5 Rotating reference frames

5.1 Introduction

As we saw in Chapter 1, Newton's second law of motion is valid only in *inertial* frames of reference. However, it is sometimes convenient to observe motion in noninertial *rotating* reference frames. For instance, it is most convenient for us to observe the motions of objects close to the Earth's surface in a reference frame that is fixed relative to this surface. Such a frame is noninertial in nature, as it accelerates with respect to a standard inertial frame as a result of the Earth's diurnal rotation. (The accelerations of this frame owing to the Earth's orbital motion about the Sun, or the Sun's orbital motion about the galactic center, and so on, are negligible compared with that associated with the Earth's diurnal rotation.) Let us investigate motion observed in a rotating reference frame.

5.2 Rotating reference frames

Suppose that a given object has position vector $\mathbf{r}$ in some inertial (i.e., nonrotating) reference frame. Let us observe the motion of this object in a noninertial reference frame that rotates with constant angular velocity $\mathbf{\Omega}$ about an axis passing through the origin of the inertial frame. Suppose, first of all, that our object appears stationary in the rotating reference frame. Hence, in the nonrotating frame, the object's position vector $\mathbf{r}$ will appear to *precess* about the origin with angular velocity $\mathbf{\Omega}$. It follows from Section A.7 that, in the nonrotating reference frame,

$$\frac{d\mathbf{r}}{dt} = \mathbf{\Omega} \times \mathbf{r}. \tag{5.1}$$

Suppose, now, that our object appears to move in the rotating reference frame with instantaneous velocity $\mathbf{v}'$. It is fairly obvious that the appropriate generalization of the preceding equation is simply

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}' + \mathbf{\Omega} \times \mathbf{r}. \tag{5.2}$$

Let d/dt and d/dt' denote apparent time derivatives in the nonrotating and rotating frames of reference, respectively. Because an object that is stationary in the rotating reference frame appears to move in the nonrotating frame, it is clear that $d/dt \neq d/dt'$. Writing the apparent velocity, $\mathbf{v}'$, of our object in the rotating reference frame as $d\mathbf{r}/dt'$, Equation (5.2) takes the form

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{dt'} + \boldsymbol{\Omega} \times \mathbf{r}, \tag{5.3}$$

or

$$\frac{d}{dt} = \frac{d}{dt'} + \boldsymbol{\Omega} \times, \tag{5.4}$$

because $\mathbf{r}$ is a general position vector. Equation (5.4) expresses the relationship between apparent time derivatives in the nonrotating and rotating reference frames.

Operating on the general position vector $\mathbf{r}$ with the time derivative in Equation (5.4), we get

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r}. \tag{5.5}$$

This equation relates the apparent velocity, $\mathbf{v} = d\mathbf{r}/dt$, of an object with position vector $\mathbf{r}$ in the nonrotating reference frame to its apparent velocity, $\mathbf{v}' = d\mathbf{r}/dt'$, in the rotating reference frame.

Operating twice on the position vector $\mathbf{r}$ with the time derivative in Equation (5.4), we obtain

$$\mathbf{a} = \left(\frac{d}{dt'} + \boldsymbol{\Omega} \times\right)(\mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r}), \tag{5.6}$$

or

$$\mathbf{a} = \mathbf{a}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\,\boldsymbol{\Omega} \times \mathbf{v}'. \tag{5.7}$$

This equation relates the apparent acceleration, $\mathbf{a} = d^2\mathbf{r}/dt^2$, of an object with position vector $\mathbf{r}$ in the nonrotating reference frame to its apparent acceleration, $\mathbf{a}' = d^2\mathbf{r}/dt'^2$, in the rotating reference frame.

Applying Newton's second law of motion in the inertial (i.e., nonrotating) reference frame, we obtain

$$m\,\mathbf{a} = \mathbf{f}. \tag{5.8}$$

Here, m is the mass of our object and $\mathbf{f}$ is the (nonfictitious) force acting on it. These quantities are the same in both reference frames. Making use of Equation (5.7), the apparent equation of motion of our object in the rotating reference frame takes the form

$$m\,\mathbf{a}' = \mathbf{f} - m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) - 2\,m\,\boldsymbol{\Omega} \times \mathbf{v}'. \tag{5.9}$$

The last two terms in this equation are so-called *fictitious forces*. Such forces are always needed to account for motion observed in noninertial reference frames. Fictitious forces can always be distinguished from nonfictitious forces in Newtonian mechanics because the former have no associated reactions. Let us now investigate the two fictitious forces appearing in Equation (5.9).

5.3 Centrifugal acceleration

Let our nonrotating inertial frame be one whose origin lies at the center of the Earth, and let our rotating frame be one whose origin is fixed with respect to some point, of

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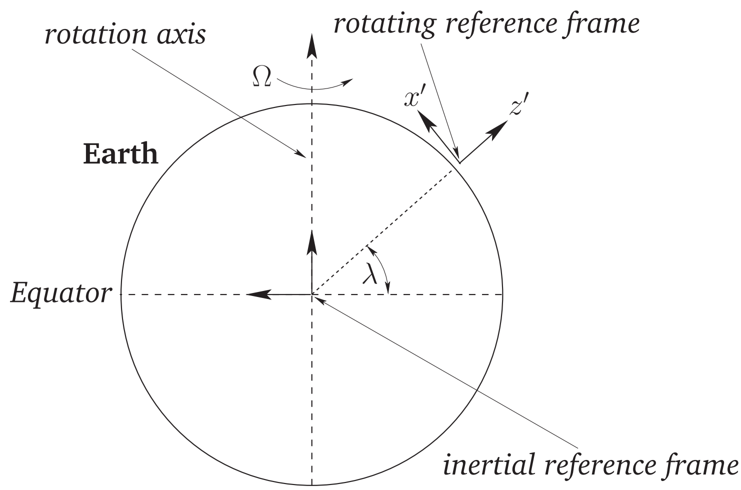

Fig. 5.1 Inertial and noninertial reference frames.

latitude λ, on the Earth's surface. (See Figure 5.1.) The latter reference frame thus rotates with respect to the former (about an axis passing through the Earth's center) with an angular velocity vector, $\mathbf{\Omega}$, which points from the center of the Earth toward its north pole and is of magnitude

$$\Omega = \frac{2\pi}{24^{\rm h}\,56^{\rm m}\,04^{\rm s}} = 7.2921\times 10^{-5}\ {\rm rad.\,s^{-1}}. \tag{5.10}$$

Here, $24^{\rm h}\,56^{\rm m}\,04^{\rm s}$ is the length of a *sidereal day*, that is, the Earth's rotation period relative to the distant stars (Yoder 1995).

Consider an object that appears stationary in our rotating reference frame, that is, an object that is stationary with respect to the Earth's surface. According to Equation (5.9), the object's apparent equation of motion in the rotating frame takes the form

$$m\,\mathbf{a}' = \mathbf{f} - m\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}). \tag{5.11}$$

Let the nonfictitious force acting on our object be the force of gravity, $\mathbf{f} = m\,\mathbf{g}$. Here, the local gravitational acceleration, $\mathbf{g}$, points directly toward the center of the Earth. It follows that the apparent gravitational acceleration in the rotating frame is written

$$\mathbf{g}' = \mathbf{g} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{R}), \tag{5.12}$$

where $\mathbf{R}$ is the displacement vector of the origin of the rotating frame (which lies on the Earth's surface) with respect to the center of the Earth. Here, we are assuming that our object is situated relatively close to the Earth's surface (i.e., $\mathbf{r}\simeq\mathbf{R}$).

It can be seen from Equation (5.12) that the apparent gravitational acceleration of a stationary object close to the Earth's surface has two components: first, the true gravitational acceleration, $\mathbf{g}$, of magnitude $g \simeq 9.82\ {\rm m\,s^{-2}}$, which always points directly toward the center of the Earth (Yoder 1995); and second, the so-called *centrifugal acceleration*, $-\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{R})$. This acceleration is normal to the Earth's axis of rotation and always points directly away from this axis. The magnitude of the centrifugal acceleration is

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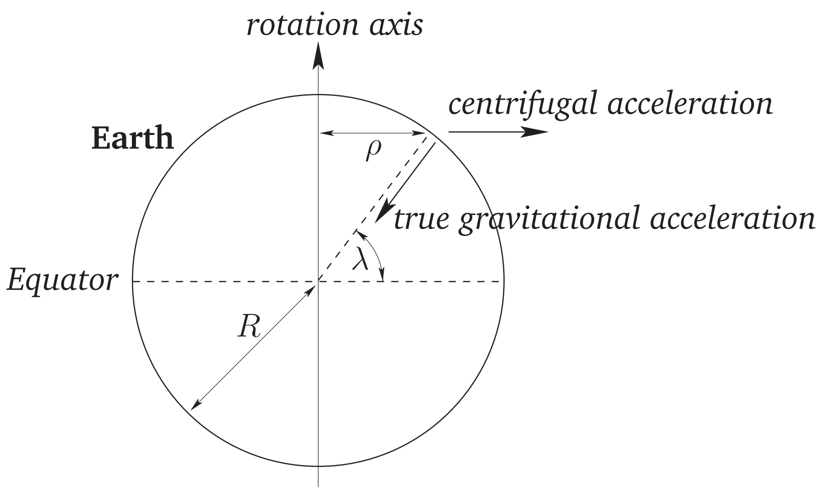

Fig. 5.2 Centrifugal acceleration.

