Wireless Communications SSY135 – Lecture 7

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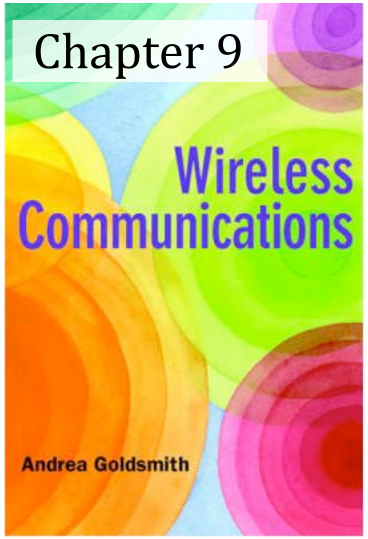
Topics for today

- Lecture learning outcomes
- Adaptation strategies
  - Rate adaptation
  - Power adaptation
  - Joint rate and power adaptation
- Statistical and deterministic water-filling

Suggested reading:
- Good time to review Chapter 4 (Capacity)

Suggested reading:
- Section 9.1
- Section 9.2.1, 9.2.2
- Section 9.3.1, 9.3.2, 9.3.4

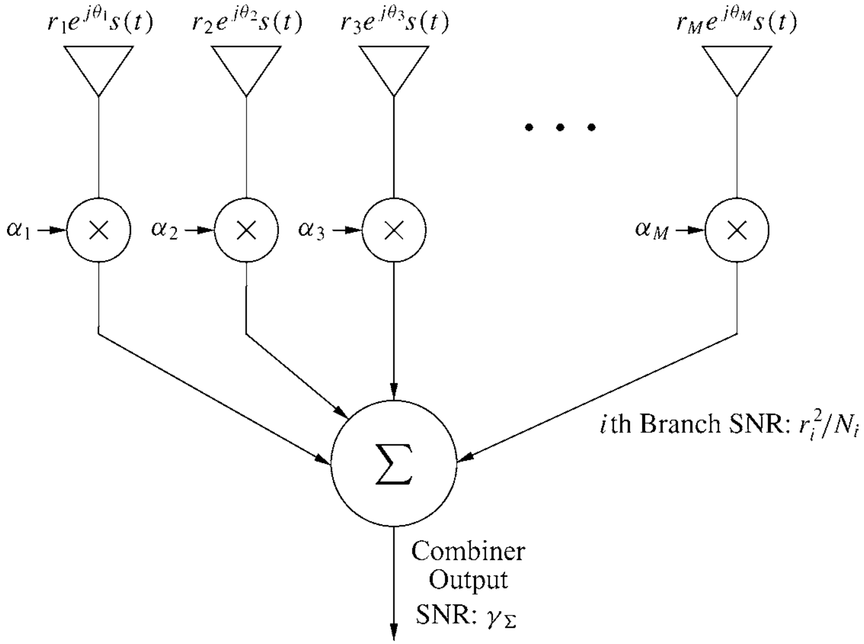


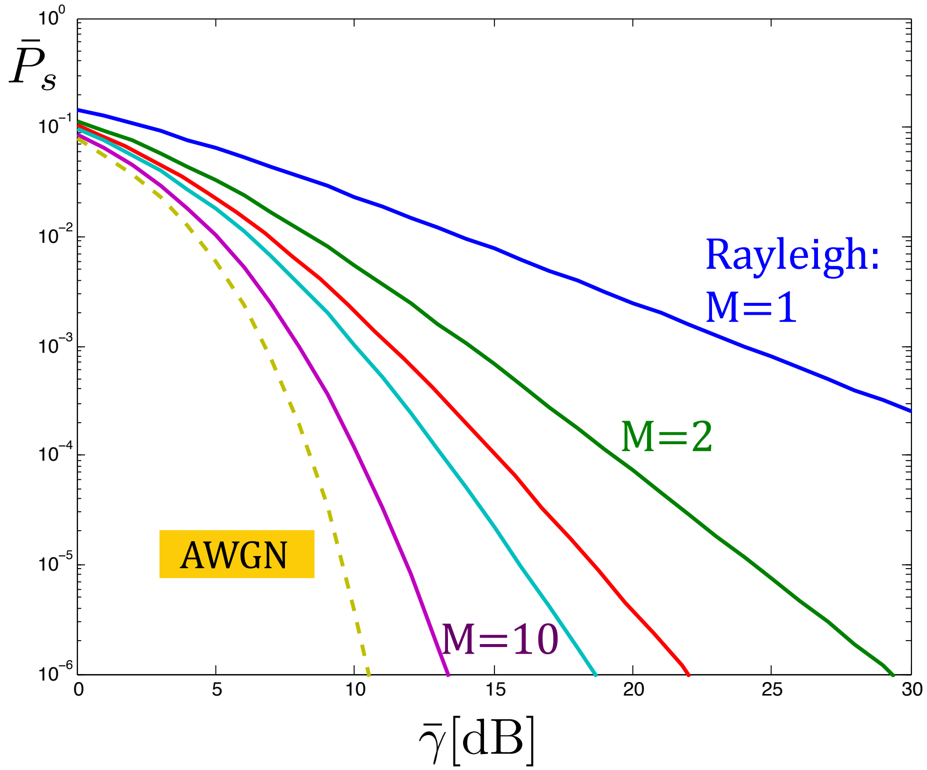
Today's learning outcomes

At the end of this lecture, you must be able to

- Explain the goal of adaptive transmission
- Explain the benefits and drawbacks of adaptive transmission
- Numerically solve rate adaptation problems
- Numerically solve power adaptation problems
- Derive and implement statistical and deterministic waterfilling

