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A Dynamical Model for the Radial Structure of Saturn's Rings

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In this paper we inquire whether the major positional features of Saturn's ring can be explained by perturbations associated with the planet's satellites. We assume a simple but observationally consistent model in which the ring is composed of a single layer of particles, and we further suppose that the system has evolved to a state in which collisions between particles may be ignored. Under these conditions, we find that only gravitational forces and the resulting perturbations by the two satellites Mimas and Titan suffice for a rather complete description of the ring system.

I. Description of a Model

For some time the astronomical literature has given the impression that the radial distribution of the particles in Saturn's ring is largely the consequence of a set of resonances with the inner satellites, particularly with Mimas. Thus, for example, perturbations associated with this satellite have been held responsible for the origin of the conspicuous Cassini division, which lies at a distance from Saturn where the mean motion of ring particles is nearly twice that of the satellite. Recently, however, Allan (1967) has called attention to the possible importance of longitudinal harmonics in the figure of Saturn to the ring structure. Dollfus (1968), with new measurements of the ring features, has again stressed the slight but apparently real displacement of the position of the Cassini division from its expected location, i.e., the distance from the primary corresponding to one-half the period of Mimas. Finally, Alfvén (1968) has, because of its low mass relative to Saturn, rejected Mimas as the dominant controller of the radial ring profile, and introduced instead a speculative process that occurred early in the history of the solar system.

In view of these comments and, in any event, because there has been so little quantitative study of the ring structure, we pose the following question: Given a simple but consistent ring model, can purely gravitational forces associated with the primary (Saturn) and the perturbers (satellites) account for the salient features of the ring structure? Figure 1 shows these features, together with certain minor ones, following the observations of Dollfus (1968).

We assume, because of its inherent simplicity, that the rings exist in a layer only one particle thick. Dynamical arguments led Jeffreys (1947) to propose such a ring model. Although photometric studies (e.g., Franklin and Cook, 1965) seemed at first to indicate a ring thickness of several and maybe many particle diameters, more recent laboratory studies of light scattering show that this need not be true. For a brief discussion of this question, see the Introduction in Cook and Franklin (1970).

To obtain the magnitude and radial dependence of the perturbations induced

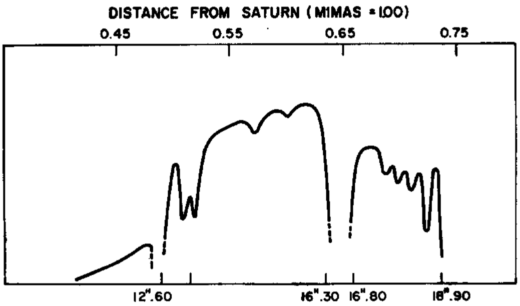

FIG. 1. Ring profile at 10 a.u., after Dollfus (1968).

on ring particles by a satellite, we follow the methods we used previously to explore orbits of particles moving under the influence of the sun and Jupiter (Colombo *et al.*, 1968). That is, we adopt the formalism of the planar restricted three-body problem with Saturn as the massive primary, Mimas as the less massive secondary, and a ring particle as a massless point. Since Mimas is assumed to move uniformly in a circular orbit, we can introduce a rotating coodinate system in which the two massive objects are stationary. We find periodic orbits for the massless body in this frame, subject to the further constraint that at $t = 0$ the massless particles lie along the radius vector joining the two massive bodies. Table I gives certain elements of several characteristic periodic orbits near the one whose period is one-half that of Mimas. Orbits with period ratios, T^*/T_M, about 25% larger or smaller than this one have osculating eccentricities at $t = 0$ that fall to very low values, of the same order as that Mimas–Saturn mass ratio, assumed here to be 0.67×10^{-7}. As has been known since the pioneering work of Schwarzschild (1903), there exists, for such orbits, near one-half the period of the perturber a range of periods in which no stable orbits are found. For the case in which Jupiter is the perturbing body, the semimajor axes of these unstable orbits define a "zone of avoidance" of about 0.1 a.u. For the present case the much smaller mass ratio (10^{-7} vs. 10^{-3}) of the perturber to the primary means that this unstable zone shrinks to a width of some 20 km, more than two orders of magnitude less than the observed Cassini division. These unstable orbits correspond to the break in the set given in Table I. Thus the hope that the division width would somehow coincide with the region of unstable orbits fails. In this sense the argument of Alfvén (1968) proves to be valid.

