ALGEBRAIC PROPERTIES OF EDGE IDEALS

A thesis submitted in partial fulfillment of the requirements for the award of degree of

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Declaration

We, hereby declare that this project entitled, **"Algebraic Properties of Edge Ideals"** and the work presented in it are our own. We confirm that:

- This work was done wholly while in candidature for a masters degree at the Central University of Kashmir.
- Where we have quoted from the work of others, the source is always given. With the exception of such quotations, this project is entirely our own work.
- Where we have consulted the published work of others, this is always clearly attributed.
- We have also fulfilled the requirements of the UGC regulations for carrying out project work in Integrated BSc-MSc.

Signed:

Date:

Certificate

This is to certify that the above mentioned students, of the Department of Mathematics, Central University of Kashmir have completed their project entitled **Algebraic Properties of Edge Ideals**. I have gone through the draft of the project and found it worthy of consideration in partial fulfillment of the requirements for the award of Integrated BSc./MSc. degree. I further certify that:-

(i). the project embodies the work of the students done by themselves;

(ii). the students worked under my guidance and supervision for the period required under the relevant ordinance;

(iii). the students have put required attendance in the Department of Mathematics, Central University of Kashmir; and

(iv). the conduct of the students remained satisfactory during the period.

Dated: ________________ **Dr. Shahnawaz Ahmad Rather**

Acknowledgment

First and foremost praise is to ALLAH, the Almighty, who has taught the use of pen and giving us opportunity, determination and strength to embark upon this project work and carry it to its successful completion. His continuous grace and mercy was with us throughout our life and ever more during the tenure of our project.

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Chapter 1

GRADED RINGS AND MODULES

1.1 Introduction

Commutative algebra evolved from problems arising in number theory and algebraic geometry. Much of the modern development of the commutative algebra emphasizes graded rings. Once the grading is considered to be trivial, the graded theory reduces to the usual module theory. So from this perspective, the theory of graded modules can be considered as an extension of module theory. Graded rings play a central role in algebraic geometry and commutative algebra. Gradings appear in many circumstances, both in elementary and advanced level.

Graded Betti numbers encode all the numerical information of the minimal graded free resolution of a homogeneous ideal. Using graded Betti numbers, one can retrieve the Hilbert function, the regularity or the projective dimension. Among these homological invariants, the regularity is one of the most important invariant which measures, roughly speaking, the complexity of a module in some sense. These homological invariants have also important geometric interpretation (see [20]) and also in graph theoretical way [40].

The class of monomial ideals, although intrinsically interesting, is a classical object in commutative algebra, which has a strong connection to combinatorics. Broadly speaking, problems in combinatorics are encoded into monomial ideals, which then allow us to use techniques and methods in commutative algebra to solve the original problem. An essential link between these two areas arises from the construction of Stanley-Reisner ring, where Stanley identified the minimal generators of a squarefree monomial ideal with the minimal non-faces of a simplicial complex. Stanley demonstrated that there are deep relations between the combinatorial properties of the simplicial complex and the algebraic properties of it associated monomial ideal. Stanley's proof of Upper Bound Conjecture [72] for simplicial spheres is seen as one of the early highlight of exploiting the connection between these two fields. There are several other celebrated results such as Reisner's criterion for Cohen-Macaulayness [62] and Hochster's formula [47] which provides a bridge between these two areas.

Moreover, any monomial ideal can be reduced to a squarefree monomial ideal by using the polarization technique which allows us to translate any problem related to monomial ideals into a problem about squarefree monomial ideals. Schwartau in [68] and Stückrad and Vogel in [73] discussed how the Stanley-Reisner theory of squarefree monomial ideals can be used to produce results about monomial ideals using polarization. There is a lot of work devoted to the study of the algebraic invariants and combinatorial properties of squarefree monomial ideals and graphs. We refer the reader to [18, 28, 30, 34, 37, 41, 42, 83] and the references therein for further details. Here we present an examples on gradings.

Following example show the applications of gradings in commutative algebra and algebraic geometry as well as in real life: 1. In the elementary school when we distribute 10 apples giving 2 apples to each person, we have 10 Apple : 2 Apple = 5 People. The psychological problem caused to many kids of how the word per People appears in the equation can be justified by correcting 10 Apple: 2 Apple/People = 5 People. This shows that already at the level of elementary school arithmetic, children are working in a much more sophisticated structure, i.e., a graded ring $\mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$ of laurant polynomial R is a commutative ring which is generated by a finite number of elements of degree 1.

The study of graded rings arises naturally out of the study of affine schemes and allows them to formalize (and unify) arguments by induction. However, this is not just an algebraic trick. The concept of grading in algebra, in particular graded modules are essential in the study of homological aspect of rings. In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concept are widely studied. As a result, graded analogue of different concepts are being developed in recent research. The objective of this paper is to study rings graded by any finitely generated abelian group, graded modules and their applications.

1.2 Prelimaries

In this section we shall collect the basics of graded rings and modules and give some examples for their use in forthcoming chapters.

Definition 1.2.1. Let G be an abelian group (written additively) and R a commutative ring. A G-grading for R is a family $(R_g)_{g\in G}$ of abelian groups of $(R, +)$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called the homogeneous elements of R of degree g. If $r \in R_g$, we write the degree of r as $\deg(r) = g$ or $|r| = g$.

Example 1.2.2. Examples on graded rings are as follows:

(i) Consider $\mathbb{K}[x] = \bigoplus_{n\in\mathbb{N}} \mathbb{K}x^n$, where $\mathbb{K}$ is a field and $\mathbb{K}x^n = 0$ if $n < 0$. Then $\mathbb{K}[x] = \ldots 0 \oplus \cdots \oplus 0 \oplus \mathbb{K} \oplus \mathbb{K}x \oplus \ldots$

(ii) Let $R = T[x]$ and G be any abelian group. Set $|x| = g$ for some $g \in G$. For $h \in G$, we see $R_h = \bigoplus_{ig=h} Tx^i$, where $T \subseteq R_0$. If $G = \mathbb{Z}$ and $|x| = 1$, then for $n \in \mathbb{Z}$

$$R_h = \begin{cases} Tx^h & \text{if } n \geq 0; \\ 0 & \text{if } n < 0. \end{cases}$$

This is N-grading.

(iii) Let $R = T[x_1, \ldots, x_d]$ and G be any abelian group. Set $|x_i| = g_i$. For $h \in G$, we have $R_h = \bigoplus_{\alpha_1 g^1 + \cdots + \alpha_d g^d = h} Tx_1\alpha^1 \cdots x_d\alpha^d$.

(a) If $R = T[x, y], G = \mathbb{Z}$, and $|x| = |y| = 1$, then for $n \in \mathbb{Z}, R_h = \bigoplus_{i+j=n, i,j\geq 0} Tx^i y^j$.

(b) If $R = T[x, y], G = \mathbb{Z}$, and $|x| = 2, |y| = 3$, then $R_m = \bigoplus_{2i+3j=m} Tx^i y^j$.

Definition 1.2.3. Let $R = \bigoplus R_n$ be a graded ring. A subring S of R is called a graded subring of R if $S = \sum_n R_n \cap S$. Equivalently, S is graded if for every element $f \in S$ all the homogeneous components of f (as an element of R) are in S.

Example 1.2.4. We can construct several examples on graded subrings which are mentioned here, e.g. :

(i) Let $R = \bigoplus R_n$ be a graded ring and $f_1, \ldots, f_d$ homogeneous elements of R of degrees $\alpha_1, \ldots, \alpha_d$ respectively. Then $S = R_0[f_1, \ldots, f_d]$ is a graded subring of R, where $S_n = \big\{ \sum_{m\in N^d} r_m f_1 m^1 \ldots f_d m^d \ / \ r_m \in R_0 \text{ and } \alpha_1 m^1 + \cdots + \alpha_d m^d = n \big\}$.

(ii) $\mathbb{K}[x^2, xy, y^2]$ is a graded subring of $\mathbb{K}[x, y]$.

(iii) $\mathbb{K}[x^3, x^4, x^5]$ is a graded subring of $\mathbb{K}[x]$.

(iv) $\mathbb{Z}[x^3, x^2 + y^3]$ is a graded subring of $\mathbb{Z}[x, y]$, where $\deg(x) = 3$ and $\deg(y) = 2$.

Definition 1.2.5. Let R be a graded ring and M an R-module. We say that M is a graded R-module (or has an R-grading) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that (i). $M = \bigoplus_{g \in G} M_g$ and

(ii). $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$.

If $a \in M \setminus \{0\}$ and $a = a_{i_1} + \cdots + a_{i_k}$ where $a_{ij} \in R_{ij} \setminus \{0\}$ then $a_{i_1}, \ldots, a_{i_k}$ are called the homogeneous components of a.

Example 1.2.6. Examples on graded modules are as follows:

(i). If R is a graded ring, then R is a graded module over itself.

(ii). Let $\{M_\lambda\}$ be a family of graded R- modules then $\bigoplus_\lambda M_\lambda$ is a graded R-module. Thus $R_n = R \oplus \cdots \oplus R$ (n times) is a graded R-module for any $n \geq 1$.

(iii). Given any graded R-module M, we can form a new graded R-module by twisting the grading on M as follows: if n is any integer, define $M(n)$ (read M twisted by n) to be equal to M as an R-module, but with it's grading defined by $M(n)_k = M_{n+k}$. (For if $M = R(-3)$ then $1 \in M_3$.) then $M(n)$ is a graded R-module. Thus, if $n_1, \ldots, n_k$ are any integers then $R(n_1), \ldots, R(n_k)$ is a graded R-module. Such modules are called free.

(iv). Let R be a graded ring and S a multiplicatively closed set of homogeneous elements of R. Then R_S is a graded ring, where $(R_S)_n = \big\{ \frac{r}{s} \in R_S \mid r$ and s are homogeneous and $\deg(r) - \deg(s) = n \big\}$.

Similiarly, if M is a graded R-module then M_S is graded both as an R-module and as an R_S-module.

Remark 1.2.7. If $S = \bigoplus_{i \in G} S_i$ is a graded ring, then S_0 is a subring of S with $1 \in S_0$ and M_i is an S_0-module for each $i \in G$. Moreover, we can identify M as direct sum of $M_i's$ as M_0-modules.

Now we shall give some examples of gradings on the polynomial ring $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ over a field $\mathbb{K}$ in n variables $x_1, x_2, \ldots, x_n$.

Example 1.2.8. (i). Let $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be the polynomial ring over a field in n variables $x_1, x_2, \ldots, x_n$ with $\deg(x_i) = 1$ for $i = 1, 2, \ldots, n$. For each vector $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ with $a_i \geq 0$ for all i, we define the monomial $\boldsymbol{x^a} = x^{a_1}x^{a_2}\ldots x^{a_n}$ and the degree of $\boldsymbol{x^a}$, written as $\deg x^a$, as follows :

$$\deg \boldsymbol{x^a} = \sum_{i=1}^{n} a_i.$$

For $d \in \mathbb{Z}$, set

$$S_d = \begin{cases} \text{k}-\text{vector space spanned by monomials } \boldsymbol{x^a} \text{ of degree } d \text{ in } S & \text{if } d \geq 0, \\ 0 & \text{if } d < 0. \end{cases}$$

Then $S = \bigoplus_{d \in \mathbb{Z}} S_d$. For, if $f \in S$ is non-zero polynomial in S, then we can write f uniquely as follows :

$$f = f_0 + f_1 + \cdots + f_d,$$

where each f_i is a homogeneous polynomial of degree i in S. Also for $i \neq j$, we have $S_i \cap S_j = \{0\}$ and if f is a homogeneous polynomial of degree i and g is a homogeneous polynomial of degree j in S, then $f.g$ is a homogeneous polynomial of degree $i + j$ in S. Thus S is a $\mathbb{Z}$-graded ring and the polynomial ring $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ together with this grading is known as the polynomial ring with standard degree grading.

(ii). Let $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$ in n variables $x_1, x_2, \ldots, x_n$. For each vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, define

$$S_d = \begin{cases} kx^{a_1}x^{a_2}\ldots x^{a_n} & \text{if } a_i \geq 0 \text{ for each } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $S = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n} S_a$; where $S_{\boldsymbol{a}}.S_{\boldsymbol{b}} = S_{\boldsymbol{a}+\boldsymbol{b}}$. For, every non-zero polynomial $f \in S$ can be uniquely written as :

$$f = \sum_{\boldsymbol{a}\in\mathbb{Z}^n} c_{\boldsymbol{a}}\boldsymbol{x}^{\boldsymbol{a}},$$

where $c_{\boldsymbol{a}} \in \mathbb{K} \mid$ is zero for all $\boldsymbol{a} \in \mathbb{Z}^n$ except finitely many $\boldsymbol{a}'s$ and $S_{\boldsymbol{a}} \bigcap S_{\boldsymbol{b}} = \{0\}$ for all $\boldsymbol{a} \neq \boldsymbol{b}$. Also, we have $x_{\boldsymbol{a}}.x_{\boldsymbol{a}} = x_{\boldsymbol{a}+\boldsymbol{b}}$ shows that $S_{\boldsymbol{a}}S_{\boldsymbol{b}} = S_{\boldsymbol{a}+\boldsymbol{b}}$. The polynomial ring $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ together with this $\mathbb{Z}^n$-grading is called the polynomial ring with fine grading.

Definition 1.2.9. Let $S = \bigoplus_{a\in\mathbb{Z}^n} S_a$ be a $\mathbb{Z}^n$-graded ring. An S-module M is called $\mathbb{Z}^n$-graded if $M = \bigoplus_{\mathbb{Z}^n} M_a$ and $S_a.M_b = M_{a+b}$ for all $a, b \in \mathbb{Z}^n$.

Let M be a $\mathbb{Z}^n$-graded S-module and N be a submodule of M. Then N is also $\mathbb{Z}^n$-graded S-module. In this case, the factor module M/N inherits a natural $\mathbb{Z}^n$-grading with components $(M/N)_a = M_a/N_a$ for all $a \in \mathbb{Z}^n$. Before proceeding further, we shall consider some examples of graded modules.

Example 1.2.10. (i). Consider the polynomial ring $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ over a field $\mathbb{K}$ with standard degree grading. Let $g_1, g_2, \ldots g_r$ be homogeneous polynomials in S and consider the ideal $I = \langle g_1, g_2 \ldots g_r$ generated by the homogeneous polynomials. Then the ideal I is a Z-graded S-module. In fact, every ideal of S is finitely generated by Hilbert Basis theorem. Also, the quotient ring $S = I$ has a natural structure as Z-graded S-module.

(ii). Consider the polynomial ring $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ with $\mathbb{Z}^n$-grading. Then S is a $\mathbb{Z}^n$-graded module over itself. Let $I \subseteq S$ be an ideal generated by finitely many monomials in S. Then I is a $\mathbb{Z}^n$-graded S-module. In fact, every $\mathbb{Z}^n$-graded submodule of S is generated by finitely many monomials in S by Hilbert Basis Theorem. The quotient ring S/I has natural $\mathbb{Z}^n$-grading given by $(S/\boldsymbol{I})_{\boldsymbol{a}} = S_{\boldsymbol{a}}/\boldsymbol{I}_{\boldsymbol{a}}$ for each $a \in \mathbb{Z}^n$.

Definition 1.2.11. Let I be a graded ideal of (R, G). Then :

(i). I is a graded prime ideal if $I \neq R$, and whenever $rs \in I, r \in I$ or $s \in I$, where $r, s \in h(R)$.

(ii). I is a graded maximal ideal if $I \neq R$ and there is no graded J of (R, G) such that $I \subset J \subset R$.

(iii). The graded radical of I is the set of all $x \in R$ such that for each $g \in G$ there exist $n_g > 0$ with ${x_g}^{n_g} \in I$. Note that, if r is a homogenous element of (R, G), then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in N$.

Definition 1.2.12. Let I be a graded ideal of (R, G). Then say that I is a graded primary ideal of (R, G) if $I \neq R$; and whenever $a, b \in h(R)$ with $ab \in I$ then $a \in I$ or $b \in I$ or $b \in Gr(I)$.

Example 1.2.13. Let $R = \mathbb{Z}[i]$ (The Gaussian integers) and let $G = \mathbb{Z}_2$. Then R is a G-graded ring with $R_0 = \mathbb{Z}, R_1 = i\mathbb{Z}$. Let $I = 2R$ be a graded prime ideal. Then I is a graded primary ideal. But I is not a primary ideal because 2 is not irreducible element of $R = \mathbb{Z}[i]$.

Definition 1.2.14. Let I be a proper graded ideal of (R, G). A graded primary G-decomposition of I is an intersection of finitely many graded primary ideals of (R, G). Such a graded primary G-decomposition $I = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ with $Gr(Q_i) = P_i$ for $i = 1, 2, \ldots, n$ of I is said to be minimal graded primary G-decomposition of I precisely when

(i). $P_1, \ldots, P_n$ are different graded prime ideals of R, and

(ii). $Q_j \not\supseteq \bigcap_{(i=1, j\neq i)} Q_i$ for all $j = 1, \ldots, n$. Say I is G-decomposable graded ideal of (R, G) precisely when it has a graded primary G-decomposition.

1.3 Monomial Ordering

Let $\mathbb{K}$ be a field and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring. We set $\boldsymbol{x}^a = x^{a_1}x^{a_2}\ldots x^{a_n}$ for $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The monomials in

$$\mathcal{P} = \{\boldsymbol{x}^{\boldsymbol{a}}/\boldsymbol{a} \in \mathbb{N}^n\}$$

are called the terms.

Definition 1.3.1. A total order $>$ of $\mathcal{P}^n$ is called a monomial order or term order if

(a) $p > I$ for all $p \in \mathcal{P}^n$, and

(b) for all $p, q, r \in \mathcal{P}^n, q > p$ implies $qr > pr$.

Definition 1.3.2. A monomial ordering $>$ on $\mathbb{K}[x_1, \ldots, x_n]$ is a relation $>$ on $\mathbb{Z}^n \geq 0$, or equivalently, a relation on the set of monomials $\boldsymbol{x}^\alpha, \alpha \in \mathbb{Z}^n \geq 0$, satisfying:

(i). $>$ is a total (or linear) ordering on $\mathbb{Z}^n \geq 0$.

(ii). If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n \geq 0$, then $\alpha + \gamma > \beta + \gamma$.

(iii). $>$ is a well-ordering on $\mathbb{Z}^n \geq 0$. This means that every nonempty subset of $\mathbb{Z}^n \geq 0$ has a smallest element under $>$. In other words, if $A \subseteq \mathbb{Z}^n \geq 0$ is nonempty, then there is $\alpha \in A$ such that $\beta > \alpha$ for every $\beta \neq \alpha$ in A.

Definition 1.3.3. (Lexicographic Order). Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be in $\mathbb{Z}^n \geq 0$. We say $\alpha >_{lex} \beta$ if the leftmost nonzero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive. We will write $x^\alpha >_{lex} x^\beta$ if $\alpha >_{lex} \beta$. Here are some examples:

(a). $(1, 2, 0) >_{lex} (0, 3, 4)$ since $\alpha - \beta = (1, -1, -4)$.

(b). $(3, 2, 4) >_{lex} (3, 2, 1)$ since $\alpha - \beta = (0, 0, 3)$.

(c). The variables $x_1, \dots, x_n$ are ordered in the usual way by the lex ordering:

$(1, 0, \dots, 0) >_{lex} (0, 1, 0, \dots, 0) >_{lex} \cdots >_{lex} (0, \dots, 0, 1)$.

so $x_1 >_{lex} x_2 >_{lex} \cdots >_{lex} x_n$.

Definition 1.3.4. (Graded Lex Order). Let $\alpha, \beta \in \mathbb{Z}^n > 0$. We say $\alpha >_{grlex} \beta$ if $|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i$, or $|\alpha| = |\beta|$ and $\alpha >_{lex} \beta$. We see that *grlex* orders by total degree first, then "break ties" using *lex* order.

Here are some examples:

(a). $(1, 2, 3) >_{grlex} (3, 2, 0)$ since $|(1, 2, 3)| = 6 > |(3, 2, 0)| = 5$.

(b). $(1, 2, 4) >_{grlex} (1, 1, 5)$ since $|(1, 2, 4)| = |(1, 1, 5)|$ and $(1, 2, 4) >_{lex} (1, 1, 5)$.

(c). The variables are ordered according to the lex order, i.e., $x_1 >_{grlex} \cdots >_{grlex} x_n$.

Definition 1.3.5. (Graded Reverse Lex Order). Let $\alpha, \beta \in \mathbb{Z}^n \geq 0$. We say $\alpha >_{grevlex} \beta$ if $|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i$, or $|\alpha| = |\beta|$ and the rightmost nonzero entry of $\alpha - \beta \in \mathbb{Z}^n$ is negative. Like *grlex*, *grevlex* orders by total degree, but it "breaks ties" in a different way.

