4 Orbits in central force fields

4.1 Introduction

This chapter examines the motion of a celestial body in a general central potential—that is, a gravitational potential that is a function of the radial coordinate, r, only, but does not necessarily vary as $1/r$.

4.2 Motion in a general central force field

Consider the motion of an object in a general (attractive) central force field characterized by the potential energy per unit mass function $V(r)$. Because the force field is central, it still remains true that

$$h = r^2\,\dot{\theta} \tag{4.1}$$

is a constant of the motion. (See Section 3.5.) As is easily demonstrated, Equation (3.28) generalizes to

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{h^2}\,\frac{dV}{du}, \tag{4.2}$$

where $u = r^{-1}$.

Suppose, for instance, that we wish to find the potential $V(r)$ that causes an object to execute the spiral orbit

$$r = r_0\,\theta^2. \tag{4.3}$$

Substitution of $u = (r_0\,\theta^2)^{-1}$ into Equation (4.2) yields

$$\frac{dV}{du} = -h^2\left(6\,r_0\,u^2 + u\right). \tag{4.4}$$

Integrating, we obtain

$$V(u) = -h^2\left(2\,r_0\,u^3 + \frac{u^2}{2}\right) \tag{4.5}$$

or

$$V(r) = -h^2\left(\frac{2\,r_0}{r^3} + \frac{1}{2\,r^2}\right). \tag{4.6}$$

In other words, the spiral orbit specified by Equation (4.3) is obtained from a mixture of an inverse-square and inverse-cube potential.

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4.3 Motion in a nearly circular orbit

In principle, a circular orbit is a possible orbit for any attractive central force. However, not all such forces result in *stable* circular orbits. Let us now consider the stability of circular orbits in a general central force field. Equation (3.25) generalizes to

$$\ddot{r} - \frac{h^2}{r^3} = f(r), \tag{4.7}$$

where $f(r) = -dV/dr$ is the radial force per unit mass. For a circular orbit, $\ddot{r} = 0$ and the above equation reduces to

$$-\frac{h^2}{r_c^3} = f(r_c), \tag{4.8}$$

where r_c is the radius of the orbit.

Let us now consider *small* departures from circularity. Suppose that

$$x = r - r_c. \tag{4.9}$$

Equation (4.7) can be written

$$\ddot{x} - \frac{h^2}{(r_c + x)^3} = f(r_c + x). \tag{4.10}$$

Expanding the two terms involving $r_c + x$ as power series in x/r_c, and keeping all terms up to first order, we obtain

$$\ddot{x} - \frac{h^2}{r_c^3}\left(1 - 3\,\frac{x}{r_c}\right) = f(r_c) + f'(r_c)\,x, \tag{4.11}$$

where $'$ denotes a derivative. Making use of Equation (4.8), we find that the preceding equation reduces to

$$\ddot{x} + \left[-\frac{3\,f(r_c)}{r_c} - f'(r_c)\right] x = 0. \tag{4.12}$$

If the term in square brackets is positive, then we obtain a simple harmonic equation, which we already know has bounded solutions (see Section 1.8)—that is, the orbit is stable to small perturbations. On the other hand, if the term in square brackets is negative, then we obtain an equation whose solutions grow exponentially in time (see Section 1.8)—that is, the orbit is *unstable* to small perturbations. Thus, the stability criterion for a circular orbit of radius r_c in a central force field characterized by a radial force (per unit mass) function $f(r)$ is

$$f(r_c) + \frac{r_c}{3}\,f'(r_c) < 0. \tag{4.13}$$

For example, consider an attractive power-law force function of the form

$$f(r) = -c\,r^n, \tag{4.14}$$

where $c > 0$. Substituting into the preceding stability criterion, we obtain

$$-c\,r_c^n - \frac{c\,n}{3}\,r_c^n < 0 \tag{4.15}$$

or

$$n > -3. \tag{4.16}$$

We conclude that circular orbits in attractive central force fields that decay faster than r^{-3} are unstable. The case $n = -3$ is special, because the first-order terms in the expansion of Equation (4.10) cancel out exactly, and it is necessary to retain the second-order terms. Doing this, we can easily demonstrate that circular orbits are also unstable for inverse-cube ($n = -3$) forces. (See Exercise 4.9.)

