TAUBIN'S METHOD

Taubin's Method is a way to fit a trivariate implicit quadric function (surface) to set of points. It performs better than least square$^{(LS)}$ method because error function is related to the geometric distance in stead of distance in z-axis. More precisely, Taubin's method uses the exact geometric distance to the first order approximation of the surface.

Let's say that we have n points and we would like to find optimal full quadric i.e. trivariate implicit quadric surface (TIQS) which can be represented by $f(x,y,z)=0$

$$\Rightarrow \quad f(x,y,z) = c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 xy + c_6 xz + c_7 y^2 + c_8 yz + c_9 z^2 = 0$$

$$\Rightarrow \quad f(x,y,z) = \underbrace{\begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 \end{bmatrix}}_{\mathbf{c}^T := \text{coefficent vector}} \underbrace{\begin{bmatrix} 1 \\ x \\ y \\ z \\ x^2 \\ xy \\ xz \\ y^2 \\ yz \\ z^2 \end{bmatrix}}_{\underline{X} := \text{basis vector}} = \underline{c}^T \underline{X} = 0 \qquad (*)$$

Note that the coefficient vector can be multiplied by any non-zero real number, the equation will still be valid. In other words $\underline{c}^T \underline{X} = 0 \Rightarrow k\,\underline{c}^T \underline{X} = 0$ where $k \in \mathbb{R} \setminus \{0\}$
It implies that we will need to use some normalization to create a representative for each coefficent family. This will be explained later.

If the point is not exactly on the surface, which is the case in most of the cases Eq $(*)$ will not be exactly equal to zero. Let's consider ith point p_i where $i \in \{1,2,\dots n\}$ and $p_i = (x_i, y_i, z_i)$

$$f(p_i) = f(x_i, y_i, z_i) = \text{Error}_i = \varepsilon_i$$

$\hookrightarrow$ Error term before normalization.

One can see that the equation can be scaled by scalar k and the error will be scalled with the same k. So, a way to normalize should be found. Taubin's idea is to use $\|\nabla f(p_i)\|$ for normalization. Thus, mean square error (MSE) $\frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2 = \frac{1}{n}\sum_{i=1}^{n} f(p_i)^2$ should be normalized by $\frac{1}{n}\sum_{i=1}^{n} \|\nabla f(p_i)\|^2$.

claim:

$$\underbrace{d^2(p_i)}_{\substack{\text{geometric distance} \\ \text{(invariant from } \underline{C})}} \approx \frac{f(p_i)^2}{\|\nabla f(p_i)\|}$$

Proof: Let's consider First order Taylor Expansion from surface $f(x,y,z)$ to point $p_i = (x_i, y_i, z_i)$

$$f(x_i, y_i, z_i) \approx \underbrace{f(x,y,z)}_{\substack{= 0 \\ \text{equation of the surface}}} + \underbrace{\frac{\partial f}{\partial x}(x-x_i) + \frac{\partial f}{\partial y}(y-y_i) + \frac{\partial f}{\partial z}(z-z_i)}_{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \langle x-x_i, y-y_i, z-z_i \rangle} + \underbrace{H.O.T}_{\to 0}$$

0 Higher Order Terms are negligible

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \underline{\nabla f}, \quad \langle x-x_i, y-y_i, z-z_i \rangle = \underline{d}$$

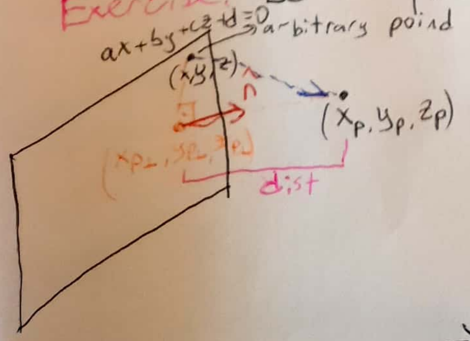
$$\|\underline{d}\| = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}$$

$$\Rightarrow f^2(x_i, y_i, z_i) \approx \left(\|\nabla f\| \, \|\underline{d}\| \underbrace{\cos\theta}_{1}\right)^2 = \|\nabla f\|^2 d_i^2$$

1 since we have orthogonal distance $\underline{\nabla f} \parallel \underline{d}$

$$\Rightarrow \boxed{d_i^2 \approx \frac{f^2(p_i)}{\|\nabla f(p_i)\|^2}} \quad \text{Q.E.D.}$$

Exercise: Let's find the distance between $\underbrace{\text{a point}}_{(x_p, y_p, z_p)}$ and $\underbrace{\text{a plane}}_{ax+by+cz+d=0}$.



