1 Newtonian mechanics

1.1 Introduction

Newtonian mechanics is a mathematical model whose purpose is to account for the motions of the various objects in the universe. The general principles of this model were first enunciated by Sir Isaac Newton in a work titled *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy). This work, which was published in 1687, is nowadays more commonly referred to as the *Principia*.[1]

Until the beginning of the twentieth century, Newtonian mechanics was thought to constitute a *complete* description of all types of motion occurring in the universe. We now know that this is not the case. The modern view is that Newton's model is only an *approximation* that is valid under certain circumstances. The model breaks down when the velocities of the objects under investigation approach the speed of light in a vacuum, and must be modified in accordance with Einstein's *special theory of relativity*. The model also fails in regions of space that are sufficiently curved that the propositions of Euclidean geometry do not hold to a good approximation, and must be augmented by Einstein's *general theory of relativity*. Finally, the model breaks down on atomic and subatomic length scales, and must be replaced by *quantum mechanics*. In this book, we shall (almost entirely) neglect relativistic and quantum effects. It follows that we must restrict our investigations to the motions of *large* (compared with an atom), *slow* (compared with the speed of light) objects moving in *Euclidean* space. Fortunately, virtually all the motions encountered in conventional celestial mechanics fall into this category.

Newton very deliberately modeled his approach in the *Principia* on that taken in Euclid's *Elements*. Indeed, Newton's theory of motion has much in common with a conventional *axiomatic system*, such as Euclidean geometry. Like all axiomatic systems, Newtonian mechanics starts from a set of terms that are *undefined* within the system. In this case, the fundamental terms are *mass*, *position*, *time*, and *force*. It is taken for granted that we understand what these terms mean, and, furthermore, that they correspond to *measurable* quantities that can be ascribed to, or associated with, objects in the world around us. In particular, it is assumed that the ideas of position in space, distance in space, and position as a function of time in space are correctly described by conventional Euclidean vector algebra and vector calculus. The next component of an axiomatic system is a set of *axioms*. These are a set of *unproven* propositions,

[1] An excellent discussion of the historical development of Newtonian mechanics, as well as the physical and philosophical assumptions that underpin this theory, is given in Barbour 2001.

involving the undefined terms, from which all other propositions in the system can be derived via logic and mathematical analysis. In the present case, the axioms are called *Newton's laws of motion* and can be justified only via experimental observation. Note, incidentally, that Newton's laws, in their primitive form, are applicable only to *point objects* (i.e., objects of negligible spatial extent). However, these laws can be applied to extended objects by treating them as collections of point objects.

One difference between an axiomatic system and a physical theory is that, in the latter case, even if a given prediction has been shown to follow necessarily from the axioms of the theory, it is still incumbent on us to test the prediction against experimental observations. Lack of agreement might indicate faulty experimental data, faulty application of the theory (for instance, in the case of Newtonian mechanics, there might be forces at work that we have not identified), or, as a last resort, incorrectness of the theory. Fortunately, Newtonian mechanics has been found to give predictions that are in excellent agreement with experimental observations in all situations in which it would be expected to hold.

In the following, it is assumed that we know how to set up a rigid Cartesian frame of reference and how to measure the positions of point objects as functions of time within that frame. It is also taken for granted that we have some basic familiarity with the laws of mechanics.

1.2 Newton's laws of motion

Newton's laws of motion, in the rather obscure language of the *Principia*, take the following form:

1. Every body continues in its state of rest, or uniform motion in a straight line, unless compelled to change that state by forces impressed on it.
2. The change of motion (i.e., momentum) of an object is proportional to the force impressed on it, and is made in the direction of the straight line in which the force is impressed.
3. To every action there is always opposed an equal reaction; or, the mutual actions of two bodies on each other are always equal, and directed to contrary parts.

Let us now examine how these laws can be applied to a system of point objects.

1.3 Newton's first law of motion

Newton's first law of motion essentially states that a point object subject to zero net external force moves in a straight line with a constant speed (i.e., it does not accelerate). However, this is true only in special frames of reference called *inertial frames*. Indeed, we can think of Newton's first law as the definition of an inertial frame: an inertial frame of reference is one in which a point object subject to zero net external force moves in a straight line with constant speed.

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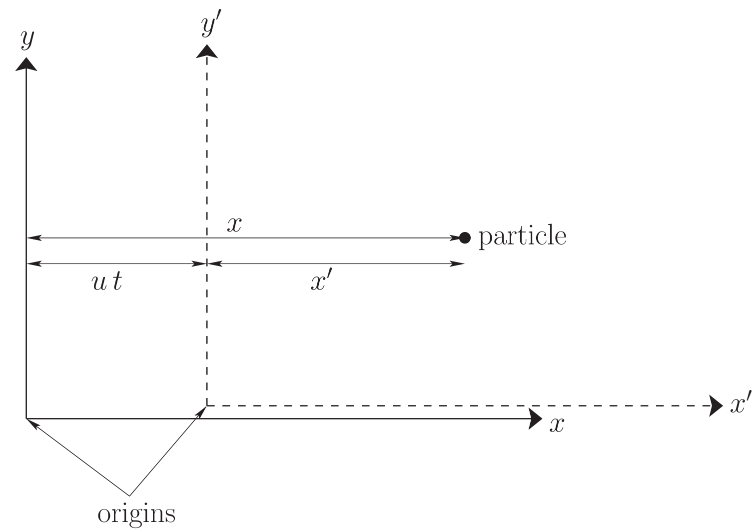

Fig. 1.1 A Galilean coordinate transformation.

Suppose that we have found an inertial frame of reference. Let us set up a Cartesian coordinate system in this frame. The motion of a point object can now be specified by giving its position vector, $\mathbf{r} \equiv (x, y, z)$, with respect to the origin of the coordinate system, as a function of time, t. Consider a second frame of reference moving with some constant velocity $\mathbf{u}$ with respect to the first frame. Without loss of generality, we can suppose that the Cartesian axes in the second frame are parallel to the corresponding axes in the first frame, that $\mathbf{u} \equiv (u, 0, 0)$, and, finally, that the origins of the two frames instantaneously coincide at $t = 0$. (See Figure 1.1.) Suppose that the position vector of our point object is $\mathbf{r}' \equiv (x', y', z')$ in the second frame of reference. It is evident, from Figure 1.1, that at any given time, t, the coordinates of the object in the two reference frames satisfy

$$x' = x - u\,t, \tag{1.1}$$

$$y' = y, \tag{1.2}$$

and

$$z' = z. \tag{1.3}$$

This coordinate transformation was first discovered by Galileo Galilei (1564–1642), and is nowadays known as a *Galilean transformation* in his honor.

