Exam in
SSY135 Wireless Communications

Department of Signals and Systems

Exam Date: March 17 2017, 14:00-18:00

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Material Allowed material is

- Chalmers-approved calculator
- L. Råde, B. Westergren. Beta, Mathematics Handbook, any edition.
- One A4 page with your own handwritten notes. Both sides of the page can be used. Photo copies, printouts, other students' notes, or any other material is not allowed.
- A dictionary.

Grading A correct, clear and well-motivated solution gives a maximum of 12 points.

An erroneous answer, unclear, incomplete or badly motivated solutions give point reductions down to a minimum of 0 points. Answers in any other language than English are ignored.

Results Results are posted no later than April 1.

Grades To pass the course, all projects and the exam must be passed. The exam is passed by securing at least 12 points. The project is passed by securing at least 8 points (4 for the report and 4 for the oral exam) in each part of the project. The final grade on the course will be decided by the homework (max score 6), project (max score 40), quizzes (max score 6), and final exam (max score 48). The sum of all scores will decide the grade according to the following table.

Total Score	0–39	40–59	60–79	≥ 80
Grade	Fail	3	4	5

PLEASE NOTE THAT THE PROBLEMS ARE NOT NECESSARILY ORDERED IN DIFFICULTY.

Good luck!

1 Question 1: Fading models

Consider a fading channel with impulse response

$$c(\tau, t) = 4\cos(t+U)\delta(\tau) + 3\beta_1(t)\delta(\tau-\tau_1) + 2\sin(t+U)\delta(\tau-\tau_2)$$

where $\tau_1 = 25$ ns, $\tau_2 = 50$ ns, $\beta_1(t)$, is an i.i.d. complex, zero-mean unit variance Gaussian random variable, and U is a random variable uniformly distributed between 0 to 2π, i.e., $U \sim \text{uniform}\,(0, 2\pi)$. Unless explicitly stated otherwise, all random variables are mutually independent. Figure 1 contains some useful relations.

1. [4 pt] Compute and draw the power delay profile. Recall that $A_c(\tau, \Delta t) = \mathbb{E}\{C(\tau, t+\Delta t)\, C^*(\tau, t)\}$.
   Hint:

$$\begin{aligned}
\cos(a)\cos(b) &= \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)\\
\sin(a)\sin(b) &= \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)\\
\sin(a)\cos(b) &= \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)\\
\cos(a)\sin(b) &= \frac{1}{2}\left(\sin(a+b) - \sin(a-b)\right)
\end{aligned}$$

2. [2 pt] Compute the RMS of the delay spread,

$$\sigma_{T_m} = \sqrt{\frac{\int_0^{+\infty} (\tau - \mu_{T_m})^2 A_c(\tau)\, d\tau}{\int_0^{+\infty} A_c(\tau)\, d\tau}}$$

   where

$$\mu_{T_m} = \sqrt{\int_0^{+\infty} \tau A_c(\tau)\, d\tau \Big/ \int_0^{+\infty} A_c(\tau)\, d\tau}.$$

   If you are unable to answer the first question, use $A_c(\tau) = 3\delta(\tau) + 7\delta(\tau-\tau_1) + 2\delta(\tau-\tau_2)$.
   Answer:

$$\begin{aligned}
&A_c(\tau, \Delta t) = E\{C(\tau, t+\Delta t)\, C^*(\tau, t)\}\\
&= 16E\{\cos(t+U+\Delta t)\cos(t+U)\}\,\delta(\tau) + 9E\left\{|\beta_1(t)|^2\right\}\delta(\tau-\tau_1) + 4E\{\sin(t+U+\Delta t)\sin(t+U)\}\,\delta(\tau-\tau_2)\\
&= 8\delta(\tau) + 9\times 1_{\Delta t=0}\delta(\tau-\tau_1) + 2\delta(\tau-\tau_2)
\end{aligned}$$

$$\lim_{\Delta t\to 0} A_c(\tau, \Delta t) = A_c(\tau) = 8\delta(\tau) + 9\delta(\tau-\tau_1) + 2\delta(\tau-\tau_2)$$

$$\begin{aligned}
\mu_{T_m} &= \tfrac{9\tau_1 + 2\tau_2}{19} = \tfrac{9\times 25 + 2\times 50}{19} = 17.1053 \text{ ns}\\
\sigma_{T_m} &= \sqrt{\tfrac{(\tau_1-\mu_{T_m})^2\tau_1 + (\tau_2-\mu_{T_m})^2\tau_2}{19}} = 7.7669 \text{ ns}
\end{aligned}$$

