6 Lagrangian mechanics

6.1 Introduction

This chapter describes an elegant reformulation of the laws of Newtonian mechanics that is due to the French-Italian scientist Joseph Louis Lagrange (1736–1813). This reformulation is particularly useful for finding the equations of motion of complicated dynamical systems.

6.2 Generalized coordinates

Let the q_i, for $i = 1, \mathcal{F}$, be a set of coordinates that uniquely specifies the instantaneous configuration of some dynamical system. Here, it is assumed that each of the q_i can vary independently. The q_i might be Cartesian coordinates, angles, or some mixture of both types of coordinate, and are therefore termed *generalized coordinates*. A dynamical system whose instantaneous configuration is fully specified by $\mathcal{F}$ independent generalized coordinates is said to have $\mathcal{F}$ *degrees of freedom*. For instance, the instantaneous position of a particle moving freely in three dimensions is completely specified by its three Cartesian coordinates, x, y, and z. Moreover, these coordinates are clearly independent of one another. Hence, a dynamical system consisting of a single particle moving freely in three dimensions has three degrees of freedom. If there are two freely moving particles then the system has six degrees of freedom, and so on.

Suppose that we have a dynamical system consisting of N particles moving freely in three dimensions. This is an $\mathcal{F} = 3N$ degree-of-freedom system whose instantaneous configuration can be specified by $\mathcal{F}$ Cartesian coordinates. Let us denote these coordinates the x_j, for $j = 1, \mathcal{F}$. Thus, x_1, x_2, x_3 are the Cartesian coordinates of the first particle, x_4, x_5, x_6 the Cartesian coordinates of the second particle, and so on. Suppose that the instantaneous configuration of the system can also be specified by $\mathcal{F}$ generalized coordinates, which we shall denote the q_i, for $i = 1, \mathcal{F}$. Thus, the q_i might be the spherical coordinates of the particles. In general, we expect the x_j to be functions of the q_i. In other words,

$$x_j = x_j(q_1, q_2, \ldots, q_{\mathcal{F}}, t) \tag{6.1}$$

for $j = 1, \mathcal{F}$. Here, for the sake of generality, we have included the possibility that the functional relationship between the x_j and the q_i might depend on the time, t, explicitly. This would be the case if the dynamical system were subject to time-varying constraints—for instance, a system consisting of a particle constrained to move on a surface that is itself moving. Finally, by the chain rule, the variation of the x_j due to a variation of the q_i (at constant t) is given by

$$\delta x_j = \sum_{i=1,\mathcal{F}} \frac{\partial x_j}{\partial q_i}\,\delta q_i \tag{6.2}$$

for $j = 1, \mathcal{F}$.

6.3 Generalized forces

The work done on the dynamical system when its Cartesian coordinates change by δx_j is simply

$$\delta W = \sum_{j=1,\mathcal{F}} f_j\,\delta x_j. \tag{6.3}$$

Here, the f_j are the Cartesian components of the forces acting on the various particles making up the system. Thus, f_1, f_2, f_3 are the components of the force acting on the first particle, f_4, f_5, f_6 the components of the force acting on the second particle, and so on. Using Equation (6.2), we can also write

$$\delta W = \sum_{j=1,\mathcal{F}} f_j \sum_{i=1,\mathcal{F}} \frac{\partial x_j}{\partial q_i}\,\delta q_i. \tag{6.4}$$

The preceding expression can be rearranged to give

$$\delta W = \sum_{i=1,\mathcal{F}} Q_i\,\delta q_i, \tag{6.5}$$

where

$$Q_i = \sum_{j=1,\mathcal{F}} f_j \frac{\partial x_j}{\partial q_i}. \tag{6.6}$$

Here, the Q_i are termed *generalized forces*. A generalized force does not necessarily have the dimensions of force. However, the product $Q_i\,q_i$ must have the dimensions of work. Thus, if a particular q_i is a Cartesian coordinate, then the associated Q_i is a force. Conversely, if a particular q_i is an angle, then the associated Q_i is a torque.

