B Appendix B Derivation of Lagrange planetary equations

B.1 Introduction

Consider a planet of mass m and relative position vector $\mathbf{r}$ that is orbiting around the Sun, whose mass is M. The planet's equation of motion is written (see Section 3.16)

$$\ddot{\mathbf{r}} + \mu\,\frac{\mathbf{r}}{r^3} = \mathbf{0}, \tag{B.1}$$

where $\mu = G\,(M + m)$. As described in Chapter 3, the solution to this equation is a Keplerian ellipse whose properties are fully determined after six integrals of the motion, known as *orbital elements*, are specified.

Now, suppose that the aforementioned Keplerian orbit is slightly perturbed—for example, by the presence of a second planet orbiting the Sun. In this case, the planet's modified equation of motion takes the general form

$$\ddot{\mathbf{r}} + \mu\,\frac{\mathbf{r}}{r^3} = \nabla\mathcal{R}, \tag{B.2}$$

where $\mathcal{R}(\mathbf{r})$ is a so-called *disturbing function* that fully describes the perturbation. Adopting the standard Cartesian coordinate system X, Y, Z, described in Section 3.12, we see that the preceding equation yields

$$\ddot{X} + \mu\,\frac{X}{r^3} = \frac{\partial\mathcal{R}}{\partial X}, \tag{B.3}$$

$$\ddot{Y} + \mu\,\frac{Y}{r^3} = \frac{\partial\mathcal{R}}{\partial Y}, \tag{B.4}$$

and

$$\ddot{Z} + \mu\,\frac{Z}{r^3} = \frac{\partial\mathcal{R}}{\partial Z}, \tag{B.5}$$

where $r = (X^2 + Y^2 + Z^2)^{1/2}$.

If the right-hand sides of Equations (B.3)–(B.5) are set to zero (i.e., if there is no perturbation), we obtain a Keplerian orbit of the general form

$$X = f_1(c_1, c_2, c_3, c_4, c_5, c_6, t), \tag{B.6}$$

$$Y = f_2(c_1, c_2, c_3, c_4, c_5, c_6, t), \tag{B.7}$$

$$Z = f_3(c_1, c_2, c_3, c_4, c_5, c_6, t), \tag{B.8}$$

$$\dot{X} = g_1(c_1, c_2, c_3, c_4, c_5, c_6, t), \tag{B.9}$$

$$\dot{Y} = g_2(c_1, c_2, c_3, c_4, c_5, c_6, t), \tag{B.10}$$

and

$$\dot{Z} = g_3(c_1, c_2, c_3, c_4, c_5, c_6, t). \tag{B.11}$$

Here, $c_1, \ldots, c_6$ are the six constant elements that determine the orbit. (See Section 3.12.) It follows that

$$g_k = \frac{\partial f_k}{\partial t} \tag{B.12}$$

for $k = 1, 2, 3$.

Let us now take the right-hand sides of Equations (B.3)–(B.5) into account. In this case, the orbital elements, $c_1, \ldots, c_6$, are no longer constants of the motion. However, provided the perturbation is sufficiently small, we would expect the elements to be relatively *slowly varying* functions of time. The purpose of this appendix is to derive evolution equations for these so-called *osculating orbital elements*. Our approach is largely based on that of Brouwer and Clemence (1961).

B.2 Preliminary analysis

According to Equation (B.6), we have

$$\frac{dX}{dt} = \frac{\partial f_1}{\partial t} + \sum_{k=1,6} \frac{\partial f_1}{\partial c_k}\frac{dc_k}{dt}. \tag{B.13}$$

If this expression, and the analogous expressions for dY/dt and dZ/dt, were differentiated with respect to time, and the results substituted into Equations (B.3)–(B.5), then we would obtain three time evolution equations for the six variables $c_1, \ldots, c_6$. To make the problem definite, three additional conditions must be introduced into the problem. It is convenient to choose

$$\sum_{k=1,6} \frac{\partial f_l}{\partial c_k}\frac{dc_k}{dt} = 0 \tag{B.14}$$

for $l = 1, 2, 3$. Hence, it follows from Equation (B.12) and (B.13) that

$$\frac{dX}{dt} = \frac{\partial f_1}{\partial t} = g_1, \tag{B.15}$$

$$\frac{dY}{dt} = \frac{\partial f_2}{\partial t} = g_2, \tag{B.16}$$

and

$$\frac{dZ}{dt} = \frac{\partial f_3}{\partial t} = g_3. \tag{B.17}$$

Differentiation of the these equations with respect to time yields

$$\frac{d^2X}{dt^2} = \frac{\partial^2 f_1}{\partial t^2} + \sum_{k=1,6} \frac{\partial g_1}{\partial c_k}\frac{dc_k}{dt}, \tag{B.18}$$

$$\frac{d^2Y}{dt^2} = \frac{\partial^2 f_2}{\partial t^2} + \sum_{k=1,6} \frac{\partial g_2}{\partial c_k}\frac{dc_k}{dt}, \tag{B.19}$$

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and

$$\frac{d^2Z}{dt^2} = \frac{\partial^2 f_3}{\partial t^2} + \sum_{k=1,6} \frac{\partial g_3}{\partial c_k}\frac{dc_k}{dt}. \tag{B.20}$$

