9 Secular perturbation theory

9.1 Introduction

The two-body orbit theory described in Chapter 3 neglects the direct gravitational interactions between the planets, while retaining those between each individual planet and the Sun. This is an excellent first approximation, as the former interactions are much weaker than the latter, as a consequence of the small masses of the planets relative to the Sun. (See Table 3.1.) Nevertheless, interplanetary gravitational interactions do have a profound influence on planetary orbits when integrated over long periods of time. In this chapter, a branch of celestial mechanics known as *orbital perturbation theory* is used to examine the *secular* (i.e., long-term) influence of interplanetary gravitational perturbations on planetary orbits. Orbital perturbation theory is also used to investigate the secular influence of planetary perturbations on the orbits of asteroids, as well as the secular influence of the Earth's oblateness on the orbits of artificial satellites.

9.2 Evolution equations for a two-planet solar system

For the moment, let us consider a simplified solar system that consists of the Sun and two planets. (See Figure 9.1.) Let the Sun be of mass M and position vector $\mathbf{R}_s$. Likewise, let the two planets have masses m and m' and position vectors $\mathbf{R}$ and $\mathbf{R}'$, respectively. Here, we are assuming that $m,\, m' \ll M$. Finally, let $\mathbf{r} = \mathbf{R} - \mathbf{R}_s$ and $\mathbf{r}' = \mathbf{R}' - \mathbf{R}_s$ be the position vectors of each planet relative to the Sun. Without loss of generality, we can assume that $r' > r$.

In an inertial reference frame, the equations of motion of the various elements of our simplified solar system are

$$M\,\ddot{\mathbf{R}}_s = G\,M\,m\,\frac{\mathbf{r}}{r^3} + G\,M\,m'\,\frac{\mathbf{r}'}{r'^3}, \tag{9.1}$$

$$m\,\ddot{\mathbf{R}} = G\,m\,m'\,\frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} - G\,m\,M\,\frac{\mathbf{r}}{r^3}, \tag{9.2}$$

and

$$m'\,\ddot{\mathbf{R}}' = G\,m'\,m\,\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} - G\,m'\,M\,\frac{\mathbf{r}'}{r'^3}. \tag{9.3}$$

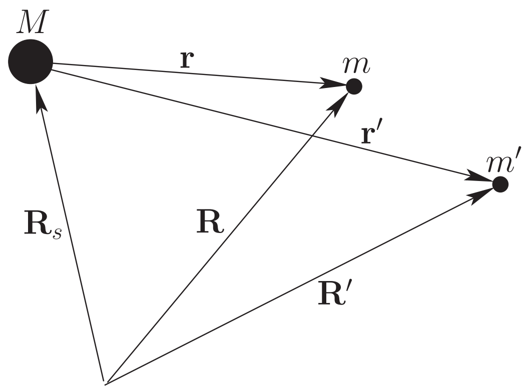

Fig. 9.1 A simplified model of the solar system.

It thus follows that

$$\ddot{\mathbf{r}} + \mu\,\frac{\mathbf{r}}{r^3} = G\,m'\left(\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} - \frac{\mathbf{r}'}{r'^3}\right) \tag{9.4}$$

and

$$\ddot{\mathbf{r}}' + \mu'\,\frac{\mathbf{r}'}{r'^3} = G\,m\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} - \frac{\mathbf{r}}{r^3}\right), \tag{9.5}$$

where $\mu = G\,(M+m)$ and $\mu' = G\,(M+m')$. The right-hand sides of these equations specify the interplanetary interaction forces that were neglected in our previous analysis. These right-hand sides can be conveniently expressed as the gradients of potentials:

$$\ddot{\mathbf{r}} + \mu\,\frac{\mathbf{r}}{r^3} = \nabla\mathcal{R} \tag{9.6}$$

and

$$\ddot{\mathbf{r}}' + \mu'\,\frac{\mathbf{r}'}{r'^3} = \nabla'\mathcal{R}', \tag{9.7}$$

where

$$\mathcal{R}(\mathbf{r},\mathbf{r}') = \tilde{\mu}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{\mathbf{r}\cdot\mathbf{r}'}{r'^3}\right) \tag{9.8}$$

and

$$\mathcal{R}'(\mathbf{r},\mathbf{r}') = \tilde{\mu}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}\right), \tag{9.9}$$

with $\tilde{\mu} = G\,m$ and $\tilde{\mu}' = G\,m'$. Here, $\mathcal{R}(\mathbf{r},\mathbf{r}')$ and $\mathcal{R}'(\mathbf{r},\mathbf{r}')$ are termed *disturbing functions*. Moreover, ∇ and ∇' are the gradient operators involving the unprimed and primed coordinates, respectively.

In the absence of the second planet, the orbit of the first planet is fully described by its six standard orbital elements (which are constants of its motion): the *major radius*, a; the *mean longitude at epoch*, $\bar{\lambda}_0$; the *eccentricity*, e; the *inclination* (to the ecliptic plane), I; the *longitude of the perihelion*, ϖ; and *the longitude of the ascending node*, Ω. (See Section 3.12.) As described in Appendix B, the perturbing influence of the second planet

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

causes these elements to slowly evolve in time. Such time-varying orbital elements are generally known as *osculating elements*.[1] Actually, when describing the aforementioned evolution, it is more convenient to work in terms of an alternative set of osculating elements, namely $a(t) = a^{(0)}[1 + \epsilon'\, a^{(1)}(t)]$, $\bar{\lambda}(t) = \bar{\lambda}_0 + n^{(0)}\, t + \lambda^{(1)}(t)$, $h = e\, \sin\varpi$, $k = e\, \cos\varpi$, $p = \sin I\, \sin\Omega$, and $q = \sin I\, \cos\Omega$. Here, $\epsilon' = \tilde{\mu}'/\mu = m'/(M+m) \ll 1$, $n(t) = n^{(0)}[1-(3/2)\,\epsilon'\, a^{(1)}(t)]$, where $n^{(0)} = (\mu/[a^{(0)}]^3)^{1/2}$ is the unperturbed mean orbital angular velocity. In the following, for ease of notation, $a^{(0)}$ and $n^{(0)}$ are written simply as a and n, respectively. Furthermore, $\bar{\lambda}$ will be used as shorthand for $\bar{\lambda}_0 + n^{(0)}\, t$. The evolution equations for the first planet's osculating orbital elements take the form (see Section C.2)

$$\frac{d\epsilon'\, a^{(1)}}{dt} = \epsilon'\, n\left[2\,\alpha\,\frac{\partial(\mathcal{S}_0+\mathcal{S}_1)}{\partial\bar{\lambda}}\right], \tag{9.10}$$

$$\frac{d\bar{\lambda}^{(1)}}{dt} = \epsilon'\, n\left[-\frac{3}{2}\,a^{(1)} - 2\,\alpha^2\,\frac{\partial(\mathcal{S}_0+\mathcal{S}_1)}{\partial\alpha} + \alpha\left(h\,\frac{\partial\mathcal{S}_1}{\partial h} + k\,\frac{\partial\mathcal{S}_1}{\partial k}\right)\right], \tag{9.11}$$

$$\frac{dh}{dt} = \epsilon'\, n\left[-\alpha\, h\,\frac{\partial\mathcal{S}_0}{\partial\bar{\lambda}} + \alpha\,\frac{\partial(\mathcal{S}_1+\mathcal{S}_2)}{\partial k}\right], \tag{9.12}$$

$$\frac{dk}{dt} = \epsilon'\, n\left[-\alpha\, k\,\frac{\partial\mathcal{S}_0}{\partial\bar{\lambda}} - \alpha\,\frac{\partial(\mathcal{S}_1+\mathcal{S}_2)}{\partial h}\right], \tag{9.13}$$

$$\frac{dp}{dt} = \epsilon'\, n\left[-\frac{\alpha}{2}\,p\,\frac{\partial\mathcal{S}_0}{\partial\bar{\lambda}} + \alpha\,\frac{\partial\mathcal{S}_2}{\partial q}\right], \tag{9.14}$$

and

$$\frac{dq}{dt} = \epsilon'\, n\left[-\frac{\alpha}{2}\,q\,\frac{\partial\mathcal{S}_0}{\partial\bar{\lambda}} - \alpha\,\frac{\partial\mathcal{S}_2}{\partial p}\right], \tag{9.15}$$

where (see Section C.3)

$$\mathcal{S}_0 = \frac{1}{2}\sum_{j=-\infty,\infty} b_{1/2}^{(j)}\,\cos[j\,(\bar{\lambda}-\bar{\lambda}')] - \alpha\,\cos(\bar{\lambda}-\bar{\lambda}'), \tag{9.16}$$

$$\begin{aligned}\mathcal{S}_1 = {} & \frac{1}{2}\sum_{j=-\infty,\infty}\Big\{k\,(-2\,j-\alpha\,D)\,b_{1/2}^{(j)}\,\cos[(1-j)\,\bar{\lambda}+j\,\bar{\lambda}'] \\ & + h\,(-2\,j-\alpha\,D)\,b_{1/2}^{(j)}\,\sin[(1-j)\,\bar{\lambda}+j\,\bar{\lambda}'] \\ & + k'\,(-1+2\,j+\alpha\,D)\,b_{1/2}^{(j-1)}\,\cos[(1-j)\,\bar{\lambda}+j\,\bar{\lambda}'] \\ & + h'\,(-1+2\,j+\alpha\,D)\,b_{1/2}^{(j-1)}\,\sin[(1-j)\,\bar{\lambda}+j\,\bar{\lambda}']\Big\} \\ & + \frac{\alpha}{2}\Big\{-k\,\cos(2\,\bar{\lambda}-\bar{\lambda}') - h\,\sin(2\,\bar{\lambda}-\bar{\lambda}') + 3\,k\,\cos\bar{\lambda}' + 3\,h\,\sin\bar{\lambda}' \\ & - 4\,k'\,\cos(\bar{\lambda}-2\,\bar{\lambda}') + 4\,h'\,\sin(\bar{\lambda}-2\,\bar{\lambda}')\Big\},\end{aligned} \tag{9.17}$$

[1] In mathematical terminology, two curves are said to osculate when they touch one another so as to have a common tangent at the point of contact. From the Latin *osculatus*, "kissed."

and

$$\mathcal{S}_2 = \frac{1}{8}(h^2 + k^2 + h'^2 + k'^2)(2\,\alpha\,D + \alpha^2\,D^2)\,b_{1/2}^{(0)} - \frac{1}{8}(p^2 + q^2 + p'^2 + q'^2)\,\alpha\,b_{3/2}^{(1)}$$
$$+ \frac{1}{4}(k\,k' + h\,h')(2 - 2\,\alpha\,D - \alpha^2\,D^2)\,b_{1/2}^{(1)} + \frac{1}{4}(p\,p' + q\,q')\,\alpha\,b_{3/2}^{(1)}. \tag{9.18}$$

Here, $\alpha = a/a'$, $D \equiv d/d\alpha$, and

$$b_s^{(j)}(\alpha) = \frac{1}{\pi}\int_0^{2\pi}\frac{\cos(j\,\psi)\,d\psi}{[1 - 2\,\alpha\,\cos\psi + \alpha^2]^s}, \tag{9.19}$$

where a', $\bar{\lambda}'$, h', k', p', q' are the osculating orbital elements of the second planet. The $b_s^{(j)}$ factors are known as *Laplace coefficients* (Brouwer and Clemence 1961). In deriving these expressions from Equations (9.6) and (9.8), we have expanded to first order in the ratio of the planetary masses to the solar mass; we have then evaluated the secular terms in the disturbing functions (i.e., the terms that are independent of $\bar{\lambda}$ and $\bar{\lambda}'$) to second order in the orbital eccentricities and inclinations. The nonsecular terms in the disturbing functions are evaluated to first order in the eccentricities and inclinations. (See Appendix C.) This expansion procedure is reasonable because the planets all have very small masses compared with that of the Sun, and they also have relatively small orbital eccentricities and inclinations.

