MAT367 pset2 Problem 4

Problem 4: Show that a smooth map $\pi : M \to N$ is a submersion if and only if for all $p \in M$, there exists an open neighborhood $V \subset N$ of $\pi(p)$ and a smooth map $\iota : V \to M$ such that:

$$\iota(\pi(p)) = p, \quad \text{and} \quad \pi \circ \iota = \mathrm{id}_V.$$

Solution:
Forward direction:

Suppose $\pi : M \to N$ is a smooth submersion.
Take any point $p = (p_1, \ldots, p_m) \in M$. By the Submersion Theorem (or Theorem 11.5 in the textbook), there exist charts (U, Φ) centered at p in M and (V, Ψ) centered at $\pi(p)$ in N such that in a neighborhood of p, π takes the form:

$$\Psi \circ \pi \circ \Phi^{-1}(r^1, \ldots, r^n, r^{n+1}, \ldots, r^m) = (r^1, \ldots, r^n).$$

Clearly, π is locally a projection since Ψ and Φ^{-1} are diffeomorphisms.

Now, we define an inclusion map $i : \mathbb{R}^n \to \mathbb{R}^m$ by:

$$i(r^1, \ldots, r^n) = (r^1, \ldots, r^n, 0, \ldots, 0),$$

which is smooth as an inclusion map between smooth manifolds (I proved this when working on the problem set 1).

Next, define the local inverse map on V:

$$\iota = \Phi^{-1} \circ i \circ \Psi.$$

Since Φ^{-1}, i, and Ψ are all smooth maps on their respective domains, it follows that ι is smooth on V.

Now we verify that ι satisfies the required properties:

$$\iota(\pi(p)) = (\Phi^{-1} \circ i \circ \Psi)(\pi(p)).$$

Since (V, Ψ) is centerted at $\pi(p)$, we get:

$$\iota(\pi(p)) = (\Phi^{-1} \circ i)(0).$$

Using the definition of i:

$$\iota(\pi(p)) = \Phi^{-1}(0) = p$$

as the chart (U, Φ) is centered at p in M. This shows that $\iota(\pi(p)) = p$ as desired.

We also define a projection map $i^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ by:

$$i^{-1}(r^1, \ldots, r^m) = (r^1, \ldots, r^n).$$

By the construction of i^{-1}, we get:

$$i^{-1} = \Psi \circ \pi \circ \Phi^{-1} \quad \Longleftrightarrow \quad \Psi^{-1} \circ i^{-1} \circ \Phi = \pi.$$

We also observe that $i^{-1} \circ i = \mathrm{id}_{\mathbb{R}^n}$.

Now, consider:

$$\pi \circ \iota(q) = \Psi^{-1} \circ i^{-1} \circ \Phi \circ \Phi^{-1} \circ i \circ \Psi(q), \quad \text{for } q \in V.$$

Expanding the composition:

$$\begin{aligned}
\pi \circ \iota(q) &= \Psi^{-1} \circ i^{-1} \circ \Phi \circ \Phi^{-1} \circ i \circ \Psi(q) \\
&= \Psi^{-1} \circ i^{-1} \circ i \circ \Psi(q) \\
&= \Psi^{-1} \circ \mathrm{id}_{\mathbb{R}^n} \circ \Psi(q) \\
&= \Psi^{-1} \circ \Psi(q) \\
&= q.
\end{aligned}$$

Thus, we obtain:

$$\pi \circ \iota = \mathrm{id}_V, \quad \text{as desired.}$$

Backward direction: Suppose that for each p, there exists an open neighborhood $V \subset N$ of $\pi(p)$ and a smooth map $\iota : V \to M$ such that:

$$\iota(\pi(p)) = p, \quad \pi \circ \iota = \mathrm{id}_V.$$

Now, differentiating both sides, we obtain:

$$d(\pi \circ \iota)_{\pi(p)} = d(\mathrm{id})_{\pi(p)}$$

$$d(\pi)_p \circ d(\iota)_{\pi(p)} = \mathrm{id}_{T_{\pi(p)}N} \text{ by chain rule}$$

Since $\mathrm{id}_{T_{\pi(p)}N}$ is the identity, $d(\pi)_p$ must be surjective. Since this holds for all points $p \in M$, π is a smooth submersion on M as required.