An *apsis* (plural, *apsides*) is a point on an orbit at which the radial distance, r, assumes either a maximum or a minimum value. Thus, the perihelion and aphelion points are the apsides of planetary orbits. The angle through which the radius vector rotates in going between two consecutive apsides is called the *apsidal angle*. Hence, the apsidal angle for elliptical orbits in an inverse-square force field is π.

For the case of stable, nearly circular orbits, we have seen that r oscillates sinusoidally about its mean value, r_c. Indeed, it is clear from Equation (4.12) that the period of the oscillation is

$$T = \frac{2\pi}{[-3\,f(r_c)/r_c - f'(r_c)]^{1/2}}. \tag{4.17}$$

The apsidal angle is the amount by which θ increases in going between a maximum and a minimum of r. The time taken to achieve this is clearly $T/2$. Now, $\dot{\theta} = h/r^2$, where h is a constant of the motion and r is almost constant. Thus, $\dot{\theta}$ is approximately constant. In fact,

$$\dot{\theta} \simeq \frac{h}{r_c^2} = \left[-\frac{f(r_c)}{r_c}\right]^{1/2}, \tag{4.18}$$

where use has been made of Equation (4.8). Thus, the apsidal angle, ψ, is given by

$$\psi = \frac{T}{2}\,\dot{\theta} = \pi\left[3 + r_c\,\frac{f'(r_c)}{f(r_c)}\right]^{-1/2}. \tag{4.19}$$

For the case of attractive power-law central forces of the form $f(r) = -c\,r^n$, where $c > 0$, the apsidal angle becomes

$$\psi = \frac{\pi}{(3+n)^{1/2}}. \tag{4.20}$$

It should be clear that if an orbit is going to close on itself, the apsidal angle needs to be a *rational* fraction of 2π. There are, in fact, only two small-integer values of the power-law index, n, for which this is the case. As we have seen, for an inverse-square force law (i.e., $n = -2$), the apsidal angle is π. For a linear force law (i.e., $n = 1$), the apsidal angle is $\pi/2$. However, for quadratic (i.e., $n = 2$) or cubic (i.e., $n = 3$) force laws, the apsidal angle is an *irrational* fraction of 2π, which means that non-circular orbits in such force fields never close on themselves.

4.4 Perihelion precession of planets

The solar system consists of eight major planets (Mercury to Neptune) moving around the Sun in slightly elliptical orbits that are approximately coplanar with one another. According to Chapter 3, if we neglect the relatively weak interplanetary gravitational interactions, the perihelia of the various planets (i.e., the points on their orbits at which they are closest to the Sun) remain *fixed* in space. However, once these interactions are taken into account, it turns out that the planetary perihelia all slowly *precess* in a prograde manner (i.e., rotate in the same direction as the orbital motion) around the Sun.[1] We can calculate the approximate rate of perihelion precession of a given planet by treating the other planets as *uniform concentric rings*, centered on the Sun, of mass equal to the planetary mass, and radius equal to the mean orbital radius. This method of calculation, which is due to Gauss, is equivalent to averaging the interplanetary gravitational interactions over the orbits of the other planets. It is reasonable to do this because the precession period in question is very much longer than the orbital period of any planet in the solar system. Thus, by treating the other planets as rings, we can calculate the mean gravitational perturbation due to these planets, and, thereby, determine the desired precession rate. (Actually, Gauss also incorporated the eccentricities, and non-uniform angular velocities, of the planetary orbits into his original calculation.)

We can conveniently index the planets in the solar system by designating Mercury as planet 1, and Neptune planet 8. Let the m_i and the a_i for $i = 1, 8$ be the planetary masses and mean orbital radii, respectively. Furthermore, let M be the mass of the Sun. It follows, from Section 2.7, that the gravitational potential generated in the vicinity of the ith planet by the Sun and the other planets is

$$\Phi_i(r) = -\frac{G\,M}{r} - \sum_{k=0,\infty} p_k \left[\sum_{j<i} \frac{G\,m_j}{a_j}\left(\frac{a_j}{r}\right)^{2k+1} + \sum_{j>i} \frac{G\,m_j}{a_j}\left(\frac{r}{a_j}\right)^{2k}\right], \tag{4.21}$$

