MAT354 Problem set 2 Q5

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Problem 5

Let $\mathbb{D}$ be the open unit disc centered at the origin in $\mathbb{C}$. Let f be a continuous function on $\overline{\mathbb{D}}$ which is analytic on $\mathbb{D}$. Suppose $f(e^{it}) = 0$ for $0 < t < \frac{\pi}{4}$. Show that $f(z) = 0$ for all $z \in \mathbb{D}$.

Proof. A sensible idea is to apply a Schwartz reflection principle and extend a function f to the region $\mathbb{C} \setminus \overline{\mathbb{D}}$. Note that for each $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, we have $|z| > 1$. Now, we define a function F on $\mathbb{C} \setminus \overline{\mathbb{D}}$ as follows:

$$F(z) = \overline{f\left(\frac{1}{\overline{z}}\right)}$$

We first need to show that f is holomorphic on $\mathbb{C}\setminus\overline{\mathbb{D}}$ and continuously extend to the boundary $I = \left\{z : z = e^{it},\ 0 < t < \frac{\pi}{4}\right\}$. We show that f is holomorphic on $\mathbb{C}\setminus\overline{\mathbb{D}}$ first. We consider $f(z)$ for an arbitrary $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$. The theorem 2.6 and 4.4 in the textbook shows that holomorphicity and analyticity of f are equivalent (complex differentiability and power series expansion). Hence, we need to show that for each $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $f(z)$ has a power series expansion.

First, $\frac{1}{\overline{z}}$ lies in $\mathbb{D}$. Let $z = x + yi$. Then:

$$\frac{1}{\overline{z}} = (\frac{x^2}{x^2+y^2}) + (\frac{y^2}{x^2+y^2})i$$

Note that $|\frac{1}{\overline{z}}| < 1$ by calculation. Now, we consider $f\left(\frac{1}{\overline{z}}\right)$. Since f is holomorphic in $\mathbb{D}$, f is also analytic and hence has a power series expansion at z, around some center, $z_0 \in \mathbb{D}$. Hence, we write:

$$f\left(\frac{1}{\overline{z}}\right) = \sum a_n(\frac{1}{\overline{z}} - z_0)^n$$

Now, taking a conjugate on $f\left(\frac{1}{\overline{z}}\right)$, we get:

$$F(z) = \overline{f\left(\frac{1}{\overline{z}}\right)} = \sum \overline{a_n}(\frac{1}{z} - \overline{z_0})^n$$

Since the power series expansion exists at each $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ for $F(z)$, F is analytic and thus holomorphic on $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Now we also show that F continuously extends to $I = \left\{z : z = e^{it},\ 0 < t < \frac{\pi}{4}\right\}$. We know that $f(z) = 0, \forall z \in I$. Since $|z| = 1, \forall z \in I$, we have:

$$F(z) = \overline{f\left(\frac{1}{\bar{z}}\right)} = \overline{f(z)}$$

Since $f(z) = 0, \forall z \in I$, and we have $f(z) = \overline{f(z)}$ when $f(z)$ is real, we have $F(z) = 0, \forall z \in I$. This shows that F continuously extends to I.

We will define an extended function as follows:

$$f_{Ex}(z) = \begin{cases} F(z) & \text{if } z \in \mathbb{C} \setminus \overline{\mathbb{D}}, \\ F(z) = f(z) = 0 & \text{if } z \in I, \\ f(z) & \text{if } z \in \mathbb{D}. \end{cases}$$

We will argue that this extended function f_{Ex} is holomorphic on the union $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup I \cup \mathbb{D}$. Our goal is to apply a Morera's theorem and show f_{Ex} is holomorphic.

