2 Newtonian gravity

2.1 Introduction

Classical gravity, which is invariably the dominant force in celestial dynamic systems, was first correctly described in Newton's *Principia*. According to Newton, any two point objects exert a gravitational force of attraction on each other. This force is directed along the straight line joining the objects, is directly proportional to the product of their masses, and is inversely proportional to the square of the distance between them. Consider two point objects of mass m_1 and m_2 that are located at position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. The gravitational force $\mathbf{f}_{12}$ that mass m_2 exerts on mass m_1 is written

$$\mathbf{f}_{12} = G\, m_1\, m_2\, \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{2.1}$$

The gravitational force $\mathbf{f}_{21}$ that mass m_1 exerts on mass m_2 is equal and opposite: $\mathbf{f}_{21} = -\mathbf{f}_{12}$. (See Figure 1.3.) Here, the constant of proportionality, G, is called the *universal gravitational constant* and takes the value (Yoder 1995)

$$G = 6.673 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}. \tag{2.2}$$

Incidentally, there is something rather curious about Equation (2.1). According to this law, the gravitational force acting on a given object is directly proportional to that object's inertial mass. Why, though, should inertia be related to the force of gravity? After all, inertia measures the reluctance of a given body to deviate from its preferred state of uniform motion in a straight line, in response to some external force. What does this have to do with gravitational attraction? This question perplexed physicists for many years; it was answered only when Albert Einstein published his general theory of relativity in 1916. According to Einstein, inertial mass acts as a sort of gravitational charge, as it is impossible to distinguish an acceleration produced by a gravitational field from an apparent acceleration generated by observing motion in a noninertial reference frame. The assumption that these two types of acceleration are indistinguishable leads directly to all the strange predictions of general relativity, such that clocks in different gravitational potentials run at different rates, mass bends space, and so on.

2.2 Gravitational potential

Consider two point masses, m and m', located at position vectors $\mathbf{r}$ and $\mathbf{r}'$, respectively. According to the preceding analysis, the acceleration $\mathbf{g}$ of mass m as a result of the

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gravitational force exerted on it by mass m' takes the form

$$\mathbf{g} = G\,m'\,\frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3}. \tag{2.3}$$

The x-component of this acceleration is written

$$g_x = G\,m'\,\frac{x' - x}{[(x' - x)^2 + (y' - y)^2 + (z' - z)^2]^{3/2}}, \tag{2.4}$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$. However, as is easily demonstrated,

$$\frac{x' - x}{[(x' - x)^2 + (y' - y)^2 + (z' - z)^2]^{3/2}} \equiv \frac{\partial}{\partial x}\left\{\frac{1}{[(x' - x)^2 + (y' - y)^2 + (z' - z)^2]^{1/2}}\right\}. \tag{2.5}$$

Hence,

$$g_x = G\,m'\,\frac{\partial}{\partial x}\left(\frac{1}{|\mathbf{r}' - \mathbf{r}|}\right), \tag{2.6}$$

with analogous expressions for g_y and g_z. It follows that

$$\mathbf{g} = -\nabla\Phi, \tag{2.7}$$

where

$$\Phi(\mathbf{r}) = -\frac{G\,m'}{|\mathbf{r}' - \mathbf{r}|} \tag{2.8}$$

is termed the *gravitational potential*. Of course, we can write $\mathbf{g}$ in the form of Equation (2.7) only because gravity is a *conservative* force. (See Section 1.4.)

It is well known that gravity is a *superposable* force. In other words, the gravitational force exerted on some point mass by a collection of other point masses is simply the vector sum of the forces exerted on the former mass by each of the latter masses taken in isolation. It follows that the gravitational potential generated by a collection of point masses at a certain location in space is the sum of the potentials generated at that location by each point mass taken in isolation. Hence, using Equation (2.8), if there are N point masses, m_i (for $i = 1, N$), located at position vectors $\mathbf{r}_i$, then the gravitational potential generated at position vector $\mathbf{r}$ is simply

$$\Phi(\mathbf{r}) = -G \sum_{i=1,N} \frac{m_i}{|\mathbf{r}_i - \mathbf{r}|}. \tag{2.9}$$

Suppose, finally, that instead of having a collection of point masses, we have a *continuous* mass distribution. In other words, let the mass at position vector $\mathbf{r}'$ be $\rho(\mathbf{r}')\,d^3\mathbf{r}'$, where $\rho(\mathbf{r}')$ is the local mass density, and $d^3\mathbf{r}'$ a volume element. Summing over all space, and taking the limit $d^3\mathbf{r}' \to 0$, we find Equation (2.9) yields

$$\Phi(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\,d^3\mathbf{r}', \tag{2.10}$$

where the integral is taken over all space. This is the general expression for the gravitational potential, $\Phi(\mathbf{r})$, generated by a continuous mass distribution, $\rho(\mathbf{r})$.

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2.3 Gravitational potential energy

Consider a collection of N point masses m_i located at position vectors $\mathbf{r}_i$ (where i runs from 1 to N). What is the gravitational potential energy stored in such a collection? In other words, how much work would we have to do to assemble the masses, starting from an initial state in which they are all at rest and very widely separated?

