10 Lunar motion

10.1 Introduction

The orbital motion of the planets around the Sun is fairly accurately described by Kepler's laws. (See Chapter 3.) Similarly, to a first approximation, the orbital motion of the Moon around the Earth can also be accounted for via these laws. However, unlike the planetary orbits, the deviations of the lunar orbit from a Keplerian ellipse are sufficiently large that they are easily apparent to the naked eye. Indeed, the largest of these deviations, which is generally known as *evection*, was discovered in ancient times by the Alexandrian astronomer Claudius Ptolemy (90 BCE–168 CE) (Pannekoek 2011). Moreover, the next largest deviation, which is called *variation*, was first observed by Tycho Brahe (1546–1601) without the aid of a telescope (Godfray 1853). Another non-Keplerian feature of the lunar orbit, which is sufficiently obvious that it was known to the ancient Greeks, is the fact that the lunar perigee (i.e., the point of closest approach to the Earth) precesses (i.e., orbits about the Earth in the same direction as the Moon) at such a rate that, on average, it completes a full circuit every 8.85 years.[1] The ancient Greeks also noticed that the lunar *ascending node* (i.e., the point at which the Moon passes through the fixed plane of the Earth's orbit around the Sun from south to north) regresses (i.e., orbits about the Earth in the opposite direction to the Moon) at such a rate that, on average, it completes a full circuit every 18.6 years (Pannekoek 2011). Of course, according to standard two-body orbit theory, the lunar perigee and ascending node should both be stationary. (See Chapter 3.)

Newton demonstrated, in Book III of his *Principia*, that the deviations of the lunar orbit from a Keplerian ellipse are due to the gravitational influence of the Sun, which is sufficiently large that it is not completely negligible compared with the mutual gravitational attraction of the Earth and the Moon. However, Newton was not able to give a full account of these deviations (in the *Principia*), because of the complexity of the equations of motion that arise in a system of three mutually gravitating bodies. (See Chapter 8.) In fact, Alexis Clairaut (1713–1765) is generally credited with the first reasonably accurate and complete theoretical explanation of the Moon's orbit to be published. His method of calculation makes use of an expansion of the lunar equations of motion in terms of various small parameters. Clairaut, however, initially experienced difficulty in accounting for the precession of the lunar perigee. Indeed, his first calculation overestimated the period of this precession by a factor of about two, leading him to question

[1] This precession rate is about 10^4 times greater than any of the planetary perihelion precession rates discussed in Sections 4.4 and 9.3.

Newton's inverse-square law of gravitation. Later, he realized that he could account for the precession in terms of standard Newtonian dynamics by continuing his expansion in small parameters to higher order. (See Section 10.6.) After Clairaut, the theory of lunar motion was further elaborated in major works by D'Alembert (1717–1783), Euler (1707–1783), Laplace (1749–1827), Damoiseau (1768–1846), Plana (1781–1864), Poisson (1781–1840), Hansen (1795–1874), De Pontécoulant (1795–1874), J. Herschel (1792–1871), Airy (1801–1892), Delaunay (1816–1872), G.W. Hill (1836–1914), and E.W. Brown (1836–1938) (Brown 1896). The fact that so many celebrated mathematicians and astronomers devoted so much time and effort to lunar theory is a tribute to its inherent difficulty, as well as its great theoretical and practical interest. Indeed, for a period of about one hundred years (between 1767 and about 1850) the so-called *method of lunar distance* was the principal means used by mariners to determine terrestrial longitude at sea (Cotter 1968). This method depends crucially on a precise knowledge of the position of the Moon in the sky as a function of time. Consequently, astronomers and mathematicians during the period in question were spurred to make ever more accurate observations of the Moon's orbit, and to develop lunar theory to greater and greater precision. An important outcome of these activities was the making of various tables of lunar motion (e.g., those of Mayer, Damoiseau, Plana, Hansen, and Brown), the majority of which were published at public expense.

This chapter contains an introduction to lunar theory in which approximate expressions for evection, variation, the precession of the perigee, and the regression of the ascending node, are derived from the laws of Newtonian mechanics. Further information on lunar theory can be obtained from Godfray (1853), Brown (1896), Adams (1900), and Cook (1988).

10.2 Preliminary analysis

Let $\mathbf{r}_E$ and $\mathbf{r}_M$ denote the position vectors of the Earth and Moon, respectively, in a non-rotating reference frame in which the Sun is at rest at the origin. Treating this reference frame as inertial (which is an excellent approximation, given that the mass of the Sun is very much greater than that of the Earth or the Moon), the Earth's equation of motion becomes (see Chapter 3)

$$\ddot{\mathbf{r}}_E + n'^{\,2}\, a'^{\,3}\, \frac{\mathbf{r}_E}{|\mathbf{r}_E|^3} = 0, \tag{10.1}$$

where $n' = 0.98560912^\circ$ per day and $a' = 149{,}598{,}261$ km are the mean angular velocity and major radius, respectively, of the terrestrial orbit about the Sun (Yoder 1995). Here, $\ddot{} \equiv d^2/dt^2$. On the other hand, the Moon's equation of motion takes the form

$$\ddot{\mathbf{r}}_M + n'^{\,2}\, a'^{\,3}\, \frac{\mathbf{r}_M}{|\mathbf{r}_M|^3} = -n^2\, a^3\, \frac{\mathbf{r}_M - \mathbf{r}_E}{|\mathbf{r}_M - \mathbf{r}_E|^3}, \tag{10.2}$$

where $n = 13.176359^\circ$ per day and $a = 384{,}399$ km are the mean angular velocity and major radius, respectively, of the lunar orbit about the Earth (Yoder 1995). We have retained the acceleration due to the Earth in the lunar equation of motion, Equation (10.2), while neglecting the acceleration due to the Moon in the terrestrial equation of motion, Equation (10.1), because the former acceleration is significantly greater [by a factor $M_E/M_M \simeq 81.3$, where M_E is the mass of the Earth, and M_M the mass of the Moon (Yoder 1995)] than the latter.

Let

$$\mathbf{r} = \mathbf{r}_M - \mathbf{r}_E \tag{10.3}$$

and

$$\mathbf{r}' = -\mathbf{r}_E \tag{10.4}$$

be the position vectors of the Moon and Sun, respectively, relative to the Earth. It follows, from Equations (10.1)–(10.4), that in a noninertial reference frame, S (say), in which the Earth is at rest at the origin but the coordinate axes point in fixed directions, the lunar and solar equations of motion take the form

$$\ddot{\mathbf{r}} + n^2\,a^3\,\frac{\mathbf{r}}{|\mathbf{r}|^3} = n'^2\,a'^3\left(\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} - \frac{\mathbf{r}'}{|\mathbf{r}'|^3}\right) \tag{10.5}$$

and

$$\ddot{\mathbf{r}}' + n'^2\,a'^3\,\frac{\mathbf{r}'}{|\mathbf{r}'|^3} = 0, \tag{10.6}$$

respectively. One obvious way of proceeding would be to express the right-hand side of the lunar equation of motion, Equation (10.5), as the gradient of a disturbing function (see Exercise 10.1), and then to use this function to determine the time evolution of the Moon's osculating orbital elements from Lagrange's planetary equations. Unfortunately, this approach is fraught with mathematical difficulties. (See Brouwer and Clemence 1961.) It is actually more straightforward to solve Equation (10.5) in a Cartesian coordinate system, centered on the Earth, that rotates about an axis normal to the ecliptic plane at the Moon's mean orbital angular velocity. This method of solution is outlined as follows.

