STT886 Homework#5

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1.

$$P_T = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

$$S = (I - P_T)^{-1} = \begin{bmatrix} 2.20689655 & 1.37931034 & 0.62068966 \\ 0.96551724 & 3.10344828 & 0.89655172 \\ 1.31034483 & 2.06896552 & 1.93103448 \end{bmatrix}$$

Thus, $(s_{13}, s_{23}, s_{33})^{\mathrm{T}} = (1.31034483, 2.06896552, 1.93103448)^{\mathrm{T}}$.

$$f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}$$

$$(f_{13}, f_{23}, f_{33})^{\mathrm{T}} = (0.6785714307908163, 1.071428574387755, 0.48214285640306126)$$

2.

Because we have $E(X_n) = \mu^n$, The number of individuals ever exist is $\sum_{n=0}^{+\infty} X_n$. Thus,

$$\begin{aligned} E(\text{the number of individuals ever exist}) &= E(\sum_{n=0}^{+\infty} X_n) \\ &= \sum_{n=0}^{+\infty} \mu^n \\ &= \frac{1}{1-\mu} \end{aligned}$$

If $X_0 = n$, we have $E(X_i) = n\mu^i$, The number of individuals ever exist is $\sum_{i=0}^{+\infty} nX_i$. Thus,

$$\begin{aligned} E(\text{the number of individuals ever exist}) &= E(\sum_{i=0}^{+\infty} X_i) \\ &= \sum_{i=0}^{+\infty} n\mu^i \\ &= \frac{n}{1-\mu} \end{aligned}$$

3.

$$\pi_0 = \frac{1}{4}\pi_0 + \frac{3}{4}\pi_0^2 \Rightarrow \pi_0 = \frac{1}{3}$$

$$\pi_0 = \frac{1}{4} + \frac{1}{2}\pi_0 + \frac{1}{4}\pi_0^2 \Rightarrow \pi_0 = 1$$

$$\pi_0 = \frac{1}{6} + \frac{1}{2}\pi_0 + \frac{1}{3}\pi_0^3 \Rightarrow \pi_0 = \frac{\sqrt{3}-1}{2}$$

4.

(a)

Yes. Because the number of white balls only depends on the last time's number of white balls.

(b)

Class:$\{0, 1, 2, \cdots, N\}$. Aperiodic. Recurrent.

(c)

$$P_{i,i+1} = p\frac{N-i}{N},\ i = 0, 1, \cdots, N-1$$

$$P_{i,i} = p\frac{i}{N} + (1-p)\frac{N-i}{N},\ i = 0, 1, \cdots, N$$

$$P_{i,i-1} = (1-p)\frac{i}{N},\ i = 1, 2, \cdots, N$$

(d)

$N = 2$. Let

$$\boldsymbol{\pi}^{\mathrm{T}} \begin{bmatrix} 1-p & p & 0 \\ (1-p)/2 & p/2 + (1-p)/2 & p/2 \\ 0 & 1-p & p \end{bmatrix} = \boldsymbol{\pi}^{\mathrm{T}}$$

$$\sum_{i=0}^{2} \pi_i = 1$$

Thus,

$$\boldsymbol{\pi}^{\mathrm{T}} = ((1-p)^2, 2p(1-p), p^2),$$

which means

$$\pi_i = \binom{N}{i} p^i (1-p)^{N-i},\ N = 2$$

(e)

So we guess

$$\pi_i = \binom{N}{i} p^i (1-p)^{N-i}.$$

(f)

When $1 \leq i \leq N$, we have

$$
\begin{aligned}
\sum_{j=1}^{N} \pi_j P_{j,i} =& P_{i-1,i} \cdot \pi_{i-1} + P_{i,i} \cdot \pi_i + P_{i+1,i} \cdot \pi_{i+1} \\
=& p\frac{N-i+1}{N}\binom{N}{i-1}p^{i-1}(1-p)^{N-i+1} + \left(p\frac{i}{N} + (1-p)\frac{N-i}{N}\right)\binom{N}{i}p^i(1-p)^{N-i} \\
& + (1-p)\frac{i+1}{N}\binom{N}{i+1}p^{i+1}(1-p)^{N-i-1} \\
=& p\frac{N-i+1}{N}\frac{N!}{(N-i+1)!(i-1)!}p^{i-1}(1-p)^{N-i+1} + \left(p\frac{i}{N} + (1-p)\frac{N-i}{N}\right)\frac{N!}{(N-i)!i!}p^i(1-p)^{N-i} \\
& + (1-p)\frac{i+1}{N}\frac{N!}{(N-i-1)!(i+1)!}p^{i+1}(1-p)^{N-i-1} \\
=& (1-p)\frac{i}{N}\frac{N!}{(N-i)!i!}p^i(1-p)^{N-i} + \left(p\frac{i}{N} + (1-p)\frac{N-i}{N}\right)\frac{N!}{(N-i)!i!}p^i(1-p)^{N-i} \\
& + p\frac{N-i}{N}\frac{N!}{(N-i)!i!}p^i(1-p)^{N-i} \\
=& \binom{N}{i}p^i(1-p)^{N-i} \\
=& \pi_i
\end{aligned}
$$

And it is obvious that $\pi_0 = P_{00} \cdot \pi_0 + P_{10} \cdot \pi_1$, $\pi_N = P_{N-1,N} \cdot \pi_{N-1} + P_{NN} \cdot \pi_N$.
Thus, $\boldsymbol{\pi}^{\mathrm{T}} = \boldsymbol{\pi}^{\mathrm{T}} P$, which indicates

$$\pi_i = \binom{N}{i}p^i(1-p)^{N-i}.$$

(g)

Suppose it takes K_j times to turn from j white to $j+1$ white. So

$$P(K_j = k) = \left(\frac{j}{N}\right)^{k-1}\frac{N-j}{N}$$

And

$$
\begin{aligned}
E(K_j) &= \sum_{k=1}^{\infty} k\left(\frac{j}{N}\right)^{k-1}\frac{N-j}{N} \\
&= \frac{N-j}{N}\sum_{k=1}^{\infty}\left[\frac{\mathrm{d}}{\mathrm{d}x}x^k\right]_{x=\frac{j}{N}} \\
&= \frac{N-j}{N}\left[\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=1}^{\infty}x^k\right]_{x=\frac{j}{N}} \\
&= \frac{N-j}{N}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1-x}\right)\right]_{x=\frac{j}{N}} \\
&= \frac{N-j}{N}\left[\frac{1}{(1-x)^2}\right]_{x=\frac{j}{N}} \\
&= \frac{N-j}{N}\frac{1}{(1-\frac{j}{N})^2} \\
&= \frac{N}{N-j}
\end{aligned}
$$

So the times are

$$
\begin{aligned}
E(\sum_{j=i}^{N-1} K_j) &= \sum_{j=i}^{N-1} E(K_j) \\
&= \sum_{j=i}^{N-1} \frac{N}{N-j} \\
&= \sum_{j=1}^{N-i} \frac{N}{j}
\end{aligned}
$$