For example:

(a). $(4, 7, 1) >_{grevlex} (4, 2, 3)$ since $|(4, 7, 1)| = 12 > |(4, 2, 3)| = 9$.

(b). $(1, 5, 2) >_{grevlex} (4, 1, 3)$ since $|(1, 5, 2)| = |(4, 1, 3)|$ and $(1, 5, 2) - (4, 1, 3) = (-3, 4, -1)$.

Note also that *lex* and *grevlex* give the same ordering on the variables. That is,

$(1, 0, \dots, 0) >_{grevlex} (0, 1, \dots, 0) >_{grevlex} \cdots >_{grevlex} (0, \dots, 0, 1)$

or $x_1 >_{grevlex} x_2 >_{grevlex} \cdots >_{grevlex} x_n$

1.4 Monomial Ideals and Simplicial Complexes

In this section, we introduce monomial ideals and combinatorics associated with them. An ideal I which is generated by monomials, a so called monomial ideal, also has a K-basis of monomials. As a consequence, a polynomial f belongs to I if and only if all monomials in f appearing with a nonzero coefficient belong to I. This is one of the reasons why algebraic operations with monomial ideals are easy to perform and are accessible to combinatorial and convex geometric arguments.

Definition 1.4.1. An ideal I of R is called a monomial ideal if there is $\mathcal{A} \in \mathbb{N}^n$ such that I is generated by $\{\boldsymbol{x}^\alpha/\alpha \in \mathcal{A}\}$. If I is a monomial ideal the quotient ring R/I is called a monomial ring.

Note that a monomial ideal is always generated by a finite set of monomials.

Example 1.4.2. Consider the polynomial ring $S = \mathbb{K}[x, y]$ in two variables, then ideal $I = <x^5, x^3y^2, x^2y^3, y^4>$ is a monomial ideal. Note that monomial ideals may contain elements other than monomials.

Definition 1.4.3. Let $I \subseteq S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be a monomial ideal. Then the graph of I, denoted by $\Omega(I)$, is defined as :

$$\Omega(I) = \{\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n \mid \boldsymbol{x}^{\boldsymbol{a}} \in I\}; \text{ where } a_i \geq 0.$$

Example 1.4.4. Consider the polynomial ring $S = \mathbb{K}[x, y]$, then the graph of the monomial ideal $I = (x^5, x^3y^2, x^2y^3, y^4)$ is pictorially given below :

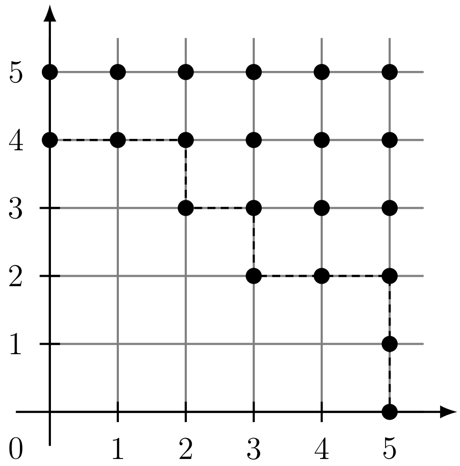

Figure 1.1: Graph of $I = (x^5, x^3y^2, x^2y^3, y^4)$.

Definition 1.4.5. A monomial $\boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1}x_2^{a_2}\dots x_n^{a_n} \in S = \mathbb{K}[x_1, x_2, \dots x_n]$ is called a squarefree monomial if each coordinate of the vector $\boldsymbol{a} \in \mathbb{Z}^n$ is either 0 or 1, that is, $\boldsymbol{a} \in \{0, 1\}$. An ideal I in S is called squarefree monomial ideal if it is generated by squarefree monomials.

Definition 1.4.6. Let $[n] = \{1, 2, \dots n\}$. A simplicial complex Δ on $[n]$ is a collection of subsets of $[n]$, called faces, such that

(i). $\{i\} \in \Delta$ for all $i \in [n]$ and

(ii). if $X \in \Delta$ and $Y \subseteq X$ then $Y \in \Delta$, that is, Δ is closed under taking the subsets. We call the set $[n] = 1, 2, \dots, n$ as the vertex set denoted by $'V'$ of the simplicial complex Δ.

If Δ is a simplicial complex on $[n]$ and every subset of $[n]$ belongs to Δ, then Δ is called a simplex. A face $X \in \Delta$ of cardinality $|X| = m + 1$ is called a face of dimension m or an m-face of Δ. Therefore the empty set ; in Δ known as empty face in any simplicial complex Δ is of dimension -1. The dimension of Δ, denoted by $\dim(\Delta)$, is the maximum of dimension of all its faces,that is,

$$\dim(\Delta) = \max\{\dim(X) : X \in \Delta\}$$

. In case $\Delta = \{\}$ is a void complex meaning which has no face, we take $\dim(\Delta) = -\infty$. A maximal face of a simplicial complex Δ (with respect to inclusion) is called a facet. We call a simplicial complex pure if all its facets have same cardinality. We denote the set of facets of a simplicial complex Δ by $\mathcal{F}(\Delta)$ and the set of minimal non-faces by $\mathcal{N}(\Delta)$. Note that a simplicial complex Δ is completely determined by $\mathcal{F}(\Delta)$.

For a face $X \in \Delta$, we define the link, denoted by $lk_\Delta(X)$, of X as follows :

$$lk_\Delta(X) = \{Y \in \Delta\} | X \cup Y \in \Delta; X \cap Y = \phi.$$

From the definition of link, it follows that $lk_\Delta(X)$ is a simplicial complex on the vertex set $[n]$ X and $lk_\Delta(X) = \Delta$.

Example 1.4.7. Let Δ be a simplicial complex on the vertex set such that $\mathcal{F}(\Delta) = \{\{v_1, v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_6\}\}$. Then Δ can be represented pictorially given as follows :

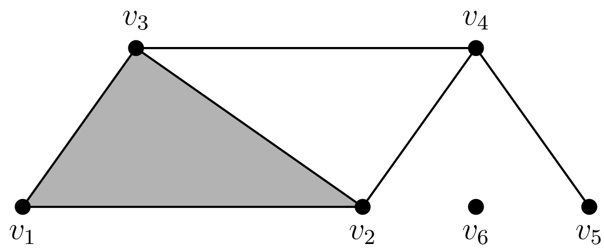

Figure 1.2: $\mathcal{F}(\Delta)$.

Definition 1.4.8. Let Δ be a simplicial complex on the vertex set $[n]$. We denote the number of 'l-dimensional faces of Δ by f_l. The sequence $f(\Delta) = (f_0, f_1, \dots f_{d-1})$ is called the $f - vector$ of Δ, where $\dim(\Delta) = d - 1$. In particular, we have $f_{-1} = 1, f_0 = n$.

Example 1.4.9. The f-vector of the simplicial complex Δ given in Example 1.4.7 is given by

$$f(\Delta) = (6, 6, 1)$$

By definition of simplicial complexes, we can specify Δ by its facets. Squarefree monomials are determined by simplicial complexes. To each subset $F \in [n]$, we correspond a squarefree monomial $\boldsymbol{x}^F = \prod_{i \in F} x^i$.

Definition 1.4.10. (i) The *Stanley − Reisner* ideal of a simplicial complex Δ is a squarefree monomial ideal I_Δ of S generated by monomials that correspond to *nonfaces* of Δ.

From this definition, we note that if $F = \{i_1, i_2, \dots i_q\}$ is a nonface of Δ, then every subset of $[n]$ containing F is also a nonface of Δ. Thus I_Δ is generated by squarefree monomials $\{x_{i_1}, x_{i_2}, \dots x_{i_q}\}$ corresponding to minimal nonfaces $i_1, i_2, \dots i_q$ of Δ. Therefore, we write

$$I_\Delta = \langle \boldsymbol{x}^F \mid F \in \mathcal{N}(\Delta) \rangle$$

The *Stanley − Reisner* ring of Δ is the quotient ring $\mathbb{K}[\Delta] = S/I_\Delta$. We can write a squarefree monomial ideal in two ways: either by its generators or as an intersection of monomial prime ideals. Monomial prime ideals are generated by subsets of $\{x_1, x_2, \dots, x_n\}$. A monomial prime ideal $\boldsymbol{m}^F$ corresponding to $F \subseteq [n]$ is given below

$$\boldsymbol{m}^F = \langle x_i \mid i \in F \rangle.$$

The following theorem describes the *Stanley − Reisner* ideal I_Δ as an intersection of monomial prime ideals in $S = \mathbb{K}[x_1, x_2, \dots, x_n]$

(ii) The *facet − ideal* of Δ is defined as :

$$\mathcal{F}(\Delta) = \prod_{x \in F} (x : \text{F is a facet of } \Delta).$$

Theorem 1.4.11. *There is a one to one correspondence between the simplicial complexes on the vertex set $[n]$ and the squarefree monomial ideals in the polynomial ring $S = \mathbb{K}[x_1, x_2, \dots, x_n]$. Moreover,*

$$I_\Delta = \bigcap_{X \in \mathcal{F}(\Delta)} \boldsymbol{m}^{\bar{X}} = \bigcap_{X \in \mathcal{F}(\Delta)} \langle x_i \mid i \notin X \rangle$$

Proof. We have

$$I_\Delta = \langle \boldsymbol{x}^F \mid F \notin \Delta \rangle$$

Therefore, the set of squarefree monomials in S that have nonzero image in the *Stanley−Reisner* ring $S = I_\Delta$ is precisely $\{\boldsymbol{x}^G | G \in \Delta\}$. Since every set has its complement, one can see that the map $\Delta \longrightarrow I_\Delta$ is a bijection. Next we note that

$$\begin{aligned}
\boldsymbol{x}^F &= \bigcap_{X \in \mathcal{F}(\Delta)} \boldsymbol{m}^{\bar{X}} \\
&\Leftrightarrow F \cap \bar{X} \neq \phi \\
&\Leftrightarrow F \text{ must be contained in no face of } X, \\
&\Leftrightarrow F \text{ is nonface of } \Delta.
\end{aligned}$$

This completes the proof □

Example 1.4.12. Consider the simplicial complex Δ given in Example 1.4.3, we can write it as intersection of monomial prime ideals as given below

$$I_\Delta = (x_4, x_5, x_6) \cap (x_1, x_3, x_5, x_6) \cap (x_1, x_2, x_5, x_6) \cap (x_1, x_2, x_3, x_6) \cap (x_1, x_2, x_3, x_4, x_5).$$

Chapter 2

Combinotorial Commutative Algebra

2.1 Free Resolutions

An important homological tool for studying the modules of a commutative ring is the minimal free resolution of a module. These objects encode much of the information about the structure of the module as well as containing several important numerical invariants of the module.

Definition 2.1.1. Consider the polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ with standard or fine grading. Let M be a finite graded R-module. A *graded free resolution* of M is a complex $\mathcal{F}$ of graded R-modules and graded homomorphisms

$$\mathcal{F}: \quad \cdots \longrightarrow F_\ell \xrightarrow{\psi_\ell} F_{\ell-1} \xrightarrow{\psi_{\ell-1}} \cdots \longrightarrow F_1 \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} M \xrightarrow{\psi_{-1}} 0$$

such that the following properties are satisfied:

1. F_i is a finite graded free R-module and ψ_i is homogeneous, for each $i \geq 0$,

2. $\mathcal{F}$ is *exact*, that is, $im(\psi_i) = \ker(\psi_{i-1})$, $\forall\ i \geq 0$.

The resolution $\mathbb{M}$ is said to be a *finite* free resolution of length ℓ if there exists a minimal index ℓ such that $F_i = 0$ for all $i > \ell$.

Example 2.1.2. Let $I = (x^2, y^3, z^6) \in Q[x, y, z] = R$. To resolve R/I, we must first find the kernel of the map $\phi_1 : R^3 \longrightarrow R$ is given by the matrix $[x^2\ y^3\ z^6]$ whose image is I. In this case, we can do this by eyeballing, and so we get $\phi_2 : R^3 \longrightarrow R^3$ is given by

$$\begin{bmatrix} y^3 & 0 & z^6 \\ -x^2 & z^6 & 0 \\ 0 & -y^3 & -x^2 \end{bmatrix}$$

Then, we may compute the kernel of the above map, to find $\phi_3 : R \longrightarrow R^3$ is $[z^6\ \ -x^2\ y^3]^T$. Then we have successfully computed this free resolution by repeatedly computing kernels of maps.

Example 2.1.3. Let $I = (x^2, xy, y^3)$. I is an ideal in the polynomial ring $R = \mathbb{K}[x, y]$. The following are both free resolutions of R/I

$$\mathcal{F}_1 :\ 0 \to R^2 \xrightarrow{\begin{bmatrix} -y & 0 \\ x & -y^2 \\ 0 & x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} R \to 0$$

$$\mathcal{F}_2 :\ 0 \to R \xrightarrow{\begin{bmatrix} y^2 \\ x \\ -1 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} -y & 0 & -y^3 \\ x & -y^2 & 0 \\ 0 & x & x^2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} R \to 0$$

In order to avoid the problem of the non-uniqueness of free resolutions we introduce the concept of minimal free resolutions.

Definition 2.1.4. Let M be an R-module. A free resolution F of M over R is said to be minimal if the ranks of the free modules in F are less than or equal to the ranks of the corresponding free modules in an arbitrary free resolution of M over R.
One forms a minimal free resolution of an R-module M by choosing a minimal set (with respect to inclusion) of generators and then a minimal set of relations on those generators, and a minimal set of relations on the relations on the generators, etc.

Furthermore, the minimal graded free resolution of graded module is unique up to isomorphism of complexes in the sense that if $\mathcal{F}$ and $\mathcal{G}$ are two minimal graded free resolutions of M, then there exist graded isomorphisms $\alpha_i : F_i \longrightarrow G_i$ such that every square in the diagram

$$\begin{tikzcd}
\mathcal{F} : \cdots \arrow[r, "\psi_2"] & F_1 \arrow[r, "\psi_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "\psi_0"] \arrow[d, "\alpha_0"] & M \arrow[d, "\mathrm{Id}_M"] \\
\mathcal{G} : \cdots \arrow[r, "\phi_2"] & G_1 \arrow[r, "\phi_1"] & G_0 \arrow[r, "\phi_0"] & M
\end{tikzcd}$$

commutes, that is, $\alpha_{i-1} \circ \psi_i = \phi_i \circ \alpha_i$, for each i.

Proposition 2.1.5. *Let M be an R-module and F a free resolution of M over R. Then F is the minimal free resolution of M over R if and only $d_i(F_i) \subseteq (x_1, \ldots, x_n)F_{i-1}$.*

Example 2.1.6. We can see by examining the differentials of $\mathcal{F}_1$ and $\mathcal{F}_2$ in Example 2.1.3 that $\mathcal{F}_1$ is minimal while $\mathcal{F}_2$ is not.

Lemma 2.1.7 (Graded Nakayama's lemma). *Let $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be a polynomial ring with standard degree grading. Let M be a finitely generated $\mathbb{Z}$-graded R-module and N be a submodule of M. Let $\mathfrak{m}$ be the unique graded maximal ideal of R such that $M = N + \mathfrak{m}M$, then $M = N$.*

Lemma 2.1.8. *Let $\{m_1, m_2, \ldots, m_k\}$ be a set of homogeneous generators of a finite $\mathbb{Z}$-*

graded R-module M with a presentation

$$0 \longrightarrow \ker(\phi) \xrightarrow{i} R^k \xrightarrow{\phi} M \longrightarrow 0, \text{ where } \phi : e_i \longmapsto m_i.$$

Then the set $\{m_1, m_2, \ldots, m_k\}$ is a minimal generating set of M if and only if $\ker(\phi) \subseteq \mathfrak{m}R^k$*, where $\mathfrak{m} = (x_1, x_2, \ldots, x_k)$ is the unique graded maximal ideal of R.*

Proof. Suppose $\ker(\phi) \subseteq \mathfrak{m}R^k$. It suffices to show that the set $\{\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_k\} \subseteq M/\mathfrak{m}M$ is linearly independent over $\mathbb{K}$. For, let $\sum\limits_{i=1}^{k} \lambda_i \overline{m}_i = 0$. Then $\sum\limits_{i=1}^{k} \lambda_i (m_i + \mathfrak{m}M) = \mathfrak{m}M$ implies that $\sum\limits_{i=1}^{k} \lambda_i m_i \in \mathfrak{m}M$ and so $\sum\limits_{i=1}^{k} \lambda_i m_i = \sum\limits_{i=1}^{k} a_i m_i$, where $a_i \in \mathfrak{m}$. Therefore, $(\lambda_1, \lambda_2, \ldots, \lambda_r) - (a_1, a_2, \ldots, a_k) \in \ker(\phi) \subseteq \mathfrak{m}R^k$, for some $a_1, a_2, \ldots, a_k$ in $\mathfrak{m}$. Hence, $\lambda_i = 0$ for all i.

Conversely, suppose $\{m_1, m_2, \ldots, m_k\}$ is a minimal set of generators for M and suppose $\ker(\phi) \nsubseteq \mathfrak{m}R^k$. Then there exist some $f = (f_1, f_2, \ldots, f_k) \in \ker(\phi)$ such that $f \notin \mathfrak{m}R^k$. This implies f_i is not in $\mathfrak{m}$, for some i and hence $f_i \in \mathbb{K}$. Therefore, we have

$$f_1 m_1 + \cdots + f_i m_i + \cdots + f_k m_k = 0.$$

That is,

$$m_i = -f_i^{-1}(f_1 m_1 + \cdots + f_{i-1} m_{i-1} + f_{i+1} m_{i+1} + \cdots + f_k m_k).$$

This shows that $\{m_1, m_2, \ldots, m_k\}$ is not a minimal generating set for M, which is a contradiction. Hence, $\ker(\phi) \subseteq \mathfrak{m}R^k$. □

Definition 2.1.9. (*Hilbert functions*). Let $R = \bigoplus R_n$ be a graded ring, and $M = \bigoplus M_n$ a graded R-module. Assume R_0 is Artinian, R is a finitely generated R_0-algebra, and M is a finitely generated R-module. Then each M_n is a finitely generated R_0-module, so is of finite length $\ell(M_n)$. We call $n \longmapsto \ell(M_n)$ the Hilbert Function of M and its generating function $\boldsymbol{H(M,t) = \sum_{n\in\mathbb{Z}} \ell(M_n) t^n}$, the Hilbert Series of M. This series is a rational function.

If $R = R_0[x_1, \ldots, x_r]$ with $x_i \in R_1$, then the Hilbert Function is, for $n >> 0$, a polynomial $h(M, n)$, is called the Hilbert Polynomial of M.

Example 2.1.10. Let $R = \mathbb{K}[x, y, z, w]$, $S = \mathbb{K}[z, w]$, $I = (x^2, y^2, xz - yw)$ and let $M = R/I$, the Hilbert function and polynomial of R/I is given by:

$\mathrm{Hilb}_M(0) = 1, \mathrm{Hilb}_M(1) = 4$, $\mathrm{Hilb}_M(2) = 7$, $\mathrm{Hilb}_M(3) = 8, \cdots, \mathrm{Hilb}_M(d) = \mathrm{Hilb}_S(d-1) + \mathrm{Hilb}_S(d) + 1 = \binom{d}{d-1} + \binom{d+1}{d} + 1 = 2d + 2$.

Remark 2.1.11. The Hilbert Function of a graded module lists the lengths of its components. The corresponding generating function is called the Hilbert Series.We now state a theorem which shows that a free resolution of finite length always exist.

Theorem 2.1.12 (Graded Hilbert's Syzygy Theorem)**.** *Let $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be a polynomial ring with standard (or fine) grading. Then every finite graded R-module M has a minimal graded free resolution of length at most n.*

Definition 2.1.13. Let M be a graded R-module. Then M has a graded S-resolution of the form

$$\cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{ij}} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{1j}} \longrightarrow \bigoplus_j R(-j)^{\beta_{0j}} \longrightarrow M \longrightarrow 0.$$

The numbers β_{ij} are called the graded Betti numbers of M.

For a finite graded R-module, the set of graded Betti numbers of M is usually arranged as rectangular array of non-negative integers, called the *Betti diagram* of M where the number of non-zero columns correspond to the length of the minimal graded free resolution of M, whereas the rows correspond to the degrees of the generators of syzygy modules of M. The Betti diagram of M is given given Figure **??**.

$$\begin{array}{rccccc} & 0 & 1 & \cdots & i & \cdots \\ \text{total:} & - & - & & - & - \\ 0: & & & & & \\ \vdots & & & & \vdots & \\ j: & & & \cdots & \beta_{i,i+j}(M) & \cdots \\ \vdots & & & & \vdots & \end{array}$$

Figure 2.1: The Betti diagram of M.