Last time: diversity gain and array gain

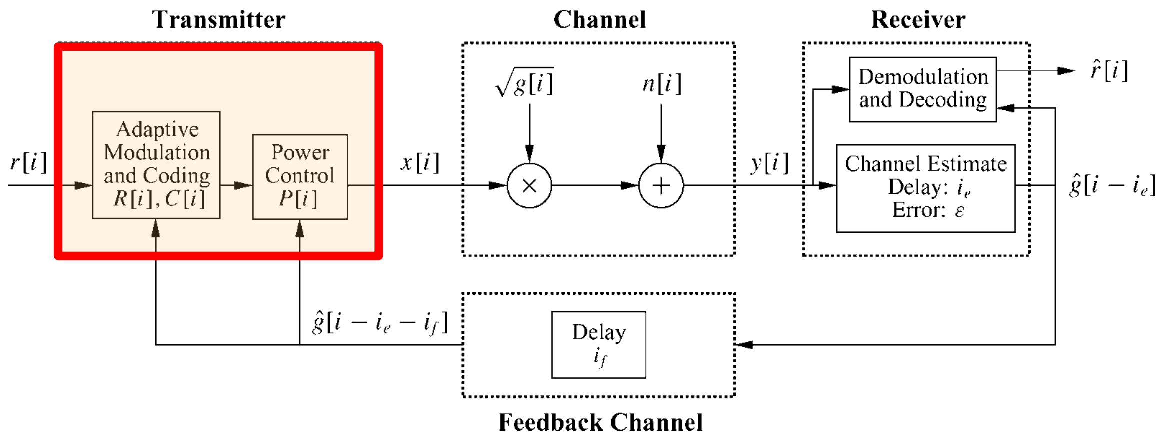




- Array gain: average SNR increases with M, for fixed SNR per branch
- Diversity gain: channel becomes more like AWGN channel

Adaptive transmission

- Goal: maintain certain quality of service (QoS), e.g., BER or data rate, in the presence of fading
- Approaches
  - Fixed transmit power and modulation: worst-case assumption require large margin
  - Variable power or modulation: adapt to varying channel
- Channel is assumed to vary slowly (so channel state information (CSI) can be fed back)
- System model



- Other types of adaptation: bandwidth, coding rate

Assumptions and preliminaries

- Discrete-time channel – each channel use corresponds to one symbol time T_s
- Channel is assumed to vary slowly
  - So channel state information (CSI) can be fed back
  - Channel has stationary[1] and ergodic[2] gain $g[k]$
- Noise is AWGN with power spectral density $N_0/2$

- The *instantaneous SNR* is

$$\gamma[k] = \frac{\bar{P}g[k]}{N_0 B}$$

  where $\bar{P}$ is the transmit power. The *average SNR* is

$$\bar{\gamma} = \frac{\bar{P}\bar{g}}{N_0 B}$$

- Since $g[k]$ is stationary and ergodic, the distribution of $\gamma[k]$ is independent of k ; we denote this distribution by $p(\gamma)$

[1] Statistics don't change over time.
[2] Statistical properties can be deduced by a single, sufficiently long sample (realization) of the process.

No adaptation

- For a fixed target BER, fixed transmit power and fixed modulation M

- Rate: $R = \log_2 M / T_s = B \log_2 M$
- Spectral efficiency: $R/B = \log_2 M$
- Transmission takes place when there is no outage
  i.e., the SNR is high enough such that the modulation meets the BER requirements
- Outage probability: probability of too low SNR, i.e., $P_{\text{out}} = \mathbb{P}(\gamma < \gamma_{\text{mod}})$
  equivalent to proportion of packets not gone through

$$P_{\text{out}} = \int_0^{\gamma_{\text{mod}}} p(\gamma) d\gamma$$

- Average spectral efficiency is

$$\begin{aligned} \bar{R} &= [1 - P_{\text{out}}] \log_2 M \\ &= \int_{\gamma_{\text{mod}}}^{\infty} p(\gamma) d\gamma \log_2 M \end{aligned}$$

Adaptive transmission

Strategy 1: adapt rate for fixed transmit power

– Idea: Use larger constellation when channel is better
– Fixed: target BER
– Performance: average rate, outage

Strategy 2: adapt transmit power for fixed rate

– Idea: Use less power when channel is better
– Fixed: average power
– Performance: achievable SNR, outage

