3 Keplerian orbits

3.1 Introduction

Newtonian mechanics was initially developed to account for the motion of the planets around the Sun. Let us now examine this problem. Suppose that the Sun, whose mass is M, is located at the origin of our coordinate system. Consider the motion of a general planet, of mass m, that is located at position vector $\mathbf{r}$. Given that the Sun and all the planets are approximately spherical, the gravitational force exerted on our planet by the Sun can be written (see Chapter 2)

$$\mathbf{f} = -\frac{G\,M\,m}{r^3}\,\mathbf{r}. \tag{3.1}$$

An equal and opposite force to that given in Equation (3.1) acts on the Sun. However, we shall assume that the Sun is so much more massive than the planet that this force does not cause the Sun's position to shift appreciably. Hence, the Sun will always remain at the origin of our coordinate system. Likewise, we shall neglect the gravitational forces exerted on our chosen planet by the other bodies in the solar system, compared with the gravitational force exerted on it by the Sun. This is reasonable because the Sun is much more massive (by a factor of at least 10^3) than any other solar system body. Thus, according to Equation (3.1) and Newton's second law of motion, the equation of motion of our planet (which can effectively be treated as a point object—see Chapter 2) takes the form

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{G\,M}{r^3}\,\mathbf{r}. \tag{3.2}$$

Note that the planet's mass, m, has canceled out on both sides of this equation.

3.2 Kepler's laws

As is well known, Johannes Kepler (1571–1630) was the first astronomer to correctly describe the motion of the planets (in works published between 1609 and 1619). This motion is summed up in three simple laws:

1. The planetary orbits are all ellipses that are confocal with the Sun (i.e., the Sun lies at one of the foci of each ellipse—see Section A.9).
2. The radius vector connecting each planet to the Sun sweeps out equal areas in equal time intervals.

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3. The square of the orbital period of each planet is proportional to the cube of its orbital major radii.

Let us now see whether we can derive Kepler's laws from Equation (3.2).

3.3 Conservation laws

As we have already seen, gravity is a *conservative force*. Hence, the gravitational force in Equation (3.1) can be written (see Section 1.4)

$$\mathbf{f} = -\nabla U, \tag{3.3}$$

where the potential energy, $U(\mathbf{r})$, of our planet in the Sun's gravitational field takes the form

$$U(\mathbf{r}) = -\frac{G\,M\,m}{r}. \tag{3.4}$$

(See Section 2.5.) It follows that the total energy of our planet is a conserved quantity. (See Section 1.4.) In other words,

$$\mathcal{E} = \frac{v^2}{2} - \frac{G\,M}{r} \tag{3.5}$$

is constant in time. Here, $\mathcal{E}$ is actually the planet's total energy per unit mass, and $\mathbf{v} = d\mathbf{r}/dt$.

Gravity is also a *central force*. Hence, the *angular momentum* of our planet is a conserved quantity. (See Section 1.5.) In other words,

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}, \tag{3.6}$$

which is actually the planet's angular momentum per unit mass, is constant in time. Assuming that $|\mathbf{h}| > 0$, and taking the scalar product of the preceding equation with $\mathbf{r}$, we obtain

$$\mathbf{h} \cdot \mathbf{r} = 0. \tag{3.7}$$

This is the equation of a plane that passes through the origin and whose normal is parallel to $\mathbf{h}$. Because $\mathbf{h}$ is a constant vector, it always points in the same direction. We therefore conclude that the orbit of our planet is *two-dimensional*—that is, it is confined to some fixed plane that passes through the origin. Without loss of generality, we can let this plane coincide with the x–y plane.

3.4 Plane polar coordinates

We can determine the instantaneous position of our planet in the x–y plane in terms of standard Cartesian coordinates, x, y, or plane polar coordinates, r, θ, as illustrated in Figure 3.1. Here, $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$. It is helpful to define two unit vectors, $\mathbf{e}_r \equiv \mathbf{r}/r$ and $\mathbf{e}_\theta \equiv \mathbf{e}_z \times \mathbf{e}_r$, at the instantaneous position of the planet. The first

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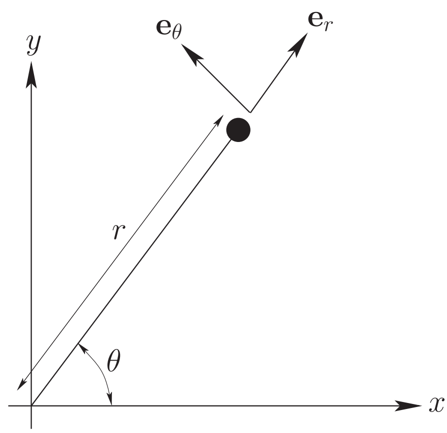

Fig. 3.1 Plane polar coordinates.

always points radially away from the origin, whereas the second is normal to the first, in the direction of increasing θ. As is easily demonstrated, the Cartesian components of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are

$$\mathbf{e}_r = (\cos\theta,\ \sin\theta) \tag{3.8}$$

and

$$\mathbf{e}_\theta = (-\sin\theta,\ \cos\theta), \tag{3.9}$$

respectively.

We can write the position vector of our planet as

$$\mathbf{r} = r\,\mathbf{e}_r. \tag{3.10}$$

Thus, the planet's velocity becomes

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r, \tag{3.11}$$

where $\dot{}$ is shorthand for d/dt. Note that $\mathbf{e}_r$ has a nonzero time derivative (unlike a Cartesian unit vector) because its direction changes as the planet moves around. As is easily demonstrated, by differentiating Equation (3.8) with respect to time, we obtain

$$\dot{\mathbf{e}}_r = \dot{\theta}\,(-\sin\theta,\ \cos\theta) = \dot{\theta}\ \mathbf{e}_\theta. \tag{3.12}$$

Thus,

$$\mathbf{v} = \dot{r}\ \mathbf{e}_r + r\,\dot{\theta}\ \mathbf{e}_\theta. \tag{3.13}$$

The planet's acceleration is written

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + (\dot{r}\,\dot{\theta} + r\,\ddot{\theta})\,\mathbf{e}_\theta + r\,\dot{\theta}\ \dot{\mathbf{e}}_\theta. \tag{3.14}$$

Again, $\mathbf{e}_\theta$ has a nonzero time derivative because its direction changes as the planet moves around. Differentiation of Equation (3.9) with respect to time yields

$$\dot{\mathbf{e}}_\theta = \dot{\theta}\,(-\cos\theta,\ -\sin\theta) = -\dot{\theta}\,\mathbf{e}_r. \tag{3.15}$$

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Hence,

$$\mathbf{a} = (\ddot{r} - r\,\dot{\theta}^2)\,\mathbf{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\mathbf{e}_\theta. \tag{3.16}$$

It follows that the equation of motion of our planet, Equation (3.2), can be written

$$\mathbf{a} = (\ddot{r} - r\,\dot{\theta}^2)\,\mathbf{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\mathbf{e}_\theta = -\frac{G\,M}{r^2}\,\mathbf{e}_r. \tag{3.17}$$

Because $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are mutually orthogonal, we can separately equate the coefficients of both, in the preceding equation, to give a *radial equation of motion*,

$$\ddot{r} - r\,\dot{\theta}^2 = -\frac{G\,M}{r^2}, \tag{3.18}$$

and a *tangential equation of motion*,

$$r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} = 0. \tag{3.19}$$

3.5 Kepler's second law

Multiplying our planet's tangential equation of motion, Equation (3.19), by r, we obtain

$$r^2\,\ddot{\theta} + 2\,r\,\dot{r}\,\dot{\theta} = 0. \tag{3.20}$$

However, this equation can be also written

$$\frac{d(r^2\,\dot{\theta})}{dt} = 0, \tag{3.21}$$

which implies that

$$h = r^2\,\dot{\theta} \tag{3.22}$$

is constant in time. It is easily demonstrated that h is the magnitude of the vector $\mathbf{h}$ defined in Equation (3.6). Thus, the fact that h is constant in time is equivalent to the statement that the angular momentum of our planet is a constant of its motion. As we have already mentioned, this is the case because gravity is a central force.