We have seen that a gravitational acceleration field can be expressed in terms of a gravitational potential:

$$\mathbf{g}(\mathbf{r}) = -\nabla\Phi. \tag{2.11}$$

We also know that the gravitational force acting on a mass m located at position $\mathbf{r}$ is written

$$\mathbf{f}(\mathbf{r}) = m\,\mathbf{g}(\mathbf{r}). \tag{2.12}$$

The work we would have to do against the gravitational force to *slowly* move the mass from point P to point Q is simply

$$U = -\int_P^Q \mathbf{f}\cdot d\mathbf{r} = -m\int_P^Q \mathbf{g}\cdot d\mathbf{r} = m\int_P^Q \nabla\Phi\cdot d\mathbf{r} = m\,[\Phi(Q) - \Phi(P)]\,. \tag{2.13}$$

The negative sign in the preceding expression comes about because we would have to exert a force $-\mathbf{f}$ on the mass to counteract the force exerted by the gravitational field. Recall, finally, that the gravitational potential generated by a point mass m' located at position $\mathbf{r}'$ is

$$\Phi(\mathbf{r}) = -\frac{G\,m'}{|\mathbf{r}' - \mathbf{r}|}. \tag{2.14}$$

Let us build up our collection of masses one by one. It takes no work to bring the first mass from infinity, as there is no gravitational field to fight against. Let us clamp this mass in position at $\mathbf{r}_1$. To bring the second mass into position at $\mathbf{r}_2$, we have to do work against the gravitational field generated by the first mass. According to Equations (2.13) and (2.14), this work is given by

$$U_2 = -\frac{G\,m_2\,m_1}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{2.15}$$

Let us now bring the third mass into position. Because gravitational fields and gravitational potentials are superposable, the work done while moving the third mass from infinity to $\mathbf{r}_3$ is simply the sum of the works done against the gravitational fields generated by masses 1 and 2 taken in isolation:

$$U_3 = -\frac{G\,m_3\,m_1}{|\mathbf{r}_1 - \mathbf{r}_3|} - \frac{G\,m_3\,m_2}{|\mathbf{r}_2 - \mathbf{r}_3|}. \tag{2.16}$$

Thus, the total work done in assembling the arrangement of three masses is given by

$$U = -\frac{G\,m_2\,m_1}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{G\,m_3\,m_1}{|\mathbf{r}_1 - \mathbf{r}_3|} - \frac{G\,m_3\,m_2}{|\mathbf{r}_2 - \mathbf{r}_3|}. \tag{2.17}$$

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This result can easily be generalized to an arrangement of N point masses, giving

$$U = -\sum_{i,j=1,N}^{j<i} \frac{G\, m_i\, m_j}{|\mathbf{r}_j - \mathbf{r}_i|}. \tag{2.18}$$

The restriction that j must be less than i makes the preceding summation rather messy. If we were to sum without restriction (other than $j \neq i$), then each pair of masses would be counted twice. It is convenient to do just this, and then to divide the result by two. Thus, we obtain

$$U = -\frac{1}{2} \sum_{i,j=1,N}^{j\neq i} \frac{G\, m_i\, m_j}{|\mathbf{r}_j - \mathbf{r}_i|}. \tag{2.19}$$

This is the *potential energy* of an arrangement of point masses. We can think of this quantity as the work required to bring the masses from infinity and assemble them in the required formation. The fact that the work is negative implies that we would gain energy during this process.

Equation (2.19) can be written

$$U = \frac{1}{2} \sum_{i=1,N} m_i\, \Phi_i, \tag{2.20}$$

where

$$\Phi_i = -G \sum_{j=1,N}^{j\neq i} \frac{m_j}{|\mathbf{r}_j - \mathbf{r}_i|} \tag{2.21}$$

is the gravitational potential experienced by the ith mass due to the other masses in the distribution. For the case of a continuous mass distribution, we can generalize the preceding result to give

$$U = \frac{1}{2} \int \rho(\mathbf{r})\, \Phi(\mathbf{r})\, d^3\mathbf{r}, \tag{2.22}$$

where

$$\Phi(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\, d^3\mathbf{r}' \tag{2.23}$$

is the familiar gravitational potential generated by a continuous mass distribution of mass density $\rho(\mathbf{r})$.

2.4 Axially symmetric mass distributions

At this point, it is convenient to adopt standard spherical coordinates, r, θ, ϕ, aligned along the z-axis. These coordinates are related to regular Cartesian coordinates as follows (see Section A.8):

$$x = r \sin\theta \cos\phi, \tag{2.24}$$

$$y = r \sin\theta \sin\phi, \tag{2.25}$$

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and

$$z = r\,\cos\theta. \tag{2.26}$$

Consider an *axially symmetric* mass distribution, that is, a $\rho(\mathbf{r})$ that is *independent* of the azimuthal angle, ϕ. We would expect such a mass distribution to generate an axially symmetric gravitational potential, $\Phi(r,\theta)$. Hence, without loss of generality, we can set $\phi = 0$ when evaluating $\Phi(\mathbf{r})$ from Equation (2.10). In fact, given that $d^3\mathbf{r}' = r'^2\,\sin\theta'\,dr'\,d\theta'\,d\phi'$ in spherical coordinates, this equation yields

$$\Phi(r,\theta) = -G\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{r'^2\,\rho(r',\theta')\,\sin\theta'}{|\mathbf{r}-\mathbf{r}'|}\,d\phi'\,d\theta'\,dr', \tag{2.27}$$

with the right-hand side evaluated at $\phi = 0$. However, because $\rho(r',\theta')$ is independent of ϕ', Equation (2.27) can also be written

$$\Phi(r,\theta) = -2\pi\,G\int_0^\infty\int_0^\pi r'^2\,\rho(r',\theta')\,\sin\theta'\,\langle|\mathbf{r}-\mathbf{r}'|^{-1}\rangle\,d\theta'\,dr', \tag{2.28}$$

where $\langle\cdots\rangle \equiv \oint(\cdots)\,d\phi'/2\pi$ denotes an average over the azimuthal angle, ϕ'.