where $p_k = [P_{2k}(0)]^2$. Here, $P_n(x)$ is a Legendre polynominal. The radial force per unit mass acting on the ith planet is written $f_i = -d\Phi_i/dr|_{r=R_i}$. Hence, it is easily demonstrated that

$$\left[3 + \frac{r\,df_i/dr}{f_i}\right]_{r=a_i}^{-1/2} \simeq 1 + \sum_{k=1,\infty} k\,(2k+1)\,p_k \left[\sum_{j<i} \frac{m_j}{M}\left(\frac{a_j}{a_i}\right)^{2k} + \sum_{j>i} \frac{m_j}{M}\left(\frac{a_i}{a_j}\right)^{2k+1}\right] \tag{4.22}$$

to first order in the ratio of the planetary masses to the solar mass. Thus, according to Equation (4.19), the apsidal angle for the ith planet is

$$\psi_i \simeq \pi + \pi \sum_{k=1,\infty} k\,(2k+1)\,p_k \left[\sum_{j<i} \frac{m_j}{M}\left(\frac{a_j}{a_i}\right)^{2k} + \sum_{j>i} \frac{m_j}{M}\left(\frac{a_i}{a_j}\right)^{2k+1}\right]. \tag{4.23}$$

[1] Precession can be either *prograde* (in the same sense as orbital motion) or *retrograde* (in the opposite sense). Retrograde precession is often called *regression*.

Table 4.1 Observed and theoretical planetary perihelion precession rates (at J2000)

Planet	$\dot{\varpi}_{obs}(''\,\mathrm{yr}^{-1})^a$	$\dot{\varpi}_{th}(''\,\mathrm{yr}^{-1})$
Mercury	5.74	5.54
Venus	2.04	12.07
Earth	11.45	12.79
Mars	16.28	17.75
Jupiter	6.55	7.51
Saturn	19.50	18.59
Uranus	3.34	2.75
Neptune	0.36	0.67

[a] Source: Standish and Williams 1992.

Hence, the perihelion of the ith planet advances by

$$\delta\varpi_i = 2\,(\psi_i - \pi) \simeq 2\pi \sum_{k=1,\infty} k\,(2k+1)\,p_k \left[\sum_{j<i} \frac{m_j}{M}\left(\frac{a_j}{a_i}\right)^{2k} + \sum_{j>i} \frac{m_j}{M}\left(\frac{a_i}{a_j}\right)^{2k+1}\right] \tag{4.24}$$

radians per revolution around the Sun. Of course, the time required for a single revolution is the orbital period, T_i. Thus, the rate of perihelion precession, in *arc seconds per year*, is given by

$$\dot{\varpi}_i \simeq \frac{1296000}{T_i(\mathrm{yr})} \sum_{k=1,\infty} k\,(2k+1)\,p_k \left[\sum_{j<i} \frac{m_j}{M}\left(\frac{a_j}{a_i}\right)^{2k} + \sum_{j>i} \frac{m_j}{M}\left(\frac{a_i}{a_j}\right)^{2k+1}\right]. \tag{4.25}$$

Table 4.1 and Figure 4.1 compare the observed perihelion precession rates of the planets with the theoretical rates calculated from Equation (4.25) and the planetary data given in Table 3.1. It can be seen that there is excellent agreement between the two, except for the planet Venus. The main reason for this is that Venus has an unusually low eccentricity ($e = 0.0068$), which renders its perihelion point extremely sensitive to small perturbations.

4.5 Perihelion precession of Mercury

If the calculation described in the previous section is carried out more accurately, taking into account the slight eccentricities of the planetary orbits, as well as their small mutual inclinations, then the perihelion precession rate of the planet Mercury is found to be 5.32 arc seconds per year (Stewart 2005). However, the observed precession rate is 5.74 arc seconds per year. It turns out that the cause of this discrepancy is a general relativistic correction to Newtonian gravity.