Suppose that the radius vector connecting our planet to the origin (i.e., the Sun) rotates through an angle $\delta\theta$ between times t and $t + \delta t$. (See Figure 3.2.) The approximately triangular region swept out by the radius vector has the area

$$\delta A \simeq \frac{1}{2}\,r^2\,\delta\theta, \tag{3.23}$$

as the area of a triangle is half its base ($r\,\delta\theta$) times its height (r). Hence, the rate at which the radius vector sweeps out area is

$$\frac{dA}{dt} = \lim_{\delta t\to 0}\frac{r^2\,\delta\theta}{2\,\delta t} = \frac{r^2}{2}\,\frac{d\theta}{dt} = \frac{h}{2}. \tag{3.24}$$

Thus, the radius vector sweeps out area at a constant rate (because h is constant in time)—this is Kepler's second law of planetary motion. We conclude that Kepler's second law is a direct consequence of *angular momentum conservation*.

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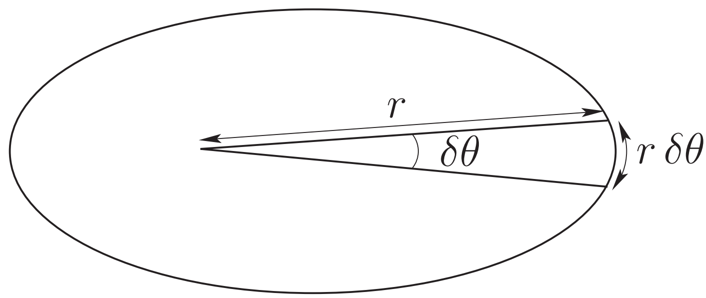

Fig. 3.2 Kepler's second law.

3.6 Kepler's first law

Our planet's radial equation of motion, Equation (3.18), can be combined with Equation (3.22) to give

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{G\,M}{r^2}. \tag{3.25}$$

Suppose that $r = u^{-1}$, where $u = u(\theta)$ and $\theta = \theta(t)$. It follows that

$$\dot{r} = -\frac{\dot{u}}{u^2} = -r^2\,\frac{du}{d\theta}\,\frac{d\theta}{dt} = -h\,\frac{du}{d\theta}. \tag{3.26}$$

Likewise,

$$\ddot{r} = -h\,\frac{d^2u}{d\theta^2}\,\dot{\theta} = -u^2\,h^2\,\frac{d^2u}{d\theta^2}. \tag{3.27}$$

Hence, Equation (3.25) can be written in the linear form

$$\frac{d^2u}{d\theta^2} + u = \frac{G\,M}{h^2}. \tag{3.28}$$

As can be seen via inspection, the general solution to the preceding equation is

$$u(\theta) = \frac{G\,M}{h^2}\,[1 + e\,\cos(\theta - \theta_0)]\,, \tag{3.29}$$

where e and θ_0 are arbitrary constants. Without loss of generality, we can set $\theta_0 = 0$ by rotating our coordinate system about the z-axis. We can also assume that $e \geq 0$. Thus, we obtain

$$r(\theta) = \frac{r_c}{1 + e\,\cos\theta}, \tag{3.30}$$

where

$$r_c = \frac{h^2}{G\,M}. \tag{3.31}$$

Equation (3.30) is the equation of a conic section that is confocal with the origin (i.e., with the Sun). (See Section A.9.) Specifically, for $e < 1$, Equation (3.30) is the equation of an *ellipse*. For $e = 1$, Equation (3.30) is the equation of a *parabola*. Finally, for $e > 1$, Equation (3.30) is the equation of a *hyperbola*. However, a planet cannot have a parabolic or a hyperbolic orbit, because such orbits are appropriate only to objects that are ultimately able to escape from the Sun's gravitational field. Thus, the orbit of our planet is an ellipse that is confocal with the Sun—this is Kepler's first law of planetary motion.

3.7 Kepler's third law

We have seen that the radius vector connecting our planet to the origin sweeps out area at the constant rate $dA/dt = h/2$ [see Equation (3.24)]. We have also seen that the planetary orbit is an ellipse. The major and minor radii of such an ellipse are $a = r_c/(1 - e^2)$ and $b = (1-e^2)^{1/2}\,a$, respectively. [See Equations (3.34) and (A.108).] The area of the ellipse is $A = \pi\, a\, b$. We expect the radius vector to sweep out the whole area of the ellipse in a single orbital period, T. Hence,

$$T = \frac{A}{dA/dt} = \frac{2\pi\, a\, b}{h} = \frac{2\pi\, a^2\,(1-e^2)^{1/2}}{h} = \frac{2\pi\, a^{3/2}\, r_c^{\,1/2}}{h}. \tag{3.32}$$

It follows from Equation (3.31) that

$$T^{\,2} = \frac{4\pi^2\, a^3}{G\, M}. \tag{3.33}$$

In other words, the square of the orbital period of our planet is proportional to the cube of its orbital major radius—this is Kepler's third law of planetary motion.

3.8 Orbital parameters

For an elliptical orbit, the closest distance to the Sun—the *perihelion distance*—is [see Equation (3.30)]

$$r_p = \frac{r_c}{1+e} = a\,(1-e). \tag{3.34}$$

This equation also holds for parabolic and hyperbolic orbits. Likewise, the furthest distance from the Sun—the *aphelion distance*—is

$$r_a = \frac{r_c}{1-e} = a\,(1+e). \tag{3.35}$$

It follows that, for an elliptical orbit, the major radius, a, is simply the mean of the perihelion and aphelion distances,

$$a = \frac{r_p + r_a}{2}. \tag{3.36}$$

The parameter

$$e = \frac{r_a - r_p}{r_a + r_p} \tag{3.37}$$

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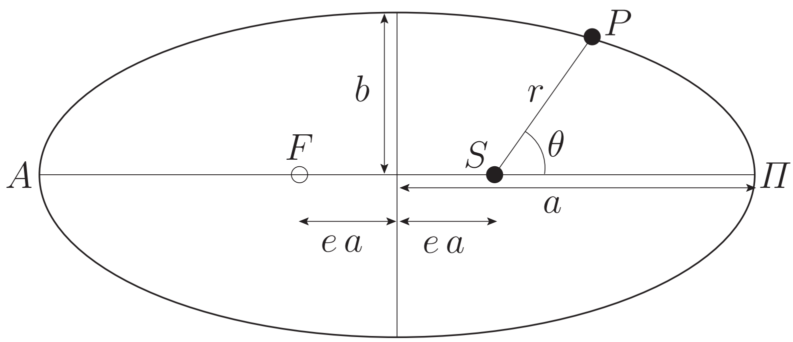

Fig. 3.3 A Keplerian elliptical orbit. S is the Sun, P the planet, F the empty focus, Π the perihelion point, A the aphelion point, a the major radius, b the minor radius, e the eccentricity, r the radial distance, and θ the true anomaly.

is called the *eccentricity*; it measures the deviation of the orbit from circularity. Thus, $e = 0$ corresponds to a circular orbit, whereas $e \to 1$ corresponds to an infinitely elongated elliptical orbit. Note that the Sun is displaced a distance $e\,a$ along the major axis from the geometric center of the orbit. (See Section A.9 and Figure 3.3.)

As is easily demonstrated from the preceding analysis, Kepler's laws of planetary motion can be written in the convenient form

$$r = \frac{a\,(1-e^2)}{1+e\,\cos\theta}, \tag{3.38}$$

$$h = r^2\,\dot{\theta} = (1-e^2)^{1/2}\,n\,a^2, \tag{3.39}$$

and

$$G\,M = n^2\,a^3, \tag{3.40}$$

where a is the mean orbital radius (i.e., the major radius), e the orbital eccentricity, and $n = 2\pi/T$ the mean orbital angular velocity.

