Homework 1
Machine Learning with Kernel Methods

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Contents

1 Function and kernel boundedness

Consider a p.d. kernel $K : \mathcal{X}^2 \to \mathbb{R}$ such that $K(x,z) \leq b^2$ for all x, z in $\mathcal{X}$. Show that $\|f\|_\infty = \sup_{x\in\mathcal{X}} |f(x)| \leq b$ for any function f in the unit ball of the corresponding RKHS.

Given that K is a p.d. kernel, it serves as the reproducing kernel of the RKHS $\mathcal{H}$. Let us define $B_\mathcal{H}$ as its unit ball. By virtue of the reproducing property, we can express:

$$\forall f \in B_\mathcal{H},\ \forall x \in \mathcal{X}, \quad f(x) = \langle f, K_x \rangle_\mathcal{H}$$

Now, applying the Cauchy-Schwarz inequality, we derive the following:

$$\forall f \in B_\mathcal{H},\ \forall x \in \mathcal{X}, \quad |f(x)| = |\langle f, K_x \rangle_\mathcal{H}| \leq \|f\|_\mathcal{H} \cdot \|K_x\|_\mathcal{H} \leq 1 \cdot \sqrt{K(x,x)} \leq b$$

This allows us to conclude:

$$\forall f \in B_\mathcal{H}, \quad \|f\|_\infty = \sup_{x\in\mathcal{X}} |f(x)| \leq b$$

2 Kernels encoding equivalence classes

Consider a similarity measure $K : \mathcal{X}^2 \to \{0,1\}$ with:

$$\forall x \in \mathcal{X}, \quad K(x,x) = 1 \tag{0}$$

Prove that K is p.d. if and only if the following two conditions hold:

$$\forall x, x' \in \mathcal{X}, \quad K(x,x') = 1 \quad \Longleftrightarrow \quad K(x',x) = 1 \tag{1}$$

$$\forall x, x', x'' \in \mathcal{X}, \quad K(x,x') = K(x',x'') = 1 \quad \Longrightarrow \quad K(x,x'') = 1 \tag{2}$$

- Let's start by assuming that K is p.d. Initially, we observe that K is symmetric, satisfying the first condition (1). Now, let's consider the inequality:

  $$\forall N \in \mathbb{N},\ \forall x_1, ..., x_N \in \mathcal{X},\ \forall a_1, ..., a_N \in \mathbb{R}, \qquad \sum_{1 \leq i,j \leq N} a_i a_j K(x_i, x_j) \geq 0$$

  Specifically, when $N = 3$, $-a_1 = a_2 = -a_3 = 1$, and $x, x', x'' \in \mathcal{X}$:

  $$(K(x,x) + K(x',x') + K(x'',x'')) + 2\left(-K(x,x') + K(x,x'') - K(x',x'')\right) \geq 0$$

  Noting that $K \in \{0,1\}^{\mathcal{X}^2}$:

  $$K(x,x') = K(x',x'') = 1 \quad \Longrightarrow \quad 2K(x,x'') - 1 \geq 0 \quad \Longrightarrow \quad K(x,x'') \geq \frac{1}{2} \quad \Longrightarrow \quad K(x,x'') = 1$$

  This shows that condition (2) is satisfied.

- Now, let's assume that conditions (1) and (2) hold. First, the first condition implies that K is symmetric. We define the relation:

  $$x \sim y \quad \Longleftrightarrow \quad K(x,y) = 1$$

  Thus, the three properties (0), (1), and (2) imply that $\sim$ is an equivalence relation. We denote $C_1, \ldots, C_K$ as the equivalence classes. For any $N \in \mathbb{N}$, $x_1, \ldots, X_N \in \mathcal{X}$, and $a_1, \ldots, a_N \in \mathbb{R}$, where $a_i = a_i(x_i)$ associates variables with the same index, we have:

  $$\sum_{1 \leq i,j \leq N} a_i a_j K(x_i, x_j) = \sum_{i=1}^{N} a_i(x_i)^2 + 2 \sum_{x_i \sim x_j} a_i(x_i) a_j(x_j) = \sum_{k=1}^{K} \left(\sum_{x \in C_k} a(x) \right)^2 \geq 0$$

  This confirms that K is indeed positive definite.