3. [2 pt] Assume that the maximum delay spread can be approximated by 8 ns, compute the coherence bandwidth of the channel. Find the maximum data rate (in Mbit/s) where the system experiences flat fading if 64-QAM with raised cosine pulse shaping with roll of factor 1 is used?

   Answer:

$$\begin{aligned}
B_c &\approx \frac{1}{\sigma_{T_m}} = 128.75 \text{ MHz}\\
\frac{1+\beta}{T_s} &< B_c\\
R_s &< \frac{B_c}{1+\beta}\\
R_b &< \frac{B_c \log_2 M}{(1+\beta)} = 386.25 \text{ MHz}
\end{aligned}$$

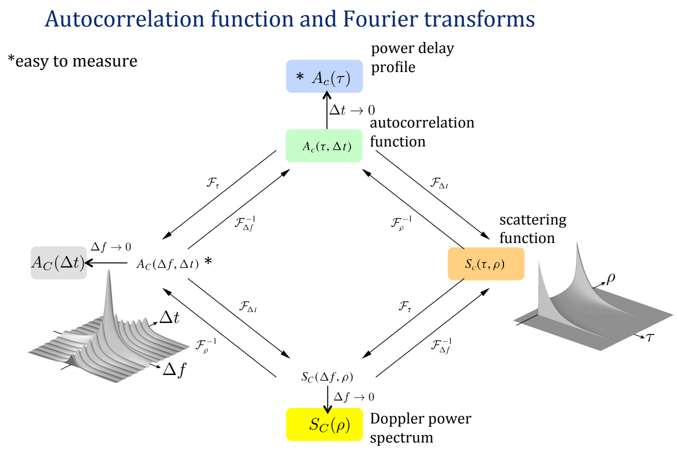

Figure 1: Fourier relationships

4. [4 pt] Now consider that the channel response is changed to a fading channel with impulse response

$$c(\tau, t) = 4\beta_0(t)\delta(\tau) + 3\beta_1(t)\delta(\tau - \tau_1) + 2\beta_2(t)\delta(\tau - \tau_2)$$

where $\tau_1 = 25$ ns, $\tau_2 = 50$ ns and $\beta_0(t)$, $\beta_1(t)$, and $\beta_2(t)$ are i.i.d complex, zero-mean unit variance Gaussian random processes with autocorrelation function $J_0(2\pi 100\Delta t)$, where $J_0(x)$ is the zeroth order Bessel function of the first kind. Compute the coherence time of the system. Find the minimum data rate where the system experiences slow fading if 64-QAM modulation is used.

Hint: The first three zero-crossings of $J_0(x)$ are for $|x| = 2.4048$, $|x| = 5.5201$, and $|x| = 8.6537$ Answer:

$$A_c(\tau, \Delta t) = 16J_0(2\pi 100\Delta t)\delta(\tau) + 9J_0(2\pi 100\Delta t)\delta(\tau - \tau_1) + 4J_0(2\pi 100\Delta t)\delta(\tau - \tau_2)$$
$$\lim_{\Delta f \to 0} \mathcal{F}_\tau(A_c(\tau, \Delta t)) = A_c(\Delta t) = 29J_0(2\pi 100\Delta t)$$
$$T_c \approx \frac{2.4048}{200\pi} = 3.8 \text{ ms}$$
$$T_s < T_c$$
$$R_s > \frac{1}{T_c}, R_b > \frac{\log_2 M}{T_c} = 1.5789 \text{ KHz}$$

