Lab0

Example 1: Symbolic Functions for FT and IFT

Symbolic functions in MATLAB: fourier() and ifourier().

```
syms t;    % define a symbol t
FT0 = fourier(cos(t))    % calculate the FT of cos t
f1 = dirac(t);   % calculate the FT of δ(t)
FT1 = fourier(f1)
f2 = heaviside(t);   % calculate the FT of u(t)
FT2 = fourier(f2)
syms t0;   % calculate the FT of u(t − t0)
FT3 = fourier(heaviside(t - t0))
```

Loop Calculation for FT-1

Consider the main-value interval $[t_1, t_2]$

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt} dt = \int_{t_1}^{t_2} f(t)\, e^{-jwt} dt \tag{3}$$

Define the interval length $T = t_2 - t_1$ and let N be the time-domain smapling number, then the smapling interval $\Delta t = \frac{T}{N}$

$$F(w) = \frac{T}{N} \sum_{n=0}^{N-1} f(t_1 + n\Delta t)\, e^{-jw(t_1 + n\Delta t)} \tag{4}$$

Consider $w \in [w_1, w_2]$ and K frequency-domain samples: $\Omega = w_2 - w_1$ and $\Delta w = \frac{\Omega}{K}$

$$F(w_1 + k\Delta w) = \frac{T}{N} \sum_{n=0}^{N-1} f(t_1 + n\Delta t)\, e^{-j(w_1 + k\Delta w)(t_1 + n\Delta t)} \tag{5}$$

Loop Calculation for FT-2

Formula for FT calculation in MATLAB

$$F(w_1 + k\Delta w) = \frac{T}{N}\sum_{n=0}^{N-1} f(t_1 + n\Delta t)\, e^{-j(w_1+k\Delta w)(t_1+n\Delta t)}$$

Pseudocode for loop calculation

for $k = 1, \cdots, K$
  for $n = 1, \cdots, N$
    $F[k, n] = F[k, n-1] + \frac{T}{N} f(t_1 + n\Delta t)\, e^{-j(w_1+k\Delta w)(t_1+n\Delta t)}$
  end
end

Loop Calculation for IFT-1

Inverse Fourier transform is given by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\, e^{jwt} dw = \frac{\Omega}{2\pi K}\sum_{k=0}^{K-1} F(w_1 + k\Delta w)\, e^{j(w_1+k\Delta w)t} \tag{6}$$

Discretize the time-domain signal

$$f(t_1 + n\Delta t) = \frac{\Omega}{2\pi K}\sum_{k=0}^{K-1} F(w_1 + k\Delta w)\, e^{j(w_1+k\Delta w)(t_1+n\Delta t)} \tag{7}$$

Formula for IFT calculation in MATLAB

$$f\left(t_1 + n\Delta t\right) = \frac{\Omega}{2\pi K} \sum_{k=0}^{K-1} F\left(w_1 + k\Delta w\right) e^{j\left(w_1 + k\Delta w\right)\left(t_1 + n\Delta t\right)}$$

Pseudocode for loop calculation

for $n = 1, \cdots, N$
    for $k = 1, \cdots, K$
        $f[n, k] = f[n, k-1] + \frac{\Omega}{2\pi K} F\left(w_1 + k\Delta w\right) e^{j\left(w_1 + k\Delta w\right)\left(t_1 + n\Delta t\right)}$
    end
end

Example 2: Loop calculation for FT and IFT

Rectangular pulse

$$f(t) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

(1) Plot $f(t)$, $t \in [-1, 1]$;
(2) Plot $F(w)$, $w \in [-8\pi, 8\pi]$;
(3) Recover $f(t)$ from $F(w)$.