This general approach to the ring structure problem is still of value if we refine slightly the question asked earlier in the paper. Suppose we impose on our idealized monolayer ring model the condition that the system has evolved to a state such that collisions between particles no longer occur. Thus we envision a situation whereby orbits of particles are separated in radius by an amount ΔR just sufficiently large to ensure that particles travelling in adjacent orbits will not collide. At the same time we also require that particles moving in the same orbit be separated by a linear distance $R\Delta\theta$ that again is just large enough so that collisions are excluded. Having postulated this extreme and simple model, we now ask how would such a ring appear to a distant observer. The orbit calculation mentioned above enables us to answer this query. It is, for example, a straightforward matter to compute the quantity ΔR, which we define as the radial range swept out during a complete periodic orbit and is equal to 2 ae, as well as the corresponding tangential quantity $R\Delta\theta$. As expected, $R\Delta\theta \simeq 2\Delta R$. Periodic orbits do not remain fixed in space but precess about the pri-

TABLE I

OSCULATING SEMIMAJOR AXES AND ECCENTRICITIES OF PERIODIC ORBITS AS A FUNCTION OF T^*/T_M, THE RATIO OF THE SIDEREAL PERIOD OF A RING PARTICLE TO THAT OF MIMAS

T^*/T_M	a_0	e_0
0. 3333. . .	0. 480 750	0. 000 000 063
0. 488 190 02	0. 620 001	0. 000 002 081
0. 497 675 72	0. 628 007	0. 000 010 733
0. 499 994 71	0. 629 956	0. 004 692
0. 499 999 59	0. 629 960	0. 052 324
0. 499 999 82	0. 629 960	0. 099 943
0. 500 144 67	0. 630 082	0. 000 173 055
0. 502 422 26	0. 631 994	0. 000 010 340
0. 511 998 38	0. 639 999	0. 000 002 087
0. 598 256 53	0. 710 000	0. 000 000 092

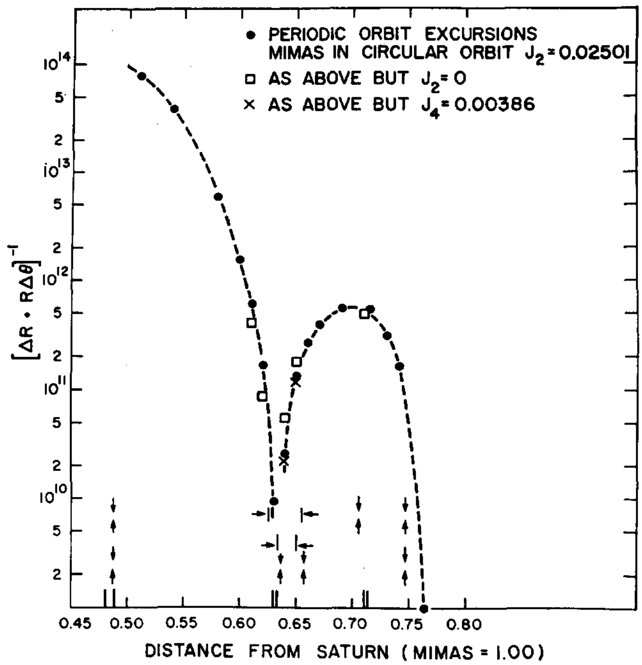

FIG. 2. Predicted density profile of the ring obtained with the perturber, Mimas, in a circular orbit. Arrows mark measured ring features. The vertical scale is drawn such that $R_{Mimas} = 1.0$. The quantity ΔR, for example, is the minimum radial separation of adjacent particles such that collisions will not occur; its value, with $R_{Mimas} = 1.86 \times 10^5$ km, at the ring A maximum is 200 meters.

mary, because of its oblateness, at rates that vary with the distance of an orbiting particle from Saturn. Thus, if collisions are to be avoided on a time scale of more than just a few revolutions, adjacent particles must be separated radially by the full amount $\Delta R = 2$ ae. Our proposed collisionless model now requires that N, the number of particles per unit area, be equal to $[\Delta R \cdot R\Delta\theta]^{-1}$.