By definition, the instantaneous velocity of the object in our first reference frame is given by $\mathbf{v} = d\mathbf{r}/dt \equiv (dx/dt, dy/dt, dz/dt)$, with an analogous expression for the velocity, $\mathbf{v}'$, in the second frame. It follows, from differentiation of Equations (1.1)–(1.3) with respect to time, that the velocity components in the two frames satisfy

$$v_x' = v_x - u, \tag{1.4}$$

$$v_y' = v_y, \tag{1.5}$$

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and

$$v_z' = v_z. \tag{1.6}$$

These equations can be written more succinctly as

$$\mathbf{v}' = \mathbf{v} - \mathbf{u}. \tag{1.7}$$

Finally, by definition, the instantaneous acceleration of the object in our first reference frame is given by $\mathbf{a} = d\mathbf{v}/dt \equiv (dv_x/dt, dv_y/dt, dv_z/dt)$, with an analogous expression for the acceleration, $\mathbf{a}'$, in the second frame. It follows, from differentiation of Equations (1.4)–(1.6) with respect to time, that the acceleration components in the two frames satisfy

$$a_x' = a_x, \tag{1.8}$$

$$a_y' = a_y, \tag{1.9}$$

and

$$a_z' = a_z. \tag{1.10}$$

These equations can be written more succinctly as

$$\mathbf{a}' = \mathbf{a}. \tag{1.11}$$

According to Equations (1.7) and (1.11), if an object is moving in a straight line with a constant speed in our original inertial frame (i.e., if $\mathbf{a} = \mathbf{0}$), then it also moves in a (different) straight line with a (different) constant speed in the second frame of reference (i.e., $\mathbf{a}' = \mathbf{0}$). Hence, we conclude that the second frame of reference is also an inertial frame.

A simple extension of the preceding argument allows us to conclude that there is an *infinite* number of different inertial frames moving with constant velocities with respect to one another. Newton thought that one of these inertial frames was special and defined an absolute standard of rest: that is, a static object in this frame was in a state of absolute rest. However, Einstein showed that this is not the case. In fact, there is no absolute standard of rest: in other words, all motion is relative—hence, the name *relativity* for Einstein's theory. Consequently, one inertial frame is just as good as another as far as Newtonian mechanics is concerned.

But what happens if the second frame of reference *accelerates* with respect to the first? In this case, it is not hard to see that Equation (1.11) generalizes to

$$\mathbf{a}' = \mathbf{a} - \frac{d\mathbf{u}}{dt}, \tag{1.12}$$

where $\mathbf{u}(t)$ is the instantaneous velocity of the second frame with respect to the first. According to this formula, if an object is moving in a straight line with a constant speed in the first frame (i.e., if $\mathbf{a} = \mathbf{0}$), then it does not move in a straight line with a constant speed in the second frame (i.e., $\mathbf{a}' \neq \mathbf{0}$). Hence, if the first frame is an inertial frame, then the second is *not*.

A simple extension of the preceding argument allows us to conclude that any frame of reference that accelerates with respect to a given inertial frame is not itself an inertial frame.

For most practical purposes, when studying the motions of objects close to the Earth's surface, a reference frame that is fixed with respect to this surface is approximately inertial. However, if the trajectory of a projectile within such a frame is measured to high precision, then it will be found to deviate slightly from the predictions of Newtonian mechanics. (See Chapter 5.) This deviation is due to the fact that the Earth is rotating, and its surface is therefore *accelerating* toward its axis of rotation. When studying the motions of objects in orbit around the Earth, a reference frame whose origin is the center of the Earth (or, to be more exact, the center of mass of the Earth–Moon system), and whose coordinate axes are fixed with respect to distant stars, is approximately inertial. However, if such orbits are measured to extremely high precision, then they will again be found to deviate very slightly from the predictions of Newtonian mechanics. In this case, the deviation is due to the Earth's orbital motion about the Sun. When studying the orbits of the planets in the solar system, a reference frame whose origin is the center of the Sun (or, to be more exact, the center of mass of the solar system), and whose coordinate axes are fixed with respect to distant stars, is approximately inertial. In this case, any deviations of the orbits from the predictions of Newtonian mechanics due to the orbital motion of the Sun about the galactic center are far too small to be measurable. It should be noted that it is impossible to identify an *absolute* inertial frame—the best approximation to such a frame would be one in which the cosmic microwave background appears to be (approximately) isotropic. However, for a given dynamic problem, it is always possible to identify an *approximate* inertial frame. Furthermore, any deviations of such a frame from a true inertial frame can be incorporated into the framework of Newtonian mechanics via the introduction of so-called fictitious forces. (See Chapter 5.)

1.4 Newton's second law of motion

Newton's second law of motion essentially states that if a point object is subject to an external force, $\mathbf{f}$, then its equation of motion is given by

$$\frac{d\mathbf{p}}{dt} = \mathbf{f}, \tag{1.13}$$

where the momentum, $\mathbf{p}$, is the product of the object's inertial mass, m, and its velocity, $\mathbf{v}$. If m is not a function of time, then Equation (1.13) reduces to the familiar equation

$$m\,\frac{d\mathbf{v}}{dt} = \mathbf{f}. \tag{1.14}$$

This equation is valid only in an *inertial frame*. Clearly, the inertial mass of an object measures its reluctance to deviate from its preferred state of uniform motion in a straight line (in an inertial frame). Of course, the preceding equation of motion can be solved only if we have an independent expression for the force, $\mathbf{f}$ (i.e., a law of force). Let us suppose that this is the case.

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An important corollary of Newton's second law is that force is a *vector quantity*. This must be the case, as the law equates force to the product of a scalar (mass) and a vector (acceleration).[2] Note that acceleration is obviously a vector because it is directly related to displacement, which is the prototype of all vectors. One consequence of force being a vector is that two forces, $\mathbf{f}_1$ and $\mathbf{f}_2$, both acting at a given point, have the same effect as a single force, $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$, acting at the same point, where the summation is performed according to the laws of vector addition. Likewise, a single force, $\mathbf{f}$, acting at a given point, has the same effect as two forces, $\mathbf{f}_1$ and $\mathbf{f}_2$, acting at the same point, provided that $\mathbf{f}_1 + \mathbf{f}_2 = \mathbf{f}$. This method of combining and splitting forces is known as the *resolution of forces*; it lies at the heart of many calculations in Newtonian mechanics.

Taking the scalar product of Equation (1.14) with the velocity, $\mathbf{v}$, we obtain

$$m\,\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \frac{m}{2}\frac{d(\mathbf{v}\cdot\mathbf{v})}{dt} = \frac{m}{2}\frac{dv^2}{dt} = \mathbf{f}\cdot\mathbf{v}. \tag{1.15}$$

This can be written

$$\frac{dK}{dt} = \mathbf{f}\cdot\mathbf{v}, \tag{1.16}$$

where

$$K = \frac{1}{2}\,m\,v^2. \tag{1.17}$$

The right-hand side of Equation (1.16) represents the rate at which the force does work on the object—that is, the rate at which the force transfers energy to the object. The quantity K represents the energy that the object possesses by virtue of its motion. This type of energy is generally known as *kinetic energy*. Thus, Equation (1.16) states that any work done on a point object by an external force goes to increase the object's kinetic energy.

