mat257 pset10 spivak 5-14

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1 Spivak 5-14 Problem

If $M \subset \mathbb{R}^n$ is an orientable $(n-1)$-dimensional manifold, show there exists an open set $A \subset \mathbb{R}^n$ and a differentiable function $g : A \to \mathbb{R}$ such that $M = g^{-1}(0)$ and $g'(x)$ has rank 1 for all $x \in M$.

Proof. I will prove one useful lemma first:

Lemma 1. *If $M \subseteq \mathbb{R}^n$ is a k-dimensional manifold and $x \in M$, then there is an open set $A \subseteq \mathbb{R}^n$ containing x and a differentiable function $g : A \to \mathbb{R}^{n-k}$ such that $A \cap M = g^{-1}(0)$ and $g'(y)$ has rank $n-k$ when $g(y) = 0$.*

Proof. Let $M \subseteq \mathbb{R}^n$ be a k-dimensional manifold and $x \in M$. By the definition of a manifold, there exists an open set $U \subseteq \mathbb{R}^n$ containing x, an open set $V \subseteq \mathbb{R}^n$, and a diffeomorphism $h : U \to V$ such that

$$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in V : y^{k+1} = \cdots = y^n = 0\}.$$

Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be the projection map defined by $\pi(x) = (x^{k+1}, \ldots, x^n)$. Define the function $g : U \to \mathbb{R}^{n-k}$ by $g = \pi \circ h$. We claim g satisfies the required conditions.

First, g is differentiable as a composition of differentiable functions; h is a diffeomorphism and a projection π between smooth manifolds $\mathbb{R}^n$ and $\mathbb{R}^{n-k}$ is smooth.

Next, observe that:

$$U \cap M = h^{-1}(V \cap (\mathbb{R}^k \times \{0\})).$$

For $y \in U$, $g(y) = 0$ if and only if $\pi(h(y)) = 0$, which occurs precisely when $h(y) \in \mathbb{R}^k \times \{0\}$. Since h maps $U \cap M$ bijectively to $V \cap (\mathbb{R}^k \times \{0\})$, it follows that:

$$g^{-1}(0) = \{y \in U : h(y) \in \mathbb{R}^k \times \{0\}\} = U \cap M.$$

Finally, consider the derivative $g'(y)$ at a point $y \in U \cap M$. Since $g = \pi \circ h$, the chain rule gives:

$$g'(y) = D\pi(h(y)) \circ Dh(y).$$

The projection π has derivative $D\pi$ of rank $n-k$ (which is full rank), and $Dh(y)$ is invertible because h is a diffeomorphism. The composition of a rank $n-k$ surjective map with an invertible map retains rank $n-k$. Thus, $g'(y)$ has maximal rank $n-k$ whenever $g(y)=0$.

Taking $A=U$, the function g satisfies all desired conditions.

$\square$

Now the proof for the Spivak 5-14 starts here.

For every $x \in M$, by the above lemma, there exists an open neighborhood $U_x \subset \mathbb{R}^n$ containing x and a differentiable function $\tilde{g}_x : U_x \to \mathbb{R}$ such that $M \cap U_x = \tilde{g}_x^{-1}(0)$ and $\tilde{g}_x'(y)$ has rank 1 (i.e., $\nabla \tilde{g}_x(y) \neq 0$) for all $y \in M \cap U_x$.

Since M is $(n-1)$ dimensional, at each $x \in M$, the tangent space T_xM is a $(n-1)$ dimensional vector space. We consider taking its orthogonal complement in each tangent space, which is a unit normal vector $\mathbf{n}(x)$.

Since M is orientable and has a co-dimension of one, we can choose an orientation μ_x aligning with these normal vectors. Equivalently, there exists a continuous unit normal vector field $\mathbf{n} : M \to \mathbb{R}^n$ consistent with the orientation.