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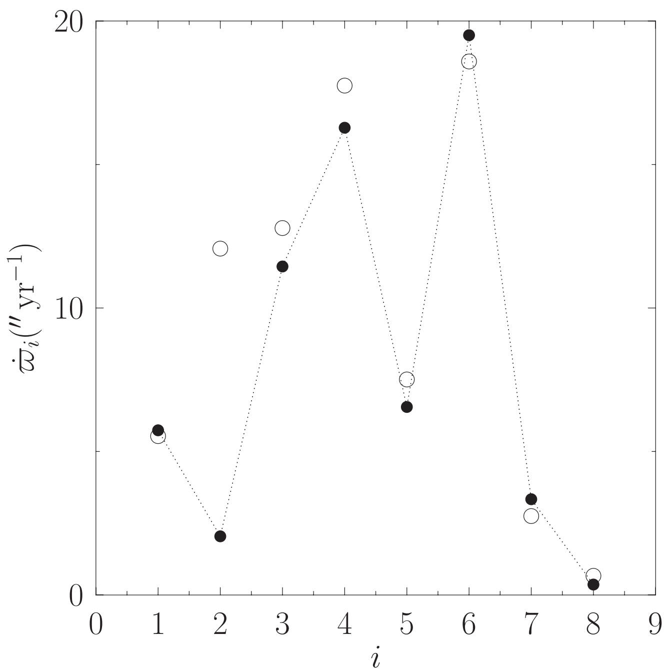

Fig. 4.1 Observed (full circles) and theoretical (empty circles) planetary perihelion precession rates (at J2000). Source (for observational data): Standish and Williams 1992.

General relativity gives rise to a small correction to the force per unit mass exerted by the Sun (mass M) on a planet in a circular orbit of radius r, and angular momentum per unit mass h. This correction is due to the curvature of space in the immediate vicinity of the Sun. In fact, the modified formula for f is (Rindler 1977)

$$f \simeq -\frac{G\,M}{r^2} - \frac{3\,G\,M\,h^2}{c^2\,r^4}, \tag{4.26}$$

where c is the velocity of light in a vacuum. It follows that

$$\frac{r\,f'}{f} \simeq -2\left(1 + \frac{3\,h^2}{c^2\,r^2} + \cdots\right), \tag{4.27}$$

to first order in $h^2/(c^2\,r^2) \ll 1$. Hence, from Equation (4.19), the apsidal angle is

$$\psi \simeq \pi\left(1 + \frac{3\,h^2}{c^2\,r^2}\right). \tag{4.28}$$

Thus, the perihelion advances by

$$\delta\varpi \simeq \frac{6\pi\,G\,M}{c^2\,r} \tag{4.29}$$

radians per revolution as a consequence of the general relativistic correction to Newtonian gravity. Here, use has been made of $h^2 = G\,M\,r$. It follows that the rate of perihelion precession due to the general relativistic correction is

$$\dot{\varpi} \simeq \frac{0.0383}{a\,T} \tag{4.30}$$

arc seconds per year, where a is the mean orbital radius in astronomical units, and T the orbital period in years. Hence, from Table 3.1, the general relativistic contribution to $\dot{\varpi}$ for Mercury is 0.41 arc seconds per year. It is easily demonstrated that the corresponding contribution is negligible for the other planets in the solar system. If the preceding calculation is carried out slightly more accurately, taking the eccentricity of Mercury's orbit into account, then the general relativistic contribution to $\delta\dot{\varpi}$ becomes 0.43 arc seconds per year. (See Exercise 9.2.) It follows that the total perihelion precession rate for Mercury is $5.32 + 0.43 = 5.75$ arc seconds per year. This is in almost exact agreement with the observed precession rate. Indeed, the ability of general relativity to explain the discrepancy between the observed perihelion precession rate of Mercury, and that calculated from Newtonian mechanics, was one of the first major successes of this theory.

Exercises

4.1 Derive Equations (4.2) and (4.7).

4.2 Prove that in the case of a central force varying inversely as the cube of the distance,

$$r^2 = A\,t^2 + B\,t + C,$$

where A, B, C are constants. (From Lamb 1923.)

4.3 The orbit of a particle moving in a central field is a circle that passes through the origin: $r = r_0 \cos\theta$, where $r_0 > 0$. Show that the force law is inverse-fifth power. (Modified from Fowles and Cassiday 2005.)