Substitution into Equations (B.3)–(B.5) gives

$$\frac{\partial^2 f_1}{\partial t^2} + \mu\frac{f_1}{r^3} + \sum_{k=1,6} \frac{\partial g_1}{\partial c_k}\frac{dc_k}{dt} = \frac{\partial \mathcal{R}}{\partial X}, \tag{B.21}$$

$$\frac{\partial^2 f_2}{\partial t^2} + \mu\frac{f_2}{r^3} + \sum_{k=1,6} \frac{\partial g_2}{\partial c_k}\frac{dc_k}{dt} = \frac{\partial \mathcal{R}}{\partial Y}, \tag{B.22}$$

and

$$\frac{\partial^2 f_3}{\partial t^2} + \mu\frac{f_3}{r^3} + \sum_{k=1,6} \frac{\partial g_3}{\partial c_k}\frac{dc_k}{dt} = \frac{\partial \mathcal{R}}{\partial Z}, \tag{B.23}$$

where $r = (f_1^{\,2} + f_2^{\,2} + f_3^{\,2})^{1/2}$. Because f_1, f_2, and f_3 are the respective solutions to Equation (B.3)–(B.5) when the right-hand sides are zero, and the orbital elements are thus constants, it follows that the first two terms in each of the preceding three equations cancel one another. Hence, writing f_1 as X, and g_1 as $\dot{X}$, and so on, we see that Equations (B.14) and (B.21)–(B.23) yield

$$\sum_{k=1,6} \frac{\partial X}{\partial c_k}\frac{dc_k}{dt} = 0, \tag{B.24}$$

$$\sum_{k=1,6} \frac{\partial Y}{\partial c_k}\frac{dc_k}{dt} = 0, \tag{B.25}$$

$$\sum_{k=1,6} \frac{\partial Z}{\partial c_k}\frac{dc_k}{dt} = 0, \tag{B.26}$$

$$\sum_{k=1,6} \frac{\partial \dot{X}}{\partial c_k}\frac{dc_k}{dt} = \frac{\partial \mathcal{R}}{\partial X}, \tag{B.27}$$

$$\sum_{k=1,6} \frac{\partial \dot{Y}}{\partial c_k}\frac{dc_k}{dt} = \frac{\partial \mathcal{R}}{\partial Y}, \tag{B.28}$$

and

$$\sum_{k=1,6} \frac{\partial \dot{Z}}{\partial c_k}\frac{dc_k}{dt} = \frac{\partial \mathcal{R}}{\partial Z}. \tag{B.29}$$

These six equations are equivalent to the three original equations of motion [(B.3)–(B.5)].

B.3 Lagrange brackets

Six new equations can be derived from Equations (B.24)–(B.29) by multiplying them successively by $-\partial\dot{X}/\partial c_j$, $-\partial\dot{Y}/\partial c_j$, $-\partial\dot{Z}/\partial c_j$, $\partial X/\partial c_j$, $\partial Y/\partial c_j$, and $\partial Z/\partial c_j$, and then

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summing the resulting equations. The right-hand sides of the new equations are

$$\frac{\partial\mathcal{R}}{\partial X}\frac{\partial X}{\partial c_j}+\frac{\partial\mathcal{R}}{\partial Y}\frac{\partial Y}{\partial c_j}+\frac{\partial\mathcal{R}}{\partial Z}\frac{\partial Z}{\partial c_j}\equiv\frac{\partial\mathcal{R}}{\partial c_j}. \tag{B.30}$$

The new equations can be written in a more compact form via the introduction of *Lagrange brackets*, which are defined as

$$[c_j,c_k]\equiv\sum_{l=1,3}\left(\frac{\partial X_l}{\partial c_j}\frac{\partial\dot{X}_l}{\partial c_k}-\frac{\partial X_l}{\partial c_k}\frac{\partial\dot{X}_l}{\partial c_j}\right), \tag{B.31}$$

where $X_1\equiv X$, $X_2\equiv Y$, and $X_3\equiv Z$. Thus, the new equations become

$$\sum_{k=1,6}[c_j,c_k]\frac{dc_k}{dt}=\frac{\partial\mathcal{R}}{\partial c_j} \tag{B.32}$$

for $j=1,6$. Note, incidentally, that

$$[c_j,c_j]=0 \tag{B.33}$$

and

$$[c_j,c_k]=-[c_k,c_j]. \tag{B.34}$$

Let

$$[p,q]=\sum_{l=1,3}\left(\frac{\partial X_l}{\partial p}\frac{\partial\dot{X}_l}{\partial q}-\frac{\partial X_l}{\partial q}\frac{\partial\dot{X}_l}{\partial p}\right), \tag{B.35}$$

where p and q are any two orbital elements. It follows that

$$\frac{\partial}{\partial t}[p,q]=\sum_{l=1,3}\left(\frac{\partial^2X_l}{\partial p\,\partial t}\frac{\partial\dot{X}_l}{\partial q}+\frac{\partial X_l}{\partial p}\frac{\partial^2\dot{X}_l}{\partial q\,\partial t}-\frac{\partial^2X_l}{\partial q\,\partial t}\frac{\partial\dot{X}_l}{\partial p}-\frac{\partial X_l}{\partial q}\frac{\partial^2\dot{X}_l}{\partial p\,\partial t}\right) \tag{B.36}$$

or

$$\frac{\partial}{\partial t}[p,q]=\sum_{l=1,3}\left[\frac{\partial}{\partial p}\left(\frac{\partial X_l}{\partial t}\frac{\partial\dot{X}_l}{\partial q}-\frac{\partial X_l}{\partial q}\frac{\partial\dot{X}_l}{\partial t}\right)-\frac{\partial}{\partial q}\left(\frac{\partial X_l}{\partial t}\frac{\partial\dot{X}_l}{\partial p}-\frac{\partial X_l}{\partial p}\frac{\partial\dot{X}_l}{\partial t}\right)\right]. \tag{B.37}$$

