Tutoring Notes For Elementary Real Analysis

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1 Introduction

In this note, we'll explore a variety of fascinating topics in real analysis.

1.1 The Existence of The Inverse Function and Its Properties

We start by defining a function $f : \mathbb{R} \to \mathbb{R}$.

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ is a collection of ordered pairs of real numbers where each element in its domain has a unique output. That is, $f = \{(x, y) : x, y \in \mathbb{R}\} \subset \mathbb{R}^2$, where for each $x \in \mathbb{R}$, there is a unique $y \in \mathbb{R}$ such that $(x, y) \in f$.

Let's consider the question: *What happens if we reverse the order of numbers in the ordered pairs of f?* While the concept of inverting ordered pairs does exist, it only applies under certain conditions. Specifically, f must be an injective function. We'll briefly revisit the definition of injectivity below.

Definition 2. The function $f : A \to B$ is injective if for each $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

We also define a potential inverse function of f as follows.

Definition 3. For any function f, the inverse of f, denoted by f^{-1}, is the set of all pairs (a, b) for which the pair (b, a) is in f.

Theorem 1. *Let $f : A \to B$ be a function. Then f^{-1} is a function if and only if f is injective.*

Proof. Suppose that f is injective. We need show that f^{-1} has a unique output for each $b \in B$. Suppose that $(b, a_1), (b, a_2) \in f^{-1}$. Then by definition, $(a_1, b), (a_2, b) \in f$. This implies that $f(a_1) = f(a_2)$. Since f is injective, $a_1 = a_2$, showing that f^{-1} is a function.

Conversely, suppose that f^{-1} is a function. For some pair of $a_1, a_2 \in A$, suppose we have $f(a_1) = f(a_2)$. Then it follows that $(a_1, f(a_1)), (a_2, f(a_2)) \in f$. Since f^{-1} is a function and $(f(a_1), a_1), (f(a_2), a_2) \in f$, we have $a_1 = a_2$. $\square$

Remark 1. Note that we have $f^{-1}[B] = A$ in this case. Note also that f is bijective onto its image $f[A]$, and f^{-1} restricted on $f[A]$ is bijective as well.

We have shown that an injective function f has an inverse. Naturally, we might wonder: if f is continuous, does it follow that f^{-1} is also continuous? To investigate this, we now recall one of the most fundamental theorems concerning the continuity of f.

Theorem 2. *(Intermediate Value Theorem) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on a closed interval $[a, b]$. Suppose without loss of generality that $f(a) < f(b)$. Then for each c such that $f(a) < c < f(b)$, there exists d such that $f(d) = c$.*

And we now come to the following theorem.

Theorem 3. *If f is continuous and injective on an interval, then f is either strictly increasing or decreasing on that interval.*

Proof. Take two points a, b from the interval with $a < b$. Without loss of generality, suppose $f(a) < f(b)$. Now consider any point c on the interval with $b < c$. We show that $f(a) < f(b) < f(c)$. Suppose to the contrary that we have $f(c) < f(b)$. Then we need to cover two cases.

Case1: Suppose $f(a) < f(c)$. Then since $f(a) < f(c) < f(b)$ and f is continuous on $[a, b]$, there exists some $d \in [a, b]$ such that $f(d) = f(c)$. But since $d \neq c$, this contradicts the injectivity of f.

Case2: Suppose $f(c) < f(a)$. The since $f(c) < f(a) < f(b)$ and f is continuous on $[b, c]$, there exists some $e \in [b, c]$ such that $f(e) = f(a)$, contradicting the injectivity of f.

Hence, for any point c on the interval with $b < c$. We show that $f(a) < f(b) < f(c)$. We also need show that for any point c on the interval with $c < a$, $f(c) < f(a) < f(b)$. Again, we can prove this case using an analogous argument. Thus, we have proved the claim. □

We now have all the tools we need to investigate the continuity of f^{-1}. Indeed, the following theorem is known to hold.

