Assignment - 1

CS348, Max. Marks: 40

NOTE: This assignment is to be done **individually**. You can discuss the assignment with each other, but the final submitted solution has to be written by each student individually. You can upload your solutions in PDF format (a scan of the handwritten solution is okay) to Moodle before the deadline.

Consider a communication system that consists of a single transmitter and a single receiver. Assume that the transmitted signal is attenuated and then corrupted by additive white Gaussian noise at the receiver. Denote the point in the two-dimensional constellation diagram (with cosine on X-axis and sine on Y-axis) corresponding to the *transmitted signal* $s(t)$ by $\underline{s} = (s_x, s_y)$. The unit vector on the x-axis is $\sqrt{\frac{2}{T}}\cos(2\pi f_0 t)$ (that is point (1,0)) and the unit vector on the y-axis (that is (0,1)) is $\sqrt{\frac{2}{T}}\sin(2\pi f_0 t)$, where T is the symbol duration (which is equal to $1/f_0$). Let us define the inner-product between two signals $g(t)$ and $h(t)$ as

$$< g(t), h(t) >= \int_0^T g(t)h(t).$$

We will assume that the *transmitted energy of any symbol* $s(t)$ is given by

$$||\underline{s}||^2 :=< s(t), s(t) >= s_x^2 + s_y^2$$

. We can also write $\underline{s}$ in polar coordinates as $\underline{s} = (||\underline{s}||, \theta_s)$ where $\theta_s = \tan^{-1}(s_y/s_x)$.

Due to the channel, the signal is attenuated by factor α (where $\alpha \in (0,1)$) and has a phase shift of ϕ (where $\phi \in (0, \pi/2)$)). Hence the received constellation point, **assuming** no additive noise, in polar coordinates is $(\alpha||\underline{s}||, \theta_s + \phi)$. However, there is always additive white Gaussian noise at the receiver, which adds n_x and n_y to the X and Y components, where n_x, n_y are i.i.d. Gaussian random variables with zero mean and variance $\frac{N_0}{2}$, where N_0 is the noise energy per symbol. Note that half the noise energy is in the X-axis direction and the other half in the Y-axis direction, which is why variance is $N_0/2$ in each direction. Effectively the X-component of the received constellation point is

$$r_x = \alpha||\underline{s}||\cos(\theta_s + \phi) + n_x \tag{1}$$

and the Y-component is

$$r_y = \alpha||\underline{s}||\sin(\theta_s + \phi) + n_y. \tag{2}$$

We define the signal-to-noise ratio (SNR) per symbol at the receiver as the ratio of the following two quantities: (i) $\alpha^2\times$(average energy per transmitted symbol), and (ii) noise energy per symbol. The average energy per transmitted symbol is just the expected value (mean) of energy of a transmitted symbol.

In the following, derive the required probabilities in terms of the $Q(.)$ function which is defined as

$$Q(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty \exp(-x^2/2)\ dx. \tag{3}$$

Write your final answers for each probability as a function of SNR per symbol. Show your working for problems 1-3.

1. (5 marks) (BPSK) Suppose the transmitter uses constellation diagram $(-A, 0)$ and $(A, 0)$ to convey bit information 1 and 0 respectively. This means that $\underline{s}$ is chosen as one of these constellation points depending on the value of the bit to be transferred. Note that here, $\theta_s = \pi$ when $\underline{s} = (-A, 0)$ and $\theta_s = 0$ when $\underline{s} = (0, A)$. Further, assume that the receiver corrects for the phase change caused by the channel, which means that we can set $\phi = 0$ in the expressions for r_x and r_y (in (1) and (2)).

   Derive an expression for the probability of incorrectly detecting the transmitted bit in terms of SNR per symbol. Assume that bits 1 and 0 are transmitted with equal probability.

2. (10 marks) (BPSK with incorrect channel estimation) Now assume that the receiver does not bother to correct for the phase change caused by the channel. This means that the received constellation points are given by (1) and (2)) for some unknown $\phi \in (0, \pi/2)$, but the receiver assumes that $\phi = 0$ and decides which bit was transmitted based on this incorrect assumption. In other words, the receiver uses the same rules used in Problem-1 above to decide which bit was transmitted.

   Derive an expression for the probability of incorrectly detecting the transmitted bit in terms of SNR per symbol. Assume that bits 1 and 0 are transmitted with equal probability.

3. (10 marks) (QPSK) Suppose the transmitter uses constellation diagram $(A, 0)$, $(0, A)$, $(-A, 0)$ and $(0, -A)$, and assigns bits 00, 01, 11, 10 to these points respectively. Assume that all constellation points are transmitted with equal probability. Calculate the probability of the first bit (of the constellation point) being received in error. Calculate the probability of the second bit being received in error. Are these two probabilities equal? Are they greater than or less than the probability calculated for BPSK (with correct estimation of channel phase distortion) above (assuming the same SNR per symbol for BPSK and QPSK)?

