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Problem 1

a:

Basic step:

n=1

left: $f(1) = 1^2 = 1$

right : $f(1) = \frac{1(1+1)(2+1)}{6} = 1$

left = right

Induction step:

The equation can be changed to:

$$f(n) = f(n-1) + n^2 \forall n > 1 \tag{1}$$

Assume that:

$$f(n-1) = \frac{(n-1)n(2n-1)}{6} \tag{2}$$

As ref (1)

$$\begin{aligned} f(n) &= \frac{(n-1)n(2n-1)}{6} + n^2 \\ &= \frac{2n^3 - 2n^2 - n^2 + n + 6n^2}{6} \\ &= \frac{2n^3 + 3n^2 + n}{6} \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Therefore, $f(n) = \frac{n(n+1)(2n+1)}{6}$

b:

$$\begin{gathered} T(1) = c \\ T(n) = 3T(\frac{n}{3}) + c \forall n \geq 3 \\ n = 3^k \forall k > 0 \\ prove\ T(n) = \frac{3c}{2}n - \frac{c}{2} \end{gathered} \tag{3}$$

Basic step

Assume

$$T(n) = \frac{3c}{2}n - \frac{c}{2} \tag{4}$$

We have

$$\begin{cases} n = 3^k \\ T(n) = 3T(\frac{n}{3}) + c \end{cases} \tag{5}$$

So

$$T(3^k) = 3T(3^{k-1}) + c \tag{6}$$

Basic step as k=0

$$T(3^0) = \frac{3c}{2}3^0 - \frac{c}{2} = c \tag{7}$$

Basic step proved

Induction step

$$\begin{aligned} T(n) = T(3^k) &= 3T(3^{k-1}) + c \\ &= 3\frac{3c}{2}3^{k-1} - \frac{3c}{2} + c \\ &= \frac{3c}{2}3^k - \frac{c}{2} \\ &= \frac{3c}{2}n - \frac{c}{2} \end{aligned}$$

Induction step proved

c:

$$\begin{cases} T(1) = c \\ T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + c \ \forall n \geq 2 \end{cases} \tag{8}$$

prove

$$T(n) \leq 2cn - c \tag{9}$$

basic step

$$T(1) = c \leq 2c - c = c \tag{10}$$

induction step

Assume

$$T(m) \leq 2cm - c \ \forall m \leq n - 1 \tag{11}$$

From (8) We can see, for $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil = \frac{n}{2}$

$$T(n) = 2T(\frac{n}{2}) + c = 2T(m) + c \leq 2(cn - c) + c = 2cn - c \tag{12}$$

for $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}, \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$

set $m = \frac{n+1}{2}$

$$
\begin{aligned}
T(n) &= T(\frac{n-1}{2}) + T(\frac{n+1}{2}) + c \\
&= T(m-1) + T(m) + c \\
&\leq 2c(\frac{n-1}{2} + \frac{n+1}{2}) - 2c + c \\
&= 2cn - c
\end{aligned}
$$

$$T(n) \leq 2cn - c \tag{13}$$

From (12) and(13), we can conclude

$$T(n) \leq 2nc - c \tag{14}$$

so the induction step is proved

d:

$$T(n) = 4T(\frac{n}{2}) + n^3 \tag{15}$$

Use the Master Theorem to find the Θ value of T(n)

By the master Theorem

$$
\begin{gathered}
T(n) = aT(\frac{n}{b}) + f(n) \ for \ n > n_0 \\
f(n) = n^3 \\
b = 2
\end{gathered}
\tag{16}
$$

So as we see from the asymptotic bounds in master theorem

$$If \ f(n) = \Theta(n^{log_b a}), then \ T(n) = \Theta(n^{\log_b a} \log n) \tag{17}$$

So

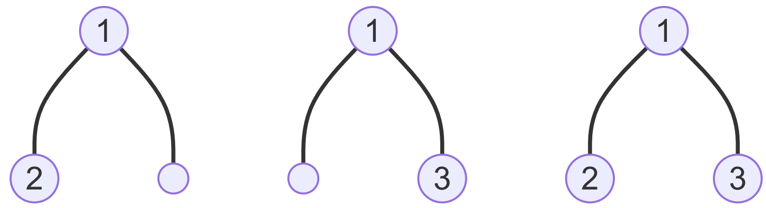
$$
\begin{gathered}
f(n^3) = \Theta(n^{log_2 8}) \\
a = 3 \\
T(n) = \Theta(n^{\log_2 8} \log n) = \Theta(n^3 \log n)
\end{gathered}
\tag{18}
$$

Problem 2

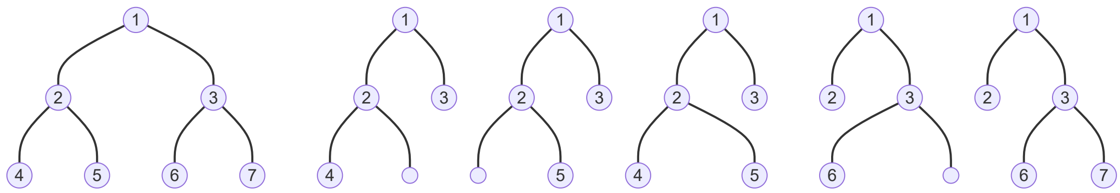
a:

- height 0

- height 1



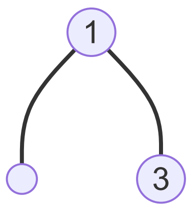
- height2



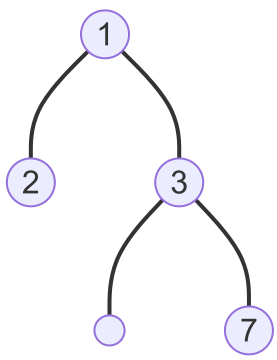
b:

- height 0

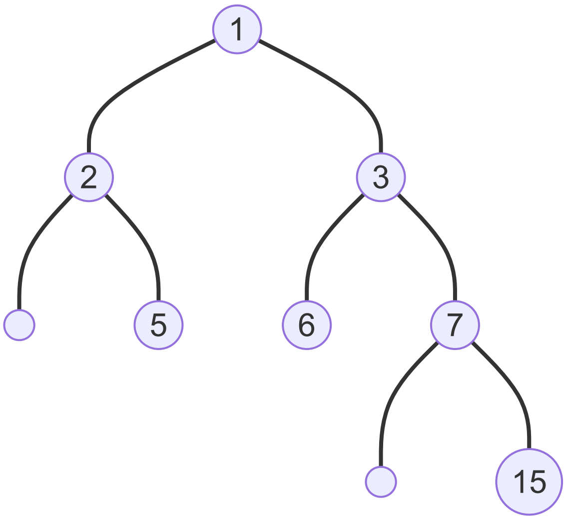
- height 1



- height2



- height3



c:

as we can see from b

the left subtree of the root of height2 is the height0, the right subtree of root of height2 is the height1

and the left subtree of the root of height3 is the height1, the right subtree of root of height3 is the height2

And we need to add the root (1node) to the sum

so we can get the equition

$$\begin{cases} N(h) = N(h-1) + N(h-2) + 1 \; \forall \; h \geq 2 \\ N(0) = 1 \\ N(1) = 2 \end{cases} \tag{19}$$

d:

for this case, the defination of

$$N(h) = N(h-1) + N(h-2) + 1 \; \forall \; h \geq 2 \tag{20}$$

matches the LRR of order2

$$x_n = ax_{n-1} + bx_{n-2} + c \; \forall \; n \geq 2 \tag{21}$$

And $c = 1, a = b = 1$ so $a^2 + 4b > 0$

So we can assume that $x_n = As_1^n + Bs_2^n + \hat{x}_n$

So firstly we need to solve $s^2 - as - b = 0$

and $s1 = \frac{1+\sqrt{5}}{2}$,$s2 = \frac{1-\sqrt{5}}{2}$

This is the same value as we've got in the hint of the problem

as $c = 1$

we set $\hat{x}_n = d$

From $\hat{x}_n = a\hat{x}_{n-1} + b\hat{x}_{n-2} + c$

we get $d = d + d + 1$

so $\hat{x}_n = d = -1$

now we can use the value of x_0 and x_1 to determin the value of A and B

$$A = \frac{2+1-\frac{1-\sqrt{5}}{2}(1+1)}{\sqrt{5}} = 1 + \frac{2}{\sqrt{5}}$$

$$B = \frac{(1+1)\frac{1+\sqrt{5}}{2}-(2+1)}{\sqrt{5}} = 1 - \frac{2}{\sqrt{5}}$$

So the final version of the equition looks like

$$N(h) = (1+\frac{2}{\sqrt{5}})(\frac{1+\sqrt{5}}{2})^h + (1-\frac{2}{\sqrt{5}})(\frac{1-\sqrt{5}}{2})^h - 1 \,\forall\, h \geq 0$$

Problem 3

a:

```
struct SampleNode{
    double data;
    SampleNode *left;
    SampleNode *right;
};

struct Record{
    double sum = 0;
    int count = 0;
}

Record travel(*SampleNode){
    Record tmpLeft,tmpRight;
    ## travel the tree
    if *SampleNode.left != null{
        tmpLeft = travel(*SampleNode.left);
    }
    if *SampleNode.right != null{
        tmpRight = travel(*SampleNode.right);
    }
    ## adding up the sum and count
    tmpLeft.sum=tmpLeft.sum+tmpRight.sum+*SampleNode.data;
    tmpLeft.count=tmpLeft.count+tmpRight.count+1;
    return tmpLeft;
}

int main(){
    SampleNode *T;
    ## populating T and it's leaf
    Record FinalOutput;
    FinalOutput = travel(T);
    printf("total nodes = %d\n average of data = %f\n",
FinalOutput.count,FinalOutput.sum/FinalOutput.count);
    return 0;
}

```

The time complexity of this algorithm is O(N), N is the number of nodes.

b:

```
struct SampleNode{
    SampleNode *left;
    SampleNode *right;
};

bool isNodeFull(*SampleNode){
    ## travel the tree
    if *SampleNode.left!= null && *SampleNode.right!=null{
        ## it is a full node with child, combine the child's result and return
        return travel(*SampleNode.left)&&travel(*SampleNode.right)
    }else if *SampleNode.left==null && *SampleNode.right==null{
        ## it is a leaf node, should be full at this point
        return true;
    }
    ## this(not matching the pervious condition) means the node is not full
    return false;
}

int main(){
    SampleNode *T;
    ## populating T and it's leaf
    ## output is for true,output is not for false return
    printf("this %s a full binary tree\n",travel(T)?"is":"is not");
    return 0;
}
```

The worst time complexity of this algorithm is O(N), N is the number of nodes.

