Example 1: Standard Discrete-Time Convolution

MATLAB functions: conv(x,h), stem(L,y)

Signal 1:

$$x[n] = 1, \quad 0 \le n \le 4 \qquad (4)$$

Signal 2:

$$h[n] = 1.1^n, \quad 0 \le n \le 6 \qquad (5)$$

(1) Plot $x[n]$ and $h[n]$;

(2) Plot $x[n] * h[n]$.

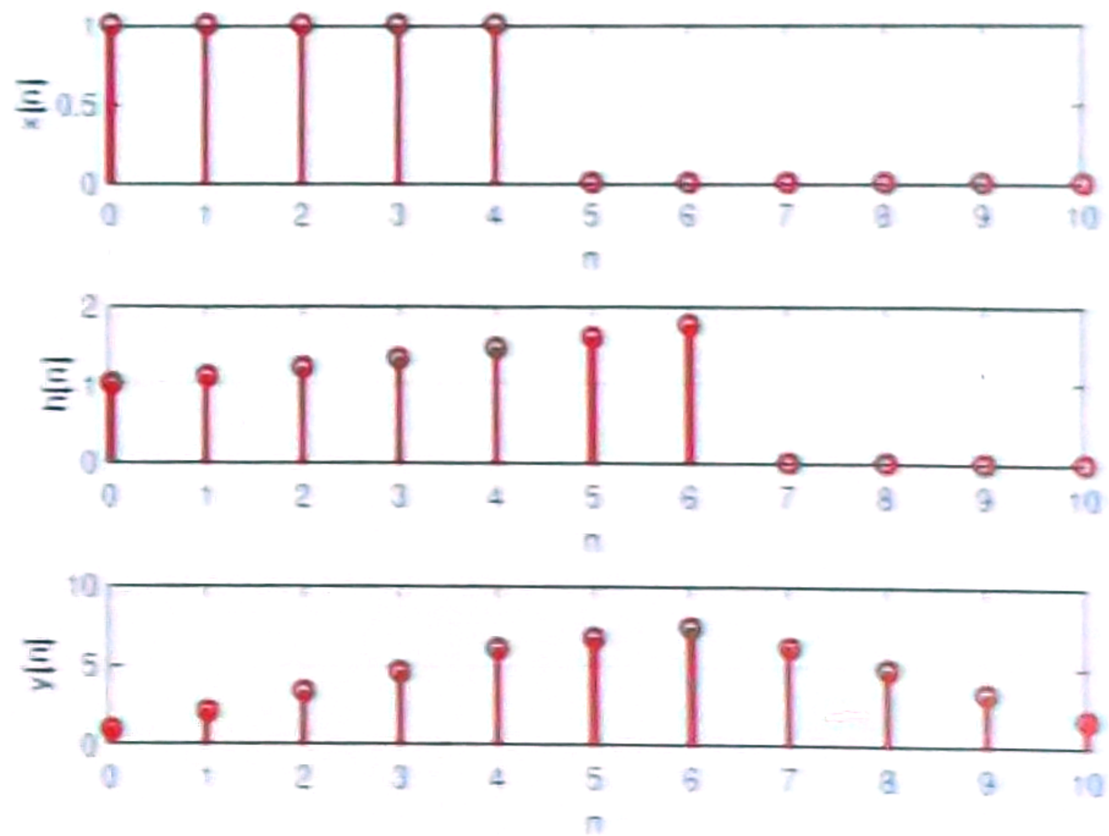

Figure 1.1: Standard discrete convolution.

Example 2: Non-Standard Discrete-Time Convolution

Signal 1:

$$x[n] = \{1, 4, 3, 5, 1, 2, 3, 2\}, -4 \leq n \leq 3 \quad (6)$$

Signal 2:

$$h[n] = \{3, 2, 4, 1, 3, 2\}, -3 \leq n \leq 2 \quad (7)$$

(1) Plot $x[n]$ and $h[n]$;

(2) Plot $x[n] * h[n]$.