Now,

$$|\mathbf{r}'-\mathbf{r}|^{-1} = (r^2 - 2\,\mathbf{r}\cdot\mathbf{r}' + r'^2)^{-1/2} \tag{2.29}$$

and

$$\mathbf{r}\cdot\mathbf{r}' = r\,r'\,F, \tag{2.30}$$

where (at $\phi = 0$)

$$F = \sin\theta\,\sin\theta'\,\cos\phi' + \cos\theta\,\cos\theta'. \tag{2.31}$$

Hence,

$$|\mathbf{r}'-\mathbf{r}|^{-1} = (r^2 - 2\,r\,r'\,F + r'^2)^{-1/2}. \tag{2.32}$$

Suppose that $r > r'$. In this case, we can expand $|\mathbf{r}'-\mathbf{r}|^{-1}$ as a convergent power series in r'/r to give

$$|\mathbf{r}'-\mathbf{r}|^{-1} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)F + \frac{1}{2}\left(\frac{r'}{r}\right)^2(3\,F^2 - 1) + \mathcal{O}\left(\frac{r'}{r}\right)^3\right]. \tag{2.33}$$

Let us now average this expression over the azimuthal angle, ϕ'. Because $\langle 1\rangle = 1$, $\langle\cos\phi'\rangle = 0$, and $\langle\cos^2\phi'\rangle = 1/2$, it is easily seen that

$$\langle F\rangle = \cos\theta\,\cos\theta', \tag{2.34}$$

and

$$\begin{aligned}\langle F^2\rangle &= \frac{1}{2}\,\sin^2\theta\,\sin^2\theta' + \cos^2\theta\,\cos^2\theta'\\ &= \frac{1}{3} + \frac{2}{3}\left(\frac{3}{2}\,\cos^2\theta - \frac{1}{2}\right)\left(\frac{3}{2}\,\cos^2\theta' - \frac{1}{2}\right).\end{aligned} \tag{2.35}$$

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Hence,

$$\left\langle |\mathbf{r}' - \mathbf{r}|^{-1} \right\rangle = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta\,\cos\theta' + \left(\frac{r'}{r}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) + \mathcal{O}\left(\frac{r'}{r}\right)^3\right]. \tag{2.36}$$

The well-known *Legendre polynomials*, $P_n(x)$, are defined (Abramowitz and Stegun 1965) as

$$P_n(x) = \frac{1}{2^n\,n!}\,\frac{d^n}{dx^n}\left[(x^2-1)^n\right], \tag{2.37}$$

for $n = 0, \infty$. It follows that

$$P_0(x) = 1, \tag{2.38}$$

$$P_1(x) = x, \tag{2.39}$$

$$P_2(x) = \frac{1}{2}\,(3\,x^2 - 1), \tag{2.40}$$

$$P_3(x) = \frac{1}{2}\,(5\,x^3 - x), \tag{2.41}$$

and so on. The Legendre polynomials are *mutually orthogonal*:

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \int_0^{\pi} P_n(\cos\theta)\,P_m(\cos\theta)\,\sin\theta\,d\theta = \frac{\delta_{nm}}{n+1/2} \tag{2.42}$$

(Abramowitz and Stegun 1965). Here, δ_{nm} is 1 if $n = m$, and 0 otherwise. The Legendre polynomials also form a *complete set*: any function of x that is well behaved in the interval $-1 \leq x \leq 1$ can be represented as a weighted sum of $P_n(x)$. Likewise, any function of θ that is well behaved in the interval $0 \leq \theta \leq \pi$ can be represented as a weighted sum of $P_n(\cos\theta)$.

A comparison of Equation (2.36) and Equations (2.38)–(2.40) makes it reasonably clear that when $r > r'$, the complete expansion of $\langle|\mathbf{r}' - \mathbf{r}|^{-1}\rangle$ is

$$\left\langle |\mathbf{r}' - \mathbf{r}|^{-1} \right\rangle = \frac{1}{r}\sum_{n=0,\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)\,P_n(\cos\theta'). \tag{2.43}$$

Similarly, when $r < r'$, we can expand in powers of r/r' to obtain

$$\left\langle |\mathbf{r}' - \mathbf{r}|^{-1} \right\rangle = \frac{1}{r'}\sum_{n=0,\infty}\left(\frac{r}{r'}\right)^n P_n(\cos\theta)\,P_n(\cos\theta'). \tag{2.44}$$