Suppose that under the action of the force, $\mathbf{f}$, our object moves from point P at time t_1 to point Q at time t_2. The net change in the object's kinetic energy is obtained by integrating Equation (1.16):

$$\Delta K = \int_{t_1}^{t_2}\mathbf{f}\cdot\mathbf{v}\,dt = \int_P^Q \mathbf{f}\cdot d\mathbf{r}, \tag{1.18}$$

because $\mathbf{v} = d\mathbf{r}/dt$. Here, $d\mathbf{r}$ is an element of the object's path between points P and Q, and the integral in $\mathbf{r}$ represents the net *work* done by the force as the object moves along the path from P to Q.

As is well known, there are basically two kinds of forces in nature: first, those for which line integrals of the type $\int_P^Q\mathbf{f}\cdot d\mathbf{r}$ depend on the end points but not on the path taken between these points; second, those for which line integrals of the type $\int_P^Q\mathbf{f}\cdot d\mathbf{r}$ depend both on the end points and the path taken between these points. The first kind of force is termed *conservative*, whereas the second kind is termed *non-conservative*. It can be demonstrated that if the line integral $\int_P^Q\mathbf{f}\cdot d\mathbf{r}$ is *path independent*, for all choices of P and Q, then the force $\mathbf{f}$ can be written as the gradient of a scalar field. (See Section A.5.)

[2] A *scalar* is a physical quantity that is invariant under rotation of the coordinate axes. A *vector* is a physical quantity that transforms in an analogous manner to a displacement under rotation of the coordinate axes.

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In other words, all conservative forces satisfy

$$\mathbf{f}(\mathbf{r}) = -\nabla U \tag{1.19}$$

for some scalar field $U(\mathbf{r})$. [Incidentally, mathematicians, as opposed to physicists and astronomers, usually write $f(\mathbf{r}) = +\nabla U$.] Note that

$$\int_P^Q \nabla U \cdot d\mathbf{r} \equiv \Delta U = U(Q) - U(P), \tag{1.20}$$

irrespective of the path taken between P and Q. Hence, it follows from Equation (1.18) that

$$\Delta K = -\Delta U \tag{1.21}$$

for conservative forces. Another way of writing this is

$$E = K + U = \text{constant}. \tag{1.22}$$

Of course, we recognize Equation (1.22) as an *energy conservation equation*: E is the object's total energy, which is conserved; K is the energy the object has by virtue of its motion, otherwise known as its *kinetic energy*; and U is the energy the object has by virtue of its position, otherwise known as its *potential energy*. Note, however, that we can write energy conservation equations only for conservative forces. Gravity is an obvious example of such a force. Incidentally, potential energy is undefined to an arbitrary additive constant. In fact, it is only the *difference* in potential energy between different points in space that is well defined.

1.5 Newton's third law of motion

Consider a system of N mutually interacting point objects. Let the ith object, whose mass is m_i, be located at position vector $\mathbf{r}_i$. Suppose that this object exerts a force $\mathbf{f}_{ji}$ on the jth object. Likewise, suppose that the jth object exerts a force $\mathbf{f}_{ij}$ on the ith object. Newton's third law of motion essentially states that these two forces are equal and opposite, irrespective of their nature. In other words,

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji}. \tag{1.23}$$

(See Figure 1.2.) One corollary of Newton's third law is that an object cannot exert a force on itself. Another corollary is that all forces in the universe have corresponding reactions. The only exceptions to this rule are the fictitious forces that arise in non-inertial reference frames (e.g., the centrifugal and Coriolis forces that appear in rotating reference frames—see Chapter 5). Fictitious forces do not generally possess reactions.

Newton's third law implies *action at a distance*. In other words, if the force that object i exerts on object j suddenly changes, then Newton's third law demands that there must be an *immediate* change in the force that object j exerts on object i. Moreover, this must be true irrespective of the distance between the two objects. However, we now know that Einstein's special theory of relativity forbids information from traveling through

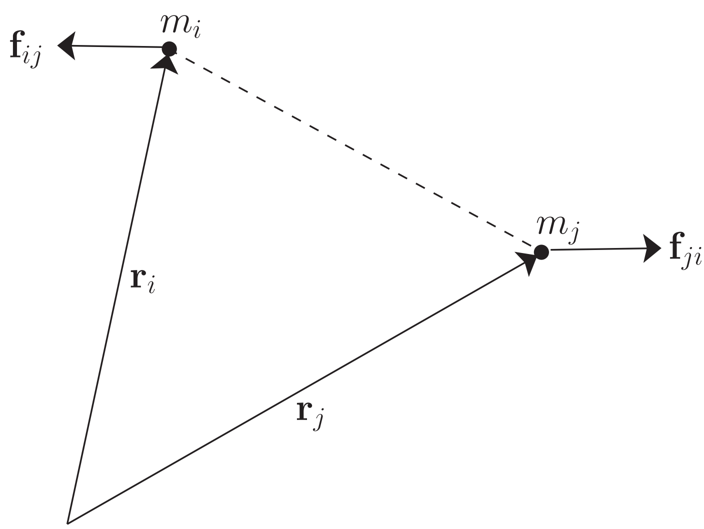

Fig. 1.2 Newton's third law.

the universe faster than the velocity of light in vacuum. Hence, action at a distance is also forbidden. In other words, if the force that object i exerts on object j suddenly changes, then there must be a *time delay*, which is at least as long as it takes a light ray to propagate between the two objects, before the force that object j exerts on object i can respond. Of course, this means that Newton's third law is not, strictly speaking, correct. However, as long as we restrict our investigations to the motions of dynamical systems over timescales that are long compared with the time required for light rays to traverse these systems, Newton's third law can be regarded as being approximately correct.

