Wireless Communications SSY135 – Lecture 6

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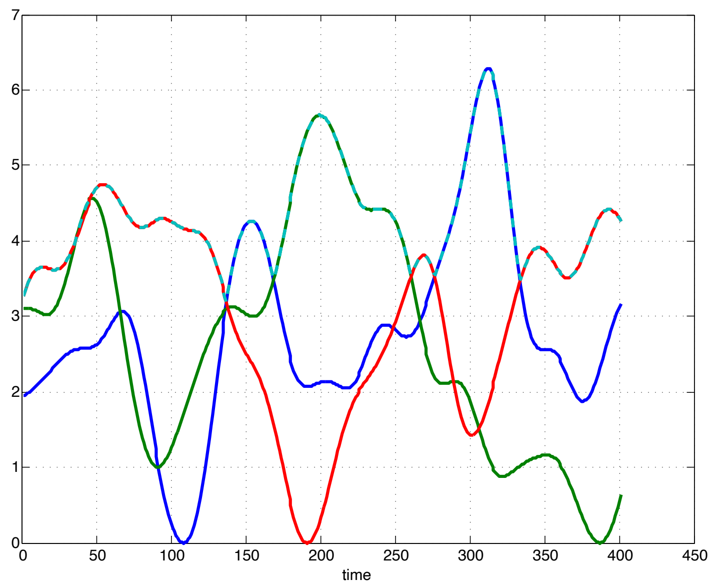

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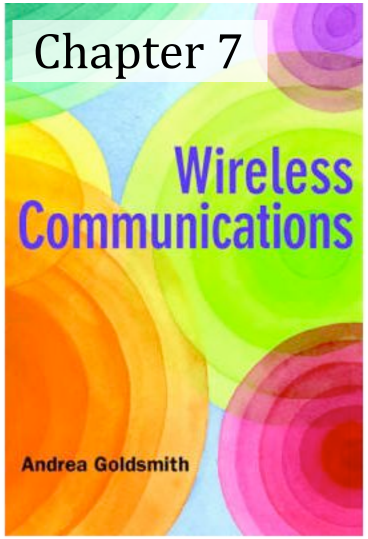
Topics for today

- Lecture learning outcomes
- Diversity: concept and realizations
- Maximal ratio combining (MRC)
- MRC: outage probability and average error probability
- Other combining methods



Suggested reading:

- Every section from Chapter 6,
- No math from 7.2.3
- Not sections 7.4.2, 7.4.3

Today's learning outcomes

At the end of this lecture, you must be able to

- describe realizations to space, time, and frequency diversity
- distinguish between linear combining, maximal ratio combining, selection combining, equal gain combining
- draw block diagrams of combing schemes and determine the output SNR
- numerically evaluate outage probability and average error probability of MRC
- explain the difference between array gain and diversity gain

Last time

$\bar{\bar{\gamma}}_s = \bar{\bar{P}}_r/(N_0 B)$
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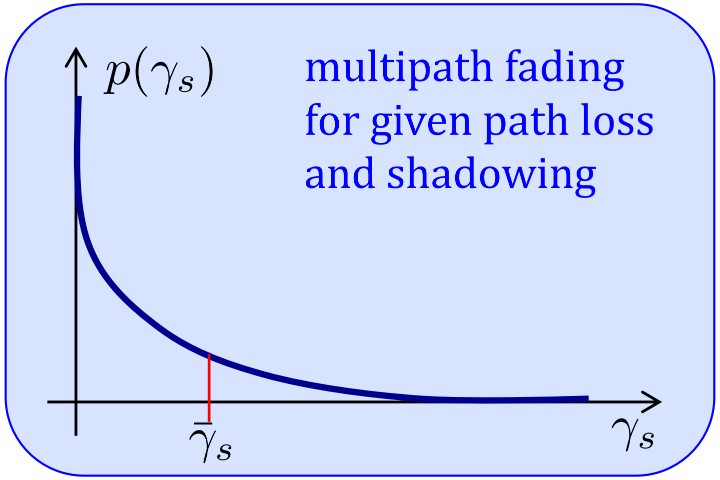
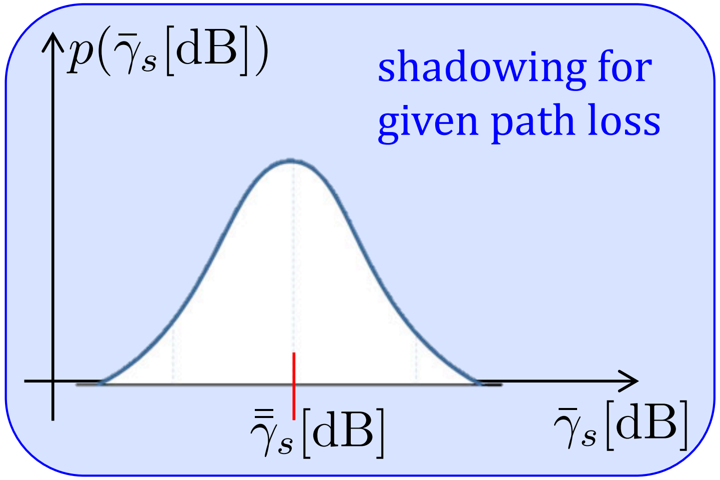
- Path loss $\bar{\bar{P}}_r = P_t K (d_0/d)^\gamma$
- Shadowing $\bar{P}_r[\mathrm{dB}] \sim \mathcal{N}(\bar{\bar{P}}_r[\mathrm{dB}], \sigma^2_{\psi,\mathrm{dB}}) = \bar{\bar{P}}_r[\mathrm{dB}] + \mathcal{N}(0, \sigma^2_{\psi,\mathrm{dB}})$
- Multipath (Rayleigh) $P_r \sim \bar{P}_r \times \exp(1) = \exp(1/\bar{P}_r)(\mathrm{mean} = \bar{P}_r)$

Averaged powers (power is always non-negative, except in dB)

- Power averaged over multipath: $\bar{P}_r = \mathbb{E}\{P_r\}$
- Power averaged over multipath and shadowing: $\bar{\bar{P}}_r[\mathrm{dB}] = \mathbb{E}\{\bar{P}_r[\mathrm{dB}]\}$