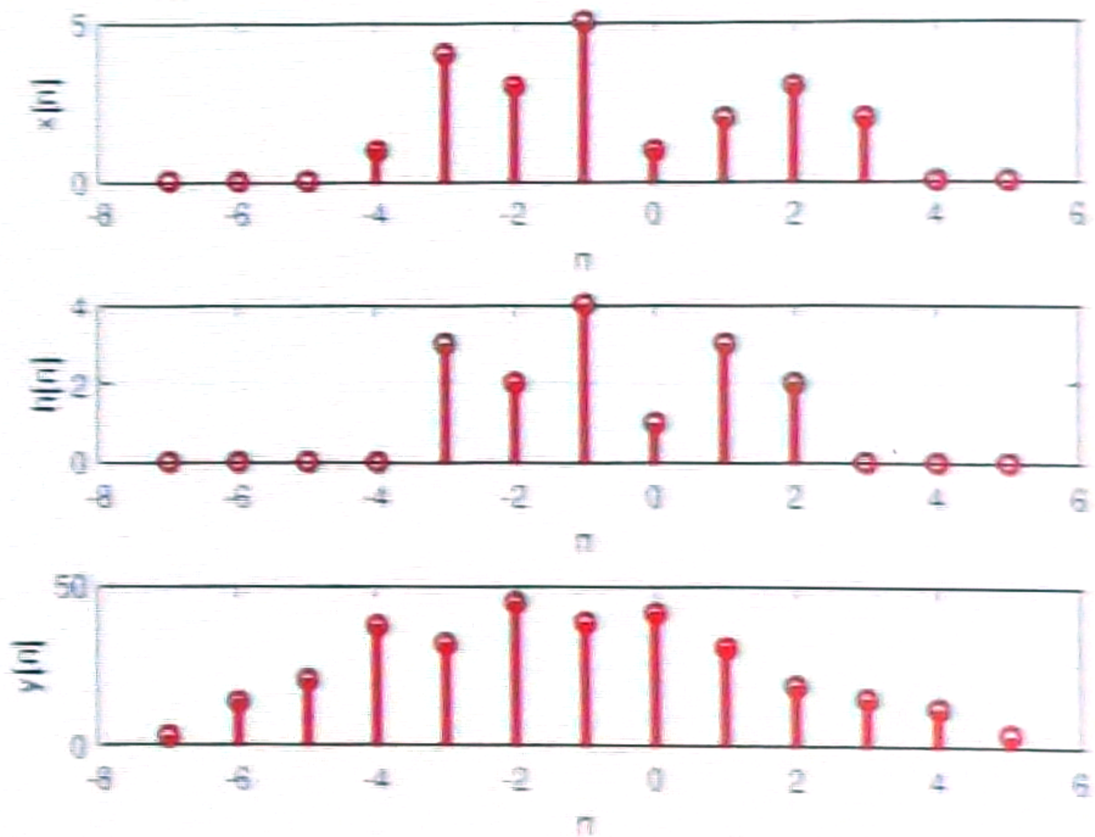

Figure 1.2: Non-standard discrete convolution.

Application 1: Polynomial Multiplication

Two polynomials are given by

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n \\ B(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1} x^{m-1} + b_m x^m \end{aligned} \tag{8}$$

Polynomial multiplication yields

$$C(x) = A(x) \cdot B(x) = \sum_{j=0}^{m+n} c_j x^j \tag{9}$$

where

$$c_j = \sum_{k=-\infty}^{\infty} a_k b_{j-k} = \sum_{k=0}^{\min\{n,j\}} a_k b_{j-k} \Rightarrow c = a * b \tag{10}$$

Convolution theory: Replacing x by $\frac{1}{z}$, $A(z)$ and $B(z)$ are the z-transforms of a and b. That is to say, time-domain convolution equals frequency-domain multiplication.

Application 2: Big Number Multiplication

Two numbers are given by

$$\begin{aligned} 789 &= 7 \times 10^2 + 8 \times 10^1 + 9 \times 10^0 \\ 345 &= 3 \times 10^2 + 4 \times 10^1 + 5 \times 10^0 \end{aligned} \tag{11}$$

Mathematical numbers can be represented by

$$\begin{aligned} A &= a_n a_{n-1} \cdots a_0 = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n = \sum_{i=0}^{n} a_j x_j \\ B &= b_m b_{m-1} \cdots b_0 = b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1} x^{m-1} + b_m x^m = \sum_{i=0}^{m} b_j x_j \end{aligned} \tag{12}$$

Multiplication yields

$$C = \sum_{j=0}^{n+m} c_j x^j = \sum_{j=0}^{n+m} \left[\sum_{k=0}^{\min\{n,j\}} a_k b_{j-k} \right] x^j \tag{13}$$

Time complexity: Exhaustive multiplication $O(n^2) \rightarrow$ FFT $O(n \log_2(n))$.