In an inertial frame, Newton's second law of motion applied to the ith object yields

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_{j=1,N}^{j\neq i} \mathbf{f}_{ij}. \tag{1.24}$$

Note that the summation on the right-hand side of this equation excludes the case $j = i$, as the ith object cannot exert a force on itself. Let us now take this equation and sum it over all objects. We obtain

$$\sum_{i=1,N} m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_{i,j=1,N}^{j\neq i} \mathbf{f}_{ij}. \tag{1.25}$$

Consider the sum over forces on the right-hand side of the preceding equation. Each element of this sum—$\mathbf{f}_{ij}$, say—can be paired with another element—$\mathbf{f}_{ji}$, in this case—that is equal and opposite, according to Newton's third law. In other words, the elements of the sum all cancel out in pairs. Thus, the net value of the sum is *zero*. It follows that Equation (1.25) can be written

$$M \frac{d^2\mathbf{r}_{cm}}{dt^2} = \mathbf{0}, \tag{1.26}$$

where $M = \sum_{i=1,N} m_i$ is the total mass. The quantity $\mathbf{r}_{cm}$ is the vector displacement of the *center of mass* of the system, which is an imaginary point whose coordinates are the mass weighted averages of the coordinates of the objects that constitute the system:

$$\mathbf{r}_{cm} = \frac{\sum_{i=1,N} m_i\, \mathbf{r}_i}{\sum_{i=1,N} m_i}. \tag{1.27}$$

According to Equation (1.26), the center of mass of the system moves in a uniform straight line, in accordance with Newton's first law of motion, irrespective of the nature of the forces acting between the various components of the system.

Now, if the center of mass moves in a uniform straight line, then the center of mass velocity,

$$\frac{d\mathbf{r}_{cm}}{dt} = \frac{\sum_{i=1,N} m_i\, d\mathbf{r}_i/dt}{\sum_{i=1,N} m_i}, \tag{1.28}$$

is a constant of the motion. However, the momentum of the ith object takes the form $\mathbf{p}_i = m_i\, d\mathbf{r}_i/dt$. Hence, the total momentum of the system is written

$$\mathbf{P} = \sum_{i=1,N} m_i \frac{d\mathbf{r}_i}{dt}. \tag{1.29}$$

A comparison of Equations (1.28) and (1.29) suggests that $\mathbf{P}$ is also a constant of the motion. In other words, the total momentum of the system is a *conserved* quantity, irrespective of the nature of the forces acting between the various components of the system. This result (which holds only if there is zero net external force acting on the system) is a direct consequence of Newton's third law of motion.

Taking the vector product of Equation (1.24) with the position vector $\mathbf{r}_i$, we obtain

$$m_i\, \mathbf{r}_i \times \frac{d^2\mathbf{r}_i}{dt^2} = \sum_{j=1,N}^{j\neq i} \mathbf{r}_i \times \mathbf{f}_{ij}. \tag{1.30}$$

The right-hand side of the this equation is the net *torque* about the origin that acts on object i as a result of the forces exerted on it by the other objects. It is easily seen that

$$m_i\, \mathbf{r}_i \times \frac{d^2\mathbf{r}_i}{dt^2} = \frac{d}{dt}\left(m_i\, \mathbf{r}_i \times \frac{d\mathbf{r}_i}{dt}\right) = \frac{d\mathbf{l}_i}{dt}, \tag{1.31}$$

where

$$\mathbf{l}_i = m_i\, \mathbf{r}_i \times \frac{d\mathbf{r}_i}{dt} \tag{1.32}$$

is the *angular momentum* of the ith object about the origin of our coordinate system. Moreover, the total angular momentum of the system (about the origin) takes the form

$$\mathbf{L} = \sum_{i=1,N} \mathbf{l}_i. \tag{1.33}$$

Hence, summing Equation (1.30) over all particles, we obtain

$$\frac{d\mathbf{L}}{dt} = \sum_{i,j=1,N}^{i\neq j} \mathbf{r}_i \times \mathbf{f}_{ij}. \tag{1.34}$$

Consider the sum on the right-hand side of Equation (1.34). A general term, $\mathbf{r}_i \times \mathbf{f}_{ij}$, in this sum can always be paired with a matching term, $\mathbf{r}_j \times \mathbf{f}_{ji}$, in which the indices

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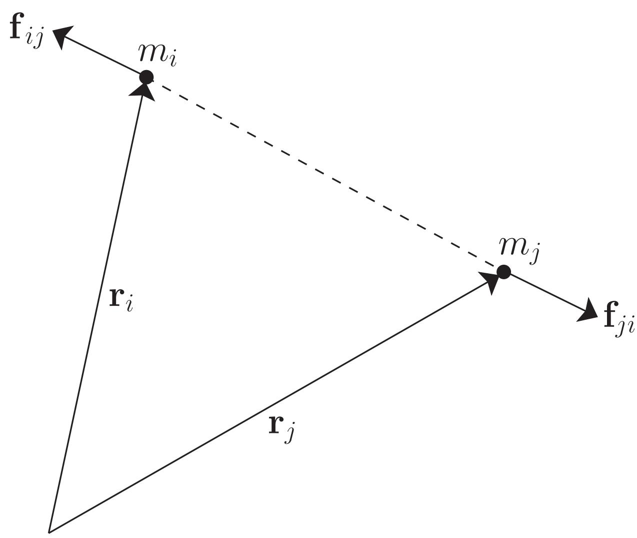

Fig. 1.3 Central forces.

have been swapped. Making use of Equation (1.23), we can write the sum of a general matched pair as

$$\mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_j \times \mathbf{f}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{f}_{ij}. \tag{1.35}$$

Let us assume that the forces acting between the various components of the system are *central* in nature, so that $\mathbf{f}_{ij}$ is parallel to $\mathbf{r}_i - \mathbf{r}_j$. In other words, the force exerted on object j by object i either points directly toward, or directly away from, object i, and vice versa. (See Figure 1.3.) This is a reasonable assumption, as virtually all the forces that we encounter in celestial mechanics are of this type (e.g., gravity). It follows that if the forces are central, then the vector product on the right-hand side of the above expression is zero. We conclude that

$$\mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_j \times \mathbf{f}_{ji} = \mathbf{0} \tag{1.36}$$

for all values of i and j. Thus, the sum on the right-hand side of Equation (1.34) is zero for any kind of central force. We are left with

$$\frac{d\mathbf{L}}{dt} = \mathbf{0}. \tag{1.37}$$

In other words, the total angular momentum of the system is a *conserved* quantity, provided that the different components of the system interact via *central* forces (and there is zero net external torque acting on the system).

1.6 Nonisolated systems

Up to now, we have considered only *isolated* dynamical systems, in which all the forces acting on the system originate from within the system itself. Let us now generalize

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our approach to deal with *nonisolated* dynamical systems, in which some of the forces originate outside the system. Consider a system of N mutually interacting point objects. Let m_i and $\mathbf{r}_i$ be the mass and position vector of the ith object, respectively. Suppose that the ith object is subject to two forces: first, an *internal force* that originates from the other objects in the system, and second, an *external force* that originates outside the system. In other words, let the force acting on the ith object take the form

$$\mathbf{f}_i = \sum_{j=1,N}^{j\neq i} \mathbf{f}_{ij} + \mathbf{F}_i, \tag{1.38}$$

where $\mathbf{f}_{ij}$ is the internal force exerted by object j on object i, and $\mathbf{F}_i$ the external force acting on object i.