$$(\vec{x} - \vec{x}_{p_\perp}) \cdot \hat{n} = 0 \quad \text{and} \quad |(\vec{x} - \vec{x}_p) \cdot \hat{n}| = \text{dist} \Rightarrow$$

$$\left| \langle x - x_p, y - y_p, z - z_p \rangle \cdot \frac{\langle a, b, c \rangle}{\sqrt{a^2+b^2+c^2}} \right| = \text{dist} \Rightarrow$$

$$\frac{|\underbrace{(ax+by+cz)}_{-d} - (ax_p + by_p + cz_p)|}{\sqrt{a^2+b^2+c^2}} = \text{dist}$$

$$\Rightarrow \text{dist}^2 = \frac{(ax_p + by_p + cz_p + d)^2}{a^2+b^2+c^2} = \frac{f^2(p_i)}{\|\nabla f(p_i)\|^2}$$

Square of geometric distance

$$\lambda := d^2 = \frac{\frac{1}{n}\sum_{i=1}^{n} f^2(p_i)}{\frac{1}{n}\sum_{i=1}^{n}\|\nabla f(p_i)\|^2} = \frac{\frac{1}{n}\sum_{i=1}^{n}\left(c^T X_i\right)^2}{\frac{1}{n}\sum\left(c^T \underbrace{D X_i}_{\text{derivative, Jacobian matrix}}\right)^2} = \frac{\frac{1}{n}\sum_{i=1}^{n} c^T X_i X_i^T c}{\frac{1}{n}\sum_{i=1}^{n} c^T D X_i (D X_i)^T c}$$

↓

Taubin's Error Metric

$$= \frac{c^T \overbrace{\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i^T\right)}^{=:M} c}{c^T \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} D X_i D X_i^T\right)}_{N} c} = \boxed{\frac{c^T M c}{c^T N c} =: \lambda} \quad (\star\star)$$

Note that

$$\underline{M}_{10\times10} := \frac{1}{n}\sum_{i=1}^{n} X_i X_i^T = \frac{1}{n}\sum_{i=1}^{n}\left(\begin{bmatrix} 1 \\ x \\ y \\ z \\ x^2 \\ xy \\ xz \\ y^2 \\ yz \\ z^2 \end{bmatrix}_{10\times1} \begin{bmatrix} 1 & x & y & z & x^2 & xy & xz & y^2 & yz & z^2 \end{bmatrix}_{1\times10} \right)$$

$$\underline{N}_{10\times10} := \frac{1}{n}\sum_{i=1}^{n} \underbrace{D X_i}_{\text{Jacobian Matrix}} D X_i^T = \frac{1}{n}\sum_{i=1}^{n}\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x & 0 & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & 2y & 0 \\ 0 & z & y \\ 0 & 0 & 2z \end{bmatrix}_{10\times3} \begin{bmatrix} 0 & 1 & 0 & 0 & 2x & y & z & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & x & 0 & 2y & z & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & x & 0 & y & 2z \end{bmatrix}_{3\times10} \right)$$

We can calculate M and N matrices for any given pointcloud only using by coordinate information. Then, we can calculate coefficient vector c. Let's manipulate Equation $(\star\star)$

$$\lambda = \frac{c^T M c}{c^T N c} \underset{c^T N c \neq 0}{\Longleftrightarrow} c^T M c = \lambda c^T N c \Longleftrightarrow \cancel{\frac{c}{c c^T}}\, c^T M c = \cancel{\frac{c}{c c^T}}\, \lambda c^T N c$$

$$\Longleftrightarrow \boxed{\underline{M}\underline{c} = \lambda \underline{N}\underline{c}} \quad (\star\star\star)$$

Note that Equation $(\star\star\star)$ represents generalized Eigenvalue problem. $\underline{M}_{10\times10}$ and $\underline{N}_{10\times10}$ are known matrices, λ is the eigenvalue and c is the eigenvector. The equation has 10 λ eigenvalue solutions and 10 corresponding c eigenvectors. The smallest λ value λ_{min} corresponds Taubin's error metric.

Let's use $\tilde{\lambda}$ for λ_{min} and $\tilde{c}$ for corresponding eigenvector c. As we have stated before c can be scaled by a scalar. We would like to have a representative of each c family.

Claim: the c eigenvector should be normalized by $\sqrt{c^T N c}$ such that $\lambda = d^2 = \frac{1}{n}\sum_{i=1}^{n} \hat{\varepsilon}^2(p_i) = \frac{1}{n}\sum_{i=1}^{n} \hat{f}^2(p_i)$" $\hat{c}^T M \hat{c}$ is true. In other words, $\hat{\varepsilon}_i$ is geometric distance approximation for point p_i without any further normalization.

Proof: Define $\hat{c} = \dfrac{\tilde{c}}{\sqrt{c^T N c}}$ and transpose of the equation

$\hat{c}^T = \dfrac{\tilde{c}^T}{\sqrt{c^T N c}}$. Note that $\sqrt{c^T N c}$ is a scalar so the traspose is itself.

Let's implement λ_{min} or $\tilde{\lambda}$ in Eq ($\star\star$)

$$\tilde{\lambda} = \frac{c^T M c}{c^T N c} = \frac{c^T}{\sqrt{c^T N c}} M \frac{c}{\sqrt{c^T N c}} = \hat{c}^T M \hat{c} \qquad \text{QED}$$