$\Omega^2 \rho = \Omega^2 R \cos\lambda$, where ρ is the perpendicular distance to the Earth's rotation axis, and $R = 6.3710 \times 10^6$ m is the Earth's radius (Yoder 1995). (See Figure 5.2.)

It is convenient to define Cartesian axes in the rotating reference frame such that the z'-axis points vertically upward and the x'- and y'-axes are horizontal, with the x'-axis pointing directly northward and the y'-axis pointing directly westward. (See Figure 5.1.) The Cartesian components of the Earth's angular velocity are thus

$$\mathbf{\Omega} = \Omega\,(\cos\lambda,\ 0,\ \sin\lambda), \tag{5.13}$$

while the vectors $\mathbf{R}$ and $\mathbf{g}$ are written

$$\mathbf{R} = (0,\ 0,\ R) \tag{5.14}$$

and

$$\mathbf{g} = (0,\ 0,\ -g), \tag{5.15}$$

respectively. It follows that the Cartesian coordinates of the apparent gravitational acceleration, from Equation (5.12), are

$$\mathbf{g}' = \left(-\Omega^2 R \cos\lambda \sin\lambda,\ 0,\ -g + \Omega^2 R \cos^2\lambda\right). \tag{5.16}$$

The magnitude of this acceleration is approximately

$$g' \simeq g - \Omega^2 R \cos^2\lambda \simeq 9.82 - 0.0338 \cos^2\lambda\ \ \mathrm{m\,s^{-2}}. \tag{5.17}$$

According to the preceding equation, the centrifugal acceleration causes the magnitude of the apparent gravitational acceleration on the Earth's surface to vary by about 0.3 percent, being largest at the poles and smallest at the equator. This variation in apparent gravitational acceleration, due (ultimately) to the Earth's rotation, causes the Earth itself to bulge slightly at the equator (see Section 5.5), which has the effect of further intensifying the variation (see Exercise 5.7), as a point on the surface of the Earth at the equator is slightly farther away from the Earth's center than a similar point at one of the poles (and, hence, the true gravitational acceleration is slightly weaker in the former case).

Another consequence of centrifugal acceleration is that the apparent gravitational acceleration on the Earth's surface has a horizontal component aligned in the north–south direction. This horizontal component ensures that the apparent gravitational acceleration does not point directly toward the center of the Earth. In other words, a plumb line on the surface of the Earth does not point vertically downward (toward the center of the Earth), but is deflected slightly away from a true vertical in the north–south direction. The angular deviation from true vertical can easily be calculated from Equation (5.16):

$$\theta_{dev} \simeq -\frac{\Omega^2 R}{2\,g}\,\sin(2\,\lambda) \simeq -0.1^\circ\,\sin(2\,\lambda). \tag{5.18}$$

Here, a positive angle denotes a northward deflection, and vice versa. Thus, the deflection is southward in the northern hemisphere (i.e., $\lambda > 0$) and northward in the southern hemisphere (i.e., $\lambda < 0$). The deflection is zero at the poles and at the equator, and it reaches its maximum magnitude (which is very small) at middle latitudes.

5.4 Coriolis force

We have now accounted for the first fictitious force, $-m\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$, appearing in Equation (5.9). Let us now investigate the second, which takes the form $-2\,m\,\mathbf{\Omega}\times\mathbf{v}'$ and is called the *Coriolis force*. Obviously, this force affects only objects that are moving in the rotating reference frame.

Consider a particle of mass m free-falling under gravity in our rotating reference frame. As before, we define Cartesian axes in the rotating frame such that the z'-axis points vertically upward and the x'- and y'-axes are horizontal, with the x'-axis pointing directly northward and the y'-axis pointing directly westward. It follows, from Equation (5.9), that the Cartesian equations of motion of the particle in the rotating reference frame take the form

$$\ddot{x}' = 2\,\Omega\,\sin\lambda\,\dot{y}', \tag{5.19}$$

$$\ddot{y}' = -2\,\Omega\,\sin\lambda\,\dot{x}' + 2\,\Omega\,\cos\lambda\,\dot{z}', \tag{5.20}$$

and

$$\ddot{z}' = -g - 2\,\Omega\,\cos\lambda\,\dot{y}'. \tag{5.21}$$

Here, g is the local acceleration due to gravity. In the preceding three equations, we have neglected the centrifugal acceleration for the sake of simplicity. This is reasonable, because the only effect of the centrifugal acceleration is to slightly modify the magnitude and direction of the local gravitational acceleration. We have also neglected air resistance, which is less reasonable.

Consider a particle that is dropped (at $t = 0$) from rest a height h above the Earth's surface. The following solution method exploits the fact that the Coriolis force is much smaller in magnitude that the force of gravity. Hence, Ω can be treated as a *small*

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parameter. To lowest order (i.e., neglecting Ω), the particle's vertical motion satisfies $\ddot{z}' = -g$, which can be solved, subject to the initial conditions, to give

$$z' = h - \frac{g\,t^2}{2}. \tag{5.22}$$

Substituting this expression into Equations (5.19) and (5.20), neglecting terms involving Ω^2, and solving subject to the initial conditions, we obtain $x' \simeq 0$ and

$$y' \simeq -g\,\Omega\,\cos\lambda\,\frac{t^3}{3}. \tag{5.23}$$

In other words, the particle is deflected eastward (i.e, in the negative y'-direction). The particle hits the ground when $t \simeq \sqrt{2\,h/g}$. Hence, the net eastward deflection of the particle as it strikes the ground is

$$d_{\rm east} \simeq \frac{\Omega}{3}\,\cos\lambda\left(\frac{8\,h^3}{g}\right)^{1/2}. \tag{5.24}$$

This deflection is in the same direction as the Earth's rotation (i.e., west to east) and is greatest at the equator and zero at the poles. A particle dropped from a height of 100 m at the equator is deflected by about 2.2 cm.

Consider a particle launched horizontally with some fairly large velocity,

$$\mathbf{v} = v_0\,(\cos\theta, -\sin\theta,\, 0). \tag{5.25}$$

Here, θ is the *compass bearing* of the velocity vector (so north is 0°, east is 90°, etc.). Neglecting any vertical motion, Equations (5.19) and (5.20) yield

$$\dot{v}_{x'} \simeq -2\,\Omega\,v_0\,\sin\lambda\,\sin\theta \tag{5.26}$$

and

$$\dot{v}_{y'} \simeq -2\,\Omega\,v_0\,\sin\lambda\,\cos\theta, \tag{5.27}$$

which can be integrated to give

$$v_{x'} \simeq v_0\,\cos\theta - 2\,\Omega\,v_0\,\sin\lambda\,\sin\theta\,t \tag{5.28}$$

and

$$v_{y'} \simeq -v_0\,\sin\theta - 2\,\Omega\,v_0\,\sin\lambda\,\cos\theta\,t. \tag{5.29}$$

To lowest order in Ω, the preceding equations are equivalent to

$$v_{x'} \simeq v_0\,\cos(\theta + 2\,\Omega\,\sin\lambda\,t) \tag{5.30}$$

and

$$v_{y'} \simeq -v_0\,\sin(\theta + 2\,\Omega\,\sin\lambda\,t). \tag{5.31}$$

It follows that the Coriolis force causes the compass bearing of the particle's velocity vector to rotate steadily as time progresses. The rotation rate is

$$\frac{d\theta}{dt} \simeq 2\,\Omega\,\sin\lambda. \tag{5.32}$$

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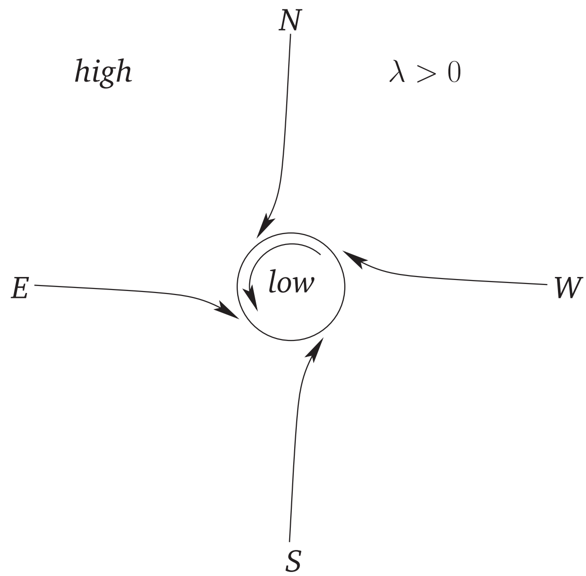

Fig. 5.3 Cyclone in Earth's northern hemisphere.

Hence, the rotation is clockwise (if we look from above) in the northern hemisphere and counterclockwise in the *southern hemisphere*. The rotation rate is zero at the equator and greatest at the poles.

