INSTITUT DE MATHÉMATIQUE D'ORSAY

A Simple Model for the Balance between Selection and Mutation

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I would like to begin by expressing my heartfelt gratitude to Mr. Olivier Henard, the assistant professor at the Institute of Mathematics, Orsay, who ignited the flame of this both thrilling and challenging journey. Thanks to you, I had the wondrous discovery of the world of probabilities during my Magistère in Mathematics, and your research project topic provided me the opportunity to dive headfirst into the intricacies of research, illuminating my eyes with a new spark for the fascinating realm of biology. This voyage into the heart of life has strengthened my profound conviction about my future vocation. Whether it's the study of biological processes or the exploration of the frontiers of medicine, my insatiable passion burns brighter than ever.

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Chapter 1

Introduction

The evolution of populations is a captivating phenomenon that has drawn the interest of various scientific disciplines, including biology, genetics, ecology, and mathematics. In this report, we draw inspiration from the innovative works of Kingman to present a mathematical model describing the temporal variation of fitness distribution within a haploid population. Our approach distinguishes itself from previous models by considering the potential collapse of the evolutionary "house of cards" caused by mutations, rather than solely accounting for minor fitness perturbations. By studying the limiting properties of this model, we unveil a threshold phenomenon: if a certain inequality is satisfied, the limiting distribution is an asymmetric version of the mutant fitness distribution; otherwise, a probability mass accumulates at the upper limit of fitness. We present the outcomes of our comprehensive study of this model and discuss their implications for understanding population evolution.

1.1 Deciphering Key Concepts

Prior to delving into the study of the model describing the temporal variation of fitness distribution within a haploid population, it is imperative to lucidly define the pivotal terms of the subject matter. Indeed, a precise understanding of these terms will provide a clear and concise insight into the subject of study and facilitate comprehension of the attained results. Thus, we commence by elucidating fundamental concepts in population genetics, such as fitness, the interplay between selection and mutation, and the notion of the evolutionary "house of cards".

1.1.1 The Dance of Selection and Mutation

In biology, selection denotes the process through which certain individuals exhibiting specific traits have an increased likelihood of survival and reproduction compared to other individuals. These traits can be physical, behavioral, or other attributes, thereby conferring a competitive advantage to these individuals in their environment. Selection can be either natural, where these traits provide an advantage in the struggle for survival, or artificial, when humans choose preferred traits in domesticated plants or animals.

Mutation, on the other hand, is a random process that brings about changes in an organism's DNA. Mutations can be caused by errors during DNA replication, mutagenic agents such as ionizing radiation or chemicals, or spontaneous events. Mutations can be beneficial, harmful, or neutral to the organism, depending on their impact on physical or behavioral traits.

Within the context of evolution, selection and mutation are two forces that collaboratively shape populations over time. Mutations generate essential genetic diversity for selection to act upon, while selection removes variants less suited to their environment. Together, these forces can lead to the emergence of new species or the adaptation of existing populations to environmental changes.

1.1.2 Fitness: The Evolutionary Elevator

Fitness, also known as selective value or adaptiveness, is a measure of an organism's ability to survive and reproduce in its environment. It is determined by an individual's physical, behavioral, and physiological traits, as well as their impact on survival and reproduction. Individuals with higher fitness have a better probability of survival and reproduction, thus enhancing the likelihood of passing on their traits to the next generation.

Consider the example of wing color in butterflies. Butterflies exhibit a range of colors, spanning from white to black, including blue, red, green, and all intermediate shades. Wing color can have a significant impact on butterfly survival and reproduction. In a forested environment with limited light, a butterfly with light-colored wings may be more easily spotted by predators such as birds compared to a butterfly with darker wings. Individuals with darker wings have a better chance of survival and reproduction, thus promoting the transmission of wing color to the next generation. Over time, this can lead to an increase in the frequency of this wing color within the butterfly population.

1.1.3 The Complexity of Haploidy

An haploid organism possesses a single set of chromosomes, unlike diploid organisms, which have two sets. Examples of haploid organisms include bacteria and yeast, while humans and most other animals are diploid.

In the realm of genetics research, haploid organisms are often favored as they allow for a simpler and quicker analysis of mutation effects on phenotypic characteristics. However, it is important to note that some characteristics, such as resistance to certain diseases, may be more challenging to study in haploid organisms due to the absence of certain protective genes present in diploid organisms.

1.1.4 The Fragile Evolutionary "House of Cards"

The concept of the evolutionary "house of cards" highlights the fragility of biological evolution and how minor shifts in environmental conditions can have substantial effects on population adaptation. This analogy draws from the idea that evolution can be perceived as the gradual construction of adaptive features within a population. When environmental conditions change or mutations occur, it can trigger a cascade of effects on individual traits, potentially leading to the collapse of the evolutionary structure.

For instance, the introduction of an invasive species into an ecosystem can disrupt the ecological balance and impact adaptations of native species. The introduction of a predator can disturb the food chain, causing changes in adaptations of both prey species and native predators.

1.2 Dynamics of Selection and Mutation

The study of biological population evolution is intricate, as it necessitates the consideration of numerous mechanisms interacting in a complex manner. To comprehensively model the evolution of a biological population, it would be necessary to account for factors such as selection, mutation, non-random mating, as well as temporal and spatial variation, among others. However, modeling all these factors exhaustively would prove excessively complex for practical use. Therefore, it is common to assume an equilibrium is reached through a balance of two predominant factors, even though other phenomena may disrupt this equilibrium.

In the realm of population genetics, selection and mutation are two significant factors that interact in a complex manner to shape the evolution of a biological population. To study this dynamic, it is assumed that these two factors act upon gametes (male or female reproductive cells containing a single chromosome) within a population with random mating, rendering the problem haploid. Additionally, the population is sufficiently large to disregard genetic drift.

Generally, mutation is depicted using a "random walk" model where possible alleles at a locus (specific location of a gene on a chromosome) are identified as points on a line, and mutation causes a small jump to the right or left. However, since most mutations are deleterious, a model has been proposed where the gene after mutation is independent of the gene before, as mutation can disrupt the biochemical "house of cards" built by evolution. This model can be seen as a selective analogy to Ohta and Kimura's "infinite alleles" neutral model.

1.3 Context and Contribution

The inception of this research project emerged within a unique context, driven by my academic journey and professional aspirations. Faced with multiple internship rejections, Mr. Olivier Henard generously provided me with the opportunity to work on a topic within the life sciences—a field I contemplated specializing in for the future.

This endeavor embarked upon an intensive initial phase, spanning from May 11th to June 16th, during which I delved into a thorough study of the seminal paper authored by Kingman. My aim was to attain a profound understanding of all presented results, as well as the requisite auxiliary outcomes. Regular dialogues were maintained with the professor from the Institute of Mathematics, Orsay, enabling weekly discussions of my progress.

Through systematic and assiduous efforts, I swiftly and effectively navigated this initial stage. Subsequent investigations centered on an in-depth analysis of condensation, as explored by Dereich and Mörters. This facet of the study was approached with equal rigor and dedication, with the goal of comprehensively grasping the discussed outcomes.

After a brief respite, the project seamlessly resumed during the two summer months. This phase marked the culmination of the preceding study, as well as the exploration of the pending result left by Kingman, concerning unbounded fitness. These developments were accompanied by the addition of original and generalizing remarks within the report, along with the creation of an appendix providing an alternative perspective to that of Kingman.

Culminating in an apotheosis, I endeavored to achieve a generalization of the pending result, formulating a comprehensive theorem presented in conclusion. However, its demonstration currently remains unfinished and under study, constituting a point of continuation for future investigations. This endeavor was characterized by an unwavering commitment to precision and in-depth exploration, reflecting my dedication to thoroughly delve into the intricate and enthralling aspects of this domain.

Chapter 2

Mathematization of Dynamics

In this report, we will delve into a mathematical model depicting the evolution of a large population, while simultaneously considering both selection and genetic mutations. More precisely, our focus will be directed towards the distribution of fitness within this population.

2.1 Modeling Evolutionary Dynamics

We posit that this distribution can be succinctly represented by a probability measure denoted as p_n over a finite interval, which we shall consider to be the unit interval $I = [0, 1]$, driven by the significance of fitness ratios.

Definition 2.1 (Viability)**.** *The viability or the average fitness of the population is defined as the weighted mean of the fitness distribution:*

$$w_n = \int_0^1 x p_n(dx)$$

2.1.1 The Era of Selection

In the event that selection were the solitary determinant influencing the fitness distribution, it would be facile to anticipate that this distribution would gravitate towards favoring individuals endowed with higher fitness. This conjecture can be articulated mathematically via the subsequent equation:

$$p_{n+1}(dx) = \frac{x p_n(dx)}{w_n} \tag{2.1}$$

Here, w_n symbolizes the population viability in the n-th generation, and p_n represents the probability measure of the fitness distribution within said generation.

2.1.2 The Play of Mutations

However, with the presence of mutations, a proportion $\beta \in]0, 1[$ of the population undergoes modifications within its fitness distribution. This reality necessitates an amendment to the selection equation in order to account for this newfound distribution. Let us hypothetically posit that the novel mutants exhibit a fitness distribution, depicted by a probability measure q, spanning the interval I. In this scenario (assuming mutations manifest subsequent to selection), the preceding selection equation necessitates the subsequent adjustment:

$$p_{n+1}(dx) = (1-\beta)\frac{xp_n(dx)}{w_n} + \beta q(dx) \tag{2.2}$$

We shall demonstrate that, in numerous instances, the fitness distribution converges towards a distribution detached from the initial fitness distribution p_0. This outcome bears pivotal importance within the context of evolutionary theory. Albeit this model's inception being tailored for population genetics, its applicability extends to encompass equilibrium comprehension across diverse non-genetic contexts. Ergo, these equations equip mathematicians with an enhanced understanding of how population fitness can metamorphose amidst the presence of diverse selection and mutation mechanisms.

2.2 Analysis of Population Dynamics

Following our presentation of the model and the evolution dynamics elucidation, our analysis proceeds to deducing multiple overarching results concerning fitness and its associated viability. It is worth highlighting that certain proofs will be proffered within the appendix, streamlining report comprehension.

2.2.1 Fitness Distribution

Initially, grounded upon equation (2.2), coupled with recursive application, it becomes viable to craft an explicit formulation for the fitness distribution at generation $n+1$, encapsulating the initial distribution p_0 alongside the mutant distribution q. This precise formulation is lucidly expounded within the ensuing proposition:

Proposition 2.1 (Explicit Formula for p_n). *For any natural number n, the fitness distribution in generation n can be expressed as follows:*

$$p_n(dx) = \beta \sum_{k=0}^{n-1} \frac{(1-\beta)^k}{\prod_{i=0}^{k-1} w_{n-1-i}} x^k q(dx) + \frac{(1-\beta)^n}{\prod_{i=0}^{n-1} w_{n-1-i}} x^n p_0(dx) \tag{2.3}$$

where w_i signifies the population viability at generation i.

2.2.2 Product of Viabilities

We shall now delve into the study of viability, enabling us to derive conclusions about the fitness distribution. To accomplish this, we introduce a quantity denoted as W_n, which signifies the product of viabilities up to generation $n-1$. This quantity is defined by the following expression:

$$W_n = \prod_{i=0}^{n-1} w_i$$

Furthermore, we shall define the n-th order moments of the initial fitness distributions:

$$m_n = \int_0^1 x^n p_0(dx) \quad \text{and} \quad \mu_n = \int_0^1 x^n q(dx)$$

These n-th order moments lack inherent significance, yet they facilitate the computation of various vital quantities during the analysis of population dynamics.

By utilizing equation (2.3), we derive a recurrent expression for the product of viabilities. The subsequent proposition conveys this relationship with precision:

Proposition 2.2 (Recurrent Formula for W_n). *The product of viabilities over the first n generations, denoted as W_n, can be expressed in terms of moments as follows:*

$$W_n = \beta \sum_{k=0}^{n-1} W_{n-k}(1-\beta)^k \mu_k + (1-\beta)^n m_n \tag{2.4}$$

$$= \beta \sum_{k=1}^{n-1} W_{n-k}(1-\beta)^{k-1} \mu_k + (1-\beta)^{n-1} m_n \tag{2.5}$$

2.2.3 Generating Series Associated with the Product

We shall employ the last derived equation to define and study the generating series associated with W_n. This approach enables us to express this series as a meromorphic function, which can possess poles, thereby facilitating its analytical development. Subsequently, this method will allow us to deduce the potential limit of W_n and, consequently, that of p_n.

Before embarking on the study of the generating series, it is essential to verify the convergence of certain power series. To achieve this, we introduce the notation D_r for any $r \in [0, 1[$, defined as:

$$D_r = D\left(0, \frac{1}{1-r}\right) = \left\{ z \in \mathbb{C} \;\middle|\; |z| < \frac{1}{1-r} \right\}$$

We thus state the following lemma:

Lemma 2.3 (Convergence of Power Series).

- *For all $z \in D_\beta$, $\sum_{n\geq 1} (1-\beta)^n m_n z^n$ and $\sum_{n\geq 1} (1-\beta)^n \mu_n z^n$ are absolutely convergent*
- *For all $z \in D_0$, $\sum_{n\geq 1} W_n z^n$ is absolutely convergent*

Having established the absolute convergence of the power series, which are therefore holomorphic functions on the associated disks, we are now prepared to formulate the following theorem, enabling us to reframe the generating series associated with W_n. For any complex number z within the disk $D_0 \subset D_\beta$, we have:

Theorem 2.4 (Generating Series Associated with W_n). *The generating series associated with W_n can be expressed as follows:*

$$\sum_{n=1}^{+\infty} W_n z^n = \sum_{n=1}^{+\infty} (1-\beta)^{n-1} m_n z^n \left[1 - \beta \sum_{n=1}^{+\infty} (1-\beta)^{n-1} \mu_n z^n \right]^{-1} \tag{2.6}$$

$$= \int_0^1 \frac{zx}{1-(1-\beta)zx} p_0(dx) \left[1 - \int_0^1 \frac{\beta zx}{1-(1-\beta)zx} q(dx) \right]^{-1} \tag{2.7}$$

where p_0 and q are the initial fitness distributions for selection and mutation, respectively.

Proof. We can begin by observing that the series

$$\sum_{n\geq 1} W_n z^n \quad \text{and} \quad \sum_{n\geq 1} (1-\beta)^n \mu_n z^n$$

are absolutely convergent as per the preceding lemma. Thus, the series with general term

$$\sum_{k=1}^{n-1} \left(W_{n-k} z^{n-k} \right) \left((1-\beta)^k \mu_k z^k \right)$$

is also absolutely convergent. Utilizing equation (2.5) to express the generating series associated with W_n, we can employ the Cauchy product to write:

$$\begin{aligned}
\sum_{n=1}^{+\infty} W_n z^n &= \beta \sum_{n=1}^{+\infty} \sum_{k=1}^{n-1} W_{n-k}(1-\beta)^{k-1}\mu_k z^n + \sum_{n=1}^{+\infty}(1-\beta)^{n-1} m_n z^n \\
&= \frac{\beta}{1-\beta} \sum_{n=1}^{+\infty} \sum_{k=1}^{n-1} \left(W_{n-k} z^{n-k}\right)\left((1-\beta)^k \mu_k z^k\right) + \sum_{n=1}^{+\infty}(1-\beta)^{n-1} m_n z^n \\
&= \frac{\beta}{1-\beta} \left(\sum_{n=1}^{+\infty} W_n z^n\right)\left(\sum_{n=1}^{+\infty}(1-\beta)^n \mu_n z^n\right) + \sum_{n=1}^{+\infty}(1-\beta)^{n-1} m_n z^n
\end{aligned}$$

Isolating the generating series in this expression yields:

$$\left(\sum_{n=1}^{+\infty} W_n z^n\right)\left(1-\beta\sum_{n=1}^{+\infty}(1-\beta)^{n-1}\mu_n z^n\right) = \sum_{n=1}^{+\infty}(1-\beta)^{n-1} m_n z^n$$

and thus equation (2.6) is satisfied.

Now, let us explicitly express the moments in terms of their probability distribution to see the integrals needed for the formulation of (2.7):

$$\begin{aligned}
\sum_{n=1}^{+\infty} W_n z^n &= \sum_{n=1}^{+\infty}(1-\beta)^{n-1} m_n z^n \left[1-\beta\sum_{n=1}^{+\infty}(1-\beta)^{n-1}\mu_n z^n\right]^{-1} \\
&= \sum_{n=1}^{+\infty}(1-\beta)^{n-1} z^n \int_0^1 x^n p_0(dx) \left[1-\beta\sum_{n=1}^{+\infty}(1-\beta)^{n-1} z^n \int_0^1 x^n q(dx)\right]^{-1} \\
&= \frac{1}{1-\beta}\sum_{n=1}^{+\infty}\int_0^1 [(1-\beta)zx]^n\, p_0(dx) \left[1-\frac{\beta}{1-\beta}\sum_{n=1}^{+\infty}\int_0^1 [(1-\beta)zx]^n\, q(dx)\right]^{-1}
\end{aligned}$$

We now seek to interchange the series and integral to obtain equation (2.7). To achieve this, we note, for any positive integer n,

$$f_n : x \in I \longmapsto [(1-\beta)zx]^n$$

- $(f_n)_n$ is thus a sequence of continuous functions on I
- For all $n \in \mathbb{N}^*$, $|f_n| \leq |(1-\beta)z|^n$ with $|(1-\beta)z| < 1$, hence $\sum |(1-\beta)z|^n$ converges as a geometric series, and $\sum f_n$ converges uniformly to

$$f : x \longmapsto \frac{(1-\beta)zx}{1-(1-\beta)zx}$$

Utilizing the theorem of series-integral inversion, with uniform convergence on a segment, we can assert that f is continuous on I, and for p_0 and q, we have:

$$\sum_{n=1}^{+\infty}\int_0^1 [(1-\beta)zx]^n\, p_0(dx) = \int_0^1 \sum_{n=1}^{+\infty} [(1-\beta)zx]^n\, p_0(dx) = \int_0^1 \frac{(1-\beta)zx}{1-(1-\beta)zx} p_0(dx)$$

Consequently, upon inserting this result into the preceding equation, we obtain:

$$\sum_{n=1}^{+\infty} W_n z^n = \frac{1}{1-\beta}\int_0^1 \frac{(1-\beta)zx}{1-(1-\beta)zx} p_0(dx) \left[1-\frac{\beta}{1-\beta}\int_0^1 \frac{(1-\beta)zx}{1-(1-\beta)zx} q(dx)\right]^{-1}$$

This directly yields equation (2.7).

$\square$

The aforementioned theorem allows us to regard the right-hand side as a meromorphic function on the disk D_0, denoted as g, since it is a quotient of holomorphic functions on the same disk with a non-identically zero denominator:

$$g : x \in D_0 \longmapsto \int_0^1 \frac{zx}{1-(1-\beta)zx} p_0(dx) \left[1 - \int_0^1 \frac{\beta zx}{1-(1-\beta)zx} q(dx)\right]^{-1}$$

In this scenario, a preliminary step toward analyzing this function involves studying its potential poles.

The following corollary asserts that the meromorphic function g can have at most one pole, which will be simple. Moreover, it provides conditions to determine whether this function has a pole or not.

Corollary 2.4.1 (Poles of the Meromorphic Function)**.** *The meromorphic function g has at most one pole, and more precisely:*

$$g \text{ has no poles} \iff \int_0^1 \frac{q(dx)}{1-x} \leq \beta^{-1} \tag{2.8}$$

$$g \text{ has a simple pole} \iff \int_0^1 \frac{q(dx)}{1-x} > \beta^{-1} \tag{2.9}$$

In this case, the pole lies within the interval $[1, (1-\beta)^{-1}[$.

Proof. To begin the proof of the corollary, we will transform the interval I into its interior $\mathring{I} =]0,1[$.

According to the integral representation in the previous theorem, if z is a pole of g, then it satisfies the following equation:

$$\int_0^1 \frac{\beta zx}{1-(1-\beta)zx} q(dx) = 1 \tag{2.10}$$

The above equation, which relates a complex number to a real number, implies that the imaginary part of the integral is zero. The imaginary part of the integral can be rewritten as follows:

$$\begin{aligned}\mathfrak{Im}\left(\int_0^1 \frac{\beta zx}{1-(1-\beta)zx} q(dx)\right) &= \mathfrak{Im}\left(\int_0^1 \frac{\beta zx(1-(1-\beta)\overline{z}x)}{|1-(1-\beta)zx|^2} q(dx)\right) \\ &= \beta\mathfrak{Im}(z) \int_0^1 \frac{x}{|1-(1-\beta)zx|^2} q(dx)\end{aligned}$$

As the integrand is strictly positive on I and β is also strictly positive, this implies that any potential pole of g satisfies:

$$\mathfrak{Im}(z) = 0$$

We will now study the integrand of equation (2.10) as a function of two variables, where the first variable belongs to the real part of the disk D_β, denoted as:

$$D_\beta^{\mathbb{R}} = \mathbb{R} \cap D_\beta$$

We introduce the following notations for the integrand and its integral:

$$\varphi : (z,x) \in D_\beta^{\mathbb{R}} \times I \longmapsto \frac{\beta zx}{1-(1-\beta)zx} \quad \text{and} \quad \psi : z \longmapsto \int_0^1 \varphi(z,x) q(dx)$$

Let's start by studying the differentiability of the latter function in order to exploit both its continuity and its derivative subsequently.

- For any $z \in D_\beta^{\mathbb{R}}$, the function $\varphi(z,.)$ is continuous and integrable on I.

- The function $\partial_z\varphi$ is well-defined on $D_\beta^{\mathbb{R}} \times I$, and can be expressed as follows:

$$\partial_z\varphi : (z,x) \longmapsto \frac{\beta x}{(1-(1-\beta)zx)^2}$$

  - For any $z \in D_\beta^{\mathbb{R}}$, the function $\partial_z\varphi(z,.)$ is piecewise continuous on I.
  - For any $x \in I$, the function $\partial_z\varphi(.,x)$ is continuous on $D_\beta^{\mathbb{R}}$.
  - For any $(z,x) \in D_\beta^{\mathbb{R}} \times I$, the following inequality holds:

$$|\partial_z\varphi(z,x)| \leq \beta\frac{x}{(1-x)^2}$$

    If we denote $\zeta(x)$ as the term on the right, then ζ is a continuous and integrable function of x on I.

Thus, by applying the theorem on parameter-differentiation, we can conclude that the function ψ is of class $\mathcal{C}^1$, with a derivative given by:

$$\psi' : z \longmapsto \int_0^1 \frac{\beta x}{(1-(1-\beta)zx)^2} q(dx) \tag{2.11}$$

Due to the $\mathcal{C}^1$ class of the function, we can utilize its continuity. Since $\varphi(.,x)$ is strictly increasing on $D_\beta^{\mathbb{R}}$ for any x in I, ψ is also strictly increasing. Furthermore, as $\psi(1) \leq 1$ due to the same inequality satisfied by the integrand, the intermediate value theorem allows us to conclude that equation (2.10) has at most one solution, i.e., g has at most one pole.

Now, we will examine the conditions under which the function has a pole or not. By using similar arguments as before, we can assert that since $\psi(1) \leq 1$, it is possible for a pole to exist in the interval $[1,(1-\beta)^{-1}[$ if the following limit is satisfied:

$$\lim_{z\to(1-\beta)^{-1}} \psi(z) > 1$$

To simplify this condition, we will verify the required conditions to exchange the limit and the integral. The conditions are as follows:

- For any $z \in D_\beta^{\mathbb{R}}$, the function $\varphi(z,.)$ is piecewise continuous on I.
- For any $x \in I$, the following limit holds:

$$\varphi(z,x) \xrightarrow[z\to(1-\beta)^{-1}]{} \frac{\beta x}{(1-\beta)(1-x)}$$

  If we denote $\xi(x)$ as the term on the right, then ξ is a piecewise continuous function of x on I.
- For any $(z,x) \in D_\beta^{\mathbb{R}} \times I$, the following inequality holds:

$$|\varphi(z,x)| \leq \frac{\beta}{1-\beta}\frac{x}{1-x}$$

  If we denote $\delta(x)$ as the term on the right, then δ is an integrable function on I.

By using the dominated convergence theorem, we can then conclude that for any $z \in D_\beta^{\mathbb{R}}$, the functions $\varphi(z,.)$ and ξ are integrable, and:

$$\lim_{z\to(1-\beta)^{-1}} \int_0^1 \varphi(z,x)dx = \int_0^1 \xi(x)dx$$

By using the aforementioned arguments, we can obtain the following chain of inequalities that leads to the final condition:

$$
\begin{aligned}
\lim_{z\to(1-\beta)^{-1}} \psi(z) > 1 &\iff \int_0^1 \frac{\beta x}{(1-\beta)(1-x)} q(dx) > 1 \\
&\iff \int_0^1 \frac{\beta x - (1-\beta)(1-x)}{(1-\beta)(1-x)} q(dx) > 0 \\
&\iff \int_0^1 \frac{\beta - (1-x)}{(1-\beta)(1-x)} q(dx) > 0 \\
&\iff \frac{\beta}{1-\beta} \int_0^1 \frac{q(dx)}{1-x} > \frac{1}{1-\beta} \\
&\iff \int_0^1 \frac{q(dx)}{1-x} > \beta^{-1}
\end{aligned}
$$

It is worth noting that, in the case where this condition is satisfied, the pole is indeed simple. By using (2.11), we observe that the integrand is strictly positive, which implies that $\psi'(z_0) > 0$, where z_0 denotes the pole.

□

In the context of the selection and mutation model, the study of the potential pole of the meromorphic function provides insight into the mechanisms governing the evolution of populations and helps determine the conditions under which a population is stable or unstable. Such understanding is crucial for the management and conservation of animal and plant species in their natural habitats. In this context, the simple pole can be interpreted as a critical parameter of the model.

Chapter 3

Evolution and Selection Regimes

In his quest to explore the mechanisms of evolution, Kingman proposes a bold classification of modes of selection. Like a modern-day scholar, he employs evocative terms such as "democracy," "meritocracy," and "aristocracy" to describe these distinct modes. This striking classification allows us to grasp how different forms of selection influence population evolution and give rise to delicate balances between mutation and selection. Like intertwined spheres, we delineate the modes of selection, separating democracy on one side and meritocracy and aristocracy on the other, based on the presence or absence of poles in the meromorphic function, in accordance with inequalities (2.8) and (2.9). However, elucidating the specific condition that distinguishes these latter modes remains a challenge to be addressed.

Equipped with knowledge, let us define the inherent properties of each selection regime, drawing inspiration from political concepts:

- In the "democratic" selection regime, each individual enjoys an equal probability of being selected for reproduction. Thus, all individuals have a fair opportunity to shape the next generation.
- In the "meritocratic" selection regime, individuals possessing particularly favorable traits have an increased probability of being selected for reproduction. In this regime, selection is based on the merit of each individual, assessed in terms of their advantageous trait.
- Finally, in the "aristocratic" selection regime, a select group of dominant individuals is granted a higher probability of being chosen for reproduction. In this regime, selection is based on social position or status of each individual, rather than specific traits.

These properties will be verified throughout our study.

3.1 Democratic Regime: Biological Equilibrium in Perspective

Let us delve into the exploration of the democratic regime, where each individual plays a crucial role. First, let us assume the verification of the following inequality:

$$\boxed{\int_0^1 \frac{q(dx)}{1-x} > \beta^{-1}}$$

This inequality implies the existence of a unique pole z_0 for the function g, situated within the interval $[1, (1-\beta)^{-1}[$. Let $s = z_0^{-1}$, which satisfies the following conditions:

$$\int_0^1 \frac{\beta x}{s-(1-\beta)x} q(dx) = 1 \quad \text{and} \quad 1-\beta < s \leq 1 \tag{3.1}$$

In this chapter, we proceed in reverse compared to the previous chapter, first studying viability and then addressing fitness.