With these arguments, we conclude:

$$\boxed{K \text{ is p.d.} \quad \Longleftrightarrow \quad (1) \text{ and } (2) \text{ hold}}$$

3 RKHS

3.1 Combining positive definite kernels

Let K_1 and K_2 be two positive definite kernels on a set $\mathcal{X}$, and α, β two positive scalars. Show that $\alpha K_1 + \beta K_2$ is positive definite, and describe its RKHS.

- Let's consider the kernel $K = \alpha K_1 + \beta K_2$. Due to the symmetry of both K_1 and K_2, we can conclude that K is symmetric as well. Now, for any natural number N, and any $x_1, \ldots, x_N$ in $\mathcal{X}$, along with $a_1, \ldots, a_N$ in $\mathbb{R}$, we can analyze the following expression:

$$\sum_{1 \leq i,j \leq N} a_i a_j K(x_i, x_j) = \underbrace{\alpha}_{>0} \underbrace{\sum_{1 \leq i,j \leq N} a_i a_j K_1(x_i, x_j)}_{\geq 0} + \underbrace{\beta}_{>0} \underbrace{\sum_{1 \leq i,j \leq N} a_i a_j K_2(x_i, x_j)}_{\geq 0} \geq 0$$

  Therefore, we can conclude that:

$$\boxed{K = \alpha K_1 + \beta K_2 \text{ is p.d.}}$$

- Let's consider a p.d. kernel K on the set $\mathcal{X}$, which corresponds to the RKHS $(\mathcal{H}, \|\cdot\|_{\mathcal{H}}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$. Now, we introduce a new kernel $\widetilde{K}$ defined as $\widetilde{K} = \alpha K$. In the context of this new kernel, we define the following objects:

$$\widetilde{\mathcal{H}} = \mathcal{H}, \quad \|\cdot\|_{\widetilde{\mathcal{H}}} = \frac{1}{\sqrt{\alpha}} \|\cdot\|_{\mathcal{H}} \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\widetilde{\mathcal{H}}} = \frac{1}{\alpha} \langle \cdot, \cdot \rangle_{\mathcal{H}}$$

  Next, we observe the following properties:

  - $\widetilde{\mathcal{H}}$ is a Hilbert space, and this can be demonstrated through the linear bijection that maps a function f in $\mathcal{H}$ to $\sqrt{\alpha} f$ in $\widetilde{\mathcal{H}}$.
  - $\left\{ \widetilde{K}_x = \widetilde{K}(x, \cdot) \;\middle|\; x \in \mathcal{X} \right\} \subset \widetilde{\mathcal{H}}$
  - For any function $\widetilde{f}$ in $\widetilde{\mathcal{H}}$ and any x in $\mathcal{X}$, we have the equality:

$$\left\langle \widetilde{f}, \widetilde{K}_x \right\rangle_{\widetilde{\mathcal{H}}} = \frac{1}{\alpha} \left\langle \widetilde{f}, \widetilde{K}_x \right\rangle_{\mathcal{H}} = \left\langle \widetilde{f}, K_x \right\rangle_{\mathcal{H}} = \widetilde{f}(x)$$

  As a result, we can conclude that $\widetilde{K}$ is a r.k. of the Hilbert space $\widetilde{\mathcal{H}}$, and $(\widetilde{\mathcal{H}}, \|\cdot\|_{\widetilde{\mathcal{H}}}, \langle \cdot, \cdot \rangle_{\widetilde{\mathcal{H}}})$ serves as the RKHS of the kernel $\widetilde{K}$.

- Let's consider two p.d. kernels, denoted as K_1 and K_2, defined on the set $\mathcal{X}$. Each of these kernels corresponds to a RKHS, which we denote as $(\mathcal{H}_1, \|\cdot\|_{\mathcal{H}_1}, \langle \cdot, \cdot \rangle_{\mathcal{H}_1})$ and $(\mathcal{H}_2, \|\cdot\|_{\mathcal{H}_2}, \langle \cdot, \cdot \rangle_{\mathcal{H}_2})$. Now, we introduce a new kernel $\widetilde{K}$, defined as the sum of these two kernels, i.e., $\widetilde{K} = K_1 + K_2$. Additionally, we define a new vector space $\widetilde{\mathcal{H}}$, which is the sum of the two Hilbert spaces, as $\widetilde{\mathcal{H}} = \mathcal{H}_1 + \mathcal{H}_2$.