4.4 The orbit of a particle moving in a central field is the cardoid $r = a\,(1 + \cos\theta)$, where $a > 0$. Show that the force law is inverse fourth power.

4.5 A particle moving in a central field describes a spiral orbit $r = r_0 \exp(k\,\theta)$, where $r_0, k > 0$. Show that the force law is inverse cube, and that θ varies logarithmically with t. Demonstrate that there are two other possible types of orbit in this force field, and give their equations. (Modified from Fowles and Cassiday 2005.)

4.6 A particle moves in the spiral orbit $r = a\,\theta$, where $a > 0$. Suppose that θ increases linearly with t. Is the force acting on the particle central in nature? If not, determine how θ would have to vary with t in order to make the force central. Assuming that the force is central, demonstrate that the particle's potential energy per unit mass is

$$V(r) = -\frac{h^2}{2}\left(\frac{a^2}{r^4} + \frac{1}{r^2}\right),$$

where h is its (constant) angular momentum per unit mass. (Modified from Fowles and Cassiday 2005.)

4.7 A particle moves under the influence of a central force of the form

$$f(r) = -\frac{k}{r^2} + \frac{c}{r^3},$$

where k and c are positive constants. Show that the associated orbit can be written

$$r = \frac{a\,(1-e^2)}{1 + e\,\cos(\alpha\,\theta)},$$

which is a closed ellipse for $e < 1$ and $\alpha = 1$. Discuss the character of the orbit for $e < 1$ and $\alpha \neq 1$. Demonstrate that

$$\alpha = \left(1 + \frac{\gamma}{1-e^2}\right)^{1/2},$$

where $\gamma = c/(k\,a)$.

4.8 A particle moves in a circular orbit of radius r_0 in an attractive central force field of the form $f(r) = -c\,\exp(-r/a)/r^2$, where $c > 0$ and $a > 0$. Demonstrate that the orbit is stable only provided that $r_0 < a$.

4.9 A particle moves in a circular orbit in an attractive central force field of the form $f(r) = -a\,r^{-3}$, where $a > 0$. Show that the orbit is unstable to small perturbations.

4.10 A particle moves in a nearly circular orbit of radius a under the action of the radial force per unit mass

$$f(r) = -\frac{\mu}{r^2}\,\mathrm{e}^{-k\,r},$$

where $\mu > 0$ and $0 < k\,a \ll 1$. Demonstrate that the so-called apse line, joining successive apse points, rotates in the same direction as the orbital motion through an angle $\pi\,k\,a$ each revolution. (From Lamb 1923.)

4.11 A particle moves in a nearly circular orbit of radius a under the action of the central potential per unit mass

$$V(r) = -\frac{\mu}{r}\,\mathrm{e}^{-k\,r},$$

where $\mu > 0$ and $0 < k\,a \ll 1$. Show that the apse line rotates in the same direction as the orbital motion through an angle $\pi\,k^2\,a^2$ each revolution. (From Lamb 1923.)

4.12 Suppose that the solar system were embedded in a tenuous uniform dust cloud. Demonstrate that the apsidal angle of a planet in a nearly circular orbit around the Sun would be

$$\pi\left(1 - \frac{3}{2}\,\frac{M_0}{M}\right),$$

where M is the mass of the Sun, and M_0 is the mass of dust enclosed by a sphere whose radius matches the major radius of the orbit. It is assumed that $M_0 \ll M$.

4.13 Consider a satellite orbiting around an idealized planet that takes the form of a uniform spheroidal mass distribution of mean radius R and ellipticity ϵ (where $0 < \epsilon \ll 1$). Suppose that the orbit is nearly circular, with a major radius a, and lies in the equatorial plane of the planet. The potential energy per unit mass of the satellite is thus (see Chapter 2)

$$V(r) = -\frac{G\,M}{r}\left(1 + \frac{\epsilon}{5}\,\frac{R^2}{r^2}\right),$$

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where r is a radial coordinate in the equatorial plane. Demonstrate that the apse line rotates in the same direction as the orbital motion at the rate

$$\dot{\varpi} = \frac{3\,\epsilon}{5}\left(\frac{R}{a}\right)^2 n,$$

where n is the mean orbital angular velocity of the satellite.

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