Theorem 4. *if f is continuous and injective on an interval, f^{-1} is also continuous.*

Proof. □

1.2 Fixed Point Theorem

Theorem 5. *Let $f : [0,1] \to [0,1]$ be a continuous function. There is at least one fixed point $c \in [0,1]$ such that $f(c) = c$.*

Proof. The proof will rely on the intermediate value theorem. First, consider the following function g, defined as $g(x) = f(x) - x$. Clearly, g is continuous since it is a sum of two continuous functions. First, it is easy to see that $0 \leq f(0)$ and $f(1) \leq 1$. If we have $0 = f(0)$ or $1 = f(1)$, we are done (since we get $g(0) = 0$ or $g(1) = 1$). Suppose this is not the case. Then, we get $g(0) > 0$ and $g(1) < 0$. By intermediate value theorem, there exists some $c \in [0,1]$ such that $g(c) = 0$. This implies that $f(c) = c$, proving the claim. $\square$

1.3 Density of Rational in $\mathbb{R}$

In this chapter, we will demonstrate a density of $\mathbb{Q}$ (rationals) and $\mathbb{I}$ (irrationals). We start by recalling the defintion of a rational number and an irratinal number.

Definition 4. A rational number is any number that can be expressed as the quotient of two integers. Formally, a number r is rational if there exist integers p and q with $q \neq 0$ such that

$$r = \frac{p}{q}.$$

The set of all irrational numbers is denoted by $\mathbb{Q}$.

Definition 5. An irrational number is any real number that is not a rational number. The set of all rational numbers is denoted by $\mathbb{I}$ or $\mathbb{R} \setminus \mathbb{Q}$.

The proof that $\mathbb{Q}$ is dense in $\mathbb{R}$ depends on a superior property of $\mathbb{R}$ called Archimedean Property and well-ordering principle.

Theorem 6. *(The Archimedean Property) For every $x \in \mathbb{R}$, there exists a natural number $n \in \mathbb{N}$ such that $n > x$.*

Theorem 7. *(The Well-Ordering Principle) Every non-empty subset of the natural numbers $\mathbb{N}$ has a least element. In other words, if $S \subseteq \mathbb{N}$ and $S \neq \emptyset$, then there exists an element $m \in S$ such that $m \leq x$ for all $x \in S$.*

Additionally, we will prove a lemma useful for showing the density of rational.

Lemma 1. *For all $x \in \mathbb{R}$, there exists an integer $m \in \mathbb{Z}$ such that $x < m \leq x+1$.*

Proof. First, we claim that for each $x \in \mathbb{R}$, there exists $k \in \mathbb{N}$ such that $x+k > 1$. The case is trivial when $x \geq 0$. Hence, suppose $x < 0$. Then we have $0 < -x$. The Archimedean property implies that there exists $k \in \mathbb{N}$ such that $-x+1 < k$. Hence, we get $1 < k + x$.

Take $x \in \mathbb{R}$ arbitrarily. Choose $k \in \mathbb{N}$ such that $x + k > 1$. Then consider the following set:

$$A = \{a \in \mathbb{N} | a > x + k > 1\} \subset \mathbb{N}$$

Now, put a well-order on $\mathbb{R}$. Then A is a well-ordered set. Note that the set A is non-empty since the Archimedean property implies that there exists some $n \in \mathbb{N}$ such that $n > x + k$. Since A is also a subset of $\mathbb{N}$, the well-ordering principle implies that A has a least element. We denote this least element to be $m + k$ for some integer $m \in \mathbb{Z}$.

Next, we claim that $m+k \leq x+k+1$. Suppose to the contrary that $m+k-1 > x+k$. However, this is a contradiction since this shows that $m+k$ is not a least element of A. Therefore, we get that $x + k < m + k \leq x + k + 1$. And then we get the following relation:

$$x < m \leq x + 1$$

proving the lemma. $\square$

Using this lemma, we will provide the proof for the density of rational below.

