# MATD94 Algebraic Topology Notes

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## Contents

# 1 Important theorems, definitions, lemmas I cited and self exercises

## 1.1 $\pi_1(S^1) \cong \mathbb{Z}$

The proof is in the textbook by Hatcher.

## 1.2 Van Kampen's Theorem

The following theorem statement is directly cited from the algebraic topology textbook by Hatcher:

**Theorem 1.20.** *If $X$ is the union of path-connected open sets $A_\alpha$ each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism*

$$\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$$

*is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of $\Phi$ is the normal subgroup $N$ generated by all elements of the form*

$$i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} \quad \text{for } \omega \in \pi_1(A_\alpha \cap A_\beta),$$

*and hence $\Phi$ induces an isomorphism*

$$\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N.$$

## 1.3 A lemma about the induced maps of homotopic paths

**Lemma 1.** *The induced maps of two homotopic paths $h, k : (X, x_0) \to (Y, y_0)$ are the same.*

*Proof.* Let $h, k : X \to Y$ be homotopic maps with a homotopy $H : X \times [0,1] \to Y$ such that $H(x,0) = h(x)$ and $H(x,1) = k(x)$. Assume the homotopy fixes the basepoint $x_0 \in X$, i.e., $H(x_0, t) = h(x_0) = k(x_0)$ for all $t$. We show $h_* = k_*$ on $\pi_1(X, x_0)$.

Let $[\gamma] \in \pi_1(X, x_0)$ be the homotopy class of a loop $\gamma : [0,1] \to X$ based at $x_0$. Define the map $G : [0,1] \times [0,1] \to Y$ by:

$$G(s,t) = H(\gamma(s), t).$$

This satisfies:

- $G(s,0) = H(\gamma(s), 0) = h(\gamma(s))$,
- $G(s,1) = H(\gamma(s), 1) = k(\gamma(s))$,
- $G(0,t) = G(1,t) = h(x_0)$ for all $t$.

Thus, $G$ is a path homotopy between $h \circ \gamma$ and $k \circ \gamma$. Hence, $[h \circ \gamma] = [k \circ \gamma]$ in $\pi_1(Y, h(x_0))$, which implies $h_*([\gamma]) = k_*([\gamma])$. Since $[\gamma]$ was arbitrary, $h_* = k_*$.

□

### 1.4 A lemma about the homotopy equivalence and their fundamental groups

**Lemma 2.** *If two spaces $X$ and $Y$ are homotopy equivalent, then $\pi_1(X) \cong \pi_1(Y)$.*

*Proof.* Let $X$ and $Y$ be topological spaces that are homotopy equivalent. By definition, there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, where $\simeq$ denotes homotopy.

We know that if $h, k : X \to Y$ are homotopic, then $h_* = k_* : \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$ from the above lemma. Applying this to $g \circ f \simeq \mathrm{id}_X$, we get:

$$(g \circ f)_* = (\mathrm{id}_X)_* \implies g_* \circ f_* = \mathrm{id}_{\pi_1(X)}.$$

Similarly, from $f \circ g \simeq \mathrm{id}_Y$, we obtain:

$$(f \circ g)_* = (\mathrm{id}_Y)_* \implies f_* \circ g_* = \mathrm{id}_{\pi_1(Y)}.$$

Thus, $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ and $g_* : \pi_1(Y, f(x_0)) \to \pi_1(X, x_0)$ are mutually inverse group homomorphisms. Therefore, $f_*$ is an isomorphism, and $\pi_1(X) \cong \pi_1(Y)$.

□

## 2 The problems assigned from the Professor Lisa Jeffrey

### 2.1 Problem 1

Find the fundamental group of a figure eight (i.e., $S^1 \vee S^1$). Use the van Kampen theorem and the fact that the fundamental group of the circle $S^1$ is $\mathbb{Z}$.

*Proof.* We will compute the fundamental group of the Figure Eight $S^1 \vee S^1$ using that $\pi_1(S^1) \cong \mathbb{Z}$ and the van Kampen theorem.

We will cover $X = S^1 \vee S^1$ using two open sets $U$ and $V$, each of which covers a circle $S^1$. Let $x_0$ be the point where two circles join. Let $U$ and $V$ be the open thickenings of the circles respectively such that the intersection of $U$ and $V$ is a disk. Then, the union $U \cup V$ covers $X$, and $U \cap V = D_\epsilon(x_0)$ for some $\epsilon > 0$. $U, V, U \cap V$ are clearly path connected.

We will check the fundamental groups of $U$, $V$, and $U \cap V$.

Firstly, we know that both $U$ and $V$ deformation retract to the circle $S^1$. Hence using inclusion and retraction maps, we can see that $U \approx S^1$ and $V \approx S^1$

where $\approx$ denotes a homotopy equivalence. Since two spaces that are homotopy equivalent to each other have the same fundamental groups by the lemma in the section 2, $\pi_1(U) \cong \pi_1(S^1)$ and $\pi_1(V) \cong \pi_1(S^1)$. Since we know from the Hatcher that $\pi_1(S^1) \cong \mathbb{Z}$, we have $\pi_1(U) \cong \pi_1(V) \cong \mathbb{Z}$.

We will show that $\pi_1(U \cap V) = 0$: The intersection $U \cap V = D_\epsilon(x_0)$ is an open disk. Let $\gamma_1, \gamma_2 : [0,1] \to D_\epsilon(x_0)$ be two paths with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. Define the *straight-line linear homotopy*:

$$H(t,s) = (1-t)\gamma_1(s) + t\gamma_2(s), \quad t \in [0,1], \quad s \in [0,1].$$

Since $D_\epsilon(x_0)$ is convex, $H(t,s)$ lies entirely within $D_\epsilon(x_0)$. Thus, $\gamma_1 \simeq \gamma_2$. This shows the disk is simply connected; since the fundamental group of a simply connected space is trivial, we get $\pi_1(U \cap V) = 0$.

We now apply the van Kampen theorem. According to the theorem:

$$\pi_1(X) \cong \pi_1(U) * \pi_1(V) \setminus N$$

where $N$ is the normal subgroup generated by elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(U \cap V)$ where $i_{\alpha\beta}$ is an induced homomorphism from the inclusion map $A_\alpha \cap A_\beta \to A_\alpha$ with $A_\alpha = U, A_\beta = V$. Since $\pi_1(U \cap V) = 0$, the only element $\omega$ is the trivial loop, so the normal subgroup $N$ is also trivial. Therefore, we get:

$$\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$$

the free group on two generators $a$ and $b$.

$\square$

## 2.2 Problem 2

Find the fundamental group of a torus $T^2$.

*Proof.* Let $T^2$ be the torus. We cover $T^2$ with two open sets $U$ and $V$: let $U = T^2 \setminus \{p\}$, where $p$ is some interior point of $T^2$, and $V$ be a small open disk around $p$.

Both $U$ and $V$ are path-connected and contain the basepoint $x_0$. Their intersection $U \cap V$ is a punctured disk, which is again path-connected. Since there are only two sets, triple intersections $U \cap V \cap A_\gamma$ are vacuous, so the conditions of the van Kampeon theorem are satisfied.

We will compute $\pi_1(U)$, $\pi_1(V)$, and $\pi_1(U \cap V)$.
*claim 1: $U$ is homotopy equivalent to $S^1 \vee S^1$.* The torus $T^2$ can be represented as a square with opposite edges identified (page 5 of Hatcher). Removing an interior point $p$, we can deformation retract $T^2 \setminus \{p\}$ onto its 1-skeleton, which is formed by the edges of the square. Using identification, the 1-skeleton becomes two circles intersecting at a single point, where the corners of the square were

identified with this single point. Therefore, this square is a wedge of two circles $S^1 \vee S^1$. In the previous problem, we saw that $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$, hence we get:

$$\pi_1(U) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}.$$

*Claim 2: $V$ is contractible.* The open disk $V$ is convex and simply connected so any two paths sharing the same endpoints are homotopic via a straight-line homotopy. Since it is simply connected, its fundamental group is trivial:

$$\pi_1(V) \cong \{1\}.$$

*Claim 3: $U \cap V$ is homotopy equivalent to $S^1$.* The intersection $U \cap V$ is a punctured disk. To show that $U \cap V$ is homotopy equivalent to $S^1$ using a deformation retraction, we construct a continuous homotopy $F : (U \cap V) \times [0,1] \to U \cap V$ as follows: for $(x, y) \in U \cap V$ (where $(a, b)$ is the center of $V$), define:

$$F((x,y),t) = \left(a + \frac{x-a}{(1-t)+t\cdot d},\ b + \frac{y-b}{(1-t)+t\cdot d}\right),$$

where $d = \sqrt{(x-a)^2 + (y-b)^2}$. We know that at $t = 0$, $F((x, y), 0) = (x, y)$, which is the identity map. At $t = 1$, $F((x, y), 1) = \left(a + \frac{x-a}{d},\ b + \frac{y-b}{d}\right)$, which lies on $S^1$. If $(x, y)$ is already on $S^1$ $(d = 1)$, then $F((x, y), t) = (x, y)$ for all $t$. This deformation retraction implies $U \cap V \simeq S^1$ where $\simeq$ denotes a homotopy equivalence, and $\pi_1(U \cap V) \cong \mathbb{Z}$.

Next, we will apply Kampen's theorem. First, we will construct a normal subgroup $N$, a kernel. By the van Kampen theorem, the kernel of $\Phi : \pi_1(U) * \pi_1(V) \to \pi_1(T^2)$ is the normal subgroup $N$ generated by elements of the form:

$$i_{UV}(\omega) \cdot i_{VU}(\omega)^{-1}, \quad \text{for } \omega \in \pi_1(U \cap V).$$

Here, $i_{UV} : \pi_1(U \cap V) \to \pi_1(U)$ is induced by the inclusion $U \cap V \hookrightarrow U$, and $i_{VU} : \pi_1(U \cap V) \to \pi_1(V)$ is induced by $U \cap V \hookrightarrow V$.

Since $\pi_1(V)$ is trivial, $i_{VU}(\omega) = 1$ for all $\omega$. The generator simplifies to:

$$i_{UV}(\omega) \cdot 1^{-1} = i_{UV}(\omega) \in N.$$

Thus, $N$ is the normal subgroup of $\mathbb{Z} * \mathbb{Z}$ generated by the image of $\pi_1(U \cap V)$ under $i_{UV}$. Under the inclusion $U \cap V \hookrightarrow U$, $\gamma$ maps to the boundary loop of the disk around $p$. As per the illustration in the page 5 of Hatcher, this boundary loop is homotopic to the loop on the square, $aba^{-1}b^{-1}$ where $a$ and $b$ are the generators of $\pi_1(S^1 \vee S^1)$. Therefore:

$$N = \langle aba^{-1}b^{-1} \rangle.$$

Quotienting $\mathbb{Z} * \mathbb{Z}$ by the commutator subgroup $N$ imposes the relation $aba^{-1}b^{-1} = 1$, i.e., $ab = ba$ since any commutator belongs to the identity element in the quotient group. This forces all generators to commute, so the group is abelian. Since we have $\pi_1(U) * \pi_1(V) \cong \pi_1(S^1 \vee S^1) * 1 \cong \mathbb{Z} * \mathbb{Z} * 1 \cong \mathbb{Z} * \mathbb{Z}$, we have:

$$\pi_1(T^2) \cong \frac{\mathbb{Z} * \mathbb{Z}}{\langle aba^{-1}b^{-1} \rangle} \cong \mathbb{Z} \times \mathbb{Z}.$$

Thus, the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.

□

## 2.3 Problem 3

Find the fundamental group of a Klein bottle.

*Proof.* We compute the fundamental group of the Klein bottle $K$ using the van Kampen theorem.

We review the definition of the Klein bottle in the quotient space representation first. The Klein bottle $K$ is constructed by identifying the sides of a square $[0, 1] \times [0, 1]$ in the following way:

- Vertical sides: $(0, y) \sim (1, y)$ for all $y \in [0, 1]$.
- Horizontal sides: $(x, 0) \sim (1 - x, 1)$ for all $x \in [0, 1]$.

The inconsistency of orientation on the horizontal sides makes it a non-orientable surface.

We will cover the Klein bottle $K$ by two open sets $U$ and $V$. Let $U = K \setminus \{p\}$, where $p$ is a point removed from the interior of the square. This punctured Klein bottle retracts to its 1-skeleton (the edges of the square ). Let $V$ be a small open disk centered at $p$, homeomorphic to $\mathbb{R}^2$. We notice that the intersection $U \cap V$ is a punctured disk around $p$, which retracts to a circle.

For $\pi_1(U)$, we may retract $U$ to the edges of the square, producing a figure-eight space with the right identification. We know that the fundamental group of figure eight is the free group on two generators:

$$\pi_1(U) \cong \langle \alpha, \beta \rangle,$$

where $\alpha$ corresponds to the vertical edges and $\beta$ to the horizontal edges. For $\pi_1(V)$, an open disk is contractible to a point; its fundamental group is trivial and $\pi_1(V) = 0$. Finally, for $\pi_1(U \cap V)$, the punctured disk clearly retracts to a circle as in the problem 2. Its fundamental group is: $\pi_1(U \cap V) \cong \mathbb{Z}$.

Next, we will apply Kampen's theorem. By the van Kampen theorem, the kernel of $\Phi : \pi_1(U) * \pi_1(V) \to \pi_1(K)$ is the normal subgroup $N$ generated by elements of the form:

$$i_{UV}(\omega) \cdot i_{VU}(\omega)^{-1}, \quad \text{for } \omega \in \pi_1(U \cap V).$$

Here, $i_{UV} : \pi_1(U \cap V) \to \pi_1(U)$ is induced by the inclusion $U \cap V \hookrightarrow U$, and $i_{VU} : \pi_1(U \cap V) \to \pi_1(V)$ is induced by $U \cap V \hookrightarrow V$.

Since $\pi_1(V)$ is trivial, $i_{VU}(\omega) = 1$ for all $\omega$. The generator simplifies to:

$$i_{UV}(\omega) \cdot 1^{-1} = i_{UV}(\omega) \in N.$$

Thus, $N$ is the normal subgroup of $\mathbb{Z} * \mathbb{Z}$ generated by the image of $\pi_1(U \cap V)$ under $i_{UV}$. Under the inclusion $U \cap V \hookrightarrow U$, $\gamma$ maps to the boundary loop of the disk around $p$. This boundary loop is homotopic to the loop on the square, $\alpha\beta\alpha\beta^{-1}$ where $\alpha$ and $\beta$ are the generators of $\pi_1(S^1 \vee S^1)$. Therefore:

$$N = \langle \alpha\beta\alpha\beta^{-1} \rangle.$$

$$\pi_1(K) \cong \frac{\pi_1(U) * \pi_1(V)}{N} = \frac{\langle \alpha, \beta \rangle * 0}{\langle \alpha\beta\alpha\beta^{-1} \rangle} \cong \frac{\langle \alpha, \beta \rangle}{\langle \alpha\beta\alpha\beta^{-1} \rangle}.$$

Since we have $\alpha\beta\alpha\beta^{-1} = e$ from the normal subgroup $N$, we obtain:

$$\beta\alpha\beta^{-1} = \alpha^{-1}.$$

The fundamental group of the Klein bottle is:

$$\left\langle \alpha, \beta \,\middle|\, \beta\alpha\beta^{-1} = \alpha^{-1} \right\rangle.$$

We also notice that the fundamental group is non-abelian, aligning with the non-orientability of the Klein bottle.

□

## 2.4 Problem 4

Prove that the fundamental group of the projective plane $\mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$.

*Proof.* We compute $\pi_1(\mathbb{RP}^2)$ using the van Kampen theorem. We will use the fact that the real projective plane can be visaulized as a union of the Mobius strip glued to the boundary of the disk.

We cover $\mathbb{RP}^2$ by two open sets $U$ and $V$ where $U$ is an open (thickened) Möbius strip and $V$ is an open (thickened) disk. Then we observe that $U \cap V$ is an open annulus (thickened boundary circle). All sets are path-connected and contain the basepoint. The intersection $U \cap V$ is also path-connected, and the triple intersections are trivial.

Next, we compute the fundamental groups of components. Since it is clear that an open thickened Mobius strip deformation retracts to the Mobius strip, we only need to compute the fundamental group of the Mobius strip since homotopy equivalent spaces have the same fundamental groups. We prove $\pi_1(U) \cong \mathbb{Z}$.

We represent $U$ as $[0,1] \times [0,1]$ with the identification $(0,t) \sim (1, 1-t)$. Define the retraction:

$$F\big((x,y),t\big) = \left(x,\ \frac{t}{2} + (1-t)y\right).$$

This linearly collapses the strip onto the central line $y = \frac{1}{2}$. Let $(0, y)$ and $(1, 1-y)$ be equivalent points under the half-twist. Applying $F$:

$$F\big((0, y), t\big) = \left(0,\ \frac{t}{2} + (1-t)y\right),$$
$$F\big((1, 1-y), t\big) = \left(1,\ \frac{t}{2} + (1-t)(1-y)\right).$$

Under the Möbius identification, these must satisfy:

$$\left(0,\ \frac{t}{2} + (1-t)y\right) \sim \left(1,\ 1 - \left(\frac{t}{2} + (1-t)y\right)\right).$$

Compute the right-hand side:

$$1 - \frac{t}{2} - (1-t)y = \frac{t}{2} + (1-t)(1-y),$$

which matches $F\big((1, 1-y), t\big)$. Thus, $F$ preserves the identification and is well-defined on $U$. At $t = 1$, $F$ collapses $U$ onto the central circle $S^1$, giving $\pi_1(U) \cong \mathbb{Z}$.

An open disk $V$: $\pi_1(V) \cong 1$ as a disk is contractible.

For the intersection $(U \cap V)$, the annulus retracts to a circle, so $\pi_1(U \cap V) \cong \mathbb{Z}$, by sending all points gradually to the points $||x|| = 1$.

Next, we will apply Kampen's theorem. By the van Kampen theorem, the kernel of $\Phi : \pi_1(U) * \pi_1(V) \to \pi_1(K)$ is the normal subgroup $N$ generated by elements of the form:

$$i_{UV}(\omega) \cdot i_{VU}(\omega)^{-1}, \quad \text{for } \omega \in \pi_1(U \cap V).$$

Here, $i_{UV} : \pi_1(U \cap V) \to \pi_1(U)$ is induced by the inclusion $U \cap V \hookrightarrow U$, and $i_{VU} : \pi_1(U \cap V) \to \pi_1(V)$ is induced by $U \cap V \hookrightarrow V$.

Now we compute the generators of the kernel of $\Phi$. For the inclusion $U \cap V \hookrightarrow U$, the loop $\gamma$ in $U \cap V$ maps to the boundary of the Möbius strip. As per the wikipedia page, due to the half-twist, this boundary wraps **twice** around the central circle. Thus:

$$\gamma \mapsto \alpha^2 \quad \text{in } \pi_1(U).$$

For the inverse of the inclusion $U \cap V \hookrightarrow V$, since $\pi_1(V)$ is trivial, we get:

$$1 \mapsto 1 \quad \text{in } \pi_1(U \cap V).$$

Thus, the generator of the kernel is given by $\alpha^2$.

By van Kampen's theorem, we get:

$$\pi_1(\mathbb{RP}^2) \cong \frac{\pi_1(U) * \pi_1(V)}{\langle\langle i_{UV}(\gamma) i_{VU}(\gamma)^{-1} \rangle\rangle} = \frac{\mathbb{Z} * 1}{\langle\langle \alpha^2 \rangle\rangle} \cong \frac{\mathbb{Z}}{\langle 2\alpha \rangle} \cong \mathbb{Z}/2\mathbb{Z}.$$

Therefore, The fundamental group of $\mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$.

$\square$

## 2.5 Problem 5

Confirm the result in Problem 4 by exhibiting the real projective plane $\mathbb{RP}^2$ as the quotient of $S^2$ by an action of $\mathbb{Z}/2\mathbb{Z}$. Specifically, note that the universal cover of $\mathbb{RP}^2$ is $S^2$, and the fundamental group $\mathbb{Z}/2\mathbb{Z}$ acts on $S^2$ via multiplication by $-1$. The quotient $S^2/(\mathbb{Z}/2\mathbb{Z})$ is homeomorphic to $\mathbb{RP}^2$, establishing $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$.

*Proof.* We prove that the fundamental group of the real projective plane $\mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$. We will also apply the proposition 1.40 of Hatcher.

We will define $\mathbb{RP}^2$ as a quotient space. The real projective plane is the set of lines through the origin in $\mathbb{R}^3$. We may construct it by identifying antipodal points on the 2-sphere $S^2$:

$$\mathbb{RP}^2 = S^2/\sim, \quad \text{where } x \sim -x \text{ for all } x \in S^2.$$

We will set up a group action as follows. Let $Y = S^2$ and $G = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. We define the action of $G$ on $Y$ by:

$$0 \cdot x = x, \text{ and } 1 \cdot x = -x \text{ (antipodal map)}$$

We notice that $S^2/\sim$ is equivalent to $S^2/G$, where $G = \mathbb{Z}/2\mathbb{Z}$ acts on $S^2$ by the antipodal map $x \mapsto -x$.

Consider the following condition
$(*)$ Each $\boldsymbol{y} \in Y$ has a neighborhood $U$ such that all the images $g(U)$ for varying $g \in G$ are disjoint. In other words, $g_1(U) \cap g_2(U) \neq \emptyset$ implies $g_1 = g_2$.

We will verify that the group action satisfies condition $(*)$. The action of $G = \mathbb{Z}/2\mathbb{Z}$ on $S^2$ satisfies that for any $x \in S^2$, choose a sufficiently small open disk $U \subset S^2$ around $x$ such that $U \cap (-U) = \emptyset$. Then we know $g(U) \cap U = \emptyset$ unless $g$ is the identity. Thus, the action meets condition $(*)$ in Proposition 1.40 of Hatcher. Moreover, $S^2$ is clearly path connected.

Now we will apply Proposition 1.40 to the covering space $S^2 \to \mathbb{RP}^2$. By the proposition, we know: (a) The quotient map $p : S^2 \to \mathbb{RP}^2$ is a normal covering space. (b) The group $G = \mathbb{Z}/2\mathbb{Z}$ is the deck transformation group of the covering space. (c) The fundamental group of $\mathbb{RP}^2$ is:

$$\mathbb{Z}/2\mathbb{Z} \cong G \cong \frac{\pi_1(\mathbb{RP}^2)}{p_*(\pi_1(S^2))}.$$

Since $S^2$ is simply connected, $\pi_1(S^2) = 0$. The induced homomorphism $p_* : \pi_1(S^2) \to \pi_1(\mathbb{RP}^2)$ maps the trivial group to the trivial subgroup:

$$p_*(0) = 0 \subset \pi_1(\mathbb{RP}^2).$$

Thus, we get:

$$\mathbb{Z}/2\mathbb{Z} \cong \frac{\pi_1(\mathbb{RP}^2)}{0} \cong \pi_1(\mathbb{RP}^2)$$

Therefore, the fundamental group of $\mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$.

