Question 2

$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \frac{1}{\theta} e^{-x/\theta} I_{(0,\infty)}(x)$, $\mathcal{X} = (0,\infty)$, $\Theta = (0,\infty)$

$f(x,\lambda) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$, $\lambda \in \Theta = (0,\infty)$

(a) $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $Y = 2n\frac{\bar{X}}{\theta} = 2n\lambda\bar{X}$ $\{$since $1/\theta = \lambda\}$

$\bar{X} \sim \text{Gamma}(n, \theta/n)$

$\rightarrow M_{\bar{x}}(t) = (M_X(t/n))^n$,

$\rightarrow M_Y(t) = M_{2n\lambda\bar{x}}(t) = \left(M_{X}\left(\frac{2\cancel{n}\lambda t}{\cancel{n}}\right)\right)^n = \left(M_X\, 2\lambda t\right)^n$

$$= \left(\frac{1}{1-\theta\cdot(2\lambda t)}\right)^n = \frac{1}{\left(1-\cancel{\theta}\cdot 2\cdot\frac{1}{\cancel{\theta}}t\right)^n} = \frac{1}{(1-2t)^n}$$

$$= \frac{1}{(1-2t)^{2n/2}}$$, $r = 2n$

$\therefore Y \sim \chi^2_{2n}$ $\rightarrow$ i.e Chi-square distribution with $2n$ degrees of freedom.

(b) $$P\left(\frac{2n\bar{X}}{\chi^2_{1-\alpha/2}} \le \theta \le \frac{2n\bar{X}}{\chi^2_{\alpha/2}}\right) = P\left(\frac{1}{\chi^2_{1-\alpha/2}} \le \frac{\theta}{2n\bar{X}} \le \frac{1}{\chi^2_{\alpha/2}}\right)$$

$$= P\left(\chi^2_{\alpha/2} \le \frac{2n\bar{X}}{\theta} \le \chi^2_{1-\alpha/2}\right) = P\left(\chi^2_{\alpha/2} \le Y \le \chi^2_{1-\alpha/2}\right)$$

$$= F_Y(\chi^2_{1-\alpha/2}) - F_Y(\chi^2_{\alpha/2})$$

$$= 1-\alpha/2 - \alpha/2 = \underline{\underline{1-\alpha}}$$

(c) Using the expression; $\dfrac{2n\bar{x}}{\chi^2_{1-\alpha/2}} \leq \theta \leq \dfrac{2n\bar{x}}{\chi^2_{\alpha/2}}$

At 95% C.I,

$$1-\alpha = 0.95, \quad \alpha = 1-0.95 = 0.05$$

$$\bar{x} = \frac{98602}{15} = 6573.5$$

$$\rightarrow \quad \frac{2 \times 15 \times 6573.5}{\chi^2_{0.975}} \leq \theta \leq \frac{2 \times 15 \times 6573.5}{\chi^2_{0.025}}$$

$$\theta = [4{,}198.5, 11{,}745]$$

* The data support the claim that the bulbs, which run 24/7, last a year on average, since the mean, $\bar{x}$ falls within the interval computed ie.

$$\bar{x} \in [4{,}198.5, 11{,}745]$$

(d) $$\bar{x} \pm \frac{2\bar{x}}{\sqrt{n}} = 6{,}573.5 \pm \frac{2 \times 6{,}573.5}{\sqrt{15}}$$

$$= 6{,}573.5 \pm 3{,}394.54$$

$$= [3{,}178.96, 9{,}968.0]$$

The interval is narrower than the interval computed in (c)