Continuous-Time Convolution

Continuous-time convolution is defined as

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau \tag{14}$$

Sample the signal by a sampling interval T

$$f(nT) = \sum_{m=-\infty}^{\infty} \int_{mT}^{mT+T} f_1(\tau) f_2(nT-\tau)\, d\tau \tag{15}$$

f T is small enough, we have

$$f(nT) \approx T \sum_{m=-\infty}^{\infty} f_1(mT) f_2(nT-mT) \tag{16}$$

f $f_1(t), t \in [t_1, t_2]$ and $f_2(t), t \in [t_3, t_4]$, then $f(t), t \in [t_1+t_3, t_2+t_4]$.

Example 2: Convolution of Two Continuous-Time Signals

Signal 1

$$x(t) = \begin{cases} 1. & -\frac{1}{2} \le t \le 1 \\ 0. & \text{otherwise} \end{cases} \tag{17}$$

Signal 2

$$h(t) = \begin{cases} \frac{t}{2}. & 0 \le t \le 2 \\ 0. & \text{otherwise} \end{cases} \tag{18}$$

(1) Plot $x(t)$ and $h(t)$, $t \in [-1, 4]$;

(2) Plot $x(t) * h(t)$.

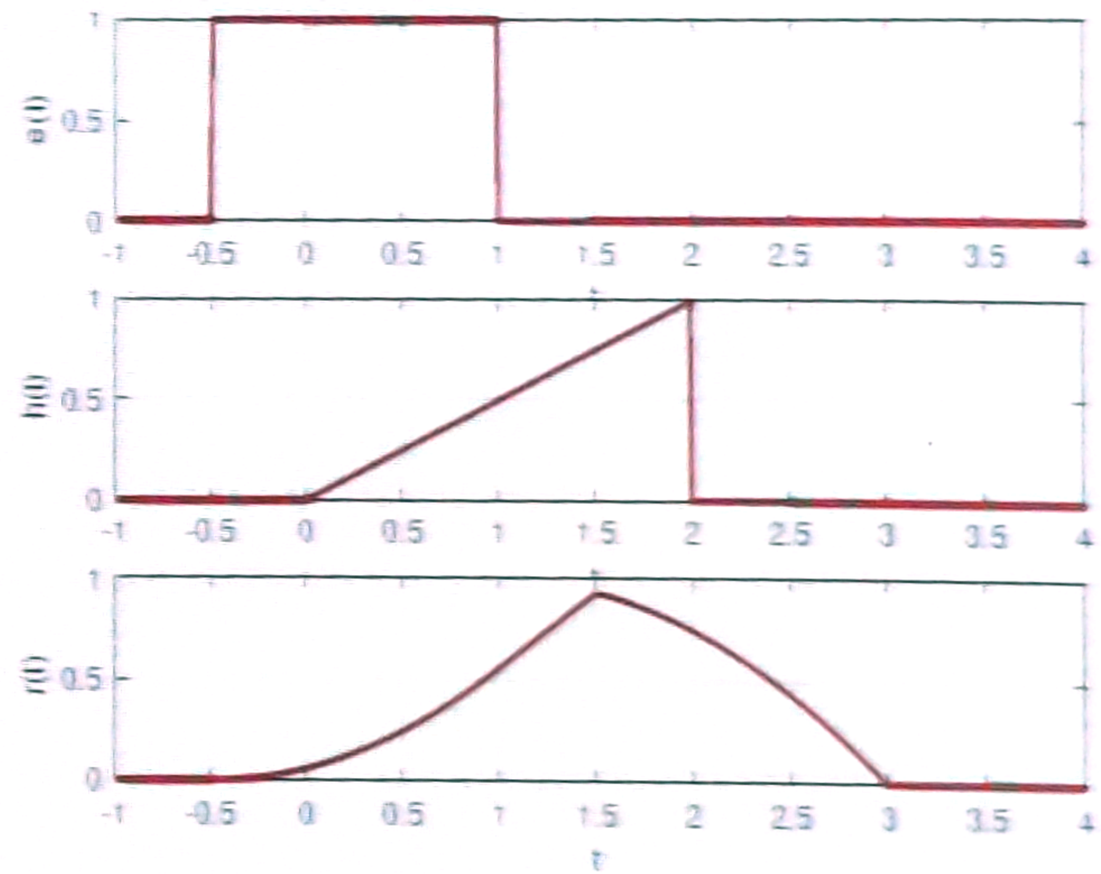

Figure 2.1: Continuous convolution.