$\square$

# 3 Selected Textbook Problems

## 3.1 Chapter 0 Problem 4

*Problem Statement.* A **deformation retraction in the weak sense** of a space $X$ to a subspace $A$ is a homotopy $f_t : X \to X$ such that $f_0 = \nVdash$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all $t$. Show that if $X$ deformation retracts to $A$ in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

*Proof.* We will show that the inclusion $i : A \hookrightarrow X$ is a homotopy equivalence when $X$ deformation retracts to $A$ in the weak sense.

Given the weak deformation retraction $f_t : X \to X$, we define $r = f_1 : X \to A$. The inclusion map is $i : A \hookrightarrow X$. We show that $i \circ r \simeq \mathrm{id}_X$ and $r \circ i \simeq \mathrm{id}_A$. For $i \circ r \simeq \mathrm{id}_X$, The homotopy $f_t$ satisfies $f_0 = \mathrm{id}_X$ and $f_1 = i \circ r$. Thus, $f_t$ provides a homotopy between $\mathrm{id}_X$ and $i \circ r$. For $r \circ i \simeq \mathrm{id}_A$, restrict $f_t$ to $A$. Since $f_t(A) \subset A$ for all $t$, the restricted homotopy $f_t|_A : A \to A$ connects $f_0|_A = \mathrm{id}_A$ to $f_1|_A = r \circ i$. Hence, $r \circ i \simeq \mathrm{id}_A$.

We observe that the maps $r$ and $i$ are homotopy inverses. Therefore, $i : A \hookrightarrow X$ is a homotopy equivalence.

□

## 3.2 Chapter 0 Problem 5

*Problem Statement.* Show that if a space $X$ deformation retracts to a point $x \in X$, then for each neighborhood $U$ of $x$ in $X$, there exists a neighborhood $V \subset U$ of $x$ such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

*Proof.* Let $F : X \times [0, 1] \to X$ be the deformation retraction of $X$ onto the point $x \in X$. By definition, $F$ satisfies the conditions: $F(x', 0) = x' \quad \text{for all } x' \in X$, $F(x', 1) = x \quad \text{for all } x' \in X$, $F(x, t) = x \quad \text{for all } t \in [0, 1]$.

Given a neighborhood $U$ of $x$ in $X$, the preimage $F^{-1}(U)$ is open in $X \times [0, 1]$ by the continuity of $F$. Since $\{x\} \times [0, 1] \subset F^{-1}(U)$ and $[0, 1]$ is compact, by the tube lemma, there exists an open neighborhood $V \subset X$ of $x$ such that

$$V \times [0, 1] \subset F^{-1}(U).$$

The restriction of $F$ to $V \times [0, 1]$ takes values in $U$ and continuous, giving a homotopy $H : V \times [0, 1] \to U$ defined by

$$H(v, t) = F(v, t).$$

By construction, $H$ satisfies $H(v, 0) = v, H(v, 1) = x$. Thus, $H$ is a homotopy between the inclusion map $V \hookrightarrow U$ and the constant map at $x$, proving that the inclusion is nullhomotopic.

Hence, there exists such a neighborhood $V \subset U$ such that the inclusion $V \hookrightarrow U$ is nullhomotopic. □

## 3.3 Chapter 0 Problem 11

*Problem Statement.* Show that $f\colon X \to Y$ is a homotopy equivalence if there exist maps $g, h\colon Y \to X$ such that $fg \simeq 1_Y$ and $hf \simeq 1_X$. More generally, show that $f$ is a homotopy equivalence if $fg$ and $hf$ are homotopy equivalences.

*Proof.* Before proving this problem, we will prove one useful lemma.

**Lemma 3.** *If $f_1, g_1\colon X \to Y$ are homotopic, and $f_2, g_2\colon Y \to Z$ are homotopic, then the compositions $f_2 \circ f_1$ and $g_2 \circ g_1$ are also homotopic.*

*Proof.* Let $f_1 \simeq g_1$ via a homotopy $H\colon X \times [0,1] \to Y$, and $f_2 \simeq g_2$ via a homotopy $K\colon Y \times [0,1] \to Z$. Define a map $L\colon X \times [0,1] \to Z$ by

$$L(x,t) = K(H(x,t),t).$$

We verify that $L$ is a homotopy between $f_2 \circ f_1$ and $g_2 \circ g_1$. Firstly, since $H$ and $K$ are continuous, the composition $L(x,t) = K(H(x,t),t)$ is continuous. Also, at $t = 0$, $L(x,0) = K(H(x,0),0) = K(f_1(x),0) = f_2(f_1(x))$. At $t = 1$, $L(x,1) = K(H(x,1),1) = K(g_1(x),1) = g_2(g_1(x))$. Thus, $L$ is a homotopy $f_2 \circ f_1 \simeq g_2 \circ g_1$ as wanted.

□

Suppose $fg \simeq 1_Y$ and $hf \simeq 1_X$. We prove $g \simeq h$, making $g, h$ a two-sided homotopy inverse of $f$.

Using $hf \simeq 1_X$, we get $hfg \simeq g$. Since $fg \simeq 1_Y$, $h \simeq g$. Similarly, $fg \simeq 1_Y$ implies $hfg \simeq h$, and $hf \simeq 1_X$ gives $g \simeq h$. Hence, $g \simeq h$ as wanted.

Since $g \simeq h$, compose $gf \simeq hf \simeq 1_X$. Thus, $g$ satisfies $fg \simeq 1_Y$ and $gf \simeq 1_X$, proving $f$ is a homotopy equivalence.

Now we prove the general case. Suppose $fg$ and $hf$ are homotopy equivalences. Let $k$ be a homotopy inverse for $fg$, so $(fg)k \simeq 1_Y$ and $k(fg) \simeq 1_Y$. Similarly, let $m$ be a homotopy inverse for $hf$, so $(hf)m \simeq 1_X$ and $m(hf) \simeq 1_X$. Define $e_r = gk$ and $e_l = mh$. We observe that $fe_r = f(gk) \simeq (fg)k \simeq 1_Y$ and $e_l f = (mh)f \simeq m(hf) \simeq 1_X$.

By the first part, $e_r \simeq e_l$, so $e_r$ is a two-sided inverse. Hence, $f$ is a homotopy equivalence. □

## 3.4 Chapter 0 Problem 13

*Problem Statement.* Show that any two deformation retractions $r_t^0$ and $r_t^1$ of a space $X$ onto a subspace $A$ can be joined by a continuous family of deformation retractions $r_t^s$, $0 \le s \le 1$, of $X$ onto $A$, where continuity means that the map $X \times I \times I \to X$ sending $(x,s,t)$ to $r_t^s(x)$ is continuous.

*Proof.* The problem seems immediate if $X$ is assumed to be convex since that allows us to consider a linear combination of deformation retractions (similar to linear homotopy). At this moment, I have no idea what if $X$ is an arbitrary topological space.

□

### 3.5 Chapter 0 Problem 21

*Problem Statement.* If $X$ is a connected Hausdorff space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that $X$ is homotopy equivalent to a wedge sum of $S^1$'s and $S^2$'s.

*Proof.* Let $X$ be a connected Hausdorff space that is a finite union of 2-spheres, where any two spheres intersect in at most one point. We show $X$ is homotopy equivalent to a wedge sum of $S^1$'s and $S^2$'s by induction.

**Base case:** If $X$ consists of a single 2-sphere, it is trivially homotopy equivalent to $S^2$.

**Inductive Step:** For an inductive hypothesis, assume that any union of $n-1$ 2-spheres (with pairwise intersections at most one point) is homotopy equivalent to a wedge sum of $S^1$'s and $S^2$'s.

For an inductive step, Let $X$ consist of $n$ spheres. We start by choosing a sphere $S \subset X$. Suppose $S$ intersects $k$ other spheres at distinct points $p_1, \ldots, p_k$. Then replace each intersection point $p_i$ with a contractible line segment $L_i$, attached to $S$. Furthermore, we contract all $k$ points on this chosen sphere to a single point, forming a contractible "tree" $T = \bigcup_{i=1}^{k} L_i$. This modification produces a new space $X'$, homotopy equivalent to $X$. This procedure worked because segments (1-cell) are contractible to the point (0-cell). The tree $T$ is a contractible subcomplex. By Hatcher's proposition on page 11, the quotient map $X' \to X'/T$ is a homotopy equivalence.

We analyze the Quotient Space $X'/T$. Let $Y$ be a union of the remaining $(n-1)$ spheres and the tree $T$ without a chosen sphere $S$. Since $Y$ is a CW complex itself and $T$ is a contactible subcomplex of $Y$, the Hatcher's proposition on page 11 and inductive hypothesis implies that $Y \simeq Y/T \simeq$ "A wedge sum of $n-1$ spheres $S^1$ and $S^2$" where the first equivalence followed from the Hatcher's proposition and the second equivalence followed from the inductive hypothesis. However, this implies that the whole space $X$ is the union of a chosen sphere $S$ and $Y/T$, or a wedge sum of $n-1$ spheres $S^1$ and $S^2$, where $S$ and $Y/T$ intersects at a point due to the contraction of $T$. By contraction of $T$, we may therefore conclude that the entire space $X$ is homotopy equivalent to the wedge sum of $n$ spheres of $S^1$ and $S^2$ as desired.

□

### 3.6 Chapter 0 Problem 23

*Problem Statement.* Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

*Proof.* Let $A$, $B$ be contractible subcomplexes whose intersection $A \cap B$ is also contractible. We will show that the union $X = A \cup B$ is contractible.

By the Hatcher's proposition on page 11, since $A \subset X$ is contractible, the quotient map $X \to X/A$ is a homotopy equivalence. Thus, $X \simeq X/A$.

The quotient $X/A$ is equivalent to $B/(A \cap B)$, as $X = A \cup B$ and $A \cap B$ is collapsed in $B$. Since $A \cap B$ is contractible, the quotient map $B \to B/(A \cap B)$ is also a homotopy equivalence. But $B$ is contractible, so:

$$B/(A \cap B) \simeq B \simeq \text{pt}.$$

Combining the results, we have:

$$X \simeq X/A \simeq \text{pt}.$$

Hence, $X$ is contractible.

□

### 3.7 Chapter 0 Problem 25

*Problem Statement.* If $X$ is a CW complex with components $X_\alpha$, show that the suspension $SX$ is homotopy equivalent to $Y \vee_\alpha SX_\alpha$ for some graph $Y$. In the case that $X$ is a finite graph, show that $SX$ is homotopy equivalent to a wedge sum of circles and 2-spheres.

*Proof.* For a space $X$, the suspension $SX$ is the quotient space:

$$SX = (X \times [0,1])/\sim$$

where $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$. This collapses $X \times \{0\}$ and $X \times \{1\}$ to two distinct points, called the north and south poles of $SX$. If $X$ is a disjoint union $\bigsqcup_\alpha X_\alpha$, then:

$$SX = \left( \bigsqcup_\alpha X_\alpha \times [0,1] \right) / \sim$$

where all $X_\alpha \times \{0\}$ are identified to one point (south pole), and all $X_\alpha \times \{1\}$ are identified to another point (north pole).

We will construct a graph $Y$ as follows. The graph $Y$ contains a base vertex $v_0$. For each cell complex $X_\alpha$, add a vertex $v_\alpha$ and an edge $e_\alpha$ that connects $v_0$ and $v_\alpha$. Moreover, let the collection of vertices $\{v_\alpha\}_{\alpha \in \Lambda}$ (excluding $v_0$). Then we will add edges so that $\{v_\alpha\}_{\alpha \in \Lambda}$ forms a tree (hence vertices are connected). Let the collection of edges which form this tree be denoted by $T$.

The graph $Y$ is a 1-dimensional CW complex clearly. Attach each $SX_\alpha$ to $Y$ by identifying the south and north poles of $SX_\alpha$ with $v_0$ and $v_\alpha$ respectively. The resulting space is a wedge sum $Y \vee_\alpha SX_\alpha$. Since a tree is contractible, collapsing the tree of the vertices $\{v_\alpha\}_{\alpha\in\Lambda}$ would result in identifying all the north poles of $X_\alpha$, but this is equivalent to the suspension of X, $SX$. Thus, $Y \vee_\alpha SX_\alpha$ is homotopy equivalent to $SX$.

Let $X$ be a finite graph. Via homotopy equivalence, its homotopy type reduces to a wedge of circles $S^1$, one for each cycle, after collapsing contractible trees. We know that $S(\text{a point}) \cong S^1$ and $S(S^1) \cong S^2$. Moreover, since suspension preserves wedge sums due to the symmetry of collapsing, we have $S(\bigvee S^1) \simeq \bigvee S(S^1) \simeq \bigvee S^2$ where the intersecting point lies at the height $\bigvee S^1 \times \{\frac{1}{2}\} \in X \times I$.

If $X$ has $k$ components, we may let the graph $Y$ add $k-1$ circles from edges connecting $v_0$ and $v_1$. For a connected graph ($k=1$), $Y$ is an interval, which is contractible, adding no additional circles. Since we already know $SX$ is homotopy equivalent to the wedge sum of $Y$ and all other cell subcomplexes, combining these results, we get:

$$SX \simeq \left(\bigvee^{k-1} S^1\right)_{\text{from } Y} \vee \left(\bigvee^{n} S^2\right)_{\text{from cycles in } X},$$

where $n$ is the number of cycles in $X$.

□

## 3.8 Chapter 1.1 Problem 3

*Problem Statement.* For a path-connected space $X$, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms $\beta_h$ depend only on the endpoints of the path $h$.

*Proof.* For a path-connected space $X$, we show that $\pi_1(X)$ is abelian if and only if all basepoint-change homomorphisms $\beta_h$ depend only on the endpoints of the path $h$. We recall the definition. Let $h : I \to X$ be a path from $x_0$ to $x_1$. The **basepoint-change homomorphism** $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is defined by:

$$\beta_h([f]) = [h \cdot f \cdot \overline{h}],$$

where $\overline{h}(s) = h(1-s)$ is the inverse path.

**Forward Direction ($\pi_1(X)$ abelian $\implies$ $\beta_h$ depends only on endpoints)**

Assume $\pi_1(X, x_0)$ is abelian. Let $h$ and $h'$ be two paths from $x_0$ to $x_1$. We show $\beta_h = \beta_{h'}$. Let $\gamma = h \cdot \overline{h'}$, which is a loop at $x_0$. Then For any $[f] \in \pi_1(X, x_1)$, $\beta_h([f]) = [h \cdot f \cdot \overline{h}]$, $\beta_{h'}([f]) = [h' \cdot f \cdot \overline{h'}]$. Now combining $\beta_h$ and $\beta_{h'}$, we get $\beta_{h'}([f]) = [\overline{\gamma} \cdot h \cdot f \cdot \overline{h} \cdot \gamma]$.

Since $\pi_1(X, x_0)$ is abelian, conjugation by $\gamma$ is trivial:

$$[\overline{\gamma} \cdot h \cdot f \cdot \overline{h} \cdot \gamma] = [h \cdot f \cdot \overline{h}].$$

Thus, $\beta_h([f]) = \beta_{h'}([f])$. Hence, $\beta_h$ depends only on $x_0$ and $x_1$.

**Reverse Direction ($\beta_h$ depends only on endpoints $\implies \pi_1(X)$ abelian)**

Assume $\beta_h$ depends only on the endpoints of $h$. Let $[f], [g] \in \pi_1(X, x_0)$. Take $h = f$, a loop at $x_0$. Then:

$$\beta_h([g]) = [f \cdot g \cdot \overline{f}] = [f \cdot g \cdot f^{-1}].$$

Since $\beta_h$ depends only on the endpoints (which are both $x_0$), $\beta_h = \beta_{\mathrm{id}_{x_0}}$, the identity map:

$$[f \cdot g \cdot f^{-1}] = [g].$$

This implies $[f][g][f^{-1}] = [g]$, so $[f][g] = [g][f]$. Hence, $\pi_1(X, x_0)$ is abelian.

Therefore, $\pi_1(X)$ is abelian if and only if all $\beta_h$ depend only on the endpoints of $h$.

$\square$

## 3.9 Chapter 1.1 Problem 9

*Problem Statement.* Let $A_1, A_2, A_3$ be compact sets in $\mathbb{R}^3$. Use the Borsuk–Ulam theorem to show that there is one plane $P \subset \mathbb{R}^3$ that simultaneously divides each $A_i$ into two pieces of equal measure.

*Proof.* Planning to revisit here.

$\square$

## 3.10 Chapter 1.1 Problem 11

*Problem Statement.* If $X_0$ is the path-component of a space $X$ containing the basepoint $x_0$, show that the inclusion $X_0 \hookrightarrow X$ induces an isomorphism $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

*Proof.* Let $X_0$ be the path-component of a space $X$ containing the basepoint $x_0$. We show that the inclusion $i : X_0 \hookrightarrow X$ induces an isomorphism $i_* : \pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

**Surjectivity of $i_*$**

*Proof.* Let $[\gamma] \in \pi_1(X, x_0)$. The loop $\gamma : [0, 1] \to X$ is based at $x_0$. Since $X_0$ is the path-component containing $x_0$, the image of $\gamma$ must lie entirely in $X_0$. Thus, $\gamma$ is a loop in $X_0$, and $i_*([\gamma]) = [\gamma]$. Therefore, $i_*$ is surjective. $\square$

**Injectivity of $i_*$**

*Proof.* Suppose $i_*([f]) = i_*([g])$ for loops $f, g : [0, 1] \to X_0$ based at $x_0$. This means $f \simeq g$ in $X$. Let $H : [0, 1] \times [0, 1] \to X$ be a homotopy between $f$ and $g$. Since $H(0, t) = H(1, t) = x_0$ for all $t$, and $X_0$ is path-connected, the homotopy $H$ must map into $X_0$. Hence, $f \simeq g$ in $X_0$, so $[f] = [g]$ in $\pi_1(X_0, x_0)$. Therefore, $i_*$ is injective. □

Since the map $i_*$ is both surjective and injective, it is an isomorphism.

□

## 3.11 Chapter 1.1 Problem 13

*Problem Statement.* Given a space $X$ and a path-connected subspace $A$ containing the basepoint $x_0$, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in $X$ with endpoints in $A$ is homotopic to a path in $A$.

*Proof.* Given a space $X$ and a path-connected subspace $A$ containing the basepoint $x_0$, we show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by inclusion $A \hookrightarrow X$ is surjective if and only if every path in $X$ with endpoints in $A$ is homotopic to a path in $A$. We note that the **inclusion map** $i : A \hookrightarrow X$ induces a homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$.

**Forward Direction**

*Proof.* Assume $i_*$ is surjective. Let $\gamma : [0, 1] \to X$ be a path with $\gamma(0) = a$, $\gamma(1) = b$, where $a, b \in A$. Since $A$ is path-connected, choose a path $\alpha : [0, 1] \to A$ from $b$ to $a$. The concatenation $\gamma \cdot \alpha$ forms a loop in $X$ based at $a$. By surjectivity of $i_*$, there exists a loop $\beta : [0, 1] \to A$ based at $a$ such that:

$$[\gamma \cdot \alpha] = [\beta] \quad \text{in } \pi_1(X, a).$$

This implies $\gamma \cdot \alpha \simeq \beta$ rel $\{0, 1\}$. Concatenating both sides with $\alpha^{-1}$, we get:

$$\gamma \simeq \beta \cdot \alpha^{-1} \quad \text{rel } \{0, 1\}.$$

Since $\beta \cdot \alpha^{-1}$ is a path in $A$, $\gamma$ is homotopic to a path in $A$, as desired. □

**Reverse Direction**

*Proof.* Assume every path in $X$ with endpoints in $A$ is homotopic to a path in $A$. Let $[\gamma] \in \pi_1(X, x_0)$. Since $\gamma$ is a loop in $X$ based at $x_0$, by assumption, there exists a loop $\tilde{\gamma} : [0, 1] \to A$ based at $x_0$ such that $\gamma \simeq \tilde{\gamma}$ rel $\{0, 1\}$. Thus:

$$[\gamma] = [\tilde{\gamma}] = i_*([\tilde{\gamma}]),$$

so $[\gamma]$ lies in the image of $i_*$. Hence, $i_*$ is surjective. □

□

## 3.12 Chapter 1.1 Problem 15

*Problem Statement.* Given a map $f : X \to Y$ and a path $h : I \to X$ from $x_0$ to $x_1$, show that

$$f_*\beta_h = \beta_{fh} f_*$$

in the diagram at the bottom.

$$\begin{tikzcd}
\pi_1(X, x_1) \arrow[r, "\beta_h"] \arrow[d, "f_*"'] & \pi_1(X, x_0) \arrow[d, "f_*"] \\
\pi_1(Y, f(x_1)) \arrow[r, "\beta_{fh}"'] & \pi_1(Y, f(x_0))
\end{tikzcd}$$

*Proof.* For a loop $[\gamma] \in \pi_1(X, x_1)$, the basepoint-change map $\beta_h$ is defined as:

$$\beta_h[\gamma] = [h \cdot \gamma \cdot \overline{h}] \in \pi_1(X, x_0),$$

where $\overline{h}(s) = h(1 - s)$.

The induced map $f_*$ sends $[\gamma]$ to $[f \circ \gamma] \in \pi_1(Y, f(x_1))$.

The basepoint-change map $\beta_{fh}$ in $Y$ uses the path $fh$ (image of $h$ under $f$) and is defined as:

$$\beta_{fh}[\delta] = [fh \cdot \delta \cdot \overline{fh}] \in \pi_1(Y, f(x_0)),$$

where $\overline{fh}(s) = f(h(1 - s))$.

We will compute $f_*\beta_h[\gamma]$:

$$f_*\beta_h[\gamma] = f_*[h \cdot \gamma \cdot \overline{h}] = [f \circ (h \cdot \gamma \cdot \overline{h})].$$

Since $f$ preserves path concatenation:

$$f \circ (h \cdot \gamma \cdot \overline{h}) = (f \circ h) \cdot (f \circ \gamma) \cdot (f \circ \overline{h}) = fh \cdot (f \circ \gamma) \cdot \overline{fh}.$$

We will compute $\beta_{fh} f_*[\gamma]$:

$$\beta_{fh} f_*[\gamma] = \beta_{fh}[f \circ \gamma] = [fh \cdot (f \circ \gamma) \cdot \overline{fh}].$$

Both expressions $f_*\beta_h[\gamma]$ and $\beta_{fh} f_*[\gamma]$ yield the same homotopy class $[fh \cdot (f \circ \gamma) \cdot \overline{fh}]$. Hence, the diagram commutes:

$$f_*\beta_h = \beta_{fh} f_*.$$

□

## 3.13 Chapter 1.1 Problem 19

*Problem Statement.*

*Proof.* □

### 3.14 Chapter 1.1 Problem 20

*Problem Statement.* Suppose $f_t : X \to X$ is a homotopy such that $f_0$ and $f_1$ are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \to X$.]

