# Solution for Exercise 10
# Massive MIMO

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1. (a) Using SVD decomposition, $\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H$. Due to unitary characteristic of $\boldsymbol{U}$ and $\boldsymbol{V}$, one can easily show that

$$\boldsymbol{H}\boldsymbol{H}^H = \boldsymbol{U}\Sigma\boldsymbol{V}^H\boldsymbol{V}\Sigma^H\boldsymbol{U}^H = \boldsymbol{U}\begin{bmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 \\ 0 & . & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\min(N_t,N_r)}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\boldsymbol{U}^H.$$

In other words,

$$\det\left(\boldsymbol{I}_{N_r} + \frac{P}{N_t}\boldsymbol{H}\boldsymbol{H}^H\right) = \prod_{l=1}^{\min(N_t,N_r)} (1 + \frac{P}{N_t}\lambda_l^2). \tag{1}$$

Substituting (**??**) in MIMO channel capacity leads to the following

$$\begin{aligned} C = \log_2\det\left(\boldsymbol{I}_{N_r} + \frac{P}{N_t}\boldsymbol{H}\boldsymbol{H}^H\right) &= \sum_{l=1}^{\min(N_t,N_r)} \log_2\left(1 + \frac{P}{N_t}\lambda_l^2\right) \qquad (2) \\ = \min(N_t, N_r) \sum_{l=1}^{\min(N_t,N_r)} \frac{1}{\min(N_t, N_r)}\log_2\left(1 + \frac{P}{N_t}\lambda_l^2\right). \end{aligned}$$

Now (**??**) can be rewritten as

$$C = \min(N_t, N_r)\mathbb{E}\left\{\log_2\left(1 + \frac{P}{N_t}\lambda_l^2\right)\right\}, \tag{3}$$

where $E.$ is the expectation function over uniformly distributed random variable $\lambda_l^2$. Since logarithm is a concave function, with applying the Jensen's inequality, we have

$$C \leq \min(N_t, N_r)\log_2\left(1 + \frac{P}{N_t}\mathbb{E}\{\lambda_l^2\}\right). \tag{4}$$

By taking the expectation, knowing that $Tr\left(\boldsymbol{H}\boldsymbol{H}^H\right) = \sum_{l=1}^{\min(N_t,N_r)} \lambda_l^2$,

$$\begin{aligned} C &\leq \min(N_t, N_r)\log_2\left(1 + \frac{P}{N_t \times \min(N_t, N_r)} \sum_{l=1}^{\min(N_t,N_r)} \lambda_l^2\right) \qquad (5) \\ &= \min\left(N_t, N_r\right)\log_2\left(1 + \frac{P.Tr\left(\boldsymbol{H}\boldsymbol{H}^H\right)}{N_t \times \min(N_t, N_r)}\right), \end{aligned}$$

follows.

(b) We should have $\frac{Tr\left(HH^H\right)}{\min(N_t,N_r)} = \max(N_t, N_r)$. This means that $Tr\left(HH^H\right) = N_t N_r$.

(c) Evaluating the upper bound for $N_t \gg N_r$: $N_r\log_2(1 + P)$.

2. (a) The distance between two consecutive elements of the array antenna is $d/(M-1)$, therefore due to to the far feild assumption, the distance of the $i$th element of array compared to the first element is $\frac{(i-1)d}{M-1}\cos(\frac{\pi}{2} - \theta_i) = \frac{(i-1)d}{M-1}\sin(\theta_i)$ smaller. This means that for example, for user one, the $i$th element of array has the phase shift of $e^{-(j\pi/(M-1))(i-1)\sin\theta_1}$ compared to the first element, i.e., the gain of the $i$th element of array for user one is $g_1 e^{-(j\pi/(M-1))(i-1)\sin\theta_1}$. Therefore,

$$\mathbf{H} = \begin{bmatrix} g_1 & g_2 & g_3 \\ g_1 e^{-(j\pi/(M-1))\sin\theta_1} & g_2 e^{-(j\pi/(M-1))\sin\theta_2} & g_3 e^{-(j\pi/(M-1))\sin\theta_3} \\ g_1 e^{-(j2\pi/(M-1))\sin\theta_1} & g_2 e^{-(j2\pi/(M-1))\sin\theta_2} & g_3 e^{-(j2\pi/(M-1))\sin\theta_3} \\ . & . & . \\ g_1 e^{-(j\pi)\sin\theta_1} & g_2 e^{-(j\pi)\sin\theta_2} & g_3 e^{-(j\pi)\sin\theta_3} \end{bmatrix}^T.$$

(b) In order to perform zero forcing, $A = ||H^H\left(HH^H\right)^{-1}||_F^{-1} H^H\left(HH^H\right)^{-1}$.

(c) $P_1 = \frac{5}{\sum\limits_{i=1}^{M}(a_{i,1})^2} = 54.593\ mW$, $P_2 = \frac{5}{\sum\limits_{i=1}^{M}(a_{i,2})^2} = 42.765\ mW$, and $P_3 = \frac{5}{\sum\limits_{i=1}^{M}(a_{i,3})^2} = 291.097\ mW$

(d) $P_1 = \frac{5}{\sum\limits_{i=1}^{M}(a_{i,1})^2} = 7.405\ mW$, $P_2 = \frac{5}{\sum\limits_{i=1}^{M}(a_{i,2})^2} = 18.975\ mW$, and $P_3 = \frac{5}{\sum\limits_{i=1}^{M}(a_{i,3})^2} = 956.320\ mW$