Tutoring Notes on Ultrafilter

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1 Preliminaries: Partially Ordered Sets and Zorn's Lemma

Definition 1. A *partial order* on a set P is a binary relation $\leq$ on P such that:

1. For all $p \in P$, $p \leq p$ (reflexivity).
2. For all $p, q \in P$, if $p \leq q$ and $q \leq p$, then $p = q$ (antisymmetry).
3. For all $p, q, r \in P$, if $p \leq q$ and $q \leq r$, then $p \leq r$ (transitivity).

If any two elements of P are *comparable*, i.e., for all $p, q \in P$, either $p \leq q$ or $q \leq p$, then we will say that the order relation $\leq$ is *total* or *linear*.

Definition 2. Let $(P, \leq)$ be a partially ordered set:

- $p \in P$ is *maximal* if there is no $q \in P$ with $p \leq q$ and $p \neq q$.
- $q \in P$ is *minimal* if there is no $p \in P$ with $p \leq q$ and $p \neq q$.
- $Q \subseteq P$ is *bounded* if there is some $r \in P$ such that $q \leq r$ for all $q \in Q$.
- $C \subseteq P$ is a *chain* if the restriction of the order relation $\leq$ to C is total.

Theorem 1 (Axiom of Choice)**.** *If $\{A_\alpha : \alpha \in \Lambda\}$ is a collection of nonempty sets, then their product*

$$\prod_{\alpha \in \Lambda} A_\alpha$$

is nonempty as well.

Theorem 2 (Zorn's Lemma)**.** *Every nonempty partially ordered set in which every chain is bounded has a maximal element.*

2 Definition of Filter and Ultrafilter

Definition 3. A collection $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is a *filter* if, and only if:

1. $\mathbb{N} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
3. If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

Moreover, if the following condition is satisfied, $\mathcal{F}$ is called an *ultrafilter*.

4. For every $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $\mathbb{N} \setminus A \in \mathcal{F}$,

The following are several motivating examples of filters and ultrafilters. One may verify these examples satisfy the conditions to be a filter or an ultrafilter.

1. $\{\mathbb{N}\}$ forms a filter.
2. Take $A \subseteq \mathbb{N}$, and let
$$\mathcal{F}_A = \{B \subseteq \mathbb{N} \mid A \subseteq B\}.$$
3. Let $\mathcal{F}_r = \{A \subseteq \mathbb{N} \mid \mathbb{N} \setminus A \text{ is finite}\}$. This is known as a Frechet filter.
4. Principal ultrafilters are defined as $\mathcal{F}_n = \{A \subseteq \mathbb{N} \mid n \in A\}$ for a fixed $n \in \mathbb{N}$. Any ultrafilter which is not principal is called non-principal ultrafilter.

Remark 1. We call a filter $\mathcal{F}$ a *proper filter* if $\emptyset \notin \mathcal{F}$.

Remark 2. An ultrafilter which does not contain any finite set is non-principal.

3 The First Cool Theorem

Theorem 3 (Ultrafilter Theorem)**.** *Every filter is included in an ultrafilter*

Proof. Let $\mathcal{F}$ be a filter.

Consider a collection of filters $\mathbb{P}_{\mathcal{F}} = \{G \subseteq \mathcal{P}(\mathbb{N}) \mid G \text{ is a filter and } \mathcal{F} \subseteq G\}$.

We will order $\mathbb{P}_{\mathcal{F}}$ by set inclusion and define a partially ordered set (hereinafter, poset), $(\mathbb{P}_{\mathcal{F}}, \leq)$. Our aim is to show that this poset admits a maximal element by invoking the Zorn's lemma.

First, clearly $\mathbb{P}_{\mathcal{F}} \neq \emptyset$. Let $\mathcal{E} \subseteq \mathbb{P}_{\mathcal{F}}$ be a chain. Let $\mathcal{H}$ be defined as follows.

$$\mathcal{H} = \bigcup \mathcal{E} = \bigcup_{\alpha \in \mathcal{E}} \alpha \quad \text{(union of the chain)}.$$

Note that each α is a filter. We claim that $\mathcal{H}$ is an element of a poset $(\mathbb{P}_{\mathcal{F}}, \leq)$, by showing that it is a filter.

We start by observing that $\mathbb{N} \in \mathcal{H}$ and $\emptyset \notin \mathcal{H}$ since for all $\alpha \in \mathcal{E}$, $\mathbb{N} \in \alpha$ and $\emptyset \notin \alpha$.