*Proof.* Given a homotopy $f_t : X \to X$ with $f_0 = \mathrm{id}_X$ and $f_1 = \mathrm{id}_X$, we show that for any $x_0 \in X$, the loop $h(t) = f_t(x_0)$ represents an element in the center of $\pi_1(X, x_0)$.

As the hint suggests, we will apply Lemma 1.19 in Hatcher. Since $f_t$ is a homotopy from $\mathrm{id}_X$ to $\mathrm{id}_X$, the path $h(t) = f_t(x_0)$ is a loop at $x_0$. By Lemma 1.19:

$$f_{0*} = \beta_h \circ f_{1*},$$

where $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the basepoint-change isomorphism. Since $f_0$ and $f_1$ are both the identity map:

$$f_{0*} = f_{1*} = \mathrm{id}_{\pi_1(X,x_0)}.$$

Substituting into Lemma 1.19, we get:

$$\mathrm{id} = \beta_h \circ \mathrm{id} \implies \beta_h = \mathrm{id}.$$

The isomorphism $\beta_h$ corresponds to conjugation by $[h]$. For any $[\gamma] \in \pi_1(X, x_0)$:

$$\beta_h([\gamma]) = [h] \cdot [\gamma] \cdot [h]^{-1}.$$

Since $\beta_h = \mathrm{id}$:

$$[h] \cdot [\gamma] \cdot [h]^{-1} = [\gamma].$$

Multiplying both sides by $[h]$:

$$[h] \cdot [\gamma] = [\gamma] \cdot [h].$$

The loop $[h]$ commutes with all elements of $\pi_1(X, x_0)$. Hence, $[h]$ lies in the center of $\pi_1(X, x_0)$.

□

### 3.15 Chapter 1.2 Problem 3

*Problem Statement.* Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

*Proof.* Hatcher proved that $\pi_1(S^n) = 0$ for $n \geq 2$. Since each $S^n$ is path-connected, $S^n$ is simply connected for $n \geq 2$. Consider removing $m$ points from $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a metric space, we can take $m$ pairwise disjoint open balls around a finite set of points. It is clear that each ball in $\mathbb{R}^n$ with a point removed can deformation retract to the sphere $S^{n-1}$. Now since $n \geq 3$, we get $\pi_1(\text{Ball}) \simeq \pi_1(S^{n-1}) \simeq 0$. Now by connecting each ball using line segments, we can further deformation retract the whole space to a wedge sum of $m$ $S^{n-1}$ spheres. Since each $S^{n-1}$ is simply connected, and a finite wedge sum of $S^{n-1}$ is simply connected, $\mathbb{R}^n \setminus \{\text{finite points}\}$ is simply connected as desired.

$\square$

### 3.16 Chapter 1.2 Problem 4

*Problem Statement.* Let $X \subset \mathbb{R}^3$ be the union of $n$ lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

*Proof.* We identify the line passing through the origin with the intersecting points on $S^2$. In particular, since each line intersects $S^2$ at two points, there are $2n$ points in $S^2 \cap X$ in total. We define a function $f : \mathbb{R}^3 \setminus X \to S^2 \setminus \{2n \text{ points}\}$ by $f(x) = \frac{x}{|x|}$. This function shows a deformation retraction from $\mathbb{R}^3 \setminus X$ to $S^2 \setminus \{2n \text{ points}\}$. We recall that there is a stereographic projection $\sigma : S^2 \setminus \{N\} \to \mathbb{R}^2$ where $N$ is a north pole or any pole of $S^2$. This map is a diffeomorphism, hence $S^2 \setminus \{2n \text{ points}\}$ is diffeomorphic to $\mathbb{R}^2 \setminus \{(2n-1) \text{ points.}\}$.

We will apply Van Kampen's theorem inductively. We compute $\pi_1(\mathbb{R}^2 \setminus \{m \text{ points}\})$ by induction on $m$.

**Base Case:** $\pi_1(\mathbb{R}^2 \setminus \{p_1\}) \cong \mathbb{Z}$, since $\mathbb{R}^2 \setminus \{p_1\}$ deformation retracts to a circle $S^1$.

**Inductive Step:** Assume the inductive hypothesis:

$$\pi_1(\mathbb{R}^2 \setminus \{p_1, \ldots, p_k\}) \cong F_k,$$

where $F_k$ is the free group on $k$ generators. To compute $\pi_1(\mathbb{R}^2 \setminus \{p_1, \ldots, p_{k+1}\})$, cover the space with two open sets:

$$U = \mathbb{R}^2 \setminus \{p_{k+1}\}, \quad \pi_1(U) \cong \mathbb{Z},$$

$$V = \mathbb{R}^2 \setminus \{p_1, \ldots, p_k\}, \quad \pi_1(V) \cong F_k.$$

Their intersection is

$$U \cap V = \mathbb{R}^2 \setminus \{p_1, \ldots, p_{k+1}\},$$

which is path-connected. By Van Kampen's Theorem,

$$\pi_1(\mathbb{R}^2 \setminus \{p_1, \ldots, p_{k+1}\}) \cong \pi_1(U) * \pi_1(V) \cong \mathbb{Z} * F_k \cong F_{k+1}.$$

Hence, by induction:

$$\pi_1(\mathbb{R}^2 \setminus \{m \text{ points}\}) \cong F_m.$$

In particular, for $m = 2n-1$, we have:

$$\pi_1(\mathbb{R}^2 \setminus \{2n-1 \text{ points}\}) \cong F_{2n-1}.$$

Since $\mathbb{R}^3 \setminus X$ is homotopy equivalent to $S^2 \setminus \{2n \text{ points}\}$, and

$$S^2 \setminus \{2n \text{ points}\} \simeq \mathbb{R}^2 \setminus \{2n-1 \text{ points}\},$$

it follows that:

$$\pi_1(\mathbb{R}^3 \setminus X) \cong F_{2n-1}.$$

□

### 3.17 Chapter 1.2 Problem 6

*Problem Statement.* Use Proposition 1.26 to show that the complement of a closed discrete subspace of $\mathbb{R}^n$ is simply connected if $n \geq 3$.

*Proof.* We will apply Proposition 1.26 to Hatcher. Suppose $S$ is a closed discrete subspace of $\mathbb{R}^n$. First, we argue that the complement $\mathbb{R}^n \setminus S$ is path connected for $n \geq 3$. Since $\mathbb{R}^n$ is second countable and $S$ is discrete, $S$ is countable. Consider any path $\gamma$ between points $x$ and $y$ in $\mathbb{R}^n$ where $x, y \notin S$. Then since the path is continuous, even though $\gamma(t) \in S$ for some $t \in [0,1]$, the path can avoid isolated points of $S$ by small perturbations while preserving continuity. Therefore, $\mathbb{R}^n \setminus S$ is path connected.

Since $S$ is discrete, for each $s \in S$, we can find a sufficiently small open neighbourhood $U_s$ such that $U_s \cap S = \{s\}$. We can then find a closed ball around $x$, $B_s$, such that $s \in B_s \subseteq U_s$ and $B_s \cap B_{s'} = \emptyset$ for any other $s' \neq s$. The interior of $B_s$ is an $n$-cell, and $\partial B_s \simeq S^{n-1}$. We note that $\partial B_s \subseteq \mathbb{R}^n \setminus S$. Let $X = \mathbb{R}^n \setminus S$. We attach each disjoint $n$-cell with the boundary $S^{n-1}$ to $X$ via an attaching map. By attaching all $B_s$ to $X$ for all $s \in S$, we can obtain $\mathbb{R}^n$.

Finally by proposition 1.26 in Hatcher, since $\mathbb{R}^n$ is obtained from $X$ by attaching $n$-cells for $n > 2$, there is an isomorphism $\pi_1(X) \cong \pi_1(\mathbb{R}^n)$. Since $\mathbb{R}^{\ltimes}$ is contractible, its fundamental group is trivial, and we get $\pi_1(X) = 0$ by an isomorphism. Thus, $X = \mathbb{R}^n \setminus S$ is simply connected.

□

### 3.18 Chapter 1.2 Problem 10

*Problem Statement.* Consider two arcs $\alpha$ and $\beta$ embedded in $D^2 \times I$ as shown in the figure. The loop $\gamma$ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of $\gamma$ in the complement of $\alpha \cup \beta$.

*Proof.* Two arcs $\alpha$ and $\beta$ are properly embedded in the cylinder in the sense that their endpoints lie on the boundary of the cylinder. We will demonstrate that the given cylinder with two arcs embedded is homotopically equivalent to the cylinder with two straight lines in it. Suppose that the arc $\alpha$ has two endpoints on the disk at the points $a$ and $b$. Without loss of generality, we may assume that the endpoints of the arc $\beta$ are the points $c$ and $d$ such that $(a \neq c)(a \neq d)$ and $(b \neq d) \wedge (b \neq c)$; for if not, we can translate the arc $\beta$ while preventing it from intersecting the arc $\alpha$, so that they have different endpoints. Now we will rotate the arc $\alpha$ and $\beta$ so that $\alpha$ is vertical and $\beta$ is horizontal. We note that a rotation of the arcs is a homeomorphism from the arc to itself. Moreover, we note that if we ensure that two arcs do not intersect in the deformation process, expanding and shrinking the arcs preserves homotopy equivalence since they are continuous motions and the composition of expansion and shrinkage is homotopic to the identity map.

We recall that $\alpha$ is placed vertically and $\beta$ is placed horizontally. Hence, we may imagine that these two arcs lie in the two-dimensional planes. In the two-dimensional plane, the arc $\alpha$ can deform to a straight line without intersecting a single point that the arc initially encloses (this point corresponds to the intersection of this plane with the arc $\beta$). This deformation works as if stretching a string. Likewise, we may apply a similar deformation to the arc $\beta$. Via these deformations, these two arcs are now the straight lines in the cylinder with the endpoints $(e,0),(e,1) \in (D^2 \times I)$ and $(f,0),(f,1) \in (D^2 \times I)$ for $e, f \in D^2$, $\alpha$ and $\beta$ respectively. This shows that $\gamma$ in $.D^2 \times I \setminus \{\alpha, \beta\}$ is homotopy equivalent to the cylinder with two straight lines removed.

Furthermore, we can deformation retract this resulting space to the disk without two points or two holes. Since a disk with two holes can deformation retract to a wedge of two circles $S^1$, whose fundamental group is a free group on two generators, the fundamental group of $D^2 \times I \setminus \{\alpha, \beta\}$ is also a free group on two generators.

The loop $\gamma$ is a loop along the boundary of the disk. Since the loop $\gamma$ encloses two holes, it is not contractible, and, in fact, the deformation retracts to a figure-eight. Therefore, there is no nullhomotopy of $\gamma$ in $.D^2 \times I \setminus \{\alpha, \beta\}$.

$\square$

## 3.19 Chapter 1.2 Problem 11

*Problem Statement.* The **mapping torus** $T_f$ of a map $f \colon X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with $f$ basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* \colon \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$. [One way to do this is to regard $T_f$ as built from $X \vee S^1$ by attaching cells.]

*Proof.* Suppose $X = S^1 \vee S^1$ and $f$ is a basepoint-preserving map, so we have $(x_0, 0) \simeq (x_0, 1)$ for each basepoint $x_0$ in the mapping torus. If we quotient the interval $I = [0,1]$ by identifying 0 and 1, we get another circle $S^1$. If we identify a point on this circle with the basepoint $x_0$, we may form $S^1 \vee S^1 \vee S^1$. Now, let $a, b$ be the generators of $X$ and let $c$ be a generator for the circle formed by the interval $I = [0,1]$. We will attach the 2-cell along the curve we describe as follows. First, the boundary curve moves along a generator $a$, and climb up to $X \times \{1\}$ via a curve $c$. As $f$ is basepoint-preserving, $f_*(a)^{-1}$, and then get back to $X \times \{0\}$ through the curve $c^-1$. Since this is a loop, we get a relation $acf_*(a)^{-1}c^{-1} = e$. Since we may do the same through a curve $b$, we get a relation $bcf_*(b)^{-1}c^{-1} = e$. Therefore, we get that $\pi_1(T_f) \simeq \langle a, b, c \mid acf_*(a)^{-1}c^{-1} = e, bcf_*(b)^{-1}c^{-1} = e\rangle$.

For $X = S^1 \times S^1$, we need to add an additional 2-cell along the relation $aba^{-1}b^{-1} = e$. Moreover, we need to attach a 3-cell into the interior of the torus. But by proposition 1.26 in Hatcher, adding a 3-cell will not change the fundamental group, thus we get $\pi_1(T_f) \simeq \langle a, b, c \mid acf_*(a)^{-1}c^{-1} = e, bcf_*(b)^{-1}c^{-1} = e, aba^{-1}b^{-1} = e\rangle$.

□

## 3.20 Chapter 1.2 Problem 14

*Problem Statement.* Consider the quotient space of a cube $I^3$ obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space $X$ is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, of order eight.

*Proof.* In this problem, we will apply the proposition 1.26 in Hatcher, the application of Van Kampen theorem to the cell complex.

We will perform an identification on the cube $I^3$ and the quotient space $I^3/\sim$ is in the Figure 1. Note that we applied a one-quarter twist in the clockwise direction for each face.

As the Figure 1 suggests, the resulting space is homotopy equivalent to the wedge sum of three circles $S^1$. So far, the resulting space has two vertices $v_1$ and $v_2$, and four edges $a, b, c, d$ with the given orientation in the Figure. We now want to attach three 2-cells along the three loops $ab$, $ac^{-1}$ and $ad$ to the interior of those loops. Finally, we attach one 3-cell along the three 2-cells, and we obtain the representation for $I^3/\sim$. By proposition 1.26 in Hatcher, adding those 2-cells and 3-cells does not change the fundamental group, thus we get $\pi_1(I^3/\sim) \simeq \pi_1(S^1 \vee S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, a free group on three generators.

As per the representation of the resulting space, let the generators of $I^3/\sim$ be denoted by $i = ab$, $j = ac^{-1}$ and $k = ad$. First, since each boundary loop on the face is homotopy equivalent to the loop 1 in $\pi_1(S^1)$, one of the faces show that $ac^{-1}d^{-1}b = 1$, which implies that $ac^{-1} = b^{-1}d$ implies $j = i^{-1}k$ implies $ij = k$. Similarly, there is another face that implies $abcd = 1$, which implies $ab = d^{-1}c^{-1}$, implies $i = k^{-1}j$, implies $ki = j$. Lastly, looking at a different face, we observe that $adb^{-1}c^{-1} = 1$, which implies $ad = cb$, implies $ad = ca^{-1}ab$, implies $k = j^{-1}i$, implies $i = jk$.

Moreover since $i = jk$, $ki = j$ and $ij = k$, we get $i^2 = j^2 = k^2 = ijk = -1$, and therefore $\pi_1(I^3/\sim) = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1\rangle$.

□

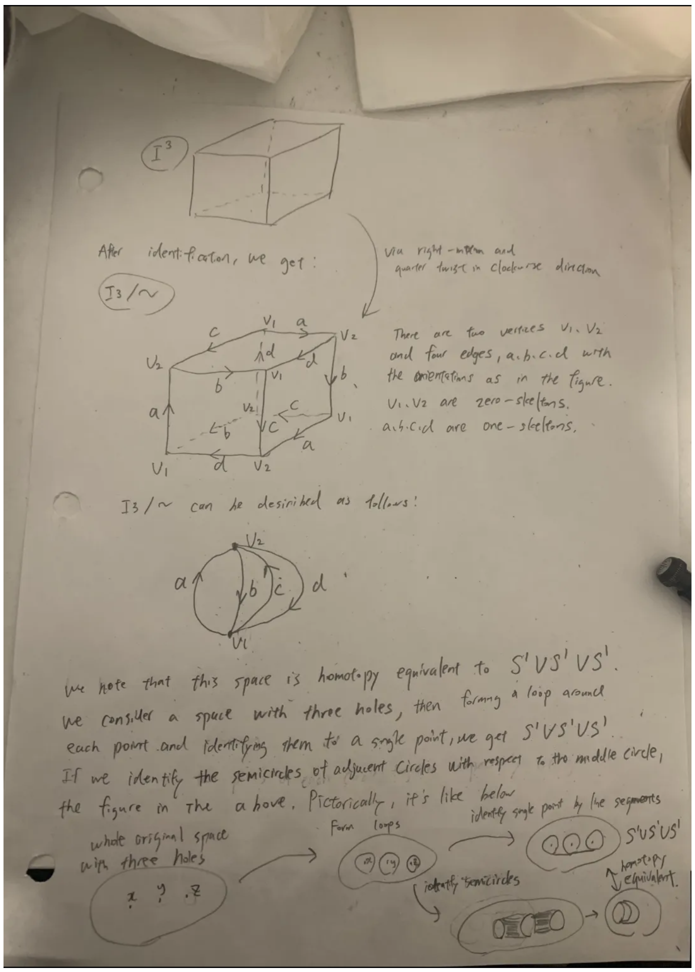



Figure 1: A diagram representation for chapter 1.2 problem 14

### 3.21 Chapter 1.2 Problem 17

*Problem Statement.* Show that $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ is uncountable.

*Proof.* First, since $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable, $\mathbb{R}^2 - \mathbb{Q}^2$ is uncountable. We will explicitly show that there are uncountably many distinct loops in the fundamental group $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2, (p,p))$ for an arbitrary $p \in \mathbb{R}^2 - \mathbb{Q}^2$.

Take any pair of distinct irrational numbers $q_1, q_2 \in \mathbb{R} - \mathbb{Q}$ with $q_1 < q_2$. For each $q_i$, we form a loop at $(p,p)$ by four line segments: $(p,p) \to (p,q_i) \to (q_i,q_i) \to (q_i,p) \to (p,p)$. By density of rational, there exists some rational $s \in \mathbb{Q}$ such that $q_1 < s < q_2$. Now, consider a subspace $\mathbb{R}^2 \setminus \{(s,s)\}$. In this subspace, the path associated with $q_2$ is a rectangular path enclosing a hole at $(s,s)$. Therefore, this path for $q_2$ is not contractible. On the other hand, the path associated with $q_1$ does not enclose a hole at $(s,s)$, hence it is contractible. This implies that these two loops are not homotopic in $\mathbb{R}^2 - \mathbb{Q}^2$; for if they are, they would be homotopic in $\mathbb{R}^2 \setminus \{(s,s)\}$ too, contradiction.

We note that we may continue this process forever in the sense that we can pick another irrational $q_3 \in \mathbb{R} \setminus \mathbb{Q}$ such that $q_2 < q_3$ and find a rational number between them. Since there are uncountably many irrational numbers, this process may repeat uncountably many times. This implies that there are uncountably many pairwise-nonhomotopic loops in $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2, (p,p))$, proving that $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2, (p,p))$ is uncountable as desired.

$\square$

### 3.22 Chapter 1.3 Problem 1

*Problem Statement.* For a covering space $p : \tilde{X} \to X$ and a subspace $A \subset X$, let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p : \tilde{A} \to A$ is a covering space.

*Proof.* To show that the restriction $p : \tilde{A} \to A$ is a covering space, we verify several properties.

We check continuity and surjectivity. The restriction $p|_{\tilde{A}}$ is continuous as the restriction of the continuous map $p$. Since $p : \tilde{X} \to X$ is surjective, every $a \in A$ has a preimage in $\tilde{A} = p^{-1}(A)$, so $p|_{\tilde{A}}$ is surjective onto $A$.

We check the property for evenly Covered Neighborhoods. For any $a \in A$, let $U \subset X$ be an open neighborhood of $a$ evenly covered by $p$. Then $p^{-1}(U) = \bigsqcup_i V_i$, where each $V_i \subset \tilde{X}$ is open and mapped homeomorphically onto $U$ by $p$. The set $U \cap A$ is an open neighborhood of $a$ in $A$ under the subspace topology. Its preimage under $p|_{\tilde{A}}$ is $p^{-1}(U) \cap \tilde{A} = \bigsqcup_i (V_i \cap \tilde{A})$. Each $V_i \cap \tilde{A}$ is open in $\tilde{A}$ (subspace topology) and disjoint. Since $p|_{V_i} : V_i \to U$ is a homeomorphism, its restriction $p|_{V_i \cap \tilde{A}} : V_i \cap \tilde{A} \to U \cap A$ is also a homeomorphism. Thus, $U \cap A$ is evenly covered by $p|_{\tilde{A}}$.

Thus, $p|_{\tilde{A}} : \tilde{A} \to A$ is a covering space.

$\square$

## 3.23 Chapter 1.3 Problem 2

*Problem Statement.* Show that if $p_1 : \tilde{X}_1 \to X_1$ and $p_2 : \tilde{X}_2 \to X_2$ are covering spaces, so is their product

$$p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2.$$

*Proof.* To show that the product $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$ is a covering space, we verify the required properties.

We first verify continuity and surjectivity. The map $p_1 \times p_2$ is continuous as the product of continuous maps $p_1$ and $p_2$. Since $p_1$ and $p_2$ are surjective, for any $(x_1, x_2) \in X_1 \times X_2$, there exist $\tilde{x}_1 \in \tilde{X}_1$ and $\tilde{x}_2 \in \tilde{X}_2$ such that $p_1(\tilde{x}_1) = x_1$ and $p_2(\tilde{x}_2) = x_2$. Thus, $(p_1 \times p_2)(\tilde{x}_1, \tilde{x}_2) = (x_1, x_2)$, proving surjectivity.

We check the property for evenly Covered Neighborhoods. Let $(a, b) \in X_1 \times X_2$. Since $p_1$ and $p_2$ are covering maps, there exist open neighborhoods $U_1 \subset X_1$ of $a$ and $U_2 \subset X_2$ of $b$, each evenly covered by $p_1$ and $p_2$, respectively. Then $U_1 \times U_2$ is an open neighborhood of $(a, b)$ in $X_1 \times X_2$. The preimage under $p_1 \times p_2$ is:

$$(p_1 \times p_2)^{-1}(U_1 \times U_2) = p_1^{-1}(U_1) \times p_2^{-1}(U_2).$$

Since $p_1^{-1}(U_1) = \bigsqcup_i V_i$ and $p_2^{-1}(U_2) = \bigsqcup_j W_j$, where each $V_i \subset \tilde{X}_1$ and $W_j \subset \tilde{X}_2$ are open and mapped homeomorphically onto $U_1$ and $U_2$, we have:

$$(p_1 \times p_2)^{-1}(U_1 \times U_2) = \bigsqcup_{i,j} (V_i \times W_j).$$

Each $V_i \times W_j$ is open in $\tilde{X}_1 \times \tilde{X}_2$ under product topology, disjoint from others, and $p_1 \times p_2$ restricts to a homeomorphism of the product $V_i \times W_j \to U_1 \times U_2$. Hence, $U_1 \times U_2$ is evenly covered.