  - First, it is important to note that $\widehat{\mathcal{H}} = \mathcal{H}_1 \times \mathcal{H}_2$ endowed with the inner product

$$\langle \cdot, \cdot \rangle_{\widehat{\mathcal{H}}} : ((f_1, f_2), (g_1, g_2)) \in \widehat{\mathcal{H}}^2 \longmapsto \langle f_1, g_1 \rangle_{\mathcal{H}_1} + \langle f_2, g_2 \rangle_{\mathcal{H}_2}$$

    is indeed a Hilbert space.

    Now, let's consider the mapping $h : (f_1, f_2) \in \widehat{\mathcal{H}} \longmapsto f_1 + f_2 \in \widetilde{\mathcal{H}}$. This mapping is linear and surjective. The kernel of this mapping is defined as:

$$\ker h = \{(f, -f) \mid f \in \mathcal{H}_1 \cap \mathcal{H}_2\}$$

If a sequence within the kernel $((f_n, -f_n))_n$ converges to (f, g) in $\widehat{\mathcal{H}}$, then it implies that $(f_n)_n$ converges to f, and $(-f_n)_n$ converges to g. Consequently, we have $-f = g$, which means that the limit (f, g) belongs to the kernel. This establishes that $\ker h$ is a closed subspace. Furthermore, we can state the following properties:

$$\widehat{\mathcal{H}} = \ker h \oplus \ker h^{\perp} \quad \text{and} \quad \widetilde{h} = h_{|_{\ker h^{\perp}}} \text{ is bijective} \tag{$\star$}$$

Given these properties, we can conclude that $\widetilde{\mathcal{H}}$, equipped with the inner product

$$\langle \cdot, \cdot \rangle_{\widetilde{\mathcal{H}}} = \left\langle \widetilde{h}^{-1}(\cdot), \widetilde{h}^{-1}(\cdot) \right\rangle_{\widehat{\mathcal{H}}}$$

is indeed a Hilbert space.

Now, let's explicitly express the norm $\|\cdot\|_{\widetilde{\mathcal{H}}}$ associated with $\widetilde{\mathcal{H}}$. We can define this norm as $\|\cdot\|_{\widetilde{\mathcal{H}}} = \|\cdot\|_{\widehat{\mathcal{H}}} \circ \widetilde{h}^{-1}$. To derive this expression, consider any (f_1, f_2) in $\widehat{\mathcal{H}}$, and by virtue of property $(\star)$, there exists $(f_1^0, f_2^0) \in \ker h$ such that $(f_1, f_2) = (f_1^0, f_2^0) + \widetilde{h}^{-1}(f_1 + f_2)$. Then, we have the following relationship:

$$\|(f_1, f_2)\|_{\widehat{\mathcal{H}}}^2 = \|(f_1^0, f_2^0)\|_{\widehat{\mathcal{H}}}^2 + \|f_1 + f_2\|_{\widetilde{\mathcal{H}}}^2$$

Now, let's apply this expression to a specific element $f = f_1 + f_2 \in \widetilde{\mathcal{H}}$:

$$\|f\|_{\widetilde{\mathcal{H}}}^2 = \|f_1 + f_2\|_{\widetilde{\mathcal{H}}}^2 \leq \|f_1\|_{\mathcal{H}_1}^2 + \|f_2\|_{\mathcal{H}_2}^2 = \|(f_1, f_2)\|_{\widehat{\mathcal{H}}}^2 = \|f\|_{\widetilde{\mathcal{H}}}^2 + \|(f_1^0, f_2^0)\|_{\widehat{\mathcal{H}}}^2$$

This inequality holds with equality if and only if $f_1^0 = f_2^0 = 0$, which implies that $(f_1, f_2) \in \ker h^{\perp}$. Consequently, this inequality provides us with an explicit expression for our norm:

$$\|f\|_{\widetilde{\mathcal{H}}} = \min_{\substack{(f_1, f_2) \in \widehat{\mathcal{H}} \\ f = f_1 + f_2}} \sqrt{\|f_1\|_{\mathcal{H}_1}^2 + \|f_2\|_{\mathcal{H}_2}^2}$$

By observing property $(\star)$, we note that the decomposition of a function is unique in $\ker h^{\perp}$, which is the space where the previous minimum is taken. Using the polarization identity, we can derive the associated inner product for $\widetilde{\mathcal{H}}$ as follows:

$$\langle \cdot, \cdot \rangle_{\widetilde{\mathcal{H}}} : (f, g) \in \widetilde{\mathcal{H}}^2 \longmapsto \min_{\substack{(f_1, f_2), (g_1, g_2) \in \widehat{\mathcal{H}} \\ f = f_1 + f_2, g = g_1 + g_2}} \langle f_1, g_1 \rangle_{\mathcal{H}_1} + \langle f_2, g_2 \rangle_{\mathcal{H}_2}$$

