Wireless Communications SSY135 – Lecture 5

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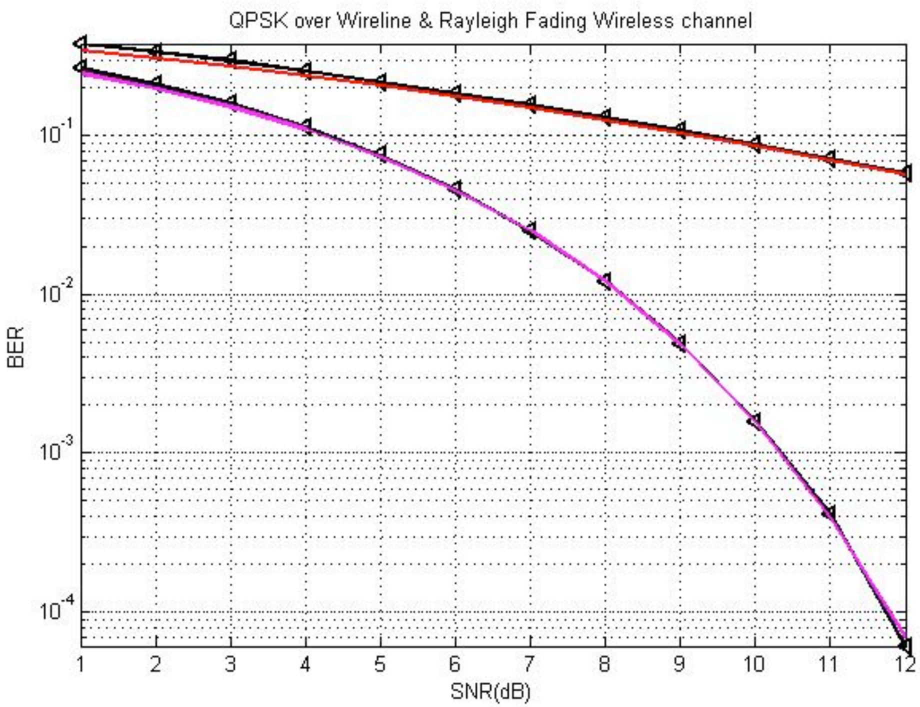
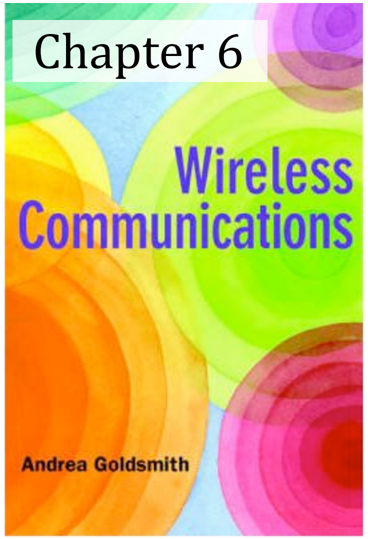


Topics for today

- Lecture learning outcomes
- Modulation formats and error probability
- Outage probability
- Average error probability
- Communication in fast fading: differential encoding / detection



Suggested reading:

- Every section from Chapter 6 …
- … Except 6.1.2, 6.1.3, 6.1.4, 6.1.5, 6.5

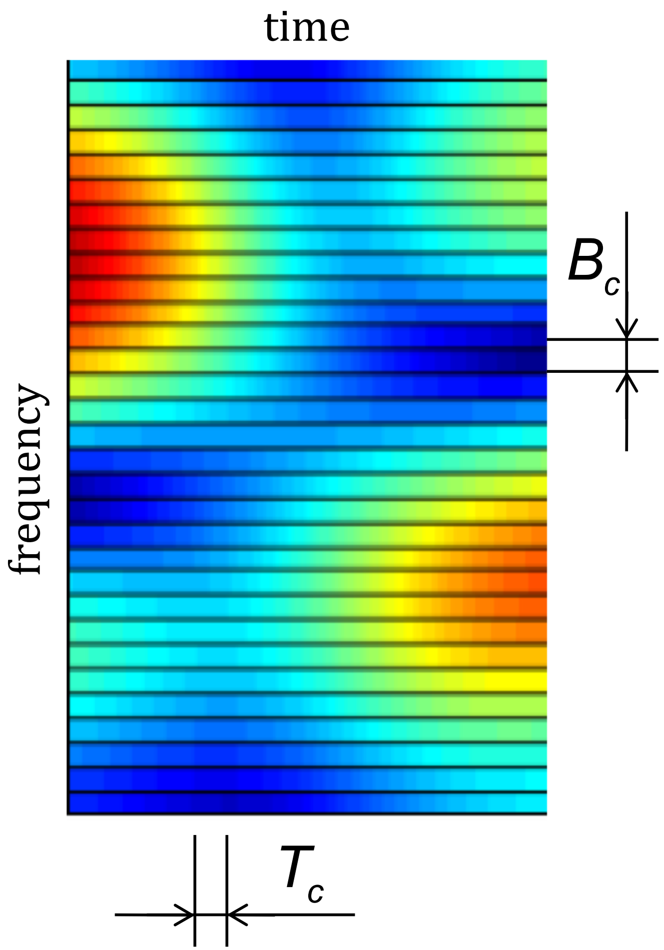
Today's learning outcomes

At the end of this lecture, you must be able to

- distinguish between SNR per symbol, SNR per bit, BER, SER
- define and compute outage probabilities to meet operational requirements
- define and compute average error probabilities
- describe the differences between coherent and differential detection
- explain why DPSK exhibits an error floor in fast fading