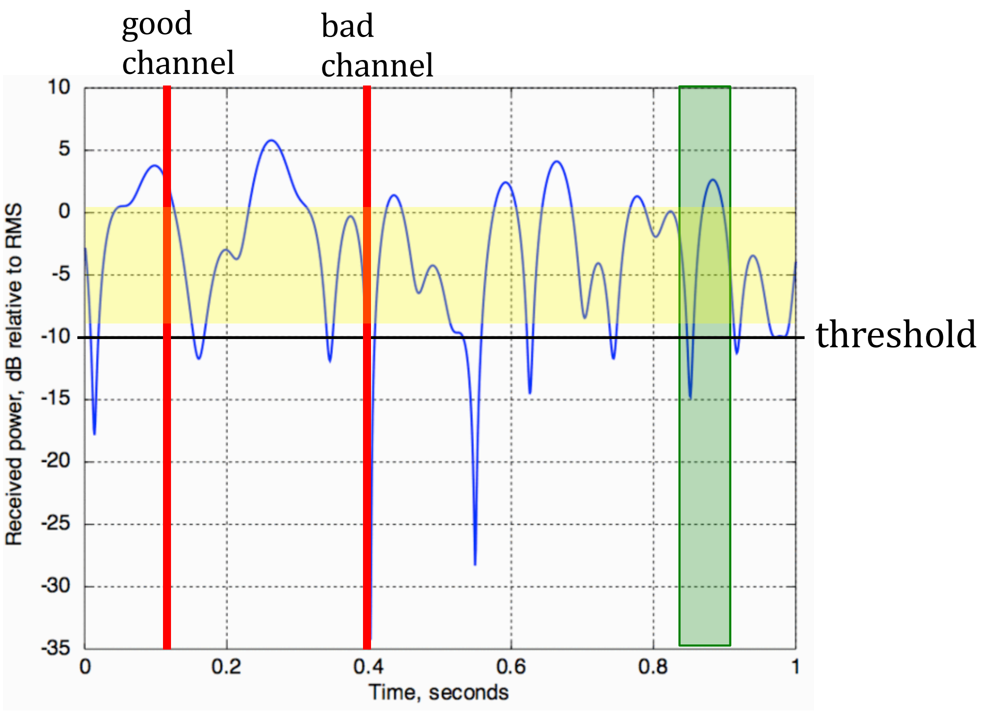
SNR distributions

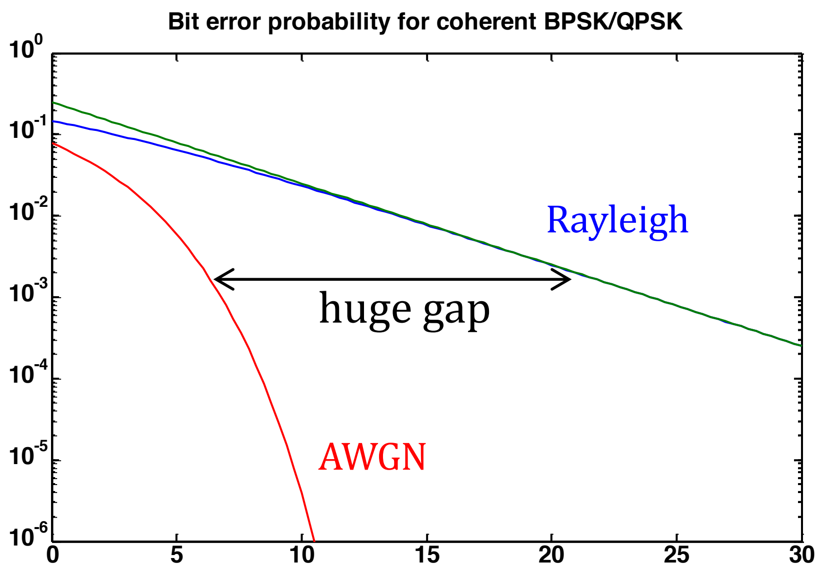




Last time

- Model: $r(t) = c(t)u(t) + n(t)$
- Performance measures
  1. Outage probability:
     $P_{\text{out}} = p(\gamma_s < \text{threshold})$
  2. Average error probability:
     $$\bar{P}_{\text{s}} = \int_0^{+\infty} P_s(\gamma)p(\gamma)\mathrm{d}\gamma$$





Technique for computing average error probability

- Moment generating function (MGF): alternative way to characterize random variables. Laplace transform of the distribution (with sign inversion).
- Definition: R.V. X

$$M_X(s) = \mathbb{E}\{e^{sX}\}$$
$$= \int p_X(x) e^{sx} \mathrm{d}x$$

Moment generating property

$$\mathbb{E}\{X^n\} = \frac{\partial^n}{\partial s^n} [M_X(s)]_{s=0}$$

- For fading distribution

$$M_\gamma(s) = \int_0^{+\infty} p_\gamma(x) e^{sx} \mathrm{d}x$$
$$M_{\gamma_s}^{\text{Rayleigh}}(s) = (1 - s\bar{\gamma}_s)^{-1}$$
$$M_{\gamma_s}^{\text{Rice}}(s) = \frac{1+K}{1+K-s\bar{\gamma}_s} \exp\left(\frac{K s\bar{\gamma}_s}{1+K-s\bar{\gamma}_s}\right)$$

Moment generating function

- For fading distribution $M_\gamma(s) = \int_0^{+\infty} p_\gamma(x) e^{sx} \mathrm{d}x$
- Average error probability

$$\bar{P}_s = \int_0^{+\infty} P_s(\gamma) p(\gamma) \mathrm{d}\gamma$$

- Using MGF

$$P_s(\gamma_s) = \alpha Q(\sqrt{2g\gamma_s})$$

$$\bar{P}_s = \int_0^{+\infty} \frac{\alpha}{\pi} \int_0^{\pi/2} \exp\left(\frac{-2g\gamma_s}{2\sin^2\phi}\right) p(\gamma_s) \mathrm{d}\phi \mathrm{d}\gamma_s$$