The entry at ith column and jth row is the ith graded Betti number of M in degree $i+j$ written as $\beta_{i,i+j}(M)$

Example 2.1.14. In the ring $R = \mathbb{K}[x, y, z, w]$, let

$$I = (x^2, xy, xz, y^3, y^2z, yz^2, z^3).$$

The corresponding *revlex* ideal is

$$C = (x^2, xy, y^2, xz^2, yz^2, z^3, xzw)$$

The Betti numbers of I and C are

$$\beta_0(I) = 7,\ \beta_1(I) = 10,\ \beta_2(I) = 4$$

$$\beta_0(C) = 7,\ \beta_1(C) = 11,\ \beta_2(C) = 6,\ \beta_3(C) = 1$$

.

Example 2.1.15. Let $I = (x^2, xy, y^3)$. I is an ideal in the polynomial ring $R = \mathbb{K}[x, y]$. We saw in the Example 2.1.3 that the minimal free resolutions of R/I over R is

$$\mathcal{F}:\ 0 \to R^2 \xrightarrow{\begin{bmatrix} -y & 0 \\ x & -y^2 \\ 0 & x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} R \to 0$$

The total Betti numbers of R/I over R are then

$$\beta_0(R/I) = 1,\ \beta_1(R/I) = 3,\ \beta_2(R/I) = 2:$$

If we want the graded Betti numbers of R/I we must look at the graded minimal free resolution of R/I.

$$0 \longrightarrow R(-3)\bigoplus R(-4) \longrightarrow S(-2)^2 \bigoplus R(-3) \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

The graded Betti numbers of R/I are given by the Betti table

$$\beta(R/I) = \begin{bmatrix} 1 & \_ & \_ \\ \_ & 2 & 1 \\ \_ & 1 & 1 \end{bmatrix}$$

We can also describe Betti numbers in terms of Tor. For an $\mathbb{N}^n$-graded module M over $R = \mathbb{K}[x_1, \ldots, x_n]$, tensoring a minimal free resolution of M,

$$0 \longrightarrow R^{\beta_h} \longrightarrow \cdots \longrightarrow R^{\beta_1},$$

with $\mathbb{K}$ (considered as an R-module via $\mathbb{K} \cong R/\langle x_1, \ldots, x_n\rangle$) produces a complex

$$0 \longrightarrow \mathbb{K}^{\beta_h} \longrightarrow \cdots \longrightarrow \mathbb{K}^{\beta_1}$$

where all the maps are zero. Taking ith homology of this complex we obtain $\beta_i(M) = \beta_i^{\mathbb{K}}(M) = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(\mathbb{K}, M)$. For a (simple, finite) graph G we may write the minimal free resolution of $\mathbb{K}[\Delta(G)]$ as

$$0 \longrightarrow R^{\beta_h} \longrightarrow \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0}$$

where $R = R_{\mathbb{K}}(G), \beta_i(\mathbb{K}[\Delta(G)]) = \dim_{\mathbb{K}}(Tor_i^R(\mathbb{K}[\Delta(G)], \mathbb{K}))$ and $\beta_h \neq 0$. Note that $(\mathbb{K}[\Delta(G)]) = 1$. Observe that here we have $\operatorname{pd}(\mathbb{K}(G)) = \operatorname{pd}(\mathbb{K}[\Delta(G)]) = h$ (and so $\operatorname{pd}^{\mathbb{K}}(I(G)) = h - 1$). Note also that $\operatorname{Tor}_i^R(\mathbb{K}[\Delta(G)], \mathbb{K})$ inherits the $\mathbb{N}^n$- grading. For $a \in \mathbb{N}^n$let $(\operatorname{Tor}_i^R(\mathbb{K}[\Delta(G)], k))$ a denote the a graded component. This allows us to define $\mathbb{N}^n$-graded Betti numbers and $\mathbb{N}$-graded Betti numbers which will in some circumstances be easier to handle than the total Betti numbers.

Definition 2.1.16. For a finite graded module M over the polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$, the *Castelnuovo-Mumford regularity* (or simply the *regularity*) of M, denoted by $\operatorname{reg}(M)$, is defined as

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

The Castelnuovo-Mumford regularity, roughly speaking, measures the width of the minimal graded free resolution of a finite graded $R-$module.

Definition 2.1.17. The *projective dimension* of a finite graded module M over the polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$, denoted by $\operatorname{pd}(M)$, is the length of its minimal graded free resolution. In other words,

$$\operatorname{pd}(M) = \max\{i \mid \beta_{i,j}(M) \neq 0 \text{ for some } j\}.$$

Remark 2.1.18. Comparing free resolutions of R/I and I for ideal I as R-modules, we have $\operatorname{pd}(R/I) = pd(I) + 1$ and $reg(R/I) = reg(I) - 1$

Definition 2.1.19. Homology of the complex

$$\mathcal{F} : \quad \cdots \longrightarrow F_2 \xrightarrow{\psi_2} F_1 \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} 0$$

is the collection of R-modules.

We should think of $\psi_0 = 0$, so $H_0(\mathcal{F}) = \ker(\psi_0)/\operatorname{im}(\psi_1) = \operatorname{coker}(\psi_1)$

Apart from Graded Betti numbers, there is an another important homological invariant in commutative algebra and algebraic geometry, known as Castelnuovo-Mumford regularity which measures the computational complexity of ideals, modules and sheaves. Having lower Castelnuovo-Mumford regularity is equivalent to having "simpler" minimal free resolution in some sense. In the case of monomial ideals related to graphs one is interested to find classes of graphs whose edge ideals or their powers have minimal regularity. The regularity of graded modules over the polynomial ring R can be defined in various ways.

Definition 2.1.20. Let a be an element of the ring R and $\mathcal{K}(a)$ be the complex defined as

$$\mathcal{K}_i = \begin{cases} R & \text{for } i = \{0,1\}, \\ 0 & \text{otherwise,} \end{cases}$$

with $d_1 : \mathcal{K}_1(a) \longrightarrow \mathcal{K}_0(a)$ being multiplication by a. Let I be an ideal of R generated by the sequence $\boldsymbol{x} = \{x_1, \ldots, x_n\}$. The ordinary Koszul complex associated to z is defined as $\mathcal{K}(\boldsymbol{x}; R) = \mathcal{K}(x_1) \otimes \cdots \otimes \mathcal{K}(x_n)$.

For an R-module M we shall write $\mathcal{K}(\boldsymbol{x}; M)$ for $\mathcal{K}(\boldsymbol{x}; R) \otimes M$. The Koszul complex $\mathcal{K}(\boldsymbol{x}; R)$ is then the exterior algebra complex associated to $E = R^n$ and the map

$$\theta : E \longrightarrow R,$$

defined as

$$\theta(z_1, \ldots, z_n) = z_1 x_1 + \cdots z_n x_n.$$

That is, θ defines a differential $\partial = \partial\theta$ on the exterior algebra $\bigwedge(E)$ of E given in degree r by

$$\partial(e_1 \wedge \cdots \wedge e_r) = \sum_{i=1}^{r} (-l)^{i-l} \theta(e_i) e_1 \wedge \cdots \wedge \tilde{e_i} \wedge \cdots \wedge e_r.$$

A consequence of the definition of the differential of $\mathcal{K}(\boldsymbol{x}; R)$ is that if w and w' are homogeneous elements of $\bigwedge(E)$, of degrees p and q respectively, then

$$\partial(w \wedge w') = (-l)^p w \wedge \partial(w') + \partial(w) \wedge w'.$$

This implies that the cycles $\mathcal{Z}(\mathcal{K})'$ form a subalgebra of $\bigwedge(E)$, and that the boundaries $\mathcal{B}(\mathcal{K})$ form a two-sided ideal of $\mathcal{Z}(\mathcal{K})$. As a consequence the homology of the Koszul complex, $\mathcal{H}(\boldsymbol{x})$, inherits a skew commutative R- algebra structure. One can also see that $\mathcal{H}(\boldsymbol{x})$ is annihilated by $I = (\boldsymbol{x})$. Indeed, if $e \in E$ and $w \in \mathcal{Z}_r(\mathcal{K})$, we have from the last formula $\partial(e \wedge w) = \theta(e)w$. The ordinary Koszul complex $\mathcal{K}(\boldsymbol{x}) = \mathcal{K}(\boldsymbol{x}; R)$ is simply the complex of free modules

$$\mathcal{K}(\boldsymbol{x}) : 0 \longrightarrow \wedge^n R^n \longrightarrow \wedge^{n-1} R^n \longrightarrow \cdots \longrightarrow \wedge^1 R^n \longrightarrow \wedge^0 R^n$$

where $\wedge^k R^n$ is the *kth* exterior power of R^n; thus $\wedge^k R^n$ is a free R-module of *rank* $\binom{n}{k}$ with basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_{i_k} \leq n\}.$$

Proposition 2.1.21. *The Koszul complex $\mathcal{K}$ is a minimal free resolution of $\mathbb{k} = S/\mathfrak{m}$ for the maximal ideal $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$.*

2.2 Simplicial Homology

Let's recall the definition of Simplicial complexes Δ, in which the set V is called the *vertex set* of the simplicial complex Δ, whereas its elements are called the *vertices* of Δ. The elements of Δ are called the *faces* of Δ. If Δ is a simplicial complex on V and every subset of V belongs to Δ, then Δ is called a *simplex.* A face $F \in \Delta$ of size $|F| = d + 1$ is called a face of dimension d or a *d-face.* However, for the empty set $\emptyset$ as an element of Δ, known as *empty face*, we adopt the convention that $\dim \emptyset = -1$. A face F in Δ is said to be a *maximal face* if it is maximal with respect to inclusion. A maximal face in a simplicial complex is also called a *facet.* We denote the set of all facets of a simplicial complex Δ by $\mathcal{F}(\Delta)$. Note that a simplicial complex Δ is completely determined by $\mathcal{F}(\Delta)$. The dimension of Δ, denoted by $\dim(\Delta)$, is the maximum of dimension of all its faces, that is,

$$\dim(\Delta) = \max\{\dim(F) \mid F \in \Delta\}.$$

In case $\Delta = \{\}$ is a void complex (which has no face), we take $\dim(\Delta) = -\infty$. We call a simplicial complex *pure* if all its facets have same size. We denote by $\text{comp}(\Delta)$, the set of all connected components of Δ. The set of minimal non-faces is denoted by $\mathcal{N}(\Delta)$.

If Δ is a simplicial complex on a vertex set V, then we call a simplicial complex Δ' a *subcomplex* of Δ if $\Delta' \subset \Delta$ and we write $\Delta' < \Delta$. However, if $W \subset V$, then the *induced* *subcomplex* on W, denoted by Δ_W, is the subcomplex of Δ given as

$$\Delta_W = \{F \in \Delta \mid F \subset W\}.$$

We denote $\Delta^i = \{F \in \Delta \mid \dim(F) = i\}$ and $\Delta^{(i)} = \{F \in \Delta \mid \dim(F) \leq i\}$. Note that Δ^i is not the simplicial complex, whereas $\Delta^{(i)}$ is a simplicial complex and is called the *i-skeleton* of Δ.

Definition 2.2.1. Let Δ be the simplicial complex on the vertex set V. The *Alexander dual* of Δ, denoted by Δ^*, is the simplicial complex over V consisting of all those faces which are complements of non-faces of Δ. That is,

$$\Delta^* = \{V \setminus F \mid F \in \mathcal{N}(\Delta)\}.$$

Example 2.2.2. The simplicial complex Δ with $V(\Delta) = \{x_1, x_2, x_3, x_4\}$ and $Facets(\Delta)$ are $\{x_1, x_2, x_3\}, \{x_2, x_4\}$ and $\{x_3, x_4\}$ is pictured below.

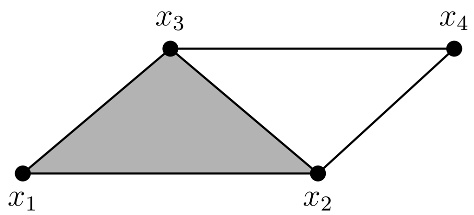

Figure 2.2: A simplicial complex with three facets.

Example 2.2.3. Consider the simplicial complex Δ in above Example 2.2.2 $\mathcal{N}(\Delta)$ and $\mathcal{F}(\Delta)$ have been computed below.

Here $\mathcal{N}(\Delta) = (x_4) \cap (x_2, x_3) \cap (x_1, x_2) = (x_1x_4, x_2x_3x_4)$ and

$\mathcal{F}(\Delta) = (x_1x_2x_3, x_2x_4, x_3x_4)$.

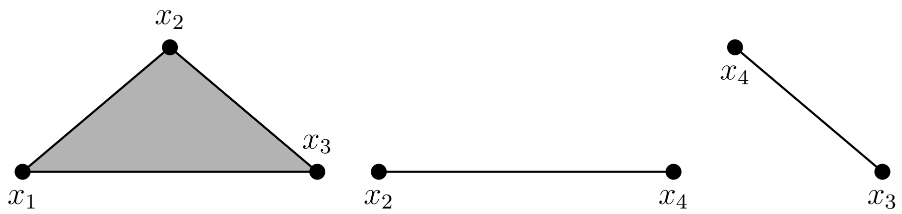

Figure 2.3: Facet ideal and Stanley-Reisner ideal

Proposition 2.2.4. *Let I be a square-free ideal. Then the square-free Alexander dual of $I = (x^{\sigma_1}, \cdots\ x^{\sigma_1})$ is*

$$I^{\vee} = \mathfrak{m}^{\sigma_1} \cap\ \cdots\ \cap \mathfrak{m}^{\sigma_{d+1}}.$$

We will now recall some basics of simplicial homology theory. Let Δ be a simplicial complex on the vertex set $[n]$. Let $\mathbb{K}^{\Delta^i}$ be a vector space over $\mathbb{K}$ whose basis elements e_F correspond to i-faces $F \in \Delta^i$.

Definition 2.2.5. The *reduced chain complex* of Δ over $\mathbb{K}$ is the complex given as

$$\widetilde{\mathcal{C}}\,(\Delta; \mathbb{K}) : 0 \longrightarrow \mathbb{K}^{\Delta^{n-1}} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow \mathbb{K}^{\Delta^i} \xrightarrow{\partial_i} \mathbb{K}^{\Delta^{i-1}} \longrightarrow \cdots \xrightarrow{\partial_0} \mathbb{K}^{\Delta^{-1}} \longrightarrow 0$$

where $\partial_i : \mathbb{K}^{\Delta^i} \longrightarrow \mathbb{K}^{\Delta^{i-1}}$ is called the *boundary map* and is defined as

$$\partial_i\,(e_F) = \sum_{j \in F} \text{sign}\,(j, F)\, e_{F \smallsetminus j}.$$

By setting $\text{sign}\,(j, F) = (-1)^{p-1}$ if j is the p^{th} element of the set $F = \{j_1, j_2, \ldots, j_i\} \subseteq [n]$ with $j_1 < j_2 < \cdots < j_i$ and $j = j_p$ for $1 \leq p \leq i$.

By definition, if $i < -1$ or $i > n - 1$, then $\mathbb{K}^{\Delta^{-1}} = 0$ and $\partial_i = 0$.

Proposition 2.2.6. *The image of the boundary map ∂_ℓ is contained in the kernel of the boundary map $\partial_{\ell-1}$. That is $\partial_{\ell-1} \circ \partial_\ell = 0$.*

Proof. Let $\{t_1, t_2, \ldots, t_{\ell+1}\} \in F^{\Delta^\ell}$ be an ℓ-dimensional face where $t_1 < t_2 < \cdots < t_\ell$.

Then we have,

$$\begin{aligned}
\partial_{\ell-1}\circ\partial_{\ell}\left(e_{\{t_1,t_2,\dots,t_{\ell+1}\}}\right) &= \partial_{\ell-1}\left(\sum_{j=1}^{\ell+1}(-1)^{j-1}\,e_{\{t_1,\dots,\widehat{t_j},\dots,t_{\ell+1}\}}\right)\\
&= \sum_{j=1}^{\ell+1}(-1)^{j-1}\,\partial_{\ell-1}\left(e_{\{t_1,\dots,\widehat{t_j},\dots,t_{\ell+1}\}}\right)\\
&= \sum_{j=1}^{\ell+1}(-1)^{j-1}\left(\sum_{\substack{k=1\\k<j}}^{l+1}(-1)^{k-1}\left(e_{\{t_1,\dots,\widehat{t_k},\dots,\widehat{t_j},\dots,t_{\ell+1}\}}\right)\right.\\
&\quad\left.+\sum_{\substack{k=1\\k>j}}^{l+1}(-1)^{k-2}\left(e_{\{t_1,\dots,\widehat{t_j},\dots,\widehat{t_k},\dots,t_{\ell+1}\}}\right)\right)\\
&= \sum_{j=1}^{\ell+1}(-1)^{j-1}\left((-1)^{k-1}+(-1)^{k-2}\right)e_{\{t_1,\dots,\widehat{t_j},\dots,\widehat{t_k},\dots,t_{\ell+1}\}}\\
&= 0.
\end{aligned}$$

where the superscript $\widehat{\ }$ means the term is omitted. □

Definition 2.2.7. The quotient space

$$\widetilde{H}_{\ell}\left(\Delta;\mathbb{K}\right)=\ker\left(\partial_{\ell}\right)/\operatorname{im}\left(\partial_{\ell+1}\right)$$

is called the ℓth reduced homology of Δ over $\mathbb{K}$ in homological degree ℓ for $-1\le\ell\le n-1$.

Observe that for $\Delta\neq\{\emptyset\}$, we have $\widetilde{H}_{\ell}\left(\Delta;\mathbb{K}\right)=0$ whenever $\ell<0$ or $\ell>n-1$ and when $\ell=n-1$, $\widetilde{H}_{n-1}\left(\Delta;\mathbb{K}\right)=\ker\left(\partial_{n-1}\right)$. The elements of $\ker(\partial_{\ell})$ are called $\ell-$*cycles* and the elements of im $(\partial_{\ell+1})$ are called $\ell-$*boundries*.

Let $R=\mathbb{K}[x_1,\dots,x_n]$ and let M be a monomial ideal over R with generating set $\{m_{\boldsymbol{j}}=\boldsymbol{x}^{\boldsymbol{a_j}}\mid j\in J\}$, for some index set J.

Definition 2.2.8. Let X be a simplicial complex with vertex set J and an incidence function $\varepsilon(F,F')$ on pairs of faces. The function ε takes values in $\{0,1,-1\}$, is such that

$\varepsilon(F, F') = 0$ except when F' is a facet of $F, \varepsilon(\{j\}, \phi) = 1$ for all $j \in J$, and

$$\varepsilon(F, F_1)\varepsilon(F_1, F') + \varepsilon(F, F_2)\varepsilon(F_2, F') = 0$$

for any codimension 2 face F' of F and F_1 and F_2 are the facets of F which contain F'. The cellular complex $\boldsymbol{F}_X$ is the R-module $\bigoplus_{F\in X, F\neq\phi} RF$ where RF is the free R-module with one generator F in degree $\boldsymbol{a}_F$ with differential

$$\partial F = \sum_{XF'\in X, F'\neq\phi} \varepsilon(F, F')\frac{mF}{mF'}F'.$$

Now we study the class of monomial ideals and face rings. Here we emphasize the connection between algebraic properties of face rings and the reduced simplicial homology of the corresponding Stanley-Reisner complex. An understated goal here is to highlight some of the work of M. Hochster, G. Reisner and R. Stanley.

Definition 2.2.9. An element in the polynomial ring R of the form $\mathbf{x}^{\mathbf{a}} = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$, where each $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ is called a *monomial* in R.

We denote the set of all monomials in the polynomial ring R by $\mathrm{Mon}(R)$. Indeed, the set $\mathrm{Mon}(R)$ is a $\mathbb{K}$-basis of R. In other words, each polynomial $p \in R$ is a unique $\mathbb{K}$-linear combination of the monomials in R as

$$p = \sum_{\mathbf{x}^{\mathbf{a}}\in\mathrm{Mon}(R)} \alpha_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}, \text{ where } \alpha_{\mathbf{a}} \in \mathbb{K} \text{ and } \alpha_{\mathbf{a}} = 0 \text{ for all but finitely many } \mathbf{a} \in \mathbb{N}^n.$$

Definition 2.2.10. An ideal I of R is called a *monomial ideal* if there is $\mathcal{A} \subseteq \mathbb{N}^n$ such that I is generated by $\{\mathbf{x}^{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathcal{A}\}$. In this case, we write

$$I = (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathcal{A}).$$

If I is a monomial ideal the quotient ring R/I is called a *monomial ring*.