There is an analogous set of equations, which can be derived from Equations (9.7) and (9.9), that describe the time evolution of the osculating orbital elements of the second planet due to the perturbing influence of the first. These take the form (see Section C.2)

$$\frac{d\epsilon' a^{(1)'}}{dt} = \epsilon\,n'\left[2\,\alpha^{-1}\,\frac{\partial(\mathcal{S}_0' + \mathcal{S}_1')}{\partial\bar{\lambda}'}\right], \tag{9.20}$$

$$\frac{d\bar{\lambda}^{(1)'}}{dt} = \epsilon\,n'\left[-\frac{3}{2}\,a^{(1)'} + 2\,\frac{\partial(\mathcal{S}_0' + \mathcal{S}_1')}{\partial\alpha} + \alpha^{-1}\left(h'\,\frac{\partial\mathcal{S}_1'}{\partial h'} + k'\,\frac{\partial\mathcal{S}_1'}{\partial k'}\right)\right], \tag{9.21}$$

$$\frac{dh'}{dt} = \epsilon\,n'\left[-\alpha^{-1}\,h'\,\frac{\partial\mathcal{S}_0'}{\partial\bar{\lambda}'} + \alpha^{-1}\,\frac{\partial(\mathcal{S}_1' + \mathcal{S}_2')}{\partial k'}\right], \tag{9.22}$$

$$\frac{dk'}{dt} = \epsilon\,n'\left[-\alpha^{-1}\,k'\,\frac{\partial\mathcal{S}_0'}{\partial\bar{\lambda}'} - \alpha^{-1}\,\frac{\partial(\mathcal{S}_1' + \mathcal{S}_2')}{\partial h'}\right], \tag{9.23}$$

$$\frac{dp'}{dt} = \epsilon\,n'\left[-\frac{\alpha^{-1}}{2}\,p'\,\frac{\partial\mathcal{S}_0'}{\partial\bar{\lambda}'} + \alpha^{-1}\,\frac{\partial\mathcal{S}_2'}{\partial q'}\right], \tag{9.24}$$

and

$$\frac{dq'}{dt} = \epsilon\,n'\left[-\frac{\alpha^{-1}}{2}\,q'\,\frac{\partial\mathcal{S}_0'}{\partial\bar{\lambda}'} - \alpha^{-1}\,\frac{\partial\mathcal{S}_2'}{\partial p'}\right], \tag{9.25}$$

where (see Section C.3)

$$\mathcal{S}'_0 = \frac{\alpha}{2}\sum_{j=-\infty,\infty} b_{1/2}^{(j)}\cos[j(\bar{\lambda}'-\bar{\lambda})] - \alpha^{-1}\cos(\bar{\lambda}'-\bar{\lambda}), \tag{9.26}$$

$$\begin{aligned}\mathcal{S}'_1 = {} & \frac{\alpha}{2}\sum_{j=-\infty,\infty}\Big\{k(-2j-\alpha D)\, b_{1/2}^{(j)}\cos[j\bar{\lambda}'+(1-j)\bar{\lambda}] \\ & + h(-2j-\alpha D)\, b_{1/2}^{(j)}\sin[j\bar{\lambda}'+(1-j)\bar{\lambda}] \\ & + k'(-1+2j+\alpha D)\, b_{1/2}^{(j-1)}\cos[j\bar{\lambda}'+(1-j)\bar{\lambda}] \\ & + h'(-1+2j+\alpha D)\, b_{1/2}^{(j-1)}\sin[j\bar{\lambda}'+(1-j)\bar{\lambda}]\Big\} \\ & + \frac{\alpha^{-1}}{2}\Big\{-k'\cos(2\bar{\lambda}'-\bar{\lambda}) - h'\sin(2\bar{\lambda}'-\bar{\lambda}) + 3k'\cos\bar{\lambda} + 3h'\sin\bar{\lambda} \\ & - 4k\cos(\bar{\lambda}'-2\bar{\lambda}) + 4h\sin(\bar{\lambda}'-2\bar{\lambda})\Big\},\end{aligned} \tag{9.27}$$

and

$$\begin{aligned}\mathcal{S}'_2 = {} & \frac{1}{8}(h^2+k^2+h'^2+k'^2)\,\alpha\,(2\alpha D+\alpha^2 D^2)\, b_{1/2}^{(0)} - \frac{1}{8}(p^2+q^2+p'^2+q'^2)\,\alpha^2\, b_{3/2}^{(1)} \\ & + \frac{1}{4}(k\,k'+h\,h')\,\alpha\,(2-2\alpha D-\alpha^2 D^2)\, b_{1/2}^{(1)} \\ & + \frac{1}{4}(p\,p'+q\,q')\,\alpha^2\, b_{3/2}^{(1)}.\end{aligned} \tag{9.28}$$

Here, $\epsilon = m/(M+m') = \tilde{\mu}/\mu' \ll 1$, and $n' = (\mu'/a'^3)^{1/2}$.

9.3 Secular evolution of planetary orbits

As a specific example of the use of orbital perturbation theory, let us determine the evolution of the osculating orbital elements of the two planets in our model solar system due to the secular terms in their disturbing functions (i.e., the terms that are independent of the mean longitudes $\bar{\lambda}$ and $\bar{\lambda}'$). This is equivalent to averaging the osculating elements over the relatively short timescales associated with the periodic terms in the disturbing functions (i.e., the terms that depend on $\bar{\lambda}$ and $\bar{\lambda}'$, and, therefore, oscillate on timescales similar to the orbital periods of the planets). From Equations (9.16)–(9.18), the secular part of the first planet's disturbing function takes the form

$$\mathcal{S}_s = \mathcal{S}_{0\,s} + \mathcal{S}_{2\,s}, \tag{9.29}$$

where

$$\mathcal{S}_{0\,s} = \frac{1}{2}\, b_{1/2}^{(0)}, \tag{9.30}$$

$$\begin{aligned}\mathcal{S}_{2\,s} = {} & \frac{1}{8}(h^2+k^2+h'^2+k'^2)\,\alpha\, b_{3/2}^{(1)} - \frac{1}{8}(p^2+q^2+p'^2+q'^2)\,\alpha\, b_{3/2}^{(1)} \\ & - \frac{1}{4}(k\,k'+h\,h')\,\alpha\, b_{3/2}^{(2)} + \frac{1}{4}(p\,p'+q\,q')\,\alpha\, b_{3/2}^{(1)},\end{aligned} \tag{9.31}$$

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

because, as can be demonstrated (Brouwer and Clemence 1961),

$$(2\,\alpha\,D + \alpha^2\,D^2)\,b_{1/2}^{(0)} \equiv \alpha\,b_{3/2}^{(1)} \tag{9.32}$$

and

$$(2 - 2\,\alpha\,D - \alpha^2\,D^2)\,b_{1/2}^{(1)} \equiv -\alpha\,b_{3/2}^{(2)}. \tag{9.33}$$

Evaluating the right-hand sides of Equations (9.10)–(9.15) to $\mathcal{O}(\epsilon'\,e\,n)$ (it is assumed that ϵ, $\epsilon' \ll e, h, k, p, q, h', k', p', q' \ll 1, \alpha$), we find that

$$\frac{da^{(1)}}{dt} = 0, \tag{9.34}$$

$$\frac{d\bar{\lambda}^{(1)}}{dt} = \epsilon'\,n\left[-\frac{3}{2}\,a^{(1)} - 2\,\alpha^2\,\frac{\partial\mathcal{S}_{0\,s}}{\partial\alpha}\right], \tag{9.35}$$

$$\frac{dh}{dt} = \epsilon'\,n\left(\alpha\,\frac{\partial\mathcal{S}_{2\,s}}{\partial k}\right), \tag{9.36}$$

$$\frac{dk}{dt} = \epsilon'\,n\left(-\alpha\,\frac{\partial\mathcal{S}_{2\,s}}{\partial h}\right), \tag{9.37}$$

$$\frac{dp}{dt} = \epsilon'\,n\left(\alpha\,\frac{\partial\mathcal{S}_{2\,s}}{\partial q}\right), \tag{9.38}$$

and

$$\frac{dq}{dt} = \epsilon'\,n\left(-\alpha\,\frac{\partial\mathcal{S}_{2\,s}}{\partial p}\right), \tag{9.39}$$

as $\partial\mathcal{S}_{0\,s}/\partial\bar{\lambda} = 0$. It follows that $a^{(1)} = 0$, and

$$\bar{\lambda} = \bar{\lambda}_0 + n\left[1 - \epsilon'\,\alpha^2\,D\,b_{1/2}^{(0)}\right]t, \tag{9.40}$$

$$\frac{dh}{dt} = \epsilon'\,n\left[\frac{1}{4}\,k\,\alpha^2\,b_{3/2}^{(1)} - \frac{1}{4}\,k'\,\alpha^2\,b_{3/2}^{(2)}\right], \tag{9.41}$$

$$\frac{dk}{dt} = \epsilon'\,n\left[-\frac{1}{4}\,h\,\alpha^2\,b_{3/2}^{(1)} + \frac{1}{4}\,h'\,\alpha^2\,b_{3/2}^{(2)}\right], \tag{9.42}$$

$$\frac{dp}{dt} = \epsilon'\,n\left[-\frac{1}{4}\,q\,\alpha^2\,b_{3/2}^{(1)} + \frac{1}{4}\,q'\,\alpha^2\,b_{3/2}^{(1)}\right], \tag{9.43}$$

and

$$\frac{dq}{dt} = \epsilon'\,n\left[\frac{1}{4}\,p\,\alpha^2\,b_{3/2}^{(1)} - \frac{1}{4}\,p'\,\alpha^2\,b_{3/2}^{(1)}\right]. \tag{9.44}$$

Note that the first planet's mean angular velocity is slightly modified in the presence of the second planet, but that its major radius remains the same.

