MAT367 pset2 Problem 10

0.1 Problem 10

For a real (2×2)-matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(n, \mathbb{R}),$$

let $\|\cdot\|$ denote its norm, defined as $\|A\|^2 = a^2 + b^2 + c^2 + d^2$.

(a) Show that the set

$$S = \{A \in M(n, \mathbb{R}) \mid \|A\| = 1, \det(A) = 0\}$$

is a 2-dimensional submanifold of $M(n, \mathbb{R})$.

(b) Show that the map

$$\pi : S \to \mathbb{RP}^1$$

taking $A \in S$ to its 1-dimensional range $\operatorname{ran}(A) \subset \mathbb{R}^2$ is smooth. Determine the fibers $\pi^{-1}(u : v)$, for $u^2 + v^2 = 1$.

(c) Prove that S is diffeomorphic to the 2-torus $S^1 \times S^1$.

0.2 Solution

0.2.1 Solution to (a):

We will use the *regular level set theorem*. Consider the function:

$$F : M(2, \mathbb{R}) \to \mathbb{R}^2, \quad F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (\|A\|^2 - 1, \det(A)),$$

where

$$\|A\|^2 = a^2 + b^2 + c^2 + d^2, \quad \det(A) = ad - bc.$$

Since the determinant function is smooth (as a polynomial of coefficients) and the norm function is smooth (as a polynomial of coefficients), we conclude that F is a smooth function. Thus, to apply the regular level set theorem, we will show that $(0, 0) \in \mathbb{R}^2$ is a regular value of F. Also, we note that $S = f^{-1}((0, 0))$.

To do this, we compute the differential dF. The Jacobian matrix J of F is given by:

$$J = \begin{bmatrix} 2a & 2b & 2c & 2d \\ d & -c & -b & a \end{bmatrix}.$$

We want to show that J has full rank (rank 2) for all $A \in S$. Each row is clearly nonzero since $\|A\| = 1$ implies that at least one of a, b, c, d is nonzero. We now show that the two rows are not scalar multiples of each other.

Suppose for contradiction that there exists a scalar λ such that:

$$\lambda d = 2a, \quad -\lambda c = 2b, \quad -\lambda b = 2c, \quad a\lambda = 2d.$$

Multiplying corresponding equations, we obtain:

$$4a^2 + 4b^2 + 4c^2 + 4d^2 = \lambda^2(a^2 + b^2 + c^2 + d^2).$$

Since $\|A\|^2 = 1$, this simplifies to:

$$\lambda^2 = 4 \Rightarrow \lambda = \pm 2.$$

Plugging in $\lambda = 2$, we get $a = d$ and $b = -c$. Since $ad - bc = 0$ must hold, we have $ad - bc = a^2 + b^2 = d^2 + c^2 = 0$. This implies that $a = b = c = d = 0$, contradicting that $a^2 + b^2 + c^2 + d^2 = 1$.

Similarly, for $\lambda = -2$, we get $a = -d$ and $b = c$. Since $ad - bc = 0$, we get $ad - bc = -d^2 - c^2 = -a^2 - b^2 = 0$. This implies that $a = b = c = d = 0$, contradicting that $a^2 + b^2 + c^2 + d^2 = 1$.

The above implies that the differential matrix J is full rank everywhere on S. Since J has full rank everywhere on S, and S is clearly nonempty, by the regular level set theorem,

$$F^{-1}(0, 0) = S$$

is a regular submanifold of $M(2, \mathbb{R})$ with dimension:

$$\dim(M(2, \mathbb{R})) - \dim(\mathbb{R}^2) = 4 - 2 = 2.$$

We clearly have a diffeomorphism $M(2, \mathbb{R}) \cong \mathbb{R}^4$, so $\dim(\mathbb{R}^4) = \dim(M(2, \mathbb{R})) = 4$. Thus, S is a 2-dimensional submanifold of $M(2, \mathbb{R})$.

0.2.2 Solution to (b):

Consider the map $\pi : S \to \mathbb{RP}^1$ that takes $A \in S$ to its 1-dimensional range $\mathrm{ran}(A) \subset \mathbb{R}^2$. We will express π as a composition of two maps.

First, we argue that the quotient map $q : \mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^1$ is smooth. We start by taking a standard (or most known, what was covered in class) C^∞ atlas for $\mathbb{RP}^1$ consisting of charts $\{(U_i, \phi_i)\}$ for $i = 0, 1$.

Take any $(a_0, a_1) \in \mathbb{R}^2 \setminus \{0\}$. Without loss of generality, suppose $a_1 \neq 0$. Then we define the local chart by letting $p = (\phi_1 \circ q)$:

$$p((a_0, a_1)) = (\phi_1 \circ q)(a_0, a_1) = \frac{a_0}{a_1}.$$

Clearly, the map $(a_0, a_1) \mapsto \frac{a_0}{a_1}$ is smooth as $a_1 \neq 0$. Since ϕ_1 is also smooth, q must be smooth.