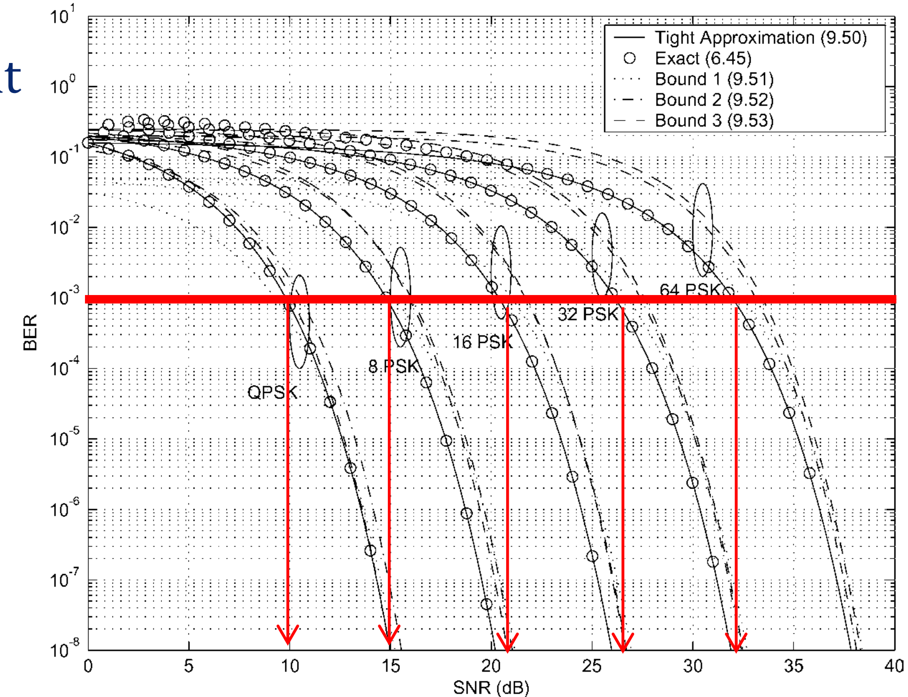
Strategy 3: adapt both transmit power and rate

– Idea: Use more power and larger constellation when channel is better
– Fixed: target BER and average power
– Performance: average rate, outage

Rate adaptation, no power adaptation

- For a fixed target BER and fixed transmit power
- Received SNR distribution: $p(\gamma)$ with mean $\bar{\gamma}$
- Set of modulation formats $\mathcal{M}_j, j = 1, \ldots, N$, with M_j points, with SNR thresholds γ_j
- Average rate: $\bar{R} = \sum_{j=1}^{N} \int_{\gamma_j}^{\gamma_{j+1}} p(\gamma) \log_2 M_j \mathrm{d}\gamma, \gamma_{N+1} = +\infty$
- Outage probability: probability of too low SNR, with no modulation format

$$P_{\text{out}} = \int_0^{\gamma_1} p(\gamma)\mathrm{d}\gamma$$



Rate adaptation example

Given

- 2 modulation formats: QPSK, 8PSK
- Rayleigh fading, average SNR of 20 dB
- Target BER of 10^{-3}. Note that for M-PSK $P_b = \frac{2}{\log_2 M} Q\left(\sqrt{2\gamma_b \log_2 M} \sin(\pi/M)\right)$

Task

- Perform rate allocation. What are the SNR ranges?
- What is the average rate (bits / transmission)?
- What is the average rate without rate allocation?
- What could you do if the SNR is very small?

Adaptive transmission

Strategy 1: adapt rate for fixed transmit power

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Strategy 2: adapt transmit power for fixed rate

- Idea: Use less power when channel is better
- Fixed: average power
- Performance: achievable SNR, outage

Strategy 3: adapt both transmit power and rate

- Idea: Use more power and larger constellation when channel is better
- Fixed: target BER and average power
- Performance: average rate, outage

Power adaptation / power control, no rate adaptation

- Fixed average power $\bar{P}$, adapt power to maintain constant SNR σ
- Fixed modulation format
- Received SNR under power control: $\gamma_{\mathrm{PC}} = \dfrac{P(\gamma)\gamma}{\bar{P}}$ (before: γ when $P(\gamma) = \bar{P}$)

- **Strategy 1**: channel inversion $P(\gamma) = \sigma \bar{P}/\gamma$

infinite for Rayleigh fading

- Average power constraint: $\int P(\gamma)p(\gamma)\mathrm{d}\gamma = \bar{P}$ so $\sigma^{-1} = \boxed{\int_0^{+\infty} \frac{1}{\gamma} p(\gamma)\mathrm{d}\gamma}$

- **Strategy 2**: channel inversion with cutoff: $P(\gamma) = \sigma \bar{P}/\gamma, \gamma > \gamma_0$
- Fix σ to meet BER, choose cutoff:

$$\sigma^{-1} = \int_{\gamma_0}^{+\infty} \frac{1}{\gamma} p(\gamma)\mathrm{d}\gamma$$

$$P_{\mathrm{out}} = \int_0^{\gamma_0} p(\gamma)\mathrm{d}\gamma$$

Adaptive transmission

Strategy 1: adapt rate for fixed transmit power

- Idea: Use larger constellation when channel is better
- Fixed: target BER
- Performance: average rate, outage