2 Question 2: MIMO beamforming

In this question, we will analyze the behavior of MIMO beamforming. Recall that when the MIMO channel is decomposed as $\mathbf{H} = \mathbf{U}\mathbf{S}\mathbf{V}^H$, where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices and $\mathbf{S}$ is a diagonal matrix with real, non-negative entries ordered from large to small. We will consider communication of the form $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$, where $\mathbf{n} \sim \mathcal{N}(0, 2\sigma^2 \mathbf{I}_{M_r})$ (i.e., real and imaginary part have variance σ^2). We will use the MIMO channel to transmit a single scalar, $x \in \mathbb{C}$, $\mathbb{E}\{|x|^2\} = 5$.

1. [4 pt] Mathematically describe how beamforming works. Draw a block diagram. What is the expression for the SNR you obtain?
Hint: Recall that unitary matrices have the property that $\mathbf{U}^H\mathbf{U} = \mathbf{U}\mathbf{U}^H = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix of suitable dimension. Answer: SNR will be $S_{max}^2 E_x/(2\sigma^2)$.

2. [3 pt] Suppose you send $\mathbf{a}x$ over the channel, where $\mathbf{a}$ is a column vector with $\|\mathbf{a}\| = 1$. You receive $\mathbf{y} = \mathbf{H}\mathbf{a}x + \mathbf{n}$. In order for the receiver to determine x, we perform maximum likelihood detection. Write down an expression of the likelihood function, starting from

$$\hat{x} = \arg\max_x p(\mathbf{y}|x, \mathbf{H}, \mathbf{a})$$

and show that $z = \mathbf{y}^H\mathbf{H}\mathbf{a} \in \mathbb{C}$ is a sufficient statistic. What is the SNR associated with z?
Hint: Decompose z is signal part and noise part. Compute expectation of absolute value squared of each part.
Answer: ML leads to

$$\hat{x} = \arg\min \|\mathbf{y} - \mathbf{H}\mathbf{a}x\|^2$$

from which the sufficient statistic is easily found. Then we can express

$$z = \mathbf{a}^H\mathbf{H}^H\mathbf{H}\mathbf{a}|x|^2 + \mathbf{n}^H\mathbf{H}\mathbf{a}$$

so that the SNR is

$$SNR = \frac{|\mathbf{a}^H\mathbf{H}^H\mathbf{H}\mathbf{a}|^2 E_x}{|\mathbf{a}^H\mathbf{H}^H\mathbf{H}\mathbf{a}|2\sigma^2}$$
$$= \frac{|\mathbf{a}^H\mathbf{H}^H\mathbf{H}\mathbf{a}|E_x}{2\sigma^2}$$

3. [4 pt] We now consider a specific channel:

$$\mathbf{U} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -0.5 & -0.5 \\ 1 & -0.5 & -0.5 \end{bmatrix} + j\begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.5 & 0.5 \\ 0 & 0.5 & -0.5 \end{bmatrix}$$

and

$$\mathbf{V} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & -j & 1 & j \\ 1 & 1 & 1 & 1 \\ -1 & j & 1 & -j \end{bmatrix}$$

and

$$\mathbf{S} = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0.0001 & 0 \end{bmatrix}$$

Let $\sigma^2 = 0.1$. What is the receive SNR under beamforming? What is the receive SNR when sending $[0.5\ 0.5\ 0.5\ 0.5]^T x$ and performing ML detection at the receiver. Is it better to send $[0.5\ 0.5\ 0.5\ 0.5]^T x$ or to perform beamforming?
Answer: The problem is set up that $\mathbf{1}$ is the direction with the very small singular value. So the SNR will be very low.

