Deductive Reasoning in Propositional Logic

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Contents

1 Deductive Systems

1. Logical Consequence:
$$A_1, ..., A_n \models C$$
   - and Logical validity:
$$\models C$$

   These are semantic notions $\rightarrow$ meaning of formulae.
   They require checking truth in all models for logic.

2. Deductive systems:
   - Are meant to capture the logical consequence and validity defined by the logical semantics, in terms of logical **logical deductions (derivations)**.
   - A Logical derivation is a mechanical procedure, not reffering the meaning of the occuring formulae.

- In deductive systems logical consequence is replaced by **deductive consequence** and valid formulae (tautologies) - by derivable formulae (theorems).

1.1 Basic Components of a deductive System

- Formal Logical language
- Axioms (Rules of inference)
- Inference (deduction) from a set of assumptions in a given deductive System D:

$$A_1, ...A_n \vdash_D C$$

- Formulae derivable from no assumptions are called **Theorems** of D:

$$\vdash_D C$$

1.2 Soundness and completeness of a deductive system

- A deductive system D is **sound** (correct) for a given logical semantics if D can **only** derive what is logically correct (valid):

$$A_1, ...A_n C \implies A_1, ...A_n \models C$$

- In particular:

$$\vdash_D C \implies \models C$$

- A system D is **Complete** for a given logical semantics if D can derive **every** valid logical consequence:

$$A_1, ...A_n \models C \implies A_1, ...A_n C$$

- In particular

$$\models C \implies \vdash_D C$$

- A deductive system is *adequte for a given semantics if it is both sounnd AND complete:

$$A_1, ...A_n \models C \iff A_1, ...A_n C$$

- in particular

$$\models C \iff \vdash_D C$$

1.3 Propositional natural deduction

- Natural Deduction (ND): System for structural logic derivation from a set of assumptions, based on rules, specific to the logical connectives.
- There are Introduction and elimination rules for each connective.
- Intro and cancelation of assumptions, assumptions can be reused many times before being discharged.
- Cancelation of assumptions: only when a rule allows it, not an obligation.
- All assumptions left at the end of a derivation must be declared.

1.3.1 NOTE:

The fewer assumptions, the stronger the claim of the derivation.

1.3.2 Rules for propositional connectives

- Refer to notes!

1. Examples in notes!

Strategies for ND proofs in propositional logic Given a set of premises Δ and the goal of trying to prove Γ:

1. Apply premises $p_i \in \Delta$ to prove Γ
2. If you need to use a $\vee$-premise, then apply $\vee$-elimination to prove Γ for each disjunct
3. Otherwise work backwards from the goal:
   (a) If the goal Γ is a conditional ($A \implies B$) then assume A and prove B using $\rightarrow$-introduction.
   (b) IF the goal Γ is negative ($\neg$ A), then assume ($\neg$ $\neg$ A) and prove contradiction; use the $\neg$-introduction.
   (c) if the goal Γ is a conjunction (A $\wedge$ B) then prove A and prove B; use $\wedge$-introduction.
   (d) If the goal Γ is a disjunction (A $\vee$ B) then prove one of A or B; use $\vee$-introduction.

1. Examples from the oxford pack - click on them to go to solutions!

    - sol

    $$\vdash P \land Q \to P \lor Q$$

    - sol

    $$\neg(P \to Q) \vdash P \land \neg Q$$

    - sol

    $$P \lor \neg Q, R \to \neg P \vdash Q \to \neg R$$

    - sol

    $$P \to Q \vdash \neg P \lor Q$$

    - sol

    $$\neg P \lor Q, P \lor \neg Q \vdash P \leftrightarrow Q$$

2 Useful Resources for Natural deductions

- Oxford logic pack