Next, let $A, B \in \mathcal{H}$, then there are filters $\mathcal{F}_A, \mathcal{F}_B \in \mathcal{E}$ such that $A \in \mathcal{F}_A$ and $B \in \mathcal{F}_B$. Since $\mathcal{E}$ is a chain, we have either $\mathcal{F}_A \subseteq \mathcal{F}_B$ or $\mathcal{F}_B \subseteq \mathcal{F}_A$. WLOG, suppose $\mathcal{F}_A \subseteq \mathcal{F}_B$. Then, we get $A, B \in \mathcal{F}_B$ and thus $A \cap B \in \mathcal{F}_B \subseteq \mathcal{H}$.

Finally, let $A \in \mathcal{H}$ and $A \subseteq B$. Then for some $\mathcal{F}_A \in \mathcal{E}$, $A \in \mathcal{F}_A$. Since $\mathcal{F}_A$ is a filter, $B \in \mathcal{F}_A \subseteq \mathcal{H}$ as desired.

Thus, the chain $\mathcal{E}$ is bounded by a filter $\bigcup_{\alpha \in \mathcal{E}} \alpha$. By Zorn's lemma, there exists a maximal element $\mathcal{U} \in \mathbb{P}_{\mathcal{F}}$.

We argue that this maximal element $\mathcal{U}$ must be an ultrafilter. Suppose not, then there exists some $A \subseteq \mathbb{N}$ such that $A \notin \mathcal{U}$ and $\mathbb{N} \setminus A \notin \mathcal{U}$. We claim that we can construct a filter that contains $\mathcal{U}$ and A, and still belongs to $\mathbb{P}_{\mathcal{F}}$. First note that for all $U \in \mathcal{U}$, $U \cap A \neq \emptyset$. Otherwise, we have $U \subseteq \mathbb{N} \setminus A$ for some $U \in \mathcal{U}$. But this implies $\mathbb{N} \setminus A \in \mathcal{U}$, contradiction.

Consider $\mathcal{U}' = \{B \subseteq \mathbb{N} \mid \exists U \in \mathcal{U} : A \cap U \subseteq B\}$. We claim that $\mathcal{U}'$ is a filter containing A and $\mathcal{U}$.

First, we can see $\mathbb{N} \in \mathcal{U}'$ by choosing $U = \mathbb{N}$. Additionally, since for all $U \in \mathcal{U}$, $U \cap A \neq \emptyset$, we get $\emptyset \notin \mathcal{U}'$. Secondly, pick $B_1, B_2 \in \mathcal{U}'$, and let $U_1, U_2 \in \mathcal{U}$ be such that $A \cap U_i \subseteq B_i$ for $i = 1, 2$. Now we notice the following:

$$A \cap (U_1 \cap U_2) = (A \cap U_1) \cap (A \cap U_2) \subseteq B_1 \cap B_2$$

Since $\mathcal{U}$ is a filter, $U_1 \cap U_2 \in \mathcal{U}$. Therefore, $B_1 \cap B_2 \in \mathcal{U}'$.
Lastly, for any $B \in \mathcal{U}'$, if $B \subseteq C \subseteq \mathbb{N}$, for some $U \in \mathcal{U}$, we have $A \cap U \subseteq B \subseteq C$, implying $C \in \mathcal{U}'$. We showed that $\mathcal{U}'$ is a filter.

It remains to show that $\mathcal{U} \subseteq \mathcal{U}'$ and $A \in \mathcal{U}'$. Clearly, $A \in \mathcal{U}'$. Next, for each $U \in \mathcal{U}$, since $A \cap U \subseteq U$, $U \in \mathcal{U}'$. Also, since $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{U}'$, $\mathcal{U}' \in \mathbb{P}_{\mathcal{F}}$.

The above result shows that $\mathcal{U} \in \mathbb{P}_{\mathcal{F}}$ is not a maximal element of $\mathbb{P}_{\mathcal{F}}$, contradicting the Zorn's lemma. Thus, we conclude that the filter $\mathcal{F}$ is included in the ultrafilter $\mathcal{U}$, as desired.

$\square$

4 The Second Cool Theorem

Caveat: This is a tough one.

Definition 4 (SFIP)**.** A collection $\mathcal{A}$ of subsets of $\mathbb{N}$ has the *strong finite intersection property (SFIP)* if, and only if, for every finite subset $\mathcal{A}_0 \subseteq \mathcal{A}$, we have that

$$\bigcap \mathcal{A}_0 \text{ is non-empty and infinite.}$$

To prove the theorem, we can introduce a useful lemma to simplify the process.