Note that a monomial ideal is always generated by a finite set of monomials by Dickson's lemma.

Definition 2.2.11. A face ideal is an ideal $\mathfrak{p}$ of R generated by a subset of the set of variables, that is, $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_k})$ for some variables x_{i_j}.

We now state a theorem which is the main tool which relates the invariants from the minimal resolution of the Stanley-Reisner ring $\mathbb{K}[\Delta]$ (considered as a module over $\mathbb{K}[x_1, x_2, \ldots, x_n]$) to the simplicial homology of Δ.

Theorem 2.2.12 (Hochster's Formula)**.** *[47] Let I be a squarefree monomial ideal in the polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ with the standard multigrading over $\mathbb{N}^n$ and let $\Delta = \Delta_{\mathcal{N}}(I)$ be the Stanley-Reisner complex of I on $[n]$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ and $i \geq 0$, then $\beta_{i,\mathbf{a}}(I) = 0$. Otherwise,*

$$\beta_{i,\mathbf{a}}(I) = \dim_{\mathbb{K}} \widetilde{H}_{|\mathbf{a}|-i-1}(\Delta_W; \mathbb{K})$$

where $W = \{j \in [n] : a_j = 1\}$. In particular, for the standard grading over $\mathbb{N}$, we have

$$\beta_{i,j}(I) = \sum_{W \subseteq [n], |W| = j} \dim_{\mathbb{K}} \widetilde{H}_{j-i-2}(\Delta_W; \mathbb{K}).$$

As a direct consequence, if $\Delta' < \Delta$, then $\beta_{i,j}(I_{\Delta'}) \leq \beta_{i,j}(I_\Delta)$ for all i, j. In particular, $\operatorname{reg}(I_{\Delta'}) \leq \operatorname{reg}(I_\Delta)$.

Chapter 3

Graph and Graph Ideals

In this unit we will study graphs in association with algebraic objects. For any (finite, simple) graph we will define a module over a polynomial ring. The intention is to find information about combinatorial objects, i.e., graphs, by studying the corresponding algebraic objects and vice versa. We do this by considering minimal free resolutions of modules. In the previous chapters we have described precisely the machinery which we will be requiring.

3.1 Basic terminology of Graph Theory

Definition 3.1.1. A graph G is a collection of vertices, which we will often write as $x_1, \ldots, x_n$ (or sometimes as $1, \ldots, n$) and say G has vertex set $V(G) = \{x_1, \ldots, x_n\}$ (or $\{1, \ldots, n\}$) along with a set of edges $E(G) \subseteq V(G) \times V(G)$. If $\{x_i, x_j\} \in E(G)$ we will say x_i and x_j are joined by an edge. Graphs will be represented pictorially in the obvious way. For example, let G be the graph with vertices x_1, x_2, x_3, x_4 and edges $\{x_1, x_2\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_2, x_4\}$.

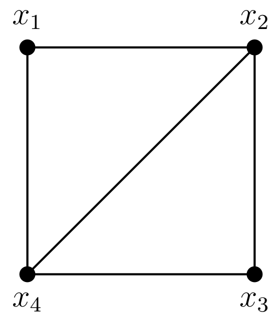

Figure 3.1: Graph G

Let $G = (V, E)$ be a graph, where V is a vertex set and E is an edge set of G.

Definition 3.1.2. A graph G is said to be *simple* if no vertex is joined to itself by an edge.

Definition 3.1.3. A graph G is said to be $finite$ if it has finitely many vertices and finitely many edges.

Definition 3.1.4. The *degree of a vertex* is the number of edges to which it is joined.

Definition 3.1.5. The *degree of a graph* is the maximum of the degrees of its vertices.

Definition 3.1.6. A *terminal vertex* is a vertex which is connected to at most one other vertex.

Definition 3.1.7. A graph G' of the graph G is said to be a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$.

Definition 3.1.8. Let G be a graph with the vertex set $V = \{x_1, \ldots, x_n\}$. For $W \subseteq V$ we define the *induced subgraph* of G with vertex set W to be the graph with vertex set W which has an edge between any two vertices if and only if there is an edge between them in G.

For example, if G is as in definition 3.1.1 and $W = x_1, x_2, x_4$ then the *induced subgraph* of G with vertex set W is

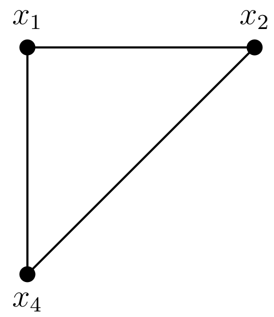

Figure 3.2: Graph W

Definition 3.1.9. *A path from a vertex x_1 to a vertex x_n is a sequence of edges,*

$$\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}.$$

Such a path is a cycle if $x_1, x_2, \ldots, x_{n-1}$ are distinct and $x_1 = x_n$.

Definition 3.1.10. A graph G is said to be *connected* if any two of its vertices are joined by a path of edges. A *connected component* is a maximal connected subgraph.

Definition 3.1.11. If $a \in V(G)$ then $G \setminus a$ denote the subgraph of G which has vertex set $V(G) \setminus a$ and all the edges of G which do not feature a.

Definition 3.1.12. A *tree* is a connected graph with no cycles.

Definition 3.1.13. A $forest$ is a graph whose connected components are all trees.

We now define, for any graph G we can associate a polynomial ring and a monomial ideal of that ring with G as follows:

Definition 3.1.14. For a graph G on vertices $x_1, \ldots, x_n$ and a field $\mathbb{K}$ we define $R_{\mathbb{K}}(G)$ to be the polynomial ring over $\mathbb{K}$ in n indeterminants which we also call $x_1, \ldots, x_n$, i.e., $R_{\mathbb{K}}(G) = \mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ is any field, and we define $I(G)$ to be the monomial ideal of $R_{\mathbb{K}}(G)$ generated by $\{x_i x_j \mid \{x_i, x_j\}$ is an edge of $G\}$. We call $I(G)$ the *graph ideal* of G.

Remark 3.1.15. Every monomial ideal of $R_{\mathbb{K}}(G) = \mathbb{K}[x_1, \ldots, x_n]$ which is generated by square free monomials of degree 2 may be interpreted as a graph ideal.

3.2 Hochster's Theorem

We now present a result of *Hochster's* which relates the Betti numbers of the Stanley-Reisner ring $\mathbb{K}[\Delta]$ to the simplicial homology of Δ. This will be of much use when calculating the Betti numbers of graph ideals. We also look at some results about Betti numbers and projective dimensions of graph ideals which are consequences of *Hochster's theorem*. See [47]

Definition 3.2.1. Let I be a monomial ideal of $R = \mathbb{K}[x_1, \ldots, x_n]$. For $\boldsymbol{b} \in \mathbb{N}^n$, define

$$K_{\boldsymbol{b}}(I) = F \subseteq \{1, \ldots, n\} \mid x^{\boldsymbol{b}-F} \in I.$$

Each face $F \in K_{\boldsymbol{b}}(I)$ is identified with its characteristic vector in $\{0,1\}^n$, i.e., F is identified with the element of $\{0,1\}^n$ which has jth component 1 if and only if $j \in F$ for $j = 1, \ldots, n$. Therefore if $\boldsymbol{b} = (b_1, \ldots, b_n)$ then the jth component of $\boldsymbol{b} - F$ is $b_j - 1$ if $j \in F$ and b_j if $j \notin F$.

The Betti numbers of I can be found from $K_{\boldsymbol{b}}(I)$ as follows.

Theorem 3.2.2. *[47] The $\mathbb{N}^n$-graded Betti numbers of I are*

$$\beta_{i,b}(I) \;=\; \dim_{\mathbb{K}} \tilde{H}_{i-1}(K_b(I); \mathbb{K}).$$

Proof. We use the Koszul complex $\mathcal{K}$ which is a minimal free resolution of the R-module $\mathbb{K} \cong R/\langle x_1, \ldots, x_n \rangle$.

$$\mathcal{K} : 0 \longrightarrow \wedge^n V \longrightarrow \cdots \longrightarrow \wedge^1 V \longrightarrow \wedge^0 V \longrightarrow 0,$$

where $V = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{K}\}$. The Betti number $\beta_{i,\boldsymbol{b}} = \dim_{\mathbb{K}}(\mathrm{Tor}_i^R(I, \mathbb{K}))_{\boldsymbol{b}}$ may be found by taking the ith homology of the degree $\boldsymbol{b}$ part of the complex

$$I \otimes \mathcal{K} : 0 \longrightarrow I \otimes \wedge^n V \longrightarrow \cdots \longrightarrow I \otimes \wedge^1 V \longrightarrow I \otimes \wedge^0 V \longrightarrow 0.$$

There is a basis of $(I \otimes \wedge^i V)_{\boldsymbol{b}}$ which consists of all the elements

$$\boldsymbol{x}^{\boldsymbol{b}}/x_{j_1} \cdots x_{j_i} \otimes x_{j_1} \wedge \cdots \wedge x_{j_i}$$

such that $\boldsymbol{x}^{\boldsymbol{b}}/x_{j_1} \cdots x_{j_i} \in I$. There is a one to one correspondence between these expressions and the $(i-1)$ faces $j_1, \cdots, j_i$ of $K_{\boldsymbol{b}}(I)$. We see that $(I \otimes \mathcal{K})_{\boldsymbol{b}}$ is the augmented chain complex of $K_{\boldsymbol{b}}(I)$ and hence that $\beta_{i,\boldsymbol{b}}(I) = \dim_{\mathbb{K}} \tilde{H}_{i-1}(K_{\boldsymbol{b}}(I); \mathbb{K})$. □

Theorem 3.2.3. *[47] Let $\mathbb{K}[\Delta] = R(\Delta)/I(\Delta)$ be the Stanley-Reisner ring of the simplicial complex Δ. The non-zero Betti numbers of $\mathbb{K}[\Delta]$ are only in squarefree degrees $\boldsymbol{b}$ and may be expressed as*

$$\beta_{i,\boldsymbol{b}}(\mathbb{K}[\Delta]) = \dim_{\mathbb{K}} \tilde{H}_{|\boldsymbol{b}|-i-1}(\Delta_{\boldsymbol{b}}; \mathbb{K}).$$

Hence the ith total Betti number may be expressed as

$$\beta_i(\mathbb{K}[\Delta]) = \sum_{V \subseteq \{1,\ldots,n\}} \dim_{\mathbb{K}} \tilde{H}_{|V|-i-1}(\Delta_V; \mathbb{K}).$$

Proof. Let $\boldsymbol{b} = (b_1, \ldots, b_n)$. If $b_j > 1$ for some j then $K_{\boldsymbol{b}}(I)$ is a cone over j and therefore $\beta_{i,\boldsymbol{b}}(\mathbb{K}[\Delta]) = \beta_{i-1,\boldsymbol{b}}(I) = \dim_{\mathbb{K}} \tilde{H}_{i-2}(K_{\boldsymbol{b}}(I); \mathbb{K}) = 0$ by Theorem (3.2.1). If $\boldsymbol{b} \in 0, 1^n$ and V is the subset of $\{1, \ldots, n\}$ identified with $\boldsymbol{b}$ (i.e., if $\boldsymbol{b} = (b_1, \ldots, b_n)$ then $j \in V$ if and only if $b_j = 1$) then $\Delta_{\boldsymbol{b}}$ is the Alexander dual of $K_{\boldsymbol{b}}(I)$. i.e., $F \in K_{\boldsymbol{b}}(I)$ if and only if $\boldsymbol{x}^{\boldsymbol{b}-F} \in I$ if and only if $F \subseteq V$ and $V \setminus F \notin \Delta$ if and only if $F \in (\Delta_{\boldsymbol{b}})^*$. We now use the fact that, for any simplicial complex X with m vertices $\tilde{H}_i(X; \mathbb{K}) \cong \tilde{H}_{m-i-3}(X^*; \mathbb{K})$ along with Theorem 3.2.2 to obtain

$$\beta_{i,\boldsymbol{b}}(\mathbb{K}[\Delta]) = \dim_{\mathbb{K}} \tilde{H}_{i-2}(K_{\boldsymbol{b}}(I); \mathbb{K}) = \dim_{\mathbb{K}} \tilde{H}_{|\boldsymbol{b}|-i-1}(\Delta_{\boldsymbol{b}}; \mathbb{K}).$$

The total Betti number is found by summing over all squarefree vectors, noting there is a one to one correspondence between these vectors and subsets of $\{1, \ldots, n\}$. □

Definition 3.2.4. The link of a face F of Δ, $\operatorname{Link}_\Delta F$, is the simplicial complex consisting of all faces, G, which satisfy

(i) $G \in \Delta$;

(ii) $G \cup F \in \Delta$;

(iii) $G \cap F = \phi$.

Theorem 3.2.5. *[66] The $\mathbb{N}$-graded Betti numbers of $\mathbb{K}[\Delta]$ may be expressed by*

$$\beta_{i,b}(\mathbb{K}[\Delta]) \;=\; \sum_{F \in \Delta^*, |F|=n-d} \dim_{\mathbb{K}} \tilde{H}_{i-2}(Link_{\Delta^*} F; \mathbb{K}).$$

and the total Betti numbers by

$$\beta_i(\mathbb{K}[\Delta]) \;=\; \sum_{F \in \Delta^*} \dim_{\mathbb{K}} \tilde{H}_{i-2}(Link_{\Delta^*} F; \mathbb{K}).$$

Proof. We use Hochster's formula 3.2.3

$$\beta_{i,\boldsymbol{b}}(\mathbb{K}[\Delta]) = \dim_{\mathbb{K}} \tilde{H}_{|\boldsymbol{b}|-i-1}(\Delta_{\boldsymbol{b}}; \mathbb{K}).$$

For $\boldsymbol{b}$ appearing in this sum, with $\boldsymbol{b}$ corresponding to $\tau \subseteq 1, \dots, n$, let $F = \{1, \dots, n\} \setminus \tau$ (so $|F| = n - d$). If τ is a face of Δ then Δ_τ is a simplex and will have no reduced homology. Hence it may be assumed that τ is not a face of Δ. From the definition of Alexander dual, we see that F is a face of Δ^*. It is therefore enough to show that $\tilde{H}_{i-2}(\mathrm{Link}_{\Delta^*} F; \mathbb{K}) \cong \tilde{H}_{|\tau|-i-1}(\Delta_\tau; \mathbb{K})$. This follows as the complexes $\mathrm{Link}_{\Delta^*} F$ and $(\Delta_\tau)^*$ are isomorphic and $\tilde{H}_{i-2}((\Delta_\tau)^*; \mathbb{K}) \cong \tilde{H}_{|\tau|-i-1}((\Delta_\tau); \mathbb{K})$. The total Betti number is found by summing all graded Betti numbers. $\square$

The above theorem finds Betti numbers from the reduced homology modules of links of faces of the Alexander Dual.

We now prove some results about Betti numbers and projective dimensions of graph ideals which are consequences of Hochster's Theorem.

3.3 Betti Numbers of Induced Subgraphs

In this section we will look that the *ith* Betti number of an induced subgraph cannot exceed the ith Betti number of the larger graph for all i.

Proposition 3.3.1. *[66] If H is an induced subgraph of G on a subset of the vertices of G then*

$$\beta_{i,d}(H) \leq \beta_{i,d}(G)$$

for all i.

Proof. Suppose that W is the vertex set of G. The vertex set of H is $W \setminus S$ for some $S \subseteq W$. Every subset of $W \setminus S$ is also a subset of W. Hence

$$\begin{aligned}\beta_{i,d}(H) &= \sum_{V \subseteq W \setminus S, |V|=d} \dim_{\mathbb{K}} \tilde{H}_{|V|-i-1}(\Delta_V; \mathbb{K}) \\ &\leq \sum_{V \subseteq W, |V|=d} \dim_{\mathbb{K}} \tilde{H}_{|V|-i-1}(\Delta_V; \mathbb{K}) \\ &= \beta_{i,d}(G).\end{aligned}$$

□

Corollary 3.3.2. *If H is an induced subgraph of G then the total Betti numbers of H do not exceed those of G.*

Proof. This follows from Proposition 3.3.1 as

$$\beta_{i,d}(H) \leq \sum_{d \in \mathbb{N}} \beta_{i,d}(H) \leq \sum_{d \in \mathbb{N}} \beta_{i,d}(G) \leq \beta_{i,d}(G).$$

□

Definition 3.3.3. For a graph G let G^c denote the (simple, finite) graph on the same vertices as G with any two vertices joined by an edge if and only if they are not joined by an edge in G.

Example 3.3.4. If G is the graph on vertices $\{x_1, x_2, x_3, x_4\}$ with edges $\{x_1, x_2\}, \{x_1, x_3\}$ and $\{x_3, x_4\}$ then G^c is the graph on the same vertices with edges $\{x_1, x_4\}, \{x_2, x_3\}$ and $\{x_2, x_4\}$.

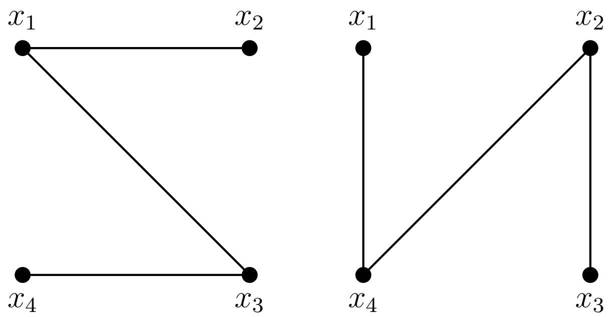

Figure 3.3: Graph G and G^c

Theorem 3.3.5. *[66] If H is an induced subgraph of G such that H^c is disconnected then*

$$pd^{\mathbb{K}}(G) \geq |V(H)| - 1.$$

Proof. As H^c is disconnected we can choose a vertex, x_1 say, in one connected component of H^c and another vertex, x_n say, in a different connected component of H^c. Now suppose that $\Delta_{V(H)}$ is connected. There must be a path of edges of $\Delta_{V(H)}$ from x_1 to x_n, i.e., there must be a sequence of edges

$$\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-2}, x_{n-1}\}, \{x_{n-1}, x_n\}$$

in $\Delta_{V(H).}$ This implies that none of $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}$ are edges of H and so they must all be edges of H^c. However this implies that x_1 and x_n are in the same connected component of H^c, a contradiction. Hence $\Delta_{V(H)}$ is not connected.

From Theorem 3.2.3

$$\begin{aligned}\beta_i(G) &= \sum_{W \subseteq V(G),} \dim_{\mathbb{K}} \tilde{H}_{|W|-i-1}(\Delta_W; \mathbb{K}) \\ &\geq \dim_{\mathbb{K}} \tilde{H}_{|V(H)|-i-1}(\Delta_{|V(H)|}; \mathbb{K})\end{aligned}$$

as $V(H) \subseteq V(G)$. When $i = |V(H)| - 1$ this becomes

$$\beta_{|V(H)|-1}(G) \geq \dim_{\mathbb{K}} \tilde{H}_0(\Delta_{V(H)}; \mathbb{K}) > 0$$

because $\Delta_{V(H)}$ is disconnected. As the $(|V(H)| - 1)$th Betti number is nonzero we must have

$$\mathrm{pd}(G) \geq |V(H)| - 1.$$

$\square$

Theorem 3.3.6. *[66] If G is a graph such that G^c is disconnected then*

$$\mathrm{pd(G) = |V(G)| - 1.}$$

Proof. Let $p = \mathrm{pd}(G)$ and let $n = |V(G)|$. By Theorem 3.3.5 $p \geq n-1$, as G is certainly an induced subgraph of itself. We may use the *Hilbert syzygy Theorem* to see that $p \leq n$, as $\mathbb{K}[\Delta(G)]$ is a finitely generated graded $R_{\mathbb{K}}(G)$-module. Therefore the only possible values p may take are n or $n-1$. Suppose it is the case that $p = n$ and hence that $\beta_p(G) \neq 0$. Using Hochster's theorem 3.2.3 we see that

$$\beta_p(G) = \sum_{V \subseteq [n],} \dim_{\mathbb{K}} \tilde{H}_{|V|-p-1}(\Delta_V; \mathbb{K}).$$

For any $V \subseteq [n]$ appearing in the above sum we must have $|V| \leq n$. If $|V| \leq n$ then $|V| - p - 1 = |V| - n - 1 < -1$. This implies that $\tilde{H}_{|V|-p-1}(\Delta_V; \mathbb{K}) = 0$. If, on the other hand, $|V| = n$ (and hence $\Delta_V = \Delta(G)$) then $|V| - p - 1 = |V| - n - 1 = -1$. The reduced homology group $\tilde{H}_{-1}(\Delta; k) \neq 0$ if and only if $\Delta(G) = \phi$. However, the graph which is such that $\Delta(G) = \phi$ is the graph with no vertices which we do not allow. Because all the summands in the above formula are zero we must have $p = 0$, a contradiction. Hence $p = n - 1$. $\square$

Definition 3.3.7. (i) The Complete Graph, K_n, (for $n > 2$) is the graph on the n vertices $x_1, \ldots, x_n$ which has edges $\{x_i, x_j\}$ for all i and j such that $1 \leq i < j \leq n$.