4. [1 pt] Can you now prove that in general sending $\mathbf{a}x$ over the channel, where $\mathbf{a}$ is a column vector with $\|\mathbf{a}\| = 1$ can never be better than beamforming?
Answer: one can express any $\mathbf{a} = \sum_{k=1}^{N_t} \alpha_k \mathbf{v}_k$, so that

$$\mathbf{H}\mathbf{a} = \mathbf{U}\mathbf{S}\mathbf{V}^H\mathbf{a}$$
$$= \mathbf{U}\begin{bmatrix} s_{11}\alpha_1 \\ s_{22}\alpha_2 \\ \vdots \end{bmatrix}$$

with SNR

$$SNR \propto \begin{bmatrix} s_{11}\alpha_1 \\ s_{22}\alpha_2 \\ \vdots \end{bmatrix}^H \begin{bmatrix} s_{11}\alpha_1 \\ s_{22}\alpha_2 \\ \vdots \end{bmatrix}$$
$$= \sum_{k=1}^{min(N_r,N_t)} s_{kk}^2|\alpha_k|^2.$$

Since $\sum|\alpha_k|^2 = 1$, this SNR is maximized when $\alpha_1 = 1$, i.e., the beamforming solution.

3 Question 3: OFDM

Consider an OFDM system with $N = 48$ subcarriers, transmission using a sinc pulse with total bandwidth of 7.5 MHz. The cyclic prefix has a duration of $1/(4\Delta f)$, where Δf is the spacing between subcarriers. The carrier frequency is 6 GHz. Transmission can use 3 modulation formats: QPSK, 16-QAM, and 64-QAM.

1. [2 p] How much delay spread can the system tolerate? Is the system designed for indoor or outdoor operation? Why? Answer: Subcarrier spacing is $7.5MHz/48 = 156kHz$, so $1/(4\Delta f) = 1.6us$, which corresponds to 480 meters, so outdoors.

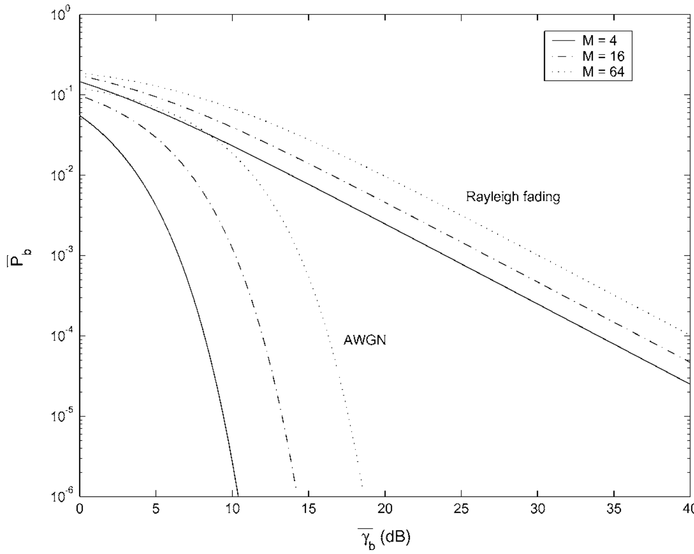

Figure 2: Performance of QPSK, 16-QAM, 64-QAM

2. [2 p] What is the maximum relative speed between transmitter and receiver that can be tolerated before there is inter-carrier-interference (due to the channel no longer being circulant during an OFDM symbol)? textcolored
Answer: Wavelength = 0.05 m. Total OFDM symbol length $1.6\mu s + 48/(7.5MHz) = 8us$. So coherence time should be about 80 us. So $v \times 80us < 0.025m$, so $v < 312.5m/s$

3. [2 p] What range of data rates does the system support? Suppose there is an existing communication system operating at subcarriers 12, 13, and 18. How could the OFDM system deal with this? How would this affect the rate?
Answer: easy answer: minimum rate is QPSK for all carriers. so $2 * 48bits/8us = 1.5Mbps$. Maximum rate is $6 * 48bits/8us = 4.5Mbps$. More complex answer: different constellation per carries, so more fine grid of rates. When certain subcarriers are already in use, the OFDM system can simply send 0 on those subcarriers. The rate would drop to $2 * (48 - 5)bits/8us = 1.4Mbps$ for QPSK.

4. [2 p] Suppose the transmitter send a sequence of 48 times the value j (before the IFFT). Compute the corresponding peak-to-average power ratio of the transmitted signal.
Hint: you can use the fact that the IFFT is a unitary transformation, with one of the columns of the corresponding matrix being a scaled all one vector.
Answer: constant value has easy Fourier transform (everything zero except first value). So the FFT would be [a 0 0 0], where $a = 48j$. The peak power is 48^2, the average power is 48.