We now define a smooth map $f : S \to \mathbb{R}^2 \setminus \{0\}$ such that $\pi = q \circ f$.

Consider:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S.$$

Define $f(A)$ by:

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{cases} \left(\frac{a}{\sqrt{a^2+c^2}}, \frac{c}{\sqrt{a^2+c^2}}\right) & \text{if } c \neq 0 \text{ or } a \neq 0, \\ \left(\frac{b}{\sqrt{b^2+d^2}}, \frac{d}{\sqrt{b^2+d^2}}\right) & \text{otherwise.} \end{cases}$$

Clearly, $\mathrm{ran}(A) = \mathrm{span}(\{(a, c), (b, d)\})$. Also, for some λ, we have $(a, c) = \lambda(b, d)$ since two columns are linearly dependent, but at least one of which is nonzero.

f maps the input matrix to the vector spanning $\mathrm{ran}(A)$. For example:

- When $(a, c) \neq 0$, $f(A)$ maps to a unit vector in the direction of (a, c).
- When $(a, c) = 0$, $f(A)$ maps to a unit vector in the direction of (b, d), where $(b, d) \neq 0$.

Clearly, f is smooth since rational functions are smooth, and the denominator of the rational functions is always nonzero by definition of f.

Since each component is smooth, f is smooth. Since the composition of smooth maps is smooth, $q \circ f = \pi$ is smooth, as required.

Determining the Fiber:
Any matrix $A \in S$ has rank 1 since $||A|| = 0$ implies the rank is not two, and $a^2 + b^2 + c^2 + d^2 = 1$ implies the rank is nonzero. Therefore, the fiber of $\pi^{-1}(u : v)$ for $u^2 + v^2 = 1$ consists of all matrices in S whose columns are multiples of (u, v). In concrete form, each matrix in the fiber has the form:

$$A = \begin{bmatrix} \alpha u & \beta u \\ \alpha v & \beta v \end{bmatrix},$$

for some scalars α, β.

Since $A \in S$, it must satisfy the norm condition $\|A\| = 1$, which gives:

$$\alpha^2 u^2 + \beta^2 u^2 + \alpha^2 v^2 + \beta^2 v^2 = 1.$$

Since $u^2 + v^2 = 1$, we obtain:

$$\alpha^2 + \beta^2 = 1.$$

Thus, the fiber consists of all matrices of the form given above, where (α, β) is any point on the unit circle S^1.

$$\pi^{-1}(u : v) = \left\{ \begin{bmatrix} \alpha u & \beta u \\ \alpha v & \beta v \end{bmatrix} \in S \mid \alpha^2 + \beta^2 = 1 \right\}.$$

0.2.3 Solution to (c):

We construct a diffeomorphism between S and $S^1 \times S^1$. We start by showing that for every matrix $A \in S$, there exist angles θ, ϕ such that:

$$a = \cos\theta\cos\phi, \quad b = \cos\theta\sin\phi, \quad c = \sin\theta\cos\phi, \quad d = \sin\theta\sin\phi,$$

for $\theta, \phi \in [0, 2\pi)$.

Take any matrix $A \in S$. Consider the norm constraint:

$$a^2 + b^2 + c^2 + d^2 = 1.$$

We can consider the following two circles:

$$a^2 + c^2 = r_1^2, \quad b^2 + d^2 = r_2^2.$$

for $r_1, r_2 \in [0, 1]$. Since $r_1^2 + r_2^2 = 1$, we may set:

$$r_1 = \cos\theta, \quad r_2 = \sin\theta, \quad \theta \in [0, 2\pi).$$

Now, each sub-circle can be parametrized by angle coordinates:

$$a = r_1\cos\phi_1, \quad c = r_1\sin\phi_1, \quad b = r_2\cos\phi_2, \quad d = r_2\sin\phi_2.$$

Next, the determinant condition $\det A = ad - bc = 0$ implies that $\phi_1 = \phi_2 = \phi$.

Thus, we obtain the desired representation:

$$a = \cos\theta\cos\phi, \quad b = \cos\theta\sin\phi, \quad c = \sin\theta\cos\phi, \quad d = \sin\theta\sin\phi.$$

This expresses every matrix in S using two independent angles (θ, ϕ).

Define the mapping:

$$f : S \to [-\pi, \pi) \times [-\pi, \pi), \quad f(A) = (\theta, \phi),$$

where A is parametrized as above.

Next, we show that f is bijective.