However, in the preceding expression, X_l and $\dot{X}_l$ stand for coordinates and velocities of Keplerian orbits calculated with $c_1,\ldots,c_6$ treated as constants. Thus, we can write $\partial X_l/\partial t\equiv\dot{X}_l$ and $\partial\dot{X}_l/\partial t\equiv\ddot{X}_l$, giving

$$\frac{\partial}{\partial t}[p,q]=\sum_{l=1,3}\left[\frac{\partial}{\partial p}\left(\frac{1}{2}\frac{\partial\dot{X}_l^2}{\partial q}-\frac{\partial F_0}{\partial X_l}\frac{\partial X_l}{\partial q}\right)-\frac{\partial}{\partial q}\left(\frac{1}{2}\frac{\partial\dot{X}_l^2}{\partial p}-\frac{\partial F_0}{\partial X_l}\frac{\partial X_l}{\partial p}\right)\right], \tag{B.38}$$

because

$$\ddot{X}_l=\frac{\partial F_0}{\partial X_l}, \tag{B.39}$$

where $F_0=\mu/r$. Equation (B.38) reduces to

$$\frac{\partial}{\partial t}[p,q]=\frac{1}{2}\frac{\partial^2v^2}{\partial p\,\partial q}-\frac{\partial^2F_0}{\partial p\,\partial q}-\frac{1}{2}\frac{\partial^2v^2}{\partial q\,\partial p}+\frac{\partial^2F_0}{\partial q\,\partial p}=0, \tag{B.40}$$

where $v^2=\sum_{l=1,3}\dot{X}_l^2$. Hence, we conclude that Lagrange brackets are functions of the osculating orbital elements, $c_1,\ldots,c_6$, but are not explicit functions of t. It follows that we can evaluate these brackets at any convenient point in the orbit.

B.4 Transformation of Lagrange brackets

The most common set of orbital elements used to parameterize Keplerian orbits consists of the *major radius*, a; the *mean longitude at epoch*, $\bar{\lambda}_0$; the *eccentricity*, e; the *inclination* (relative to some reference plane), I; the *longitude of the perihelion*, ϖ; and *the longitude of the ascending node*, Ω. (See Section 3.12.) The mean orbital angular velocity is $n = (\mu/a^3)^{1/2}$ [see Equation (3.116)].

Consider how a particular Lagrange bracket transforms under a rotation of the coordinate system X, Y, Z about the Z-axis (if we look along the axis). We can write

$$[p, q] = \frac{\partial(X, \dot{X})}{\partial(p, q)} + \frac{\partial(Y, \dot{Y})}{\partial(p, q)} + \frac{\partial(Z, \dot{Z})}{\partial(p, q)}, \tag{B.41}$$

where

$$\frac{\partial(a, b)}{\partial(c, d)} \equiv \frac{\partial a}{\partial c}\frac{\partial b}{\partial d} - \frac{\partial a}{\partial d}\frac{\partial b}{\partial c}. \tag{B.42}$$

Let the new coordinate system be x', y', z'. A rotation about the Z-axis through an angle Ω brings the ascending node to the x'-axis. (See Figure 3.6.) The relation between the old and new coordinates is (see Section A.6)

$$X = \cos\Omega\, x' - \sin\Omega\, y', \tag{B.43}$$

$$Y = \sin\Omega\, x' + \cos\Omega\, y', \tag{B.44}$$

and

$$Z = z'. \tag{B.45}$$

The partial derivatives with respect to p can be written

$$\frac{\partial X}{\partial p} = A_1\, \cos\Omega - B_1\, \sin\Omega, \tag{B.46}$$

$$\frac{\partial Y}{\partial p} = B_1\, \cos\Omega + A_1\, \sin\Omega, \tag{B.47}$$

$$\frac{\partial \dot{X}}{\partial p} = C_1\, \cos\Omega - D_1\, \sin\Omega, \tag{B.48}$$

and

$$\frac{\partial \dot{Y}}{\partial p} = D_1\, \cos\Omega + C_1\, \sin\Omega, \tag{B.49}$$

where

$$A_1 = \frac{\partial x'}{\partial p} - y'\,\frac{\partial \Omega}{\partial p}, \tag{B.50}$$

$$B_1 = \frac{\partial y'}{\partial p} + x'\,\frac{\partial \Omega}{\partial p}, \tag{B.51}$$

$$C_1 = \frac{\partial \dot{x}'}{\partial p} - \dot{y}'\,\frac{\partial \Omega}{\partial p}, \tag{B.52}$$

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and

$$D_1 = \frac{\partial \dot{y}'}{\partial p} + \dot{x}'\,\frac{\partial \Omega}{\partial p}. \tag{B.53}$$

