Solution to Exercise 9

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1. [E2011-Mar] rank($\mathbf{H}_2$) = 2 > rank($\mathbf{H}_1$) = 1. $\mathbf{H}_2$ allows for more spatial channels than $\mathbf{H}_1$. Hence system 2 will allow for better spectral efficiency. However, when different modulation formats can be used for these systems, then depending on the SNRs of the systems, either of these channels can be spectrally efficient.

2. [E2010-Aug-Q.3]

   (a) We get three spatial channel by using $\mathbf{A} = \mathbf{V}(:, 1:3)$ and $\mathbf{B} = \mathbf{U}(:, 1:3)^H$
   $\Rightarrow \tilde{\mathbf{Z}} = \mathbf{B}\mathbf{y} = \mathbf{B}(\mathbf{H}\mathbf{A}\tilde{\mathbf{x}} + \mathbf{n})$
   $= \mathbf{B}\mathbf{H}\mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{n}$, where $\mathbf{B}\mathbf{n} \sim \mathcal{CN}(0, N_0\mathbf{I}_3)$
   $\mathbf{B}\mathbf{H}\mathbf{A} = [\mathbf{U}(:, 1:3)]^H \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H\mathbf{V}(:, 1:3) = \mathbf{\Sigma}(1:3, 1:3)$. Hence, max. data rate is $R_{b,max} = 3\log_2 M\frac{1}{T_s} = 3 \cdot 2 \cdot \frac{1}{10^{-6}} = 6\,\text{Mbps}$.

   (b) Spatial beamforming. $\mathbf{A} = \mathbf{V}(:, 1)$,$\mathbf{B} = \mathbf{U}(:, 1)^H$. Therefore, $R_{b,BF} = \log_2 M = 2\,\text{Mbps}$.

   (c) $B_n \sim \mathcal{CN}(0, N_0\mathbf{I}_3)$ where $N_0 = 1$.
   Max. rate: we have 3 parallel channels
   $\gamma_{b,1} = \frac{1}{log_2 M}\gamma_{s,1} = \frac{1}{2} \cdot 1 \cdot \sigma_1^2 \cdot \frac{1}{1}$, where $\frac{1}{2}$ is due to QPSK, 1 is due to unit-energy of transmitted symbols, σ_1^2 is the power gain of the first spatial channel and finally the last term is the noice variance.
   $\gamma_{b,1} = \frac{\sigma_1^2}{2} = \frac{16}{2} = 8 \Rightarrow P_{b,1} = Q(\sqrt{2\gamma_{b,1}}) = Q(\sqrt{16}) = 3.1 \times 10^{-5}$.
   In the same manner, $P_{b,2} = Q(\sqrt{\sigma_2^2}) = 3.1 \times 10^{-3}$ and $P_{b,3} = Q(\sqrt{\sigma_3^2}) = 1.4 \times 10^{-2}$. Therefore, $P_b = \frac{1}{3}[P_{b,1} + P_{b,2} + P_{b,3}] = 5.8 \times 10^{-3}$. Beamforming $P_b = P_{b,1} = 3.1 \times 10^{-5}$ as above.

3. Different ways of using multiple antennas in MIMO

   (a) Power $\rho = 10$ and sicne equal power allocation is done, each channel gets $\rho/2$
   $\therefore \gamma_1 = (3.6)^2\frac{\rho}{2} = 64.8$ and $\gamma_2 = (1.4)^2\frac{\rho}{2} = 9.8$

The tight BER bound, $P_b \leq 0.2e^{-1.5\gamma/(M-1)}$, can be rewritten as $M \leq 1 + \frac{1.5\gamma}{-ln(5P_b)}$. This gives
$M_1 \leq 19.34 M_1 = 16$ bits on the first channel
$M_2 \leq 3.77 M_2 = 2$ bits on the first channel
$R = (4+1) \times 100$ Hz $= 500$ kbps $= 0.5$ Mbps
$A_{SVD} = V$ and $B_{SVD} = U^T$

(b) When the transmitter uses only the first antenna, the channel is $h = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Since the receiver uses MRC, the SNR at the combiner will simply be $\gamma_1 + \gamma_2$, where
$\gamma_1 = (2)^2\rho = 40$ and $\gamma_2 = (1)^2\rho = 10$. $\therefore \gamma_\Sigma = \gamma_1 + \gamma_2 = 50$
The required data rate is 5 bps per Hz from part (a), therefore $M = 2^5 = 32$. Using the BER bound, $P_b \leq 0.2e^{-1.5\gamma/(M-1)}$, with $M = 32$, $\gamma = 50$, we get $P_b = 17.8 \times 10^{-3}$
If we used beamforming, $\gamma = (3.6)^2 10 = 129.6 P_b = 3.7 \times 10^{-4} \ll 17.8 \times 10^{-3}$. Hence by choosing on transmit antenna arbitrarily as compared to beamforming, we get high performance loss.

(c) The transmit filter to be employed is H^{-1}, which gives

$$A_{BC} = H^{-1} = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.4 \end{bmatrix}$$

The power user 1 gets is $|(0.6)^2 + (-0.2)^2|P_1$, where P_1 is the power in x_1 i.e. $P_1 = \mathbb{E}[x_1^2]$. This should be equal to 5 as there is equal power allocated to both users
$|(0.6)^2 + (-0.2)^2|P_1 = 5 P_1 = 12.5$. Similarly $P_2 = \mathbb{E}[x_2^2] = 25$.
Now the received power at user 1 is simply P_1as the effective channel is 1 (after H and H^{-1} are multiplied), hence $\gamma_1 = 12.5, \quad \gamma_2 = 25$
$M_1 \leq 4.5 M_1 = 4$ kbps is transmitted to user 1 and
$M_2 \leq 8.1 M_2 = 8$ kbps is transmitted to user 2.