Appendix A Useful mathematics

A.1 Calculus

$$\frac{d}{dx}\,\mathrm{e}^x = \mathrm{e}^x \tag{A.1}$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \tag{A.2}$$

$$\frac{d}{dx}\sin x = \cos x \tag{A.3}$$

$$\frac{d}{dx}\cos x = -\sin x \tag{A.4}$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} \tag{A.5}$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \tag{A.6}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \tag{A.7}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \tag{A.8}$$

$$\frac{d}{dx}\sinh x = \cosh x \tag{A.9}$$

$$\frac{d}{dx}\cosh x = \sinh x \tag{A.10}$$

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} \tag{A.11}$$

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}} \tag{A.12}$$

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}} \tag{A.13}$$

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2} \tag{A.14}$$

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A.2 Series expansions

Notation: $k! = k\,(k-1)\,(k-2)\ldots 2.1$, $f^{(n)}(x) = d^n f(x)/dx^n$.

$$f(x) = f(a) + \frac{(x-a)}{1!}\,f^{(1)}(a) + \frac{(x-a)^2}{2!}\,f^{(2)}(a) + \cdots \frac{(x-a)^n}{n!}\,f^{(n)}(a) + \cdots \tag{A.15}$$

$$(1+x)^\alpha = 1 + \alpha\,x + \frac{\alpha\,(\alpha-1)}{2!}\,x^2 + \frac{\alpha\,(\alpha-1)\,(\alpha-2)}{3!}\,x^3 + \cdots \tag{A.16}$$

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{A.17}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{A.18}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{A.19}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{A.20}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \frac{17\,x^7}{315} + \cdots \tag{A.21}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \tag{A.22}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \tag{A.23}$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2\,x^5}{15} - \frac{17\,x^7}{315} + \cdots \tag{A.24}$$

A.3 Trigonometric identities

$$\sin(-\alpha) = -\sin\alpha \tag{A.25}$$

$$\cos(-\alpha) = +\cos\alpha \tag{A.26}$$

$$\tan(-\alpha) = -\tan\alpha \tag{A.27}$$

$$\sin^2\alpha + \cos^2\alpha = 1 \tag{A.28}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\,\cos\beta \pm \cos\alpha\,\sin\beta \tag{A.29}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\,\cos\beta \mp \sin\alpha\,\sin\beta \tag{A.30}$$

$$\tan(\alpha \pm \beta) = = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\,\tan\beta} \tag{A.31}$$

$$\sin\alpha + \sin\beta = 2\,\sin\left(\frac{\alpha+\beta}{2}\right)\,\cos\left(\frac{\alpha-\beta}{2}\right) \tag{A.32}$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{A.33}$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{A.34}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{A.35}$$

$$\sin\alpha\,\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] \tag{A.36}$$

$$\cos\alpha\,\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right] \tag{A.37}$$

$$\sin\alpha\,\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right] \tag{A.38}$$

$$\sin(\alpha/2) = \pm\left(\frac{1-\cos\alpha}{2}\right)^{1/2} \tag{A.39}$$

$$\cos(\alpha/2) = \pm\left(\frac{1+\cos\alpha}{2}\right)^{1/2} \tag{A.40}$$

$$\tan(\alpha/2) = \pm\left(\frac{1-\cos\alpha}{1+\cos\alpha}\right)^{1/2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha} \tag{A.41}$$

$$\sin(2\alpha) = 2\sin\alpha\,\cos\alpha \tag{A.42}$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \tag{A.43}$$

$$\sin(3\alpha) = -4\sin^3\alpha + 3\sin\alpha \tag{A.44}$$

$$\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha \tag{A.45}$$

$$\sin(4\alpha) = (-8\sin^3\alpha + 4\sin\alpha)\cos\alpha \tag{A.46}$$

$$\cos(4\alpha) = 8\cos^4\alpha - 8\cos^2\alpha + 1 \tag{A.47}$$

$$\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha) \tag{A.48}$$

$$\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha) \tag{A.49}$$

$$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha) \tag{A.50}$$

$$\cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha) \tag{A.51}$$

$$\sin^4\alpha = \frac{1}{8}(3 - 4\cos 2\alpha + \cos 4\alpha) \tag{A.52}$$

$$\cos^4\alpha = \frac{1}{8}(3 + 4\cos 2\alpha + \cos 4\alpha) \tag{A.53}$$

$$\sinh(-\alpha) = -\sinh\alpha \tag{A.54}$$

$$\cosh(-\alpha) = +\cosh\alpha \tag{A.55}$$

$$\tanh(-\alpha) = -\tanh\alpha \tag{A.56}$$

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$$\cosh^2\alpha - \sinh^2\alpha = 1 \tag{A.57}$$

$$\sinh(\alpha \pm \beta) = \sinh\alpha\,\cosh\beta \pm \cosh\alpha\,\sinh\beta \tag{A.58}$$

$$\cosh(\alpha \pm \beta) = \cosh\alpha\,\cosh\beta \pm \sinh\alpha\,\sinh\beta \tag{A.59}$$

$$\tanh(\alpha \pm \beta) = \frac{\tanh\alpha \pm \tanh\beta}{1 \pm \tanh\alpha\,\tanh\beta} \tag{A.60}$$