Lemma 1. *Suppose that $\mathcal{F}$ is a filter on a set $\mathbb{X}$ and $A \subseteq \mathbb{X}$ such that $A \notin \mathcal{F}$ and $\mathbb{X} \setminus A \notin \mathcal{F}$. Then $\mathcal{F} \cup \{A\}$ can be extended to a filter.*

Proof. Suppose for some $A \in \mathbb{X}$, $A \notin \mathcal{F}$ and $\mathbb{X} \setminus A \notin \mathcal{F}$. We claim that we can construct a filter that contains $\mathcal{F}$ and A. First note that for all $F \in \mathcal{F}$, $F \cap A \neq \emptyset$. Otherwise, we have $F \subseteq \mathbb{X} \setminus A$ for some $F \in \mathcal{F}$. But this implies $\mathbb{X} \setminus A \in \mathcal{F}$, contradiction.

Consider $\mathcal{F}' = \{B \subseteq \mathbb{X} \mid \exists F \in \mathcal{F} : A \cap F \subseteq B\}$. We claim that $\mathcal{F}'$ is a filter containing A and $\mathcal{F}$.

First, we can see $\mathbb{X} \in \mathcal{F}'$ by choosing $F = \mathbb{X}$. Additionally, since for all $F \in \mathcal{F}$, $F \cap A \neq \emptyset$, we get $\emptyset \notin \mathcal{F}'$. Secondly, pick $B_1, B_2 \in \mathcal{F}'$, and let $F_1, F_2 \in \mathcal{F}$ be such that $A \cap F_i \subseteq B_i$ for $i = 1, 2$. Now we notice the following:

$$A \cap (F_1 \cap F_2) = (A \cap F_1) \cap (A \cap F_2) \subseteq B_1 \cap B_2$$

Since $\mathcal{F}$ is a filter, $F_1 \cap F_2 \in \mathcal{F}$. Therefore, $B_1 \cap B_2 \in \mathcal{F}'$.
Lastly, for any $B \in \mathcal{F}'$, if $B \subseteq C \subseteq \mathbb{X}$, for some $F \in \mathcal{F}$, we have $A \cap F \subseteq B \subseteq C$, implying $C \in \mathcal{F}'$. We showed that $\mathcal{F}'$ is a filter.

We observe that $A \in \mathcal{F}'$ by choosing $F = \mathbb{X}$. For each $F \in \mathcal{F}$, since $A \cap F \subseteq F$, $F \in \mathcal{F}'$. Thus $\mathcal{F}'$ is a filter containing A and $\mathcal{F}$.

□

Theorem 4. *If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ has the SFIP, then it is included in a non-principal ultrafilter.*

Proof. Suppose $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ has the SFIP. Consider the following:

$$\mathcal{F}_{\mathcal{A}} := \{B \subseteq \mathbb{N} \mid \exists A_1, \ldots, A_n \in \mathcal{A} \text{ such that } A_1 \cap \cdots \cap A_n \subseteq B\}$$

We start by showing that $\mathcal{F}_{\mathcal{A}}$ is a filter. Clearly, $\mathbb{N} \in \mathcal{F}_{\mathcal{A}}$ since we can take any $A_i \in \mathcal{A}$ to see $A_i \subseteq \mathbb{N}$. Also, since $\mathcal{A}$ has the SFIP, $\emptyset \notin \mathcal{F}_{\mathcal{A}}$. Secondly, let $B_1, B_2 \in \mathcal{F}_{\mathcal{A}}$. Then there exist two finite collections of subsets of $\mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2$ such that

$$\bigcap \mathcal{A}_1 \subseteq B_1 \text{ and } \bigcap \mathcal{A}_2 \subseteq B_2$$

Since $\mathcal{A}_1 \cup \mathcal{A}_2$ is a finite set, and

$$\bigcap(\mathcal{A}_1 \cup \mathcal{A}_2) = (\bigcap \mathcal{A}_1) \cap (\bigcap \mathcal{A}_2) \subseteq B_1 \cap B_2$$

it follows that $B_1 \cap B_2 \in \mathcal{F}_{\mathcal{A}}$.
Finally, let $B \in \mathcal{F}_{\mathcal{A}}$ and $B \subseteq C$. Then for some finite collection of subsets of $\mathbb{N}$, namely $\mathcal{A}_i$, we know $\bigcap \mathcal{A}_i \subseteq B$, thus $\bigcap \mathcal{A}_i \subseteq C$ and $C \in \mathcal{F}_{\mathcal{A}}$. Thus, $\mathcal{F}_{\mathcal{A}}$ is a filter.