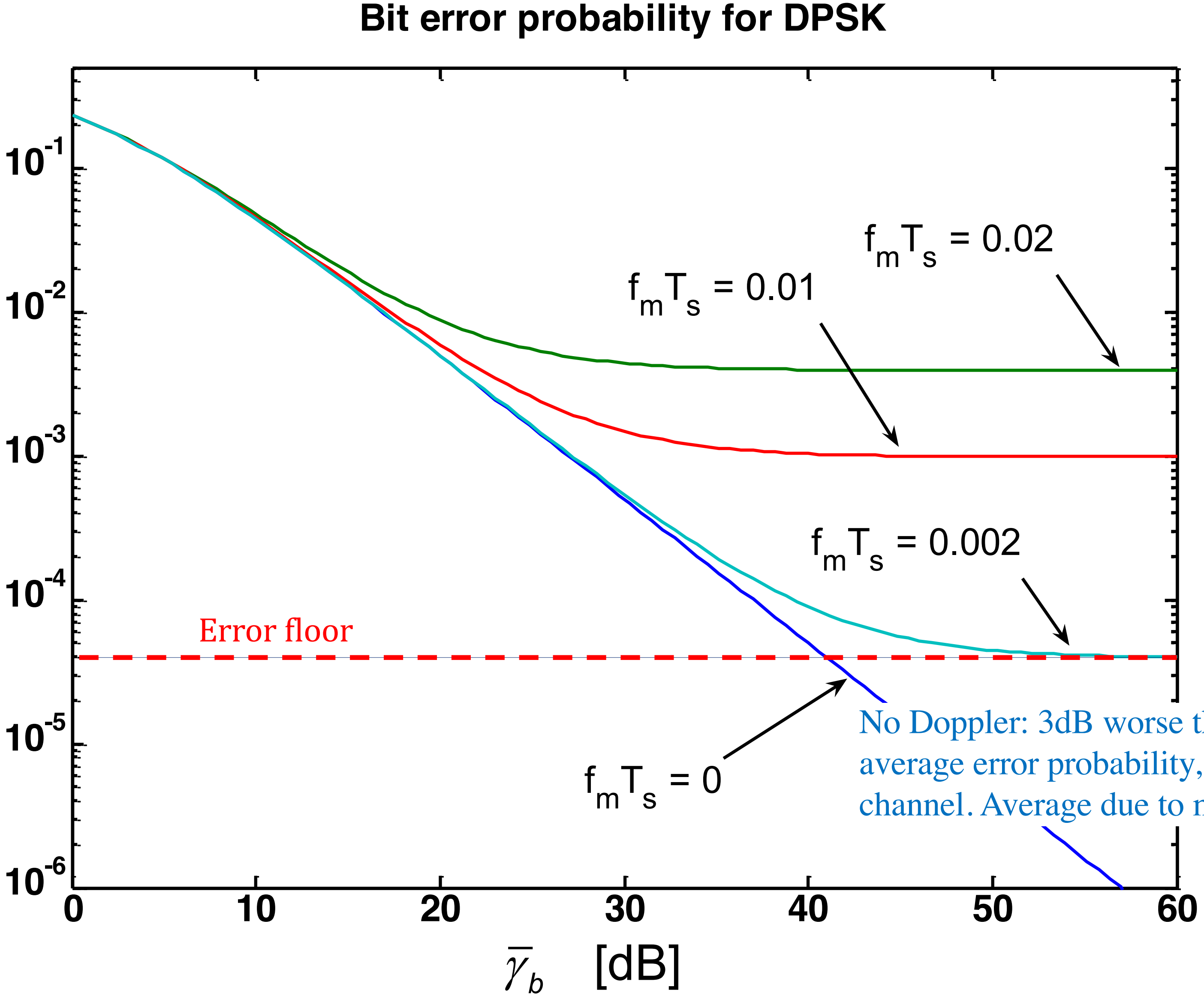
$$= \frac{\alpha}{\pi} \int_0^{\pi/2} M_{\gamma_s}\left(\frac{-g}{\sin^2\phi}\right) \mathrm{d}\phi$$

- "Easy" to calculate numerically

Craig's formula:

$$Q(z) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(\frac{-z^2}{2\sin^2\phi}\right) \mathrm{d}\phi$$

Error floor due to fast fading in DPSK (binary DPSK)

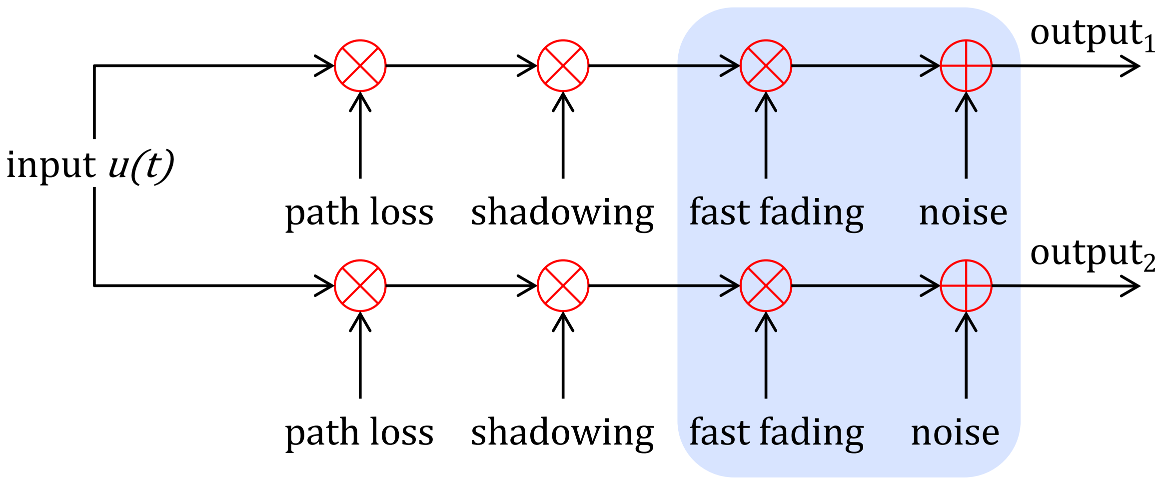


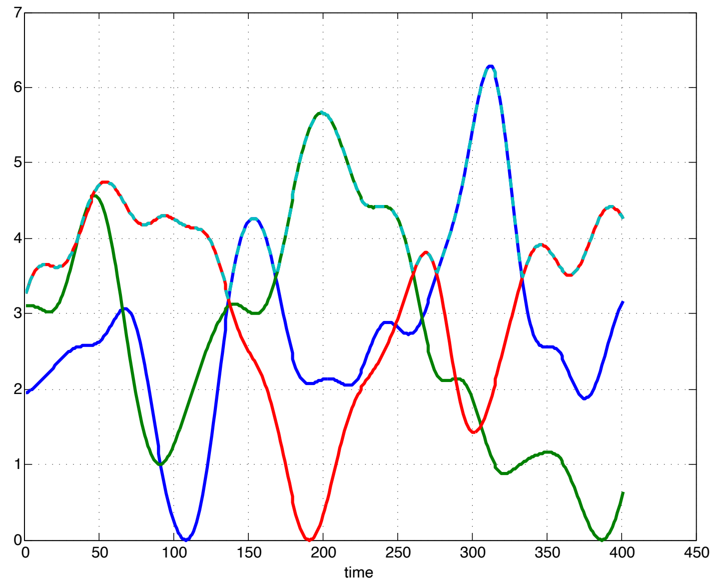
$$f_m = f_{D,\max} = \frac{v}{\lambda}$$

No Doppler: 3dB worse than Rayleigh average error probability, assuming known channel. Average due to many experiments

Diversity

- Using multiple signal paths





- Realizations for *independent* signals
  - space: using multiple receive antennas (how far apart?)
  - time: repeat same signal multiple times (how far apart?)
  - frequency: repeat same signal over different frequencies (how far apart?)