3.1.1 Convergence toward Harmonious Equilibrium

The value of s we have defined holds significant meaning, as demonstrated by the following proposition:

Proposition 3.1 (Limit Viability in Democratic Regime)**.** *The sequence of viabilities $(w_n)_n$ converges, and the limit viability is equal to the inverse of the pole of g:*

$$w_n \xrightarrow[n \to +\infty]{} s$$

Proof. Recall that

$$g : x \longmapsto \int_0^1 \frac{zx}{1-(1-\beta)zx} p_0(dx) \left[1 - \int_0^1 \frac{\beta zx}{1-(1-\beta)zx} q(dx)\right]^{-1}$$

is a holomorphic function on the punctured disk $D_\beta \backslash z_0$, with a simple pole at z_0. According to the theorem on classification of isolated singularities, the holomorphic function h, defined as:

$$h : z \in D_\beta \backslash \{z_0\} \longmapsto g(z) - \frac{\mathrm{Res}(g, z_0)}{z - z_0}$$

has a removable singularity at z_0. Consequently, it has a unique holomorphic extension over the entire disk D_β.

By denoting g_1 and g_2 as the numerator and denominator of g respectively, the residue of g at z_0 can be expressed as:

$$\mathrm{Res}(g, z_0) = \lim_{z \to z_0} (z - z_0) g(z) = \frac{g_1(z_0)}{g_2'(z_0)}$$

Combining this equation with the inequality $s(1-\beta)^{-1} > 1 \geq x$ for all $x \in I$, we obtain:

$$\mathrm{Res}(g, s^{-1}) = \int_0^1 \frac{x}{s-(1-\beta)x} p_0(dx) \left[- \int_0^1 \frac{\beta s^2 x}{(s-(1-\beta)x)^2} q(dx)\right]^{-1} < 0$$

Let $a = -s \cdot \mathrm{Res}(g, s^{-1}) > 0$. Thus, according to Equation (2.7), for all $z \in D_0$, as $|zs| < 1$, we have:

$$h(z) = \sum_{n=1}^{+\infty} W_n z^n - \frac{a}{1-zs} = \sum_{n=1}^{+\infty} W_n z^n - a \sum_{n=0}^{+\infty} s^n z^n = -a + \sum_{n=1}^{+\infty} (W_n - a s^n) z^n$$

Furthermore, this equation indicates that h is expandable in a power series in the vicinity of 0. Due to the analyticity of holomorphic functions and the uniqueness of the power series expansion of h, we deduce that:

$$\forall n \in \mathbb{N}^*, \quad W_n - a s^n = \frac{h^{(n)}(0)}{n!} \quad \text{and} \quad h(0) = -a$$

Consider $\theta_0 > 0$. Then, h is a holomorphic function defined in the vicinity of

$$\overline{D\left(0, \frac{1}{\theta_0(1-\beta)}\right)} \subsetneq D_\beta$$

By Cauchy's estimates, we have:

$$\forall n \in \mathbb{N}^*, \quad \left|\frac{h^{(n)}(0)}{n!}\right| \leq \frac{M_{r_0}}{r_0^n}, \quad \text{with} \quad r_0 = \frac{1}{\theta_0(1-\beta)} \quad \text{and} \quad M_{r_0} = \sup_{|z|=r_0} |h(z)| < +\infty$$

This implies that:

$$\forall n \in \mathbb{N}^*, \quad |W_n - a s^n| \leq \frac{M_{r_0}}{r^n}$$

By choosing $\theta > \theta_0$ and setting $r = [\theta(1-\beta)]^{-1} < r_0$, we obtain:

$$|W_n - as^n|r^n \leq M_{r_0} \underbrace{\left(\frac{r}{r_0}\right)}_{<1}^{n} \xrightarrow[n\to+\infty]{} 0$$

Therefore, since this property holds for any $\theta_0 > 1$, by letting θ_0 tend to 1, we arrive at the following statement:

$$\forall \theta > 1, \quad W_n = as^n + o\left((1-\beta)^n \theta^n\right) \tag{3.2}$$

Let us utilize this result concerning the product of viabilities to deduce a similar property for viability. Let $\delta = (1-\beta)s^{-1}$. Then we have:

$$w_n = \frac{W_{n+1}}{W_n} = \frac{as^{n+1} + o((1-\beta)^{n+1}\theta^{n+1})}{as^n + o((1-\beta)^n\theta^n)} = \frac{s + o(s^{-n}(1-\beta)^{n+1}\theta^{n+1})}{1 + o(s^{-n}(1-\beta)^n\theta^n)} = \frac{s + o(\delta^n\theta^n)}{1 + o(\delta^n\theta^n)}$$

Now denote $x_n = s + o(\delta^n\theta^n)$ and $y_n = 1 + o(\delta^n\theta^n)$ as the numerator and denominator of w_n respectively. Then, there exist two sequences $(\varepsilon_n^x)_n$ and $(\varepsilon_n^y)_n$ such that from a certain index N onwards, we have the following inequalities:

$$|x_n - s| \leq \varepsilon_n^x \delta^n \theta^n \quad \text{and} \quad |y_n - 1| \leq \varepsilon_n^y \delta^n \theta^n$$

Starting from this integer N, and enlarging it if necessary, the sequence $(y_n)_n$ is non-zero, meaning there exists a constant $c > 0$ such that for all $n \geq N$, we have $|y_n| \geq c$. Hence, for all $n \geq N$, we obtain:

$$|w_n - s| = \left|\frac{x_n - s - s(y_n - 1)}{y_n}\right| \leq \frac{|x_n - s| + s|y_n - 1|}{|y_n|} \leq \left(\frac{\varepsilon_n^x + s\varepsilon_n^y}{c}\right)\delta^n\theta^n$$

Letting $\varepsilon_n^w = (\varepsilon_n^x + s\varepsilon_n^y)c^{-1}$, we observe that the sequence ε^w converges to 0. Thus, we arrive at the following statement concerning limit viabilities:

$$\forall \theta > 1, \quad w_n = s + o(\delta^n\theta^n) \tag{3.3}$$

Since this result holds for any $\theta > 1$, and $\delta < 1$, we choose θ such that $\delta\theta < 1$ to ensure that the term within the small o converges to 0. This allows us to deduce the proposition regarding the limit viability.

□

3.1.2 Convergence to a Stable Distribution

We have now established the limit viability. Let us now state a theorem that describes the distribution of the limit fitness in the democratic regime, along with its remarkable properties.

Theorem 3.2 (Limit Fitness in Democratic Regime)**.** *In the democratic regime where*

$$\int_0^1 \frac{q(dx)}{1-x} > \beta^{-1}$$

the sequence of fitness distributions $(p_n)_n$ converges in total variation. The limit distribution is characterized by the following expression:

$$p(dx) = \frac{\beta s}{s - (1-\beta)x} q(dx)$$

This limit distribution is absolutely continuous with respect to the mutant distribution q and independent of the initial distribution p_0.

Proof. To prove this theorem, let us revisit the fundamentals of this study. From Equation (2.3), we can deduce the following expression:

$$p_n(dx) = \beta \sum_{k=0}^{n-1} \frac{W_{n-k}}{W_n}(1-\beta)^k x^k q(dx) + \frac{1}{W_n}(1-\beta)^n x^n p_0(dx)$$

Given that $1-\beta < s$, we can express p similarly:

$$p(dx) = \beta \sum_{k=0}^{+\infty} \left(\frac{1-\beta}{s}\right)^k x^k q(dx)$$

Let $f \in \mathcal{M}_I$, then we have:

$$\begin{aligned}
&f(x)p_n(dx) - f(x)p(dx) \\
&= \beta \sum_{k=0}^{n-1} \frac{W_{n-k}}{W_n}(1-\beta)^k f(x) x^k q(dx) \;... \\
&\qquad ... + \frac{1}{W_n}(1-\beta)^n f(x) x^n p_0(dx) \;... \\
&\qquad\qquad ... - \beta \sum_{k=0}^{+\infty} \left(\frac{1-\beta}{s}\right)^k f(x) x^k q(dx) \\
&= \beta \sum_{k=0}^{n-1} \left(\frac{W_{n-k}}{W_n} - s^{-k}\right)(1-\beta)^k f(x) x^k q(dx) \;... \\
&\qquad ... + \frac{1}{W_n}(1-\beta)^n f(x) x^n p_0(dx) - \beta \sum_{k=n}^{+\infty} \delta^k f(x) x^k q(dx)
\end{aligned}$$

Thus, we obtain the following inequality:

$$\|p_n - p\|_{TV} \leq \beta \sum_{k=0}^{n-1} \left|\frac{W_{n-k}}{W_n} - s^{-k}\right| (1-\beta)^k + \frac{(1-\beta)^n}{W_n} + \beta \sum_{k=n}^{+\infty} \delta^k$$

To study this inequality, let's focus on the three terms one by one. For this purpose, take $\theta > 1$ to use Equations (3.2) and (3.3). Let's tackle them from right to left, from simplest to more complex.

- The last term can be directly studied without any issue:

$$\beta \sum_{k=n}^{+\infty} \delta^k = \beta \frac{\delta^n}{1-\delta} = o(\delta^n \theta^n)$$

- The second term is also straightforward to study using (3.2). By potentially increasing θ, we have:

$$\frac{(1-\beta)^n}{W_n} = \frac{(1-\beta)^n}{as^n + o\left((1-\beta)^n \theta^n\right)} = \frac{\delta^n}{a + o\left(\delta^n \theta^n\right)} = o(\delta^n \theta^n)$$

- As for the first term, note first that using (3.3), there exists a sequence $(\varepsilon_{k,n})_n$ converging to 0 and an integer N such that for n beyond this index:

$$\frac{W_{n-k}}{W_n} = (s + \varepsilon_{k,n}\delta^n\theta^n)^{-k} > 0$$

Thus, we can split the sum into two parts using this integer. Abusing notation, we write:

$$\beta \sum_{k=0}^{n-1} \left|\frac{W_{n-k}}{W_n} - s^{-k}\right| (1-\beta)^k = \beta \left(\sum_{k=0}^{n-N} + \sum_{k=n-N+1}^{n-1}\right) \left|\frac{W_{n-k}}{W_n} - s^{-k}\right| (1-\beta)^k$$

First Part: For the smaller values of k, where W_{n-k} remains "close" to W_n, examine this sum using the asymptotic property:

$$\frac{W_{n-k}}{W_n} - s^{-k} = (s + \varepsilon_{k,n}\delta^n\theta^n)^{-k} - s^{-k}$$

The function $x \longmapsto x^{-k}$ is continuous and differentiable over $\mathbb{R}_+$, with a derivative $x \longmapsto -kx^{-k-1}$. Hence, by the mean value theorem, there exists a sequence $(\alpha_{k,n})_n$ belonging to the interval $[0,1]$ such that the previous equality can be written as:

$$\frac{W_{n-k}}{W_n} - s^{-k} = -k\frac{\varepsilon_{k,n}\delta^n\theta^n}{(s + \alpha_{k,n}\varepsilon_{k,n}\delta^n\theta^n)^{k+1}} = o(\delta^n\theta^n)$$

Thus, the sum can be rewritten as:

$$\beta\sum_{k=0}^{n-N}\left|\frac{W_{n-k}}{W_n} - s^{-k}\right|(1-\beta)^k = \beta o(\delta^n\theta^n)\sum_{k=0}^{n-N}\underbrace{(1-\beta)^k}_{<1} \leq \beta o(\delta^n\theta^n)\frac{1-\beta}{\beta} = o(\delta^n\theta^n)$$

Second Part: For larger values of k, split the sum again into two parts using the triangle inequality, to show that, with a change of index, the terms W_{n-k}, which are fixed and finite in number, do not pose a problem:

$$\beta\sum_{k=n-N+1}^{n-1}\left|\frac{W_{n-k}}{W_n} - s^{-k}\right|(1-\beta)^k \leq \beta\sum_{j=1}^{N-1}\frac{W_j}{W_n}(1-\beta)^{n-j} + \beta\sum_{k=n-N+1}^{n-1}\delta^k$$

Using the previous result showing the smallness of the second term, and the fact that the first sum is finite, we obtain:

$$\beta\sum_{j=1}^{N-1}\frac{W_j}{W_n}(1-\beta)^{n-j} \leq \beta\frac{(1-\beta)^n}{W_n}\sum_{j=1}^{N-1}\left(\frac{1}{1-\beta}\right)^j = o(\delta^n\theta^n)$$

Moreover, the second sum directly leads to:

$$\beta\sum_{k=n-N+1}^{n-1}\delta^k \leq \beta\frac{\delta^{n-N+1}}{1-\delta} = o(\delta^n\theta^n)$$

In summary, by decomposing into three terms, we have shown that:

$$\|p_n - p\|_{TV} = o(\delta^n\theta^n) \tag{3.4}$$

As in the previous proof, let's choose θ in such a way that $\delta\theta < 1$ to ensure the term within the small o converges to 0. Thus, we can conclude regarding the limit fitness theorem.

□

It is crucial to emphasize that the limit fitness distribution does not depend on the initial distribution p_0, implying that the initial composition of the population does not affect the final shape of the fitness distribution. Regardless of the initial state of the population, thanks to the absolute continuity of the limit distribution with respect to that of mutants, all individuals have an equal chance to reproduce and pass on their heritage to future generations. The effect of selection simply translates to a rightward shift of the distribution.

However, it is important to note that the distribution of mutants q plays a crucial role in determining the limit fitness distribution. It brings genetic diversity that can influence the long-term dynamics of the population. While the exponentially fast convergence is promising, it's worth noting that, in the proof, the convergence parameter $\delta = (1-\beta)s^{-1}$ can be very close to unity when β is small, which is intuitive as the proportion of mutants will be smaller.

Furthermore, before studying the case where g has no poles, it is remarkable that the definition of p through the limit fitness theorem gives us the value of the limit viability, using the equation satisfied by s mentioned at the beginning of the section:

$$p(dx) = \frac{\beta s}{s-(1-\beta)x}q(dx) \Longrightarrow \int_0^1 xp(dx) = s\int_0^1 \frac{\beta x}{s-(1-\beta)x}q(dx) = s$$

3.2 Non-Democratic Regimes: Concentration of Power

We now delve into the study of the case where the function g has no poles, which means that the following inequality is satisfied:

$$\int_0^1 \frac{q(dx)}{1-x} \leq \beta^{-1}$$

Consequently, we deduce that:

$$\int_0^1 \sum_{n=0}^{+\infty} x^n q(dx) \leq \beta^{-1} \Longrightarrow \sum_{n=0}^{+\infty} \mu_n \leq \beta^{-1} \Longrightarrow \sum_{n=1}^{+\infty} \mu_n \leq \frac{1-\beta}{\beta}$$

Under this new assumption, our function g no longer has poles. Therefore, the limit viability will differ from that obtained in the case of the democratic regime. To address this new scenario, we will adopt a different approach using renewal sequences defined in Appendix B.

Let's define $(f_n)_n$ as follows:

$$\forall n \in \mathbb{N}, \quad f_n = \frac{\beta}{1-\beta}\mu_n \tag{3.5}$$

Thus, this sequence satisfies the conditions of the definition in the appendix, and let $(u_n)_n$ be the associated renewal sequence according to (B.1). This sequence exhibits several interesting properties for our study. Let's first focus on the limit of the ratio of successive terms. This limit will allow us to divide the study into two distinct selection regimes: the "meritocratic" regime and the "aristocratic" regime, depending on whether it is equal to 1 or strictly less than this value, respectively:

$$\lim_{n\to+\infty} \frac{u_n}{u_{n-1}} = 1 \quad \text{or} \quad \lim_{n\to+\infty} \frac{u_n}{u_{n-1}} = \sigma < 1$$

As per Remark B.1 stated in the appendix, when the limit of the sequence of ratios of successive terms is equal to 1, it signifies that as time progresses, the probability of returning to the initial state at time n is equivalent to the probability of returning at time $n-1$. In other words, the chances of selection or returning to the initial state do not depend on elapsed time. Thus, this behavior is often associated with a "meritocratic" regime.

Conversely, when the limit of the sequence of ratios of successive terms is strictly less than 1, it indicates that the probability of returning to the initial state gradually decreases over time. This can be interpreted as a gradual concentration of power or opportunities among a privileged few individuals. Therefore, this behavior is often associated with an "aristocratic" regime.

It is important to note that these paragraphs aim to explain the intuition behind the distinction between the two cases. A more detailed explanation of the concentration of advantages within a restricted group of individuals will be addressed in the following sections.

3.3 Rise of Meritocracy

We will now study the first case of a non-democratic regime, assuming the following hypotheses:

$$\boxed{\int_0^1 \frac{q(dx)}{1-x} \leq \beta^{-1} \quad \text{and} \quad \lim_{n\to+\infty} \frac{u_n}{u_{n-1}} = 1}$$

To analyze the viability and limit fitness of this regime, we introduce a new sequence $(v_n)_n$ defined from the nth-order moments of the initial distribution p_0 and the renewal sequence as follows:

$$\forall n \in \mathbb{N}^*, \quad v_n = \sum_{k=1}^{n} m_k u_{n-k} \tag{3.6}$$

3.3.1 Reduction of Diversity

To study the limit viability in the meritocratic regime, we need to utilize specific lemmas based on the sequence $(v_n)_n$ that we have introduced. The first lemma involves examining the limit of the ratios of successive terms of this sequence, which happens to be the same as that of the ratios of successive terms of the sequence $(u_n)_n$ mentioned in the hypotheses.

Lemma 3.3 (Successive Term Ratios). *The sequence of ratios of successive terms of $(v_n)_n$ converges, and we have:*

$$\lim_{n\to+\infty} \frac{v_n}{v_{n-1}} = 1$$

Let's recall the viability product defined in the previous chapter and remind ourselves of equation (2.5) that we will use:

$$W_n = \beta \sum_{k=1}^{n-1} W_{n-k}(1-\beta)^{k-1}\mu_k + (1-\beta)^{n-1}m_n$$

Using this equation, we can formulate our second lemma as follows:

Lemma 3.4 (Viability Product). *The viability product can be expressed in terms of the sequence $(v_n)_n$ as follows:*

$$\forall n \in \mathbb{N}^*, \quad W_n = (1-\beta)^{n-1}v_n \tag{3.7}$$

Using these two lemmas, we can easily deduce the limit viability in the case of a meritocratic regime. The following proposition states our desired result.

Proposition 3.5 (Meritocratic Limit Viability). *The sequence of viabilities $(w_n)_n$ converges, and the limit viability corresponds to the proportion of the non-mutant population:*

$$w_n \xrightarrow[n\to+\infty]{} 1-\beta$$

Proof. The proof of this result is straightforward:

$$w_n = \frac{W_{n+1}}{W_n} = (1-\beta)\frac{v_{n+1}}{v_n} \xrightarrow[n\to+\infty]{} 1-\beta$$

□

3.3.2 Convergence to Excellence

We now turn to the study of the limit fitness distribution. In the case of a non-democratic regime, we adopt a different approach by proceeding in two steps. We start by proving a result for Lebesgue-measurable intervals strictly less than 1 within I, before deducing the overall result for the entire interval. The following lemma represents the first step of this approach.

Lemma 3.6 (Partial Limit Fitness). *Let $\xi \in]0,1[$. The sequence of fitness distributions converges in total variation on the restricted interval $[0,\xi]$. The partial limit distribution is characterized by the following expression:*

$$\tilde{p}(dx) = \frac{\beta}{1-x}q(dx)$$

Proof. To prove this result, we draw inspiration from the same proof as in the democratic case. Thus, we begin with the same expression:

$$p_n(dx) = \beta \sum_{k=0}^{n-1} \frac{W_{n-k}}{W_n}(1-\beta)^k x^k q(dx) + \frac{1}{W_n}(1-\beta)^n x^n p_0(dx)$$

The difference here lies in the introduction of a new sequence $(v_n)_n$ which proves useful in simplifying our study. This expression can then be rewritten as:

$$p_n(dx) = \beta \sum_{k=0}^{n-1} \frac{v_{n-k}}{v_n} x^k q(dx) + \frac{1}{v_n}(1-\beta)x^n p_0(dx)$$

Since $\xi < 1$, we can express $\tilde{p}$ similarly:

$$\tilde{p}(dx) = \beta \sum_{k=0}^{+\infty} x^k q(dx)$$

For any $f \in \mathcal{M}_I$, we then have:

$$\begin{aligned} f(x)p_n(dx) - f(x)\tilde{p}(dx) &= \beta \sum_{k=0}^{n-1} \frac{v_{n-k}}{v_n} f(x)x^k q(dx) + \frac{1}{v_n}(1-\beta)f(x)x^n p_0(dx) \ ... \\ &\quad ... - \beta \sum_{k=0}^{+\infty} f(x)x^k q(dx) \\ &= \beta \sum_{k=0}^{n-1} \left(\frac{v_{n-k}}{v_n} - 1 \right) f(x)x^k q(dx) \ ... \\ &\quad ... + \frac{1}{v_n}(1-\beta)f(x)x^n p_0(dx) - \beta \sum_{k=n}^{+\infty} f(x)x^k q(dx) \end{aligned}$$

Thus, we arrive at the following inequality:

$$\|p_n - \tilde{p}\|_{TV,\xi} \leq \beta \sum_{k=0}^{n-1} \left| \frac{v_{n-k}}{v_n} - 1 \right| \xi^k + \frac{(1-\beta)\xi^n}{v_n} - \beta \sum_{k=n}^{+\infty} \xi^k$$

To study this inequality, let's focus on the three terms one by one, starting from the simplest term and progressing to the more complex one.

- The last term can be directly studied without difficulty:

$$\beta \sum_{k=n}^{+\infty} \xi^k = \beta \frac{\xi^n}{1-\xi} = o(1)$$

- The second term is also straightforward to study. Let's denote it Ξ_n. We have the following result:

$$\frac{\Xi_{n+1}}{\Xi_n} = \xi \frac{v_n}{v_{n+1}} \xrightarrow[n \to +\infty]{} \xi < 1 \quad \Longrightarrow \quad \Xi_n = o(1)$$

- Regarding the first term, first note that there exists a sequence $(\varepsilon_{k,n})_n$ converging to 0 and an integer N such that for n beyond this index:

$$\frac{v_{n-k}}{v_n} = (1+\varepsilon_{k,n})^{-k} > 0$$

  We can then split the sum into two parts using this integer. For the sake of simplicity, let's write:

$$\beta \sum_{k=0}^{n-1} \left| \frac{v_{n-k}}{v_n} - 1 \right| \xi^k = \beta \left(\sum_{k=0}^{n-N} + \sum_{k=n-N+1}^{n-1} \right) \left| \frac{v_{n-k}}{v_n} - 1 \right| \xi^k$$

  First Part: For smaller values of k, where v_{n-k} remains "close" to v_n, let's examine this sum using the asymptotic property:

$$\frac{v_{n-k}}{v_n} - 1 = (1+\varepsilon_{k,n})^{-k} - 1^{-k}$$

By the mean value theorem, using arguments similar to those in the democratic case, there exists a sequence $(\alpha_{k,n})_n$ belonging to the interval $[0,1]$ such that the above equality can be written as:

$$\frac{v_{n-k}}{v_n} - 1 = -k\frac{\varepsilon_{k,n}}{(1+\alpha_{k,n}\varepsilon_{k,n})^{k+1}} = o(1)$$

Thus, the sum can be rewritten as:

$$\beta\sum_{k=0}^{n-N}\left|\frac{v_{n-k}}{v_n} - 1\right|\xi^k = \beta o(1)\sum_{k=0}^{n-N}\xi^k \leq \beta o(1)\frac{\xi}{1-\xi} = o(1)$$

Second Part: For larger values of k, let's split the sum again into two parts using the triangle inequality, to show that, with a change of index, the terms v_{n-k}, which are a fixed finite number, pose no problem:

$$\beta\sum_{k=n-N+1}^{n-1}\left|\frac{v_{n-k}}{v_n} - 1\right|\xi^k \leq \beta\sum_{j=1}^{N-1}\frac{v_j}{v_n}\xi^{n-j} + \beta\sum_{k=n-N+1}^{n-1}\xi^k$$

Using the previous result showing the smallness of the second term, and the fact that the first sum is finite, we obtain:

$$\beta\sum_{j=1}^{N-1}\frac{v_j}{v_n}\xi^{n-j} \leq \beta\frac{\xi^n}{v_n}\sum_{j=1}^{N-1}\left(\frac{1}{\xi}\right)^j = o(1)$$

Furthermore, the second sum directly leads to:

$$\beta\sum_{k=n-N+1}^{n-1}\xi^k \leq \beta\frac{\xi^{n-N+1}}{1-\xi} = o(1)$$

In summary, by breaking down into three terms, we have shown that:

$$\|p_n - \tilde{p}\|_{TV,\xi} = o(1) \tag{3.8}$$

Consequently, we can conclude this lemma on the partial limit fitness.

□

Using the appendix on the convergence of measures, we can state the long-awaited result.

Theorem 3.7 (Meritocratic Limit Fitness)**.** *In the meritocratic regime:*

$$\int_0^1 \frac{q(dx)}{1-x} \leq \beta^{-1} \quad \textit{and} \quad \lim_{n\to+\infty}\frac{u_n}{u_{n-1}} = 1$$

the sequence of fitness distributions $(p_n)_n$ converges weakly. The limiting distribution is characterized by the following expression:

$$p(dx) = \frac{\beta}{1-x}q(dx) + \left(1 - \int_0^1 \frac{\beta}{1-y}q(dy)\right)\delta_1(dx)$$

This limit is no longer absolutely continuous with respect to the mutant distribution q, but it remains independent of the initial distribution p_0.

Proof. We established in the appendix that the space of probability measures on I, denoted $\mathcal{P}_I$, equipped with the weak topology, is a compact metric space. Therefore, by possibly extracting a subsequence, the sequence of fitness distributions $(p_n)_n$ converges weakly to a probability measure that we denote as $\hat{p}$.

Moreover, we demonstrated that convergence in total variation implies weak convergence. Thus, according to the previous lemma, for any real $\xi \in]0,1[$, the sequence $(p_n)_n$ converges weakly to $\tilde{p}$ on the interval $[0,\xi]$. Due to the uniqueness of the limit, the measures $\hat{p}$ and $\tilde{p}$ coincide on these restricted intervals.

Given that $\hat{p}$ is a probability measure, we obtain the following equalities:

$$\int_0^\xi \hat{p}(dy) = \int_0^\xi \frac{\beta}{1-y} q(dy) \quad \text{and} \quad \int_0^1 \hat{p}(dy) = \int_0^1 \frac{\beta}{1-y} q(dy) + \left(1 - \int_0^1 \frac{\beta}{1-y} q(dy)\right)$$

Thus, the probability measure $\hat{p}$ can only be given by the following expression:

$$\hat{p}(dx) = \frac{\beta}{1-x} q(dx) + \left(1 - \int_0^1 \frac{\beta}{1-y} q(dy)\right) \delta_1(dx)$$

This allows us to conclude the theorem on the limiting fitness.

□

It is noteworthy that by comparing results (3.4) and (3.8), we observe that convergence now occurs at a rate of $o(1)$ instead of being exponential as in the democratic case. In this situation, the limiting distribution still does not depend on the initial distribution p_0, but it is no longer absolutely continuous with respect to the mutant distribution q. This implies that regardless of the initial composition of the population, all individuals have the opportunity to demonstrate their merit and have the best chance of perpetuating their legacy. The appearance of the atom in the limiting fitness, compared to the previous case, signifies the concentration of survival probability on the side of the most deserving individuals.

Furthermore, before addressing the second non-democratic case, it is noteworthy that the definition of p by the theorem of limiting fitness once again provides us with the value of the limiting viability:

$$\int_0^1 xp(dx) = \beta \sum_{n=1}^{+\infty} \mu_n + \left(1 - \beta \sum_{n=0}^{+\infty} \mu_n\right) = 1 - \beta$$

3.4 Hegemony of the Elites

We now delve into the second case of the non-democratic regime, assuming the following hypotheses:

$$\int_0^1 \frac{q(dx)}{1-x} \leq \beta^{-1} \quad \text{and} \quad \lim_{n \to +\infty} \frac{u_n}{u_{n-1}} = \sigma < 1$$

In this non-democratic regime, the presented results will be quite similar to those of the previous case, with some subtle differences.