3.9 Orbital energies

Let us now generalize our analysis to take into account the orbits of asteroids and comets about the Sun. Such orbits satisfy Equation (3.30) but can be parabolic ($e = 1$) or even hyperbolic ($e > 1$), as well as elliptical ($e < 1$). According to Equations (3.5) and (3.13), the total energy per unit mass of an object in a general orbit around the Sun is given by

$$\mathcal{E} = \frac{\dot{r}^2 + r^2\,\dot{\theta}^2}{2} - \frac{G\,M}{r}. \tag{3.41}$$

It follows from Equations (3.22), (3.26), and (3.31) that

$$\mathcal{E} = \frac{h^2}{2}\left[\left(\frac{du}{d\theta}\right)^2 + u^2 - 2\,u\,u_c\right], \tag{3.42}$$

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where $u = r^{-1}$ and $u_c = r_c^{-1}$. However, according to Equation (3.30),

$$u(\theta) = u_c\,(1 + e\,\cos\theta). \tag{3.43}$$

The previous two equations can be combined with Equations (3.31) and (3.34) to give

$$\mathcal{E} = \frac{u_c^2\,h^2}{2}\,(e^2 - 1) = \frac{G\,M}{2\,r_p}\,(e - 1). \tag{3.44}$$

We conclude that elliptical orbits ($e < 1$) have *negative* total energies, whereas parabolic orbits ($e = 1$) have *zero* total energies, and hyperbolic orbits ($e > 1$) have *positive* total energies. This makes sense because in a conservative system in which the potential energy at infinity is set to zero [see Equation (3.4)], we expect *bounded* orbits to have negative total energies, and *unbounded* orbits to have positive total energies. (See Section 1.7.) Thus, elliptical orbits, which are clearly bounded, should indeed have negative total energies, whereas hyperbolic orbits, which are clearly unbounded, should indeed have positive total energies. Parabolic orbits are marginally bounded (i.e, an object executing a parabolic orbit only just manages to escape from the Sun's gravitational field), and thus have zero total energy. For the special case of an elliptical orbit, whose major radius a is finite, we can write

$$\mathcal{E} = -\frac{G\,M}{2\,a}. \tag{3.45}$$

It follows that the energy of such an orbit is completely determined by its *major radius*.

3.10 Transfer orbits

Consider an artificial satellite in an elliptical orbit around the Sun (the same considerations also apply to satellites in orbit around the Earth). At perihelion, $\dot{r} = 0$, and Equations (3.41) and (3.44) can be combined to give

$$\frac{v_t}{v_c} = \sqrt{1 + e}. \tag{3.46}$$

Here, $v_t = r\,\dot{\theta}$ is the satellite's tangential velocity and $v_c = \sqrt{G\,M/r_p}$ is the tangential velocity that it would need to maintain a circular orbit at the perihelion distance. Likewise, at aphelion,

$$\frac{v_t}{v_c} = \sqrt{1 - e}, \tag{3.47}$$

where $v_c = \sqrt{G\,M/r_a}$ is now the tangential velocity that the satellite would need to maintain a circular orbit at the aphelion distance.

Suppose that our satellite is initially in a circular orbit of radius r_1 and that we wish to transfer it into a circular orbit of radius r_2, where $r_2 > r_1$. We can achieve this by temporarily placing the satellite in an elliptical orbit whose perihelion distance is r_1 and whose aphelion distance is r_2. It follows, from Equation (3.37), that the required eccentricity of the elliptical orbit is

$$e = \frac{r_2 - r_1}{r_2 + r_1}. \tag{3.48}$$

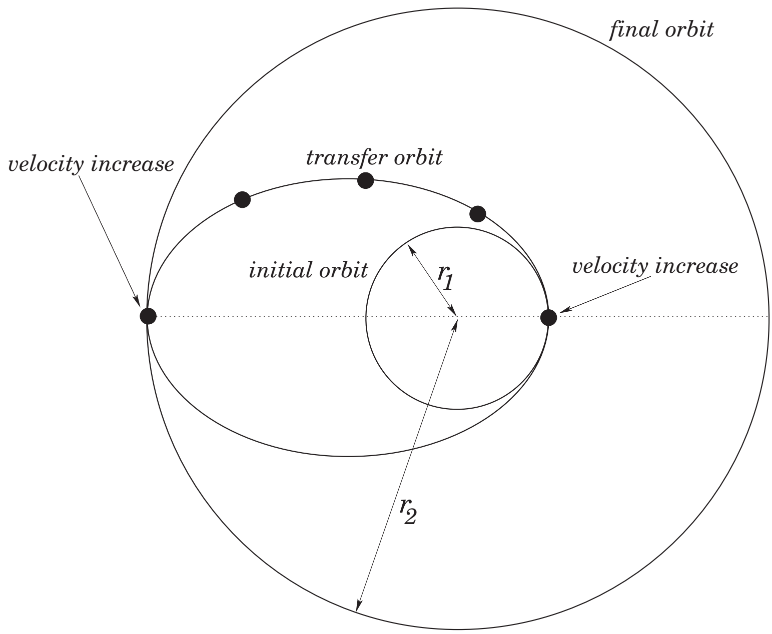

Fig. 3.4 A transfer orbit between two circular orbits.

According to Equation (3.46), we can transfer our satellite from its initial circular orbit into the temporary elliptical orbit by increasing its tangential velocity (by briefly switching on the satellite's rocket motor) by a factor

$$\alpha_1 = \sqrt{1+e}. \tag{3.49}$$

We must next allow the satellite to execute half an orbit, so it attains its aphelion distance, and then boost the tangential velocity by a factor [see Equation (3.47)]

$$\alpha_2 = \frac{1}{\sqrt{1-e}}. \tag{3.50}$$

The satellite will now be in a circular orbit at the aphelion distance, r_2. This process is illustrated in Figure 3.4. Obviously, we can transfer our satellite from a larger to a smaller circular orbit by performing the preceding process in reverse. Note, finally, from Equation (3.46) that if we increase the tangential velocity of a satellite in a circular orbit about the Sun by a factor greater than $\sqrt{2}$, then we will transfer it into a hyperbolic orbit ($e > 1$), and it will eventually escape from the Sun's gravitational field.

3.11 Elliptical orbits

Let us determine the radial and angular coordinates, r and θ, respectively, of a planet in an elliptical orbit about the Sun as a function of time. Suppose the planet passes through its perihelion point, $r = r_p$ and $\theta = 0$, at $t = \tau$. The constant τ is termed the *time*

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of perihelion passage. It follows from the previous analysis that

$$r = \frac{r_p\,(1+e)}{1+e\,\cos\theta}, \tag{3.51}$$

and

$$\mathcal{E} = \frac{\dot{r}^{\,2}}{2} + \frac{h^2}{2\,r^2} - \frac{G\,M}{r}, \tag{3.52}$$

where e, $h = \sqrt{G\,M\,r_p\,(1+e)}$, and $\mathcal{E} = G\,M\,(e-1)/(2\,r_p)$ are the orbital eccentricity, angular momentum per unit mass, and energy per unit mass, respectively. The preceding equation can be rearranged to give

$$\dot{r}^{\,2} = (e-1)\,\frac{G\,M}{r_p} - (e+1)\,\frac{r_p\,G\,M}{r^2} + \frac{2\,G\,M}{r}. \tag{3.53}$$

Taking the square root and integrating, we obtain

$$\int_{r_p}^{r} \frac{r\,dr}{[2\,r + (e-1)\,r^2/r_p - (e+1)\,r_p]^{1/2}} = \sqrt{G\,M}\,(t-\tau). \tag{3.54}$$

Consider an elliptical orbit characterized by $0 < e < 1$. Let us write

$$r = \frac{r_p}{1-e}\,(1 - e\,\cos E), \tag{3.55}$$

where E is termed the *eccentric anomaly*. In fact, E is an angle that varies between $-\pi$ and $+\pi$. Moreover, the perihelion point corresponds to $E = 0$, and the aphelion point to $E = \pi$. Now,

$$dr = \frac{r_p}{1-e}\,e\,\sin E\,dE, \tag{3.56}$$

whereas

$$2\,r + (e-1)\,\frac{r^2}{r_p} - (e+1)\,r_p = \frac{r_p}{1-e}\,e^2\,(1-\cos^2 E) = \frac{r_p}{1-e}\,e^2\,\sin^2 E. \tag{3.57}$$

Thus, Equation (3.54) reduces to

$$\int_0^E (1 - e\,\cos E)\,dE = \left(\frac{G\,M}{a^3}\right)^{1/2}(t-\tau), \tag{3.58}$$

where $a = r_p/(1-e)$. This equation can immediately be integrated to give

$$E - e\,\sin E = \mathcal{M}. \tag{3.59}$$

Here,

$$\mathcal{M} = n\,(t-\tau) \tag{3.60}$$

is termed the *mean anomaly*, $n = 2\pi/T$ is the mean orbital angular velocity, and $T = 2\pi\,(a^3/GM)^{1/2}$ is the orbital period. The mean anomaly is an angle that increases uniformly in time at the rate of 2π radians every orbital period. Moreover, the perihelion point corresponds to $\mathcal{M} = 0$, and the aphelion point to $\mathcal{M} = \pi$. Incidentally, the angle θ, which determines the true angular location of the planet relative to its perihelion point, is called the *true anomaly*. Equation (3.59), which is known as *Kepler's equation*, is