- Also: polarization, angle, coding (BICM)
- Questions: how to combine? what is performance?

Diversity combining

Given

- Consider a discrete-time communication system with M independent fading channels with equal statistics, a common random input s, and outputs $y_1 \ldots y_M$
- Let $y_i = a_i \exp(j\theta_i)\, s + n_i$ where $n_i \sim \mathcal{CN}(0,N_0)$, iid , $a_i \exp(j\theta_i)$ is known

Task

- What is the instantaneous SNR per channel?
- What is the average SNR per channel?(average w.r.t. channel)
- how would you process $\mathbf{y}=[y_1,\ldots,y_M]^T$ to form an ML estimate of s?
- What is the instantaneous SNR after processing?
- What is the average SNR after processing?

Diversity combining

- Instantaneous SNR per channel $\gamma_i = \frac{|a_i|^2 \mathbb{E}\{|s|^2\}}{N_0}$
- Average SNR per channel $\bar{\gamma} = \frac{\mathbb{E}\{|s|^2 |a_i|^2\}}{N_0} = \frac{\mathbb{E}\{|s|^2\}\mathbb{E}\{|a_i|^2\}}{N_0}$
- ML detector:

$$
\begin{aligned}
\hat{s} &= \arg\max_s p(\mathbf{y}|s, \boldsymbol{\beta}) \\
&= \arg\max_s -\sum_i |y_i - a_i s e^{j\theta_i}|^2 \\
&= \arg\max_s -\sum_i |y_i|^2 - |a_i s|^2 + 2\Re\{y_i a_i s^* e^{-j\theta_i}\} \\
&= \arg\max_s -\|\mathbf{y}\|^2 - \|\mathbf{a}\|^2 |s|^2 + 2\Re\{s^* \boldsymbol{\beta}^H \mathbf{y}\}
\end{aligned}
$$

$$\log p(\mathbf{y}|s, \boldsymbol{\beta}) \propto -\|\mathbf{y} - \boldsymbol{\beta} s\|^2$$
$$\boldsymbol{\beta} = [a_1 e^{j\theta_1} \ldots, a_M e^{j\theta_M}]^{\mathrm{T}}$$

Notice that $\boldsymbol{\beta}^H \mathbf{y}$ is a sufficient statistic (knowing this is sufficient to make an ML decision).

- Instantaneous SNR of $\boldsymbol{\beta}^H \mathbf{y} = \boldsymbol{\beta}^H (\boldsymbol{\beta} s + \mathbf{n})$:

$$\gamma_\Sigma = \frac{(\boldsymbol{\beta}^H \boldsymbol{\beta})^2 \mathbb{E}\{|s|^2\}}{\mathbb{E}\{|\boldsymbol{\beta}^H \mathbf{n}|^2\}} = \frac{(\boldsymbol{\beta}^H \boldsymbol{\beta})^2 \mathbb{E}\{|s|^2\}}{N_0 \boldsymbol{\beta}^H \boldsymbol{\beta}} = \frac{(\sum_i a_i^2)\mathbb{E}\{|s|^2\}}{N_0} = \sum_{i=1}^{M} \gamma_i$$

Diversity combining

- Instantaneous SNR per channel $\gamma_i = \frac{|a_i|^2 \mathbb{E}\{|s|^2\}}{N_0}$
- Average SNR per channel

$$\bar{\gamma} = \frac{\mathbb{E}\{|s|^2 |a_i|^2\}}{N_0} = \frac{\mathbb{E}\{|s|^2\}\mathbb{E}\{|a_i|^2\}}{N_0}$$

- ML detector:

$$\begin{aligned}\hat{s} &= \arg\max_s p(\mathbf{y}|s,\boldsymbol{\beta}) \\ &= \arg\max_s \log p(\mathbf{y}|s,\boldsymbol{\beta}) \\ &= \arg\min_s \|\mathbf{y} - \boldsymbol{\beta} s\|^2 \\ &= \arg\min_s \|\mathbf{y}\|^2 + \|\boldsymbol{\beta}\|^2 |s|^2 - 2\Re\{s^* \boldsymbol{\beta}^H \mathbf{y}\}\end{aligned}$$

$$\boxed{\boldsymbol{\beta} = [a_1 e^{j\theta_1} \ldots, a_M e^{j\theta_M}]^{\mathrm{T}}}$$

Notice that $\boldsymbol{\beta}^H \mathbf{y}$ is a sufficient statistic (knowing this is sufficient to make an ML decision).

- Instantaneous SNR of $\boldsymbol{\beta}^H \mathbf{y} = \boldsymbol{\beta}^H(\boldsymbol{\beta} s + \mathbf{n})$:

$$\gamma_\Sigma = \frac{|\boldsymbol{\beta}^H \boldsymbol{\beta}|^2 |s|^2}{\mathbb{E}\{|\boldsymbol{\beta}^H \mathbf{n}|^2\}} = \frac{\|\boldsymbol{\beta}\|^4 |s|^2}{\|\boldsymbol{\beta}\|^2 N_0} = \frac{\sum_{i=1}^M a_i^2 |s|^2}{N_0} = \sum_{i=1}^M \gamma_i$$

Maximum likelihood (also maximal-ratio combining - MRC)

- Weigh each branch with complex conjugate of channel ("better branches get larger weight")
- Co-phasing + multiply with amplitude
- Output SNR: $$\gamma_\Sigma = \sum_{i=1}^{M} \gamma_i$$
- Distribution of γ_Σ depends on distribution of γ_i
- Rayleigh fading: sum of M i.i.d. exponential R.V.: Chi-squared R.V. (equivalent: Gamma distribution)