The predicted ring profile of this model is displayed in Fig. 2, in which we plot $[\Delta R \cdot R\Delta\theta]^{-1}$ on a logarithmic vertical scale against the distance from Saturn, where the Mimas–Saturn distance is 1.0. We include measured values of the chief ring features in the lower portion of Fig. 2. The upper set of arrows shows measurements compiled from several papers (See, 1902; Lowell and Slipher, 1915; Cook and Franklin, 1958). These values are characterized by an attempt to remove the effects of irradiation. A minimum measured Cassini gap width lies on this figure just beneath the most probable one, while the final set of arrows denotes ring features as recently measured by Dollfus (1968) and collaborators. The vertical lines just above the lower horizontal scale mark the location of the resonances: the left line of any set with the oblateness of the primary equal to 0, and the right with the observed value $J_2 = 0.02501$ (Jeffreys, 1954). Figure 2 shows that the ring model just discussed is in several respects similar to the observed profile. The surface density, hence the brightness, of ring B much exceeds that of ring A, a fact previously found missing in a model based only upon gravitational forces (Alfvén, 1968). (We shall see presently that a slightly improved model predicts the ring B/ring A brightness ratio within the observed limits.) The present model also predicts a rapid decrease in the surface density in the neighbourhood of the observed Cassini division and at the outer edge of ring A; in fact, Fig. 2 shows that the density there and at the two gap boundaries is essentially the same.

Despite these encouraging features, three obvious discrepancies remain: (1) We have only been able to define the boundaries of the Cassini division somewhat vaguely as regions where the density gradient becomes sufficiently steep. It seems likely that the area density of particles at the gap boundaries should be at least an order of magnitude less than its value in ring A. Thus the division appears rather narrower than is observed, and it is displaced from its measured position. (2) The brightest observed portion of ring B lies just inside the Cassini division, whereas the predicted density distribution continues to rise gradually throughout this ring. (3) Neither the inner boundary of ring B nor any structure in ring A (the Encke "division") is apparent.

We suspect that (2) results from a weakness of the collisionless, steady-state model. Point (3) has a clear, removable cause, the proper treatment of which may lead to at least a partial lessening of the first two drawbacks. The absences referred to under (3) originated because the perturber, Mimas, was confined, by assumption, to a circular orbit. Allowing Mimas to move in an eccentric orbit introduces the complication that a rotating coordinate system in which Mimas and Saturn are stationary can, in general, no longer be used. It thus becomes impossible to obtain, as before, a representative class of orbits from which the quantities ΔR and $R\Delta\theta$ can be calculated. However, it is still possible to obtain a set of two similar quantities by the following means. We use a fixed coordinate system and once more consider, one at a time, a field of particles distributed along the radius vector to Mimas. At $t = 0$ each particle begins to move with a velocity V_{0c}, which is the local Keplerian velocity corrected for the oblateness of the primary. The integrations continue until each particle has completed a sufficient number of revolutions (about 100) so that the pattern of maximum and minimum radial and tangential excursions becomes clear.

For any distance from the primary, except at or exceedingly close to a resonance, it is therefore possible to determine a ΔR_{max}, which we define as the largest radial excursion. The quantity ΔR_{max} is essentially the amplitude of the nearly periodic radial excursions. Only very near resonance is the beat period between Mimas' motion and that of a ring particle so long that ΔR_{max} cannot be determined in, say, 100 revolutions. This is no drawback, because we know ΔR_{max} is very large here and hence $[\Delta R \cdot R\Delta\theta]^{-1}$ is many orders of magnitude less than its value outside of resonance. There is no permanent coherence in the radial dependence of the phase of the maximum radial excursions; that is, the density distribution will show no permanent large-scale longitudinal variation.

Corresponding values of $R\Delta\theta$ are likewise obtainable. Not-too-close-to resonance values of $[\Delta R \cdot R\Delta\theta]^{-1}$ are smaller than corresponding ones of the first calculation by a factor of about 2, but this factor rises somewhat as a resonance is approached. The velocity V_{0c}, defined above, gives within a few percent the smallest set of ΔR_{max}.

Figure 3 presents the density profile resulting from these new values of ΔR and $R\Delta\theta$. This model does show density minima at the location of the resonances at 1/3 and 3/5 Mimas' period. Other features remain relatively unchanged, although the predicted Cassini gap is somewhat wider than before and ring B now has a clear maximum. At 2/5 Mimas' period this calculation predicts a density minimum so narrow that it is not included in Fig. 3. It lies some 0".6 further from the planet than the double minimum near the inner boundary of ring B as drawn by Dollfus and shown in Fig. 1. Resonances of the form $[n/(n+2)]\,P$, where n is an integer and P the period of the perturber, are associated with the first power of the eccentricity of the latter, while those of the form $[n/(n+3)]\,P$ with the square of the eccentricity. Since Mimas' eccentricity is 0.02, the insignificance of the 2/5 resonance is not surprising. Figure 3 also shows the expected result that $[n/(n+1)]\,P$ resonances, associated with the zero power of the eccentricity, are more important than $[n/(n+2)]\,P$ ones.