Theorem 8. *(The Density Of Rational) Given two real numbers x, y with $x < y$, there is a rational number $r = \frac{p}{q}$ such that $x < r < y$.*

Proof. Suppose $x < y$, then $y - x > 0$. Since $\mathbb{R}$ is a field, there exists a multiplicative inverse of $(y-x)$, denote it as $(y-x)^{-1}$. By Archimedean property, there exists some $n \in \mathbb{N}$ such that $(y-x)^{-1} < n$. Now consider the following:

$$\begin{aligned} (y-x)^{-1} &< n, \\ (y-x)(y-x)^{-1} &< n(y-x), \\ 1 &< yn - xn, \\ xn + 1 &< yn. \end{aligned}$$

Now applying the lemma, for some $m \in \mathbb{Z}$, we get:

$$\begin{aligned} xn < m \leq xn + 1 &< yn, \\ xn < m &< yn, \\ x < \frac{m}{n} &< y \end{aligned}$$

Let $r = \frac{m}{n}$. Then since $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, r is a rational number. And $x < r < y$. Thus, we have proved the density of rational.

$\square$

Using the density of rational, we can also prove the density of *irrational*. Note that the proof relies on a few lemmas, whose proofs are left to the reader as exercise.

Lemma 2. *Let $p \in \mathbb{Z}$. If p^2 is even, then p is even.*

Proof. Exercise. $\square$

Lemma 3. *$\sqrt{2}$ is irrational.*

Proof. Exercise. $\square$

Lemma 4. *The sum of a rational number and irrational number is irrational number*

Proof. Exercise. $\square$

Theorem 9. *(The Density Of Irrational) Given two rational numbers r, s with $r < s$, there is an irrational number a such that $r < a < s$.*

Proof. Consider the fact that $-\sqrt{2}$ is irrational since $\sqrt{2}$ is irrational. Since $\mathbb{R}$ is a field, $r - \sqrt{2}$ and $s - \sqrt{2}$ are in $\mathbb{R}$. We also have $r - \sqrt{2} < s - \sqrt{2}$. Then by density of rational, there exists some rational number $b \in \mathbb{Q}$ such that $r - \sqrt{2} < b < s - \sqrt{2}$. Then we get that $r < b + \sqrt{2} < s$. Since the sum of rational number and irrational number is irrational, $b + \sqrt{2}$ is the desired irrational number.

$\square$

1.4 Deep Into Sequence

In this chapter, we will cover many core and intriguing theorems in the study of sequences. We begin by recalling the definition of a sequence.

Definition 6. An infinite sequence of real numbers is a function whose domain is $\mathbb{N}$.

In general, the sequence is a function $a : \mathbb{N} \to \mathbb{R}$. We often consider the infinite sequence of numbers $a_1, a_2, \dots a_n, \dots$. Let's also recall the definition of sequence convergence.

Definition 7. A sequence $\{a_n\}$ is said to converge to a limit L if, for every $\epsilon > 0$, there exists a positive integer N such that for all $n \geq N$, $|a_n - L| < \epsilon$.

Definition 8. (Supremum Axiom) Every non-empty subset $S \subseteq \mathbb{R}$ that is bounded above has a least upper bound, or supremum, in $\mathbb{R}$. In other words, there exists a real number $\sup S$ such that:

- $\sup S \geq s$ for all $s \in S$ (i.e., $\sup S$ is an upper bound of S),
- for any $\epsilon > 0$, there exists an $s \in S$ such that $\sup S - \epsilon < s$.
- let B be a collection of upper bounds for S. Then $\sup S \leq b$ for all $b \in B$.

Now, we dive into the first fundamental theorem relating to the sequence convergence.

Theorem 10. *The Monotone Convergence Theorem states that if $\{a_n\}$ is a sequence that is monotonic (either increasing or decreasing) and bounded, then $\{a_n\}$ converges. Specifically:*

- *If $\{a_n\}$ is monotone increasing and bounded above, then $\{a_n\}$ converges to its supremum:* $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.
- *If $\{a_n\}$ is monotone decreasing and bounded below, then $\{a_n\}$ converges to its infimum:* $\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.