Suppose that the dynamical system in question is *conservative*. It follows that

$$f_j = -\frac{\partial U}{\partial x_j} \tag{6.7}$$

for $j = 1, \mathcal{F}$, where $U(x_1, x_2, \ldots, x_{\mathcal{F}}, t)$ is the system's potential energy. Hence, according to Equation (6.6),

$$Q_i = -\sum_{j=1,\mathcal{F}} \frac{\partial U}{\partial x_j}\frac{\partial x_j}{\partial q_i} = -\frac{\partial U}{\partial q_i} \tag{6.8}$$

for $i = 1, \mathcal{F}$.

https://doi.org/10.1017/CBO9781139152310.007 Published online by Cambridge University Press

6.4 Lagrange's equation

The Cartesian equations of motion of our system take the form

$$m_j\,\ddot{x}_j = f_j \tag{6.9}$$

for $j = 1, \mathcal{F}$, where m_1, m_2, m_3 are each equal to the mass of the first particle; m_4, m_5, m_6 are each equal to the mass of the second particle; and so forth. Furthermore, the kinetic energy of the system can be written

$$K = \frac{1}{2}\sum_{j=1,\mathcal{F}} m_j\,\dot{x}_j^{\,2}. \tag{6.10}$$

Because $x_j = x_j(q_1, q_2, \ldots, q_{\mathcal{F}}, t)$, we can write

$$\dot{x}_j = \sum_{i=1,\mathcal{F}} \frac{\partial x_j}{\partial q_i}\,\dot{q}_i + \frac{\partial x_j}{\partial t} \tag{6.11}$$

for $j = 1, \mathcal{F}$. Hence, it follows that $\dot{x}_j = \dot{x}_j(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{\mathcal{F}}, q_1, q_2, \ldots, q_{\mathcal{F}}, t)$. According to the preceding equation,

$$\frac{\partial \dot{x}_j}{\partial \dot{q}_i} = \frac{\partial x_j}{\partial q_i}, \tag{6.12}$$

where we are treating the $\dot{q}_i$ and the q_i as *independent* variables.

Multiplying Equation (6.12) by $\dot{x}_j$, and then differentiating with respect to time, we obtain

$$\frac{d}{dt}\left(\dot{x}_j\,\frac{\partial \dot{x}_j}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\dot{x}_j\,\frac{\partial x_j}{\partial q_i}\right) = \ddot{x}_j\,\frac{\partial x_j}{\partial q_i} + \dot{x}_j\,\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right). \tag{6.13}$$

Now,

$$\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right) = \sum_{k=1,\mathcal{F}} \frac{\partial^2 x_j}{\partial q_i\,\partial q_k}\,\dot{q}_k + \frac{\partial^2 x_j}{\partial q_i\,\partial t}. \tag{6.14}$$

Furthermore,

$$\frac{1}{2}\,\frac{\partial \dot{x}_j^{\,2}}{\partial \dot{q}_i} = \dot{x}_j\,\frac{\partial \dot{x}_j}{\partial \dot{q}_i} \tag{6.15}$$

and

$$\begin{aligned}\frac{1}{2}\,\frac{\partial \dot{x}_j^{\,2}}{\partial q_i} = \dot{x}_j\,\frac{\partial \dot{x}_j}{\partial q_i} &= \dot{x}_j\,\frac{\partial}{\partial q_i}\left(\sum_{k=1,\mathcal{F}} \frac{\partial x_j}{\partial q_k}\,\dot{q}_k + \frac{\partial x_j}{\partial t}\right)\\ &= \dot{x}_j\left(\sum_{k=1,\mathcal{F}} \frac{\partial^2 x_j}{\partial q_i\,\partial q_k}\,\dot{q}_k + \frac{\partial^2 x_j}{\partial q_i\,\partial t}\right) = \dot{x}_j\,\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right),\end{aligned} \tag{6.16}$$

where use has been made of Equation (6.14). Thus, it follows from Equations (6.13), (6.15), and (6.16) that

$$\frac{d}{dt}\left(\frac{1}{2}\,\frac{\partial \dot{x}_j^{\,2}}{\partial \dot{q}_i}\right) = \ddot{x}_j\,\frac{\partial x_j}{\partial q_i} + \frac{1}{2}\,\frac{\partial \dot{x}_j^{\,2}}{\partial q_i}. \tag{6.17}$$

https://doi.org/10.1017/CBO9781139152310.007 Published online by Cambridge University Press

