Appendix C Expansion of orbital evolution equations

C.1 Introduction

The purpose of this appendix is to derive simplified evolution equations for the osculating orbital elements of a two-planet solar system, starting from the Lagrange planetary equations, Equations (B.125)–(B.130), and exploiting the fact that the planetary masses are all very small compared with the solar mass, as well as the fact that the planetary orbital eccentricities and inclinations (in radians) are small compared with unity. Our approach is mostly based on that of Murray and Dermott 1999.

Let the first planet have position vector $\mathbf{r}$, mass m, and the standard osculating elements a, $\bar{\lambda}_0$, e, I, ϖ, Ω. (See Section 3.12.) It is convenient to define the alternative elements $\bar{\lambda} = \bar{\lambda}_0 + \int_0^t n(t')\,dt'$, $h = e\,\sin\varpi$, $k = e\,\cos\varpi$, $p = \sin I\,\sin\Omega$, and $q = \sin I\,\cos\Omega$, where $n = (\mu/a^3)^{1/2}$ is the mean orbital angular velocity, $\mu = G\,(M+m)$, and M is the solar mass. Thus, the osculating elements of the first planet become a, $\bar{\lambda}$, h, k, p, q. Let a', $\bar{\lambda}'$, h', k', p', q' be the corresponding osculating elements of the second planet. Furthermore, let the second planet have position vector $\mathbf{r}'$, mass m', and mean orbital angular velocity $n' = (\mu'/a'^{\,3})^{1/2}$, where $\mu' = G\,(M + m')$.

C.2 Expansion of Lagrange planetary equations

The first planet's disturbing function can be written in the form [see Equation (9.8)]

$$\mathcal{R} = \frac{\tilde{\mu}'}{a'}\,\mathcal{S}, \tag{C.1}$$

where $\tilde{\mu}' = G\,m'$, and $\mathcal{S}$ is $\mathcal{O}(1)$. Thus, because $\mu = n^2\,a^3$, the Lagrange planetary equations, Equations (B.125)–(B.130), applied to the first planet, reduce to

$$\frac{d\ln a}{dt} = 2\,n\,\epsilon'\alpha\,\frac{\partial\mathcal{S}}{\partial\bar{\lambda}}, \tag{C.2}$$

$$\frac{d\bar{\lambda}}{dt} = n - 2\,n\,\epsilon'\alpha^2\,\frac{\partial\mathcal{S}}{\partial\alpha} + \frac{n\,\epsilon'\alpha\,(1-e^2)^{1/2}}{[1+(1-e^2)^{1/2}]}\left(h\,\frac{\partial\mathcal{S}}{\partial h} + k\,\frac{\partial\mathcal{S}}{\partial k}\right)$$
$$+ \frac{n\,\epsilon'\alpha\,\cos I}{2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\left(p\,\frac{\partial\mathcal{S}}{\partial p} + q\,\frac{\partial\mathcal{S}}{\partial q}\right), \tag{C.3}$$

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$$\frac{dh}{dt} = -\frac{n\,\epsilon'\alpha\,(1-e^2)^{1/2}}{[1+(1-e^2)^{1/2}]}\,h\,\frac{\partial\mathcal{S}}{\partial\bar{\lambda}} + n\,\epsilon'\alpha\,(1-e^2)^{1/2}\,\frac{\partial\mathcal{S}}{\partial k}$$

$$+\frac{n\,\epsilon'\alpha\,\cos I}{2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,k\left(p\,\frac{\partial\mathcal{S}}{\partial p} + q\,\frac{\partial\mathcal{S}}{\partial q}\right), \tag{C.4}$$

$$\frac{dk}{dt} = -\frac{n\,\epsilon'\alpha\,(1-e^2)^{1/2}}{[1+(1-e^2)^{1/2}]}\,k\,\frac{\partial\mathcal{S}}{\partial\bar{\lambda}} - n\,\epsilon'\alpha\,(1-e^2)^{1/2}\,\frac{\partial\mathcal{S}}{\partial h}$$

$$-\frac{n\,\epsilon'\alpha\,\cos I}{2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,h\left(p\,\frac{\partial\mathcal{S}}{\partial p} + q\,\frac{\partial\mathcal{S}}{\partial q}\right), \tag{C.5}$$

$$\frac{dp}{dt} = -\frac{n\,\epsilon'\alpha\,\cos I}{2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,p\left(\frac{\partial\mathcal{S}}{\partial\bar{\lambda}} + k\,\frac{\partial\mathcal{S}}{\partial h} - h\,\frac{\partial\mathcal{S}}{\partial k}\right)$$

$$+\frac{n\,\epsilon'\alpha\,\cos I}{(1-e^2)^{1/2}}\,\frac{\partial\mathcal{S}}{\partial q}, \tag{C.6}$$

and

$$\frac{dq}{dt} = -\frac{n\,\epsilon'\alpha\,\cos I}{2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,q\left(\frac{\partial\mathcal{S}}{\partial\bar{\lambda}} + k\,\frac{\partial\mathcal{S}}{\partial h} - h\,\frac{\partial\mathcal{S}}{\partial k}\right)$$

$$-\frac{n\,\epsilon'\alpha\,\cos I}{(1-e^2)^{1/2}}\,\frac{\partial\mathcal{S}}{\partial p}, \tag{C.7}$$

where

$$\epsilon' = \frac{\tilde{\mu}'}{\mu} = \frac{m'}{M+m} \tag{C.8}$$

and

$$\alpha = \frac{a}{a'}. \tag{C.9}$$

The Sun is much more massive than any planet in the solar system. It follows that the parameter ϵ' is very small compared with unity. Expansion of Equations (C.2)–(C.7) to first order in ϵ' yields

$$\bar{\lambda}(t) = \bar{\lambda}_0 + n^{(0)}\,t + \bar{\lambda}^{(1)}(t), \tag{C.10}$$