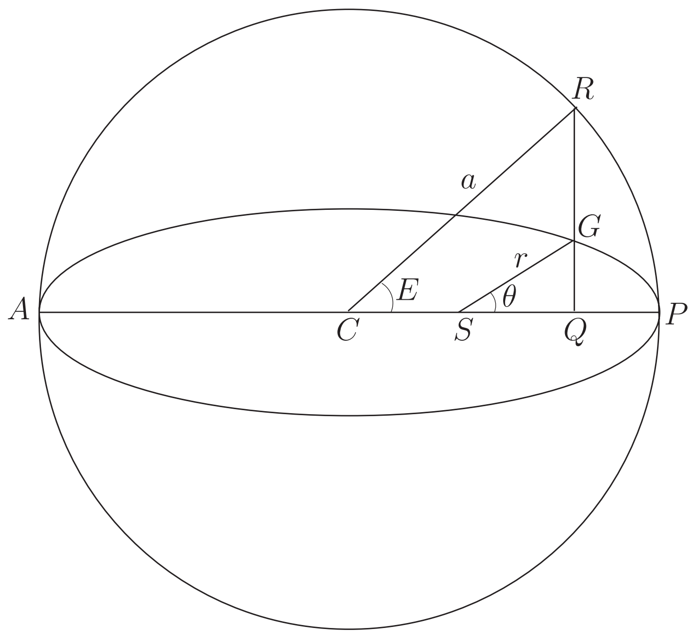

Fig. 3.5 Eccentric anomaly.

a transcendental equation that does not possess a convenient analytic solution. Fortunately, it is fairly straightforward to solve numerically. For instance, when we use an iterative approach, if E_n is the nth guess, then

$$E_{n+1} = \mathcal{M} + e\,\sin E_n. \tag{3.61}$$

This iteration scheme converges very rapidly when $0 \leq e \ll 1$ (as is the case for planetary orbits).

Equations (3.51) and (3.55) can be combined to give

$$r\,\cos\theta = a\,(\cos E - e). \tag{3.62}$$

This expression allows us to give a simple geometric interpretation of the eccentric anomaly, E. Consider Figure 3.5. Let PGA represent the elliptical orbit of a planet, G, about the Sun, S. Let ACP be the major axis of the orbit, where P is the perihelion point, A the aphelion point, and C the geometric center. It follows that $CA = CP = a$ and $CS = e\,a$ (see Section A.9), where a is the orbital major radius and e the eccentricity. Moreover, the distance SG and the angle GSP correspond to the radial distance, r, and the true anomaly, θ, respectively. Let PRA be a circle of radius a centered on C. It follows that AP is a diameter of this circle. Let RGQ be a line, perpendicular to AP, that passes through G and joins the circle to the diameter. It follows that $CR = a$. Let us denote the angle RCS as E. Simple trigonometry reveals that $SQ = r\,\cos\theta$ and $CQ = a\,\cos E$. But $CQ = CS + SQ$, or $a\,\cos E = e\,a + r\,\cos\theta$, which can be rearranged to give $r\,\cos\theta = a\,(\cos E - e)$, which is identical to Equation (3.62). We thus conclude that the eccentric anomaly, E, can be identified with the angle RCS in Figure 3.5.

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Equations (3.51) and (3.55) can be combined to give

$$\cos\theta = \frac{\cos E - e}{1 - e\,\cos E}. \tag{3.63}$$

Thus,

$$1 + \cos\theta = 2\,\cos^2(\theta/2) = \frac{2\,(1-e)\,\cos^2(E/2)}{1 - e\,\cos E}, \tag{3.64}$$

and

$$1 - \cos\theta = 2\,\sin^2(\theta/2) = \frac{2\,(1+e)\,\sin^2(E/2)}{1 - e\,\cos E}. \tag{3.65}$$

The previous two equations imply that

$$\tan(\theta/2) = \left(\frac{1+e}{1-e}\right)^{1/2}\tan(E/2). \tag{3.66}$$

The eccentric anomaly, E, and the true anomaly, θ, always lie in the same quadrant (i.e., if $0 \leq E \leq \pi/2$, then $0 \leq \theta \leq \pi/2$, etc.) We conclude that in the case of a planet in an elliptical orbit around the Sun, the radial distance, r, and the true anomaly, θ, are specified as functions of time via the solution of the following set of equations:

$$\mathcal{M} = n\,(t - \tau), \tag{3.67}$$

$$E - e\,\sin E = \mathcal{M}, \tag{3.68}$$

$$r = a\,(1 - e\,\cos E), \tag{3.69}$$

and

$$\tan(\theta/2) = \left(\frac{1+e}{1-e}\right)^{1/2}\tan(E/2). \tag{3.70}$$

Here, $n = 2\pi/T$, $T = 2\pi\,(a^3/G\,M)^{1/2}$, and $a = r_p/(1-e)$. Incidentally, it is clear that if $t \to t + T$, then $\mathcal{M} \to \mathcal{M} + 2\pi$, $E \to E + 2\pi$, and $\theta \to \theta + 2\pi$. In other words, the motion is periodic with period T.

3.12 Orbital elements

The previous analysis suffices when considering a single planet orbiting around the Sun. However, it becomes inadequate when dealing with multiple planets whose orbital planes and perihelion directions do not necessarily coincide. Incidentally, for the time being, we are neglecting interplanetary gravitational interactions, which allows us to assume that each planet executes an independent Keplerian elliptical orbit about the Sun.

Let us characterize all planetary orbits using a common Cartesian coordinate system X, Y, Z, centered on the Sun. (See Figure 3.6.) The X–Y plane defines a *reference plane*, which is chosen to be the *ecliptic plane* (i.e., the plane of the Earth's orbit), with the Z-axis pointing toward the ecliptic north pole (i.e., the direction normal to the ecliptic plane in a northward sense). Likewise, the X-axis defines a *reference direction*, which is chosen to point in the direction of the vernal equinox (i.e., the point in the Earth's

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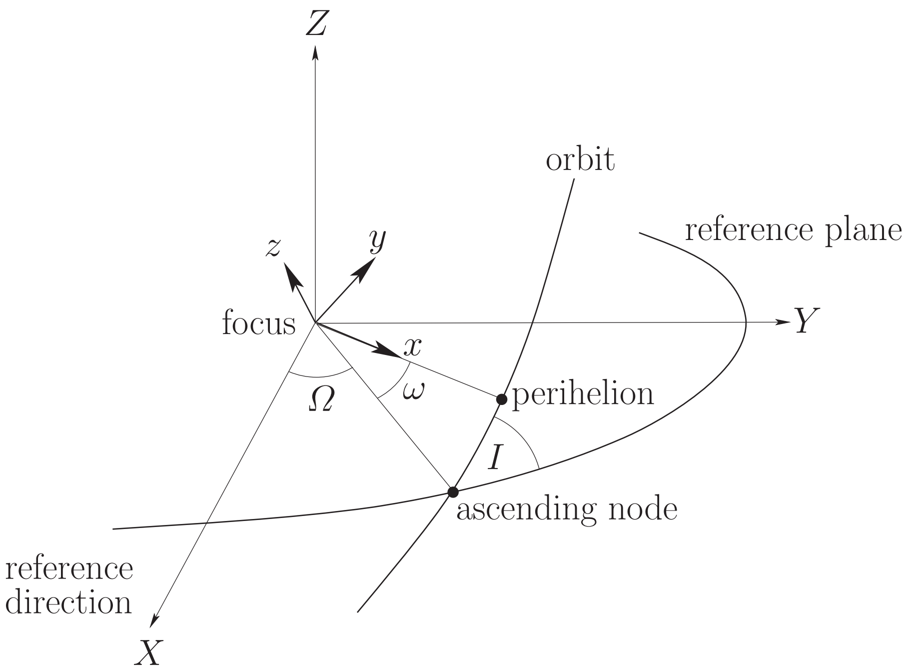

Fig. 3.6 A general planetary orbit.

sky at which the apparent orbit of the Sun passes through the extension of the Earth's equatorial plane from south to north). Suppose that the plane of a given planetary orbit is inclined at an angle I to the reference plane. The point at which this orbit crosses the reference plane in the direction of increasing Z is termed its *ascending node*. The angle Ω subtended between the reference direction and the direction of the ascending node is termed the *longitude of the ascending node*. Finally, the angle, ω, subtended between the direction of the ascending node and the direction of the orbit's perihelion, is termed the *argument of the perihelion*.

