MAT367 pset2 Problem 2

Problem 2: (Tangent vectors to a sphere) The unit sphere S^n in $\mathbb{R}^{n+1}$ is defined by the equation:

$$\sum_{i=1}^{n+1} (x^i)^2 = 1.$$

For $p = (p^1, \ldots, p^{n+1}) \in S^n$, show that a necessary and sufficient condition for

$$X_p = \sum a^i \frac{\partial}{\partial x^i}\Big|_p \in T_p\mathbb{R}^{n+1}$$

to be tangent to S^n at p is:

$$\sum a^i p^i = 0.$$

Solution:
Consider $X_p = \sum a^i \frac{\partial}{\partial x^i}\Big|_p \in T_p(\mathbb{R}^{n+1})$, where $T_p(\mathbb{R}^{n+1})$ is an $(n+1)$-dimensional vector space centered at $p = (p^1, \ldots, p^{n+1})$.

Since $x^i(X_p) = \sum a^i \frac{\partial x^i}{\partial x^i} = a^i$ for a coordinate function x^i, we see that $X_p = (a^1, a^2, \ldots, a^{n+1})$.

Now, suppose $(a^1, \ldots, a^{n+1}) \in X_p$ is tangent to S^n at p. Then, by Proposition 8.16 in the textbook by Loring Tu, there exists a curve $c : (-\epsilon, \epsilon) \to S^n$ such that

$$c(0) = p, \quad c'(0) = X_p.$$

Write $c(t) = (x^1(t), x^2(t), \ldots, x^{n+1}(t))$. Since $c(t) \in S^n$ for all t, we have:

$$\sum_{i=1}^{n+1} (x^i(t))^2 = 1.$$

Differentiating both sides with respect to t and evaluating at $t = 0$ gives:

$$\sum_{i=1}^{n+1} 2x^i(t)x^{i\prime}(t)\Big|_{t=0} = 0.$$

Since $x^i(0) = p^i$ and $x^{i\prime}(0) = a^i$, this simplifies to:

$$2\sum_{i=1}^{n+1} p^i a^i = 0 \quad \Rightarrow \quad \sum_{i=1}^{n+1} p^i a^i = 0.$$

Now, consider the subspace of $\mathbb{R}^{n+1}$:

$$N = \{(a^1, \dots, a^{n+1}) \in \mathbb{R}^{n+1} \mid \sum a^i p^i = 0\}.$$

To confirm that N is a subspace, we check the conditions:

- The zero vector $\mathbf{0} = (0, 0, \dots, 0)$ is in N since $\sum 0 \cdot p^i = 0$.
- Let $a = (a^1, \dots, a^{n+1}) \in N$ and $b = (b^1, \dots, b^{n+1}) \in N$. Then,

$$\sum (a^i + cb^i) p^i = \sum a^i p^i + c \sum b^i p^i = 0 + c \cdot 0 = 0.$$

  This shows that $ca + b \in N$ for any scalar $c \in \mathbb{R}$.

Thus, N is a subspace of $\mathbb{R}^{n+1}$.

From the previous derivation, we have $T_p(S^n) \subseteq N$.

We also observe that $p \notin N$ since $\sum (p^i)^2 = 1 \neq 0$. Since S^n is an n-dimensional manifold, $T_p(S^n)$ is an n-dimensional subspace of $\mathbb{R}^{n+1}$. Moreover, since $T_p(S^n) \subseteq N$ and both are subspaces of $\mathbb{R}^{n+1}$, their dimensions satisfy:

$$\dim(T_p(S^n)) = n \leq \dim(N) \leq n + 1 = \dim(\mathbb{R}^{n+1}).$$

Since $p \notin N$ and $\dim(\mathbb{R}^{n+1}) = n + 1$, we obtain $\dim(N) = n$. Since $T_p(S^n)$ and N have the same dimension, we conclude:

$$T_p(S^n) = N.$$

Thus, any X_p tangent to S^n at p is in N, and any $(a^1, \dots, a^{n+1}) \in N$ is also tangent to S^n at p, as required.