It is clear that the gradient $\nabla g_x(y)$ is normal to the tangent space locally, and therefore parallel to $\mathbf{n}(y)$. Since we want to make the orientation imposed by g consistent with the orientation μ_x of M, we define each $\tilde{g}_x$ as follows:

$$g_x = \begin{cases} \tilde{g}_x & \text{if } \nabla \tilde{g}_x \text{ aligns with } \mathbf{n}, \\ -\tilde{g}_x & \text{otherwise.} \end{cases}$$

This makes sure that $\nabla g_x(y)$ points in the direction of $\mathbf{n}(y)$ for all $y \in M \cap U_x$.

Next, we will construct a global function using a partition of unity. Let $A = \bigcup_{x \in M} U_x$ and $\{U_x\}_{x \in M}$ cover M. Let $\{\phi_i\}$ be a smooth partition of unity for A, subordinate to $\{U_{x_i}\}$, where each $\phi_i \geq 0$, $\text{supp}(\phi_i) \subset U_{x_i}$ (compactly supported), and $\sum_i \phi_i = 1$ on A. Also, we smoothly extend g_{x_i} to all of A, for example, by using a C^∞ bump function supported on a large enough open set inside U_{x_i}. For the purpose of this problem, we must make sure this bump function is 1 (or at least positive) at some point $p \in U_{x_i}$ where $g_{x_i}(p) \neq 0$. By using this C^∞ bump function, we may smoothly let $g_{x_i}(x) = 0$ for all $x \in A \backslash U_{x_i}$. Define the global function:

$$g = \sum_i \phi_i \cdot g_{x_i}.$$

Note that $A = \bigcup_{x \in M} U_x$ is the open set on which g is defined. We will check that g satisfies the desired conditions.

We first check that $M = g^{-1}(0)$. Firstly, let $y \in M$. Since the open cover $\{U_{x_i}\}$ contains y, there exists at least one U_{x_j} such that $y \in U_{x_j}$. By construction,

$g_{x_j}(y) = 0$ for all j where $y \in U_{x_j}$. By subordinate property, $\phi_i(y) = 0$ for any i such that $y \notin U_{x_i}$. Thus, we get:

$$g(y) = \sum_i \phi_i(y) \cdot g_{x_i}(y) = \sum_{i, y \in U_{x_i}} \phi_i(y) \cdot 0 = 0.$$

Thus, $y \in g^{-1}(0)$, proving $M \subseteq g^{-1}(0)$.

Conversely, Suppose $y \in g^{-1}(0)$, i.e., $g(y) = \sum_i \phi_i(y) \cdot g_{x_i}(y) = 0$. By construction, if $y \notin M$, there exists at least one U_{x_j} containing y where $g_{x_j}(y) \neq 0$. By orientability and construction of g_x, all nonzero $g_{x_i}(y)$ outside M share the same sign. Therefore, the weighted sum $\sum_i \phi_i(y) g_{x_i}(y)$ cannot equal zero, a contradiction. Hence, $y \in M$, and $g^{-1}(0) \subseteq M$.

Finally, we check that $g'(x)$ has rank 1 for all $x \in M$. For $y \in M$, compute the derivative $Dg(y)$. By the product rule:

$$Dg(y) = \sum_i \phi_i(y) \cdot Dg_{x_i}(y) + \sum_i g_{x_i}(y) \cdot D\phi_i(y).$$

Since $g_{x_i}(y) = 0$ on M, the second term vanishes:

$$Dg(y) = \sum_i \phi_i(y) \cdot Dg_{x_i}(y).$$

Each $Dg_{x_i}(y)$ has rank 1 by the above lemma, hence $\nabla g_{x_i}(y) \neq 0$. By construction, the gradients $\nabla g_{x_i}(y)$ are all parallel to the unit normal $\mathbf{n}(y)$ due to consistent orientation. Thus:

$$\nabla g(y) = \sum_i \phi_i(y) \cdot \nabla g_{x_i}(y) = \left(\sum_i \phi_i(y) c_i \right) \mathbf{n}(y),$$

where $c_i \neq 0$ are constants of the same sign. Because $\sum_i \phi_i(y) = 1$ and c_i do not cancel, the coefficient $\sum_i \phi_i(y) c_i \neq 0$. Therefore, $\nabla g(y) \neq 0$, implying $Dg(y)$ retains rank 1 as desired.

$\square$