EE120 - Fall'19 - Lecture 22 Notes[1]

Murat Arcak

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Discrete-Time LTI Systems

Section 10.7 *in* Oppenheim & Willsky

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = (h * x)[n]$$

From the convolution property:

$$\boxed{Y(z) = H(z)X(z)}$$

where $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$ is called the transfer function.

Determining Stability from the Transfer function $H(z)$

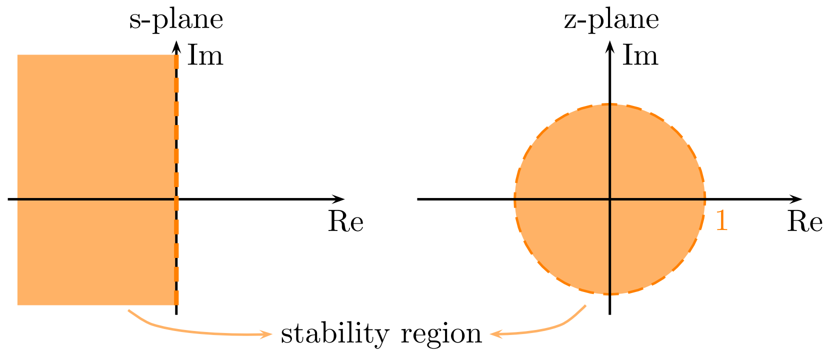
> A causal LTI discrete-time system with transfer function $H(z)$ is stable if and only if all poles of $H(z)$ lie inside the unit circle.

The stability criterion above follows from two observations:

1) Causality means $h[n] = 0 \ \forall n < 0$, which implies that the ROC of the z-transform $H(z)$ extends from the outermost pole to infinity.

2) The stability criterion $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ is equivalent to the statement that the ROC includes the unit circle.

Combining the two, we conclude that all eigenvalues must be strictly inside the unit circle.



Note that, by this criterion, if $H(z)$ has a pole on the unit circle then the system is unstable. As an example consider $H(z) = \frac{1}{1-z^{-1}}$, which has a pole at $z = 1$. This system is indeed unstable because its impulse response is the unit step and, when the input is also a step, the convolution of the two is a sequence that grows unbounded.

From Difference Equations to Transfer Functions

$$\begin{aligned}
\sum_{k=0}^{N} a_k y[n-k] &= \sum_{k=0}^{M} b_k x[n-k] \\
\sum_{k=0}^{N} a_k z^{-k} Y(z) &= \sum_{k=0}^{M} b_k z^{-k} X(z) \\
H(z) = \frac{Y(z)}{X(z)} &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}
\end{aligned}$$

Example:

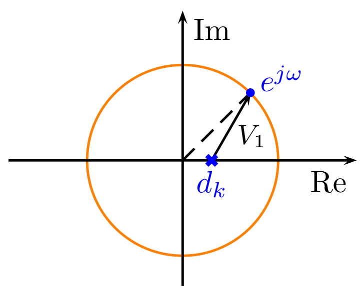
$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1] \implies H(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Geometric Evaluation of the Frequency Response $H(e^{j\omega})$:

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad \begin{array}{l} c_k : \text{zeros} \\ d_k : \text{poles} \end{array}$$

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}|1-c_k e^{-j\omega}|}{\prod_{k=1}^{N}|1-d_k e^{-j\omega}|}$$

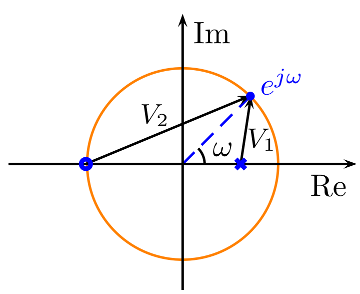
$$|1-d_k e^{-j\omega}| = |e^{j\omega} - d_k| = |V_1|$$