$$\sinh\alpha + \sinh\beta = 2\,\sinh\left(\frac{\alpha+\beta}{2}\right)\cosh\left(\frac{\alpha-\beta}{2}\right) \tag{A.61}$$

$$\sinh\alpha - \sinh\beta = 2\,\cosh\left(\frac{\alpha+\beta}{2}\right)\sinh\left(\frac{\alpha-\beta}{2}\right) \tag{A.62}$$

$$\cosh\alpha + \cosh\beta = 2\,\cosh\left(\frac{\alpha+\beta}{2}\right)\cosh\left(\frac{\alpha-\beta}{2}\right) \tag{A.63}$$

$$\cosh\alpha - \cosh\beta = 2\,\sinh\left(\frac{\alpha+\beta}{2}\right)\sinh\left(\frac{\alpha-\beta}{2}\right) \tag{A.64}$$

$$\sinh\alpha\,\sinh\beta = \frac{1}{2}\,[\cosh(\alpha+\beta) - \cosh(\alpha-\beta)] \tag{A.65}$$

$$\cosh\alpha\,\cosh\beta = \frac{1}{2}\,[\cosh(\alpha+\beta) + \cosh(\alpha-\beta)] \tag{A.66}$$

$$\sinh\alpha\,\cosh\beta = \frac{1}{2}\,[\sinh(\alpha+\beta) + \sinh(\alpha-\beta)] \tag{A.67}$$

$$\sinh(\alpha/2) = \left(\frac{\cosh\alpha - 1}{2}\right)^{1/2} \tag{A.68}$$

$$\cosh(\alpha/2) = \left(\frac{\cosh\alpha + 1}{2}\right)^{1/2} \tag{A.69}$$

$$\tanh(\alpha/2) = \left(\frac{\cosh\alpha - 1}{\cosh\alpha + 1}\right)^{1/2} = \frac{\cosh\alpha - 1}{\sinh\alpha} = \frac{\sinh\alpha}{\cosh\alpha + 1} \tag{A.70}$$

$$\sinh(2\alpha) = 2\,\sinh\alpha\,\cosh\alpha \tag{A.71}$$

$$\cosh(2\alpha) = \cosh^2\alpha + \sinh^2\alpha = 2\,\cosh^2\alpha - 1 = 2\,\sinh^2\alpha + 1 \tag{A.72}$$

A.4 Vector identities

Notation: $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are general vectors; ϕ, ψ are general scalar fields; $\mathbf{a} = a_x\,\mathbf{e}_x + a_y\,\mathbf{e}_y + a_x\,\mathbf{e}_z = (a_x, a_y, a_z)$, and so on, where $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are right-handed Cartesian basis vectors.

$$|\mathbf{a}| = \left(a_x^2 + a_y^2 + a_z^2\right)^{1/2} \tag{A.73}$$

$$\mathbf{a}\cdot\mathbf{b} = a_x\,b_x + a_y\,b_y + a_z\,b_z \tag{A.74}$$

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\ &= (a_y\,b_z - a_z\,b_y)\,\mathbf{e}_x + (a_z\,b_x - a_x\,b_z)\,\mathbf{e}_y + (a_x\,b_y - a_y\,b_x)\,\mathbf{e}_z\end{aligned} \tag{A.75}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} \tag{A.76}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\, \mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\, \mathbf{a} \tag{A.77}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\, (\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\, (\mathbf{b} \cdot \mathbf{c}) \tag{A.78}$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\, \mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\, \mathbf{d} \tag{A.79}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, \mathbf{e}_x + \frac{\partial\phi}{\partial y}\, \mathbf{e}_y + \frac{\partial\phi}{\partial z}\, \mathbf{e}_z \tag{A.80}$$

$$\nabla(\phi\, \psi) = \phi\, \nabla\psi + \psi\, \nabla\phi \tag{A.81}$$

A.5 Conservative fields

Consider a vector field $\mathbf{A}(\mathbf{r})$. In general, the line integral $\int_P^Q \mathbf{A} \cdot d\mathbf{r}$ depends on the path taken between the end points, P and Q. However, for some special vector fields, the integral is path independent. Such fields are called *conservative* fields. It can be shown that if $\mathbf{A}$ is a conservative field, then $\mathbf{A} = \nabla V$ for some scalar field $V(\mathbf{r})$. The proof of this is straightforward. Keeping P fixed, we have

$$\int_P^Q \mathbf{A} \cdot d\mathbf{r} = V(Q), \tag{A.82}$$

where $V(Q)$ is a well-defined function, owing to the path-independent nature of the line integral. Consider moving the position of the end point by an infinitesimal amount dx in the x-direction. We have

$$V(Q + dx) = V(Q) + \int_Q^{Q+dx} \mathbf{A} \cdot d\mathbf{r} = V(Q) + A_x\, dx. \tag{A.83}$$

Hence,

$$\frac{\partial V}{\partial x} = A_x, \tag{A.84}$$

with analogous relations for the other components of $\mathbf{A}$. It follows that

$$\mathbf{A} = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) \equiv \nabla V. \tag{A.85}$$

