# Solution to Exercise 5

Feb 11, 2020

1. [G 7.17]

   By using BPSK modulation, the bit error probability for an AWGN channel with SNR/bit $x$ is

$$P_{\text{b,AWGN}}(\gamma) = Q(\sqrt{2\gamma}) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{\gamma}{\sin^2\phi}\right] d\phi \tag{1}$$

   Let $\gamma_1$ and $\gamma_2$ denote the SNR for the first branch and the second branch, with $\bar{\gamma_1} = \bar{\gamma_2} = \bar{\gamma} = 10$ dB. The combiner SNR with MRC combining is

$$\gamma_\Sigma = \gamma_1 + \gamma_2 \tag{2}$$

   Since the brance SNRs are independent, the joint distribution is a product of the individual distributions, that is,

$$p_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2) = p_{\gamma_1}(\gamma_1)p_{\gamma_2}(\gamma_2) \tag{3}$$

   Therefore, the average error probability is

$$\begin{aligned}
\bar{P}_b &= \int_0^\infty P_{\text{b,AWGN}}(\gamma)p_{\gamma_\Sigma}(\gamma)d\gamma \\
&= \int_0^\infty \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{\gamma}{\sin^2\phi}\right] d\phi p_{\gamma_\Sigma}(\gamma)d\gamma \\
&= \int_0^\infty\int_0^\infty \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{\gamma_1+\gamma_2}{\sin^2\phi}\right] d\phi p_{\gamma_1}(\gamma_1)p_{\gamma_2}(\gamma_2)d\gamma_1 d\gamma_2 \\
&= \frac{1}{\pi}\int_0^{\pi/2}\left(\left(\int_0^\infty \exp\left[-\frac{\gamma_1}{\sin^2\phi}\right] p_{\gamma_1}(\gamma_1)d\gamma_1\right)\left(\int_0^\infty \exp\left[-\frac{\gamma_2}{\sin^2\phi}\right] p_{\gamma_2}(\gamma_2)d\gamma_2\right)\right) d\phi
\end{aligned}$$

   Since $p_{\gamma_1}(\gamma_1) = p_{\gamma_2}(\gamma_2) = p(\gamma)$ and $\int_0^\infty p(\gamma)\exp(-x\gamma)d\gamma = 0.01\bar{\gamma}/\sqrt{x}$, we have

$$\bar{P}_b = \frac{1}{\pi}\int_0^{\pi/2} \left(0.01\bar{\gamma}\sin\phi\right)^2 d\phi$$

$$
\begin{aligned}
&= \frac{(0.01\bar{\gamma})^2}{\pi}\int_0^{\pi/2}\sin^2\phi d\phi \\
&= \frac{(0.01\bar{\gamma})^2}{\pi}\int_0^{\pi/2}\left(\frac{1-\cos(2\phi)}{2}\right)d\phi \\
&= \frac{(0.01\bar{\gamma})^2}{4} = 0.0025
\end{aligned}
$$

2. (a) $\gamma_i \sim U[0,10],\ i=1,2$, so the probability density function is:

$$p_{\gamma_i}(x) = \begin{cases} \frac{1}{10} & 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

and the cumulative distribution function is:

$$P_{\gamma_i}(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{10} & 0 \le x < 10 \\ 1 & x \ge 10 \end{cases}$$

In selection combining, $\gamma_\Sigma = \max\{\gamma_1,\gamma_2\}$ and $P^{\text{SC}}_{\gamma_\Sigma}(x) = P_{\gamma_1}(x)\times P_{\gamma_2}(x)$, so

$$P^{\text{SC}}_{\gamma_\Sigma}(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{100} & 0 \le x < 10 \\ 1 & x \ge 10 \end{cases}$$

and the pdf is

$$p^{\text{SC}}_{\gamma_\Sigma}(x) = \begin{cases} \frac{x}{50} & 0 \le x < 10 \\ 0 & \text{otherwise} \end{cases}$$