Problem 4

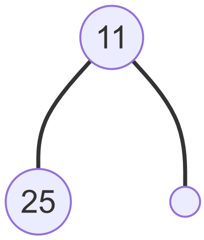
a:

intert 25, 11, 54, 35, 46, 5, 14, 65, 2, 59, 3 to min heap, first 9 show result, last 3 show step by step

- insert25

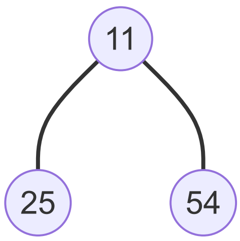
1	2	3	4	5	6	7	8	9	10	11	12
25											

- insert 11



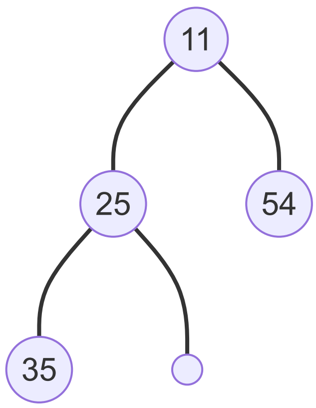
1	2	3	4	5	6	7	8	9	10	11	12
11	25										

- insert 54



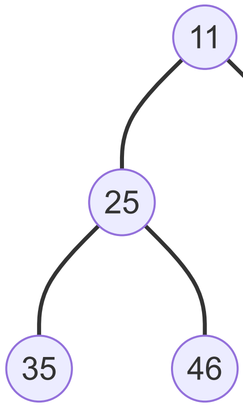
1	2	3	4	5	6	7	8	9	10	11	12
11	25	54									

- insert 35



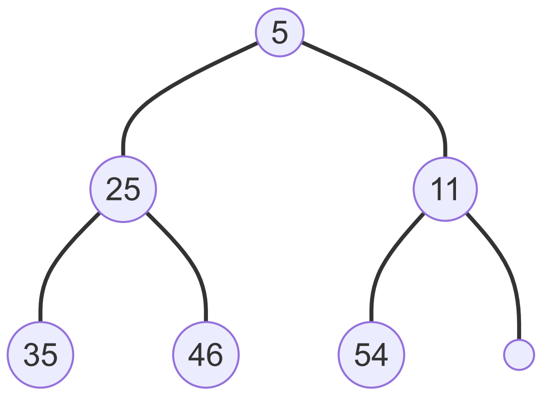
1	2	3	4	5	6	7	8	9	10	11	12
11	25	54	35								

- insert 46



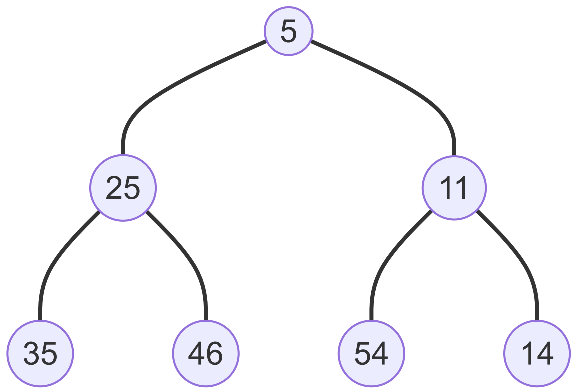
1	2	3	4	5	6	7	8	9	10	11	12
11	25	54	35	46							

- insert 5



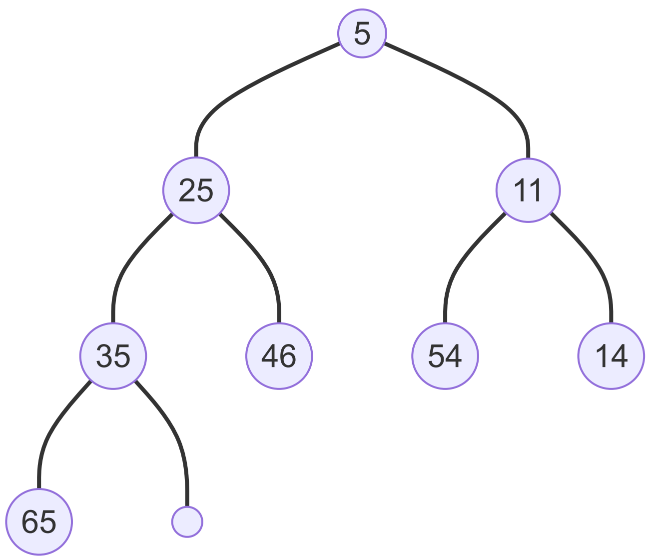
1	2	3	4	5	6	7	8	9	10	11	12
5	25	11	35	46	54						

- insert 14



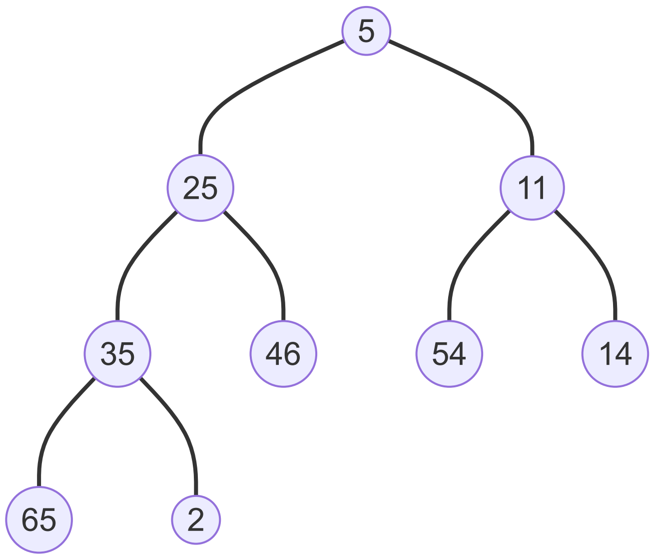
1	2	3	4	5	6	7	8	9	10	11	12
5	25	11	35	46	54	14					

- insert 65



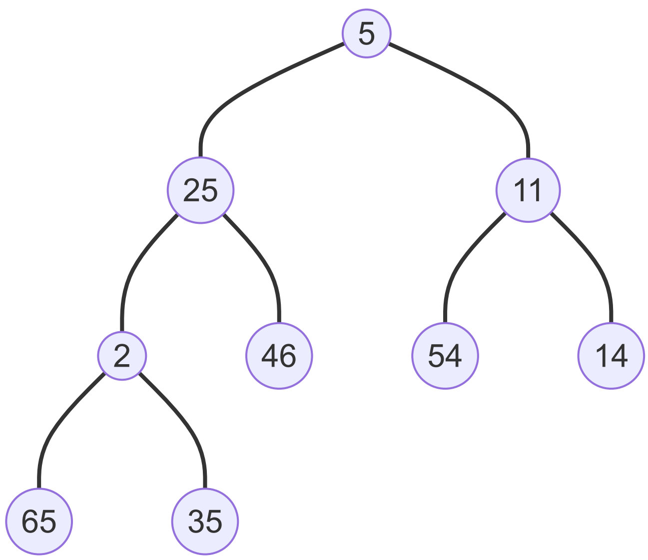
1	2	3	4	5	6	7	8	9	10	11	12
5	25	11	35	46	54	14	65				

- insert 2
    - first step



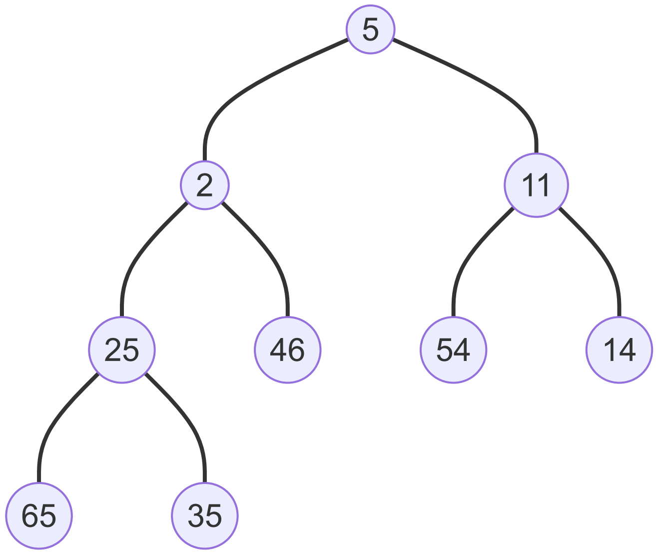
1	2	3	4	5	6	7	8	9	10	11	12
5	25	11	35	46	54	14	65	2			

- - second step



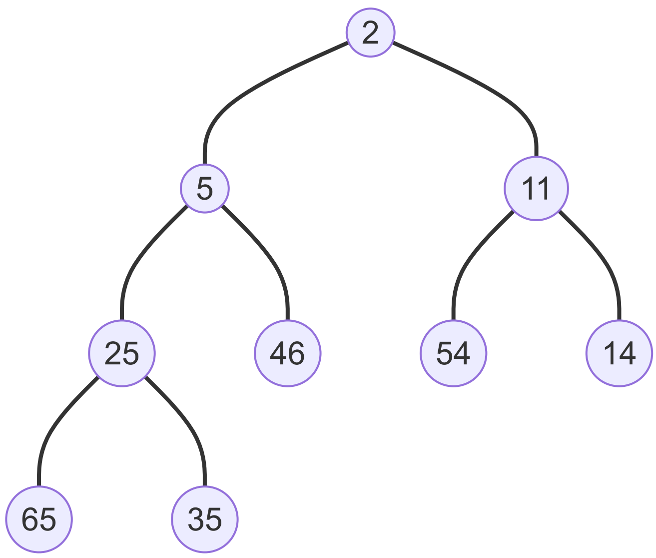
1	2	3	4	5	6	7	8	9	10	11	12
5	25	11	2	46	54	14	65	35			