From Equations (9.26)–(9.28), the secular part of the second planet's disturbing function takes the form

$$\mathcal{S}_s' = \mathcal{S}_{0\,s}' + \mathcal{S}_{2\,s'}', \tag{9.45}$$

where

$$\mathcal{S}_{0\,s}' = \frac{1}{2}\,\alpha\,b_{1/2}^{(0)} \tag{9.46}$$

and

$$
\begin{aligned}
\mathcal{S}'_{2\,s} &= \frac{1}{8}\,(h^2+k^2+h'^{\,2}+k'^{\,2})\,\alpha^2\,b_{3/2}^{(1)} - \frac{1}{8}\,(p^2+q^2+p'^{\,2}+q'^{\,2})\,\alpha^2\,b_{3/2}^{(1)} \\
&\quad -\frac{1}{4}\,(k\,k'+h\,h')\,\alpha^2\,b_{3/2}^{(2)} + \frac{1}{4}\,(p\,p'+q\,q')\,\alpha^2\,b_{3/2}^{(1)}.
\end{aligned} \tag{9.47}
$$

Evaluating the right-hand sides of Equations (9.20)–(9.25) to $\mathcal{O}(\epsilon\,e\,n')$, we find that

$$
\frac{da^{(1)'}}{dt} = 0, \tag{9.48}
$$

$$
\frac{d\bar{\lambda}^{(1)'}}{dt} = \epsilon\,n'\left(-\frac{3}{2}\,a^{(1)'} + 2\,\frac{\partial\mathcal{S}'_{0\,s}}{\partial\alpha}\right), \tag{9.49}
$$

$$
\frac{dh'}{dt} = \epsilon\,n'\left(\alpha^{-1}\,\frac{\partial\mathcal{S}'_{2\,s}}{\partial k'}\right), \tag{9.50}
$$

$$
\frac{dk'}{dt} = \epsilon\,n'\left(-\alpha^{-1}\,\frac{\partial\mathcal{S}'_{2\,s}}{\partial h'}\right), \tag{9.51}
$$

$$
\frac{dp'}{dt} = \epsilon\,n'\left(\alpha^{-1}\,\frac{\partial\mathcal{S}'_{2\,s}}{\partial q'}\right), \tag{9.52}
$$

and

$$
\frac{dq'}{dt} = \epsilon\,n'\left(-\alpha^{-1}\,\frac{\partial\mathcal{S}'_{2\,s}}{\partial p'}\right). \tag{9.53}
$$

It follows that $a^{(1)'} = 0$, and

$$
\bar{\lambda}' = \bar{\lambda}'_0 + n'\left[1 - \epsilon\,D\left(\alpha\,b_{1/2}^{(0)}\right)\right]t, \tag{9.54}
$$

$$
\frac{dh'}{dt} = \epsilon\,n'\left[\frac{1}{4}\,k'\,\alpha\,b_{3/2}^{(1)} - \frac{1}{4}\,k\,\alpha\,b_{3/2}^{(2)}\right], \tag{9.55}
$$

$$
\frac{dk'}{dt} = \epsilon\,n'\left[-\frac{1}{4}\,h'\,\alpha\,b_{3/2}^{(1)} + \frac{1}{4}\,h\,\alpha\,b_{3/2}^{(2)}\right], \tag{9.56}
$$

$$
\frac{dp'}{dt} = \epsilon\,n'\left[-\frac{1}{4}\,q'\,\alpha\,b_{3/2}^{(1)} + \frac{1}{4}\,q\,\alpha\,b_{3/2}^{(1)}\right], \tag{9.57}
$$

and

$$
\frac{dq'}{dt} = \epsilon\,n'\left[\frac{1}{4}\,p'\,\alpha\,b_{3/2}^{(1)} - \frac{1}{4}\,p\,\alpha\,b_{3/2}^{(1)}\right]. \tag{9.58}
$$

The second planet's mean angular velocity is also slightly modified in the presence of the first planet, but its major radius remains the same.

Let us now generalize the preceding analysis to take all eight of the major planets in the solar system into account. Let planet i (where i runs from 1 to 8) have mass m_i, major radius a_i, eccentricity e_i, longitude of the perihelion ϖ_i, inclination I_i, and longitude of the ascending node Ω_i. As before, it is convenient to introduce the alternative orbital elements $h_i = e_i\,\sin\varpi_i$, $k_i = e_i\,\cos\varpi_i$, $p_i = \sin I_i\,\sin\Omega_i$, and $q_i = \sin I_i\,\cos\Omega_i$. It is also

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

helpful to define the following parameters:

$$\alpha_{ij} = \begin{cases} a_i/a_j & a_j > a_i \\ a_j/a_i & a_j < a_i \end{cases}, \tag{9.59}$$

and

$$\bar{\alpha}_{ij} = \begin{cases} a_i/a_j & a_j > a_i \\ 1 & a_j < a_i \end{cases}, \tag{9.60}$$

as well as

$$\epsilon_{ij} = \frac{m_j}{M + m_i} \tag{9.61}$$

and

$$n_i = [G\,(M + m_i)/a_i^3]^{1/2}, \tag{9.62}$$

where M is the mass of the Sun. It then follows, from the preceding analysis, that the secular terms in the planetary disturbing functions cause the h_i, k_i, p_i, and q_i to vary in time as

$$\frac{dh_i}{dt} = \sum_{j=1,8} A_{ij}\,k_j, \tag{9.63}$$

$$\frac{dk_i}{dt} = -\sum_{j=1,8} A_{ij}\,h_j, \tag{9.64}$$

$$\frac{dp_i}{dt} = \sum_{j=1,8} B_{ij}\,q_j, \tag{9.65}$$

and

$$\frac{dq_i}{dt} = -\sum_{j=1,8} B_{ij}\,p_j, \tag{9.66}$$

where

$$A_{ii} = \sum_{j\neq i} \frac{n_i}{4}\,\epsilon_{ij}\,\alpha_{ij}\,\bar{\alpha}_{ij}\,b_{3/2}^{(1)}(\alpha_{ij}), \tag{9.67}$$

$$A_{ij} = -\frac{n_i}{4}\,\epsilon_{ij}\,\alpha_{ij}\,\bar{\alpha}_{ij}\,b_{3/2}^{(2)}(\alpha_{ij}), \tag{9.68}$$

$$B_{ii} = -\sum_{j\neq i} \frac{n_i}{4}\,\epsilon_{ij}\,\alpha_{ij}\,\bar{\alpha}_{ij}\,b_{3/2}^{(1)}(\alpha_{ij}), \tag{9.69}$$

and

$$B_{ij} = \frac{n_i}{4}\,\epsilon_{ij}\,\alpha_{ij}\,\bar{\alpha}_{ij}\,b_{3/2}^{(1)}(\alpha_{ij}), \tag{9.70}$$

for $j \neq i$. Here, Mercury is planet 1, Venus is planet 2, and so on, and Neptune is planet 8. Note that the time evolution of the h_i and the k_i, which determine the eccentricities of the planetary orbits, is decoupled from that of the p_i and the q_i, which determine the inclinations. Let us search for normal mode solutions to Equations (9.63)–(9.66) of the form

$$h_i = \sum_{l=1,8} e_{il} \sin(g_l t + \beta_l), \tag{9.71}$$

$$k_i = \sum_{l=1,8} e_{il} \cos(g_l t + \beta_l), \tag{9.72}$$

$$p_i = \sum_{l=1,8} I_{il} \sin(f_l t + \gamma_l), \tag{9.73}$$

and

$$q_i = \sum_{l=1,8} I_{il} \cos(f_l t + \gamma_l). \tag{9.74}$$

It follows that

$$\sum_{j=1,8} \left(A_{ij} - \delta_{ij}\, g_l\right) e_{jl} = 0 \tag{9.75}$$

and

$$\sum_{j=1,8} \left(B_{ij} - \delta_{ij}\, f_l\right) I_{jl} = 0. \tag{9.76}$$

At this stage, we have effectively reduced the problem of determining the secular evolution of the planetary orbits to a pair of matrix eigenvalue equations (Gradshteyn and Ryzhik 1980e) that can be solved via standard numerical techniques (Press et al. 1992). Once we have determined the eigenfrequencies, g_l and f_l, and the corresponding eigenvectors, e_{il} and I_{il}, we find the phase angles β_l and γ_l by demanding that, at $t = 0$, Equations (9.71)–(9.74) lead to the values of e_i, I_i, ϖ_i, and Ω_i given in Table 3.1.

The theory outlined here is generally referred to as *Laplace-Lagrange secular evolution theory*. The eigenfrequencies, eigenvectors, and phase angles obtained from this theory are listed in Tables 9.1–9.3. Note that the largest eigenfrequency is of magnitude 25.90 arc seconds per year, which translates to an oscillation period of about 5×10^4 years. In other words, the typical timescale over which the secular evolution of the solar system predicted by Laplace-Lagrange theory takes place is at least 5×10^4 years, and is, therefore, much longer than the orbital period of any planet.

Figure 9.2 compares the observed perihelion and ascending node precession rates of the Planets at $t = 0$ (which corresponds to the epoch J2000) with those calculated from the theory described previously. It can be seen that, generally, there is good agreement between the theoretical and observed precession rates, which gives us some degree of confidence in the theory. On the whole, the degree of agreement exhibited in the left-hand panel of Figure 9.2 is better than that exhibited in Figure 4.1, indicating that the Laplace-Lagrange secular evolution theory described in this chapter is an improvement on the (highly simplified) Gaussian secular evolution theory outlined in Section 4.4.

