(1) 若 f 连续 (f 连续是为了保证积分换序的正确性), 证明:

$$\int_a^b \mathrm{d}x \int_a^x \mathrm{d}y \int_a^y f(x,y,z)\mathrm{d}z = \int_a^b \mathrm{d}z \int_z^b \mathrm{d}y \int_y^b f(x,y,z)\mathrm{d}x$$

(2) 仔细分析下题的解题步骤, 并证明:

$$\int_0^a \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y f(x)f(y)f(z)\mathrm{d}z = \frac{1}{3!}\left(\int_0^a f(t)\mathrm{d}t\right)^3$$

例:若 f 连续, 证明:

$$\int_0^a \mathrm{d}x \int_0^x f(x)f(y)\mathrm{d}y = \frac{1}{2}\left(\int_0^a f(t)\mathrm{d}t\right)^2$$

解:

$$\int_0^a \mathrm{d}x \int_0^x f(x)f(y)\mathrm{d}y = \int_0^a f(x)\left(\int_0^x f(y)\mathrm{d}y\right)\mathrm{d}x \tag{1}$$

由于 $f(x)$ 的一个原函数为 $\int_0^x f(y)\mathrm{d}y$, 所以有

$$f(x)\mathrm{d}x = \mathrm{d}\left(\int_0^x f(y)\mathrm{d}y\right)$$

带入到 (1) 中有

$$\begin{aligned}\int_0^a \mathrm{d}x \int_0^x f(x)f(y)\mathrm{d}y &= \int_0^a \left(\int_0^x f(y)\mathrm{d}y\right)\mathrm{d}\left(\int_0^x f(y)\mathrm{d}y\right)\\ &= \frac{1}{2}\int_0^a \mathrm{d}\left(\int_0^x f(y)\mathrm{d}y\right)^2\\ &= \frac{1}{2}\left(\int_0^a f(y)\mathrm{d}y\right)^2\end{aligned}$$

此处不要被复杂的积分号弄乱思路, 时刻谨记 $\int_0^x f(y)\mathrm{d}y$ 只是一个关于 x 的函数 $F(x)$, 如果作换元 $F(x) = \int_0^x f(y)\mathrm{d}y$, 则 $f(x) = F'(x), F(0) = 0$ 就很

容易得到

$$
\begin{aligned}
\int_0^a \mathrm{d}x \int_0^x f(x)f(y)\mathrm{d}y &= \int_0^a F'(x)\mathrm{d}x \int_0^x F'(y)\mathrm{d}y \\
&= \int_0^a F'(x)F(x)\mathrm{d}x \\
&= \int_0^a F(x)\mathrm{d}F(x) \\
&= \frac{1}{2}\int_0^a \mathrm{d}F^2(x) \\
&= \frac{1}{2}F^2(a) = \frac{1}{2}\left(\int_0^a f(y)\mathrm{d}y\right)^2
\end{aligned}
$$