Strategy 2: adapt transmit power for fixed rate

- Idea: Use less power when channel is better
- Fixed: average power
- Performance: achievable SNR, outage

Strategy 3: adapt both transmit power and rate

- Idea: Use more power and larger constellation when channel is better
- Fixed: target BER and average power
- Performance: average rate, outage

Review of some basic concepts

- Optimization

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g(\mathbf{x}) \leq 0 \\ & h(\mathbf{x}) = 0 \end{aligned}$$

- Solution (for convex f, g, linear h) procedure*:
  1. Write down Lagrangian $\mathcal{L}(\mathbf{x}, \lambda, \mu) = f(\mathbf{x}) + \lambda g(\mathbf{x}) + \mu h(\mathbf{x}), \lambda \geq 0$
  2. Set partial derivative to zero

     $\partial \mathcal{L}(\mathbf{x}, \lambda, \mu) / \partial \mathbf{x} = \mathbf{0}$

     $\partial \mathcal{L}(\mathbf{x}, \lambda, \mu) / \partial \mu = 0$
  3. Additional conditions

     $\lambda \geq 0, g(\mathbf{x}) \leq 0$

     $\lambda g(\mathbf{x}) = 0$
  4. Solve for $(\mathbf{x}, \lambda, \mu)$

*Other technical conditions may apply

Review of some basic concepts

Given

- minimize $x^2 + y^2$, with $x+y=0.5$

Task

- What is the optimal (x,y)?
- What if we add the constraint $x \geq 1$?

Review of some basic concepts

Given

- minimize $x^2 + y^2$, with x+y=0.5

Task

- What is the optimal (x,y)?

Solution

$$\mathcal{L}(x, y, \mu) = x^2 + y^2 + \mu(x + y - 5)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + \mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y + \mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = x + y - 5 = 0$$

$$\rightarrow \mu = -5, x = y = 2.5$$

Waterfilling: a general approach

- Given a_l, b_l, c, P: $\max_{\mathbf{x}} \sum_{l=1}^{L} a_l \cancel{\log_2}(c + x_l b_l)$ Replace by ln (natural logarithm)

$$\text{s.t.} \sum_{l=1}^{L} a_l x_l = P,\ x_l \geq 0$$

- Lagrangian

$$\mathcal{L}(\mathbf{x}, \lambda, \boldsymbol{\mu}) = -\sum_l a_l \ln(c + x_l b_l) + \lambda(\mathbf{x}^T\mathbf{a} - P) - \sum_l \mu_l x_l$$

- Solution

$$\left[\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial x_l} = -a_l b_l \dfrac{1}{c + x_l b_l} + \lambda a_l - \mu_l = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = \mathbf{x}^T\mathbf{a} - P = 0 \\ \mu_l \geq 0 \\ x_l \geq 0 \\ \mu_l x_l = 0 \end{array}\right. \Rightarrow \left[\begin{array}{r} -\dfrac{b_l}{c + x_l b_l} + \lambda \geq 0 \\ \mathbf{x}^T\mathbf{a} - P = 0 \\ \mu_l \geq 0 \\ x_l \geq 0 \\ \left(-\dfrac{b_l}{c + x_l b_l} + \lambda\right) x_l = 0 \end{array}\right. \Rightarrow \left[\begin{array}{r} \lambda \geq \dfrac{1}{d_l + x_l}, d_l = c/b_l \\ \mathbf{x}^T\mathbf{a} - P = 0 \\ \mu_l \geq 0 \\ x_l \geq 0 \\ \left(\lambda - \dfrac{1}{d_l + x_l}\right) x_l = 0 \end{array}\right.$$

- So

$$\left.\begin{array}{l} \lambda < \dfrac{1}{d_l} \Rightarrow x_l > 0 \Rightarrow \lambda = \dfrac{1}{d_l + x_l} \\ \lambda \geq \dfrac{1}{d_l} \Rightarrow x_l = 0 \end{array}\right\} x_l = \max(1/\lambda - d_l, 0)$$

Choose so that $\sum_{k=1}^{L} a_k x_k = P$

- "Waterfilling" solution: rare closed-form; interpretation comes soon
- Result also holds when sums are replaced with integrals

Statistical waterfilling: discrete case

- Fixed target BER, fixed average power: aim to maximize expected rate
- Focus on M-QAM, express rate as function of power allocation:

$$P_b(\gamma) \leq \frac{1}{5}\exp\left[\frac{-1.5\gamma}{M-1}\frac{P(\gamma)}{\bar{P}}\right] \Longrightarrow M(\gamma) = 1 + K\gamma\frac{P(\gamma)}{\bar{P}}, \quad K = -\frac{1.5}{\log(5P_b)}$$