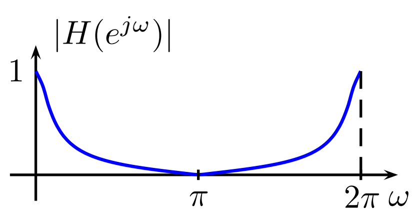


Example:

$$H(z) = \underbrace{0.05}_{\text{s.t. } H(1)=1 \text{ (dc gain)}} \frac{1+z^{-1}}{1-0.9z^{-1}}$$

$$|H(e^{j\omega})| = 0.05\frac{|V_2|}{|V_1|}$$

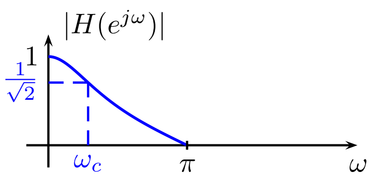




Simple Filters

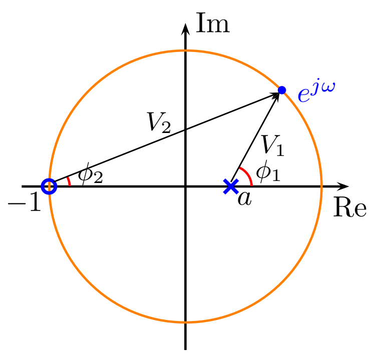
<u>Low-Pass:</u> generalization of the previous example

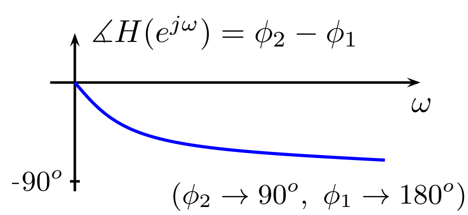
$$H(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}, \quad |\alpha| < 1 \text{ for stability}$$



3dB cutoff frequency ω_c related to α by:

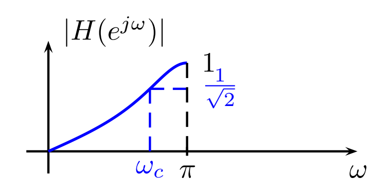
$$\alpha = \frac{1 - sin(\omega_c)}{cos(\omega_c)}$$





<u>High-Pass:</u>

$$H(z) = \frac{1+\alpha}{2}\frac{1-z^{-1}}{1-\alpha z^{-1}}, \quad H(1) = 0 \text{ and } H(-1) = H(e^{j\pi}) = 1$$

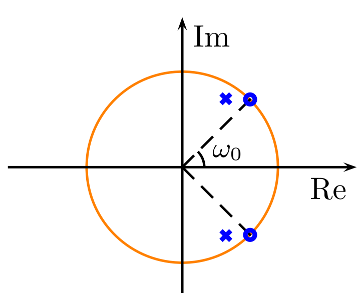


$$\alpha = \frac{1 - sin(\omega_c)}{cos(\omega_c)}$$

<u>Band-Stop (Notch):</u>

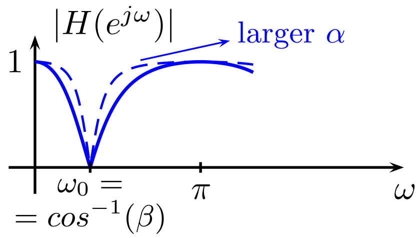
$$H(z) = \frac{1+\alpha}{2}\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \quad |\beta| < 1 \ \ |\alpha| < 1$$

Note: $1-2\beta z^{-1}+z^{-2} = (1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})$ where $cos\omega_0 = \beta$



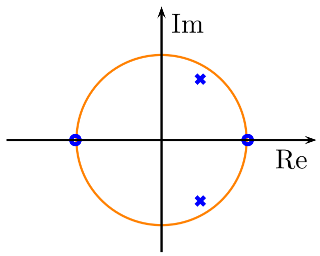
zeros on the unit circle: $e^{\mp j\omega_0}$
poles approach zeros as $\alpha \to 1$.

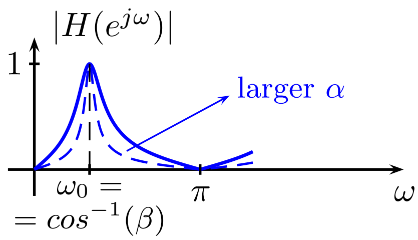
$$H(\mp 1) = \frac{1+\alpha}{2}\frac{2 \pm 2\beta}{(1+\alpha)(1 \pm \beta)} = 1$$



Band-Pass:

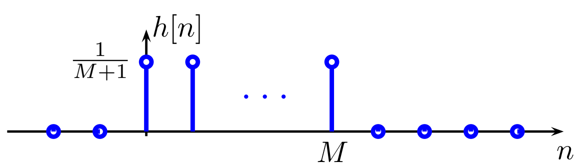
$$H(z) = \frac{1-\alpha}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}, \quad |\alpha| < 1 \ \ |\beta| < 1$$