Let us set up a conventional Cartesian coordinate system in S that is such that the (apparent) orbit of the Sun about the Earth lies in the x–y plane. This implies that the x–y plane corresponds to the so-called ecliptic plane. Accordingly, in S, the Sun appears to orbit the Earth at the mean angular velocity $\boldsymbol{\omega}' = n'\,\mathbf{e}_z$ (assuming that the z-axis points toward the so-called north ecliptic pole), whereas the projection of the Moon onto the ecliptic plane orbits the Earth at the mean angular velocity $\boldsymbol{\omega} = n\,\mathbf{e}_z$.

In the following, for the sake of simplicity, we shall neglect the small eccentricity, $e' = 0.016711$, of the Sun's apparent orbit about the Earth (which is actually the eccentricity of the Earth's orbit about the Sun), and approximate the solar orbit as a circle, centered on the Earth. Thus, if x', y', z' are the Cartesian coordinates of the Sun in S, then an appropriate solution of the solar equation of motion, Equation (10.6), is

$$x' = a'\,\cos(n'\,t), \tag{10.7}$$

$$y' = a'\,\sin(n'\,t), \tag{10.8}$$

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and

$$z' = 0. \tag{10.9}$$

10.3 Lunar equations of motion

As we have already mentioned, it is convenient to solve the lunar equation of motion, Equation (10.5), in a geocentric frame of reference, S_1 (say), that rotates with respect to S at the fixed angular velocity ω. Thus, if the lunar orbit were a circle, centered on the Earth and lying in the ecliptic plane, then the Moon would appear stationary in S_1. In fact, the small eccentricity of the lunar orbit, $e = 0.05488$, combined with its slight inclination to the ecliptic plane, $I = 5.16^\circ$, causes the Moon to execute a small periodic orbit about the stationary point (Yoder 1995).

Let x, y, z and x_1, y_1, z_1 be the Cartesian coordinates of the Moon in S and S_1, respectively. It is easily demonstrated that (see Section A.6)

$$x = \cos(n\,t)\,x_1 - \sin(n\,t)\,y_1, \tag{10.10}$$

$$y = \sin(n\,t)\,x_1 + \cos(n\,t)\,y_1, \tag{10.11}$$

and

$$z = z_1. \tag{10.12}$$

Moreover, if x_1', y_1', z_1' are the Cartesian components of the Sun in S_1, then (see Section A.6)

$$x_1' = \cos(n\,t)\,x' + \sin(n\,t)\,y', \tag{10.13}$$

$$y_1' = -\sin(n\,t)\,x' + \cos(n\,t)\,y', \tag{10.14}$$

and

$$z_1' = z', \tag{10.15}$$

giving

$$x_1' = a'\,\cos[(n - n')\,t], \tag{10.16}$$

$$y_1' = -a'\,\sin[(n - n')\,t], \tag{10.17}$$

and

$$z_1' = 0, \tag{10.18}$$

where use has been made of Equations (10.7)–(10.9).

In the rotating frame S_1, the lunar equation of motion, Equation (10.5), transforms to (see Section 5.2)

$$\ddot{\mathbf{r}} + 2\,\boldsymbol{\omega}\times\dot{\mathbf{r}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + n^2\,a^3\,\frac{\mathbf{r}}{|\mathbf{r}|^3} = n'^2\,a'^3\left(\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} - \frac{\mathbf{r}'}{|\mathbf{r}'|^3}\right), \tag{10.19}$$

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where $\dot{} \equiv d/dt$. Furthermore, expanding the final term on the right-hand side of Equation (10.19) to lowest order in the small parameter $a/a' = 0.00257$, we obtain

$$\ddot{\mathbf{r}} + 2\,\boldsymbol{\omega}\times\dot{\mathbf{r}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + n^2\,a^3\,\frac{\mathbf{r}}{|\mathbf{r}|^3} \simeq \frac{n'^2\,a'^3}{|\mathbf{r}'|^3}\left[\frac{(3\,\mathbf{r}\cdot\mathbf{r}')\,\mathbf{r}'}{|\mathbf{r}'|^2} - \mathbf{r}\right]. \tag{10.20}$$

When written in terms of Cartesian coordinates, this equation yields

$$\ddot{x}_1 - 2\,n\,\dot{y}_1 - \left(n^2 + n'^2/2\right)x_1 + n^2\,a^3\,\frac{x_1}{r^3} \simeq \frac{3}{2}\,n'^2\,\cos[2\,(n-n')\,t]\,x_1 - \frac{3}{2}\,n'^2\,\sin[2\,(n-n')\,t]\,y_1, \tag{10.21}$$

$$\ddot{y}_1 + 2\,n\,\dot{x}_1 - \left(n^2 + n'^2/2\right)y_1 + n^2\,a^3\,\frac{y_1}{r^3} \simeq -\frac{3}{2}\,n'^2\,\sin[2\,(n-n')\,t]\,x_1 - \frac{3}{2}\,n'^2\,\cos[2\,(n-n')\,t]\,y_1, \tag{10.22}$$

and

$$\ddot{z}_1 + n'^2\,z_1 + n^2\,a^3\,\frac{z_1}{r^3} \simeq 0, \tag{10.23}$$

where $r = (x_1^2 + y_1^2 + z_1^2)^{1/2}$ and use has been made of Equations (10.16)–(10.18).

It is convenient, at this stage, to normalize all lengths to a, and all times to n^{-1}. Accordingly, let

$$X = x_1/a, \tag{10.24}$$

$$Y = y_1/a, \tag{10.25}$$

and

$$Z = z_1/a; \tag{10.26}$$

let $r/a = R = (X^2 + Y^2 + Z^2)^{1/2}$, and $T = n\,t$. In normalized form, Equations (10.21)–(10.23) become

$$\ddot{X} - 2\,\dot{Y} - (1 + m^2/2)\,X + \frac{X}{R^3} \simeq \frac{3}{2}\,m^2\,\cos[2\,(1-m)\,T]\,X - \frac{3}{2}\,m^2\,\sin[2\,(1-m)\,T]\,Y, \tag{10.27}$$

$$\ddot{Y} + 2\,\dot{X} - (1 + m^2/2)\,Y + \frac{Y}{R^3} \simeq -\frac{3}{2}\,m^2\,\sin[2\,(1-m)\,T]\,X - \frac{3}{2}\,m^2\,\cos[2\,(1-m)\,T]\,Y, \tag{10.28}$$

and

$$\ddot{Z} + m^2\,Z + \frac{Z}{R^3} \simeq 0, \tag{10.29}$$

respectively, where $m = n'/n = 0.07480$ is a measure of the perturbing influence of the Sun on the lunar orbit. Here, $\ddot{} \equiv d^2/dT^2$ and $\dot{} \equiv d/dT$.