Under the aforementioned assumptions, the support of the mutant fitness distribution must necessarily be contained within a bounded set. The following lemma explicates this observation.

Lemma 3.8 (Support of Mutant Fitness Distribution). *The support of the mutant fitness distribution is restricted by the limit of the ratios of successive terms of the renewal sequence:*

$$supp(q) \subset [0, \sigma]$$

Since only fitness ratios are relevant, it is possible to rescale the fitness using a factor so that 1 becomes the upper limit of the support of q (which would correspond to either the democratic or meritocratic cases), unless the upper limit of the support of p_0 exceeds that of q.

Thus, the only new possibility is one where the mutant fitness distribution is concentrated on an interval $[0, \sigma]$, where σ is strictly less than the upper limit of the fitness of the original population, which can be taken, without loss of generality, to be equal to 1:

$$\max \operatorname{supp}(p_0) = 1$$

This situation is described as an "aristocracy"; some non-mutant descendants of the original generation are inherently fitter than all possible mutants.

3.4.1 Genetic Monopoly

In this new case, it turns out that the limiting viability remains unchanged compared to the previous case. Since survival chances are mainly concentrated on a small dominant group of individuals, the viability limit remains the same, as demonstrated by the following proposition.

Proposition 3.9 (Aristocratic Limit Viability). *The sequence of viabilities $(w_n)_n$ converges, and the limiting viability corresponds to the proportion of non-mutant population:*

$$w_n \xrightarrow[n\to+\infty]{} 1-\beta$$

3.4.2 Dominant Lineage

Similarly, since the proofs remain unchanged, the limiting fitness distribution given will be the same as in the meritocratic case. However, in this situation, it is its properties that will be modified. The following theorem explicates these modifications.

Theorem 3.10 (Aristocratic Limit Fitness). *In the aristocratic regime:*

$$\int_0^1 \frac{q(dx)}{1-x} \leq \beta^{-1} \quad \textit{and} \quad \lim_{n\to+\infty} \frac{u_n}{u_{n-1}} = \sigma < 1$$

the sequence of fitness distributions $(p_n)_n$ converges weakly. One of the limiting distributions is characterized by the following expression:

$$p(dx) = \frac{\beta}{1-x} q(dx) + \left(1 - \int_0^1 \frac{\beta}{1-y} q(dy)\right) \delta_1(dx)$$

This limit remains non-absolutely continuous with respect to the mutant distribution q, but it now becomes dependent on the initial distribution p_0.

Most of the remarks stated in the previous case remain applicable here. However, it is worth noting that not all individuals will have the same opportunity to perpetuate their legacy over time. Rather, it is those who were initially in positions of power who will continue to be so. Nevertheless, it is important to highlight that the distribution p now depends on the initial distribution p_0, but solely through the upper limit of its support.

3.5 Comparative Analysis of Results

After this thorough study of various selection regimes, it is time to recapitulate the obtained results.

In the democratic case, all individuals have equal chances to perpetuate their legacy over time, favoring an equitable fitness distribution. In contrast, in the non-democratic cases, the distribution of survival chances differs.

In the meritocratic regime, survival is based on individual aptitudes, thus favoring the most adapted individuals. This approach can be considered beneficial, as it encourages the selection of traits most favorable for survival.

Conversely, in the aristocratic regime, survival is primarily reserved for a small group of dominant individuals. This results in reduced genetic diversity within the population, as only descendants of the initially favorable state continue to thrive. This concentration of survival may yield short-term benefits for the dominant group but could also limit adaptability and resilience of the population in changing environments.

The following table summarizes the asymptotic behavior of our sequences in the three selection regimes:

Selection Regime	**Limiting Viability**	**Limiting Fitness - Properties**	**Limiting Fitness - Indep. of p_0**
Democracy	s	absolutely continuous w.r.t. q	yes
Meritocracy	$1 - \beta$	atom at maximum of mutant fitness distribution	yes
Aristocracy	$1 - \beta$	atom at maximum of non-mutant fitness distribution	no

Table 3.1: Asymptotic Behavior of $(w_n)_n$ and $(p_n)_n$

It may appear paradoxical that convergence of the integral leads to the emergence of an atom at 1, thus creating a "meritocracy" when the mutant fitness distribution decreases rapidly. The idea is that the existence of a slower decreasing tail, causing the integral to diverge, can contain sufficient fitness to keep the average above $1 - \beta$, without sacrificing absolute continuity. The designation of the critical situation where there is equality between the integral and β^{-1} in the non-democratic cases rather than in the democratic case is somewhat arbitrary but is justified by the difference in analysis method and the failure (in general) of exponentially fast convergence.

However, it is important to note that despite these differences, the limiting viability always remains above the mutant population proportion $1 - \beta$, with equality occurring in the non-democratic cases. Thus, considering the well-being of the population, opting for a democratic regime is preferable, which seems intuitive given its name. Favoring equity in selection best preserves the overall heritage and ensures the sustainability of the entire population. Valuing the most meritorious individuals is not detrimental, especially since it occurs independently of the initial organization. On the other hand, the last case, that of aristocracy, is objectively indefensible based on the arguments proposed.

These results highlight the fundamental differences between selection regimes and their implications on fitness distribution, genetic diversity, and individual survival. They illustrate how different survival strategies can lead to distinct fitness distributions and population proportions. Therefore, it is important to consider these aspects when studying selection systems and their impact on biological populations.

3.6 General Initial Support

Let us now consider the general case where the maximum of the initial distribution's support is not fixed at 1. In this scenario, the following theorem helps to characterize the limiting distribution and elucidates the dependence on the initial distribution p_0.

Theorem 3.11 (Invariant Measures)**.** *For a support maximum* $\max supp(p_0)$ *belonging to the following set:*

$$\left\{ x_0 \in \mathbb{R} \;\middle|\; \delta \leq x_0 \leq 1 \text{ and } \int_0^1 \frac{\beta}{1-\frac{x}{x_0}} q(dx) < 1 \right\}$$

an invariant measure with an atom at this real number is associated. This measure is characterized by the expression:

$$p_{x_0}(dx) = \frac{\beta}{1-\frac{x}{x_0}} q(dx) + \left(1 - \int_0^1 \frac{\beta}{1-\frac{y}{x_0}} q(dy)\right) \delta_{x_0}(dx)$$

Proof. If the sequence of viabilities $(w_n)_n$ converges to a real number w, then the limit of the fitness distributions p must satisfy the following equation:

$$p(dx) = (1-\beta)\frac{xp(dx)}{w} + \beta q(dx)$$

By imposing an atom at x_0, we necessarily obtain:

$$p(x_0) = (1-\beta)\frac{x_0 p(x_0)}{w}, \quad \text{i.e.} \quad \frac{1-\beta}{w} = \frac{1}{x_0}$$

Furthermore, for any $x < \delta$, we have:

$$p(dx) = \frac{\beta}{1-\frac{1-\beta}{w}x} q(dx) = \frac{\beta}{1-\frac{x}{x_0}} q(dx)$$

This concludes the proof of the theorem. □

Let us introduce a significant result regarding the limit viability, thereby confirming the observation discussed in the preceding section. Considering that the sequence of viabilities $(w_n)_n$ converges to a real value w, the limiting fitness distribution p must satisfy the following equation:

$$p(dx) = (1-\beta)\frac{xp(dx)}{w} + \beta q(dx), \quad \text{i.e.} \quad p(dx) = \frac{\beta}{1-\frac{1-\beta}{w}x} q(dx)$$

It is desirable that the following series is convergent:

$$\beta \sum_{k\geq 0} \left(\frac{1-\beta}{w}x\right)^k q(dx)$$

This can be achieved only if

$$(1-\beta)\max\operatorname{supp}(q) \leq w$$

Therefore, by adopting the preliminary assumption that measures were defined over the interval $I = [0,1]$, our result naturally follows.

In parallel, a similar result can be demonstrated concerning the support of the initial distribution p_0. Indeed, let $\theta = \max\operatorname{supp}(p_0)$ and assume that $(1-\beta)\theta > w$. In this case,

$$\forall \varepsilon > 0,\ \exists \theta' < \theta,\ \exists N \in \mathbb{N} \ / \ \forall i \geq N,\ \frac{1-\beta}{w}\theta' > (1+\varepsilon)$$

Now, using the explicit formula of $p_n(dx)$ given by (2.3), we can write for any integer n, there exists a positive measure r_n such that

$$p_n(dx) = r_n(dx) + \frac{(1-\beta)^n}{w_{n-1}...w_0} x^n p_0(dx)$$

Applying this, we have:

$$\forall n \geq N, \quad p_n(]\theta', \theta[) = r_n(]\theta', \theta[) + \frac{(1-\beta)^{n-N} x^{n-N}}{w_{n-1} ... w_N} \frac{(1-\beta)^N x^N}{w_{N-1} ... w_0} p_0(]\theta', \theta[)$$

This allows us to conclude that:

$$\exists C_{N,\theta'} > 0 \ / \quad 1 \geq p_n(]\theta', \theta[) \geq C_{N,\theta'} (1+\varepsilon)^{n-N} \xrightarrow[n \to +\infty]{} +\infty$$

This conclusion is absurd, thus we deduce that the inverse inequality to our assumption holds. Consequently, we have demonstrated two results that generalize the one stated by Kingman:

$$\boxed{(1-\beta) \max \operatorname{supp}(q) \leq w \quad \text{and} \quad (1-\beta) \max \operatorname{supp}(p_0) \leq w}$$

Chapter 4

Emergence of Condensation

In the preceding chapter, we examined the fitness distribution at a given time and in the long term under different regimes. Two phases were observed: when "mutation" is favored over selection, the limiting distribution is an asymmetric version of the mutant fitness distribution. Conversely, if "selection" is favored over mutation, a condensation phenomenon occurs. We then observe a positive proportion of the late-generation population having fitness very close to the optimal value, leading to the emergence of an atom at the maximum fitness value in the limiting distribution. Thus, Dereich and Mörters focused on the model proposed by Kingman and investigated the shape of the fitness distribution for the portion of the population that eventually constitutes the atom in the limiting distribution.

In the context of their article titled "Emergence of condensation in Kingman's model of selection and mutation", the term "condensation" refers to a phenomenon of concentration or accumulation of certain elements or characteristics of the selection and mutation model proposed by Kingman. This implies that specific elements or individuals tend to cluster or concentrate significantly, which can have a significant impact on the overall dynamics of the model.

4.1 Context and Premise of the Study

Throughout our study, we assume that the distribution of mutants near its tail is stochastically larger than the fitness distribution in the initial population. This is expressed through the following moment condition:

$$\lim_{n \to +\infty} \frac{m_n}{\mu_n} = 0 \tag{4.1}$$

Under this assumption (or a slightly weaker one), Kingman demonstrated that $(p_n)_n$ converges to a limiting distribution p, independent of p_0. Furthermore, the limiting distribution p is absolutely continuous with respect to q if and only if

$$\beta \int_0^1 \frac{q(dx)}{1-x} \geq 1$$

Otherwise,

$$\gamma(\beta) = 1 - \beta \int_0^1 \frac{q(dx)}{1-x} > 0 \tag{4.2}$$

and it is the meritocratic regime that interests us. In this regime, the limiting distribution p always exhibits an atom at the optimal fitness 1, referred to as "condensation". The limiting distribution p is then expressed as:

$$p(dx) = \beta \frac{q(dx)}{1-x} + \gamma(\beta)\delta_1(dx)$$

The main result of Dereich and Mörters describes the dynamics of condensation in terms of a scaling limit theorem that focuses on the neighborhood of the maximum fitness value and reveals the shape of the "wave" that eventually forms the condensate.

The authors specify that the total mass within the "wave" moving toward the maximum fitness value corresponds to the mass of the atom in the limiting distribution $p(dx)$. The rescaled shape of this mass follows a "gamma distribution" with a shape parameter α.

4.2 Renewal Theory

Before delving into the study of our asymptotic wave, we establish that the sequences we previously defined satisfy properties of renewal theory as outlined in the appendix.

Firstly, consider the sequence $(v_n)_n$ that we introduced using the sequences $(m_n)_n$ and $(u_n)_n$ in a first renewal equation:

$$\forall n \in \mathbb{N}^*, \quad v_n = \sum_{k=1}^{n} m_k u_{n-k}$$

It is shown that this sequence satisfies another renewal equation (B.4), connecting the moments $(m_n)_n$ and $(\mu_n)_n$ through the sequence $(f_n)_n$ defined in (3.5). This is stated in the following lemma:

Lemma 4.1 (Renewal Equation)**.** *The sequence $(v_n)_n$ satisfies the following renewal equation:*

$$\forall n \in \mathbb{N}^*, \quad v_n = m_n + \sum_{k=1}^{n-1} f_k v_{n-k}$$

Next, we intend to apply the well-known Renewal Theorem B.3 to our sequences. Its application will lead to a more interesting result, albeit one that remains anecdotal. The corresponding theorem is formulated as follows.

Theorem 4.2 (Renewal Theorem)**.** *The series with terms given by v_n converges. Moreover, the sum of this series satisfies the following equation:*

$$\sum_{n=1}^{+\infty} v_n = \frac{1-\beta}{\gamma(\beta)} \sum_{n=1}^{+\infty} m_n < +\infty$$

Proof. Let's verify the required assumptions of the renewal theorem by Feller. Recall that, based on (4.1) and (4.2), we have the following relations:

$$m_n = o(\mu_n) \quad \text{and} \quad \sum_{n=1}^{+\infty} \mu_n \leq \frac{\beta}{1-\beta} < +\infty$$

According to the theorem of summation for comparison relations, this implies that the series with terms given by m_n converges, and that the remainders of the series are comparable:

$$\sum_{k=n+1}^{+\infty} m_k = o\left(\sum_{k=n+1}^{+\infty} \mu_k\right)$$

Our definition of $\gamma(\beta)$ gives us a value for the sum in the preceding inequality:

$$\sum_{n=1}^{+\infty} \mu_n = \frac{1}{\beta}\left(1-\gamma(\beta)\right)$$

This allows us to deduce a value for the sum of the series of terms f_n defined in (3.5), thus demonstrating its convergence and strict inferiority to 1:

$$\sum_{n=1}^{+\infty} f_n = 1 - \frac{\gamma(\beta)}{1-\beta} \tag{4.3}$$

We can now apply the theorem presented in the appendix to obtain:

$$\sum_{n=1}^{+\infty} v_n = \sum_{n=1}^{+\infty} m_n \left[1 - \sum_{n=1}^{+\infty} f_n\right]^{-1}$$

And the previous equation (4.3) allows us to conclude.

□

4.3 Fitness Distribution Approaching the Atom

We now turn to the study of the fitness distribution before the emergence of the atom, by stating assumptions about the behavior of mutants. These assumptions will have an impact on the behavior of the overall population. In the context of the study conducted by Dereich and Mörters, we make the following assumption:

$$q(dy) = \alpha(1-y)^{\alpha-1}dy$$

where $\alpha > 1$ is a real number. In particular, we have the following equivalence:

$$q\left(]1-h, 1[\right) \underset{h \to 0}{\sim} h^{\alpha}$$

The condition that α is greater than 1 is essential in the context of this study due to its crucial role in modeling and understanding the asymptotic behavior of the population. This condition is intimately linked to the properties of mutants and how they contribute to the evolution of the population.

Indeed, the choice of $\alpha > 1$ has significant implications for the fitness distribution of mutants. This condition implies that the probability for a mutant to have slightly higher fitness than the reference fitness $(1-h)$ decays more slowly than exponentially as the gap h approaches 0. In other words, the probability of observing mutants that are moderately better than the reference individual does not decrease too rapidly.

This characteristic is crucial to ensure sufficient diversity in the population and allow the coexistence of different evolutionary strategies. If α were less than 1, the rapid decrease in the probabilities of more fit mutants could lead to a quick convergence to a single dominant type of individual, thereby limiting genetic diversity and the ability to explore new evolutionary solutions.

By choosing $\alpha > 1$, the authors enable the population to maintain a variety of evolutionary traits, thereby promoting a dynamic balance between different strategies. This provides a realistic framework for studying evolution in a meritocratic regime and contributes to a better understanding of the underlying mechanisms that influence stability and diversity within evolving populations.

Before presenting the long-awaited theorem on the scaling limit, let us introduce two lemmas. The first concerns the behavior near the support boundary of q of the sequence of moments of this distribution.

Lemma 4.3 (Mutants' Moments)**.** *For any $x > 0$, the following equivalence holds:*

$$\int_{1-x/n}^{1} y^n q(dy) \underset{n \to +\infty}{\sim} \frac{\alpha}{n^{\alpha}} \int_0^x t^{\alpha-1} e^{-t} dt \tag{4.4}$$

In particular, we have:

$$\mu_n \underset{n \to +\infty}{\sim} \frac{\Gamma(\alpha+1)}{n^{\alpha}} \tag{4.5}$$

The second lemma concerns the asymptotic behavior of the sequence $(v_n)_n$ that we studied previously:

$$\forall n \in \mathbb{N}^*, \quad v_n = \sum_{k=1}^{n} m_k u_{n-k} = m_n + \sum_{k=1}^{n-1} f_k v_{n-k}$$

This sequence also scales like $n^{-\alpha}$, similar to the moments of the mutants, but with a different associated constant.

Lemma 4.4. *The sequence $(v_n)_n$ satisfies the following equivalence:*

$$v_n \underset{n\to+\infty}{\sim} \frac{c}{n^\alpha}$$

where

$$c = \frac{\beta}{\gamma(\beta)}\Gamma(\alpha+1)\sum_{n=1}^{+\infty} v_n$$

Finally, with the help of these two lemmas, we are ready to present the theorem formulated by the authors, which addresses the asymptotic behavior of the population in a meritocratic regime, under a specific condition on mutants.

Theorem 4.5 (Gamma Condensation)**.** *Consider the following assumptions:*

$$\gamma(\beta) = 1 - \beta\int_0^1 \frac{q(dx)}{1-x} > 0 \quad \textit{and} \quad q\left(]1-h,1[\right) \underset{h\to 0}{\sim} h^\alpha$$

For any $x > 0$, the following equivalence holds:

$$p_n\left(\left]1-\frac{x}{n},1\right[\right) \underset{n\to+\infty}{\sim} \frac{\gamma(\beta)}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}e^{-y}dy \tag{4.6}$$

Proof. Let $x > 0$. To prove this result, let's start with the explicit formula for fitness:

$$p_n(dx) = \beta\sum_{k=0}^{n-1}\frac{v_{n-k}}{v_n}x^k q(dx) + \frac{1}{v_n}(1-\beta)x^n p_0(dx)$$

Now integrate this formula:

$$p_n\left(\left]1-\frac{x}{n},1\right[\right) = \beta\sum_{k=0}^{n-1}\frac{v_{n-k}}{v_n}\int_{1-x/n}^1 y^k q(dy) + \frac{1}{v_n}(1-\beta)\int_{1-x/n}^1 y^n p_0(dy)$$

Let's analyze the two terms obtained on the right-hand side of the equation, starting with the second term.

- The second term can be quickly analyzed using the previous lemma:

$$\frac{1}{v_n}(1-\beta)\int_{1-x/n}^1 y^n p_0(dy) \underset{n\to+\infty}{\sim} m_n n^\alpha\left[\frac{1-\beta}{c}\frac{\int_{1-x/n}^1 y^n p_0(dy)}{\int_0^1 y^n p_0(dy)}\right]$$

  The term in square brackets is finite, and we noticed during the proof that the remaining term converges to 0. Thus, this term will be negligible in the study of asymptotic fitness.

- For the first term, we again use the equivalence from the previous lemma:

$$\beta\sum_{k=0}^{n-1}\frac{v_{n-k}}{v_n}\int_{1-x/n}^1 y^k q(dy) \underset{n\to+\infty}{\sim} \frac{\beta n^\alpha}{c}\sum_{k=0}^{n-1} v_{n-k}\int_{1-x/n}^1 y^k q(dy)$$

As we have integrals involving the mutant distribution, we are tempted to use the first lemma on the mutants' moments. However, this lemma applies to integrals of the form y^n rather than y^k. To handle this, we split the sum into terms of small and large values of k. To do this, we choose a large real number $M > 0$ that satisfies the following properties:

- For all $k \geq n - Mn^{1/\alpha}$, we have:

$$\int_{1-x/n}^{1} y^k q(dy) \underset{n\to+\infty}{\sim} \int_{1-x/n}^{1} y^n q(dy)$$

- We have:

$$\frac{1}{M^\alpha} \ll \frac{\gamma(\beta)}{\Gamma(\alpha)} \int_0^x y^{\alpha-1} e^{-y} dy$$

in the sense that the sum of the two terms can be considered equal to the second term only.

We can then study these two parts separately. Let us note, by abuse of notation:

$$\frac{\beta n^\alpha}{c} \sum_{k=0}^{n-1} v_{n-k} \int_{1-x/n}^{1} y^k q(dy) = \frac{\beta n^\alpha}{c} \left(\sum_{k=0}^{n-Mn^{1/\alpha}} + \sum_{k=n-Mn^{1/\alpha}+1}^{n-1} \right) v_{n-k} \int_{1-x/n}^{1} y^k q(dy)$$

First Part: The first part of this separation is negligible. Using the previous lemma, there exists a real number $K > 0$ such that:

$$\begin{aligned}\frac{\beta n^\alpha}{c} \sum_{k=0}^{n-Mn^{1/\alpha}} v_{n-k} \int_{1-x/n}^{1} y^k q(dy) &\leq \frac{\beta n^\alpha}{c} q\left(\left]1 - \frac{x}{n}, 1\right[\right) \sum_{k=Mn^{1/\alpha}}^{n} v_k \\ &\leq K \left(n - Mn^{1/\alpha}\right) \left(\frac{n}{Mn^{1/\alpha}}\right)^\alpha q\left(\left]1 - \frac{x}{n}, 1\right[\right)\end{aligned}$$

Using the hypothesis formulated at the beginning of the section, we obtain:

$$\frac{\beta n^\alpha}{c} \sum_{k=0}^{n-Mn^{1/\alpha}} v_{n-k} \int_{1-x/n}^{1} y^k q(dy) \underset{n\to+\infty}{\sim} \frac{n - Mn^{1/\alpha}}{M^\alpha n} \underset{n\to+\infty}{\sim} \frac{1}{M^\alpha}$$

Second Part: The entire contribution to the asymptotic fitness is in this second part. Using the assumption made when choosing M and the equivalence (4.4), we obtain:

$$\frac{\beta n^\alpha}{c} \sum_{k=n-Mn^{1/\alpha}+1}^{n-1} v_{n-k} \int_{1-x/n}^{1} y^k q(dy) \underset{n\to+\infty}{\sim} \frac{\alpha\beta}{c} \sum_{n=1}^{+\infty} v_n \int_0^x y^{\alpha-1} e^{-y} dy$$

In summary, by combining all our previous results and using the constant c obtained in the previous lemma, we can conclude this theorem as follows:

$$p_n\left(\left]1 - \frac{x}{n}, 1\right[\right) \underset{n\to+\infty}{\sim} \frac{\gamma(\beta)}{\Gamma(\alpha)} \int_0^x y^{\alpha-1} e^{-y} dy$$

□

The Kingman model presents two significant aspects. On the one hand, it is a simple model in which we can observe the effect of condensation. On the other hand, it is complex enough to allow the study of the emergence of condensation as a dynamic phenomenon. Its simplicity allows for rigorous analysis using elementary methods. However, the calculations performed have broad implications, as many more complex models from various scientific domains exhibit similar characteristics.

The authors of the condensation emergence article argue that the Kingman model shares many features with other models, such as Bose-Einstein condensation in physics, wealth condensation in macroeconomics, or even the emergence of traffic jams. Thus, their main conjecture is that, in a broad class of universal models where effects similar to mutation and selection interact in a limited and continuous state space, the spread of the "wave" reaching the maximal condensation state follows a Gamma distribution form.

This study is based on a specific assumption regarding the distribution of mutants. A question then arises: if a different assumption is formulated, could we obtain another well-known "wave" form, such as a Gaussian distribution? To explore this question, we turn our attention to an assumption related to the question left unanswered by Kingman in his article:

$$q\left(]1-h,1[\right) \underset{h\to 0}{\sim} e^{-1/(1-h)}$$

with the difference that his "wave" appears at infinity. We will address this question in the following chapter.

Chapter 5

Evolutionary Dynamics with Unbounded Fitness

Up to this point, our analysis has heavily relied on the assumption that all fitness values are bounded, which allowed us to observe a typical behavior of partial accumulation at the higher fitness level or near it. This boundedness assumption of fitness values made the Kingman model simple and amenable to analysis using elementary methods. However, we now wish to explore what happens in the equation (2.2) as stated below:

$$p_{n+1}(dx) = (1-\beta)\frac{xp_n(dx)}{w_n} + \beta q(dx), \quad \text{where} \quad w_n = \int_0^1 xp_n(dx)$$

when q and p_0 are not restricted to a finite interval. This question will be at the heart of this chapter and will allow us to generalize our previous analysis.

5.1 General Results

Suppose now that fitness distributions are not restricted to a finite interval. In this case, the definition of the sequence $(w_n)_n$ will be modified to replace the term 1 with $+\infty$, but the equation (2.2) remains unchanged:

$$p_{n+1}(dx) = (1-\beta)\frac{xp_n(dx)}{w_n} + \beta q(dx)$$

For this equation to be well-defined, given its recursive definition, it is necessary for the moments of order n to be finite:

$$\forall n \in \mathbb{N}, \quad \mu_n < +\infty \quad \text{and} \quad m_n < +\infty$$

Most of our previous arguments remain valid in this case, except for the finiteness of the series of f_n. This leads us to study a new generalized theory, presented in the appendix, which concerns generalized renewal sequences.

5.1.1 Generalized Renewal Sequences

Recall that the sequence $(f_n)_n$ was defined as follows:

$$\forall n \in \mathbb{N}^*, \quad f_n = \alpha\mu_n, \quad \text{with} \quad \alpha = \frac{\beta}{1-\beta}$$

Furthermore, the sequence $(u_n)_n$ is recursively defined as follows:

$$u_0 = 1 \quad \text{and} \quad \forall n \in \mathbb{N}^*, \; u_n = f_n + \sum_{k=1}^{n-1} f_k u_{n-k}$$

Using the appendix on generalized renewal sequences, we can deduce the following proposition in the context of our study.

Proposition 5.1 (Nature of the Sequence)**.** *The sequence $(u_n)_n$ thus defined is a generalized renewal sequence of wild nature.*

Proof. We prove this assertion in two steps, starting by showing that the sequence $(u_n)_n$ is a generalized renewal sequence by studying the series of f_n, then using the definition, we prove its wild nature.

Consider a real number $z > 0$ and a natural number $n \geq 1$. Then, we have:

$$f_n z^n = \alpha \mu_n z^n = \frac{\beta}{1-\beta}\left[\int_0^{1/z}(xz)^n q(dx) + \int_{1/z}^{+\infty}(xz)^n q(dx)\right] \underset{n\to+\infty}{\not\longrightarrow} 0$$

The limit conclusion follows from the fact that the second integral is strictly positive. Thus, $\sum f_n z^n = +\infty$, and in particular $\sum f_n = +\infty$. This demonstrates that $(u_n)_n$ is a generalized renewal sequence.

