Solution to Exercise 6

Topic: Rate and Power Adaptation

Problem 1

QPSK $\longrightarrow$ $M = 4$

$P_b = 2 \times 10^{-3}$

$P_b \approx \frac{2}{\log_2 M} Q\left(\sqrt{2\gamma_b \log_2 M} \sin\left(\frac{\pi}{M}\right)\right)$

$\longrightarrow P_b = Q\left(\sqrt{2\gamma_b}\right) = 2 \times 10^{-3}$

$\hookrightarrow$ Required SNR : $\gamma_b = 4.14190$

$\gamma \sim \exp(\bar{\gamma})$

$\bar{\gamma} = 20$ dB $= 100$

$\longrightarrow$ We need to find γ^* in a way that : $E_{\gamma^*}\left\{\frac{1}{\gamma}\right\} = \frac{1}{\text{Required SNR}}$

$$\hookrightarrow \int_{\gamma^*}^{\infty} \frac{1}{\gamma} \underbrace{\left(\frac{1}{100} e^{-\frac{\gamma}{100}}\right)}_{\text{pdf of } \gamma} d\gamma = \frac{1}{4.1419}$$

γ^* can be found numerically using MATLAB : $\gamma^* = 2 \times 10^{-9}$

Then the optimal power adaptation scheme would be as follows :

$$\frac{P(\gamma)}{\bar{P}} = \begin{cases} \frac{4.1419}{\gamma} & \gamma \geqslant 2 \times 10^{-9} \\ 0 & \gamma < 2 \times 10^{-9} \end{cases}$$

Finally , the outage probability equals to :

$$P_{out} = Pr\left\{\gamma < 2 \times 10^{-9}\right\} = \int_0^{2 \times 10^{-9}} \frac{1}{100} e^{-\frac{\gamma}{100}} d\gamma \simeq 2 \times 10^{-11}$$

Problem 2

(a) For MQAM we know :

$$P_b \approx 0.2\, e^{\frac{-1.5\gamma}{M(\gamma)-1}\left(\frac{P(\gamma)}{\bar{P}}\right)}$$

$$\hookrightarrow M(\gamma) \simeq 1 + k\gamma\frac{P(\gamma)}{\bar{P}} \quad , \quad k = \frac{-1.5}{\ln(5P_b)}$$

for $P_b = 10^{-3}$

$$k = 0.2831$$

So, the optimal power and rate adaptation schemes are as follows :

$$\frac{P(\gamma)}{\bar{P}} = \begin{cases} \frac{1}{k}\left(\frac{1}{\gamma^*} - \frac{1}{\gamma}\right) & \gamma \geq \gamma^* \\ 0 & \gamma < \gamma^* \end{cases} \quad , \quad M(\gamma) = \begin{cases} \frac{\gamma}{\gamma^*} & \gamma \geq \gamma^* \\ 0 & \gamma < \gamma^* \end{cases}$$

Then, we need to find γ^* in a way that $E\{P(\gamma)\} = \bar{P}$. In order to do that, first we assume that it is possible to assign power to transmitted symbols in all 4 channel states, which means that we assume that γ^* is less than all possible values of received SNR, then we have :

$$E\{P(\gamma)\} = \sum_{i=1}^{4} P(\gamma_i)\, Pr\{\gamma = \gamma_i\} = \sum_{i=1}^{4} \frac{\cancel{\bar{P}}}{k}\left(\frac{1}{\gamma^*} - \frac{1}{\gamma_i}\right) Pr\{\gamma = \gamma_i\} = \cancel{\bar{P}}$$

$$\hookrightarrow \frac{1}{k}\Big(\frac{1}{\gamma^*} - \underbrace{\frac{1}{3.16}}_{\gamma_1 = 5\text{ dB}}\Big)\underbrace{(0.4)}_{Pr\{\gamma = 5\text{ dB}\}} + \frac{1}{k}\Big(\frac{1}{\gamma^*} - \underbrace{\frac{1}{10}}_{\gamma_2 = 10\text{ dB}}\Big)\underbrace{(0.2)}_{Pr\{\gamma = 10\text{ dB}\}} + \frac{1}{k}\Big(\frac{1}{\gamma^*} - \underbrace{\frac{1}{31.62}}_{\gamma_3 = 15\text{ dB}}\Big)\underbrace{(0.2)}_{Pr\{\gamma = 15\text{ dB}\}} + \frac{1}{k}\Big(\frac{1}{\gamma^*} - \underbrace{\frac{1}{100}}_{\gamma_4 = 20\text{ dB}}\Big)\underbrace{(0.2)}_{Pr\{\gamma = 20\text{ dB}\}} = 1$$

$$\Longrightarrow \gamma^* = 2.2835 \rightarrow$$

γ^* is less than γ_i for $i = 1,2,3,4$, so our assumption was correct and this value of γ^* is acceptable.