(ii) The Complete Bipartite graph, $K_{n,m}$, $(n, m > 1)$ is the graph which consists of vertices $x_1, \ldots, x_n, y_1, \ldots, y_m$ and the edges $\{x_i, y_j\}$ for all i and j such that $1 \leq i \leq n$ and $1 \leq j \leq m$.

Corollary 3.3.8. *Let G and H be graphs. Let K_n denote the complete graph on n vertices and let $K_{n,m}$ denote the complete bipartite graph on $n + m$ vertices.*

(i) If K_n is an induced subgraph of G then $\mathrm{pd}(G) \geq n - 1$.

(ii) If $K_{n,m}$ is an induced subgraph of H then $\mathrm{pd}(H) \geq n + m - 1$.

Proof. (i) The graph K_n^c is disconnected (it is comprised of n isolated vertices) so the result follows from Theorem 3.3.5.

(ii) The graph $K_{n,m}$ is also disconnected (it is the graph with two components, K_n and K_m) so again the result follows from Theorem 3.3.5.

□

Corollary 3.3.9. *(i) For the complete graph on n vertices, K_n, we have* $\mathrm{pd}(K_n) = n - 1$.

(ii) For the complete bipartite graph on $n + m$ vertices, K_{n+m}, we have $\mathrm{pd}(K_{n,m}) = n + m - 1$.

Proof. As we observed above K_n^c and $K_{n,m}^c$ are disconnected. The results follow from Theorem 3.3.5. □

3.4 Betti Numbers of Complete and Bipartite Graphs

In this section we find explicit descriptions of the Betti numbers of complete graphs and complete bipartite graphs. We will also see that the Betti numbers (and therefore the projective dimensions) do not depend on our choice of base field of the polynomial ring $R_{\mathbb{K}}(G)$.

Theorem 3.4.1. *[66] The $\mathbb{N}$-graded Betti numbers of the complete graph with n vertices are independent of the characteristic of $\mathbb{K}$ and may be written*

$$\beta_{i,d}(K_n) = \begin{cases} i\binom{n}{i+1} & \text{if d = i + 1} \\ 0 & \text{if d} \neq \text{i + 1.} \end{cases}$$

Proof. From Theorem 3.2.3 we have,

$$\beta_{i,d}(K_n) = \sum_{V \subseteq X \setminus S, |V|=d} \dim_{\mathbb{K}} \tilde{H}_{|V|-i-1}(\Delta_V; \mathbb{K})$$

where $X = \{x_1, \ldots, x_n\}$ and $\Delta = \Delta(K_n) = \{\{x_1\}, \ldots, \{x_n\}\}$. For any $V \subseteq X$, such that $V \neq \phi, \Delta_V$ is a simplicial complex which comprises of isolated zero dimensional faces. The only possibly non-zero reduced homology groups for such simplicial complexes are those in the $0th$ position, $\tilde{H}_0(\Delta_V; \mathbb{K})$. Because, regardless of our choice of the field $\mathbb{K}, \tilde{H}_0(\Delta_V; \mathbb{K})$ is one less than the number of connected components of Δ we obtain $\tilde{H}_0(\Delta_V; \mathbb{K}) = |V| - 1$. So the only contribution to $\beta_{i,d}(K_n)$ is of $|V| - 1 = d - 1$ whenever $d - i - 1 = 0$, i.e., when $d = i + 1$. There are $\binom{n}{d} = \binom{n}{i+1}$ choices for subsets of X of size d, each contributing $d - 1 = i$ to $\beta_{i,d}(K_n)$. Hence $\beta_{i,d}(K_n) = i\binom{n}{i+1}$ when $d = i + 1$ and $\beta_{i,d}(K_n)$ is zero otherwise. □

Corollary 3.4.2. *The total Betti numbers of the complete graph are*

$$\beta_i(K_n) = i\binom{n}{i+1}$$

Proof. This follows from 3.4.1 by summing all the $\mathbb{N}$-graded ith Betti numbers. □

Remark 3.4.3. For a field $\mathbb{K}$ let $R_{\mathbb{K}}(K_{n,m}) = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ and

$$I = I(K_{n,m}) = \langle x_i y_j | 1 \leq i \leq n,\ 1 \leq j \leq m \rangle.$$

The simplicial complex $\Delta(K_{n,m})$ is the disjoint union of two simplices, one of dimension $n-1$, the other of dimension $m-1$. As in the case of the complete graph we use this simple description of the associated simplicial complex with (3.2.2) to find the Betti numbers.

Theorem 3.4.4. *[47] The $\mathbb{N}$-graded Betti numbers of the complete bipartite graph with $n+m$ vertices are independent of the characteristic of $\mathbb{K}$ and may be written*

$$\beta_{i,d}(K_{n,m}) = \begin{cases} \sum_{j+l=i+1,j,l\geq 1} \binom{n}{j}\binom{m}{l} & \text{if } d = i+1 \\ 0 & \text{if } d \neq i+1. \end{cases}$$

Proof. From Theorem 3.2.3, we see that

$$\beta_{i,d}(K_{n,m}) = \sum_{V\subseteq X\cup Y,|V|=d} \dim_{\mathbb{K}} \tilde{H}_{|V|-i-1}(\Delta_V;\mathbb{K})$$

where $X = x_1,\ldots,x_n, Y = \{y_1,\ldots,y_m\}$ and $\Delta = \Delta(K_{n,m})$, the disjoint union of an $(n-1)$-dimensional simplex and an $(m-1)$-dimensional simplex. Suppose $V \neq \phi$. If $V \subseteq X$ or $V \subseteq Y$ then Δ_V is a simplex and hence has zero reduced homology everywhere. Consequently we may suppose that $V \cap X \neq \phi$ and $V \cap Y \neq \phi$ and therefore Δ_V is the disjoint union of two simplices. This implies that

$$\tilde{H}_j(\Delta_V;\mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}$$

So we will have a contribution (of 1) to $\beta_{i,d}(K_{n,m})$ if and only if $|V|-i-1 = d-i-1$, i.e., if $d = i+1$. Therefore the only non-zero $\mathbb{N}$-graded Betti numbers are of the form $\beta_{i,i+1}$ and to find their value we must count the subsets of $X\cup Y$, which are disjoint from neither X nor Y, and contain $i+1$ elements. We obtain a suitable subset by choosing 1 element from X and i from Y. There are $\binom{n}{1}\binom{m}{i}$ subsets of this type. Similarly we may choose 2 from X and $i-1$ from Y, providing a further $\binom{n}{2}\binom{m}{i-1}$ and so on to obtain

$$\binom{n}{1}\binom{m}{i} + \binom{n}{2}\binom{m}{i-1} + \cdots + \binom{n}{i}\binom{m}{1} = \sum_{j+l=i+1,j,l\geq 1} \binom{n}{j}\binom{m}{l}.$$

□

Corollary 3.4.5. *The total Betti numbers of the complete bipartite graph are independent of choice of field and are*

$$\beta_i(K_{n,m}) = \sum_{j+l=i+1, j,l\geq 1} \binom{n}{j}\binom{m}{l}.$$

Proof. Follows from above Theorem 3.4.4 by summing the graded Betti numbers. $\square$

Definition 3.4.6. We define the complete multipartite graph, $K_{n_1,\dots,n_t}$, as follows. The vertices of $K_{n_1,\dots,n_t}$ are $x_j^{(i)}$ for $1 \leq i \leq t$ and $1 \leq j \leq n_i$. The edges of $K_{n_1,\dots,n_t}$ are $\{x_j^{(i)}, x_l^{(k)}\}$ for all i, j, k, l such that $i \neq k$.

We have $R = R(K_{n_1,\dots,n_t}) = \mathbb{K}[x_j^{(i)} \mid i = 1, \dots, t, j = 1, \dots, n_i]$ and

$$I = I(K_{n_1,\dots,n_t}) = \langle x_j^{(i)} x_l^{(k)} \mid i \neq k \rangle.$$

The simplicial complex $\Delta(K_{n_1,\dots,n_t})$ is defined by its maximal faces which are

$$\{x_1^{(1)}, \dots, x_{n_1}^{(1)}\}, \dots, \{x_1^{(1)}, \dots, x_{n_t}^{(1)}\}.$$

i.e., $\Delta(K_{n_1,\dots,n_t})$ is the disjoint union of t simplices of dimensions $n_1 - 1, \dots, n_t - 1$.

Theorem 3.4.7. *[66] The $\mathbb{N}$-graded Betti numbers of the complete multipartite graph $K_{n_1,\dots,n_t}$ are independent of the characteristic of $\mathbb{K}$ and may be written*

$$\beta_{i,i+1}(K_{n_1,\dots,n_t}) = \sum_{l=2}^{t} l - 1 \sum_{\alpha_1+\dots+\alpha_l=i+1, j_1<\dots<j_l, \alpha_1,\dots,\alpha_l\geq 1} \binom{n_{j_1}}{\alpha_1} \cdots \binom{n_{j_l}}{\alpha_l}$$

and $\beta_{i,d}(K_{n_1,\dots,n_t}) = 0$ for $d \neq 0$.

Proof. Let $X^(i) = \{x_j^{(i)} \mid j = 1, \dots, n_i\}$ for $i = 1, \dots, t$ and let $X = \bigcup_{i=1}^{t} X^(i)$. For $\phi \neq V \subseteq X$ the simplicial complex ΔV is the disjoint union of at most t simplices. So, as in the case for bipartite graphs, the only homology module which may possibly be non-zero is the $0th$ one.

$$\dim \tilde{H}_j(\Delta_V; \mathbb{K}) = \begin{cases} (\text{no. of connected components of } \Delta_V) - 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases}$$

Using Theorem 3.2.3 we see that the only possibly non-zero Betti numbers are those of the form

$\beta_{i,i+1}(K_{n_1,\dots,n_t})$. We consider the contribution from those choices of $V \subseteq X$ for which ΔV has l connected components. The $i+1$ elements of V must be chosen from precisely l of the sets $X^(1), \dots, X^(t)$. Counting all the possibilities we obtain

$$\sum_{\alpha_1+\ldots+\alpha_l=i+1, j_1<\cdots<j_l, \alpha_1,\ldots,\alpha_l\geq 1} \binom{n_{j_1}}{\alpha_1} \cdots \binom{n_{j_l}}{\alpha_l}$$

such subsets of X. Each of these contribute $(l-1)$ to $\beta_i(K_{n_1,\dots,n_t})$. Summing from $l=2$ to t provides the result. □

Corollary 3.4.8. *The total Betti numbers of the complete bipartite graph are independent of choice of field and are*

$$\beta_i(K_{n_1,\dots,n_t}) = \sum_{l=2}^{t} l-1 \sum_{\alpha_1+\ldots+\alpha_l=i+1, j_1<\cdots<j_l, \alpha_1,\ldots,\alpha_l\geq 1} \binom{n_{j_1}}{\alpha_1} \cdots \binom{n_{j_l}}{\alpha_l}$$

Proof. Follows from above theorem 3.4.7 by summing the graded Betti numbers. □

Definition 3.4.9. The Star Graph, S_n, is the graph with the $n+1$ vertices $x, y_1, \dots, y_n$ and the n edges $\{x, y_1\}, \dots, \{x, y_n\}$.

Theorem 3.4.10. *[66] The Betti numbers for the star graph with n vertices, S_n, are independent of $\mathbb{K}$ and may be expressed as*

$$\beta_i(S_n) = \binom{n}{i}.$$

It follows that $\mathrm{pd}(S_n) = n$.

Proof. The star graph, S_n, is in fact just the complete bipartite graph $K_{1,n}$. By Theorem 3.4.4

$$\beta_i(S_n) = \beta_i(K_{1,n}) = \sum_{j+l=i+1, j,l\geq 1} \binom{1}{j}\binom{n}{l} = \binom{1}{1}\binom{n}{i} = \binom{n}{i}.$$

□

Chapter 4

Links of Subgraphs

In this chapter we will reinterpret Theorem 3.2.3 for graph ideals which will make use of the structure of graphs. We first need a useful way to describe the links of faces of the Alexander Dual. This will be achieved by establishing a correspondence between induced subgraphs of a graph and the faces of the Alexander Dual.

4.1 Links and Induced Subgraphs

Definition 4.1.1. Let $a_1, \ldots, a_s$ be subsets of the finite set V. Define $\epsilon(a_1, \ldots, a_s; V)$ to be the simplicial complex which has vertex set $\bigcup_{i=1}^{s}(V \setminus a_i)$ and maximal faces $V \setminus a_1, \ldots, V \setminus a_s$.

Notice that, with the above notation, the simplicial complex $\epsilon(\{1,2\}; \{1,2\})$ has no vertices and is in fact the complex which has only one face, namely the empty set (we write this simplicial complex simply as ϕ). In what follows $I = I(G)$ will be a graph ideal (for some graph G) of the polynomial ring $R_{\mathbb{K}}(G) = \mathbb{K}[x_1, \ldots, x_n], \Delta = \Delta(I)$, the Stanley-Reisner complex of I. The field $\mathbb{K}$ will remain unspecified throughout this chapter. Although as noted before Betti numbers will in general be dependent on the characteristic of $\mathbb{K}$, this will not affect the results which follow.

Remark 4.1.2. We will sometimes write $\Delta(G)$ for $\Delta(I(G))$ and $\Delta^*(G)$ for $(\Delta(G))^* = (\Delta(I(G)))^*$.

Definition 4.1.3. For a face, F, of the simplicial complex Γ we write $\Gamma \setminus F$ to denote the subcomplex of Γ which comprises all the faces of Γ which do not intersect F. Note that $V(\Gamma \setminus F) = V(\Gamma) \setminus F$ (where $V(-)$ denotes the vertex set of a simplicial complex).

Lemma 4.1.4. *[47] Let G be a graph with vertex set V and let $\Delta = \Delta(G)$. For a subset F of V we have $F \in \Delta^*$ if and only if $V \setminus F$ contains an edge of G.*

Proof. This follows from the definition of $\Delta(G)$ and the definition of the Alexander Dual $\Delta^*(G)$. We see that F is a face of Δ^* if and only if $V \setminus F$ is not a face of Δ which is in turn true if and only if $V \setminus F$ contains an edge of G. $\square$

Proposition 4.1.5. *[64] Let $F \in \Delta^* = \Delta^*(G)$. Suppose that $e_1, \ldots, e_r$ are all the edges of G which are disjoint from F. Then $\mathrm{Link}_{\Delta^*} F = \epsilon(e_1, \ldots, e_r; W)$ where $W = V(G) \setminus F$.*

Proof. We first show that the set $(V(G) \setminus F) \setminus e_i$ is a face of $\mathrm{Link}_{\Delta^*} F$ for $i = 1, \ldots, r$, using Lemma 4.1.4.

(i) $(V(G) \setminus F)\ e_i \in \Delta^*$ because $V(G) \setminus ((V(G) \setminus F) \setminus e_i) = F \cup e_i$ contains the edge e_i of G.

(ii) $((V(G) \setminus F) \setminus e_i) \cup F \in \Delta^*$ because $V(G) \setminus (((V(G) \setminus F) \setminus e_i) \cup F) = e_i$ contains an edge of G.

(iii) $((V(G) \setminus F) \setminus e_i) \cap F = \phi$ By (38) $(V(G) \setminus F) \setminus e_i$ is a face of $\mathrm{Link}_{\Delta^*} F$. So all the maximal faces of $\epsilon(e_1, \ldots, e_r; V(G) \setminus F)$ are faces of $\mathrm{Link}_{\Delta^*} F$. As $\mathrm{Link}_{\Delta^*} F$ is a simplicial complex all the faces of $\epsilon(e_1, \ldots, e_r; V(G) \setminus F)$ must be faces of $\mathrm{Link}_{\Delta^*} F$. Therefore $\epsilon(e_1, \ldots, e_r; V(G) \setminus F) \subseteq \mathrm{Link}_{\Delta^*} F$.

Now suppose that $J \in \mathrm{Link}_{\Delta^*} F$. Therefore $V(G) \setminus (F \cup J)$ contains an edge of G (which is disjoint from F), i.e., it contains e_i for some $i \in 1, \ldots, r$. Also $F \cap J = \phi$, so we see that $J \subseteq (V(G) \setminus F) \setminus e_i$ which is a face of $\epsilon(e_1, \ldots, e_r; V(G) \setminus F)$. Hence $J \in \epsilon(e_1, \ldots, e_r; V(G) \setminus F)$ and therefore $\mathrm{Link}_{\Delta^*} F \subseteq \epsilon(e_1, \ldots, e_r; V(G) \setminus F)$ and we conclude $\mathrm{Link}_{\Delta^*} F = \epsilon(e_1, \ldots, e_r; V(G) \setminus F)$.

$\square$

Proposition 4.1.6. *[58] There is a bijection between the faces of $\Delta^*(G)$ and the set of induced subgraphs of G which have at least one edge.*

$$F \in \Delta^*(G) \xrightarrow{\phi} \textit{The induced subgraph of } G \textit{ on vertices } V(G) \setminus F.$$

Proof. Suppose that H is an induced subgraph of G and that e is an edge of H (we are assuming that H has at least one edge). Let $F = V(G) \setminus V(H)$. Notice that $F \in \Delta^*(G)$ because $V(G) \setminus F = V(G) \setminus (V(G) \setminus V(H)) = V(H)$ contains the edge e of G. Now we observe that $H = \psi(F)$. Therefore ψ is surjective.

Now suppose that $F \in \Delta^*(G)$ and $F' \in \Delta^*$ are such that $\psi(F) = \psi(F') = H$. Thus H must be the induced subgraph of G on vertices $V(G) \setminus F = V(G) \setminus F'$. Hence $F = F'$ and ψ is injective. $\square$

Example 4.1.7. let G be the following graph,

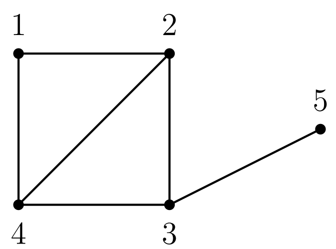

Figure 4.1: Graph G

If we now consider the face $F = \{3, 5\}$ of the Alexander Dual (this is so as $\{1, 2, 3, 4, 5\} \setminus F$ contains an edge of G). Therefore F corresponds to the induced subgraph of G on vertices

$$\{1, 2, 3, 4, 5\} \setminus \{3, 5\} = \{1, 2, 4\}.$$

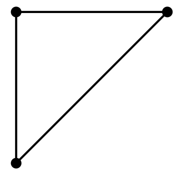

Figure 4.2: Induced subgraph of G

We can write the link of F as

$$\mathrm{Link}_{\Delta^*}(G)F = \varepsilon(1, 2, 1, 4, 2, 4; 1, 2, 4)$$

(this is the simplicial complex which consists of three isolated vertices).

Definition 4.1.8. Let H be an induced subgraph of the graph G. If H is associated to the face F of the Alexander Dual as described above we write $\varepsilon(H)$ for $\varepsilon(e_1, \ldots, e_s; V)$, where $e_1, \ldots, e_s$ are the edges of H and V is the vertex set of $V(G) \setminus F$ (or equivalently the vertex set of H).

Theorem 4.1.9. *[66] For a graph G the* $\mathbb{N}$*-graded Betti numbers are*

$$\beta_{i,d}(G) = \sum_{H \subseteq G, |V(H)| = d} \dim_{\mathbb{K}} \tilde{H}_{i-2}(\varepsilon(H); \mathbb{K})$$

and the total Betti numbers are

$$\beta_i(G) = \sum_{H \subseteq G} \dim_{\mathbb{K}} \tilde{H}_{i-2}(\varepsilon(H); \mathbb{K})$$

where the sum is over all induced subgraphs of G which contain at least one edge.

Proof. This is just Theorem 3.2.3 with the $\mathrm{Link}_{\Delta^*} F$ replaced by $\varepsilon(H)$ where H is the subgraph of G associated to the face F of Δ^* as described above. Note that $|F| = n - d \Longleftrightarrow |V(H)| = n - |F| = d$. □

4.2 Reduced Homology of Links

In this section we prove few results about the reduced homology of the type of simplicial complexes.