Now suppose that $(u_n)_n$ is of tame nature, i.e., that:

$$\exists c > 0 \ / \ \forall n \in \mathbb{N}^*, \quad u_n \leq c^n$$

Then, by Abel's lemma, we have:

$$\forall z \in \left]0, c^{-1}\right[, \quad \sum u_n z^n < +\infty$$

However, we have seen that:

$$\sum_{n=0}^{+\infty} u_n z^n = \left[1 - \sum_{n=1}^{+\infty} f_n z^n\right]^{-1}$$

Thus, the generating series associated with $(u_n)_n$ must necessarily be zero, implying the nullity of its terms. In other words, for any real $z \in \left]0, c^{-1}\right[$, we have:

$$\sum_{n=0}^{+\infty} u_n z^n = 0 \quad \Longrightarrow \quad \forall n \in \mathbb{N}, \ u_n z^n = 0 \quad \Longrightarrow \quad \forall n \in \mathbb{N}, \ u_n = 0$$

However, this last property is absurd. Therefore, we conclude that $(u_n)_n$ is actually of wild nature. □

As mentioned in the appendix, we wish to use the second part of the powerful Theorem C.6 characterizing a wide class of generalized renewal sequences. This part allows us to formulate the following proposition.

Proposition 5.2 (Growth and Divergence)**.** *The sequence of ratios of successive terms of the generalized renewal sequence is increasing and divergent:*

$$\frac{u_n}{u_{n-1}} \xrightarrow[n\to+\infty]{} +\infty$$

Proof. The proof of the growth and divergence of the sequence of ratios of successive terms of the generalized renewal sequence is rigorously established. First, under the definition of the sequences, it is evident that the generalized renewal sequence $(u_n)_n$ is strictly positive. Furthermore, since the sequence $(f_n)_n$ is a sequence of moments associated with a finite Borel measure by assumption, the generalized renewal sequence $(u_n)_n$ represents a sequence of moments of a probability measure. Thus, the growth of the sequence of ratios of its successive terms is a direct result.

Furthermore, by applying the aforementioned powerful theorem, which holds in our case due to the wild nature of the sequence, we can conclude that the sequence of ratios diverges. □

5.1.2 Impact on Viability

In light of the study of non-democratic regimes in the context of bounded fitness, we now wish to explore the case of unbounded fitness. For this purpose, we again make use of the renewal sequence defined earlier by (3.6):

$$\forall n \in \mathbb{N}, \quad v_n = m_n + \sum_{k=1}^{n-1} m_k u_{n-k}$$

Interestingly, the sequence of ratios of successive terms of $(v_n)_n$ will evolve in the same way as that associated with $(u_n)_n$. This is demonstrated by the following lemma.

Lemma 5.3 (Divergence). *The sequence of ratios of successive terms of the sequence $(v_n)_n$ is divergent:*

$$\frac{v_n}{v_{n-1}} \xrightarrow[n\to+\infty]{} +\infty$$

Using this lemma, similarly to the study of non-democratic regimes in the case of bounded fitness, we can deduce the limit viability in the case of unbounded fitness.

Proposition 5.4 (Limit Viability). *The sequence of viabilities $(w_n)_n$ diverges:*

$$w_n \xrightarrow[n\to+\infty]{} +\infty$$

Proof. It is recalled that the product of viabilities can be expressed in terms of $(v_n)_n$ as follows:

$$\forall n \in \mathbb{N}^*, \quad W_n = (1-\beta)^{n-1} v_n$$

Therefore, it follows that:

$$w_n = \frac{W_{n+1}}{W_n} = (1-\beta)\frac{v_{n+1}}{v_n} \xrightarrow[n\to+\infty]{} +\infty$$

□

Unfortunately, it is no longer possible to analyze the limit fitness in the same way as before. We have already exhausted all available general information on this subject. However, this does not mean that we have to stop here. Indeed, there exists a famous probability distribution called the exponential distribution, which has support over the positive real numbers and is known for its memoryless property. This distribution offers interesting properties that simplify its use and study. In the next section, we will therefore explore the application of this distribution in our context.

5.2 Exponential Initial Distributions

The equation (2.2), recalled at the beginning of this chapter, suggests a form of geometric evolution of the population. In this perspective, we can use the exponential distribution, which is a continuous version of the geometric distribution, to provide a smoother and more regular dimension to our study.

In this situation, we assume that the initial distributions are probability density functions, specifically exponential distributions with parameter 1. We also denote by the same letter their density, given by the following form:

$$\forall x \in \mathbb{R}_+, \quad p_0(x) = q(x) = e^{-x}$$

To conduct this specific study, we will divide the work into two distinct steps to gain a better understanding of the behavior of our population in different situations. Initially, we will assume the absence of mutations, and then we will reintroduce them later to deepen our analysis.

5.2.1 In the Absence of Mutations

Initially, we consider the absence of mutants, i.e., $\beta = 0$, to obtain a basic approach to the evolution of populations according to the model by Kingman. In this case, we revert to the first equation (2.1) of the model, which is expressed as follows:

$$p_{n+1}(dx) = \frac{xp_n(dx)}{w_n}, \quad \text{where} \quad w_n = \int_0^{+\infty} xp_n(dx)$$

By performing a recurrence on this equation, we can formulate the following proposition:

Proposition 5.5 (Fitness and Viability)**.** *The notions of fitness and viability in the n-th generation are explicitly expressed as follows:*

$$p_n(dx) = \frac{x^n e^{-x}}{n!}dx \quad \textit{and} \quad w_n = n+1$$

Consequently, the density of the n-th generation is differentiable, and its derivative satisfies:

$$\forall x \in \mathbb{R}_+, \quad p_n'(x) = \frac{(n-x)x^{n-1}e^{-x}}{n!} = \frac{n-x}{n}p_{n-1}(x)$$

Thus, the derivative vanishes at n and at 0; the latter condition is satisfied only if $n \geq 2$. However, regardless of this, we can generally state that the density always exhibits a peak at n and is primarily concentrated around this peak, over a range that can be estimated.

Indeed, the density obtained previously corresponds to that of a gamma distribution $\Gamma(n+1, 1)$. Consequently, we know its mean (viability) as well as its standard deviation σ_n, given by:

$$w_n = n+1 \quad \text{and} \quad \sigma_n = \sqrt{n+1}$$

However, a random variable S_n following the $\Gamma(n+1,1)$ distribution can be considered as the sum of $n+1$ independent and identically distributed random variables $(X_0, ..., X_n)$ following an exponential distribution $\mathcal{E}(1)$. Therefore, according to the central limit theorem, we have:

$$\sqrt{\frac{n}{n+1}}\left[\frac{S_n - n}{\sqrt{n}} - \frac{1}{\sqrt{n}}\right] = \frac{1}{\sqrt{n+1}}\left[\sum_{k=0}^{n} X_k - (n+1)\right] \xrightarrow[n\to+\infty]{d} \mathcal{N}(0,1)$$

Thus, asymptotically, S_n is similar to a Gaussian random variable $\mathcal{N}(n, n)$. In this context, we can list in a table how the information is asymptotically contained under the curve of p_n.

Interval Width	Percentage of Information
$2\sqrt{n}$	68.3
$4\sqrt{n}$	95.5
$6\sqrt{n}$	99.7

Table 5.1: Distribution of Information under p_n

The presented results can be numerically verified using the following graphs. The first graph depicts the behavior of the density p_n, while the second graph compares the curves of Gaussian and gamma densities over the interval $[-3\sqrt{n}, 3\sqrt{n}]$, corresponding to the last row of the table. This allows us to observe that the similarities between the two densities are present in almost all the information.

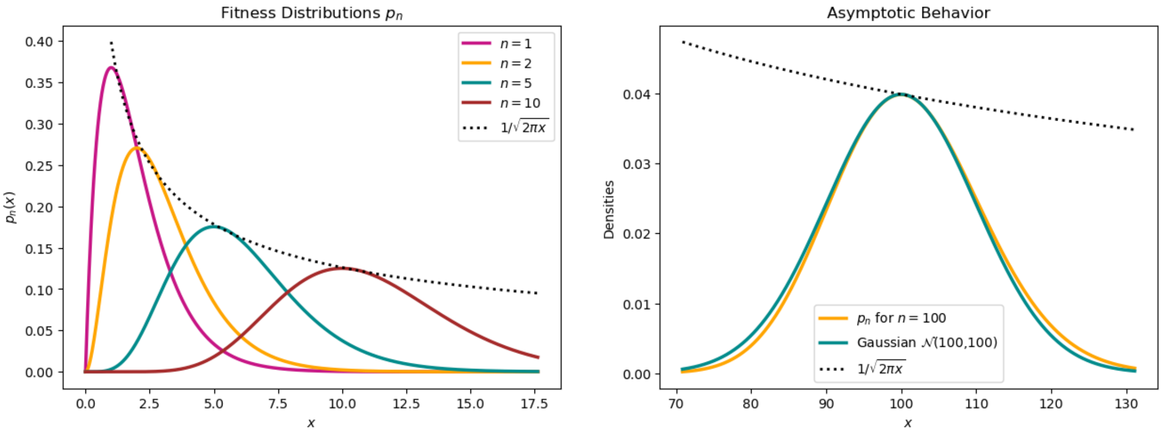

Figure 5.1: Behavior of Fitness p_n

Thus, in the absence of mutants in the population, we observe the emergence of a single wave of information, which asymptotically follows a Gaussian distribution moving towards infinity as time progresses. This wave of information is characterized by a density peak

$$\frac{n^n e^{-n}}{n!} \underset{n\to+\infty}{\sim} \frac{1}{\sqrt{2\pi n}}$$

decreasing at a rate of $\sqrt{n}$ (Stirling's formula), and a spread that increases with n, as previously expressed. However, it is important to note that these results are limited to the absence of mutants. In the next subsection, we will present everything we can say when mutations are present.

5.2.2 In the Presence of Mutations

Now, let's consider the case where we reintroduce mutants into our study, meaning $\beta > 0$. This situation makes the analysis more complex, but we will study each quantity to obtain as much information as possible.

Thanks to the simplicity of our specific model, we can easily express the moments of order n:

$$\forall n \in \mathbb{N}, \quad \mu_n = m_n = \int_0^{+\infty} x^n e^{-x} dx = \Gamma(n+1) = n!$$

Thus, we can deduce that:

$$\forall n \in \mathbb{N}, \quad f_n = \alpha n!, \quad \text{where} \quad \alpha = \frac{\beta}{1-\beta}$$

To obtain results on asymptotic viability, let us now study our renewal sequences $(u_n)_n$ and $(v_n)_n$ defined by:

$$\forall n \in \mathbb{N}^*, \quad u_n = f_n + \sum_{k=1}^{n-1} f_k u_{n-k} \quad \text{and} \quad v_n = m_n + \sum_{k=1}^{n-1} m_k u_{n-k} = m_n + \sum_{k=1}^{n-1} f_k v_{n-k} \tag{5.1}$$

Before that, let's state three lemmas that will facilitate our cumbersome study of the sequences. The first lemma deals with binomial coefficients, which appear following the definition of $(f_n)_n$.

Lemma 5.6 (Binomial Coefficients)**.**

$$\forall n \geq 5, \ \forall k \in [\![2, n-2]\!], \ \binom{n}{k} \geq 2n$$

The second lemma bounds some sequences specifically defined using binomial coefficients.

Lemma 5.7 (Bound)**.** *For a real number $\alpha > 0$, consider the sequence $(a_n)_n$ defined recursively as follows:*

$$a_0 = 1 \quad \text{and} \quad \forall n \in \mathbb{N}^*,\ a_n = a_0 + \alpha \sum_{k=1}^{n-1} \frac{a_k}{\binom{n}{k}}$$

Then the sequence $(a_n)_n$ is bounded:

$$\forall n \in \mathbb{N}, \quad a_n \leq C$$

where the bound C is given by:

$$C = \max\left\{\frac{5}{5-3\alpha}, 1 + \frac{2\alpha}{3} + \frac{\alpha^2}{4} + \frac{\alpha^3}{24}, 1 + \frac{3\alpha}{5} + \frac{\alpha^2}{4} + \frac{\alpha^3}{15} + \frac{\alpha^4}{120}\right\}$$

The third lemma indicates that the sequences defined as previously are equivalent to their first term.

Lemma 5.8 (Asymptotic Expansion)**.** *For a real number $\alpha > 0$, consider the sequence $(a_n)_n$ defined recursively as follows:*

$$a_0 = 1 \quad \text{and} \quad \forall n \in \mathbb{N}^*,\ a_n = a_0 + \alpha \sum_{k=1}^{n-1} \frac{a_k}{\binom{n}{k}}$$

Then the sequence $(a_n)_n$ can be asymptotically expanded as follows:

$$a_n = a_0 + O(n^{-1})$$

The three lemmas play an essential role in our analysis. Indeed, the sequences $(u_n/f_n)_n$ and $(v_n/m_n)_n$ satisfy the conditions of the last two lemmas. Thus, we can precisely formulate the following proposition:

Proposition 5.9 (Renewal Sequences)**.** *The sequences $(u_n)_n$ and $(v_n)_n$, defined as indicated in equation* (5.1)*, can be asymptotically expanded as follows:*

$$u_n = \alpha n!\left(1 + O(n^{-1})\right) \quad \text{and} \quad v_n = n!\left(1 + O(n^{-1})\right)$$

Let's reintroduce the notation from the previous subsection for the standard deviation:

$$\forall n \in \mathbb{N}, \quad \sigma_n = \sqrt{\int_0^{+\infty} x^2 p_n(dx) - w_n^2}$$

Although we cannot establish an exact equality for viability and standard deviation as in the case without mutants, we can provide precise equivalents for these values through the following proposition:

Proposition 5.10 (Viability and Standard Deviation)**.**

$$w_n \underset{n\to+\infty}{\sim} (1-\beta)n \quad \text{and} \quad \sigma_n \underset{n\to+\infty}{\sim} \sqrt{\beta(1-\beta)}n$$

It is noteworthy that the result on the standard deviation is, apart from a constant, different from that stated by Kingman in his article.

To conclude this section, let's examine the asymptotic behavior of the sequence of fitness distributions. First of all, let's note that, thanks to the definition provided in the appendix, the total variation distance between two absolutely continuous measures with respect to the Lebesgue measure corresponds to the L^1 distance between their respective densities with respect to the Lebesgue measure. This observation allows us to notice that, unlike the case without mutants where the asymptotic information was contained under a single "bump", in the presence of mutations, asymptotic information is contained under two "bumps". Indeed, let Γ_n denote the measure and density associated with a random variable following the $\Gamma(n, 1)$ distribution. With this, we can state the following theorem:

Theorem 5.11 (Asymptotic Fitness)**.** *The sequence of fitness distributions $(p_n)_n$ is equivalent to the weighted sum of an exponential and a gamma:*

$$\|p_n - \beta q - (1-\beta)\Gamma_{n+1}\|_{TV} = \|p_n - \beta q - (1-\beta)\Gamma_{n+1}\|_1 \xrightarrow[n\to+\infty]{} 0$$

We also want to obtain a graphical representation of the stated result. Since we do not have an explicit formula for the fitness distribution in this case, we will plot the curve of the equivalent distribution according to the previous theorem for two values of β.

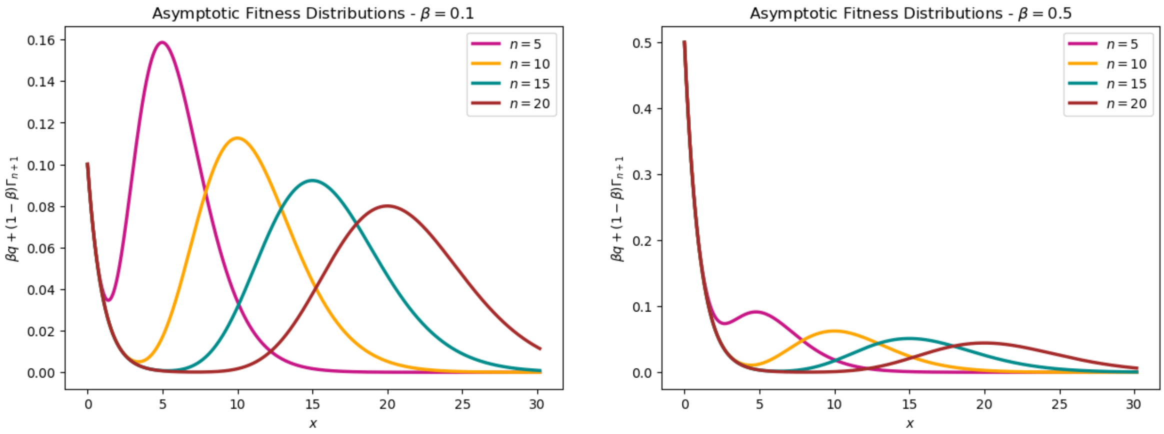

Figure 5.2: Behavior of the Equivalent Fitness

Thus, in the presence of mutants in the population, we observe the emergence of two waves of information. The first wave becomes increasingly distinct as n increases, while the second wave resembles that found in the case without mutants. However, this second wave quickly loses importance as β increases.

Chapter 6

Conclusion

In summary, this in-depth study of evolutionary dynamics based on the model of Kingman has illuminated the essential mechanisms underlying the evolution of biological populations. Through meticulous analysis of moments and renewal sequences, we have delved into various regimes of selection, from democratic to hegemonic, examining how fitness distributions and viabilities evolve in each scenario. This has highlighted trends toward harmonious biological equilibria and genetic power concentrations within elites.

Furthermore, by focusing on a specific meritocratic case, we explored convergence towards a well-known limiting fitness, following a gamma distribution. The thorough investigation of the unbounded fitness case has raised new questions and revealed intriguing new concepts.

In essence, this research has enriched our understanding of evolutionary mechanisms by shedding light on diverse selection regimes and unveiling the subtle relationships among viabilities, fitness distributions, and moments. It underscores the crucial importance of mathematical tools in exploring and deciphering complex biological phenomena while paving the way for intriguing new questions in the realm of evolutionary biology.

As a gateway to new avenues of research, we consider generalizing the obtained result for the renewal sequence $(u_n)_n$ in the context of the specific case of unbounded fitness in the presence of mutations. The initial result we have derived is stated as follows:

Proposition 6.1. *For a real $\alpha > 0$, consider the sequence $(f_n)_n$ explicitly defined as follows:*

$$\forall n \in \mathbb{N}, \quad f_n = \alpha n!$$

and the sequence $(u_n)_n$ recursively defined as follows:

$$u_0 = 1 \quad \text{and} \quad \forall n \in \mathbb{N}^*, \; u_n = f_n + \sum_{k=1}^{n-1} f_k u_{n-k}$$

Then the sequence $(u_n)_n$ can be asymptotically expanded as follows:

$$u_n = \alpha n! \left(1 + O(n^{-1})\right)$$

We intend to generalize this result in the form of the following theorem:

Theorem 6.2. *Consider an increasing sequence of positive real numbers $(f_n)_n$ and the sequence $(u_n)_n$ recursively defined as follows:*

$$u_0 = 1 \quad \textit{and} \quad \forall n \in \mathbb{N}^*,\ u_n = f_n + \sum_{k=1}^{n-1} f_k u_{n-k}$$

Then

$$\frac{f_{n-1}}{f_n} \xrightarrow[n\to+\infty]{} 0 \quad \Longleftrightarrow \quad u_n \underset{n\to+\infty}{\sim} f_n$$

Proof. Let's consider each direction of the equivalence separately:

(⇒) This part is currently under investigation and requires a thorough inquiry to establish its validity.

(⇐) For the reverse direction, let's assume that

$$u_n \underset{n\to+\infty}{\sim} f_n$$

This implies that the ratio u_n/f_n tends to 1 as n tends to infinity. Expressing it as a sum, we get:

$$\frac{u_n}{f_n} = 1 + \sum_{k=1}^{n-1} \frac{f_k u_{n-k}}{f_n} \xrightarrow[n\to+\infty]{} 1$$

Consequently, the right-hand sum must converge to 0 as n tends to infinity. As it is a sum of positive terms, this implies in particular that the last term of this sum must tend to 0:

$$\frac{f_{n-1} u_1}{f_n} \xrightarrow[n\to+\infty]{} 0$$

Thus, we can conclude that the ratio f_{n-1}/f_n tends to 0 as n tends to infinity.

□

However, the proof of this theorem remains currently incomplete and is under study, paving the way for new avenues of research in the future.

Appendix A

Convergence of Measures

The interval $I = [0, 1]$, equipped with the usual distance in $\mathbb{R}$, forms a compact metric space. We endow it with its Borel σ-algebra, denoted by $\mathcal{B}_I$. Moreover, let $\mathcal{P}_I$ denote the set of probability measures on $(I, \mathcal{B}_I)$.

A.1 Weak Convergence

Let's begin by defining a primary notion of convergence, known as weak convergence, for measures in $\mathcal{P}_I$, drawing from the work of Dictionnaire de mathématiques - Bibmath. There exist several equivalent definitions of this convergence due to the portmanteau theorem for measures. Here, we state the most common form.

Definition A.1 (Weak Convergence)**.** *Consider a sequence $(\mu_n)_n$ of measures from $\mathcal{P}_I$, and let μ be another measure in this space. We say that $\underline{(\mu_n)_n \text{ converges weakly to } \mu}$ if, for every function $f \in \mathcal{C}(I, \mathbb{R})$, the following holds:*

$$\int_I f d\mu_n \xrightarrow[n \to +\infty]{} \int_I f d\mu$$

In reality, this definition is formulated for continuous and bounded functions on the metric space I. However, given that we are dealing with a compact space in the context of this study, the continuity condition suffices.

Moreover, from this convergence, we can induce a topology on $\mathcal{P}_I$, known as the weak topology, rendering it a compact metric space (though not explored here).

A.2 Total Variation Convergence

The space $\mathcal{P}_I$ is a topological space and can be equipped with different distances. However, drawing from Wikipedia - L'encyclopédie libre, we define a specific distance called total variation distance on this space, which is stronger than the previously defined weak convergence. For this purpose, let $\mathcal{M}_I$ denote the set of measurable functions from I to the interval $[-1, 1]$.

Definition A.2 (Total Variation Distance)**.** *Let μ and ν be two measures from $\mathcal{P}_I$. The <u>total variation distance</u> between these measures is defined as follows:*

$$\|\mu - \nu\|_{TV} = \sup_{f \in \mathcal{M}_I} \left\{ \int_I f d\mu - \int_I f d\nu \right\}$$

Now, we can define total variation convergence, which corresponds to convergence under the aforementioned distance.

Definition A.3 (Total Variation Convergence). *Consider a sequence $(\mu_n)_n$ of measures from $\mathcal{P}_I$, and let μ be another measure in this space. We say that $(\mu_n)_n$ converges in total variation to μ if:*

$$\|\mu_n - \mu\|_{TV} = \sup_{f \in \mathcal{M}_I} \left\{ \int_I f d\mu_n - \int_I f d\mu \right\} \xrightarrow[n \to +\infty]{} 0$$

Based on the above definitions, we can deduce the following result:

Proposition A.1 (Hierarchy of Convergences). *Total variation convergence implies weak convergence.*

Proof. Let $(\mu_n)_n$ be a sequence of measures from $\mathcal{P}_I$ that converges in total variation to another measure μ in this space, and let f be a continuous function on I. Without loss of generality, assume that f takes values in $[-1, 1]$. As every continuous function is measurable, we have $f \in \mathcal{M}_I$. Hence, we have directly:

$$\left| \int_I f d\mu_n - \int_I f d\mu \right| \leq \|\mu_n - \mu\|_{TV} \xrightarrow[n \to +\infty]{} 0$$

This implies that $(\mu_n)_n$ converges weakly to μ.

□

Appendix B

Renewal Sequences

We dedicate an appendix to renewal sequences based on the article by Kingman titled “Powers of Renewal Sequences” for the general concepts, the main paper for the properties that will be discussed later, and the book by Feller for the underlying theory. These sequences will facilitate the analysis in cases where the meromorphic function does not possess a pole.

B.1 Fundamentals

Let’s start by defining these sequences.

Definition B.1 (Renewal Sequence)**.** *A* <u>*renewal sequence*</u> *is a sequence* $(u_n)_n$ *defined by the following recurrence relation:*

$$u_0 = 1 \quad \textit{and} \quad \forall n \in \mathbb{N}^*,\ u_n = \sum_{k=1}^{n} f_k u_{n-k} \tag{B.1}$$

where $(f_n)_n$ *is a sequence of positive real numbers such that the series with general term* f_n *is convergent and bounded by* 1 *:*

$$\forall n \in \mathbb{N}^*,\ f_n \geq 0 \quad \textit{and} \quad \sum_{n=1}^{+\infty} f_n \leq 1$$

Let’s add a note to better understand the context and reasons behind introducing renewal sequences.

Remark B.1 (Introduction of Renewal Sequences)**.** *Renewal sequences were introduced by Feller in the 20th century to study discrete-time Markov processes. They quantify the probabilities of returning to an initial state at different time instances. Specifically, the value* u_n *represents the probability of returning to the initial state exactly at time* n*, while the value* f_n *represents the probability of first-time return at time* n*.*

Let’s also state a property that holds for the set of these sequences.

Remark B.2 (Set of Renewal Sequences)**.** *The set of all these renewal sequences forms a commutative semigroup under multiplication, where each element of the product sequence is obtained by multiplying the corresponding elements of the two sequences.*

In their article, Kingman seeks to extend this property by showing that for any renewal sequence u, the sequence u^α is also a renewal sequence, where $\alpha > 1$ is a real number.

The use of renewal sequences may evoke the concept of regularization through convolution, which relies on the use of approximations of the identity. Indeed, the recurrence relation of renewal sequences exhibits similarities with discrete convolution. In the process of convolutional regularization, an input function is combined with a smoothing sequence, which acts as an approximation of the identity function. This operation smoothens and regularizes the input function by considering local information from its neighborhood.

Similarly, renewal sequences combine previous terms using the coefficients f_k to generate subsequent terms. This procedure can be interpreted as a form of discrete regularization, where each term in the sequence is influenced by preceding terms according to a specific pattern.

B.2 Moment Problem

The moment problem in mathematical analysis is an inverse problem that aims to reconstruct a real measure on a given interval from its moments. Specifically, given a real interval I and a sequence $(u_n)_n$ of real numbers, the question arises whether a positive Borel measure ν exists on I such that for every natural number n:

$$u_n = \int_I x^n \nu(dx)$$

If such a measure exists, it represents the probability distribution of a real random variable whose moments are given by the numbers u_n. The moment problem comes in various variants depending on the form of the interval:

- Hamburger moment problem: The interval I is the set of real numbers, $\mathbb{R}$.
- Stieltjes moment problem: The interval I is semi-open to the right, $[0, +\infty[$.
- Hausdorff moment problem: The interval I is a segment, $[a, b]$.

In our study, we focus on the specific case of the third variant, where the interval I is defined as $I = [0, 1]$. While the proof of this case can be approached in various ways, our primary objective here is different. We aim to present a theorem of particular significance in our study:

Theorem B.1 (Renewal Sequences as Moments). *Consider a finite Borel measure μ on the interval I and its sequence of moments $(f_n)_n$:*

$$\forall n \in \mathbb{N}^*, \quad f_n = \int_0^1 x^n \mu(dx)$$

Suppose that the latter satisfies the following condition:

$$\sum_{n=1}^{+\infty} f_n \leq 1, \quad \textit{i.e.} \quad \int_0^1 \frac{x}{1-x} \mu(dx) \leq 1$$

We then introduce the associated renewal sequence $(u_n)_n$, defined as:

$$u_0 = 1 \quad \textit{and} \quad \forall n \in \mathbb{N}^*,\ u_n = \sum_{k=1}^{n} f_k u_{n-k}$$

Under these conditions, there exists a probability measure ν on I such that $(u_n)_n$ represents its sequence of moments, i.e.:

$$\exists \nu \in \mathcal{P}_I \ / \ \forall n \in \mathbb{N}, \quad u_n = \int_0^1 x^n \nu(dx)$$

Proof. To prove this theorem, we proceed in two steps. The first step is to examine the particular case where the measure μ is a discrete measure, while the second step is to generalize this result to the continuous case by taking the limit of the discrete case.