Observe that one of the inclination eigenfrequencies, f_5, takes the value zero. This is a consequence of the conservation of angular momentum. Because the solar system is effectively an isolated dynamical system, its net angular momentum vector, $\mathbf{L}$, is constant in both magnitude and direction. The plane normal to $\mathbf{L}$ that passes through

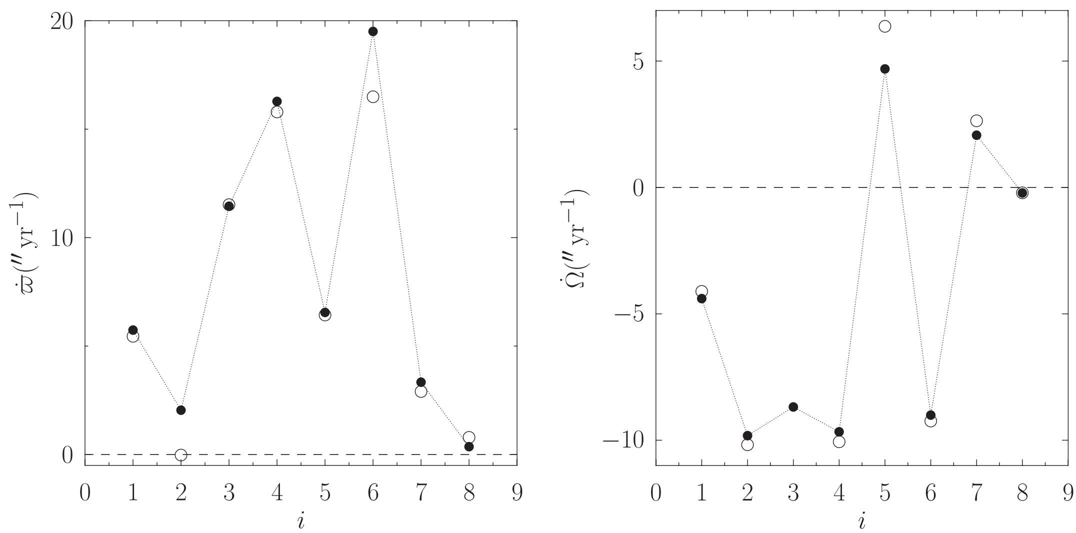

Fig. 9.2 The filled circles show the observed planetary perihelion precession rates (left-hand panel) and ascending node precession rates (right-hand panel) at J2000. All ascending nodes are measured relative to the mean ecliptic at J2000. The empty circles show the theoretical precession rates calculated from Laplace-Lagrange secular evolution theory. Source (for observational data): Standish and Williams 1992.

the center of mass of the solar system (which lies very close to the Sun) is known as the *invariable plane*. If all the planetary orbits were to lie in the invariable plane, the net angular velocity vector of the solar system would be parallel to its fixed net angular momentum vector. Moreover, the angular momentum vector would be parallel to one of the solar system's principal axes of rotation. In this situation, we would expect the Solar System to remain in the invariable plane. (See Chapter 7.) In other words, we would not expect any time evolution of the planetary inclinations. (Of course, lack of time variation implies an eigenfrequency of zero.) According to Equations (9.73) and (9.74), and the data shown in Tables 9.1 and 9.3, if the solar system were in the inclination eigenstate associated with the null eigenfrequency, f_5, then we would have

$$p_i = 2.751 \times 10^{-2}\ \sin \gamma_5 \tag{9.77}$$

and

$$q_i = 2.751 \times 10^{-2}\ \cos \gamma_5, \tag{9.78}$$

for $i = 1, 8$. Because $p_i = \sin I_i\, \sin \Omega_i$ and $q_i = \sin I_i\, \cos \Omega_i$, it follows that all the planetary orbits would lie in the same plane, and this plane—which is, of course, the invariable plane—is inclined at $I_5 = \sin^{-1}(2.751 \times 10^{-2}) = 1.576°$ to the ecliptic plane. Furthermore, the longitude of the ascending node of the invariable plane, with respect to the ecliptic plane, is $\Omega_5 = \gamma_5 = 107.5°$. Actually, it is generally more convenient to measure the inclinations of the planetary orbits with respect to the invariable plane, rather than the ecliptic plane, as the inclination of the latter plane varies in time. We can achieve this goal by simply omitting the fifth inclination eigenstate when calculating orbital inclinations from Equations (9.73) and (9.74).

Table 9.1 Eigenfrequencies and phase angles obtained from Laplace-Lagrange secular evolution theory

l	g_l(″ yr^{-1})	f_l(″ yr^{-1})	β_l(°)	γ_l(°)
1	5.462	−5.201	89.65	20.23
2	7.346	−6.570	195.0	318.3
3	17.33	−18.74	336.1	255.6
4	18.00	−17.64	319.0	296.9
5	3.724	0.000	30.12	107.5
6	22.44	−25.90	131.0	127.3
7	2.708	−2.911	109.9	315.6
8	0.6345	−0.6788	67.98	202.8

Table 9.2 Components of eccentricity eigenvectors e_{il} obtained from Laplace-Lagrange secular evolution theory. All components multiplied by 10^5

	i = 1	2	3	4	5	6	7	8
l = 1	18128	629	404	66	0	0	0	0
2	−2331	1919	1497	265	−1	−1	0	0
3	154	−1262	1046	2979	0	0	0	0
4	−169	1489	−1485	7281	0	0	0	0
5	2446	1636	1634	1878	4331	3416	−4388	159
6	10	−51	242	1562	−1560	4830	−180	−13
7	59	58	62	82	207	189	2999	−322
8	0	1	1	2	6	6	144	954

Table 9.3 Components of inclination eigenvectors I_{il} obtained from Laplace-Lagrange secular evolution theory. All components multiplied by 10^5

	i = 1	2	3	4	5	6	7	8
l = 1	12548	1180	850	180	−2	−2	2	0
2	−3548	1006	811	180	−1	−1	0	0
3	409	−2684	2446	−3595	0	0	0	0
4	116	−685	451	5021	0	−1	0	0
5	2751	2751	2751	2751	2751	2751	2751	2751
6	27	14	279	954	−636	1587	−69	−7
7	−333	−191	−173	−125	−95	−77	1757	−206
8	−144	−132	−129	−122	−116	-112	109	1181

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

Table 9.4 Maximum/minimum eccentricities and inclinations of planetary orbits, and mean perihelion/nodal precession rates, from Laplace-Lagrange secular evolution theory. All inclinations relative to invariable plane

Planet	$e_{\min}$	$e_{\max}$	$\langle\dot{\varpi}\rangle$	$I_{\min}(^\circ)$	$I_{\max}(^\circ)$	$\langle\dot{\Omega}\rangle$
Mercury	0.130	0.233	g_1	4.57	9.86	f_1
Venus	0.000	0.0705	–	0.000	3.38	–
Earth	0.000	0.0638	–	0.000	2.95	–
Mars	0.0444	0.141	g_4	0.000	5.84	–
Jupiter	0.0256	0.0611	g_5	0.241	0.489	f_6
Saturn	0.0121	0.0845	g_6	0.797	1.02	f_6
Uranus	0.0106	0.0771	g_5	0.902	1.11	f_7
Neptune	0.00460	0.0145	g_8	0.554	0.800	f_8

Consider the ith planet. Suppose one of the e_{il} coefficients—say, e_{ik}—is sufficiently large that

$$|e_{ik}| > \sum_{l=1,8}^{l\neq k} |e_{il}|. \tag{9.79}$$

This is known as the *Lagrange condition* (Hagihara 1971). As can be demonstrated, if the Lagrange condition is satisfied, the eccentricity of the ith planet's orbit varies between the minimum value,

$$e_{i\,\min} = |e_{ik}| - \sum_{l=1,8}^{l\neq k} |e_{il}|, \tag{9.80}$$

and the maximum value,

$$e_{i\,\max} = |e_{ik}| + \sum_{l=1,8}^{l\neq k} |e_{il}|. \tag{9.81}$$

Moreover, on average, the ith planet's perihelion point precesses at the associated eigenfrequency, g_k. The precession is prograde (i.e., in the same direction as the orbital motion) if the frequency is positive, and retrograde (i.e., in the opposite direction) if the frequency is negative. If the Lagrange condition is not satisfied, all we can say is that the maximum eccentricity is given by Equation (9.81), and there is no minimum eccentricity (i.e., the eccentricity can vary all the way down to zero). Furthermore, no mean precession rate of the perihelion point can be identified. It can be seen from Table 9.2 that the Lagrange condition for the orbital eccentricities is satisfied for all planets except Venus and Earth. The maximum and minimum eccentricities, and mean perihelion precession rates, of the planets (when they exist) are given in Table 9.4. Note that Jupiter and Uranus have the same mean perihelion precession rates, and that all planets that possess mean precession rates exhibit prograde precession.

There is also a Lagrange condition associated with the inclinations of the planetary orbits (Hagihara 1971). This condition is satisfied for the ith planet if one of the I_{il}—say,

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

I_{ik}—is sufficiently large that

$$|I_{ik}| > \sum_{l=1,8}^{l\neq k, l\neq 5} |I_{il}|. \tag{9.82}$$

The fifth inclination eigenmode is omitted from this summation because we are now measuring inclinations relative to the invariable plane. If the Lagrange condition is satisfied, the inclination of the ith planet's orbit with respect to the invariable plane varies between the minimum value

$$I_{i\,\min} = \sin^{-1}\left(|I_{ik}| - \sum_{l=1,8}^{l\neq k, l\neq 5} |I_{il}|\right) \tag{9.83}$$

and the maximum value

$$I_{i\,\max} = \sin^{-1}\left(|I_{ik}| + \sum_{l=1,8}^{l\neq k, l\neq 5} |I_{il}|\right). \tag{9.84}$$

Moreover, on average, the ascending node precesses at the associated eigenfrequency, f_k. The precession is prograde (i.e., in the same direction as the orbital motion) if the frequency is positive, and retrograde (i.e., in the opposite direction) if the frequency is negative. If the Lagrange condition is not satisfied, all we can say is that the maximum inclination is given by Equation (9.84), and there is no minimum inclination (i.e., the inclination can vary all the way down to zero). Furthermore, no mean precession rate of the ascending node can be identified. It can be seen from Table 9.3 that the Lagrange condition for the orbital inclinations is satisfied for all planets except Venus, Earth, and Mars. The maximum and minimum inclinations, and mean nodal precession rates, of the planets (when they exist) are given in Table 9.4. The four outer planets, which possess most of the mass of the solar system, all have orbits whose inclinations to the invariable plane remain small. On the other hand, the four relatively light inner planets have orbits whose inclinations to the invariable plane can become relatively large. Observe that Jupiter and Saturn have the same mean nodal precession rates, and that all planets that possess mean precession rates exhibit retrograde precession.