Let A_2, B_2, C_2, and D_2 be the equivalent quantities obtained by replacing p by q in the above equations. It thus follows that

$$\begin{aligned}\frac{\partial(X,\dot{X})}{\partial(p,q)} &= (A_1\,C_2 - A_2\,C_1)\,\cos^2\Omega + (B_1\,D_2 - B_2\,D_1)\,\sin^2\Omega \\ &\quad + (-A_1\,D_2 - B_1\,C_2 + A_2\,D_1 + B_2\,C_1)\,\sin\Omega\,\cos\Omega,\end{aligned} \tag{B.54}$$

and

$$\begin{aligned}\frac{\partial(Y,\dot{Y})}{\partial(p,q)} &= (B_1\,D_2 - B_2\,D_1)\,\cos^2\Omega + (A_1\,C_2 - A_2\,C_1)\,\sin^2\Omega \\ &\quad + (A_1\,D_2 + B_1\,C_2 - A_2\,D_1 - B_2\,C_1)\,\sin\Omega\,\cos\Omega.\end{aligned} \tag{B.55}$$

Hence,

$$[p,q] = A_1\,C_2 - A_2\,C_1 + B_1\,D_2 - B_2\,D_1 + \frac{\partial(Z,\dot{Z})}{\partial(p,q)}. \tag{B.56}$$

Now,

$$\begin{aligned}A_1\,C_2 - A_2\,C_1 &= \left(\frac{\partial x'}{\partial p} - y'\,\frac{\partial \Omega}{\partial p}\right)\left(\frac{\partial \dot{x}'}{\partial q} - \dot{y}'\,\frac{\partial \Omega}{\partial q}\right) - \left(\frac{\partial x'}{\partial q} - y'\,\frac{\partial \Omega}{\partial q}\right)\left(\frac{\partial \dot{x}'}{\partial p} - \dot{y}'\,\frac{\partial \Omega}{\partial p}\right) \\ &= \frac{\partial(x',\dot{x}')}{\partial(p,q)} \\ &\quad + \left(-y'\,\frac{\partial \dot{x}'}{\partial q} + \dot{y}'\,\frac{\partial x'}{\partial q}\right)\frac{\partial \Omega}{\partial p} + \left(-\dot{y}'\,\frac{\partial x'}{\partial p} + y'\,\frac{\partial \dot{x}'}{\partial p}\right)\frac{\partial \Omega}{\partial q}.\end{aligned} \tag{B.57}$$

Similarly,

$$\begin{aligned}B_1\,D_2 - B_2\,D_1 &= \left(\frac{\partial y'}{\partial p} + x'\,\frac{\partial \Omega}{\partial p}\right)\left(\frac{\partial \dot{y}'}{\partial q} + \dot{x}'\,\frac{\partial \Omega}{\partial q}\right) - \left(\frac{\partial y'}{\partial q} + x'\,\frac{\partial \Omega}{\partial q}\right)\left(\frac{\partial \dot{y}'}{\partial p} + \dot{x}'\,\frac{\partial \Omega}{\partial p}\right) \\ &= \frac{\partial(y',\dot{y}')}{\partial(p,q)} + \left(x'\,\frac{\partial \dot{y}'}{\partial q} - \dot{x}'\,\frac{\partial y'}{\partial q}\right)\frac{\partial \Omega}{\partial p} + \left(\dot{x}'\,\frac{\partial y'}{\partial p} - x'\,\frac{\partial \dot{y}'}{\partial p}\right)\frac{\partial \Omega}{\partial q}.\end{aligned} \tag{B.58}$$

Let

$$[p,q]' = \frac{\partial(x',\dot{x}')}{\partial(p,q)} + \frac{\partial(y',\dot{y}')}{\partial(p,q)} + \frac{\partial(z',\dot{z}')}{\partial(p,q)}. \tag{B.59}$$

Because $Z = z'$ and $\dot{Z} = \dot{z}'$, it follows that

$$\begin{aligned}[p,q] &= [p,q]' + \left(x'\,\frac{\partial \dot{y}'}{\partial q} + \dot{y}'\,\frac{\partial x'}{\partial q} - y'\,\frac{\partial \dot{x}'}{\partial q} - \dot{x}'\,\frac{\partial y'}{\partial q}\right)\frac{\partial \Omega}{\partial p} \\ &\quad - \left(x'\,\frac{\partial \dot{y}'}{\partial p} + \dot{y}'\,\frac{\partial x'}{\partial p} - y'\,\frac{\partial \dot{x}'}{\partial p} - \dot{x}'\,\frac{\partial y'}{\partial p}\right)\frac{\partial \Omega}{\partial q} \\ &= [p,q]' + \frac{\partial(\Omega,\,x'\,\dot{y}' - y'\,\dot{x}')}{\partial(p,q)}.\end{aligned} \tag{B.60}$$

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However,

$$x'\,\dot{y}' - y'\,\dot{x}' = h\,\cos I = [\mu\,a\,(1-e^2)]^{1/2}\,\cos I \equiv \mathcal{G}, \tag{B.61}$$

because the left-hand side is the component of the angular momentum per unit mass parallel to the z'-axis. Of course, this axis is inclined at an angle I to the z-axis, which is parallel to the angular momentum vector. Thus, we obtain

$$[p,q] = [p,q]' + \frac{\partial(\Omega,\mathcal{G})}{\partial(p,q)}. \tag{B.62}$$

Consider a rotation of the coordinate system about the x'-axis. Let the new coordinate system be x'', y'', z''. A rotation through an angle I brings the orbit into the x''–y'' plane. (See Figure 3.6.) Let

$$[p,q]'' = \frac{\partial(x'',\dot{x}'')}{\partial(p,q)} + \frac{\partial(y'',\dot{y}'')}{\partial(p,q)} + \frac{\partial(z'',\dot{z}'')}{\partial(p,q)}. \tag{B.63}$$