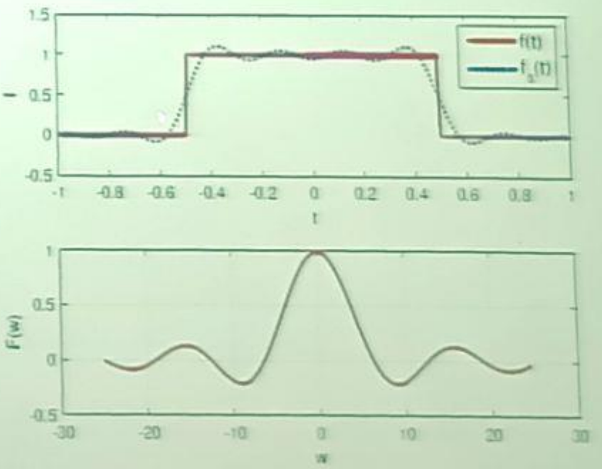

Figure 1.1: Waveform and spectrum

Vector Product for FT

Formula for FT calculation in MATLAB

$$F\left(w_1 + k\Delta w\right) = \frac{T}{N}\sum_{n=0}^{N-1} f\left(t_1 + n\Delta t\right) e^{-j(w_1+k\Delta w)(t_1+n\Delta t)}$$

Vector form for FT calculation

$$F(w_1 + k\Delta w) = \frac{T}{N}\left[e^{-j(w_1+k\Delta w)t_1} \quad e^{-j(w_1+k\Delta w)(t_1+\Delta t)} \quad \cdots \quad e^{-j(w_1+k\Delta w)(t_2-\Delta t)}\right]\begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t)\end{bmatrix} \tag{9}$$

Pseudocode for vector product

for $k = 1, \cdots, K$
  $F_k = \frac{T}{N}\boldsymbol{a}_k^\top \boldsymbol{f}$
end

Vector Product for IFT

Formula for IFT calculation in MATLAB

$$f\left(t_1 + n\Delta t\right) = \frac{\Omega}{2\pi K}\sum_{k=0}^{K-1} F\left(w_1 + k\Delta w\right) e^{j(w_1+k\Delta w)(t_1+n\Delta t)}$$

Vector form for IFT calculation

$$f(t_1 + n\Delta t) = \frac{\Omega}{2\pi K}\left[e^{jw_1(t_1+n\Delta t)} \quad e^{j(w_1+\Delta w)(t_1+n\Delta t)} \quad \cdots \quad e^{j(w_2-\Delta w)(t_1+n\Delta t)}\right]\begin{bmatrix} F(w_1) \\ F(w_1+\Delta w) \\ \vdots \\ F(w_2-\Delta w)\end{bmatrix} \tag{10}$$

Pseudocode for vector product

for $n = 1, \cdots, N$
  $f_n = \frac{\Omega}{2\pi K}\boldsymbol{b}_n^\top \boldsymbol{F}$
end

Matrix Product for FT and IFT-1

Matrix form for the Fourier transform

$$\begin{bmatrix} F(w_1) \\ F(w_1+\Delta w) \\ \vdots \\ f(w_2-\Delta w) \end{bmatrix} = \frac{T}{N} \begin{bmatrix} e^{-jw_1 t_1} & e^{-jw_1(t_1+\Delta t)} & \cdots & e^{-jw_1(t_2-\Delta t)} \\ e^{-j(w_1+\Delta w)t_1} & e^{-j(w_1+\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_1+\Delta w)(t_2-\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{-j(w_2-\Delta w)t_1} & e^{-j(w_2-\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_2-\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t) \end{bmatrix} \quad (12)$$

Matrix form for the inverse Fourier transform

$$\begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t) \end{bmatrix} = \frac{\Omega}{2\pi K} \begin{bmatrix} e^{jw_1 t_1} & e^{j(w_1+\Delta w)t_1} & \cdots & e^{j(w_2-\Delta w)t_1} \\ e^{jw_1(t_1+\Delta t)} & e^{j(w_1+\Delta w)(t_1+\Delta t)} & \cdots & e^{j(w_2-\Delta w)(t_1+\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{jw_1(t_2-\Delta t)} & e^{j(w_1+\Delta w)(t_2-\Delta t)} & \cdots & e^{j(w_2-\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} F(w_1) \\ F(w_1+\Delta w) \\ \vdots \\ f(w_2-\Delta w) \end{bmatrix} \quad (13)$$

Matrix form for the Fourier transform

$$\begin{bmatrix} F(w_1) \\ F(w_1+\Delta w) \\ \vdots \\ f(w_2-\Delta w) \end{bmatrix} = \frac{T}{N} \begin{bmatrix} e^{-jw_1 t_1} & e^{-jw_1(t_1+\Delta t)} & \cdots & e^{-jw_1(t_2-\Delta t)} \\ e^{-j(w_1+\Delta w)t_1} & e^{-j(w_1+\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_1+\Delta w)(t_2-\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{-j(w_2-\Delta w)t_1} & e^{-j(w_2-\Delta w)(t_1+\Delta t)} & \cdots & e^{-j(w_2-\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t) \end{bmatrix} \quad (1$$