The equation of motion of the ith object is

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{f}_i = \sum_{j=1,N}^{j\neq i} \mathbf{f}_{ij} + \mathbf{F}_i. \tag{1.39}$$

Summing over all objects, we obtain

$$\sum_{i=1,N} m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_{i,j=1,N}^{j\neq i} \mathbf{f}_{ij} + \sum_{i=1,N} \mathbf{F}_i, \tag{1.40}$$

which reduces to

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}, \tag{1.41}$$

where

$$\mathbf{F} = \sum_{i=1,N} \mathbf{F}_i \tag{1.42}$$

is the net external force acting on the system. Here, the sum over the internal forces has canceled out in pairs as a result of Newton's third law of motion. (See Section 1.5.) We conclude that if there is a net external force acting on the system, then the total linear momentum evolves in time according to Equation (1.41) but is completely unaffected by any internal forces. The fact that Equation (1.41) is similar in form to Equation (1.13) suggests that the center of mass of a system consisting of many point objects has analogous dynamics to a single point object whose mass is the total system mass, moving under the action of the net external force.

Taking $\mathbf{r}_i \times$ Equation (1.39), and summing over all objects, we obtain

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}, \tag{1.43}$$

where

$$\boldsymbol{\tau} = \sum_{i=1,N} \mathbf{r}_i \times \mathbf{F}_i \tag{1.44}$$

is the net external torque (about the origin) acting on the system. Here, the sum over the internal torques has canceled out in pairs, assuming that the internal forces are central in nature. (See Section 1.5.) We conclude that if there is a net external torque acting

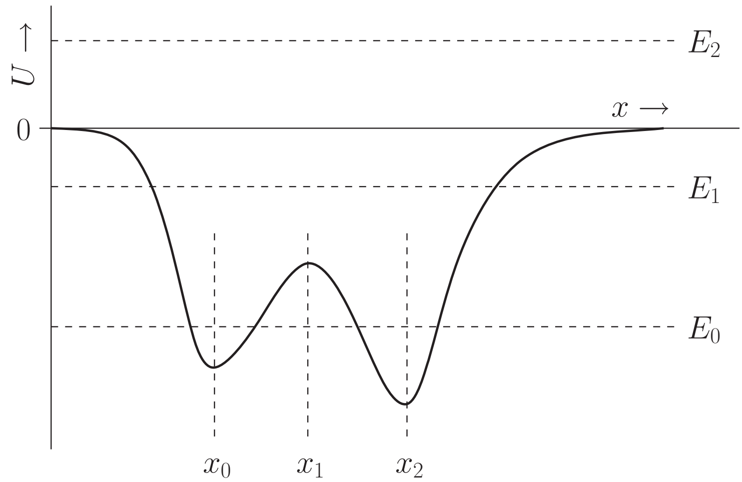

Fig. 1.4 A potential energy curve.

on the system, then the total angular momentum evolves in time according to Equation (1.43) but is completely unaffected by any internal torques.

1.7 Motion in one-dimensional potential

As a simple illustration of the application of Newton's laws of motion, consider a point particle of mass m moving in the x-direction, say, under the action of some x-directed force $f(x)$. Suppose that $f(x)$ is a conservative force, such as gravity. In this case, according to Equation (1.19), we can write

$$f(x) = -\frac{dU(x)}{dx}, \tag{1.45}$$

where $U(x)$ is the potential energy of the particle at position x.

Let the curve $U(x)$ take the form shown in Figure 1.4. For instance, this curve might represent the gravitational potential energy of a cyclist freewheeling in a hilly region. Observe that we have set the potential energy at infinity to zero (which we are generally free to do, as potential energy is undefined to an arbitrary additive constant). This is a fairly common convention. What can we deduce about the motion of the particle in this potential?

Well, we know that the total energy, E—which is the sum of the kinetic energy, K, and the potential energy, U—is a *constant* of the motion [see Equation (1.22)]. Hence, we can write

$$K(x) = E - U(x). \tag{1.46}$$

However, we also know that a kinetic energy can never be negative, because $K = (1/2)\, m\, v^2$, and neither m nor v^2 can be negative. Hence, the preceding expression tells

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us that the particle's motion is restricted to the region (or regions) in which the potential energy curve $U(x)$ falls below the value E. This idea is illustrated in Figure 1.4. Suppose that the total energy of the system is E_0. It is clear, from the figure, that the particle is trapped inside one or other of the two dips in the potential—these dips are generally referred to as *potential wells*. Suppose that we now raise the energy to E_1. In this case, the particle is free to enter or leave each of the potential wells, but its motion is still *bounded* to some extent, as it clearly cannot move off to infinity. Finally, let us raise the energy to E_2. Now the particle is *unbounded*: that is, it can move off to infinity. In conservative systems in which it makes sense to adopt the convention that the potential energy at infinity is zero, bounded systems are characterized by $E < 0$, whereas unbounded systems are characterized by $E > 0$.

The preceding discussion suggests that the motion of a particle moving in a potential generally becomes less bounded as the total energy E of the system increases. Conversely, we would expect the motion to become more bounded as E decreases. In fact, if the energy becomes sufficiently small, then it appears likely that the system will settle down in some *equilibrium state* in which the particle remains stationary. Let us try to identify any prospective equilibrium states in Figure 1.4. If the particle remains stationary, then it must be subject to zero force (otherwise, it would accelerate). Hence, according to Equation (1.45), an equilibrium state is characterized by

$$\frac{dU}{dx} = 0. \tag{1.47}$$

In other words, an equilibrium state corresponds to either a *maximum* or a *minimum* of the potential energy curve, $U(x)$. It can be seen that the $U(x)$ curve shown in Figure 1.4 has three associated equilibrium states located at $x = x_0$, $x = x_1$, and $x = x_2$.

Let us now make a distinction between *stable* equilibrium points and *unstable* equilibrium points. When the particle is displaced slightly from a stable equilibrium point, then the resultant force acting on it must always be such as to return it to this point. In other words, if $x = x_0$ is an equilibrium point, then we require

$$\left.\frac{df}{dx}\right|_{x_0} < 0 \tag{1.48}$$

for stability: that is, if the particle is displaced to the right, so that $x - x_0 > 0$, then the force must act to the left, so that $f < 0$, and vice versa. Likewise, if

$$\left.\frac{df}{dx}\right|_{x_0} > 0 \tag{1.49}$$

then the equilibrium point $x = x_0$ is unstable. It follows, from Equation (1.45), that stable equilibrium points are characterized by

$$\frac{d^2U}{dx^2} > 0. \tag{1.50}$$

In other words, a stable equilibrium point corresponds to a *minimum* of the potential energy curve, $U(x)$. Likewise, an unstable equilibrium point corresponds to a *maximum* of the $U(x)$ curve. Hence we conclude that, in Figure 1.4, $x = x_0$ and $x = x_2$ are stable equilibrium points, whereas $x = x_1$ is an unstable equilibrium point. Of course, this makes perfect sense if we think of $U(x)$ as a gravitational potential energy curve, so that