By analogy with the previous analysis,

$$[p,q]' = [p,q]'' + \frac{\partial(I,\,y''\,\dot{z}'' - z''\,\dot{y}'')}{\partial(p,q)}. \tag{B.64}$$

However, z'' and $\dot{z}''$ are both zero, as the orbit lies in the x''–y'' plane. Hence,

$$[p,q]' = [p,q]''. \tag{B.65}$$

Consider, finally, a rotation of the coordinate system about the z''-axis. Let the final coordinate system be x, y, z. A rotation through an angle $\varpi - \Omega$ brings the perihelion to the x-axis. (See Figure 3.6.) Let

$$[p,q]''' = \frac{\partial(x,\dot{x})}{\partial(p,q)} + \frac{\partial(y,\dot{y})}{\partial(p,q)}. \tag{B.66}$$

By analogy with the previous analysis,

$$[p,q]'' = [p,q]''' + \frac{\partial(\varpi - \Omega,\,x\,\dot{y} - y\,\dot{x})}{\partial(p,q)}. \tag{B.67}$$

However,

$$x\,\dot{y} - y\,\dot{x} = h = [\mu\,a\,(1-e^2)]^{1/2} \equiv H, \tag{B.68}$$

so, from Equations (B.62) and (B.65),

$$[p,q] = [p,q]''' + \frac{\partial(\varpi - \Omega, H)}{\partial(p,q)} + \frac{\partial(\Omega,\mathcal{G})}{\partial(p,q)}. \tag{B.69}$$

It thus remains to calculate $[p,q]'''$.

The coordinates $x = r\,\cos\theta$ and $y = r\,\sin\theta$—where r represents radial distance from the Sun, and θ is the true anomaly—are functions of the major radius, a, the eccentricity, e, and the mean anomaly, $\mathcal{M} = \bar{\lambda}_0 - \varpi + n\,t$. Because the Lagrange brackets are independent of time, it is sufficient to evaluate them at $\mathcal{M} = 0$, that is, at the perihelion point. It is easily demonstrated from Equations (3.85) and (3.86) that

$$x = a\,(1-e) + \mathcal{O}(\mathcal{M}^2), \tag{B.70}$$

$$y = a\,\mathcal{M}\left(\frac{1+e}{1-e}\right)^{1/2} + \mathcal{O}(\mathcal{M}^3), \tag{B.71}$$

$$\dot{x} = -a\,n\,\frac{\mathcal{M}}{(1-e)^2} + \mathcal{O}(\mathcal{M}^3), \tag{B.72}$$

and

$$\dot{y} = a\,n\left(\frac{1+e}{1-e}\right)^{1/2} + \mathcal{O}(\mathcal{M}^2) \tag{B.73}$$

at small $\mathcal{M}$. Hence, at $\mathcal{M} = 0$,

$$\frac{\partial x}{\partial a} = 1-e, \tag{B.74}$$

$$\frac{\partial x}{\partial e} = -a, \tag{B.75}$$

$$\frac{\partial y}{\partial(\bar{\lambda}_0 - \varpi)} = a\left(\frac{1+e}{1-e}\right)^{1/2}, \tag{B.76}$$

$$\frac{\partial \dot{x}}{\partial(\bar{\lambda}_0 - \varpi)} = -\frac{a\,n}{(1-e)^2}, \tag{B.77}$$

$$\frac{\partial \dot{y}}{\partial a} = -\frac{n}{2}\left(\frac{1+e}{1-e}\right)^{1/2}, \tag{B.78}$$

and

$$\frac{\partial \dot{y}}{\partial e} = a\,n\,(1+e)^{-1/2}\,(1-e)^{-3/2}, \tag{B.79}$$

because $n \propto a^{-3/2}$. All other partial derivatives are zero. Because the orbit in the x, y, z coordinate system depends only on the elements a, e, and $\bar{\lambda}_0 - \varpi$, we can write

$$\begin{aligned}
[p,q]''' = {} & \frac{\partial(a,e)}{\partial(p,q)}\left[\frac{\partial(x,\dot{x})}{\partial(a,e)} + \frac{\partial(y,\dot{y})}{\partial(a,e)}\right] \\
& + \frac{\partial(e,\bar{\lambda}_0 - \varpi)}{\partial(p,q)}\left[\frac{\partial(x,\dot{x})}{\partial(e,\bar{\lambda}_0 - \varpi)} + \frac{\partial(y,\dot{y})}{\partial(e,\bar{\lambda}_0 - \varpi)}\right] \\
& + \frac{\partial(\bar{\lambda}_0 - \varpi, a)}{\partial(p,q)}\left[\frac{\partial(x,\dot{x})}{\partial(\bar{\lambda}_0 - \varpi, a)} + \frac{\partial(y,\dot{y})}{\partial(\bar{\lambda}_0 - \varpi, a)}\right].
\end{aligned} \tag{B.80}$$

