STT886 Homework#3

Haiyang Yu

September 26, 2019

1.

(1)

Because the tick cannot remain static, so $p_1 = 0$,

$$p_2 = P(O_2) = P(O|A)P(A) + P(O|B)P(B) + P(O|C)P(C) + P(O|D)P(D) = \frac{1}{3}$$

(2)

Apparently, $P(A_1) = P(B_1) = P(C_1) = P(D_1) = 1/4$, assume $P(A_n) = P(B_n) = P(C_n) = P(D_n) = \alpha, P(O_n) = 1 - 4\alpha$. So,

$$\begin{aligned}
P(A_{n+1}) &= P(B_n)P(A_{n+1}|B_n) + P(D_n)P(A_{n+1}|D_n) + P(O_n)P(A_{n+1}|O_n) \\
&= \frac{1}{3}\alpha + \frac{1}{3}\alpha + \frac{1}{4}(1 - 4\alpha) \\
&= \frac{1}{4} - \frac{1}{3}\alpha
\end{aligned}$$

In the same way, we have $P(B_{n+1}) = P(C_{n+1}) = P(D_{n+1}) = 1/4 - \alpha/3$. So $P(A_{n+1}) = P(B_{n+1}) = P(C_{n+1}) = P(D_{n+1})$, which means $P(A_n) = P(B_n) = P(C_n) = P(D_n), \forall n \geq 1$

(3)

$$\begin{aligned}
p_{n+1} &= P(O_{n+1}) \\
&= P(A_n)P(O_{n+1}|A_n) + P(B_n)P(O_{n+1}|B_n) + P(C_n)P(O_{n+1}|C_n) + P(D_n)P(O_{n+1}|D_n) \\
&= \frac{1}{3}\left(P(A_n) + P(B_n) + P(C_n) + P(D_n)\right) \\
&= \frac{1}{3}(1 - p_n)
\end{aligned}$$

$$\begin{aligned}
p_{n+1} = \frac{1}{3}(1 - p_n) &\Leftrightarrow p_{n+1} - \frac{1}{4} = -\frac{1}{3}\left(p_n - \frac{1}{4}\right) \\
&\Leftrightarrow p_n - \frac{1}{4} = \left(-\frac{1}{3}\right)^n \left(p_0 - \frac{1}{4}\right) \\
&\Leftrightarrow p_n = \frac{1}{4} + \frac{3}{4}\left(-\frac{1}{3}\right)^n
\end{aligned}$$

(4)

Let $n \to \infty$, $p_n \to 1/4$,

$$\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P(C_n) = \lim_{n\to\infty} P(D_n) = \frac{1}{4}(1 - \lim_{n\to\infty} p_n) = \frac{3}{16}$$

So the proportion of A, B, C, D, O is $3/16, 3/16, 3/16, 3/16, 1/4$.

2.

(1)

$$\begin{aligned}E(S_n|X_i) &= \sum_{j=1}^{n} E(X_j|X_i) \\ &= (n-1)E(X_1) + E(X_i|X_i) \\ &= (n-1)E(X_1) + X_i\end{aligned}$$

(2)

$$\begin{aligned}E(X_i|S_n) &= \frac{1}{n}\sum_{j=1}^{n} E(X_j|S_n) \\ &= \frac{1}{n}E(S_n|S_n) \\ &= \frac{1}{n}S_n\end{aligned}$$

(3)

$p_{x_1,x_2}(x_1, x_2) = f(x_1)f(x_2)$, so

$$p_{x_1,S_2}(x_1, S_2) = p_{x_1,x_2}(x_1, S_2 - x_1)\left|\frac{\partial(x_1, x_2)}{\partial(x_1, S_2)}\right| = f(x_1)f(S_2 - x_1)$$

$$\begin{aligned}p_{x_1|S_2}(x_1|S_2) &= \frac{p_{x_1,S_2}(x_1, S_2)}{p_{S_2}(S_2)} \\ &= \frac{f(x_1)f(S_2 - x_1)}{\int_{-\infty}^{+\infty} f(x_1)f(S_2 - x_1)\mathrm{d}x_1}\end{aligned}$$

3.