Finally, let us write

$$X = X_0 + \delta X, \tag{10.30}$$

$$Y = \delta Y, \tag{10.31}$$

and

$$Z = \delta Z, \tag{10.32}$$

where $X_0 = (1 + m^2/2)^{-1/3}$, and $|\delta X|, |\delta Y|, |\delta Z| \ll X_0$. Thus, if the lunar orbit were a circle, centered on the Earth, and lying in the ecliptic plane, then, in the rotating frame S_1, the Moon would appear stationary at the point $(X_0, 0, 0)$. Expanding Equations (10.27)–(10.29) to second order in δX, δY, δZ, and neglecting terms of order m^4 and $m^2\,\delta X^2$, and so on, we obtain

$$\begin{aligned} \delta\ddot{X} - 2\,\delta\dot{Y} - 3\,(1 + m^2/2)\,\delta X \;\simeq\; & \frac{3}{2}\,m^2\,\cos[2\,(1-m)\,T] + \frac{3}{2}\,m^2\,\cos[2\,(1-m)\,T]\,\delta X \\ & - \frac{3}{2}\,m^2\,\sin[2\,(1-m)\,T]\,\delta Y - 3\,\delta X^2 \\ & + \frac{3}{2}\,(\delta Y^2 + \delta Z^2), \end{aligned} \tag{10.33}$$

$$\begin{aligned} \delta\ddot{Y} + 2\,\delta\dot{X} \;\simeq\; & -\frac{3}{2}\,m^2\,\sin[2\,(1-m)\,T] \\ & - \frac{3}{2}\,m^2\,\sin[2\,(1-m)\,T]\,\delta X \\ & - \frac{3}{2}\,m^2\,\cos[2\,(1-m)\,T]\,\delta Y + 3\,\delta X\,\delta Y, \end{aligned} \tag{10.34}$$

and

$$\delta\ddot{Z} + (1 + 3\,m^2/2)\,\delta Z \simeq 3\,\delta X\,\delta Z. \tag{10.35}$$

After the preceding three equations have been solved for δX, δY, and δZ, the Cartesian coordinates, x, y, z, of the Moon in the nonrotating geocentric frame S are obtained from Equations (10.10)–(10.12), (10.24)–(10.26), and (10.30)–(10.32). However, it is more convenient to write $x = r\,\cos\lambda$, $y = r\,\sin\lambda$, and $z = r\,\sin\beta$, where r is the radial distance between the Earth and Moon and λ and β are termed the Moon's *geocentric* (i.e., centered on the Earth) *ecliptic longitude* and *ecliptic latitude*, respectively. Moreover, it is easily seen that, to second order in δX, δY, δZ, and neglecting terms of order m^3 and $m^2\,\delta X$, and so on,

$$\frac{r}{a} - 1 + \frac{m^2}{6} \;\simeq\; \delta X + \frac{1}{2}\,\delta Y^2 + \frac{1}{2}\,\delta Z^2, \tag{10.36}$$

$$\lambda - n\,t \;\simeq\; \delta Y - \delta X\,\delta Y, \tag{10.37}$$

and

$$\beta \simeq \delta Z - \delta X\,\delta Z. \tag{10.38}$$

10.4 Unperturbed lunar motion

Let us, first of all, neglect the perturbing influence of the Sun on the Moon's orbit by setting $m = 0$ in the lunar equations of motion, Equations (10.33)–(10.35). For the sake of simplicity, let us also neglect nonlinear effects in these equations by setting $\delta X^2 = \delta Y^2 = \delta Z^2 = \delta X\,\delta Y = \delta X\,\delta Z = 0$. In this case, the equations reduce to

$$\delta\ddot{X} - 2\,\delta\dot{Y} - 3\,\delta X \simeq 0, \tag{10.39}$$

$$\delta\ddot{Y} + 2\,\delta\dot{X} \simeq 0, \tag{10.40}$$

and

$$\delta\ddot{Z} + \delta Z \simeq 0. \tag{10.41}$$

By inspection, appropriate solutions are

$$\delta X \simeq -e\,\cos(T - \alpha_0), \tag{10.42}$$

$$\delta Y \simeq 2\,e\,\sin(T - \alpha_0), \tag{10.43}$$

$$\delta Z \simeq I\,\sin(T - \gamma_0), \tag{10.44}$$

where e, α_0, I, and γ_0 are arbitrary constants. Recalling that $T = n\,t$, it follows from Equations (10.36)–(10.38) that

$$r \simeq a\,[1 - e\,\cos(n\,t - \alpha_0)]\,, \tag{10.45}$$

$$\lambda \simeq n\,t + 2\,e\,\sin(n\,t - \alpha_0), \tag{10.46}$$

and

$$\beta \simeq I\,\sin(n\,t - \gamma_0). \tag{10.47}$$

However, Equations (10.45) and (10.46) are simply first-order (in e) approximations to the familiar Keplerian laws (see Chapter 3)

$$r = \frac{a\,(1 - e^2)}{1 + e\,\cos\theta} \tag{10.48}$$

and

$$r^2\,\dot{\theta} = (1 - e^2)^{1/2}\,n\,a^2, \tag{10.49}$$

where $\theta = \lambda - \alpha_0$ is the Moon's true anomaly and $\dot{} \equiv d/dt$. (See Section 3.13.) Of course, these two laws describe a body that executes an elliptical orbit, confocal with the Earth, of major radius a, mean angular velocity n, and eccentricity e, such that the radius vector connecting the body to the Earth sweeps out equal areas in equal time intervals. We conclude, unsurprisingly, that the unperturbed lunar orbit is a Keplerian ellipse. Note that the lunar perigee lies at the fixed ecliptic longitude $\lambda = \alpha_0$. Equation (10.47) is the first-order approximation to

$$\beta = I\,\sin(\lambda - \gamma_0). \tag{10.50}$$

(See Section 3.13.) This expression implies that the unperturbed lunar orbit is coplanar but is inclined at an angle I to the ecliptic plane. Moreover, the ascending node lies at the fixed ecliptic longitude $\lambda = \gamma_0$. Incidentally, the neglect of nonlinear terms in Equations (10.39)–(10.41) is valid only as long as e, $I \ll 1$, in other words, provided the unperturbed lunar orbit is only slightly elliptical and slightly inclined to the ecliptic plane. In fact, the observed values of e and I are 0.05488 and 0.09008 radians, respectively (Yoder 1995; Standish and Williams 1992), so this is a good approximation.