Particular Case: Suppose that μ is a discrete measure with support contained in the interior of the interval I. In this case, we can express μ as a weighted sum of Dirac delta functions, i.e.:

$$\exists N \in \mathbb{N}^*,\ \exists (a_k)_{1\leq k\leq N} \in \mathbb{R}_+^N \text{ not all zero}, \exists (x_k)_{1\leq k\leq N} \in \mathring{I}^N \ / \quad \mu = \sum_{k=1}^{N} a_k \delta_{x_k}$$

The assumptions of the theorem can then be formulated as follows:

$$\forall n \in \mathbb{N}^*,\ f_n = \sum_{k=1}^{N} a_k x_k^n \quad \text{and} \quad \sum_{k=1}^{N} \frac{a_k x_k}{1-x_k} \leq 1 \tag{B.2}$$

By using Abel's lemma and the fact that the sequences $(f_n)_n$ and $(u_n)_n$ are bounded by 1, we can see that the series $\sum f_n z^n$ and $\sum u_n z^n$ converge absolutely in the disk D_0. We thus define the function U as follows:

$$U : z \in D_0 \longmapsto \sum_{n=0}^{+\infty} u_n z^n$$

For any complex number z within the disk D_0, we can write the following equalities using the Cauchy product:

$$U(z) = 1 + \sum_{n=1}^{+\infty} u_n z^n = 1 + \sum_{n=1}^{+\infty} \left(\sum_{k=1}^{n} f_k u_{n-k} \right) z^n = 1 + \left(\sum_{n=1}^{+\infty} f_n z^n \right) \left(\sum_{n=0}^{+\infty} u_n z^n \right)$$

However, the generating series associated with $(f_n)_n$ can be expressed as follows:

$$\sum_{n=1}^{+\infty} f_n z^n = \sum_{n=1}^{+\infty} \left(\sum_{k=1}^{N} a_k x_k^n \right) z^n = \sum_{k=1}^{N} \frac{a_k x_k z}{1 - x_k z}$$

We can then simplify the above expression using the following equality:

$$U(z) = \left[1 - \sum_{k=1}^{N} \frac{a_k x_k z}{1 - x_k z} \right]^{-1}$$

Thus, we have reduced the problem to studying a meromorphic function, with the above denominator being non-identically zero. Let's denote this function by V and analyze its poles. These poles satisfy the equation:

$$\sum_{k=1}^{N} \frac{a_k x_k z}{1 - x_k z} = 1 \tag{B.3}$$

By taking the imaginary part of this equation, we obtain:

$$\mathfrak{Im} \left(\sum_{k=1}^{N} \frac{a_k x_k z}{1 - x_k z} \right) = \mathfrak{Im}(z) \sum_{k=1}^{N} \frac{a_k x_k}{|1 - x_k z|^2} = 0$$

Since the terms in the above sum are not all zero, we can conclude that the poles of the function V are real. We can observe that the inverse of V is well-defined and strictly decreasing on the interval $]-\infty, 1]$, and it is positive at 1 according to equation (B.2). Therefore, the potential poles of V lie on the other side of the real line. Moreover, the poles cannot be equal to any of the points x_k^{-1}, as the function defined by equation (B.3) diverges in the vicinity of these points. Outside of these points, the inverse of V is differentiable, and its derivative satisfies:

$$\left(\frac{1}{V} \right)'(z) = - \sum_{k=1}^{N} \frac{a_k x_k}{(1 - x_k z)^2} < 0$$

Hence, the potential poles of V are all simple, and we denote them as $(\zeta_1, ..., \zeta_N)$. Therefore, the function V is holomorphic on $\mathbb{C}\backslash\{\zeta_1, ..., \zeta_N\}$. By virtue of the theorem on the classification of isolated singularities, we can define a holomorphic function W as follows:

$$W : z \in \mathbb{C}\backslash\{\zeta_1, ..., \zeta_N\} \longmapsto V(z) - \sum_{k=1}^{N} \frac{\mathrm{Res}(V, \zeta_k)}{z - \zeta_k}$$

where the residues are given by:

$$\forall k \in [\![1, N]\!], \quad \mathrm{Res}(V, \zeta_k) = \lim_{z \to \zeta_k} (z - \zeta_k) V(z) = \frac{1}{\left(\frac{1}{V}\right)'(\zeta_k)} = -\left[\sum_{k=1}^{N} \frac{a_k x_k}{(1 - x_k z)^2}\right]^{-1} < 0$$

The function W has removable singularities at the poles of V, which implies that it can be extended into a unique entire function. Furthermore, we have the following result:

$$\lim_{|z| \to +\infty} |W(z)| = \lim_{|z| \to +\infty} |V(z)| = \lim_{|z| \to +\infty} \left| 1 - \sum_{k=1}^{N} \frac{a_k}{(x_k z)^{-1} - 1} \right|^{-1} = \left[1 + \sum_{k=1}^{N} a_k\right]^{-1}$$

Consequently, W is a bounded entire function, which, according to Liouville's theorem, implies that it is constant and equal to its limit. Letting $b_k = -\mathrm{Res}(V, \zeta_k) > 0$ for all $k \in [\![1, N]\!]$, we can then reexpress the function V as follows:

$$V(z) = \left[1 + \sum_{k=1}^{N} a_k\right]^{-1} + \sum_{k=1}^{N} \frac{b_k}{\zeta_k} \frac{1}{1 - z\zeta_k^{-1}}$$

Thus, by returning to the study on the disk D_0, where the poles lie outside of it, we obtain the following equality:

$$\sum_{n=0}^{+\infty} u_n z^n = \left[1 + \sum_{k=1}^{N} a_k\right]^{-1} + \sum_{n=0}^{+\infty} \left(\sum_{k=1}^{N} \frac{b_k}{\zeta_k} \left(\frac{1}{\zeta_k}\right)^n \right) z^n$$

By applying the principle of uniqueness of power series representation, we arrive at the following equalities:

$$\left[1 + \sum_{k=1}^{N} a_k\right]^{-1} + \sum_{k=1}^{N} \frac{b_k}{\zeta_k} = 1 \quad \text{and} \quad \forall n \geq 1, \ u_n = \sum_{k=1}^{N} \frac{b_k}{\zeta_k} \left(\frac{1}{\zeta_k}\right)^n$$

We can then define the measure ν as follows:

$$\nu = \left[1 + \sum_{k=1}^{N} a_k\right]^{-1} \delta_0 + \sum_{k=1}^{N} \frac{b_k}{\zeta_k} \delta_{\zeta_k^{-1}}$$

It is easy to verify that ν is a probability measure, as:

$$\int_0^1 \nu(dx) = \left[1 + \sum_{k=1}^{N} a_k\right]^{-1} + \sum_{k=1}^{N} \frac{b_k}{\zeta_k} = 1$$

Moreover, the sequence $(u_n)_n$ is indeed the sequence of moments of the measure ν, as:

$$\forall n \geq 1, \quad \int_0^1 x^n \nu(dx) = \sum_{k=1}^{N} \frac{b_k}{\zeta_k} \left(\frac{1}{\zeta_k}\right)^n = u_n$$

General Case: Now let's proceed to the study of the general case. Our goal is to construct a sequence $(f^{(l)})_l$ that satisfies the particular case with its associated renewal sequences $(u^{(l)})_l$, converging respectively to the sequences $(f_n)_n$ and $(u_n)_n$ mentioned in the statement. Consider a fixed integer $l \in \mathbb{N}^*$ and define the following quantities for $k \in [\![1, l-1]\!]$ and $n \in \mathbb{N}^*$:

$$a_k^{(l)} = \mu\left(\left]\frac{k}{l}, \frac{k+1}{l}\right]\right), \quad x_k^{(l)} = \frac{k}{l} \quad \text{and} \quad f_n^{(l)} = \sum_{k=1}^{l-1} a_k^{(l)} \left(x_k^{(l)}\right)^n$$

Let's begin by showing that this sequence, along with the associated renewal sequences, converges before proving that they satisfy the particular case. For $l, n \in \mathbb{N}^*$, since the function $x \longmapsto x^n$ is increasing on I, we can use the comparison between sum and integral to obtain the following inequalities:

$$\sum_{k=1}^{l-1} \mu\left(\left]\frac{k}{l}, \frac{k+1}{l}\right]\right)\left(\frac{k}{l}\right)^n \leq \int_0^1 x^n \mu(dx) \leq \sum_{k=1}^{l-1} \mu\left(\left]\frac{k}{l}, \frac{k+1}{l}\right]\right)\left(\frac{k+1}{l}\right)^n$$

This allows us to obtain the following inequalities:

$$0 \leq f_n - f_n^{(l)} \leq \sum_{k=1}^{l-1} a_k^{(l)} \left(\left(\frac{k+1}{l}\right)^n - \left(\frac{k}{l}\right)^n\right)$$

Furthermore, the power function is not only increasing but also continuous and differentiable on I, with a derivative bounded by n on this interval. By using the mean value theorem, we can write:

$$0 \leq f_n - f_n^{(l)} \leq \sum_{k=1}^{l-1} a_k^{(l)} \frac{n}{l} \leq \frac{n}{l} \mu\left(]0,1]\right) \xrightarrow[l \to +\infty]{} 0$$

Thus, by induction on n, we can conclude that:

$$u_n^{(l)} = \sum_{k=1}^{n} f_k^{(l)} u_{n-k}^{(l)} \xrightarrow[l \to +\infty]{} \sum_{k=1}^{n} f_k u_{n-k} = u_n$$

Now we are ready to show that the used sequences satisfy the particular case and that the limit renewal sequence indeed represents a sequence of moments of a probability measure on I. For a fixed integer $l \in \mathbb{N}^*$, the real numbers $a_k^{(l)}$ are not all zero, and by using the growth of the function $x \mapsto x(1-x)^{-1}$ on I, we can once again employ a comparison between sum and integral to obtain inequality (B.2) with:

$$\sum_{k=1}^{l-1} \frac{a_k^{(l)} x_k^{(l)}}{1 - x_k^{(l)}} \leq \int_0^1 \frac{x}{1-x} \mu(dx) \leq 1$$

Hence, there exists a probability measure $\nu^{(l)}$ on I such that $u^{(l)}$ is its sequence of moments. Due to the first section of the previous appendix, we know that $\mathcal{P}_I$ is a compact metric space for the weak topology. Therefore, by possibly extracting a subsequence, the sequence $(\nu^{(l)})_l$ weakly converges to a measure $\nu \in \mathcal{P}_I$. Using the continuity of the power function on I, we can conclude, by weak convergence, that:

$$\forall n \in \mathbb{N}^*, \quad u_n = \lim_{l \to +\infty} u_n^{(l)} = \lim_{l \to +\infty} \int_0^1 x^n \nu^{(l)}(dx) = \int_0^1 x^n \nu(dx)$$

This concludes the proof of this theorem.

□

B.3 Monotonicity of Renewal Sequences

In this section of the appendix, we will study the monotonicity of renewal sequences and the ratios of successive terms in these sequences. These results will be essential for analyzing the case where the meromorphic function has no poles. We will use the previous theorem to state the following corollary:

Corollary B.1.1 (Monotonicity of Renewal Sequences)**.** *Under the assumptions of the previous theorem, the renewal sequence $(u_n)_n$ satisfies the following properties:*

- *The sequence $(u_n)_n$ is decreasing.*
- *The sequence of ratios of successive terms $\left(\frac{u_n}{u_{n-1}}\right)_n$ is increasing.*

Proof. Under the assumptions of the previous theorem, we can rewrite the renewal sequence as follows:

$$\forall n \in \mathbb{N}, \quad u_n = \int_0^1 x^n \nu(dx)$$

where ν is a probability measure on I.

- The decrease of the sequence $(u_n)_n$ is immediate. Indeed, since $(x^n)_n$ is a decreasing sequence for any $x \in I$, it follows that $(u_n)_n$ is also decreasing.

- Let $n \geq 1$ be a natural number. To show that the sequence of ratios of successive terms is increasing, we start by rewriting u_n^2 to reveal the product of the preceding and succeeding terms in the sequence:

$$u_n^2 = \left(\int_0^1 x^n \nu(dx)\right)^2 = \left(\int_0^1 \sqrt{x^{n-1}}\sqrt{x^{n+1}}\nu(dx)\right)^2$$

We can now apply the Cauchy-Schwarz inequality to isolate the two desired terms:

$$\left(\int_0^1 \sqrt{x^{n-1}}\sqrt{x^{n+1}}\nu(dx)\right)^2 \leq \left(\int_0^1 x^{n-1}\nu(dx)\right)\left(\int_0^1 x^{n+1}\nu(dx)\right)$$

Thus, we obtain:

$$\frac{u_n}{u_{n-1}} \leq \frac{u_{n+1}}{u_n}$$

This demonstrates the increase of the sequence of ratios of successive terms.

□

By proving these results, we can conclude that the sequence of ratios of successive terms is convergent. Indeed, this sequence is increasing and bounded by 1, since the sequence $(u_n)_n$ is decreasing. Therefore, its limit can be equal to 1 or to a real number $\sigma < 1$. This distinction of cases will be essential in our study of non-democratic regimes.

B.4 Renewal Theory

In addition, we introduce a final section based on the renewal theory presented in Feller's book on probabilities. The previously presented equation can be formulated in a more general way. From two given sequences, we can uniquely define a third sequence by recurrence, regardless of the situation, through an equation known as the renewal equation.

Definition B.2 (Renewal Equation). *Consider two sequences of real numbers $(f_n)_n$ and $(m_n)_n$. We uniquely define the sequence $(v_n)_n$ using the following recurrence relation:*

$$\forall n \in \mathbb{N}^*, \quad v_n = m_n + \sum_{k=1}^{n-1} f_k v_{n-k} \tag{B.4}$$

This relation is known as the renewal equation.

To facilitate the use of this definition in various situations, we now wish to adapt it using generating series. The following proposition states a relation between series that is equivalent to our definition.

Proposition B.2 (Associated Generating Series)**.** *Suppose we have two sequences of positive real numbers $(f_n)_n$ and $(m_n)_n$ that satisfy the following conditions:*

$$\sum_{n\geq 1} f_n < +\infty \quad \textit{and} \quad \sum_{n\geq 1} m_n < +\infty$$

In this case, the generating series F and M, associated with these sequences, converge within the disk D_0 and define a new function expandable into a power series, denoted as V, which also converges within this disk, provided that D_0 is stable under F. Moreover, these series are related by the following equation:

$$\forall z \in D_0, \quad V(z) = \frac{M(z)}{1 - F(z)}$$

Proof. Under the assumptions of convergence, we can deduce that the sequences $(f_n)_n$ and $(m_n)_n$ converge to 0, which implies their boundedness. Therefore, according to Abel's lemma, their generating series are well-defined. Similarly, we can conclude that the sequence $(v_n)_n$ also satisfies this property.

Let's take $z \in D_0$. Using Cauchy's product similarly to the way it was used in dealing with the generating series associated with the viability product, we can establish the following equalities using equation (B.4):

$$\sum_{n=1}^{+\infty} v_n z^n = \sum_{n=1}^{+\infty} m_n z^n + \sum_{n=1}^{+\infty}\sum_{k=1}^{n-1} f_k v_{n-k} z^n = \sum_{n=1}^{+\infty} m_n z^n + \left(\sum_{n=1}^{+\infty} f_n z^n\right)\left(\sum_{n=1}^{+\infty} v_n z^n\right)$$

Thus, we obtain:

$$\left(1 - \sum_{n=1}^{+\infty} f_n z^n\right)\left(\sum_{n=1}^{+\infty} v_n z^n\right) = \sum_{n=1}^{+\infty} m_n z^n$$

And due to the stability of the open disk D_0 under F, we can conclude.

□

This proposition, originally presented as a remark valid on the open unit disk, can be extended to the value 1 to obtain a direct result stemming from the definition. This is what we state in the following theorem.

Theorem B.3 (Renewal Theorem)**.** *Suppose we have two sequences of positive real numbers $(f_n)_n$ and $(m_n)_n$ that satisfy the following conditions:*

$$\sum_{n\geq 1} f_n < +\infty \textit{ with sum } f \quad \textit{and} \quad \sum_{n\geq 1} m_n < +\infty \textit{ with sum } m$$

Suppose $f < 1$. Then the sequence $(v_n)_n$ defined by the renewal equation

$$\forall n \in \mathbb{N}^*, \quad v_n = m_n + \sum_{k=1}^{n-1} f_k v_{n-k}$$

converges to 0. Moreover, it is the general term of a convergent series with sum v, and this series satisfies the following equation:

$$v = \frac{m}{1 - f}$$

This theorem, originally formulated by Feller, is actually more extensive as it covers all cases of f values. However, in our study, we have confined ourselves to this specific case, as it's the only one we will use. The proof of this simplified version is identical to that of the previous proposition. We start with the renewal equation and apply Cauchy's product to arrive at the result.

Appendix C

Generalized Renewal Sequences

In the previous chapter, we studied renewal sequences, yet they possess certain limitations, notably the constraint that the series with general term f_n must be bounded by 1. In order to surpass this restriction, we shall delve into a generalization of these sequences, primarily drawing from the article titled "Semi-p-Functions" by Kingman, as well as the article by Kaluza for theorem formulation and a chapter from Zucker's book for philosophical insights in the third section. Subsequently, we will employ these generalized sequences in our study of unbounded fitness.

In the commutative ring $(\mathbb{R}, +, \times)$, we can define formal power series with real coefficients. Thus, a sequence $(u_n)_n$ can also be expressed in the following form:

$$\sum_{n=0}^{+\infty} u_n X^n \in \mathbb{R}[[X]] \quad \text{or} \quad \sum_{n=0}^{+\infty} u_n z^n$$

We shall employ the latter notation to simplify and draw closer to the representation of power series (when they are defined).

C.1 Initial Approach

Consider a sequence of real numbers $(u_n)_n$ with $u_0 = 1$. Let

$$U = \sum_{n \geq 0} u_n X^n$$

Then $-U$ is invertible, with inverse

$$V = -\sum_{n \geq 0} (1 - U)^n = -U^{-1}$$

We also define $F = 1 + V$, removing the constant term from V.

Now, let's define the sequence $(f_n)_n$ as the coefficients of the formal power series V. Thus, each f_n is a polynomial $\phi_n(u)$ in $u_1, ..., u_n$, and conversely, each u_n is a polynomial in $f_1, ..., f_n$. The relationship between these two sequences is described by the following proposition:

Proposition C.1 (Renewal Equations)**.** *The renewal equations below describe the recursive relationship between the terms of the two sequences:*

$$\forall n \in \mathbb{N}^*, \quad u_n = f_n + \sum_{k=1}^{n-1} f_k u_{n-k} \quad \textit{and} \quad f_n = u_n - \sum_{k=1}^{n-1} u_k f_{n-k} = \phi_n(u) \qquad \text{(C.1)}$$

Proof. Using formal power series introduction, the proof of this proposition is straightforward through the Cauchy product. By observing the relations

$$UV = -1 \quad \text{and} \quad F = 1 + V \quad \Longrightarrow \quad U - 1 = FU$$

, we deduce that:

$$\sum_{n=1}^{+\infty} u_n X^n = \left(\sum_{n=1}^{+\infty} f_n X^n\right)\left(\sum_{n=1}^{+\infty} u_n X^n\right) = \sum_{n=1}^{+\infty}\left(\sum_{k=1}^{n} f_k u_{n-k}\right) X^n$$

Thus, we obtain equations (C.1), which are clearly equivalent.

□

The above proposition enables us to calculate the values of u_n and f_n for any natural number $n \geq 1$. However, we make the following remark:

Remark C.1 (Initial Terms)**.** *Using the initial value for u_0 and the definition of V for f_0, we have:*

$$u_0 = 1 \quad \text{and} \quad f_0 = -1$$

However, in practice, the first term of the sequence $(f_n)_n$ is rarely used.

Let's restate the definition of renewal sequences using the aforementioned polynomials ϕ_n.

Definition C.1 (Renewal Sequence)**.** *A sequence of real numbers $(u_n)_n$ with $u_0 = 1$ is a renewal sequence if the following conditions are satisfied:*

$$\forall n \in \mathbb{N}^*,\ \phi_n(u) \geq 0 \quad \text{and} \quad \sum_{n=1}^{+\infty} \phi_n(u) \leq 1$$

We will denote this set of sequences as $\mathcal{RS}$, also abbreviated.

The second assumption of the preceding definition is specific and may not always be attainable. Therefore, we can define a new type of sequence as follows:

Definition C.2 (Generalized Renewal Sequence)**.** *A sequence of real numbers $(u_n)_n$ with $u_0 = 1$ is a generalized renewal sequence if the following condition is satisfied:*

$$\forall n \in \mathbb{N}^*, \quad \phi_n(u) \geq 0$$

We will denote this set of sequences as $\mathcal{GRS}$, also abbreviated.

As a result, Remark B.1, which illustrates the interpretation of renewal sequences according to Feller, is no longer valid in this context. However, it is still relevant to inquire under which conditions a sequence can be considered a generalized renewal sequence. Kaluza has formulated an initial sufficient condition in the following theorem, while further cases will be presented later.

Theorem C.2 (Kaluza's Theorem)**.** *Let $(u_n)_n$ be a sequence of real numbers with initial term $u_0 = 1$, where the second term is strictly positive, and the sequence of quotients of its successive terms is increasing:*

$$u_1 > 0 \quad \text{and} \quad \forall n \in \mathbb{N}^*, \begin{vmatrix} u_{n-1} & u_n \\ u_n & u_{n+1} \end{vmatrix} \geq 0$$

In this case, the coefficients of the formal power series F are all nonnegative:

$$\forall n \in \mathbb{N}^*, \quad f_n \geq 0$$

Proof. The proof of this theorem relies on various types of recurrences, all of which are quite straightforward. Here are the detailed steps of the proof:

- First, by double induction, we show that $u_n > 0$ for all natural numbers n. We use the initial datum $u_0 = 1$ and the condition $u_1 > 0$. The heredity can be written as:

$$u_{n+2} \geq \frac{u_{n+1}^2}{u_n} > 0$$

- Second, we restate the increasing condition of the quotients of successive terms:

$$\forall n \in \mathbb{N}^*,\ \forall k \in [\![1, n]\!], \quad \frac{u_{n-k+1}}{u_{n-k}} \geq \ldots \geq \frac{u_{n+1}}{u_n}$$

- Using the two previous results, we perform another recurrence to establish the theorem's result. We begin by noticing that for all natural numbers $n \geq 1$, we have the following equations:

$$\begin{cases} 0 = u_n - \sum_{k=1}^{n} f_k u_{n-k} & (1) \\ f_{n+1} = u_{n+1} - \sum_{k=1}^{n} f_k u_{n+1-k} & (2) \end{cases}$$

By linearly combining the two equations, we obtain:

$$-u_{n+1} \times (1) + u_n \times (2) \quad \Longrightarrow \quad u_n f_{n+1} = \sum_{k=1}^{n} f_k (u_{n-k} u_{n+1} - u_{n-k+1} u_n)$$

This allows us to write the following recurrence relation:

$$f_{n+1} = \frac{1}{u_n} \sum_{k=1}^{n} f_k \begin{vmatrix} u_{n-k} & u_{n-k+1} \\ u_n & u_{n+1} \end{vmatrix}$$

Through a strong induction combined with the two previous results, we can conclude this proof.

□

Next, we consider the study of relationships between the two types of sequences, with the aim of eventually utilizing existing results. This idea is developed in the following section.

C.2 Relation with Renewal Sequences

Despite the apparent connection between the definitions, let's examine some properties that relate generalized renewal sequences to ordinary ones. The following proposition presents an initial and relatively straightforward link.

Proposition C.3 (Specialization of Sequences)**.** *Consider $(u_n)_n \in \mathcal{GRS}$. Then we have the following equivalence:*

$$(u_n)_n \in \mathcal{RS} \quad \Longleftrightarrow \quad \forall n \in \mathbb{N},\ u_n \leq 1$$

Proof. Let's establish both directions of the equivalence, with the first direction being relatively straightforward.

($\Rightarrow$) By virtue of equation (C.1), this first direction can be rapidly proven by strong induction. The initialization is given by definition, and the heredity is established as follows:

$$\forall n \in \mathbb{N}^*, \quad u_n = \sum_{k=1}^{n} f_k u_{n-k} \leq \sum_{k=1}^{+\infty} f_k \leq 1$$

($\Leftarrow$) For the converse direction, note that the sequence of general terms $f_n = \phi_n(u)$ is positive by definition, and using a recurrence discussed previously in our general study, the sequence $(u_n)_n$ is also positive. Thus, through equation (C.1), we obtain:

$$\forall n \in \mathbb{N}^*, \quad 0 \leq f_n \leq u_n \leq 1$$

Consequently, the formal power series defined in the previous section can be seen as power series converging in the disk D_0. Hence, for any $z \in [0, 1[$, we have:

$$\sum_{n=1}^{+\infty} f_n z^n = 1 - \left[\sum_{n=0}^{+\infty} u_n z^n\right]^{-1} \quad \Longrightarrow \quad \sum_{n=1}^{+\infty} f_n z^n \leq 1$$

And through continuity, we ultimately obtain:

$$\sum_{n=1}^{+\infty} f_n \leq 1$$

□

Next, let's observe through the following lemma that applying a specific scaling factor to a generalized renewal sequence preserves the property of being a generalized renewal sequence.

Lemma C.4 (Stability)**.** *Let $(u_n)_n$ be a sequence of real numbers with $u_0 = 1$ and $b \geq 0$ a real number. Define $(v_n)_n$ as a sequence of real numbers using the following relation:*

$$\forall n \in \mathbb{N}, \quad v_n = u_n b^n$$

We can state the following result:

$$(u_n)_n \in \mathcal{GRS} \quad \Longleftrightarrow \quad (v_n)_n \in \mathcal{GRS}$$

Proof. The proof of this lemma relies on a simple observation. Using the formal power series representation, we have:

$$\sum_{n=1}^{+\infty} \phi_n(v) X^n = 1 - \left[\sum_{n=0}^{+\infty} u_n b^n X^n\right]^{-1} = \sum_{n=1}^{+\infty} \phi_n(u)(bX)^n$$

Thus, the polynomials ϕ_n applied to the sequences satisfy the same relation as those sequences:

$$\forall n \in \mathbb{N}^*, \quad \phi_n(v) = \phi_n(u) b^n$$

This confirms the lemma's result.

□

Hence, if we assume having a real number $c \geq 0$ and a sequence $(u_n)_n \in \mathcal{GRS}$ satisfying the following relation:

$$\forall n \in \mathbb{N}, \quad u_n \leq c^n \tag{C.2}$$

then combining the previous lemma with the earlier proposition indicates that if we set $b = c^{-1}$, the sequence $(v_n)_n$ defined by the following relation:

$$\forall n \in \mathbb{N}, \quad v_n = u_n b^n \leq 1$$

defines a renewal sequence. Thus, these particular generalized renewal sequences are not so different from ordinary renewal sequences. A minor specialization allows us to transition from one type to the other.

Conversely, if our generalized renewal sequence grows so rapidly that the inequality (C.2) becomes unattainable for any positive real c, then we face a challenge. However, this is actually a "non-issue".

The finite segments of such sequences exhibit no new phenomenon, as illustrated by the following result:

Theorem C.5 (Partial Specialization)**.** *Let $(u_n)_n$ be a sequence of real numbers with initial term $u_0 = 1$. The following equivalence holds:*

$$(u_n)_n \in \mathcal{GRS} \iff \forall N \geq 1, \exists (v_n)_n \in \mathcal{RS}, \exists c \geq 0 \ / \ \forall n \in [\![1, N]\!],\ u_n = v_n c^n$$

Proof. Let's establish both directions of the equivalence one by one.

($\Rightarrow$) To demonstrate the existence of both elements, let's construct them. Let N be a natural number. Choose a real number $c \geq 0$ sufficiently large to satisfy the following inequality:

$$\sum_{n=1}^{N} \phi_n(u) c^{-n} \leq 1$$

Additionally, define $(v_n)_n$ as a renewal sequence satisfying:

$$\forall n \in \mathbb{N}^*, \quad \phi_n(v) = \begin{cases} \phi_n(u) c^{-n} & \text{if } n \leq N \\ 0 & \text{if } n > N \end{cases}$$

Then, the relation $\overline{u}_n = v^n c^n$ for every natural number n defines a sequence satisfying

$$\forall n \leq N, \quad \phi_n(\overline{u}) = \phi_n(v) c^n = \phi_n(u)$$

Thus, for all $n \leq N$, we have $\overline{u}_n = u_n$.