Last time

Realization of channel

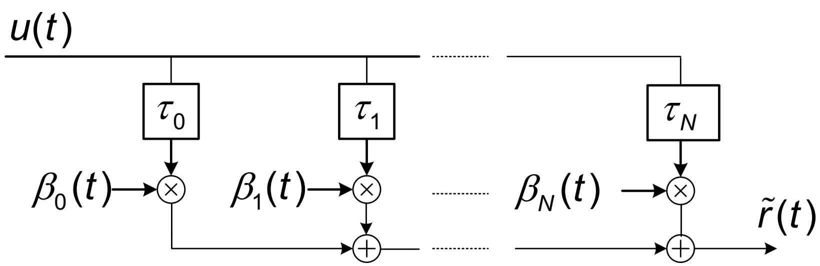


Channel has different components

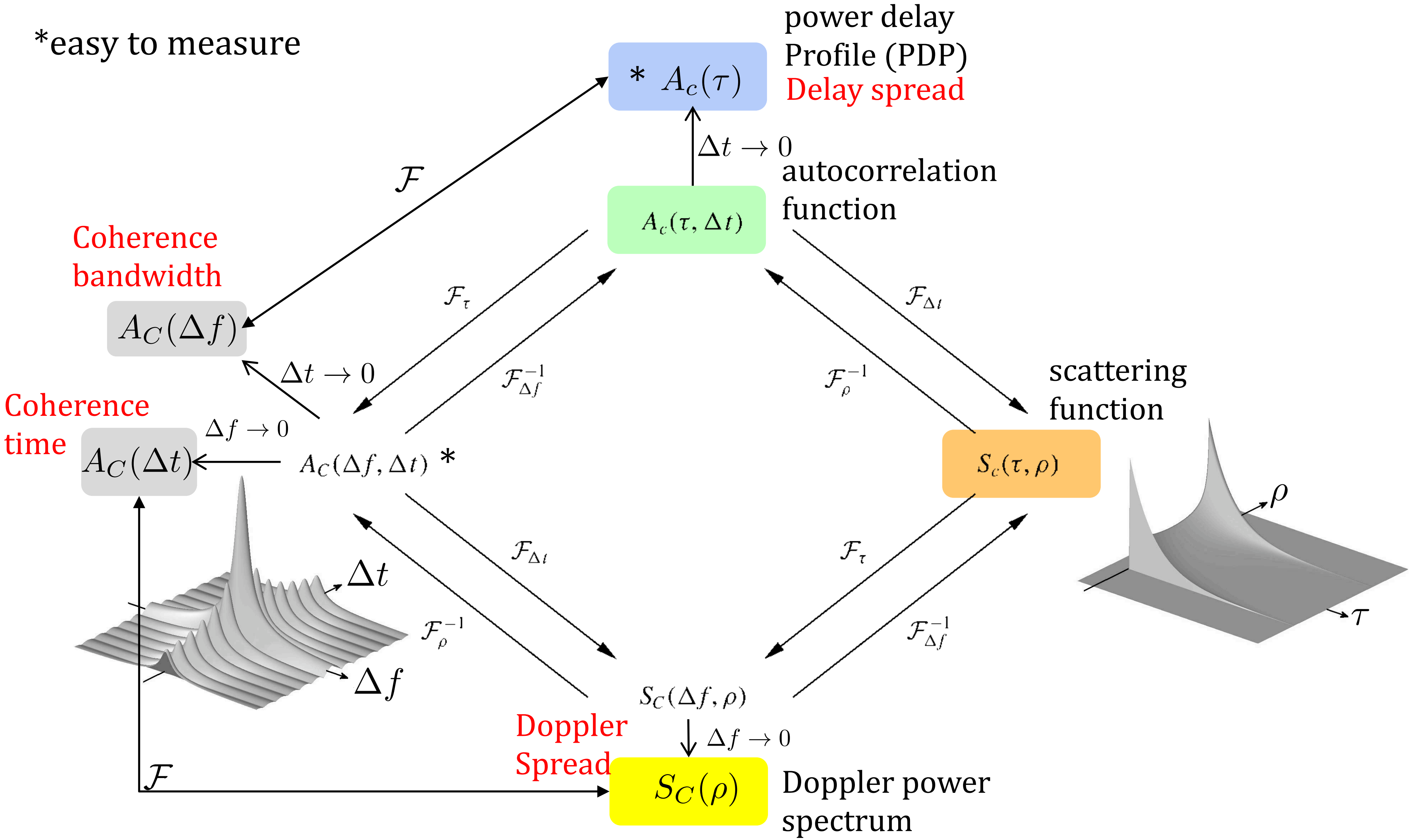
- path loss
- shadowing
- multipath

Note:

- Each varies over time, with specific autocorrelation
- Shadowing and multipath are random with specific distribution



Autocorrelation function and Fourier transforms



Matz, Gerald, and Franz Hlawatsch. "Fundamentals of time-varying communication channels." *Wireless Communications over Rapidly Time-Varying Channels* (2011): 1-63.