10.5 Perturbed lunar motion

The perturbed nonlinear lunar equations of motion, Equations (10.33)–(10.35), take the general form

$$\delta\ddot{X} - 2\,\delta\dot{Y} - 3\,(1 + m^2/2)\,\delta X \simeq R_X, \tag{10.51}$$

$$\delta\ddot{Y} + 2\,\delta\dot{X} \simeq R_Y, \tag{10.52}$$

and

$$\delta\ddot{Z} + (1 + 3\,m^2/2)\,\delta Z \simeq R_Z, \tag{10.53}$$

where (see Tables 10.1 and 10.2)

$$R_X = a_0 + \sum_{j=1,5} a_j\,\cos(\omega_j\,T - \alpha_j), \tag{10.54}$$

$$R_Y = \sum_{j=1,5} b_j\,\sin(\omega_j\,T - \alpha_j), \tag{10.55}$$

and

$$R_Z = \sum_{j=1,5}^{j\neq 4} c_j\,\sin(\Omega_j\,T - \gamma_j). \tag{10.56}$$

Let us search for solutions of the general form

$$\delta X = x_0 + \sum_{j=1,5} x_j\,\cos(\omega_j\,T - \alpha_j), \tag{10.57}$$

$$\delta Y = \sum_{j=1,5} y_j\,\sin(\omega_j\,T - \alpha_j), \tag{10.58}$$

and

$$\delta Z = \sum_{j=1,5}^{j\neq 4} z_j\,\sin(\Omega_j\,T - \gamma_j). \tag{10.59}$$

Substituting Equations (10.54)–(10.59) into Equations (10.51)–(10.53), we can easily demonstrate that

$$x_0 = -\frac{a_0}{3\,(1 + m^2/2)}, \tag{10.60}$$

$$x_j = \frac{\omega_j\,a_j - 2\,b_j}{\omega_j\,(1 - 3\,m^2/2 - \omega_j^2)}, \tag{10.61}$$

$$y_j = \frac{(\omega_j^2 + 3 + 3\,m^2/2)\,b_j - 2\,\omega_j\,a_j}{\omega_j^2\,(1 - 3\,m^2/2 - \omega_j^2)}, \tag{10.62}$$

and

$$z_j = \frac{c_j}{1 + 3\,m^2/2 - \Omega_j^2}, \tag{10.63}$$

where $j = 1, 5$.

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Table 10.1 Angular frequencies and phase shifts associated with principal periodic driving terms appearing in perturbed nonlinear lunar equations of motion

j	ω_j	α_j	Ω_j	γ_j
1	$1 + c\, m^2$	α_0	$1 + g\, m^2$	γ_0
2	$2\,(1 + c\, m^2)$	$2\,\alpha_0$	$(c - g)\, m^2$	$\alpha_0 - \gamma_0$
3	$2\,(1 + g\, m^2)$	$2\,\gamma_0$	$2 + (c + g)\, m^2$	$\alpha_0 + \gamma_0$
4	$2 - 2\, m$	0		
5	$1 - 2\, m - c\, m^2$	$-\alpha_0$	$1 - 2\, m - g\, m^2$	$-\gamma_0$

The angular frequencies, ω_j, Ω_j, and phase shifts, α_j, γ_j, of the principal periodic driving terms that appear on the right-hand sides of the perturbed nonlinear lunar equations of motion, Equations (10.51)–(10.53), are specified in Table 10.1. Here, c and g are, as yet, unspecified $\mathcal{O}(1)$ constants associated with the precession of the lunar perigee and the regression of the ascending node, respectively. Note that ω_1 and Ω_1 are the frequencies of the Moon's unforced motion in ecliptic longitude and latitude, respectively. Moreover, ω_4 is the forcing frequency associated with the perturbing influence of the Sun. All other frequencies appearing in Table 10.1 are combinations of these three fundamental frequencies. In fact, $\omega_2 = 2\,\omega_1$, $\omega_3 = 2\,\Omega_1$, $\omega_5 = \omega_4 - \omega_1$, $\Omega_2 = \omega_1 - \Omega_1$, $\Omega_3 = \omega_1 + \Omega_1$, and $\Omega_5 = \omega_4 - \Omega_1$. Note that there is no Ω_4.

A comparison of Equations (10.33)–(10.35), (10.51)–(10.53), and Table 10.1 reveals that

$$R_X = \frac{3}{2}\, m^2 \cos(\omega_4\, T - \alpha_4) + \frac{3}{2}\, m^2 \cos(\omega_4\, T - \alpha_4)\,\delta X - \frac{3}{2}\, m^2 \sin(\omega_4\, T - \alpha_4)\,\delta Y - 3\,\delta X^{\,2} + \frac{3}{2}\,(\delta Y^{\,2} + \delta Z^{\,2}), \tag{10.64}$$

$$R_Y = -\frac{3}{2}\, m^2 \sin(\omega_4\, T - \alpha_4) - \frac{3}{2}\, m^2 \sin(\omega_4\, T - \alpha_4)\,\delta X - \frac{3}{2}\, m^2 \cos(\omega_4\, T - \alpha_4)\,\delta Y + 3\,\delta X\,\delta Y, \tag{10.65}$$

and

$$R_Z = 3\,\delta X\,\delta Z. \tag{10.66}$$

Substitution of Equations (10.57)–(10.59) into these equations, followed by a comparison with Equations (10.54)–(10.56), yields the amplitudes a_j, b_j, and c_j specified in Table 10.2. In calculating these amplitudes, we have neglected all contributions to the periodic driving terms appearing in Equations (10.51)–(10.53) that involve cubic, or higher order, combinations of e, I, m^2, x_j, y_j, and z_j, because we expanded Equations (10.33)–(10.35) only to *second order* in δX, δY, and δZ.

For $j = 0$, it follows from Equation (10.60) and Table 10.2 that

$$x_0 \simeq -\frac{1}{2}\, e^2 - \frac{1}{4}\, I^{\,2}. \tag{10.67}$$

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Table 10.2 Amplitudes of periodic driving terms appearing in perturbed nonlinear lunar equations of motion

j	a_j	b_j	c_j
0	$\frac{3}{2}\,e^2+\frac{3}{4}\,I^2$		
1	$\frac{3}{4}\,m^2\,x_5-\frac{3}{4}\,m^2\,y_5-3\,x_4\,x_5+\frac{3}{2}\,y_4\,y_5$	$-\frac{3}{4}\,m^2\,x_5+\frac{3}{4}\,m^2\,y_5+\frac{3}{2}\,y_4\,x_5-\frac{3}{2}\,y_5\,x_4$	$-\frac{3}{2}\,x_4\,z_5$
2	$-\frac{9}{2}\,e^2$	$-3\,e^2$	$\frac{3}{2}\,e\,I$
3	$-\frac{3}{4}\,I^2$	0	$-\frac{3}{2}\,e\,I$
4	$\frac{3}{2}\,m^2$	$-\frac{3}{2}\,m^2$	
5	$-\frac{9}{4}\,m^2\,e+3\,e\,x_4+3\,e\,y_4$	$\frac{9}{4}\,m^2\,e-3\,e\,x_4-\frac{3}{2}\,e\,y_4$	$-\frac{3}{2}\,I\,x_4$