A.6 Rotational coordinate transformations

Consider a conventional right-handed Cartesian coordinate system, x, y, z. Suppose that we transform to a new coordinate system, x', y', z', that is obtained from the x, y, z system by rotating the coordinate axes through an angle θ about the z-axis. (See Figure A.1.) Let the coordinates of a general point P be (x, y, z) in the first coordinate system, and (x', y', z') in the second. According to simple trigonometry, these two sets

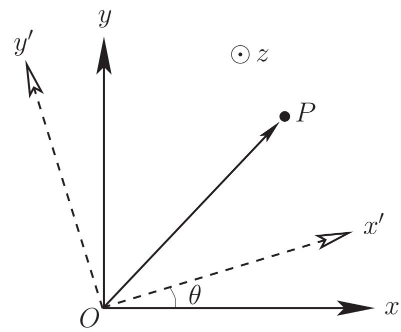

Fig. A.1 Rotation of the coordinate axes about the z-axis.

of coordinates are related to one another via the transformation

$$x' = \cos\theta\; x + \sin\theta\; y, \tag{A.86}$$

$$y' = -\sin\theta\; x + \cos\theta\; y, \tag{A.87}$$

and

$$z' = z. \tag{A.88}$$

When expressed in matrix form, this transformation becomes

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{A.89}$$

The reverse transformation is accomplished by rotating the coordinate axes through an angle $-\theta$ about the z'-axis:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{A.90}$$

It follows that the matrix appearing in Equation (A.89) is the inverse of that appearing in Equation (A.90), and vice versa. However, because these two matrices are clearly also the transposes of each other, we deduce that both matrices are unitary. In fact, it is easily demonstrated that all rotation matrices must be unitary; otherwise, they would not preserve the lengths of the vectors on which they act.

A rotation through an angle ϕ about the x'-axis transforms the x', y', z' coordinate system into the x'', y'', z'' system, where, by analogy with the previous analysis,

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{A.91}$$

Thus, from Equations (A.89) and (A.91), a rotation through an angle θ about the z-axis, followed by a rotation through an angle ϕ about the x'-axis, transforms the x, y, z

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coordinate system into the x'', y'', z'' system, where

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{A.92}$$

A.7 Precession

Suppose that some position vector $\mathbf{r}$ precesses (i.e., rotates) about the z-axis at the angular velocity Ω. If $x(t)$, $y(t)$, $z(t)$ are the Cartesian components of $\mathbf{r}$ at time t then it follows from the analysis in the previous section that

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos(\Omega\,t) & \sin(\Omega\,t) & 0 \\ -\sin(\Omega\,t) & \cos(\Omega\,t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}. \tag{A.93}$$

Hence, making use of the small angle approximations to the sine and cosine functions, we obtain

$$\begin{pmatrix} x(\delta t) \\ y(\delta t) \\ z(\delta t) \end{pmatrix} - \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} \simeq \begin{pmatrix} 0 & \Omega\,\delta t & 0 \\ -\Omega\,\delta t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}, \tag{A.94}$$

which immediately implies that

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{A.95}$$

or

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\Omega} \times \mathbf{r}, \tag{A.96}$$

where $\boldsymbol{\Omega} = \Omega\,\mathbf{e}_z$ is the angular velocity of precession. Because vector equations are coordinate independent, we deduce that the preceding expression is the general equation for the time evolution of a position vector $\mathbf{r}$ that precesses at the angular velocity $\boldsymbol{\Omega}$.

A.8 Curvilinear coordinates

In the *cylindrical* coordinate system, the standard Cartesian coordinates x and y are replaced by $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$. Here, r is the perpendicular distance from the z-axis, and θ the angle subtended between the perpendicular radius vector and the x-axis. (See Figure A.2.) A general vector $\mathbf{A}$ is thus written

$$\mathbf{A} = A_r\,\mathbf{e}_r + A_\theta\,\mathbf{e}_\theta + A_z\,\mathbf{e}_z, \tag{A.97}$$

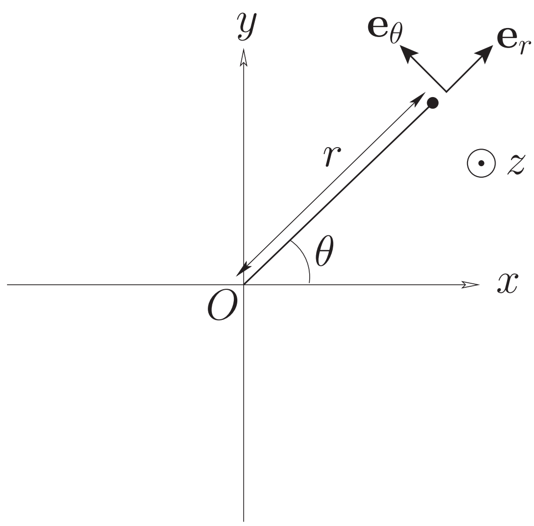

Fig. A.2 Cylindrical coordinates.