Channel vs communication

nature

Channel

From autocorrelation function $A_c(\tau, \Delta t)$

- τ for which $A_c(\tau)$ is close to zero: delay spread
- Δf for which $A_C(\Delta f)$ is close to zero: coherence bandwidth
- Δt for which $A_C(\Delta t)$ is close to zero: coherence time
- ρ for which $S_C(\rho)$ is close to zero: Doppler spread

Relations:

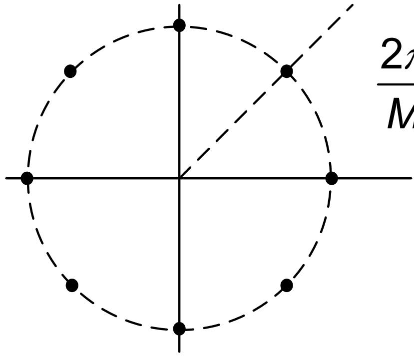
- $A_C(\Delta f)$ = Fourier transform($A_c(\tau)$), so delay spread ≈ 1/ coherence bandwidth
- $S_C(\rho)$ = Fourier transform($A_C(\Delta t)$), so Doppler spread ≈ 1/ coherence time

engineering

Communication (symbols slot T, bandwidth B=1/T):

- Flat (no ISI): B « coherence bandwidth (same as: T » delay spread)
- Dispersive (ISI): B > coherence bandwidth (same as: T < delay spread)
- Fast (each symbol different channel): T ≈ coherence time (same: B ≈ Doppler spread)
- Slow (many symbols same channel): T « coherence time (same: B » Doppler spread)

Baseband communication



- Transmitted signal, bandwidth B

$$u(t) = \sum_k a_k p(t - kT_s)$$

- Received signal $r(t) = \underbrace{u(t)}_{\text{power } P_r} + n(t)$ PSD $N_0/2$ per real dimension
- Receiver processing, after matched filter and synchronized sampling

$$y_k = \sqrt{E_s} a_k + n_k, n_k \sim \mathcal{CN}(0, N_0)$$

- Signal-to-noise ratio – analog version $\mathrm{SNR} = P_r/(N_0 B)$
- Signal-to-noise ratio – digital version $\mathrm{SNR} = E_s/N_0$ $\quad T_s = \mathrm{constant}/B$
- SNR per bit and SNR per symbol: symbol = $\log_2(M)$ bits

$$\gamma_s = E_s/N_0$$

$$\gamma_b = E_b/N_0 = E_s/(N_0 \log_2 M)$$

- Performance of modulation format

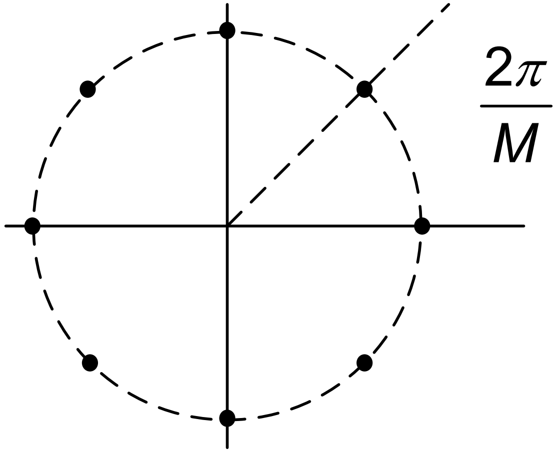
$$\mathrm{SER} = P_{\mathrm{s}}(M, \mathrm{mod}, \gamma_b)$$

$$\mathrm{BER} = P_{\mathrm{b}}(M, \mathrm{mod}, \mathrm{mapping}, \gamma_b)$$

- Comment: gamma (γ) used for both SNR and path loss exponent

Example

- M-ary phase shift keying



$$P_s = \frac{1}{\pi}\int_0^{\pi-\pi/M} \exp\left(-\frac{\sin^2(\pi/M)}{\sin^2(\phi)}\frac{E_s}{N_0}\right)d\phi$$

$$= \begin{cases} Q\left(\sqrt{\frac{2E_s}{N_0}}\right), & M=2 \\ 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right), & M=4 \\ \leq 2Q\left(\sin\frac{\pi}{M}\sqrt{\frac{2E_s}{N_0}}\right), & M>4 \end{cases}$$

- For PSK, QAM, we have approximately $P_s \approx \alpha_M Q\left(\sqrt{\beta_M \frac{E_s}{N_0}}\right)$

