MIMO and Multi-user Communications

You are given a 2×2 real MIMO channel of the following form:

$$\mathbf{H} = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix},$$

where $a \in [0, 1]$. You communicate over this channel in discrete time, with an average transmission energy E_s and add complex Gaussian noise at the receiver with variance σ^2 for real and imaginary dimension. The observation model is thus of the form

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n},$$

with $\mathbb{E}\{\|\mathbf{x}\|^2\} = E_s$ and $\mathbf{n} \sim \mathcal{CN}(0, 2\sigma^2)$. The energy of the channel is defined as $E_H = 2 + 2a^2$. The MIMO SNR is defined as $E_s E_H/(4\sigma^2)$.

1. [3 pt] Determine the the MIMO SNR as a function of a. Applying a singular value decomposition (SVD) to $\mathbf{H}$ indicates that the singular values of the channel are given by $1 + a$ and $1 - a$ (so $\mathbf{H} = \mathbf{U}\mathbf{S}\mathbf{V}^H$, in which $\mathbf{S} = [1 + a\, 0;\, 0\, 1 - a]$ and $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices). Using suitable precoding at the transmitter and shaping at the receiver, we can obtain two parallel, non-interfering channels of the form

$$z_i = x_i + w_i,\ i \in \{1, 2\},$$

where w_i is an AWGN sample with variance σ_i^2 for real and imaginary dimension. Determine the SNR of each of the channels and relate to the MIMO SNR. Assume you are able to transmit at a rate of $\log_2(1 + \text{SNR})$ bits per channel use on each of the channels, what is the total data rate? What is the total rate or $a = 0$ and $a = 1$? Is the rate maximized when the MIMO SNR is maximized (with respect to a)? Why or why not?
Answer: The MIMO SNR is

$$SNR = \frac{2E_s(1 + a^2)}{4\sigma^2}.$$

The SNR of channel 1 is

$$SNR_1 = \frac{E_s(1 + a)^2}{4\sigma^2}$$

while the SNR of channel 2 is

$$SNR_2 = \frac{E_s(1 - a)^2}{4\sigma^2}.$$

The sum of the SNRs is the MIMO SNR. The total data rate is

$$R(a) = \log_2\left(1 + \frac{E_s(1 + a)^2}{4\sigma^2}\right) + \log_2\left(1 + \frac{E_s(1 - a)^2}{4\sigma^2}\right).$$

Hence $R(0) = 2\log_2(1 + E_s/4\sigma^2)$ *and* $R(1) = \log_2(1 + E_s/\sigma^2)$. *The rate is not necessarily maximized for the maximum SNR, due to the non-linear rate expression.*
Grading: 1 point for SNR computation, 1 point for rate computation, one for discussion for $a \in \{0, 1\}$.

2. [3 pt] Derive the zero-forcing (ZF) receiver. Show that after applying the ZF receiver, you again obtain 2 parallel channels of the form

$$z_i = x_i + w_i,\ i \in \{1, 2\}$$

where w_i is an AWGN sample with variance σ_i^2 for real and imaginary dimension. Determine the SNR of each of the channels and relate to the MIMO SNR. Assume you are able to transmit at a rate of $\log_2(1+\text{SNR})$ bits per channel use on each of the channels, what is the total data rate? Comment on the differences between the SVD approach. What is the total rate or $a = 0$ and $a = 1$? What are the benefits that ZF has over SVD?
Answer: the ZF receiver is obtained by computing the inverse of $\mathbf{H}$. *We find that*

$$\mathbf{H}^{-1} = \frac{1}{1 - a^2}\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}.$$

Applying this inverse, we find that

$$\mathbf{z} = \mathbf{H}^{-1}\mathbf{y} = \mathbf{x} + \mathbf{H}^{-1}\mathbf{n}$$

so

$$z_i = x_i + w_i.$$

The variance of w_i is given by (same for both dimensions)

$$\begin{aligned} 2\sigma_i^2 &= E\left\{|w_1|^2\right\} = E\left\{w_1 w_1^*\right\} \\ &= \frac{1}{(1-a^2)^2} E\left\{(n_1 - a n_2)(n_1 - a n_2)^*\right\} \\ &= \frac{1+a^2}{(1-a^2)^2} 2\sigma^2. \end{aligned}$$

Hence the SNR on each channel is

$$SNR_i = \frac{E_s(1-a^2)^2}{4\sigma^2(1+a^2)},$$

while the sum SNR is $2SNR_i$. You can quickly verify that the sum SNR for ZF is always less than for SVD, except when $a = 0$. The total data rate is

$$R = 2\log_2\left(1 + \frac{E_s(1-a^2)^2}{4\sigma^2(1+a^2)}\right).$$

This rate is zero when $a = 1$ and the same as SVD for $a = 0$. Hence, the SVD approach is always better, but comes at a cost of needing channel knowledge at the transmitter.
grading: 1 point for ZF, 1 point for SNR per channel and rate, 1 point on discussion for $a \in \{0, 1\}$ and comparison with SVD.