Let us define a second Cartesian coordinate system x, y, z, also centered on the Sun. Let the x–y plane coincide with the plane of a particular planetary orbit, and let the x-axis point toward the orbit's perihelion point. Clearly, we can transform from the x, y, z system to the X, Y, Z system via a series of three rotations of the coordinate system: first, a rotation through an angle ω about the z-axis (looking down the axis); second, a rotation through an angle I about the new x-axis; and finally, a rotation through an angle Ω about the new z-axis. It thus follows from standard coordinate transformation theory (see Section A.6) that

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix} \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{3.71}$$

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Table 3.1 Planetary data for J2000

Planet	a(AU)	$\bar{\lambda}_0(^\circ)$	e	$I(^\circ)$	$\varpi(^\circ)$	$\Omega(^\circ)$	T(yr)	m/M
Mercury	0.3871	252.25	0.20564	7.006	77.46	48.34	0.241	1.659×10^{-7}
Venus	0.7233	181.98	0.00676	3.398	131.77	76.67	0.615	2.447×10^{-6}
Earth	1.0000	100.47	0.01673	0.000	102.93	–	1.000	3.039×10^{-6}
Mars	1.5237	355.43	0.09337	1.852	336.08	49.71	1.881	3.226×10^{-7}
Jupiter	5.2025	34.33	0.04854	1.299	14.27	100.29	11.87	9.542×10^{-4}
Saturn	9.5415	50.08	0.05551	2.494	92.86	113.64	29.47	2.857×10^{-4}
Uranus	19.188	314.20	0.04686	0.773	172.43	73.96	84.05	4.353×10^{-5}
Neptune	30.070	304.22	0.00895	1.770	46.68	131.79	164.9	5.165×10^{-5}

a – major radius; $\bar{\lambda}_0$ – mean longitude at epoch; e – eccentricity; I – inclination to ecliptic; ϖ – longitude of perihelion; Ω – longitude of ascending node; T – orbital period; m/M – planetary mass / solar mass.

[a] Source: Standish and Williams 1992.

However, $x = r\cos\theta$, $y = r\sin\theta$, and $z = 0$. Hence,

$$X = r\left[\cos\Omega\,\cos(\omega+\theta) - \sin\Omega\,\sin(\omega+\theta)\,\cos I\right], \tag{3.72}$$

$$Y = r\left[\sin\Omega\,\cos(\omega+\theta) + \cos\Omega\,\sin(\omega+\theta)\,\cos I\right], \tag{3.73}$$

$$Z = r\,\sin(\omega+\theta)\,\sin I. \tag{3.74}$$

Thus, a general planetary orbit is determined by Equations (3.67)–(3.70) and (3.72)–(3.74) and is therefore parameterized by six orbital elements: the major radius, a; the time of perihelion passage, τ; the eccentricity, e; the inclination (to the ecliptic plane), I; the argument of the perihelion, ω; and the longitude of the ascending node, Ω. [The mean orbital angular velocity, in radians per year, is $n = 2\pi/a^{3/2}$, where a is measured in astronomical units. Here, an astronomical unit is the mean Earth–Sun distance, and corresponds to 1.496×10^{11} m (Yoder 1995).]

In low-inclination orbits, the argument of the perihelion is usually replaced by

$$\varpi = \Omega + \omega, \tag{3.75}$$

which is termed the *longitude of the perihelion*. Likewise, the time of perihelion passage, τ, is often replaced by the mean longitude at $t = 0$—otherwise known as the *mean longitude at epoch*—where the mean longitude is defined as

$$\bar{\lambda} = \varpi + \mathcal{M} = \varpi + n\,(t-\tau). \tag{3.76}$$

Thus, if $\bar{\lambda}_0$ denotes the mean longitude at epoch ($t = 0$), then

$$\bar{\lambda} = \bar{\lambda}_0 + n\,t, \tag{3.77}$$

where $\bar{\lambda}_0 = \varpi - n\,\tau$. The orbital elements of the major planets at the epoch J2000 (i.e., at 00:00 UT on January 1, 2000) are given in Table 3.1.

The *heliocentric* (i.e., as seen from the Sun) position of a planet is most conveniently expressed in terms of its ecliptic longitude, λ, and ecliptic latitude, β. This type of longitude and latitude is referred to the ecliptic plane, with the Sun as the origin. Moreover,

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the vernal equinox is defined to be the zero of longitude. It follows that

$$\tan\lambda = \frac{Y}{X} \tag{3.78}$$

and

$$\sin\beta = \frac{Z}{\sqrt{X^2+Y^2}}, \tag{3.79}$$

where (X, Y, Z) are the heliocentric Cartesian coordinates of the planet.

3.13 Planetary orbits

According to Table 3.1, the planets all have low-inclination orbits characterized by $I \ll 1$ (when I is expressed in radians). In this case, making use of the small angle approximations $\cos I \simeq 1$ and $\sin I \simeq I$, as well as some trigonometric identities (see Section A.3), we find that Equations (3.72)–(3.74) simplify to give

$$X \simeq r\,\cos(\theta + \varpi), \tag{3.80}$$

$$Y \simeq r\,\sin(\theta + \varpi), \tag{3.81}$$

and

$$Z \simeq r\,I\,\sin(\theta + \varpi - \Omega), \tag{3.82}$$

where we have made use of Equation (3.75). It thus follows from Equations (3.78) and (3.79) that

$$\lambda \simeq \theta + \varpi \tag{3.83}$$

and

$$\beta \simeq I\,\sin(\theta + \varpi - \Omega). \tag{3.84}$$

According to Table 3.1, the planets also have *low-eccentricity* orbits, characterized by $0 < e \ll 1$. In this situation, Equations (3.68)–(3.70) can be usefully solved via series expansion in e to give

$$\theta = \mathcal{M} + 2\,e\,\sin\mathcal{M} + \frac{5\,e^2}{4}\,\sin 2\mathcal{M} + \frac{e^3}{12}\,(13\,\sin 3\mathcal{M} - 3\,\sin\mathcal{M}) + \mathcal{O}(e^4), \tag{3.85}$$

$$\frac{r}{a} = 1 - e\,\cos\mathcal{M} + \frac{e^2}{2}\,(1 - \cos 2\mathcal{M}) + \frac{3\,e^3}{8}\,(\cos\mathcal{M} - \cos 3\mathcal{M}) + \mathcal{O}(e^4). \tag{3.86}$$

(See Section A.10.)

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The preceding expressions can be combined with Equations (3.67), (3.77), (3.83), and (3.84) to produce

$$n = \frac{2\pi}{a^{3/2}}, \tag{3.87}$$

$$\bar{\lambda} = \bar{\lambda}_0 + n\,t, \tag{3.88}$$

$$\frac{r}{a} \simeq 1 - e\,\cos(\bar{\lambda} - \varpi), \tag{3.89}$$

$$\lambda \simeq \bar{\lambda} + 2\,e\,\sin(\bar{\lambda} - \varpi), \tag{3.90}$$

and

$$\beta \simeq I\,\sin(\bar{\lambda} - \Omega). \tag{3.91}$$

Here, n is expressed in radians per year, and a in astronomical units. These equations, which are valid up to first order in small quantities (i.e., e and I), illustrate how a planet's six orbital elements—a, $\bar{\lambda}_0$, e, I, ϖ, and Ω—can be used to determine its approximate position relative to the Sun as a function of time. The planet reaches its perihelion point when the mean ecliptic longitude, $\bar{\lambda}$, becomes equal to the longitude of the perihelion, ϖ. Likewise, the planet reaches its aphelion point when $\bar{\lambda} = \varpi + \pi$. Furthermore, the ascending node corresponds to $\bar{\lambda} = \Omega$, and the point of furthest angular distance north of the ecliptic plane (at which $\beta = I$) corresponds to $\bar{\lambda} = \Omega + \pi/2$.