Lemma 4.2.1. *[66] Let $e_1, \ldots, e_t$ be subsets of the finite set V. Let $E = \varepsilon(e_1, \ldots, e_t; V)$. If $V \setminus \bigcup_{i=1}^{t} e_i \neq \phi$, i.e., if there is an element of V which is in none of $e_1, \ldots, e_t$, then $\tilde{H}_i(E) = 0$ for all i.*

Proof. If $V \setminus \bigcup_{i=1}^{t} e_i \neq \phi$, then there exists some vertex common to all maximal faces of E, i.e., E is a cone and so has zero reduced homology everywhere. □

Corollary 4.2.2. *Let H be a graph with at least one edge and at least one isolated vertex. Then $\dim_{\mathbb{K}} \tilde{H}_i(\varepsilon(H); \mathbb{K}) = 0$ for all i.*

Proof. Let a be an isolated vertex of H and let $e_1, \ldots, e_r$ be the edges of H. Then $\varepsilon(H) = \varepsilon(e_1, \ldots, e_r; V(H))$. As $a \in V(H) \setminus \bigcup_{j=1}^{r} e_j$ we have $\dim_{\mathbb{K}} \tilde{H}_i(\varepsilon(H); \mathbb{K}) = 0$ for all i by Lemma 4.2.1. □

Lemma 4.2.3. *[66] Suppose the simplicial complexes E_1 and E_2 are defined as follows. Let $E_1 = \varepsilon(e_1, \ldots, e_t; V)$ and let $E_2 = \varepsilon(f; V)$. The intersection may be expressed as $E_1 \cap E_2 = \varepsilon(f \cup e_1, \ldots, f \ldots e_t; V)$. If $f \cap \bigcup_{i=1}^{t} e_i = \phi$ then $E_1 \cap E_2 = \varepsilon(e_1, \ldots, e_t; V \setminus f)$.*

Proof. The maximal faces of $E_1 \cap E_2$ are the intersections of the maximal faces of E_1, that is $V \setminus e_1, \ldots, V \setminus e_t$, with the maximal face of E_2, $V \setminus f$. These are the sets $V \setminus (e_i \cup f)$

for $i = 1, \dots, t$. Hence we may write

$$E_1 \cap E_2 = \varepsilon(f \cup e_1, \dots, f \cup e_t; V).$$

If $f \cap \bigcup_{t=1}^{t} e_i = \phi$ then we have $E_1 \cap E_2 = \varepsilon((e_1 \cup f) \setminus f, \dots, (e_t \cup f) \setminus f; V \setminus f) = \varepsilon(e_1, \dots, e_t; V \setminus f)$. $\square$

Lemma 4.2.4. *If $f \subseteq g$ then $\varepsilon(f, g, e_1, \dots, e_t; V) = \varepsilon(f, e_1, \dots, e_t; V)$.*

Proof. The simplicial complex $\varepsilon(f, g, e_1, \dots, e_t; V)$ has faces

$$V \setminus f, V \setminus g, V \setminus e_1, \dots, V \setminus e_t$$

and all of their subsets. If $f \subseteq g$ then $V \setminus g$ is a subset of $V \setminus f$, as are all the subsets of $V \setminus g$, so we may write the simplicial complex as $\varepsilon(f, e_1, \dots, e_t; V)$. $\square$

Theorem 4.2.5. *(Mayer-Vietoris Sequence). Let Δ be a simplicial complex, Δ_1 and Δ_2 be subcomplexes of Δ such that $\Delta = \Delta_1 \cup \Delta_2$ and denote $\Delta_0 = \Delta_1 \cap \Delta_2$. Then, there is a long exact sequence*

$$\cdots \longrightarrow \tilde{H}_i(\Delta_0) \longrightarrow \tilde{H}_i(\Delta_1) \oplus \tilde{H}_i(\Delta_2) \longrightarrow \tilde{H}_i(\Delta) \longrightarrow \tilde{H}_{i-1}(\Delta_0) \longrightarrow \cdots$$

which is called the Mayer-Vietoris sequence, provided $\Delta_0 \neq \phi$, there is an analogous long exact sequence for reduced homology groups.

Example 4.2.6. Let Δ be a simplicial complex and $v \in V(\Delta)$. Then,

$$\Delta = \mathrm{del}_\Delta(v) \cup \mathrm{star}_\Delta(v) \text{ and } \mathrm{del}_\Delta(v) \cap \mathrm{star}_\Delta(v) = \mathrm{link}_\Delta(v)$$

and hence we can apply Theorem 4.2.5 taking into account that $\mathrm{star}_\Delta(v)$ is acyclic. We obtain the following long exact sequence:

$$\cdots \longrightarrow H_i(\mathrm{link}_\Delta(v)) \longrightarrow H_i(\mathrm{del}_\Delta(v)) \longrightarrow H_i(\Delta) \longrightarrow H_{i-1}(\mathrm{link}_\Delta(v)) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_0(\mathrm{link}_\Delta(v)) \longrightarrow H_0(\mathrm{del}_\Delta(v)) \longrightarrow H_0(\Delta) \longrightarrow 0.$$

and the analogous sequence for reduced homology groups is also exact if $\mathrm{link}_\Delta(v) \neq \phi$.

Lemma 4.2.7. *[66] We define the simplicial complex, E, as follows. Let $E = \varepsilon(\{a\}, e_1, \dots, e_t; V)$ where $\{a\}, e_1, \dots, e_t$ are subsets of the set V. If $a \neq \bigcup_{i=1}^t e_i$ then*

$$\tilde{H}_i(E) = \tilde{H}_{i-1}(\varepsilon(e_1, \dots, e_t; V\ \ \{a\}))$$

for all i.

Proof. Let $\varepsilon_1 = \varepsilon(\{a\}; V)$ and let $\varepsilon_2 = \varepsilon(e_1, \dots, e_t; V)$. It is easily seen that $E = \varepsilon_1 \cup \varepsilon_2$. The simplicial complex ε_1 is in fact just a simplex (the $|V| - 2$ dimensional simplex on the vertices $V \setminus \{a\}$) and so has zero reduced homology everywhere. Also we note that $a \in V \setminus \bigcup_{i=1}^t e_i$ so by Lemma 4.2.1 $\tilde{H}_i(\varepsilon_2) = 0$ for all i. We now make use of the Mayer-Vietoris sequence

$$\cdots \longrightarrow \tilde{H}_i(\varepsilon_1) \oplus \tilde{H}_i(\varepsilon_2) \longrightarrow \tilde{H}_i(E) \longrightarrow \tilde{H}_{i-1}(\varepsilon_1 \cap \varepsilon_2) \longrightarrow \cdots$$

which becomes

$$\cdots \longrightarrow 0 \longrightarrow \tilde{H}_i(E) \longrightarrow \tilde{H}_{i-1}(\varepsilon_1 \cap \varepsilon_2) \longrightarrow 0 \longrightarrow \cdots$$

and therefore we have the isomorphism of reduced homology modules $\tilde{H}_i(E) \cong \tilde{H}_{i-1}(\varepsilon_1 \cap \varepsilon_2)$ for all i. By Lemma 4.2.3

$$\varepsilon_1 \cap \varepsilon_2 = \varepsilon(e_1 \cup \{a\}, \dots, e_t \cup \{a\}; V) = \varepsilon(e_1, \dots, e_t; V \setminus \{a\})$$

as $a \notin \bigcup_{i=1}^t e_i$. Therefore we conclude that

$$\tilde{H}_i(E) \cong \tilde{H}_{i-1}(\varepsilon(e_1, \dots, e_t; V \setminus \{a\}))$$

for all i. □

Corollary 4.2.8. *Let $a_1, \dots, a_s$ be s distinct elements of V and let $E = \varepsilon(\{a_1\}, \dots, \{a_s\}, e_1, \dots, e_t; V)$. If*

$$\{a_1, \dots, a_s\} \cap \bigcup_{i=1}^t e_i = \phi$$

then $\tilde{H}_i(E) = \tilde{H}_{i-s}(\varepsilon(e_1, \dots, e_t; V'))$, where $V' = V \setminus \{a_1, \dots, a_s\}$, for all i.

Proof. Because $a_j \notin (\bigcup i = j+1^s\{a_i\}) \cup (\bigcup i = 1^t e_i)$ for $j = 1, \ldots, s$ we may repeatedly apply Lemma 4.2.7 to obtain

$$\begin{aligned}
\tilde{H}_i(E) &= \tilde{H}_{i-1}(\varepsilon(\{a_2\}, \ldots, \{a_s\}, e_1, \ldots, e_t; V \setminus \{a_1\})) \\
&= \tilde{H}_{i-2}(\varepsilon(\{a_3\}, \ldots, \{a_s\}, e_1, \ldots, e_t; V \setminus \{a_1, a_2\})) \\
&= \cdots \\
&= \tilde{H}_{i-s}(\varepsilon(e_1, \ldots, e_t; V')).
\end{aligned}$$

□

Definition 4.2.9. Let $W \subseteq V = \{1, \ldots, n\}$. The induced subgraph of C_n on the elements of W will consist of one or more connected components which are straight line graphs. We will call such a connected subgraph which has m vertices a *run of length m*.

Example 4.2.10. If $n = 8$ and $W = \{1, 2, 4, 5, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\} = V(C_8)$, then the induced subgraph of C_8 on W consists of one run of length 2 and one run of length 3.

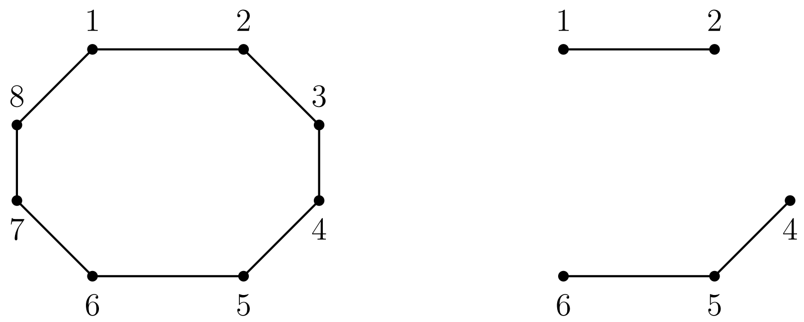

Figure 4.3: C_8 and an Induced Subgraph

4.3 Homology of Links

In this section we show how to find the reduced homology of links associated to subgraphs of cycles. We will see that these reduced homology modules are a shift of the reduced homology modules of links of 'smaller' subgraphs of C_n. Note that we will use the symbol k instead of $\mathbb{K}$

Theorem 4.3.1. *[66] For $F \in \Delta^*(C_n)$, with $F \neq \phi$, such that $Link^*_\Delta(F) = E(s_1, \ldots, s_r)$ with $r \geq 1$ and $s_r \geq 5$, we have the isomorphism of reduced homology modules $\tilde{H}_i(E(s_1, \ldots, s_r)) \cong \tilde{H}_{i-2}(E(s_1, \ldots, s_r - 3))$ for all i.*

Proof. Suppose the edges of G which are disjoint from F are $e_1, \ldots, e_t$, ordered such that $e_1, \ldots, e_{s_r} - 1$ are the edges in the run of length s_r. Let $E = E(s_1, \ldots, s_r) = \varepsilon(e_1, \ldots, e_t; V)$. For convenience we label the first five vertices in the run $1, 2, 3, 4, 5$ and so write $e_1 = \{1, 2\}, e_2 = \{2, 3\}, e_3 = \{3, 4\}$ and $e_4 = \{4, 5\}$. We can write $E = E_1 \cup E_2$ where $E_1 = \varepsilon(e_1; V)$ and $E_2 = \varepsilon(e_2, \ldots, e_t; V)$. We now use the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_i(E_1) \oplus \tilde{H}_i(E_2) \to \tilde{H}_i(E) \to \tilde{H}_{i-1}(E_1 \cap E_2) \to \tilde{H}_{i-1}(E_1) \oplus \tilde{H}_{i-1}(E_2) \to \cdots$$

We know that $\tilde{H}_i(E_1) = 0$ for all i because E_1 is a simplex. Also because $1 \in V \setminus \bigcup_{i=2}^t e_i$ we can use Lemma 4.2.1 to see that $\tilde{H}_i(E_2) = 0$ for all i. Hence the Mayer-Vietoris sequence becomes

$$\cdots \to 0 \to \tilde{H}_i(E) \to \tilde{H}_{i-1}(E_1 \cap E_2) \to \tilde{H}_{i-1}(E_1 \cap E_2) \to 0 \to \cdots$$

which implies $\tilde{H}_i(E) \cong \tilde{H}_{i-1}(E_1 \cap E_2)$ for all i. We now examine the simplicial complex $E_1 \cap E_2$. By Lemma (4.2.3)

$$\begin{aligned}
E_1 \cap E_2 &= \varepsilon(e_1 \cup e_2, e_1 \cup e_3, \ldots, e_1 \cup e_t; V) \\
&= \varepsilon(\{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \ldots, \{1, 2\} \cup e_t; V) \\
&= \varepsilon(\{3\}, \{3, 4\}, \{4, 5\}, \ldots, e_t; V') \quad \text{(using Lemma (4.2.4) as} \{3\} \subseteq \{3, 4\}) \\
&= \varepsilon(\{3\}, \{4, 5\}, \ldots, e_t; V') \\
&= \varepsilon(\{3\}, e_4, \ldots, e_t; V')
\end{aligned}$$

where $V' = V \setminus e_1$. Because $3 \notin \bigcup_{i=4}^t e_i$ we use Lemma 4.2.7 to see that $\tilde{H}_i(\varepsilon(\{3\}, e_4, \ldots, e_t; V')) = \tilde{H}_{i-1}(\varepsilon(e_4, \ldots, e_t; V''))$ where $V'' = V \setminus \{1, 2, 3\}$. Putting these results together

$$\begin{aligned}\tilde{H}_i(E) &\cong \tilde{H}_{i-1}(\varepsilon(\{3\}, e_4, \ldots, e_t; V'))\\ &\cong \tilde{H}_{i-2}(\varepsilon(e_4, \ldots, e_t; V''))\\ &\cong \tilde{H}_{i-2}(E(s_1, \ldots, s_r - 3)).\end{aligned}$$

□

Remark 4.3.2. There is no ordering of $s_1, \ldots, s_r$ in $E(s_1, \ldots, s_r)$, i.e.,

$$E(s_1, \ldots, s_r) = E(\sigma(s_1), \ldots, \sigma(s_r))$$

for any permutation, σ, of $s_1, \ldots, s_r$. So long as $r \geq 1$ and $s_j \geq 5$, with $1 \leq j \leq r$, then we have an isomorphism of reduced homology groups $\tilde{H}_i(E(s_1, \ldots, s_j, \ldots, s_r)) \cong \tilde{H}_{i-2}(E(s_1, \ldots, s_j - 3, \ldots, s_r))$ for all i.

We now consider what happens when we have runs of lengths less than 5.

Theorem 4.3.3. *[66] Let $E = E(s_1, \ldots, s_r)$.*

(i) If $s_r = 2$ and $r \geq 2$ then $\tilde{H}_i(E(s_1, \ldots, s_r)) = \tilde{H}_{i-1}(E(s_1, \ldots, s_{r-1}))$.

(ii) If $s_r = 3$ and $r \geq 2$ then $\tilde{H}_i(E(s_1, \ldots, s_r)) = \tilde{H}_{i-2}(E(s_1, \ldots, s_{r-1}))$.

(iii) If $s_r = 4$ and $r \geq 1$ then $\tilde{H}_i(E(s_1, \ldots, s_r)) = 0$.

Proof. (i). Let $E = E(s_1, \ldots, s_{r-1}, 2) = \varepsilon(\{1, 2\}, e_2, \ldots, e_t; V)$ for some edges $e_2, \ldots, e_t$ and where $\{1, 2\}$ is the run of length 2. Let $E_1 = \varepsilon(\{1, 2\}; V)$ and $E_2 = \varepsilon(e_2, \ldots, e_t; V)$ so that $E = E_1 \cup E_2$. We have $\tilde{H}_i(E_1) = 0$ for all i because E_1 is a simplex, and $\tilde{H}_i(E_2) = 0$ for all i by Lemma 4.2.1 as $1, 2 \in V \setminus \bigcup_{i=2}^t e_i$. Use of the Mayer-Vietoris sequence implies that $\tilde{H}_i(E) = \tilde{H}_{i-1}(E_1 \cap E_2)$ for all i. We now see that

$$\begin{aligned}E_1 \cap E_2 &= \varepsilon(\{1, 2\} \cup e_2, \ldots, \{1, 2\} \cup e_t; V) \quad \text{(using Lemma (6.2.11))}\\ &= \varepsilon(e_2, \ldots, e_t; V')\end{aligned}$$

where $V' = V \setminus \{1,2\}$. Hence $\tilde{H}_i(E(s_1,\ldots,s_r)) = \tilde{H}_{i-1}(\varepsilon(e_2,\ldots,e_t;V'))$ for all i. The simplicial complex $\varepsilon(e_2,\ldots,e_t;V') = E(s_1,\ldots,s_{r-1})$, as this is the complex associated with the induced subgraph of C_n which is composed of runs of lengths $s_1,\ldots,s_{r-1}$. Hence $\tilde{H}_i(E(s_1,\ldots,s_r)) = \tilde{H}_{i-1}(E(s_1,\ldots,s_{r-1}))$ for all i.

(ii). Now let $E = E(s_1,\ldots,s_{r-1},3) = \varepsilon(\{1,2\},\{2,3\},e_3,\ldots,e_t;V)$ where $\{1,2\},\{2,3\}$ is a run of length 3 and so none of 1,2 or 3 is in any of $e_3,\ldots,e_t$. Let $E_1 = \varepsilon(\{1,2\};V)$ and $E_2 = \varepsilon(\{2,3\},e_3,\ldots,e_t;V)$. As before $\tilde{H}_i(E_1) = \tilde{H}_i(E_2) = 0$ for all i (E_1 is a simplex and $1 \in V \setminus (\{2,3\}\bigcup_{i=3}^t e_i)$ so Lemma 4.2.1 applies). The Mayer-Vietoris sequence implies $\tilde{H}_i(E) \cong \tilde{H}_{i-1}(E_1 \cap E_2)$, for all i. We have

$$\begin{aligned} E_1 \cap E_2 &= \varepsilon(\{1,2,3\},\{1,2\}\cup e_3,\ldots,\{1,2\}\cup e_t;V) \\ &= \varepsilon(\{3\},e_3,\ldots,e_t;V') \end{aligned}$$

where $V' = V \setminus \{1,2\}$. As $3 \notin \bigcup_{i=3}^t e_i$ we can make use of Lemma 4.2.7 to obtain, for all i,

$$\begin{aligned} \tilde{H}_i(E_1 \cap E_2) &= \tilde{H}_i(\varepsilon(\{3\},e_3,\ldots,e_t;V')) \\ &= \tilde{H}_{i-1}(\varepsilon(e_3,\ldots,e_t;V'')) \end{aligned}$$

where $V' = V' \setminus \{3\}$. We note that $\varepsilon(e_3,\ldots,e_t;V'') = E(s_1,\ldots,s_{r-1})$, the simplicial complex associated with the induced subgraph of C_n which is $r-1$ runs of lengths $s_1,\ldots,s_{r-1}$. Hence $\tilde{H}_i(E(s_1,\ldots,s_{r-1},3)) = \tilde{H}_{i-2}(\varepsilon(e_3,\ldots,e_t;V'')) = \tilde{H}_{i-2}(E(s_1,\ldots,s_{r-1}))$.

(iii). Finally suppose that

$$E = E(s_1,\ldots,s_{r-1},4) = \varepsilon(\{1,2\},\{2,3\},\{3,4\},e_4,\ldots,e_t;V).$$

with $\{1,2\},\{2,3\},\{3,4\}$ being the run of length 4. Let $E_1 = \varepsilon(\{1,2\};V)$ and $E_2 = \varepsilon(\{2,3\},\{3,4\},e_4,\ldots,e_t;V)$. For similar reasons to the previous parts we have $\tilde{H}_i(E_1) = \tilde{H}_i(E_2) = 0$ for all i and so $\tilde{H}_i(E) = \tilde{H}_{i-1}(E_1 \cap E_2)$ for all i. We see that

$$\begin{aligned} E_1 \cap E_2 &= \varepsilon(\{1,2,3\},\{1,2,3,4\},\{1,2\}\cup e_4,\dots,\{1,2\}\cup e_t;V) \\ &= \varepsilon(\{3\},\{3,4\},e_4,\dots,e_t;V') \\ &= \varepsilon(\{3\},e_4,\dots,e_t;V') \qquad \text{(using Lemma 4.2.4)} \end{aligned}$$

where $V' = V \setminus \{1,2\}$. Since $4 \in V' \setminus (\{3\} \cup \bigcup_{i=4}^{t} e_i)$ we can use Lemma 4.2.1 to see that $\tilde{H}_i(\varepsilon(\{3\},e_4,\dots,e_t;V')) = 0$ for all i. Therefore we also have $\tilde{H}_i(E) = 0$ for all i.