where $\mathbf{e}_r = \nabla r/|\nabla r|$ and $\mathbf{e}_\theta = \nabla\theta/|\nabla\theta|$. (See Figure A.2.) The unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ are mutually orthogonal. Hence, $A_r = \mathbf{A}\cdot\mathbf{e}_r$, etc. The volume element in this coordinate system is $d^3\mathbf{r} = r\,dr\,d\theta\,dz$. Moreover, the gradient of a general scalar field $V(\mathbf{r})$ takes the form

$$\nabla V = \frac{\partial V}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\,\mathbf{e}_\theta + \frac{\partial V}{\partial z}\,\mathbf{e}_z. \tag{A.98}$$

In the *spherical* coordinate system, the Cartesian coordinates x, y, and z are replaced by $r = (x^2 + y^2 + z^2)^{1/2}$, $\theta = \cos^{-1}(z/r)$, and $\phi = \tan^{-1}(y/x)$. Here, r is the radial distance from the origin, θ the angle subtended between the radius vector and the z-axis, and ϕ the angle subtended between the projection of the radius vector onto the x–y plane and the x-axis. (See Figure A.3.) Note that r and θ in the spherical system are *not* the same as their counterparts in the cylindrical system. A general vector $\mathbf{A}$ is written

$$\mathbf{A} = A_r\,\mathbf{e}_r + A_\theta\,\mathbf{e}_\theta + A_\phi\,\mathbf{e}_\phi, \tag{A.99}$$

where $\mathbf{e}_r = \nabla r/|\nabla r|$, $\mathbf{e}_\theta = \nabla\theta/|\nabla\theta|$, and $\mathbf{e}_\phi = \nabla\phi/|\nabla\phi|$. The unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ are mutually orthogonal. Hence, $A_r = \mathbf{A}\cdot\mathbf{e}_r$, and so on. The volume element in this coordinate system is $d^3\mathbf{r} = r^2\,\sin\theta\,dr\,d\theta\,d\phi$. Moreover, the gradient of a general scalar field $V(\mathbf{r})$ takes the form

$$\nabla V = \frac{\partial V}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\,\mathbf{e}_\theta + \frac{1}{r\,\sin\theta}\frac{\partial V}{\partial\phi}\,\mathbf{e}_\phi. \tag{A.100}$$

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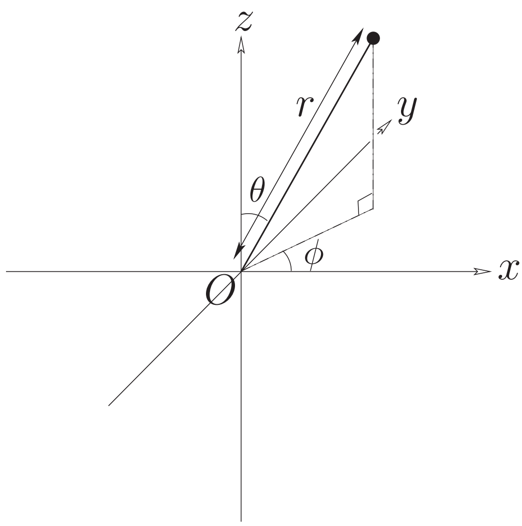

Fig. A.3 Spherical coordinates.

A.9 Conic sections

The ellipse, the parabola, and the hyperbola are collectively known as *conic sections*, as these three types of curve can be obtained by taking various different plane sections of a right cone.

An *ellipse*, centered on the origin, of major radius a and minor radius b, which are aligned along the x- and y-axes, respectively (see Figure A.4), satisfies the following

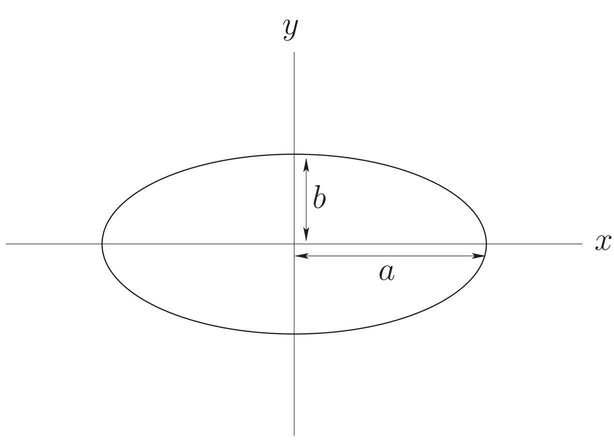

Fig. A.4 An ellipse.

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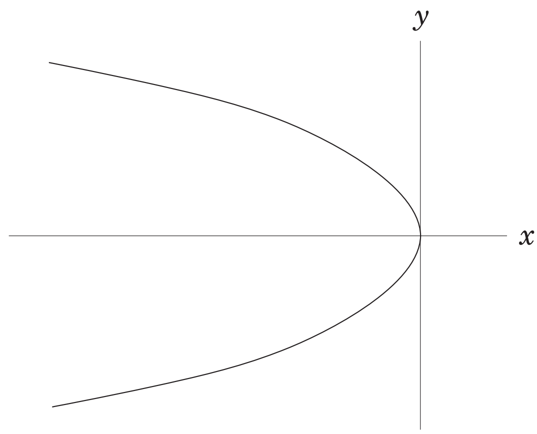

Fig. A.5 A parabola.

well-known equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{A.101}$$

where $0 < b \leq a$.