- $\left\{ \widetilde{K}_x = K_{1,x} + K_{2,x} \;\middle|\; x \in \mathcal{X} \right\} \subset \{K_{1,x} \mid x \in \mathcal{X}\} + \{K_{2,x} \mid x \in \mathcal{X}\} \subset \mathcal{H}_1 + \mathcal{H}_2 = \widetilde{\mathcal{H}}$
- For any function $\widetilde{f}$ in $\widetilde{\mathcal{H}}$ and any x in $\mathcal{X}$, utilizing property $(\star)$ once more, we can establish the existence of unique pairs (f_1, f_2) and $(\widetilde{K}_{1,x}, \widetilde{K}_{2,x})$ in $\ker h^{\perp}$ such that:

$$\widetilde{f} = \widetilde{h}(f_1, f_2) = f_1 + f_2 \quad \text{and} \quad \widetilde{K}_x = \widetilde{h}\left(\widetilde{K}_{1,x}, \widetilde{K}_{2,x}\right) = \widetilde{K}_{1,x} + \widetilde{K}_{2,x}$$

This implies that:

$$\left\langle \widetilde{f}, \widetilde{K}_x \right\rangle_{\widetilde{\mathcal{H}}} = \left\langle (f_1, f_2), \left(\widetilde{K}_{1,x}, \widetilde{K}_{2,x}\right) \right\rangle_{\widehat{\mathcal{H}}} = \left\langle (f_1, f_2), (K_{1,x}, K_{2,x}) + \left(\widetilde{K}_{1,x} - K_{1,x}, \widetilde{K}_{2,x} - K_{2,x}\right) \right\rangle_{\widehat{\mathcal{H}}}$$

However, it is important to note that $\widetilde{K}_{1,x} - K_{1,x} + \widetilde{K}_{2,x} - K_{2,x} = \widetilde{K}_x - \widetilde{K}_x = 0$. Therefore, the pair $\widetilde{K}_{1,x} - K_{1,x}, \widetilde{K}_{2,x} - K_{2,x}$ is an element of $\ker h$, leading to the following equalities:

$$\left\langle \widetilde{f}, \widetilde{K}_x \right\rangle_{\widetilde{\mathcal{H}}} = \langle (f_1, f_2), (K_{1,x}, K_{2,x}) \rangle_{\widehat{\mathcal{H}}} = \langle f_1, K_{x,1} \rangle_{\mathcal{H}_1} + \langle f_2, K_{x,2} \rangle_{\mathcal{H}_2} = f_1(x) + f_2(x) = \widetilde{f}(x)$$

As a result, we can conclusively affirm that $\widetilde{K}$ serves as a r.k. of the Hilbert space $\widetilde{\mathcal{H}}$, and that $(\widetilde{\mathcal{H}}, \|\cdot\|_{\widetilde{\mathcal{H}}}, \langle \cdot, \cdot \rangle_{\widetilde{\mathcal{H}}})$ operates as the RKHS associated with the kernel $\widetilde{K}$.

In the given exercise scenario, we are working with two positive scalars, α and β, and two p.d. kernels, K_1 and K_2, defined on the set $\mathcal{X}$. Each of these kernels corresponds to a RKHS, which we label as $(\mathcal{H}_1, \|\cdot\|_{\mathcal{H}_1}, \langle\cdot,\cdot\rangle_{\mathcal{H}_1})$ and $(\mathcal{H}_2, \|\cdot\|_{\mathcal{H}_2}, \langle\cdot,\cdot\rangle_{\mathcal{H}_2})$.

Now, we introduce a new kernel, denoted as $\widetilde{K}$, which is formed by taking a linear combination of these two kernels: $\widetilde{K} = \alpha K_1 + \beta K_2$. Additionally, we define a novel vector space, $\widetilde{\mathcal{H}}$, as the direct sum of these two Hilbert spaces: $\widetilde{\mathcal{H}} = \mathcal{H}_1 + \mathcal{H}_2$.

