$$
\begin{aligned}
I &= \int_{x^2+y^2=R^2} \ln\frac{1}{\sqrt{(x-h)^2+y^2}}\ \mathrm{d}s \\
&= -\frac{1}{2}\int_{x^2+y^2=R^2} \ln((x-h)^2+y^2)\ \mathrm{d}s \\
&= -\frac{1}{2}\int_0^{2\pi} R\ \ln(R^2-2Rh\ \cos\ t+h^2)\ \mathrm{d}t \\
&= -R\int_0^{\pi} \ln(R^2-2Rh\ \cos\ t+h^2)\ \mathrm{d}t \qquad (1)
\end{aligned}
$$

不妨设 $R>h$, 令 $R/h=a$, 则 $a>1$, 由 (1) 有

$$
\begin{aligned}
I &= -R\int_0^{\pi} \ln(h^2)+\ln((\frac{R}{h})^2-2(\frac{R}{h})\ \cos\ t+1)\ \mathrm{d}t \\
&= -2\pi R\ \ln\ h - R\int_0^{\pi} \ln(a^2-2a\ \cos\ t+1)\ \mathrm{d}t
\end{aligned}
$$

令 $f(a)=\int_0^{\pi} \ln(a^2-2a\ \cos\ t+1)\ \mathrm{d}t$

$$
\begin{aligned}
f'(a) &= \int_0^{\pi} \frac{2a-2\ \cos\ t}{a^2-2a\ \cos\ t+1}\ \mathrm{d}t \\
&= \int_0^{\pi} \frac{1}{a}+\frac{a-\frac{1}{a}}{a^2-2a\ \cos\ t+1}\ \mathrm{d}t \\
&= \frac{\pi}{a}+(a-\frac{1}{a})\int_0^{\pi} \frac{\mathrm{d}t}{a^2-2a\ \cos\ t+1}
\end{aligned}
$$

令 $\tan\ \frac{t}{2}=u$, 则 $\mathrm{d}t=\frac{2}{1+u^2}\ \mathrm{d}u$,$\cos\ t=\frac{1-u^2}{1+u^2}$

$$
\begin{aligned}
f'(a) &= \frac{\pi}{a}+2(a-\frac{1}{a})\int_0^{+\infty} \frac{\mathrm{d}u}{(a^2+1)(u^2+1)+2a(u^2-1)} \\
&= \frac{\pi}{a}+\frac{2(a-\frac{1}{a})}{(a+1)^2}\int_0^{+\infty} \frac{\mathrm{d}u}{u^2+(\frac{a-1}{a+1})^2} \\
&= \frac{2\pi}{a}
\end{aligned}
$$

而

$$
\begin{aligned}
f(1) &= \int_0^{\pi} \ln(2-2\,\cos\,t)\ \mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \ln(2-2\,\cos\,t)\ \mathrm{d}t + \int_{\frac{\pi}{2}}^{\pi} \ln(2+2\,\cos\,(\pi-t))\ \mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \ln(2-2\,\cos\,t)\ \mathrm{d}t + \int_0^{\frac{\pi}{2}} \ln(2+2\,\cos\,t)\ \mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \ln(4\,\sin^2\,t)\ \mathrm{d}t \\
&= \frac{\pi \ln\,4}{2} + 2\int_0^{\frac{\pi}{2}} \ln\,\sin\,t\ \mathrm{d}t
\end{aligned}
$$

并且

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \ln\,\sin\,t\ \mathrm{d}t &= \int_0^{\frac{\pi}{4}} \ln\,\sin\,t\ \mathrm{d}t + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\,\cos(\frac{\pi}{2}-t)\ \mathrm{d}t \\
&= \int_0^{\frac{\pi}{4}} \ln\,\sin\,t\ \mathrm{d}t + \int_0^{\frac{\pi}{4}} \ln\,\cos\,t\ \mathrm{d}t \\
&= \int_0^{\frac{\pi}{4}} \ln\frac{\sin\,2t}{2}\ \mathrm{d}t \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\,\sin\,t\ \mathrm{d}t - \frac{\pi \ln\,2}{4}
\end{aligned}
$$

得到

$$
\int_0^{\frac{\pi}{2}} \ln\,\sin\,t\ \mathrm{d}t = -\frac{\pi \ln\,2}{2}
$$

所以 $f(1) = 0$

$$
\begin{aligned}
f(a) &= f(1) + \int_1^{a} f'(s)\ \mathrm{d}s \\
&= 2\pi \ln\,a
\end{aligned}
$$

所以

$$
\begin{aligned}
I &= -2\pi R\,\ln\,h - R\,f(a) \\
&= -2\pi R\,\ln\,R
\end{aligned}
$$

同理当 $R < h$ 时, 令 $h/R = a$, 则 $a > 1$ 由 (1) 有

$$
\begin{aligned}
I &= -R\int_0^{\pi} \ln(R^2) + \ln((\frac{h}{R})^2 - 2(\frac{h}{R})\cos t + 1)\ \mathrm{d}t \\
&= -2\pi R\ln R - R\int_0^{\pi} \ln(a^2 - 2a\cos t + 1)\ \mathrm{d}t \\
&= -2\pi R\ln R - R\ f(a) \\
&= -2\pi R\ln h
\end{aligned}
$$

所以 $I = -2\pi R\ln(\max(h, R))$