Figure 9.3 shows the time variation of the eccentricity, inclination, perihelion precession rate, and ascending node precession rate of Mercury, as predicted by the Laplace-Lagrange secular perturbation theory described earlier. It can be seen that the eccentricity and inclination do indeed oscillate between the upper and lower bounds specified in Table 9.4. Moreover, the perihelion and ascending node precession rates do appear to oscillate about the mean values (g_1 and f_1, respectively) specified in the same table.

According to Laplace-Lagrange secular perturbation theory, the mutual gravitational interactions between the various planets in the solar system cause their orbital eccentricities and inclinations to oscillate between *fixed bounds* on timescales that are long compared with their orbital periods. Recall, however, that these results depend on a great many approximations: the neglect of all nonsecular terms in the planetary disturbing functions, and the neglect of secular terms beyond first order in the planetary masses and beyond second order in the orbital eccentricities and inclinations. It turns out that when the neglected terms are included in the analysis, the largest correction to standard

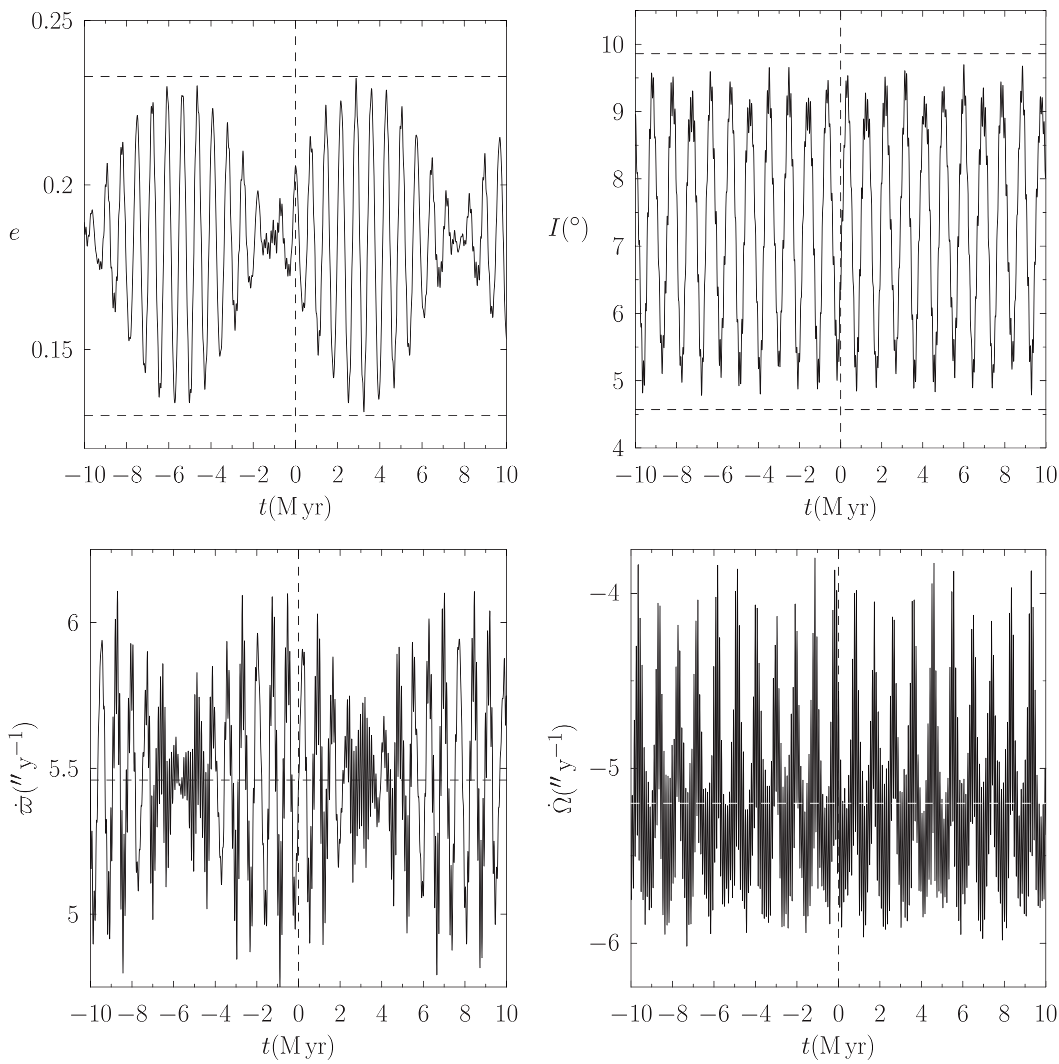

Fig. 9.3 Time variation of the eccentricity (top left), inclination (top right), perihelion precession rate (bottom left), and ascending node precession rate (bottom right) of Mercury, as predicted by Laplace-Lagrange secular perturbation theory. Time is measured in millions of years relative to J2000. All inclinations are relative to the invariable plane. The horizontal dashed lines in the top panels indicate the predicted minimum and maximum eccentricities and inclinations from Table 9.4. The horizontal dashed lines in the bottom panels indicate the predicted mean perihelion and ascending node precession rates from the same table.

Laplace-Lagrange theory is a second-order (in the planetary masses) effect caused by periodic terms in the disturbing functions of Jupiter and Saturn that oscillate on a relatively long timescale (i.e., almost 900 years), because the orbital periods of these two planets are almost commensurable (i.e., five times the orbital period of Jupiter is almost equal to two times the orbital period of Saturn). In 1950, Brouwer and van Woerkom worked out a modified version of Laplace-Lagrange secular perturbation theory that

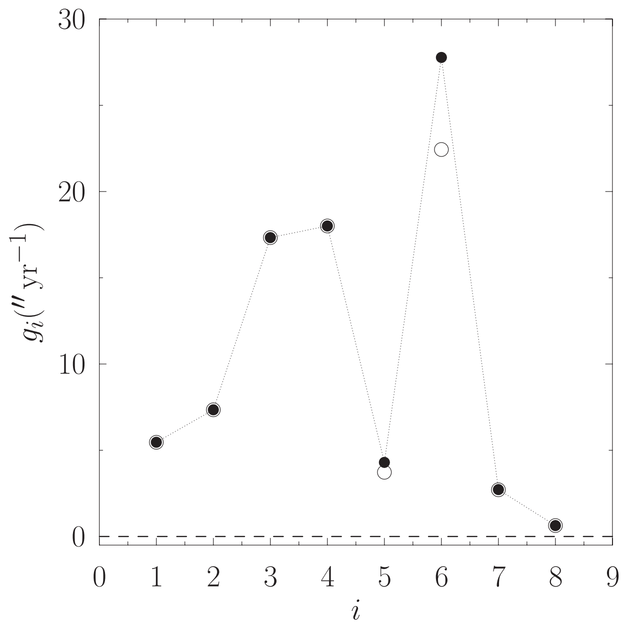

Fig. 9.4 The eccentricity eigenfrequencies obtained from Brouwer and van Woerkom's refinement of standard Laplace-Lagrange secular evolution theory (filled circles), compared with the corresponding eigenfrequencies from Table 9.1 (open circles).

takes the aforementioned correction into account (Brouwer and van Woerkom 1950). This refined secular evolution theory is described, in detail, in Murray and Dermott (1999). As is illustrated in Figure 9.4, the values of the eccentricity eigenfrequencies g_5 and g_6 in the Brouwer-van Woerkom theory differ somewhat from those specified in Table 9.1. The corresponding eigenvectors are also somewhat modified. The Brouwer-van Woerkom theory also contains two additional, relatively small-amplitude, eccentricity eigenmodes that oscillate at the eigenfrequencies $g_9 = 2\,g_5 - g_6$ and $g_{10} = 2\,g_6 - g_5$. On the other hand, the Brouwer-van Woerkom theory does not give rise to any significant modifications to the inclination eigenmodes. Figure 9.5 shows the maximum and minimum orbital eccentricities predicted by the Brouwer-van Woerkom theory, compared with the corresponding limits from Table 9.4. It can be seen that the refinements introduced by Brouwer and van Woerkom modify the oscillation limits for the orbital eccentricities of Mercury and Uranus somewhat, but do not significantly change the limits for the other planets. Of course, the oscillation limits for the orbital inclinations are unaffected by these refinements (because the inclination eigenmodes are unaffected).

It must be emphasized that the Brouwer-van Woerkom secular evolution theory is only approximate in nature. In fact, the theory is capable of predicting the secular evolution of the solar system with reasonable accuracy up to a million or so years into the future or the past. However, over longer timescales, it becomes inaccurate because the true long-term dynamics of the solar system contain *chaotic* elements. These elements originate from two secular resonances among the planets: $(g_4 - g_3) - (f_4 - f_3) \simeq 0$, which is

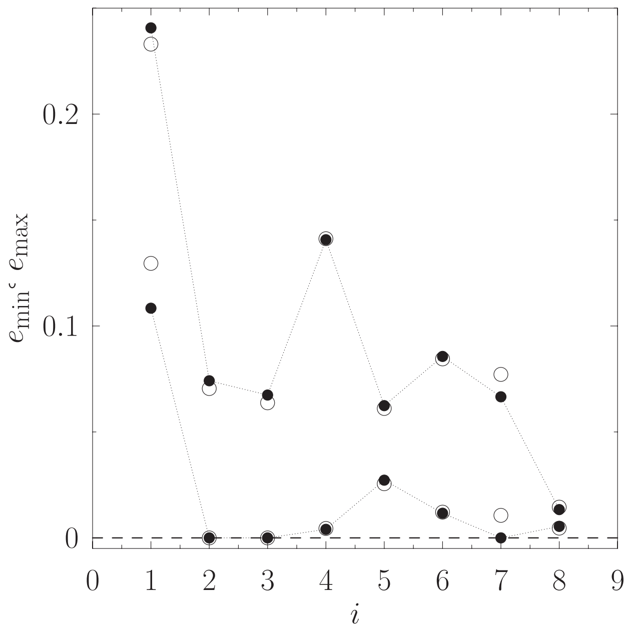

Fig. 9.5 The maximum and minimum orbital eccentricities predicted by Brouwer and van Woerkom's refinement of standard Laplace-Lagrange secular evolution theory (filled circles), compared with the corresponding limits from Table 9.4 (open circles).

related to the gravitational interaction of Mars and the Earth; and $(g_1 - g_5) - (f_1 - f_2) \simeq 0$, which is related to the interaction of Mercury, Venus, and Jupiter (Laskar 1990).

