1logic

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August 6, 2024

Contents

1 Propositional Logic

$A \implies B$ This evalutes true if A is <u>false</u> **OR** B is <u>true</u>

1.1 Logical Constants

$\top$: True propostion
$\bot$: False propstion

1.2 Variables

- p, q, r : lower case roman letters.

1.3 Construction Trees

- Taking any subtree is a valid logical formulae itself
- the size of the graph is the number of nodes (variables + connectives)
- The main connective is the one applied last, non-leaf node
- create the tree from the inside

1.4 Parsing Trees

- Start constructing these from the main connective
- Top to bottom

1.5 Truth Tables

- ternary function mapping $\{0, 1\} \rightarrow \{0, 1\}$

1.6 Tautologies

- propositionnaly valid formula that is always True.

1.6.1 Notation

- $\models A$

1. Examples
    - $\models$ p $\vee$ $\neg$ p - Law of excluded middle

- $\models \neg(p \land \neg\, p)$ - Law of non-contradiction
- $\models ((p \land (p \implies q)) \implies q)$ - modus ponents
  - Set q false, left must be true so all true.

1.7 Contradictions

- always false formula

1. If a formula is not a contradiction, then it is satisfiable

1.8 Propositional Logical Consequence

$$A_1, A_2, ..., A_n \models C$$

- if C is true whenever all A_i are true.
- **Tautology** is a special case of consequence

$$\phi \models C = \models C$$

1.8.1 Consiquence is reducible to validity

$$A_1, ..., A_n \models B$$

is equivalent to: $A_1 \land ... \land A_n \models B$
then

$$\models (A_1 \land ... \land A_n) \implies B$$

1.9 Sound Rules of propositional inference

- A rule of propositional inference is a scheme where P_i are the premises we assume and C is the conclusion

$$\frac{P_1, ..., P_n}{C}$$

- An inference rule is **logically sound** if

$$P_1, ..., P_n \models C$$

- Putting concrete propostions (substituting for variables) is a propositional inference.

1.10 How to check correctness

Check if a rule is sound :
$$\frac{S, S \implies h}{h}$$

so check if $S, S \implies h \overset{?}{\models} h$
$\therefore$ the inference is correct based on a sound rule, but can't say rules is sound

1.11 Fallacies of Implication (Derivative Implication)

- converse: $B \implies A$
- inverse: ¬ A $\implies$ ¬ B
- contrapositve: ¬ B $\implies$ ¬ A this is **sound**.

1.12 Truth assignments in propostional logical

- PVAR : propostional variables (countably infinite)
- FOR : set of all propositional formulas

A truth assignment is a map \ v : PVAR → {T, F} and
$\bar{v}$:FORM →{T,F}

1. Note the following:
   - $\bar{v}(\top)$ = T
   - $\bar{v}(\bot)$ = F
   - $\bar{v}(p)$ = v(p)

* $\bar{v}(\not A)$ =T,$(A) = F$

1.13 Logical Equivalence

A ≡ B both obtain the same truth value under all truth valuations

$$A \equiv B \iff \bar{v}(A) = \bar{v}(B)$$

for all truth assignments if v

1.13.1 De' Morgans Law

¬(p∧ q) ≡ ¬ p ∨ ¬ q

1.13.2 this one

p $\wedge$ (p $\vee$ q) $\equiv$ p $\wedge$ p $\equiv$ p\

1.14 Relating Logical Equivalence, consequence and validity

1. Reducible to validity

$$A \equiv B \; iff \models A \iff B$$

   (a) Proof in notes Suppose A $\equiv$ B
   Let v:PVAR$\rightarrow$ {T,F} be arbitrary
   since A $\equiv$ B, $\bar{v}$(A) = $\bar{v}$(B)
   $\therefore$ $\bar{v}$(A $\iff$ B) = T
   Since v was arbitrary we conclude
   $\models$ A $\iff$ B

2. Reducible to Logical Consequence A $\equiv$ B iff both A $\models$ B and $B \models A$

1.15 $\equiv$ is an equivalence relation

1. reflexive A$\equiv$ A
2. symmetric if A$\equiv$ B then B $\equiv$ A
3. transitive if A $\equiv$ B and B $\equiv$ C then A $\equiv$ C