Matrix form for the inverse Fourier transform

$$\begin{bmatrix} f(t_1) \\ f(t_1+\Delta t) \\ \vdots \\ f(t_2-\Delta t) \end{bmatrix} = \frac{\Omega}{2\pi K} \begin{bmatrix} e^{jw_1 t_1} & e^{j(w_1+\Delta w)t_1} & \cdots & e^{j(w_2-\Delta w)t_1} \\ e^{jw_1(t_1+\Delta t)} & e^{j(w_1+\Delta w)(t_1+\Delta t)} & \cdots & e^{j(w_2-\Delta w)(t_1+\Delta t)} \\ \vdots & \vdots & & \vdots \\ e^{jw_1(t_2-\Delta t)} & e^{j(w_1+\Delta w)(t_2-\Delta t)} & \cdots & e^{j(w_2-\Delta w)(t_2-\Delta t)} \end{bmatrix} \begin{bmatrix} F(w_1) \\ F(w_1+\Delta w) \\ \vdots \\ f(w_2-\Delta w) \end{bmatrix} \quad ($$

Simplified matrix form

$$\boldsymbol{F} = \frac{T}{N}\boldsymbol{U}\boldsymbol{f}, \quad \boldsymbol{f} = \frac{\Omega}{2\pi K}\boldsymbol{V}\boldsymbol{F} \quad ($$

How to obtain $\boldsymbol{U}$ and $\boldsymbol{V}$: Kronecker product

$$\begin{bmatrix} w_1 \\ w_1+\Delta w \\ \vdots \\ w_2-\Delta w \end{bmatrix} \otimes \begin{bmatrix} t_1 & t_1+\Delta t & \cdots & t_2-\Delta t \end{bmatrix}$$
$$= \begin{bmatrix} w_1 t_1 & w_1(t_1+\Delta t) & \cdots & w_1(t_2-\Delta t) \\ (w_1+\Delta w)t_1 & (w_1+\Delta w)(t_1+\Delta t) & \cdots & (w_1+\Delta w)(t_2-\Delta t) \\ \vdots & \vdots & & \vdots \\ (w_2-\Delta w)t_1 & (w_2-\Delta w)(t_1+\Delta t) & \cdots & (w_2-\Delta w)(t_2-\Delta t) \end{bmatrix} \tag{15}$$

$\otimes$ denotes Kronecker tensor product: kron() in MATLAB.

Final problem

Problem 1: Observe the Gibbs Phenomenon

Rectangular pulse

$$f(t) = \begin{cases} E, & |x| < \tau \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

with $E = 1$ and $\tau = \frac{1}{2}$. Set the sampling numbers $N = 500$ and $K = 1000$

(1) Plot $f(t)$, $t \in [-1, 1]$;

(2) Plot $F(w)$, $w \in [-8\pi, 8\pi]$;

(3) Compare the time costs of 3 methods;

(4) Recover $f(t)$ from $F(w)$;

(5) Observe the Gibbs Phenomenon.

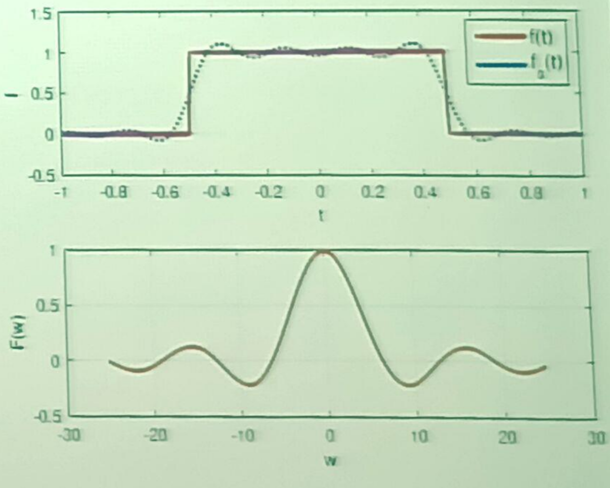

Figure 1.4: Waveform and spectrum