- - third step



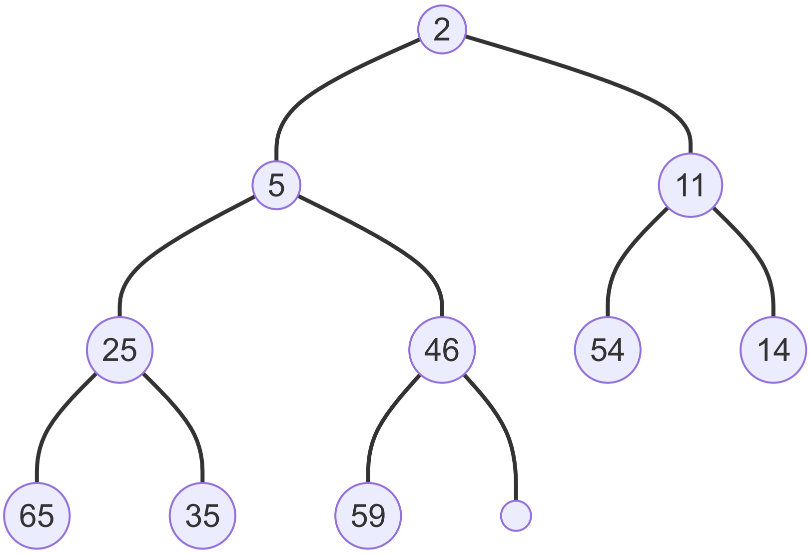
1	2	3	4	5	6	7	8	9	10	11	12
5	2	11	25	46	54	14	65	35			

-
  - fouth step



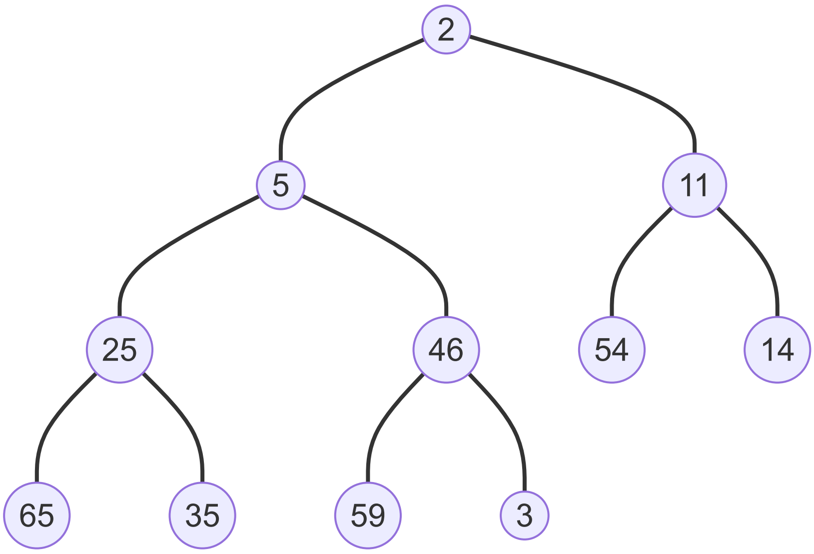
1	2	3	4	5	6	7	8	9	10	11	12
2	5	11	25	46	54	14	65	35			

- insert 59
-
  - first step



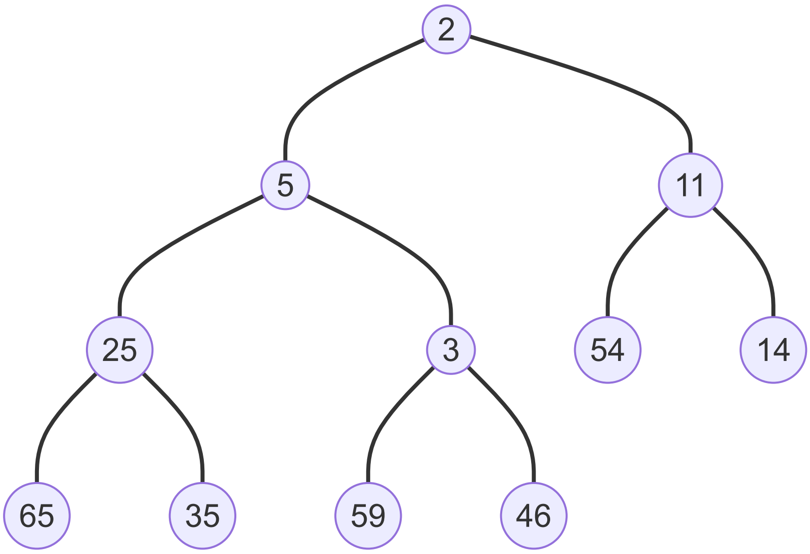
1	2	3	4	5	6	7	8	9	10	11	12
2	5	11	25	46	54	14	65	35	59		

- insert 3
- - first step



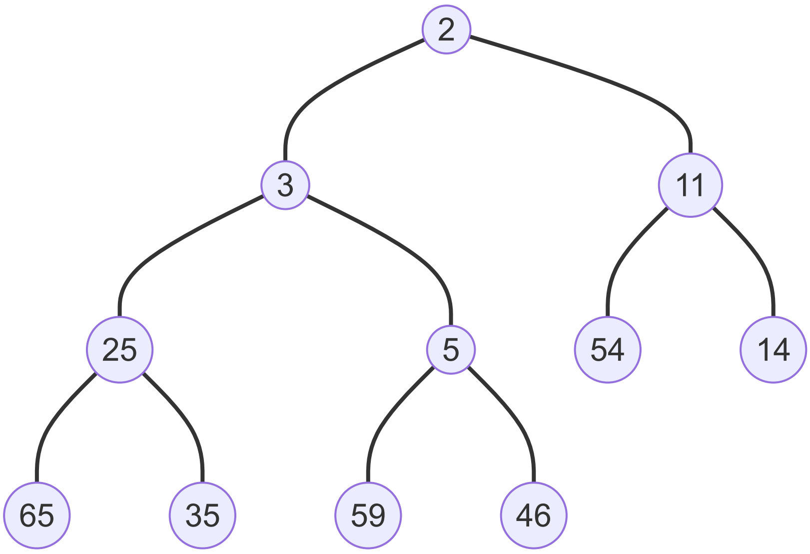
1	2	3	4	5	6	7	8	9	10	11	12
2	5	11	25	46	54	14	65	35	59	3	

- - second step



1	2	3	4	5	6	7	8	9	10	11	12
2	5	11	25	3	54	14	65	35	59	46	

-
    - third step

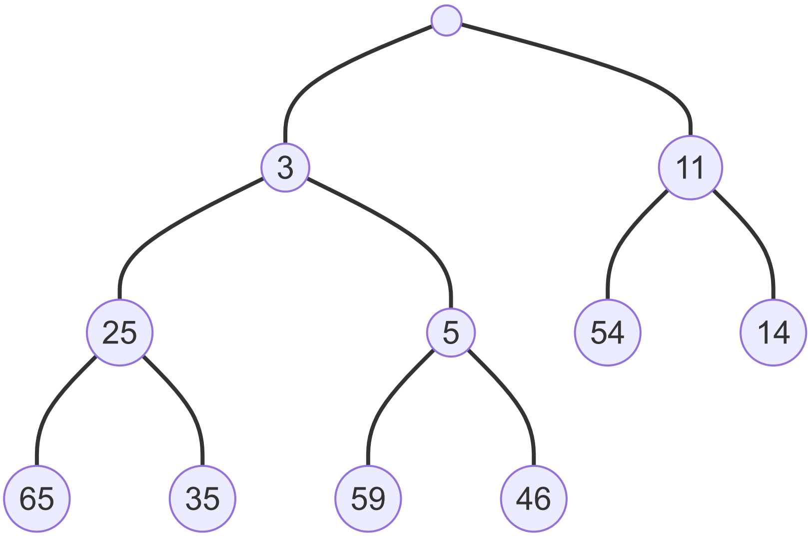


1	2	3	4	5	6	7	8	9	10	11	12
2	3	11	25	5	54	14	65	35	59	46	

Now the insert completes

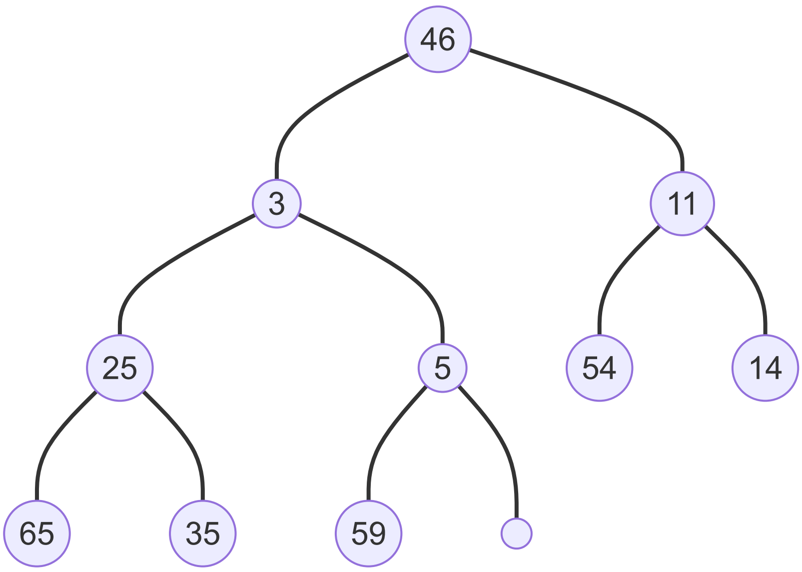
b:

- Deletemin
-
    - first step



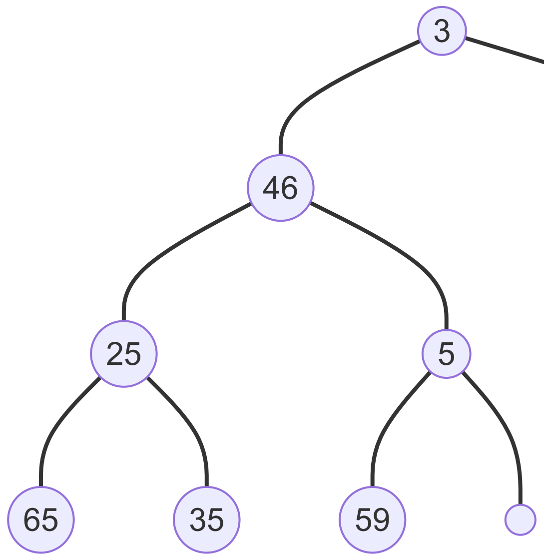
1	2	3	4	5	6	7	8	9	10	11	12
	3	11	25	5	54	14	65	35	59	46	

- - second step



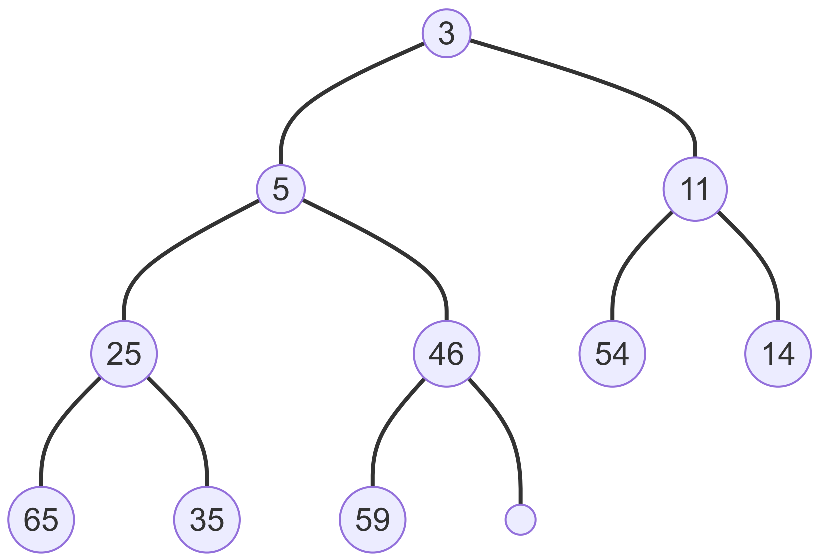
1	2	3	4	5	6	7	8	9	10	11	12
46	3	11	25	5	54	14	65	35	59		

- - third step



1	2	3	4	5	6	7	8	9	10	11	12
3	46	11	25	5	54	14	65	35	59		

- - fourth step



1	2	3	4	5	6	7	8	9	10	11	12
3	5	11	25	46	54	14	65	35	59		

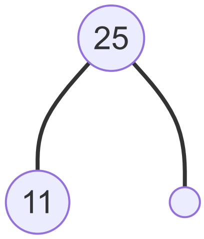
Now the delete progress is finished

c:

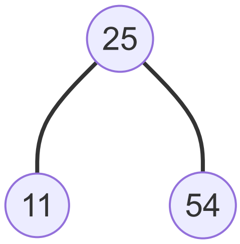
Insert 25, 11, 54, 35, 46, 5, 14, 65, 2, 59, 3, 12, 13, 7, 10, 18, 17, 15 into binary search tree