□

We now find the reduced homology of the simplicial complexes $E(2)$ and $E(3)$. All the simplicial complexes we are interested in here (those of the form $E(s_1,\dots,s_r)$) will have reduced homology which is some shift of the reduced homology of $E(2)$ or $E(3)$.

Proposition 4.3.4. *For $E = E(2)$ we have*

$$\tilde{H}_i(E(2)) = \begin{cases} k & \text{if } i = -1 \\ 0 & \text{if } i \neq -1. \end{cases}$$

Proof. The result follows from observing that E is in fact the empty simplicial complex. We may write

$$\begin{aligned} E &= \varepsilon(\{1,2\};\{1,2\}) \qquad \text{(this is the simplicial complex with one maximal face} \{1,2\}\setminus\{1,2\}) \\ &= \{\phi\}. \end{aligned}$$

□

Proposition 4.3.5. *For $E = E(3)$ we have*

$$\tilde{H}_i(E(3)) = \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

Proof. We may write E in the following way

$$\begin{aligned} E &= \varepsilon(\{1,2\};\{2,3\};\{1,2,3\}) \\ &= \{\{3\},\{1\}\}. \end{aligned}$$

This is the simplicial complex which is made up of two disjoint 0 dimensional faces hence the result. $\square$

Proposition 4.3.6. *[66] If $E = E(s_1,\ldots,s_r,3m+1)$, for some $m > 1$, then $\tilde{H}_i(E) = 0$ for all i.*

Proof. We may describe E as follows. Let

$$E = \varepsilon(\{v_1,v_2\},\{v_2,v_3\},\ldots,v_{3m},v_{3m+1},e_1,\ldots,e_t;V).$$

where $v_1,\ldots,v_{3m+1}$ is the run of length $3m+1$. We will split E into two complexes and use the Mayer-Vietoris sequence. Let $E_1 = \varepsilon(\{v_{3m},v_{3m+1}\};V)$ and let

$$E_2 = \varepsilon(\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{3m-1},v_{3m}\},e_1,\ldots,e_t;V).$$

We have $\tilde{H}_i(E_1) = 0$ for all i because E_1 is a simplex and $\tilde{H}_i(E_2) = 0$ for all i by Lemma 4.2.1 because v_{3m+1} is in every maximal face of E_2. We also have $E = E_1 \cup E_2$ and so we may use the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_i(E_1) \oplus \tilde{H}_i(E_2) \to \tilde{H}_i(E) \to \tilde{H}_{i-1}(E_1 \cap E_2) \to \cdots$$

which becomes

$$\cdots \to 0 \to \tilde{H}_i(E) \to \tilde{H}_{i-1}(E_1 \cap E_2) \to 0 \to \cdots$$

from which we see that $\tilde{H}_i(E) \cong \tilde{H}_{i-1}(E_1 \cap E_2)$ for all i. So we now examine the complex $E_1 \cap E_2$.

$$\begin{aligned} E_1 \cap E_2 = \varepsilon(&\{v_1,v_2,v_{3m},v_{3m+1}\},\{v_2,v_3,v_{3m},v_{3m+1}\},\ldots \\ &\ldots,\{v_{3m-2},v_{3m-1},v_{3m},v_{3m+1}\},\{v_{3m-1},v_{3m},v_{3m+1}\},e_1,\ldots,e_t;V) \\ = \varepsilon(&\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{3m-3},v_{3m-2}\},\{v_{3m-2},v_{3m-1}\},\{v_{3m-1}\},e_1,\ldots,e_t;V') \\ = \varepsilon(&\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{3m-3},v_{3m-2}\},\{v_{3m-1}\},e_1,\ldots,e_t;V') \end{aligned}$$

where $V' = V \setminus \{v_{3m}, v_{3m+1}\}$. By Lemma 4.2.7, since

$$v_{3m-1} \notin (\bigcup_{j=1}^{t} e_j) \cup (\{v_1, v_2\} \cup \cdots \cup \{v_{3m-3}, v_{3m-2}\}).$$

we deduce that, for all i,

$$\begin{aligned}\tilde{H}_i(E) &\cong \tilde{H}_{i-1}(E_1 \cap E_2)\\ &\cong \tilde{H}_{i-2}(\varepsilon(\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{3m-3}, v_{3m-2}\}, e_1, \ldots, e_t; V' \setminus \{v_{3m-1}\})\\ &= \tilde{H}_{i-2}(\varepsilon(\{v_1, v_2\}, \{v_2, v_3\}, \ldots, v_{3(m-1)}, v_{3(m-1)+1}, e_1, \ldots, e_t; V') \setminus \{v_{3m-1}\})\end{aligned}$$

The simplicial complex

$$\varepsilon(\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{3(m-1)}, v_{3(m-1)+1}\}, e_1, \ldots, e_t; V' \setminus \{v_{3m-1}\})$$

may be described as $E(s_1, \ldots, s_r, 3(m-1)+1)$. We have shown that the reduced homology of E is a shift of the homology of

$$E(s_1, \ldots, s_r, 3(m-1)+1),$$

i.e., $\tilde{H}_i(E(s_1, \ldots, s_r, 3m+1)) \cong \tilde{H}_{i-2}(E(s_1, \ldots, s_r, 3(m-1)+1))$ for all i. We can make repeated use of the above method to obtain

$$\tilde{H}_i(E) \cong \tilde{H}_{i-2m+2}(E(s_1, \ldots, s_r, 4))$$

for all i. And so, by Proposition 4.3.3, we have $\tilde{H}_i(E) = 0$ for all i. □

We have now deduced that if F is a non-empty face of $\Delta^*(C_n)$ such that $\mathrm{Link}_{\Delta^*(C_n)}F$ has some non-zero reduced homology modules then

$$\mathrm{Link}_{\Delta^*(C_n)}F = E(3p_1, \ldots, 3p_\alpha, 3q_1 + 2, \ldots, 3q_\beta + 2).$$

That is to say $\mathrm{Link}_{\Delta^*(C_n)}F$ corresponds to a subgraph of C_n with runs only of lengths 0 and 2 mod 3. If we use the the formula

$$\beta_i(C_n) = \sum_{H \subset C_n} \dim_k \tilde{H}_{i-2}(\varepsilon(H))$$

(where the sum is over induced subgraphs of C_n which contain at least one edge) we may exclude all the induced subgraphs of C_n with any runs of length 1 mod 3 from consideration as these will contribute 0.

Lemma 4.3.7. *[66] Write $E(3^\alpha, 2^\beta)$ to denote the simplicial complex associated with the graph which is α runs of length 2 and β runs of length 3.*

$$\tilde{H}_i(E(3^\alpha, 2^\beta)) = \begin{cases} k & \textit{if } i = 2\alpha + \beta - 2 \\ 0 & \textit{if } i \neq 2\alpha + \beta - 2. \end{cases}$$

Proof. We first make repeated use of 4.3.3 part (i) to see that $\tilde{H}_i(E(3^\alpha, 2^\beta)) \cong \tilde{H}_{i-\beta}(E(3^\alpha))$ for all i. Now we use 4.3.3 part (ii) to obtain

$$\begin{aligned} \tilde{H}_i(E(3,\ldots,3,2,\ldots,2)) &\cong \tilde{H}_{i-\beta-2(\alpha-1)}(E(3)) \\ &= \begin{cases} k & \text{if } i = 2\alpha + \beta - 2 \\ 0 & \text{if } i \neq 2\alpha + \beta - 2 \end{cases} \end{aligned}$$

using proposition 4.3.5, which gives the reduced homology of $E(3)$. □

Proposition 4.3.8. *[66] Let $E = E(3p_1, \ldots, 3p_\alpha, 3q_1 + 2, \ldots, 3q_\beta + 2)$, for integers $p_1, \ldots, p_\alpha, q_1, \ldots, q$, and let $P = \sum_{j=1}^{\alpha} p_j$ and $Q = \sum_{j=1}^{\beta} q_j$. Then*

$$\dim_k(\tilde{H}_i(E)) = \begin{cases} 1 & \textit{if } i = 2(P+Q) + \beta - 2 \\ 0 & \textit{if } i \neq 2(P+Q) + \beta - 2. \end{cases}$$

Proof. Repeatedly applying 4.3.1, we see that

$$
\begin{aligned}
\tilde{H}_i(E) &= \tilde{H}_{i-2}(E(3p_1-3,\dots,3p_\alpha,3q_1+2,\dots,3q_\beta+2)) \\
&= \tilde{H}_{i-2}(p_1-1)(E(3,3p_2,\dots,3p_\alpha,3q_1+2,\dots,3q_\beta+2)) \\
&\quad \text{(Applying 4.3.1 } p_1-1 \text{ times to the first run)} \\
&= \tilde{H}_{i-2(p_1-1)-2(p_2-1)}(E(3,3,3p_3,\dots,3p_\alpha,3q_1+2,\dots,3q_\beta+2)) \\
&\quad \text{(Applying 4.3.1 } p_2-1 \text{ times to the second run)} \\
&= \tilde{H}_{i-2P+2\alpha}(E(3^\alpha,3q_1+2,\dots,3q_\beta+2)) \\
&= \tilde{H}_{i-2P+2\alpha-2}(E(3^\alpha,3(q_1-1)+2,\dots,3q_\beta+2)) \quad \text{(Applying 4.3.1 to the next run)} \\
&= \tilde{H}_{i-2P+2\alpha-2q_1}(E(3^\alpha,2,\dots,3q_\beta+2)) \\
&= \tilde{H}i-2(P+Q)+2\alpha(E(3^\alpha,2^\beta)).
\end{aligned}
$$

We now use this together with Lemma 4.3.7, which shows us that $\tilde{H}_i(E(3^\alpha,2^\beta)) = k$ if and only if $i = 2\alpha+\beta-2$ (and is zero otherwise), to conclude that

$$
\begin{aligned}
\tilde{H}_i(E) &= \begin{cases} k & \text{if } i-2(P+Q)+2\alpha = 2\alpha+\beta-2 \\ 0 & \text{if } i-2(P+Q)+2\alpha \neq 2\alpha+\beta-2. \end{cases} \\
&= \begin{cases} k & \text{if } i = 2(P+Q)+\beta-2 \\ 0 & \text{if } i \neq 2(P+Q)+\beta-2. \end{cases}
\end{aligned}
$$

and the result is proved. □

4.4 Betti Numbers from Counting Induced Subgraphs

To find the Betti numbers of C_n we use the formula

$$\beta_i(C_n) = \sum_{F\in\Delta^*(C_n)} \dim_k \tilde{H}_{i-2}(Link_{\Delta^*}(C_n)F;k)$$

from 3.2.5. Along with the above result this shows that $\beta_1(C_n)$ is equal to the number of faces of $\Delta^*(C_n)$ such that $\dim_k \hat{H}_{d-2}\left(\mathrm{Link}_{\Delta^*(C_\alpha)} F\right) = 1$. This is the number of faces of $\Delta^*(C_\mathrm{n})$ with link of the form

$$E\left(3p_1 \ldots, 3p_a, 3q_1 + 2, \ldots, 3q_\beta + 2\right)$$

such that $i = 2(P+Q)+\beta.$, where $P = \sum_{j=1}^{a} P_y$ and $Q = \sum_{j=1}^{\beta} q_j$. It is now convenient to consider the N-graded Betti numbers of C_n separately. This is because when using the formula of (EIR) the sum which gives $\beta_{1,d}$ is taken aver all induced subgroups which have precisely d vertices.

Remark 4.4.1. We will use Theorem 4.1.9. The *ith* N-graded Betti number of degree d of I is $= \sum_{H \subset C_\mathrm{n}, V(H_x = d} \dim_k \hat{H}_{i-2}([(H))$ Note that

$$\beta_1(C_n) = \sum_{d \in \mathbb{N}} \beta_{1,d}(C_n)$$

Lemma 4.4.2. *[66] The ith Betti number of degree $d, \beta_{1,d}$, may be obtained by counting the number of faces of $\Delta^*(C_n)$ which hase link of the form*

$$E\left(3p_1 \ldots, 3p_{a_1} 3q_1 + 2, \ldots, 3q_a + 2\right)$$

for integers $p_1, \ldots, p_a, q_1, \ldots, q_\beta$, such that

(i). $\quad i = 2(P+Q)+\beta$

(ii). $d = 3(p+Q)+2\beta$ *(where $P = \sum_{j=1}^{a} p_j$ and $Q = \sum_{j=1}^{\beta}$ qy.*

Equivalently, $\beta_{y,d}$ is the number of induced subgraph is of C_n which are composed of runs of lengths

$$3p_1, \ldots, 3p_a, 3q_1 + 2, \ldots, 3q_\beta + 2$$

satisfying the above two conditions.

Proof. As previously noticed the only links of faces of $\Delta^*(C_\mathrm{n})$ which may have non-zero reduced homology are of the form $E(3p_1 \ldots, 3p_a, 3q_1 + 2, \ldots, 3q\beta + 2)$ which corresponds to a subgraph of C_n which has runs of length is

$$3p_1, \ldots, 3p_a, 3q_1 + 2, \ldots, 3q_\beta + 2$$

Such a subgraph has d vertices, i.e., $d = 3p_2 + \cdots + 3p_a + 3q_1 + 2 + \cdots + 3q_\beta + 2 = 3(P+Q) + 2\beta,$ which prowidess the condition (ii). By Proposition 4.3.8 reduced homology module, $\tilde{H}_{1-2}(E(3p_1 \ldots, 3p_a, 3q_1 + 2, \ldots, 3qs + 2))$, is nonzero (and 1-dimensional) if and only if $i = 2(P+Q) + \beta$, which is condition (i). Therefore each such induced subgraph of C_n contributes ' 1 ' to $\beta_{1,d}$ in the formula $\beta_{1,d} = \sum_{H \subset C_n, |V(H)|=d} \dim_k \tilde{H}_{i-2}(\varepsilon(H))$. □

Proposition 4.4.3. *[66] The ith Betti number of degree $d = 2i, \beta_{i,2i}(C_n)$, is the number of induced subgraphs of C_n which are comprised of i runs of length* 2.

Proof. We consider when a face of $\Delta^*(C_\mathrm{n})$ will contribute 1 to $\beta_{i,2i}$ in the sum from 4.1.9. From Lemma 4.4.2 the face F of $\Delta^*(C_n)$ will contribute to $\beta_{4,2x}(C_n)$ if and only if Link $_{\Delta^*(C_a)}F = E(\mathrm{Sp}_1 \ldots, 3p_{a1}, 3q_1 + 2, \ldots, 3q_\beta + 2)$, for some integers $p_1, \ldots, p_a, q_1, \ldots, q_\beta$ such that $i = 2(p+Q) + \beta$ and $2i = 3(P+Q) + 2\beta$ where $P = \sum_{j=1}^{a} P_j, Q = \sum_{j=1}^{\beta} q_j$. These conditions imply that $2(P+Q) + \beta = i = 2i - i = P + Q + \beta$. A.s all these numbers are nonnegative integers, this is only possible if $P = Q = 0$ which in turn implies that $p_1 = \cdots = p_a = q_1 = \cdots = q_\beta = 0$ and $i = \beta$. Hence Link $_{\Delta^*}(C_n)F = E(2^1)$. The *ith* Betti number of degree $2i$ is thus obtained by counting all the induced subgraphs of C_n which correspond to the links of such faces. These are the subgraphs which are i rans of length 2. □

4.5 Counting Runs

As shown above calculating Betti numbers of cycles is equivalent to counting particular types of induced subgraphs. Here we demonstrate how this may be done.

Definition 4.5.1. *(i). Define $B(n, l, m)$ to be the number of ways of choosing an induced subgraph of C_n which consists of m runs of length l.*

(ii). Define $C(n, l, m)$ to be the number of such subgraphs in which neither 1 nor n feature in a run.

Lemma 4.5.2. *The number of choices of* m *runs of length l in C_{m} such that neither I nor* n *are fentured in any run is*

$$C(n, l, m) = \binom{n - Im - 1}{m}$$

Proof. We can find $C(n, l, m)$ by considering the number of ways of separating the vertices which are not in any run by the m runs. There are $n - lm$ vertices (including the vertices 1 and n) which cannot be in any runs. We can construct all the appropriate subgraphs by placing the m runs of length l in the spaces between the adjacent vertices which are not in any run (except between 1 and n). There are $n - lm - 1$ spaces into each we can choose to place, or not place, a run. There are $\binom{n - lm - 1}{m}$ ways of doing this. □

Lemma 4.5.3. *[66] The number of choices of m runs of length l in C_n is*

$$B(n, l, m) = \binom{n - lm}{m} + l \binom{n - lm - 1}{m - 1}$$

Proof. We first count those subgraphs which have at least one of 1 and n in a run. There are $l + 1$ possible runs of length l which contain 1 or n (or both). The other $m - 1$ runs are selected from the remaining $n - l$ vertices. Since the vertices adjacent to the run containing 1 or n cannot be included we have $C(n-l,l,m-1)$ choices. Hence

$$\begin{aligned} B(n,l,m) &= C(n,l,m) + (l+1)C(n-l,l,m-1) \\ &= \begin{pmatrix} n-lm-1 \\ m \end{pmatrix} + (l+1)\begin{pmatrix} n-lm-1 \\ m-1 \end{pmatrix} \\ &= \begin{pmatrix} n-lm \\ m \end{pmatrix} + l\begin{pmatrix} n-lm-1 \\ m-1 \end{pmatrix} \end{aligned}$$

□

Corollary 4.5.4. *The ith Betti number of degree $2i$ of C_n is*

$$\beta_{i,2i}(C_n) = \binom{n-2i}{i} + 2\binom{n-2i-1}{i-1}$$

Proof. By 4.5.3 $\beta_{i,2i}(C_n)$ is the number of subgraphs of C_n which are i runs of length 2 . That is, $\beta_{i,2i}(C_n) = B(n,2,i)$. Using 4.5.2

$$\beta_{i,2i}(C_n) = B(n,2,i) = \begin{pmatrix} n-2i \\ i \end{pmatrix} + 2\begin{pmatrix} n-2i-1 \\ i-1 \end{pmatrix}.$$

□

Lemma 4.5.5. *Let A be an induced subgraph of C_n composed of i runs of length 2 and let $j \leqslant i$. Select j of the vertices of C_n which are adjacent to, and clockwise from, the runs of A. Let A' be the induced subgraph of C_n which has the j vertices selected together with the vertices of A. The graph A' is comprised of runs of lengths* $0 \bmod 3$ *and* $2 \bmod 3$ *only.*

Proof. A run of A' will be of one of the following forms.

(i) It was constructed from a string of r runs of length 2 of A which were separated by single vertices. The new run includes these formerly separating vertices along with all the vertices of the r runs of length 2 . It will have length $2r+(r-1) = 3r-1 \equiv 2 \bmod 3$.

(ii) It was constructed as in part (i) but also includes the vertex immediately clockwise of the clockwise most of the r runs. In this case the new run will be of length $2r + r = 3r \equiv 0 \bmod 3$.

$\square$

Note the case $r = 1$ in the above gives a run of length 2 or 3 .

Proposition 4.5.6. *[66] For $2i + j \neq n$*

$$\begin{aligned}\beta_{i+j,2i+j}\left(C_n\right) &= \left(\begin{array}{c} i \\ j \end{array}\right)\beta_{i,2i} \\ &= \left(\begin{array}{c} i \\ j \end{array}\right)\left\{\left(\begin{array}{c} n-2i \\ i \end{array}\right) + 2\left(\begin{array}{c} n-2i-1 \\ i-1 \end{array}\right)\right\}\end{aligned}$$

Proof. First observe that any N-graded Betti number can be written in the above way since $\beta_{l,d} = \beta_{i+j,2i+j}$ where $i = d-l$ and $j = 2l-d$. By Lemma 4.4.2 $\beta_i + j, 2i + j$ is the number of faces of $\Delta^*\left(C_n\right)$ which have link of the form $E\left(3p_1, \ldots, 3p_\alpha, 3q_1 + 2, \ldots, 3q_\beta + 2\right)$, for some integers $p_1, \ldots, p_\alpha, q_1, \ldots, q_\beta$, such that

$$\left\{\begin{array}{l} i + j = 2(P+Q) + \beta \\ 2i + j = 3(P+Q) + 2\beta \end{array}\right.$$

where $P = \sum_{j-1}^{\alpha} p_j$ and $Q = \sum_{j-1}^{\beta} q_j$. This is equivalent to counting the induced subgraphs of C_n which consist of runs of lengths $3p_1, \ldots, 3p_\alpha, 3q_1 + 2, \ldots, 3q_\beta + 2$, for some $p_1, \ldots, p_\alpha, q_1, \ldots, q_\beta$, satisfying the above conditions. We can construct any such subgraph starting from a subgraph of i runs of length 2 in C_n. Let $A \subset C_n$ denote a subgraph which consists of i runs of length 2 .

