# Measure Theory

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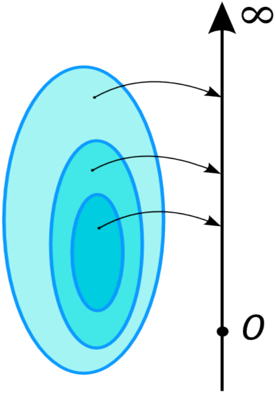
# Introduction

In mathematical analysis, a measure on a set is a systematic way to assign a number to each suitable subset of that set, intuitively interpreted as its size. In this sense, a measure is a generalization of the concepts of length, area, and volume.(Picture taken from Wikipedia)



# Definition

Let $X$ be a set and $\Sigma$ a $\sigma$-algebra over $X$. A function $\mu$ from $\Sigma$ to the extended real number line is called a measure if it satisfies the following properties:

- Non-negativity: For all $E$ in $\Sigma$: $\mu(E) \geq 0$.
- Null empty set: $\mu(\varnothing) = 0$.
- Countable additivity (or $\sigma$-additivity): For all countable collections $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\Sigma$:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty}(E_k)$$

## Examples

Some important measures are listed here.

- The counting measure is defined by $\mu(S) =$ number of elements in $S$.
- The Lebesgue measure on $\mathbf{R}$ is a complete translation-invariant measure on a $\sigma$-algebra containing the intervals in $\mathbf{R}$ such that $\mu([0,1]) = 1$; and every other measure with these properties extends Lebesgue measure.
- Circular angle measure is invariant under rotation, and hyperbolic angle measure is invariant under squeeze mapping.

# Properties

Let $\mu$ be a measure.

- Monotonicity: If $E_1$ and $E_2$ are measurable sets with $E_1 \subseteq E_2$ then $\mu(E_1) \leq \mu(E_2)$.
- Subadditivity: For any countable sequence $E_1, E_2, E_3$, ... of (not necessarily disjoint) measurable sets $E_n$ in $\Sigma$:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

- Continuity from below: If $E_1, E_2, E_3$, ... are measurable sets and $E_n$ is a subset of $E_{n+1}$ for all $n$, then the union of the sets $E_n$ is measurable, and

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{i\to\infty} \mu(E_i)$$

Thank you!