The Coriolis force has a significant effect on terrestrial weather patterns. Near equatorial regions, the Sun's intense heating of the Earth's surface causes hot air to rise. In the northern hemisphere, this causes cooler air to move in a southerly direction toward the equator. The Coriolis force deflects this moving air in a clockwise sense (if we look from above), resulting in the *trade winds*, which blow toward the southwest. In the southern hemisphere, the cooler air moves northward and is deflected by the Coriolis force in a counterclockwise sense, resulting in trade winds that blow toward the northwest. Furthermore, as air flows from high- to low-pressure regions, the Coriolis force deflects the air in a clockwise/counterclockwise manner in the northern/southern hemisphere, producing *cyclonic* rotation. (See Figure 5.3.) It follows that cyclonic rotation is counterclockwise (seen from above) in the northern hemisphere, and clockwise in the southern hemisphere. Thus, this is the direction of rotation of tropical storms (e.g., hurricanes, typhoons) in each hemisphere.

5.5 Rotational flattening

Consider the equilibrium configuration of a self-gravitating celestial body, composed of incompressible fluid, that is rotating steadily and uniformly about some fixed axis passing through its center of mass. Let us assume that the outer boundary of the body is spheroidal. (See Section 2.6.) Let M be the body's total mass, R its mean radius, ϵ its

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ellipticity, and Ω its angular rotation velocity. Suppose, finally, that the body's axis of rotation coincides with its axis of symmetry, which is assumed to run along the z-axis.

Let us transform to a noninertial frame of reference that co-rotates with the body about the z-axis, and in which the body consequently appears to be stationary. From Section 5.3, the problem is now analogous to that of a nonrotating body, except that the acceleration is written $\mathbf{g} = \mathbf{g}_g + \mathbf{g}_c$, where $\mathbf{g}_g = -\nabla\Phi(r,\theta)$ is the gravitational acceleration, $\mathbf{g}_c$ the centrifugal acceleration, and Φ the gravitational potential. The latter acceleration is of magnitude $r\,\sin\theta\,\Omega^2$ and is everywhere directed away from the axis of rotation. (See Section 5.2.) Here, r and θ are spherical coordinates whose origin is the body's geometric center and whose symmetry axis coincides with the axis of rotation. The centrifugal acceleration is thus

$$\mathbf{g}_c = r\,\Omega^2\,\sin^2\theta\,\mathbf{e}_r + r\,\Omega^2\,\sin\theta\,\cos\theta\,\mathbf{e}_\theta. \tag{5.33}$$

It follows that $\mathbf{g}_c = -\nabla\chi$, where

$$\chi(r,\theta) = -\frac{\Omega^2\,r^2}{2}\,\sin^2\theta = \frac{\Omega^2\,r^2}{3}\,[P_2(\cos\theta) - 1] \tag{5.34}$$

can be thought of as a sort of centrifugal potential. Thus, the total acceleration is

$$\mathbf{g} = -\nabla(\Phi + \chi). \tag{5.35}$$

It is convenient to write the centrifugal potential in the form

$$\chi(r,\theta) = \frac{G\,M}{R}\left(\frac{r}{R}\right)^2 \zeta\,[P_2(\cos\theta) - 1]\,, \tag{5.36}$$

where the dimensionless parameter

$$\zeta = \frac{\Omega^2\,R^3}{3\,G\,M} \tag{5.37}$$

is the typical ratio of the centrifugal acceleration to the gravitational acceleration at $r \sim R$. Let us assume that this ratio is small: $\zeta \ll 1$.

As before (see Section 2.6), the criterion for an equilibrium state is that the total potential be uniform over the body's surface, to eliminate any tangential forces that cannot be balanced by internal pressure. Let us assume that the surface satisfies [see Equation (2.56)]

$$r = R_\theta(\theta) = R\left[1 - \frac{2}{3}\,\epsilon\,P_2(\cos\theta)\right], \tag{5.38}$$

where

$$\epsilon = \frac{R_e - R_p}{R}. \tag{5.39}$$

Here, R is the body's mean radius, $R_p = R\,(1 - 2\,\epsilon/3)$ the radius at the poles (i.e., along the axis of rotation), and $R_e = R\,(1 + \epsilon/3)$ the radius at the equator (i.e., perpendicular to the axis of rotation). (See Figure 5.4.) It is assumed that $|\epsilon| \ll 1$, so the body is almost spherical. The external (to the body) gravitational potential can be written [see Equation (2.66)]

$$\Phi(r,\theta) \simeq -\frac{G\,M}{r} + J_2\,\frac{G\,M\,R^2}{r^3}\,P_2(\cos\theta), \tag{5.40}$$

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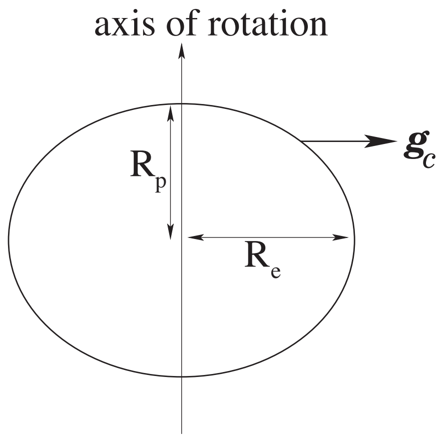

Fig. 5.4 Rotational flattening.

where $J_2 \sim \mathcal{O}(\epsilon)$. The equilibrium configuration is specified by

$$\Phi(R_\theta, \theta) + \chi(R_\theta, \theta) = c, \tag{5.41}$$

where c is a constant. It follows from Equations (5.36), (5.38), and (5.40) that, to first order in ϵ and ζ,

$$-\frac{G\,M}{R}\left[1 + \left(\frac{2}{3}\,\epsilon - J_2\right) P_2(\cos\theta)\right] + \frac{G\,M}{R}\,\zeta\,\left[P_2(\cos\theta) - 1\right] \simeq c, \tag{5.42}$$

which yields

$$\epsilon = \frac{3}{2}\,(J_2 + \zeta)\,. \tag{5.43}$$

For the special case of a *uniform density* body, we have $J_2 = (2/5)\,\epsilon$ [see Equation (2.64)]. Hence, the previous equation simplifies to

$$\epsilon = \frac{15}{4}\,\zeta, \tag{5.44}$$

or

$$\frac{R_e - R_p}{R} = \frac{5}{4}\,\frac{\Omega^2\,R^3}{G\,M}. \tag{5.45}$$

We conclude, from the preceding expression, that the equilibrium configuration of a (relatively slowly) rotating self-gravitating fluid mass is an *oblate spheroid*—a sphere that is slightly flattened along its axis of rotation. The degree of flattening is proportional to the square of the rotation rate.

The result of Equation (5.44) was derived on the assumption that there is zero shear stress at the surface of a uniform-density, rotating, self-gravitating celestial body. This is certainly true for a *fluid* body, as fluids (by definition) are unable to withstand shear stresses. Solids, on the other hand, can withstand such stresses to a limited extent. Hence, it is not necessarily true that there is zero shear stress at the surface of a *solid* rotating body, such as the Earth. Let us investigate whether Equation (5.44) needs to be modified for such a body.

In the presence of the centrifugal potential specified in Equation (5.36), the normal stress at the surface of a spheroidal body, of mean radius R, ellipticity ϵ, and uniform density γ, can be written $\sigma = -X\,P_2(\cos\theta) + p_0$, where p_0 is a constant,

$$X = \sigma_c \left(\frac{R}{R_c}\right)^2 \left(\zeta - \frac{4}{15}\,\epsilon\right), \tag{5.46}$$

and σ_c is the *yield stress* of the material from which the body is composed (i.e., the critical shear stress above which the material flows like a liquid; Love 2011). The shear stress is proportional to $\partial\sigma/\partial\theta$. Furthermore,

$$R_c = \left(\frac{3}{4\pi}\,\frac{\sigma_c}{G\,\gamma^2}\right)^{1/2}. \tag{5.47}$$

For the rock that makes up the Earth's mantle, $\sigma_c \simeq 2\times 10^8\,\mathrm{N\,m^{-2}}$ and $\gamma \simeq 5\times 10^3\,\mathrm{kg\,m^{-3}}$, giving $R_c \simeq 169\,\mathrm{km}$ (de Pater and Lissauer 2010). Let us assume that $R \gg R_c$, which implies that, in the absence of the centrifugal potential, the self-gravity of the body in question is sufficiently strong to force it to adopt a spherical shape. (See Section 2.6.) If the surface shear stress is less than the yield stress (i.e., if $|X| < \sigma_c$) then the body responds *elastically* to the stress in such a manner that

$$\epsilon = \frac{15}{38}\,\frac{X}{\mu}, \tag{5.48}$$

where μ is the *shear modulus*, or *rigidity*, of the body's constituent material (Love 2011). For the rock that makes up the Earth's mantle, $\mu \simeq 1\times 10^{11}\,\mathrm{N\,m^{-2}}$ (de Pater and Lissauer 2010). It follows that

$$\epsilon = \frac{15}{4}\,\frac{\zeta}{1+\tilde{\mu}}, \tag{5.49}$$

where

$$\tilde{\mu} = \frac{57}{8\pi}\,\frac{\mu}{G\,\gamma^2\,R^2} \tag{5.50}$$

is a dimensionless quantity that is termed the body's *effective rigidity*. On the other hand, if the surface shear stress is greater than the yield stress, the body flows like a liquid until the stress becomes zero. So, it follows from Equation (5.46) that

$$\epsilon = \frac{15}{4}\,\zeta, \tag{5.51}$$

which is identical to Equation (5.44). Hence, we deduce that the rotational flattening of a solid, uniform-density, celestial body is governed by Equation (5.49) if the surface shear stress does not exceed the yield stress, and by Equation (5.44) otherwise. In the former case, the condition $|X| < \sigma_c$ is equivalent to $\zeta < \zeta_c$, where (assuming that $R \ll R_c$)

$$\zeta_c = \frac{2}{19}\,\frac{\sigma_c}{\mu} + \left(\frac{R_c}{R}\right)^2 \simeq \frac{2}{19}\,\frac{\sigma_c}{\mu}. \tag{5.52}$$