($\Leftarrow$) For the converse, consider a natural number $N \geq 1$. We have $(v_n)_n \in \mathcal{RS}$ and $c \geq 0$ such that:

$$\forall n \in [\![1, N]\!], \quad \phi_n(u) = \phi_n(v) c^n \geq 0$$

This equality holds because, for a given n, $\phi_n(u)$ depends only on the values of $u_1, \ldots, u_n$. Since this relation holds for every $N \geq 1$, we can infer that the above inequality holds for every natural number $n \geq 1$.

□

We have just observed that generalized renewal sequences differ significantly from ordinary renewal sequences only when they grow faster than any geometric progression. Hence, in the next section, we will examine both cases.

C.3 Wildness and Tameness

The hypothesis (C.2) introduced earlier holds particular significance and allows us to divide our study of generalized renewal sequences into two distinct cases.

One of the most fundamental and common distinctions found in literature is the differentiation between wild animals and domesticated animals. Kingman, after drawing parallels with political regimes in his previous study, also employs this distinction. These two classes of animals are familiar to all of us, which is why the author uses them in the following definition:

Definition C.3 (Classes of GRS)**.** *Let $(u_n)_n$ be a generalized renewal sequence. We say that:*

$$(u_n)_n \text{ is } \underline{\text{tame}} \iff \exists c \geq 0 \ / \ \forall n \in \mathbb{N}^*,\ u_n \leq c^n$$

and conversely:

$$(u_n)_n \text{ is } \underline{\text{wild}} \iff \forall c \geq 0 \ / \ \exists n \in \mathbb{N}^*,\ u_n > c^n$$

Let's attempt to relate the sequences of Kingman with terms related to animals using the following remark:

Remark C.2. *In the realm of animals, a wild animal is generally considered to live in freedom, independently, and following its own rules and instincts. In contrast, a domesticated animal is accustomed to human proximity and influence, making it more predictable and subject to a certain degree of control.*

Similarly, generalized renewal sequences can exhibit different behaviors. A sequence is called "tame" if its terms grow in a limited manner, following a geometric progression with a bounded factor. This implies a certain regularity and predictability in the growth of sequence terms. On the other hand, a sequence is deemed "wild" if its terms grow faster than any geometric progression, resulting in less predictable and potentially chaotic behavior.

Thus, by employing the terms "wild" and "tame" to describe generalized renewal sequences, Kingman seeks to establish an analogy with the distinction between more regular and predictable behaviors (analogous to domesticated animals) and more erratic and unpredictable behaviors (analogous to wild animals). This analogy aids in better grasping the characteristics and properties of the studied sequences.

What is remarkable is that the term "wildness" is not a technical term specific to a group of specialists, unlike many other terms. It is a widely understood category. This category, universally relevant and adaptable, is present in Homeric poems, Hippocratic medical texts, Platonic philosophical writings, Aristotelian biological studies, and Philonic theological texts.

Let's focus on Aristotle's perspective due to the nature of our study topic. The Aristotelian Problems draw an analogy between taming animals and educating children, where it is argued that a wild animal is to a tamed animal what a child is to an adult. The ancient Greek philosopher actually distinguishes three categories of animals in terms of wildness: naturally wild animals, domesticable animals, and domesticated animals. However, one could also consider that all tamed species are also wild. According to Aristotle, the wild state is the only natural state for an animal, and all domesticated animals have a wild form.

The last proposition from the previous section states that every wild sequence agrees on any finite interval with a tame sequence. Thus, the mathematical results presented align with the philosophical commentary stated. Therefore, we can employ this theory to study a wide family of generalized renewal sequences, as demonstrated by the following theorem.

Theorem C.6. *Let $(u_n)_n$ be a sequence of strictly positive real numbers with an initial term equal to 1:*

$$u_0 = 1 \quad \text{and} \quad \forall n \in \mathbb{N}^*,\ u_n > 0$$

such that the sequence of ratios of its successive terms $(r_n)_n$ is increasing, defined by:

$$\forall n \in \mathbb{N}, \quad r_n = \frac{u_{n+1}}{u_n}$$

Then, $(u_n)_n$ is a generalized renewal sequence. Furthermore,

$$(u_n)_n \text{ is tame} \quad \Longleftrightarrow \quad r = \lim_{n\to+\infty} r_n < +\infty$$

Proof. Let N be a natural number greater than or equal to 1, and set $c = r_N > 0$. Define a sequence $(v_n)_n$ as follows:

$$\forall n \in \mathbb{N}, \quad v_n = \begin{cases} u_n c^{-n} & \text{if } n \leq N \\ u_N c^{-N} & \text{if } n > N \end{cases}$$

As $(r_n)_n$ is increasing, it is observed that:

$$\forall n \in \mathbb{N}, \quad 0 < v_n \leq 1 = v_0$$

Using the same reasoning, combined with the fact that:

$$u_N \leq cu_{N-1} \quad \Longrightarrow \quad v_{N+1}v_{N-1} = u_N c^{-N} u_{N-1} c^{-N+1} \geq u_N^2 c^{-2N} = v_N^2$$

for the intermediate case, we notice that:

$$\forall n \in \mathbb{N}^*, \quad v_{n+1}v_{n-1} \geq v_n^2$$

Thus, Kaluza's theorem guarantees that $(v_n)_n$ is a generalized renewal sequence, and proposition C.3 ensures that it is, in fact, an ordinary renewal sequence. Since the choice of N is arbitrary, theorem C.5 allows us to conclude that $(u_n)_n$ is also a generalized renewal sequence.

Let $r = \lim_{n\to+\infty} r_n$ denote, by abuse of notation, the limit of the sequence of ratios of successive terms of $(u_n)_n$, whether it is finite or not. Thus, for any natural number n, we have $u_n \leq r^n$, allowing us to write:

$$\exists c \geq 0 \;/\; \forall n \in \mathbb{N},\ u_n \leq c^n \quad \Longleftrightarrow \quad r \leq c$$

This conclusion completes the proof of the theorem.

□

Thus, we have demonstrated a powerful theorem that describes a family of generalized renewal sequences. However, in practice, we generally focus on the second part of the theorem's conclusion concerning generalized renewal sequences whose sequence of ratios of successive terms is increasing. We will characterize some of these sequences in the next section.

C.4 Moment Problem and Monotonicity

Utilizing the concepts and results from the chapter on renewal sequences, we can express generalized renewal sequences in terms of moments to determine the growth of the sequences of ratios of their associated successive terms.

Theorem C.7 (Generalized Renewal Sequences as Moments)**.** *Consider a finite Borel measure μ defined on the set of positive real numbers $\mathbb{R}_+$. For this measure, define its sequence of moments $(f_n)_n$ as follows:*

$$\forall n \in \mathbb{N}^*, \quad f_n = \int_0^{+\infty} x^n \mu(dx)$$

Next, we introduce the associated generalized renewal sequence $(u_n)_n$, defined as follows:

$$u_0 = 1 \quad \textit{and} \quad \forall n \in \mathbb{N}^*,\ u_n = f_n + \sum_{k=1}^{n-1} f_k u_{n-k}$$

Then, there exists a probability measure ν defined on the set of positive real numbers $\mathbb{R}_+$ such that $(u_n)_n$ is its sequence of moments, i.e.,

$$\exists \nu \in \mathcal{P}_{\mathbb{R}_+} \;/\; \forall n \in \mathbb{N}, \quad u_n = \int_0^{+\infty} x^n \nu(dx)$$

Proof. To prove this theorem, we will follow a similar approach as used for ordinary renewal sequences, dividing the proof into two steps. The first step involves examining the particular case where the measure μ has support within a finite interval, while the second step generalizes this result to the case where the support is unbounded, by taking a limit of the particular case.

Particular Case: Suppose μ has support contained in the interval $[0, A]$, where $A > 1$ is a sufficiently large real number. In this case, the moments f_n can be expressed as follows:

$$\forall n \in \mathbb{N}^*, \quad f_n = \int_0^A x^n \mu(dx)$$

We can then choose a sufficiently small real number α such that $\alpha < A^{-1}$ and:

$$\sum_{n=1}^{+\infty} f_n \alpha^n \leq 1$$

For instance, we can take $\alpha = (f_1 + A)^{-1}$. Thus, if $(u_n)_n$ is the associated generalized renewal sequence to $(f_n)_n$, then $(u_n \alpha^n)_n$ is the renewal sequence associated to $(f_n \alpha^n)_n$. Applying theorem B.1 that expresses renewal sequences as moments, and making renormalizations if necessary, we obtain that $(u_n)_n$ is the sequence of moments of a probability measure ν on the interval $]0, \alpha^{-1}]$.

General Case: To handle the general case, we rely on the observation that in the context of renewal sequences, we can define a sequence $(f^{(l)})_l$ that satisfies the particular case above, with its corresponding generalized renewal sequences $(u^{(l)})_l$ converging respectively to the sequences $(f_n)_n$ and $(u_n)_n$ mentioned in the statement. For any $l \in \mathbb{N}^*$, we define the sequence $f^{(l)}$ as follows:

$$\forall n \in \mathbb{N}^*, \quad f_n^{(l)} = \int_0^l x^n \mu(dx)$$

This sequence satisfies the conditions of the particular case, and due to dominated convergence, we can establish that $(f^{(l)})_l$ converges to $(f_n)_n$. This convergence also extends to the associated generalized renewal sequences by induction.

To prove that the limit $(u_n)_n$ indeed represents a sequence of moments of a probability measure, we note that $\mathbb{R}_+$ is a metric space in which closed balls are compact. Therefore, our argument of weak convergence remains valid when applied to the space of probability measures on $\mathbb{R}_+$. This concludes the proof of the theorem.

□

It is worth mentioning that this result is well-known, but it is still presented in the article titled "Compacité faible de l'ensemble des probabilités" by Buzzi, in case an additional reference is required.

As a consequence of this theorem, we are able to study the sequence of ratios of successive terms of generalized renewal sequences. Since the proof is identical to that of the ordinary case, we can state the following corollary without proof:

Corollary C.7.1 (Monotonicity)**.** *Under the assumptions of the preceding theorem, the sequence of ratios of successive terms of the generalized renewal sequence* $\left(\frac{u_n}{u_{n-1}}\right)_n$ *is increasing.*

Unlike renewal sequences, where we can directly deduce an interval in which the potential limit lies, in the case of generalized renewal sequences, we have no information about monotonicity. Thus, each case must be treated individually to determine the interval in which the limit may reside. However, generally, the sequence of ratios diverges.

Appendix D

Probabilistic Approach

In this fourth appendix chapter, we adopt a probabilistic perspective to reexamine equation (2.3), which describes the fitness distribution. Let us revisit this equation from the outset of our study and examine it from a probability standpoint. We recall that we initially had:

$$p_n(dx) = \beta \sum_{k=0}^{n-1} \frac{W_{n-k}}{W_n}(1-\beta)^k x^k q(dx) + \frac{1}{W_n}(1-\beta)^n x^n p_0(dx) = \beta \sum_{k=0}^{n-1} \frac{v_{n-k}}{v_n} x^k q(dx) + \frac{1}{v_n}(1-\beta) x^n p_0(dx)$$

To achieve our goal, we introduce the following quantities:

$$\forall k \in [\![0, n-1]\!],\ \theta_{k,n} = \beta \frac{v_{n-k}}{v_n}\mu_k \quad \text{and} \quad \theta_{n,n} = (1-\beta)\frac{1}{v_n} m_n$$

Additionally, we define the following probability measures:

$$\forall k \in \mathbb{N},\ q_k(dx) = \frac{x^k q(dx)}{\mu_k} \quad \text{and} \quad p_{0,n}(dx) = \frac{x^n p_0(dx)}{m_n}$$

Using these new definitions, we can establish a probability on the set $[\![0, n]\!]$. To do this, we introduce a random variable T_n such that:

$$\forall k \in [\![0, n]\!], \quad \mathbb{P}(T_n = k) = \theta_{k,n}$$

This formulation of probability allows us to satisfy the normalization condition:

$$\sum_{k=0}^{n} \mathbb{P}(T_n = k) = \sum_{k=0}^{n} \theta_{k,n} = 1$$

This probabilistic approach thus opens new perspectives for understanding fitness distribution and exploring its fundamental properties.

D.1 Exploration of the Unbounded Fitness Case

Let us return to the fifth chapter where we deepened our analysis by considering evolutionary dynamics with unbounded fitness, particularly by studying exponential initial distributions. Thanks to our new probabilistic perspective, we can now highlight an interesting observation. It becomes clear that the final theorem 5.11, which deals with asymptotic fitness, could have been anticipated and demonstrated using our new equation:

$$p_n(dx) = \sum_{k=0}^{n-1} \mathbb{P}(T_n = k) q_k(dx) + \mathbb{P}(T_n = n) p_{0,n}(dx)$$

By recalling that we had

$$m_n = \mu_n = n! \quad \text{and} \quad v_n \underset{n\to+\infty}{\sim} n!$$

We can conclude that the result naturally follows from the following observation:

$$\mathbb{P}(T_n = 0) = \beta,\ \mathbb{P}(T_n = 1) \underset{n\to+\infty}{\sim} \frac{\beta}{n},\ \dots\ ,\ \mathbb{P}(T_n = n-1) \underset{n\to+\infty}{\sim} \frac{\beta}{n},\ \mathbb{P}(T_n = n) \xrightarrow[n\to+\infty]{} 1-\beta$$

This analysis illustrates how our probabilistic approach enriches our understanding of asymptotic behavior in the context of unbounded fitness, paving the way for a better appreciation of the underlying mechanisms governing population evolution.

D.2 Significant Example: Upward Trajectory of Viabilities

In this section, we will examine in detail a specific case associated with an unusual initial distribution p_0. To do this, consider the following values:

$$p_0 = \delta_0 \quad \text{and} \quad p_1 = \beta q + (1-\beta)\delta_0$$

Under these assumptions, for any integer $n \geq 2$, the explicit formula for p_n simplifies to:

$$p_n(dx) = \sum_{k=0}^{n-1} \theta_{k,n} q_k(dx)$$

where:

$$\forall k \in [\![0, n-1]\!], \quad \theta_{k,n} = \beta \frac{(1-\beta)^k}{w_{n-1}\dots w_{n-k}} \mu_k \quad \text{and} \quad q_k(dx) = \frac{x^k q(dx)}{\mu_k}$$

We will then undertake a comprehensive study of the sequence of viabilities $(w_n)_n$ in this specific context. To facilitate the understanding and realization of this analysis, we will begin with a preliminary subsection on stochastic order. The results presented in this section are attributed to the work of Camille Coron, a lecturer at the Laboratoire de Mathématiques d'Orsay.

D.2.1 A Relevant Order Relation

In this subsection, we focus on a new order relation that holds special significance in our study. We begin by formally defining this relation.

Definition-Proposition D.1 (Stochastic Order)**.** *Let $\preccurlyeq$ be the relation defined on the set $\mathcal{P}_I$ of probability measures on I by:*

$$\forall p, q \in \mathcal{P}_I, \quad p \preccurlyeq q \quad \iff \quad \int_0^1 xp(dx) \leq \int_0^1 xq(dx)$$

The relation $\preccurlyeq$ is then a specific order relation, called the stochastic order.

Proof. The proof of this result follows a standard approach. The reflexivity and transitivity of the relation are evident, and its antisymmetry follows from the Riesz representation theorem.

□

We now use this definition to deduce a significant result. In fact, $(\mathcal{P}_I, \preccurlyeq)$ forms a partially ordered set. An example of comparable measures can be highlighted through the following proposition:

Proposition D.1 (Stochastic Growth)**.** *The sequence of probability measures $(q_k)_k$ exhibits growth under the stochastic order.*

Proof. The proof of this result is not particularly complex. Let k be a natural number. The Cauchy-Schwarz inequality indicates that:

$$\left[\int_0^1 xq_k(dx)\right]^2 \leq \int_0^1 x^2 q_k(dx)$$

Consequently, we deduce that:

$$\int_0^1 xq_k(dx) \leq \int_0^1 x\frac{xq_k(dx)}{\int_0^1 yq_k(dy)} = \int_0^1 xq_{k+1}(dx)$$

This observation allows us to conclude the proof of the proposition.

□

D.2.2 Ascending Evolution of Viabilities

Let us now delve into the study of the trajectory of the sequence of viabilities, by presenting the central theorem that governs it. Before reaching that point, let us observe that the stochastic growth of the probability measures $(q_k)_k$ is actually equivalent to the ordinary growth of the sequence of successive quotients $(\mu_{k+1}/\mu_k)_k$. Therefore, since $(\mu_n)_n$ forms a sequence of moments, it would have been possible to avoid the previous subsection, presented in an anecdotal manner, by proving this result as in Appendix B. This realization allows us to state the following theorem:

Theorem D.2 (by Coron)**.** *Suppose the initial distributions are defined as:*

$$p_0 = \delta_0 \quad \textit{and} \quad p_1 = \beta q + (1-\beta)\delta_0$$

where q is a probability measure on the interval $I = [0,1]$. In this context, the sequence of viabilities $(w_n)_n$ exhibits an upward trajectory under the usual order.

Proof. Consider $n \geq 1$ as a natural integer and introduce the proposition $A(n)$ defined as follows:

$$w_{n-1} \leq w_n$$

Let us prove this proposition by strong induction on n, starting with initialization for the first two terms.

Initialization: For $1 \leq n \leq 2$, it is evident that:

$$w_1 = \int_0^1 \beta xq(dx) \geq 0 = w_0 \quad \text{and} \quad w_2 = \int_0^1 x\left[\beta q(dx) + \theta_{1,2}q_1(dx)\right] \geq w_1$$

Thus, properties $A(1)$ and $A(2)$ are satisfied.

Induction Step: Let $n \geq 2$ be a natural integer, assuming that $A(1), ..., A(n-1)$ are true. We now demonstrate that $A(n)$ is also verified. Notice that for a fixed k, the numbers $(\theta_{k,n})_n$ satisfy a certain order:

$$\forall k \in [\![0, n-2]\!], \quad \theta_{k,n} \leq \theta_{k,n-1}$$

This observation follows from the induction hypothesis, as:

$$\frac{\theta_{k,n}}{\theta_{k,n-1}} = \frac{w_{n-1-k}}{w_n} \leq 1$$

Then, using the previously stated explicit formula, we have:

$$p_n(dx) = \sum_{k=0}^{n-1} \theta_{k,n}q_k(dx) \quad \text{and} \quad p_{n-1}(dx) = \sum_{k=0}^{n-2} \theta_{k,n-1}q_k(dx)$$

Given that:

$$\sum_{k=0}^{n-2} \theta_{k,n} = 1 - \theta_{n-1,n} \quad \text{and} \quad \sum_{k=0}^{n-1} \theta_{k,n-1} = 1$$

we can deduce, using our observation and the stochastic growth, that:

$$\begin{aligned} w_n - w_{n-1} &= \int_0^1 x p_n(dx) - \int_0^1 x p_{n-1}(dx) \\ &= \sum_{k=0}^{n-2} (\theta_{k,n} - \theta_{k,n-1}) \frac{\mu_{k+1}}{\mu_k} + \theta_{n-1,n} \frac{\mu_n}{\mu_{n-1}} \\ &= \sum_{k=0}^{n-2} (\theta_{k,n} - \theta_{k,n-1}) \left(\frac{\mu_{k+1}}{\mu_k} - \frac{\mu_n}{\mu_{n-1}} \right) \\ &\geq 0 \end{aligned}$$

This proves that $A(n)$ is also satisfied.

Conclusion: In summary, we have demonstrated that $A(1)$ and $A(2)$ hold, and that $A(1), ..., A(n-1) \implies A(n)$ for any natural integer $n \geq 2$. Thus, by strong induction, we have established that the proposition $A(n)$ is valid for any natural integer $n \geq 1$.

□

D.2.3 Favorable Consequences

The statement of this theorem, attributed to Camille Coron, brings substantial simplification to the study of fitness in a specific case. Assuming that we start from p_1 as the initial fitness, a significant relationship connects the nth order moments:

$$\forall n \in \mathbb{N}^*, \quad m_n = \beta \mu_n = (1-\beta) f_n$$

By recalling the definitions of the sequences given in (5.1), we arrive at a concise relation between these two elements:

$$v_n = \sum_{k=1}^{n} m_k u_{n-k} = (1-\beta) \sum_{k=1}^{n} f_k u_{n-k} = (1-\beta) u_n$$

Using (3.7), we thus obtain the relation linking the sequence of viabilities to our sequence $(u_n)_n$:

$$w_n = \frac{W_{n+1}}{W_n} = (1-\beta) \frac{u_{n+1}}{u_n}$$

This connection between the growth of viabilities and that of the sequence of successive quotients eliminates the need for the appendix on the moment problem associated with renewal sequences, thereby simplifying our analysis of fitness.

Let us further explore this investigation. By definition, the sequence of viabilities is bounded by 1, which implies that the sequence of successive quotients is bounded by $(1-\beta)^{-1}$. The addition of its growth leads to its convergence. Three potential cases of convergence emerge:

$$\frac{u_{n+1}}{u_n} \xrightarrow[n \to +\infty]{} \frac{1}{\delta} > 1 \quad \text{or} \quad \frac{u_{n+1}}{u_n} \xrightarrow[n \to +\infty]{} 1 \quad \text{or} \quad \frac{u_{n+1}}{u_n} \xrightarrow[n \to +\infty]{} \sigma < 1$$

- ⋆ The first case corresponds to the democratic regime. In this situation, the sequence of viabilities converges to $(1-\beta)\delta^{-1}$. Denoting this limit as s, this constant corresponds to the same s as in the democratic case of our main study.
- ⋆ The second case is well-recognized and corresponds to the meritocratic regime. The sequence of viabilities indeed converges to the value $1-\beta$.
- ⋆ However, the last case never occurs. Thus, in this population, the aristocratic regime cannot manifest. This assumption is supported by lemma 3.8, which demonstrates that having such a limit for the sequence of successive quotients is impossible, lest it compromise the probabilistic nature of p_1: the support of this distribution must necessarily have a maximum fixed at 1.

Appendix E

Graphical Representations

In this section of the appendix, our aim is to address one of the major challenges inherent in fundamental mathematics, particularly within the framework of the analysis undertaken in this research. This area of mathematics often yields results that are abstract, and while we can develop intuitions and explanations in hindsight, nothing replaces the clarity provided by a graphical, numerical, or manual representation.

E.1 Key Points

It is in this spirit that we dedicate a chapter to crafting graphical representations of fitness limits in democratic and non-democratic scenarios. It is crucial to recall that these two situations are distinguished by the inequalities (2.9) and (2.8):

$$\int_0^1 \frac{q_d(dx)}{1-x} > \beta^{-1} \quad \text{and} \quad \int_0^1 \frac{q_{nd}(dx)}{1-x} \leq \beta^{-1}$$

This fundamental distinction gives rise to two possible fitness limits:

$$p_d(dx) = \frac{\beta s}{s-(1-\beta)x} q_d(dx) \quad \text{and} \quad p_{nd}(dx) = \frac{\beta}{1-x} q_{nd}(dx) + \left(1 - \int_0^1 \frac{\beta}{1-y} q_{nd}(dy)\right) \delta_1(dx)$$

It is important to note that in the first limit, the constant s is chosen such that it satisfies the conditions (3.1):

$$\int_0^1 \frac{\beta x}{s-(1-\beta)x} q_d(dx) = 1 \quad \text{and} \quad 1-\beta < s \leq 1$$

E.2 Mutation Profiles

We now turn our attention to the examination of the mutant fitness distributions that we will employ in this chapter. We draw inspiration from the section on condensation emergence. Let us opt for distributions that take the following form:

$$q^\alpha(dx) = \alpha(1-x)^{\alpha-1}, \quad \text{where} \quad \alpha > 1$$

This form simplifies the calculation of the distinction integral:

$$\int_0^1 \frac{q^\alpha(dx)}{1-x} = \frac{\alpha}{\alpha-1}$$

Hence, we can establish a distinction solely based on the value of α, while fixing β beforehand:

$$\alpha < \frac{1}{1-\beta} \quad \text{and} \quad \alpha \geq \frac{1}{1-\beta}$$

Let's choose $\beta = 0.7$, which favors the incorporation of a significant proportion of mutants in each generation. Next, let's pick specific constants α for each regime: set $\alpha = 2$ for the first case, and take two values, $\alpha \in \{5, 10\}$, for the second case. Therefore, the mutant fitness distributions we consider are expressed by the following formulas:

$$q_d(dx) = 2(1-x)dx \quad \text{and} \quad q_{nd}^{\alpha}(dx) = \alpha(1-x)^{\alpha-1}dx, \text{ where } \alpha \in \{5, 10\}$$

E.3 In Search of the Inverse of the Pole

Our next goal is to determine the formulas for fitness limits, beginning with the democratic case. The determination of the constant s, the inverse of the pole of the meromorphic function in (2.7), is essential for this endeavor.

E.3.1 Equation to Solve

Based on the previous sections, the sought-after constant s must satisfy the following conditions:

$$\int_0^1 \frac{2\beta x(1-x)}{s-(1-\beta)x}dx = 1 \quad \text{and} \quad 1-\beta < s \leq 1$$

Let $\mathfrak{I}$ be the integral in the above equation, and let's expand its expression. But before that, let's consider two useful equations:

$$\forall \theta \in \mathbb{R}\backslash\{1\}, \quad \frac{\theta^2}{1-\theta} = -\left(\frac{-\theta}{1-\theta} + \theta\right) \tag{E.1}$$

$$\forall \theta \in \mathbb{R}\backslash\{1\}, \quad \frac{-\theta}{1-\theta} = 1 - \frac{1}{1-\theta} \tag{E.2}$$

Now let's proceed with the expansion of $\mathfrak{I}$:

$$\begin{aligned}
\mathfrak{I} &= \frac{2\beta}{s}\left\{\int_0^1 \frac{x}{1-\frac{1-\beta}{s}x}dx - \int_0^1 \frac{x^2}{1-\frac{1-\beta}{s}x}dx\right\} \\
&= \frac{2\beta}{s}\left\{\left(-\frac{s}{1-\beta}\right)\int_0^1 \frac{\left(-\frac{1-\beta}{s}x\right)}{1-\frac{1-\beta}{s}x}dx - \left(\frac{s}{1-\beta}\right)^2\int_0^1 \frac{\left(\frac{1-\beta}{s}x\right)^2}{1-\frac{1-\beta}{s}x}dx\right\} \\
&= -\frac{2\beta}{1-\beta}\left\{\int_0^1 \frac{\left(-\frac{1-\beta}{s}x\right)}{1-\frac{1-\beta}{s}x}dx - \frac{s}{1-\beta}\left[\int_0^1 \frac{\left(-\frac{1-\beta}{s}x\right)}{1-\frac{1-\beta}{s}x}dx + \frac{1-\beta}{s}\int_0^1 xdx\right]\right\}, \quad \text{by (E.1)} \\
&= -\frac{2\beta}{1-\beta}\left\{\left(1-\frac{s}{1-\beta}\right)\int_0^1 \frac{\left(-\frac{1-\beta}{s}x\right)}{1-\frac{1-\beta}{s}x}dx - \frac{1}{2}\right\} \\
&= -\frac{2\beta}{1-\beta}\left\{\left(1-\frac{s}{1-\beta}\right)\left[1+\frac{s}{1-\beta}\int_0^1 \frac{\left(-\frac{1-\beta}{s}\right)}{1-\frac{1-\beta}{s}x}dx\right] - \frac{1}{2}\right\}, \quad \text{by (E.2)} \\
&= -\frac{2\beta}{1-\beta}\left\{\left(1-\frac{s}{1-\beta}\right)\left[1+\frac{s}{1-\beta}\ln\left(1-\frac{1-\beta}{s}\right)\right] - \frac{1}{2}\right\}
\end{aligned}$$

Thus, by eliminating the integral, we can solve the equation $\mathfrak{I} = 1$. Let's consider $\mathfrak{I} - 1$ as a function of s to determine its zero:

$$f : s \longmapsto -\frac{2\beta}{1-\beta}\left\{\left(1-\frac{s}{1-\beta}\right)\left[1+\frac{s}{1-\beta}\ln\left(1-\frac{1-\beta}{s}\right)\right] - \frac{1}{2}\right\} - 1 \tag{E.3}$$

E.3.2 Numerical Solution

We have various methods to find the zeros of a function. Our function is clearly of class $\mathcal{C}^{\infty}$ on the interval $]1-\beta, 1]$, which allows us to apply the Newton-Raphson algorithm, known for its quadratic convergence in around ten iterations.

