EE 120: Signals and Systems
Bonus Review: Root Locus
December 12, 2019

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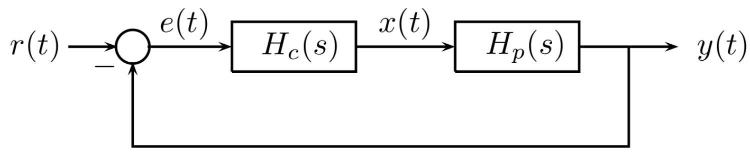
Suppose we have a feedback system like the one shown below:



in which we have taken a "plant" $H_p(s)$ and closed a feedback look around it with a controller $H_c(s)$. The transfer function of this "closed-loop" system is

$$H_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}.$$

We will choose a controller of the form $H_c(s) = KG(s)$, where K is the *controller gain*. One we have chosen a $G(s)$, the act of "designing" the controller will just be choosing K. We would like to know how our choice of K we can affect the closed-loop poles. This is equivalent to asking: What values can the roots of the equation

$$1 + KG(s)H_p(s) = 0$$

have for different values of K? The *root locus*, which is the set of all possible closed-loop poles for $K \in [0, \infty)$, answers this question.

Sketching the Root Locus

The root locus turns out to be a collection of line segments on the plane, called *branches*. There's one branch for each closed-loop pole. We can make a rough sketch of the branches with a few simple rules:

1. Each branch *starts* on a pole of $G(s)H_p(s)$. Each branch *ends* in one of two ways: on a zero of $G(s)H_p(s)$, or going to infinity.

2. The branches that go to infinity tend to straight-line asymptotes. The asymptotes all meet at the point $(\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{m} \beta_k)/(n-m)$, where α_k and β_k are the poles and zeros of $G(s)H_p(s)$, and leave this point at angles

   $(\pi + 2(k-1)\pi)/n - m$, $k = 1, \ldots, n-m$.

3. Parts of the real line to the left of an odd number of real poles and zeros of $H_p(s)$ must be on one of the branches.

4. Branches between two real poles must leave the real axis. These "break-away points" occur when the branches meet, which will be at the roots of $\frac{d}{ds}(G(s)H_p(s)) = 0$. This is the only way that two branches can intersect.

5. The root locus is *conjugate symmetric*: if s is on a branch, so is s^*.

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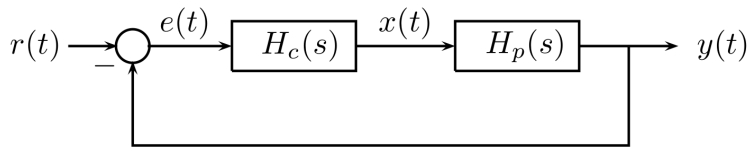

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These rules all come from the properties of the equation $G(s)H_p(s) = -\frac{1}{K}$ when it is split into its magnitude and phase parts:

$$|G(s)H_p(s)| = \frac{\prod_{k=1}^{m} |s-\beta_k|}{\prod_{k=1}^{n} |s-\alpha_k|} = \frac{1}{K}$$

$$\angle G(s) + \angle H_p(s) = \sum_{k=1}^{m} \angle(s-\beta_i) - \sum_{k=1}^{n} \angle(s-\alpha_k) = -\pi + 2\pi w, \text{ for integer } w.$$

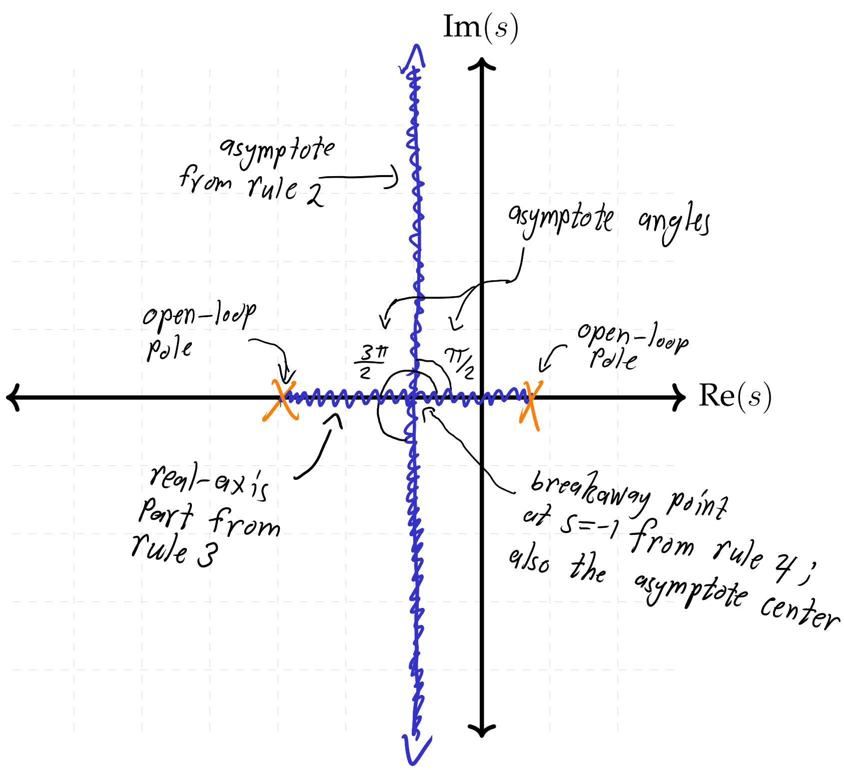
In addition, the "break-away point" condition in Rule 4 comes from the observation that if $G(s)H_p(s)$ has a repeated root, then $\frac{d}{ds}(G(s)H_p(s))$ has the same root. Proofs for the other rules are available in Lecture Note 19. More details and examples are available in Oppenheim & Willsky, Section 11.3.