- For PSK, QAM with gray coding $P_b \approx \frac{P_s}{\log_2 M}$

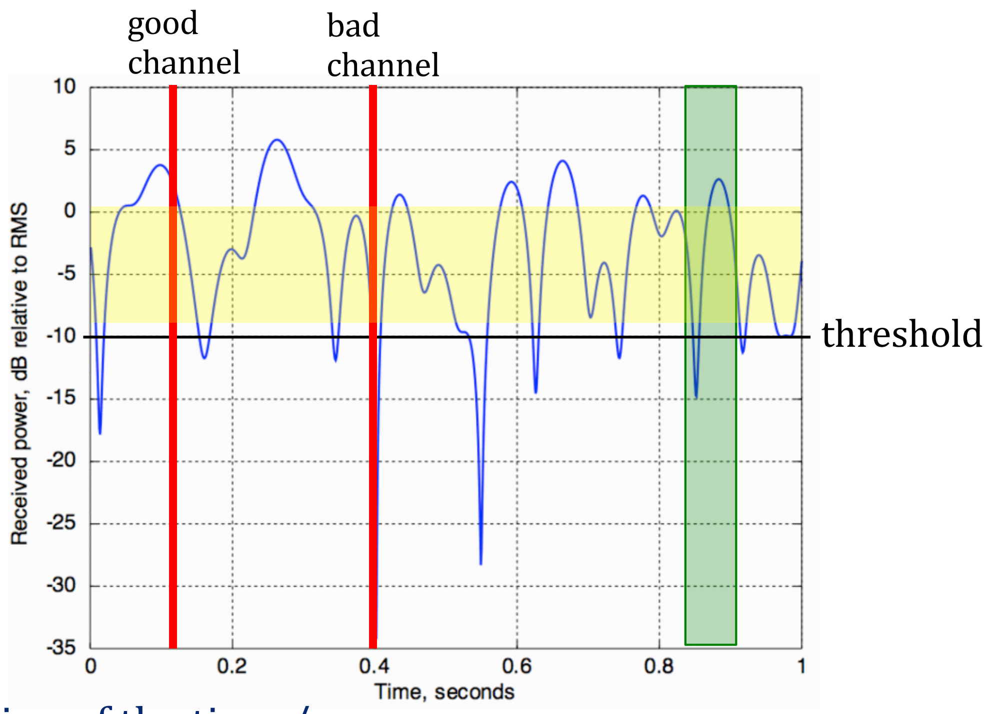
- In MATLAB: `qfunc`

General cases

Table 6.1: Approximate symbol and bit error probabilities for coherent modulations

Modulation	$P_s(\gamma_s)$	$P_b(\gamma_b)$
BFSK		$P_b = Q\left(\sqrt{\gamma_b}\right)$
BPSK		$P_b = Q\left(\sqrt{2\gamma_b}\right)$
QPSK, 4-QAM	$P_s \approx 2Q\left(\sqrt{\gamma_s}\right)$	$P_b \approx Q\left(\sqrt{2\gamma_b}\right)$
MPAM	$P_s = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\bar{\gamma}_s}{M^2-1}}\right)$	$P_b \approx \frac{2(M-1)}{M\log_2 M} Q\left(\sqrt{\frac{6\bar{\gamma}_b \log_2 M}{M^2-1}}\right)$
MPSK	$P_s \approx 2Q\left(\sqrt{2\gamma_s}\sin\left(\frac{\pi}{M}\right)\right)$	$P_b \approx \frac{2}{\log_2 M} Q\left(\sqrt{2\gamma_b \log_2 M}\sin\left(\frac{\pi}{M}\right)\right)$
Rectangular MQAM	$P_s \approx 4Q\left(\sqrt{\frac{3\bar{\gamma}_s}{M-1}}\right)$	$P_b \approx \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\bar{\gamma}_b \log_2 M}{M-1}}\right)$
Nonrectangular MQAM	$P_s \approx 4Q\left(\sqrt{\frac{3\bar{\gamma}_s}{M-1}}\right)$	$P_b \approx \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\bar{\gamma}_b \log_2 M}{M-1}}\right)$

Performance in flat fading



- Model: $r(t) = c(t)u(t) + n(t)$
- Performance measures
  1. Outage probability:
     $P_{\text{out}} = p(\gamma_s < \text{threshold})$
  2. Average error probability:
     $$\bar{P}_{\text{s}} = \int_0^{+\infty} P_s(\gamma)p(\gamma)\mathrm{d}\gamma$$
  3. Combination of 1 & 2: achieve certain average error prob. some fraction of the time / space
- Three regimes
  1. symbol time ≪ coherence time: outage probability, use of power control
  2. symbol time ≫ coherence time: fading is averaged out, not considered
  3. symbol time ≈ coherence time: average error probability, use diversity

- **Performance in frequency selective fading:** See equalization and OFDM lectures

Three different notions of power and SNR

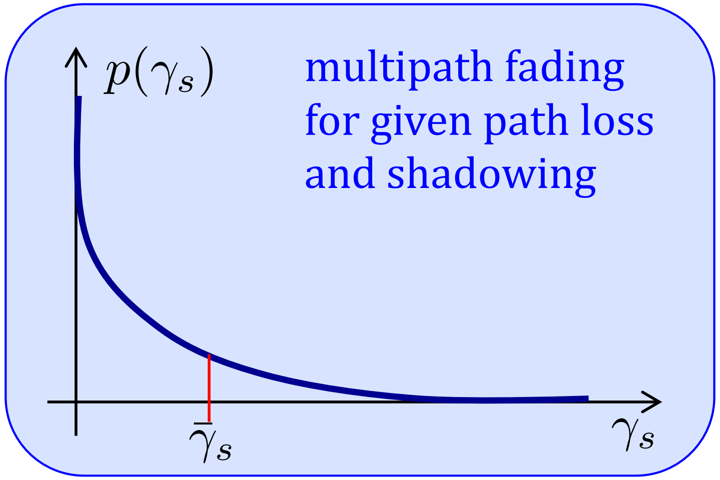
$$\bar{\bar{\gamma}}_s = \bar{\bar{P}}_r/(N_0 B)$$
$$\bar{\gamma}_s = \bar{P}_r/(N_0 B)$$
$$\gamma_s = P_r/(N_0 B)$$