For the rock that makes up the Earth's mantle [for which $\sigma_c \simeq 2\times 10^8\,\mathrm{N\,m^{-2}}$ and $\mu \simeq 1\times 10^{11}\,\mathrm{N\,m^{-2}}$ (de Pater and Lissauer 2010)], we find that

$$\zeta_c \simeq 2\times 10^{-4}. \tag{5.53}$$

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Thus, if $\zeta < \zeta_c$, the rotational flattening of a uniform body made up of such rock is governed by Equation (5.49), but if $\zeta > \zeta_c$, the flattening is governed by Equation (5.44).

For the case of the Earth itself, $R = 6.37 \times 10^6$ m, $\Omega = 7.29 \times 10^{-5}$ rad. s^{-1}, and $M = 5.97 \times 10^{24}$ kg (Yoder 1995). It follows that

$$\zeta = 1.15 \times 10^{-3}. \tag{5.54}$$

Because $\zeta \gg \zeta_c$, we deduce that the Earth's centrifugal potential is sufficiently strong to force its constituent rock to flow like a liquid. Hence, the rotational flattening is governed by Equation (5.44), which implies that

$$\epsilon = 4.31 \times 10^{-3}. \tag{5.55}$$

This corresponds to a difference between the Earth's equatorial and polar radii of

$$\Delta R = R_e - R_p = \epsilon R = 27.5 \text{ km}. \tag{5.56}$$

In fact, the observed degree of rotational flattening of the Earth is $\epsilon = 3.35 \times 10^{-3}$ (Yoder 1995), corresponding to a difference between equatorial and polar radii of 21.4 km. Our analysis has overestimated the Earth's rotational flattening because, for the sake of simplicity, we assumed that the terrestrial interior is of uniform density. In reality, the Earth's core is much denser than its crust. (See Exercise 5.6.) Incidentally, the observed value of the parameter J_2, which measures the strength of the Earth's quadrupole gravitational field, is 1.08×10^{-3} (Yoder 1995). Hence, $(3/2)\,(J_2 + \zeta) = 3.35 \times 10^{-3}$. In other words, the Earth's rotational flattening satisfies Equation (5.43) extremely accurately. This confirms that although the Earth is not a uniform density body, its response to the centrifugal potential is indeed fluidlike [because Equation (5.43) was derived on the assumption that the response is fluidlike.]

For the planet Jupiter, $R = 6.92 \times 10^7$ m, $\Omega = 1.76 \times 10^{-4}$ rad. s^{-1}, and $M = 1.90 \times 10^{27}$ kg (Yoder 1995; Seidelmann et al. 2007). Hence,

$$\zeta = 2.70 \times 10^{-2}. \tag{5.57}$$

Because Jupiter is largely composed of liquid, its rotation flattening is governed by Equation (5.44), which yields

$$\epsilon = 0.101. \tag{5.58}$$

This degree of flattening is much larger than that of the Earth, owing to Jupiter's relatively large radius (about ten times that of Earth), combined with its relatively short rotation period (about 0.4 days). In fact, the rotational flattening of Jupiter is clearly apparent from images of this planet. (See Figure 5.5.) The observed degree of rotational flattening of Jupiter is actually $\epsilon = 0.065$ (Yoder 1995). Our estimate for ϵ is slightly too large because Jupiter has a mass distribution that is strongly concentrated at its core. (See Exercise 5.6.) Incidentally, the measured value of J_2 for Jupiter is 1.47×10^{-2} (Yoder 1995). Hence, $(3/2)\,(J_2 + \zeta) = 0.063$. Thus, Jupiter's rotational flattening also satisfies Equation (5.43) fairly accurately, confirming that its response to the centrifugal potential is fluidlike.

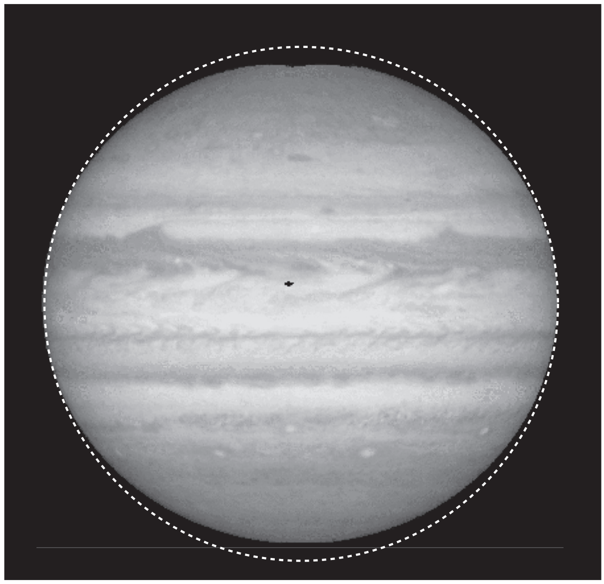

Fig. 5.5 Jupiter. Photograph taken by the Hubble Space Telescope. A circle is superimposed on the image to make the rotational flattening more clearly visible. The axis of rotation is vertical. Credit: NASA.

5.6 Tidal elongation

Consider two point masses, m and m', executing circular orbits about their common center of mass, C, with angular velocity ω. Let a be the distance between the masses and ρ the distance between point C and mass m. (See Figure 5.6.) We know from Section 3.16, that

$$\omega^2 = \frac{G\,M}{a^3}, \tag{5.59}$$

and

$$\rho = \frac{m'}{M}\,a, \tag{5.60}$$

where $M = m + m'$.

Let us transform to a noninertial frame of reference that rotates, about an axis perpendicular to the orbital plane and passing through C, at the angular velocity ω. In this

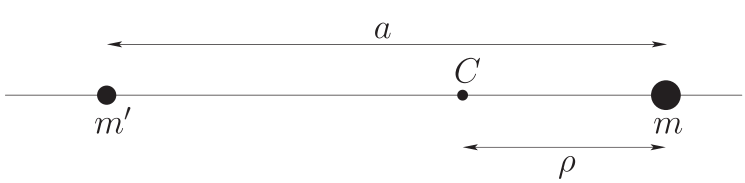

Fig. 5.6 Two orbiting masses.

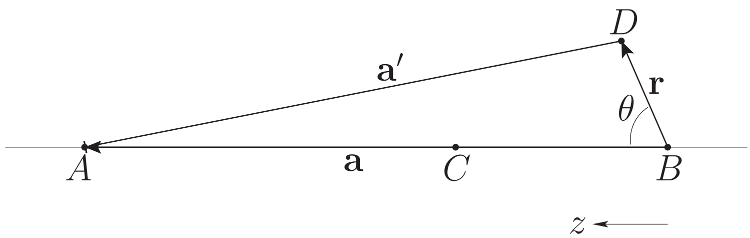

Fig. 5.7 Calculation of tidal forces.

reference frame, both masses appear to be stationary. Consider mass m. In the rotating frame, this mass experiences a gravitational acceleration

$$a_g = \frac{G\, m'}{a^2} \tag{5.61}$$

directed toward the center of mass, and a centrifugal acceleration (see Section 5.3)

$$a_c = \omega^2 \rho \tag{5.62}$$

directed away from the center of mass. However, it is easily demonstrated, using Equations (5.59) and (5.60), that

$$a_c = a_g. \tag{5.63}$$

In other words, the gravitational and centrifugal accelerations balance, as must be the case if mass m is to remain stationary in the rotating frame. Let us investigate how this balance is affected if the masses m and m' have finite spatial extents.

Let the center of the mass distribution m' lie at A, the center of the mass distribution m at B, and the center of mass at C. (See Figure 5.7.) We wish to calculate the centrifugal and gravitational accelerations at some point D in the vicinity of point B. It is convenient to adopt spherical coordinates, centered on point B and aligned such that the z-axis coincides with the line BA.

