1 Introduction to Solitons and Vorticies

Something or other

2 Question 1

3 Question 2

Consider the following potential

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$$

where $\lambda > 0$ and $v > 0$. Plot this potential, and show that the minima of this potential is at $\phi_{min} = \pm v$.

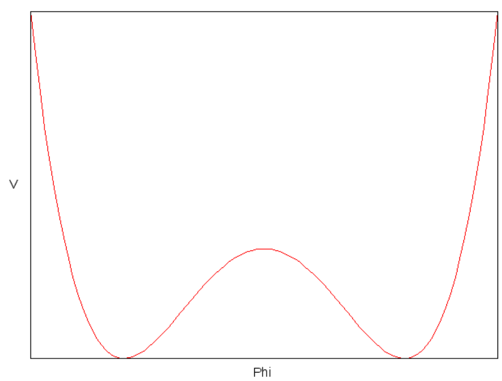

Figure 1: Potential

Turning points of a function may be found when the first derivative of the function is equal to zero.

$$\frac{\partial V(\phi)}{\partial \phi} = \frac{\lambda}{4}2(\phi^2 - v^2)2\phi = \lambda(\phi^2 - v^2)\phi$$

It can be seen that when this derivative is equal to zero that there are three possible solutions at $\phi = 0$ and $\phi = \pm v^2$. In order to identify which points are minima it is necessary to evaluate the second derivative of the potential at each of these points.

$$\frac{\partial^2 V(\phi)}{\partial \phi^2} = \lambda(\phi^2 - v^2) + 2\lambda\phi$$

Evaluating the second derivateive at these points shows that the minima of this function are at $\phi = \pm v^2$ and that there is a maximum at $\phi = 0$.

Considering the static solutions, where $\dot{\phi} = \ddot{\phi} = 0$. The Klein-Gordon Equation in this limit is then

$$\frac{\partial^2 \phi}{\partial x^2} - \lambda(\phi^2 - v^2)\phi = 0$$

Showing that the *vacuum* solution, where $\phi(x,t) = \pm v$, satisfies the static Klein-Gordon Equation is as simple as substituting ϕ into the equation. As the value for ϕ is a constant it is clear that $\partial_x \phi$ is zero along with all higher order derivatives. This simplifies the solving of the equation.

$$(0) - \lambda\bigg((\pm v)^2 - v^2\bigg)(\pm v) = 0$$

Changing ϕ to the following where m $= \sqrt{\lambda}v$ and x_0 are constants is known as the kink solution.

$$\phi^{+}(x) = vtanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right]$$

Having shown that the kink solultion does indeed satisfy the static Klein-Gordon equation one wonders what other equations will do so. It is possible to check a for a few more solutions simply by changing the parity of the solution. If this is possible it should be possible to show that the static Klein-Gordon equation is invariant under the parity transform.

$$\frac{\partial^2 \phi}{\partial x^2} - \lambda(\phi^2 - v^2)\phi = 0$$

In the instance of the static Klein-Gordon equation ϕ is the only parameter that depends upon x. Therefore the parity transform can be implemented by changing $\phi \to -\phi$.

$$\frac{\partial^2 \phi}{\partial x^2} - \lambda(\phi^2 - v^2)\phi = 0 \to$$

$$\frac{\partial^2 (-\phi)}{\partial x^2} - \lambda\Big((-\phi)^2 - v^2\Big)(-\phi) = 0$$

$$-\frac{\partial^2 \phi}{\partial x^2} + \lambda(\phi^2 - v^2)\phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} - \lambda(-\phi^2 - v^2)\phi = 0$$

As the kink solution is an odd function, this means that under the parity transform it gains a minus sign. Plugging this in to the Klein-Gordon equation is the same as using $-\phi$ and is therefore exactly the same.

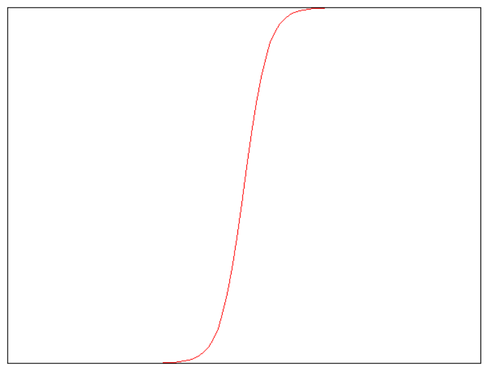

Figure 2: Kink Solution

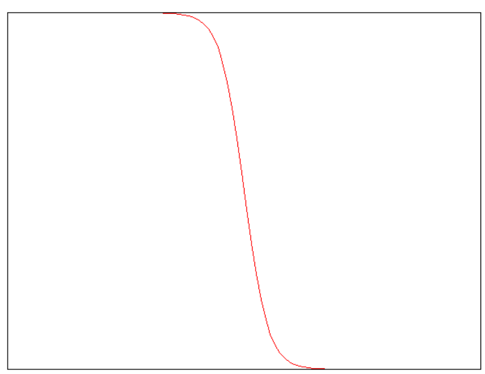

Figure 3: Anti-Kink Solution

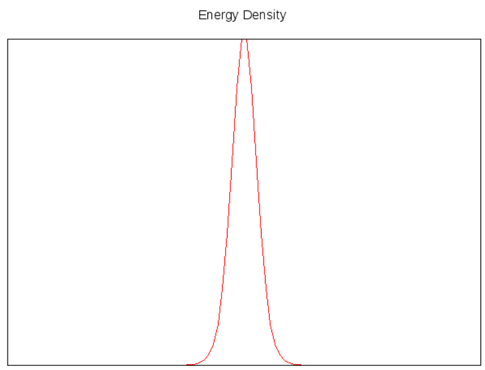

Figure 4: Energy Density