Proof. We only do a proof for the increasing case. The proof for the decreasing case follows similarly. Suppose that the sequence $\{a_n\}$ is bounded and monotonically increasing. We will claim that $\{a_n\}$ converges to its supremum, $\sup(\{a_n\})$. Let $c = \sup(\{a_n\})$. Let $\epsilon > 0$ be arbitrary. By supremum axiom, there exists some $N \in \mathbb{N}$ such that $c - \epsilon < a_N < c$. Since $\{a_n\}$ is monotonically increasing, for all $m \in \mathbb{N}$ with $m > N$, $a_N \leq a_m$, implying that for all $m > N$, $|a_m - c| < \epsilon$. Thus, we see that the sequence $\{a_n\}$ converges to its supremum. The similar argument shows that the sequence that is bounded and monotonically decreasing converges to its infimum.

□

Using the monotone convergence theorem, we can prove another interesting theorem, called nested interval theorem.

Theorem 11. *The Nested Interval Theorem states that for any sequence of closed, nested intervals $\{[a_n, b_n]\}$ in $\mathbb{R}$, where each interval $[a_n, b_n]$ satisfies $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and $\lim_{n\to\infty}(b_n - a_n) = 0$, there exists exactly one point $x \in \mathbb{R}$ contained in all intervals. That is,*

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{x\}.$$

There are some key observations arising from this statement of the theorem. First, we can infer that the sequence $\{a_n\}$ is monotonically increasing and $\{b_n\}$ is monotonically decreasing. Moreover, both $\{a_n\}$ and $\{b_n\}$ are bounded by a_1 and b_1. Also since $\lim_{n\to\infty}(b_n - a_n) = 0$, both sequences converge to the same limit. Using these observations, we can finally write a complete proof for this theorem.

Proof. Consider two sequences $\{a_n\}$ and $\{b_n\}$. We observe that $\{a_n\}$ is monotonically increasing and bounded by a_1 and b_1. Similarly, $\{b_n\}$ is monotonically decreasing and bounded by a_1 and b_1. By monotone convergence theorem, we know that $\{a_n\}$ converges to $\sup(\{a_n\})$ and $\{b_n\}$ converges to $\inf(\{b_n\})$. Since $\lim_{n\to\infty}(b_n - a_n) = 0$, we have $\sup(\{a_n\}) = \inf(\{b_n\})$. Let $x = \sup(\{a_n\}) = \inf(\{b_n\})$. Clearly, $x \in \bigcap_{n=1}^{\infty}[a_n, b_n]$.

Now, we need to show that $\bigcap_{n=1}^{\infty}[a_n, b_n]$ contains a unique point $x = \sup(\{a_n\}) = \inf(\{b_n\})$. Suppose to the contrary that it contains two distinct points $x \neq y$. We need to consider two cases.

Case 1: Suppose that $x < y$. Then there exists some $\epsilon > 0$ such that $y - x > \epsilon > 0$. Since $x = \inf(\{b_n\})$, there exists some $N \in \mathbb{N}$ such that $x < b_N < x + \epsilon$. Thus, $y \notin [a_N, b_N]$, and $y \notin \bigcap_{n=1}^{\infty}[a_n, b_n]$.

Case 2: Suppose that $x > y$. Then there exists some $\epsilon > 0$ such that $x - y > \epsilon > 0$. Since $x = \sup(\{a_n\})$, there exists some $N \in \mathbb{N}$ such that $x > a_N > x - \epsilon$. Thus, $y \notin [a_N, b_N]$, and $y \notin \bigcap_{n=1}^{\infty}[a_n, b_n]$.

Therefore, $\bigcap_{n=1}^{\infty}[a_n, b_n]$ contains a unique point x.

□

Finally, we will prove one of the most fundamental theorems in the study of sequence using nested interval theorem, namely, Bolzano-Weierstrass Theorem.

Theorem 12. *The Bolzano-Weierstrass Theorem states that every bounded sequence in $\mathbb{R}$ has a convergent subsequence. In other words, if $\{a_n\}$ is a bounded sequence in $\mathbb{R}$, then there exists a subsequence $\{a_{n_k}\}$ and a limit $L \in \mathbb{R}$ such that*

$$\lim_{k\to\infty} a_{n_k} = L.$$

There are two main approaches to proving this theorem, and I will introduce both methods. The first proof is presented below.

Proof. Suppose that the sequence $\{a_n\}$ is bounded. Then there exists some $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Therefore, $\{a_n\} \subseteq [-M, M]$. Our goal is to construct a convergent subsequence of $\{a_n\}$ by repeatedly bisecting the interval.