Let us assume that the mass distribution m is orbiting around C, but is *not* rotating about an axis passing through its center of mass, to exclude rotational flattening from our analysis. If this is the case, it is easily seen that each constituent point of m executes circular motion of angular velocity ω and radius ρ. (See Figure 5.8.) Hence, each point experiences the *same* centrifugal acceleration:

$$\mathbf{g}_c = -\omega^2 \rho\, \mathbf{e}_z. \tag{5.64}$$

It follows that

$$\mathbf{g}_c = -\nabla \chi', \tag{5.65}$$

where

$$\chi' = \omega^2 \rho\, z \tag{5.66}$$

is the centrifugal potential and $z = r\, \cos\theta$. The centrifugal potential can also be written

$$\chi' = \frac{G\, m'}{a} \frac{r}{a} P_1(\cos\theta). \tag{5.67}$$

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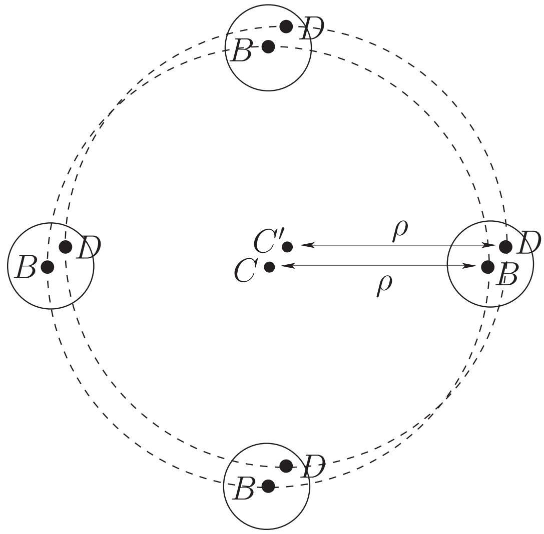

Fig. 5.8 The center B of mass distribution m orbits about the center of mass C in a circle of radius ρ. If m is nonrotating, then a noncentral point D maintains a constant spatial relationship to B, such that D orbits some point C' that has the same spatial relationship to C that D has to B, in a circle of radius ρ.

The gravitational acceleration at point D due to mass m' is given by

$$\mathbf{g}_g = -\nabla\Phi', \tag{5.68}$$

where the gravitational potential takes the form

$$\Phi' = -\frac{G\,m'}{a'}. \tag{5.69}$$

Here, a' is the distance between points A and D. The gravitational potential generated by the mass distribution m' is the same as that generated by an equivalent point mass at A, as long as the distribution is spherically symmetric, which we shall assume to be the case.

Now,

$$\mathbf{a}' = \mathbf{a} - \mathbf{r}, \tag{5.70}$$

where $\mathbf{a}'$ is the vector $\overrightarrow{DA}$, and $\mathbf{a}$ the vector $\overrightarrow{BA}$. (See Figure 5.7.) It follows that

$$a'^{-1} = \left(a^2 - 2\,\mathbf{a}\cdot\mathbf{r} + r^2\right)^{-1/2} = \left(a^2 - 2\,a\,r\,\cos\theta + r^2\right)^{-1/2}. \tag{5.71}$$

Expanding in powers of r/a, we obtain

$$a'^{-1} = a^{-1}\sum_{n=0,\infty}\left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{5.72}$$

Hence,

$$\Phi' \simeq -\frac{G\,m'}{a}\left[1 + \frac{r}{a}\,P_1(\cos\theta) + \frac{r^2}{a^2}\,P_2(\cos\theta)\right], \tag{5.73}$$

to second order in r/a, where the $P_n(x)$ are Legendre polynomials.

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Adding χ' and Φ', we find that

$$\chi = \chi' + \Phi' \simeq -\frac{G\,m'}{a}\left[1 + \frac{r^2}{a^2}\,P_2(\cos\theta)\right], \tag{5.74}$$

to second order in r/a. Note that χ is the potential due to the net externally generated force acting on the mass distribution m. This potential is constant up to first order in r/a, because the first-order variations in χ' and Φ' cancel each other. The cancellation is a manifestation of the balance between the centrifugal and gravitational accelerations in the equivalent point mass problem discussed earlier. However, this balance is exact only at the center of the mass distribution m. Away from the center, the centrifugal acceleration remains constant, whereas the gravitational acceleration increases with increasing z. Hence, at positive z, the gravitational acceleration is larger than the centrifugal acceleration, giving rise to a net acceleration in the $+z$ direction. Likewise, at negative z, the centrifugal acceleration is larger than the gravitational giving rise to a net acceleration in the $-z$ direction. It follows that the mass distribution m is subject to a residual acceleration, represented by the second-order variation in Equation (5.74), that acts to elongate it along the z-axis. This effect is known as *tidal elongation*.

Suppose that the mass distribution m is a sphere of radius R and uniform density γ, made up of rock similar to that found in the Earth's mantle. Let us estimate the elongation of this distribution due to the *tidal potential* specified in Equation (5.74), which (neglecting constant terms) can be written

$$\chi(r,\theta) = \frac{G\,m}{R}\left(\frac{r}{R}\right)^2 \zeta\,P_2(\cos\theta). \tag{5.75}$$

Here, the dimensionless parameter

$$\zeta = -\frac{m'}{m}\left(\frac{R}{a}\right)^3 \tag{5.76}$$

is (minus) the typical ratio of the tidal acceleration to the gravitational acceleration at $r \sim R$. Let us assume that $|\zeta| \ll 1$. By analogy with the analysis in the previous section, in the presence of the tidal potential, the distribution becomes slightly spheroidal in shape, such that its outer boundary satisfies Equation (5.38). Moreover, the induced ellipticity, ϵ, of the distribution is related to the normalized amplitude, ζ, of the tidal potential according to Equation (5.49) if $|\zeta| < \zeta_c \simeq 2\times 10^{-4}$, and according to Equation (5.44) if $|\zeta| > \zeta_c$. In the former case, the distribution responds elastically to the tidal potential, whereas in the latter case it responds as a liquid.

Consider the tidal elongation of the Earth due to the Moon. In this case, we have $R = 6.37\times10^6$ m, $a = 3.84\times10^8$ m, $m = 5.97\times10^{24}$ kg, and $m' = 7.35\times10^{22}$ kg (Yoder 1995). Hence, we find that

$$\zeta = -5.62\times10^{-8}. \tag{5.77}$$

Note that $|\zeta| \ll \zeta_c$. We conclude that the Earth responds elastically to the tidal potential of the Moon, rather than deforming like a liquid. For the rock that makes up the Earth's mantle, $\mu \simeq 1\times10^{11}\,\mathrm{N\,m^{-2}}$ and $\gamma \simeq 5\times10^3\,\mathrm{kg\,m^{-3}}$ (de Pater and Lissauer 2010). Thus, it follows from Equation (5.50) that

$$\tilde{\mu} \simeq 3.35. \tag{5.78}$$

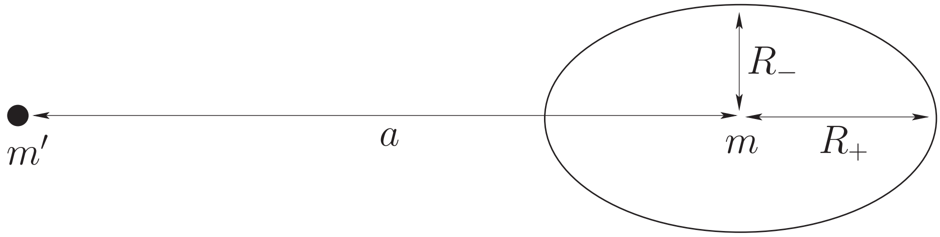

Fig. 5.9 Tidal elongation.

Hence, according to Equation (5.49), the ellipticity of the Earth induced by the tidal effect of the Moon is

$$\epsilon = \frac{15}{4}\left(\frac{\zeta}{1+\tilde{\mu}}\right) \simeq -4.8 \times 10^{-8}. \tag{5.79}$$

The fact that ϵ is negative implies that the Earth is elongated along the z-axis, that is, along the axis joining its center to that of the Moon. See Equation (5.38). If R_+ and R_- are the greatest and least radii of the Earth, respectively, due to this elongation (see Figure 5.9), then

$$\Delta R = R_+ - R_- = -\epsilon R = 0.31\,\mathrm{m}. \tag{5.80}$$

Thus, we predict that the tidal effect of the Moon (which is actually due to spatial gradients in the Moon's gravitational field) causes the Earth to elongate along the axis joining its center to that of the Moon by about 31 centimeters. This elongation is only about a quarter of that which would result were the Earth a nonrigid (i.e., liquid) body. The true tidal elongation of the Earth due to the Moon is about 35 centimeters [assuming a Love number $h_2 \simeq 0.6$ (Bertotti et al. 2003)]. We have slightly underestimated this elongation because, for the sake of simplicity, we treated the Earth as a uniform-density body.

Consider the tidal elongation of the Earth due to the Sun. In this case, we have $R = 6.37 \times 10^6$ m, $a = 1.50 \times 10^{11}$ m, $m = 5.97 \times 10^{24}$ kg, and $m' = 1.99 \times 10^{30}$ kg. Hence, we calculate that $\zeta = -2.55 \times 10^{-8}$ and $\epsilon = -2.2 \times 10^{-8}$, or

$$\Delta R = R_+ - R_- = -\epsilon R = 0.14\,\mathrm{m}. \tag{5.81}$$

Thus, the tidal elongation of the Earth due to the Sun is about half that due to the Moon. The true tidal elongation of the Earth due to the Sun is about 16 centimeters [assuming a Love number $h_2 \sim 0.6$ (Bertotti et al. 2003)]. Again, we have slightly underestimated the elongation because we treated the Earth as a uniform-density body.

