EE120 - Fall'19 - Lecture 23 Notes[1]

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The Unilateral z-Transform

Section 10.9 *in* Oppenheim & Willsky

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots \tag{1}$$

Most properties of the bilateral z-transform hold for the unilateral transform, with the following exceptions:

Convolution:

$$(x_1 * x_2)[n] \overset{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}_1(z)\mathcal{X}_2(z) \quad \text{if } x_1[n] = x_2[n] = 0\ \forall n < 0.$$

Time Delay:

$$x[n-1] \overset{\mathcal{UZ}}{\longleftrightarrow} z^{-1}\mathcal{X}(z) + x[-1]$$

Contrast to: $x[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-1}\mathcal{X}(z)$

Proof:

$$\sum_{n=0}^{\infty} x[n-1]z^{-n} = x[-1] + \underbrace{x[0]z^{-1} + x[1]z^{-2} + x[2]z^{-3} + \ldots}_{=z^{-1}\underbrace{(x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots)}_{=\mathcal{X}(z)}}$$

Applying repeatedly:

$$\begin{aligned}
x[n-2] &\overset{\mathcal{UZ}}{\longleftrightarrow} z^{-1}(z^{-1}\mathcal{X}(z) + x[-1]) + x[-2] \\
&\quad = z^{-2}\mathcal{X}(z) + x[-1]z^{-1} + x[-2] \\
x[n-3] &\overset{\mathcal{UZ}}{\longleftrightarrow} z^{-1}(z^{-2}\mathcal{X}(z) + x[-1]z^{-1} + x[-2]) + x[-3] \\
&\quad = z^{-3}\mathcal{X}(z) + x[-1]z^{-2} + x[-2]z^{-1} + x[-3]
\end{aligned}$$

Solving Difference Equations using the Unilateral z-Transform

Example 1: $y[n] - 0.6y[n-1] = (0.5)^n u[n]$

Take unilateral z-transforms on both sides:

$$Y(z) - 0.6(z^{-1}Y(z) + y[-1]) = \frac{1}{1-0.5z^{-1}}$$

$$(1-0.6z^{-1})Y(z) = 0.6y[-1] + \frac{1}{1-0.5z^{-1}} = \frac{1+0.6y[-1] - 0.3y[-1]z^{-1}}{1-0.5z^{-1}}$$

$$Y(z) = \frac{(1+0.6y[-1]) - 0.3y[-1]z^{-1}}{(1-0.6z^{-1})(1-0.5z^{-1})} = \frac{A}{1-0.6z^{-1}} + \frac{B}{1-0.5z^{-1}}$$

$$\left.\begin{aligned} A + B &= 1 + 0.6y[-1] \\ 0.5A + 0.6B &= 0.3y[-1] \end{aligned}\right\} \quad \begin{aligned} B &= -5 \\ A &= 6 + 0.6y[-1] \end{aligned}$$

$$y[n] = (6 + 0.6y[-1])(0.6)^n u[n] - 5(0.5)^n u[n]$$

Compare to the time domain method:
1) Homogenous solution: $A(0.6)^n$
2) Particular solution: $y_p[n] = B(0.5)^n$
Substitute in difference equation to find B:

$$B(0.5)^n - 0.6B(0.5)^{n-1} = (0.5)^n$$

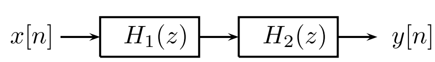
$$B - 0.6(0.5)^{-1}B = 1 \implies -0.2B = 1 \implies B = -5$$

3) The complete solution is $y[n] = A(0.6)^n - 5(0.5)^n$ and
A is determined from the initial condition:

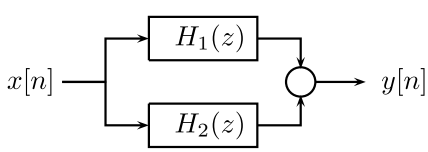
$$y[-1] = A(0.6)^{-1} - 5(0.5)^{-1}$$

$$A = (0.6)(y[-1] + 10) = 6 + 0.6y[-1]$$

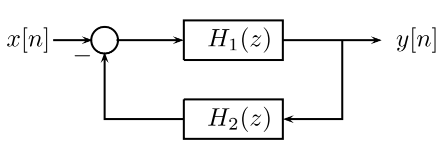
Interconnections of Discrete-Time LTI Systems