Included also in Fig. 3 are a few sample results of density minima due to Janus,

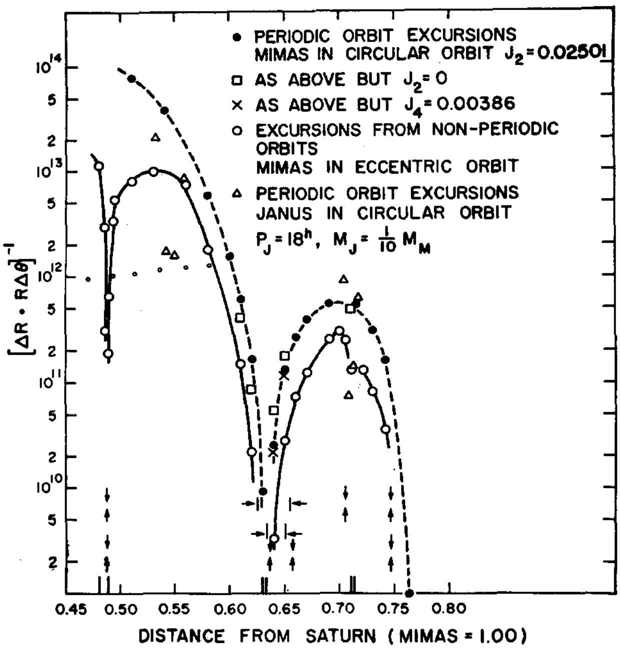

FIG. 3. Predicted density profile obtained with Mimas in an eccentric orbit.

whose mass is assumed to be 1/10 that of Mimas. This mass ratio is an estimate based upon the observation (Dollfus, 1968) that Janus is 1 to 2 magnitudes fainter than Mimas. The predicted width of the resonance at 1/2 Janus' period, which lies near 0.54 on the horizontal scale of Fig. 3, is very nearly identical with that at 1/3 Mimas' period. The former might be associated with the small depression in the middle of ring B shown in Fig. 1. Why these two resonances of similar strength should produce such disparate results remains a mystery. The discrepancy could be used as an argument against the approach taken in this paper, or it may be that Janus' mass is much less than the estimated value of 1/10 that of Mimas. At present even Janus' existence seems in question (Rosino and Stagni, 1969). Other $[n/(n+1)]\ P$ resonances of Janus are too close to features associated with Mimas' period to be seen independently. The same comment applies in part to the resonance at 1/3 Enceladus' period. While criticizing the role of resonances in determining the ring structure, Alfvén (1968) points out that the ring shows no feature corresponding to it. Effects of this resonance fail to appear for two reasons: First, because the eccentricity of Enceladus is a factor of more than 4 less than that of Mimas, and second, because this resonance falls at 0.62 on the horizontal scale of Fig. 3 and therefore would be invisible in the wing of the 1/2 Mimas' period resonance.

Among the other satellites, Titan, the most massive, cannot be ignored. While no important resonances with Titan exist within the boundaries of the ring, perturbations by Titan, not Mimas, determine the minimum radial and tangential separations for a system of noncolliding particles in the inner part of ring B. The appropriate limit on the quantity $[\Delta R \cdot R\Delta\theta]^{-1}$ due to Titan perturbations is given by the nearly horizontal dotted line in Fig. 3.