Because the Earth's oceans are liquid, their tidal elongation is significantly larger than that of the underlying land. (See Exercise 5.10.) Hence, the oceans rise, relative to the land, in the region of the Earth closest to the Moon, and also in the region farthest away. Because the Earth is rotating, whereas the tidal bulge of the oceans remains relatively stationary, the Moon's tidal effect causes the ocean at a given point on the Earth's surface to rise and fall twice daily, giving rise to the phenomenon known as the *tides*. There is also an oceanic tidal bulge due to the Sun that is about half as large as that due to the Moon. Consequently, ocean tides are particularly high when the Sun, the Earth, and the Moon lie approximately in a straight line, so the tidal effects of the Sun and the Moon reinforce one another. This occurs at a new moon, or at a full moon. These types of tides are called *spring tides* (the name has nothing to do with the season). Conversely, ocean tides are particularly low when the Sun, the Earth, and the Moon form a right angle, so that the tidal effects of the Sun and the Moon partially cancel one another. These type of tides are called *neap tides*. Generally, we would expect two spring tides and two neap tides per month.

We can roughly calculate the vertical displacement of the oceans, relative to the underlying land, by treating the oceans as a shallow layer of negligible mass, covering the surface of the Earth. The Earth's external gravitational potential is written [see Equation (2.65)]

$$\Phi(r,\theta) = -\frac{G\,m}{r} + \frac{2}{5}\,\epsilon\,\frac{G\,m\,R^2}{r^3}\,P_2(\cos\theta), \tag{5.82}$$

where ϵ is given by Equation (5.79). Let the ocean surface satisfy

$$r = R'(\theta) = R\left[1 - \frac{2}{3}\,\epsilon'\,P_2(\cos\theta)\right]. \tag{5.83}$$

Because fluids cannot withstand shear stresses, we expect this surface to be an equipotential:

$$\Phi(R'_\theta,\theta) + \chi(R'_\theta,\theta) = c. \tag{5.84}$$

It follows that, to first order in ϵ' and ζ,

$$\epsilon' = \frac{3}{5}\,\epsilon + \frac{3}{2}\,\zeta = \frac{3}{2}\left(\frac{5/2+\tilde{\mu}}{1+\tilde{\mu}}\right)\zeta. \tag{5.85}$$

Thus, the maximum vertical displacement of the ocean relative to the underlying land is

$$\Delta R = -(\epsilon'-\epsilon)\,R = -\frac{3}{2}\left(\frac{\tilde{\mu}}{1+\tilde{\mu}}\right)\zeta\,R. \tag{5.86}$$

As we saw earlier, $\tilde{\mu} \simeq 3.35$ for the Earth. Moreover, the tidal potential due to the Moon is such that $\zeta = -5.62\times 10^{-8}$. We thus conclude that the Moon causes the oceans to rise a maximum vertical distance of 0.41 m relative to the land. Likewise, the tidal potential due to the Sun is such that $\zeta = -2.55\times 10^{-8}$. Hence, we predict that the Sun causes the oceans to rise a maximum vertical distance of 0.19 m relative to the land.

In reality, the relationship between ocean tides and the Moon and Sun is much more complicated than that indicated in the previous discussion. This is partly because of the presence of the continents, which impede the flow of the oceanic tidal bulge around the Earth, and partly because of the finite inertia of the oceans.

Note, finally, that as a consequence of friction within the Earth's crust and friction between the oceans and the underlying land, there is a time lag of roughly 12 minutes between the Moon (or Sun) passing directly overhead (or directly below) and the corresponding maximum in the net tidal elongation of the Earth and the oceans (Bertotti et al. 2003).

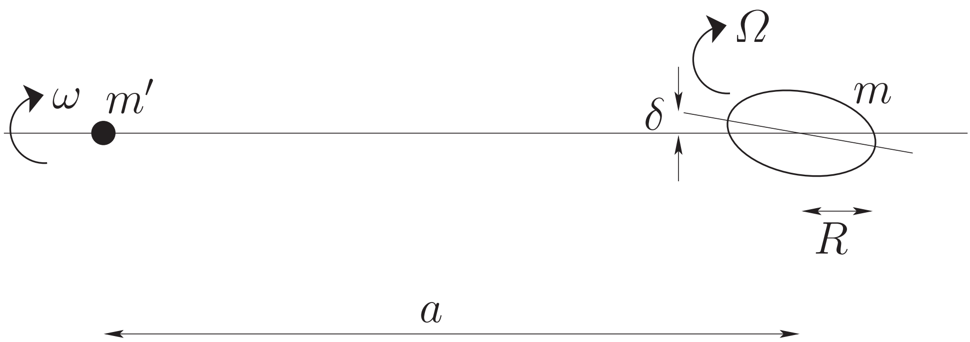

Fig. 5.10 Origin of tidal torque.

5.7 Tidal torques

The fact that there is a time lag between the Moon passing overhead and the corresponding maximum net tidal elongation of the Earth and the oceans suggests the physical scenario illustrated in Figure 5.10. According to this scenario, the Moon, which is of mass m' and which is treated as a point particle, orbits the Earth (it actually orbits the center of mass of the Earth–Moon system, but this amounts to almost the same thing) in an approximately circular orbit of radius a. Moreover, the orbital angular velocity of the Moon is [see Equation (5.59)]

$$\omega = \frac{(G\,M)^{1/2}}{a^{3/2}}, \tag{5.87}$$

where $M = m + m' \simeq m$ is the total mass of the Earth–Moon system. The Earth (including the oceans) is treated as a uniform sphere of mass m and radius R that rotates daily about its axis (which is approximately normal to the orbital plane of the Moon) at the angular velocity Ω. Note, incidentally, that the Earth rotates in the same sense that the Moon orbits, as indicated in the figure. As we saw in the previous section, spatial gradients in the Moon's gravitational field produce a slight tidal elongation of the Earth. However, because of frictional effects, this elongation does not quite line up along the axis joining the centers of the Earth and Moon. In fact, because $\Omega > \omega$, the tidal elongation is carried ahead (in the sense defined by the Earth's rotation) of this axis by some small angle δ (say), as shown in the figure.

Defining a spherical coordinate system, r, θ, ϕ, whose origin is the center of the Earth, and which is orientated such that the Earth–Moon axis always corresponds to $\theta = 0$ (see Figure 5.7), we find the Earth's external gravitational potential is [*cf.* Equation (2.65)]

$$\Phi(r,\theta) = -\frac{G\,m}{r} + \frac{\epsilon}{5}\,\frac{G\,m\,R^2}{r^3}\left[3\,\cos^2(\theta-\delta) - 1\right], \tag{5.88}$$

where ϵ is the ellipticity induced by the tidal field of the Moon. Note that the second term on the right-hand side of this expression is the contribution of the Earth's tidal bulge, which attains its maximum amplitude at $\theta = \delta$, rather than $\theta = 0$, because of the aforementioned misalignment between the bulge and the Earth–Moon axis. Equations (5.76), (5.79), and (5.88) can be combined to give

$$\Phi(r, \theta) = -\frac{G\,m}{r} - \frac{3}{4}\,\frac{G\,m'\,R^2}{(1+\tilde{\mu})\,r^3}\left(\frac{R}{a}\right)^3\left[3\,\cos^2(\theta-\delta)-1\right]. \tag{5.89}$$

From Equation (2.7), the torque about the Earth's center that the terrestrial gravitational field exerts on the Moon is

$$\tau = -m'\,\frac{\partial\Phi}{\partial\theta}\bigg|_{\theta=0,\,r=a} \simeq \frac{9}{2}\,\frac{G\,m'^{\,2}}{R}\left(\frac{R}{a}\right)^6\frac{\delta}{1+\tilde{\mu}}, \tag{5.90}$$

where use has been made of Equation (5.89), as well as the fact that δ is a small angle. There is zero torque in the absence of a misalignment between the Earth's tidal bulge and the Earth–Moon axis. The torque τ acts to increase the Moon's orbital angular momentum. By conservation of angular momentum, an equal and opposite torque, $-\tau$, is applied to the Earth; it acts to decrease its rotational angular momentum. Incidentally, if the Moon were sufficiently close to the Earth that its orbital angular velocity exceeded the Earth's rotational angular velocity (i.e., if $\omega > \Omega$), then the phase lag between the Earth's tidal elongation and the Moon's tidal field would cause the tidal bulge to fall slightly behind the Earth–Moon axis (i.e., $\delta < 0$). In this case, the gravitational torque would act to reduce the Moon's orbital angular momentum and to increase the Earth's rotational angular momentum.