$$Y(z) = \underbrace{H_2(z)H_1(z)}_{H(z)}X(z)$$

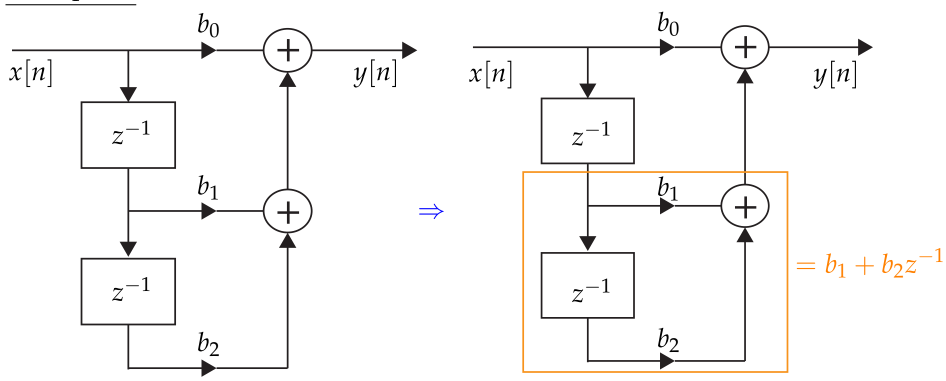


$$Y(z) = (H_1(z) + H_2(z))X(z)$$



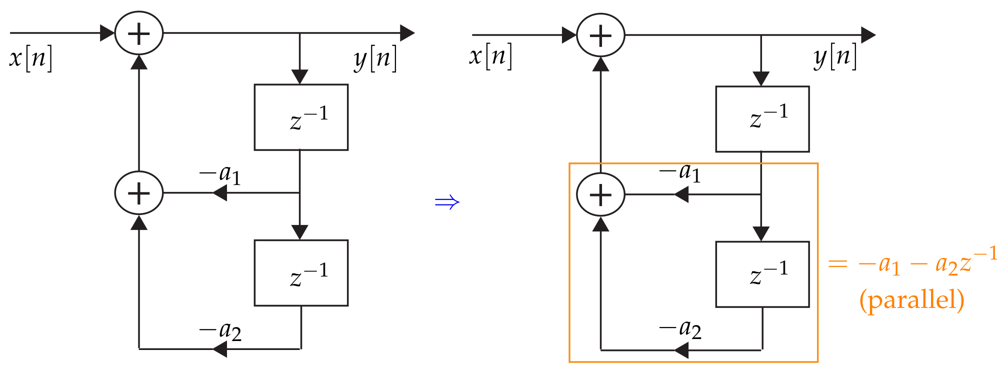
$$\frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

Example 2:



$$\Rightarrow \quad H(z) = b_0 + z^{-1}(b_1 + b_2 z^{-1}) = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

Example 3:



Then, from the feedback interconnection formula:

$$H(z) = \frac{1}{1 - z^{-1}(-a_1 - a_2 z^{-1})} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Example 4: Consider now the interconnection shown below. It follows from a generalization of Example 2 that the blue block implements the transfer function:

$$H_1(z) = b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}.$$

Likewise, the orange block implements the transfer function:

$$H_2(z) = \frac{1}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

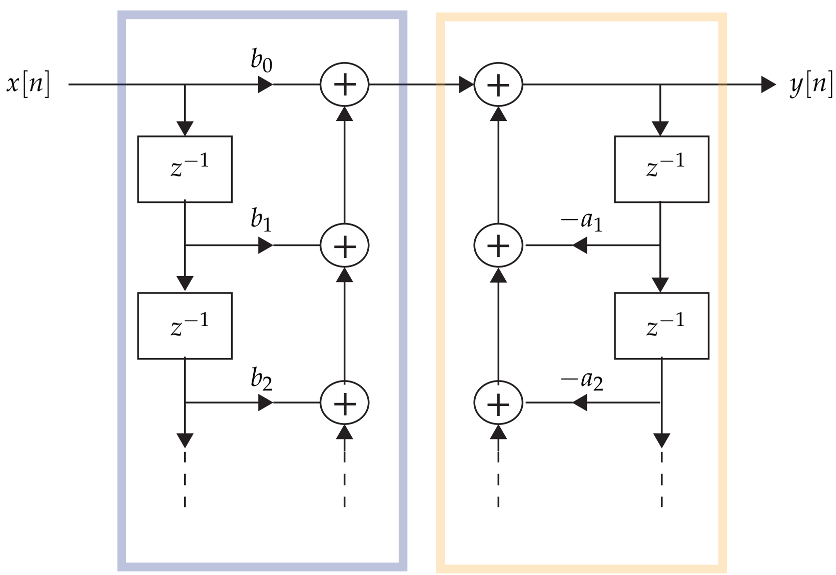
as in Example 3 above. Thus, their series interconnection gives:

$$H(z) = H_1(z) H_2(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} \qquad (2)$$

which, as shown in Lecture 22, describes the LTI system defined by the difference equation:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

Indeed the block diagram below is an implementation of this difference equation, where $\sum_{k=0}^{M} b_k x[n-k]$ is the output of the blue block and $y[n]$ is obtained by adding to this output the feedback terms $-\sum_{k=1}^{N} a_k y[n-k]$ in the orange block.



The implementation above requires $N + M$ delay elements (memory registers). For an implementation with fewer memory registers, first note that changing the order of $H_1(z)$ and $H_2(z)$ does not change the product; that is, we can swap the blue and orange blocks above. Next, note that in this swapped form we can merge two delay elements into one as shown on the next page.

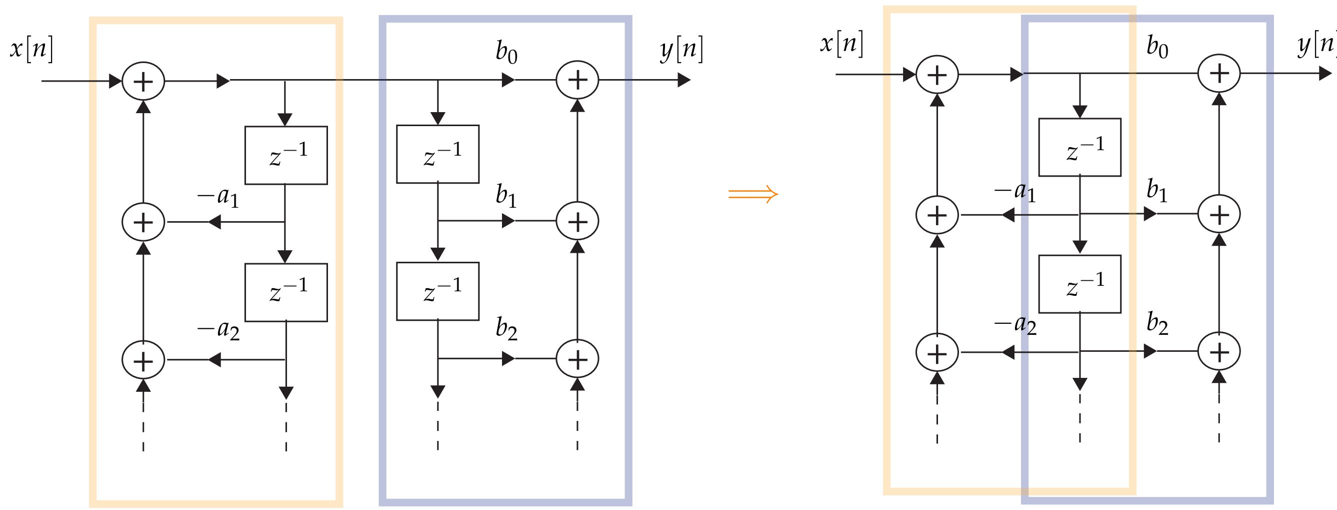
Transfer Functions from State Space Models

Consider the state space representation where the "state vector" $\vec{w}[n]$ evolves according to the vector difference equation

$$\vec{w}[n+1] = A\vec{w}[n] + Bx[n] \tag{3}$$

where $x[n]$ is the input. The output $y[n]$ is a linear function of $\vec{w}[n]$ and $x[n]$:

$$y[n] = C\vec{w}[n] + Dx[n]. \tag{4}$$



To obtain the transfer function from this representation we apply the z-Transform to both sides of (3)-(4) and obtain:

$$z\vec{W}(z) = A\vec{W}(z) + BX(z) \tag{5}$$
$$Y(z) = C\vec{W}(z) + DX(z). \tag{6}$$