1.16 $\equiv$ is a congruence wrt to logical connectives

- if A $\equiv$ B then $\neg$ A $\equiv$ $\neg$ B

- if $A_1 \equiv B_1$ and $A_2 \equiv B_2$ then ($A_1 \cdot A_2$) $\equiv$ ($B_1 \cdot B_2$) for all $\cdot$ = {$\vee$, $\wedge$, $\implies$, $\iff$ }

1. Theorem Equivalent replacement if A $\equiv$ B then C(A/p) $\equiv$ C(B/p)
   We replace p with A all at once, the same for B
   (a) Proof in notes $\rightarrow$ by induction on C By Induction on C:
      i. C is $\perp$ Then $C(A/p)$ = $\perp$ and C(B/p) = $\perp$

$$\therefore C(A/p) = \perp \equiv \perp = C(B/p)$$

i. C is $\top$ same as above.
ii. 1. C is q where $q \neq p$, then $C(A/p) = q$ and $C(B/p) = q$

SO $C(A/p) = q \equiv q = C(B/p)$
i. 2. C is p , then

$C(A/p) = A \equiv B = C(B/p)$
Inductive Hypothesis: Suppose holds for formulas C_1 and C_2
$C_1(A/p) \equiv C_1(B/p)$ and
$C_2(A/p) \equiv C_2(B/p)$
i. C is ¬ C :

Then $C(A/p) = \not C(A/p) \overset{\star}{\equiv} \not C(B/p) = C(B/p)$,
Where $\star$ holds by the Inductive Hypothesis and the fact that $\equiv$ is a congruence wrt to propositional connectives.
i. C is $(C_1 \cdot C_2)$ for some $\cdot = \{\wedge \vee \implies \iff\}$

Then $C(A/p) = (C_1(A/p) \cdot C_2(A/p))$ but by inductive hypothesis
$C_1(A/p) \equiv C_1(B/p)$ and
$C_2(A/p) \equiv C_2(B/p)$
So by the fact that $\equiv$ us a congruence wrt to propositional connectives;
$C(A/p) = ((C_1(A/p) \cdot C_2(A/p)) \equiv (C(A/p) = (C_1(B/p) \cdot C_2(B/p)) = C(B/p)$
∴ C(A/p) ≡ C(B/p)

1.16.1 Important Logical Equivalence

- Idempotency : applying more than once makes no difference p∧ p and p∨ p
- Commutativity:
- Associativity:
- Distributivity:
- BUNCH OF STUFF AT THE END OF SECTION 1.3

1. Know important ≡ for negation how it affects $\implies$ and $\iff$

1.17 Inductive Definitions

1.17.1 Structured induction

Important when an infinite set of structured objects are to be defined
BNF

$$w := \epsilon | wa$$

, where $a \in A$

1. Example proof The number of right parenthesis is the same as left in logical formalae ATTACH

   l : **FOR** $\rightarrow \mathbb{N}$
   r : **FOR** $\rightarrow \mathbb{N}$
   To prove /l/(ψ)= /r/(ψ) $\forall \psi \in$ **FORM**

   (a) if ψ is $\top$, then $l\ (\psi) = l\ (\top) = 0 = r\ (\top) = r\ (\psi)$
   if ψ is $\perp$, then $l\ (\psi) = l\ (\perp) = 0 = r\ (\perp) = r\ (\psi)$

   (b) if ψ is p then $l\ (\psi) = l$ (p) $= 0 = r$ (p) $= r\ (\backslash\psi)$

   (c) Suppose l (A) $= r$ (A) and A is ¬ A:

   then $l\ (\psi) = l$ (¬ A) $= l$ (A) =/r/ (A) $= r$ (¬ A) $= r\ (\psi)$

   (a) Suppose *l/(A)* $=$ /*r* (A) and l (B) $= r$ (B) and ψ is (A · B), for $\cdot = \{\wedge, \vee, \implies, \iff\}$

   Then $l\ (\psi) = l$ ((A · B) $= l$ (A) $+\ l$ (B) $+ 1$
   Now by IH $= r$ (A) $+\ r$ (B) +1 $= r$ ((A · B)) $= r\ (\psi)$

1.18 Complex Proof

We will prove $A \wedge B \implies B \implies A$ using natural deduction.

$$\frac{\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}}{B \wedge A}$$