5. [4 p] Suppose the noise PSD has $N_0 = 0.1333 \times 10^{-20}$ W/Hz and we want to guarantee at least 16-QAM transmission on each of the 48 subcarriers with a BER of less than 10^{-6} (see Figure 3). What is the minimum received power on each subcarrier that is required for each received OFDM symbol?
Answer: we need around 15 dB per bit, so 60 dB per symbol. This mean $P_r/(N_0W) > 60dB$, so

$$Pr[dB] - N_0W[dB] > 60$$

With $N_0W = 0.1333 \times 10^{-20} \times 7.5 * 10^6 = 10^{-14} = -140dB$ so $P_r > 60 - 140 = -80dB$

4 Question 4: Miscellaneous topics

1. [2 p] Describe one role that the FCC (Federal Communications Commission) plays in the reduction of harmful effects of electro-magnetic radiation? Answer is 20 words or less. Answer: FCC sets minimum guidelines for safe human exposure from wireless devices.

2. [2 p] Consider a channel that has a doppler spread of 5 Hz. Assume a QPSK modulated voice signal is transmitted over this channel at 30 Kbps. Is outage probability or average probability of error a better performance metric and why?

Answer: Symbol duration $T_s = 2/30k = 0.067ms$. Coherence time $T_c \approx 1/B_D = 0.2s$. Since $T_s \ll T_c$, outage probability is a better performance measure as a single deep fade can affect a lot of symbols.

3. [1 p] What is the difference between outage capacity and ergodic capacity? When would you use which capacity notion?

   Answer: Slow fading outage, fast fading: ergodic

4. [1 p] What is channel reuse in cellular systems? What are the trade-offs involved in setting a reuse factor?

   Answer: Number of cells per cluster, lower N leads to more efficiency, but with more interference

5. [2 p] Consider a flat fading channel where the complex envelope of the received signal is $r(t) = \alpha(t)s(t) + n(t)$, where $n(t)$ is complex white Gaussian noise with variance $sigma^2$ per real dimension, and $s(t)$ is the complex envelope of the transmitted signal. For Rayleigh-fading, what are the distributions of $\alpha(t)$, $|\alpha(t)|$, and $|\alpha(t)|^2$?

   Answer: complex normal, Rayleigh, exponential

6. [2 p] Consider a flat fading channel where the complex envelope of the received signal is $r(t) = \alpha(t)s(t) + n(t)$, where $n(t)$ is complex white Gaussian noise with variance $sigma^2$ per real dimension, and $s(t)$ is the complex envelope of the transmitted signal. For Rayleigh-fading, what is the distribution of $\mathbb{E}\{n(t)\}$, $\mathbb{E}\{|n(t)|^2\}$? Answer: both are delta's, one at zero, one at $2sigma^2$

7. [2 p] Assume a cellular system with dedicated 100 MHz bandwidth for each cluster where the pathloss is the dominated fading. If the pathloss exponent is 2, the required SINR of each cell is 10 dB, and the each user has 50 KHz requested bandwidth, find the maximum user capacity of the cellular system?

   Answer: $$\begin{aligned} &N \geq \tfrac{1}{3}(6SINR_{\min})^{\frac{2}{\gamma}} \\ &N_{\min} = 20 \\ &N_s^{\max} = \tfrac{B_{cluster}}{N_{\min}B_{user}} = 100 \end{aligned}$$

8. [1 p] Describe the difference between array gain and diversity gain.

   Answer: Array gain means that you capture more power. Diversity gain means that the channel behaves more like an AWGN channel (slope of BER vs SNR is increased)

9. [1 p] Describe how orthogonality among users can be maintained for TDMA, FDMA, and CDMA in the downlink.

   Answer: TDMA and FDMA are inherently orthogonal, since the transmitter is the same. For CDMA, users are orthogonal when the spreading codes are orthogonal.)