For $j=2$, making the approximation $\omega_2\simeq 2$ (see Table 10.1), we see from Equations (10.61) and (10.62) and Table 10.2 that

$$x_2\simeq\frac{1}{2}\,e^2 \tag{10.68}$$

and

$$y_2\simeq\frac{1}{4}\,e^2. \tag{10.69}$$

Likewise, making the approximation $\Omega_2\simeq 0$ (see Table 10.1), we see from Equation (10.63) and Table 10.2 that

$$z_2\simeq\frac{3}{2}\,e\,I. \tag{10.70}$$

For $j=3$, making the approximation $\omega_3\simeq 2$ (see Table 10.1), we see from Equations (10.61) and (10.62) and Table 10.2 that

$$x_3\simeq\frac{1}{4}\,I^2 \tag{10.71}$$

and

$$y_3\simeq-\frac{1}{4}\,I^2. \tag{10.72}$$

Likewise, making the approximation $\Omega_3\simeq 2$ (see Table 10.1), we see from Equation (10.63) and Table 10.2 that

$$z_3\simeq\frac{1}{2}\,e\,I. \tag{10.73}$$

For $j=4$, making the approximation $\omega_4\simeq 2$ (see Table 10.1), we see from Equations (10.61) and (10.62) and Table 10.2 that

$$x_4\simeq-m^2 \tag{10.74}$$

and

$$y_4\simeq\frac{11}{8}\,m^2. \tag{10.75}$$

Thus, according to Table 10.2,

$$a_5 \simeq -\frac{9}{8}\,m^2\,e, \tag{10.76}$$

$$b_5 \simeq \frac{51}{16}\,m^2\,e, \tag{10.77}$$

and

$$c_5 \simeq \frac{3}{2}\,m^2\,I. \tag{10.78}$$

For $j = 5$, making the approximation $\omega_5 \simeq 1 - 2\,m$ (see Table 10.1), we see from Equations (10.61), (10.62), (10.76), and (10.77) that

$$x_5 \simeq -\frac{15}{8}\,m\,e \tag{10.79}$$

and

$$y_5 \simeq \frac{15}{4}\,m\,e. \tag{10.80}$$

Likewise, making the approximation $\Omega_5 \simeq 1 - 2\,m$ (see Table 10.1), we see from Equations (10.63) and (10.78) that

$$z_5 \simeq \frac{3}{8}\,m\,I. \tag{10.81}$$

Thus, according to Table 10.2,

$$a_1 \simeq -\frac{135}{64}\,m^3\,e, \tag{10.82}$$

$$b_1 \simeq \frac{765}{128}\,m^3\,e, \tag{10.83}$$

and

$$c_1 \simeq \frac{9}{16}\,m^3\,I. \tag{10.84}$$

Finally, for $j = 1$, by analogy with Equations (10.42)–(10.44), we expect

$$x_1 \simeq -e, \tag{10.85}$$

$$y_1 \simeq 2\,e, \tag{10.86}$$

and

$$z_1 \simeq I. \tag{10.87}$$

Thus, because $\omega_1 = 1 + c\,m^2$ (see Table 10.1), it follows from Equations (10.61), (10.82), (10.83), and (10.85) that

$$-e \simeq \frac{-(225/16)\,m^3\,e}{-(3/2)\,m^2 - 2\,c\,m^2}, \tag{10.88}$$

which yields

$$c \simeq -\frac{3}{4} - \frac{225}{32}\,m + \mathcal{O}(m^2). \tag{10.89}$$

Likewise, because $\Omega_1 = 1 + g\,m^2$ (see Table 10.1), it follows from Equations (10.63), (10.84), and (10.87) that

$$I \simeq \frac{(9/16)\,m^3\,I}{(3/2)\,m^2 - 2\,g\,m^2}, \tag{10.90}$$

which yields

$$g \simeq \frac{3}{4} - \frac{9}{32}\,m + \mathcal{O}(m^2). \tag{10.91}$$

According to this analysis, our final expressions for δX, δY, and δZ are

$$\begin{aligned}
\delta X \;\simeq\; & -\frac{1}{2}\,e^2 - \frac{1}{4}\,I^2 - e\,\cos[(1 + c\,m^2)\,T - \alpha_0] \\
& + \frac{1}{2}\,e^2\,\cos[2\,(1 + c\,m^2)\,T - 2\,\alpha_0] \\
& + \frac{1}{4}\,I^2\,\cos[2\,(1 + g\,m^2)\,T - 2\,\gamma_0] - m^2\,\cos[2\,(1 - m)\,T] \\
& - \frac{15}{8}\,m\,e\,\cos[(1 - 2\,m - c\,m^2)\,T + \alpha_0],
\end{aligned} \tag{10.92}$$

$$\begin{aligned}
\delta Y \;\simeq\; & 2\,e\,\sin[(1 + c\,m^2)\,T - \alpha_0] + \frac{1}{4}\,e^2\,\sin[2\,(1 + c\,m^2)\,T - 2\,\alpha_0] \\
& - \frac{1}{4}\,I^2\,\sin[2\,(1 + g\,m^2)\,T - 2\,\gamma_0] + \frac{11}{8}\,m^2\,\cos[2\,(1 - m)\,T] \\
& + \frac{15}{4}\,m\,e\,\sin[(1 - 2\,m - c\,m^2)\,T + \alpha_0],
\end{aligned} \tag{10.93}$$

and

$$\begin{aligned}
\delta Z \;\simeq\; & I\,\sin[(1 + g\,m^2)\,T - \gamma_0] + \frac{3}{2}\,e\,I\,\sin[(c - g)\,m^2\,T - \alpha_0 + \gamma_0] \\
& + \frac{1}{2}\,e\,I\,\sin[(2 + c\,m^2 + g\,m^2)\,T - \alpha_0 - \gamma_0] \\
& + \frac{3}{8}\,m\,I\,\sin[(1 - 2\,m - g\,m^2)\,T + \gamma_0].
\end{aligned} \tag{10.94}$$

Thus, making use of Equations (10.36)–(10.38), we find that

$$\begin{aligned}
\frac{r}{a} \;\simeq\; & 1 - e\,\cos[(1 + c\,m^2)\,T - \alpha_0] \\
& + \frac{1}{2}\,e^2 - \frac{1}{6}\,m^2 - \frac{1}{2}\,e^2\,\cos[2\,(1 + c\,m^2)\,T - 2\,\alpha_0] \\
& - m^2\,\cos[2\,(1 - m)\,T] - \frac{15}{8}\,m\,e\,\cos[(1 - 2\,m - c\,m^2)\,T + \alpha_0],
\end{aligned} \tag{10.95}$$

$$\begin{aligned}
\lambda \;\simeq\; & T + 2\,e\,\sin[(1 + c\,m^2)\,T - \alpha_0] + \frac{5}{4}\,e^2\,\sin[2\,(1 + c\,m^2)\,T - 2\,\alpha_0] \\
& - \frac{1}{4}\,I^2\,\sin[2\,(1 + g\,m^2)\,T - 2\,\gamma_0] + \frac{11}{8}\,m^2\,\sin[2\,(1 - m)\,T] \\
& + \frac{15}{4}\,m\,e\,\sin[(1 - 2\,m - c\,m^2)\,T + \alpha_0],
\end{aligned} \tag{10.96}$$

and

$$\beta \simeq I \sin[(1+g\,m^2)\,T-\gamma_0]+e\,I \sin[(c-g)\,m^2\,T-\alpha_0+\gamma_0]$$
$$+\,e\,I \sin[(2+c\,m^2+g\,m^2)\,T-\alpha_0-\gamma_0]$$
$$+\,\frac{3}{8}\,m\,I \sin[(1-2\,m-g\,m^2)\,T+\gamma_0]. \tag{10.97}$$

These expressions are accurate up to second order in the small parameters e, I, and m.