By combining all the previously established results, we can conclude that K is indeed p.d. and functions as a r.k. of the Hilbert space $\widetilde{\mathcal{H}}$. Moreover, $(\widetilde{\mathcal{H}}, \|\cdot\|_{\widetilde{\mathcal{H}}}, \langle\cdot,\cdot\rangle_{\widetilde{\mathcal{H}}})$ serves as the RKHS for the kernel $\widetilde{K}$. The essential properties of this new Hilbert space can be succinctly summarized as follows:

$$\widetilde{\mathcal{H}} = \mathcal{H}_1 + \mathcal{H}_2$$

$$\|\cdot\|_{\widetilde{\mathcal{H}}} : f \in \widetilde{\mathcal{H}} \longmapsto \min_{\substack{(f_1,f_2)\in\widehat{\mathcal{H}} \\ f=f_1+f_2}} \sqrt{\frac{1}{\alpha}\|f_1\|^2_{\mathcal{H}_1} + \frac{1}{\beta}\|f_2\|^2_{\mathcal{H}_2}}$$

$$\langle\cdot,\cdot\rangle_{\widetilde{\mathcal{H}}} : (f,g) \in \widetilde{\mathcal{H}}^2 \longmapsto \min_{\substack{(f_1,f_2),(g_1,g_2)\in\widehat{\mathcal{H}} \\ f=f_1+f_2, g=g_1+g_2}} \frac{1}{\alpha}\langle f_1, g_1\rangle_{\mathcal{H}_1} + \frac{1}{\beta}\langle f_2, g_2\rangle_{\mathcal{H}_2}$$

3.2 Characterizing the RKHS of kernel K

Let $\mathcal{X}$ be a set and $\mathcal{F}$ be a Hilbert space. Let $\Psi : \mathcal{X} \to \mathcal{F}$, and $K : \mathcal{X}^2 \to \mathbb{R}$ be:

$$\forall x, x' \in \mathcal{X}, \quad K(x, x') = \langle \Psi(x), \Psi(x') \rangle_{\mathcal{F}}$$

Show that K is a p.d. kernel on $\mathcal{X}$, and describe its RKHS.

- The function K is symmetric because it relies on the inner product of the Hilbert space, which is inherently symmetric.
- Let $N \in \mathbb{N}$, $a_1, \ldots, a_N \in \mathbb{R}$, and $x_1, \ldots, x_N \in \mathcal{X}$. We consider the sum:

$$\sum_{1 \leq i,j \leq N} a_i a_j K(x_i, x_j) = \left\langle \sum_{i=1}^{N} a_i \Psi(x_i), \sum_{j=1}^{N} a_j \Psi(x_j) \right\rangle_{\mathcal{F}} = \left\| \sum_{i=1}^{N} a_i \Psi(x_i) \right\|_{\mathcal{F}}^2 \geq 0$$

Thus, we can confidently assert (or establish through Aronszajn's theorem) that:

$$\boxed{K \text{ is a p.d. kernel on } \mathcal{X}}$$

With the positive definiteness of K established, we can now proceed to describe its RKHS. Consider the space $\mathcal{H} = \left\{ \widetilde{K}_z = \langle z, \Psi(\cdot) \rangle_{\mathcal{F}} \;\middle|\; z \in \mathcal{F} \right\}$ endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \left(\widetilde{K}_z, \widetilde{K}_{z'} \right) \in \mathcal{H}^2 \longmapsto \langle z, z' \rangle_{\mathcal{F}}$.

- The mapping $\xi : z \in \mathcal{F} \longmapsto \widetilde{K}_z \in \mathcal{H}$ is linear, surjective, and isometric. This mapping ensures that $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ itself is a Hilbert space.
- $\{K_x = \langle \Psi(x), \Psi(\cdot) \rangle_{\mathcal{F}} \mid x \in \mathcal{X}\} \subset \{\langle z, \Psi(\cdot) \rangle_{\mathcal{F}} \mid z \in \mathcal{F}\} = \mathcal{H}$
- For any x in $\mathcal{X}$ and z in $\mathcal{F}$, we find:

$$\left\langle \widetilde{K}_z, K_x \right\rangle_{\mathcal{H}} = \left\langle \widetilde{K}_z, \widetilde{K}_{\Psi(x)} \right\rangle_{\mathcal{H}} = \langle z, \Psi(x) \rangle_{\mathcal{F}} = \widetilde{K}_z(x)$$