Likewise, a parabola, which is aligned along the $-x$-axis and passes through the origin (see Figure A.5), satisfies

$$y^2 + b\,x = 0, \tag{A.102}$$

where $b > 0$.

Finally, a hyperbola, which is aligned along the $-x$-axis and whose asymptotes intersect at the origin (see Figure A.6), satisfies

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{A.103}$$

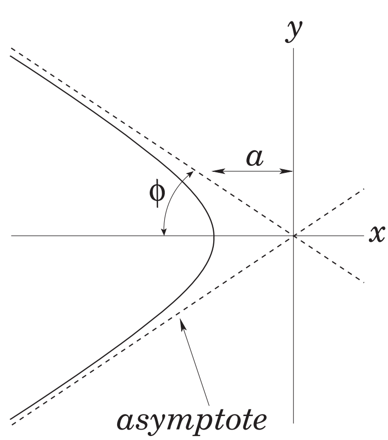

Fig. A.6 A hyperbola.

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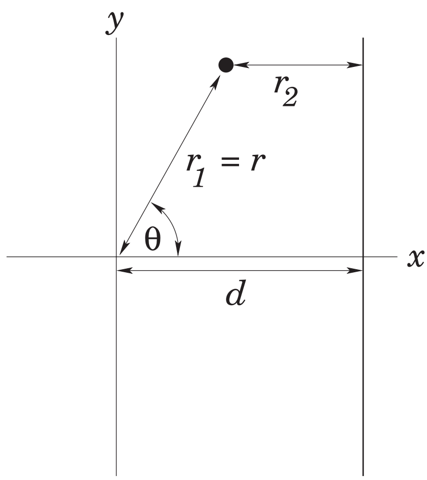

Fig. A.7 Conic sections in polar coordinates.

where $a, b > 0$. Here, a is the distance of closest approach to the origin. The asymptotes subtend an angle $\phi = \tan^{-1}(b/a)$ with the $-x$-axis.

It is not obvious, from the preceding formulae, what the ellipse, the parabola, and the hyperbola have in common. It turns out, in fact, that these three curves can all be represented as the locus of a movable point whose distance from a fixed point is in a constant ratio to its perpendicular distance to some fixed straight line. Let the fixed point—which is termed the *focus*—lie at the origin, and let the fixed line—which is termed the *directrix*—correspond to $x = d$ (with $d > 0$). Thus, the distance of a general point (x, y) (which lies to the left of the directrix) from the focus is $r_1 = (x^2 + y^2)^{1/2}$, whereas the perpendicular distance of the point from the directrix is $r_2 = d - x$. (See Figure A.7.) In polar coordinates, $r_1 = r$ and $r_2 = d - r\cos\theta$. Hence, the locus of a point for which r_1 and r_2 are in a fixed ratio satisfies the following equation:

$$\frac{r_1}{r_2} = \frac{(x^2 + y^2)^{1/2}}{d - x} = \frac{r}{d - r\cos\theta} = e, \tag{A.104}$$

where $e \geq 0$ is a constant. When expressed in terms of polar coordinates, the preceding equation can be rearranged to give

$$r = \frac{r_c}{1 + e\cos\theta}, \tag{A.105}$$

where $r_c = e\,d$.

When written in terms of Cartesian coordinates, Equation (A.104) can be rearranged to give

$$\frac{(x - x_c)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{A.106}$$

for $e < 1$. Here,

$$a = \frac{r_c}{1-e^2}, \tag{A.107}$$

$$b = \frac{r_c}{\sqrt{1-e^2}} = \sqrt{1-e^2}\,a, \tag{A.108}$$

and

$$x_c = -\frac{e\,r_c}{1-e^2} = -e\,a. \tag{A.109}$$

Equation (A.106) can be recognized as the equation of an ellipse whose center lies at $(x_c,\ 0)$, and whose major and minor radii, a and b, are aligned along the x- and y-axes, respectively [see Equation (A.101)]. Note, incidentally, that an ellipse actually possesses two foci located on the major axis ($y = 0$) a distance $e\,a$ on either side of the geometric center (i.e., at $x = 0$ and $x = -2\,e\,a$). Likewise, an ellipse possesses two directrices located at $x = a\,(1 \pm e^2)/e$.

When again written in terms of Cartesian coordinates, Equation (A.104) can be rearranged to give

$$y^2 + 2\,r_c\,(x - x_c) = 0 \tag{A.110}$$

for $e = 1$. Here, $x_c = r_c/2$. This is the equation of a *parabola* that passes through the point $(x_c,\ 0)$, and which is aligned along the $-x$-direction [see Equation (A.102)].

Finally, when written in terms of Cartesian coordinates, Equation (A.104) can be rearranged to give

$$\frac{(x-x_c)^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{A.111}$$

for $e > 1$. Here,

$$a = \frac{r_c}{e^2-1}, \tag{A.112}$$

$$b = \frac{r_c}{\sqrt{e^2-1}} = \sqrt{e^2-1}\,a, \tag{A.113}$$

and

$$x_c = \frac{e\,r_c}{e^2-1} = e\,a. \tag{A.114}$$

Equation (A.111) can be recognized as the equation of a *hyperbola* whose asymptotes intersect at $(x_c,\ 0)$, and which is aligned along the $-x$-direction. The asymptotes subtend an angle

$$\phi = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\sqrt{e^2-1}\right) \tag{A.115}$$

with the $-x$-axis [see Equation (A.103)].

