Assignment 0

Arnav Garg

22B1021

1 Elegance

I was able to solve the Oddtown problem after taking a small hint (the matrix representation of the system).
I was unable to solve the Eventown problem by myself and had to look up the solution.

2 Weaker Suffices!

Let us prove that the condition $\langle Ax\ ,\ x\rangle = \langle x\ ,\ Ax\rangle$ is necessary. This is trivial by putting $y = x$ in the base condition of Hermitian that is $\langle Ax\ ,\ y\rangle = \langle x\ ,\ Ay\rangle$. Hence this is proved.
Now we prove the sufficiency. Let us assume that $\langle Ax\ ,\ x\rangle = \langle x\ ,\ Ax\rangle$ holds for some operator A. We shall prove that A is Hermitian, thus, $\langle Ax\ ,\ y\rangle = \langle x\ ,\ Ay\rangle$. As x is arbitrary we may represent x as $x = u + v$ where u and v are vectors belonging to the complex field.

$$\begin{aligned}
\langle u+v\ ,\ A(u+v)\rangle &= \langle A(u+v)\ ,\ u+v\rangle \\
\implies \langle Au\ ,\ v\rangle + \langle Av\ ,\ u\rangle &= \langle u\ ,\ Av\rangle + \langle v\ ,\ Au\rangle \\
\implies \langle (A-A^*)u\ ,\ v\rangle &= \langle (A^*-A)v\ ,\ u\rangle \\
&= \langle u\ ,\ (A^*-A)v\rangle^* \\
&= \langle (A-A^*)u\ ,\ v\rangle^*
\end{aligned}$$

A complex number is only equal to its conjugate when it is real. However as u and v are arbitrary vectors in the complex vector space we may multiply either v or u by i, however the result must remain real. The only real number which when multiplied by i stays real is 0.
Hence, $\langle (A-A^*)u\ ,\ v\rangle = 0$
Setting $v = (A-A^*)u$ we observe $(A-A^*)u = 0$. As u is an arbitrary vector, $A - A^* = O$. Hence, A is hermitian.

3 Higher Dimensions

a) We may notice that

$$\begin{bmatrix} 0 & 5 & 0 & 4 \\ 5 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \bigotimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For matrices A (with eigenvalues $v_i : 1 \leq i \leq j$) and B (with eigenvalues $w_k : 1 \leq k \leq l$) we have the property that the eigenvalues of $C = A \otimes B$ are $v_i w_k : 1 \leq i \leq j, 1 \leq k \leq l$.
Thus the eigenvalues of the given matrix are:

$$\lambda = \frac{7+\sqrt{57}}{2}$$

$$\lambda = \frac{7-\sqrt{57}}{2}$$

$$\lambda = \frac{-7+\sqrt{57}}{2}$$

$$\lambda = \frac{-7-\sqrt{57}}{2}$$

b)Similarly

$$\begin{bmatrix} 0 & 0 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 5 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigotimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

As the constituent matrices are the same in the tensor product as the last part, we observe that the eigenvalues remain the same!
c)We notice that

$$\begin{bmatrix} 25 & 20 & 20 & 16 \\ 15 & 10 & 12 & 8 \\ 15 & 12 & 10 & 8 \\ 9 & 6 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \bigotimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

Thus the obtained eigenvalues would be-

$$\lambda = \frac{53+7\sqrt{57}}{2}$$

$$\lambda = \frac{53-7\sqrt{57}}{2}$$

$$\lambda = -2$$

$\lambda = -2$ has an algebraic multiplicity of 2 here.

4 Algebra and Technicalities

a) Let us first list the desired conditions for a norm function:

1. $p(x+y) \leq p(x) + p(y)$

$$\begin{aligned}\sqrt{\langle x+y\ ,\ x+y\rangle} &= \sqrt{\langle x+y\ ,\ x\rangle + \langle x+y\ ,\ y\rangle} \\ &= \sqrt{\langle x\ ,\ x\rangle + \langle x\ ,\ y\rangle + \langle y\ ,\ x\rangle + \langle y\ ,\ y\rangle} \\ &\leq \sqrt{\langle x\ ,\ x\rangle} + \sqrt{\langle y\ ,\ y\rangle}\end{aligned}$$

2. $p(sx) = |s|p(x)$

$$\begin{aligned}\sqrt{\langle sx\ ,\ sx\rangle} &= \sqrt{s^2\langle x\ ,\ x\rangle} \\ &= |s|\sqrt{\langle x\ ,\ x\rangle}\end{aligned}$$

3. $p(x) = 0 \iff x = 0$ (Follows from definition)

Thus $|x| = \sqrt{\langle x\ ,\ x\rangle}$ is a valid norm function.
b) Now lets see the conditions for a metric:

1. $d(x,x) = 0$
   This follows from the fact that $\sqrt{\langle 0\ ,\ 0\rangle} = 0$.
2. If $x \neq y$ then $d(x,y) > 0$
   This follows from the fact that $\langle a\ ,\ a\rangle > 0\ \forall a \neq 0$
3. $d(x,y) = d(y,x)$ (This again trivially follows)
4. $d(x,z) \leq d(y,z) + d(x,y)$
   This follows from the fact that $x - z = (x-y) + (y-z)$ and then applying the AM-GM inequality.

Thus $|x-y|$ is a valid metric.
c) The set of all continuous functions over $[-1,1]$ is a vector space as it satisfies both the normal equations $(f+g)(t) = f(t) + g(t)$ and $(\alpha f)(t) = \alpha f(t)$.
We prove the first by proving

$$\lim_{x \to a} f(x) + g(x) = f(a) + g(a)$$

Let $\varepsilon > 0$ be given

$$(x-a) < \delta_f \implies |f(x) - f(a)| < \frac{\varepsilon}{2}$$

$$(x-a) < \delta_g \implies |g(x) - g(a)| < \frac{\varepsilon}{2}$$

Taking $\delta = \min(\delta_g, \delta_f)$,

$$(x-a) < \delta \implies |f(x) + g(x) - (f(a) + g(a))| < \varepsilon$$

The proof would proceed similarly for the second case proving the continuity for $\frac{\varepsilon}{\alpha}$ and then using it to prove the limit.
Thus the given set of vectors is a vector space.
d)

$$\begin{aligned}\langle f_n\ ,\ f_m\rangle &= \int_{-1}^{1} f(x)g(x)\,dx \\ &= 1-\frac{m+n}{2(\max(m,n))^2}+\frac{mn}{3(\max(m,n))^3}\end{aligned}$$

For cauchy, $|f_n - f_m| = \frac{1-\frac{n}{m}}{\sqrt{m}}$ which allows us to choose a suitable m for any and all n.
This sequence converges to $f(x) = \frac{1-sgn(x)}{2}$ which is not a part of the mentioned vector space and hence, the inner product space is not complete.

5 Hilbert-Schmidt and Vectorization

The set $\mathcal{L}(A, B)$ is a vector space as it being a space of operators follows the standard equations.

$$\langle \alpha v_i \otimes w_i \mid \beta v_j \otimes w_j\rangle = \alpha^*\beta\langle v_i|v_j\rangle\langle w_i|w_j\rangle$$

$$\begin{aligned}\langle U|V\rangle_{HS} &= \langle TU|TV\rangle \\ &= \sum_{i,j,k,l}\alpha^*\beta\langle v_i|v_k\rangle\langle w_j|w_l\rangle \\ &= \sum\alpha^*\beta\end{aligned}$$

Thus due to orthonormality of the bases chosen we can say that $\langle U|V\rangle_{HS} = tr(A^+B)$