It follows from Equations (2.28), (2.43), and (2.44) that

$$\Phi(r,\theta) = \sum_{n=0,\infty}\Phi_n(r)\,P_n(\cos\theta), \tag{2.45}$$

where

$$\Phi_n(r) = -\frac{2\pi\,G}{r^{n+1}}\int_0^r\int_0^{\pi} r'^{\,n+2}\rho(r',\theta')\,P_n(\cos\theta')\,\sin\theta'\,d\theta'\,dr' - 2\pi\,G\,r^n\int_r^{\infty}\int_0^{\pi} r'^{\,1-n}\rho(r',\theta')\,P_n(\cos\theta')\,\sin\theta'\,d\theta'\,dr'. \tag{2.46}$$

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Given that $P_n(\cos\theta)$ form a complete set, we can always write

$$\rho(r,\theta) = \sum_{n=0,\infty} \rho_n(r)\, P_n(\cos\theta). \tag{2.47}$$

This expression can be inverted, with the aid of Equation (2.42), to give

$$\rho_n(r) = (n+1/2)\int_0^\pi \rho(r,\theta)\, P_n(\cos\theta)\,\sin\theta\, d\theta. \tag{2.48}$$

Hence, Equation (2.46) reduces to

$$\Phi_n(r) = -\frac{2\pi\, G}{(n+1/2)\, r^{n+1}}\int_0^r r'^{\,n+2}\,\rho_n(r')\, dr' - \frac{2\pi\, G\, r^n}{n+1/2}\int_r^\infty r'^{\,1-n}\,\rho_n(r')\, dr'. \tag{2.49}$$

Thus, we now have a general expression for the gravitational potential, $\Phi(r,\theta)$, generated by an axially symmetric mass distribution, $\rho(r,\theta)$.

2.5 Potential due to a uniform sphere

Let us calculate the gravitational potential generated by a sphere of *uniform* mass density γ and radius R, whose center coincides with the origin. Expressing $\rho(r,\theta)$ in the form of Equation (2.47), we find it clear that

$$\rho_0(r) = \begin{cases} \gamma & \text{for } r \le R \\ 0 & \text{for } r > R \end{cases} \tag{2.50}$$

with $\rho_n(r) = 0$ for $n > 0$. Thus, from Equation (2.49),

$$\Phi_0(r) = -\frac{4\pi\, G\,\gamma}{r}\int_0^r r'^{\,2}\, dr' - 4\pi\, G\,\gamma\int_r^R r'\, dr' \tag{2.51}$$

for $r \le R$, and

$$\Phi_0(r) = -\frac{4\pi\, G\,\gamma}{r}\int_0^R r'^{\,2}\, dr' \tag{2.52}$$

for $r > R$, with $\Phi_n(r) = 0$ for $n > 0$. Hence,

$$\Phi(r) = -\frac{2\pi\, G\,\gamma}{3}\,(3\, R^2 - r^2) = -G\, M\,\frac{(3\, R^2 - r^2)}{2\, R^3} \tag{2.53}$$

for $r \le R$, and

$$\Phi(r) = -\frac{4\pi\, G\,\gamma}{3}\,\frac{R^3}{r} = -\frac{G\, M}{r} \tag{2.54}$$

for $r > R$. Here, $M = (4\pi/3)\, R^3\,\gamma$ is the total mass of the sphere.

According to Equation (2.54), the gravitational potential outside a uniform sphere of mass M is the same as that generated by a point mass M located at the sphere's center. It turns out that this is a general result for *any* finite spherically symmetric mass distribution. Indeed, from the preceding analysis, it is clear that $\rho(r,\theta) = \rho_0(r)$ and

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$\Phi(r,\theta) = \Phi_0(r)$ for such a distribution. Suppose that the distribution extends out to $r = R$. It immediately follows, from Equation (2.49), that

$$\Phi_0(r) = -\frac{G}{r}\int_0^R 4\pi\, r'^{\,2}\,\rho_0(r')\,dr' = -\frac{G\,M}{r} \tag{2.55}$$

for $r > R$, where M is the total mass of the distribution.

Consider a point mass m that lies a distance r from the center of a spherically symmetric mass distribution of mass M (where r exceeds the outer radius of the distribution). Because the external gravitational potential generated by the distribution is the same as that of a point mass M located at its center, the force exerted on the mass m by the distribution is the same as that due to a point mass M located at the center of the distribution. In other words, the force is of magnitude $G\,M\,m/r^2$ and is directed from the mass toward the center of the distribution. Assuming that the system is isolated, the resultant force that the mass exerts on the distribution is of magnitude $G\,M\,m/r^2$ and has a line of action directed from the center of the distribution toward the mass. (See Exercise 1.3.) However, this is the same as the force that the mass would exert on a point mass M located at the center of the distribution. Because gravitational fields are superposable, we conclude that the resultant gravitational force acting on a spherically symmetric mass distribution of mass M situated in the gravitational field generated by many point masses is the same as that which would act on a point mass M located at the center of the distribution.

The center of mass of a spherically symmetric mass distribution lies at the geometric center of the distribution. Moreover, the translational motion of the center of mass is analogous to that of a point particle whose mass is equal to that of the whole distribution, moving under the action of the resultant external force. (See Section 1.6.) If the external force is due to a gravitational field, then the resultant force is the same as that exerted by the field on a point particle, whose mass is that of the distribution, located at the center of mass. We thus conclude that Newton's laws of motion, in their primitive form, apply not only to point masses, but also to the translational motions of extended spherically symmetric mass distributions interacting via gravity (e.g., the Sun and the planets).