Finally, the optimal power and rate adaptation schemes are as follow :

$$\frac{P(\gamma)}{\bar{P}} = \begin{cases} 0.4298 & \gamma = 5^{dB} \\ 1.1936 & \gamma = 10^{dB} \\ 1.4351 & \gamma = 15^{dB} \\ 1.51 & \gamma = 20^{dB} \end{cases} \quad , \qquad M(\gamma) = \begin{cases} 1.384 & \gamma = 5^{dB} \\ 4.379 & \gamma = 10^{dB} \\ 13.84 & \gamma = 15^{dB} \\ 43.79 & \gamma = 20^{dB} \end{cases}$$

(b)

According to the optimal rate adaptation scheme, the average spectral efficiency equals to :

$$E\left\{ \log_2 (M(\gamma)) \right\} = \sum_{i=1}^{4} \log_2 (M(\gamma_i)) \, Pr\left\{ \gamma = \gamma_i \right\}$$

$$= \log_2(1.384)(0.4) + \log_2(4.379)(0.2) + \log_2(13.84)(0.2) + \log_2(43.79)(0.2)$$

$$= 2.4629$$

Problem 3

The main difference between this problem and the previous one is that now the channel state is not random anymore and we are supposed to allocate power using deterministic water-filling.

(a)

The optimal power and rate adaptation schemes are :

$$\frac{P(\gamma)}{\bar{P}} = \begin{cases} \frac{1}{K}\left(\frac{1}{\gamma^*} - \frac{1}{\gamma}\right) & \gamma \geq \gamma^* \\ 0 & \gamma < \gamma^* \end{cases} \quad , \quad \mu(\gamma) = \begin{cases} \frac{\gamma}{\gamma^*} & \gamma \geq \gamma^* \\ 0 & \gamma < \gamma^* \end{cases}$$

Assumption 1: we can assign power to all different channel states.

Then, we need to find γ^* such that $\sum_{i=1}^{4} P(\gamma_i) = \bar{P}$:

$$\sum_{i=1}^{4} P(\gamma_i) = \sum_{i=1}^{4} \frac{\cancel{\bar{P}}}{K}\left(\frac{1}{\gamma^*} - \frac{1}{\gamma_i}\right) = \cancel{\bar{P}}$$

$$\rightarrow \frac{4}{\gamma^*} - \sum_{i=1}^{4} \frac{1}{\gamma_i} = K \rightarrow \gamma^* = \frac{4}{K + \sum_{i=1}^{4} \frac{1}{\gamma_i}} = 5.3967 \rightarrow \text{not acceptable}$$

It's remarkable that the resulted value of γ^* is larger than $5^{dB} = 3.16$, which means that we can not allocate power to this channel state. This is in contrast with Assumption 1 so this assumption is wrong and we can not assign power to all channel states.

Assumption 2 : we can just assign power to the 3 best channel states.

In this case, we do not assign power to the channel state with the received SNR of $5^{dB} = 3.16$. The we need to find $\gamma^* > 3.16$ such that $\sum_{i=2}^{4} P(\gamma_i) = \bar{P}$:

$$\sum_{i=2}^{4} P(\gamma_i) = \sum_{i=2}^{k} \frac{\cancel{\bar{P}}}{k}\left(\frac{1}{\gamma^*} - \frac{1}{\gamma_i}\right) = \cancel{\bar{P}}$$

$$\frac{3}{\gamma^*} - \sum_{i=2}^{4} \frac{1}{\gamma_i} = k \rightsquigarrow \gamma^* = \frac{3}{k + \sum_{i=2}^{4} \frac{1}{\gamma_i}} = 7.063 \checkmark$$

The resulted γ^* is acceptable, since it is larger than $5^{dB} = 3.16$ and less than $10^{dB} = 10$, which means that we can just assign power to channel state with received SNR of 10^{dB}, 15^{dB} and 20^{dB}. Therefore, Assumption 2 is correct and the optimal power and rate adaptation schemes are :

$$\frac{P(\gamma)}{\bar{P}} = \begin{cases} 0 & \gamma = 5^{dB} \\ 0.1468 & \gamma = 10^{dB} \\ 0.3884 & \gamma = 15^{dB} \\ 0.4648 & \gamma = 20^{dB} \end{cases} , \qquad M(\gamma) = \begin{cases} 0 \quad \text{No transmission} & \gamma = 5^{dB} \\ 1.4158 & \gamma = 10^{dB} \\ 4.4768 & \gamma = 15^{dB} \\ 14.1582 & \gamma = 20^{dB} \end{cases}$$

(b)

According to the proposed optimal rate adaptation, the spectral efficiency equals to :

$$\text{Total spectral efficiency} = \sum_{i=2}^{4} \log_2(M(\gamma_i)) = 6.4876$$