A Hausdorff Noetherian Space is Finite.

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1 Introduction

Lemma 1. *Let X be an infinite Hausdorff space. Then there exists an open subset U of X such that $X \setminus \bar{U}$ is infinite.*

Proof. Suppose to the contrary that there does not exist such an open set. Take any open subset U with $\bar{U} \neq X$, then $X \setminus \bar{U}$ is finite by assumption. Since any subspace of Hausdorff space is again Hausdorff, $X \setminus \bar{U}$ is T_2 (and T_1), hence it has a discrete topology. Since $X \setminus \bar{U}$ is an open subspace of X, a singleton $\{x\}$ where $x \in X \setminus \bar{U}$ is open in X. Since X is Hausdorff, every singleton including $\{x\}$ is closed. Since the closure of $\{x\}$ is again $\{x\}$, $\{x\}$ is an open subset of X such that $X \setminus \{x\}$ is infinite, but this is contradictory. Therefore, there must exists some open subset U of X such that $X \setminus \bar{U}$ is infinite.

$\square$

Corollary 1. Every infinite Hausdorff space contains an infinite discrete subspace.

Proof. We inductively construct a collection of open sets that satisfies the following properties. Consider a collection of open subsets $\{U_\alpha \mid \alpha \in \Lambda\}$ such that $X \setminus \bigcup_{\alpha=1}^{n} \overline{U_\alpha}$ is infinite and $U_{n+1} \subseteq X \setminus \bigcup_{\alpha=1}^{n} \overline{U_\alpha}$. This inductive construction is possible by the lemma above. Then the set $\{x \mid x \in U_\alpha \text{ for some } \alpha \in \Lambda, \text{ and if } x, y \in U_\alpha, \text{ then } x = y\}$ is the desired infinite discrete subspace.

$\square$

Lemma 2. *Every subspace of a Noetherian space is compact.*

Proof. Suppose X is Noetherian, and let $A \subseteq X$ be a subspace. Let $\mathcal{O} = \{U_i \mid i \in I\}$ be an open cover of A, such that $A \subseteq \bigcup_{i \in I} U_i$. Since an arbitrary union of open sets is open, $\bigcup_{i \in I} U_i$ is open. Consider an infinite ascending chain of open sets $\{V_i \mid i \in I\}$ such that $V_i = \bigcup_{i=1}^{n} U_i$. Clearly, $V_i \subseteq V_{i+1}$ for all $i \in I$, but since X is Noetherian, there is an integer M such that $V_n = V_m$ for any $n, m \geq M$. This implies that A has a finite subcover of size M. Therefore, a subspace A is compact.

$\square$

Theorem 1. *A Hausdorff Noetherian Space is finite.*

Proof. Let X be a Hausdorff Noetherian space. Suppose to the contrary that X is infinite. Then from the corollary above, X contains an infinite discrete subspace, call it $U \subseteq X$. Since X is Noetherian, U is compact. Consider an open cover $\mathcal{O} = \{U_i \mid i \in I\}$ of U such that each U_i contains a single point in U. Since U is infinite, this open cover $\mathcal{O}$ does not have a finite subcover, contradicting the fact that U is compact. Hence, X must be finite.

$\square$