$$a(t) = a^{(0)}\left[1 + \epsilon'\,a^{(1)}(t)\right], \tag{C.11}$$

and

$$n(t) = n^{(0)}\left[1 - (3/2)\,\epsilon'\,a^{(1)}(t)\right], \tag{C.12}$$

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where $\bar{\lambda}^{(1)} \sim \mathcal{O}(\epsilon')$, $a^{(1)} \sim \mathcal{O}(1)$, $n^{(0)} = \{\mu/[a^{(0)}]^3\}^{1/2}$, and

$$\frac{d\epsilon' a^{(1)}}{dt} = \epsilon'\, n^{(0)} \left[2\,\alpha\,\frac{\partial \mathcal{S}}{\partial \bar{\lambda}^{(0)}}\right], \tag{C.13}$$

$$\frac{d\bar{\lambda}^{(1)}}{dt} = \epsilon'\, n^{(0)} \left\{-\frac{3}{2}\,a^{(1)} - 2\,\alpha^2\,\frac{\partial \mathcal{S}}{\partial \alpha} + \frac{\alpha\,(1-e^2)^{1/2}}{[1+(1-e^2)^{1/2}]}\left(h\,\frac{\partial \mathcal{S}}{\partial h} + k\,\frac{\partial \mathcal{S}}{\partial k}\right)\right.$$
$$\left. + \frac{\alpha\,\cos I}{2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\left(p\,\frac{\partial \mathcal{S}}{\partial p} + q\,\frac{\partial \mathcal{S}}{\partial q}\right)\right\}, \tag{C.14}$$

$$\frac{dh}{dt} = \epsilon'\, n^{(0)} \left\{-\frac{\alpha\,(1-e^2)^{1/2}}{[1+(1-e^2)^{1/2}]}\,h\,\frac{\partial \mathcal{S}}{\partial \bar{\lambda}^{(0)}} + \alpha\,(1-e^2)^{1/2}\,\frac{\partial \mathcal{S}}{\partial k}\right.$$
$$\left. + \frac{\alpha\,\cos I}{2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,k\left(p\,\frac{\partial \mathcal{S}}{\partial p} + q\,\frac{\partial \mathcal{S}}{\partial q}\right)\right\}, \tag{C.15}$$

$$\frac{dk}{dt} = \epsilon'\, n^{(0)} \left\{-\frac{\alpha\,(1-e^2)^{1/2}}{[1+(1-e^2)^{1/2}]}\,k\,\frac{\partial \mathcal{S}}{\partial \bar{\lambda}^{(0)}} - \alpha\,(1-e^2)^{1/2}\,\frac{\partial \mathcal{S}}{\partial h}\right.$$
$$\left. - \frac{\alpha\,\cos I}{2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,h\left(p\,\frac{\partial \mathcal{S}}{\partial p} + q\,\frac{\partial \mathcal{S}}{\partial q}\right)\right\}, \tag{C.16}$$

$$\frac{dp}{dt} = \epsilon'\, n^{(0)} \left[-\frac{\alpha\,\cos I}{2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,p\left(\frac{\partial \mathcal{S}}{\partial \bar{\lambda}^{(0)}} + k\,\frac{\partial \mathcal{S}}{\partial h} - h\,\frac{\partial \mathcal{S}}{\partial k}\right)\right.$$
$$\left. + \frac{\alpha\,\cos I}{(1-e^2)^{1/2}}\,\frac{\partial \mathcal{S}}{\partial q}\right], \tag{C.17}$$

and

$$\frac{dq}{dt} = \epsilon'\, n^{(0)} \left[-\frac{\alpha\,\cos I}{2\,\cos^2(I/2)\,(1-e^2)^{1/2}}\,q\left(\frac{\partial \mathcal{S}}{\partial \bar{\lambda}^{(0)}} + k\,\frac{\partial \mathcal{S}}{\partial h} - h\,\frac{\partial \mathcal{S}}{\partial k}\right)\right.$$
$$\left. - \frac{\alpha\,\cos I}{(1-e^2)^{1/2}}\,\frac{\partial \mathcal{S}}{\partial p}\right], \tag{C.18}$$

with $\bar{\lambda}^{(0)} = \bar{\lambda}_0 + n^{(0)}\,t$, and $\alpha = (a/a')^{(0)}$. In the following, for ease of notation, $\bar{\lambda}^{(0)}$, $a^{(0)}$, and $n^{(0)}$ are written simply as $\bar{\lambda}$, a, and n, respectively.

According to Table 3.1, the planets in the solar system all possess orbits whose eccentricities, e, and inclinations, I (in radians), are small compared with unity but large compared with the ratio of any planetary mass to that of the Sun. It follows that

$$\epsilon' \ll e,\, I \ll 1, \tag{C.19}$$

which is our fundamental ordering of small quantities. Assuming that $I, e', I' \sim \mathcal{O}(e)$, we can perform a secondary expansion in the small parameter e. It turns out that when the normalized disturbing function, $\mathcal{S}$, is expanded to second order in e it takes the general form (see Section C.3)

$$\mathcal{S} = \mathcal{S}_0(\alpha, \bar{\lambda}, \bar{\lambda}') + \mathcal{S}_1(\alpha, \bar{\lambda}, \bar{\lambda}', h, h', k, k') + \mathcal{S}_2(\alpha, \bar{\lambda}, \bar{\lambda}', h, h', k, k', p, p', q, q'), \tag{C.20}$$