The Earth's rotational equation of motion is

$$\mathcal{I}_{\parallel}\,\dot{\Omega} = -\tau, \tag{5.91}$$

where $\mathcal{I}_{\parallel}$ is its moment of inertia about its axis of rotation. Very crudely approximating the Earth as a uniform sphere, we have $\mathcal{I}_{\parallel} = (2/5)\,m R^2$. Hence, the previous two equations can be combined to give

$$\frac{\dot{\Omega}}{\Omega} \simeq -\frac{45}{4}\,\frac{\omega^2}{\Omega}\left(\frac{m'}{m}\right)^2\left(\frac{R}{a}\right)^3\frac{\delta}{1+\tilde{\mu}}, \tag{5.92}$$

where use has been made of Equation (5.87), as well as the fact that $m \gg m'$. A time lag of 12 minutes between the Moon being overhead and a maximum of the Earth's tidal elongation implies a lag angle of $\delta \sim 0.05$ radians (i.e., $\delta \sim 3°$). Hence, employing the observed values $m = 5.97 \times 10^{24}$ kg, $m' = 7.35 \times 10^{22}$ kg, $R = 6.37 \times 10^6$ m, $a = 3.84 \times 10^8$ m, $\Omega = 7.29 \times 10^{-5}$ rad. s^{-1}, and $\omega = 2.67 \times 10^{-6}$ rad. s^{-1} (Yoder 1995), as well as the estimate $\tilde{\mu} \simeq 3.35$ (from the previous section), we find that

$$\frac{\dot{\Omega}}{\Omega} \simeq -8.7 \times 10^{-18}\ \mathrm{s}^{-1}. \tag{5.93}$$

It follows that under the influence of the tidal torque, the Earth's axial rotation is gradually decelerating. Indeed, according to this estimate, the length of a day should be increasing at the rate of about 2.3 milliseconds per century. An analysis of ancient and medieval solar and lunar eclipse records indicates that the length of the day is actually increasing at the rate of 1.7 milliseconds per century (Stephenson and Morrison 1995). The timescale for the tidal torque to significantly reduce the Earth's rotational angular

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velocity is estimated to be

$$T \simeq \frac{\Omega}{|\dot{\Omega}|} \simeq 4 \times 10^9. \tag{5.94}$$

This timescale is comparable with the Earth's age, which is thought to be 4.5×10^9 years. Hence, we conclude that, although the Earth is certainly old enough for the tidal torque to have significantly reduced its rotational angular velocity, it is plausible that it is not sufficiently old for the torque to have driven the Earth–Moon system to a final steady state. Such a state, in which the Earth's rotational angular velocity matches the Moon's orbital angular velocity, is termed *synchronous*. In a synchronous state, the Moon would appear stationary to an observer on the Earth's surface, and, hence, there would be no tides (from the observer's perspective), no phase lag, and no tidal torque.

Up to now, we have concentrated on the effect of the tidal torque on the rotation of the Earth. Let us now examine its effect on the orbit of the Moon. The total angular momentum of the Earth–Moon system is

$$L \simeq \frac{2}{5}\, m\, R^2\, \Omega + m'\, a^2\, \omega, \tag{5.95}$$

where the first term on the right-hand side is the rotational angular momentum of the Earth and the second the orbital angular momentum of the Moon. Of course, L is a conserved quantity. Moreover, ω and a are related according to Equation (5.87). It follows that

$$\frac{\dot{a}}{a} \simeq -\frac{4}{5}\,\frac{m}{m'}\left(\frac{R}{a}\right)^2 \frac{\dot{\Omega}}{\omega} \simeq 4.3 \times 10^{-18}\,\mathrm{s}^{-1}, \tag{5.96}$$

where use has been made of Equation (5.93). In other words, the tidal torque causes the radius of the Moon's orbit to gradually increase. According to this estimate, this increase should take place at the rate of about 5 cm a year. The observed rate, which is obtained from lunar laser ranging data, is 3.8 cm a year (Chapront et al. 2002). This suggests that, despite the numerous approximations we have made, our calculation remains reasonably accurate. We also have

$$\frac{\dot{\omega}}{\omega} = -\frac{3}{2}\,\frac{\dot{a}}{a} \simeq -6.5 \times 10^{-18}\,\mathrm{s}^{-1}. \tag{5.97}$$

In other words, the tidal torque produces a gradual angular deceleration in the Moon's orbital motion. According to the above estimate, this deceleration should take place at the rate of 35 arc seconds per century squared. The measured deceleration is about 26 arc seconds per century squared (Yoder 1995).

The net rate at which the tidal torques acting on the Moon and the Earth do work is

$$\dot{E} = \tau\,(\omega - \Omega). \tag{5.98}$$

Note that $\dot{E} < 0$, because $\tau > 0$ and $\Omega > \omega$. This implies that the deceleration of the Earth's rotation and that of the Moon's orbital motion induced by tidal torques are necessarily associated with the dissipation of energy. This dissipation manifests itself as frictional heating of the Earth's crust and the oceans (Bertotti et al. 2003).

Of course, we would expect spatial gradients in the gravitational field of the Earth to generate a tidal bulge in the Moon. We would also expect dissipative effects to produce a phase lag between this bulge and the Earth. This would allow the Earth to exert a gravitational torque that acts to drive the Moon toward a synchronous state in which its rotational angular velocity matches its orbital angular velocity. By analogy with the previous analysis, the de-spinning rate of the Moon is estimated to be

$$\frac{\dot{\Omega}'}{\Omega'} \simeq -\frac{45}{4}\,\frac{\omega^2}{\Omega'}\,\frac{m}{m'}\left(\frac{R'}{a}\right)^3 \frac{\delta}{1+\tilde{\mu}} \sim -2.2\times 10^{-14}\,\mathrm{s}^{-1}, \tag{5.99}$$

where

$$\tilde{\mu} = \frac{57}{8\pi}\,\frac{\mu}{G\,\gamma^2 R'^2} \simeq 51.5 \tag{5.100}$$

is the Moon's effective rigidity, Ω' its rotational angular velocity, $R' = 1.74\times 10^6\,\mathrm{m}$ its radius, $\gamma = 3.3\times 10^3\,\mathrm{m\,s^{-1}}$ its density (Yoder 1995), $\mu \sim 5\times 10^{10}\,\mathrm{N\,m^{-2}}$ its shear modulus (Zhang 1992), and δ the lag angle. The above numerical estimate is made with the guesses $\Omega' = 2\,\omega$ and $\delta = 0.01$ radians. Thus, the time required for the Moon to achieve a synchronous state is

$$T' \simeq \frac{\Omega'}{|\dot{\Omega}'|} \simeq 1.5\times 10^6\ \text{years}. \tag{5.101}$$

This is considerably less than the age of the Moon. Hence, it is not surprising that the Moon has actually achieved a synchronous state, as evidenced by the fact that the same side of the Moon is always visible from the Earth. (See Section 7.11.)

5.8 Roche radius

Consider a spherical moon of mass m and radius R that is in a circular orbit of radius a about a spherical planet of mass m' and radius R'. (Strictly speaking, the moon and the planet execute circular orbits about their common center of mass. However, if the planet is much more massive than the moon, the center of mass lies very close to the planet's center.) According to the analysis in Section 5.6, a constituent element of the moon experiences a force per unit mass, due to the gravitational field of the planet, that takes the form

$$\mathbf{g}' = -\nabla\chi, \tag{5.102}$$

where

$$\chi = -\frac{G\,m'}{a^3}\,(z^2 - x^2/2 - y^2/2) + \text{const}. \tag{5.103}$$

Here, x, y, z is a Cartesian coordinate system whose origin is the center of the moon, and whose z-axis always points toward the center of the planet. It follows that

$$\mathbf{g}' = \frac{2\,G\,m'}{a^3}\left(-\frac{x}{2}\,\mathbf{e}_x - \frac{y}{2}\,\mathbf{e}_y + z\,\mathbf{e}_z\right). \tag{5.104}$$

This so-called *tidal force* is generated by the spatial variation of the planet's gravitational field over the interior of the moon and acts to elongate the moon along an axis joining its center to that of the planet and to compress it in any direction perpendicular to this axis. Note that the magnitude of the tidal force increases strongly as the radius, a, of the moon's orbit decreases. In fact, if the tidal force becomes sufficiently strong, it can overcome the moon's self-gravity and thereby rip the moon apart. It follows that there is a minimum radius, generally referred to as the *Roche radius*, at which a moon can orbit a planet without being destroyed by tidal forces.