II. Comparison with Observation

In Fig. 4, we have drawn a profile in ring A and near the Cassini division in terms of a more directly measurable quantity, the optical thickness τ. For a monolayer the optical thickness of the medium, the radius of a particle, and the average separation

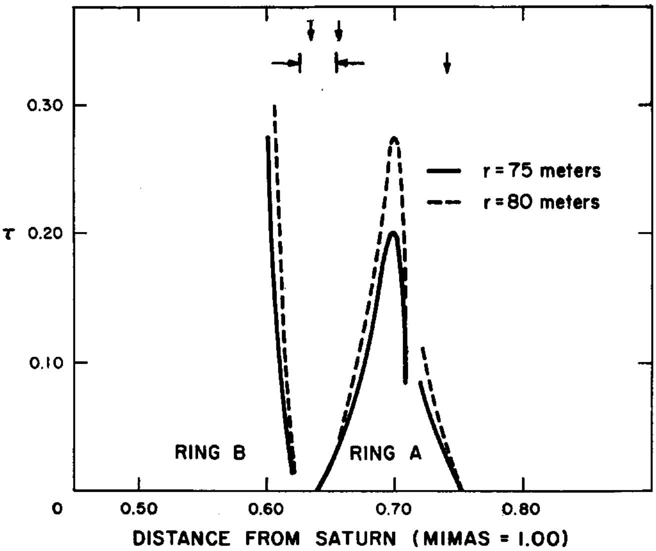

FIG. 4. Predicted values of the optical thickness, τ, near the Cassini division.

between particles are related. The photometrically determined value of τ for ring A lies near 0.3 for the inner part and near 0.15 for the outer (Franklin and Cook, 1965). The first of these, coupled with values of $(\Delta R \cdot R\Delta\theta]^{-1}$ from Fig. 3, requires that particles in ring A have radii that approach 100 meters. The minimum separation between particles in the brightness part of ring B, as depicted in Fig. 3, is slightly less than twice this value. If particles in ring B are identical in size with those in ring A, then there is a small inconsistency. The density level of the bright portion of ring B has been set by perturbations of Titan, on the assumption that it moves in a circular orbit. The slightly larger perturbations resulting from the motion of Titan in an eccentric orbit would remove this disparity, and still maintain the area density of material in ring B at a factor of about 3 greater than its value in the bright portion of ring A. This is in good agreement with observed values (Franklin and Cook, 1965). At the same time, the relatively high space density of material seems consistent with ring stability. Cook and Franklin (1965), in revising the work of Maxwell, show that the rings will be stable against infinitesimal disturbances for space densities $\gtrsim 0.18$ gm/cm^{-3} and can be stable for densities $\gtrsim 1.04$ gm/cm^{-3}.

Measurements of the ring thickness obtained by Kiladze (1967) when the Earth passed through the ring plane gave 0.9 ± 0.6 km; a slightly larger value was found by Focas and Dollfus (1969). Our model suggests a ring thickness of ~0.2 km. In view of its idealizations, we do not regard this discrepancy as serious.

One of the more unusual phenomena exhibited by the ring system is the appearance of a set of two small bright regions on on each ansa during those occasions, near their respective passages through the ring plane, when the Earth and Sun are on opposide sides of the plane. Bond and his staff (1857) were among the first to observe and measure these knots or condensations. They were rediscovered by Barnard (1908), who observed them for several months, measuring their positions repeatedly and making photometric estimates. The condensations have been observed on other favorable occasions before the most recent passage (cf. Hepburn, 1922). It is possible to link the probable cause of the condensations, which was essentially given by Barnard himself, with the ring model discussed in Section I.

In order to investigate the origin of the condensations, Barnard concentrated chiefly upon establishing their location and size. We give in Table II certain relevant

TABLE II

RING DIMENSIONS AND LOCATION OF CONDENSATIONS[a]

	Positions in arc seconds at 10 a.u.
Equatorial radius of Saturn	8″.49
Inner radius, ring A	16.71
Outer radius, ring B	16.21
Inner radius, ring B	12.23
Inner radius, ring C	9.79
Inner edge, inner condensation	11.07
Outer edge, inner condensation	12.31
Inner edge, outer condensation	14.50
Center of outer condensation	15.59
Outer edge, outer condensation	16.67

[a] After Barnard (1908).

data from Barnard's 1908 paper. Measures in the upper part of Table II result from observations made well before ring plane passage. Visual and photographic observations by other observers show that such measures are subject to systematic effects, most especially irradiation. On the basis of these measurements, Barnard suggested that ring C and the Cassini division, both regions of low relative space density, produced the two sets of condensations. In modern terms, his argument could be expanded as follows: In the Cassini division and at least in part of ring C, the optical thickness τ is much less than 1. However, it effectively increases as the elevation angle B of the Earth above the ring plane approaches zero. When the Earth and Sun are on the same side of the ring and B and B' (the elevation angle of the Sun) are small, the entire ring is uniformly bright. However, when the two angles are of opposite sign, and necessarily small, the effective τ for all parts of the ring, except for the Cassini division and ring C, is so high that an observer on the Earth sees a ring surface lit only by higher order scattering of sunlight and the disk of Saturn. Under these circumstances most of the ring becomes exceedingly faint, while the two regions specifically referred to attain an effective optical thickness large enough to be visible but not so large that higher order scattering reduces their brightness. These comments are particularly appropriate for a ring composed of several layers of particles. For a monolayer the sunlight is effectively blocked by the densest parts of the ring, but particles illuminated by the Sun are visible when the area density is sufficiently small.