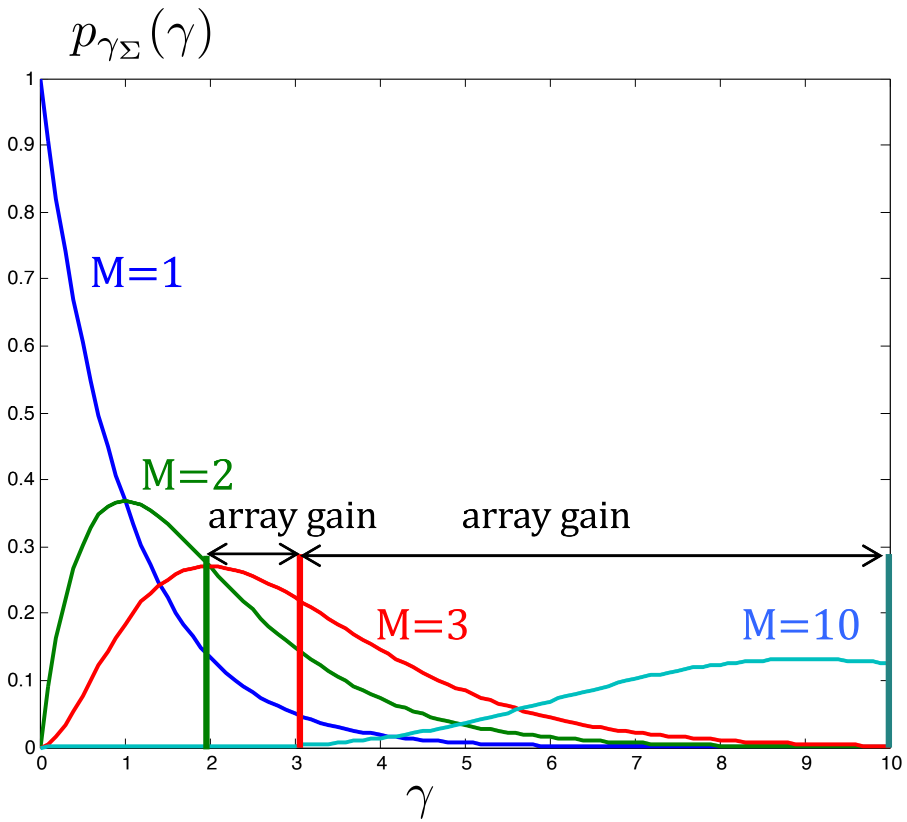
$$p_{\gamma_\Sigma}(\gamma) = \frac{\gamma^{M-1} e^{-\gamma/\bar{\gamma}}}{\bar{\gamma}^M (M-1)!} \qquad P_{\gamma_\Sigma}(\gamma) = 1 - e^{-\gamma/\bar{\gamma}} \sum_{k=1}^{M} \frac{(\gamma/\bar{\gamma})^{k-1}}{(k-1)!}$$

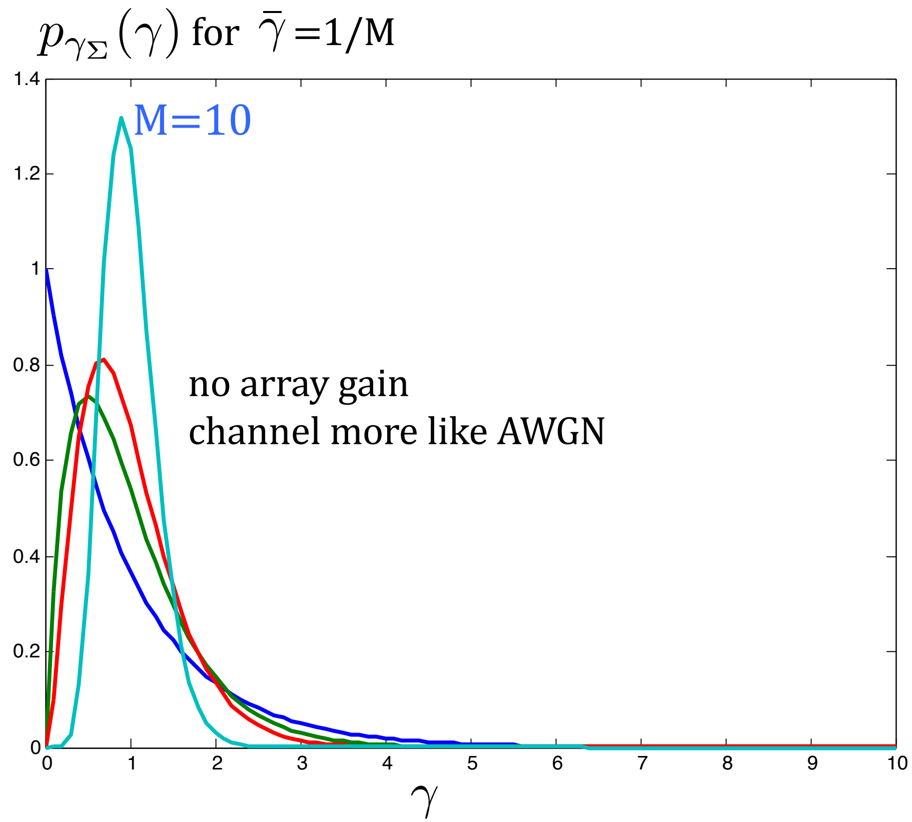
array gain: we collect more power

leads to diversity gain: channel looks more like AWGN

- Mean of γ_Σ : $\bar{\gamma}_\Sigma = \boxed{M}\bar{\gamma}$
- Variance of γ_Σ : $M\bar{\gamma}^2$
- Divide SNR per branch by M → mean $\bar{\gamma}$ and variance $\bar{\gamma}^2\boxed{/M}$
- We can compute outage probability and average error probability
- Diversity shows up as slope in outage prob. and average error prob.