□

## 3.24 Chapter 1.3 Problem 3

*Problem Statement.* Let $p : \tilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite and nonempty for all $x \in X$.
Show that $\tilde{X}$ is compact Hausdorff if and only if $X$ is compact Hausdorff.

*Proof.* We will show that $\tilde{X}$ is compact Hausdorff if and only if $X$ is compact Hausdorff.

**Forward Direction ($\tilde{X}$ compact Hausdorff $\Longrightarrow X$ compact Hausdorff):**

For compactness, the continuous image of a compact space is compact. Since $p$ is surjective and continuous, $X = p(\tilde{X})$ is compact.

For Hansdorffness, Let $x, y \in X$ with $x \neq y$. For any $\tilde{x} \in p^{-1}(x), \tilde{y} \in p^{-1}(y)$, there exist disjoint open neighborhoods $U_{\tilde{x}}, U_{\tilde{y}} \subset \tilde{X}$. Since $p$ is a local homeomorphism, $p(U_{\tilde{x}})$ and $p(U_{\tilde{y}})$ are disjoint open neighborhoods of $x$ and $y$ in $X$. Thus, $X$ is Hausdorff.

**Converse Direction ($X$ compact Hausdorff $\Longrightarrow \tilde{X}$ compact Hausdorff):**

For compactness, Since $X$ is compact, cover it with finitely many evenly covered open sets $\{U_i\}_{i=1}^n$. Each $p^{-1}(U_i)$ is a finite disjoint union $\bigsqcup_j V_{ij}$, where $V_{ij} \subset \tilde{X}$ are open. The collection $\{V_{ij}\}$ forms a finite open cover of $\tilde{X}$, proving $\tilde{X}$ is compact.

For Hausdorffness, For distinct $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$, if $p(\tilde{x}_1) \neq p(\tilde{x}_2)$, use Hausdorffness of $X$ to separate them via $p$. If $p(\tilde{x}_1) = p(\tilde{x}_2)$, use disjoint sheets in the covering space to separate $\tilde{x}_1, \tilde{x}_2$ (this is possible by a homeomorphism property).

□

## 3.25 Chapter 1.3 Problem 5

*Problem Statement.*

*Proof.* □

## 3.26 Chapter 1.3 Problem 6

*Problem Statement.* Let $X$ be the shrinking wedge of circles in Example 1.25, and let $\widetilde{X}$ be its covering space shown in the figure below. Construct a two-sheeted covering space $Y \to \widetilde{X}$ such that the composition $Y \to \widetilde{X} \to X$ of the two covering spaces is **not** a covering space. Note that a composition of two covering spaces does have the unique path lifting property, however.

*Proof.* We defined a two-sheeted covering space $Y \to \widetilde{X}$ in the Figure 2. We first show that $Y$ is a covering space of $\widetilde{X}$. It is clear that we can find two sheets for any "non-connecting" point (or not in the intersection of all circles in the Hawaiian Earring), since we may take arbitrarily small open neighborhoods that only intersect the circle on which this point lies. Therefore, we only consider the "connecting point." For a connecting point of any Hawaiian Earring in $\widetilde{X}$, we may take a small enough open neighborhood of it so that the circles lying inside of this neighborhood are all connected in $Y$. In this case, the sheets are homeomorphic to this neighhorhood as required.

Next, we show that the composition of two covering spaces from $Y$ to $X$ is not a covering space. We first take a connecing point of the Hawaiian Earring in $X$. Then there are infinite sheets in $\widetilde{X}$, corresponding to each Hawaiian Earring in $\widetilde{X}$. Take *any* open neighborhood of the connecting point $X$. Then there are infinite sheets that are homeomorphic to this neighborhood in $\widetilde{X}$. in $Y$, there exist Hawaiian Earrings in which this open neighborhood would contain disconnected circles. In this case, there could not be two sheets in $Y$, corresponding to this open neighborhood in $\widetilde{X}$. Hence, the composition is not a covering space.

□

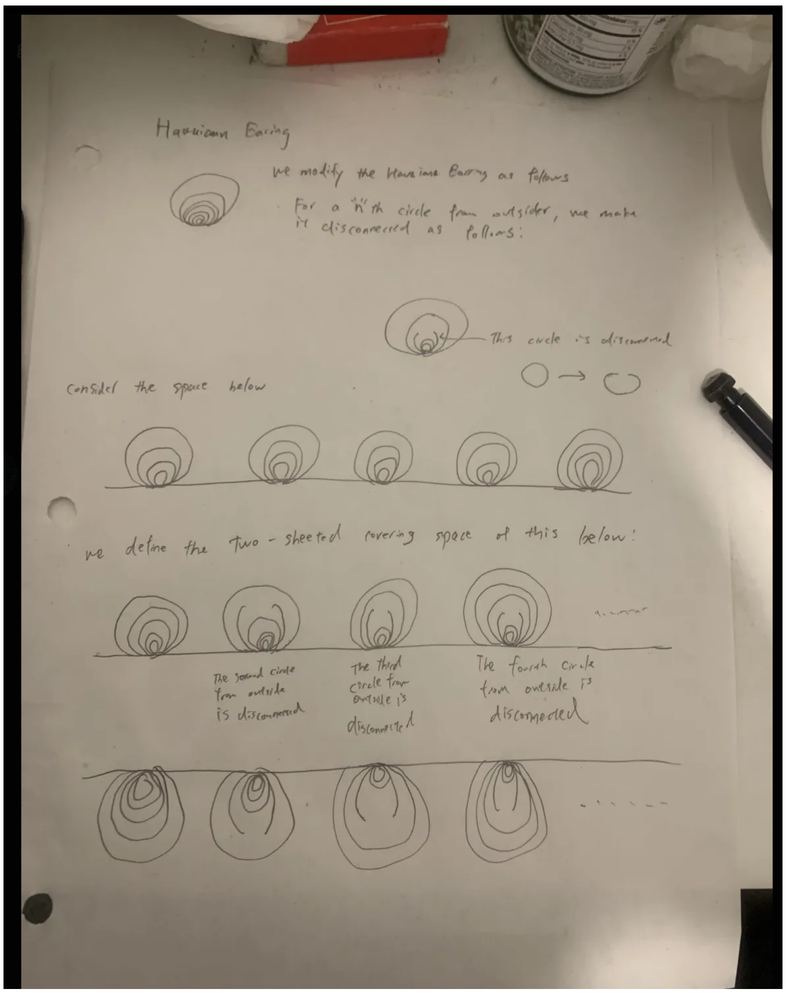



Figure 2: Diagram for chapter 1.3 Problem 6

## 3.27 Chapter 1.3 Problem 8

*Problem Statement.* Let $\widetilde{X}$ and $\widetilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces $X$ and $Y$. Show that if $X \simeq Y$ then $\widetilde{X} \simeq \widetilde{Y}$. [Exercise 11 in Chapter 0 may be helpful.]

*Proof.* By the Lifting Criterion for covering spaces, if $Z$ is path-connected, locally path-connected, and $h_*(\pi_1(Z)) \subset p_*(\pi_1(\widetilde{X}))$, then a lift $\tilde{h} : Z \to \widetilde{X}$ exists. Since $\widetilde{X}$ and $\widetilde{Y}$ are simply-connected, $\pi_1(\widetilde{X}) = \pi_1(\widetilde{Y}) = 0$. For the maps $f \circ p_{\widetilde{X}} : \widetilde{X} \to Y$ and $g \circ p_{\widetilde{Y}} : \widetilde{Y} \to X$, the induced fundamental group images are trivial. Thus, unique lifts $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ and $\tilde{g} : \widetilde{Y} \to \widetilde{X}$ exist.

Given $f \circ g \simeq \mathrm{id}_Y$, let $H : Y \times [0,1] \to Y$ be the homotopy with $H_0 = f \circ g$ and $H_1 = \mathrm{id}_Y$. Fix the initial lift $\widetilde{H}_0 = \tilde{f} \circ \tilde{g}$. By the Homotopy Lifting Property of covering spaces, there exists a unique homotopy $\widetilde{H} : \widetilde{Y} \times [0,1] \to \widetilde{Y}$ lifting $H$. At $t = 1$, $\widetilde{H}_1$ must lift $\mathrm{id}_Y$, which is $\mathrm{id}_{\widetilde{Y}}$. Thus, $\tilde{f} \circ \tilde{g} \simeq \mathrm{id}_{\widetilde{Y}}$. Similarly, we have $\tilde{g} \circ \tilde{f} \simeq \mathrm{id}_{\widetilde{X}}$.

The lifted maps $\tilde{f}$ and $\tilde{g}$ satisfy $\tilde{f} \circ \tilde{g} \simeq \mathrm{id}_{\widetilde{Y}}$ and $\tilde{g} \circ \tilde{f} \simeq \mathrm{id}_{\widetilde{X}}$. Hence, $\tilde{f}$ and $\tilde{g}$ are homotopy equivalences.

□

## 3.28 Chapter 1.3 Problem 9

*Problem Statement.* Show that if a path-connected, locally path-connected space $X$ has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic. [Use the covering space $\mathbb{R} \to S^1$.]

*Proof.* The universal covering space of $S^1$ is $p : \mathbb{R} \to S^1$, defined by $p(t) = e^{2\pi i t}$. Since $X$ is path-connected and locally path-connected, we may apply the Lifting Criterion. A map $f : X \to S^1$ lifts to $\tilde{f} : X \to \mathbb{R}$ if and only if $f_*(\pi_1(X)) \subset p_*(\pi_1(\mathbb{R}))$. As $\pi_1(\mathbb{R})$ is trivial and $\pi_1(X)$ is finite, the only homomorphism $\pi_1(X) \to \pi_1(S^1) \cong \mathbb{Z}$ is the trivial map. Thus, $f_* = 0$, and the lift $\tilde{f}$ exists.

The space $\mathbb{R}$ is contractible, so $\tilde{f}$ is homotopic to a constant map. Thus, we define a homotopy:

$$\tilde{H}(x,t) = (1-t)\tilde{f}(x) + t \cdot c \quad \text{for some constant } c \in \mathbb{R}.$$

and:

$$H(x,t) = p(\tilde{H}(x,t)) = e^{2\pi i((1-t)\tilde{f}(x) + t \cdot c)}.$$

At $t = 0$, $H(x,0) = f(x)$, and at $t = 1$, $H(x,1) = p(c)$, a constant map. Thus, $H$ is a nullhomotopy for $f$.

Therefore, every map $X \to S^1$ is nullhomotopic.

□

## 3.29 Chapter 1.3 Problem 15

*Problem Statement.* Let $p : \widetilde{X} \to X$ be a simply-connected covering space of $X$ and let $A \subset X$ be a path-connected, locally path-connected subspace, with $\widetilde{A} \subset \widetilde{X}$ a path-component of $p^{-1}(A)$. Show that $p : \widetilde{A} \to A$ is the covering space corresponding to the kernel of the map $\pi_1(A) \to \pi_1(X)$.

*Proof.* We wil show that $p : \widetilde{A} \to A$ corresponds to the kernel of $i_* : \pi_1(A) \to \pi_1(X)$.

The map $p|_{\widetilde{A}} : \widetilde{A} \to A$ is a covering map. By the lifting property of covering spaces, a loop $\gamma$ in $A$ lifts to a loop $\widetilde{\gamma}$ in $\widetilde{A}$ if and only if $[\gamma] \in p_*(\pi_1(\widetilde{A}))$.

The map $i_* : \pi_1(A) \to \pi_1(X)$ is induced by the inclusion $i : A \hookrightarrow X$. The kernel $\ker(i_*)$ consists of loops $\gamma$ in $A$ that are contractible in $X$ (i.e., $[\gamma] = 0$ in $\pi_1(X)$). Since $\widetilde{X}$ is simply connected, any loop $\gamma$ in $X$ lifts to a loop $\widetilde{\gamma}$ in $\widetilde{X}$ if and only if $\gamma$ is contractible in $X$. If $\gamma \in \ker(i_*)$, then $\gamma$ is contractible in $X$, so its lift $\widetilde{\gamma}$ is a loop in $\widetilde{X}$.

We know that $\widetilde{A}$ is a path-component of $p^{-1}(A)$. If $\gamma$ is a loop in $A$, its lift $\widetilde{\gamma}$ starts at a basepoint in $\widetilde{A}$. Since $\widetilde{A}$ is path-connected, $\widetilde{\gamma}$ must stay entirely within $\widetilde{A}$.

By the lifting property, $p_*(\pi_1(\widetilde{A}))$ consists of loops in $A$ that lift to loops in $\widetilde{A}$. But these loops are precisely $\ker(i_*)$. Thus:

$$p_*(\pi_1(\widetilde{A})) = \ker(i_*).$$

as desired.

□

## 3.30 Chapter 1.3 Problem 16

*Problem Statement.* Given maps $X \to Y \to Z$ such that both $Y \to Z$ and the composition $X \to Z$ are covering spaces, show that $X \to Y$ is a covering space if $Z$ is locally path-connected, and show that this covering space is normal if $X \to Z$ is a normal covering space.

*Proof.* □

## 3.31 Chapter 1.3 Problem 17

*Problem Statement.* Given a group $G$ and a normal subgroup $N$, show that there exists a normal covering space $\widetilde{X} \to X$ with $\pi_1(X) \cong G$, $\pi_1(\widetilde{X}) \cong N$, and deck transformation group $G(\widetilde{X}) \cong G/N$.

*Proof.* By corollary 1.28 in Hatcher, we may construct a two-dimensional cell complex $X$ such that $\pi_1(X) \cong G$. By the classification of covering spaces, for a "nice" space $X$ (path-connected, locally path-connected, and semilocally simply connected), there is a bijection between conjugacy classes of subgroups of $\pi_1(X)$ and covering spaces of $X$. Since $N$ is a normal subgroup of $G \cong \pi_1(X)$, the corresponding covering space $\tilde{X} \to X$ is a normal (regular) covering.

Clearly, we have $p_*(\pi_1(\tilde{X})) = N$. Since $p_*$ is injective, we have $\pi_1(\tilde{X}) \cong N$. For a normal covering, a deck transformation group is isomorphic to the quotient group $\pi_1(X)/\pi_1(\tilde{X})$, therefore, we obtain $G(\widetilde{X}) \cong G/N$ as desired.

□

## 3.32 Chapter 1.A Problem 4

*Problem Statement.* If $X$ is a finite graph and $Y$ is a subgraph homeomorphic to $S^1$ and containing the basepoint $x_0$, show that $\pi_1(X, x_0)$ has a basis in which one element is represented by the loop $Y$.

*Proof.* Suppose $X$ is a finite graph and $Y \subseteq X$ is a subgraph homeomorphic to $S^1$ and containing the basepoint $x_0$. Then we can infer that $Y$ is a cycle in $X$ (otherwise, $Y$ would be contractible). Suppose $|V(X)| = n$ where $V(X)$ is a set of vertices of $X$. Then any spanning tree would contain $n$ vertices with $(n-1)$ edges. Suppose also that $|V(Y)| = m$ where $V(Y)$ is a set of vertices of $Y$. Then the cycle contains $m$ vertices and $m$ edges. Removing any edge from $Y$ would create a spanning tree of $Y$. We call this edge removed $e$.

Now consider a spanning tree $T$ of $X$ such that $e \notin T$ and $E(Y) \setminus \{e\} \subseteq T$ where $E(Y)$ is a set of edges in $Y$. By proposition 1A.2 in Hatcher, $\pi_1(X)$ is a free group containing a basis element class $[f_e]$. However, since all the other edges in $Y$ are contained in the spanning tree $T$, the loop represented by $[f_e]$ is equivalent to the subgraph $Y$.

□

## 3.33 Chapter 1.A Problem 14

*Problem Statement.* Show that the existence of maximal trees is equivalent to the Axiom of Choice.

*Proof.* **Axiom of Choice implies the existence of maximal trees.**

Suppose Axiom of Choice holds. For the purpose of this problem, we pretend that we know Axiom of Choice is equivalent to Zorn's lemma. Let $X$ be a connected graph. Define a poset of all subtrees of $X$, ordered by the set inclusion. It is clear that every chain has an upper bound, which is a union of all elements in the chain. By Zorn's lemma, there exists a maximal element in the poset, which is a spanning tree; for if it is not a spanning tree, then $X$ would not be connected, contradiction. This maximal element in the poset is, in fact, a maximal tree of the connected graph $X$.

**The existence of maximal trees implies Axiom of Choice.** Given a family of nonempty sets $\{A_i \mid i \in I\}$ where $I$ is an index set, we construct a connected graph $G$ with a base vertex $v_0$ and all vertices in $\bigcup_{i\in I} A_i$. We connect a base vertex $v_0$ with all other vertices in $\bigcup_{i\in I} A_i$ and make each $A_i$ a clique. By assumption, the spanning tree $T$ exists for this connected graph. We also observe that since each clique formed by $A_i$ is not pairwise connected, $T$ must connect a base vertex $v_0$ with at least one vertex in each $A_i$. Moreover, since each $A_i$ is a clique, $T$ must not connect $v_0$ with more than one vertex in $A_i$. Now, selecting the vertex $v_i$ of $A_i$ that is connected to the base vertex $v_0$, we may define the choice function $f : A_i \mapsto v_i$. This implies Axiom of Choice as desired.

$\square$

# 4 MAT1301 Homework 1

## 4.1 Problem 1

*Problem Statement.* Let $e$ denote the constant loop at the basepoint $x_0$, and let $f\bar{f}$ denote the concatenation of a loop $f$ with its reverse $\bar{f}$, where $\bar{f}(s) = f(1-s)$ . Construct an explicit homotopy $h(t,s)$ between $e$ and $f\bar{f}$, adhering to the algebraic properties $[F] \cdot [\overline{F}] = [e]$ and $[F]^{-1} = [F]$. The homotopy must be expressed as a piecewise function with three cases.

*Proof.* Define the homotopy $h(t,s)$ as follows:

$$h(t,s) = \begin{cases} f\left(\dfrac{2t}{s}\right) & \text{if } 0 \leq t \leq \dfrac{s}{2}, \text{ and } s \neq 0, \\ x_0 & \text{if } \dfrac{s}{2} < t < 1 - \dfrac{s}{2}, \\ f\left(\dfrac{2(1-t)}{s}\right) & \text{if } 1 - \dfrac{s}{2} \leq t \leq 1, \text{ and } s \neq 0 \end{cases}$$

We check if this construction is fine. Firstly, $h$ is clearly well defined on all of its domain. When $s = 0$, the conditions collapse to $h(t,0) = x_0$ for all $t$, which matches the identity $e$. When $s = 1$, the homotopy becomes:

$$h(t,1) = \begin{cases} f(2t) & 0 \leq t \leq \dfrac{1}{2}, \\ f(2(1-t)) & \dfrac{1}{2} \leq t \leq 1, \end{cases}$$

which is the concatenation $f\bar{f}(t)$. Continuity at $t = s/2$ and $t = 1 - s/2$ holds clearly because $f(1) = x_0$ and $f(0) = x_0$. Since $f$ is continuous on $[0,1]$, it follows that $h(t,s)$ is continuous on $[0,1] \times [0,1]$. $\square$

$$h(t,s) = \begin{cases} f\left(\dfrac{2t}{s}\right) & 0 \leq t \leq \dfrac{s}{2}, \\ x_0 & \dfrac{s}{2} < t < 1 - \dfrac{s}{2}, \\ f\left(\dfrac{2(1-t)}{s}\right) & 1 - \dfrac{s}{2} \leq t \leq 1. \end{cases}$$

## 4.2 Problem 2

*Problem Statement.* Let $X$ be a path-connected topological space. Prove the equivalence of the following statements:

1. $\pi_1(X, x_0) = 0$ for some (hence any) $x_0 \in X$.
2. Any two paths $f_0, f_1$ in $X$ with shared endpoints $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1) = x_1$ are homotopic relative to the endpoints.
3. Any two maps $S^1 \to X$ are homotopic.

*Proof.* We prove the equivalences cyclically.

**(1) $\implies$ (2):**

**Lemma 4.** *Let $X$ be a path-connected space and $a, b \in X$. There is a bijection*

$$\pi_0\Omega_{[a,b]}(X) \cong \pi_0\Omega_a(X).$$

*Proof.* Since $X$ is path-connected, choose a path $\rho\colon [0,1] \to X$ from $b$ to $a$. Define a map

$$\rho_*\colon \pi_0\Omega_{[a,b]}(X) \to \pi_0\Omega_a(X)$$

by $\rho_*([\gamma]) = [\gamma \cdot \rho]$, where $\cdot$ denotes path concatenation.

We check that $\rho*$ is well defined. Concatenation is well-defined up to homotopy relative endpoints, so $\rho_*$ is well-defined.

Next, $\rho*$ is surjective. For any loop $\lambda$ based at $a$, we have: $\lambda \simeq (\lambda \cdot \rho^{-1}) \cdot \rho$. Thus $[\lambda] = \rho_*([\lambda \cdot \rho^{-1}])$.

Finally, $\rho*$ is injective. Suppose $\rho_*([\gamma]) = \rho_*([\gamma'])$. Then there is a homotopy: $H\colon \gamma \cdot \rho \simeq \gamma' \cdot \rho$. By reparameterizing $H$ so that we constrain it to act only on the $\gamma$-portion when $t \leq \frac{1}{2}$, we may obtain a homotopy $\gamma \simeq \gamma'$.

□

Now, we argue that if $\pi_1(X; a)$ is trivial, then any two paths $\gamma, \gamma' \in \Omega_{[a,b]}(X)$ are homotopic relative to endpoints. By the above lemma, $\pi_0\Omega_{[a,b]}(X) \cong \pi_0\Omega_a(X) = \pi_1(X; a)$. If $\pi_1(X; a)$ is trivial, then $\pi_0\Omega_{[a,b]}(X)$ has exactly one element. Hence all paths from $a$ to $b$ belong to the same path-component in $\Omega_{[a,b]}(X)$, so all of the paths are homotopic relative to endpoints.