New observations of the condensations reported by Rosino and Stagni (1969) are in qualitative agreement with those of Barnard, but place the center of the outer condensations at a distance of 15″.9 ± 0″.12 from the planet's center. (All distances used here refer to values at 10 a.u.)

For Rosino and Stagni the appearance of the condensations was such that no precise value of their radial extent could be determined. This was perhaps a consequence of $B \gtrsim 0^\circ.2$, when B' was of the opposite sign. The apparent discrepancy in the two locations of the center of the outer condensation can possibly be resolved as a result of irradiation. Barnard measured the position of the condensation relative to the adjacent limb of the plant. Irradiation would in this case make the disk appear too large, and the separation between the limb and the condensation would consequently be underestimated. In fact, on two occasions during observations of edge-on phenomena when Barnard measured the radius of the planet, he obtained an average value 0″.30 greater than that quoted in Table II. Thus we can suppose that the positional measures of the condensations in Table II are too small by ~0″.3. Such a correction would give agreement with the new observations, but at the same time shift the outer edge of the condensations as observed by Barnard well into the body of ring A. If this is truly the case, the above explanation for the origin of the condensations fails. Because Barnard measured the size of condensations on only two nights and because of the inherent difficulty of such a measurement—an "ill-defined satellite" (Barnard), possibly enlarged by irradiation—we feel that values of the size of the condensations must be viewed with some suspicion. The observations of Bond (1857), which probably also suffer from the effects of irradiation, describe a set of two small outer condensations whose inner boundaries coincide with the outer boundary of ring B. Bond makes no estimate of the radial extent or position of maximum light of the condensations. Probably we can say with some certainty only that the brightness maxima of the two outer condensations lie very nearly at the outer boundary of ring B.

We have seen that a small amount of material remaining at least near the borders of the Cassini division, coupled with the geometry of the ring and its position relative to the Earth, could produce the knots. This explanation is consistent with our model that shows a steep density gradient as the region of a resonance is approached. Until now we have viewed the inability of the model to predict sharp boundaries as something of a liability where tapering ones are in fact required to account for the knots.

To investigate this question further we suppose the area density $\rho(r)$ of particles remaining in the division to be represented by a relation of the form

$$\rho(r) \propto \mid r^2\text{-}[b + \alpha(c - b)]^2|^\beta, \qquad (1)$$

where r is the distance from the center of the planet and α, $0 \leqslant \alpha \leqslant 1$, is a parameter that locates the position of exact resonance in terms of the division width. Figure 3 shows that there is a slight displacement in the location of the predicted division with respect to the position of the 1/2 Mimas' period resonance (corrected for the oblateness of the primary) so that $\alpha = 0.6$. The quantities b and c stand, respectively, for the inner and outer effective boundaries of the gap as required for use in Eq. (1). As mentioned, we have no way of defining their precise values. We do have the guideline that the density at a boundary must correspond to an optical thickness at least one order of magnitude less than a characteristic value for ring A. We can fulfill this requirement while at the same time obtaining a predicted width roughly equal to the observed one if $b = 15''.90 \pm 0''.05$ and $c = 16''.44 \pm 0''.05$. These values apply at an Earth–Saturn distance of 10 a.u. If the exponent $\beta = 0$ in Eq. (1), the density of particles in the division is constant, while $\beta = 2$ closely represents the density dependence, as shown on Fig. 3, near the suggested gap boundaries. We also suppose that the brightness distribution $I(x)$ may be obtained from $\rho(r)$, i.e.,

$$I(x) \propto \int \rho(r)\, dy,$$

where $r = (x^2 + y^2)^{1/2}$. The relative brightness curves for $1 \leqslant \beta \leqslant 3$ are so similar that only the case $\beta = 2$ is shown in Fig. 5. We therefore expect a bright condensation, several tenths of a second in width and centered at the outer boundary of ring B. This prediction is in good agreement with the observation of Rosino and Stagni, but less so with those of Barnard, especially as the latter measured a condensation width of $\sim$2 sec. Seeing and irradiation would both operate to make this value too large.