4. (10 marks) Draw a single graph with SNR on the x-axis and $\log_{10}$ of the various probabilities in (1) and (3). For problem-2 above, generate 3 different random numbers between $(0, \pi/2)$ (note: we expect each student to generate different random numbers). Set ϕ to be equal to each of these values and plot $\log_{10}$ of the probability in problem-2 on the same graph as you plotted for problems (1) and (3). You should have a total of 5 plots on your graph. You can use any scientific calculator application (including matlab or one of its clones) and mark which line corresponds to BPSK, QPSK, and the different BPSK with incorrect phase-estimation (with corresponding values of ϕ). (You can use an online graphing tool such as: Desmos . You can try typing in the left panel $Q(x) = 0.5 - 0.5 * erf(x/\sqrt(2))$ etc. as shown in Figure 1. You can add more plots just by entering more functions (e.g. $y = x^2$) in the rows in the left panel.)

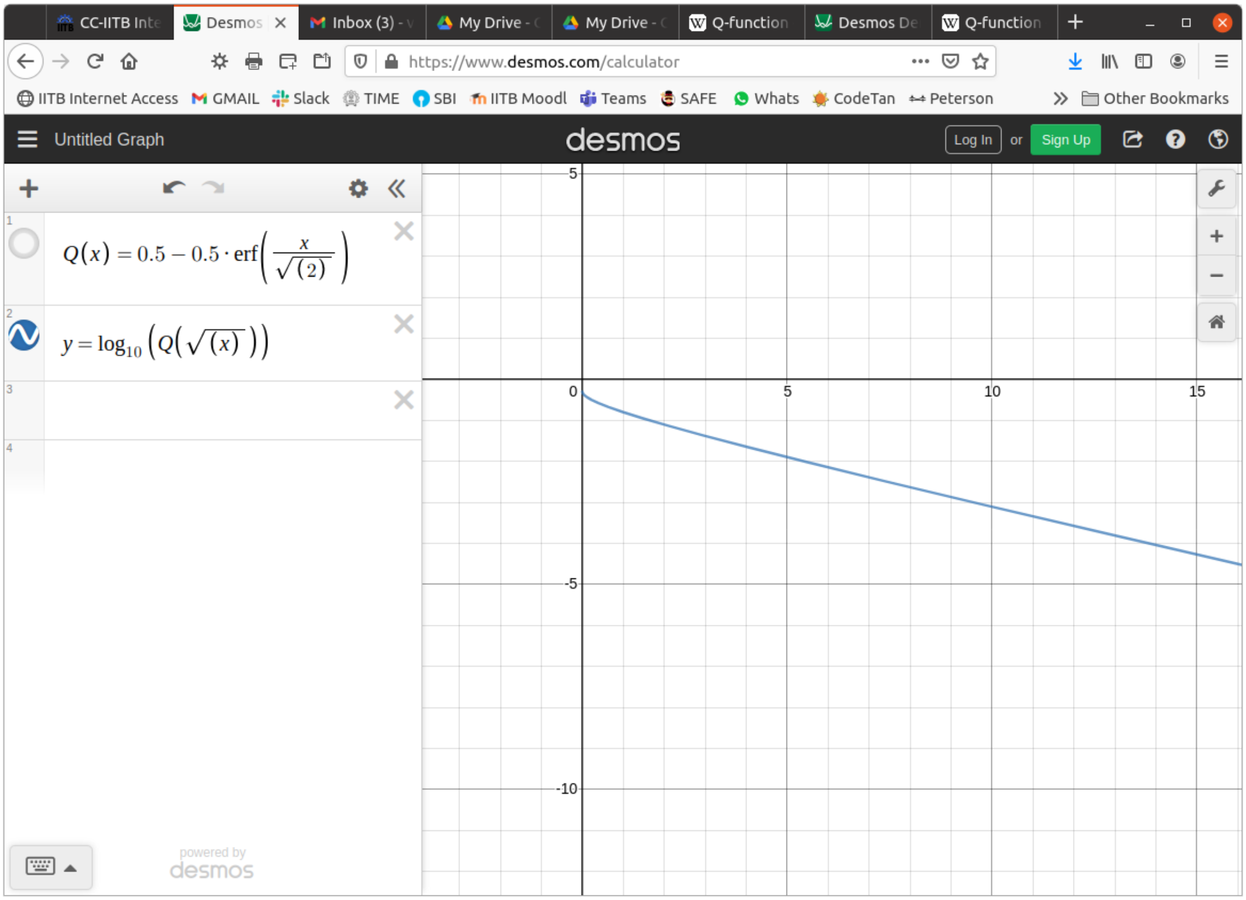

Figure 1: Example using Desmos

5. (5 marks) What is the SNR required to get a probability of bit error less than 10^{-4} for each of the 5 plots in problem-4 (just write the 5 SNR values)? In addition, comment on how the value of ϕ (in problem-2, as observed in the plot from problem-4) affects the probability of bit error in general.