Let us take Equation (6.17), multiply by m_j, and then sum over all j. We obtain

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = \sum_{j=1,\mathcal{F}} f_j\,\frac{\partial x_j}{\partial q_i} + \frac{\partial K}{\partial q_i}, \tag{6.18}$$

where we have made use of Equations (6.9) and (6.10). Thus, it follows from Equation (6.6) that

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = Q_i + \frac{\partial K}{\partial q_i}. \tag{6.19}$$

Finally, making use of Equation (6.8), we get

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = -\frac{\partial U}{\partial q_i} + \frac{\partial K}{\partial q_i}. \tag{6.20}$$

It is helpful to introduce a function $\mathcal{L}$, called the *Lagrangian*, which is defined as the difference between the kinetic and potential energies of the dynamical system under investigation:

$$\mathcal{L} = K - U. \tag{6.21}$$

Becuase the potential energy U is clearly independent of the $\dot{q}_i$, it follows from Equation (6.20) that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{6.22}$$

for $i = 1, \mathcal{F}$. This equation is known as *Lagrange's equation*.

According to the preceding analysis, if we can express the kinetic and potential energies of our dynamical system solely in terms of our generalized coordinates and their time derivatives, then we can immediately write down the equations of motion of the system, expressed in terms of the generalized coordinates, using Lagrange's equation, Equation (6.22). Unfortunately, this scheme works only for conservative systems.

As an example, consider a particle of mass m moving in two dimensions in the central potential $U(r)$. This is clearly a two-degree-of-freedom dynamical system. As described in Section 3.4, the particle's instantaneous position is most conveniently specified in terms of the plane polar coordinates r and θ. These are our two generalized coordinates. According to Equation (3.13), the square of the particle's velocity can be written

$$v^2 = \dot{r}^2 + (r\,\dot{\theta})^2. \tag{6.23}$$

Hence, the Lagrangian of the system takes the form

$$\mathcal{L} = \frac{1}{2}\,m\,(\dot{r}^2 + r^2\,\dot{\theta}^2) - U(r). \tag{6.24}$$

Note that

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\,\dot{r}, \qquad \frac{\partial \mathcal{L}}{\partial r} = m\,r\,\dot{\theta}^2 - \frac{dU}{dr}, \tag{6.25}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\,r^2\,\dot{\theta}, \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \theta} = 0. \tag{6.26}$$

https://doi.org/10.1017/CBO9781139152310.007 Published online by Cambridge University Press

Now, Lagrange's equation, Equation (6.22), yields the equations of motion,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \tag{6.27}$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0. \tag{6.28}$$

Hence, we obtain

$$\frac{d}{dt}(m\,\dot{r}) - m\,r\,\dot{\theta}^2 + \frac{dU}{dr} = 0 \tag{6.29}$$

and

$$\frac{d}{dt}\left(m\,r^2\,\dot{\theta}\right) = 0, \tag{6.30}$$

or

$$\ddot{r} - r\,\dot{\theta}^2 = -\frac{dV}{dr} \tag{6.31}$$

and

$$r^2\,\dot{\theta} = h, \tag{6.32}$$

where $V = U/m$ and h is a constant. We recognize Equations (6.31) and (6.32) as the equations that we derived in Chapter 3 for motion in a central potential. The advantage of the Lagrangian method of deriving these equations is that we avoid having to express the acceleration in terms of the generalized coordinates r and θ.

6.5 Generalized momenta

Consider the motion of a single particle moving in one dimension. The kinetic energy is

$$K = \frac{1}{2}\,m\,\dot{x}^2, \tag{6.33}$$

where m is the mass of the particle and x its displacement. The particle's linear momentum is $p = m\,\dot{x}$. However, this can also be written

$$p = \frac{\partial K}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial \dot{x}}, \tag{6.34}$$

because $\mathcal{L} = K - U$ and the potential energy U is independent of $\dot{x}$.