**(2) $\implies$ (3):** Let $\gamma_1, \gamma_2 : S^1 \to X$ be continuous. Fix $x_0 \in X$. Using path-connectedness, the paths $\gamma_1, \gamma_2$ form loops at $x_0$. By (2), these loops are homotopic via endpoint-fixed homotopies, hence $\gamma_1 \simeq \gamma_2$.

**(3) $\implies$ (1):** Let $\gamma$ be a loop at $x_0$. By (3), $\gamma \simeq c_{x_0}$, so $\pi_1(X, x_0) = 0$.

Thus, all three statements are equivalent under path-connectedness.

□

## 4.3 Problem 3

*Problem Statement.* If $X$ is path-connected, the set of homotopy classes of maps $S^1 \to X$ can be put in a bijection with the set of conjugacy classes in the fundamental group of $X$.

*Proof.* Let $[S^1, X]$ denote the set of free homotopy classes of maps $S^1 \to X$. Define the map:

$$\Phi : \pi_1(X, x_0) \to [S^1, X]$$

by forgetting the basepoint (so we use free homotopy instead). We show $\Phi$ induces a bijection with conjugacy classes in $\pi_1(X, x_0)$.

We show the surjectivity of $\Phi$ first. Let $[f] \in [S^1, X]$. Choose a representative $f : S^1 \to X$, and let $x_1 = f(1)$ where 1 is chosen to be a basepoint of $S^1$. Since $X$ is path-connected, we can choose a path $h : [0,1] \to X$ from $x_0$ to $x_1$. Define the loop by:

$$\alpha = h \cdot (f \circ \gamma) \cdot \bar{h},$$

where $\gamma : [0,1] \to S^1$ generates $\pi_1(S^1, 1)$. Construct a free homotopy $H : S^1 \times [0,1] \to X$ as follows:

$$H(s,t) = \begin{cases} h(\frac{3s}{1-t}) & \text{if } 0 \leq s \leq \frac{1-t}{3}, t \neq 1, \\ f(\gamma(\frac{3s-(1-t)}{1+2t})) & \text{if } \frac{1-t}{3} \leq s \leq \frac{2+t}{3}, \\ h(\frac{3(1-s)}{1-t}) & \text{if } \frac{2+t}{3} \leq s \leq 1, t \neq 1. \end{cases}$$

At $t = 0$, $H(s,0) = \alpha(s)$. At $t = 1$, $H(s,1) = (f \circ \gamma)(s)$. Since $H(s,t)$ is clearly continuous in $t, s$, $\alpha$ is freely homotopic to $f$. Thus, $\Phi([\alpha]) = [f]$, proving $\Phi$ is surjective.

Next, we will show that $\Phi$ is injective modulo the conjugacy class. Suppose $\Phi([f]) = \Phi([g])$. Then there exists a free homotopy $\varphi_t : S^1 \to X$ between $f$ and $g$. By Lemma 1.19 in the textbook by Hatcher, if we let $h(t) = \varphi_t(1)$, we obtain:

$$[g] = \beta_h([f]) = [h \cdot f \cdot \bar{h}],$$

where $\beta_h$ is a change of basepoint map. So $[f]$ and $[g]$ are conjugate whenever $\Phi([f]) = \Phi([g])$.

Therefore, the map $\Phi$ is surjective and injective (thus bijective) with respect to the conjugacy classes in $\pi_1(X, x_0)$. Hence, there is a natural bijection between the set of homotopy classes of maps $S^1 \to X$ and the set of conjugacy classes in the fundamental group of $X$ as desired.

$\square$

# 5 MAT1301 Homework 2

## 5.1 Problem 1

*Problem Statement.* A topological space $X$ is called "locally path connected" if every point in it has arbitrarily small neighborhoods that are path connected. Namely, if $x \in X$ and if $U$ is a neighborhood of $x$, then there is a path connected open set $V$ such that $x \in V \subset U$.

*On the subject of liftings:* Prove that if $X$ is path connected, locally path connected, and simply connected, and if $p : (E, e_0) \to (B, b_0)$ is a covering map, then every $f : (X, x_0) \to (B, b_0)$ has a unique lift to a map $\tilde{f} : (X, x_0) \to (E, e_0)$ such that $\tilde{f}/p = f$.

*Hint.* Lift paths, lift endpoints of paths, worry about well-definedness, worry about continuity.

*Proof.* I review important definitions first. A covering map is a continuous surjective map $p : E \to B$ where every $b \in B$ has a neighborhood $U$ such that $p^{-1}(U)$ is a disjoint union of open sets in $E$, each homeomorphic to $U$. The lift is a map $\hat{f} : X \to E$ such that $p \circ \hat{f} = f$. The path lifting property says for any path $\gamma : [0, 1] \to B$ and $e_0 \in p^{-1}(\gamma(0))$, there exists a unique path $\tilde{\gamma} : [0, 1] \to E$ with $\tilde{\gamma}(0) = e_0$ and $p \circ \tilde{\gamma} = \gamma$.

Suppose that $X$ is path-connected, locally path-connected, and simply connected. Let $p : (E, e_0) \to (B, b_0)$ be a covering map and $f : (X, x_0) \to (B, b_0)$ be continuous.

We need to show that there exists a unique lift $\hat{f} : (X, x_0) \to (E, e_0)$ with $p \circ \hat{f} = f$.

We start by defining $\hat{f}$. For $x \in X$, choose a path $\gamma_x$ from $x_0$ to $x$. Define:

$$\hat{f}(x) = \tilde{\gamma}_x(1),$$

where $\tilde{\gamma}_x$ is the unique lift of $f \circ \gamma_x$ starting at $e_0 = p^{-1}(\gamma_x(0))$. The value $\tilde{\gamma}_x(1)$ is the endpoint of the lifted path, corresponding to $x$.

We will show $\hat{f}$ is well defined. If $\gamma'_x$ is another path from $x_0$ to $x$, then $\gamma_x \simeq \gamma'_x$ (homotopic) because $X$ is simply connected. By the homotopy lifting property, their lifts $\tilde{\gamma}_x, \tilde{\gamma}'_x$ have the same endpoint: $\tilde{\gamma}_x(1) = \tilde{\gamma}'_x(1)$. Thus, $\hat{f}(x)$ is independent of the path choice.

We will show the Continuity of $\hat{f}$. Let $x \in X$. Since $X$ is locally path-connected, take a path-connected neighborhood $U$ of $x$. For $y \in U$, choose a path $\gamma_y = \gamma_x \cdot \eta$, where $\eta$ is a path in $U$ from $x$ to $y$. The lift $\tilde{\gamma}_y$ decomposes as $\tilde{\gamma}_x \cdot \tilde{\eta}$, where $\tilde{\eta}$ is the lift of $f \circ \eta$. Since $f \circ \eta$ lies in an evenly covered neighborhood $V \subset B$ around $f(x)$, its lift $\tilde{\eta}$ stays in a single sheet of $p^{-1}(V)$.

Finally, we will show the uniqueness. If $\hat{f}_1, \hat{f}_2$ are two lifts, then for any path $\gamma_x$, both $\hat{f}_1 \circ \gamma_x$ and $\hat{f}_2 \circ \gamma_x$ lift $f \circ \gamma_x$. By the unique path lifting property, they must coincide. Hence, $\hat{f}_1 = \hat{f}_2$.

□

## 5.2 Problem 2

*Problem Statement.* With the obvious assumptions and definitions, prove that $\pi_1((X, x_0) \times (Y, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

*Proof.* We will construct an explicit isomorphism between these groups.

Consider the map $\Phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ defined by sending the homotopy class of a loop $\gamma = (\gamma_1, \gamma_2)$ in $X \times Y$ to $([\gamma_1], [\gamma_2])$.

We will verify that $\Phi$ is a homomorphism. For loops $\gamma, \delta$ in $X \times Y$, their concatenation satisfies $\gamma * \delta = (\gamma_1 * \delta_1, \gamma_2 * \delta_2)$. Then $\Phi([\gamma * \delta]) = ([\gamma_1 * \delta_1], [\gamma_2 * \delta_2]) = ([\gamma_1] * [\delta_1], [\gamma_2] * [\delta_2]) = \Phi([\gamma]) * \Phi([\delta])$, preserving the group operation.

We show that $\Phi$ is injective. If $\Phi([\gamma]) = ([c_{x_0}], [c_{y_0}])$, then $\gamma_1 \simeq c_{x_0}$ and $\gamma_2 \simeq c_{y_0}$ via homotopies $H_1, H_2$. The product homotopy $H = (H_1, H_2)$ shows $\gamma \simeq c_{(x_0,y_0)}$, so $[\gamma]$ is trivial, and $\Phi$ is injective.

Lastly, we show that $\Phi$ is surjective. For any $([\alpha], [\beta]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$, define $\gamma(t) = (\alpha(t), \beta(t))$. Then $\Phi([\gamma]) = ([\alpha], [\beta])$, proving surjectivity.

Since $\Phi$ is a bijective homomorphism, it is an isomorphism:

$$\pi_1((X, x_0) \times (Y, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

□

## 5.3 Problem 3

*Problem Statement.* Let $M$ denote the Möbius band.

1. Show that $\pi_1(M) \cong \mathbb{Z}$.
2. Is there a retraction from $M$ to its boundary?

Here is a proof for 1.

*Proof.* We will adhere to the square representation of the Möbius band.

We construct a deformation retraction from the Möbius band to the central circle. Define the **central circle** $C \subset M$ as the image of the line $[0, 1] \times \{1/2\}$ after gluing. This loop runs along the "middle" of the Möbius band. The deformation retraction $r_t : M \to M$ can be constructed by continuously shrinking the width of the Möbius band towards $C$ from both sides. For $t \in [0, 1]$, let $r_t$ map $(x, y) \mapsto (x, (1-t)y + t \cdot \frac{1}{2})$. We notice that $r_0$ is an identity map, the image of $r_1$ lies in the central circle, and any $r_t$ is an identity map on the central circle for all $t$. Since a retraction is clearly continuous, $r_t$ is the desired deformation retraction from the Möbius band to the central circle. This implies that $S^1$ and $M$ are homotopy equivalent, hence $\pi_1(M) \simeq \pi_1(S^1) \simeq \mathbb{Z}$.

□

Here is a proof for 2.

*Proof.* We start by showing that the boundary of the Möbius band is homeomorphic to the circle $S^1$.

First, we note that the boundary of the Möbius band can be described as the quotient:

$$\partial M = (I \times \{0, 1\}) / \sim$$

where $\sim$ is the identification from the square representation of the Möbius band construction.

We define a function as follows:

$$f : I \times \{0, 1\} \to S^1$$

$$f(x, y) = \exp(\pi i(x + y))$$

This continuous function maps $I \times \{0\}$ onto the first half of the circle and $I \times \{1\}$ onto the second half. With a slight modification, this map is actually a homeomorphism from $\partial M$ to $S^1$.

We construct a homeomorphism as follows:

$$F : \partial M \to S^1, \quad F([x, y]) = f(x, y)$$

By the theorem in quotient topology, this map $F$ is well-defined and continuous as a composition of a continuous $f$ and a quotient map. Moreover, this map is a bijection between $\partial M$ and $S^1$. Since $\partial M$ is compact and $S^1$ is Hausdorff, a continuous bijection from $\partial M$ to $S^1$ is a homeomorphism. Thus, $\partial M \cong S^1$.

Although the boundary $\partial M$ is a single circle, this boundary loop *wraps around $M$ twice* when followed continuously as per the explanation on Wikipedia.

A retraction $r : M \to \partial M$ would satisfy $r|_{\partial M} = \mathrm{id}_{\partial M}$. Let $i : \partial M \hookrightarrow M$ be the inclusion map. Then $r \circ i = \mathrm{id}_{\partial M}$. On fundamental groups, this implies $r_* \circ i_* = \mathrm{id}_{\pi_1(\partial M)}$.

We observe that $\pi_1(\partial M) \cong \mathbb{Z}$, with generator $[\gamma]$, where $\gamma$ loops once around $\partial M$, and $\pi_1(M) \cong \mathbb{Z}$, with generator $[\alpha]$, where $\alpha$ loops once around the central circle $C$. The inclusion $i_*$ maps $[\gamma] \in \pi_1(\partial M)$ to $[\alpha]^2 \in \pi_1(M)$, since $\partial M$ wraps around $C$ twice. Thus,

$$i_*([\gamma]) = 2[\alpha].$$

For $r_* \circ i_* = \mathrm{id}$, we would need $r_*(2[\alpha]) = [\gamma]$. Suppose $r_*([\alpha]) = k[\gamma]$, then

$$r_*(2[\alpha]) = 2k[\gamma] = [\gamma] \Rightarrow 2k = 1,$$

which implies $k = \frac{1}{2}$, a contradiction since we must have $k \in \mathbb{Z}$.

We conclude that no retraction exists because the equation $2k = 1$ has no integer solution.

□

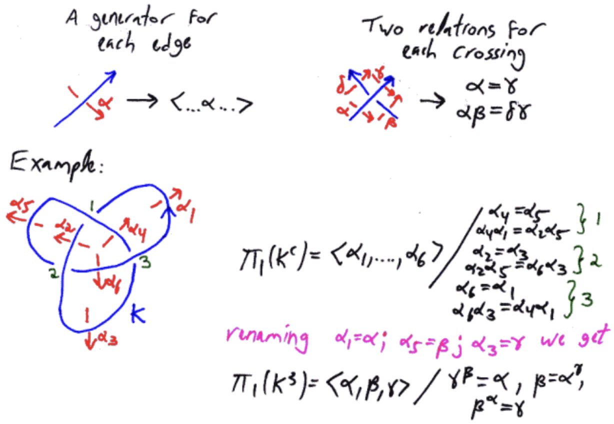



Figure 3: The figure for Homework 3 Problem 1

# 6 MAT1301 Homework 3

## 6.1 Problem 1

*Problem Statement.* Let $K$ be a knot in $\mathbb{R}^3$ presented by a planar diagram $D$. With a massive use of van Kampen's theorem, show that the fundamental group of the complement of $K$ has a presentation (the "Wirtinger presentation", as discussed in class) with one generator for each edge of $D$ and two relations for each crossing of $D$, as indicated in the Figure 3.

*Proof.* We can make lots of observations from the Figure 3. First, there are three crossings on which we can derive relations, and we can decompose the trefoil knot into six edges whose orientations are going outward from the center of the knot. We will compute the fundamental group of the complement of the trefoil knot $K$, $\pi_1(K^c)$, using the van Kampen's theorem.

Let $X = \mathbb{R}^3 \setminus K$. We will cover the space $X$ by a collection of path-connected open sets. First, we will define open sets $U_1, U_2, U_3$ as follows: for each crossing $i$ of the knot, let $U_i$ be a tubular neighborhood of the crossing $i$ of the trefoil knot $K$. We will define one more open set $V$:

$$V = X \setminus \{\text{the union of three small closed neighborhoods of the crossings.}\}$$

We need to ensure that those small closed neighborhoods of the crossings are contained in the tubular neighborhoods of them, $U_i$'s. We note that each open set is path connected since removing an 1 dimensional submanifold $K$ from $\mathbb{R}^3$ would disconnect the ambient space (the trefoil knot is a smooth embedding from $S^1$ to $\mathbb{R}^3$, so a 1 dimensioanl submanifold ). We also note that the intersections of open sets are path connected. For example, the intersection $U_i \cap U_j$ for $i \neq j$ is disjoint, so trivially path-connected. For the intersection

$V \cap U_i$, it would look like a loop surrounding the crossing since a small closed neighborhood is contained in $U_i$. Therefore, each $V \cap U_i$ is path-connected. Any intersection of three open sets is empty since it must contain at least two distinct $U_i$'s which we know are disjoint.

First, in a sense, the removal of each closed neighborhood of the crossing would join two understrand edges of the trefoil knot going through the crossing. Since this holds at each crossing, the union of three closed neighborhoods of the crossings and the trefoil knot would be one single connected subset of $\mathbb{R}^3$. This implies that the complement of it has a fundamental group isomorphic to that of a circle $S^1$. Thus, we get $\pi_1(V) \simeq \mathbb{Z}$, a free group with a single generator, where the generator can be one of $\alpha_i$ for $i = 1, 2, 3, 4, 5, 6$.

For each $U_i$, since each crossing contains one overstrand edge and two understrand edges, $U_i \setminus K$ deformation retracts to the wedge sum of three circles $S^1$, or $S^1 \vee S^1 \vee S^1$. However, there are extra two relations for us to consider at each crossing. We will compute the explicit expression later, but I informally describe how the relations look like here. At a crossing 1, we notice that there are one overstrand and two understrands. It is clear that we have the relations: $\alpha_4\alpha_1 = \alpha_2\alpha_5$. Through the overstrand, we may identify that $\alpha_4 = \alpha_5$. Using the same argument, at a crossing 2, we have the relations: $\alpha_2\alpha_5 = \alpha_6\alpha_3$ and $\alpha_2 = \alpha_3$. Likewise, at a crossing 3, we get the relations: $\alpha_6\alpha_3 = \alpha_4\alpha_1$ and $\alpha_6 = \alpha_1$. For convenience, we would denote the fundamental group of each $U_i$ as follows:

$$\pi_1(U_1) = \langle \alpha_1, \alpha_2, \alpha_5 \rangle$$

$$\pi_1(U_2) = \langle \alpha_3, \alpha_5, \alpha_6 \rangle$$

$$\pi_1(U_3) = \langle \alpha_1, \alpha_3, \alpha_4 \rangle$$

Finally, we examine the fundamental group of the intersection of open sets. For each $\pi_1(V \cap U_i)$, since it is a tubular neighborhood of the crossing $U_i$ minus the trefoil knot and the small enough closed neighborhood of the crossing, $V \cap U_i$ would look like a thickened loop surrounding around the crossing. This implies that $\pi_1(V \cap U_i) \simeq \pi_1(S^1) \simeq \mathbb{Z}$. This is actually a free group with a single generator.

Now, we examine the generator of the kernel of the homomorphism from the free group of components to the fundamental group of $X$. It suffices to check the homomorphisms induced by the inclusions: $i_{VU_i} : \pi_1(V \cap U_i) \to \pi_1(V)$ and $i_{U_iV} : \pi_1(V \cap U_i) \to \pi_1(U_i)$. We will compute it for $i = 1$ first. For the inclusion homomorphism, $i_{U_1V} : \pi_1(V \cap U_1) \to \pi_1(U_1)$, the generator can be computed as: $i_{U_1V}(w) = \alpha_4\alpha_1\alpha_2^{-1}$ and $i_{VU_1}(w)^{-1} = \alpha_5^{-1}$. This induces a relation: $\alpha_4\alpha_1 = \alpha_2\alpha_5$. By identifying the edges lying on an overstrand at the crossing, we get $\alpha_4 = \alpha_5$.

Now doing the same task at the crossings 2 and 3, we obtain the relations: $\alpha_2 = \alpha_3$, $\alpha_2\alpha_5 = \alpha_6\alpha_3$, $\alpha_6 = \alpha_1$, and $\alpha_6\alpha_3 = \alpha_4\alpha_1$.

By van kampen theorem, we have $\pi_1(\mathbb{R}^3 \backslash K) \simeq \pi_1(U_1) * \pi_1(U_2) * \pi_1(U_3) * \pi_1(V)/N$ where $N$ is a normal subgroup generated by six relations, $\alpha_4\alpha_1 = \alpha_2\alpha_5$, $\alpha_4 = \alpha_5$, $\alpha_2 = \alpha_3$, $\alpha_2\alpha_5 = \alpha_6\alpha_3$, $\alpha_6 = \alpha_1$, and $\alpha_6\alpha_3 = \alpha_4\alpha_1$. Therefore, we conclude that:

$$\pi_1(\mathbb{R}^3 \backslash K) \simeq \frac{\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle}{\langle \alpha_4\alpha_1 = \alpha_2\alpha_5, \alpha_4 = \alpha_5, \alpha_2 = \alpha_3, \alpha_2\alpha_5 = \alpha_6\alpha_3, \alpha_6 = \alpha_1, \alpha_6\alpha_3 = \alpha_4\alpha_1 \rangle}$$

□

## 6.2 Problem 2

*Problem Statement.* The trefoil knot above, whose fundamental group is

$$G_1 = \langle \alpha, \beta, \gamma : \alpha = \gamma^\beta,\ \beta = \alpha^\gamma,\ \gamma = \beta^\alpha \rangle$$

is in fact the torus knot $T_{3/2}$, whose fundamental group, as computed in class, is

$$G_2 = \langle \lambda, \mu : \lambda^2 = \mu^3 \rangle.$$

Prevent the collapse of mathematics by showing that these two groups are isomorphic.

*Proof.* We need to show two groups $G_1 = \langle \alpha, \beta, \gamma : \alpha = \gamma^\beta,\ \beta = \alpha^\gamma,\ \gamma = \beta^\alpha \rangle$ and $G_2 = \langle \lambda, \mu : \lambda^2 = \mu^3 \rangle$ are isomorphic to each other.

We first observe the following. The relations in $G_1$ are conjugations: $\alpha = \beta^{-1}\gamma\beta$, $\beta = \gamma^{-1}\alpha\gamma$ and $\gamma = \alpha^{-1}\beta\alpha$. From these relations, we get $\gamma = \beta\alpha\beta^{-1}$. By substituting this into the relation, we get $\beta\alpha\beta^{-1} = \alpha^{-1}\beta\alpha \implies \alpha\beta\alpha = \beta\alpha\beta$. Thus, $G_1$ reduces to $\langle \alpha, \beta \mid \alpha\beta\alpha = \beta\alpha\beta \rangle$, which is the braid group $B_3$.

We will define a forward and inverse homomorphism that preserves the relations between $G_1$ and $G_2$. We will construct a map $\phi : G_2 \to G_1$ first. Define the map $\phi$ by

$$\phi(\lambda) = \alpha\beta\alpha, \quad \phi(\mu) = \alpha\beta$$

We will verify $\phi(\lambda^2) = \phi(\mu^3)$. Using the braid relation $\alpha\beta\alpha = \beta\alpha\beta$:

$$\lambda^2 = (\alpha\beta\alpha)^2 = \alpha\beta\alpha\alpha\beta\alpha = \alpha\beta\alpha^2\beta\alpha.$$
$$\mu^3 = (\alpha\beta)^3 = \alpha\beta\alpha\beta\alpha\beta.$$

Substitute $\alpha\beta\alpha = \beta\alpha\beta$, we observe that $\phi(\lambda^2) = \phi(\mu^3)$ as desired, so it preserves the relation.