We will define a sequence of nested intervals $\{I_n\}$ with $I_1 = [-M, M]$. For each $n \in \mathbb{N}$, if $I_n = [a, b]$, we divide I_n at its midpoint, choosing the next interval I_{n+1} such that it contains infinitely many terms of $\{a_n\}$. Specifically, we set:

$$I_{n+1} = \begin{cases} [a, \frac{a+b}{2}] & \text{if this subinterval contains infinitely many terms of } \{a_n\}, \\ [\frac{a+b}{2}, b] & \text{otherwise.} \end{cases}$$

Note that each I_n contains infinitely many points of $\{a_n\}$. Moreover, we have $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, and $length(I_n) = \frac{2M}{2^{n-1}}$. Hence it is clear that $\lim_{n\to\infty} length(I_n) = 0$. Now, we construct a subsequence $\{a_{n_k}\}$ where $a_{n_k} \in I_k$ for each $k \in \mathbb{N}$. Now applying a nested interval theorem to $\{I_n\}$, we observe that there is exactly one point $x \in \mathbb{R}$ such that $x \in \bigcap_{n=1}^{\infty} I_n$. Now, take $\epsilon > 0$. Then since $\lim_{n\to\infty} length(I_n) = 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $length(I_n) < \epsilon$. Thus taking the same N, by definition of convergence, the subsequence a_{n_k} converges to $x \in \bigcap_{n=1}^{\infty} I_n$.

Since we have constructed a convergent subsequence, we proved the claim. □

Note that this is only one proof of the Bolzano-Weierstrass Theorem, among several possible approaches. I will introduce an additional proof below.

Lemma 5. *Any sequence $\{a_n\}$ contains a subsequence which is either non-increasing or non-decreasing.*

Proof. Call a natural number n a "peak point" of the sequence $\{a_n\}$ if $a_m < a_n$ for all $m > n$.

Case 1. The sequence has infinitely many peak points. In this case, if $n_1 < n_2 < n_3 < \ldots$ are the peak points, then $a_{n_1} > a_{n_2} > a_{n_3} > \ldots$, so $\{a_{n_k}\}$ is the desired (nonincreasing) subsequence.

Case 2. The sequence has only finitely many peak points. In this case, let n_1 be greater than all peak points. Since n_1 is not a peak point, there is some $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. Since n_2 is not a peak point (it is greater than n_1, and hence greater than all peak points), there is some $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. Continuing in this way, we obtain the desired (nondecreasing) subsequence.

$\square$

Corollary 1. (Bolzano-Weierstrass Theorem) Every bounded sequence has a convergent subsequence.

Proof. Suppose that the sequence is bounded. By the lemma above, this sequence contains a monotonic subsequence, which is again bounded. Hence by monotone convergence theorem, this subsequence is convergent. $\square$

Remark 2. The first proof of the Bolzano-Weierstrass Theorem uses the nested interval theorem, while the second proof directly relies on the monotone convergence theorem. Since the nested interval theorem is derived from the monotone convergence theorem, both proofs ultimately depend on it. Which approach do you find simpler?

Corollary 2. This is a corollary from a nested interval theorem. Consider $\bigcap_{n=1}^{\infty}[-1/n, 1/n]$. By nested interval theorem, $\bigcap_{n=1}^{\infty}[-1/n, 1/n]$ contains exactly one point, and $0 \in \bigcap_{n=1}^{\infty}[-1/n, 1/n]$. Now consider $\bigcap_{n=1}^{\infty}(-1/n, 1/n)$, then we can deduce that $\bigcap_{n=1}^{\infty}(-1/n, 1/n)$ contains exactly one point and $0 \in \bigcap_{n=1}^{\infty}(-1/n, 1/n)$. Since a singleton $\{0\}$ is closed, this shows that an infinite intersection of open sets is not open necessarily.

Finally, I will present a proof for the intermediate value theorem. However, the proof requires a lemma concerning the sequential continuity. Hence, we will prove the following lemma first.