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where $\mathcal{S}_n$ is $\mathcal{O}(e^n)$. If we expand the right-hand sides of Equations (C.13)–(C.18) to first order in e, we obtain

$$\frac{d\epsilon' a^{(1)}}{dt} = \epsilon' n \left[2\,\alpha\, \frac{\partial(\mathcal{S}_0+\mathcal{S}_1)}{\partial\bar{\lambda}} \right], \tag{C.21}$$

$$\frac{d\bar{\lambda}^{(1)}}{dt} = \epsilon' n \left[-\frac{3}{2}\, a^{(1)} - 2\,\alpha^2\, \frac{\partial(\mathcal{S}_0+\mathcal{S}_1)}{\partial\alpha} + \alpha \left(h\, \frac{\partial\mathcal{S}_1}{\partial h} + k\, \frac{\partial\mathcal{S}_1}{\partial k} \right) \right], \tag{C.22}$$

$$\frac{dh}{dt} = \epsilon' n \left[-\alpha\, h\, \frac{\partial\mathcal{S}_0}{\partial\bar{\lambda}} + \alpha\, \frac{\partial(\mathcal{S}_1+\mathcal{S}_2)}{\partial k} \right], \tag{C.23}$$

$$\frac{dk}{dt} = \epsilon' n \left[-\alpha\, k\, \frac{\partial\mathcal{S}_0}{\partial\bar{\lambda}} - \alpha\, \frac{\partial(\mathcal{S}_1+\mathcal{S}_2)}{\partial h} \right], \tag{C.24}$$

$$\frac{dp}{dt} = \epsilon' n \left[-\frac{\alpha}{2}\, p\, \frac{\partial\mathcal{S}_0}{\partial\bar{\lambda}} + \alpha\, \frac{\partial\mathcal{S}_2}{\partial q} \right], \tag{C.25}$$

and

$$\frac{dq}{dt} = \epsilon' n \left[-\frac{\alpha}{2}\, q\, \frac{\partial\mathcal{S}_0}{\partial\bar{\lambda}} - \alpha\, \frac{\partial\mathcal{S}_2}{\partial p} \right]. \tag{C.26}$$

Note that h, k, p, q are $\mathcal{O}(e)$, whereas α and $\bar{\lambda}$ are $\mathcal{O}(1)$.

By analogy, writing the second planet's disturbing function as [see Equation (9.9]

$$\mathcal{R}' = \frac{\tilde{\mu}}{a}\, \mathcal{S}', \tag{C.27}$$

where $\tilde{\mu} = G\, m$ and $\mathcal{S}'$ is $\mathcal{O}(1)$, and assuming that $\mathcal{S}'$ takes the form

$$\mathcal{S}' = \mathcal{S}_0'(\alpha, \bar{\lambda}, \bar{\lambda}') + \mathcal{S}_1'(\alpha, \bar{\lambda}, \bar{\lambda}', h, h', k, k') + \mathcal{S}_2'(\alpha, \bar{\lambda}, \bar{\lambda}', h, h', k, k', p, p', q, q'), \tag{C.28}$$

where $\mathcal{S}_n'$ is $\mathcal{O}(e^n)$, we see that the Lagrange planetary equations, applied to the second planet, yield

$$\frac{d\epsilon' a^{(1)'}}{dt} = \epsilon\, n' \left[2\,\alpha^{-1}\, \frac{\partial(\mathcal{S}_0'+\mathcal{S}_1')}{\partial\bar{\lambda}'} \right], \tag{C.29}$$

$$\frac{d\bar{\lambda}^{(1)'}}{dt} = \epsilon\, n' \left[-\frac{3}{2}\, a^{(1)'} + 2\, \frac{\partial(\mathcal{S}_0'+\mathcal{S}_1')}{\partial\alpha} + \alpha^{-1} \left(h'\, \frac{\partial\mathcal{S}_1'}{\partial h'} + k'\, \frac{\partial\mathcal{S}_1'}{\partial k'} \right) \right], \tag{C.30}$$

$$\frac{dh'}{dt} = \epsilon\, n' \left[-\alpha^{-1}\, h'\, \frac{\partial\mathcal{S}_0'}{\partial\bar{\lambda}'} + \alpha^{-1}\, \frac{\partial(\mathcal{S}_1'+\mathcal{S}_2')}{\partial k'} \right], \tag{C.31}$$

$$\frac{dk'}{dt} = \epsilon\, n' \left[-\alpha^{-1}\, k'\, \frac{\partial\mathcal{S}_0'}{\partial\bar{\lambda}'} - \alpha^{-1}\, \frac{\partial(\mathcal{S}_1'+\mathcal{S}_2')}{\partial h'} \right], \tag{C.32}$$

$$\frac{dp'}{dt} = \epsilon\, n' \left[-\frac{\alpha^{-1}}{2}\, p'\, \frac{\partial\mathcal{S}_0'}{\partial\bar{\lambda}'} + \alpha^{-1}\, \frac{\partial\mathcal{S}_2'}{\partial q'} \right], \tag{C.33}$$

and

$$\frac{dq'}{dt} = \epsilon\, n' \left[-\frac{\alpha^{-1}}{2}\, q'\, \frac{\partial\mathcal{S}_0'}{\partial\bar{\lambda}'} - \alpha^{-1}\, \frac{\partial\mathcal{S}_2'}{\partial p'} \right], \tag{C.34}$$

where

$$\epsilon = \frac{\tilde{\mu}}{\mu} = \frac{m}{M+m'}. \tag{C.35}$$

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C.3 Expansion of planetary disturbing functions

Equations (9.8), (9.9), (C.1), and (C.27) give

$$\mathcal{S} = \mathcal{S}_D + \alpha\,\mathcal{S}_E, \tag{C.36}$$

$$\mathcal{S}' = \alpha\,\mathcal{S}_D + \alpha^{-1}\mathcal{S}_I, \tag{C.37}$$

where

$$\mathcal{S}_D = \frac{a'}{|\mathbf{r}' - \mathbf{r}|}, \tag{C.38}$$

and

$$\mathcal{S}_E = -\left(\frac{r}{a}\right)\left(\frac{a'}{r'}\right)^2 \cos\psi, \tag{C.39}$$

$$\mathcal{S}_I = -\left(\frac{r'}{a'}\right)\left(\frac{a}{r}\right)^2 \cos\psi. \tag{C.40}$$

In the preceding equations, ψ is the angle subtended between the directions of $\mathbf{r}$ and $\mathbf{r}'$.