3. [3 pt] Now we associate the two transmit antennas with difference users. We also consider a different channel

$$\mathbf{H} = \begin{bmatrix} 1 & a_2 \\ a_1 & 1 \end{bmatrix},$$

where $a_1, a_2 \in \mathbb{R}$ with $a_1 > a_2$. So there is a channel $\mathbf{h}_1 = [1\, a_1]^T$ between user 1 and the receiver and a channel $\mathbf{h}_2 = [a_2\, 1]^T$ between user 2 and the receiver. We consider TDMA transmission, where only one user is allowed to transmit at a time and they alternate in using the channel. Assume each user is able to transmit data symbols with energy E_s at a rate of $\log_2(1 + \text{SNR})$ bits per channel use. What is the *effective* per-user rate and sum rate, under maximum ratio combining? Which user has the highest rate?
Answer: for user i, the observation is (after suitable permutation of the antenna order)

$$\mathbf{y}_i = \begin{bmatrix} 1 \\ a_i \end{bmatrix} x_i + \mathbf{n}_i.$$

Under MRC, the observation for user i is thus

$$z_i = (1 + a_i^2)x_i + \underbrace{n_1 + a_i n_2}_{w_i}$$

where

$$E\{|w_i|^2\} = (1 + a_i^2)2\sigma^2.$$

The rate of user i is

$$R_i = \frac{1}{2}\log_2\left(1 + \frac{(1+a_i^2)^2 E_s}{(1+a_i^2)2\sigma^2}\right) = \frac{1}{2}\log_2\left(1 + \frac{(1+a_i^2)E_s}{2\sigma^2}\right)$$

The factor 0.5 is due to the fact that each user uses the channel half of the time. The sum rate is $R_1 + R_2$. User 1 has the highest rate.
grading: 2 point for determining the observation and SNR expression, 1point for rate.

4. [3 pt] We now use a slightly more sophisticated version of TDMA, where user 1 transmits with a probability $p \in [0, 1]$. If user 1 transmits, user 2 is quiet and vice versa. What is the per-user rate and the average rate? Can you recover the effective per-user rate and the sum rate for TDMA from the previous sub-question?

Suppose we want to maximize the sum rate, how would you choose p? What can you say about the fairness of this solution? Finally, consider a_1 and a_2 to be independent random variables uniformly distributed over the interval $[0, 1]$. User i now no longer coordinate, but transmit when $a_i > 0.5$. When both users transmit, there is a packet collision and the information is lost. What is the expected effective per user rate, expressed as an integral?

Answer: The rate of user i is now

$$R_i = p \underbrace{\log_2\left(1 + \frac{(1+a_i^2)E_s}{2\sigma^2}\right)}_{=\alpha_i}$$

We want to find the value of p to maximize

$$R_1 + R_2 = p\alpha_1 + (1-p)\alpha_2,$$

where $\alpha_1 > \alpha_2$. This objective is maximized when $p = 1$, i.e., only user 1 uses the channel. This approach is not fair. For the last point, user 1 will transmit when $a_1 > 0.5$ and $a_2 < 0.5$. This happens 25% of the time. Same for user 2. The remaining 50% of the time, either no user transmits or both transmit. In either case, the effective rate is zero. For the 25% of the time that $a_1 > 0.5$ and $a_2 < 0.5$, the expected rate will be

$$R_1 \quad = \quad 0.25 \times \int_{0.5}^{1} \log_2\left(1 + \frac{(1+a_1^2)E_s}{2\sigma^2}\right) da_1$$

grading: 1 point for finding the rate, 1 point for finding the optimal p, 1 point for the expected rate expression

Various questions

1. [1 pt] something about capacity lecture
2. [1 pt] something about hardware impairments lecture (phase noise, nonlinearities)
3. [1 pt] health impact of EM radiation
4. [1 pt] FCC spectrum masks
5. [1 pt] question about OFDM
6. [1 pt] question about cellular networks
7. [1 pt] question about Doppler
8. [1 pt] question about AWGN vs Rayleigh vs Ricean
9. [2 pt] question on ML detection
10. [2 pt] question on CDMA