Extension: Numerical Integral

One of numerical methods for integral

$$F(a, b) = \int_a^b f(t)\, dt \approx \frac{b-a}{N} \sum_{n=0}^{N-1} f\left(a + n\frac{b-a}{N}\right) \tag{19}$$

Advanced numerical methods include interpolation and fitting: fit().

Problem: Convolution and Convolution Theory

Triangular pulse:

$$f(t) = \begin{cases} E\left(1 - \frac{2|t|}{\tau}\right), & t \leq \frac{\tau}{2} \\ 0, & t > \frac{\tau}{2} \end{cases} \qquad (20)$$

with $E = 1$ and $\tau = 1$. For $t \in [-1, 1]$ and $w \in [-50, 50]$:

(1) Determine $g(t)$ that satisfies $f(t) = g(t) * g(t)$;

(2) Plot $f(t)$ and $g(t)$;

(3) Plot $F(w)$, $G(w)$, and $G_e(w) = G(w) \cdot G(w)$;

(4) Plot $F_e(w) = \mathrm{FT}\{g(t) * g(t)\}$;

(5) Compare $F(w)$, $G_e(w)$, and $F_e(w)$.

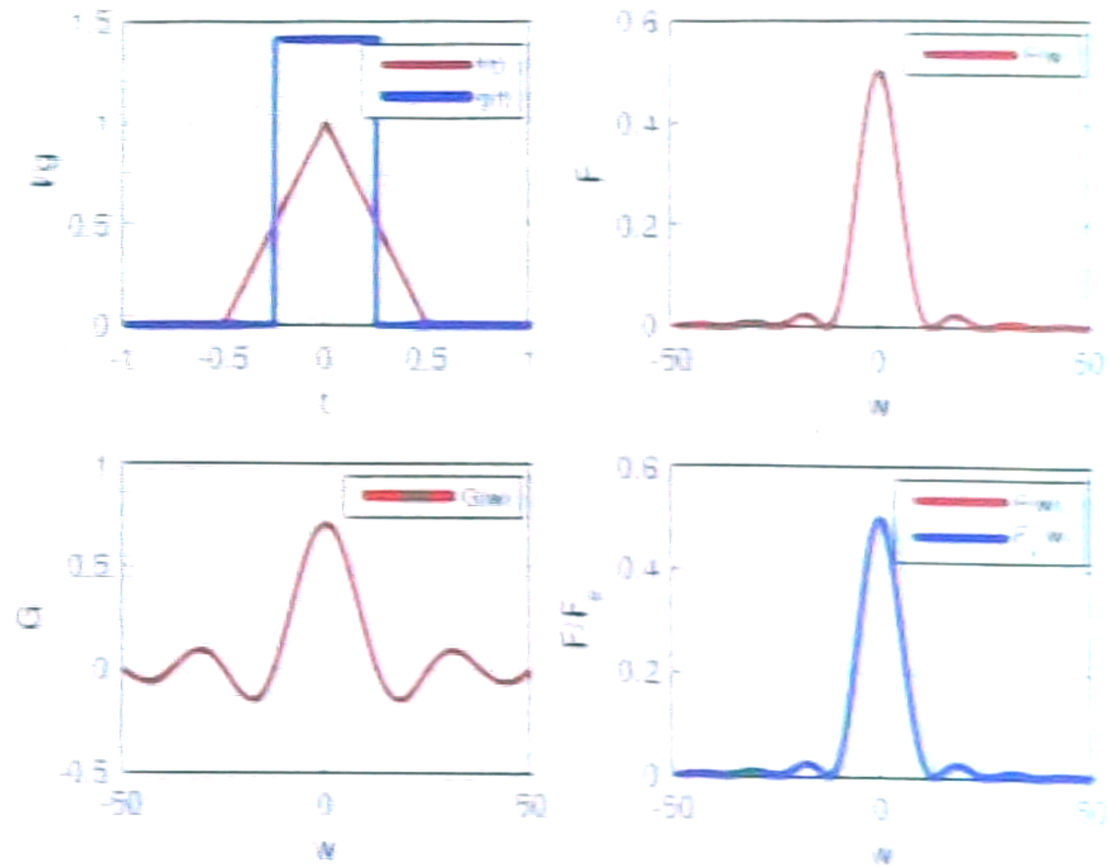

Figure 2.2: Verification of the convolution property