Now, we will consider a collection of filters below:

$\mathcal{D} = \{\mathcal{F} \subseteq \mathcal{P}(\mathbb{N}) \mid \mathcal{F}$ is a proper filter, $\mathcal{A} \subseteq \mathcal{F}$, and $\mathcal{F}$ does not contain any finite subset of $\mathbb{N}\}$

We define a poset $(\mathcal{D}, \leq)$ where elements are ordered by the set inclusion. Note that $\mathcal{D}$ is non-empty since $\mathcal{F}_{\mathcal{A}} \in \mathcal{D}$. To see this, $\mathcal{F}_{\mathcal{A}}$ is clearly a proper filter since it does not contain an empty set. Also, $\mathcal{A} \subseteq \mathcal{F}_{\mathcal{A}}$ since for each $A_i \in \mathcal{A}$, we have $A_i \subseteq A_i$ trivially. Finally, since $\mathcal{A}$ has the SFIP, $\mathcal{F}_{\mathcal{A}}$ cannot contain any finite set.

Consider an arbitrary chain $\mathcal{C} \subseteq \mathcal{D}$. Denote each filter in $\mathcal{C}$ by $\mathcal{F}_\alpha$ where $\alpha \in \Lambda$. We show that the chain $\mathcal{C}$ is bounded by the union $\bigcup \mathcal{C} = \bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$.

First, since each $\mathcal{F}_\alpha$ is a filter, $\emptyset \notin \mathcal{F}_\alpha$ and $\mathbb{N} \in \mathcal{F}_\alpha$ for all $\alpha \in \Lambda$. Thus, $\emptyset \notin \bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$ and $\mathbb{N} \in \bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$.

Secondly, let $A, B \in \bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$. Then, $A \in \mathcal{F}_{\alpha_1}$ and $B \in \mathcal{F}_{\alpha_2}$ for some $\mathcal{F}_{\alpha_1}, \mathcal{F}_{\alpha_2} \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, WLOG, suppose that $\mathcal{F}_{\alpha_1} \subseteq \mathcal{F}_{\alpha_2}$. Then, $A, B \in \mathcal{F}_{\alpha_2}$ and since $\mathcal{F}_{\alpha_2}$ is a filter, $A \cap B \in \mathcal{F}_{\alpha_2}$. Therefore, $A \cap B \in \bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$.

Finally, let $A \in \bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$ and $A \subseteq B \subseteq \mathbb{N}$. Then $A \in \mathcal{F}_{\alpha_1}$ for some $\mathcal{F}_{\alpha_1} \in \mathcal{C}$. Since $\mathcal{F}_{\alpha_1}$ is a filter, $B \in \mathcal{F}_{\alpha_1}$, and therefore $B \in \bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$.

We have just shown that $\bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$ is a filter. We need still show that this filter belongs to $\mathcal{D}$. Needless to say, $\bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$ is a proper filter. Since $\mathcal{A} \subseteq \mathcal{F}_\alpha \in \mathcal{C}$ for each $\alpha \in \Lambda$, $\mathcal{A} \subseteq \bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$. Finally, since each $\mathcal{F}_\alpha \in \mathcal{C}$ does not contain any finite subset of $\mathbb{N}$, neither does $\bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha$. Thus, the union $\bigcup_{\mathcal{F}_\alpha \in \mathcal{C}} \mathcal{F}_\alpha \in \mathcal{D}$ is the upper bound of the chain $\mathcal{C}$.

By Zorn's lemma, we observe that the poset $\mathcal{D}$ admits a maximal element, call this maximal element $\mathcal{F}_{max}$. Now, it only remains to show that $\mathcal{F}_{max}$ is indeed a non-principal ultrafilter.

We first show that $\mathcal{F}_{max}$ is an ultrafilter. Suppose not, then there exists some $A \subseteq \mathbb{N}$ such that $A \notin \mathcal{F}_{max}$ and $\mathbb{N} \setminus A \notin \mathcal{F}_{max}$. WLOG, suppose A is infinite. Now is the time for us to invoke the above lemma, and we see that $\mathcal{F}_{max} \cup \{A\}$

can be extended to a filter containing A and $\mathcal{F}_{max}$. We notice that the filter extended by $\mathcal{F}_{max} \cup \{A\}$ belongs to $\mathcal{D}$, contradicting the Zorn's maximality of $\mathcal{F}_{max}$ on $\mathcal{D}$. If A is finite, then we just need to consider a filter extended by $\mathcal{F}_{max} \cup \{\mathbb{N} \setminus A\}$ and can make analogous arguments. Thus, $\mathcal{F}_{max}$ is an ultrafilter.

Clearly, since $\mathcal{F}_{max} \in \mathcal{D}$ does not contain any finite subset of $\mathbb{N}$, $\mathcal{F}_{max}$ is a non-principal ultrafilter (since any principal ultrafilter must contain some finite subsets of $\mathbb{N}$). The proof is now complete.

$\square$