In conclusion, Equation (A.105) is the polar equation of a general conic section that is confocal with the origin (i.e., the origin lies at a focus). For $e < 1$, the conic section is an ellipse. For $e = 1$, the conic section is a parabola. Finally, for $e > 1$, the conic section is a hyperbola.

A.10 Elliptic expansions

The well-known *Bessel functions of the first kind*, $J_n(x)$, where n is an integer, are defined as the Fourier coefficients in the expansion of $\exp(\mathrm{i}\,x\,\sin\phi)$:

$$\mathrm{e}^{\,\mathrm{i}\,x\,\sin\phi} \equiv \sum_{n=-\infty,\infty} J_n(x)\,\mathrm{e}^{\,\mathrm{i}\,n\,\phi}. \tag{A.116}$$

It follows that

$$J_n(x) = \frac{1}{2\pi}\oint \mathrm{e}^{-\mathrm{i}\,(n\,\phi - x\,\sin\phi)}\,d\phi = \frac{1}{\pi}\int_0^{\pi}\cos\left(x\,\sin\phi - n\,\phi\right)\,d\phi \tag{A.117}$$

(Gradshteyn and Ryzhik 1980b). The Taylor expansion of $J_n(x)$ about $x = 0$ is

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0,\infty} \frac{(-x^2/4)^k}{k!\,(n+k)!} \tag{A.118}$$

for $n \geq 0$ (Gradshteyn and Ryzhik 1980c). Moreover,

$$J_{-n}(x) = (-1)^n\,J_n(x), \tag{A.119}$$

$$J_{-n}(-x) = J_n(x). \tag{A.120}$$

In particular,

$$J_0(x) = 1 - \frac{x^2}{4} + \mathcal{O}(x^4), \tag{A.121}$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{16} + \mathcal{O}(x^5), \tag{A.122}$$

$$J_2(x) = \frac{x^2}{8} + \mathcal{O}(x^4), \tag{A.123}$$

and

$$J_3(x) = \frac{x^3}{48} + \mathcal{O}(x^5). \tag{A.124}$$

Let us write

$$\mathrm{e}^{\,\mathrm{i}\,E} = \sum_{n=-\infty,\infty} A_n\,\mathrm{e}^{\,\mathrm{i}\,n\,\mathcal{M}}, \tag{A.125}$$

where E is the *eccentric anomaly*, and $\mathcal{M}$ the *mean anomaly*, of a Keplerian elliptic orbit. (See Section 3.11.) It follows that

$$A_n = \frac{1}{2\pi}\oint \mathrm{e}^{\,\mathrm{i}\,(E-n\,\mathcal{M})}\,d\mathcal{M}. \tag{A.126}$$

Integrating by parts, we obtain

$$A_n = \frac{1}{2\pi\,n}\oint \mathrm{e}^{\,\mathrm{i}\,(E-n\,\mathcal{M})}\,dE. \tag{A.127}$$

However, according to Equation (3.59), the relationship between the eccentric and the mean anomalies is

$$E - e\,\sin E = \mathcal{M}, \tag{A.128}$$

where e is the orbital eccentricity. Hence,

$$A_n = \frac{1}{2\pi\, n}\oint \mathrm{e}^{-\mathrm{i}\,[(n-1)\,E - n\,e\,\sin E]}\,dE. \tag{A.129}$$

Comparison with Equation (A.117) reveals that

$$A_n = \frac{J_{n-1}(n\,e)}{n}. \tag{A.130}$$

For the special case $n = 0$, L'Hôpital's rule, together with Equations (A.119) and (A.122), yields

$$A_0 = e\,J_{-1}'(0) = -e\,J_1'(0) = -\frac{e}{2}, \tag{A.131}$$

where $'$ denotes a derivative.

The real part of Equation (A.125) gives

$$\begin{aligned}\cos E &= A_0 + \sum_{n=1,\infty}(A_n + A_{-n})\,\cos(n\,\mathcal{M})\\ &= -\frac{e}{2} + \sum_{n=1,\infty}\left[\frac{J_{n-1}(n\,e) - J_{-n-1}(-n\,e)}{n}\right]\cos(n\,\mathcal{M})\\ &= -\frac{e}{2} + \sum_{n=1,\infty}\left[\frac{J_{n-1}(n\,e) - J_{n+1}(n\,e)}{n}\right]\cos(n\,\mathcal{M}),\end{aligned} \tag{A.132}$$

where use has been made of Equations (A.120), (A.130), and (A.131). Likewise, the imaginary part of (A.125) yields

$$\sin E = \sum_{n=1,\infty}\left[\frac{J_{n-1}(n\,e) + J_{n+1}(n\,e)}{n}\right]\sin(n\,\mathcal{M}). \tag{A.133}$$