- insert 25

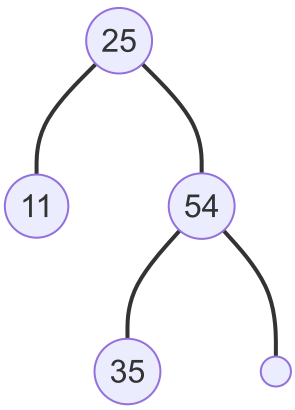
- insert 11



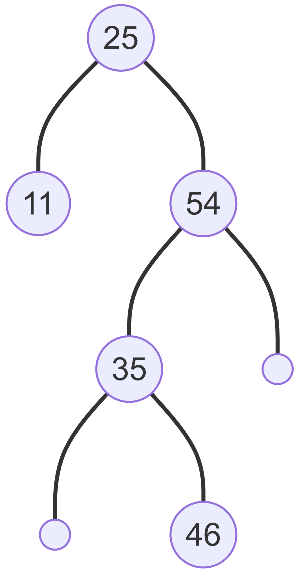
- insert 54



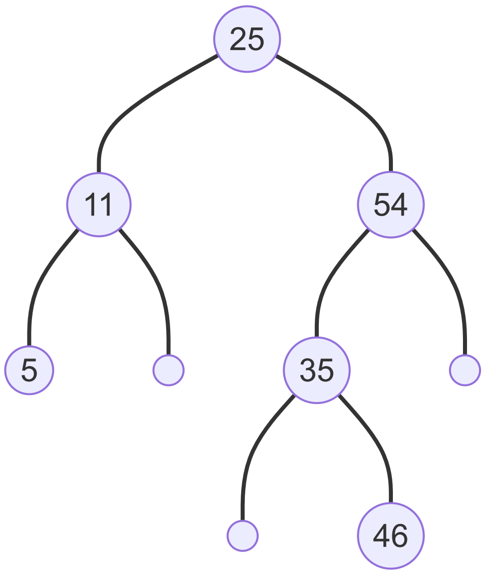
- insert 35



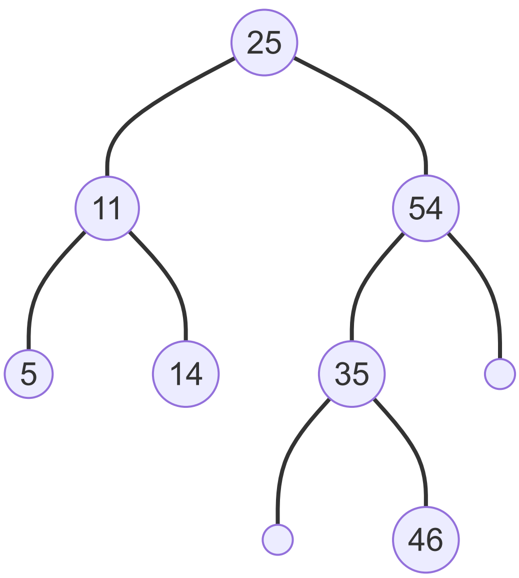
- insert 46



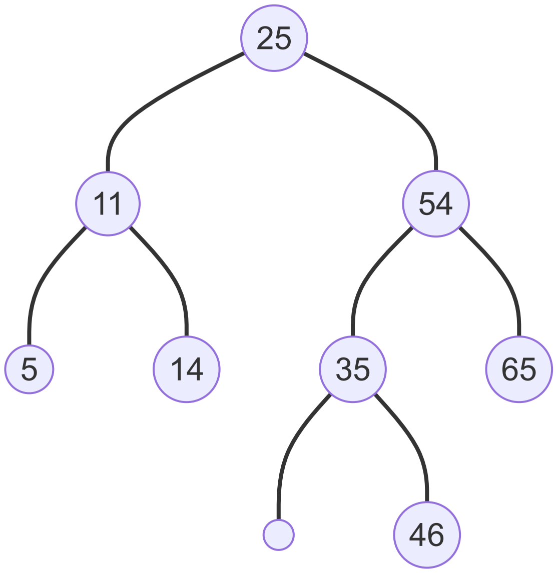
- insert 5



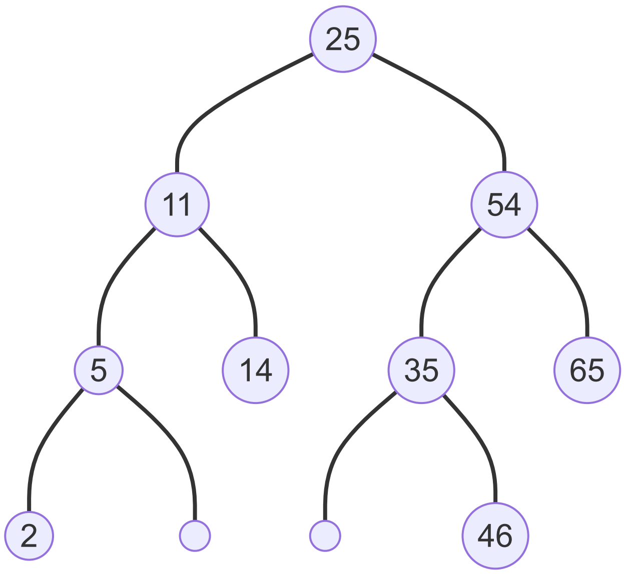
- insert 14



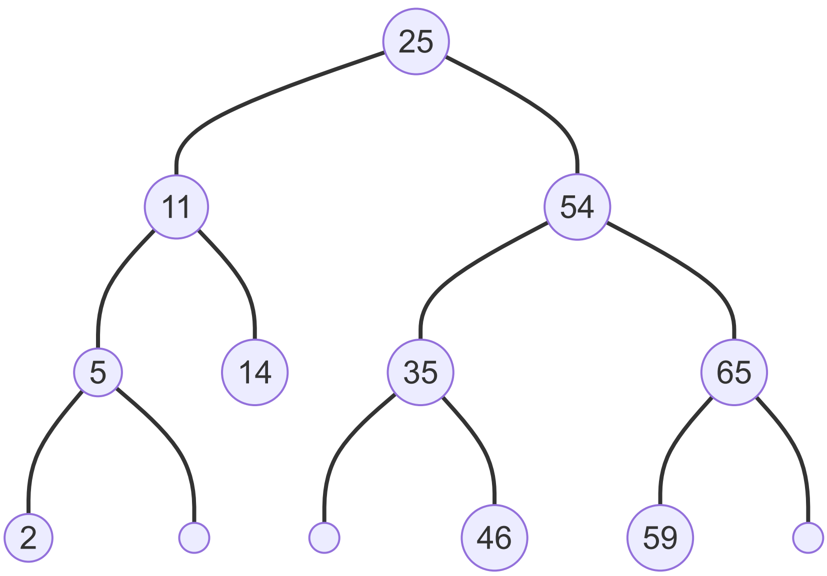
- insert 65



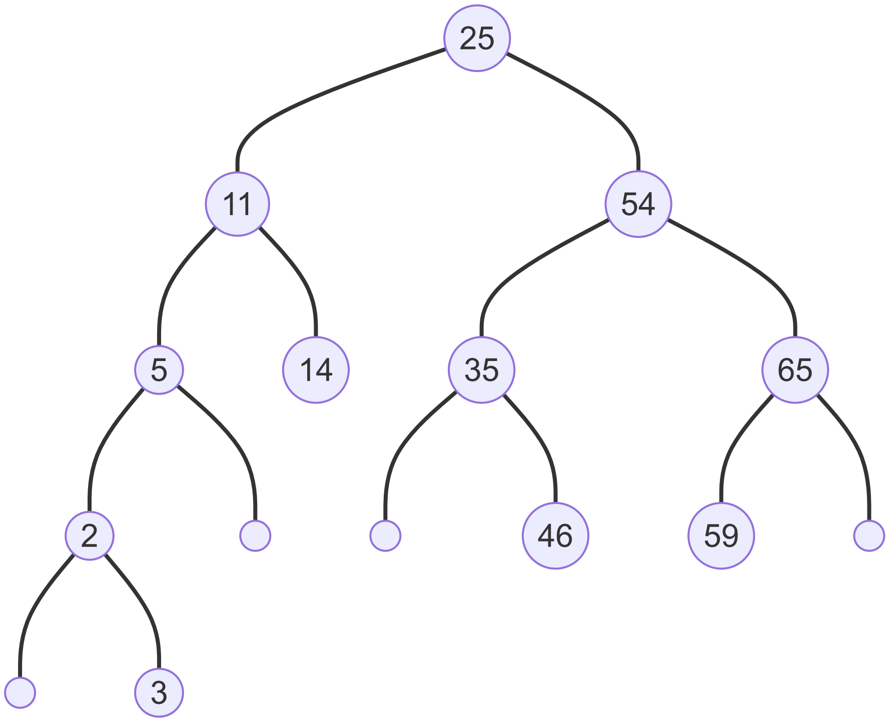
- insert 2



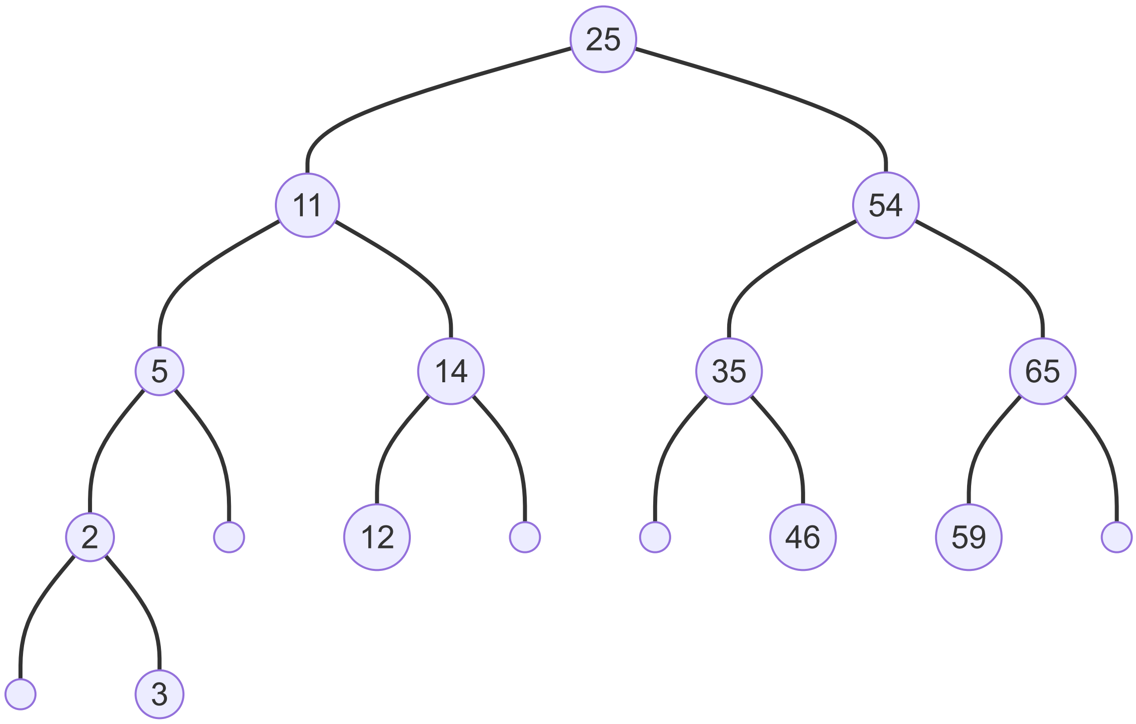
- insert 59