Consider an arbitrary point $z \in I$. Then consider a sufficiently small epsilon ball centered at z. Obviously, f_{Ex} is continuous on this epsilon ball since f is continuous on $\overline{\mathbb{D}}$. Then we take any triangle T inside this epsilon ball. If T does not intersect I, then:

$$\int_T f_{Ex}(z)\, dz = 0$$

Next, suppose that a triangle T intersects I at one point. Let T_δ is a triangle obtained from T by slightly raising the edge or vertex in the direction perpendicular to the tangent on the arc towards $\mathbb{C} \setminus \overline{\mathbb{D}}$ at the intersecting point. Then, $T_\delta \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Since f_{Ex} is holomorphic on $\mathbb{C} \setminus \overline{\mathbb{D}}$, $\int_{T_\delta} f_{Ex}(z)\, dz = 0$. If we let $\delta \to 0$, then by continuity, we can conclude that $\int_T f_{Ex}(z)\, dz = 0$

Finally, we consider a case where a triangle intersects the arc in the sense that the triangle T contains part of the arc. Then we will decompose the triangle as follows (we will call this a *process* later). Consider the subset of the arc of the circle contained in the triangle, and let a and b be two intersecting points. Next, we consider a midpoint w on the subset of the arc contained in the triangle T. Let w be such that $|w - a| = |w - b|$. We are going to draw from three to five lines. First, we draw three lines from the vertices of the triangle T to a point w. Then we draw two lines from a to w, and b to w. Note that if two vertices of the triangle are on the arc, then the latter two lines correspond to the two of the first three lines. In this case, we need only draw three lines. Similarly, if one vertex of the triangle is on the arc, then one of the latter two lines correspond to one of the first three lines.

Then after drawing lines in the triangle, we notice that the integration over the combination of all these sub-triangles is the same as the integration over the original triangle. Moreover, there will only be two sub-triangles that intersect both region $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $\mathbb{D}$. Hence, the integration over all other triangles T_i with $T_i \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ or $T_i \subset \mathbb{D}$, will be zero by the holomorphicity of f on $\mathbb{D}$ and F on $\mathbb{C} \setminus \overline{\mathbb{D}}$.

We now focus on the two sub-triangles that intersect both regions $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $\mathbb{D}$. The essential idea is to repeat the process described above on these two sub-triangles forever. Let n be a number of times we do and repeat this process. Then as $n \to \infty$, those sub-triangles eventually converge to a line, from a vertex of the triangle (call it v_1 for now) to the point on the arc (call it w' for now), with two orientations, where one orientation is from v_1 to w' and another orientation is from w' to v_1. Thus, the integration over each sub-triangle will eventually become zero as we take $n \to \infty$. Since the sum of all the sub-triangles that intersect both $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $\mathbb{D}$, and all other sub-trinalges T_i where $T_i \subset \mathbb{D}$ or $T_i \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, is the same as the original triangle T, we have $\int_T f_{Ex}(z)\,dz = 0$ as $n \to \infty$.

Now, since f_{Ex} is continuous everywhere in $\mathbb{C}$, in particular, on the epsilon disk centered at a point $z \in I$, and for any triangle T in this epsilon disk, we have $\int_T f_{Ex}(z)\,dz = 0$, f_{Ex} is holomorphic in this epsilon disk by **Morera's** theorem.

We have shown that an extended function is holomorphic on the union $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup I \cup \mathbb{D}$, thus we can apply an identity theorem on a region $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup I \cup \mathbb{D}$.

Now, consider a sequence of distinct points $\{a_n\}_{n \in \mathbb{N}}$ on the arc from $e^{i\frac{\pi}{8}}$ to $e^{i\frac{\pi}{16}}$ where, $\mathrm{Im}(a_n) \geq \mathrm{Im}(a_{n+1})$ (Im is an imagenary part of complex numbers). There is a bijection between a closed interval in $\mathbb{R}$ and the set of points on the arc from $e^{i\frac{\pi}{8}}$ to $e^{i\frac{\pi}{16}}$, we can indeed take uncountably many points to form such a sequence. Obviously, f_{Ex} vanishes on this sequence since $f(e^{it}) = 0$ for $0 < t < \frac{\pi}{4}$ and the limit point $e^{i\frac{\pi}{16}}$ is contained in I. Hence we apply an identity theorem, and conclude that f_{Ex} is identically zero on $\mathbb{D}$.

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