Next we check that $\phi$ is a bijection. Consider $\phi^{-1}(e)$ where $e$ is an identity element in $B^3$. in $B^3$, we have $e = \alpha^{-1}\beta^{-1}\alpha^{-1}\beta\alpha\beta$. The kernel is clearly trivial, so $\phi$ is injective. To show surjectivity, we just need to show that there exists elements in $G^2$ that maps to the generators of $B^3$. We find that: $\phi(u^{-1}\lambda) = \alpha$ and $\phi(\lambda^{-1}u^2) = \beta$, so $\phi$ is surjective.

Since $G^1$ is the same as the braid group $B^3$ and $G^2$ is isomorphic to $B^3$, we have shown that $G^1 \cong G^2$ as wanted.

□

# 7 MAT1301 Homework 4

## 7.1 Problem 1

*Problem Statement.* Let $X$ be the "Olympic Rings" covering of the figure 8 space, $8^a_b$, whose basepoint is taken to be at the quadrivalent vertex in its centre and whose fundamental group is the free group on two letters $a$ and $b$: $G := \pi_1(8^a_b) = F(a,b)$.

A. Describe the right $G$-set $S$ corresponding to $8^a_b$: it is a set with __ elements, and $a$ and $b$ act on it as the permutations ___ and ___.

B. Taking the basepoint $x_1$ of $X$ to be the point marked as "1" on the right, write a set of generators for the image $H$ of $\pi_1(X, x_1)$ within $G := \pi_1(8^a_b)$.

C. Is $H$ a normal subgroup of $G$?

*Proof.* □

## 7.2 Problem 2

*Problem Statement.* Describe all the 2-sheeted and 3-sheeted connected coverings of the figure 8 space, $8^a_b$. (Meanings, all the connected coverings that are 2 to 1 or 3 to 1).

*Proof.* The answer is shown in the Figure 4.

□

## 7.3 Problem 3

*Problem Statement.* Prove Corollary 11 from the Covering Spaces handout: If $X$ is a connected covering of a nice space $B$ (meaning, $B$ is connected, locally connected and semi-locally simply connected) and $H := p_*\pi_1(X) < G := \pi_1(B)$, then

$$\mathrm{Aut}(X) = N_G(H)/H \quad \text{where } N_G(H) := \{g \in G : H = g^{-1}Hg\} \text{ is the normalizer of } H \text{ in } G.$$

*Proof.* Let $p : X \to B$ be a connected covering space, where $B$ is a nice space. Let $G = \pi_1(B, b_0)$ and $H = p_*\pi_1(X, x_0)$ where $p*$ is a covering projection. $\mathrm{Aut}(X)$ consists of deck transformations, which are homeomorphisms $f : X \to X$ such that $p \circ f = p$.

We note that the fiber $p^{-1}(b_0)$ is a G-set, so $G$ acts on it. Let us fix a basepoint $x_0 \in p^{-1}(b_0)$. By classification theorem, we know that $H = p_*\pi_1(X, x_0)$ is a stabilizer subgroup of $x_0$, hence $H = \{g \in G \mid x_0 \cdot g = x_0\}$. We are given that $H = p_*\pi_1(X, x_0)$ is a normal subgroup of $G$, hence by the proposition in Hatcher, we know that the covering map $p : X \to B$ is normal. Since the covering map $p : X \to B$ is normal, for any two points $x_1, x_2 \in p^{-1}(b_0)$, there exists a deck transformation $f \in \mathrm{Aut}(X)$ such that $f(x_1) = x_2$.

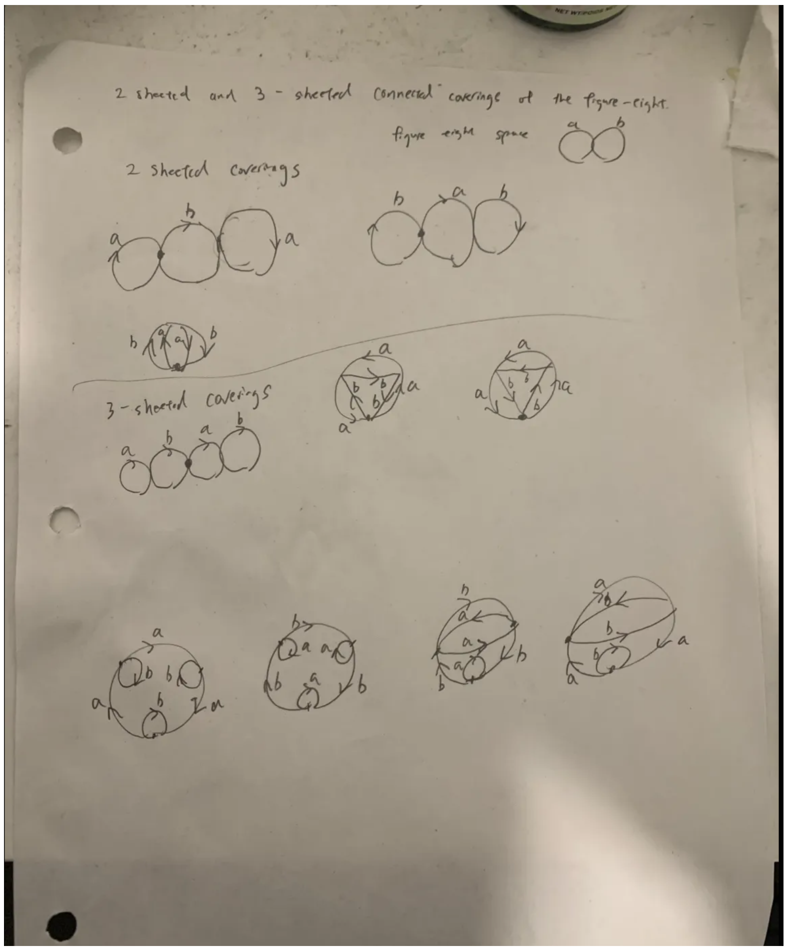



Figure 4: The answer for problem 2 in homework 4

We know that a deck transformation $f \in \text{Aut}(X)$ permutes the points in the fiber $p^{-1}(b_0)$; for any $x_0 \in p^{-1}(b_0)$, $f(x_0) = x_0 \cdot g$ for some $g \in G$. We observe that since a normal subgroup $H$ is closed under conjugation, the stabilizer of $f(x_0) = x_0 \cdot g$ is the same as the stabilizer of $x_0$; the stabilizer of $x_0 \cdot g$ is $g^{-1}Hg$ and $H = g^{-1}Hg$. Therefore, such $g$ is in $N_G(H)$.

Now we define the following equivalence relation in the group $N_G(H)$. We define an equivalence relation $\sim$ by

$$g_1 \sim g_2 \text{ if and only if } g_1 = g_2 \cdot h \text{ for some } h \in H$$

To check this, we may note that for any two elements $g_1, g_2 \in N_G(H)$, if $g_1 = g_2 \cdot h$, for a basepoint $x_0 \in p^{-1}(b_0)$, $x_0 \cdot g_1 = x_0 \cdot (g_2 \cdot h) = x_0 \cdot g_2$ since $h \in H$ is a stabilizing element. Thus, they define the same deck transformations.

Since each deck transformation $f \in \text{Aut}(X)$ is uniquely determined by where it sends a point $x_0$, if $f(x_0) = x_0 \cdot g$, then $g \in N_G(H)$. Therefore, we have obtained that:

$$\text{Aut}(X) \cong \frac{\text{elements of } N_G(H)}{\text{elements of } H} = N_G(H)/H.$$

□

## 7.4 Problem 4

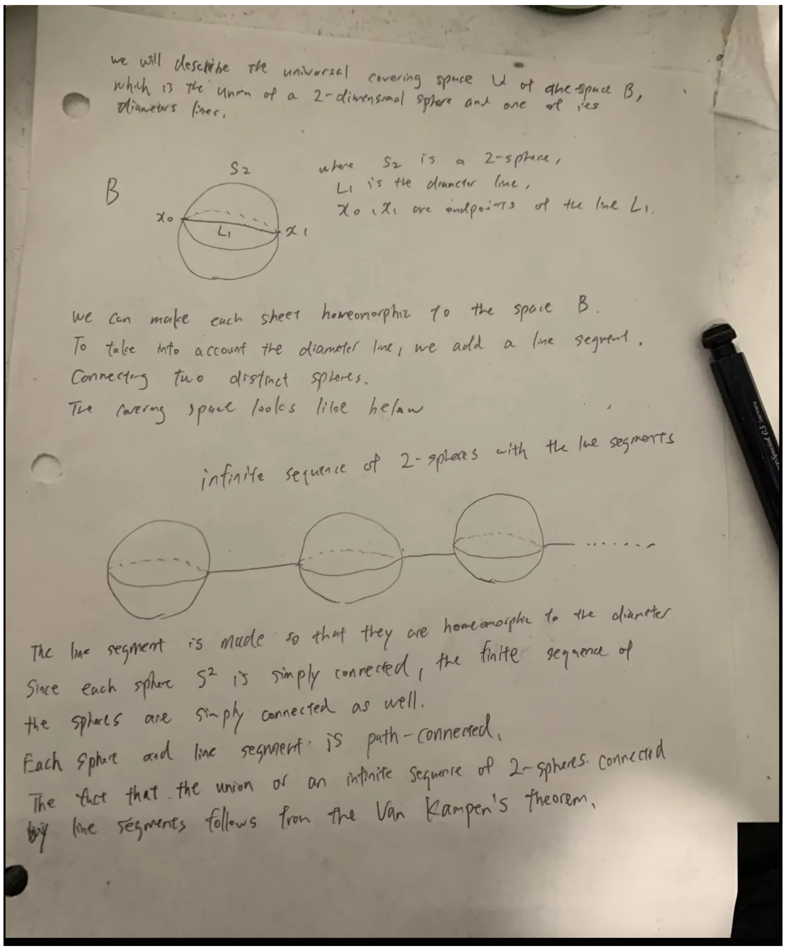
*Problem Statement.* Describe the universal covering space $U$ of the space $B$ which is the union of a 2-dimensional sphere and one of its diameter lines. (Don't say "it's the space of spelunkers" – you are expected to give a concrete description of $U$ as some familiar space or as a simple subset of some familiar space).

*Proof.* The answer is in the Figure 5. As shown in the figure, we can map each sphere and attached line segment to the space $B$ homeomorphically. We just need to identify the endpoints of the diameters with the endpoints of the line segment in the covering space. The covering space is an infinite sequence of 2-spheres where each pair of two adjacent spheres is connected by the line segment. We may apply the van kampen's theorem to see that this covering space is simply connected in the following sense: We take an open neighbourhood of each line segment and sphere. Clearly, these open neighborhoods are contractible, thus simply connected. The intersection of those open sets would look like a disk, which is contractile. Therefore, the van kampen's theorem implies that the covering space is simply connected as required.

□

## 7.5 Problem 5

*Problem Statement.* If $B$ is a nice space and $U$ its universal cover, show that $U$ is a covering of every connected covering $X$ of $B$.



we will describe the universal covering space $U$ of the space $B$, which is the union of a 2-dimensional sphere and one of its diameters lines.

$B$

$S_2$

$x_0$ $L_1$ $x_1$

where $S_2$ is a 2-sphere,
$L_1$ is the diameter line,
$x_0$, $x_1$ are endpoints of the line $L_1$.

we can make each sheet homeomorphic to the space $B$.
To take into account the diameter line, we add a line segment, connecting two distinct spheres.
The covering space looks like below

infinite sequence of 2-spheres with the line segments

The line segment is made so that they are homeomorphic to the diameter.
Since each sphere $S^2$ is simply connected, the finite sequence of the spheres are simply connected as well.
Each sphere and line segment is path-connected.
The fact that the union of an infinite sequence of 2-spheres connected by line segments follows from the Van Kampen's theorem.

Figure 5: The answer for problem 4 of homework 4

*Proof.* Let $\pi : U \to B$ be the universal covering map. Also, let $p : X \to B$ be any connected covering of $B$. We show there exists a covering map $q : U \to X$.

Since $U$ is simply connected, $\pi_1(U) = 0$. For the map $\pi : U \to B$, we have:

$$\pi_*(\pi_1(U)) = \pi_*(0) = 0 \subseteq p_*(\pi_1(X)).$$

By the lifting criterion, there exists a lift $q : U \to X$ such that:

$$p \circ q = \pi.$$

Next we will show that $q$ is a covering map. To prove $q$ is a covering map, we prove that the lift $q : U \to X$ is a surjective local homeomorphism with evenly covered neighborhoods.

For surjectivity, we first show that $q$ is a Local Homeomorphism. Since $\pi$ and $p$ are covering maps, they are local homeomorphisms. The lift $q$ inherits this property: For every $u \in U$, there exists an open neighborhood $W \subset U$ such that $q|_W : W \to q(W)$ is a homeomorphism. Hence, $q$ is an open map.

We notice that $q(U)$ is Open in $X$ since

$$q(U) = \bigcup_{u \in U} q(W_u),$$

where each $q(W_u)$ is open in $X$. A union of open sets is open. Thus, $q(U)$ is open in $X$.

We also notice that $q(U)$ is Closed in $X$. In a connected space $X$, the only clopen subsets are $\emptyset$ and $X$. Let $x \in \overline{q(U)}$ (the closure of $q(U)$). Since $X$ is locally path-connected, take a path-connected neighborhood $V \subset X$ of $x$. By the definition of closure, $V \cap q(U) \neq \emptyset$. Pick a point $y \in V \cap q(U)$. Connect $y$ to $x$ by a path $\gamma$ in $V$.

Let $\tilde{y} \in U$ such that $q(\tilde{y}) = y$. Since $U$ is simply connected and $q$ is a local homeomorphism, the path $\gamma$ lifts uniquely to a path $\tilde{\gamma}$ in $U$ starting at $\tilde{y}$. The endpoint $\tilde{x}$ of $\tilde{\gamma}$ satisfies $q(\tilde{x}) = x$. Thus, $x \in q(U)$, proving $\overline{q(U)} \subseteq q(U)$. Hence, $q(U)$ is closed.

Since $X$ is connected and $q(U)$ is non-empty, open, and closed, it must equal $X$. Thus, $q$ is surjective.

For evenly covered neighborhoods, for any $x \in X$, let $V \subseteq B$ be an evenly covered neighborhood of $p(x)$. Then:

$p^{-1}(V) = \bigsqcup_i W_i$, where $W_i \subseteq X$ are open and $p|_{W_i} : W_i \to V$ is a homeomorphism.

$\pi^{-1}(V) = \bigsqcup_j \widetilde{W}_j \subseteq U$, where $\pi|_{\widetilde{W}_j} : \widetilde{W}_j \to V$ is a homeomorphism.

Since $p \circ q = \pi$, the map $q$ restricts to $\widetilde{W}_j \to W_i$, which is a homeomorphism (as both $\pi$ and $p$ are covering maps).

Thus, $q^{-1}(W_i) = \bigsqcup_j \widetilde{W}_j$, showing $W_i$ is evenly covered.

The lift $q : U \to X$ is a surjective local homeomorphism with evenly covered neighborhoods. Therefore, $q$ is a covering map, and $U$ is a covering space of $X$.

$\square$

# 8 MAT1301 Homework 5

## 8.1 Problem 1

*Problem Statement.* For any based space $(X, x_0)$ define a natural non-zero map $\alpha_{(X,x_0)} : \pi_1(X, x_0) \to H_1(X)$. The challenge will be to show that your definition is well-defined.

What means "non-zero"? We don't have the tools yet to prove that $H_1$ is ever non-zero! So I will be happy enough with a map that is non-zero as per our intuitive notion of $H_1(S^1)$.

What means "natural"? That if $f : (X, x_0) \to (Y, y_0)$, then the following diagram is commutative:

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\alpha_{(X,x_0)}"] \arrow[d, "f_*"'] & H_1(X) \arrow[d, "f_*"] \\
\pi_1(Y, y_0) \arrow[r, "\alpha_{(Y,y_0)}"'] & H_1(Y)
\end{tikzcd}$$

(In other words, $\alpha$ should be a "natural transformation".)

*Proof.* We recall that $\pi_1(X, x_0)$ consists of homotopy classes of loops $\gamma : [0, 1] \to X$ based at $x_0$, with group operation $\gamma \cdot \eta = \gamma$ followed by $\eta$ (concetenation). The first homology group $H_1(X)$ is defined as:

$$H_1(X) = \frac{\text{1-cycles}}{\text{1-boundaries}}.$$

where a 1-cycle is a formal sum of paths $\sum n_i \sigma_i$ with $\partial(\sum n_i \sigma_i) = 0$.

We will define a homomorphism $\alpha : \pi_1(X, x_0) \to H_1(X)$. For a loop $\gamma \in \pi_1(X, x_0)$, define:

$$\alpha_{(X,x_0)}([\gamma]) = \text{homology class of } \gamma \text{ in } H_1(X).$$

Since a loop $\gamma$ is a 1-cycle ($\partial \gamma = 0$), so it represents a class in $H_1(X)$.

We will show that $\alpha$ is well defined. The proof is actually available in the chapter 2.A of Hatcher, but I will summarize it here.

We start by recalling that $f \simeq g$ denotes a homotopy (fixing endpoints), and $f \sim g$ means $f - g$ is the boundary of some 2-chain.

(i) If $f$ is a constant path, then $H_1(\text{point}) = 0$, so it is a boundary. Consider a constant 2-simplex $\sigma$ with the same image as $f$, then:

$$\partial \sigma = \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]} = f - f + f = f.$$

Next we will show that if $f \simeq g$, then $f \sim g$. A homotopy $F : I \times I \to X$ from $f$ to $g$ subdivides the square into 2-simplices $\sigma_1$ and $\sigma_2$. The boundary

$\partial(\sigma_1 - \sigma_2)$ cancels along the diagonal, leaving $f - g$ plus two constant paths (since endpoints are fixed), hence $f - g$ is a boundary. This shows that the mapping from a loop to its homology class is well defined.

Finally, we show that $f \cdot g \sim f + g$, where $f \cdot g$ is the path product. Let $\sigma : \Delta^2 \to X$ be the composition of the projection of $\Delta^2 = [v_0, v_1, v_2]$ onto $[v_0, v_2]$, followed by $f \cdot g$. Then

$$\partial\sigma = g - f \cdot g + f.$$

Rearranging gives $f \cdot g \sim f + g$. This shows that the mapping from a loop to its homology class is a homomorphism since it preserves group operations.

The theorem or proposition in the chapter 2.A in Hatcher further asserts that $\alpha$ is the abelianization map:

$$\alpha : \pi_1(X, x_0) \to \pi_1(X, x_0)^{\mathrm{ab}} = \frac{\pi_1(X, x_0)}{[\pi_1, \pi_1]},$$

where $[\pi_1, \pi_1]$ is the commutator subgroup.

For a continuous map $f : (X, x_0) \to (Y, y_0)$, the diagram:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\alpha_{(X,x_0)}} & H_1(X) \\ \downarrow f_* & & \downarrow f_* \\ \pi_1(Y, y_0) & \xrightarrow{\alpha_{(Y,y_0)}} & H_1(Y) \end{array}$$

commutes because:

$$f_*([\gamma]) = [f \circ \gamma] \text{ in } \pi_1,$$
$$f_*(\alpha([\gamma])) = f_*([\gamma]) = [f \circ \gamma] \text{ in } H_1,$$
$$\alpha(f_*([\gamma])) = \alpha([f \circ \gamma]) = [f \circ \gamma] \text{ in } H_1.$$

For $X = S^1$, $\pi_1(S^1) \cong \mathbb{Z}, \quad H_1(S^1) \cong \mathbb{Z}$, $\alpha$ is an isomorphism: $\alpha(1) = 1 \neq 0$. For spaces with non-trivial $\pi_1$, $\alpha$ is non-zero on loops that survive abelianization.

We will argue that without loss of generality, we may assume that $X$ is path-connected. Therefore, we may apply the theorem in the chapter 2.A of Hatcher is directly applicable in our situation. If $X = \bigsqcup_i X_i$ (disjoint union of path components), we have the following properties:

$$\pi_1(X, x_0) = \pi_1(X_i, x_0) \quad \text{where } x_0 \in X_i.$$
$$H_1(X) = \bigoplus_i H_1(X_i).$$

Here is how $\alpha$ works. We know that $\alpha : \pi_1(X_i, x_0) \to H_1(X_i)$ for the component $X_i$ containing $x_0$. On other path components, $\pi_1(X_j)$ is irrelevant (since $x_0 \notin X_j$), but $H_1(X_j)$ still contributes to $H_1(X)$.

Therefore, if $X$ is path-connected already, $\alpha$ is surjective with kernel $[\pi_1, \pi_1]$, so $H_1(X) \cong \pi_1^{\text{ab}}$. if $X$ is not path-connected, then we have:

$$H_1(X) = \bigoplus_i H_1(X_i), \quad \text{where } X_i \text{ are path components.}$$

and $\alpha$ maps $\pi_1(X, x_0)$ to $H_1(X_0)$.

Therefore, the map $\alpha_{(X,x_0)} \colon \pi_1(X, x_0) \to H_1(X)$ sends a loop to its homology class, and it is natural, nonzero and well defined as the abelianization of $\pi_1$.

□

## 8.2 Problem 2

*Problem Statement.* Show that the set

$$\Delta'_n := \{(s_1, s_2, \ldots, s_n) : 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq 1\} \subset I^n \subset \mathbb{R}^n$$

is homeomorphic to the standard $n$ simplex $\Delta_n$ via a map of the form $[v_0, \ldots, v_n]$ : $\Delta_n \to \Delta'_n$, where $v_0, \ldots, v_n \in I^n$.

*Proof.* To show that the set

$$\Delta'_n = \{(s_1, \ldots, s_n) : 0 \leq s_1 \leq \cdots \leq s_n \leq 1\}$$

is homeomorphic to the standard $n$-simplex $\Delta_n$, we construct a map using vertices in $I^n$. The vertices of $\Delta'_n$ are points where coordinates transition from 0 to 1 in order. For example, in $\Delta'_2$, the vertices are $(0,0)$, $(0,1)$, and $(1,1)$. For $\Delta'_n$, there are $n+1$ vertices $v_0, \ldots, v_n$, where $v_k = (0, \ldots, 0, 1, \ldots, 1)$ with $k$ trailing 1s.