Consider a dynamical system described by $\mathcal{F}$ generalized coordinates q_i for $i = 1, \mathcal{F}$. By analogy with the above expression, we can define *generalized momenta* of the form

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{6.35}$$

for $i = 1, \mathcal{F}$. Here, p_i is sometimes called the momentum *conjugate* to the coordinate q_i. Hence, Lagrange's equation, Equation (6.22), can be written

$$\frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i} \tag{6.36}$$

for $i = 1, \mathcal{F}$. Note that a generalized momentum does not necessarily have the dimensions of linear momentum.

Suppose that the Lagrangian $\mathcal{L}$ does not depend explicitly on some coordinate q_k. It follows from Equation (6.36) that

$$\frac{dp_k}{dt} = \frac{\partial \mathcal{L}}{\partial q_k} = 0. \tag{6.37}$$

Hence,

$$p_k = \text{const.} \tag{6.38}$$

The coordinate q_k is said to be *ignorable* in this case. Thus, we conclude that the generalized momentum associated with an ignorable coordinate is a constant of the motion.

For example, the Lagrangian [Equation (6.24)] for a particle moving in a central potential is independent of the angular coordinate θ. Thus, θ is an ignorable coordinate, and

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\, r^2\, \dot{\theta} \tag{6.39}$$

is a constant of the motion. Of course, p_θ is the angular momentum about the origin. This is conserved because a central force exerts no torque about the origin.

Exercises

6.1 A horizontal rod AB rotates with constant angular velocity ω about its midpoint O. A particle P is attached to it by equal strings AP and BP. If θ is the inclination of the plane APB to the vertical, prove that

$$\frac{d^2\theta}{dt^2} - \omega^2 \sin\theta \cos\theta = -\frac{g}{l} \sin\theta,$$

where $l = OP$. Deduce the condition that the vertical position of OP should be stable.

6.2 A double pendulum consists of two simple pendula, with one pendulum suspended from the bob of the other. Suppose that the two pendula have equal lengths, l, and bobs of equal mass, m, and are confined to move in the same vertical plane. Let θ and ϕ—the angles that the upper and lower pendula make with the downward vertical (respectively)—be the generalized coordinates. Demonstrate that Lagrange's equations of motion for the system are

$$2\,\ddot{\theta} + \cos(\theta - \phi)\,\ddot{\phi} + \sin(\theta - \phi)\,\dot{\phi}^2 + \frac{2\,g}{l} \sin\theta = 0$$

https://doi.org/10.1017/CBO9781139152310.007 Published online by Cambridge University Press

and

$$\ddot{\phi} + \cos(\theta - \phi)\,\ddot{\theta} - \sin(\theta - \phi)\,\dot{\theta}^2 + \frac{g}{l}\,\sin\phi = 0.$$

6.3 Consider an elastic pendulum consisting of a bob of mass m attached to a light elastic string of stiffness k and unstretched length l. Let x be the extension of the string, and θ the angle that the string makes with the downward vertical. Assume that any motion is confined to a vertical plane. Demonstrate that Lagrange's equations of motion for the system are

$$\ddot{x} - (l + x)\,\dot{\theta}^2 - g\,\cos\theta + \frac{k}{m}\,x = 0$$

and

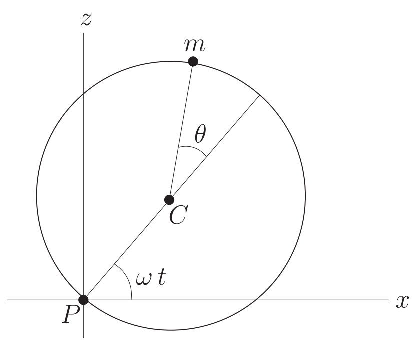
$$\ddot{\theta} + \frac{2\,\dot{x}\,\dot{\theta}}{l + x} + \frac{g}{l + x}\,\sin\theta = 0.$$