2.6 Potential outside a uniform spheroid

Let us now calculate the gravitational potential generated outside a spheroid of uniform mass density γ and mean radius R. A *spheroid* is the solid body produced by rotating an ellipse about a major or a minor axis. Let the axis of rotation coincide with the z-axis, and let the outer boundary of the spheroid satisfy

$$r = R_\theta(\theta) = R\left[1 - \frac{2}{3}\,\epsilon\,P_2(\cos\theta)\right], \tag{2.56}$$

where ϵ is termed the *ellipticity*. In fact, the radius of the spheroid at the poles (i.e., along the axis) is $R_p = R\,(1 - 2\,\epsilon/3)$, whereas the radius at the equator (i.e., in the bisecting

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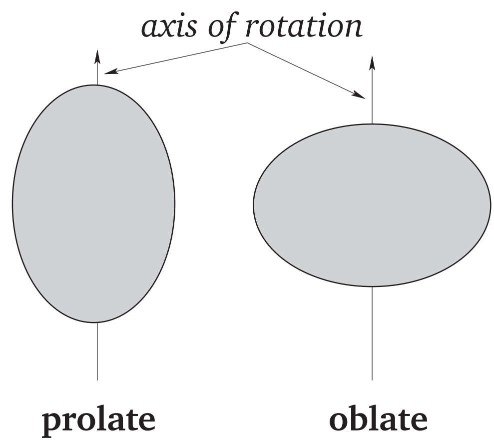

Fig. 2.1 Prolate and oblate spheroids.

plane perpendicular to the axis) is $R_e = R\,(1 + \epsilon/3)$. Hence,

$$\epsilon = \frac{R_e - R_p}{R}. \tag{2.57}$$

Let us assume that $|\epsilon| \ll 1$, so that the spheroid is very close to being a sphere. If $\epsilon > 0$, then the spheroid is slightly squashed along its symmetry axis and is termed *oblate*. Likewise, if $\epsilon < 0$, then the spheroid is slightly elongated along its symmetry axis and is termed *prolate*. (See Figure 2.1.) Of course, if $\epsilon = 0$ then the spheroid reduces to a sphere.

Now, according to Equations (2.45) and (2.46), the gravitational potential generated outside an axially symmetric mass distribution can be written

$$\Phi(r, \theta) = \frac{G\,M}{R} \sum_{n=0,\infty} J_n \left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta), \tag{2.58}$$

where M is the total mass of the distribution, and

$$J_n = -\frac{2\pi\,R^3}{M} \int\!\!\int \left(\frac{r}{R}\right)^{2+n} \rho(r, \theta)\, P_n(\cos\theta)\, \sin\theta\, d\theta\, \frac{dr}{R}. \tag{2.59}$$

Here, the integral is taken over the whole cross section of the distribution in r–θ space.

It follows that for a uniform spheroid, for which $M = (4\pi/3)\,\gamma\,R^3$,

$$J_n = -\frac{3}{2} \int_0^{\pi} P_n(\cos\theta) \int_0^{R_\theta(\theta)} \frac{r^{\,2+n}\,dr}{R^{\,3+n}}\, \sin\theta\, d\theta. \tag{2.60}$$

Hence,

$$J_n = -\frac{3}{2\,(3+n)} \int_0^{\pi} P_n(\cos\theta) \left[\frac{R_\theta(\theta)}{R}\right]^{3+n} \sin\theta\, d\theta, \tag{2.61}$$

giving

$$J_n \simeq -\frac{3}{2\,(3+n)} \int_0^{\pi} P_n(\cos\theta) \left[P_0(\cos\theta) - \frac{2}{3}\,(3+n)\,\epsilon\,P_2(\cos\theta)\right] \sin\theta\, d\theta, \tag{2.62}$$

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to first order in ϵ. It is thus clear, from Equation (2.42), that, to first order in ϵ, the only nonzero J_n are

$$J_0 = -1 \tag{2.63}$$

and

$$J_2 = \frac{2}{5}\,\epsilon. \tag{2.64}$$

Thus, the gravitational potential outside a uniform spheroid of total mass M, mean radius R, and ellipticity ϵ is

$$\Phi(r,\theta) = -\frac{G\,M}{r} + \frac{2}{5}\,\epsilon\,\frac{G\,M\,R^2}{r^3}\,P_2(\cos\theta) + \mathcal{O}(\epsilon^2). \tag{2.65}$$

By analogy with the preceding analysis, the gravitational potential outside a general (i.e., axisymmetric, but not necessarily uniform) spheroidal mass distribution of mass M, mean radius R, and ellipticity ϵ (where $|\epsilon| \ll 1$) can be written

$$\Phi(r,\theta) = -\frac{G\,M}{r} + J_2\,\frac{G\,M\,R^2}{r^3}\,P_2(\cos\theta) + \mathcal{O}(\epsilon^2), \tag{2.66}$$

where $J_2 \sim \mathcal{O}(\epsilon)$. (See Exercise 2.9.) In particular, the gravitational potential on the surface of the spheroid is

$$\Phi(R_\theta,\theta) = -\frac{G\,M}{R_\theta} + J_2\,\frac{G\,M\,R^2}{R_\theta^3}\,P_2(\cos\theta) + \mathcal{O}(\epsilon^2), \tag{2.67}$$

which yields

$$\Phi(R_\theta,\theta) \simeq -\frac{G\,M}{R}\left[1 + \left(\frac{2}{3}\,\epsilon - J_2\right)P_2(\cos\theta) + \mathcal{O}(\epsilon^2)\right], \tag{2.68}$$

where use has been made of Equation (2.56). For the case of a uniform-density spheroid, for which $J_2 = (2/5)\,\epsilon$ [see Equation (2.64)], the preceding expression simplifies to

$$\Phi(R_\theta,\theta) \simeq -\frac{G\,M}{R}\left[1 + \frac{4}{15}\,\epsilon\,P_2(\cos\theta) + \mathcal{O}(\epsilon^2)\right]. \tag{2.69}$$