- Maximize rate (for continuous M, discrete SNR values $\gamma \in \{\gamma_1, \gamma_2, \ldots, \gamma_L\}$)

$$\begin{aligned} \text{maximize}_{\mathbf{P}} \quad & \sum_{l=1}^{L} p(\gamma=\gamma_l)\log_2(1+K\gamma_l\frac{P_l}{\bar{P}}) \\ \text{s.t.} \quad & \sum_{l=1}^{L} p(\gamma=\gamma_l)P_l = \bar{P} \\ & P_l \geq 0, l = 1, \ldots, L \end{aligned}$$

- Solution

$$P(\gamma) = \begin{cases} \bar{P}/K(1/\gamma_K - 1/\gamma) & \gamma \geq \gamma_K \\ 0 & \gamma < \gamma_K \end{cases}$$

$$M(\gamma) = \begin{cases} \gamma/\gamma_K & \gamma \geq \gamma_K \\ 0 & \gamma < \gamma_K \end{cases}$$

Mapping from general formulation:

$a_l \leftrightarrow p(\gamma = \gamma_l)$

$x_l \leftrightarrow P_l$

$c \leftrightarrow 1$

$b_l \leftrightarrow K\frac{\gamma_l}{\bar{P}}$

Statistical waterfilling: continuous case

- Fixed target BER, fixed average power: aim to maximize rate
- Focus on M-QAM, express rate as function of power allocation:

$$P_b(\gamma) \leq \frac{1}{5} \exp\left[\frac{-1.5\gamma}{M-1}\frac{P(\gamma)}{\bar{P}}\right] \Longrightarrow M(\gamma) = 1 + K\gamma\frac{P(\gamma)}{\bar{P}}, \quad K = -\frac{1.5}{\log(5P_b)}$$

- Maximize rate (for continuous M)

$$\text{maximize}_{P(\gamma)} \int_0^{+\infty} \log_2 M(\gamma)p(\gamma)\mathrm{d}\gamma$$

$$\text{s.t.} \int_0^{+\infty} P(\gamma)p(\gamma)\mathrm{d}\gamma = \bar{P}$$

- Solution

$$P(\gamma) = \begin{cases} \bar{P}/K(1/\gamma_K - 1/\gamma) & \gamma \geq \gamma_K \\ 0 & \gamma < \gamma_K \end{cases}$$

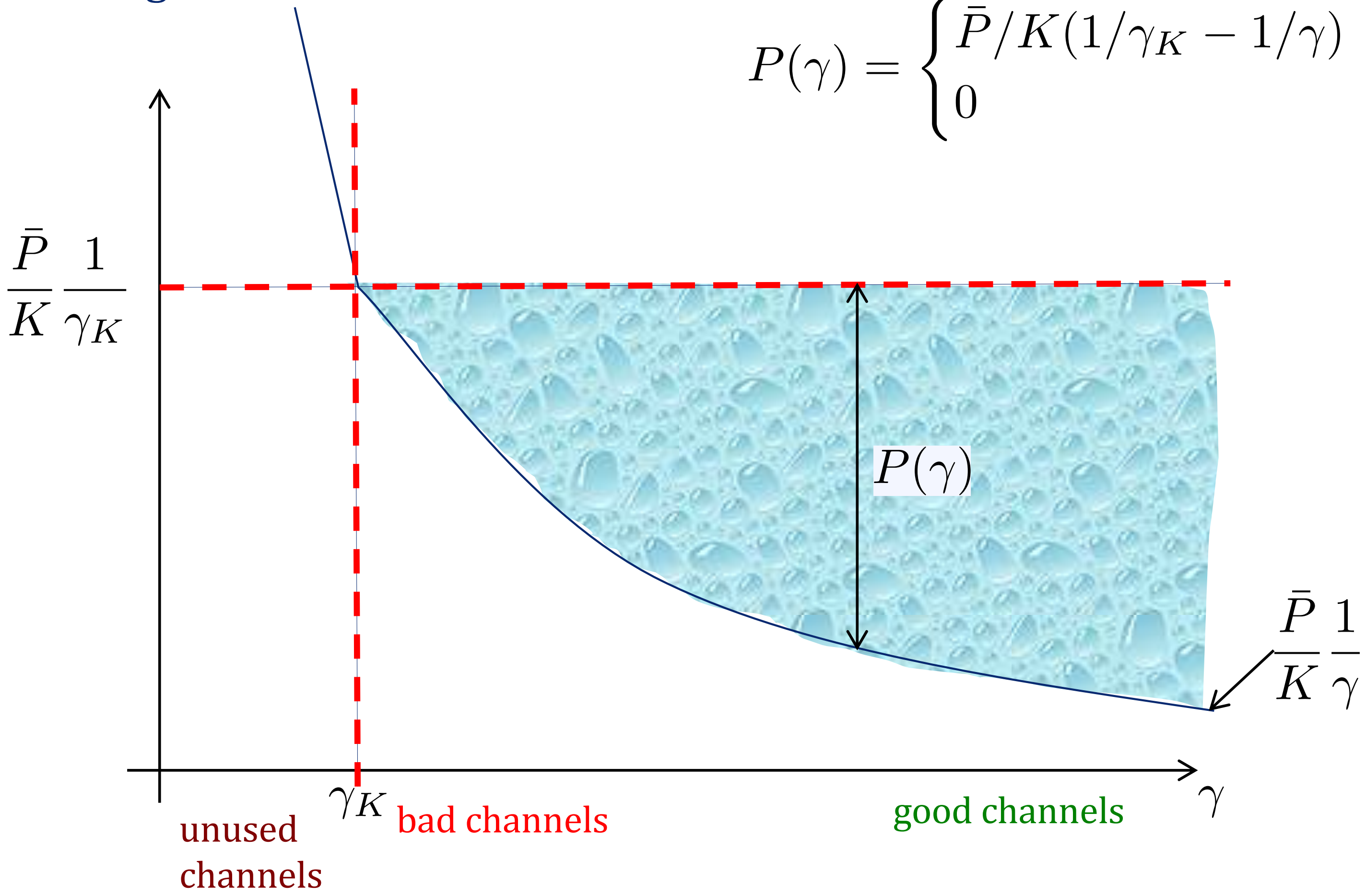
$$\int_{\gamma_K}^{+\infty} P(\gamma)p(\gamma)\mathrm{d}\gamma = \bar{P}$$