It follows from Equations (A.121)–(A.124) that

$$\begin{aligned}\sin E &= \left(1 - \frac{e^2}{8}\right)\sin\mathcal{M} + \frac{e}{2}\left(1 - \frac{e^2}{3}\right)\sin 2\mathcal{M} + \frac{3\,e^2}{8}\,\sin 3\mathcal{M}\\ &\quad + \frac{e^3}{3}\,\sin 4\mathcal{M} + \mathcal{O}(e^4),\end{aligned} \tag{A.134}$$

and

$$\begin{aligned}\cos E &= -\frac{e}{2} + \left(1 - \frac{3\,e^2}{8}\right)\cos\mathcal{M} + \frac{e}{2}\left(1 - \frac{2\,e^2}{3}\right)\cos 2\mathcal{M}\\ &\quad + \frac{3\,e^2}{8}\,\cos 3\mathcal{M} + \frac{e^3}{3}\,\cos 4\mathcal{M} + \mathcal{O}(e^4).\end{aligned} \tag{A.135}$$

Hence, from (A.128),

$$\begin{aligned}E &= \mathcal{M} + e\,\sin\mathcal{M} + \frac{e^2}{2}\,\sin 2\mathcal{M} + \frac{e^3}{8}\,(3\,\sin 3\mathcal{M} - \sin\mathcal{M})\\ &\quad + \frac{e^4}{6}\,(2\,\sin 4\mathcal{M} - \sin 2\mathcal{M}) + \mathcal{O}(e^5).\end{aligned} \tag{A.136}$$

According to Equation (3.69),

$$\frac{r}{a} = 1 - e\,\cos E, \tag{A.137}$$

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where r is is the radial distance from the focus of the orbit and a is the orbital major radius. Thus,

$$\begin{aligned}\frac{r}{a} &= 1 - e\,\cos\mathcal{M} - \frac{e^2}{2}\,(\cos 2\mathcal{M} - 1) - \frac{3\,e^3}{8}\,(\cos 3\mathcal{M} - \cos\mathcal{M})\\ &\quad - \frac{e^4}{3}\,(\cos 4\mathcal{M} - \cos 2\mathcal{M}) + \mathcal{O}(e^5).\end{aligned} \tag{A.138}$$

Equations (3.39) and (3.67) imply that

$$\frac{d\theta}{d\mathcal{M}} = (1 - e^2)^{1/2}\left(\frac{a}{r}\right)^2, \tag{A.139}$$

where θ is the true anomaly. Hence, it follows from Equations (A.128) and (A.137), and the fact that $\theta = 0$ when $\mathcal{M} = 0$, that

$$\theta = (1 - e^2)^{1/2}\int_0^{\mathcal{M}}\left(\frac{dE}{d\mathcal{M}}\right)^2 d\mathcal{M}. \tag{A.140}$$

From Equation (A.136),

$$\begin{aligned}\frac{dE}{d\mathcal{M}} &= 1 + e\left(1 - \frac{e^2}{8}\right)\cos\mathcal{M} + e^2\left(1 - \frac{e^2}{3}\right)\cos 2\mathcal{M} + \frac{9\,e^3}{8}\,\cos 3\mathcal{M}\\ &\quad + \frac{4\,e^4}{3}\,\cos 4\mathcal{M} + \mathcal{O}(e^5).\end{aligned} \tag{A.141}$$

Thus,

$$\begin{aligned}\left(\frac{dE}{d\mathcal{M}}\right)^2 &= 1 + \frac{e^2}{2} + \frac{3\,e^4}{8} + 2\,e\left(1 + \frac{3\,e^2}{8}\right)\cos\mathcal{M} + \frac{5\,e^2}{2}\left(1 + \frac{2\,e^2}{15}\right)\cos 2\mathcal{M}\\ &\quad + \frac{13\,e^3}{4}\,\cos 3\mathcal{M} + \frac{103\,e^4}{24}\,\cos 4\mathcal{M} + \mathcal{O}(e^5)\end{aligned} \tag{A.142}$$

and

$$\begin{aligned}\theta &= \mathcal{M} + 2\,e\,\sin\mathcal{M} + \frac{5\,e^2}{4}\,\sin 2\mathcal{M} + e^3\left(\frac{13}{12}\,\sin 3\mathcal{M} - \frac{1}{4}\,\sin\mathcal{M}\right)\\ &\quad + e^4\left(\frac{103}{96}\,\sin 4\mathcal{M} - \frac{11}{24}\,\sin 2\mathcal{M}\right) + \mathcal{O}(e^5).\end{aligned} \tag{A.143}$$

A.11 Matrix eigenvalue theory

Suppose that $\mathbf{A}$ is a *real symmetric* square matrix of dimension n. If follows that $\mathbf{A}^* = \mathbf{A}$ and $\mathbf{A}^T = \mathbf{A}$, where * denotes a complex conjugate, and ^{T} denotes a transpose. Consider the matrix equation

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}. \tag{A.144}$$

Any column vector $\mathbf{x}$ that satisfies this equation is called an *eigenvector* of $\mathbf{A}$. Likewise, the associated number λ is called an *eigenvalue* of $\mathbf{A}$ (Gradshteyn and Ryzhik 1980c).

Let us investigate the properties of the eigenvectors and eigenvalues of a real symmetric matrix.