Consider a self-gravitating spheroid of mass M, mean radius R, and ellipticity ϵ, such as a star or a planet. Assuming, for the sake of simplicity, that the spheroid is composed of uniform-density incompressible fluid, it follows that the gravitational potential on its surface is given by Equation (2.69). However, the condition for an equilibrium state is that the potential be *constant* over the surface. If this is not the case, then there will be gravitational forces acting *tangential* to the surface. Such forces cannot be balanced by internal fluid pressure, which acts only *normal* to the surface. Hence, from Equation (2.69), it is clear that the condition for equilibrium is $\epsilon = 0$. In other words, the equilibrium configuration of a uniform-density, self-gravitating, fluid, mass distribution is a *sphere*. Deviations from this configuration can be caused only by forces in addition to self-gravity and internal fluid pressure, such as internal tensile forces, centrifugal forces due to rotation, or tidal forces due to orbiting masses. The same is true for a self-gravitating mass distribution of non-uniform density. (See Chapter 5.)

Fig. 2.2 The martian moon Phobos (mean radius 11.1 km). Photograph taken by the European Space Agency's Mars Express spacecraft in 2010. Credit: G. Neukum (ESA/DLR/FU Berlin).

We can estimate how small a rocky asteroid, say, needs to be before its material strength is sufficient to allow it to retain a significantly nonspherical shape. The typical density of rocky asteroids in the solar system is $\gamma \sim 3.5 \times 10^3\,\mathrm{kg\,m^{-3}}$. Moreover, the critical pressure above which the rock out of which such asteroids are composed ceases to act as a rigid material, and instead deforms and flows like a liquid, is $p_c \sim 2 \times 10^8\,\mathrm{N\,m^{-2}}$ (de Pater and Lissauer 2010). We must compare this critical pressure with the pressure at the center of the asteroid. Assuming, for the sake of simplicity, that the asteroid is roughly spherical, of radius R, and of uniform density γ, the central pressure is

$$p_0 = \int_0^R \gamma\, g(r)\, dr, \tag{2.70}$$

where $g(r) = (4\pi/3)\, G\, \gamma\, r$ is the gravitational acceleration at radius r [see Equation (2.53)]. This result is a simple generalization of the well-known formula $\rho\, g\, h$ for the pressure a depth h below the surface of a fluid. It follows that

$$p_0 = \frac{2\pi}{3}\, G\, \gamma^2 R^2. \tag{2.71}$$

If $p_0 \ll p_c$, then the internal pressure in the asteroid is not sufficiently high to cause its constituent rock to deform like a liquid. Such an asteroid can therefore retain a significantly nonspherical shape. On the other hand, if $p_0 \gg p_c$, then the internal pressure is large enough to render the asteroid fluidlike. Such an asteroid cannot withstand the tendency of self-gravity to make it adopt a spherical shape. The same applies to any rocky body in the solar system. The condition $p_0 \ll p_c$ is equivalent to $R \ll R_c$, where

$$R_c = \left(\frac{3}{2\pi}\, \frac{p_c}{G\, \gamma^2}\right)^{1/2} \simeq 230\,\mathrm{km}. \tag{2.72}$$

It follows that only a rocky body whose radius is significantly less than about 230 km—for instance, the two moons of Mars, Phobos (see Figure 2.2) and Deimos—can retain a highly nonspherical shape. On the other hand, a rocky body whose radius is significantly

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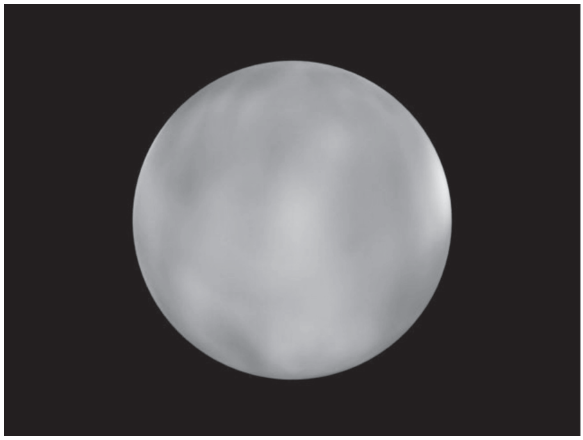

Fig. 2.3 The asteroid Ceres (mean radius 470 km). Photograph taken by the Hubble Space Telescope. Credit: NASA, ESA, J.-Y. Li (University of Maryland) and G. Bacon (STScI).

greater than about 230 km—for instance, the asteroid Ceres (see Figure 2.3) and the Earth's Moon—is forced by gravity to be essentially spherical.