$$M(\gamma) = \begin{cases} \gamma/\gamma_K & \gamma \geq \gamma_K \\ 0 & \gamma < \gamma_K \end{cases}$$

Statistical waterfilling: interpretation

- Idea: taking advantage of good channel conditions by using more power, not using channel below cutoff value

$$P(\gamma) = \begin{cases} \bar{P}/K(1/\gamma_K - 1/\gamma) & \gamma \geq \gamma_K \\ 0 & \gamma < \gamma_K \end{cases}$$

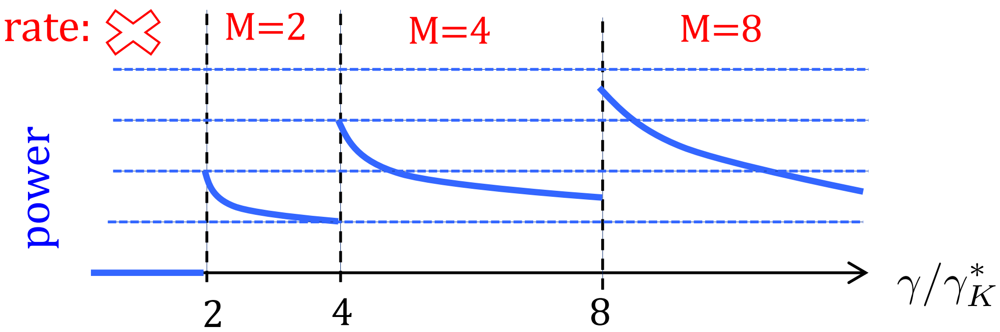


Continuous vs. discrete modulation order

- Optimal modulation: $M = \gamma/\gamma_K$

$$M(\gamma) = 1 + K\gamma\frac{P(\gamma)}{\bar{P}}, \quad K = -\frac{1.5}{\log(5P_b)} \quad P(\gamma) = \begin{cases} \bar{P}/K(1/\gamma_K - 1/\gamma) & \gamma \geq \gamma_K \\ 0 & \gamma < \gamma_K \end{cases}$$

- For discrete M (say, {2,4,8})
  1. choose a γ_K^* (water level)
  2. define fading regions $M = \gamma/\gamma_K^*$
  3. determine power adaptation for each region $P(\gamma) = \dfrac{\bar{P}(M-1)}{\gamma K}$
  4. compute rate $R(\gamma_K^*)$
  5. Find γ_K^* to maximize rate s.t. power constraint



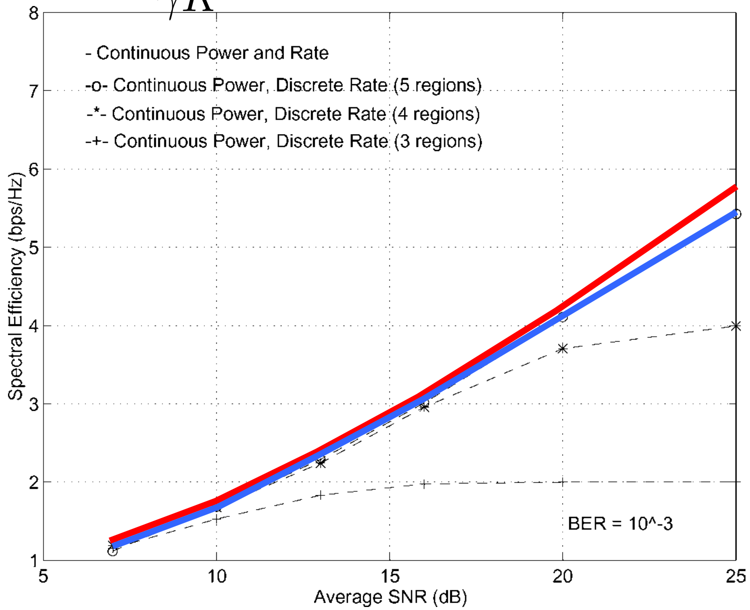


Figure 9.6: Discrete-rate efficiency in Rayleigh fading.

