Exponential and Sinusoidal Forms

Exponential form for the Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{jkw_1 t} \tag{1}$$

$$F_k = \frac{1}{T_1} \int_{t_0}^{t_0+T_1} f(t) e^{-jkw_1 t} dt \tag{2}$$

Sinusoidal form for the Fourier series

$$f(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(kw_1 t) + b_k \sin(kw_1 t)] \tag{3}$$

$$a_0 = F_0. \quad \begin{cases} a_k = F_k + F_{-k} \\ b_k = j(F_k - F_{-k}) \end{cases} \quad (k = 1, 2, \cdots) \tag{4}$$

Matrix Product for ICTFS-1

Approximate $f(t)$ from k_1 to k_2

$$f(t) = \sum_{k=k_1}^{k_2} F_k e^{jkw_1 t} \tag{5}$$

Interested time interval $[t_0, t_0 + T]$ and N samples

$$f(t_0 + n\Delta t) = \sum_{k=k_1}^{k_2} F_k e^{jkw_1(t_0+n\Delta t)} \tag{6}$$

Matrix form for the inverse Fourier series

$$\begin{bmatrix} f(t_0) \\ f(t_0+\Delta t) \\ \vdots \\ f(t_0+T_1-\Delta t) \end{bmatrix} = \begin{bmatrix} e^{jk_1 w_1 t_0} & e^{j(k_1+1)w_1 t_0} & \dots & e^{jk_2 w_1 t_0} \\ e^{jk_1 w_1(t_0+\Delta t)} & e^{j(k_1+1)w_1(t_0+\Delta t)} & \dots & e^{jk_2 w_1(t_0+\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{jk_1 w_1(t_0+T_1-\Delta t)} & e^{j(k_1+1)w_1(t_0+T_1-\Delta t)} & \dots & e^{jk_2 w_1(t_0+T_1-\Delta t)} \end{bmatrix} \begin{bmatrix} F_{k_1} \\ F_{k_1+1} \\ \vdots \\ F_{k_2} \end{bmatrix} \tag{7}$$

Matrix Product for ICTFS-2

Matrix form for the inverse Fourier series

$$\begin{bmatrix} f(t_0) \\ f(t_0+\Delta t) \\ \vdots \\ f(t_0+T_1-\Delta t) \end{bmatrix} = \begin{bmatrix} e^{jk_1w_1t_0} & e^{j(k_1+1)w_1t_0} & \cdots & e^{jk_2w_1t_0} \\ e^{jk_1w_1(t_0+\Delta t)} & e^{j(k_1+1)w_1(t_0+\Delta t)} & \cdots & e^{jk_2w_1(t_0+\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{jk_1w_1(t_0+T_1-\Delta t)} & e^{j(k_1+1)w_1(t_0+T_1-\Delta t)} & \cdots & e^{jk_2w_1(t_0+T_1-\Delta t)} \end{bmatrix} \begin{bmatrix} F_{k_1} \\ F_{k_1+1} \\ \vdots \\ F_{k_2} \end{bmatrix}$$

Simplified as

$$\boldsymbol{f} = \boldsymbol{UF} \tag{8}$$

Matrix Product for CTFS-1

$$F_k = \frac{1}{T_1}\int_{t_0}^{t_0+T_1} f(t)\, e^{-jkw_1 t} dt$$

Discretize the coefficients

$$F_k = \frac{1}{T_1}\frac{T_1}{N}\sum_{n=0}^{N-1} f(t_0 + n\Delta t)\, e^{-jkw_1(t_0+n\Delta t)} = \frac{1}{N}\sum_{n=0}^{N-1} f(t_0 + n\Delta t)\, e^{-jkw_1(t_0+n\Delta t)} \tag{9}$$

Matrix form for the Fourier series

$$\begin{bmatrix} F_{k_1} \\ F_{k_1+1} \\ \vdots \\ F_{k_2} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} e^{-jk_1 w_1 t_0} & e^{-jk_1 w_1(t_0+\Delta t)} & \cdots & e^{-jk_1 w_1(t_0+T_1-\Delta t)} \\ e^{-j(k_1+1)w_1 t_0} & e^{-j(k_1+1)w_1(t_0+\Delta t)} & \cdots & e^{-j(k_1+1)w_1(t_0+T_1-\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{-jk_2 w_1 t_0} & e^{-jk_2 w_1(t_0+\Delta t)} & \cdots & e^{-jk_2 w_1(t_0+T_1-\Delta t)} \end{bmatrix} \begin{bmatrix} f(t_0) \\ f(t_0+\Delta t) \\ \vdots \\ f(t_0+T_1-\Delta t) \end{bmatrix} \tag{10}$$