Algorithm 1: Newton-Raphson Algorithm

Input: initial point s_0 and stopping criterion threshold ε
Output: the inverse of the pole s

provisional solution: $s = s_0$;
number of iterations: $n = 0$;
stopping criterion variable: $d = \frac{f(s_0)}{f'(s_0)}$;
while $n < 10$ *and* $d > \varepsilon$ **do**
    $s = s - \frac{f(s)}{f'(s)}$;
    $n = n + 1$;
end

To apply this algorithm, we need to compute the derivative of the function f and verify the existence of a zero. Let s be a real number in the interval $]1-\beta, 1]$.

$$\begin{aligned}
f'(s) = -\frac{2\beta}{1-\beta}&\left\{-\frac{1}{1-\beta}\left[1+\frac{s}{1-\beta}\ln\left(1-\frac{1-\beta}{s}\right)\right] \; ...\right.\\
&\left. ... + \left(1-\frac{s}{1-\beta}\right)\left[\frac{1}{1-\beta}\ln\left(1-\frac{1-\beta}{s}\right)+\frac{s}{1-\beta}\frac{\frac{1-\beta}{s^2}}{1-\frac{1-\beta}{s}}\right]\right\}\\
= -\frac{2\beta}{1-\beta}&\left\{-\frac{1}{1-\beta}\left[1+\frac{s}{1-\beta}\ln\left(1-\frac{1-\beta}{s}\right)\right] \; ...\right.\\
&\left. ... + \left(1-\frac{s}{1-\beta}\right)\left[\frac{1}{1-\beta}\ln\left(1-\frac{1-\beta}{s}\right)+\frac{1}{s}\frac{1}{1-\frac{1-\beta}{s}}\right]\right\}\\
= -\frac{2\beta}{1-\beta}&\left\{-\frac{1}{1-\beta}\left[1-\left(1-2\frac{s}{1-\beta}\right)\ln\left(1-\frac{1-\beta}{s}\right)\right]-\frac{1}{1-\beta}\right\}
\end{aligned}$$

The derivative of f is given by the following expression:

$$f' : s \longmapsto -\frac{2\beta}{(1-\beta)^2}\left\{\left(1-2\frac{s}{1-\beta}\right)\ln\left(1-\frac{1-\beta}{s}\right)-2\right\} \tag{E.4}$$

It is remarkable that f' is strictly increasing with specific limits and values at the endpoints (recalling that $\beta = 0.7$):

$$f'(s) \xrightarrow[s\to 1-\beta]{} -\infty \quad \text{and} \quad f'(1) = -\frac{2\beta}{(1-\beta)^2}\left\{\left(1-\frac{2}{1-\beta}\right)\ln(\beta)-2\right\} \simeq -0.329$$

Since these values are strictly negative, the function f is strictly decreasing over its domain, with specific limits and values at the endpoints:

$$f(s) \xrightarrow[s\to 1-\beta]{} \frac{\beta}{1-\beta}-1 \simeq 1.333 \quad \text{and} \quad f(1) = 2\left(\frac{\beta}{1-\beta}\right)^2\left(1+\frac{\ln\beta}{1-\beta}\right)+\frac{\beta}{1-\beta}-1 \simeq -0.724$$

These values sandwich zero, which, by the Intermediate Value Theorem, implies the existence of a unique solution to the equation $f(s) = 0$ in the interval $]1-\beta, 1]$.

After a graphical analysis of the function, it is notable that the solution is around 0.4. Therefore, we choose this value as the initial point in the Newton-Raphson algorithm and numerically solve our equation with a stopping criterion threshold of $\varepsilon = 10^{-15}$. The following figure illustrates the results obtained:

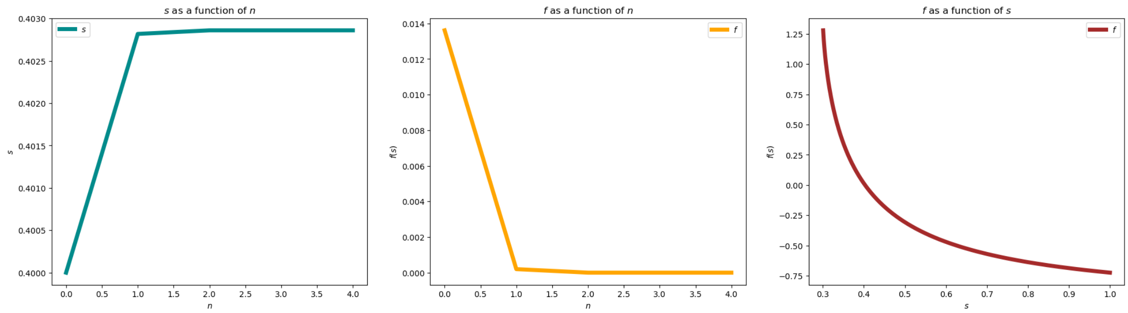

Figure E.1: Determination of s using the Newton-Raphson method

Remarkably, with only 4 iterations, we achieve a remarkably accurate estimate of the inverse pole s, as shown in the following table:

s	**f(s)**
0.40286047677014253	$-1.3322676295501878 \times 10^{-15}$

Table E.1: Precise estimation of the obtained solution

This determined value will then be used to establish the expression of the fitness limit in the democratic case.

E.4 Liminal Fitness Configurations

We are now ready to describe the profiles of limit fitness distributions, both in the democratic and non-democratic regimes. All the necessary elements have been carefully laid out in the previous sections. As a result, we can express the limit distributions as follows:

$$p_d(dx) = \frac{2\beta s(1-x)}{s-(1-\beta)x}dx \quad \text{and} \quad p_{nd}^{\alpha}(dx) = \left\{\alpha\beta(1-x)^{\alpha-2} + \left(1-\beta\frac{\alpha}{\alpha-1}\right)\delta_1(x)\right\}dx \tag{E.5}$$

However, it's not sufficient to merely visualize the limit distributions. It's crucial to track the evolution of fitness profiles across generations to fully understand their convergence behavior. This dynamic approach will provide deeper insights into these limit distributions.

E.5 Selection at Each Generation

In the framework of the assumptions established in this chapter, we merge them with the premises from the previous chapter, where the initial distributions were specified as follows:

$$p_0 = \delta_0 \quad \text{and} \quad p_1 = \beta q + (1-\beta)\delta_0$$

Consequently, fitness values for each generation, for $n \geq 2$, are expressed using the following formula:

$$p_n(dx) = \beta \sum_{k=0}^{n-1} \frac{(1-\beta)^k}{\prod_{i=1}^{k} w_{n-i}} x^k q(dx) = \alpha\beta(1-x)^{\alpha}\left\{1 + \sum_{k=1}^{n-1} \frac{(1-\beta)^k x^k}{\prod_{i=1}^{k} w_{n-i}}\right\}dx \tag{E.6}$$

This formula applies generally to both cases of selection. Regarding the associated viability, it is null in the initial state, but becomes inoperative subsequently. For the subsequent generations, a general recursive formula is stated:

$$w_n = \beta \sum_{k=0}^{n-1} \frac{(1-\beta)^k}{\prod_{i=1}^{k} w_{n-i}} \mu_{k+1}$$

The latter depends on moments related to mutants, the expression of which is easily obtained through repeated integration by parts:

$$\begin{aligned}
\mu_n &= \int_0^1 x^n q(dx) \\
&= \alpha \int_0^1 x^n (1-x)^{\alpha-1} dx \\
&= [x^n(1-x)^\alpha]_0^1 + n \int_0^1 x^{n-1}(1-x)^\alpha dx \\
&= \frac{n(n-1)}{\alpha+1} \int_0^1 x^{n-2}(1-x)^{\alpha+1} dx \\
&= \frac{n(n-1)\dots 1}{(\alpha+1)\dots(\alpha+n-1)} \int_0^1 (1-x)^{\alpha+n-1} dx \\
&= \frac{n!}{\prod_{i=1}^{n}(\alpha+i)}
\end{aligned}$$

Remark E.1 (Equivalent of Moments)**.** *Considering α as an integer, it is remarkable that we can immediately recover the equivalent presented in* (4.5) *in the chapter on the emergence of condensation:*

$$\mu_n = \frac{n!}{\prod_{i=1}^{n}(\alpha+i)} = \frac{\alpha!}{\prod_{i=n+1}^{n+\alpha} i} \underset{n\to+\infty}{\sim} \frac{\Gamma(\alpha+1)}{n^\alpha}$$

As a result, we are now able to provide an expression for the viability values of each generation:

$$w_n = \beta \left\{ \frac{1}{\alpha+1} + \sum_{k=1}^{n-1} \frac{(1-\beta)^k}{\prod_{i=1}^{k} w_{n-i}} \frac{(k+1)!}{\prod_{i=1}^{k+1}(\alpha+i)} \right\} \tag{E.7}$$

E.6 Visualization

We now have all the necessary components to create captivating graphical representations, which form the heart of this appendix. Our main objective is to visually depict the distributions p_n, where the recursive formula is defined by (E.6), using the viability values stated in (E.7). Due to computational constraints, we will focus solely on the density plots of generations 3 and 20, a selection that is more than sufficient to illustrate our point. We will also overlay these curves with their corresponding limit fitness distributions, defined by (E.5). This approach will be performed for both the democratic and non-democratic cases. In the latter case, we will repeat this procedure twice, choosing α equal to 5 and 10, to study the impact of the value of α. The following figure presents the results of this analysis:

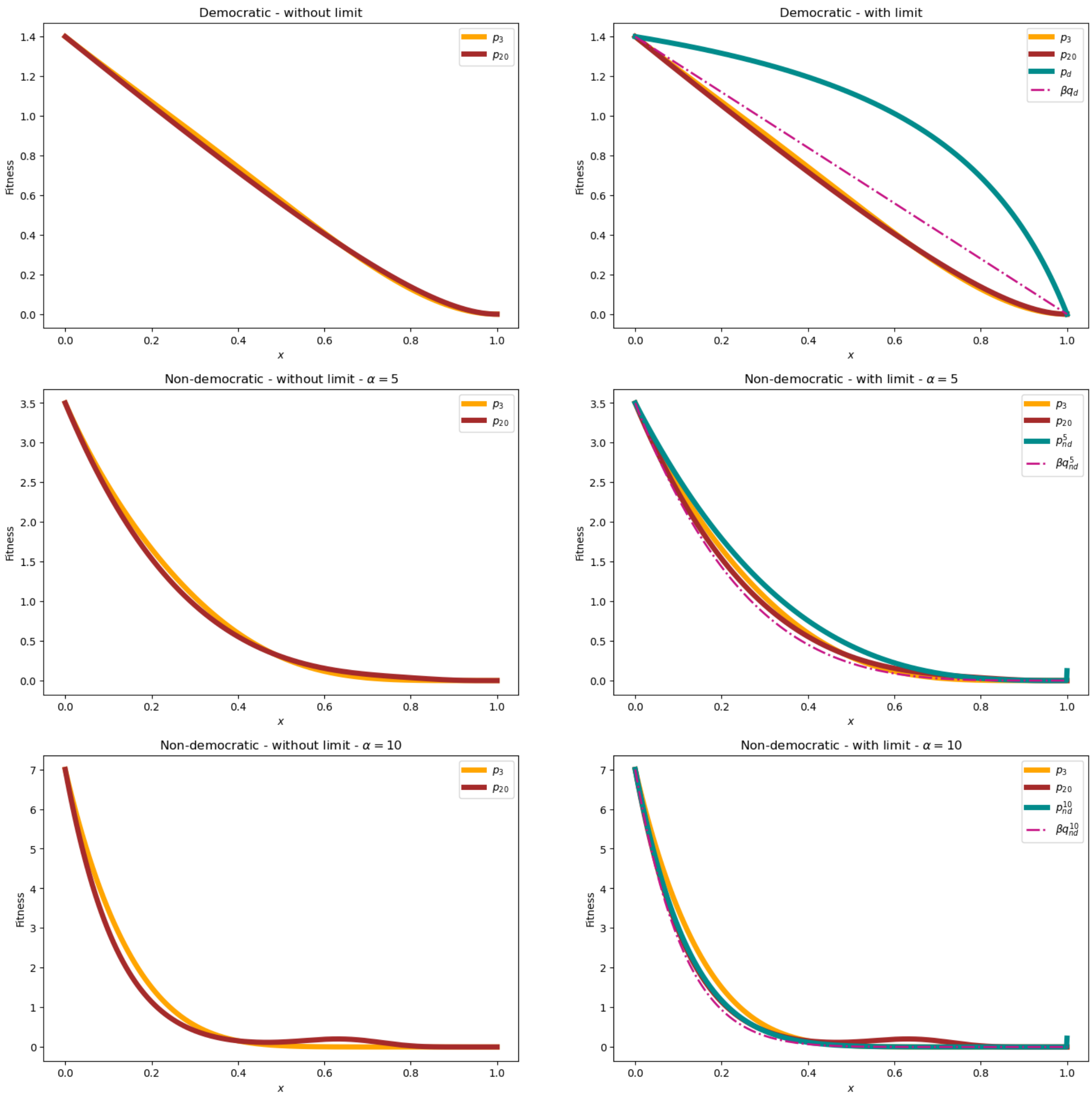

Figure E.2: Fitness distributions at generations 3, 20, and limits

Let's now make some pertinent observations based on these graphs. In the democratic scenario, it is noteworthy that despite a gap of 17 generations, the distributions exhibit minimal change. This result aligns with the expectations of this regime. The inherent egalitarianism in this model persists over time, suggesting that convergence to the limit state will be extremely slow (in the sense that the total variation of the distributions will remain below a threshold for the "unreachable" generations). Conversely, in the non-democratic case, it can be observed that the distributions vary a bit more with the passage of generations, in addition to the fact that the densities are already quite close to the limit density.

It is also remarkable that on the left side of all curves, an increase in generations leads to a decrease in density. In contrast, on the right side, a slight elevation is observed, contributing to the formation of the democratic wave or the non-democratic atom. This phenomenon is more pronounced during the generational jump when α has a large value.

As α increases, the limit density (as well as the densities of each generation, in reality) adheres closely to the first term, βq. This convergence can be explained in the democratic case by the predominance of mutants in the initial population. In the second case, this tendency can be demonstrated analytically using approximations. This increasing proximity leads to the formation of an atomic spike with a growing height as α increases, but this spike remains bounded by $1-\beta$. The values of these atomic spikes for the selected cases are indicated in the following table:

$\alpha = 5$	$\alpha = 10$
0.125	0.222

Table E.2: Atomic Spikes in the Non-Democratic Case

These observations enhance our understanding of evolutionary dynamics in both selection regimes and provide valuable insights into the impact of model parameters.

E.7 Additional Graphical Exploration

To satisfy our curiosity, we wish to explore new graphical representations. Thanks to the rapid convergence of the Newton method, we can study the relationship between the inverse of the pole s and the proportion of the population subject to selection, denoted as $1-\beta$. While retaining the previous parameters, with $\alpha = 2$, we know that β must lie strictly between 0 and 1, while also being greater than the quotient $(\alpha-1)/\alpha$. This leads us to examine how s evolves with respect to β over the interval $]1/2, 1[$, while superimposing this study with that of $1-\beta$ as a function of β. The graph below presents the results of this exploration:

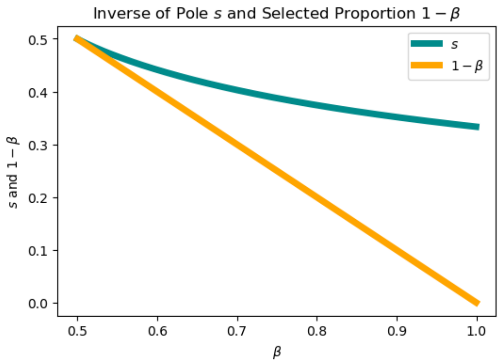

Figure E.3: Inverse of the Pole s as a Function of β

We can observe that s decreases as β increases, gradually approaching $1/3$ as β approaches the upper bound of its interval. On the other hand, s appears to converge to the value $1-\beta$ as β decreases. However, it is noteworthy that these values were actually analytically determinable through the integral equation satisfied by s, restated below:

$$\int_0^1 \frac{2\beta x(1-x)}{s-(1-\beta)x} dx = 1 \quad \text{and} \quad 1-\beta < s \leq 1$$

Appendix F

Missing Proofs

In this chapter, we present the proofs that were omitted from the main part of the report for various reasons. Some were excluded due to their simplicity, while others were left out because the key steps were already provided by the authors. We have decided to gather these proofs in an appendix to avoid unnecessarily burdening our main study. Each section in this part corresponds to a chapter, and the results of the proofs are presented sequentially.

F.1 Mathematization of Dynamics

Proposition F.1 (Explicit Formula for p_n)**.** *For any natural number n, the fitness distribution at generation n can be expressed as follows:*

$$p_n(dx) = \beta \sum_{k=0}^{n-1} \frac{(1-\beta)^k}{\prod_{i=0}^{k-1} w_{n-1-i}} x^k q(dx) + \frac{(1-\beta)^n}{\prod_{i=0}^{n-1} w_{n-1-i}} x^n p_0(dx) \tag{F.1}$$

where w_i represents the viability of the population at generation i.

Proof. Let n be a natural number, and let $P(n)$ be the following proposition:

$$p_n(dx) = \beta \sum_{k=0}^{n-1} \frac{(1-\beta)^k}{\prod_{i=0}^{k-1} w_{n-1-i}} x^k q(dx) + \frac{(1-\beta)^n}{\prod_{i=0}^{n-1} w_{n-1-i}} x^n p_0(dx)$$

We will prove this proposition by induction on n.

Initialization: For $n = 0$, we have $p_0(dx) = 0 + p_0(dx)$, so $P(0)$ is true.

Inductive Step: Let n be a natural number, and assume that $P(n)$ is true. We will show that $P(n+1)$ is also true. Starting from the equation (2.2):

$$p_{n+1}(dx) = \beta q(dx) + (1-\beta) \frac{x p_n(dx)}{w_n}$$

By substituting the expression for $p_n(dx)$ from $P(n)$, we obtain:

$$\begin{aligned}
p_{n+1}(dx) &= \beta q(dx) + (1-\beta)\frac{x}{w_n}\left(\beta\sum_{k=0}^{n-1}\frac{(1-\beta)^k}{\prod_{i=0}^{k-1} w_{n-1-i}}x^k q(dx) + \frac{(1-\beta)^n}{\prod_{i=0}^{n-1} w_{n-1-i}}x^n p_0(dx)\right)\\
&= \beta q(dx) + (1-\beta)\frac{x}{w_n}\left(\beta\sum_{k=1}^{n}\frac{(1-\beta)^{k-1}}{\prod_{i=1}^{k-1} w_{n-i}}x^{k-1} q(dx) + \frac{(1-\beta)^n}{\prod_{i=1}^{n} w_{n-i}}x^n p_0(dx)\right)\\
&= \beta q(dx) + \beta\sum_{k=1}^{n}\frac{(1-\beta)^k}{\prod_{i=1}^{k-1} w_{n-i}}x^k q(dx) + \frac{(1-\beta)^{n+1}}{\prod_{i=0}^{n} w_{n-i}}x^{n+1} p_0(dx)\\
&= \beta\sum_{k=0}^{n}\frac{(1-\beta)^k}{\prod_{i=1}^{k-1} w_{n-i}}x^k q(dx) + \frac{(1-\beta)^{n+1}}{\prod_{i=0}^{n} w_{n-i}}x^{n+1} p_0(dx)
\end{aligned}$$

This proves that $P(n+1)$ is true.

Conclusion: Thus, since $P(0)$ is true and $P(n) \implies P(n+1)$ for any natural number n, we have shown by induction that the proposition holds.

□

Proposition F.2 (Recursive Formula for W_n)**.** *The product of viabilities over the first n generations, denoted as W_n, can be expressed in terms of moments as follows:*

$$\begin{aligned}
W_n &= \beta\sum_{k=0}^{n-1} W_{n-k}(1-\beta)^k\mu_k + (1-\beta)^n m_n && \text{(F.2)}\\
&= \beta\sum_{k=1}^{n-1} W_{n-k}(1-\beta)^{k-1}\mu_k + (1-\beta)^{n-1} m_n && \text{(F.3)}
\end{aligned}$$

Proof. First, we rearrange the equation (**F.1**) to isolate W_n:

$$p_n(dx) = \beta\sum_{k=0}^{n-1}\frac{(1-\beta)^k}{\prod_{i=0}^{k-1} w_{n-1-i}}x^k q(dx) + \frac{(1-\beta)^n}{\prod_{i=0}^{n-1} w_{n-1-i}}x^n p_0(dx)$$

This is equivalent to:

$$\begin{aligned}
\left(\prod_{i=0}^{n-1} w_{n-1-i}\right)p_n(dx) &= \beta\sum_{k=0}^{n-1}\left(\prod_{i=k}^{n-1} w_{n-1-i}\right)(1-\beta)^k x^k q(dx) + (1-\beta)^n x^n p_0(dx)\\
&= \beta\sum_{k=0}^{n-1}\left(\prod_{i=0}^{n-k-1} w_{n-k-1-i}\right)(1-\beta)^k x^k q(dx) + (1-\beta)^n x^n p_0(dx)\\
&= \beta\sum_{k=0}^{n-1}\left(\prod_{i=0}^{n-k-1} w_i\right)(1-\beta)^k x^k q(dx) + (1-\beta)^n x^n p_0(dx)
\end{aligned}$$

It follows that:

$$W_n p_n(dx) = \beta\sum_{k=0}^{n-1} W_{n-k}(1-\beta)^k x^k q(dx) + (1-\beta)^n x^n p_0(dx) \tag{F.4}$$

Next, we integrate the equation (F.4) to bring out the moments μ_k and m_n, which leads to obtaining equation (F.2):

$$W_n = \beta \sum_{k=0}^{n-1} W_{n-k}(1-\beta)^k \mu_k + (1-\beta)^n m_n$$

By factoring out the first term in the summation, we obtain:

$$W_n = \beta W_n + \beta \sum_{k=1}^{n-1} W_{n-k}(1-\beta)^k \mu_k + (1-\beta)^n m_n$$

which is equivalent to

$$(1-\beta) W_n = \beta \sum_{k=1}^{n-1} W_{n-k}(1-\beta)^k \mu_k + (1-\beta)^n m_n$$

Dividing by $(1-\beta)$ gives us equation (F.3):

$$W_n = \beta \sum_{k=1}^{n-1} W_{n-k}(1-\beta)^{k-1} \mu_k + (1-\beta)^{n-1} m_n$$

□

Lemma F.3 (Convergence of Power Series)**.**

- *For all $z \in D_\beta$, $\sum_{n\geq 1} (1-\beta)^n m_n z^n$ and $\sum_{n\geq 1} (1-\beta)^n \mu_n z^n$ are absolutely convergent.*
- *For all $z \in D_0$, $\sum_{n\geq 1} W_n z^n$ is absolutely convergent.*

Proof. The proofs of these properties naturally follow from Abel's lemma, using the boundedness of the sequences. Indeed, the sequences of terms

$$(1-\beta)^n m_n \left(\frac{1}{1-\beta}\right)^n, \quad (1-\beta)^n \mu_n \left(\frac{1}{1-\beta}\right)^n,$$

and W_n are all bounded by 1. Thus, by applying Abel's lemma, we can conclude that the power series are absolutely convergent for any $z \in D_\beta$ and $z \in D_0$.

□

F.2 Evolution and Selection Regimes

Lemma F.4 (Successive Term Ratio)**.** *The sequence of successive term ratios of $(v_n)_n$ converges, and we have:*

$$\lim_{n\to+\infty} \frac{v_n}{v_{n-1}} = 1$$

Proof. To prove this lemma, we will proceed in three steps before reaching the conclusion. First, we rewrite the sequence $(v_n)_n$ in the form of an integral of a sequence of functions, which allows us to study the upper and lower limits of the successive term ratios of our sequence.

Rewriting: Let n be a natural number. We recall that $(u_n)_n$ represents a sequence of moments, as does $(m_n)_n$, where:

$$u_n = \int_0^1 x^n \nu(dx) \quad \text{and} \quad m_n = \int_0^1 x^n p_0(dx)$$

Define $\lambda = \nu \otimes p_0$ as the product measure of these two measures. Thus, the sequence $(v_n)_n$ can be rewritten as follows:

$$v_n = \sum_{k=1}^{n} \int_{I^2} x^{n-k} y^k \lambda(dxdy) = \int_{I^2} y \sum_{k=0}^{n-1} x^{n-1-k} y^k \lambda(dxdy)$$

Now let's define the sequence of functions $(g_n)_n$ as follows:

$$g_n : (x,y) \in I^2 \mapsto \begin{cases} y\frac{x^n - y^n}{x-y} & \text{if } x \neq y \\ nx^n & \text{if } x = y \end{cases}$$

Thus, we can rewrite the sequence $(v_n)_n$ as an integral of the sequence $(g_n)_n$:

$$v_n = \int_{I^2} g_n(x,y) \lambda(dxdy)$$

This new representation will allow us to more easily study the properties of the sequence of successive term ratios of $(v_n)_n$.

Limit Superior: With this new representation, we can now study the limit superior of our sequence. Consider $x, y \in I$.

- Suppose, without loss of generality, that $x > y$. Using the decreasing property of the sequence $(t^n)_n$ for all $t \in I$, we can establish the following implications:

$$\int_y^x t^{n-1} dt \geq \int_y^x t^{n-1} dt \Longrightarrow \frac{x^{n-1} - y^{n-1}}{n-1} \geq \frac{x^n - y^n}{n} \Longrightarrow \frac{g_{n-1}(x,y)}{n-1} \geq \frac{g_n(x,y)}{n}$$

- Now suppose that $x = y$. By the same argument, we obtain the following result:

$$x^{n-1} \geq x^n \Longrightarrow \frac{g_{n-1}(x,x)}{n-1} \geq \frac{g_n(x,x)}{n}$$

This allows us to conclude with the following implications:

$$\frac{v_{n-1}}{n-1} \geq \frac{v_n}{n} \Longrightarrow \frac{v_n}{v_{n-1}} \leq \frac{n}{n-1} \Longrightarrow \limsup_{n \to +\infty} \frac{v_n}{v_{n-1}} \leq 1$$

Limit Inferior: Establishing the limit inferior of the sequence is slightly more involved. To find it, consider a real number $\alpha \in]0,1[$ along with our two real numbers $x, y \in I$.