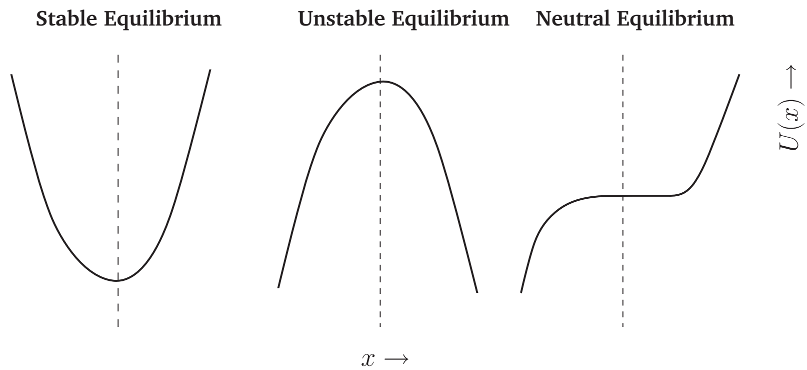

Fig. 1.5 Different types of equilibrium points.

U is directly proportional to height. In this case, all we are saying is that it is easy to confine a low energy mass at the bottom of a smooth valley, but very difficult to balance the same mass on the top of a smooth hill (because any slight displacement of the mass will cause it to slide down the hill). Finally, if

$$\frac{dU}{dx} = \frac{d^2U}{dx^2} = 0 \tag{1.51}$$

at any point (or in any region), then we have what is known as a *neutral equilibrium* point. We can move the particle slightly away from such a point and it will still remain in equilibrium (i.e., it will neither attempt to return to its initial state, nor will it continue to move). A neutral equilibrium point corresponds to a *flat spot* in a $U(x)$ curve. See Figure 1.5.

The equation of motion of a particle moving in one dimension under the action of a conservative force is, in principle, integrable. Because $K = (1/2)\,m\,v^2$, the energy conservation equation, Equation (1.46), can be rearranged to give

$$v = \pm\left\{\frac{2\,[E - U(x)]}{m}\right\}^{1/2}, \tag{1.52}$$

where the $\pm$ signs correspond to motion to the left and to the right, respectively. However, because $v = dx/dt$, this expression can be integrated to give

$$t = \pm\left(\frac{m}{2\,E}\right)^{1/2}\int_{x_0}^{x}\frac{dx'}{\sqrt{1 - U(x')/E}}, \tag{1.53}$$

where $x(t = 0) = x_0$. For sufficiently simple potential functions, $U(x)$, Equation (1.53) can be solved to give x as a function of t. For instance, if $U = (1/2)\,k\,x^2$, $x_0 = 0$, and the plus sign is chosen, then

$$t = \left(\frac{m}{k}\right)^{1/2}\int_0^{(k/2\,E)^{1/2}\,x}\frac{dy}{\sqrt{1 - y^2}} = \left(\frac{m}{k}\right)^{1/2}\sin^{-1}\left[\left(\frac{k}{2\,E}\right)^{1/2}x\right], \tag{1.54}$$

which can be inverted to give

$$x = a\,\sin(\omega\,t), \tag{1.55}$$

where $a = \sqrt{2\,E/k}$ and $\omega = \sqrt{k/m}$. Note that the particle reverses direction each time it reaches one of the so-called *turning points* ($x = \pm a$) at which $U = E$ (and, so, $K = 0$).

1.8 Simple harmonic motion

Consider the motion of a point particle of mass m, moving in one dimension, that is slightly displaced from a *stable* equilibrium point located at $x = 0$. Suppose that the particle is moving in the conservative force field $f(x)$. According to the preceding analysis, for $x = 0$ to correspond to a stable equilibrium point, we require both

$$f(0) = 0 \tag{1.56}$$

and

$$\frac{df(0)}{dx} < 0. \tag{1.57}$$

Our particle obeys Newton's second law of motion,

$$m\,\frac{d^2x}{dt^2} = f(x). \tag{1.58}$$

Let us assume that the particle always stays fairly close to its equilibrium point. In this case, to a good approximation, we can represent $f(x)$ via a truncated Taylor expansion about this point. In other words,

$$f(x) \simeq f(0) + \frac{df(0)}{dx}\,x + \mathcal{O}(x^2). \tag{1.59}$$

However, according to Equations (1.56) and (1.57), the preceding expression can be written

$$f(x) \simeq -m\,\omega_0^2\,x, \tag{1.60}$$

where $df(0)/dx = -m\,\omega_0^2$. Hence, we conclude that our particle satisfies the following approximate equation of motion:

$$\frac{d^2x}{dt^2} + \omega_0^2\,x \simeq 0, \tag{1.61}$$

provided that it does not stray too far from its equilibrium point: in other words, provided $|x|$ does not become too large.

Equation (1.61) is called the *simple harmonic equation*; it governs the motion of all one-dimensional conservative systems that are slightly perturbed from some stable equilibrium state. The solution of Equation (1.61) is well known:

$$x(t) = a\,\sin(\omega_0\,t - \phi_0). \tag{1.62}$$

The pattern of motion described by this expression, which is called *simple harmonic motion*, is *periodic* in time, with repetition period $T_0 = 2\pi/\omega_0$, and oscillates between $x = \pm a$. Here, a is called the *amplitude* of the motion. The parameter ϕ_0, known as the *phase angle*, simply shifts the pattern of motion backward and forward in time.

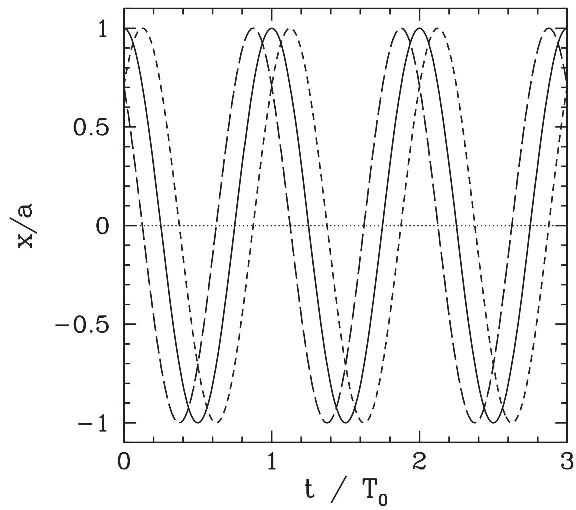

Fig. 1.6 Simple harmonic motion.