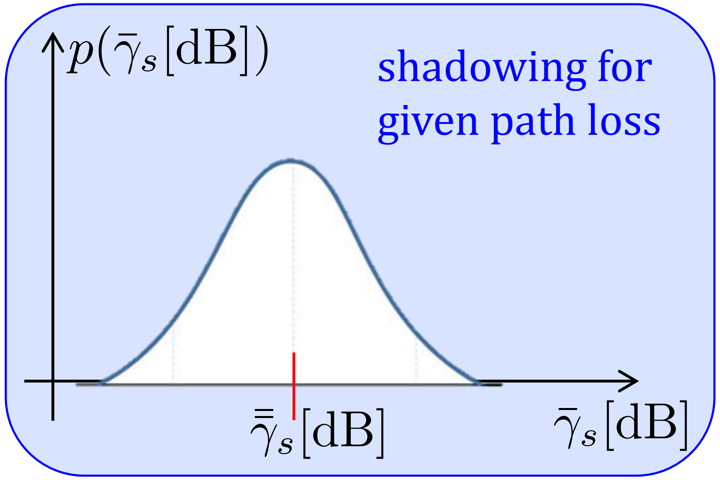
- Path loss $\bar{\bar{P}}_r = K(d_0/d)^\gamma$
- Shadowing $\bar{P}_r[\mathrm{dB}] \sim \mathcal{N}(\bar{\bar{P}}_r[\mathrm{dB}], \sigma^2_{\psi,\mathrm{dB}}) = \bar{\bar{P}}_r[\mathrm{dB}] + \mathcal{N}(0, \sigma^2_{\psi,\mathrm{dB}})$
- Multipath (Rayleigh) $P_r \sim \bar{P}_r \times \exp(1) = \exp(1/\bar{P}_r)(\mathrm{mean} = \bar{P}_r)$

Averaged powers (power is always non-negative, except in dB)

- Power averaged over multipath: $\bar{P}_r = \mathbb{E}\{P_r\}$
- Power averaged over multipath and shadowing: $\bar{\bar{P}}_r[\mathrm{dB}] = \mathbb{E}\{\bar{P}_r[\mathrm{dB}]\}$

SNR distributions





Measure 1: Outage probability

- *Outage*: event where performance is below threshold (e.g., BER above 10^{-3})
- In terms of SNR: $P_{\text{out}} = p(\gamma_s < \text{threshold})$
- Rayleigh fading: SNR has exponential distribution

$$P_{\text{out}} = p(\gamma_s < \gamma_0)$$
$$= \int_0^{\gamma_0} \frac{1}{\bar{\gamma}_s} e^{-\gamma_s/\bar{\gamma}_s} \mathrm{d}\gamma_s = 1 - e^{-\gamma_0/\bar{\gamma}_s}$$

- Implies value for average SNR (fade margin)

$$\bar{\gamma}_s = \frac{\gamma_0}{-\ln(1 - P_{\text{out}})}$$

- Outage can also be defined for shadowing

$$P_{\text{out}} = p(\bar{\gamma}_s < \text{threshold})$$

Measure 2: Average error probability

- Time [t_0,t_1] during which $\bar{\gamma}_s$ is constant and many symbols are sent
- Each symbol sees different fading
- Average symbol error probability γ_s

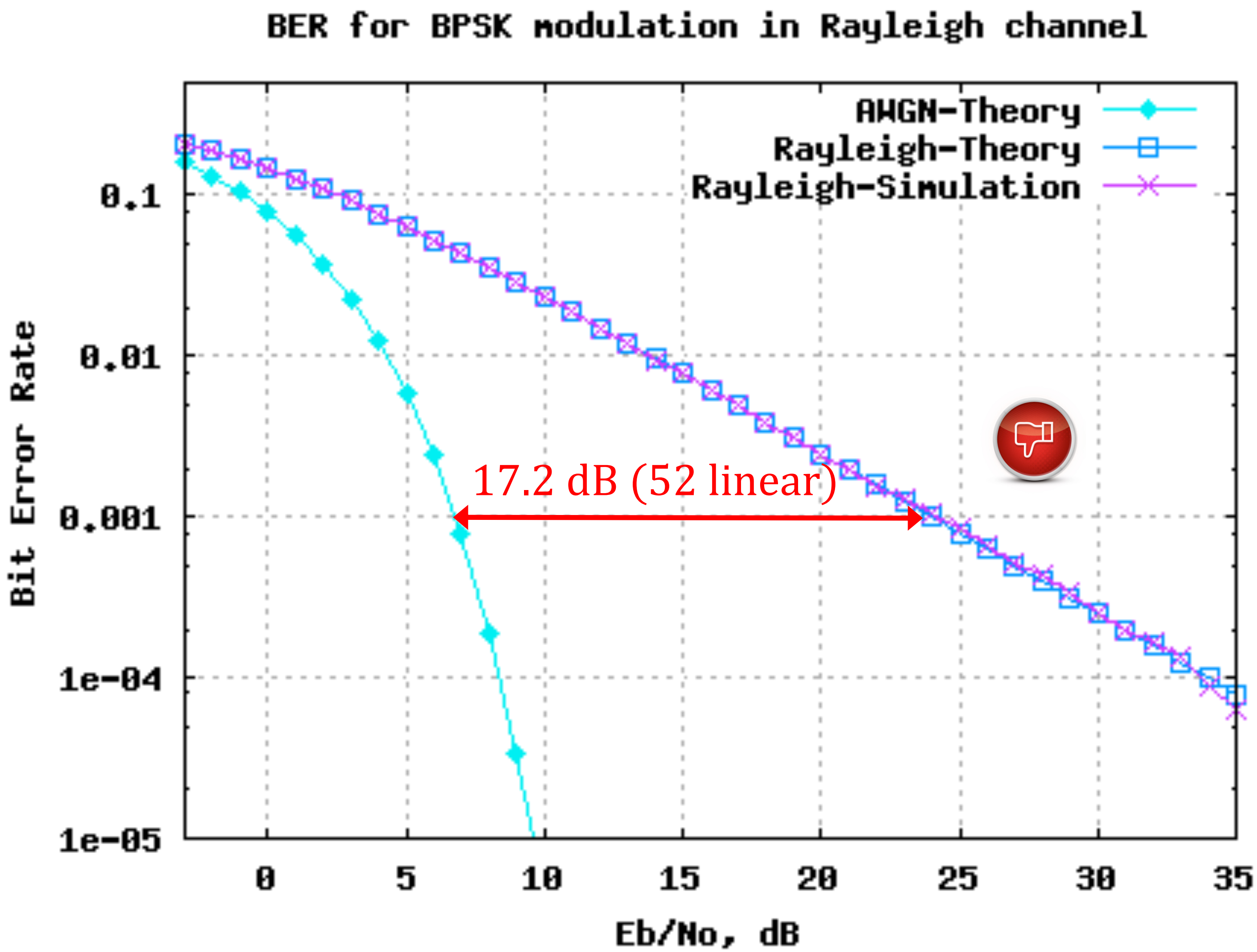
$$\bar{P}_s = \int_0^{+\infty} P_s(\gamma)\boxed{p(\gamma)}\mathrm{d}\gamma$$