10.6 Description of lunar motion

To better understand the expressions for perturbed lunar motion derived in the previous section, it is helpful to introduce the concept of the *mean moon*. This is an imaginary body that orbits the Earth, in the ecliptic plane, at a steady angular velocity equal to the Moon's mean orbital angular velocity, n. Likewise, the *mean sun* is a second imaginary body that orbits the Earth, in the ecliptic plane, at a steady angular velocity equal to the Sun's mean (apparent) orbital angular velocity, n'. Thus, the geocentric ecliptic longitudes of the mean moon and the mean sun are

$$\bar{\lambda} = n\,t \tag{10.98}$$

and

$$\bar{\lambda}' = n'\,t, \tag{10.99}$$

respectively. Here, for the sake of simplicity, and also for the sake of consistency with our previous analysis, we have assumed that both objects are located at ecliptic longitude 0 at time $t = 0$.

From Equation (10.95), to first order in small parameters, the lunar perigee corresponds to $(1+c\,m^2)\,n\,t-\alpha_0 = j\,2\pi$, where j is an integer. However, this condition can also be written $\bar{\lambda} = \alpha$, where

$$\alpha = \alpha_0 + \alpha'\,n'\,t, \tag{10.100}$$

and, making use of Equation (10.89), together with the definition $m = n'/n$,

$$\alpha' = \frac{3}{4}\,m + \frac{225}{32}\,m^2 + \mathcal{O}(m^3). \tag{10.101}$$

Thus, we can identify α as the mean ecliptic longitude of the perigee. Moreover, according to Equation (10.100), the perigee precesses (i.e., its longitude increases in time) at the mean rate of $360\,\alpha'$ degrees per year. (Of course, a year corresponds to $\Delta t = 2\pi/n'$.) Furthermore, it is clear that this precession is entirely due to the perturbing influence of the Sun, because it depends only on the parameter m, which is a measure of this influence. Given that $m = 0.07480$, we find that the perigee advances by 34.36° per year. Hence, we predict that the perigee completes a full circuit about the Earth every $1/\alpha' = 10.5$ years. In fact, the lunar perigee completes a full circuit every 8.85 years. Our prediction is somewhat inaccurate because our previous analysis neglected $\mathcal{O}(m^2)$, and smaller, contributions to the parameter c [see Equation (10.89)], and these turn out to be significant.

From Equation (10.97), to first order in small parameters, the Moon passes through its ascending node when $(1 + g\,m^2)\,n\,t - \gamma_0 = j\,2\pi$, where j is an integer. However, this condition can also be written $\bar{\lambda} = \gamma$, where

$$\gamma = \gamma_0 - \gamma'\,n'\,t, \tag{10.102}$$

and, making use of Equation (10.91),

$$\gamma' = \frac{3}{4}\,m - \frac{9}{32}\,m^2 + \mathcal{O}(m^3). \tag{10.103}$$

Thus, we can identify γ as the mean ecliptic longitude of the ascending node. Moreover, according to Equation (10.102), the ascending node regresses (i.e., its longitude decreases in time) at the mean rate of $360\,\gamma'$ per year. As before, it is clear that this regression is entirely due to the perturbing influence of the Sun. Moreover, we find that the ascending node retreats by 19.63° per year. Hence, we predict that the ascending node completes a full circuit about the Earth every $1/\gamma' = 18.3$ years. In fact, the lunar ascending node completes a full circuit every 18.6 years, so our prediction is fairly accurate.

It is interesting to note that Clairaut's initial lunar theory, produced in 1747, neglected the $\mathcal{O}(m^2)$ contributions to α' and γ'—see Equations (10.101) and (10.103), respectively—leading to the prediction that the lunar perigee should precess at the same rate at which the ascending node regresses, and that both the perigee and the node should complete full circuits around the Earth every 17.8 years (Taton and Wilson 1995). Of course, this prediction was in serious disagreement with observations, according to which the lunar perigee precesses at about twice the rate at which the ascending node regresses. This discrepancy between theory and observations lead Clairaut to briefly doubt the inverse-square nature of Newtonian gravity. Fortunately, Clairaut realized in 1748 that the discrepancy could be resolved by carrying his expansion to higher order in m. Indeed, as is clear from Equations (10.101) and (10.103), the $\mathcal{O}(m^2)$ contributions to α' and γ' give rise to a significant increase in the precession rate of the lunar perigee relative to the regression rate of the ascending node. In fact, the complete expression for α' (Delaunay 1867),

$$\alpha' = \frac{3}{4}\,m + \frac{225}{32}\,m^2 + \frac{4071}{128}\,m^3 + \frac{265493}{2048}\,m^4 + \frac{12822631}{24576}\,m^5 + \cdots, \tag{10.104}$$

takes the form of a power series in m. Despite the fact that, for the case of the Moon, m takes the relatively small value 0.07480, this series is slowly converging, and many terms must be retained to get an accurate value for the precession rate of the perigee. The complete expression for γ' (Delaunay 1867),

$$\gamma' = \frac{3}{4}\,m - \frac{9}{32}\,m^2 - \frac{273}{128}\,m^3 - \frac{9797}{2048}\,m^4 + \cdots, \tag{10.105}$$

also takes the form of a power series in m. Fortunately, for the case of the Moon, this series converges relatively quickly, which accounts for the fact that our prediction for the regression rate of the lunar ascending node is considerably more accurate than that for the precession rate of the perigee.

It is helpful to introduce the lunar *mean anomaly*,

$$\mathcal{M} = \bar{\lambda} - \alpha, \tag{10.106}$$

which is defined as the angular distance (in geocentric ecliptic longitude) between the mean Moon and the perigee. It is also helpful to introduce the lunar *mean argument of latitude*,

$$F = \bar{\lambda} - \gamma, \tag{10.107}$$

which is defined as the angular distance (in geocentric ecliptic longitude) between the mean Moon and the ascending node. Finally, it is helpful to introduce the *mean elongation* of the Moon,

$$D = \bar{\lambda} - \bar{\lambda}', \tag{10.108}$$

which is defined as the difference between the geocentric ecliptic longitudes of the mean Moon and the mean Sun.

When expressed in terms of $\mathcal{M}$, F, and D, our previous expression for the true geocentric ecliptic longitude of the Moon, Equation (10.96), becomes

$$\lambda = \bar{\lambda} + \delta\lambda, \tag{10.109}$$

where

$$\delta\lambda \;=\; 2e\,\sin\mathcal{M} + \frac{5}{4}\,e^2\,\sin 2\mathcal{M} - \frac{1}{4}\,I^2\,\sin 2F + \frac{11}{8}\,m^2\,\sin 2D + \frac{15}{4}\,m\,e\,\sin(2D - \mathcal{M}) \tag{10.110}$$

is the angular distance (in geocentric ecliptic longitude) between the Moon and the mean moon.

The first three terms on the right-hand side of Equation (10.110) are Keplerian in origin (i.e., they are independent of the perturbing action of the Sun). In fact, the first is due to the eccentricity of the lunar orbit (i.e., the fact that the geometric center of the orbit is slightly shifted from the center of the Earth), the second is due to the ellipticity of the orbit (i.e., the fact that the orbit is slightly noncircular), and the third is due to the slight inclination of the orbit to the ecliptic plane. The first and third terms are usually called the *major inequality* and the *reduction to the ecliptic*, respectively.