Now

$$\mathcal{S}_D = a'\left[r'^{\,2} - 2\,r'\,r\,\cos\psi + r^2\right]^{-1/2}. \tag{C.41}$$

Let

$$\zeta = \frac{r-a}{a}, \tag{C.42}$$

$$\zeta' = \frac{r'-a'}{a'}, \tag{C.43}$$

$$\delta = \cos\psi - \cos(\vartheta - \vartheta'), \tag{C.44}$$

where $\vartheta = \varpi + \theta$ and $\vartheta' = \varpi' + \theta'$. Here, θ and θ' are the true anomalies of the first and second planets, respectively. We expect ζ and ζ' to both be $\mathcal{O}(e)$ [see Equation (C.54)], and δ to be $\mathcal{O}(e^2)$ [see Equation (C.64)]. We can write

$$\mathcal{S}_D = (1+\zeta')^{-1}\left[1 - 2\,\tilde{\alpha}\,\cos(\vartheta-\vartheta') + \tilde{\alpha}^2 - 2\,\tilde{\alpha}\,\delta\right]^{-1/2}, \tag{C.45}$$

where

$$\tilde{\alpha} = \left(\frac{1+\zeta}{1+\zeta'}\right)\alpha. \tag{C.46}$$

Expanding in e, and retaining terms only up to $\mathcal{O}(e^2)$, we obtain

$$\mathcal{S}_D \simeq (1+\zeta')^{-1}\left[F + (\tilde{\alpha}-\alpha)\,D\,F + \frac{1}{2}\,(\tilde{\alpha}-\alpha)^2\,D^2\,F\right] + \delta\,\alpha\,F^3, \tag{C.47}$$

where $D \equiv \partial/\partial\alpha$, and

$$F(\alpha, \vartheta - \vartheta') = \frac{1}{[1 - 2\,\alpha\,\cos(\vartheta-\vartheta') + \alpha^2]^{1/2}}. \tag{C.48}$$

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Hence,

$$\mathcal{S}_D \simeq \left[(1-\zeta'+\zeta'^2)+(\zeta-\zeta'-2\,\zeta\,\zeta'+2\,\zeta'^2)\,\alpha\,D \right. \\ \left. +\frac{1}{2}(\zeta^2-2\,\zeta\,\zeta'+\zeta'^2)\,\alpha^2\,D^2\right]F+\delta\,\alpha\,F^3. \tag{C.49}$$

Now, we can expand F and F^3 as Fourier series in $\vartheta-\vartheta'$:

$$F(\alpha,\vartheta-\vartheta') = \frac{1}{2}\sum_{j=-\infty,\infty} b_{1/2}^{(j)}(\alpha)\,\cos[j\,(\vartheta-\vartheta')], \tag{C.50}$$

$$F^3(\alpha,\vartheta-\vartheta') = \frac{1}{2}\sum_{j=-\infty,\infty} b_{3/2}^{(j)}(\alpha)\,\cos[j\,(\vartheta-\vartheta')], \tag{C.51}$$

where

$$b_s^{(j)}(\alpha)=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos(j\,\psi)\,d\psi}{[1-2\,\alpha\,\cos\psi+\alpha^2]^s}. \tag{C.52}$$

Incidentally, the $b_s^{(j)}$ are known as *Laplace coefficients*. Thus,

$$\mathcal{S}_D \simeq \frac{1}{2}\sum_{j=-\infty,\infty}\left\{\left[(1-\zeta'+\zeta'^2)+(\zeta-\zeta'-2\,\zeta\,\zeta'+2\,\zeta'^2)\,\alpha\,D\right.\right. \\ \left.\left.+\frac{1}{2}(\zeta^2-2\,\zeta\,\zeta'+\zeta'^2)\,\alpha^2\,D^2\right]b_{1/2}^{(j)}(\alpha)+\delta\,\alpha\,b_{3/2}^{(j)}(\alpha)\right\}\cos[j\,(\vartheta-\vartheta')], \tag{C.53}$$

where D now denotes $d/d\alpha$.

Equation (3.86) gives

$$\zeta \equiv \frac{r}{a}-1\simeq -e\,\cos\,\mathcal{M}+\frac{e^2}{2}\,(1-\cos\,2\mathcal{M}) \\ \simeq -e\,\cos(\bar{\lambda}-\varpi)+\frac{e^2}{2}\,\left[1-\cos(2\,\bar{\lambda}-2\,\varpi)\right] \tag{C.54}$$

to $\mathcal{O}(e^2)$. Here, $\mathcal{M}=\bar{\lambda}-\varpi$ is the first planet's mean anomaly. Obviously, there is an analogous equation for ζ'. Moreover, from Equation (3.85), we have

$$\sin\theta \simeq \sin\mathcal{M}+e\,\sin 2\mathcal{M}+\frac{e^2}{8}\,(9\,\sin 3\mathcal{M}-7\,\sin\mathcal{M}), \tag{C.55}$$

$$\cos\theta \simeq \cos\mathcal{M}+e\,(\cos 2\mathcal{M}-1)+\frac{e^2}{8}\,(9\,\cos 3\mathcal{M}-9\,\cos\mathcal{M}). \tag{C.56}$$

Hence,

$$\cos(\omega+\theta) \equiv \cos\omega\,\cos\theta-\sin\omega\,\sin\theta \\ \simeq \cos(\omega+\mathcal{M})+e\,[\cos(\omega+2\mathcal{M})-\cos\omega] \\ +\frac{e^2}{8}\,[-8\,\cos(\omega+\mathcal{M})-\cos(\omega-\mathcal{M})+9\,\cos(\omega+3\mathcal{M})], \tag{C.57}$$