Recall: Gaussian channel taps, Rayleigh envelope, exponential SNR

- For BPSK (note: $\bar{\gamma}_s = \bar{\gamma}_b$)

$$\begin{aligned}\bar{P}_s &= \int_0^{+\infty} Q(\sqrt{2\gamma})\frac{1}{\bar{\gamma}_s}e^{-\gamma/\bar{\gamma}_s}\mathrm{d}\gamma \\ &= \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\gamma}_s}{1+\bar{\gamma}_s}}\right) \\ &\approx \frac{1}{4\bar{\gamma}_s}\end{aligned}$$

Measure 2: Average error probability



Measure 3: Combined outage and average probability

Concept:

- Outage when $\bar{\gamma}_s$ falls below threshold
- When not in outage: target average error probability

Example:

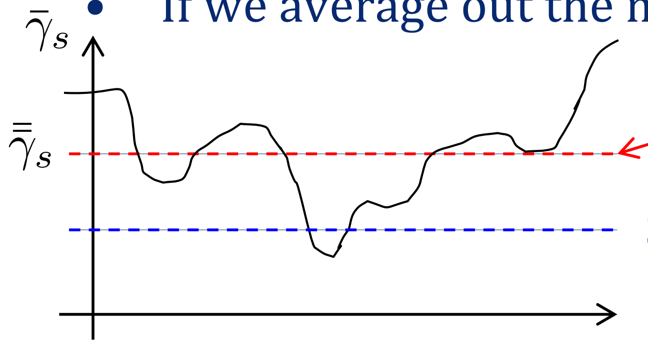
- BPSK signaling over a channel with Rayleigh fading and shadowing (8 dB standard deviation)
- Average SER should be below 10^{-4} most of the time (99%)

Task

- How should we set $\bar{\bar{\gamma}}_s$ to achieve the target SER 99% of the time?
- How should we set $\bar{\bar{\gamma}}_s$ to achieve the target SER 99.999% of the time?

Solution

- For a given $\bar{\bar{\gamma}}_s$ the SNR will vary due to shadowing and multipath fading
- If we average out the multipath fading, we get



- We want to make sure that $\bar{\gamma}_s$ is such that the average BER (averaged due to Rayleigh fading) is 10^{-4}. This means that we want $\bar{\gamma}_s \geq 34\text{dB}$ (slide 13)
- So we want that $\bar{\gamma}_s$ is larger than 34 dB, 99% of the time. Since $\bar{\gamma}_s$ is log-normal with mean $\bar{\bar{\gamma}}_s$ [dB] and standard deviation 8, we find that

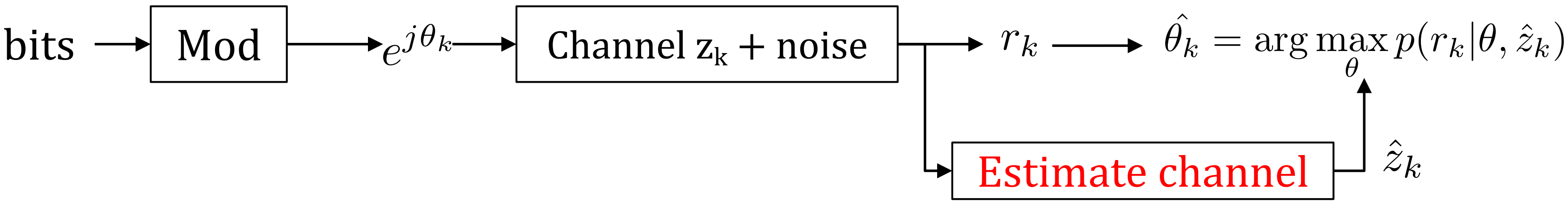
$$p(\bar{\gamma}_s[\text{dB}] < 34 \text{ dB}) \leq 0.01$$