Deterministic waterfilling (aka adaptive loading)

- Consider transmission over N parallel channel with total power P.
  - Channel *i*: Given fading envelope α_i, bandwidth B, noise PSD N_0
  - Select power P_i and modulation order M_i
- Power allocation to maximize capacity

$$\text{maximize}_{\mathbf{P}} \quad \sum_{i=1}^{N} \log_2\left(1+\frac{\alpha_i^2 P_i}{N_0 B}\right) = \sum_{i=1}^{N} \log_2\left(1+\frac{\gamma_i P_i}{P}\right)$$

$$\text{s.t.} \quad \sum_i P_i = P$$

- Solution:

$$P_i/P = \max(\boxed{1/\gamma_c} - 1/\gamma_i, 0)$$

chosen to satisfy
total power constraint

- M-QAM modulation from (or in combination with code)

$$P_b \leq \frac{1}{5}\exp\left[\frac{-1.5\gamma_i}{M_i-1}\frac{P_i}{P}\right]$$

Mapping from general formulation:

$$a_k \leftrightarrow 1$$

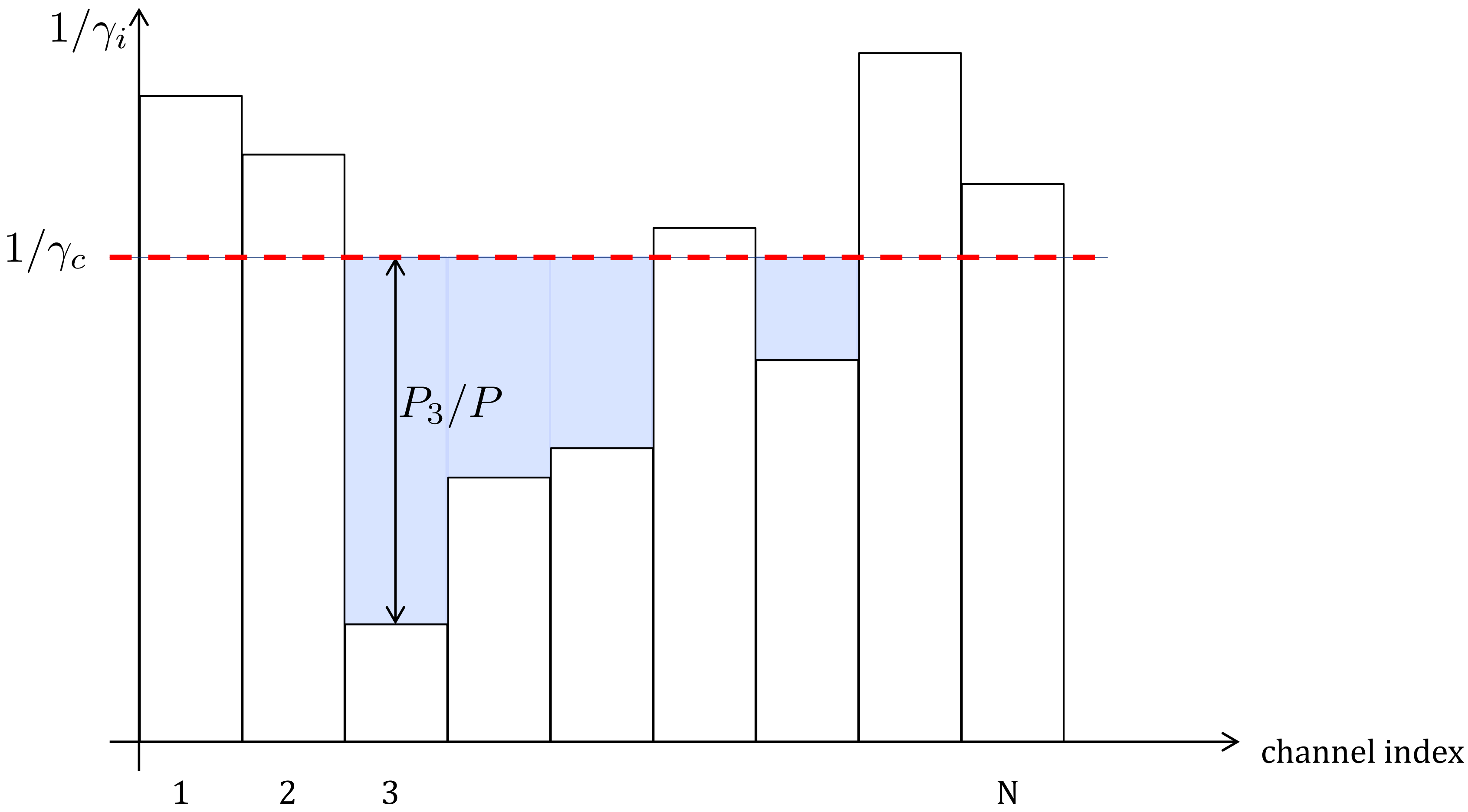
$$x_k \leftrightarrow P_k$$

$$c \leftrightarrow 1$$

$$b_k \leftrightarrow \frac{\gamma_k}{P}$$

Adaptive loading: interpretation

- Idea: taking advantage of good channel conditions by using more power, not using channel below cutoff value