The fourth term on the right-hand side of Equation (10.110) corresponds to variation; it is clearly due to the perturbing influence of the Sun (because it depends only on the parameter m, which is a measure of this influence). Variation attains its maximal amplitude around the so-called *octant points*, at which the Moon's disk is either one-quarter or three-quarters illuminated (i.e., when $D = 45^\circ$, 135°, 225°, or 315°). Conversely, the amplitude of variation is zero around the so-called *quadrant points*, at which the Moon's disk is either fully illuminated, half illuminated, or not illuminated at all (i.e., when $D = 0^\circ$, 90°, 180°, or 270°). Variation generates a perturbation in the lunar ecliptic longitude that oscillates sinusoidally with a period of half a synodic month.[2] This oscillation period is in good agreement with observations. However, the amplitude of the oscillation (calculated using $m = 0.07480$) is 1,630 arc seconds, which is considerably

[2] A synodic month, which is 29.53 days, is the mean period between successive new moons.

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less than the observed amplitude of 2,370 arc seconds (Chapront-Touzé and Chapront 1988). This discrepancy between theory and observation is due to the fact that, for the sake of simplicity, we have calculated only the lowest order (in m) contribution to variation.

The fifth term on the right-hand side of Equation (10.110) corresponds to evection and is due to the combined action of the Sun and the eccentricity of the lunar orbit. In fact, evection can be thought of as causing a slight reduction in the eccentricity of the lunar orbit around the times of the new moon and the full moon (i.e., $D = 0^\circ$ and $D = 180^\circ$), and a corresponding slight increase in the eccentricity around the times of the first and last quarter moons (i.e., $D = 90^\circ$ and $D = 270^\circ$). (See Exercise 10.2.) This follows because the evection term in Equation (10.110) augments the eccentricity term, $2e\,\sin\mathcal{M}$, when $\cos 2D = -1$, and reduces the term when $\cos 2D = +1$. Evection generates a perturbation in the lunar ecliptic longitude that oscillates sinusoidally with a period of 31.8 days. This oscillation period is in good agreement with observations. However, the amplitude of the oscillation (calculated using $m = 0.07480$ and $e = 0.05488$) is 3,218 arc seconds, which is considerably less than the observed amplitude of 4,586 arc seconds (Chapront-Touzé and Chapront 1988). Again, this discrepancy between theory and observation exists because we have calculated only the lowest order (in m and e) contribution to evection.

Recall that we previously neglected the slight eccentricity, $e' = 0.016711$, of the Sun's apparent orbit about the Earth in our calculation. In fact, the eccentricity of the solar orbit gives rise to a small addition term on the right-hand side of Equation (10.110), which, to lowest order, takes the form $-3\,m\,e'\,\sin\mathcal{M}'$. Here, $\mathcal{M}'$ is the Sun's mean anomaly. This term, which is known as the *annual inequality*, generates a perturbation in the lunar ecliptic longitude that oscillates with a period of a solar year and has an amplitude of 772 arc seconds. As before, the oscillation period is in good agreement with observations, whereas the amplitude is inaccurate [it should be 666 arc seconds (Chapront-Touzé and Chapront 1988)] because of the omission of higher order (in m and e') contributions.

When written in terms of D and F, our previous expression for the geocentric ecliptic latitude of the Moon, Equation (10.97) becomes

$$\beta = I\,\sin(F + \delta\lambda) + \frac{3}{8}\,m\,I\,\sin(2D - F). \tag{10.111}$$

The first term on the right-hand side of Equation (10.111) is Keplerian in origin (i.e., it is essentially independent of the perturbing influence of the Sun). The second term, which is known as *evection in latitude*, is due to the combined action of the Sun and the inclination of the lunar orbit to the ecliptic. Evection in latitude can be thought of as causing a slight increase in the inclination of the lunar orbit at the times of the first and last quarter moons, and a slight decrease at the times of the new moon and the full moon. (See Exercise 10.2.) Evection in latitude generates a perturbation in the lunar ecliptic latitude that oscillates sinusoidally with a period of 32.3 days; it has an amplitude of 521 arc seconds. As before, the oscillation period is in good agreement with observations, but the amplitude is inaccurate (it should be 624 arc seconds; Chapront-Touzé and Chapront 1988) due to the omission of higher-order (in m and I) contributions.

Exercises

10.1 Demonstrate that the lunar equation of motion, Equation (10.5), can be written in the canonical form

$$\ddot{\mathbf{r}} + n^2\, a^3\, \frac{\mathbf{r}}{r^3} = \nabla\mathcal{R},$$

where

$$\mathcal{R} = n'^{\,2}\, a'^{\,3}\left(\frac{1}{|\mathbf{r}' - \mathbf{r}|} - \frac{\mathbf{r}\cdot\mathbf{r}'}{r'^{\,3}}\right)$$

is the disturbing function due to the gravitational influence of the Sun. Show that if the Earth's orbit about the Sun is approximated as a circle of radius a', then, to lowest order in a/a', the solar disturbing function can be written

$$\mathcal{R} \simeq n'^{\,2}\, r^2\, P_2(\cos\psi),$$

where ψ is the angle subtended between the vectors $\mathbf{r}$ and $\mathbf{r}'$.

10.2 Demonstrate that the evection term

$$\frac{15}{4}\, m\, e\, \sin(2D - \mathcal{M}),$$

appearing in Equation (10.110), can be represented as the combined effect of periodic variations in the eccentricity, e, of the lunar orbit, and the mean longitude, α, of the lunar perigee; in other words, $e \rightarrow e\,[1 - (15/8)\, m\, \cos 2D]$ and $\alpha \rightarrow \alpha - (15/8)\, m\, \sin 2D$. Likewise, show that the evection term

$$\frac{3}{8}\, m\, I\, \sin(2D - F),$$

appearing in Equation (10.111), can be represented as the combined effect of periodic variations in the inclination, I, of the lunar orbit, and the mean longitude, γ, of the lunar ascending node; in other words, $I \rightarrow I\,[1 - (3/8)\, m\, \cos 2D]$ and $\gamma \rightarrow \gamma - (3/8)\, m\, \sin 2D$.

10.3 Suppose that the major radius of the lunar orbit were reduced by a multiplicative factor ζ: that is, $a \rightarrow \zeta\, a$, where $0 < \zeta < 1$. Assuming that the masses of the Earth and Sun and the major radius of the terrestrial orbit remain constant, demonstrate that the parameter m, which measures the perturbing influence of the Sun on the lunar orbit, would be reduced by a factor $\zeta^{3/2}$: that is, $m \rightarrow \zeta^{3/2}\, m$. Given that $m = 0.07480$ for the true lunar orbit, how small would ζ have to be before the (theoretical) precession rate of the lunar perigee became equal to the regression rate of the ascending node to within 1 percent? What is the corresponding major radius of the lunar orbit in units of mean Earth radii? (The true major radius of the lunar orbit is 60.9 mean Earth radii.)