According to Equations (2.58) and (2.63), the gravitational potential outside a general axisymmetric body of mass M and mean radius R can be written

$$\Phi(r,\theta) = -\frac{G\,M}{r}\left[1 - \sum_{n=1,\infty} J_n\,\frac{R^n}{r^n}\,P_n(\cos\theta)\right], \tag{2.73}$$

where the J_n are $\mathcal{O}(1)$ (or smaller) dimensionless parameters that depend on the exact shape of the body. However, a long way from the body ($r \gg R$), the right-hand side of the preceding expression is clearly dominated by the first term inside the round brackets, so that

$$\Phi(r,\theta) \simeq -\frac{G\,M}{r}. \tag{2.74}$$

However, this is just the gravitational potential of a point particle, located at the center of the body, whose mass is equal to that of the body. This suggests that the gravitational interaction between two irregularly shaped bodies in the solar system can be approximated as the interaction of two point masses, provided that the distance between the bodies is much larger than the sum of their radii.

2.7 Potential due to a uniform ring

Consider a uniform ring of mass M, radius a, and negligible cross-sectional area, centered on the origin, and lying in the x–y plane. Let us consider the gravitational potential

$\Phi(r)$ generated by such a ring in the x–y plane (which corresponds to $\theta = 90°$). It follows, from Section 2.4, that for $r > a$,

$$\Phi(r) = -\frac{G\,M}{a} \sum_{n=0,\infty} [P_n(0)]^2 \left(\frac{a}{r}\right)^{n+1}. \tag{2.75}$$

However, $P_0(0) = 1$, $P_1(0) = 0$, $P_2(0) = -1/2$, $P_3(0) = 0$, $P_4(0) = 3/8$, $P_5(0) = 0$, $P_6(0) = -5/16$, $P(7) = 0$, and $P_8(0) = 35/128$. Hence,

$$\Phi(r) = -\frac{G\,M}{r}\left[1 + \frac{1}{4}\left(\frac{a}{r}\right)^2 + \frac{9}{64}\left(\frac{a}{r}\right)^4 + \frac{25}{256}\left(\frac{a}{r}\right)^6 + \frac{1225}{16384}\left(\frac{a}{r}\right)^8 + \cdots\right]. \tag{2.76}$$

Likewise, for $r < a$,

$$\Phi(r) = -\frac{G\,M}{a} \sum_{n=0,\infty} [P_n(0)]^2 \left(\frac{r}{a}\right)^{n}, \tag{2.77}$$

giving

$$\Phi(r) = -\frac{G\,M}{a}\left[1 + \frac{1}{4}\left(\frac{r}{a}\right)^2 + \frac{9}{64}\left(\frac{r}{a}\right)^4 + \frac{25}{256}\left(\frac{r}{a}\right)^6 + \frac{1225}{16384}\left(\frac{r}{a}\right)^8 + \cdots\right]. \tag{2.78}$$

Exercises

2.1 A particle is projected vertically upward from the Earth's surface with a velocity that would, were gravity uniform, carry it to a height h. Show that if the variation of gravity with height is allowed for, but the resistance of air is neglected, then the height reached will be greater by $h^2/(R - h)$, where R is the Earth's radius. (From Lamb 1923.)

2.2 A particle is projected vertically upward from the Earth's surface with a velocity just sufficient for it to reach infinity (neglecting air resistance). Prove that the time needed to reach a height h is

$$\frac{1}{3}\left(\frac{2\,R}{g}\right)^{1/2}\left[\left(1 + \frac{h}{R}\right)^{3/2} - 1\right],$$

where R is the Earth's radius, and g its surface gravitational acceleration. (From Lamb 1923.)

2.3 Assuming that the Earth is a sphere of radius R, and neglecting air resistance, show that a particle that starts from rest a distance R from the Earth's surface will reach the surface with speed $\sqrt{R\,g}$ after a time $(1 + \pi/2)\,\sqrt{R/g}$, where g is the surface gravitational acceleration. (Modified from Smart 1951.)

2.4 Demonstrate that if a narrow shaft were drilled though the center of a uniform self-gravitating sphere, then a test mass moving in this shaft executes simple harmonic motion about the center of the sphere with period

$$T = 2\pi\,\sqrt{\frac{R}{g}},$$

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where R is the radius of the sphere, and g the gravitational acceleration at its surface.

2.5 Consider an isolated system consisting of N point objects interacting via gravity. The equation of motion of the ith object is

$$m_i\,\ddot{\mathbf{r}}_i = \sum_{j=1,N}^{j\neq i} G\,m_i\,m_j\,\frac{\mathbf{r}_j-\mathbf{r}_i}{|\mathbf{r}_j-\mathbf{r}_i|^3},$$

where m_i and $\mathbf{r}_i$ are the mass and position vector of this object, respectively. Moreover, the total potential energy of the system takes the form

$$U = -\frac{1}{2}\sum_{i,j=1,N}^{i\neq j}\frac{G\,m_i\,m_j}{|\mathbf{r}_j-\mathbf{r}_i|}.$$

Write an expression for the total kinetic energy, K. Demonstrate from the equations of motion that $K+U$ is constant in time.