We will construct the Linear Map $f : \Delta_n \to \Delta'_n$. The standard $n$-simplex $\Delta_n$ is the convex hull of the basis vectors $e_0, e_1, \ldots, e_n$ in $\mathbb{R}^{n+1}$. Define $f$ by mapping vertices:

$$f(e_k) = v_k \quad \text{for } k = 0, 1, \ldots, n.$$

For any point $(t_0, t_1, \ldots, t_n) \in \Delta_n$ where $t_i \geq 0$ and $\sum_{i=0}^n t_i = 1$:

$$f(t_0, t_1, \ldots, t_n) = \sum_{i=0}^{n} t_i v_i \in \Delta'_n.$$

Then, the $k$-th coordinate of $f(t_0, \ldots, t_n)$ is:

$$s_k = t_k + t_{k+1} + \cdots + t_n, \quad \text{ensuring } 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq 1.$$

We will construct the inverse map $g : \Delta'_n \to \Delta_n$. For $(s_1, s_2, \ldots, s_n) \in \Delta'_n$, define $t_i$ as:

$$t_0 = 1 - s_n, \quad t_1 = s_n - s_{n-1}, \quad t_2 = s_{n-1} - s_{n-2}, \quad \ldots, \quad t_n = s_1.$$

This construction is valid: since $s_1 \leq s_2 \leq \cdots \leq s_n \leq 1$, all $t_i \geq 0$, and:

$$t_0 + t_1 + \cdots + t_n = (1 - s_n) + (s_n - s_{n-1}) + \cdots + s_1 = 1.$$

Both $f$ and $g$ are linear (affine) transformations. In finite-dimensional spaces, linear maps are continuous. For bijectivity, we have the following.

Injectivity: If $f(t) = f(t'),$ then $t_i = t'_i$ for all $i$ by the uniqueness of the inverse map.

and

Surjectivity: Every $(s_1, \ldots, s_n) \in \Delta'_n$ is mapped to by some $(t_0, \ldots, t_n) \in \Delta_n$.

Since $f$ and $g$ are continuous, linear, and inverse to each other, $\Delta_n$ and $\Delta'_n$ are homeomorphic.

□

## 8.3 Problem 3

*Problem Statement.* Using the alternative model of the $n$-simplex presented in the previous problem, show that

1. The $n$-cube $I^n$ can be presented as a union of size $n!$ of $n$-simplices.
2. The product $\Delta_p \times \Delta_q$ can be presented as a union of size $\binom{p+q}{q}$ of $(p+q)$-simplices.

Here is the proof for the first part.

*Proof.* We note that $I^n = [0,1]^n$ is the $n$-dimensional unit cube, consisting of all points $(x_1, x_2, \ldots, x_n)$ where each $x_i \in [0,1]$.

A permutation $\sigma \in S_n$ rearranges the coordinates $(x_1, \ldots, x_n)$.
For each permutation $\sigma$, define a region in $I^n$:

$$\Delta_\sigma = \left\{(x_1, \ldots, x_n) \in I^n : x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}\right\}.$$

This region is isomorphic to $\Delta'_n$ from Problem 2, which is homeomorphic to the standard $n$-simplex $\Delta_n$.

There are $n!$ permutations of $n$ coordinates. Each permutation defines a distinct ordering of coordinates, corresponding to a distinct simplex $\Delta_\sigma$.

Every point in $I^n$ lies in at least one $\Delta_\sigma$. Hence we have $I_n = \bigcup_{\sigma \in S_n} \Delta_\sigma$ where the interior of each $\Delta_\sigma$ is disjoint and $|S_n| = n!$. Thus, simplices overlap only on boundaries where coordinates are equal. Each $\Delta_\sigma$ is an $n$-Simplex from problem 2.

□

Here is the proof for the second part.

*Proof.* We will show that the product $\Delta_p \times \Delta_q$ as a Union of $\binom{p+q}{q}$ $(p+q)$-Simplices.

$\Delta_p \times \Delta_q$ is the set of pairs $(s,t)$, where $s \in \Delta'_p$ (coordinates $0 \leq s_1 \leq \cdots \leq s_p \leq 1$) and $t \in \Delta'_q$ (coordinates $0 \leq t_1 \leq \cdots \leq t_q \leq 1$). This product lies in $I^{p+q}$.

Shuffling interleaves the coordinates of $s$ and $t$. The number of ways to interleave $p$ ordered coordinates (from $\Delta'_p$) and $q$ ordered coordinates (from $\Delta'_q$) is the binomial coefficient $\binom{p+q}{q}$. This is because we may choose $q$ positions for the $t$-coordinates in the combined sequence of $p+q$ coordinates, while preserving order.

As a product, each shuffle corresponds to a $(p+q)$-simplex $\Delta'_{p+q} \subset I^{p+q}$. Just like the previous problem, every point $(s,t) \in \Delta'_p \times \Delta'_q$ lies in exactly one simplex up to boundaries.

$$\Delta_p \times \Delta_q = \bigcup_{\text{shuffles}} \Delta_{p+q}, \quad \text{with } \binom{p+q}{q} \text{ simplices.}$$

□

## 8.4 Problem 4

*Problem Statement.* "Homotopies between maps" define an "ideal" within the category of topological spaces and continuous maps between them: the homotopy relation is an equivalence relation, and if $f_1 \sim f_2$, then $f_1 \circ g \sim f_2 \circ g$ and $g \circ f_1 \sim g \circ f_2$ whenever these compositions make sense. Show that the same is true for the notion "homotopy of morphisms between chain complexes", within the category Kom of chain complexes.

*Proof.* I will review the important definitions first before I proceed.

A *chain complex* $(C_\bullet, \partial_\bullet)$ is a sequence of abelian groups $C_n$ and homomorphisms $\partial_n : C_n \to C_{n-1}$, called *boundary maps*, such that:

$$\partial_n \circ \partial_{n+1} = 0 \quad \text{for all } n.$$

This implies the image of $\partial_{n+1}$ must be contained in the kernel of $\partial_n$.

A *chain map* $f : C_\bullet \to D_\bullet$ between two chain complexes is a collection of homomorphisms $f_n : C_n \to D_n$ that commute with boundary maps:

$$f_{n-1} \circ \partial_n^{(C)} = \partial_n^{(D)} \circ f_n \quad \text{for all } n.$$

Visually we have the following commutative diagram:

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ \downarrow \partial_n^{(C)} & & \downarrow \partial_n^{(D)} \\ C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} \end{array}$$

Two chain maps $f, g : C_\bullet \to D_\bullet$ are *homotopic* (which we denote by $f \sim g$) if there exists a collection of homomorphisms $h_n : C_n \to D_n$ such that:

$$f_n - g_n = \partial_n^{(D)} \circ h_n + h_{n-1} \circ \partial_n^{(C)}.$$

This equation means that the difference $f - g$ is "trivial up to boundaries".

We will show that homotopy is an equivalence relation.
For reflexivity, for any chain map $f$, take $h_n = 0$. Then:

$$f_n - f_n = 0 = \partial h_n + h_{n-1}\partial.$$

Thus, $f \sim f$.

For symmetry, if $f \sim g$ via homotopy $h_n$, then $g \sim f$ via homotopy $-h_n$:

$$g_n - f_n = -(\partial h_n + h_{n-1}\partial) = \partial(-h_n) + (-h_{n-1})\partial.$$

For transitivity, if $f \sim g$ via $h_n$ and $g \sim k$ via $h'_n$, then $f \sim k$ via $h_n + h'_n$:

$$f_n - k_n = (f_n - g_n) + (g_n - k_n) = \partial(h_n + h'_n) + (h_{n-1} + h'_{n-1})\partial.$$

Lastly, we will check the compatibility with composition from both sides.

*The first case:*
Let $f_1 \sim f_2 : C_\bullet \to D_\bullet$ via homotopy $h_n$, and let $g : B_\bullet \to C_\bullet$ be a chain map. Then:

$$f_1 \circ g - f_2 \circ g = (f_1 - f_2) \circ g = (\partial h + h\partial) \circ g = \partial(h \circ g) + h \circ (\partial g).$$

Since $g$ is a chain map ($\partial g = g\partial$), we can simplify this to:

$$\partial(h \circ g) + (h \circ g)\partial.$$

Thus, $f_1 \circ g \sim f_2 \circ g$.

*The second case:*
Let $f_1 \sim f_2 : C_\bullet \to D_\bullet$ via homotopy $h_n$, and let $g : D_\bullet \to E_\bullet$ be a chain map. Then:

$$g \circ f_1 - g \circ f_2 = g \circ (f_1 - f_2) = g \circ (\partial h + h\partial) = \partial(g \circ h) + (g \circ h)\partial.$$

Thus, $g \circ f_1 \sim g \circ f_2$.

Therefore, in the category **Kom** of chain complexes, homotopy is an equivalence relation and the composition properties hold below:

$$f_1 \sim f_2 \Longrightarrow f_1 \circ g \sim f_2 \circ g \quad \text{(precomposition)},$$
$$f_1 \sim f_2 \Longrightarrow g \circ f_1 \sim g \circ f_2 \quad \text{(postcomposition)}.$$

□

# 9 MAT1301 Homework 6

## 9.1 Problem 1

*Problem Statement.* Just in case you thought commutative diagrams cannot get any worse, here's a question to prove you wrong (though actually, it is surprisingly easy).

1. Define a category $S$ of "short exact sequences of chain complexes".
2. Define a category $L$ of "long exact sequences".
3. Construct a functor $\mathcal{H} : S \to L$. (We've constructed this functor on objects already, so you don't need to do that again. The challenge is to do it on morphisms and to verify that starting from a morphism $\mathcal{F}$ in $S$, its image $\mathcal{H}(\mathcal{F})$ is indeed a morphism in $L$.)

The proof for part 1:

*Proof.* We will define the Category $\mathcal{S}$ of Short Exact Sequences of Chain Complexes.

*Objects:* A short exact sequence of chain complexes is a diagram:

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

where $A_\bullet, B_\bullet, C_\bullet$ are chain complexes, and for each $n$,

$$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$$

is exact (i.e., $\mathrm{Im}(f_n) = \ker(g_n)$).

*Morphisms:* A morphism between two objects $\mathcal{E}$ and $\mathcal{E}'$ in $\mathcal{S}$ is a triple of chain maps:

$$(\alpha : A_\bullet \to A'_\bullet, \quad \beta : B_\bullet \to B'_\bullet, \quad \gamma : C_\bullet \to C'_\bullet)$$

such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "f"] \arrow[d, "\alpha"] & B_\bullet \arrow[r, "g"] \arrow[d, "\beta"] & C_\bullet \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A'_\bullet \arrow[r, "f'"] & B'_\bullet \arrow[r, "g'"] & C'_\bullet \arrow[r] & 0
\end{tikzcd}$$

Next, we will show that this defines a category structure.

We show the closure under the composition operation. For morphisms:

$$(\alpha : A_\bullet \to A'_\bullet, \beta : B_\bullet \to B'_\bullet, \gamma : C_\bullet \to C'_\bullet)$$

and

$$(\alpha' : A'_\bullet \to A''_\bullet, \beta' : B'_\bullet \to B''_\bullet, \gamma' : C'_\bullet \to C''_\bullet),$$

their composition is:

$$(\alpha' \circ \alpha, \beta' \circ \beta, \gamma' \circ \gamma).$$

We show that the diagram extended by the compositions is still commutative. For the left squares, we have:

$$f'' \circ (\alpha' \circ \alpha) = (\beta' \circ f') \circ \alpha = \beta' \circ (f' \circ \alpha) = \beta' \circ (\beta \circ f) = (\beta' \circ \beta) \circ f.$$

For the right squares, we have:

$$g'' \circ (\beta' \circ \beta) = (\gamma' \circ g') \circ \beta = \gamma' \circ (g' \circ \beta) = \gamma' \circ (\gamma \circ g) = (\gamma' \circ \gamma) \circ g.$$

Thus, the composition preserves commutativity of the diagram.

Next, we will prove the associativity of composition. We notice that chain map composition is associative:

$$(\alpha'' \circ (\alpha' \circ \alpha), \beta'' \circ (\beta' \circ \beta), \gamma'' \circ (\gamma' \circ \gamma)) = ((\alpha'' \circ \alpha') \circ \alpha, (\beta'' \circ \beta') \circ \beta, (\gamma'' \circ \gamma') \circ \gamma).$$

Hence, associativity holds.

For each object $\mathcal{E}$, the identity morphism is given by

$$(\mathrm{id}_{A_\bullet}, \mathrm{id}_{B_\bullet}, \mathrm{id}_{C_\bullet}).$$

We clearly have:

$$(\alpha, \beta, \gamma) \circ (\mathrm{id}_{A_\bullet}, \mathrm{id}_{B_\bullet}, \mathrm{id}_{C_\bullet}) = (\alpha, \beta, \gamma),$$
$$(\mathrm{id}_{A'_\bullet}, \mathrm{id}_{B'_\bullet}, \mathrm{id}_{C'_\bullet}) \circ (\alpha, \beta, \gamma) = (\alpha, \beta, \gamma).$$

□

The proof for part 2:

*Proof.* We will define the category $\mathcal{L}$ of Long Exact Sequences as follows.

*Objects:* A long exact sequence is an infinite sequence of abelian groups (or modules) connected by homomorphisms:

$$\cdots \to H_n \xrightarrow{\phi_n} H_{n-1} \xrightarrow{\phi_{n-1}} H_{n-2} \to \cdots$$

where $\mathrm{Im}(\phi_n) = \ker(\phi_{n-1})$ for all $n$.

*Morphisms:* A morphism between two objects $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{L}$ is a family of homomorphisms:

$$\{h_n : H_n \to H'_n\}$$

such that the following diagram commutes for all $n$:

$$\begin{array}{ccc} H_n & \xrightarrow{\phi_n} & H_{n-1} \\ \downarrow h_n & & \downarrow h_{n-1} \\ H'_n & \xrightarrow{\phi'_n} & H'_{n-1} \end{array}$$

Next, we will show that this defines a category structure.

We will show that it is closed under composition. Given morphisms:

$$\{h_n : H_n \to H'_n\} \quad \text{and} \quad \{h'_n : H'_n \to H''_n\},$$

their composition $\{h''_n = h'_n \circ h_n : H_n \to H''_n\}$ satisfies:

$$\phi''_n \circ h''_n = \phi''_n \circ h'_n \circ h_n = h'_{n-1} \circ \phi'_n \circ h_n = h'_{n-1} \circ h_{n-1} \circ \phi_n = h''_{n-1} \circ \phi_n.$$

Thus, composition preserves commutativity.

We will show the associativity of the morphisms. Composition is associative because homomorphism composition is associative:

$$(h'''_n \circ (h''_n \circ h'_n)) = ((h'''_n \circ h''_n) \circ h'_n).$$

For each object $\mathcal{L}$, the identity morphism is $\{\mathrm{id}_{H_n} : H_n \to H_n\}$. It commutes because:

$$\phi_n \circ \mathrm{id}_{H_n} = \mathrm{id}_{H_{n-1}} \circ \phi_n.$$

$\square$

The proof for part 3:

*Proof.* Construction of the Functor $\mathcal{H}$ is as follows.

**On Objects:** Given a short exact sequence of chain complexes in $S$:

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0,$$

apply the *snake lemma* to obtain the long exact sequence in homology:

$$\cdots \to H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \to \cdots$$

Here, $\delta$ is the connecting homomorphism.

**On Morphisms:** Given a morphism $(\alpha, \beta, \gamma) : \mathcal{E} \to \mathcal{E}'$ in $S$, define:

$$\mathcal{H}(\alpha, \beta, \gamma) = \{\alpha_*, \beta_*, \gamma_*\},$$

where $\alpha_* : H_n(A) \to H_n(A')$, $\beta_* : H_n(B) \to H_n(B')$, and $\gamma_* : H_n(C) \to H_n(C')$ are the induced maps on homology.

We will check the functorial properties. First, we check if it preserves identity morphisms. For the identity morphism $(\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C)$ in $S$:

$$\mathcal{H}(\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C) = \{\mathrm{id}_{H_n(A)}, \mathrm{id}_{H_n(B)}, \mathrm{id}_{H_n(C)}\},$$

which is the identity morphism in $\mathcal{L}$.

We will check if it preserves composition. Given morphisms $(\alpha, \beta, \gamma) : \mathcal{E} \to \mathcal{E}'$ and $(\alpha', \beta', \gamma') : \mathcal{E}' \to \mathcal{E}''$ in $S$, their composition is:

$$(\alpha' \circ \alpha, \beta' \circ \beta, \gamma' \circ \gamma).$$

Applying $\mathcal{H}$:

$$\mathcal{H}(\alpha' \circ \alpha, \beta' \circ \beta, \gamma' \circ \gamma) = \{\alpha'_* \circ \alpha_*, \beta'_* \circ \beta_*, \gamma'_* \circ \gamma_*\}.$$

By functoriality of homology:

$$\alpha'_* \circ \alpha_* = (\alpha' \circ \alpha)_*, \quad \beta'_* \circ \beta_* = (\beta' \circ \beta)_*, \quad \gamma'_* \circ \gamma_* = (\gamma' \circ \gamma)_*,$$

so we have:

$$\mathcal{H}((\alpha', \beta', \gamma') \circ (\alpha, \beta, \gamma)) = \mathcal{H}(\alpha', \beta', \gamma') \circ \mathcal{H}(\alpha, \beta, \gamma).$$

Regarding the naturality, for the square involving $f_*$, $g_*$, by the naturality of homology functors,

$$\begin{array}{ccc} H_n(A) & \xrightarrow{f_*} & H_n(B) \\ \alpha_* \downarrow & & \downarrow \beta_* \\ H_n(A') & \xrightarrow{f'_*} & H_n(B') \end{array} \quad \text{and} \quad \begin{array}{ccc} H_n(B) & \xrightarrow{g_*} & H_n(C) \\ \beta_* \downarrow & & \downarrow \gamma_* \\ H_n(B') & \xrightarrow{g'_*} & H_n(C') \end{array}$$

commute. This follows from the original chain maps commuting with $f$, $g$.

For the square involving $\delta$,

$$\begin{array}{ccc} H_n(C) & \xrightarrow{\delta} & H_{n-1}(A) \\ \gamma_* \downarrow & & \downarrow \alpha_* \\ H_n(C') & \xrightarrow{\delta'} & H_{n-1}(A') \end{array}$$

Let $[c] \in H_n(C)$. Represent $c$ as $c = g(b)$ for $b \in B_n$.
Then:

$$\delta([c]) = [\partial b] \in H_{n-1}(A).$$

Apply $\gamma_*$:

$$\gamma_*([c]) = [g'(\beta(b))] \in H_n(C').$$

Apply $\delta'$: Lift $g'(\beta(b))$ to $\beta(b) \in B'_n$, so:

$$\delta'([g'(\beta(b))]) = [\partial\beta(b)] \in H_{n-1}(A').$$

By commutativity of $\beta \circ f = f' \circ \alpha$, $\partial\beta(b) = \beta(\partial b)$.
Since $f'$ is injective,

$$[\partial\beta(b)] = [\alpha(\partial b)] = \alpha_*([\partial b]).$$

Thus,

$$\delta' \circ \gamma_*([c]) = \alpha_* \circ \delta([c]).$$

$\square$

## 9.2 Problem 2

*Problem Statement.* Given a morphism $f : (X, A) \to (Y, B)$, show that the diagram

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{\delta} & H_{n-1}(A) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ H_n(Y,B) & \xrightarrow{\delta} & H_{n-1}(B) \end{array}$$

is commutative.

*Proof.* To prove commutativity, we verify that both paths through the diagram agree on an arbitrary homology class $[\sigma] \in H_n(X, A)$. We are given that $f$ is a morphism of pairs, i.e., $f(A) \subset B$.

Let $[\sigma] \in H_n(X, A)$ be represented by a relative cycle $\sigma \in C_n(X)$ with $\partial\sigma \in C_{n-1}(A)$. The *boundary map* $\delta$ is defined by

$$\delta([\sigma]) = [\partial\sigma] \in H_{n-1}(A).$$

For the *induced chain map* $f_\#$, since $f(A) \subset B$, $f_\# : C_n(X) \to C_n(Y)$ satisfies $f_\#(C_n(A)) \subset C_n(B)$. This allows $f_\#$ to descend to a map on relative chains:

$$f_\# : C_n(X, A) = \frac{C_n(X)}{C_n(A)} \longrightarrow \frac{C_n(Y)}{C_n(B)} = C_n(Y, B).$$

The *induced homomorphism* $f_*$ is defined on homology by:

$$f_*([\sigma]) = [f_\#(\sigma)] \in H_n(Y, B).$$

This is well-defined because $\partial f_\#(\sigma) = f_\#(\partial\sigma) \in C_{n-1}(B)$, as $\partial\sigma \in C_{n-1}(A)$ and $f(A) \subset B$.

Next, we compute both paths in the diagram. For the first path right then down, we have:

$$\delta(f_*([\sigma])) = \delta([f_\#(\sigma)]) = [\partial f_\#(\sigma)] \in H_{n-1}(B).$$

For the second path down then right, we have:

$$f_*(\delta([\sigma])) = f_*([\partial\sigma]) = [f_\#(\partial\sigma)] \in H_{n-1}(B).$$

Since $f_\#$ is a chain map, we have:

$$\partial \circ f_\# = f_\# \circ \partial.$$

Thus:

$$\partial f_\#(\sigma) = f_\#(\partial\sigma).$$

Mapping to the same homology classes:

$$[\partial f_\#(\sigma)] = [f_\#(\partial\sigma)] \in H_{n-1}(B).$$

Therefore, we have:

$$\delta \circ f_* = f_* \circ \delta.$$

Hence, the diagram commutes.

□

## 9.3 Problem 3

*Problem Statement.* Given a triple of spaces $B \subset A \subset X$, construct a long exact sequence relating $H_*(X,A)$, $H_*(A,B)$, and $H_*(X,B)$.

*Proof.* We start by defining relative chain complexes. For a pair $(X, A)$, the relative chain group $C_n(X, A)$ is defined as the quotient $C_n(X)/C_n(A)$. Similarly, $C_n(A,B) = C_n(A)/C_n(B), \quad C_n(X,B) = C_n(X)/C_n(B)$. These are valid chain complexes because $\partial^2 = 0$ holds before passing to the quotient groups. Hence, we obtain maps

$$\partial : C_n(X,A) \to C_{n-1}(X,A), \quad \text{etc., with } \partial^2 = 0.$$

We will construct a short exact sequence of chain complexes. We claim that the following is a short exact sequence of chain complexes:

$$0 \longrightarrow C_\bullet(A,B) \xrightarrow{\iota} C_\bullet(X,B) \xrightarrow{q} C_\bullet(X,A) \longrightarrow 0$$

We consider the inclusion map $\iota : C_n(A,B) \hookrightarrow C_n(X,B)$. Since $A \subset X$, every singular chain in $C_n(A)$ naturally sits inside $C_n(X)$. Thus, passing to quotients by $C_n(B)$ yields:

$$\iota : C_n(A)/C_n(B) \hookrightarrow C_n(X)/C_n(B)$$

This is injective because distinct equivalence classes in $C_n(A,B)$ remain distinct in $C_n(X,B)$ by construction.