In MRC, $\gamma_\Sigma = \gamma_1+\gamma_2$, so due to i.i.d. fading distribution of each branches:

$$
\begin{aligned}
P^{\text{MRC}}_{\gamma_\Sigma}(x) = \Pr\{\gamma_1+\gamma_2 \le x\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{x-u} p_{\gamma_1,\gamma_2}(u,v)du\ dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{x-u} p_{\gamma_1}(u)p_{\gamma_2}(v)du\ dv \\
&= \int_{-\infty}^{\infty} P_{\gamma_2}(x-u)p_{\gamma_1}(u)du
\end{aligned}
$$

so

$$p^{\text{MRC}}_{\gamma_\Sigma}(x) = \frac{d}{dx}P^{\text{MRC}}_{\gamma_\Sigma}(x) = \int_{-\infty}^{\infty} p_{\gamma_2}(x-u)p_{\gamma_1}(u)du = (p_{\gamma_1} * p_{\gamma_2})(x)$$

$$p^{\text{MRC}}_{\gamma_\Sigma}(x) = \frac{1}{10}\int_0^{10} p_{\gamma_2}(x-u)du = \frac{1}{10}\int_{x-10}^{x} p_{\gamma_2}(y)dy$$

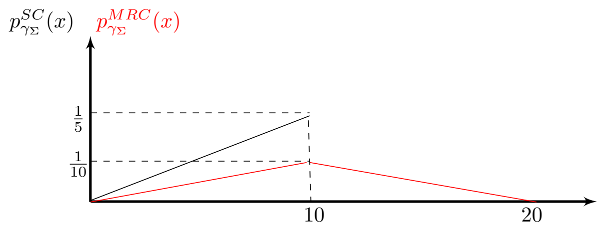


Figure 1: The distribution of the output SNR under selection combining and maximum ratio combining.

$$p^{\mathrm{MRC}}_{\gamma_\Sigma}(x) = \begin{cases} 0 & x < 0 \text{ or } x \geq 20 \\ \frac{x}{100} & 0 \leq x < 10 \\ \frac{20-x}{100} & 10 \leq x < 20 \end{cases}$$

(b) For DPSK, $P_{b,AWGN}(\gamma_b) = \frac{1}{2}e^{-\gamma_b} = 0.1 \Rightarrow \gamma_b = 1.609$

Selection combining: $P^{SC}_{out} = \Pr\{\gamma^{\mathrm{SC}}_\Sigma < 1.609\} = 2.50\%$

Maximum ratio combining: $P^{MRC}_{out} = \Pr\{\gamma^{\mathrm{MRC}}_\Sigma < 1.609\} = 1.29\%$

3. (a)

$$\gamma_{MRC} = \gamma_1 + \gamma_2$$
$$f_{\gamma_{MRC}}(\gamma_{MRC}) = f_{\gamma_1}(\gamma_1) * f_{\gamma_2}(\gamma_2)$$
$$f_{\gamma_{MRC}}(\gamma_{MRC}) = \left\{ \begin{array}{c} \frac{3}{200}x \; 0 \leq x \leq 5 \\ \frac{1}{200}x + \frac{1}{20} \; 5 \leq x \leq 10 \\ \frac{-3}{200}x + \frac{5}{20} \; 10 \leq x \leq 15 \\ \frac{-1}{200}x + \frac{2}{20} \; 15 \leq x \leq 20 \end{array} \right\}$$
$$F_{\gamma_{SC}}(\gamma_{SC}) = F_{\gamma_1}(\gamma_1)\, F_{\gamma_2}(\gamma_2) = \left\{ \begin{array}{c} \frac{3}{200}x^2 \; 0 \leq x \leq 5 \\ \frac{1}{200}x^2 + \frac{1}{20}x \; 5 \leq x \leq 10 \end{array} \right\}$$
$$f_{\gamma_{SC}}(\gamma_{SC}) = \frac{\partial}{\partial \gamma_{SC}} F_{\gamma_{SC}}(\gamma_{SC}) = \left\{ \begin{array}{c} \frac{6}{200}x \; 0 \leq x \leq 5 \\ \frac{1}{100}x + \frac{1}{20} \; 5 \leq x \leq 10 \end{array} \right\}$$

(b)

$$P^{\gamma_{MRC}}_{out} = \Pr(\gamma_{MRC} \leq 1) = \frac{3}{400}$$
$$P^{\gamma_{SC}}_{out} = \Pr(\gamma_{SC} \leq 1) = \frac{3}{200}$$