9.4 Secular evolution of asteroid orbits

Let us now consider the perturbing influence of the planets on the orbit of an asteroid. Because asteroids have much smaller masses than planets, it is reasonable to suppose that the perturbing influence of the asteroid on the planetary orbits is negligible. Let the asteroid have the standard osculating orbital elements a, $\bar{\lambda}_0$, e, I, ϖ, Ω, and the alternative elements $h = e \sin\varpi$, $k = e \cos\varpi$, $p = \sin I \sin\Omega$, and $q = \sin I \cos\Omega$. Thus, the mean orbital angular velocity of the asteroid is $n = (G\,M/a^3)^{1/2}$, where M is the solar mass. Likewise, let the eight planets have the standard osculating orbital elements a_i, $\bar{\lambda}_{0\,i}$, e_i, I_i, ϖ_i, Ω_i, and the alternative elements $h_i = e_i \sin\varpi_i$, $k_i = e_i \cos\varpi_i$, $p_i = \sin I_i \sin\Omega_i$, and $q_i = \sin I_i \cos\Omega_i$, for $i = 1, 8$. It is helpful to define the following parameters:

$$\alpha_i = \begin{cases} a/a_i & a_i > a \\ a_i/a & a_i < a \end{cases} \tag{9.85}$$

and

$$\bar{\alpha}_i = \begin{cases} a/a_i & a_i > a \\ 1 & a_i < a \end{cases}, \tag{9.86}$$

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

as well as

$$\epsilon_i = \frac{m_i}{M}. \tag{9.87}$$

By analogy with the analysis in the previous section, the secular terms in the disturbing function of the asteroid, generated by the perturbing influence of the planets, cause the asteroid's osculating orbital elements to evolve in time as

$$\frac{dh}{dt} = A\,k + \sum_{i=1,8} A_i\,k_i, \tag{9.88}$$

$$\frac{dk}{dt} = -A\,h - \sum_{i=1,8} A_i\,h_i, \tag{9.89}$$

$$\frac{dp}{dt} = B\,q + \sum_{i=1,8} B_i\,q_i, \tag{9.90}$$

and

$$\frac{dq}{dt} = -B\,p - \sum_{i=1,8} B_i\,p_i, \tag{9.91}$$

where

$$A = \sum_{i=1,8} \frac{n}{4}\,\epsilon_i\,\alpha_i\,\bar{\alpha}_i\,b_{3/2}^{(1)}(\alpha_i), \tag{9.92}$$

$$A_i = -\frac{n}{4}\,\epsilon_i\,\alpha_i\,\bar{\alpha}_i\,b_{3/2}^{(2)}(\alpha_i), \tag{9.93}$$

$$B = -\sum_{i=1,8} \frac{n}{4}\,\epsilon_i\,\alpha_i\,\bar{\alpha}_i\,b_{3/2}^{(1)}(\alpha_i), \tag{9.94}$$

and

$$B_i = \frac{n}{4}\,\epsilon_i\,\alpha_i\,\bar{\alpha}_i\,b_{3/2}^{(1)}(\alpha_i). \tag{9.95}$$

However, as we have already seen, the planetary osculating elements themselves evolve in time as

$$h_i = \sum_{l=1,8} e_{il}\,\sin(g_l\,t + \beta_l), \tag{9.96}$$

$$k_i = \sum_{l=1,8} e_{il}\,\cos(g_l\,t + \beta_l), \tag{9.97}$$

$$p_i = \sum_{l=1,8} I_{il}\,\sin(f_l\,t + \gamma_l), \tag{9.98}$$

and

$$q_i = \sum_{l=1,8} I_{il}\,\cos(f_l\,t + \gamma_l). \tag{9.99}$$

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

Equations (9.88)–(9.91) can be solved to give

$$h(t) = e_{\rm free}\,\sin(A\,t+\beta_{\rm free}) + h_{\rm forced}(t), \tag{9.100}$$

$$k(t) = e_{\rm free}\,\cos(A\,t+\beta_{\rm free}) + k_{\rm forced}(t), \tag{9.101}$$

$$p(t) = \sin I_{\rm free}\,\sin(B\,t+\gamma_{\rm free}) + p_{\rm forced}(t), \tag{9.102}$$

and

$$q(t) = \sin I_{\rm free}\,\cos(B\,t+\gamma_{\rm free}) + q_{\rm forced}(t), \tag{9.103}$$

where

$$h_{\rm forced} = -\sum_{l=1,8}\frac{\nu_l}{A-g_l}\,\sin(g_l\,t+\beta_l), \tag{9.104}$$

$$k_{\rm forced} = -\sum_{l=1,8}\frac{\nu_l}{A-g_l}\,\cos(g_l\,t+\beta_l), \tag{9.105}$$

$$p_{\rm forced} = -\sum_{l=1,8}\frac{\mu_l}{B-f_l}\,\sin(f_l\,t+\gamma_l), \tag{9.106}$$

and

$$q_{\rm forced} = -\sum_{l=1,8}\frac{\mu_l}{B-f_l}\,\cos(f_l\,t+\gamma_l), \tag{9.107}$$

as well as

$$\nu_l = \sum_{i=1,8} A_i\,e_{il} \tag{9.108}$$

and

$$\mu_l = \sum_{i=1,8} B_i\,I_{il}. \tag{9.109}$$

The parameters $e_{\rm free}$ and $I_{\rm free}$ appearing in Equations (9.100)–(9.103) are the eccentricity and inclination, respectively, that the asteroid orbit would possess were it not for the perturbing influence of the planets. These parameters are usually called the *free*, or *proper*, eccentricity and inclination, respectively. Roughly speaking, the planetary perturbations cause the osculating eccentricity, $e = (h^2+k^2)^{1/2}$, and inclination, $I = \sin^{-1}\{[p^2+q^2]^{1/2}\}$, to oscillate about the corresponding free quantities, $e_{\rm free}$ and $I_{\rm free}$, respectively.

Figure 9.6 shows the osculating eccentricity plotted against the sine of the osculating inclination for the orbits of the first 100,000 numbered asteroids (asteroids are numbered in order of their discovery). No particular patten is apparent. Figures 9.7 and 9.8 show the *free* eccentricity plotted against the sine of the *free* inclination for the same 100,000 orbits. In Figure 9.7, the free orbital elements are determined from standard Laplace-Lagrange secular evolution theory, whereas in Figure 9.8 they are determined from Brouwer and van Woerkom's refinement of this theory. It can be seen that many of the points representing the asteroid orbits have condensed into clumps. These clumps, which are somewhat clearer in Figure 9.8 than in Figure 9.7, are known as *Hirayama*

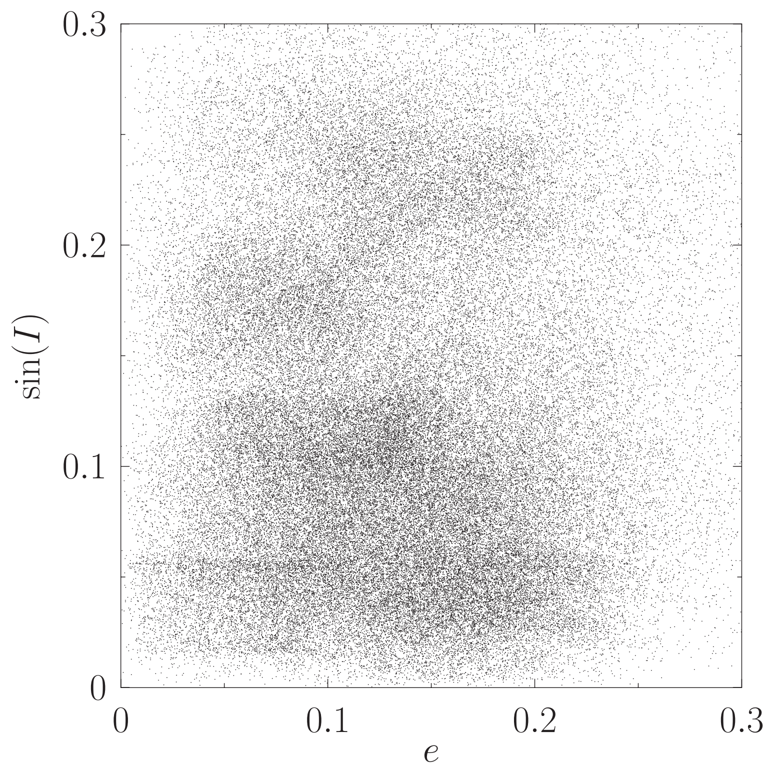

Fig. 9.6 The osculating eccentricity plotted against the sine of the osculating inclination (relative to J2000 equinox and ecliptic) for the orbits of the first 100,000 numbered asteroids at MJD 55400. Raw data from JPL Small-Body Database.

families after their discoverer, the Japanese astronomer Kiyotsugu Hirayama (1874–1943). It is thought that the asteroids making up a given family had a common origin—most likely due to the break up of some much larger body (Bertotti et al. 2003). As a consequence of this origin, the asteroids originally had similar orbital elements. However, as time progressed, these elements were jumbled by the perturbing influence of the planets. Thus, only when this influence is removed does the commonality of the orbits becomes apparent. Hirayama families are named after their largest member. The most prominent families are the (4) Vesta, (15) Eunomia, (24) Themis, (44) Nysa, (158) Koronis, (221) Eos, and (1272) Gefion families. (The number in parentheses is that of the corresponding asteroid.)