Today's learning outcomes

At the end of this lecture, you must be able to

- Explain the goal of adaptive transmission
- Explain the benefits and drawbacks of adaptive transmission
- Numerically solve rate adaptation problems
- Numerically solve power adaptation problems
- Derive and implement statistical and deterministic waterfilling

Solutions

Rate adaptation example

Given

- 2 modulation formats: QPSK, 8PSK
- Rayleigh fading, average SNR of 20 dB
- Target BER of 10^{-3}. Note that for M-PSK $P_b = \frac{2}{\log_2 M} Q\left(\sqrt{2\gamma_b \log_2 M} \sin(\pi/M)\right)$

Task

- Perform rate allocation. What are the SNR ranges?
- What is the average rate (bits / transmission)?
- What is the average rate without rate allocation?
- What could you do if the SNR is very small?

Solution

- From the figures in the slides, we can see that to have the desired BER, we need an SNR of 9.8 dB (=9.55) for QPSK and 14.8 dB (=30) for 8PSK. So the regions are
  - SNR below 9.55: transmit nothing
  - SNR between 9.55 and 30: use Q-PSK
  - SNR above 30: use 8-PSK

The average rate depends on how often the different SNR regions occur. The CDF of the SNR is $P(\gamma) = 1 - \exp(-\gamma/\bar{\gamma})$ where $\bar{\gamma} = 100$. So

$$p(\gamma < 9.55) = 1 - \exp(-9.55/100) = 0.09$$

$$p(\gamma > 30) = \exp(-30/100) = 0.74$$

$$p(\gamma \in [9.55]) = 1 - 0.09 - 0.74 = 0.17$$

This means that the average rate is $0 \times 0.09 + 2 \times 0.17 + 3 \times 0.74 = 2.56 \text{ bits/channel use}$

If we had always used QPSK, the average rate would be

$0 \times 0.09 + 2 \times (0.17 + 0.74) = 1.82 \text{ bits/channel use}$

Power and rate adaptation for M-QAM

- Focusing on M-QAM:

$$P_b(\gamma) \approx \frac{1}{5}\exp\left[\frac{-1.5\gamma}{M(\gamma)-1}\frac{P(\gamma)}{\bar{P}}\right], \quad M(\gamma) = 1 + K\gamma\frac{P(\gamma)}{\bar{P}}, \quad K = -\frac{1.5}{\ln(5P_b)}$$

- Optimization problem: $\text{maximize} \int_0^{+\infty} \log_2\left(1 + K\gamma\frac{P(\gamma)}{\bar{P}}\right) p(\gamma)\mathrm{d}\gamma$

$$\text{s.t.} \int_0^{+\infty} P(\gamma)p(\gamma)\mathrm{d}\gamma = \bar{P}, P(\gamma) > 0$$

- Lagrangian: objective + constraint

$$\mathcal{L}(P(\gamma),\lambda) = \int_0^{+\infty} \log_2\left(1 + K\gamma\frac{P(\gamma)}{\bar{P}}\right) p(\gamma)\mathrm{d}\gamma - \lambda\left(\int_0^{+\infty} P(\gamma)p(\gamma)\mathrm{d}\gamma - \bar{P}\right)$$

- Optimal value satisfies $\frac{\partial \mathcal{L}(P(\gamma),\lambda)}{\partial P(\gamma)} = 0$. Using functional derivative*:

$$\frac{\partial \mathcal{L}(P(\gamma),\lambda)}{\partial P(\gamma)} = \frac{\partial}{\partial P(\gamma)}\left(\log_2\left(1 + K\gamma\frac{P(\gamma)}{\bar{P}}\right) - \lambda P(\gamma)\right)p(\gamma)$$

$$= \left(\frac{1}{\log 2}\frac{1}{1 + K\gamma\frac{P(\gamma)}{\bar{P}}}K\gamma\frac{1}{\bar{P}} - \lambda\right)p(\gamma)$$

- So, with properly chosen λ

$$P(\gamma) = \max\left(\frac{1}{\lambda\log 2} - \frac{\bar{P}}{K}\frac{1}{\gamma}, 0\right) = \max\left(\frac{\bar{P}}{K}\frac{1}{\gamma_K} - \frac{\bar{P}}{K}\frac{1}{\gamma}, 0\right)$$

*http://en.wikipedia.org/wiki/Functional_derivative