Problem: CTFS and ICTFS

Periodic Rectangular pulse

$$f(t) = \begin{cases} \frac{E}{2}, & |t| \le \frac{T_1}{4} \\ -\frac{E}{2}, & \frac{T_1}{4} \le |t| \le \frac{T_1}{2}. \end{cases} \quad (12)$$

with $E = 1$ and $T_1 = 1$.

(1) Plot coefficients $F_{-10}, \cdots, F_{10}$;

(2) Plot coefficients $a_0, \cdots, a_{10}$;

(3) Recover $f_s(t)$ by using the first 5 coefficients $a_0, \cdots, a_5$.

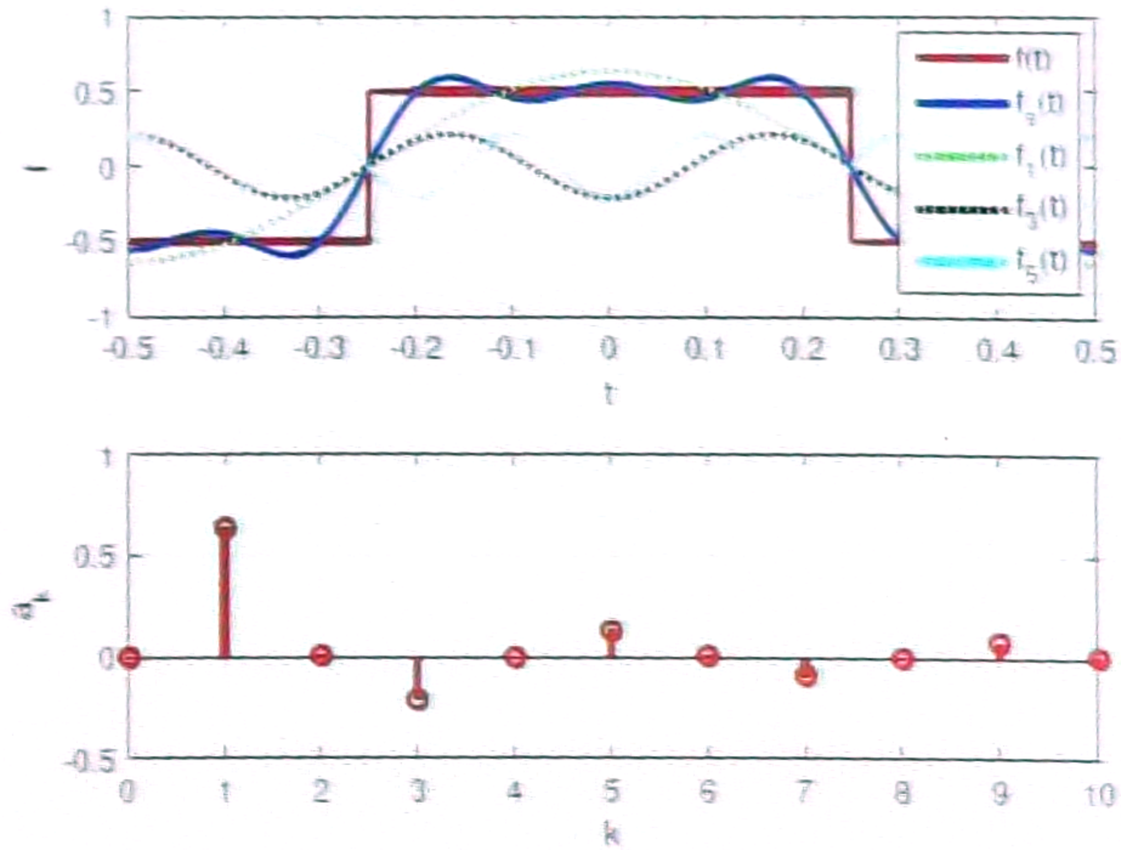

Figure 0.1: Waveform and Fourier coefficient