Substitution of the values of the derivatives evaluated at $\mathcal{M} = 0$ into this expression yields

$$\frac{\partial(x,\dot{x})}{\partial(a,e)} + \frac{\partial(y,\dot{y})}{\partial(a,e)} = 0, \tag{B.81}$$

$$\frac{\partial(x,\dot{x})}{\partial(e,\bar{\lambda}_0 - \varpi)} + \frac{\partial(y,\dot{y})}{\partial(e,\bar{\lambda}_0 - \varpi)} = 0, \tag{B.82}$$

$$\frac{\partial(x,\dot{x})}{\partial(\bar{\lambda}_0 - \varpi, a)} + \frac{\partial(y,\dot{y})}{\partial(\bar{\lambda}_0 - \varpi, a)} = \frac{a\,n}{2}, \tag{B.83}$$

and

$$[p,q]''' = \frac{\partial(\bar{\lambda}_0-\varpi,a)}{\partial(p,q)}\,\frac{n\,a}{2} = \frac{\partial(\bar{\lambda}_0-\varpi,a)}{\partial(p,q)}\,\frac{\mu^{1/2}}{2\,a^{1/2}} = \frac{\partial(\bar{\lambda}_0-\varpi,L)}{\partial(p,q)}, \tag{B.84}$$

where $L=(\mu\,a)^{1/2}$. Hence, from Equation (B.69), we obtain

$$[p,q] = \frac{\partial(\bar{\lambda}_0-\varpi,L)}{\partial(p,q)} + \frac{\partial(\varpi-\Omega,H)}{\partial(p,q)} + \frac{\partial(\Omega,\mathcal{G})}{\partial(p,q)}. \tag{B.85}$$

B.5 Lagrange planetary equations

Now,

$$L = (\mu\,a)^{1/2}, \tag{B.86}$$

$$H = [\mu\,a\,(1-e^2)]^{1/2}, \tag{B.87}$$

$$\mathcal{G} = [\mu\,a\,(1-e^2)]^{1/2}\,\cos I, \tag{B.88}$$

and $n\,a=(\mu/a)^{1/2}$. Hence,

$$\frac{\partial L}{\partial a} = \frac{n\,a}{2}, \tag{B.89}$$

$$\frac{\partial H}{\partial a} = \frac{n\,a}{2}\,(1-e^2)^{1/2}, \tag{B.90}$$

$$\frac{\partial H}{\partial e} = -n\,a^2\,e\,(1-e^2)^{-1/2}, \tag{B.91}$$

$$\frac{\partial \mathcal{G}}{\partial a} = \frac{n\,a}{2}\,(1-e^2)^{1/2}\,\cos I, \tag{B.92}$$

$$\frac{\partial \mathcal{G}}{\partial e} = -n\,a^2\,e\,(1-e^2)^{-1/2}\,\cos I, \tag{B.93}$$

and

$$\frac{\partial \mathcal{G}}{\partial I} = -n\,a^2\,(1-e^2)^{1/2}\,\sin I, \tag{B.94}$$

with all other partial derivatives zero. Thus, from Equation (B.85), the only nonzero Lagrange brackets are

$$[\bar{\lambda}_0,a] = -[a,\bar{\lambda}_0] = \frac{n\,a}{2}, \tag{B.95}$$

$$[\varpi,a] = -[a,\varpi] = -\frac{n\,a}{2}\,[1-(1-e^2)^{1/2}], \tag{B.96}$$

$$[\Omega,a] = -[a,\Omega] = -\frac{n\,a}{2}\,(1-e^2)^{1/2}\,(1-\cos I), \tag{B.97}$$

$$[\varpi,e] = -[e,\varpi] = -n\,a^2\,e\,(1-e^2)^{-1/2}, \tag{B.98}$$

$$[\Omega,e] = -[e,\Omega] = n\,a^2\,e\,(1-e^2)^{-1/2}\,(1-\cos I), \tag{B.99}$$

and

$$[\Omega, I] = -[I, \Omega] = -n\,a^2\,(1-e^2)^{1/2}\,\sin I. \tag{B.100}$$

Hence, Equations (B.32) yield

$$\begin{aligned}
[a, \bar{\lambda}_0]\,\frac{d\bar{\lambda}_0}{dt} + [a, \varpi]\,\frac{d\varpi}{dt} + [a, \Omega]\,\frac{d\Omega}{dt} &= \frac{\partial\mathcal{R}}{\partial a}, && \text{(B.101)}\\
[e, \varpi]\,\frac{d\varpi}{dt} + [e, \Omega]\,\frac{d\Omega}{dt} &= \frac{\partial\mathcal{R}}{\partial e}, && \text{(B.102)}\\
[\bar{\lambda}_0, a]\,\frac{da}{dt} &= \frac{\partial\mathcal{R}}{\partial\bar{\lambda}_0}, && \text{(B.103)}\\
[I, \Omega]\,\frac{d\Omega}{dt} &= \frac{\partial\mathcal{R}}{\partial I}, && \text{(B.104)}\\
[\Omega, a]\,\frac{da}{dt} + [\Omega, e]\,\frac{de}{dt} + [\Omega, I]\,\frac{dI}{dt} &= \frac{\partial\mathcal{R}}{\partial\Omega}, && \text{(B.105)}
\end{aligned}$$

and

$$[\varpi, a]\,\frac{da}{dt} + [\varpi, e]\,\frac{de}{dt} = \frac{\partial\mathcal{R}}{\partial\varpi}. \tag{B.106}$$