2.6 Consider a function of many variables $f(x_1, x_2, \ldots, x_n)$. Such a function that satisfies

$$f(t\,x_1, t\,x_2, \ldots, t\,x_n) = t^a\,f(x_1, x_2, \ldots, x_n)$$

for all $t>0$, and all values of x_i, is termed a *homogeneous function of degree a*. Prove the following theorem regarding homogeneous functions:

$$\sum_{i=1,n} x_i\,\frac{\partial f}{\partial x_i} = a\,f.$$

2.7 Consider an isolated system consisting of N point particles interacting via attractive central forces. Let the mass and position vector of the ith particle be m_i and $\mathbf{r}_i$, respectively. Suppose that magnitude of the force exerted on particle i by particle j is $k_i\,k_j\,|\mathbf{r}_i-\mathbf{r}_j|^{-n}$. Here, k_i measures some constant physical property of the ith particle (e.g., its electric charge). Show that the total potential energy U of the system is written

$$U = -\frac{1}{2}\,\frac{1}{n-1}\sum_{i,j=1,N}^{j\neq i}\frac{k_i\,k_j}{|\mathbf{r}_j-\mathbf{r}_i|^{n-1}}.$$

Is this a homogeneous function? If so, what is its degree? Demonstrate that the equation of motion of the ith particle can be written

$$m_i\,\ddot{\mathbf{r}}_i = -\frac{\partial U}{\partial \mathbf{r}_i}.$$

(This is shorthand for $m_i\,\ddot{x}_i = -\partial U/\partial x_i$, $m_i\,\ddot{y}_i = -\partial U/\partial y_i$, etc., where the x_i, y_i, z_i, for $i = 1, N$ are treated as *independent* variables.) Use the mathematical theorem from the previous exercise to show that

$$\frac{1}{2}\,\frac{d^2\mathcal{I}}{dt^2} = 2\,K + (n-1)\,U,$$

where $\mathcal{I} = \sum_{i=1,N} m_i\,r_i^2$, and K is the total kinetic energy. This result is known as the *virial theorem*. Demonstrate that when $n \geq 3$, the system possesses no virial equilbria (i.e., states for which $\mathcal{I}$ does not evolve in time) that are bound.

2.8 Demonstrate that the gravitational potential energy of a spherically symmetric mass distribution of mass density $\rho(r)$ that extends out to $r = R$ can be written

$$U = -16\pi^2 G \int_0^R \int_0^r r'^2 \rho(r')\, r\, \rho(r)\, dr'\, dr.$$

Hence, show that if the mass distribution is such that

$$\rho(r) = \begin{cases} \rho_0\, r^{-\alpha} & r \leq R \\ 0 & r > R \end{cases}$$

where $\alpha < 5/2$, then

$$U = -\frac{(3-\alpha)}{(5-2\alpha)} \frac{G M^2}{R},$$

where M is the total mass.

2.9 Consider a spheroidal mass distribution of mass M, mean radius R, and ellipticity ϵ (where $|\epsilon| \ll 1$). Let the density distribution within the spheroid be of the form specified in the previous exercise. Demonstrate that the gravitational potential outside the spheroid takes the form

$$\Phi(r,\theta) = -\frac{G M}{r} + J_2 \frac{G M R^2}{r^3} P_2(\cos\theta) + \mathcal{O}(\epsilon^2),$$

where

$$J_2 = \frac{2}{5}\left(\frac{3-\alpha}{3}\right)\epsilon.$$

Here, r and θ are spherical coordinates whose origin lies at the geometric center of the distribution, and whose symmetry axis coincides with that of the distribution. Let $\mathcal{I}_{\|}$ be the moment of inertia of the distribution about its symmetry axis. Show that

$$\mathcal{I}_{\|} = \frac{2}{3}\left(\frac{3-\alpha}{5-\alpha}\right) M R^2 + \mathcal{O}(\epsilon).$$

2.10 A globular star cluster can be approximated as an isolated self-gravitating virial equilibrium consisting of a great number of equal mass stars. Demonstrate, from the virial theorem, that

$$K = -\frac{1}{2}\, U$$

for such a cluster. Suppose that the stars in a given cluster are uniformly distributed throughout a spherical volume. Show that

$$\bar{v} = \sqrt{\frac{3}{10}}\, v_{\rm esc},$$

where $\bar{v}$ is the mean stellar velocity and $v_{\rm esc}$ is the escape speed (i.e., the speed a star at the edge of the cluster would require in order to escape to infinity.) See Exercise 2.7.

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2.11 A star can be thought of as a spherical system that consists of a very large number of particles of mass m_i and position vector $\mathbf{r}_i$ interacting via gravity. Show that, for such a system, the virial theorem implies that

$$\frac{d^2\mathcal{I}}{dt^2} = -2\,U + c,$$

where c is a constant, $\mathcal{I} = \sum m_i\,r_i^2$, and the r_i are measured from the geometric center. Hence, deduce that the angular frequency of small-amplitude radial pulsations of the star (in which the radial displacement is directly proportional to the radial distance from the center) takes the form

$$\omega = \left(\frac{|U_0|}{\mathcal{I}_0}\right)^{1/2},$$

where U_0 and $\mathcal{I}_0$ are the equilibrium values of U and $\mathcal{I}$. Finally, show that if the mass density within the star varies as $r^{-\alpha}$, where r is the radial distance from the geometric center and where $\alpha < 5/2$, then

$$\omega = \left(\frac{5-\alpha}{5-2\,\alpha}\,\frac{G\,M}{R^3}\right)^{1/2},$$

where M and R are the stellar mass and radius, respectively. See Exercises 2.7 and 2.8.

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