Consider the Earth's orbit about the Sun. As has already been mentioned, ecliptic longitude is measured relative to a point on the ecliptic circle—the circular path that the Sun appears to trace out against the backdrop of the stars—known as the *vernal equinox*. When the Sun reaches the vernal equinox, which it does every year on about March 20, day and night are equally long everywhere on the Earth (because the Sun lies in the Earth's equatorial plane). Likewise, when the Sun reaches the opposite point on the ecliptic circle, known as the *autumnal equinox*, which it does every year on about September 22, day and night are again equally long everywhere on the Earth. The points on the ecliptic circle halfway between the equinoxes are known as the *solstices*. When the Sun reaches the *summer solstice*, which it does every year on about June 21, this marks the longest day in the Earth's northern hemisphere and the shortest day in the southern hemisphere. Likewise, when the Sun reaches the *winter solstice*, which it does every year on about December 21, this marks the shortest day in the Earth's northern hemisphere and the longest day in the southern hemisphere. The period between (the Sun reaching) the vernal equinox and the summer solstice is known as *spring*, that between the summer solstice and the autumnal equinox as *summer*, that between the autumnal equinox and the winter solstice as *autumn*, and that between the winter solstice and the next vernal equinox as *winter*.

Let us calculate the approximate lengths of the seasons. It follows, from the preceding discussion, that the ecliptic longitudes of the Sun, relative to the Earth, at the (times at which the Sun reaches the) vernal equinox, summer solstice, autumnal equinox, and winter solstice are 0°, 90°, 180°, and 270°, respectively. Hence, the ecliptic longitudes, λ, of the Earth, relative to the Sun, at the same times are 180°, 270°, 0°, and 90°, respectively. The mean longitude, $\bar{\lambda}$, of the Earth increases *uniformly* in time at the rate of 360° per year. Thus, the length of a given season is simply the fraction $\Delta\bar{\lambda}/360°$ of a year, where $\Delta\bar{\lambda}$ is the change in mean longitude associated with the season. Equation (3.90) can be inverted to give

$$\bar{\lambda} \simeq \lambda - 2\,e\,\sin(\lambda - \varpi), \tag{3.92}$$

to first order in e. Hence, the mean longitudes associated with the autumnal equinox, winter solstice, vernal equinox, and summer solstice are

$$\bar{\lambda}_{AE} \simeq 0^\circ + 2\,e\,\sin\varpi = 1.87^\circ, \tag{3.93}$$

$$\bar{\lambda}_{WS} \simeq 90^\circ - 2\,e\,\cos\varpi = 90.43^\circ, \tag{3.94}$$

$$\bar{\lambda}_{VE} \simeq 180^\circ - 2\,e\,\sin\varpi = 178.13^\circ, \tag{3.95}$$

and

$$\bar{\lambda}_{SS} \simeq 270^\circ + 2\,e\,\cos\varpi = 269.57^\circ, \tag{3.96}$$

respectively. (Recall that, according to Table 3.1, $e = 0.01673$ radians and $\varpi = 102.93^\circ$ for the Earth.) Thus,

$$\Delta\bar{\lambda}_{\rm spring} \simeq 91.44^\circ, \tag{3.97}$$

$$\Delta\bar{\lambda}_{\rm summer} \simeq 92.30^\circ, \tag{3.98}$$

$$\Delta\bar{\lambda}_{\rm autumn} \simeq 88.56^\circ, \tag{3.99}$$

and

$$\Delta\bar{\lambda}_{\rm winter} \simeq 87.70^\circ. \tag{3.100}$$

(See Figure 3.7.) Given that the length of a tropical year (i.e., the mean period between successive vernal equinoxes) is 365.24 days, we deduce that spring, summer, autumn, and winter last 92.8, 93.6, 89.8, and 89.0 days, respectively. Clearly, although the deviations of the Earth's orbit from a uniform circular orbit that is concentric with the Sun seem relatively small, they are still large enough to cause a noticeable difference between the lengths of the various seasons. The preceding calculation was used, in reverse, by ancient Greek astronomers, such as Hipparchus, to determine the eccentricity, and the longitude of the perigee, of the Sun's apparent orbit about the Earth from the observed lengths of the seasons (Evans 1998).

3.14 Parabolic orbits

For the case of a *parabolic* orbit about the Sun, characterized by $e = 1$, similar analysis to that in Section 3.11 yields

$$P + \frac{P^3}{3} = \left(\frac{G\,M}{2\,r_p^3}\right)^{1/2} (t - \tau), \tag{3.101}$$

$$r = r_p\,(1 + P^2), \tag{3.102}$$

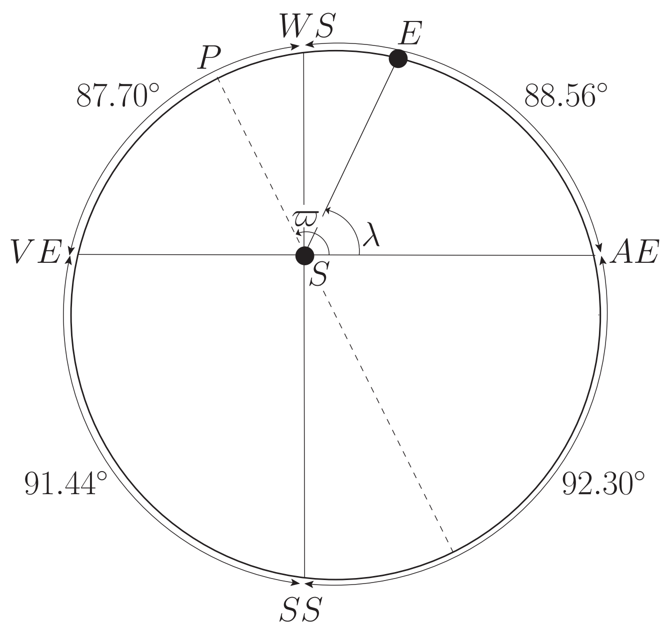

Fig. 3.7 A schematic diagram showing the orbit of the Earth, E, about the Sun, S, as well as the vernal equinox (VE), summer solstice (SS), autumnal equinox (AE), and winter solstice (WS). Here, λ is the Earth's ecliptic longitude, and ϖ the longitude of its perihelion (P).

and

$$\tan(\theta/2) = P. \tag{3.103}$$

Here, P is termed the *parabolic anomaly* and varies between $-\infty$ and $+\infty$, with the perihelion point corresponding to $P = 0$. Note that Equation (3.101) is a cubic equation, possessing a single real root, which can, in principle, be solved analytically. (See Exercise 3.18.) However, a numerical solution is generally more convenient.

3.15 Hyperbolic orbits

For the case of a *hyperbolic* orbit about the Sun, characterized by $e > 1$, similar analysis to that in Section 3.11 gives

$$e \sinh H - H = \left(\frac{G\,M}{a^3}\right)^{1/2} (t - \tau), \tag{3.104}$$

$$r = a\,(e \cosh H - 1), \tag{3.105}$$

and

$$\tan(\theta/2) = \left(\frac{e+1}{e-1}\right)^{1/2} \tanh(H/2). \tag{3.106}$$

Here, H is termed the *hyperbolic anomaly* and varies between $-\infty$ and $+\infty$, with the perihelion point corresponding to $H = 0$. Moreover, $a = r_p/(e-1)$. As in the elliptical case, Equation (3.104) is a transcendental equation that is most easily solved numerically.

3.16 Binary star systems

Approximately half the stars in our galaxy are members of so-called *binary star systems*. Such systems consist of two stars, of mass m_1 and m_2, and position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, orbiting about their common center of mass. The distance separating the stars is generally much less than the distance to the nearest neighbor star. Hence, a binary star system can be treated as a two-body dynamical system to a very good approximation.