6.4 A disk of mass M and radius R rolls without slipping down a plane inclined at an angle α to the horizontal. The disk has a short weightless axle of negligible radius. From this axle is suspended a simple pendulum of length $l < R$ whose bob is of mass m. Assume that the motion of the pendulum takes place in the plane of the disk. Let s be the displacement of the center of mass of the disk down the slope, and let θ be the angle subtended between the pendulum and the downward vertical. Demonstrate that Lagrange's equations of motion for the system are

$$\left(\frac{3}{2}\,M + m\right)\ddot{s} + m\,l\,\cos(\alpha + \theta)\,\ddot{\theta} - m\,l\,\sin(\alpha + \theta)\,\dot{\theta}^2 - (M + m)\,g\,\sin\alpha = 0$$

and

$$\ddot{\theta} + \cos(\alpha + \theta)\,\frac{\ddot{s}}{l} + \frac{g}{l}\,\sin\theta = 0.$$



6.5 A vertical circular hoop of radius a is rotated in a vertical plane about a point P on its circumference at the constant angular velocity ω. A bead of mass m slides without friction on the hoop. Let the generalized coordinate be the angle θ shown in the diagram. Here, x is a horizontal Cartesian coordinate, z a vertical Cartesian coordinate, and C the center of the hoop. Demonstrate that the equation of motion of the system is

$$\ddot{\theta} + \omega^2\,\sin\theta + \frac{g}{a}\,\cos(\omega\,t + \theta) = 0.$$

(Modified from Fowles and Cassiday 2005.)

6.6 The kinetic energy of a rotating rigid object with an axis of symmetry can be written

$$K = \frac{1}{2}\left[\mathcal{I}_\perp\,\dot{\theta}^2 + (\mathcal{I}_\perp\,\sin^2\theta + \mathcal{I}_\parallel\,\cos^2\theta)\,\dot{\phi}^2 + 2\,\mathcal{I}_\parallel\,\cos\theta\,\dot{\phi}\,\dot{\psi} + \mathcal{I}_\parallel\,\dot{\psi}^2\right],$$

where $\mathcal{I}_\parallel$ is the moment of inertia about the symmetry axis, $\mathcal{I}_\perp$ is the moment of inertia about an axis perpendicular to the symmetry axis, and θ, ϕ, ψ are the three Euler angles. (See Chapter 7.) Suppose that the object is rotating freely. Find the momenta conjugate to the Euler angles. Which of these momenta are conserved? Find Lagrange's equations of motion for the system. Demonstrate that if the system is precessing steadily (which implies that θ, $\dot{\phi}$, and $\dot{\psi}$ are constants), then

$$\dot{\psi} = \left(\frac{\mathcal{I}_\perp - \mathcal{I}_\parallel}{\mathcal{I}_\parallel}\right)\cos\theta\,\dot{\phi}.$$

6.7 Demonstrate that the components of acceleration in the spherical coordinate system are

$$\begin{aligned} a_r &= \ddot{r} - r\,\dot{\theta}^2 - r\,\sin^2\theta\,\dot{\phi}^2, \\ a_\theta &= \frac{1}{r}\frac{d}{dt}(r^2\,\dot{\theta}) - r\,\sin\theta\,\cos\theta\,\dot{\phi}^2, \\ a_\phi &= \frac{1}{r\,\sin\theta}\frac{d}{dt}(r^2\,\sin^2\theta\,\dot{\phi}). \end{aligned}$$

(From Lamb 1923.)

6.8 A particle is constrained to move on a smooth spherical surface of radius a. Suppose that the particle is projected with velocity v along the horizontal great circle. Demonstrate that the particle subsequently falls a vertical height $a\,\mathrm{e}^{-u}$, where

$$\sinh u = \frac{v^2}{4\,g\,a}.$$

Show that if v^2 is large compared with $4\,g\,a$, then this height becomes approximately $2\,g\,a^2/v^2$. (From Lamb 1923.)

6.9 Consider a nonconservative system in which the dissipative forces take the form $f_i = -k_i\,\dot{x}_i$, where the x_i are Cartesian coordinates, and the k_i are all positive. Demonstrate that the dissipative forces can be incorporated into the Lagrangian formalism provided that Lagrange's equations of motion are modified to read

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) - \frac{\partial\mathcal{L}}{\partial q_i} + \frac{\partial\mathcal{R}}{\partial\dot{q}_i} = 0,$$

where

$$\mathcal{R} = \frac{1}{2}\sum_i k_i\,\dot{x}_i^2$$

is termed the *Rayleigh dissipation function*.