MAT367 pset3 Problem 3

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1 Problem 3

Suppose M is a smooth manifold and $X \in \mathfrak{X}(M)$. If $\gamma : J \to M$ is a maximal integral curve of X whose domain J has a finite least upper bound b, then for any $t_0 \in J$, the set $\gamma([t_0, b))$ is not contained in any compact subset of M.

1.1 Solution

Before we write the proof, I will use one useful lemma.

Lemma 1. *Let M be a smooth manifold, and let $K \subset M$ be a compact subset. Then there exist precompact open sets U and W such that*

$$K \subset U \quad \text{and} \quad \overline{U} \subset W,$$

where $\overline{U}$ denotes the closure of U in M.

Proof. Since M is a smooth manifold, it is locally compact and Hausdorff. To see this, at each point $x \in M$, there is a coordinate chart (U, ψ), which is a diffeomorphism. We take an open ball $B_{\psi(x)}$ of $\psi(x)$, which is precompact since its closure (closed ball) $\overline{B_{\psi(x)}}$ is compact in $\mathbb{R}^m$. By continuity of ψ^{-1} and ψ, $\psi^{-1}(B_{\psi(x)})$ and $\psi^{-1}(\overline{B_{\psi(x)}}) = \overline{\psi^{-1}(B_{\psi(x)})}$ are the desired open and compact neighbourhoods of x. This also means that for every point $x \in K$, there exists an open neighborhood U_x such that its closure $\overline{U_x}$ is compact.

The collection of precompact neighbourhoods $\{U_x\}_{x \in K}$ forms an open cover of K. Since K is compact, we can extract a finite subcover:

$$K \subset U_1 \cup U_2 \cup \cdots \cup U_n.$$

Define

$$U = U_1 \cup U_2 \cup \cdots \cup U_n.$$

Since the finite union of precompact sets is precompact, it follows that U is precompact.

Next, we consider the closure of U:

$$\overline{U} = \overline{U_1} \cup \overline{U_2} \cup \cdots \cup \overline{U_n}.$$

Since each $\overline{U_i}$ is compact, their finite union is also compact, implying that $\overline{U}$ is compact.

Repeating the same procedure for $\overline{U}$, we can claim that there exists a precompact set W such that $\overline{U} \subset W$. $\square$

Now the real proof starts here.

Proof. Suppose for contradiction that there exists t_0 such that the set $\gamma([t_0, b))$ is contained in some compact subset K of M. By the above lemma, there exists a precompact set U, W such that $K \subset U$ and $\overline{U} \subset W$. From the chapter 13 of the textbook by Loring Tu, we can construct a C^∞ bump function $\psi : M \to \mathbb{R}$, which is one on U, non-vanishing (or supported) on W, and zero outside of W.

It is clear that ψX is a smooth vector field as a composition of C^∞ functions. Moreover, since the support of ψX is $\overline{W}$ where W is precompact, ψX has a compact support. Since a smooth vector field with a compact support is complete, ψX is a complete vector field; its flow is defined on all of $M \times \mathbb{R}$. Since ψ is one on U, X and ψX agree on U.

By the uniqueness of the integral curve or ODE solution, ψX and X have the same flow over U, the same set of integral curves through $\gamma(t_0)$ over U . Let $\theta : \mathbb{R} \times M \to M$ be the global flow of the vector field ψX. We define $u(t) = \theta(t - t_0, \gamma(t_0))$ for $t \in \mathbb{R}$. Thus, u be an integral curve of ψX satisfying $u(t_0) = \gamma(t_0)$. We have $u(t) = \gamma(t)$ for $t \in [t_0, b)$ by the uniqueness of the ODE solution (or by the Picard–Lindelöf theorem). This shows that the curve u is also an integral curve of X. Since ψX is a complete vector field, the integral curve u is defined on all of $\mathbb{R}$. In particular, u extends γ to $t \geq b$, contradicting the maximality of γ.

Hence, we conclude that for any $t_0 \in J$, the set $\gamma([t_0, b))$ is not contained in any compact subset of M.

$\square$