In conclusion, we confirm that K serves as a r.k. of the Hilbert space $\mathcal{H}$, and $(\mathcal{H}, \|\cdot\|_{\mathcal{H}}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ functions as the RKHS associated with the kernel K. The key characteristics of this RKHS are summarized as follows:

$$\mathcal{H} = \left\{ \widetilde{K}_z = \langle z, \Psi(\cdot) \rangle_{\mathcal{F}} \;\middle|\; z \in \mathcal{F} \right\}$$

$$\|\cdot\|_{\mathcal{H}} : \widetilde{K}_z \in \mathcal{H} \longmapsto \|z\|_{\mathcal{F}}$$

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} : \left(\widetilde{K}_z, \widetilde{K}_{z'} \right) \in \mathcal{H}^2 \longmapsto \langle z, z' \rangle_{\mathcal{F}}$$

3.3 RKHS inclusion criteria

Prove that for any p.d. kernel K on a space $\mathcal{X}$, a function $f : \mathcal{X} \to \mathbb{R}$ belongs to the RKHS $\mathcal{H}$ with kernel K if and only if there exists $\lambda > 0$ such that $K(x, x') - \lambda f(x)f(x')$ is p.d.

- Let $\lambda > 0$, and define $\widehat{K} : (x, x') \longmapsto K(x, x') - \lambda f(x)f(x')$. It's worth noting that when $f = 0$, $\widehat{K}$ is evidently positive definite. Thus, let's consider the case where $f \neq 0$. We can establish the symmetry of $\widehat{K}$ due to the symmetry of its terms.

  Now, let's choose $N \in \mathbb{N}$, $a_1, ..., a_N \in \mathbb{R}$, and $x_1, ..., x_N \in \mathcal{X}$. By leveraging the reproducing property and Cauchy-Schwarz inequality, we derive the following chain of (in)equalities:

$$\begin{aligned}
\sum_{1 \leq i,j \leq N} a_i a_j \left(K(x_i, x_j) - \lambda f(x_i) f(x_j)\right) &= \sum_{1 \leq i,j \leq N} a_i a_j \left(\langle K_{x_i}, K_{x_j} \rangle_{\mathcal{H}} - \lambda \langle f, K_{x_i} \rangle_{\mathcal{H}} \langle f, K_{x_j} \rangle_{\mathcal{H}}\right) \\
&= \left\| \sum_{i=1}^N a_i K_{x_i} \right\|_{\mathcal{H}}^2 - \lambda \left| \left\langle f, \sum_{i=1}^N a_i K_{x_i} \right\rangle_{\mathcal{H}} \right|^2 \\
&\geq \left\| \sum_{i=1}^N a_i K_{x_i} \right\|_{\mathcal{H}}^2 - \lambda \|f\|_{\mathcal{H}}^2 \left\| \sum_{i=1}^N a_i K_{x_i} \right\|_{\mathcal{H}}^2 \\
&= \left(1 - \lambda \|f\|_{\mathcal{H}}^2\right) \left\| \sum_{i=1}^N a_i K_{x_i} \right\|_{\mathcal{H}}^2
\end{aligned}$$

  Since $f \neq 0$, the interval $\left(0, \|f\|_{\mathcal{H}}^{-2}\right)$ is non-empty. Therefore, by selecting λ from this interval, we can confidently affirm the positive definiteness of $\widehat{K}$.

- Now, assuming the positive definiteness of $\widehat{K}$, we can assert that it has an associated RKHS, denoted as $\widehat{H}$. This space is a Hilbert space, which naturally includes the zero vector. Additionally, considering that $\mathbb{R}$, equipped with the standard inner product, forms a Hilbert space, we can affirm that the mapping $\widetilde{K} : (x, x') \longmapsto f(x)f(x')$, as proven in the previous question, is a positive definite kernel. Its associated RKHS is given by $\widetilde{H} = \{\zeta f \mid \zeta \in \mathbb{R}\}$. Therefore, it includes the function f itself.

  Notably, recognizing that $K = \widehat{K} + \lambda \widetilde{K}$, we can utilize the result obtained in the first question to deduce that the associated RKHS, denoted as $\mathcal{H}$, is the direct sum of the two previously mentioned spaces. Consequently, we can conclude that $f = 0 + f$ belongs to $\widehat{\mathcal{H}} + \widetilde{\mathcal{H}} = \mathcal{H}$.

In summary, we can confidently state that:

$$f \in \mathcal{H} \quad \Longleftrightarrow \quad \exists \lambda > 0 \ / \ \widehat{K} : (x, x') \longmapsto K(x, x') - \lambda f(x)f(x') \text{ is p.d.}$$