Now, these 6 rules will allow you to make a pretty good sketch of the root locus. Specifically, you can get the starting points and the parts of the locus on the real line exactly right, and you can estimate where the branches will go for large values of K with the help of the asymptotes. You can also get the "break-away points" exactly right as well, but these rules don't tell you what happens after the branches leave the real axis.

Problem 1: Let

$$H_p(s) = \frac{1}{s^2 + 2s - 3}, \quad H_c(s) = K.$$

Sketch the root locus of the closed-loop system. Then, using the locus, find a value of K that stabilizes the system.



- First, factor the open-loop transfer function to find the open-loop poles where the branches start. $H_p(s)\,G(s) = \frac{1}{s^2+2s-3} = \frac{1}{(s+3)(s-1)}$.
  open-loop poles: -3, $+1$
  open-loop zeros: none
  $n=2$, $m=0$
- Then, rule 3 tells us that the part of the real axis between -3 and $+1$ (that is, between the poles) is on the locus.
- Since there are two poles and no zeros, we know there must be two asymptotes. The asymptotes have center $\frac{\sum \alpha_k - \sum \beta_k}{n-m} = \frac{(-3+1)}{2-0} = -1$, and have angles $\frac{\pi + (0)2\pi}{2} = \frac{\pi}{2}$, $\frac{\pi + (1)2\pi}{2} = \frac{3\pi}{2}$.
- The breakaway points are the zeros of

$$\frac{d}{ds}\left(H_p(s)\,G(s)\right) = \frac{d}{ds}\frac{1}{(s+3)(s-1)} = \frac{s+3+s-1}{(s+3)^2(s-1)^2} = \frac{2s+2}{(s+3)^2(s-1)^2}$$

; only one zero at $s=-1$, so that's the breakaway point.

Solution to 1, continued:

We now have the full root locus, so now we only need to find a K that will put the closed-loop poles on the left half of the plane. To do that, we'll pick a point on each branch that's stable, and find the K value corresponding to those points.

How do we figure out what K value corresponds to a specific point on a branch? Suppose a point p is on a branch: then we know

$$1 + K\,G(p)\,H_p(p) = 0$$

$$\Rightarrow G(p)H_p(p) = \frac{(p-\beta_1)\times\cdots\times(p-\beta_m)}{(p-\alpha_1)\times\cdots\times(p-\alpha_n)} = -\frac{1}{K}$$

$$\Rightarrow |G(p)H_p(p)| = \frac{|(p-\beta_1)|\times\cdots\times|(p-\beta_m)|}{|(p-\alpha_1)|\times\cdots\times|(p-\alpha_n)|} = \frac{1}{K}. \quad \text{("magnitude equation")}$$

Since we will choose a specific p, and since we know the α_i and β_i, we can directly solve for K.

You might also be wondering: How do I know if points on two (or more) branches "belong" to the same K? This is a pertinent question for us because we're trying to find <u>two</u> stable poles with <u>one</u> K. The general answer is that you can't: you would pick one branch you care most about (the unstable branch, for instance), pick a point on it, solve for K, and then solve $1 + K\,G(p)\,H_p(s)$ for that value of K to find out where the other roots are. If they all look fine (e.g. stable), then you're done! If they don't, then you'd have to try again with a new point.

There are also two cases where you can be <u>sure</u> that two poles belong to the same K:

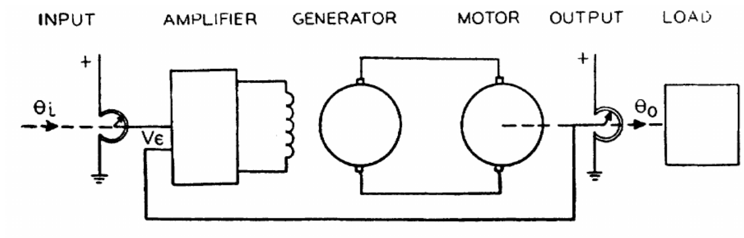
- When they're complex conjugates, since the poles for a given K (that is, for a specific feedback system) must all be conjugate symmetric.
- At a breakaway point, since the poles have to "meet" before they break away.