There are i vertices of C_n which are adjacent to, and clockwise from, the runs of A. Select any j of these vertices and add them the run (or runs) to which they are adjacent.

By Lemma 4.5.5 the new subgraph of C_n which this defines, A', say, will be comprised of runs of lengths 0 mod 3 and 2 mod 3 only.

Say the runs are of lengths $3p_1, \ldots, 3p_\alpha, 3q_1 + 2, \ldots, 3q_\beta + 2$. The number of vertices in A' is clearly $2i + j = 3P + 3Q + 2\beta$ where $P = \sum_{i=1}^{\alpha} p_i$ and $Q = \sum_{i-1}^{\beta} q_i$. To show A' is a subgraph of the type contributing to $\beta_{i+j,2i+j}$ it remains to demonstrate that $i + j = 2(P + Q) + \beta$.

When $j = 0$ we see that $P + Q = 0$ as we have only runs of length 2 . In this case $i + j = i = \beta = 2(P + Q) + \beta$. Now suppose that $j > 0$. Each selection of one of the j vertices either forms a run of length 3 or else increases a run by 3. Whichever of these happens $P + Q$ increases by one. Adding j vertions will increase $P + Q$ by j. Because $P + Q = 0$ when $j = 0$ this implies that $j = P + Q$. Hence

$$\begin{aligned} 2i &= 3P + 3Q + 2\beta - j \\ &= 3P + 3Q + 2\beta - P - Q \\ &= 2P + 2Q + 2\beta \end{aligned}$$

And so $i = p + Q + \beta$. Therefore we have $i + j = 2(p + Q) + \beta$ as required. We now show that every subgroup of C_{n} which contributes to $\beta_{t+j,2x+j}$ can be constructed in this way. Suppose that

$$E = E\left(3p_1, \ldots, 3p_{a-}3q_1 + 2, \ldots, 3q_\beta + 2\right)$$

is the link of a face of $\Delta^*\left(C_{\mathrm{n}}\right)$ satisfying the above conditions. Consider the associated subgraph of C_{n}, which is comprised of runs of lengths

$$3p_1, \ldots, 3p_a, 3q_1 + 2, \ldots, 3q\beta + 2$$

Far a run of length $I \equiv 0$ mod 3 with vertices $a_1, a_2, \ldots, a_1$ remove the vertices $a_3, a_{6,\ldots,}$ al. Repeat this with every run of length 0 mod 3 . For a run of length $m \equiv 2$ mod 3 with vertices $b_1, b_2, \ldots, b_m$ remove the vertices $b_{\mathrm{s}}, b_{\mathrm{G}}, b_2, \ldots, b_{\mathrm{m}-2}$. This leaves a graph of i runs of length 2 from which our earlier construction gives the graph associated with E

We count the graphs we can construct in this way. For any graph which is i runs of length 2 we are choosing j of the i vertices which are adjacent and clockwise to the runs. So we have $\begin{pmatrix} 1 \\ j \end{pmatrix}$ graphs constructed far each arrangement of i runs of length 2, of which there are $\beta_{1,2}(C_n)$, by Proposition 4.4.3

$$\begin{aligned} \beta_{i+j,2i+j}(C_n) &= \begin{pmatrix} i \\ j \end{pmatrix} \beta_{i,2i}(C_n) \\ &= \begin{pmatrix} i \\ j \end{pmatrix} \left\{ \begin{pmatrix} n-2i \\ i \end{pmatrix} + 2 \begin{pmatrix} n-2i-1 \\ i-1 \end{pmatrix} \right\} \end{aligned}$$

$\square$

4.6 Betti Numbers of Degree n

Now it only remains to calculate the reduced homology of $\mathrm{Link}_{\Delta^*}(C_n)^{\phi}$. It can be easily seen fromthat Link $_{\Delta^*(C_n)}\emptyset = \Delta^*(C_n)$. Note that, using the correspondence between faces of $\Delta^*(C_n)$ and induced subgraphs of C_n, the face $\emptyset$ is associated with C_n. This is the only subgraph of C_n which cannot be thought of as a collection of runs and so must be considered separately.

Lemma 4.6.1. *Define the simplicial complex* Γ *as follous*

$$\Gamma = \varepsilon(\{1,2\},\{2,3\},\ldots,\{m-1,m\},\{m+1\},\{m+2\};[m+2])$$

Then Γ *has reduced homology which is a shift of the reduced homology of* $E(m)$*, the simplicial complex associated with the graph which is a single run of length* m*, as follows*

$$\begin{aligned} \tilde{H}_i(\Gamma) &= \tilde{H}_{i-2}(\varepsilon(\{1,2\},\{2,3\},\ldots,\{m-1,m\};[m]) \\ &= \tilde{H}_{i-2}(E(m)) \end{aligned}$$

Proof. This is just Lemma 4.2.7 used twice. Note that the vertex $m+2$ is in none of the sets $\{1,2\},\{2,3\},\ldots,\{m-1,m\},\{m+1\}$ so Lemma 4.2.7 implies

$$\begin{aligned}\widetilde{H}_i(\Gamma) &= \widetilde{H}_i(\varepsilon(\{1,2\},\{2,3\},\ldots,\{m-1,m\},\{m+1\},\{m+2\};[m+2]) \\ &= \widetilde{H}_{i-1}(\varepsilon(\{1,2\},\{2,3\},\ldots,\{m-1,m\},\{m+1\};[m+1])\end{aligned}$$

for all i. Now we note that the vertex $m+1$ is not in any of the sets $\{1,2\},\{2,3\},\ldots,\{m-1,m\}$ and we apply Lemma 4.2.7 once more to obtain

$$\begin{aligned}\widetilde{H}_i(\Gamma) &= \widetilde{H}_{i-1}(\varepsilon(\{1,2\},\{2,3\},\ldots,\{m-1,m\},\{m+1\};[m+1]) \\ &= \widetilde{H}_{i-2}(\varepsilon(\{1,2\},\{2,3\},\ldots,\{m-1,m\};[m]) \\ &= \widetilde{H}_{i-2}(E(m)).\end{aligned}$$

for all i. □

Lemma 4.6.2. *The ith Betti number of degree n is*

$$\begin{aligned}\beta_{i,n}\left(C_n\right) &= \dim_k \widetilde{H}_{i-2}\left(\Delta^*\left(C_n\right);k\right) \\ &= \dim_k \widetilde{H}_{i-2}\left(\varepsilon\left(C_n\right);k\right).\end{aligned}$$

Proof. Using the formula from 3.2.5 we have

$$\beta_{i,n}\left(C_n\right) = \sum_{F\in\Delta^*(C_n):|V(C_n)\setminus F|-n} \dim_k \widetilde{H}_{i-2}\left(\mathrm{Link}_{\Delta^*(C_n)} F;k\right) = \sum_{H\subset C_n} \dim_k \widetilde{H}_{i-2}(\varepsilon(H);k).$$

The only face of $\Delta^*(C_n)$ which is such that its complement is of size n is the empty set $\emptyset$. Also Link $\Delta^*\left(C_n\right)\theta = \Delta^*\left(C_n\right)$ so we have $\beta_{i,n}\left(C_n\right) = \dim_k \widetilde{H}_{i-2}\left(\Delta^*\left(C_n\right)\right)$. We can also write $\Delta^*\left(C_n\right)$ as $\varepsilon\left(C_n\right)$ □

Proposition 4.6.3. *[66] The non-zero Betti numbers of C_n of degree n are as follows:*

(i) *If* $n \equiv 1 \bmod 3$, $\beta_{\frac{2n+1}{3},n}(C_n) = 1$

(ii) *If* $n \equiv 2 \bmod 3$, $\beta_{\frac{2n-1}{3},n}(C_n) = 1$

(iii) *If* $n \equiv 0 \bmod 3$, $\beta_{\frac{2n}{3},n}(C_n) = 2$

Proof. As commented above Link $\Delta^*(C_n)^0 = \Delta^*(C_n)$. We can write the Alexander dual as

$$\begin{aligned}\Delta^*(C_n) &= \varepsilon(\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\};[n])\\ &= E_1 \cup E_2\end{aligned}$$

where $E_1 = \varepsilon(\{1,2\},\{2,3\},\ldots,\{n-1,n\};[n])$ and $E_2 = \varepsilon(\{n,1\};[n])$. The intersection of these simplicial complexes is

$$\begin{aligned}E_1 \cap E_2 =& \varepsilon(\{1,2,n\},\{1,2,3,n\},\{3,4,1,n\},\ldots\\ &\ldots,\{n-3,n-2,1,n\},\{n-2,n-1,n,1\}\{n-1,n,1\};[n])\\ =&\varepsilon(\{2\},\{3,4\},\ldots,\{n-3,n-2\},\{n-1\};[n]\backslash\{1,n\})\\ &\text{(Using Lemmas (4.2.3) and (4.2.4)).}\end{aligned}$$

By Lemma (4.6.1 $\widetilde{H}_i(E_1 \cap E_2) = \widetilde{H}_{i-2}(E(n-4))$

(i) We now consider the first of these cases. Here we have $n = 3m+1$ for some m. By Proposition 4.3.6 $\tilde{H}_{\mathrm{i}}(E_1) = 0$ for all i. We also have $\widetilde{H}_{\mathrm{i}}(E_2) = 0$ for all i since E_2 is a simplex. Hence we can use the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_i(E_1)\oplus\tilde{H}_i(E_2) \to \widetilde{H}_i(\Delta^*) \to \tilde{H}_{i-1}(E_1 \cap E_2) \to \tilde{H}_{i-1}(E_1)\oplus\tilde{H}_{i-1}(E_2) \to \ldots$$

to see that

$$\begin{aligned}\widetilde{H}_i(\Delta^*(C_n)) &= \widetilde{H}_{i-1}(E_1 \cap E_2)\big)\\ &= \widetilde{H}_{i-3}(E(n-4))\\ &= \widetilde{H}_{i-3}(E(3(m-1)))\end{aligned}$$

By Proposition 4.3.8

$$\begin{aligned}\widetilde{H}_i(\Delta^*(C_n)) &= \widetilde{H}_{i-3}(E(3(m-1)))\\ &= \begin{cases} k & \text{if} \qquad i-3 = 2(m-1)-2\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases} k & \text{if} \qquad i = 2m-1\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We used this along with Lemma 4.6.2 to find $\beta_{i,n}(C_n)$:

$$\begin{aligned}\beta_{i,n}(C_n) &= \dim_k \widetilde{H}_{i-2}\left(\mathrm{Link}_{\Delta^*(C_n)}\emptyset\right)\\ &= \dim_k \widetilde{H}_{i-2}\left(\Delta^*(C_n)\right)\\ &= \begin{cases} 1 & \text{if} \quad i = 2m+1 \\ 0 & \text{if} \quad i \neq 2m+1 \end{cases}\end{aligned}$$

(ii) In this case $n = 3m+2$ for some m. Lemma 4.6.1 and Proposition 4.3.6 imply

$$\begin{aligned}\widetilde{H}_i(E_1 \cap E_2) &= \widetilde{H}_{i-2}(E(n-4))\\ &= \widetilde{H}_{i-2}(E(3(m-1)+1))\\ &= 0\end{aligned}$$

for all i. The Mayer-Vietoris sequence

$$\cdots \to \widetilde{H}_i(E_1) \oplus \widetilde{H}_i(E_2) \to \widetilde{H}_i(\Delta^*) \to \widetilde{H}_{i-1}(E_1 \cap E_2) \to \widetilde{H}_{i-1}(E_1) \oplus \widetilde{H}_{i-1}(E_2) \to \dots$$

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$$\cdots \to 0 \to \widetilde{H}_i(E_1) \oplus \tilde{H}_i(E_2) \to \widetilde{H}_i(\Delta^*(C_n)) \to 0 \to \dots$$

and so we obtain

$\widetilde{H}_i(E_1) \oplus \widetilde{H}_i(E_2) = \widetilde{H}_i(E_1)$

(as E_2 is a simplex) (By Proposition 4.3.8)

$= \widetilde{H}_i(\Delta^*(C_n))$.

$$\begin{aligned}\widetilde{H}_i(\Delta^*(C_n)) &= \widetilde{H}_i(E_1)\\ &= \begin{cases} k & \text{if} \quad i = 2m-1 \\ 0 & \text{if} \quad i \neq 2m-1. \end{cases}\end{aligned}$$

As in the previous case

$$\begin{aligned}\beta_{i,n} &= \dim_k \widetilde{H}_{i-2}\left(\Delta^*(C_n)\right)\\ &= \begin{cases} 1 & \text{if} \quad i = 2m+1 \\ 0 & \text{if} \quad i \neq 2m+1 \end{cases}\end{aligned}$$

(iii) Finally we have the case where $n = 3m$ for some m. We now use the Mayer-Vietoris sequence

$$\cdots \to \widetilde{H}_i\left(E_1 \cap E_2\right) \to \widetilde{H}_i\left(E_1\right) \oplus \widetilde{H}_i\left(E_2\right) \to \widetilde{H}_i\left(\Delta^*\right) \to \dots$$

As before $\widetilde{H}_i\left(E_2\right) = 0$ for all i and $\widetilde{H}_i\left(E_1 \cap E_2\right) = \widetilde{H}_{i-2}(E(n-4)) = \widetilde{H}_{i-2}(E(3m-4)$ for all i. So the sequence becomes

$$\cdots \to \widetilde{H}_{i-2}(E(3m-4)) \to \widetilde{H}_i(E(3m)) \to \widetilde{H}_i\left(\Delta^*\right) \to \dots$$

From Theorem 4.3.1 we obtain

$$\begin{aligned} \widetilde{H}_i(E(3m)) &= \widetilde{H}_{i-2(m-1)}(E(3)) \\ &= \begin{cases} k & \text{if} \quad i-2(m-1) = 0 \\ 0 & \text{if} \quad i-2(m-1) \neq 0 \end{cases} \\ &= \begin{cases} k & \text{if} \quad i = 2m-2 \\ 0 & \text{if} \quad i \neq 2m-2 \end{cases} \end{aligned}$$

and also using 4.3.1

$$\begin{aligned} \widetilde{H}_i(E(3m-4)) &= \widetilde{H}_i(E(3(m-2)+2)) \\ &= \widetilde{H}_{i-2(m-2)}(E(2)) \\ &= \begin{cases} k & \text{if} \quad i-2(m-2) = -1 \\ 0 & \text{if} \quad i-2(m-2) \neq -1 \end{cases} \\ &= \begin{cases} k & \text{if} \quad i = 2m-5 \\ 0 & \text{if} \quad i \neq 2m-5. \end{cases} \end{aligned}$$

If we now look at the appropriate part of the Mayer-Vietoris sequence we find

$$\begin{aligned} \cdots \to \widetilde{H}_{2m-4}(E(3m-4)) \to \widetilde{H}_{2m-2}(E(3m)) \to \widetilde{H}_{2m-2}\left(\Delta^*\left(C_n\right)\right) \to \\ \to \widetilde{H}_{2m-5}(E(3m-4)) \to \widetilde{H}_{2m-3}(E(3m)) \to \dots \end{aligned}$$

From the above comments about the reduced homology of $E(3m)$ and $E(3m-4)$ we convert this into a short exact sequence of k vector spaces

$$0 \to k \to \widetilde{H}_{2m-2}\left(\Delta^*\left(C_n\right)\right) \to k \to 0$$

So we must have $\widetilde{H}_{2m-2}\left(\Delta^*\left(C_n\right)\right) = k^2$. We also have that $\widetilde{H}_i\left(\Delta^*\right) = 0$ for all $i \neq 2m-2$ because the Mayer-Vietoris sequence is

$$\cdots \to 0 \to \widetilde{H}_i\left(\Delta^*\left(C_n\right)\right) \to 0 \to \ldots$$

wherever $i \neq 2m-2$. Hence the ith Betti number of degree n is

$$\begin{aligned}
\beta_{i,n}\left(C_n\right) &= \dim_k \tilde{H}_{i-2}\left(\mathrm{Link}_{\Delta^*(C_n)}\right); k\right) \\
&= \dim_k \tilde{H}_{i-2}\left(\Delta^*\left(C_n\right)\right) \\
&= \begin{cases} 2 & \text{if } i = 2m \\ 0 & \text{if } i \neq 2m \end{cases}
\end{aligned}$$

□

4.7 Betti numbers of Cycles

We now combine these results to obtain

Theorem 4.7.1. *[47] The non-zero N-graded Betti numbers of C_n are all in degree less than or equal to n and for $2i + j < n$*

$$\begin{aligned}
\beta_{i+j,2i+j}\left(C_n\right) &= \begin{pmatrix} i \\ j \end{pmatrix} \beta_{i,2i} \\
&= \begin{pmatrix} i \\ j \end{pmatrix} \left\{ \begin{pmatrix} n-2i \\ i \end{pmatrix} + 2 \begin{pmatrix} n-2i-1 \\ i-1 \end{pmatrix} \right\} \\
&= \frac{n}{n-2i} \begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} n-2i \\ i \end{pmatrix}
\end{aligned}$$

Equivalently, for $d < n$ and $2l \geqslant d$,

$$\beta_{l,d}\left(C_n\right) = \frac{n}{n-2(d-l)} \begin{pmatrix} d-l \\ 2l-d \end{pmatrix} \begin{pmatrix} n-2(d-l) \\ d-l \end{pmatrix}$$

and

$$\beta_{2m+1,n}(C_n) = 1 \text{ if } \quad n = 3m+1$$

$$\beta_{2m+1,n}(C_n) = 1 \quad \text{if} \quad n = 3m+2$$

$$\beta_{2m,n}(C_n) = 2 \quad \text{if} \quad n = 3m.$$

Proof. This is a combination of Propositions 4.5.5 and 4.6.3 together with $i = d - l$ and $j = 2l - d$. □

Remark 4.7.2. *Notice that the Betti numbers of cycles do not depend on our choice of field.*

Corollary 4.7.3. *The projective dimension of the cycle graph is independent of the characteristic of the chosen field and is*

$$\mathrm{pd}(C_n) = \begin{cases} \frac{2n}{3} & \text{if} \quad n \equiv 0 \mod 3 \\ \frac{2n+1}{3} & \text{if} \quad n \equiv 1 \mod 3 \\ \frac{2n-1}{3} & \text{if} \quad n \equiv 2 \mod 3 \end{cases}$$

Proof. The expression given in 4.7.1,

$$\frac{n}{n-2(d-l)} \begin{pmatrix} d-l \\ 2l-d \end{pmatrix} \begin{pmatrix} n-2(d-l) \\ d-l \end{pmatrix}$$

is non-zero only if $d - l \geqslant 2l - d$,ie., if $2d \geqslant 3l$.

We first consider the case $n = 3m$ for some integer m. By Theorem 4.7.1 $\beta_{2m}(C_n) \neq 0$ and so $\mathrm{pd}(C_n) \geqslant 2m$. Suppose that $l \geqslant 2m+1$. Assume that $\beta_{l,d}(C_n) \neq 0$. We have $2d \geqslant 3l \geqslant 3(2m+1) = 6m+3$. Therefore $d \geqslant 3m + \frac{3}{2} > n$ which is not possible. Hence $\beta_{l,d}(C_n) = 0$ and so $\mathrm{pd}(C_n) = 2m = \frac{2n}{3}$.

Now suppose that $n = 3m+1$ for some integer m. Theorem 4.7.1 shows that $\beta_{2m+1}(C_n) \neq 0$ and so $\mathrm{pd}(C_n) \geqslant 2m+1$. Suppose that $l \geqslant 2m+2$. Assume that $\beta_{l,d}(C_n) \neq 0$. This implies that $2d \geqslant 3l \geqslant 6m+6$ and so $d \geqslant 3m+3 > n$, a contradiction. Hence $\beta_{l,d}(C_n) = 0$ and $\mathrm{pd}(C_n) = 2m+1 = \frac{2n+1}{3}$.

Finally we consider the case $n = 3m + 2$ for some m. Theorem 4.7.1 shows that $2m + 1(C_n) \neq 0$ and so $\mathrm{pd}(C_n) \geq 2m + 1$. Now suppose that $l \geqslant 2m + 2$ and that $\beta_{l,d}(C_n) \neq 0$. This implies that $d \geqslant 3m+3 > n$. which is not possible. Hence $\beta_{l,d}(C_n) = 0$ and $\mathrm{pd}(C_n) = 2m + 1 = \frac{2n-1}{3}$. $\square$

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