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and

$$\begin{aligned}\sin(\omega+\theta) &\equiv \sin\omega\,\cos\theta+\cos\omega\,\sin\theta\\ &\simeq \sin(\omega+\mathcal{M})+e\,[\sin(\omega+2\mathcal{M})-\sin\omega]\\ &\quad+\frac{e^2}{8}\,[-8\,\sin(\omega+\mathcal{M})+\sin(\omega-\mathcal{M})+9\,\sin(\omega+3\mathcal{M})]\,.\end{aligned}\tag{C.58}$$

Thus, Equations (3.72)–(3.74) yield

$$\begin{aligned}\frac{X}{r} &\simeq \cos(\omega+\Omega+\mathcal{M})+e\,[\cos(\omega+\Omega+2\mathcal{M})-\cos(\omega+\Omega)]\\ &\quad+\frac{e^2}{8}\,[9\,\cos(\omega+\Omega+3\mathcal{M})-\cos(\omega+\Omega-\mathcal{M})-8\,\cos(\omega+\Omega+\mathcal{M})]\\ &\quad+s^2\,[\cos(\omega-\Omega+\mathcal{M})-\cos(\omega+\Omega+\mathcal{M})]\,,\end{aligned}\tag{C.59}$$

$$\begin{aligned}\frac{Y}{r} &\simeq \sin(\omega+\Omega+\mathcal{M})+e\,[\sin(\omega+\Omega+2\mathcal{M})-\sin(\omega+\Omega)]\\ &\quad+\frac{e^2}{8}\,[9\,\sin(\omega+\Omega+3\mathcal{M})-\sin(\omega+\Omega-\mathcal{M})-8\,\sin(\omega+\Omega+\mathcal{M})]\\ &\quad-s^2\,[\sin(\omega-\Omega+\mathcal{M})+\sin(\omega+\Omega+\mathcal{M})]\,,\end{aligned}\tag{C.60}$$

and

$$\frac{Z}{r} \simeq 2\,s\,\sin(\omega+\mathcal{M})+2\,e\,s\,[\sin(\omega+2\mathcal{M})-\sin\omega)]\,,\tag{C.61}$$

where $s\equiv\sin(I/2)$ is assumed to be $\mathcal{O}(e)$. Here, X, Y, Z are the Cartesian components of $\mathbf{r}$. There are, of course, completely analogous expressions for the Cartesian components of $\mathbf{r}'$.

Now,

$$\cos\psi = \frac{X}{r}\,\frac{X'}{r'}+\frac{Y}{r}\,\frac{Y'}{r'}+\frac{Z}{r}\,\frac{Z'}{r'},\tag{C.62}$$

so

$$\begin{aligned}\cos\psi &\simeq (1-e^2-e'^{\,2}-s^2-s'^{\,2})\,\cos(\bar\lambda-\bar\lambda')+e\,e'\,\cos(2\,\bar\lambda-2\,\bar\lambda'-\varpi+\varpi')\\ &\quad+e\,e'\,\cos(\varpi-\varpi')+2\,s\,s'\,\cos(\bar\lambda-\bar\lambda'-\Omega+\Omega')\\ &\quad+e\,\cos(2\,\bar\lambda-\bar\lambda'-\varpi)-e\,\cos(\bar\lambda'-\varpi)+e'\,\cos(\bar\lambda-2\,\bar\lambda'+\varpi')\\ &\quad-e'\,\cos(\bar\lambda-\varpi')+\frac{9\,e^2}{8}\,\cos(3\,\bar\lambda-\bar\lambda'-2\,\varpi)-\frac{e^2}{8}\,\cos(\bar\lambda+\bar\lambda'-2\,\varpi)\\ &\quad+\frac{9\,e'^{\,2}}{8}\,\cos(\bar\lambda-3\,\bar\lambda'+2\,\varpi')-\frac{e'^{\,2}}{8}\,\cos(\bar\lambda+\bar\lambda'-2\,\varpi')\\ &\quad-e\,e'\,\cos(2\,\bar\lambda-\varpi-\varpi')-e\,e'\,\cos(2\,\bar\lambda'-\varpi-\varpi')\\ &\quad+s^2\,\cos(\bar\lambda+\bar\lambda'-2\,\Omega)+s'^{\,2}\,\cos(\bar\lambda+\bar\lambda'-2\,\Omega')\\ &\quad-2\,s\,s'\,\cos(\bar\lambda+\bar\lambda'-\Omega-\Omega').\end{aligned}\tag{C.63}$$

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It is easily demonstrated that $\cos(\vartheta - \vartheta')$ represents the value taken by $\cos\psi$ when $s = s' = 0$. Hence, from Equations (C.44) and (C.63),

$$\begin{aligned}\delta \simeq\; & s^2\left[\cos(\bar{\lambda} + \bar{\lambda}' - 2\,\Omega) - \cos(\bar{\lambda} - \bar{\lambda}')\right]\\ & + 2\,s\,s'\left[\cos(\bar{\lambda} - \bar{\lambda}' - \Omega + \Omega') - \cos(\bar{\lambda} + \bar{\lambda}' - \Omega - \Omega')\right]\\ & + s'^{\,2}\left[\cos(\bar{\lambda} + \bar{\lambda}' - 2\,\Omega') - \cos(\bar{\lambda} - \bar{\lambda}')\right]. \qquad \text{(C.64)}\end{aligned}$$

Now,

$$\cos[j\,(\vartheta - \vartheta')] \equiv \cos(j\,\vartheta)\,\cos(j\,\vartheta') + \sin(j\,\vartheta)\,\sin(j\,\vartheta'). \qquad \text{(C.65)}$$