$$1 - Q\left(\frac{34 - \bar{\bar{\gamma}}_s[\text{dB}]}{8}\right) \leq 0.01$$

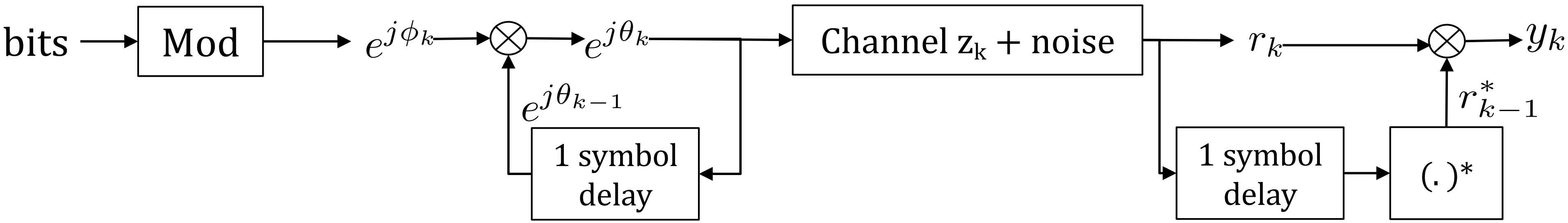
$$\bar{\bar{\gamma}}_s[\text{dB}] \geq 52.6 \text{ dB}$$

Coherent vs noncoherent PSK communication

- Discrete-time model: $r_k = z_k e^{j\theta_k} + n_k$ where θ_k contains the data, channel z_k
- Coherent communication: encode information in absolute phase



