

# Assignment 0

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## 1 Elegance

I was able to solve the Oddtown problem after taking a small hint (the matrix representation of the system).

I was unable to solve the Eventown problem by myself and had to look up the solution.

## 2 Weaker Suffices!

Let us prove that the condition  $\langle Ax, x \rangle = \langle x, Ax \rangle$  is necessary. This is trivial by putting  $y = x$  in the base condition of Hermitian that is  $\langle Ax, y \rangle = \langle x, Ay \rangle$ . Hence this is proved.

Now we prove the sufficiency. Let us assume that  $\langle Ax, x \rangle = \langle x, Ax \rangle$  holds for some operator  $A$ . We shall prove that  $A$  is Hermitian, thus,  $\langle Ax, y \rangle = \langle x, Ay \rangle$ . As  $x$  is arbitrary we may represent  $x$  as  $x = u + v$  where  $u$  and  $v$  are vectors belonging to the complex field.

$$\begin{aligned}\langle u + v, A(u + v) \rangle &= \langle A(u + v), u + v \rangle \\ \implies \langle Au, v \rangle + \langle Av, u \rangle &= \langle u, Av \rangle + \langle v, Au \rangle \\ \implies \langle (A - A^*)u, v \rangle &= \langle (A^* - A)v, u \rangle \\ &= \langle u, (A^* - A)v \rangle^* \\ &= \langle (A - A^*)u, v \rangle^*\end{aligned}$$

A complex number is only equal to its conjugate when it is real. However as  $u$  and  $v$  are arbitrary vectors in the complex vector space we may multiply either  $v$  or  $u$  by  $i$ , however the result must remain real. The only real number which when multiplied by  $i$  stays real is 0.

Hence,  $\langle (A - A^*)u, v \rangle = 0$

Setting  $v = (A - A^*)u$  we observe  $(A - A^*)u = 0$ . As  $u$  is an arbitrary vector,  $A - A^* = 0$ . Hence,  $A$  is hermitian.

### 3 Higher Dimensions

a) We may notice that

$$\begin{bmatrix} 0 & 5 & 0 & 4 \\ 5 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For matrices  $A$  (with eigenvalues  $v_i : 1 \leq i \leq j$ ) and  $B$  (with eigenvalues  $w_k : 1 \leq k \leq l$ ) we have the property that the eigenvalues of  $C = A \otimes B$  are  $v_i w_k : 1 \leq i \leq j, 1 \leq k \leq l$ .

Thus the eigenvalues of the given matrix are:

$$\lambda = \frac{7 + \sqrt{57}}{2}$$

$$\lambda = \frac{7 - \sqrt{57}}{2}$$

$$\lambda = \frac{-7 + \sqrt{57}}{2}$$

$$\lambda = \frac{-7 - \sqrt{57}}{2}$$

b) Similarly

$$\begin{bmatrix} 0 & 0 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 5 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

As the constituent matrices are the same in the tensor product as the last part, we observe that the eigenvalues remain the same!

c) We notice that

$$\begin{bmatrix} 25 & 20 & 20 & 16 \\ 15 & 10 & 12 & 8 \\ 15 & 12 & 10 & 8 \\ 9 & 6 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

Thus the obtained eigenvalues would be-

$$\lambda = \frac{53 + 7\sqrt{57}}{2}$$

$$\lambda = \frac{53 - 7\sqrt{57}}{2}$$

$$\lambda = -2$$

$\lambda = -2$  has an algebraic multiplicity of 2 here.

## 4 Algebra and Technicalities

a) Let us first list the desired conditions for a norm function:

1.  $p(x + y) \leq p(x) + p(y)$

$$\begin{aligned}\sqrt{\langle x + y, x + y \rangle} &= \sqrt{\langle x + y, x \rangle + \langle x + y, y \rangle} \\ &= \sqrt{\langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle} \\ &\leq \sqrt{\langle x, x \rangle} + \sqrt{\langle y, y \rangle}\end{aligned}$$

2.  $p(sx) = |s|p(x)$

$$\begin{aligned}\sqrt{\langle sx, sx \rangle} &= \sqrt{s^2 \langle x, x \rangle} \\ &= |s| \sqrt{\langle x, x \rangle}\end{aligned}$$

3.  $p(x) = 0 \iff x = 0$  (Follows from definition)

Thus  $|x| = \sqrt{\langle x, x \rangle}$  is a valid norm function.

b) Now let's see the conditions for a metric:

1.  $d(x, x) = 0$

This follows from the fact that  $\sqrt{\langle 0, 0 \rangle} = 0$ .

2. If  $x \neq y$  then  $d(x, y) > 0$

This follows from the fact that  $\langle a, a \rangle > 0 \forall a \neq 0$

3.  $d(x, y) = d(y, x)$  (This again trivially follows)

4.  $d(x, z) \leq d(y, z) + d(x, y)$

This follows from the fact that  $x - z = (x - y) + (y - z)$  and then applying the AM-GM inequality.

Thus  $|x - y|$  is a valid metric.

c) The set of all continuous functions over  $[-1, 1]$  is a vector space as it satisfies both the normal equations  $(f + g)(t) = f(t) + g(t)$  and  $(\alpha f)(t) = \alpha f(t)$ .

We prove the first by proving

$$\lim_{x \rightarrow a} f(x) + g(x) = f(a) + g(a)$$

Let  $\varepsilon > 0$  be given

$$(x - a) < \delta_f \implies |f(x) - f(a)| < \frac{\varepsilon}{2}$$

$$(x - a) < \delta_g \implies |g(x) - g(a)| < \frac{\varepsilon}{2}$$

Taking  $\delta = \min(\delta_g, \delta_f)$ ,

$$(x - a) < \delta \implies |f(x) + g(x) - (f(a) + g(a))| < \varepsilon$$

The proof would proceed similarly for the second case proving the continuity for  $\frac{\varepsilon}{\alpha}$  and then using it to prove the limit.

Thus the given set of vectors is a vector subspace.

## 5 Hilbert-Schmidt and Vectorization