However, from Equation (3.85),

$$\begin{aligned}\cos(j\,\vartheta) \;\equiv\;& \cos[j\,(\varpi + \theta)]\\ \simeq\;& (1 - j^2\,e^2)\,\cos(j\,\bar{\lambda}) + e^2\left(\frac{j^2}{2} - \frac{5\,j}{8}\right)\cos[(2 - j)\,\bar{\lambda} - 2\,\varpi]\\ & + e^2\left(\frac{j^2}{2} + \frac{5\,j}{8}\right)\cos[(2 + j)\,\bar{\lambda} - 2\,\varpi]\\ & - j\,e\,\cos[(1 - j)\,\bar{\lambda} - \varpi] + j\,e\,\cos[(1 + j)\,\bar{\lambda} - \varpi)], \qquad \text{(C.66)}\end{aligned}$$

because $\mathcal{M} = \bar{\lambda} - \varpi$. Likewise,

$$\begin{aligned}\sin(j\,\vartheta) \;\equiv\;& \sin[j\,(\varpi + \theta)]\\ \simeq\;& (1 - j^2\,e^2)\,\sin(j\,\bar{\lambda}) + e^2\left(\frac{5\,j}{8} - \frac{j^2}{2}\right)\sin[(2 - j)\,\bar{\lambda} - 2\,\varpi]\\ & + e^2\left(\frac{5\,j}{8} + \frac{j^2}{2}\right)\sin[(2 + j)\,\bar{\lambda} - 2\,\varpi]\\ & + j\,e\,\sin[(1 - j)\,\bar{\lambda} - \varpi] + j\,e\,\sin[(1 + j)\,\bar{\lambda} - \varpi]. \qquad \text{(C.67)}\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\cos[j\,(\vartheta - \vartheta')] \;\simeq\;& (1 - j^2\,e^2 - j^2\,e'^{\,2})\,\cos[j\,(\bar{\lambda} - \bar{\lambda}')]\\ & + e^2\left(\frac{5\,j}{8} + \frac{j^2}{2}\right)\cos[(2 + j)\,\bar{\lambda} - j\,\bar{\lambda}' - 2\,\varpi]\\ & + e^2\left(\frac{j^2}{2} - \frac{5\,j}{8}\right)\cos[(2 - j)\,\bar{\lambda} + j\,\bar{\lambda}' - 2\,\varpi]\\ & + j\,e\,\cos[(1 + j)\,\bar{\lambda} - j\,\bar{\lambda}' - \varpi] - j\,e\,\cos[(1 - j)\,\bar{\lambda} + j\,\bar{\lambda}' - \varpi]\\ & + e'^{\,2}\left(\frac{j^2}{2} - \frac{5\,j}{8}\right)\cos[j\,\bar{\lambda} + (2 - j)\,\bar{\lambda}' - 2\,\varpi']\\ & + e'^{\,2}\left(\frac{5\,j}{8} + \frac{j^2}{2}\right)\cos[j\,\bar{\lambda} - (2 + j)\,\bar{\lambda}' + 2\,\varpi']\\ & - j\,e'\,\cos[j\,\bar{\lambda} + (1 - j)\,\bar{\lambda}' - \varpi'] + j\,e'\,\cos[j\,\bar{\lambda} - (1 + j)\,\bar{\lambda}' + \varpi']\\ & - j^2\,e\,e'\,\cos[(1 + j)\,\bar{\lambda} + (1 - j)\,\bar{\lambda}' - \varpi - \varpi']\end{aligned}$$

$$
\begin{aligned}
&- j^2\, e\, e'\, \cos[(1-j)\,\bar{\lambda} + (1+j)\,\bar{\lambda}' - \varpi - \varpi'] \\
&+ j^2\, e\, e'\, \cos[(1+j)\,\bar{\lambda} - (1+j)\,\bar{\lambda}' - \varpi + \varpi'] \\
&+ j^2\, e\, e'\, \cos[(1-j)\,\bar{\lambda} - (1-j)\,\bar{\lambda}' - \varpi + \varpi'].
\end{aligned}
\tag{C.68}
$$

Equations (C.53), (C.54), (C.64), and (C.68) yield

$$
\mathcal{S}_D = \sum_{j=-\infty,\infty} \mathcal{S}^{(j)},
\tag{C.69}
$$

where

$$
\begin{aligned}
\mathcal{S}^{(j)} \simeq\ & \left[\frac{b_{1/2}^{(j)}}{2} + \frac{1}{8}\left(e^2 + e'^2\right)\left(-4\,j^2 + 2\,\alpha\,D + \alpha^2\,D^2\right) b_{1/2}^{(j)}\right. \\
& \left. - \frac{\alpha}{4}\left(s^2 + s'^2\right)\left(b_{3/2}^{(j-1)} + b_{3/2}^{(j+1)}\right)\right]\cos[j\,(\bar{\lambda}' - \bar{\lambda})] \\
& + \frac{e\,e'}{4}\left(2 + 6\,j + 4\,j^2 - 2\,\alpha\,D - \alpha^2\,D^2\right) b_{1/2}^{(j+1)}\,\cos(j\,\bar{\lambda}' - j\,\bar{\lambda} + \varpi' - \varpi) \\
& + s\,s'\,\alpha\,b_{3/2}^{(j+1)}\,\cos(j\,\bar{\lambda}' - j\,\bar{\lambda} + \Omega' - \Omega) \\
& + \frac{e}{2}\,(-2\,j - \alpha\,D)\,b_{1/2}^{(j)}\,\cos[j\,\bar{\lambda}' + (1-j)\,\bar{\lambda} - \varpi] \\
& + \frac{e'}{2}\,(-1 + 2\,j + \alpha\,D)\,b_{1/2}^{(j-1)}\,\cos[j\,\bar{\lambda}' + (1-j)\,\bar{\lambda} - \varpi'] \\
& + \frac{e^2}{8}\left(-5\,j + 4\,j^2 - 2\,\alpha\,D + 4\,j\,\alpha\,D + \alpha^2\,D^2\right) b_{1/2}^{(j)}\,\cos[j\,\bar{\lambda}' + (2-j)\,\bar{\lambda} - 2\,\varpi] \\
& + \frac{e\,e'}{4}\left(-2 + 6\,j - 4\,j^2 + 2\,\alpha\,D - 4\,j\,\alpha\,D - \alpha^2\,D^2\right) b_{1/2}^{(j-1)}\,\cos[j\,\bar{\lambda}' + (2-j)\,\bar{\lambda} - \varpi' - \varpi] \\
& + \frac{e'^2}{8}\left(2 - 7\,j + 4\,j^2 - 2\,\alpha\,D + 4\,j\,\alpha\,D + \alpha^2\,D^2\right) b_{1/2}^{(j-2)}\,\cos[j\,\bar{\lambda}' + (2-j)\,\bar{\lambda} - 2\,\varpi'] \\
& + \frac{s^2}{2}\,\alpha\,b_{3/2}^{(j-1)}\,\cos[j\,\bar{\lambda}' + (2-j)\,\bar{\lambda} - 2\,\Omega] \\
& - s\,s'\,\alpha\,b_{3/2}^{(j-1)}\,\cos[j\,\bar{\lambda}' + (2-j)\,\bar{\lambda} - \Omega' - \Omega] \\
& + \frac{s'^2}{2}\,\alpha\,b_{3/2}^{(j-1)}\,\cos[j\,\bar{\lambda}' + (2-j)\,\bar{\lambda} - 2\,\Omega'].
\end{aligned}
\tag{C.70}
$$