9.5 Secular evolution of artificial satellite orbits

Consider a nonrotating (with respect to the distant stars) frame of reference whose origin coincides with the center of the Earth. We can regard such a frame as approximately inertial when we consider orbital motion in the Earth's immediate vicinity. Let X, Y, Z be a Cartesian coordinate system in the said reference frame that is oriented such that

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

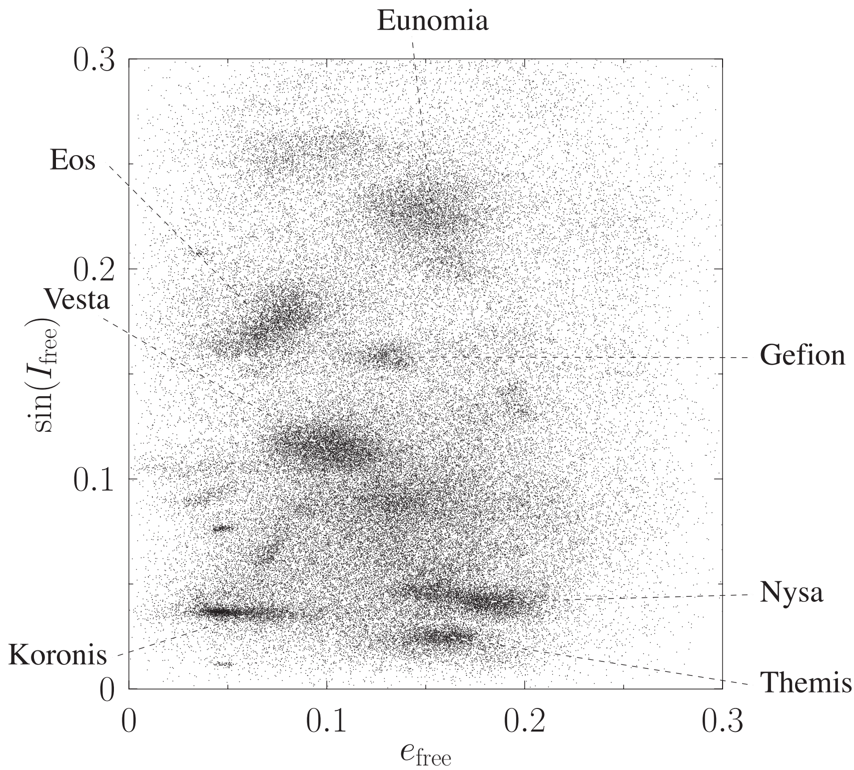

Fig. 9.7 The free eccentricity plotted against the sine of the free inclination (relative to J2000 equinox and ecliptic) for the orbits of the first 100,000 numbered asteroids at MJD 55400. The free orbital elements are determined from standard Laplace-Lagrange secular evolution theory. The most prominent Hirayama families are labeled. Raw data from JPL Small-Body Database.

its Z-axis is aligned with the Earth's (approximately) constant axis of rotation (with the terrestrial north pole lying at positive Z). As we saw in Section 7.9, the gravitational potential in the immediate vicinity of the Earth can be written

$$\Phi(r, \vartheta) \simeq -\frac{G\,M}{r} + \frac{G\,(\mathcal{I}_{\parallel} - \mathcal{I}_{\perp})}{r^3}\,P_2(\cos\vartheta), \tag{9.110}$$

where M is the Earth's mass, $\mathcal{I}_{\parallel}$ its moment of inertia about the Z-axis, and $\mathcal{I}_{\perp}$ its moment of inertia about an axis lying in the X–Y plane. Here, $r = (X^2 + Y^2 + Z^2)^{1/2}$ and $\vartheta = \cos^{-1}(Z/r)$ are standard spherical coordinates. The first term on the right-hand side of the preceding expression is the monopole gravitational potential that would result were the Earth spherically symmetric. The second term is the small quadrupole correction to this potential generated by the Earth's slight oblateness; see Section 5.5. It is conventional to parameterize this correction in terms of the dimensionless quantity (Yoder 1995)

$$J_2 = \frac{\mathcal{I}_{\parallel} - \mathcal{I}_{\perp}}{M\,R^2} = 1.083 \times 10^{-3}, \tag{9.111}$$

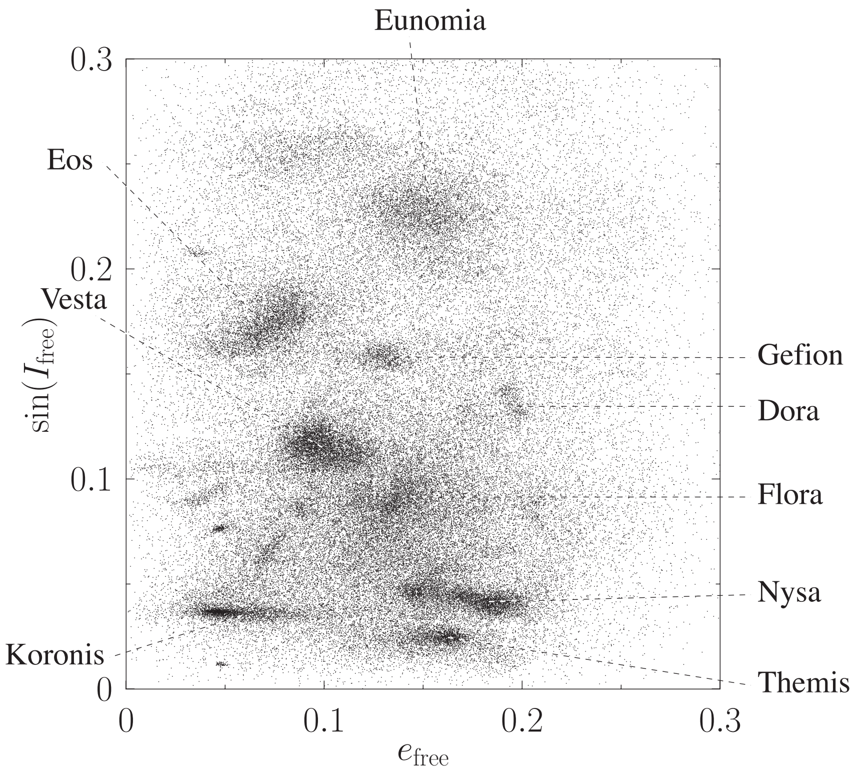

Fig. 9.8 The free eccentricity plotted against the sine of the free inclination (relative to J2000 equinox and ecliptic) for the orbits of the first 100,000 numbered asteroids at MJD 55400. The free orbital elements are determined from Brouwer and van Woerkom's improved secular evolution theory. The most prominent Hirayama families are labeled. Raw data from JPL Small-Body Database.

where R is the Earth's equatorial radius. Hence, Equation (9.110) can be written

$$\Phi(r, \vartheta) \simeq -\frac{G\,M}{r} + J_2\,\frac{G\,M\,R^2}{r^3}\,P_2(\cos\vartheta). \tag{9.112}$$

Consider an artificial satellite in orbit around the Earth. The satellite's equation of motion in our approximately inertial geocentric reference frame takes the form

$$\ddot{\mathbf{r}} = -\nabla\Phi. \tag{9.113}$$

This can be combined with Equation (9.112) to give

$$\ddot{\mathbf{r}} + \mu\,\frac{\mathbf{r}}{r^3} = \nabla\mathcal{R}, \tag{9.114}$$

where $\mu = G\,M$, and

$$\mathcal{R}(r, \vartheta) = -J_2\,\frac{\mu\,R^2}{r^3}\,P_2(\cos\vartheta). \tag{9.115}$$

Note that the preceding expression has exactly the same form as the canonical equation of motion, Equation (B.2), that is the starting point for orbital perturbation theory.

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

In particular, $\mathcal{R}(r, \vartheta)$ is the disturbing function that describes the perturbation to the Keplerian orbit of the satellite due to the Earth's small quadrupole gravitational field.

Let the satellite's osculating orbital elements be the major radius, a; the time of perigee passage, τ; the eccentricity, e; the inclination (to the Earth's equatorial plane), I; the argument of the perigee, ω; and the longitude of the ascending node, Ω. (See Section 3.12.) Actually, it is more convenient to replace τ by the mean anomaly, $\mathcal{M} = n\,(t - \tau)$, where $n = (\mu/a^3)^{1/2}$ is the (unperturbed) mean orbital angular velocity. According to standard orbital perturbation theory, the time evolution of the satellite's orbital elements is governed by the *Lagrange planetary equations*, which, for the particular set of elements under consideration, take the form (see Section B.6)

$$\frac{da}{dt} = \frac{2}{n\,a}\,\frac{\partial\mathcal{R}}{\partial\mathcal{M}}, \tag{9.116}$$

$$\frac{d\mathcal{M}}{dt} = n - \frac{2}{n\,a}\,\frac{\partial\mathcal{R}}{\partial a} - \frac{1-e^2}{n\,a^2\,e}\,\frac{\partial\mathcal{R}}{\partial e}, \tag{9.117}$$

$$\frac{de}{dt} = \frac{1-e^2}{n\,a^2\,e}\,\frac{\partial\mathcal{R}}{\partial\mathcal{M}} - \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\,\frac{\partial\mathcal{R}}{\partial\omega}, \tag{9.118}$$

$$\frac{dI}{dt} = \frac{\cot I}{n\,a^2\,(1-e^2)^{1/2}}\,\frac{\partial\mathcal{R}}{\partial\omega} - \frac{(1-e^2)^{-1/2}}{n\,a^2\,\sin I}\,\frac{\partial\mathcal{R}}{\partial\Omega}, \tag{9.119}$$

$$\frac{d\omega}{dt} = \frac{(1-e^2)^{1/2}}{n\,a^2\,e}\,\frac{\partial\mathcal{R}}{\partial e} - \frac{\cot I}{n\,a^2\,(1-e^2)^{1/2}}\,\frac{\partial\mathcal{R}}{\partial I}, \tag{9.120}$$

and

$$\frac{d\Omega}{dt} = \frac{(1-e^2)^{-1/2}}{n\,a^2\,\sin I}\,\frac{\partial\mathcal{R}}{\partial I}. \tag{9.121}$$

According to Equation (3.74),

$$\cos\vartheta = \frac{Z}{r} = \sin(\omega + \theta)\,\sin I, \tag{9.122}$$

where θ is the satellite's true anomaly. Thus, Equation (9.115) can be written

$$\mathcal{R} = \frac{J_2}{2}\,\frac{\mu\,R^2}{r^3}\left[1 - \frac{3}{2}\,\sin^2 I + \frac{3}{2}\,\sin^2 I\,\cos(2\,\omega + 2\,\theta)\right]. \tag{9.123}$$

We are interested primarily in the *secular* evolution of the satellite's orbital elements, that is, the evolution on timescales much longer than the orbital period. We can concentrate on this evolution, and filter out any relatively short-term oscillations in the elements, by averaging the disturbing function over an orbital period. In other words, in Equations (9.116)–(9.121), we need to replace $\mathcal{R}$ by

$$\overline{\mathcal{R}} = \frac{1}{T}\int_0^T \mathcal{R}\,dt = \frac{n}{h}\oint r^2\,\mathcal{R}\,\frac{d\theta}{2\pi} = (1-e^2)^{-1/2}\oint \frac{r^2}{a^2}\,\mathcal{R}\,\frac{d\theta}{2\pi}, \tag{9.124}$$

where $T = 2\pi/n$. Here, we have made use of the fact that $r^2\,\dot{\theta} = h = (1-e^2)^{1/2}\,n\,a^2$ is a constant of the motion in a Keplerian orbit. (See Chapter 3.) A Keplerian orbit is also characterized by $r/a = (1-e^2)\,(1 + e\,\cos\theta)^{-1}$. Hence, the previous two equations can