We rearrange (5) as

$$(zI - A)\vec{W}(z) = BX(z),$$

where the scalar z is multiplied with the identity matrix to match the dimension of A, and obtain

$$\vec{W}(z) = (zI - A)^{-1}BX(z).$$

Substituting in (6) we get

$$Y(z) = C(zI - A)^{-1}BX(z) + DX(z) = [C(zI - A)^{-1}B + D]X(z),$$

which means that the transfer function is

$$\boxed{H(z) = C(zI - A)^{-1}B + D.} \tag{7}$$

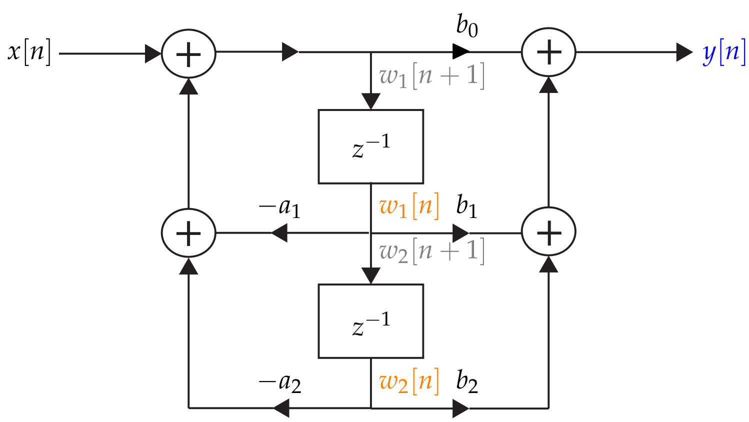
Note that $(zI - A)^{-1}$ is well-defined when $zI - A$ is full-rank, thus invertible. Since the values of z for which $zI - A$ drops rank are the eigenvalues of A, $H(z)$ has a singularity at each eigenvalue. Thus:

$\boxed{\text{The poles of } H(z) \text{ are the eigenvalues of } A.}$

Obtaining State Equations from Block Diagrams

In the examples above a repeated application of the series, parallel, and feedback interconnection formulas gave the transfer function. However, this is not an efficient approach in general and may not be possible for every interconnection. A more systematic procedure is to first derive state equations from the block diagram and then to apply the formula above to obtain the transfer function.

Step 1: Label the output of each delay element as a function of time: $w_1[n], w_2[n], \ldots$ These are the state variables.



Step 2: Note that the input signal to the ith delay element is $w_i[n+1]$ so that its output is $w_i[n]$. By inspecting the interconnection, express these inputs in terms of $x[n]$ and state variables $w_1[n], w_2[n], \ldots$ These are the state equations and can be brought to the vector form (3). In the block diagram above:

$$\begin{aligned} w_1[n+1] &= -a_1 w_1[n] - a_2 w_2[n] + x[n] \\ w_2[n+1] &= w_1[n] \end{aligned}$$

therefore,

$$\begin{bmatrix} w_1[n+1] \\ w_2[n+1] \end{bmatrix} = A = \underbrace{\begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} w_1[n] \\ w_2[n] \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} x[n]. \tag{8}$$

Step 3: Express $y[n]$ in terms of $x[n]$ and $w_1[n]$, $w_2[n]$, ..., again using the interconnection, and bring this equation to the vector form (4). In the block diagram above:

$$\begin{aligned} y[n] &= b_0 w_1[n+1] + b_1 w_1[n] + b_2 w_2[n] \\ &= b_0(-a_1 w_1[n] - a_2 w_2[n] + x[n]) + b_1 w_1[n] + b_2 w_2[n] \\ &= (b_1 - a_1 b_0) w_1[n] + (b_2 - a_2 b_0) w_2[n] + b_0 x[n] \end{aligned}$$

thus,

$$C = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 \end{bmatrix} \quad D = b_0. \tag{9}$$

Step 4: Apply the formula (7) to find the transfer function. Do this as an exercise for the running example with A, B, C, D as in (8)-(9). You should find:

$$H(z) = C(zI - A)^{-1}B + D = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}.$$

<u>Exercise:</u> Find the transfer function for the block diagram below.