In a binary star system, the gravitational force that the first star exerts on the second is

$$\mathbf{f} = -\frac{G\,m_1\,m_2}{r^3}\,\mathbf{r}, \tag{3.107}$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. As we have seen in Section 1.9, a two-body system can be reduced to an equivalent one-body system whose equation of motion is of the form given in Equation (1.76), where $\mu = m_1\,m_2/(m_1 + m_2)$. Hence, in this particular case, we can write

$$\frac{m_1\,m_2}{m_1 + m_2}\,\frac{d^2\mathbf{r}}{dt^2} = -\frac{G\,m_1\,m_2}{r^3}\,\mathbf{r}, \tag{3.108}$$

which gives

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{G\,M}{r^3}\,\mathbf{r}, \tag{3.109}$$

where

$$M = m_1 + m_2. \tag{3.110}$$

Equation (3.109) is identical to Equation (3.2), which we have already solved. Hence, we can immediately write down the solution:

$$\mathbf{r} = (r\,\cos\theta,\ r\,\sin\theta,\ 0), \tag{3.111}$$

where

$$r = \frac{a\,(1-e^2)}{1 + e\,\cos\theta} \tag{3.112}$$

and

$$\frac{d\theta}{dt} = \frac{h}{r^2}, \tag{3.113}$$

with

$$a = \frac{h^2}{(1-e^2)\,G\,M}. \tag{3.114}$$

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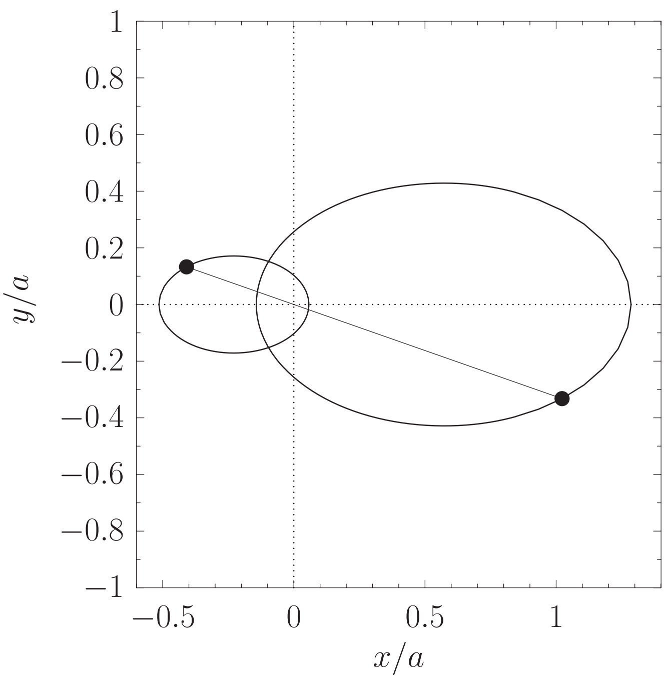

Fig. 3.8 An example binary star orbit.

Here, h is a constant, and we have aligned our Cartesian axes such that the plane of the orbit coincides with the x–y plane. According to this solution, the second star executes a Keplerian elliptical orbit, with major radius a and eccentricity e, relative to the first star, and vice versa. From Equation (3.33), the period of revolution, T, is given by

$$T = \sqrt{\frac{4\pi^2\, a^3}{G\, M}}. \tag{3.115}$$

Moreover, if $n = 2\pi/T$, then

$$n = \frac{\sqrt{G\, M}}{a^{3/2}}. \tag{3.116}$$

In the *inertial* frame of reference whose origin always coincides with the center of mass—the so-called *center of mass frame*—the position vectors of the two stars are

$$\mathbf{r}_1 = -\frac{m_2}{m_1 + m_2}\,\mathbf{r} \tag{3.117}$$

and

$$\mathbf{r}_2 = \frac{m_1}{m_1 + m_2}\,\mathbf{r}, \tag{3.118}$$

where $\mathbf{r}$ is specified in Equation (3.111). Figure 3.8 shows an example binary star orbit in the center of mass frame, calculated with $m_1/m_2 = 0.4$ and $e = 0.8$. It can be seen that both stars execute elliptical orbits about their common center of mass and, at any point in time, are diagrammatically opposite one another, relative to the origin.

Binary star systems have been very useful to astronomers, as it is possible to determine the masses of both stars in such a system by careful observation. The sum of the masses of the two stars, $M = m_1 + m_2$, can be found from Equation (3.115) after a measurement of the major radius, a (which is the mean of the greatest and smallest distance apart of the two stars during their orbit), and the orbital period, T. The ratio of the masses of the two stars, m_1/m_2, can be determined from Equations (3.117) and (3.118) by observing the fixed ratio of the relative distances of the two stars from the common center of mass about which they both appear to rotate. Obviously, given the sum of the masses and the ratio of the masses, the individual masses themselves can then be calculated.

Exercises

3.1 Demonstrate that if a particle moves in a central force field with zero angular momentum, so that $\mathbf{h} \equiv \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{0}$, then the particle's trajectory lies on a fixed straight line that passes through the origin. [Hint: Show that $d/dt(\mathbf{r}/r) = \mathbf{0}$.]

3.2 Demonstrate that $h = r^2\,\dot{\theta}$ is the magnitude of the angular momentum (per unit mass) vector $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$. Here, r and θ are plane polar coordinates.

3.3 Consider a planet in a Keplerian elliptical orbit about the Sun. Let $\mathbf{r}$ be the planet's position vector, relative to the Sun, and let $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ be its angular momentum per unit mass. Demonstrate that the so-called *Laplace-Runge-Lenz* vector,

$$\mathbf{l} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{G\,M} - \frac{\mathbf{r}}{r},$$

can be written

$$\mathbf{l} = e\,\cos\theta\;\mathbf{e}_r - e\,\sin\theta\;\mathbf{e}_\theta,$$

where M is the solar mass, e the orbital eccentricity, and r, θ are plane polar coordinates in the orbital plane (with the perihelion corresponding to $\theta = 0$). Hence, show that $\mathbf{l}$ is a *constant* vector, of length e, that is directed from the Sun toward the perihelion point.

3.4 Given the Sun's mean apparent radius seen from the Earth ($16'$), the Earth's mean apparent radius seen from the Moon ($57'$), and the mean number of lunar revolutions in a year (13.4), show that the ratio of the Sun's mean density to that of the Earth is 0.252. (From Lamb 1923.)

3.5 Prove that the orbital period of a satellite close to the surface of a spherical planet depends on the mean density of the planet, but not on its size. Show that if the mean density is that of water, the period is 3 h. 18 m. (From Lamb 1923.)

3.6 Jupiter's satellite Ganymede has an orbital period of 7 d. 3 h. 43 m. and a mean orbital radius that is 15.3 times the mean radius of the planet. The Moon has an orbital period of 27 d. 7 h. 43 m. and a mean orbital radius that is 60.3 times the Earth's mean radius. Show that the ratio of Jupiter's mean density to that of the Earth is 0.238. (From Lamb 1923.)

3.7 Halley's comet has an orbital eccentricity of 0.967 and a perihelion distance of 55,000,000 miles. Find the orbital period and the comet's speed at perihelion and aphelion.

3.8 Show that the velocity at any point on a Keplerian elliptical orbit can be resolved into two constant components: a velocity $n\,a/\sqrt{1-e^2}$ at right angles to the radius vector, and a velocity $n\,a\,e/\sqrt{1-e^2}$ at right angles to the major axis. Here, n is the mean orbital angular velocity, a the major radius, and e the eccentricity. (From Lamb 1923.)

3.9 The *latus rectum* of a conic section is a chord that passes through a focus; it is perpendicular to the major axis (or the symmetry axis, in the case of a parabola or a hyperbola). Show that, for a body in a Keplerian orbit around the Sun, the maximum value of the radial speed occurs at the points where the latus rectum (associated with the non-empty focus, in the case of an ellipse) intersects the orbit, and that this maximum value is $e\,h/[r_p\,(1+e)]$. Here, h is the angular momentum per unit mass, e the orbital eccentricity, and r_p the perihelion distance.

3.10 A comet is observed a distance R astronomical units from the Sun, traveling at a speed that is V times the Earth's mean orbital speed. Show that the orbit of the comet is hyperbolic, parabolic, or elliptical, depending on whether the quantity $V^2 R$ is greater than, equal to, or less than 2, respectively. (Modified from Fowles and Cassiday 2005.)