Likewise, Equations (C.39), (C.40), (C.54), and (C.63) give

$$
\begin{aligned}
\mathcal{S}_E \simeq\ & \left(-1 + \frac{e^2}{2} + \frac{e'^2}{2} + s^2 + s'^2\right)\cos(\bar{\lambda}' - \bar{\lambda}) \\
& - e\,e'\,\cos(2\,\bar{\lambda}' - 2\,\bar{\lambda} - \varpi' + \varpi) - 2\,s\,s'\,\cos(\bar{\lambda}' - \bar{\lambda} - \Omega' + \Omega) \\
& - \frac{e}{2}\,\cos(\bar{\lambda}' - 2\,\bar{\lambda} + \varpi) + \frac{3\,e}{2}\,\cos(\bar{\lambda}' - \varpi) - 2\,e'\,\cos(2\,\bar{\lambda}' - \bar{\lambda} - \varpi') \\
& - \frac{3\,e^2}{8}\,\cos(\bar{\lambda}' - 3\,\bar{\lambda} + 2\,\varpi) - \frac{e^2}{8}\,\cos(\bar{\lambda}' + \bar{\lambda} - 2\,\varpi) \\
& + 3\,e\,e'\,\cos(2\,\bar{\lambda} - \varpi' - \varpi) - \frac{e'^2}{8}\,\cos(\bar{\lambda}' + \bar{\lambda} - 2\,\varpi') \\
& - \frac{27\,e'^2}{8}\,\cos(3\,\bar{\lambda}' - \bar{\lambda} - 2\,\varpi') - s^2\,\cos(\bar{\lambda}' + \bar{\lambda} - 2\,\Omega) \\
& + 2\,s\,s'\,\cos(\bar{\lambda}' + \bar{\lambda} - \Omega' - \Omega) - s'^2\,\cos(\bar{\lambda}' + \bar{\lambda} - 2\,\Omega'),
\end{aligned}
\tag{C.71}
$$

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and

$$
\begin{aligned}
\mathcal{S}_I \simeq\ & \left(-1 + \frac{e^2}{2} + \frac{e'^2}{2} + s^2 + s'^2\right)\cos(\bar{\lambda}' - \bar{\lambda}) \\
& - e\,e'\,\cos(2\,\bar{\lambda}' - 2\,\bar{\lambda} - \varpi' + \varpi) - 2\,s\,s'\,\cos(\bar{\lambda}' - \bar{\lambda} - \Omega' + \Omega) \\
& - 2\,e\,\cos(\bar{\lambda}' - 2\,\bar{\lambda} + \varpi) + \frac{3\,e'}{2}\,\cos(\bar{\lambda} - \varpi') - \frac{e'}{2}\,\cos(2\,\bar{\lambda}' - \bar{\lambda} - \varpi') \\
& - \frac{27\,e^2}{8}\,\cos(\bar{\lambda}' - 3\,\bar{\lambda} + 2\,\varpi) - \frac{e^2}{8}\,\cos(\bar{\lambda}' + \bar{\lambda} - 2\,\varpi) \\
& + 3\,e\,e'\,\cos(2\,\bar{\lambda} - \varpi' - \varpi) - \frac{e'^2}{8}\,\cos(\bar{\lambda}' + \bar{\lambda} - 2\,\varpi') \\
& - \frac{3\,e'^2}{8}\,\cos(3\,\bar{\lambda}' - \bar{\lambda} - 2\,\varpi') - s^2\,\cos(\bar{\lambda}' + \bar{\lambda} - 2\,\Omega) \\
& + 2\,s\,s'\,\cos(\bar{\lambda}' + \bar{\lambda} - \Omega' - \Omega) - s'^2\,\cos(\bar{\lambda}' + \bar{\lambda} - 2\,\Omega'). \qquad \text{(C.72)}
\end{aligned}
$$

We can distinguish two different types of term that appear in our expansion of the disturbing functions. *Periodic terms* vary *sinusoidally* in time as our two planets orbit the Sun (i.e., they depend on the mean ecliptic longitudes, $\bar{\lambda}$ and $\bar{\lambda}'$), whereas *secular terms* remain *constant* in time (i.e., they do not depend on $\bar{\lambda}$ or $\bar{\lambda}'$). We expect the periodic terms to give rise to relatively short-period (i.e., of order a typical orbital period) oscillations in the osculating orbital elements of our planets. On the other hand, we expect the secular terms to produce an initially linear increase in these elements with time. Because such an increase can become significant over a long period of time, even if the rate of increase is small, it is necessary to evaluate the secular terms in the disturbing function to higher order than the periodic terms. Hence, in the following, we shall evaluate periodic terms to $\mathcal{O}(e)$ and secular terms to $\mathcal{O}(e^2)$.