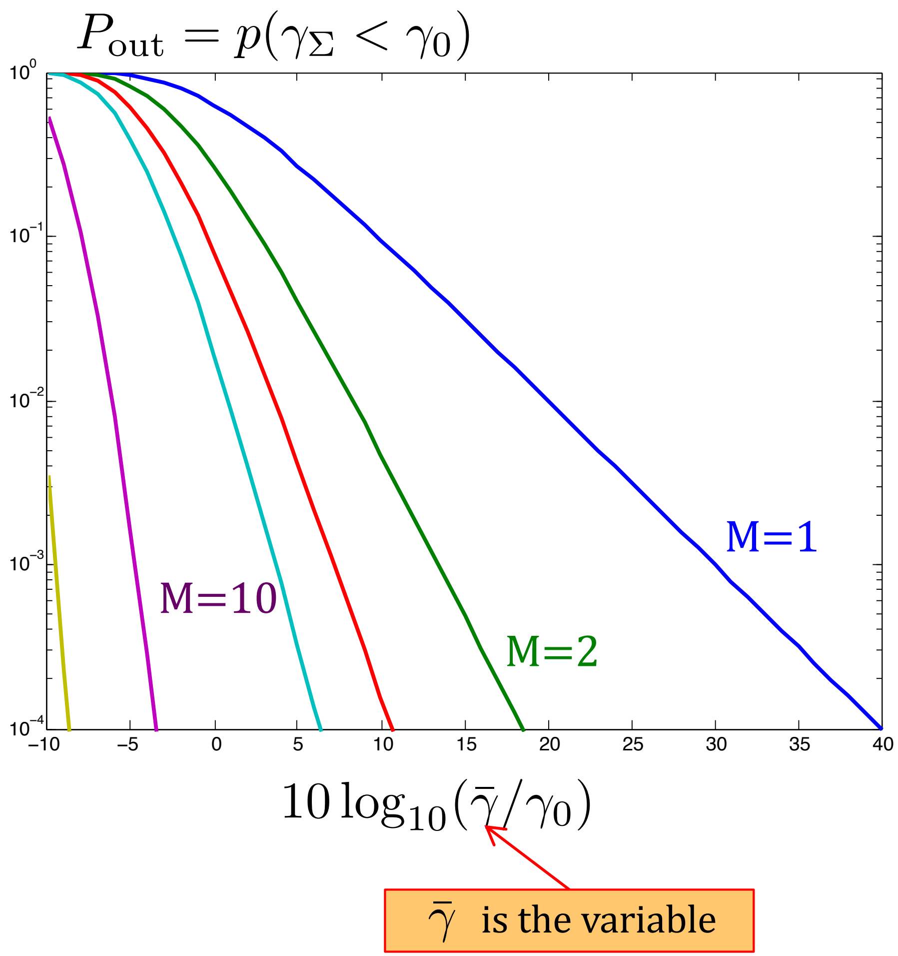
MRC: output SNR distribution ($\bar{\gamma} = 1$ per branch as an example)

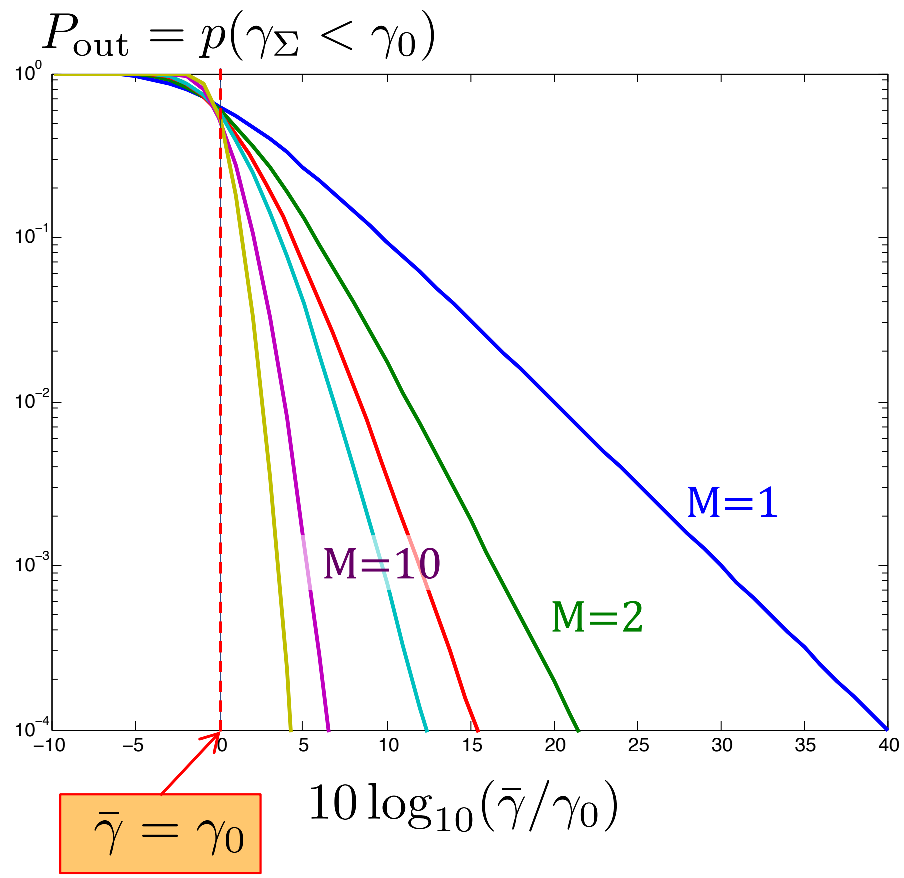




MRC: outage probability (target γ_0, some fixed value)

Per branch SNR $\bar{\gamma}/M$: shift by $10\log_{10}(M)$





Array gain (shift) & diversity gain (slope)

diversity gain (slope)

