over this channel. What is the data rate? What is the spectral efficiency?
- What would change if the coherence time was only 2 ms?

Solution

Single carrier

- The symbol slot is 0.5 us, so ISI would stretch over 20 symbols
- The data rate is 8 Mbit/s and the spectral efficiency 4/bits/s/Hz

OFDM for coherence time of 20 ms

- We can design an OFDM system with L=20 symbols slots (=10 us)
- To determine N, we need the channel to remain constant over (N+L)*0.5us, so we want (N+L)*0.5us < 2 ms (coherence time/10). We find that N should be below 4000. The largest power of 2 below 4000 is 2048. So we set L=20, N=2048.
- The rate is 2048x4 bits / (2068 x 0.5us) = 7.92 Mbit/s, corresponding to a spectral efficiency of 3.96 bits/s/Hz

OFDM for coherence time of 2 ms

- When the coherence time is reduced by a factor of 10, we must set N < 400, so we set N= 256.
- The rate is 7.42 Mbit/s and the spectral efficiency 3.7 bits/s/Hz

OFDM properties

Task

- Given a certain propagation channel, how would you design the receiver front end filter?
- Prove that the noise is white after DFT
- What happens if the data symbols are all equal?

Solution

Receiver filter

- The receive filter should be a square root Nyquist filter for the sampling rate

White noise

- The noise before DFT is **n** (a vector of length N) with covariance N_0 $\mathbf{I}_N$
- The noise after DFT is **w=Qn.**
- Now $\mathbb{E}\{\mathbf{w}\mathbf{w}^H\} = \mathbb{E}\{\mathbf{Q}\mathbf{n}\mathbf{n}^H\mathbf{Q}^H\}$

$$
\begin{aligned}
&= \mathbf{Q}\mathbb{E}\{\mathbf{n}\mathbf{n}^H\}\mathbf{Q}^H \\
&= \mathbf{Q}N_0\mathbf{I}_N\mathbf{Q}^H \\
&= N_0\mathbf{I}_N\mathbf{Q}\mathbf{Q}^H \\
&= N_0\mathbf{I}_N
\end{aligned}
$$

All symbols equal

- It is easy to verify that Q**1** (where **1** is a vector of all ones) gives a vector $[\sqrt{N}\ 0\ldots\ 0\ 0 0\ ]^T$