3.11 Consider a planet in an elliptical orbit of major radius a and eccentricity e about the Sun. Suppose that the eccentricity of the orbit is small ($0 < e \ll 1$), as is indeed the case for all of the planets. Demonstrate that, to first order in e, the orbit can be approximated as a circle whose center is shifted a distance $e\,a$ from the Sun, and that the planet's angular motion appears uniform when viewed from a point (called the equant) that is shifted a distance $2\,e\,a$ from the Sun, in the same direction as the center of the circle. [This theorem is the basis of Ptolemy's model of planetary motion (Evans 1998).]

3.12 How long (in days) does it take the Sun–Earth radius vector to rotate through 90°, starting at the perihelion point? How long does it take starting at the aphelion point? The period and eccentricity of the Earth's orbit are $T = 365.24$ days and $e = 0.01673$ radians, respectively.

3.13 If θ is the Sun's ecliptic longitude, measured from the perigee (the point of closest approach to the Earth), show that the Sun's apparent diameter is given by

$$D \simeq D_1\,\cos^2(\theta/2) + D_2\,\sin^2(\theta/2),$$

where D_1 and D_2 are the greatest and least values of D. (From Lamb 1923.)

3.14 Show that the time-averaged apparent diameter of the Sun, as seen from a planet describing a low-eccentricity elliptical orbit, is approximately equal to the apparent diameter when the planet's distance from the Sun equals the major radius of the orbit. (From Lamb 1923.)

3.15 Consider an asteroid orbiting the Sun. Demonstrate that, at fixed orbital energy, the orbit that maximizes the orbital angular momentum is circular.

3.16 Derive Equations (3.101)–(3.103).

3.17 Derive Equations (3.104)–(3.106).

3.18 A parabolic Keplerian orbit is specified by Equation (3.101), which can be written

$$P + \frac{P^3}{3} = \mathcal{M},$$

where P is the parabolic anomaly and $\mathcal{M} = (G\,M/2\,r_p^3)^{1/2}\,(t-\tau)$ is termed the *parabolic mean anomaly*. Here, M is the solar mass, r_p the perihelion distance, and τ the time of perihelion passage. Demonstrate that the preceding equation has the analytic solution

$$P = \frac{1}{2}\,Q^{1/3} - 2\,Q^{-1/3},$$

where

$$Q = 12\,\mathcal{M} + 4\,\sqrt{4 + 9\,\mathcal{M}^2}.$$

3.19 Consider a comet in an elliptical orbit about the Sun. Let x and y be Cartesian coordinates in the orbital plane, such that $x = y = 0$ corresponds to the Sun and the x-axis is parallel to the orbital major axis. Demonstrate that

$$x = a\,(\cos E - e)$$

and

$$y = a\,(1-e^2)^{1/2}\,\sin E,$$

where a is the orbital major radius, e the eccentricity, and E the eccentric anomaly.

3.20 Consider a comet in a parabolic orbit about the Sun. Let x and y be Cartesian coordinates in the orbital plane, such that $x = y = 0$ corresponds to the Sun and the x-axis is parallel to the orbital symmetry axis. Demonstrate that

$$\begin{aligned} x &= r_p\,(1-P^2), \\ y &= 2\,r_p\,P, \end{aligned}$$

where r_p is the perihelion distance and P the parabolic anomaly.

3.21 Consider a comet in an hyperbolic orbit about the Sun. Let x and y be Cartesian coordinates in the orbital plane, such that $x = y = 0$ corresponds to the Sun and the x-axis is parallel to the orbital symmetry axis. Demonstrate that

$$\begin{aligned} x &= a\,(e - \cosh H), \\ y &= a\,(e^2-1)^{1/2}\,\sinh H, \end{aligned}$$

where a is the orbital major radius, e the eccentricity, and H the hyperbolic anomaly.

3.22 Consider a comet in an elliptical orbit about the Sun (*Lambert's Theorem*). If r_1 and r_2 are the radial distances from the Sun of two neighboring points, C_1 and C_2, on the orbit, and if s is the length of the straight line joining these two points, prove that the time, t, required for the comet to move from C_1 to C_2 is

$$\Delta t = \frac{T}{2\pi}\left[(\eta - \sin\eta) - (\xi - \sin\xi)\right],$$

where

$$\sin(\eta/2) = \frac{1}{2}\left(\frac{r_1 + r_2 + s}{a}\right)^{1/2},$$
$$\sin(\xi/2) = \frac{1}{2}\left(\frac{r_1 + r_2 - s}{a}\right)^{1/2}.$$

Here, T and a are the period and the major radius of the orbit, respectively.

3.23 Consider a comet in a parabolic orbit about the Sun (*Euler's Theorem*). If r_1 and r_2 are the radial distances from the Sun of two neighboring points, C_1 and C_2, on the orbit, and if s is the length of the straight line joining these two points, prove that the time required for the comet to move from C_1 to C_2 is

$$\Delta t = \frac{T}{12\pi}\left[\left(\frac{r_1 + r_2 + s}{a}\right)^{3/2} - \left(\frac{r_1 + r_2 - s}{a}\right)^{3/2}\right],$$

where T and a are the period and the major radius of the Earth's orbit, respectively.

3.24 Consider a comet in a hyperbolic orbit about the Sun. If r_1 and r_2 are the radial distances from the Sun of two neighboring points, C_1 and C_2, on the orbit, and if s is the length of the straight line joining these two points, prove that the time, t, required for the comet to move from C_1 to C_2 is

$$\Delta t = \frac{T}{2\pi}\left[(\sinh\eta - \eta) - (\sinh\xi - \xi)\right],$$

where

$$\sinh(\eta/2) = \frac{1}{2}\left(\frac{r_1 + r_2 + s}{a}\right)^{1/2}$$

and

$$\sinh(\xi/2) = \frac{1}{2}\left(\frac{r_1 + r_2 - s}{a}\right)^{1/2}.$$

Here, a is major radius of the orbit and T is the period of an elliptical orbit with the same major radius. (From Smart 1951.)

3.25 A comet is in a parabolic orbit that lies in the plane of the Earth's orbit. Regarding the Earth's orbit as a circle of radius a, show that the points at which the comet intersects the Earth's orbit are given by

$$\cos\theta = -1 + \frac{2\,r_p}{a},$$

where r_p is the perihelion distance of the comet, defined at $\theta = 0$. Demonstrate that the time interval that the comet remains inside the Earth's orbit is the fraction

$$\frac{2^{1/2}}{3\pi}\left(\frac{2\,r_p}{a} + 1\right)\left(1 - \frac{r_p}{a}\right)^{1/2}$$

of a year, and that the maximum value of this time interval is $2/3\pi$ year, or about 11 weeks.

3.26 The orbit of a comet around the Sun is a hyperbola of eccentricity e, lying in the ecliptic plane, whose least distance from the Sun is $1/n$ times the radius of the Earth's orbit (which is approximated as a circle). Prove that the time that the

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comet remains within the Earth's orbit is $(T/\pi)\,(e\,\sinh\phi - \phi)$, where $e\,\cosh\phi = 1 - n\,(1-e)$, and T is the periodic time of a planet describing an elliptic orbit whose major radius is equal to that of the hyperbolic orbit. (From Smart 1951.)

3.27 Consider a comet in a hyperbolic orbit focused on the Sun. The *impact parameter*, b, is defined as the the distance of closest approach in the absence of any gravitational attraction between the comet and the Sun. Demonstrate that $b = h/(2\,\mathcal{E})^{1/2}$, where h is the comet's angular momentum per unit mass and $\mathcal{E}$ its energy per unit mass. Show that the relationship between the impact parameter, b, and the true distance of closest approach, r_p, is

$$r_p = \frac{2\,b}{\alpha + \sqrt{\alpha^2+4}},$$

where $\alpha = G\,M/(\mathcal{E}\,b)$ and M is the solar mass. Hence, deduce that if the comet is to avoid hitting the Sun, then

$$\alpha < \frac{b}{R} - \frac{R}{b}$$

(assuming that $b > R$), where R is the solar radius.

3.28 Spectroscopic analysis has revealed that Spica is a double star whose components revolve around one another with a period of 4.1 days, the greatest relative orbital velocity being 36 miles per second. Show that the mean distance between the components of the star is 2.03×10^6 miles, and that the total mass of the system is 0.083 times that of the Sun. The mean distance of the Earth from the Sun is 92.75 million miles. (From Lamb 1923.)