Let us derive an expression for the Roche radius. Consider a small mass element at the point on the surface of the moon that lies closest to the planet, and at which the tidal force is consequently largest (i.e., $x = y = 0$, $z = R$). According to Equation (5.104), the mass experiences an upward (from the moon's surface) tidal acceleration due to the gravitational attraction of the planet of the form

$$\mathbf{g}' = \frac{2\,G\,m'\,R}{a^3}\,\mathbf{e}_z. \tag{5.105}$$

The mass also experiences a downward gravitational acceleration due to the gravitational influence of the moon, which is written

$$\mathbf{g} = -\frac{G\,m}{R^2}\,\mathbf{e}_z. \tag{5.106}$$

Thus, the effective surface gravity at the point in question is

$$g_{\rm eff} = \frac{G\,m}{R^2}\left(1 - 2\,\frac{m'}{m}\,\frac{R^3}{a^3}\right). \tag{5.107}$$

If $a < a_c$, where

$$a_c = \left(2\,\frac{m'}{m}\right)^{1/3} R, \tag{5.108}$$

then the effective gravity is negative; in other words, the tidal force due to the planet is strong enough to overcome surface gravity and lift objects off the moon's surface. If this is the case, and the tensile strength of the moon is negligible, then it is fairly clear that the tidal force will start to break the moon apart. Hence, a_c is the Roche radius. Now, $m'/m = (\rho'/\rho)\,(R'/R)^3$, where ρ and ρ' are the mean mass densities of the moon and planet, respectively. Thus, the above expression for the Roche radius can also be written

$$a_c = 1.41\left(\frac{\rho'}{\rho}\right)^{1/3} R'. \tag{5.109}$$

The previous calculation is somewhat inaccurate, as it fails to take into account the inevitable distortion of the moon's shape in the presence of strong tidal forces. (In fact, the calculation assumes that the moon always remains spherical.) A more accurate calculation, which treats the moon as a self-gravitating incompressible fluid body, yields (Chandrasekhar 1969)

$$a_c = 2.44\left(\frac{\rho'}{\rho}\right)^{1/3} R'. \tag{5.110}$$

It follows that if the planet and the moon have the same mean density, then the Roche radius is 2.44 times the planet's radius. Note that small bodies such as rocks, or even very small moons, can survive intact within the Roche radius because they are held together by internal tensile forces rather than by gravitational attraction. However, this mechanism becomes progressively less effective as the size of the body in question increases. (See Section 2.6.) Not surprisingly, virtually all large planetary moons found in the solar system have orbital radii that exceed the relevant Roche radius, whereas virtually all planetary ring systems (which consist of myriads of small orbiting rocks) have radii that lie inside the relevant Roche radius.

Exercises

5.1 A ball bearing is dropped down an elevator shaft in the Empire State Building (h = 381 m, latitude = 41° N). Find the ball bearing's horizontal deflection (magnitude and direction) at the bottom of the shaft due to the Coriolis force. Neglect air resistance. (Modified from Fowles and Cassiday 2005.)

5.2 A projectile is fired due east, at an elevation angle β, from a point on the Earth's surface whose latitude is $+\lambda$. Demonstrate that the projectile strikes the ground with a lateral deflection $4\,\Omega\,v_0^3\,\sin\lambda\,\sin^2\beta\,\cos\beta/g^2$. Is the deflection northward or southward? Here, Ω is the Earth's angular velocity, v_0 the projectile's initial speed, and g the acceleration due to gravity. Neglect air resistance. (Modified from Thornton and Marion 2004.)

5.3 A particle is thrown vertically upward, reaches some maximum height, and falls back to the ground. Show that the horizontal Coriolis deflection of the particle when it returns to the ground is opposite in direction, and four times greater in magnitude, than the Coriolis deflection when it is dropped at rest from the same maximum height. Neglect air resistance. (From Goldstein et al. 2002.)

5.4 A ball of mass m rolls without friction over a horizontal plane located on the surface of the Earth. Show that in the northern hemisphere it rolls in a clockwise sense (seen from above) around a circle of radius

$$r = \frac{v}{2\,\Omega\,\sin\lambda},$$

where v is the speed of the ball, Ω the Earth's angular velocity, and λ the terrestrial latitude.

5.5 A satellite is in a circular orbit of radius a about the Earth. Let us define a set of co-moving Cartesian coordinates, centered on the satellite, such that the x-axis always points toward the center of the Earth, the y-axis in the direction of the satellite's orbital motion, and the z-axis in the direction of the satellite's orbital angular velocity, $\boldsymbol{\omega}$. Demonstrate that the equations of motion of a small mass in orbit about the satellite are

$$\ddot{x} = 3\,\omega^2\,x + 2\,\omega\,\dot{y}$$

and

$$\ddot{y} = -2\,\omega\,\dot{x},$$

assuming that $|x|/a \ll 1$ and $|y|/a \ll 1$. Neglect the gravitational attraction between the satellite and the mass. Show that the mass executes a retrograde (i.e., in the opposite sense to the satellite's orbital rotation) elliptical orbit about the satellite whose period matches that of the satellite's orbit, and whose major and minor axes are in the ratio 2:1, and are aligned along the y- and x-axes, respectively.

5.6 Show that

$$\epsilon = \frac{5}{2\,(2+\alpha)}\,\frac{\Omega^2\,R^3}{G\,M}$$

for a self-gravitating, rotating, fluidlike, spheroid of ellipticity $\epsilon \ll 1$, mass M, mean radius R, and angular velocity Ω, whose mass density varies as $r^{-\alpha}$ (where $\alpha < 3$). Demonstrate that this formula matches the observed rotational flattening of the Earth when $\alpha = 0.575$, and of Jupiter when $\alpha = 1.12$. (See Exercise 2.9.)

5.7 Treating the Earth as a uniform-density, liquid spheroid, and taking rotational flattening into account, show that the variation of the surface acceleration, g, with terrestrial latitude, λ, is

$$g \simeq \frac{G\,M}{R^2} + \frac{1}{6}\,\Omega^2\,R - \frac{5}{4}\,\Omega^2\,R\,\cos^2\lambda.$$

Here, M is the terrestrial mass, R the mean terrestrial radius, and Ω the terrestrial axial angular velocity.

5.8 The Moon's orbital period about the Earth is approximately 27.32 days, and is in the same direction as the Earth's axial rotation (whose period is 24 hours). Use these data to show that high tides at a given point on the Earth occur every 12 hours and 27 minutes.

5.9 Demonstrate that the mean time interval between successive spring tides is 14.76 days.

5.10 Let us model the Earth as a completely rigid sphere that is covered by a shallow ocean of negligible density. Demonstrate that the tidal elongation of the ocean layer due to the Moon is

$$\frac{\Delta R}{R} = \frac{3}{2}\,\frac{m'}{m}\left(\frac{R}{a}\right)^3,$$

where m is the mass of the Earth, m' the mass of the Moon, R the radius of the Earth, and a the radius of the lunar orbit. Show that $\Delta R = 0.54$ m, and also that the tidal elongation of the ocean layer due to the Sun is such that $\Delta R = 0.24$ m.

5.11 Estimate the tidal heating rate of the Earth due to the Moon. Is this rate significant compared with the net heating rate from solar radiation?

5.12 Estimate the tidal elongation of the Sun due to the Earth.

5.13 An approximately spherical moon of uniform density, mass m, radius R, and effective rigidity $\tilde{\mu}$ is in a circular orbit of major radius a about a spherical planet of mass $m_p \gg m$. The moon rotates about an axis passing through its center of mass that is directed normal to the orbital plane. Suppose that the moon is in a synchronous state such that its rotational angular velocity, Ω, matches its orbital angular velocity, $\omega \simeq (G\,m_p/a^3)^{1/2}$. Let x, y, z be a set of Cartesian coordinates, centered on the moon, such that the z-axis is normal to the orbital plane, and the x-axis is directed from the center of the moon to the center of the planet. Assuming that the moon responds elastically to the centrifugal and tidal potentials, show

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that the changes in radius, parallel to the three coordinate axes, induced by the centrifugal potential are

$$\frac{\delta R_x}{R} = \frac{5}{12}\,\frac{\omega^2 R^3}{G\,m}\,\frac{1}{1+\tilde{\mu}},$$
$$\frac{\delta R_y}{R} = \frac{5}{12}\,\frac{\omega^2 R^3}{G\,m}\,\frac{1}{1+\tilde{\mu}},$$

and

$$\frac{\delta R_z}{R} = -\frac{5}{6}\,\frac{\omega^2 R^3}{G\,m}\,\frac{1}{1+\tilde{\mu}},$$

whereas the corresponding changes induced by the tidal potential are

$$\frac{\delta R_x}{R} = \frac{5}{2}\,\frac{\omega^2 R^3}{G\,m}\,\frac{1}{1+\tilde{\mu}},$$
$$\frac{\delta R_y}{R} = -\frac{5}{4}\,\frac{\omega^2 R^3}{G\,m}\,\frac{1}{1+\tilde{\mu}},$$

and

$$\frac{\delta R_z}{R} = -\frac{5}{4}\,\frac{\omega^2 R^3}{G\,m}\,\frac{1}{1+\tilde{\mu}}.$$

Assuming that these changes are additive, deduce that

$$\frac{\delta R_y - \delta R_z}{\delta R_x - \delta R_z} = \frac{1}{4}. \tag{5.111}$$

Estimate δR_x, δR_y, and δR_z for the Earth's Moon (which is in a synchronous state).

5.14 An artificial satellite consists of two point objects of mass $m/2$ connected by a light rigid rod of length l. The satellite is placed in a circular orbit of radius $a \gg l$ (measured from the midpoint of the rod) around a planet of mass m'. The rod is oriented such that it always points toward the center of the planet. Demonstrate that the tension in the rod is

$$T = \frac{3}{4}\,\frac{G\,m\,m'\,l}{a^3} - \frac{1}{4}\,\frac{G\,m^2}{l^2}.$$

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