Several observers, most recently Feibelman (1967), have found evidence of a faint ring D, external to ring A. Its existence, or detectability, is still considered very doubtful by most observers (e.g., Rosino and Stagni, 1969). The maximum area density of particle allowed by our model (Mimas perturbations only) in the region outside ring A is more than three orders of magnitude less than the maximum density in ring A. To record such a ring of visual or photographic means at the Earth seems very unlikely.

We conclude by indicating the positive and negative aspects of this study. The collisionless model discussed here makes it likely that gravitational effects alone can account for the mean features of Saturn's ring. More specifically, the appearance, maybe even the existence of the ring, is controlled by Mimas. The model seems consistent with the major features of Fig. 1, as well as the other observations discussed above. That it does not reproduce all the minor features of Fig. 1 should not, we feel, be sufficient grounds for rejection.

Two clear difficulties do however remain: (1) the predicted Cassini division is displaced inward from its measured position by $\sim 0''.2$, and (2) although we can define an inner boundary to ring B, we are unable to account for the fact that the conspicuous part of the ring system terminates there. The first discrepancy could be either theoretical or observational in origin.

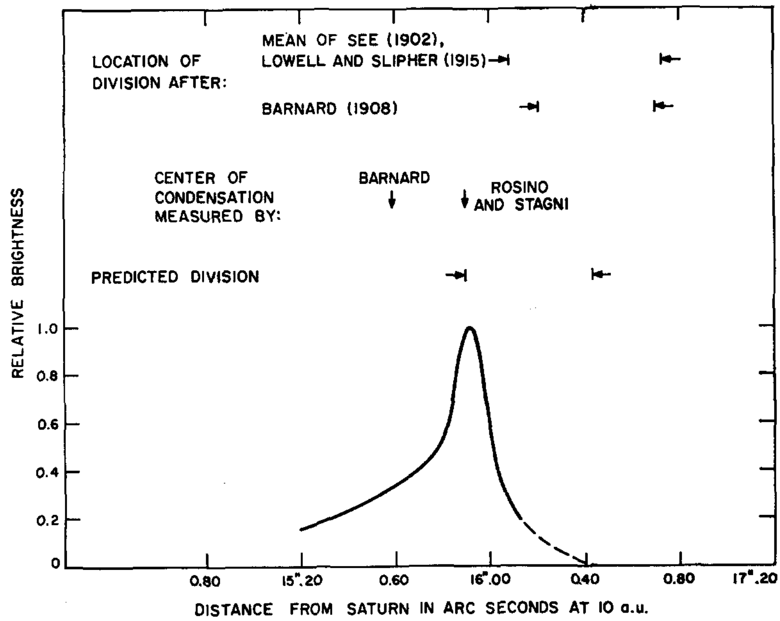

FIG. 5. Relative brightness of the Cassini division region when the Earth and Sun are on opposite sides of the ring plane.

Among possible causes of the first category, it is unlikely that self-gravitation of the ring is important. It is true that the addition of an effective ring of material closer to the primary than a ring particle would displace a resonance, and consequently a predicted gap, outwards. However, the mass of ring B, though observationally undetermined, is still so much less than Saturn's that such a ring could produce only a minute fraction of the apparent displacement. Nor is it at all clear that a model in which collisions are included would provide the desired shift. For reasons mentioned earlier in this paper, we suspect that at least part of the discrepancy may result from observational difficulties. We therefore await with much interest new positional measurements or photometric scans of the ring, especially from balloons.

We now turn to the second difficulty. We have seen that our model predicts an outer boundary to ring A. Actually this outer boundary could be regarded as the inner boundary of a Cassini type gap near 2/3 Mimas' period. So little material remains in noncolliding orbits at distances further from the planet that no outer boundary of this hypothetical gap, or other faint ring, can be found. Thus this model can account for the outer termination of the ring system. It does not, however, do so for the inner one. There are reasonable drag processes that might deplete the population of ring C from its predicted level, as shown in Fig. 3, to its observed lower value. Or it may have been that only a small amount of material was put here when the ring came into being. We only remark here that the simple gravitational effects thus far considered do not seem able to account for the characteristics of ring C.

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