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

be combined to give

$$\overline{\mathcal{R}} = \frac{J_2}{2}\,\frac{\mu R^2}{a^3}\,(1-e^2)^{-3/2}\oint (1+e\,\cos\theta)\left[1-\frac{3}{2}\,\sin^2 I+\frac{3}{2}\,\sin^2 I\,\cos(2\,\omega+2\,\theta)\right]\frac{d\theta}{2\pi}, \tag{9.125}$$

which evaluates to

$$\overline{\mathcal{R}} = \frac{J_2}{2}\,\frac{\mu R^2}{a^3}\,(1-e^2)^{-3/2}\left(1-\frac{3}{2}\,\sin^2 I\right). \tag{9.126}$$

Substitution of this expression into Equations (9.116), (9.118), and (9.119) (recalling that we are replacing $\mathcal{R}$ by $\overline{\mathcal{R}}$) reveals that there is no secular evolution of the satellite's orbital major radius, a, eccentricity, e, and inclination, I, due to the Earth's oblateness (because $\overline{\mathcal{R}}$ does not depend on $\mathcal{M}$, ω, or Ω). On the other hand, according to Equations (9.120) and (9.121), the oblateness causes the satellite's perigee and ascending node to precess at the constant rates

$$\dot{\omega} = \frac{3\,J_2}{4}\,n\,\frac{R^2}{a^2}\,\frac{(5\,\cos^2 I-1)}{(1-e^2)^2} \tag{9.127}$$

and

$$\dot{\Omega} = -\frac{3\,J_2}{2}\,n\,\frac{R^2}{a^2}\,\frac{\cos I}{(1-e^2)^2}, \tag{9.128}$$

respectively. These formulae suggest that the precession of the ascending node is always in the opposite sense to the orbital motion: that is, it is retrograde. Note, however, that the ascending node remains fixed in the special case of a so-called polar orbit that passes over the terrestrial poles (i.e., $I = \pi/2$). The formulae also suggest that the perigee precesses in a prograde fashion when $\cos I > 1/\sqrt{5}$, precesses in a retrograde fashion when $\cos I < 1/\sqrt{5}$, and remains fixed when $\cos I = 1/\sqrt{5}$. In other words, the perigee of an orbit lying in the Earth's equatorial plane precesses in the same direction as the orbital motion, the perigee of a polar orbit precesses in the opposite direction, and the perigee of an orbit that is inclined at the critical angle of 63.4° to the Earth's equatorial plane does not precess at all.

Exercises

9.1 Consider the secular evolution of two planets moving around a star in coplanar orbits of low eccentricity. Let a, e, and ϖ be the orbital major radius, eccentricity, and longitude of the periastron (i.e., the point of closest approach to the star) of the first planet, respectively, and let a', e', and ϖ' be the the corresponding parameters for the second planet. Suppose that $a' > a$. Let $h = e\,\sin\varpi$, $k = e\,\cos\varpi$, $h' = e'\,\sin\varpi'$, and $k' = e'\,\cos\varpi'$. Consider normal mode solutions of the two planets' secular evolution equations of the form $h(t) = \hat{e}\,\sin(g\,t+\beta)$, $k(t) = \hat{e}\,\cos(g\,t+\beta)$, $h'(t) = \hat{e}\,'\,\sin(g\,t+\beta)$, and $k'(t) = \hat{e}'\,\cos(g\,t+\beta)$, where $\hat{e}$, $\hat{e}\,'$, g, and β are constants.

https://doi.org/10.1017/CBO9781139152310.010 Published online by Cambridge University Press

Demonstrate that

$$\begin{pmatrix} \hat{g} - q\,\alpha & q\,\alpha\,\beta \\ \alpha^{3/2}\beta & \hat{g} - \alpha^{3/2} \end{pmatrix} \begin{pmatrix} \hat{e} \\ \hat{e}' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $\hat{g} = g/[(1/4)\,\epsilon\,n\,\alpha\,b_{3/2}^{(1)}(\alpha)]$, $\epsilon = m/M$, $n = (G\,M/a^3)^{1/2}$, $\alpha = a/a'$, $q = m'/m$, and $\beta = b_{3/2}^{(2)}(\alpha)/b_{3/2}^{(1)}(\alpha)$. Here, m, m', and M are the masses of the first planet, second planet, and star, respectively. It is assumed that $M \gg m, m'$. Hence, deduce that the general time variation of the osculating orbital elements $h(t)$, $k(t)$, $h'(t)$, and $k'(t)$ is a linear combination of two normal modes of oscillation, which are characterized by

$$\hat{g} = \frac{1}{2}\left\{q\,\alpha + \alpha^{3/2} \pm \left[(q\,\alpha - \alpha^{3/2})^2 + 4\,q\,\alpha^{5/2}\,\beta^2\right]^{1/2}\right\},$$

and

$$\frac{\hat{e}\,'}{\hat{e}} = \frac{q\,\alpha - \hat{g}}{q\,\alpha\,\beta} = \frac{\alpha^{3/2}\,\beta}{\alpha^{3/2} - \hat{g}}.$$

Demonstrate that in the limit $\alpha \ll 1$, in which $b_{3/2}^{(1)}(\alpha) \simeq 3\,\alpha$ and $b_{3/2}^{(2)}(\alpha) \simeq (15/4)\,\alpha^2$, the first normal mode is such that $g \simeq (3/4)\,\epsilon'\,n\,\alpha^3$ and $\hat{e}\,'/\hat{e} \simeq -(5/4)\,\alpha^{3/2}/q$ (assuming that $q \gg \alpha^{1/2}$), whereas the second mode is such that $g \simeq (3/4)\,\epsilon\,n'\,\alpha^2$ and $\hat{e}\,'/\hat{e} \simeq (4/5)\,\alpha^{-1}$. Here, $\epsilon' = m'/M$ and $n' = (G\,M/a'^{\,3})^{1/2}$. (Modified from Murray and Dermott 1999.)

9.2 The gravitational potential of the Sun in the vicinity of the planet Mercury can be written

$$\Phi(r) = -\frac{G\,M}{r} - \frac{G\,M\,h^2}{c^2\,r^3},$$

where M is the mass of the Sun, r the radial distance of Mercury from the center of the Sun, h the conserved angular momentum per unit mass of Mercury, and c the velocity of light in vacuum. The second term on the right-hand side of the preceding expression comes from a small general relativistic correction to Newtonian gravity (Rindler 1977). Show that Mercury's equation of motion can be written in the standard form

$$\ddot{\mathbf{r}} + \mu\,\frac{\mathbf{r}}{r^3} = \nabla\mathcal{R},$$

where $\mu = G\,M$, and

$$\mathcal{R} = \frac{\mu\,h^2}{c^2\,r^3}$$

is the disturbing function due to the general relativistic correction. Demonstrate that when the disturbing function is averaged over an orbital period it becomes

$$\overline{\mathcal{R}} = \frac{\mu\,h^2}{c^2\,a^3\,(1 - e^2)^{3/2}},$$

where a and e are the major radius and eccentricity, respectively, of Mercury's orbit. Hence, deduce from Lagrange's planetary equations that the general relativistic correction causes the argument of the perigee of Mercury's orbit to precess at the rate

$$\dot{\omega} = \frac{3\,\mu\,h^2}{c^2\,n\,a^5\,(1-e^2)^2} = \frac{3\,\mu^{3/2}}{c^2\,a^{5/2}\,(1-e^2)},$$

where n is Mercury's mean orbital angular velocity. Finally, show that the preceding expression evaluates to $0.43''\ \mathrm{yr}^{-1}$.

9.3 The gravitational potential in the immediate vicinity of the Earth can be written

$$\Phi(r,\vartheta) = -\frac{\mu}{r}\left[1 - J_2\left(\frac{R}{r}\right)^2 P_2(\cos\vartheta) - J_3\left(\frac{R}{r}\right)^3 P_3(\cos\vartheta) + \cdots\right],$$

where $\mu = G\,M$, M is the terrestrial mass, r, ϑ, ϕ are spherical coordinates that are centered on the Earth and aligned with its axis of rotation, R is the Earth's equatorial radius, and $J_2 = 1.083\times10^{-3}$, $J_3 = -2.112\times10^{-6}$ (Yoder 1995). In the preceding expression, the term involving J_2 is caused by the Earth's small oblateness, and the term involving J_3 is caused by the Earth's slightly asymmetric mass distribution between its northern and southern hemispheres. Consider an artificial satellite in orbit around the Earth. Let a, e, I, and ω be the orbital major radius, eccentricity, inclination (to the Earth's equatorial plane), and argument of the perigee, respectively. Furthermore, let $n = (\mu/a^3)^{1/2}$ be the unperturbed mean orbital angular velocity.

Demonstrate that, when averaged over an orbital period, the disturbing function due to the J_3 term takes the form

$$\overline{\mathcal{R}} = -\frac{3\,J_3}{2}\,\frac{\mu\,R^3}{a^4}\,\frac{e}{(1-e^2)^{5/2}}\left(\frac{5}{4}\,\sin^2 I - 1\right)\sin I\,\sin\omega.$$

Hence, deduce that the J_3 term causes the eccentricity and inclination of the satellite orbit to evolve in time as

$$\frac{de}{d(n\,t)} = -\frac{3\,J_3}{8}\left(\frac{R}{a}\right)^3\frac{(5\,\cos^2 I - 1)}{(1-e^2)^2}\,\sin I\,\cos\omega,$$

$$\frac{dI}{d(n\,t)} = \frac{3\,J_3}{8}\left(\frac{R}{a}\right)^3\frac{e\,(5\,\cos^2 I - 1)}{(1-e^2)^3}\,\cos I\,\cos\omega,$$

respectively. Given that the (much larger) J_2 term causes the argument of the perigee to precess at the approximately constant (assuming that the variations in e and I are small) rate

$$\frac{d\omega}{d(n\,t)} = \frac{3\,J_2}{4}\left(\frac{R}{a}\right)^2\frac{(5\,\cos^2 I - 1)}{(1-e^2)^2},$$

deduce that the variations in the orbital eccentricity and inclination induced by the J_3 term can be written

$$e \simeq e_0 - \frac{J_3}{2\,J_2}\,\frac{R}{a}\,\sin I_0\,\sin\omega,$$

$$I \simeq I_0 + \frac{J_3}{2\,J_2}\,\frac{R}{a}\,\frac{e_0}{1-e_0^2}\,\cos I_0\,\sin\omega,$$

respectively, where e_0 and I_0 are constants. (Modified from Murray and Dermott 1999.)