- **f is surjective:** Given any $(\theta, \phi) \in [-\pi, \pi) \times [-\pi, \pi)$, we can just take a matrix A with entries $a = \cos\theta\cos\phi, b = \cos\theta\sin\phi, c = \sin\theta\cos\phi, d = \sin\theta\sin\phi$.
- **f is injective:** Suppose $(\theta_1, \phi_1) = (\theta_2, \phi_2)$. Then each entry of the matrix is determined as follows:

$$\begin{aligned} a_1 &= \cos\theta_1\cos\phi_1 = \cos\theta_2\cos\phi_2 = a_2, \\ b_1 &= \cos\theta_1\sin\phi_1 = \cos\theta_2\sin\phi_2 = b_2, \\ c_1 &= \sin\theta_1\cos\phi_1 = \sin\theta_2\cos\phi_2 = c_2, \\ d_1 &= \sin\theta_1\sin\phi_1 = \sin\theta_2\sin\phi_2 = d_2 \end{aligned}$$

  Hence, we get $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$.

Thus, f is a bijection.

Now, we show that f is smooth. We explicitly construct θ and ϕ in terms of a, b, c, d as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{bmatrix}$$

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \left(\theta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right), \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right)$$

$$\phi = \begin{cases} \arctan\left(\frac{b}{a}\right), & \text{if } \theta \neq \frac{\pi}{2}, -\frac{\pi}{2}, \phi \neq \frac{\pi}{2}, -\frac{\pi}{2} \\ \arctan\left(\frac{d}{c}\right), & \text{if } \theta = \frac{\pi}{2}, -\frac{\pi}{2}, \phi \neq \frac{\pi}{2}, -\frac{\pi}{2} \\ \frac{\pi}{2}, & \text{if } \phi = \frac{\pi}{2} \\ -\frac{\pi}{2}, & \text{if } \phi = -\frac{\pi}{2} \end{cases}$$

$$\theta = \begin{cases} \arcsin(\sqrt{c^2 + d^2}), & \text{if } \theta \in [0, \pi) \setminus \{\frac{\pi}{2}\} \\ \frac{\pi}{2}, & \text{if } \theta = \frac{\pi}{2} \\ -\frac{\pi}{2}, & \text{if } \theta = -\frac{\pi}{2} \\ \arcsin(-\sqrt{c^2 + d^2}), & \text{if } \theta \in (-\pi, 0) \setminus \{\frac{-\pi}{2}\} \end{cases}$$

The component functions ϕ and θ map the matrix to the angles ϕ and θ respectively. Since arcsin, arctan, constant function and the square root function are all smooth in their respective domains, each component function ϕ and θ are also smooth. Therefore, f is smooth.

The inverse of f is also smooth since the trigonometric functions $\sin(x)$, $\cos(x)$ and their products such as $\sin(x)\cos(x)$ on $[-\pi, \pi) \times [-\pi, \pi)$ are smooth, and thus each component of the matrix is smooth. Therefore, f is a diffeomorphism between S and $[-\pi, \pi) \times [-\pi, \pi)$.

Now, we show that a diffeomorphism exists between $[-\pi, \pi) \times [-\pi, \pi)$ and $S^1 \times S^1$. Define:

$$g : [-\pi, \pi) \times [-\pi, \pi) \to S^1 \times S^1, \quad g(\theta, \phi) = (e^{i\theta}, e^{i\phi}).$$

This map is clearly smooth since exponentiation e^x is smooth. Next, we show that g is bijective.

g **is surjective:** It is clear that every pair $(e^{i\theta}, e^{i\phi}) \in S^1 \times S^1$ corresponds to a unique $(\theta, \phi) \in [-\pi, \pi) \times [-\pi, \pi)$.

g **is injective:** If $g(\theta_1, \phi_1) = g(\theta_2, \phi_2)$, then $e^{i\theta_1} = e^{i\theta_2}$ and $e^{i\phi_1} = e^{i\phi_2}$. Since the complex logarithm is not analytic on the principal branch at $\theta = -\pi$, we define the inverse function as follows:

$$g^{-1}(z) = \begin{cases} \frac{\log z}{i}, & \text{if } z \neq -1, \\ -\pi, & \text{if } z = -1. \end{cases}$$

When $z = -1$, which corresponds to $e^{i\theta} = -1$, the logarithm is not applied since the complex logarithm is analytic only on the principal branch $\mathbb{C} \setminus (-\infty, 0]$.

Since the inverse of g is a component-wise complex logarithm function, which is analytic (smooth) on $g((-\pi, \pi) \times (-\pi, \pi)) \subseteq \mathbb{C} \setminus (-\infty, 0]$ and smooth at $z = -1$, we can conclude that g is a diffeomorphism.

Since the composition of diffeomorphisms is again a diffeomorphism, $g \circ f$ is the desired diffeomorphism between S and $S^1 \times S^1$. we conclude that S is diffeomorphic to $S^1 \times S^1$, as required.