$q : C_n(X,B) \twoheadrightarrow C_n(X,A)$ is a quotient map. We define $q$ via:

$$q([\sigma + C_n(B)]) = [\sigma + C_n(A)]$$

This is well-defined because $C_n(B) \subset C_n(A)$. This is a group homomorphism as it's linear on formal sums of simplices, and it is surjective because every element of $C_n(X,A) = C_n(X)/C_n(A)$ has a representative $\sigma \in C_n(X)$, and $[\sigma + C_n(B)] \in C_n(X,B)$ maps to it. Hence, $q$ is a surjective homomorphism, so a quotient map. Furthermore, we apply the Third Isomorphism Theorem for Abelian Groups:

$$\frac{C_n(X)/C_n(B)}{C_n(A)/C_n(B)} \cong C_n(X)/C_n(A) \Rightarrow \frac{C_n(X,B)}{C_n(A,B)} \cong C_n(X,A)$$

We notice that at $C_n(A,B)$, injectivity of $\iota$ implies exactness at $0 \to C_n(A,B)$. At $C_n(X,A)$, surjectivity of $q$ implies exactness at the final term. At $C_n(X,B)$, We have $\mathrm{Im}(\iota) = \ker(q)$. To see this, observe that $\ker(q)$ consists of those $[\sigma + C_n(B)] \in C_n(X,B)$ such that $[\sigma] \in C_n(X,A)$ is trivial, i.e., $\sigma \in C_n(A)$. But also $[\sigma] \in C_n(X,B)$ is coming from an element in $C_n(A)/C_n(B) = C_n(A,B)$ via the inclusion, thus the image of $\iota$ consists of $\sigma \in C_n(A)$. Hence, the kernel of $q$ is precisely the image of $\iota$. Therefore,

$$0 \longrightarrow C_\bullet(A,B) \xrightarrow{\iota} C_\bullet(X,B) \xrightarrow{q} C_\bullet(X,A) \longrightarrow 0$$

is a short exact sequence of chain complexes as desired.

We now visualize the short exact sequence of chain complexes as follows:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 \arrow[r] & C_{n+1}(A,B) \arrow[r, "i"] \arrow[d, "\partial"'] & C_{n+1}(X,B) \arrow[r, "q"] \arrow[d, "\partial"'] & C_{n+1}(X,A) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_n(A,B) \arrow[r, "i"] \arrow[d, "\partial"'] & C_n(X,B) \arrow[r, "q"] \arrow[d, "\partial"'] & C_n(X,A) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_{n-1}(A,B) \arrow[r, "i"] & C_{n-1}(X,B) \arrow[r, "q"] & C_{n-1}(X,A) \arrow[r] & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Columns are chain complexes, and each row is a short exact sequence.

Now we apply the theorem 2.16 in Hatcher or Snake Lemma. A short exact sequence of chain complexes induces a long exact sequence in homology:

$$\cdots \to H_n(A,B) \xrightarrow{i_*} H_n(X,B) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to \cdots$$

where for $[\sigma] \in H_n(X,A)$, the boundary map sends

$$[\sigma] \mapsto [\partial\sigma] \in H_{n-1}(A,B)$$

We obtained the long exact sequence, and this gives the desired relationship among the homology groups of the triple $B \subset A \subset X$.

$\square$

### 9.4 Problem 4

*Problem Statement.* Khovanov homology is an invariant of knots obtained by first defining a chain complex $\mathcal{C}$, and then taking its homology. To show that Khovanov homology does not change when the knot moves, one has to show that pieces of $\mathcal{C}$ can be cancelled off — be removed without changing the homology.

1. Suppose $\mathcal{C}'$ is an acyclic subcomplex of $\mathcal{C}$ ("acyclic" is a different word for "exact", or "having no homology"). Show that $\mathcal{C}'$ can be cancelled off. Namely, that the quotient $\mathcal{C}/\mathcal{C}'$ has the same homology as the original complex $\mathcal{C}$.
2. Suppose $\mathcal{C}'$ is a subcomplex of $\mathcal{C}$, and that the quotient complex $\mathcal{C}/\mathcal{C}'$ is acyclic. Then $\mathcal{C}'$ has the same homology as $\mathcal{C}$ (roughly, the complement of $\mathcal{C}'$ can be cancelled off).
3. There is a third theorem along these lines. What does it say?

The proof for part 1:

*Proof.* We need to show that if $C'$ is an acyclic subcomplex of $C$, show $H_*(C/C') \cong H_*(C)$.

The inclusion $C' \to C$ and quotient map $C \to C/C'$ form a short exact sequence:

$$0 \longrightarrow C_n' \longrightarrow C_n \longrightarrow C/C_n' \longrightarrow 0.$$

To check that this is a short exact sequence, we note that an inclusion is injective and quotient map is surjective. Moreover, it is clear that $\mathrm{Im}(\iota) = \ker(q) = C'$. Thus, it is a short exact sequence.

By theorem 2.16 in Hatcher, this short exact sequence, induces a long exact sequence:

$$\cdots \longrightarrow H_n(C') \longrightarrow H_n(C) \longrightarrow H_n(C/C') \longrightarrow H_{n-1}(C') \longrightarrow \cdots$$

Since $C'$ is acyclic, $H_n(C') = 0$ for all $n$. Substituting into the sequence:

$$0 \longrightarrow H_n(C) \longrightarrow H_n(C/C') \longrightarrow 0.$$

This implies that $\mathrm{Im}(\iota^*) = 0$, hence $\ker(q^*) = 0$, implying that the induced map of the quotient $C \to C/C'$ is injective. Hence, $H_n(C) \cong H_n(C/C')$ for all $n$. Consequently, we conclude quotienting by an acyclic subcomplex preserves homology.

□

The proof for part 2:

*Proof.* As before, we have the short exact sequence of chain complexes:

$$0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{j} C/C' \longrightarrow 0$$

which induces the long exact sequence in homology:

$$\cdots \longrightarrow H_n(C') \xrightarrow{i_*} H_n(C) \xrightarrow{j_*} H_n(C/C') \xrightarrow{\partial} H_{n-1}(C') \longrightarrow \cdots$$

Now, carefully:

Since $C/C'$ is acyclic,

$$H_n(C/C') = 0 \quad \text{for all } n$$

Using this, the long exact sequence becomes:

$$\cdots \longrightarrow H_n(C') \xrightarrow{i_*} H_n(C) \xrightarrow{0} 0 \xrightarrow{\partial} H_{n-1}(C') \xrightarrow{i_*} H_{n-1}(C) \longrightarrow 0 \longrightarrow \cdots$$

Hence, the map $j_*$ is a zero map because $H_n(C/C') = 0$.

Exactness at $H_n(C)$ tells us:

$$\ker(j_*) = \operatorname{im}(i_*)$$

but $j_*$ is the zero map, so:

$$\ker(j_*) = H_n(C)$$

thus:

$$\operatorname{im}(i_*) = H_n(C)$$

Since $\operatorname{im}(i_*)$ is an isomorphism, we get $H_n(C') \cong H_n(C)$ for all $n$.

□

The proof for part 3:

*Proof.* One possible theorem is as follows.

If we have a short exact sequence,

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and any two of $C', C, C''$ are acyclic, then so is the third.

If $C'$ and $C$ are acyclic, then $C''$ is acyclic.

If $C$ and $C''$ are acyclic, then $C'$ is acyclic.

If $C'$ and $C''$ are acyclic, then $C$ is acyclic.

□

## 10 MAT1301 Homework 7

The pictorial details are available in the Figure 6.

### 10.1 Problem 1

*Problem Statement.* On page 177 of your textbook, Hocking and Young display an embedded interval in $\mathbb{R}^3$ whose complement $X$ is not simply connected. I took the liberty of adding a little red circle to the picture, which represents a class in $H_1(X)$. But by a theorem from class, $H_1(X) = 0$, so $\gamma$ must be the boundary of some 2D object $B$ in $X$. Draw it!

If you need scratch paper, I've left multiple paper copies of the above picture in an envelope near my office door (Bahen 6178). Feel free to take some (yet leave some for others).

Not for credit, ponder the following: Everything we did in class was in-principle constructive: the prism construction, barycentric subdivisions, the long exact sequence of a short exact sequence, etc. How exactly did these relatively benign constructions "discover" the relatively sophisticated surface that you must have discovered when you answered this problem?

*Proof.* □

### 10.2 Problem 2

*Problem Statement.* Search your memories and I'm sure you can go back to these times when you were lying in a crib looking up at a baby mobile, a lovely toy such as in the picture on the right. Little did you expect that twenty-something years later baby mobiles will come back to haunt you in an algebraic topology homework assignment.

If $(X_i, x_i)$ are connected based topological spaces for $i = 1, \ldots, n$, we let $BM((X_i, x_i))$ be the topological space obtained by connecting each of the $X_i$'s by a string to some central point $y_0$. In formulas, let $Y$ be a star-shaped tree with centre $y_0$ and leaves $y_1, \ldots, y_n$, and let

$$BM((X_i, x_i)) = (Y \cup X_1 \cup \cdots \cup X_n)/(\forall i, x_i \sim y_i).$$

Using the Mayer-Vietoris sequence and/or whatever else we studied, compute the homology of $BM((X_i, x_i))$ in terms of the homologies of the individual $X_i$'s.

*Proof.* □

Dror Bar-Natan: Classes:
2024-25: MAT 1301 Algebraic Topology:

## Homework Assignment 7

http://drorbn.net/25-1301
Due March 26 at 11:59pm

**Hocking and Young "Topology"**

4–6] KNOTS AND RELATED IMBEDDING PROBLEMS 177

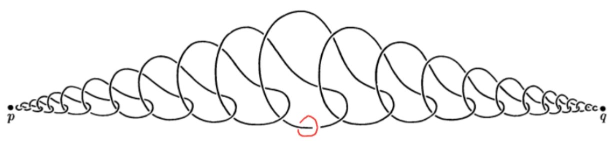


**Fig. 4–12. A simple arc in $E^3$ whose complement is not simply connected.**

**Problem 1.** On page 177 of their topology textbook, Hocking and Young display an embedded interval in $\mathbb{R}^3$ whose complement $X$ is not simply connected. I took the liberty of adding a little red circle to the picture, which represents a class $\gamma$ in $H_1(X)$. But by a theorem from class, $H_1(X) = 0$, so $\gamma$ must be the boundary of some 2D object $\beta$ in $X$. Draw it!

If you need scratch paper, I've left multiple paper copies of the above picture in an envelope near my office door (Bahen 6178). Feel free to take some (yet leave some for others).

Not for credit, ponder the following: Everything we did in class was in-principle constructive: the prizm construction, barycentric subdivisions, the long exact sequence of a short exact sequence, etc. How exactly did these relatively benign constructions "discover" the relatively sophisticated surface that you must have discovered when you answered this problem?

**Problem 2.** Search your memories and I'm sure you can go back to these times when you were lying in a crib looking up at a baby mobile, a lovely toy such as in the picture on the right. Little did you expect that twenty-something years later baby mobiles will come back to haunt you in an algebraic topology homework assignment.

If $(X_i, x_i)$ are connected based topological spaces for $i = 1, \dots, n$, we let $BM((X_i, x_i))$ be the topological space obtained by connecting each of the $X_i$'s by a string to some central point $y_0$. In formulas, let $Y$ be a star-shaped tree with centre $y_0$ and leafs $y_1, \dots, y_n$, and let

$$BM((X_i, x_i)) \coloneqq (Y \sqcup X_1 \sqcup \cdots \sqcup X_n) \,/\, (\forall i \; x_i \sim y_i).$$

https://www.canadiantire.ca/en/pdp/tiny-love-tiny-princess-soothe-n-groove-mobile-2744079p.html

Using the Mayer-Vietoris sequence and/or whatever else we studied, compute the homology of $BM((X_i, x_i))$ in terms of the homologies of the individual $X_i$'s.

**Problem 3.** The suspension $\Sigma X$ of a topological space $X$ is $X$ multiplied by an interval, with the top and the bottom sides crashed into points $S$ and $N$ (that are not in $X$):

$$\Sigma X \coloneqq (X \times [-1, 1] \sqcup \{S, N\}) \,/\, (\forall x \; (x, 1) \sim S, \; (x, -1) \sim N).$$

1. (0 points) Identify the colonial roots of the discomfort you felt regarding the choice of directions, signs, and poles used in this definition.

2. (20 points) Using the same tools as in the previous question, compute the homology of $\Sigma X$ in terms of the homology of $X$.

**Problem 4.**

1. Compute the homology groups of the torus $T^2 = S^1 \times S^1$.

2. (Hatcher's problem 28a on page 157). Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained from a torus $T^2$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

Figure 6: The homework 7 of MAT1301

## 10.3 Problem 3

*Problem Statement.*

The suspension $\Sigma X$ of a topological space $X$ is $X$ multiplied by an interval, with the top and the bottom sides crashed into points $S$ and $N$ (that are not in $X$):

$$\Sigma X = (X \times [-1, 1] \cup \{S, N\})/(\forall x, (x, 1) \sim S, (x, -1) \sim N).$$

1. (0 points) Identify the colonial roots of the discomfort you felt regarding the choice of directions, signs, and poles used in this definition.
2. (20 points) Using the same tools as in the previous question, compute the homology of $\Sigma X$ in terms of the homology of $X$.

*Proof.* □

## 10.4 Problem 4

*Problem Statement.*

1. Compute the homology groups of the torus $T^2 = S^1 \times S^1$.
2. (Hatcher's problem 28a on page 157). Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained from a torus $T^2$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

The proof for part 1:

*Proof.* We recall that the Mayer-Vietoris sequence is a long exact sequence in homology that relates the homology of a space $X$ to two open subsets $U$ and $V$ covering $X$.

$U$ and $V$ are open subsets of $X$ and $X = U \cup V$ or the interiors of $U$ and $V$ cover $X$.

We have a following sequence structure:

$$\cdots \to H_n(U \cap V) \xrightarrow{(i_*, j_*)} H_n(U) \oplus H_n(V) \xrightarrow{k_* - l_*} H_n(X) \xrightarrow{\partial} H_{n-1}(U \cap V) \to \cdots$$

where $i : U \cap V \hookrightarrow U$, $j : U \cap V \hookrightarrow V$: inclusion maps.
$k : U \hookrightarrow X$, $l : V \hookrightarrow X$: inclusion maps.
$\partial$: Boundary map connecting homology groups.

By exactness, The image of one map equals the kernel of the next. The homology of $X$ is derived by by analyzing the maps between $H_n(U)$, $H_n(V)$, $H_n(U \cap V)$, and $H_n(X)$.

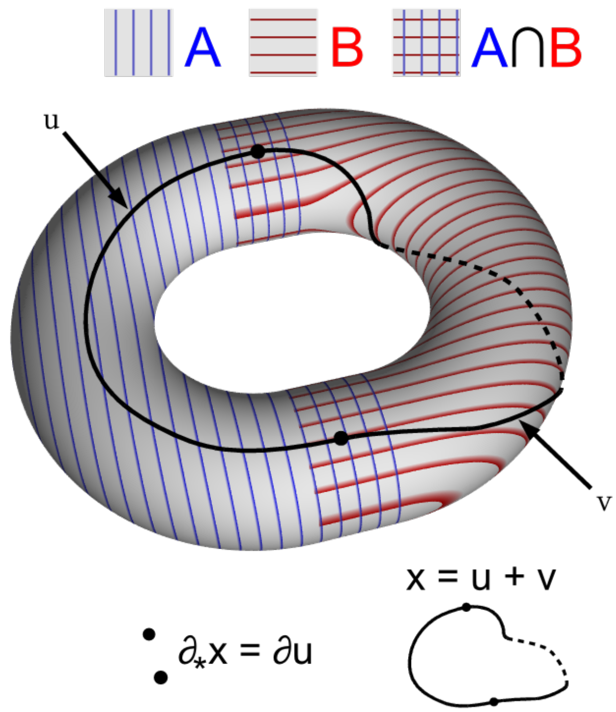



Figure 7: The Open Covering of The Torus From Wikipedia

For the homology of $S^1$, we know from Hatcher that the circle $S^1$ has homology:

$$H_k(S^1) = \begin{cases} \mathbb{Z}, & k = 0, 1, \\ 0, & k \geq 2. \end{cases}$$

$H_0(S^1) = \mathbb{Z}$ since $S^1$ is path-connected. $H_1(S^1) = \mathbb{Z}$ is generated by the loop around the circle. The higher homology groups vanish because $S^1$ is 1-dimensional.

We will compute the homology of the torus $T^2$.

We will decompose $T^2$ into $U$ and $V$. Let $U$ and $V$ be the open sets as in the Figure 7. We let $U$ consist of the vertical bands on the square representation of the torus. Similarly, we let $V$ consist of the horizontal bands on the square representation. We note that $U, V$ are open sets covering roughly a little more than the half of the torus on each side.

We observe that each open set $U, V$ is homotopy equivalent to the cylinder $S^1 \times I$, which can deformation retract to a circle $S^1$. Therefore, we have $U \simeq V \simeq S^1$.

Their intersection $U \cap V$ is homotopy equivalent to two disjoint circles: $U \cap V \simeq S^1 \sqcup S^1$. This is because the part of $U \cap V$ coming from the open set

$U$ (the vertical bands on the square representation) deformation retracts to a corresponding circle, and the part of $U \cap V$ coming from the open set $V$ (the horizontal bands on the square representation) retracts to another circle. Since two circles $S^1$ are disjoint, we have $U \cap V \simeq S^1 \sqcup S^1$.

We sort out the homology of components.

$U$ and $V$:

$$H_k(U) = H_k(V) = \begin{cases} \mathbb{Z}, & k = 0, 1, \\ 0, & k \geq 2. \end{cases}$$

$U \cap V$ (two disjoint circles):

$$H_k(U \cap V) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & k = 0, 1, \\ 0, & k \geq 2. \end{cases}$$

Finally, we will apply the Mayer-Vietoris sequence. For $T^2 = U \cup V$, the sequence becomes:

$$\cdots \to H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(T^2) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(T^2) \to 0$$

There are several cases for us to consider.

*Case $n = 0$:*

The diagonal map $\Delta : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ sends $a \mapsto (a, a)$. In homology, the map $H_0(U \cap V) \to H_0(U) \oplus H_0(V)$ acts similarly:

$$(a, b) \mapsto (a + b, a + b).$$

This is equivalent to embedding $\mathbb{Z} \oplus \mathbb{Z}$ into itself via the diagonal subgroup $\{(k, k) \mid k \in \mathbb{Z}\}$.

The cokernel $(\mathbb{Z} \oplus \mathbb{Z}) / \operatorname{im}(\Delta)$ identifies elements that differ by a diagonal element. This collapses $\mathbb{Z} \oplus \mathbb{Z}$ to a single $\mathbb{Z}$, so:

$$(\mathbb{Z} \oplus \mathbb{Z}) / \operatorname{im}(\Delta) \cong \mathbb{Z}.$$

Hence using exactness and first isomorphism theorem, we get that $H_0(T^2) \cong \mathbb{Z}$.

*Case $n = 1$:*
We have $H_1(U \cap V) = \mathbb{Z} \oplus \mathbb{Z}$, $H_1(U) \oplus H_1(V) = \mathbb{Z} \oplus \mathbb{Z}$, and $H_0(U \cap V) \cong \mathbb{Z} \oplus \mathbb{Z}$. By property of exactness, we obtain $H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$.

*Case $n = 2$:*
First, we examine the kernel of $H_1(U \cap V) \to H_1(U) \oplus H_1(V)$. The map sends $(a, b) \mapsto (a + b, a + b)$. The kernel consists of elements $(a, b)$ such that $a + b = 0$, i.e., $(k, -k)$. These are generated by $(1, -1)$, so the kernel is $\mathbb{Z}$.

By Mayer-Vietoris Sequence:

$$0 \to H_2(T^2) \xrightarrow{\partial} H_1(U \cap V) \to H_1(U) \oplus H_1(V).$$

Since $H_2(U) \oplus H_2(V) = 0$, the boundary map $\partial : H_2(T^2) \to H_1(U \cap V)$ is injective.

The kernel of the map $H_1(U \cap V) \to H_1(U) \oplus H_1(V)$ is $\mathbb{Z}$ (generated by $(1, -1)$).

By exactness, $\partial$ maps $H_2(T^2)$ isomorphically onto this kernel. Hence,

$$H_2(T^2) \cong \mathbb{Z}.$$

*Case $n \geq 3$:*
The torus is a 2-dimensional manifold. Homology groups $H_n(X)$ detect "holes" in dimension $n$, and manifolds have no holes in dimensions exceeding their own. Moreover in the Mayer-Vietoris Sequence, for $n \geq 3$:

$$H_n(U) = H_n(V) = 0 \quad \text{and} \quad H_{n-1}(U \cap V) = 0,$$

Exactness forces $H_n(T^2) = 0$. Hence, for $n \geq 3$, the homology groups of the torus $T^2 = S^1 \times S^1$ vanish:

$$H_n(T^2) = 0 \quad \text{for all } n \geq 3.$$

Hence, we obtained the homology groups of the torus below:

$$H_0(T^2) = \mathbb{Z}, \quad H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}, \quad H_2(T^2) = \mathbb{Z}, \quad H_n(T^2) = 0 \quad \text{for} \quad n \geq 3.$$

□

The proof for part 2:

*Proof.* □

# 11 MAT1301 Homework 8

## 11.1 Problem 1

*Problem Statement.*

1. Formulate and prove a naturality property for the Mayer-Vietoris sequence. Your property must be at least strong enough to answer part 2 of this question.

2. Use part 1 of this question to prove that if $f : S^n \to S^n$ then $\deg(f) = \deg(\Sigma f)$ where $\Sigma$ is the suspension functor, mentioned previously both in class and in HW7.

*Proof.* □

## 11.2 Problem 2

*Problem Statement.* Suppose $n$ is even.

1. Show that for any continuous map $f : S^n \to S^n$ there is a point $x$ such that $f(x) = \pm x$.

2. Show that any continuous map $f : \mathbb{R}P^n \to \mathbb{R}P^n$ has a fixed point.

*Proof.* □

## 11.3 Problem 3

*Problem Statement.*

1. Compute the homology over $\mathbb{Z}$ of a the space $X$ obtained from the 2D disk $D^2$ by identifying each of its boundary points with the point you get from it by applying a 1/3 rotation counterclockwise.

2. Same question, but over $\mathbb{Z}/3$.

*Proof.* □

## 11.4 Problem 4

*Problem Statement.* The statement "all reasonable everyday spaces are at least homotopy equivalent to CW complexes" sounds completely reasonable. At least until you hit the first example where it's hard. Show that the complement $X$ of the trefoil knot in $\mathbb{R}^3$ is homotopy equivalent to a 3-dimensional CW complex.

*Proof.* □