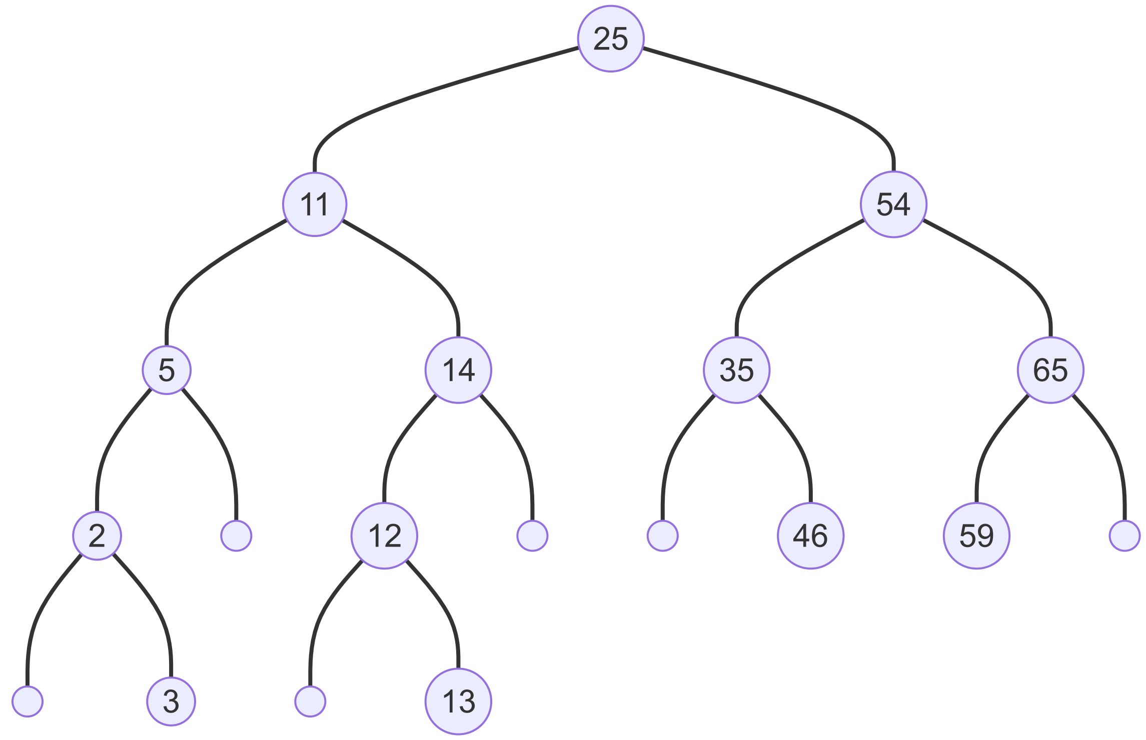
- insert 3



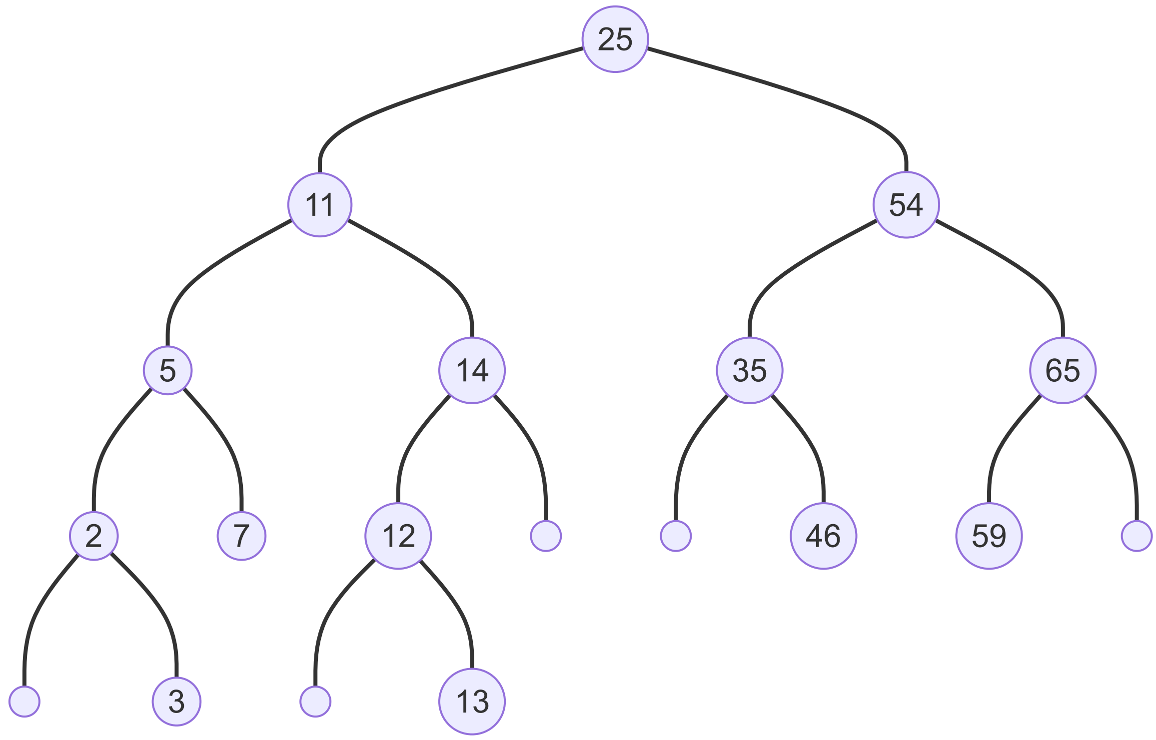
- insert 12



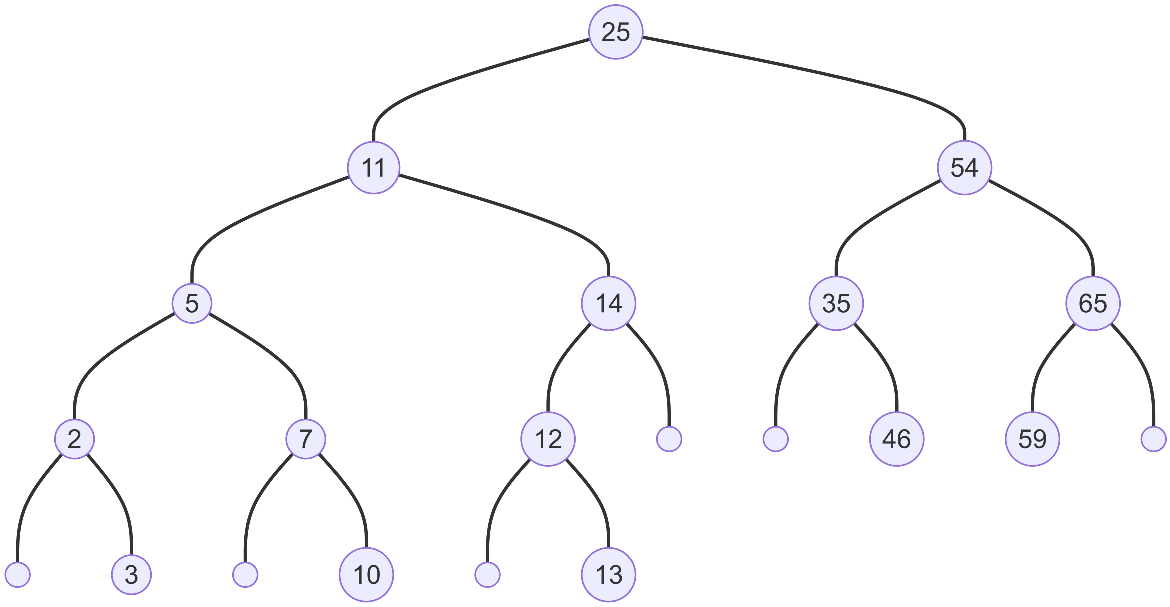
- insert 13



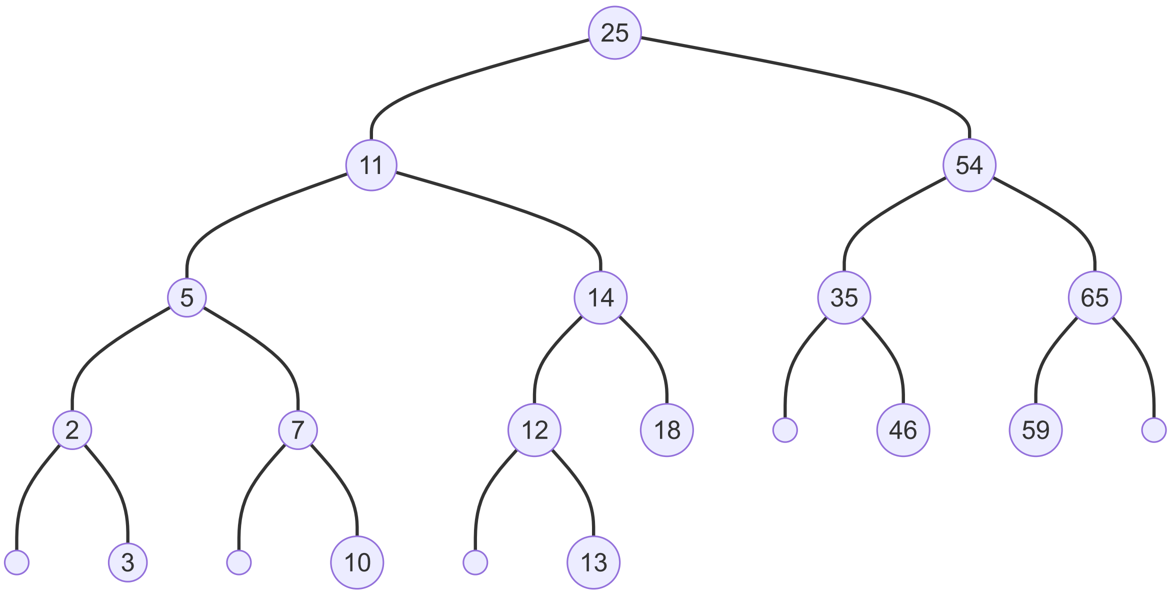
- insert 7



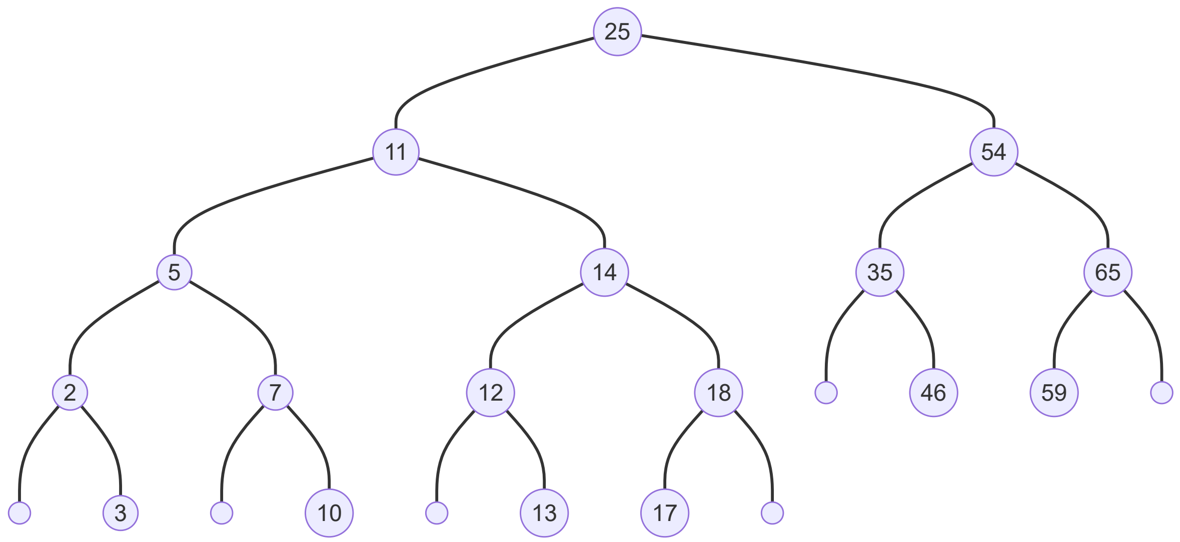
- insert 10



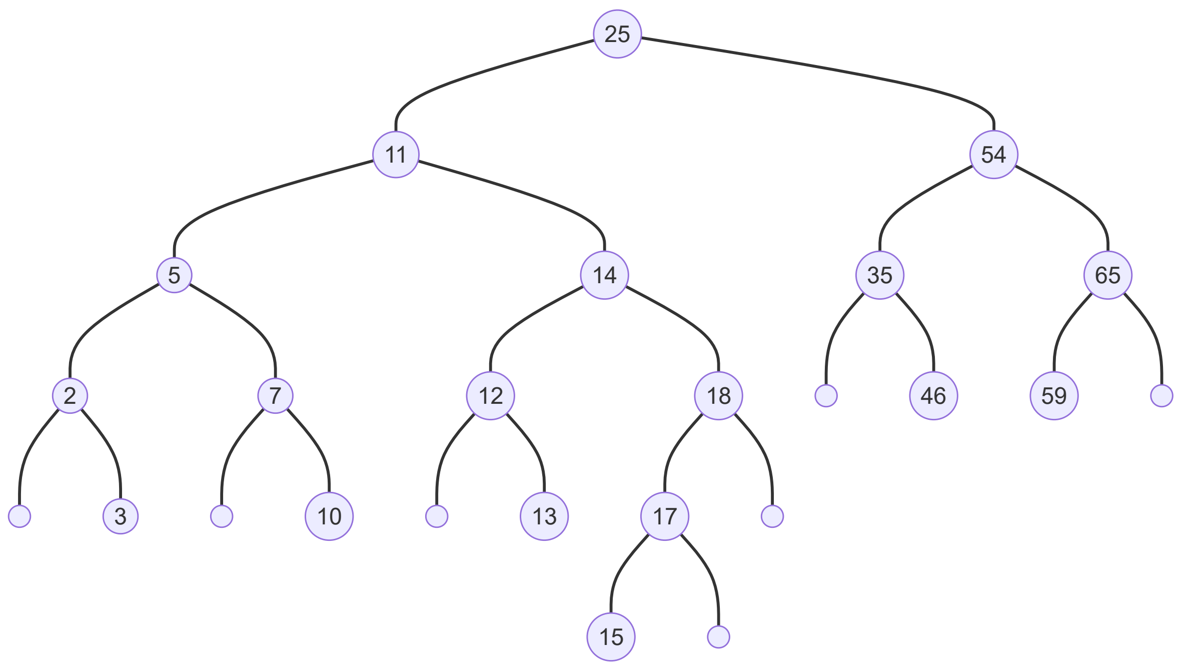
- insert 18



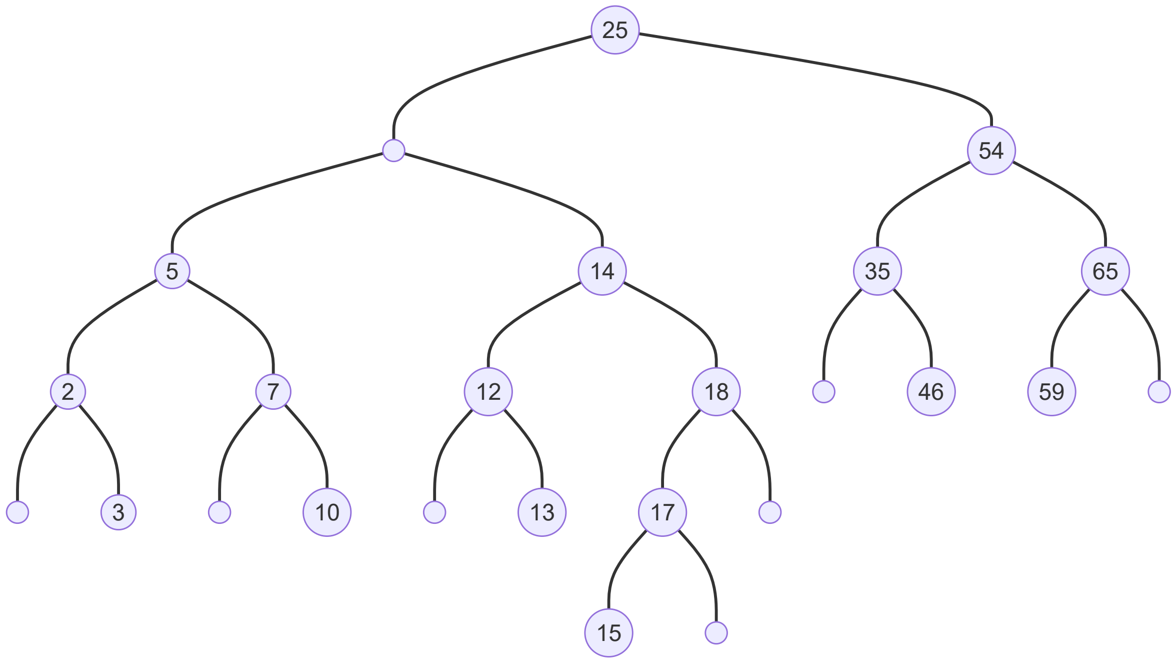
- insert 17



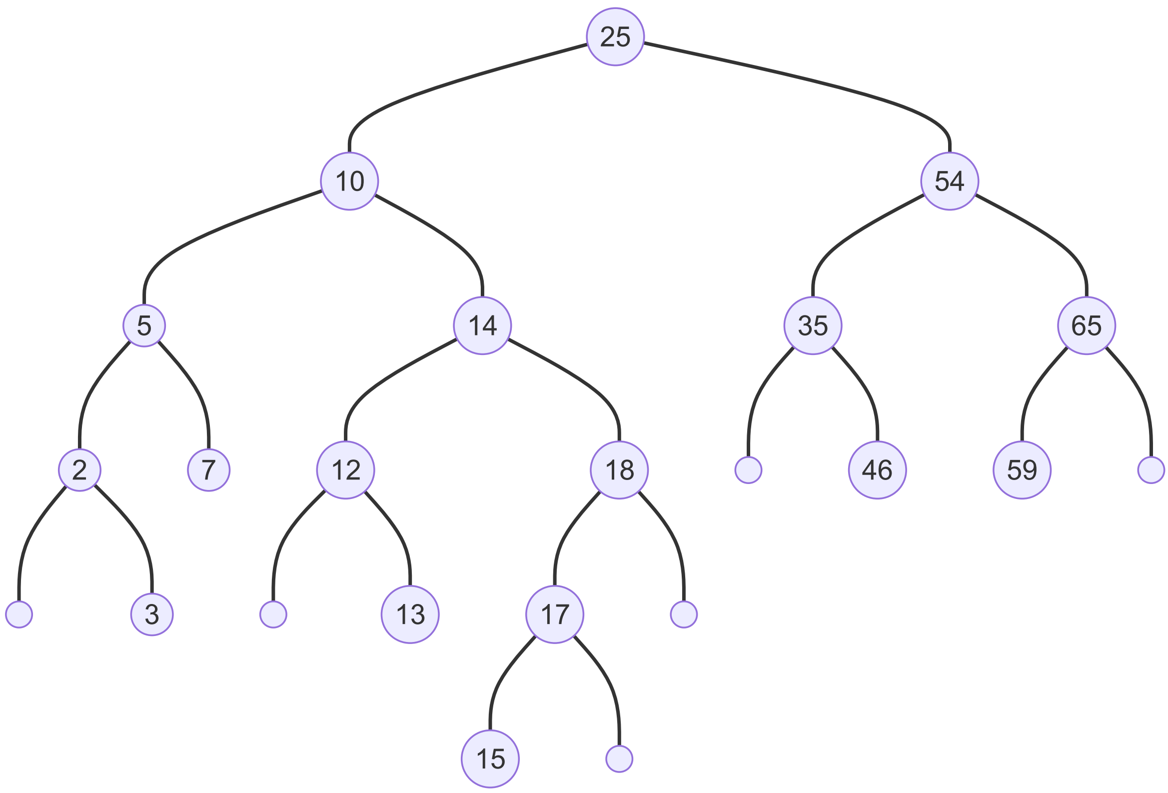
- insert 15



- delete 11
    - firstly 11 is gone