Finally, Equations (B.95)–(B.106) can be rearranged to give

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{n\,a}\,\frac{\partial\mathcal{R}}{\partial\bar{\lambda}_0}, && \text{(B.107)}\\
\frac{d\bar{\lambda}_0}{dt} &= -\frac{2}{n\,a}\,\frac{\partial\mathcal{R}}{\partial a} + \frac{(1-e^2)^{1/2}\,[1-(1-e^2)^{1/2}]}{n\,a^2\,e}\,\frac{\partial\mathcal{R}}{\partial e}\\
&\quad + \frac{\tan(I/2)}{n\,a^2\,(1-e^2)^{1/2}}\,\frac{\partial\mathcal{R}}{\partial I}, && \text{(B.108)}\\
\frac{de}{dt} &= -\frac{(1-e^2)^{1/2}}{n\,a^2\,e}\,[1-(1-e^2)^{1/2}]\,\frac{\partial\mathcal{R}}{\partial\bar{\lambda}_0} - \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\,\frac{\partial\mathcal{R}}{\partial\varpi}, && \text{(B.109)}\\
\frac{dI}{dt} &= -\frac{\tan(I/2)}{n\,a^2\,(1-e^2)^{1/2}}\left(\frac{\partial\mathcal{R}}{\partial\bar{\lambda}_0} + \frac{\partial\mathcal{R}}{\partial\varpi}\right) - \frac{(1-e^2)^{-1/2}}{n\,a^2\,\sin I}\,\frac{\partial\mathcal{R}}{\partial\Omega}, && \text{(B.110)}\\
\frac{d\varpi}{dt} &= \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\,\frac{\partial\mathcal{R}}{\partial e} + \frac{\tan(I/2)}{n\,a^2\,(1-e^2)^{1/2}}\,\frac{\partial\mathcal{R}}{\partial I}, && \text{(B.111)}
\end{aligned}$$

and

$$\frac{d\Omega}{dt} = \frac{1}{n\,a^2\,(1-e^2)^{1/2}\,\sin I}\,\frac{\partial\mathcal{R}}{\partial I}. \tag{B.112}$$

Equations (B.107)–(B.112), which specify the time evolution of the osculating orbital elements of our planet under the action of the disturbing function, are known collectively as the *Lagrange planetary equations* (Brouwer and Clemence 1961).

In fact, the orbital element $\bar{\lambda}_0$ always appears in the disturbing function in the combination $\bar{\lambda}_0 + \int_0^t n(t')\,dt'$. This combination is known as the *mean longitude* and is denoted $\bar{\lambda}$. It follows that

$$\frac{\partial \mathcal{R}}{\partial \bar{\lambda}_0} = \frac{\partial \mathcal{R}}{\partial \bar{\lambda}}, \tag{B.113}$$

$$\frac{\partial \mathcal{R}}{\partial a} = \frac{\partial \mathcal{R}}{\partial a} + \frac{\partial \mathcal{R}}{\partial \bar{\lambda}} \int_0^t \frac{dn}{da}\,dt'. \tag{B.114}$$

The integral appearing in the previous equation is problematic. Fortunately, it can easily be eliminated by replacing the variable $\bar{\lambda}_0$ by $\bar{\lambda}$. In this case, the Lagrange planetary equations become

$$\frac{da}{dt} = \frac{2}{n\,a}\,\frac{\partial \mathcal{R}}{\partial \bar{\lambda}}, \tag{B.115}$$

$$\begin{aligned}\frac{d\bar{\lambda}}{dt} &= n - \frac{2}{n\,a}\,\frac{\partial \mathcal{R}}{\partial a} + \frac{(1-e^2)^{1/2}\,[1-(1-e^2)^{1/2}]}{n\,a^2\,e}\,\frac{\partial \mathcal{R}}{\partial e} \\ &\quad + \frac{\tan(I/2)}{n\,a^2\,(1-e^2)^{1/2}}\,\frac{\partial \mathcal{R}}{\partial I},\end{aligned} \tag{B.116}$$

$$\frac{de}{dt} = -\frac{(1-e^2)^{1/2}}{n\,a^2\,e}\,[1-(1-e^2)^{1/2}]\,\frac{\partial \mathcal{R}}{\partial \bar{\lambda}} - \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\,\frac{\partial \mathcal{R}}{\partial \varpi}, \tag{B.117}$$

$$\frac{dI}{dt} = -\frac{\tan(I/2)}{n\,a^2\,(1-e^2)^{1/2}}\left(\frac{\partial \mathcal{R}}{\partial \bar{\lambda}} + \frac{\partial \mathcal{R}}{\partial \varpi}\right) - \frac{(1-e^2)^{-1/2}}{n\,a^2\,\sin I}\,\frac{\partial \mathcal{R}}{\partial \Omega}, \tag{B.118}$$

$$\frac{d\varpi}{dt} = \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\,\frac{\partial \mathcal{R}}{\partial e} + \frac{\tan(I/2)}{n\,a^2\,(1-e^2)^{1/2}}\,\frac{\partial \mathcal{R}}{\partial I}, \tag{B.119}$$

and

$$\frac{d\Omega}{dt} = \frac{1}{n\,a^2\,(1-e^2)^{1/2}\,\sin I}\,\frac{\partial \mathcal{R}}{\partial I}, \tag{B.120}$$

where $\partial/\partial\bar{\lambda}$ is taken at constant a, and $\partial/\partial a$ at constant $\bar{\lambda}$ (Brouwer and Clemence 1961).