MRC: average error probability

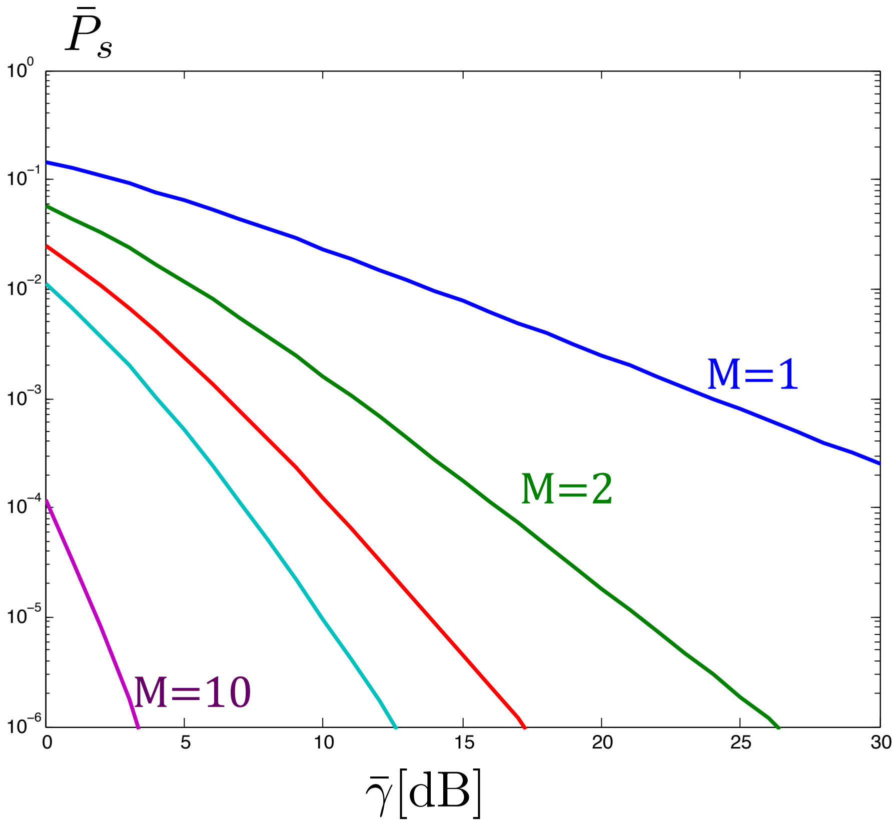
- Recall $\bar{P}_s = \int_0^{+\infty} P_s(\gamma) p(\gamma) \mathrm{d}\gamma$ now of output SNR after MRC
- Consider $P_s(\gamma) = \int_A^B c_1 \exp(-c_2(x)\gamma)\mathrm{d}x$ in which $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_M$
- Due to independence: MGF of sum is product of MGFs

$$
\begin{aligned}
\bar{P}_s &= c_1 \int_A^B \prod_{i=1}^{M} \mathcal{M}_{\gamma_i}(-c_2(x))\mathrm{d}x \\
&= c_1 \int_A^B \mathcal{M}_{\gamma_i}^{M}(-c_2(x))\mathrm{d}x
\end{aligned}
$$

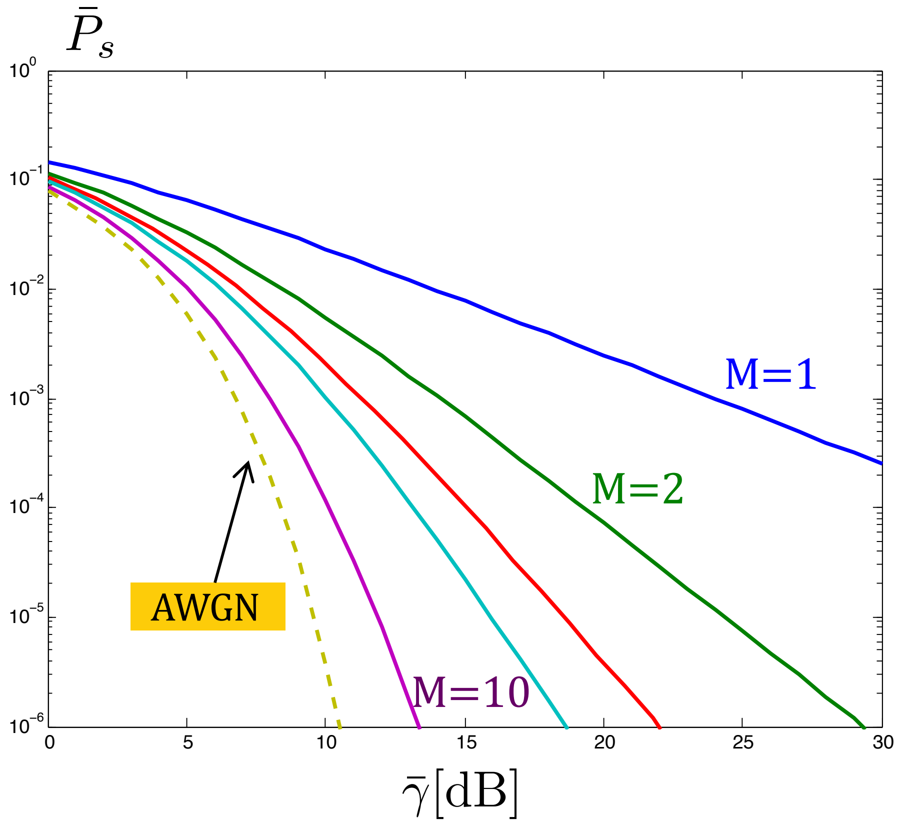
- Known expression for BPSK and Rayleigh fading, most other cases: numerical evaluation of integral

$$
\bar{P}_s \approx \alpha_M \left(\frac{\beta_M \bar{\gamma}}{2}\right)^{-M}.
$$

MRC: average error probability for BPSK



after dividing $\bar{\gamma}$ by M: shift by $10\log_{10}$(M)



Array gain (shift) & diversity gain (slope)

diversity gain (slope)

Other combining methods

- Equal gain combining (EGC):
  - only co-phasing
  - no closed-form distribution of γ_Σ in general

- Selection combining (SC):
  - choose best current branch, so
    $$P_{\gamma_\Sigma}(\gamma) = p(\gamma_\Sigma < \gamma) = p(\max_i \gamma_i < \gamma) = \prod_{i=1}^{M} p(\gamma_i < \gamma)$$
  - For i.i.d. Rayleigh
    $$P_{\gamma_\Sigma}(\gamma) = \left(1 - e^{-\gamma/\bar{\gamma}}\right)^M$$
  - Mean known, but no known MGF
    $$\bar{\gamma}_\Sigma^{\mathrm{SC}} = \bar{\gamma} \sum_{i=1}^{M} 1/i \boxed{<} M\bar{\gamma} = \bar{\gamma}_\Sigma^{\mathrm{MRC}}$$

less array gain

- Threshold combining: similar to SC but less switching

Diversity combining

Given

- Diversity system with M branches, average SNR per branch of 15 dB
- Target SNR of 7 dB
- Rayleigh fading

Task

- What is the outage probability of selection combining for M=1, M=2, M=3?