Figure 1.6 shows some examples of simple harmonic motion. Here, $\phi_0 = 0$, $+\pi/4$, and $-\pi/4$ correspond to the solid, short-dashed, and long-dashed curves, respectively.

Note that the frequency, ω_0—and, hence, the period, T_0—of simple harmonic motion is determined by the parameters appearing in the simple harmonic equation, Equation (1.61). However, the amplitude, a, and the phase angle, ϕ_0, are the two integration constants of this second-order ordinary differential equation, and are thus determined by the initial conditions: the particle's initial displacement and velocity.

From Equations (1.45) and (1.60), the potential energy of our particle at position x is approximately

$$U(x) \simeq \frac{1}{2}\, m\, \omega_0^2\, x^2. \tag{1.63}$$

Hence, the total energy is written

$$E = K + U = \frac{1}{2}\, m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\, m\, \omega_0^2\, x^2, \tag{1.64}$$

giving

$$E = \frac{1}{2}\, m\, \omega_0^2\, a^2\, \cos^2(\omega_0\, t - \phi_0) + \frac{1}{2}\, m\, \omega_0^2\, a^2\, \sin^2(\omega_0\, t - \phi_0) = \frac{1}{2}\, m\, \omega_0^2\, a^2, \tag{1.65}$$

where use has been made of Equation (1.62), and the trigonometric identity $\cos^2\theta + \sin^2\theta \equiv 1$. Note that the total energy is *constant* in time, as is to be expected for a conservative system, and is proportional to the amplitude squared of the motion.

Consider the motion of a point particle of mass m that is slightly displaced from an *unstable* equilibrium point at $x = 0$. The fact that the equilibrium is unstable implies that

$$f(0) = 0 \tag{1.66}$$

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and

$$\frac{df(0)}{dx} > 0. \tag{1.67}$$

As long as $|x|$ remains small, our particle's equation of motion takes the approximate form

$$m\,\frac{d^2x}{dt^2} \simeq f(0) + \frac{df(0)}{dx}\,x, \tag{1.68}$$

which reduces to

$$\frac{d^2x}{dt^2} \simeq k^2\,x, \tag{1.69}$$

where $df(0)/dx = m\,k^2$. The most general solution to the preceding equation is

$$x(t) = A\,\mathrm{e}^{k\,t} + B\,\mathrm{e}^{-k\,t}, \tag{1.70}$$

where A and B are arbitrary constants. Thus, unless the initial conditions are such that A is *exactly* zero, the particle's displacement from the unstable equilibrium point grows *exponentially* in time.

1.9 Two-body problem

An isolated dynamical system consisting of two freely moving point objects exerting forces on one another is conventionally termed a *two-body problem*. Suppose that the first object is of mass m_1 and is located at position vector $\mathbf{r}_1$. Likewise, the second object is of mass m_2 and is located at position vector $\mathbf{r}_2$. Let the first object exert a force $\mathbf{f}_{21}$ on the second. By Newton's third law, the second object exerts an equal and opposite force, $\mathbf{f}_{12} = -\mathbf{f}_{21}$, on the first. Suppose that there are no other forces in the problem. The equations of motion of our two objects are thus

$$m_1\,\frac{d^2\mathbf{r}_1}{dt^2} = -\mathbf{f} \tag{1.71}$$

and

$$m_2\,\frac{d^2\mathbf{r}_2}{dt^2} = \mathbf{f}, \tag{1.72}$$

where $\mathbf{f} = \mathbf{f}_{21}$.

The center of mass of our system is located at

$$\mathbf{r}_{cm} = \frac{m_1\,\mathbf{r}_1 + m_2\,\mathbf{r}_2}{m_1 + m_2}. \tag{1.73}$$

Hence, we can write

$$\mathbf{r}_1 = \mathbf{r}_{cm} - \frac{m_2}{m_1 + m_2}\,\mathbf{r} \tag{1.74}$$

and

$$\mathbf{r}_2 = \mathbf{r}_{cm} + \frac{m_1}{m_1 + m_2}\,\mathbf{r}, \tag{1.75}$$

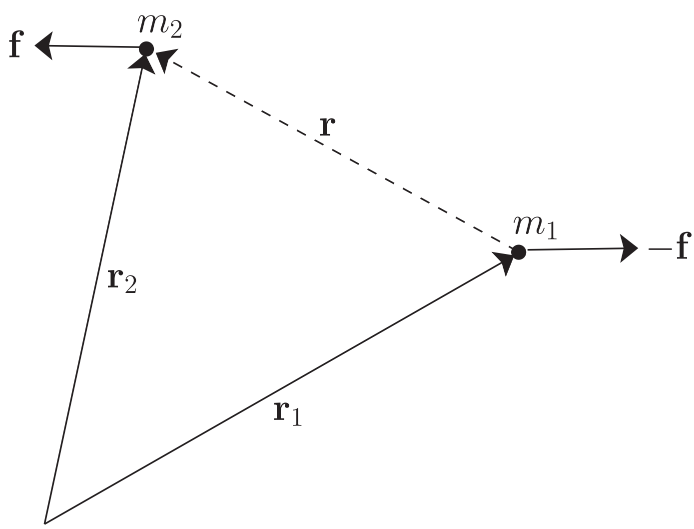

Fig. 1.7 Two-body problem.

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. (See Figure 1.7.) Substituting the preceding two equations into Equations (1.71) and (1.72), and making use of the fact that the center of mass of an isolated system *does not accelerate* (see Section 1.5), we find that both equations yield

$$\mu \frac{d^2\mathbf{r}}{dt^2} = \mathbf{f}, \tag{1.76}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{1.77}$$

is called the *reduced mass*. Hence, we have effectively converted our original two-body problem into an equivalent one-body problem. In the equivalent problem, the force $\mathbf{f}$ is the *same* as that acting on both objects in the original problem (except for a minus sign). However, the mass, μ, is *different*, and it is less than either of m_1 or m_2 (which is why it is called the "reduced" mass). We conclude that the dynamics of an isolated system consisting of two interacting point objects can always be reduced to that of an equivalent system consisting of a single point object moving in a fixed potential.

Exercises

1.1 Derive Equation (1.12).

1.2 Consider a system consisting of N point particles. Let $\mathbf{r}_i$ be the position vector of the ith particle, and let $\mathbf{F}_i$ be the external force acting on this particle. Any internal forces are assumed to be central in nature. The resultant force and torque

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(about the origin) acting on the system are

$$\mathbf{F} = \sum_{i=1,N} \mathbf{F}_i$$

and

$$\boldsymbol{\tau} = \sum_{i=1,N} \mathbf{r}_i \times \mathbf{F}_i,$$

respectively. A *point of action* of the resultant force is defined as a point whose position vector $\mathbf{r}$ satisfies

$$\mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}.$$

Demonstrate that there are an infinite number of possible points of action lying on the straight line

$$\mathbf{r} = \frac{\mathbf{F} \times \boldsymbol{\tau}}{F^2} + \lambda \frac{\mathbf{F}}{F},$$

where λ is arbitrary. This straight line is known as the *line of action* of the resultant force.