10.4 An artificial satellite orbits the Moon in a low-eccentricity orbit whose major radius is twice the lunar radius. The plane of the satellite orbit is slightly inclined to the plane of the Moon's orbit about the Earth. Given that the mass of the Earth is 81.3 times that of the Moon, and the major radius of the lunar orbit is 221.3 times the lunar radius, estimate the precession period of the satellite orbit's perilune (i.e., its point of closest approach to the Moon) in months due to the perturbing influence of the Earth. Likewise, estimate the regression rate of the satellite orbit's ascending node (with respect to the plane of the lunar orbit) in months. (Assume that the Moon is a perfect sphere.)

10.5 The mean ecliptic longitudes (measured with respect to the vernal equinox at a fixed epoch) of the Moon and the Sun increase at the rates 13.176359° per day and 0.98560912° per day, respectively. However, the vernal equinox regresses in such a manner that, on average, it completes a full circuit every 25,772 years. Furthermore, the lunar perigee precesses in such a manner that, on average, it completes a full circuit every 8.848 years, whereas the lunar ascending mode regresses in such a manner that, on average, it completes a full circuit every 18.615 years (Yoder 1995). A *sidereal month* is the mean period of the Moon's orbit with respect to the fixed stars, a *tropical month* is the mean time required for the Moon's ecliptic longitude (with respect to the true vernal equinox) to increase by 360°, a *synodic month* is the mean period between successive new moons, an *anomalistic month* is the mean period between successive passages of the Moon through its perigee, and a *draconic month* is the mean period between successive passages of the Moon through its ascending node. Use the preceding information to demonstrate than the lengths of a sidereal, tropical, synodic, anomalistic, and draconic month are 27.32166, 27.32158, 29.5306, 27.5546, and 27.2123 days, respectively.

10.6 To first order in the Moon's orbital eccentricity and inclination, the geocentric ecliptic longitude and latitude of the Moon, relative to the Sun, are written

$$\Delta\lambda \simeq (n - n')\,t + 2\,e\,\sin[(n - n_p)\,t]$$

and

$$\Delta\beta \simeq I\,\sin[(n - n_n)\,t],$$

respectively, where $e = 0.05488$ and $I = 0.9008$ radians. Here, n and n' are the mean geocentric orbital angular velocities of the Moon and Sun, respectively, n_p is the mean orbital angular velocity of the lunar perigee, and n_n is the mean orbital angular velocity of the lunar ascending node. Note that $2\pi/(n - n') = T_{\rm synodic}$, $2\pi/(n - n_p) = T_{\rm anomalistic}$, and $2\pi/(n - n_n) = T_{\rm draconic}$, where $T_{\rm synodic}$, $T_{\rm anomalistic}$, and $T_{\rm draconic}$ are the lengths of a synodic, anomalistic, and draconic month, respectively. At $t = 0$, we have $\Delta\lambda = \Delta\beta = 0°$. In other words, at $t = 0$, the Moon and Sun have exactly the same geocentric ecliptic longitudes and latitudes, which implies that a solar eclipse occurs at this time. Suppose we can find some time period T that satisfies $T = j_1\,T_{\rm synodic} = j_2\,T_{\rm anomalistic} = j_3\,T_{\rm draconic}$, where j_1, j_2, j_3 are positive integers. Demonstrate that $\Delta\lambda = \Delta\beta = 0°$ at $t = T$. Thus, if the period T, which is known as the *saros*, existed, then solar (and lunar) eclipses would occur in infinite sequences spaced j_1 synodic months apart (Roy 2005). Show that for $0 < j_1, j_2, j_3 < 1{,}000$, the closest approximation to the saros is obtained when $j_1 = 223$, $j_2 = 239$, and $j_3 = 242$. Demonstrate that if $\Delta\lambda = \Delta\beta = 0°$ at $t = 0$ (i.e., if there is a solar eclipse at $t = 0$) then, exactly 223 synodic months later, $\Delta\lambda = -0.3°$ and $\Delta\beta = -0.1°$. It turns out that these values of $\Delta\lambda$

and $\Delta\beta$ are sufficiently small that the eclipse recurs. In fact, because 223 synodic months almost satisfies the saros condition, solar (and lunar) eclipses occur in series of about 70 eclipses spaced 223 synodic months, or 18 years and 11 days, apart.

10.7 Let the x-, y-, and z-axes be the lunar principal axes of rotation passing through the lunar center of mass. Because the Moon is not quite spherically symmetric, its principal moments of inertia are not exactly equal to one another. Let us label the principal axes such that $\mathcal{I}_{zz} > \mathcal{I}_{yy} > \mathcal{I}_{xx}$. To a first approximation, the Moon is spinning about the z-axis, which is oriented normal to its orbital plane. Moreover, the Moon spins in such a manner that the x-axis always points approximately in the direction of the Earth. Let η be the (small) angle subtended between the x-axis and the line joining the centers of the Moon and the Earth. A slight generalization of the analysis in Section 7.11 reveals that

$$\eta = \eta_o + \eta_p,$$

where $\eta_o = \delta\lambda$ and η_p are the Moon's optical and physical libration (in ecliptic longitude), respectively, $\delta\lambda$ is defined in Equation (10.109), and

$$\ddot{\eta}_p + n_0^2\,\eta_p = -n_0^2\,\eta_o.$$

Here, $n_0 = [3\,(\mathcal{I}_{yy} - \mathcal{I}_{xx})/\mathcal{I}_{zz}]^{1/2}\,n = 0.3446^\circ$ per day is the Moon's free libration rate, whereas $n = 13.1764^\circ$ per day is the lunar mean sidereal orbital angular velocity.

We can write

$$\delta\lambda \simeq \sum_{i=1,5} A_i\,\sin(n_i\,t - \gamma_i),$$

where the $i = 1$, 2, 3, 4, and 5 terms correspond to the major inequality, the reduction to the ecliptic, variation, evection, and the annual inequality, respectively. Furthermore, $A_1 = 22640''$, $A_2 = 418''$, $A_3 = 2370''$, $A_4 = 4586''$, $A_5 = 666''$, and $n_1 = n$, $n_2 = 2\,n_{\rm dr}$, $n_3 = 2\,n_{\rm sy}$, $n_4 = 2\,n_{\rm sy} - n_{\rm an}$, $n_5 = n'$. Here, $n_{\rm dr} = 13.2293^\circ$ per day, $n_{\rm sy} = 12.1908^\circ$ per day, and $n_{\rm an} = 13.0650^\circ$ per day are the lunar mean draconic, synodic, and anomalistic orbital angular velocities, respectively, and $n' = 0.9856^\circ$ per day is the Earth's mean sidereal orbital angular velocity. Demonstrate that

$$\eta_p \simeq \sum_{i=1,5} A_i'\,\sin(n_i\,t - \gamma_i),$$

where $A_1' = 15.7''$, $A_2' = 0.07''$, $A_3' = 0.48''$, $A_4' = 4.3''$, and $A_5' = 94.0''$ are the forced libration amplitudes associated with the major inequality, reduction to the ecliptic, variation, evection, and the annual inequality, respectively. [The observed values of A_1', A_3', A_4', and A_5' are 16.8″, 0.50″, 4.1″, and 90.7″, respectively. A_2' is too small to measure. (Meeus 2005).]