- Non-coherent communication: encode information in *relative* phase

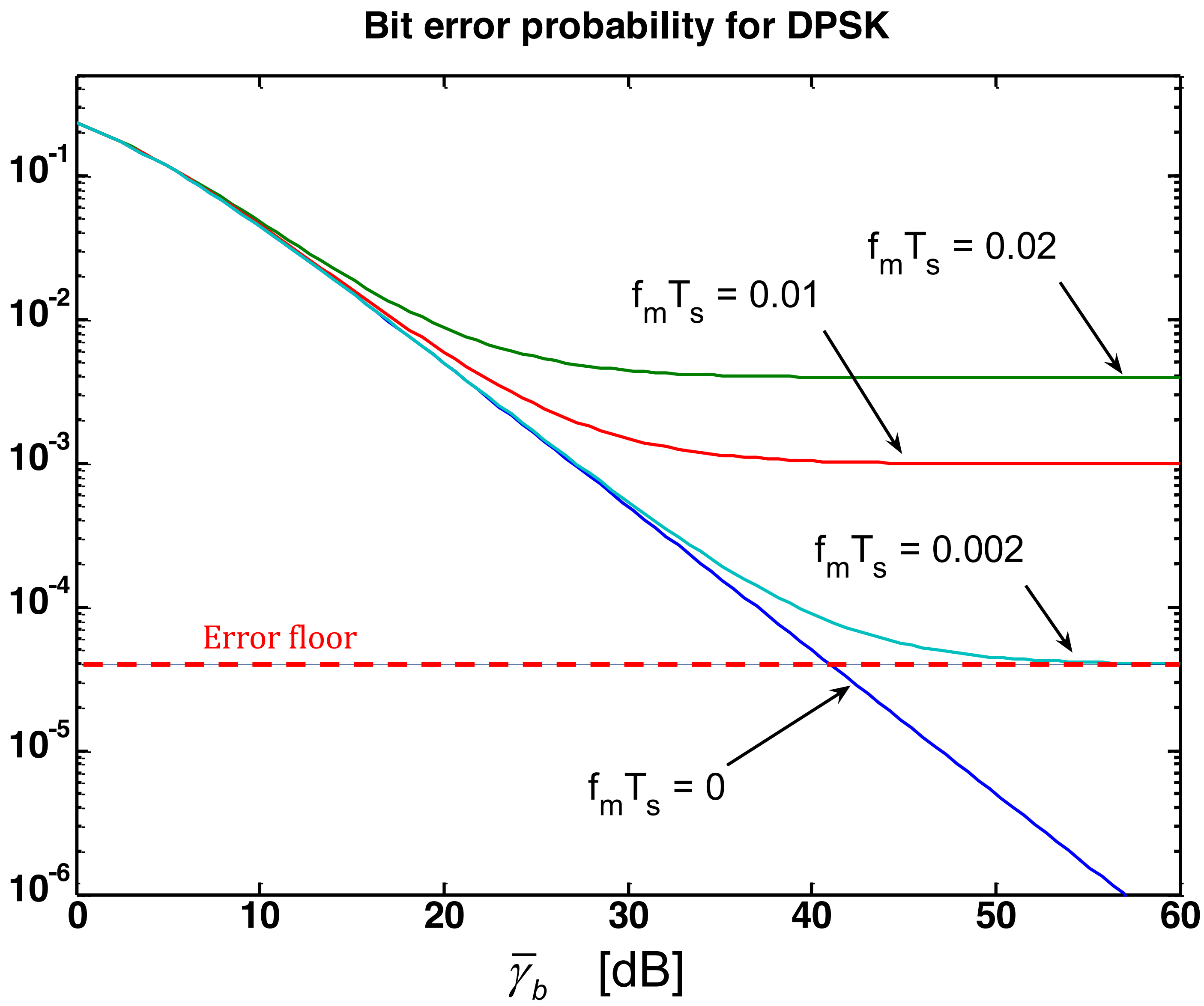


- Observation in noise-free case: $y_k = r_k r^*_{k-1} = z_k z^*_{k-1} e^{j(\theta_k - \theta_{k-1})} = \boxed{z_k z^*_{k-1}} e^{j\phi_k}$

Real number for constant channel
Almost real number for slowly varying channel
Any complex number for fast varying channel

- BER for DBPSK: $\bar{P}_b = \frac{1}{2}\left(\frac{1+\bar{\gamma}_b(1-\rho_C)}{1+\bar{\gamma}_b}\right).$
- With $\rho_C = A_c(T)/A_c(0)$ small for fast varying channels, 1 for constant channel
- Jakes: $\rho_C = J_0(2\pi f_D T)$

Error floor due to fast fading in DPSK (binary DPSK)



$$f_m = f_{D,\max} = \frac{v}{\lambda}$$

Today's learning outcomes

At the end of this lecture, you must be able to

- distinguish between SNR per symbol, SNR per bit, BER, SER
- define and compute outage probabilities to meet operational requirements
- define and compute average error probabilities
- describe the differences between coherent and differential detection
- explain why DPSK exhibits an error floor in fast fading

Technique for computing average error probability: moment generating function

- Moment generating function (MGF): alternative way to characterize random variables. Laplace transform of the distribution (with sign inversion).
- Definition: R.V. X

$$M_X(s) = \mathbb{E}\{e^{sX}\}$$
$$= \int p_X(x)e^{sx}\mathrm{d}x$$

Moment generating property

$$\mathbb{E}\{X^n\} = \frac{\partial^n}{\partial s^n}\left[M_X(s)\right]_{s=0}$$

- For fading distribution

$$M_\gamma(s) = \int_0^{+\infty} p_\gamma(x)e^{sx}\mathrm{d}x$$
$$M_{\gamma_s}^{\mathrm{Rayleigh}}(s) = (1 - s\bar{\gamma}_s)^{-1}$$
$$M_{\gamma_s}^{\mathrm{Rice}}(s) = \frac{1+K}{1+K-s\bar{\gamma}_s}\exp\left(\frac{Ks\bar{\gamma}_s}{1+K-s\bar{\gamma}_s}\right)$$

Average error probability using the MFG

- Average error probability $\bar{P}_s = \int_0^{+\infty} P_s(\gamma)p(\gamma)\mathrm{d}\gamma$
- For many formats

  $P_s(\gamma_s) = \alpha Q(\sqrt{2g\gamma_s})$

- Using MGF

  $P_s(\gamma_s) = \alpha Q(\sqrt{2g\gamma_s})$

$$\bar{P}_s = \int_0^{+\infty} \frac{\alpha}{\pi} \int_0^{\pi/2} \exp\left(\frac{-2g\gamma_s}{2\sin^2\phi}\right) p(\gamma_s)\mathrm{d}\phi\mathrm{d}\gamma_s$$

$$= \frac{\alpha}{\pi} \int_0^{\pi/2} M_{\gamma_s}\left(\frac{-g}{\sin^2\phi}\right) \mathrm{d}\phi$$

Craig's formula:

$$Q(z) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(\frac{-z^2}{2\sin^2\phi}\right) \mathrm{d}\phi$$

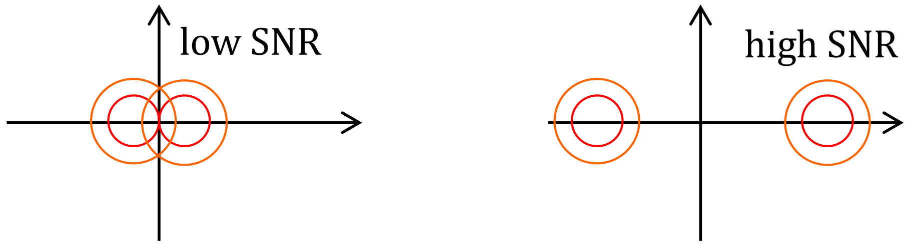
- Easy to calculate numerically, since finite boundaries

- Write an expression for the error probability for M-QAM in Rayleigh fading

Solutions

Solution

- If you assume the noise variance is fixed, the scatter plots should look (very roughly) like this



- For channel 1 the BER for 10 dB is around 10^{-5}, and for 0 dB around 10^{-1}. You can see this from the BER plot. The average BER will be $0.5*10^{-5} + 0.5*10^{-1}$, which is approximately 0.05. Hence, the bad channel condition dominates.
- Channel 2 will also be 50% of the time bad, and 50% of the time good (due to the symmetry in the state diagram). This can be exploited by sending more data (e.g., using 16QAM) when the channel is good, and less data (e.g., using a code) when the channel is bad.

Average bit / symbol error probability