- ◦ second find the largest on the left of the deleted node, which is 10, to replace deleted node



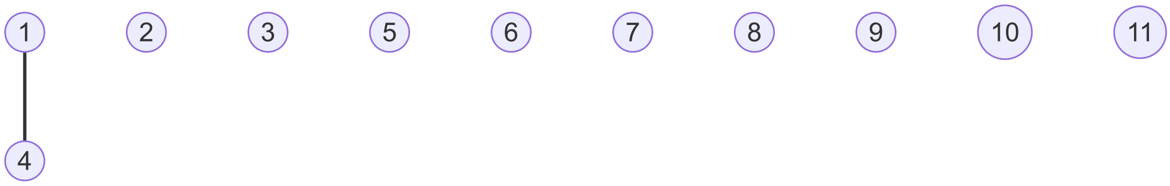
10 has no leaf originally, so this is the final state.

problem 5

- initial state

1	2	3	4	5	6	7	8	9	10	11
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

- U(1,4)

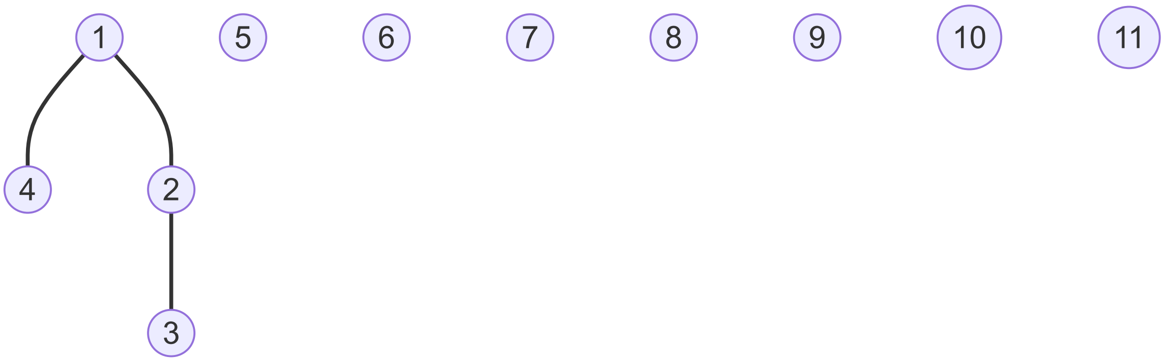


1	2	3	4	5	6	7	8	9	10	11
-2	-1	-1	1	-1	-1	-1	-1	-1	-1	-1

- U(2,3)

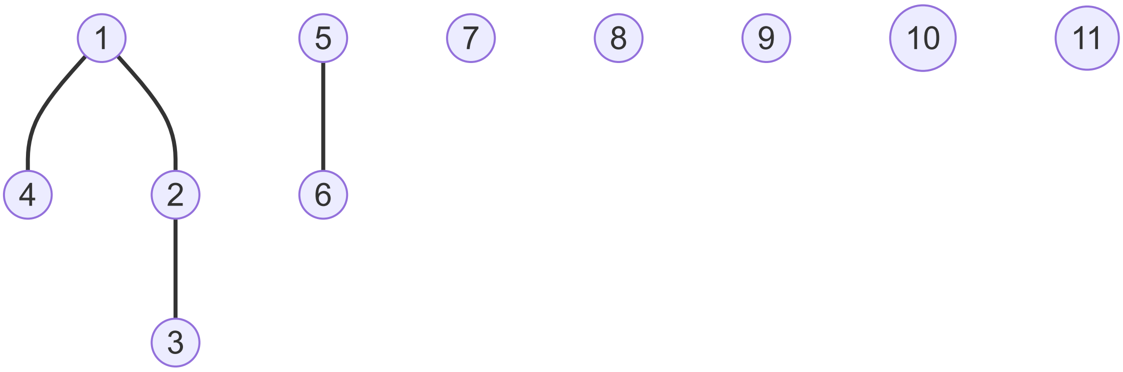
1	2	3	4	5	6	7	8	9	10	11
-2	-2	2	1	-1	-1	-1	-1	-1	-1	-1

- U(1,2)