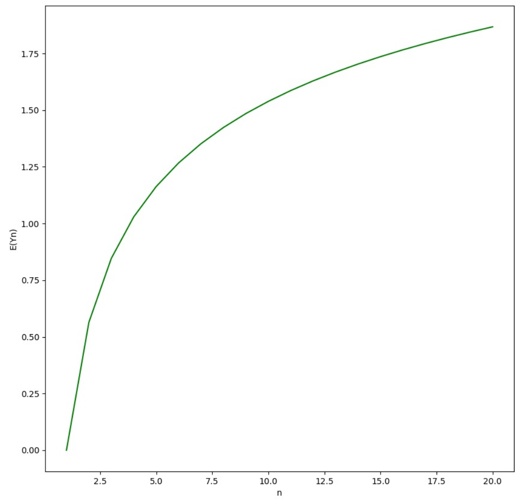
Suppose the joint density of X, Y is $f(x, y)$. Let $t = x+y$, We have $f_{x,t}(x, t) = f(x, t-x)$, $f_{y,t}(y, t) = f(y, t - y)$.

$$\begin{aligned}E(X - Y|X + Y) &= E(X|t) - E(Y|t) \\ &= \int_{-\infty}^{+\infty} x f_{x|t}(x|t)\mathrm{d}x - \int_{-\infty}^{+\infty} y f_{y|t}(y|t)\mathrm{d}y \\ &= \int_{-\infty}^{+\infty} x\frac{f_{x,t}(x, t)}{\int_{-\infty}^{+\infty} f_{x,t}(x, t)\mathrm{d}x}\mathrm{d}x - \int_{-\infty}^{+\infty} y\frac{f_{y,t}(y, t)}{\int_{-\infty}^{+\infty} f_{y,t}(y, t)\mathrm{d}y}\mathrm{d}y \\ &= \int_{-\infty}^{+\infty} x\frac{f(x, t - x)}{\int_{-\infty}^{+\infty} f(x, t - x)\mathrm{d}x}\mathrm{d}x - \int_{-\infty}^{+\infty} y\frac{f(y, t - y)}{\int_{-\infty}^{+\infty} f(y, t - y)\mathrm{d}y}\mathrm{d}y \\ &= 0\end{aligned}$$

4.

$$\begin{aligned}F_Y(y) &= P(Y \le y) \\ &= P(\max_{i=1,2,\cdots,n}\{X_1, X_2, \cdots, X_n\} \le y) \\ &= P(X_1 \le y, X_2 \le y, \cdots, X_n \le y) \\ &= \Pi_{i=1}^{n} P(X_i \le y) \\ &= (F(y))^n\end{aligned}$$

So, $f_Y(y) = n(F(y))^{n-1}f(y)$. Suppose $X_i \sim N(0,1)$, we get




```
import numpy as np
import matplotlib.pyplot as plt
import math
from scipy import integrate
mu=0
sigma=1
pi=2*math.acos(0)

def px(x):
    return 1/(math.sqrt(2*pi*sigma))*math.exp(-((x-mu)**2)/(2*sigma**2))

def Fx(x):
    res = integrate.quad(px,-np.inf,x)
    return res[0]

y=[]
for i in range(1,21):
    ypy=lambda y:y*i*px(y)*Fx(y)**(i-1)
    res=integrate.quad(ypy,-np.inf,np.inf)[0]
    y.append(res)

x=list(range(1,21))
fig, ax=plt.subplots(figsize=(10,10))
ax.plot(x,y,color='g')
plt.xlabel("n")
plt.ylabel("E(Yn)")
plt.savefig("STT886HomeWork3.jpg")
plt.show()
```