$(M+1)$-point Moving Average Filter:

$$y[n] = \frac{1}{M+1}\left(x[n] + x[n-1] + \ldots + x[n-M]\right)$$

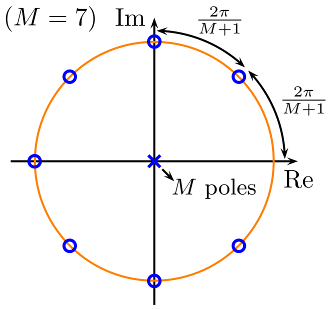


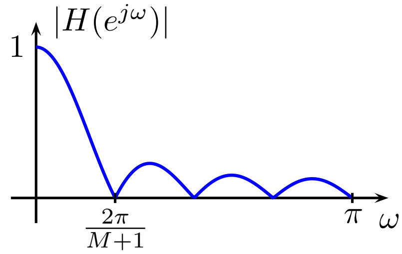
$$\begin{aligned} H(z) &= \frac{1}{M+1}\left(1 + z^{-1} + \ldots + z^{-M}\right) \\ &= \frac{1}{M+1} \frac{z^M + z^{M-1} + \ldots + 1}{z^M} \end{aligned}$$

All poles at $z = 0$. (Note this is true for any FIR system.)
Zeros: roots of $z^M + z^{M-1} + \ldots + 1$.

From the identity $z^{M+1} - 1 = (z-1)(z^M + z^{M-1} + \ldots + 1)$, the roots of $z^M + z^{M-1} + \ldots + 1$ are the roots of $z^{M+1} - 1$ except for $z = 1$.

$z^{M+1} = 1 \implies z = e^{j\frac{2\pi}{M+1}k}$ $k = 1, 2, \ldots, M$ ($k = 0$, *i.e.*, $z = 1$ excluded)



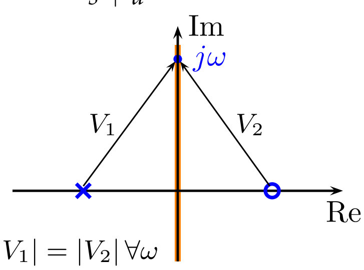


All-Pass Filters

All-pass filters pass all frequencies with unit gain, that is $|H(j\omega)| = 1$ in continuous time and $|H(e^{j\omega})| = 1$ in discrete-time. They are used as phase compensators, since they alter the phase of the input but not its magnitude: $|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})| = |X(e^{j\omega})|$.

Continuous-time:

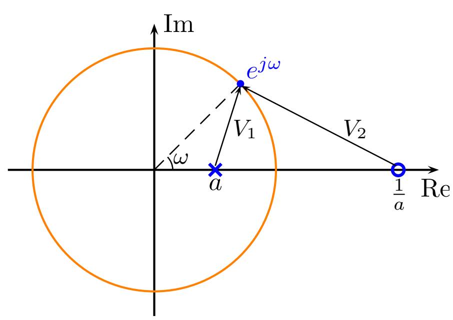
$H(s) = \dfrac{a-s}{s+a}$



$|V_1| = |V_2| \, \forall \omega$

Discrete-time:

$H(z) = \dfrac{z^{-1}-a}{1-az^{-1}} = -a\dfrac{z-1/a}{z-a}$, a real



$$
\begin{aligned}
|V_1|^2 &= (cos\omega - a)^2 + sin^2\omega = 1 - 2acos\omega + a^2 \\
|V_2|^2 &= \left(\frac{1}{a} - cos\omega\right)^2 + sin^2\omega = \frac{1}{a^2} - \frac{2}{a}cos\omega + 1
\end{aligned}
$$

then, $a^2|V_2|^2 = |V_1|^2 \Rightarrow |a|\frac{|V_2|}{|V_1|} = 1.$

$$
|H(e^{j\omega})| = |a|\frac{|e^{j\omega} - 1/a|}{|e^{j\omega} - a|} = |a|\frac{|V_2|}{|V_1|} = 1 \ \forall\omega
$$

General form of a discrete-time all-pass system:

$$
H(z) = \prod_{k=1}^{N}\left(\frac{z^{-1} - a_k}{1 - a_k z^{-1}}\right)
$$

Each pole a_k accompanied by a zero at $1/a_k$.