Theorem 13. *(Sequential Continuity) Suppose $r < a < t$, and $f(x)$ is defined on (r, t). Then $f(x)$ is continuous at $x = a$ if and only if for every sequence $\{a_n\}$ of elements of (r, t) that converges to a, we have that the sequence $\{f(a_n)\}$ converges to $f(a)$.*

Proof. Suppose f is continuous at $a \in (r, t)$. Then $\lim_{x\to a} f(x) = f(a)$. Take any sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = a$. Consider some $\epsilon > 0$. By the continuity of f, there exists $\delta > 0$ such that for $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$. By the definition of sequence convergence $\{a_n\} \to a$, for $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - a| < \delta$. For $n > N$, $|f(a_n) - f(a)| < \epsilon$, since $|a_n - a| < \delta$. Thus, $\{f(a_n)\}$ converges to $f(a)$.

Conversely, suppose for all sequences $\{a_n\}$ with $\lim_{n\to\infty} a_n = a$, we have $f(a_n) \to f(a)$. Suppose f is not continuous at a. Then there exists $\epsilon > 0$ such that $\forall \delta > 0$, $|f(x) - f(a)| \geq \epsilon$ for $|x - a| < \delta$. Take a sequence $a_n \to a$. Then $f(a_n) \to f(a)$. Now, consider a sequence $a_n = a + \frac{1}{n}$. Then a_n converges to a. However, $f(a_n)$ never converges to $f(a)$, since $|f(a_n) - f(a)| \geq \epsilon$, $\forall n$. This is a contradiction.

Thus, we have proved the claim. $\square$

Finally, the proof of the intermediate value theorem is as follows.

Theorem 14. *(Intermediate Value Theorem) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on a closed interval $[a, b]$. Suppose $f(a) < 0 < f(b)$. Then there exists a point $c \in (a, b)$ such that $f(c) = 0$.*

Proof. Suppose $f(a) < 0 < f(b)$. We aim to construct a sequence of closed, nested intervals $\{[a_n, b_n]\}$ such that each interval $[a_n, b_n]$ satisfies $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and $\lim_{n\to\infty}(b_n - a_n) = 0$.

First, set $a_1 = a$ and $b_1 = b$. To define the subsequent terms a_{n+1} and b_{n+1}, bisect the interval $[a_n, b_n]$ into two subintervals: $[a_n, m_n]$ and $[m_n, b_n]$, where $m_n = \frac{a_n+b_n}{2}$ is the midpoint of $[a_n, b_n]$.

Next, we select the subinterval that contains the sign change for f. Without loss of generality, suppose $f(a_n) < 0$ and $f(m_n) > 0$. Then we set $a_{n+1} = a_n$ and $b_{n+1} = m_n$. This process ensures that $f(a_n) < 0 < f(b_n)$ for all n.

If we find an n for which $f(a_n) = 0$ or $f(b_n) = 0$, we are done. Otherwise, this process continues indefinitely, yielding a sequence of nested intervals $\{[a_n, b_n]\}$ with lengths $b_n - a_n = \frac{b-a}{2^n}$, which converges to zero as $n \to \infty$.

Since $\lim_{n\to\infty}(b_n - a_n) = 0$, both sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit L. By the Monotone Convergence Theorem, these sequences are convergent, so let $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

By the (sequential) continuity of f, we have

$$\lim_{n\to\infty} f(a_n) = f(L) = \lim_{n\to\infty} f(b_n).$$

Since $f(a_n) < 0$ for all n and $f(b_n) > 0$ for all n, it follows that $f(L) \leq 0$ and $f(L) \geq 0$.

Thus, we conclude that $f(L) = 0$, as required.

$\square$

1.5 Cardinality of Sets

Theorem 15. *The Schröder–Bernstein Theorem states that if there exist injective functions $f : A \to B$ and $g : B \to A$ between two sets A and B, then there exists a bijective function $h : A \to B$. In other words, if there is an injection from A to B and an injection from B to A, then A and B have the same cardinality.*

Example 1. A collection of open intervals in $\mathbb{R}$ is countable.

Proof. □

1.6 Reference

References

[1] Michael Spivak, *Calculus*, 4th ed., Publish or Perish, 2008.