Making use of Equations (C.20), (C.36), and (C.69)–(C.71), as well as the definitions (B.121)–(B.124) (with $p \simeq I\,\sin\Omega$ and $q \simeq I\,\cos\omega$), we see that the order-by-order expansion (in e) of the first planet's disturbing function becomes $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2$, where

$$
\mathcal{S}_0 = \frac{1}{2}\sum_{j=-\infty,\infty} b_{1/2}^{(j)}\,\cos[j\,(\bar{\lambda} - \bar{\lambda}')] - \alpha\,\cos(\bar{\lambda} - \bar{\lambda}'), \qquad \text{(C.73)}
$$

$$
\begin{aligned}
\mathcal{S}_1 =\ & \frac{1}{2}\sum_{j=-\infty,\infty} \Big\{k\,(-2\,j - \alpha\,D)\,b_{1/2}^{(j)}\,\cos[(1-j)\,\bar{\lambda} + j\,\bar{\lambda}'] \\
& + h\,(-2\,j - \alpha\,D)\,b_{1/2}^{(j)}\,\sin[(1-j)\,\bar{\lambda} + j\,\bar{\lambda}'] \\
& + k'\,(-1 + 2\,j + \alpha\,D)\,b_{1/2}^{(j-1)}\,\cos[(1-j)\,\bar{\lambda} + j\,\bar{\lambda}'] \\
& + h'\,(-1 + 2\,j + \alpha\,D)\,b_{1/2}^{(j-1)}\,\sin[(1-j)\,\bar{\lambda} + j\,\bar{\lambda}']\Big\} \\
& + \frac{\alpha}{2}\Big\{-k\,\cos(2\,\bar{\lambda} - \bar{\lambda}') - h\,\sin(2\,\bar{\lambda} - \bar{\lambda}') + 3\,k\,\cos\bar{\lambda}' + 3\,h\,\sin\bar{\lambda}' \\
& - 4\,k'\,\cos(\bar{\lambda} - 2\,\bar{\lambda}') + 4\,h'\,\sin(\bar{\lambda} - 2\,\bar{\lambda}')\Big\}, \qquad \text{(C.74)}
\end{aligned}
$$

and

$$\mathcal{S}_2 = \frac{1}{8}(h^2+k^2+h'^2+k'^2)(2\,\alpha\,D+\alpha^2\,D^2)\,b_{1/2}^{(0)} - \frac{1}{8}(p^2+q^2+p'^2+q'^2)\,\alpha\,b_{3/2}^{(1)}$$
$$+\frac{1}{4}(k\,k'+h\,h')(2-2\,\alpha\,D-\alpha^2\,D^2)\,b_{1/2}^{(1)} + \frac{1}{4}(p\,p'+q\,q')\,\alpha\,b_{3/2}^{(1)}. \quad \text{(C.75)}$$

Likewise, the order by order expansion of the second planet's disturbing function becomes $\mathcal{S}' = \mathcal{S}'_0 + \mathcal{S}'_1 + \mathcal{S}'_2$, where

$$\mathcal{S}'_0 = \frac{\alpha}{2}\sum_{j=-\infty,\infty} b_{1/2}^{(j)}\cos[j\,(\bar{\lambda}'-\bar{\lambda})] - \alpha^{-1}\cos(\bar{\lambda}'-\bar{\lambda}), \quad \text{(C.76)}$$

$$\mathcal{S}'_1 = \frac{\alpha}{2}\sum_{j=-\infty,\infty}\Big\{k\,(-2\,j-\alpha\,D)\,b_{1/2}^{(j)}\cos[j\,\bar{\lambda}'+(1-j)\,\bar{\lambda}]$$
$$+\,h\,(-2\,j-\alpha\,D)\,b_{1/2}^{(j)}\sin[j\,\bar{\lambda}'+(1-j)\,\bar{\lambda}]$$
$$+\,k'\,(-1+2\,j+\alpha\,D)\,b_{1/2}^{(j-1)}\cos[j\,\bar{\lambda}'+(1-j)\,\bar{\lambda}]$$
$$+\,h'\,(-1+2\,j+\alpha\,D)\,b_{1/2}^{(j-1)}\sin[j\,\bar{\lambda}'+(1-j)\,\bar{\lambda}]\Big\}$$
$$+\,\frac{\alpha^{-1}}{2}\Big\{-k'\cos(2\,\bar{\lambda}'-\bar{\lambda}) - h'\sin(2\,\bar{\lambda}'-\bar{\lambda}) + 3\,k'\cos\bar{\lambda} + 3\,h'\sin\bar{\lambda}$$
$$-\,4\,k\cos(\bar{\lambda}'-2\,\bar{\lambda}) + 4\,h\sin(\bar{\lambda}'-2\,\bar{\lambda})\Big\}, \quad \text{(C.77)}$$

and

$$\mathcal{S}'_2 = \frac{1}{8}(h^2+k^2+h'^2+k'^2)\,\alpha\,(2\,\alpha\,D+\alpha^2\,D^2)\,b_{1/2}^{(0)}$$
$$-\,\frac{1}{8}(p^2+q^2+p'^2+q'^2)\,\alpha^2\,b_{3/2}^{(1)}$$
$$+\,\frac{1}{4}(k\,k'+h\,h')\,\alpha\,(2-2\,\alpha\,D-\alpha^2\,D^2)\,b_{1/2}^{(1)}$$
$$+\,\frac{1}{4}(p\,p'+q\,q')\,\alpha^2\,b_{3/2}^{(1)}. \quad \text{(C.78)}$$