Transmitter vs. receiver diversity

- M branches can be created at transmitter or receiver
- Example
  - Receive diversity: M receive antennas, 1 transmit antenna
  - Transmit diversity: M transmit antennas, 1 receive antenna
- *Case 1:* Channel known at transmitter: diversity order M, array gain M. But challenging in practice.
- *Case 2:* Channel not known at transmitter: Alamouti scheme

Homework problem

Given the following optimization problem, for convex $f_k(x_k)$: minimize $\sum_{k=1}^{N} f_k(x_k)$

s.t. $x_k \geq 0$

$\mathbf{a}^T\mathbf{x} + b = 0$

then the solution must satisfy

$$x_k \geq 0 \quad \text{and} \quad \frac{\partial f_k(x_k)}{\partial x_k} + \lambda \frac{\partial(\mathbf{a}^T\mathbf{x} + b)}{\partial x_k} = 0$$

Show that for the following problem minimize $-\sum_{k=1}^{N} a_k \log_2(1 + \gamma_k x_k)$

s.t. $\mathbf{a}^T\mathbf{x} + b = 0$

$x_i \geq 0,$

the solution can be written as $x_k = \max(0, 1/\gamma_c - 1/\gamma_k)$ for a suitable value γ_c

Today's learning outcomes

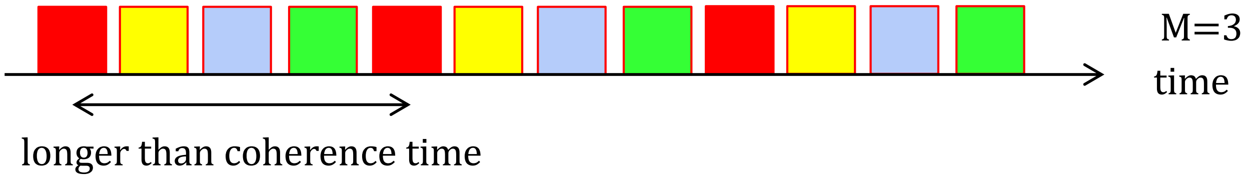
At the end of this lecture, you must be able to

- describe realizations to space, time, and frequency diversity
- distinguish between linear combining, maximal ratio combining, selection combining, equal gain combining
- draw block diagrams of combing schemes and determine the output SNR
- numerically evaluate outage probability and average error probability of MRC
- explain the difference between array gain and diversity gain

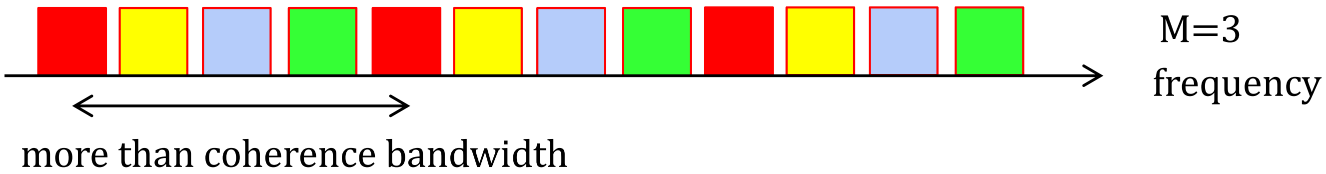
Solutions

Creating diversity

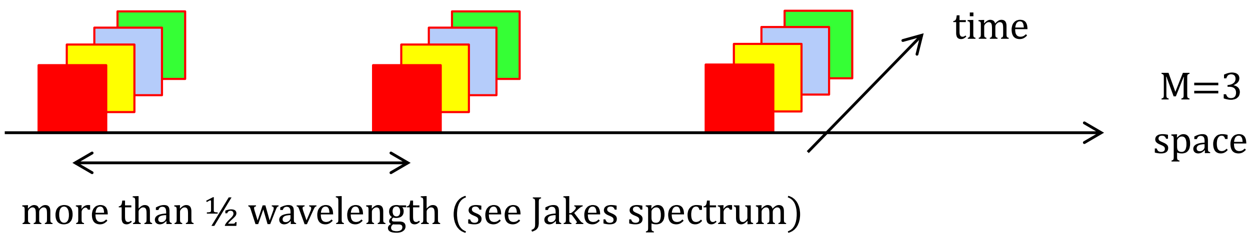
- Time: send same symbol M times, wait at least coherence time in between



- Frequency: send same symbol M times, each on different frequency, with at least coherence bandwidth in between



- Space: send symbol 1 time, but use M antennas at receiver, spaced at least 0.5 wavelength apart



Diversity combining

Given

- Diversity system with M branches, average SNR per branch of 15 dB
- Target SNR of 7 dB
- Rayleigh fading

Task

- What is the outage probability of selection combining for M=1, M=2, M=3?

Solution

- The outage probability is $P_{\text{out}}(\gamma_0) = (1 - \exp(-\gamma_0/\bar{\gamma}))^M$
  with $\gamma_0 = 7\text{dB} \approx 5$
  $\bar{\gamma} = 15\text{dB} \approx 31.6,$
- Substitution for M=1,2,3 gives outages of 0.15, 0.01, and 0.003, respectively

Homework problem

Given the following optimization problem, for convex $f_k(x_k)$: minimize $\sum_{k=1}^{N} f_k(x_k)$

s.t. $x_k \geq 0$

$\mathbf{a}^T \mathbf{x} + b = 0$

then the solution must satisfy

$$x_k \geq 0 \quad \text{and} \quad \frac{\partial f_k(x_k)}{\partial x_k} + \lambda \frac{\partial(\mathbf{a}^T\mathbf{x} + b)}{\partial x_k} = 0$$

Show that for the following problem minimize $-\sum_{k=1}^{N} a_k \log_2(1 + \gamma_k x_k)$

s.t. $\mathbf{a}^T \mathbf{x} + b = 0, x_i \geq 0, \forall i$

the solution can be written as $x_k = \max(0, 1/\gamma_c - 1/\gamma_k)$ for a suitable value γ_c

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Solution:

$$\frac{\partial f_k(x_k)}{\partial x_k} + \lambda \frac{\partial(\mathbf{a}^T\mathbf{x} + b)}{\partial x_k} = 0$$

$$-\frac{1}{\ln 2} \frac{a_k \gamma_k}{1 + \gamma_k x_k} + \lambda a_k = 0, x_k \geq 0$$

$$x_k = 1/(\lambda \ln 2) - 1/\gamma_k, x_k \geq 0$$

$$x_k = \max(0, 1/(\lambda \ln 2) - 1/\gamma_k)$$

choose λ such that
$\mathbf{a}^T \mathbf{x} + b = 0$