Equation (A.144) can be rearranged to give

$$(\mathbf{A} - \lambda\,\mathbf{1})\,\mathbf{x} = \mathbf{0}, \tag{A.145}$$

where $\mathbf{1}$ is the unit matrix. The preceding matrix equation is essentially a set of n homogeneous simultaneous algebraic equations for the n components of $\mathbf{x}$. A well-known property of such a set of equations is that it has a nontrivial solution only when the determinant of the associated matrix is set to zero (Gradshteyn and Ryzhik 1980e). Hence, a necessary condition for the preceding set of equations to have a nontrivial solution is that

$$|\mathbf{A} - \lambda\,\mathbf{1}| = 0, \tag{A.146}$$

where $|\;|$ denotes a *determinant*. This formula is essentially an nth-order *polynomial* equation for λ. We know that such an equation has n (possibly complex) roots. Hence, we conclude that there are n eigenvalues, and n associated eigenvectors, of the n-dimensional matrix $\mathbf{A}$.

Let us now demonstrate that the n eigenvalues and eigenvectors of the real symmetric matrix $\mathbf{A}$ are all *real*. We have

$$\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i, \tag{A.147}$$

and, taking the transpose and complex conjugate,

$$\mathbf{x}_i^{*\,T}\,\mathbf{A} = \lambda_i^{*}\,\mathbf{x}_i^{*\,T}, \tag{A.148}$$

where $\mathbf{x}_i$ and λ_i are the ith eigenvector and eigenvalue of $\mathbf{A}$, respectively. Left multiplying Equation (A.147) by $\mathbf{x}_i^{*\,T}$, we obtain

$$\mathbf{x}_i^{*\,T}\,\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i^{*\,T}\mathbf{x}_i. \tag{A.149}$$

Likewise, right multiplying Equation (A.148) by $\mathbf{x}_i$, we get

$$\mathbf{x}_i^{*\,T}\,\mathbf{A}\,\mathbf{x}_i = \lambda_i^{*}\,\mathbf{x}_i^{*\,T}\mathbf{x}_i. \tag{A.150}$$

The difference of the previous two equations yields

$$(\lambda_i - \lambda_i^{*})\,\mathbf{x}_i^{*\,T}\mathbf{x}_i = 0. \tag{A.151}$$

It follows that $\lambda_i = \lambda_i^{*}$, because $\mathbf{x}_i^{*\,T}\mathbf{x}_i$ (which is $\mathbf{x}_i^{*}\cdot\mathbf{x}_i$ in vector notation) is real and positive definite. Hence, λ_i is real. It immediately follows that $\mathbf{x}_i$ is real.

Next, let us show that two eigenvectors corresponding to two *different* eigenvalues are *mutually orthogonal*. Let

$$\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i \tag{A.152}$$

and

$$\mathbf{A}\,\mathbf{x}_j = \lambda_j\,\mathbf{x}_j, \tag{A.153}$$

where $\lambda_i \neq \lambda_j$. Taking the transpose of the first equation and right multiplying by $\mathbf{x}_j$, and left multiplying the second equation by $\mathbf{x}_i^T$, we obtain

$$\mathbf{x}_i^T\,\mathbf{A}\,\mathbf{x}_j = \lambda_i\,\mathbf{x}_i^T\,\mathbf{x}_j \tag{A.154}$$

and

$$\mathbf{x}_i^T\,\mathbf{A}\,\mathbf{x}_j = \lambda_j\,\mathbf{x}_i^T\,\mathbf{x}_j. \tag{A.155}$$

Taking the difference of these two equations, we get

$$(\lambda_i - \lambda_j)\,\mathbf{x}_i^T\,\mathbf{x}_j = 0. \tag{A.156}$$

Because, by hypothesis, $\lambda_i \neq \lambda_j$, it follows that $\mathbf{x}_i^T\,\mathbf{x}_j = 0$. In vector notation, this is the same as $\mathbf{x}_i \cdot \mathbf{x}_j = 0$. Hence, the eigenvectors $\mathbf{x}_i$ and $\mathbf{x}_j$ are mutually orthogonal.

Suppose that $\lambda_i = \lambda_j = \lambda$. In this case, we cannot conclude that $\mathbf{x}_i^T\,\mathbf{x}_j = 0$ by the preceding argument. However, it is easily seen that any linear combination of $\mathbf{x}_i$ and $\mathbf{x}_j$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ. Hence, it is possible to define two new eigenvectors of $\mathbf{A}$, with the eigenvalue λ, which are mutually orthogonal. For instance,

$$\mathbf{x}_i' = \mathbf{x}_i \tag{A.157}$$

and

$$\mathbf{x}_j' = \mathbf{x}_j - \left(\frac{\mathbf{x}_i^T\,\mathbf{x}_j}{\mathbf{x}_i^T\,\mathbf{x}_i}\right)\mathbf{x}_i. \tag{A.158}$$

It should be clear that this argument can be generalized to deal with any number of eigenvalues that take the same value.

In conclusion, a real symmetric n-dimensional matrix possesses n real eigenvalues, with n associated real eigenvectors, which are, or can be chosen to be, mutually orthogonal.