1.3 Consider an isolated system consisting of two extended bodies (which can, of course, be modeled as collections of point particles), A and B. Let $\mathbf{F}_A$ be the resultant force acting on A due to B, and let $\mathbf{F}_B$ be the resultant force acting on B due to A. Demonstrate that $\mathbf{F}_B = -\mathbf{F}_A$, and that both forces have the same line of action.

1.4 An extended body is acted on by two resultant forces, $\mathbf{F}_1$ and $\mathbf{F}_2$. Show that these forces can be only replaced by a single equivalent force, $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, provided:

a. $\mathbf{F}_1$ and $\mathbf{F}_2$ are parallel (or antiparallel). In this case, the line of action of $\mathbf{F}$ is parallel to those of $\mathbf{F}_1$ and $\mathbf{F}_2$.

b. $\mathbf{F}_1$ and $\mathbf{F}_2$ are not parallel (or antiparallel), but their lines of action cross at a point. In this case, the line of action of $\mathbf{F}$ passes through the crossing point.

1.5 Deduce that if an isolated system consists of three extended bodies, A, B, and C, where $\mathbf{F}_A$ is the resultant force acting on A (due to B and C), $\mathbf{F}_B$ is the resultant force acting on B, and $\mathbf{F}_C$ is the resultant force acting on C; then $\mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_C = \mathbf{0}$, and the forces either all have parallel lines of action or have lines of action that cross at a common point.

1.6 A particle of mass m moves in one dimension and has an instantaneous displacement x. The particle is released at rest from $x = a$, subject to the force $f(x) = -c\,x^{-2}$, where $a, c > 0$. Demonstrate that the time needed for the particle to reach $x = 0$ is

$$\pi \left(\frac{m\,a^3}{8\,c} \right)^{1/2}.$$

(Modified from Fowles and Cassiday 2005.)

1.7 A particle of mass m moves in one dimension and has an instantaneous displacement x. The particle is released at rest from $x = a$, subject to the force

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$f(x) = -m\mu\,(a^5/x^2)^{1/3}$, where $a, \mu > 0$. Show that the particle will reach the origin with a speed $a\sqrt{6\mu}$ after a time $(8/15)\,(6/\mu)^{1/2}$ has elapsed. (Modified from Smart 1951.)

1.8 A particle moves in one dimension and has an instantaneous displacement x. The particle is released at rest from $x = a$ and accelerates such that $\ddot{x} = \mu\,(x + a^4/x^3)$, where $a > 0$. Show that the particle will reach the origin after a time $\pi/(4\sqrt{\mu})$ has elapsed, and that its speed is then infinite. (Modified from Smart 1951.)

1.9 A particle of mass m, moving in one dimension with an initial (i.e., at $t = -\infty$) velocity v_0, is subject to a force

$$f(t) = \frac{p_0\,\delta t}{\pi}\,\frac{1}{(t - t_0)^2 + (\delta t)^2}.$$

Find the velocity as a function of time. Show that, as $\delta t \to 0$, the motion approaches motion at constant velocity, with an abrupt change in velocity, by an amount p_0/m, at $t = t_0$.

1.10 A particle of mass m moving in one dimension is subject to a force

$$f(x) = -k\,x + \frac{a}{x^3},$$

where $k, a > 0$. Find the potential energy, $U(x)$. Find the equilibrium points. Are they stable or unstable? Determine the angular frequency of small-amplitude oscillations about any stable equilibrium points.

1.11 A particle moving in one dimension with simple harmonic motion has speeds u and v at displacements a and b, respectively, from its mean position. Show that the period of the motion is

$$T = 2\pi\left(\frac{b^2 - a^2}{u^2 - v^2}\right)^{1/2}.$$

Find the amplitude. (From Smart 1951.)

1.12 The potential energy for the force between two atoms in a diatomic molecule has the approximate form

$$U(x) = -\frac{a}{x^6} + \frac{b}{x^{12}},$$

where x is the distance between the atoms, and a, b are positive constants. Find the force.

a. Assuming that one of the atoms is relatively heavy and remains at rest while the other, whose mass is m, moves in a straight line, find the equilibrium distance and the period of small oscillations about the equilibrium position.

b. Assuming that both atoms have the same mass m and move in a straight line, find the equilibrium distance and the period of small oscillations about the equilibrium position.

1.13 Two light springs have spring constants k_1 and k_2, respectively, and are used in a vertical orientation to support an object of mass m. Show that the angular frequency of oscillation is $[(k_1 + k_2)/m]^{1/2}$ if the springs are connected in parallel, and $[k_1\,k_2/(k_1 + k_2)\,m]^{1/2}$ if the springs are connected in series.

1.14 A body of uniform cross-sectional area A and mass density ρ floats in a liquid of density ρ_0 (where $\rho < \rho_0$), and at equilibrium displaces a volume V. Show that the period of small oscillations about the equilibrium position is

$$T = 2\pi \sqrt{\frac{V}{g\,A}}.$$

1.15 A particle of mass m executes one-dimensional simple harmonic oscillation under the action of a conservative force such that its instantaneous displacement is

$$x(t) = a\,\cos(\omega\,t - \phi).$$

Find the average values of x, x^2, $\dot{x}$, and $\dot{x}^2$ over a single cycle of the oscillation. Here, $\dot{} \equiv d/dt$. Find the average values of the kinetic and potential energies of the particle over a single cycle of the oscillation.

1.16 Using the notation of Section 1.9, show that the total momentum and angular momentum of a two-body system take the form

$$\mathbf{P} = M\,\dot{\mathbf{r}}_{cm}$$

and

$$\mathbf{L} = M\,\mathbf{r}_{cm} \times \dot{\mathbf{r}}_{cm} + \mu\,\mathbf{r} \times \dot{\mathbf{r}},$$

respectively, where $M = m_1 + m_2$, and $\dot{} \equiv d/dt$.

a. If the force acting between the bodies is conservative, such that $\mathbf{f} = -\nabla U$, demonstrate that the total energy of the system is written

$$E = \frac{1}{2}\,M\,\dot{r}_{cm}^{\,2} + \frac{1}{2}\,\mu\,\dot{r}^{\,2} + U.$$

Show, from the equation of motion, $\mu\,\ddot{\mathbf{r}} = -\nabla U$, that E is constant in time.

b. If the force acting between the particles is central, so that $\mathbf{f} \propto \mathbf{r}$, demonstrate, from the equation of motion, $\mu\,\ddot{\mathbf{r}} = \mathbf{f}$, that $\mathbf{L}$ is constant in time.