B.6 Alternative forms of Lagrange planetary equations

It can be seen, from Equations (B.115)–(B.120), that in the limit of small eccentricity, e, and small inclination, I, certain terms on the right-hand sides of the Lagrange planetary equations become singular. This problem can be alleviated by defining the alternative orbital elements,

$$h = e\,\sin\varpi, \tag{B.121}$$

$$k = e\,\cos\varpi, \tag{B.122}$$

$$p = \sin I\,\sin\Omega, \tag{B.123}$$

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and

$$q = \sin I \, \cos \Omega. \tag{B.124}$$

If we write the Lagrange planetary equations in terms of these new elements, we obtain

$$\frac{da}{dt} = \frac{2}{n\,a}\,\frac{\partial \mathcal{R}}{\partial \bar{\lambda}}, \tag{B.125}$$

$$\begin{aligned}\frac{d\bar{\lambda}}{dt} &= n - \frac{2}{n\,a}\,\frac{\partial \mathcal{R}}{\partial a} + \frac{(1-e^2)^{1/2}}{n\,a^2\,[1+(1-e^2)^{1/2}]}\left(h\,\frac{\partial \mathcal{R}}{\partial h} + k\,\frac{\partial \mathcal{R}}{\partial k}\right)\\ &\quad + \frac{\cos I}{2\,n\,a^2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\left(p\,\frac{\partial \mathcal{R}}{\partial p} + q\,\frac{\partial \mathcal{R}}{\partial q}\right),\end{aligned} \tag{B.126}$$

$$\begin{aligned}\frac{dh}{dt} &= -\frac{(1-e^2)^{1/2}}{n\,a^2\,[1+(1-e^2)^{1/2}]}\,h\,\frac{\partial \mathcal{R}}{\partial \bar{\lambda}} + \frac{(1-e^2)^{1/2}}{n\,a^2}\,\frac{\partial \mathcal{R}}{\partial k}\\ &\quad + \frac{\cos I}{2\,n\,a^2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,k\left(p\,\frac{\partial \mathcal{R}}{\partial p} + q\,\frac{\partial \mathcal{R}}{\partial q}\right),\end{aligned} \tag{B.127}$$

$$\begin{aligned}\frac{dk}{dt} &= -\frac{(1-e^2)^{1/2}}{n\,a^2\,[1+(1-e^2)^{1/2}]}\,k\,\frac{\partial \mathcal{R}}{\partial \bar{\lambda}} - \frac{(1-e^2)^{1/2}}{n\,a^2}\,\frac{\partial \mathcal{R}}{\partial h}\\ &\quad - \frac{\cos I}{2\,n\,a^2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,h\left(p\,\frac{\partial \mathcal{R}}{\partial p} + q\,\frac{\partial \mathcal{R}}{\partial q}\right),\end{aligned} \tag{B.128}$$

$$\begin{aligned}\frac{dp}{dt} &= -\frac{\cos I}{2\,n\,a^2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,p\left(\frac{\partial \mathcal{R}}{\partial \bar{\lambda}} + k\,\frac{\partial \mathcal{R}}{\partial h} - h\,\frac{\partial \mathcal{R}}{\partial k}\right)\\ &\quad + \frac{\cos I}{n\,a^2\,(1-e^2)^{1/2}}\,\frac{\partial \mathcal{R}}{\partial q},\end{aligned} \tag{B.129}$$

and

$$\begin{aligned}\frac{dq}{dt} &= -\frac{\cos I}{2\,n\,a^2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,q\left(\frac{\partial \mathcal{R}}{\partial \bar{\lambda}} + k\,\frac{\partial \mathcal{R}}{\partial h} - h\,\frac{\partial \mathcal{R}}{\partial k}\right)\\ &\quad - \frac{\cos I}{n\,a^2\,(1-e^2)^{1/2}}\,\frac{\partial \mathcal{R}}{\partial p}.\end{aligned} \tag{B.130}$$

Note that the new equations now contain no singular terms in the limit $e, I \to 0$.

It is sometimes convenient to write the Lagrange planetary equations in terms of the *mean anomaly*, $\mathcal{M} = \bar{\lambda} - \varpi$, and the *argument of the perigee*, $\omega = \varpi - \Omega$, rather than $\bar{\lambda}$ and ω. Making the appropriate substitutions, we see that the equations take the form (Brouwer and Clemence 1961)

$$\frac{da}{dt} = \frac{2}{n\,a}\,\frac{\partial \mathcal{R}}{\partial \mathcal{M}}, \tag{B.131}$$

$$\frac{d\mathcal{M}}{dt} = n - \frac{2}{n\,a}\,\frac{\partial \mathcal{R}}{\partial a} - \frac{1-e^2}{n\,a^2\,e}\,\frac{\partial \mathcal{R}}{\partial e}, \tag{B.132}$$

$$\frac{de}{dt} = \frac{1-e^2}{n\,a^2\,e}\,\frac{\partial\mathcal{R}}{\partial\mathcal{M}} - \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\,\frac{\partial\mathcal{R}}{\partial\omega}, \tag{B.133}$$

$$\frac{dI}{dt} = \frac{\cot I}{n\,a^2\,(1-e^2)^{1/2}}\,\frac{\partial\mathcal{R}}{\partial\omega} - \frac{(1-e^2)^{-1/2}}{n\,a^2\,\sin I}\,\frac{\partial\mathcal{R}}{\partial\Omega}, \tag{B.134}$$

$$\frac{d\omega}{dt} = \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\,\frac{\partial\mathcal{R}}{\partial e} - \frac{\cot I}{n\,a^2\,(1-e^2)^{1/2}}\,\frac{\partial\mathcal{R}}{\partial I}, \tag{B.135}$$

and

$$\frac{d\Omega}{dt} = \frac{(1-e^2)^{-1/2}}{n\,a^2\,\sin I}\,\frac{\partial\mathcal{R}}{\partial I}. \tag{B.136}$$