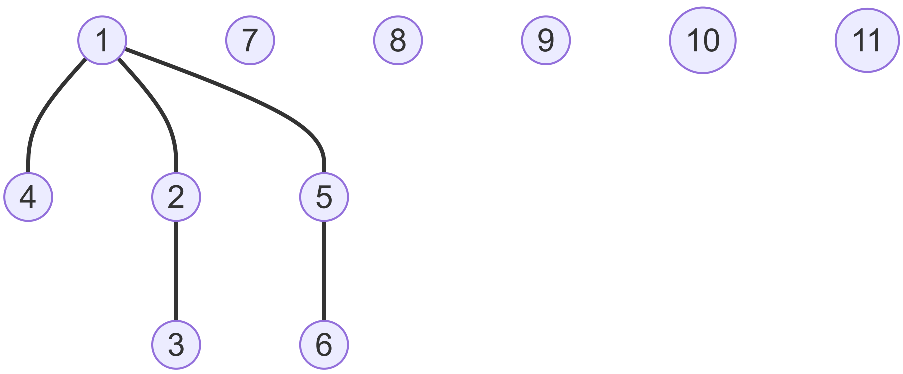
1	2	3	4	5	6	7	8	9	10	11
-4	1	2	1	-1	-1	-1	-1	-1	-1	-1

- U(5,6)



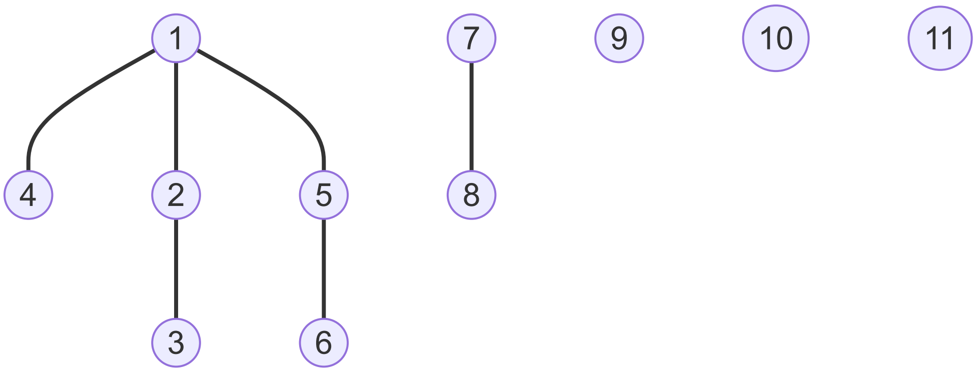
1	2	3	4	5	6	7	8	9	10	11
-4	1	2	1	-2	5	-1	-1	-1	-1	-1

- U(1,5)



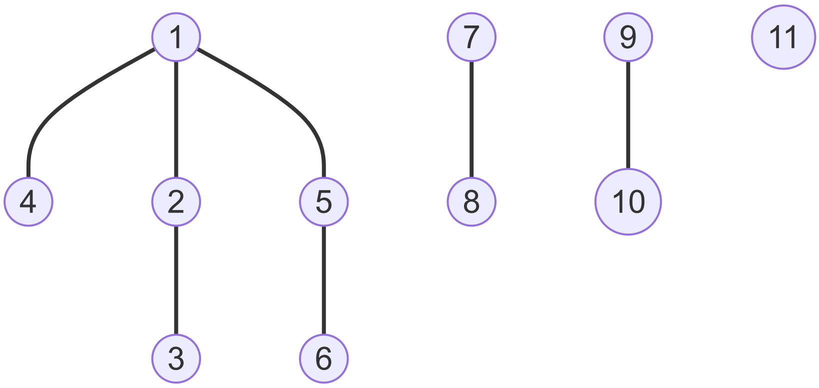
1	2	3	4	5	6	7	8	9	10	11
-6	1	2	1	1	5	-1	-1	-1	-1	-1

- U(7,8)



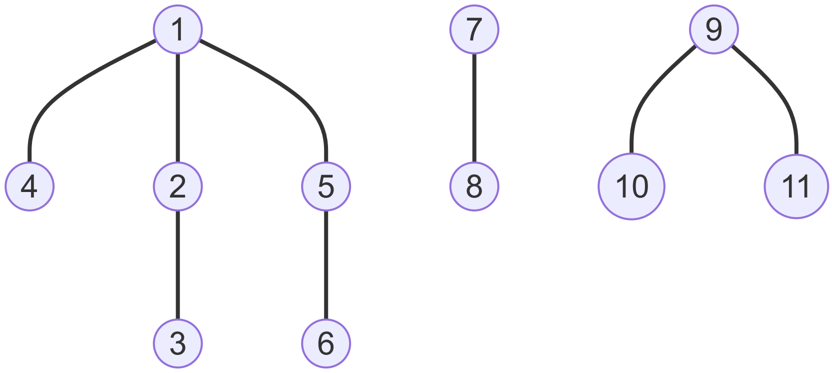
1	2	3	4	5	6	7	8	9	10	11
-6	1	2	1	1	5	-2	7	-1	-1	-1

- U(9,10)



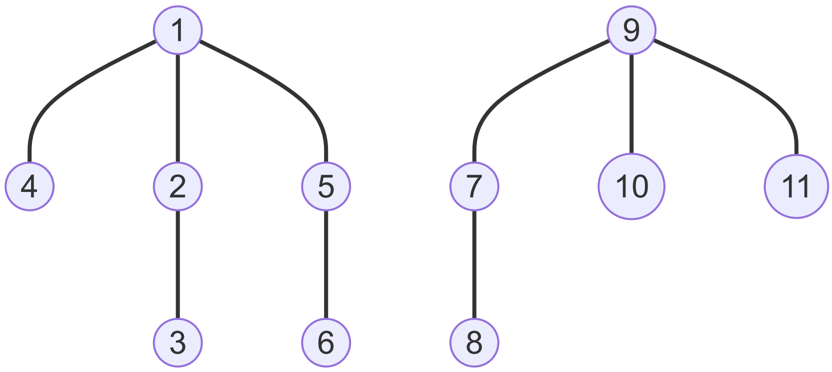
1	2	3	4	5	6	7	8	9	10	11
-6	1	2	1	1	5	-2	7	-2	9	-1

- U(9,11)



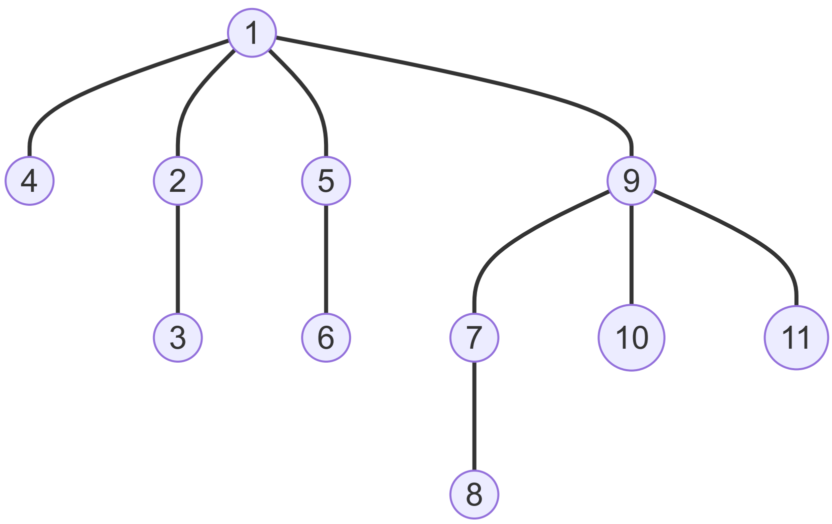
1	2	3	4	5	6	7	8	9	10	11
-6	1	2	1	1	5	-2	7	-3	9	9

- U(7,9)



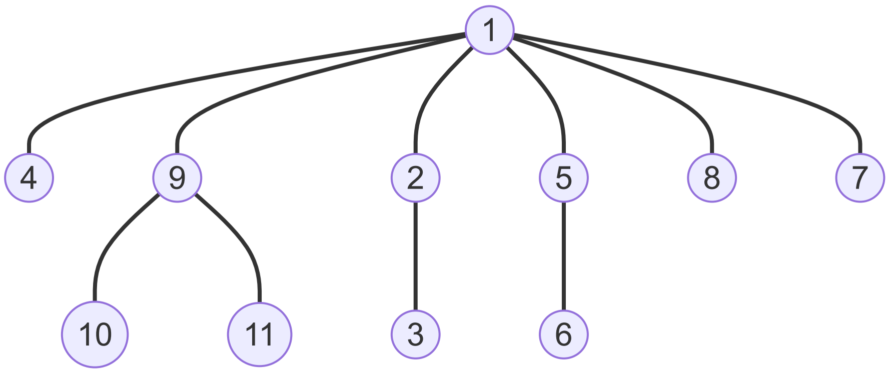
1	2	3	4	5	6	7	8	9	10	11
-6	1	2	1	1	5	9	7	-5	9	9

- U(9,1)



1	2	3	4	5	6	7	8	9	10	11
-11	1	2	1	1	5	9	7	1	9	9

- F(8) Using path compression
    - find 8, 8->7->9->1, all these three nodes are set to 1's direct child



1	2	3	4	5	6	7	8	9	10	11
-11	1	2	1	1	5	1	1	1	9	9

finished

Bonus Problem

Assume that $\forall n \geq 2, T(n) \leq c + \log\log n$

Baisc step:

$T(2) = c \leq c + \log\log 2$

Induction step:

Assume that $T(m) \leq c + \log\log m \forall m \leq n - 1$

set $m = \lfloor\sqrt{n}\rfloor$

as $\lfloor\sqrt{n}\rfloor \leq \sqrt{n}$

We have

$$\begin{aligned} T(n) &= T(\lfloor\sqrt{n}\rfloor) + 1 \\ &= T(m) + 1 \\ &\leq c + \log\log\sqrt{n} + 1 \\ &\leq c + \log\log n \end{aligned}$$

so the induction step is proved