MAT354 Problem Set 2 Q4

October 2024

1 Q4

Problem 4: Let f and g be two holomorphic functions in a region Ω in $\mathbb{C}$, which have no zeroes in Ω. Suppose there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ of points in Ω such that

$$\lim_{n\to\infty} a_n = a, \quad a \in \Omega \quad \text{and} \quad a_n \neq a \quad \text{for all } n,$$

and

$$\frac{f'(a_n)}{f(a_n)} = \frac{g'(a_n)}{g(a_n)} \quad \text{for all } n.$$

Show that there exists a constant c such that

$$f(z) = cg(z) \quad \text{for all } z \in \Omega.$$

Proof. Let f and g be two holomorphic functions in a region Ω in $\mathbb{C}$, which have no zeroes in Ω. The key is to show that the function $\frac{f}{g}$ is a constant function in Ω. First note that $\frac{f}{g}$ is holomorphic in Ω since f and g are holomorphic in Ω and g has no zeroes in Ω. Next, we want to show that the derivative of $\frac{f}{g}$ is identically zero in Ω. In order to show this, we need to find a sequence of distinct points in Ω that has a limit point in Ω, such that the derivative of $\frac{f}{g}$ vanishes on this sequence of points. If we focus on the numerator of the derivative of $\frac{f}{g}$, we have $f'g - fg'$. Now, consider a given sequence $\{a_n\}_{n\in\mathbb{N}} \subset \Omega$, $a \in \Omega$ and $\lim_{n\to\infty} a_n = a$ with $a_n \neq a$, $\forall n \in \mathbb{N}$. Since we have for all n,

$$\frac{f'(a_n)}{f(a_n)} = \frac{g'(a_n)}{g(a_n)}$$

we get:

$$f'(a_n)g(a_n) - f(a_n)g'(a_n) = 0$$

Notice that the derivative of $\frac{f}{g}$ vanishes on the sequence of points $\{a_n\}_{n\in\mathbb{N}}$ in Ω since the numerator vanishes. Note that the denominator is just g^2, where g is non-zero in Ω, so we do not worry about the case where the denominator is zero. Now, applying an identity theorem, we can say that $'(\frac{f}{g})$ is identically zero in Ω, $'(\frac{f}{g}) = 0$. Since $\frac{f}{g}$ is holomorphic in Ω and $'(\frac{f}{g}) = 0$, $\frac{f}{g}$ is a constant function in Ω by corollary 3.4 in the textbook, complex analysis by Stein, hence we write $\frac{f(z)}{g(z)} = c$ for some constant c and for all $z \in \Omega$. Therefore, we have found a constant c such that $f(z) = cg(z)$ for all $z \in \Omega$.

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