- Suppose that one of the two numbers is greater than or equal to α, i.e., $\max(x,y) \geq \alpha$.
  - In the case where $x > y$, we obtain the following result:

$$x^x - y^n \geq x^n - xy^{n-1} \geq \alpha \left(x^{n-1} - y^{n-1}\right) \Longrightarrow g_n(x,y) \geq \alpha g_{n-1}(x,y)$$

  - Similarly, for $x = y$, the inequality holds:

$$nx^n \geq \alpha n x^{n-1} \geq \alpha(n-1)x^{n-1} \Longrightarrow g_n(x,x) \geq \alpha g_{n-1}(x,x)$$

- Now consider the opposite case where both numbers are strictly less than α, i.e., $\max(x,y) < \alpha$.
  - In the case where $x > y$, we obtain the following result:

$$y \sum_{k=0}^{n-1} x^{n-1-k} y^k \leq y \sum_{k=0}^{n-1} x^{n-1} \left(\frac{y}{x}\right)^k < n\alpha^n \Longrightarrow g_n(x,y) < n\alpha^n$$

  - Similarly, for $x = y$, the inequality holds:

$$nx^n < n\alpha^n \Longrightarrow g_n(x,y) < n\alpha^n$$

Thus, using these properties, we obtain the following sequence of inequalities:

$$\begin{aligned}
v_n &= \int_{I^2} g_n(x,y)\lambda(dxdy) \\
&\geq \int_{\max(x,y)\geq\alpha} g_n(x,y)\lambda(dxdy) \\
&\geq \alpha \int_{\max(x,y)\geq\alpha} g_{n-1}(x,y)\lambda(dxdy) \\
&= \alpha v_{n-1} - \alpha \int_{\max(x,y)<\alpha} g_{n-1}(x,y)\lambda(dxdy) \\
&\geq \alpha v_{n-1} - n\alpha^n
\end{aligned}$$

Consequently, we can establish a lower bound for the successive term ratio of our sequence:

$$\frac{v_n}{v_{n-1}} \geq \alpha - \frac{n\alpha^n}{v_{n-1}} \geq \alpha - \frac{n\alpha^n}{m_1 u_{n-2}}$$

Define the latter term as M_n and study its limit. Notice that the sequence $(u_n)_n$ is decreasing and positive, hence it converges. Therefore, we get the following result:

$$\frac{M_{n+1}}{M_n} = \alpha \frac{n+1}{n} \frac{u_{n-2}}{u_{n-1}} \xrightarrow[n\to+\infty]{} \alpha < 1$$

Thus, we deduce the limit of $(M_n)_n$:

$$M_n \underset{n\to+\infty}{\sim} \alpha^{n-2} M_2 \quad \Longrightarrow \quad M_n \xrightarrow[n\to+\infty]{} 0$$

Returning to the previous inequality, we obtain:

$$\liminf_{n\to+\infty} \frac{v_n}{v_{n-1}} \geq \alpha$$

Since this property holds for any real $\alpha \in]0,1[$, we finally conclude:

$$\liminf_{n\to+\infty} \frac{v_n}{v_{n-1}} \geq 1$$

Conclusion: By combining our two results, we arrive at:

$$\liminf_{n\to+\infty} \frac{v_n}{v_{n-1}} = \limsup_{n\to+\infty} \frac{v_n}{v_{n-1}} = 1$$

Thus, we conclude that the sequence of successive term ratios converges, and its limit is given by:

$$\lim_{n\to+\infty} \frac{v_n}{v_{n-1}} = 1$$

□

Lemma F.5 (Product of Viabilities)**.** *The product of viabilities can be expressed in terms of the sequence $(v_n)_n$ as follows:*

$$\forall n \in \mathbb{N}^*, \quad W_n = (1-\beta)^{n-1} v_n$$

Proof. Let $n \geq 1$ be a natural number. To prove the result, we will proceed by induction on n. Let $Q(n)$ be the following proposition:

$$W_n = (1-\beta)^{n-1} v_n = (1-\beta)^{n-1} \sum_{k=1}^{n} m_k u_{n-k}$$

Initialization: For $n = 1$, we have $W_1 = m_1$ as per the previously mentioned equation. Thus, $Q(1)$ is true.

Induction Step: Suppose that proposition $Q(n)$ is true for a certain natural number $n \geq 1$. We will now prove that proposition $Q(n+1)$ is also true. Starting from equation (**F.3**), we will use the induction hypothesis and summation reversal to simplify the expression, and then apply the definition of renewal sequences to reach the desired conclusion.

$$\begin{aligned}
W_{n+1} &= \beta \sum_{k=1}^{n} W_{n+1-k}(1-\beta)^{k-1}\mu_k + (1-\beta)^n m_{n+1} \\
&= \beta \sum_{k=1}^{n} \left((1-\beta)^{n-k} \sum_{j=1}^{n+1-k} m_j u_{n+1-k-j} \right) (1-\beta)^{k-1}\mu_k + (1-\beta)^n m_{n+1} \\
&= (1-\beta)^n \sum_{j=1}^{n} m_j \sum_{k=1}^{n+1-j} \frac{\beta}{1-\beta}\mu_k u_{n+1-j-k} + (1-\beta)^n m_{n+1} \\
&= (1-\beta)^n \sum_{j=1}^{n} m_j u_{n+1-j} + (1-\beta)^n m_{n+1} \\
&= (1-\beta)^n \sum_{j=1}^{n+1} m_j u_{n+1-j}
\end{aligned}$$

Thus, we have proved that $Q(n+1)$ is true.

Conclusion: Since $Q(1)$ is true and $Q(n) \Longrightarrow Q(n+1)$ for all natural numbers $n \geq 1$, we have proven by induction that the proposition is true.

□

Lemma F.6 (Support of Mutant Fitness Distribution)**.** *The support of the mutant fitness distribution is constrained by the limit of the successive term ratios of the renewal sequence:*

$$\mathit{supp}(q) \subset [0, \sigma]$$

Proof. Let $n \geq 1$ be a natural number. First, note that we have the immediate inequality:

$$f_n \leq \sum_{k=1}^{n} f_k u_{n-k} = u_n$$

Furthermore, since the sequence of successive term ratios of the renewal sequence is increasing and converges to σ, we can deduce the following immediate result by induction:

$$u_n \leq \sigma^n$$

Combining these two inequalities, we obtain:

$$f_n \leq \sigma^n$$

Now, let's prove the lemma by contradiction using this inequality. Suppose there exists a real number $\eta \in]0, 1[$ such that $q([\eta, 1]) > 0$. In that case, we have the following result:

$$\frac{f_n}{\sigma^n} = \frac{\beta}{1-\beta}\frac{\mu_n}{\sigma^n} \geq \frac{\beta}{1-\beta}\frac{1}{\sigma^n}\int_{\eta}^{1} x^n q(dx) \geq \frac{\beta}{1-\beta}\left(\frac{\eta}{\sigma}\right)^n q([\eta, 1]) \xrightarrow[n\to+\infty]{} +\infty$$

This final limit is absurd, leading to a contradiction. Therefore, we conclude the desired result.

□

Proposition F.7 (Limiting Aristocratic Viability). *The sequence of viabilities $(w_n)_n$ converges, and the limiting viability corresponds to the proportion of non-mutant population:*

$$w_n \xrightarrow[n\to+\infty]{} 1-\beta$$

Proof. The proof of this proposition in the aristocratic case is almost identical to that of the meritocratic case, with the exception that, since the support of q is modified, the part of the proof concerning the limit inferior of the sequence of successive term ratios of the new sequence $(v_n)_n$ will be adjusted. Thus, consider the results of this proof and take a real number $\xi \in]\alpha, 1[$. Then we have the following inequalities:

$$\forall n \in \mathbb{N}^*, \quad m_n \geq \int_\xi^1 x^n p_0(dx) \geq \xi^n p_0([\xi, 1]) > 0$$

As a consequence, we obtain the following limit by comparison of growth rates:

$$\frac{v_n}{v_{n-1}} \geq \alpha - \frac{n\alpha^n}{v_{n-1}} \geq \alpha - \frac{n\alpha^n}{m_{n-1}} \geq \alpha - \frac{n\xi}{p_0([\xi,1])}\left(\frac{\alpha}{\xi}\right)^n \xrightarrow[n\to+\infty]{} \alpha$$

This allows us to conclude:

$$\liminf_{n\to+\infty} \frac{v_n}{v_{n-1}} \geq \alpha$$

The rest of the proof remains the same. □

F.3 Emergence of Condensation

Lemma F.8 (Renewal Equation). *The sequence $(v_n)_n$ satisfies the following renewal equation:*

$$\forall n \in \mathbb{N}^*, \quad v_n = m_n + \sum_{k=1}^{n-1} f_k v_{n-k}$$

Proof. The proof of this result is indeed concise. For $n \in \mathbb{N}^*$, by substituting the renewal equation satisfied by $(u_n)_n$:

$$u_n = \sum_{j=1}^{n} f_j u_{n-j}$$

into the definition of $(v_n)_n$ and rearranging terms through a summation reversal, we obtain:

$$v_n = m_n + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} f_j m_k u_{n-k-j} = m_n + \sum_{j=1}^{n-1} f_j \sum_{k=1}^{n-j} m_k u_{n-j-k} = m_n + \sum_{j=1}^{n-1} f_j v_{n-j}$$

□

Lemma F.9 (Mutant Moments). *For any $x > 0$, we have the following equivalence:*

$$\int_{1-x/n}^1 y^n q(dy) \underset{n\to+\infty}{\sim} \frac{\alpha}{n^\alpha}\int_0^x t^{\alpha-1}e^{-t}dt \tag{F.5}$$

In particular, we have:

$$\mu_n \underset{n\to+\infty}{\sim} \frac{\Gamma(\alpha+1)}{n^\alpha} \tag{F.6}$$

Proof. By assumption, we have $q(dy) = \alpha(1-y)^{\alpha-1}dy$, so a simple affine change of variable leads to the following expression for the integral:

$$\int_{1-x/n}^{1} y^n q(dy) = \int_{1-x/n}^{1} y^n \alpha(1-y)^{\alpha-1}dy = \frac{\alpha}{n^\alpha}\int_0^n \left(1-\frac{t}{n}\right)^n t^{\alpha-1}\mathbf{1}_{t\leq x}dt$$

Let's define

$$\gamma_n^x : t \mapsto \left(1-\frac{t}{n}\right)^n t^{\alpha-1}\mathbf{1}_{t\leq \min\{x,n\}}$$

The sequence of functions $(\gamma_n^x)_n$ satisfies the following properties:

- For any $t \in \mathbb{R}_+$, the following limit holds:

$$\gamma_n^x(t) \xrightarrow[n\to+\infty]{} e^{-t}t^{\alpha-1}\mathbf{1}_{t\leq x}$$

- For any $(n,t) \in \mathbb{N}^* \times \mathbb{R}_+$, the following inequality holds:

$$|\gamma_n^x(t)| \leq e^{-t}t^{\alpha-1}$$

Let $\gamma^\star(t)$ be the right-hand side, then $\gamma^\star$ is an integrable function on $\mathbb{R}_+$ (with integral $\Gamma(\alpha)$).

Applying the dominated convergence theorem, we obtain:

$$\int_0^{+\infty} \gamma_n^x(t)dt \xrightarrow[n\to+\infty]{} \int_0^{+\infty} e^{-t}t^{\alpha-1}\mathbf{1}_{t\leq x}dt$$

This establishes the first result of the lemma:

$$\int_{1-x/n}^{1} y^n q(dy) \underset{n\to+\infty}{\sim} \frac{\alpha}{n^\alpha}\int_0^x t^{\alpha-1}e^{-t}dt$$

The second result, obtained by setting "$x = n$" in the first result, is proven similarly, with the x-dependent indicator vanishing, leading to the expression:

$$\int_0^1 \alpha y^n(1-y)^{\alpha-1}dy \underset{n\to+\infty}{\sim} \frac{\alpha}{n^\alpha}\int_0^{+\infty} e^{-t}t^{\alpha-1}dt$$

This concludes the proof of the lemma.

□

Lemma F.10. *The sequence $(v_n)_n$ satisfies the following equivalence:*

$$v_n \underset{n\to+\infty}{\sim} \frac{c}{n^\alpha}$$

with

$$c = \frac{\beta}{\gamma(\beta)}\Gamma(\alpha+1)\sum_{n=1}^{+\infty} v_n$$

Proof. Let $\delta > 0$ and $0 < \varepsilon < \eta < 1$. We begin by noticing that according to the previous lemma, there exists an integer $N \geq 1$ such that for all $n \geq N$, $n-1 \geq n\eta$, and:

$$\forall k \geq (1-\eta)n \geq 1,\ \frac{\Gamma(\alpha+1)-\delta}{k^\alpha} \leq \mu_k \leq \frac{\Gamma(\alpha+1)+\delta}{k^\alpha} \tag{F.7}$$

We now introduce the proposition $R(n)$ for all $n \in \mathbb{N}^*$:

$$\exists c_n, \tilde{c}_n > 0 \ /\ \frac{\tilde{c}_n}{n^\alpha} \leq v_n \leq \frac{c_n}{n^\alpha}$$

We will prove this property by strong induction, focusing on the right-hand inequality as the proofs are similar.

Initialization: For $n = 1$, we have $v_1 = m_1$, so we can choose $c_1 = m_1$, which satisfies $R(1)$.

Induction Step: Suppose that $R(1), ..., R(n-1)$ are true for some natural number $n \geq 2$. We show that $R(n)$ is also true.

$$\forall k \in [\![\lfloor \varepsilon n \rfloor + 1, n-1]\!],\ \exists c_k > 0 \ / \ v_k \leq \frac{c_k}{k^\alpha}$$

Now we evaluate $v_k \mu_{n-k}$ in three cases for the values of k. Using the inequalities above, we obtain:

- If $k \geq \lfloor \eta n \rfloor + 1$, then $k \geq \eta n$, and we have:

$$v_k \mu_{n-k} \leq \frac{c_k}{(\eta n)^\alpha} \mu_{n-k}$$

- If $\lfloor \varepsilon n \rfloor + 1 \leq k \leq \lfloor \eta n \rfloor$, then $(1-\eta)n \leq n-k \leq (1-\varepsilon)n$, and we have:

$$v_k \mu_{n-k} \leq \frac{c_k}{k^\alpha} \frac{\Gamma(\alpha+1)+\delta}{(n-k)^\alpha}$$

- If $k \leq \lfloor \varepsilon n \rfloor$, then $(1-\varepsilon)n \leq n-k$, and we have:

$$v_k \mu_{n-k} \leq v_k \frac{\Gamma(\alpha+1)+\delta}{((1-\varepsilon)n)^\alpha}$$

Thus, we obtain:

$$v_n = m_n + \sum_{k=1}^{n-1} v_k \mu_{n-k} \leq \frac{c_n}{n^\alpha}$$

where c_n is given by the following expression:

$$\begin{aligned} c_n = \frac{\beta}{1-\beta} \Bigg[& \frac{\Gamma(\alpha+1)+\delta}{(1-\varepsilon)^n} \sum_{k=1}^{\lfloor \varepsilon n \rfloor} v_k \ \ ... \\ & ... + \frac{\Gamma(\alpha+1)+\delta}{n^{\alpha-1}} \left(\frac{1}{n} \sum_{k=\lfloor \varepsilon n \rfloor+1}^{\lfloor \eta n \rfloor} c_k \left(\frac{k}{n}\right)^{-\alpha} \left(1-\frac{k}{n}\right)^{-\alpha} \right) \ ... \\ & ... + \frac{1}{\eta^\alpha} \sum_{k=\lfloor \eta n \rfloor+1}^{n-1} c_k \mu_{n-k} \Bigg] + m_n n^\alpha \end{aligned}$$

This shows that $R(n)$ is true.

Conclusion: Thus, since $R(1)$ is true and $R(1), ..., R(n-1) \Longrightarrow R(n)$ for all natural numbers $n \geq 2$, we have shown by strong induction that the proposition is true.

This recurrence allowed us to construct the sequences $(c_n)_n$ and $(\tilde{c}_n)_n$ that bound from above and below the sequence $(v_n n^\alpha)_n$. Let's continue our study of the $(c_n)_n$ sequence, as that of $(\tilde{c}_n)_n$ will be similar. As the sequence $(m_n n^\alpha)_n$ converges to 0 according to hypothesis (4.1):

$$m_n n^\alpha = o(\mu_n n^\alpha) = o\left(\Gamma(\alpha+1)\right) = o(1)$$

it follows that the sequence $(c_n)_n$ is bounded. Moreover, the sequence with general term S_n defined as the second sum in the expression for c_n is an O of a Riemann integral:

$$\exists K > 0 \ / \ S_n \leq \frac{K}{n^{\alpha-1}} \int_\varepsilon^\eta t^{-\alpha}(1-t)^{-\alpha} dt \xrightarrow[n \to +\infty]{} 0$$

Consequently, the sequence $(c_n)_n$ converges to a constant $c_\infty = c_\infty(\delta, \varepsilon, \eta)$, which is the unique solution of the following equation:

$$c_\infty = \frac{\beta}{1-\beta}\left[\frac{\Gamma(\alpha+1)+\delta}{(1-\varepsilon)^\alpha}\sum_{n=1}^{+\infty} v_n + \frac{c_\infty}{\eta^\alpha}\sum_{n=1}^{+\infty}\mu_n\right]$$

By letting δ, ε, and η tend to 0, we obtain that c_∞ converges to the constant c which is the unique solution of the following equation:

$$c = \frac{\beta}{1-\beta}\Gamma(\alpha+1)\sum_{n=1}^{+\infty} v_n + c\sum_{n=1}^{+\infty} f_n$$

Using equation (4.3), we finally obtain:

$$c = \frac{\beta}{\gamma(\beta)}\Gamma(\alpha+1)\sum_{n=1}^{+\infty} v_n$$

By performing a similar analysis on the sequence $(\tilde{c}_n)_n$, we find that this sequence converges to a constant $\tilde{c}_\infty(\delta, \varepsilon, \eta)$, and this tends to c as $(\delta, \varepsilon, \eta)$ approach $(0, 0, 1)$. By the sandwich theorem, we conclude that the sequence $(v_n n^\alpha)_n$ also converges to c, which proves the desired result.

□

F.4 Evolutionary Dynamics with Unbounded Fitness

Lemma F.11 (Divergence)**.** *The sequence of ratios of successive terms of the sequence $(v_n)_n$ is divergent:*

$$\frac{v_n}{v_{n-1}} \xrightarrow[n\to+\infty]{} +\infty$$

Proof. The proof of this lemma is nearly identical to that of Lemma E.4, which deals with the same sequence in the case of bounded fitness. However, in this situation, the study of the limit superior is unnecessary. Thus, we focus on the limit inferior by considering a real number $A > 1$ sufficiently large instead of $\alpha \in]0, 1[$. We resume from the following step:

$$\frac{v_n}{v_{n-1}} \geq A - \frac{nA^n}{m_1 u_{n-2}}$$

and examine the last term, which we define as M_n. According to the previous proposition, the inverse of the sequence of ratios of successive terms of $(u_n)_n$ converges to 0. Therefore, we deduce that:

$$\frac{M_{n+1}}{M_n} = A\frac{n+1}{n}\frac{u_{n-2}}{u_{n-1}} \xrightarrow[n\to+\infty]{} 0$$

There exists an integer $N \geq 1$ such that for all $n \geq N$:

$$\frac{M_{n+1}}{M_n} \leq \frac{1}{2} \quad\Longrightarrow\quad M_n \leq \frac{2^N M_N}{2^n} \xrightarrow[n\to+\infty]{} 0$$

Returning to the previous inequality, we obtain:

$$\liminf_{n\to+\infty}\frac{v_n}{v_{n-1}} \geq A$$

Since this property holds for any real $A > 1$, we finally deduce the conclusion of our lemma:

$$\lim_{n\to+\infty}\frac{v_n}{v_{n-1}} = +\infty$$

□

Proposition F.12 (Fitness and Viability)**.** *The concepts of fitness and viability in the n-th generation are explicitly expressed as follows:*

$$p_n(dx) = \frac{x^n e^{-x}}{n!} dx \quad \text{and} \quad w_n = n+1$$

Proof. Let's prove this proposition by induction, but first note that the formula for viabilities is obtained directly, provided we have proven the formula for fitness. Thus, let's introduce the proposition $S(n)$ for all natural numbers n:

$$p_n(dx) = \frac{x^n e^{-x}}{n!} dx$$

Initialization: For $n = 0$, we have $p_0(dx) = e^{-x}dx$ by definition, which verifies $S(0)$.

Induction Step: Suppose $S(n)$ is true for a certain natural number n and let's show that $S(n+1)$ is also true. First notice that by the induction hypothesis:

$$w_n = \int_0^{+\infty} x p_n(dx) = \frac{1}{n!} \int_0^{+\infty} x^{n+1} e^{-x} dx = \frac{(n+1)!}{n!} = n+1$$

Similarly, we obtain:

$$p_{n+1}(dx) = \frac{x p_n(dx)}{w_n} = \frac{x}{n+1} \frac{x^n e^{-x}}{n!} dx = \frac{x^{n+1} e^{-x}}{(n+1)!} dx$$

This proves that $S(n+1)$ is satisfied.

Conclusion: Thus, since $S(0)$ is true and $S(n) \Longrightarrow S(n+1)$ for any natural number n, we have proven by induction that the proposition is true.

□

Lemma F.13 (Binomial Coefficients)**.**

$$\forall n \geq 5,\ \forall k \in [\![2, n-2]\!],\ \binom{n}{k} \geq 2n$$

Proof. Let $n \geq 5$ be a natural number, we have:

$$n - 1 \geq 4 \quad \Longrightarrow \quad \frac{n(n-1)}{2} \geq 2n \quad \Longrightarrow \quad \binom{n}{2} = \binom{n}{n-2} \geq 2n$$

Let $k \in [\![2, n-2]\!]$, then finally:

$$\binom{n}{k} \geq \binom{n}{2} \geq 2n$$

□

Lemma F.14 (Bound)**.** *For a real $\alpha > 0$, consider the sequence $(a_n)_n$ defined recursively as follows:*

$$a_0 = 1 \quad \text{and} \quad \forall n \in \mathbb{N}^*,\ a_n = a_0 + \alpha \sum_{k=1}^{n-1} \frac{a_k}{\binom{n}{k}}$$

Then the sequence $(a_n)_n$ is bounded:

$$\forall n \in \mathbb{N}, \quad a_n \leq C$$

where the bound C is given by:

$$C = \max\left\{\frac{5}{5-3\alpha}, 1 + \frac{2\alpha}{3} + \frac{\alpha^2}{4} + \frac{\alpha^3}{24}, 1 + \frac{3\alpha}{5} + \frac{\alpha^2}{4} + \frac{\alpha^3}{15} + \frac{\alpha^4}{120}\right\}$$

Proof. Let n be a natural number, and consider the proposition $T(n)$ defined as follows:

$$a_n \leq C$$

We will prove this proposition by strong induction on n, starting with the initialization for the first 6 terms.

Initialization: For $0 \leq n \leq 5$, we observe that:

- $a_0 = a_1 = 1 \leq C$
- $a_2 = 1 + \frac{\alpha}{2} \leq C$
- $a_3 = 1 + \frac{\alpha}{3} + \alpha\frac{1+\frac{\alpha}{2}}{3} = 1 + \frac{2\alpha}{3} + \frac{\alpha^2}{6} \leq C$
- $a_4 = 1 + \frac{\alpha}{4} + \alpha\frac{1+\frac{\alpha}{2}}{6} + \alpha\frac{1+\frac{2\alpha}{3}+\frac{\alpha^2}{6}}{4} = 1 + \frac{2\alpha}{3} + \frac{\alpha^2}{4} + \frac{\alpha^3}{24} \leq C$
- $a_5 = 1 + \frac{\alpha}{5} + \alpha\frac{1+\frac{\alpha}{2}}{10} + \alpha\frac{1+\frac{2\alpha}{3}+\frac{\alpha^2}{6}}{10} + \alpha\frac{1+\frac{2\alpha}{3}+\frac{\alpha^2}{4}+\frac{\alpha^3}{24}}{5} = 1 + \frac{3\alpha}{5} + \frac{\alpha^2}{4} + \frac{\alpha^3}{15} + \frac{\alpha^4}{120} \leq C$

Thus, we have verified that $T(0), ..., T(5)$ are true.

Induction Step: Let $n \geq 5$ be a natural number, and assume that $T(0), ..., T(n-1)$ are true. Now, let's prove that $T(n)$ is also true. Starting from the definition of $(a_n)_n$ and using the result from the previous lemma, we can establish the following bound:

$$a_0 \leq a_n \leq a_0 + 2\frac{\alpha C}{n} + (n-3)\frac{\alpha C}{\frac{n(n-1)}{2}} \leq a_0 + \frac{4\alpha C}{n}$$

This proves the lemma.

□

Lemma F.15 (Asymptotic Expansion)**.** *For a real $\alpha > 0$, consider the sequence $(a_n)_n$ defined recursively as follows:*

$$a_0 = 1 \quad \textit{and} \quad \forall n \in \mathbb{N}^*,\ a_n = a_0 + \alpha \sum_{k=1}^{n-1} \frac{a_k}{\binom{n}{k}}$$

Then the sequence $(a_n)_n$ can be asymptotically expanded as follows:

$$a_n = a_0 + O(n^{-1})$$

Proof. Let $n \geq 3$ be a natural number. Starting from the definition of $(a_n)_n$ and using the previous result, we can establish the following bound:

$$a_0 \leq a_n \leq a_0 + 2\frac{\alpha C}{n} + (n-3)\frac{\alpha C}{\frac{n(n-1)}{2}} \leq a_0 + \frac{4\alpha C}{n}$$

This proves the lemma.

□

Proposition F.16 (Viability and Standard Deviation)**.**

$$w_n \underset{n\to+\infty}{\sim} (1-\beta)n \quad \textit{and} \quad \sigma_n \underset{n\to+\infty}{\sim} \sqrt{\beta(1-\beta)n}$$

Proof. Using Proposition 5.9 which establishes that:

$$v_n \underset{n\to+\infty}{\sim} n!$$

we can immediately deduce the first result about viability:

$$w_n = \frac{W_{n+1}}{W_n} = (1-\beta)\frac{v_{n+1}}{v_n} \underset{n\to+\infty}{\sim} (1-\beta)n$$

For the second result concerning the standard deviation, we return to equation (2.2):

$$p_{n+1}(dx) = (1-\beta)\frac{xp_n(dx)}{w_n} + \beta q(dx)$$

By multiplying this equation by the variable x before integrating, we get:

$$w_{n+1} = \frac{1-\beta}{w_n}\int_0^{+\infty} x^2 p_n(dx) + \beta\mu_1$$

The variance can then be expressed as:

$$\int_0^{+\infty} x^2 p_n(dx) - w_n^2 = \frac{w_n}{1-\beta}(w_{n+1} - \beta\mu_1) - w_n^2 \underset{n\to+\infty}{\sim} (1-\beta)n^2 - (1-\beta)^2 n^2 = \beta(1-\beta)n^2$$

This allows us to conclude this proof.

□

Theorem F.17 (Asymptotic Fitness)**.** *The sequence of fitness distributions $(p_n)_n$ is equivalent to the weighted sum of an exponential and a gamma distribution:*

$$\|p_n - \beta q - (1-\beta)\Gamma_{n+1}\|_{TV} = \|p_n - \beta q - (1-\beta)\Gamma_{n+1}\|_1 \xrightarrow[n\to+\infty]{} 0$$

Proof. To prove this theorem, we start by recalling equation (F.1), adapted to densities and displaying the sequence $(v_n)_n$:

$$p_n(x) = \beta\sum_{k=0}^{n-1}\frac{v_{n-k}}{v_n}x^k q(x) + \frac{1}{v_n}(1-\beta)x^n p_0(x)$$

Isolating the two terms of interest, we have:

$$p_n(x) = \beta e^{-x} + (1-\beta)\frac{x^n e^{-x}}{n!} + \beta\sum_{k=1}^{n-1}\frac{1}{\binom{n}{k}}\frac{x^k}{k!}e^{-x}$$

Now we focus on the remaining sum:

$$\beta\sum_{k=1}^{n-1}\frac{1}{\binom{n}{k}}\frac{x^k}{k!}e^{-x} \leq \frac{\beta e^{-x}}{n}\left[x + \frac{x^{n-1}}{(n-1)!} + 2\frac{\lfloor x\rfloor^{\lfloor x\rfloor}}{\lfloor x\rfloor!}\right] \xrightarrow[n\to+\infty]{} 0$$

Using the corollary of the Scheffé lemma, which applies here because the measures involved are probability measures, we can conclude the theorem.

□

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