Since we have a breakaway point on the left half of the plane, I'll choose to put the closed-loop poles there. Putting p=1 in the modular equation above yields

$$\frac{1}{|-1-(-3)|\times|-1-1|} = \frac{1}{4} = \frac{1}{K},$$

So a gain of K=4 will stabilize the system by placing both closed-loop poles at -1.

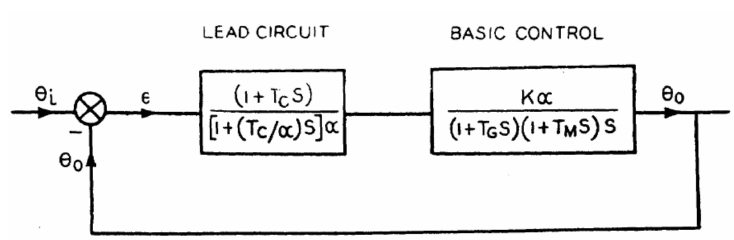
Problem 2: [1] Consider the motor servomechanism shown below:



This system's transfer function is

$$H_p(s) = \frac{1}{s(1 + T_g s)(1 + T_M s)},$$

with $T_G = \frac{1}{4}$, $T_m = 1$. The pole at zero models a *time delay* in the system, meaning that the servo will take some time to respond to a command. To get rid of the time delay, we will put the system in the following feedback loop:



where we've used a *lead controller* of the form

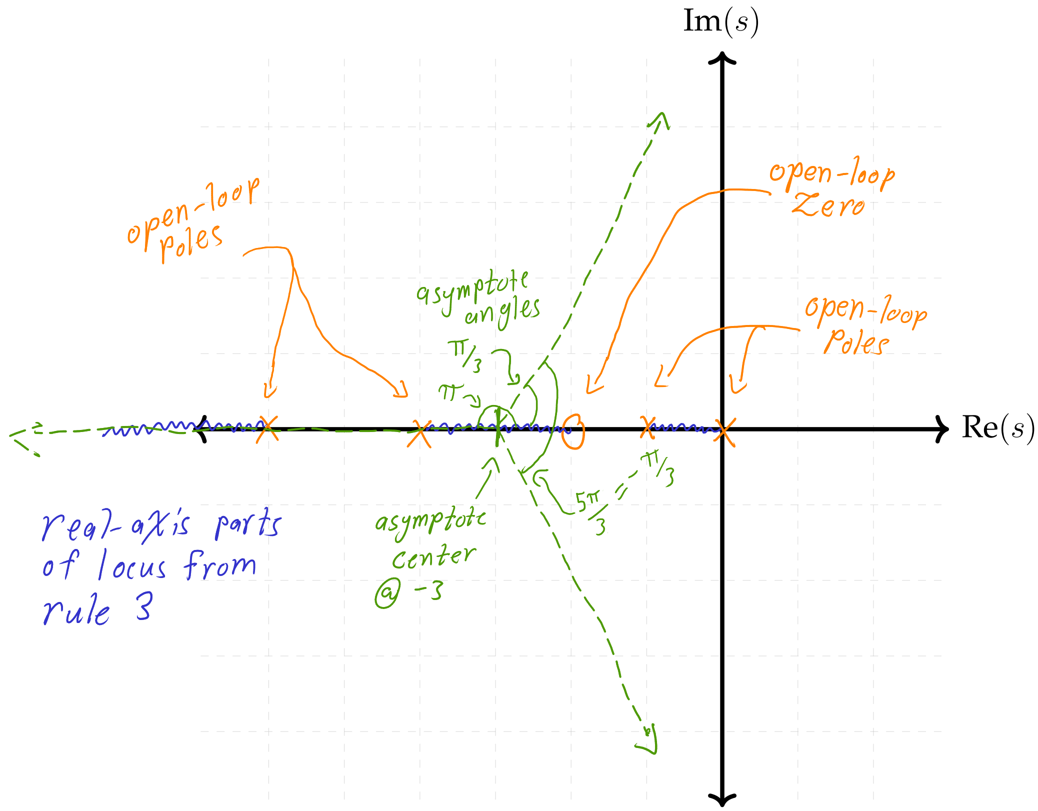
$$H_c(s) = K \frac{1 + T_c s}{(1 + (T_c/\alpha)s)\alpha}, \tag{1}$$

with $T_c = \frac{1}{2}$, $\alpha = 3$.

For this closed-loop system, sketch the part of the root locus *on the real axis*, and sketch the asymptotes.

Can a high K destabilize this system?

[1] Adapted from an example in Evans, "Graphical Analysis of Control Systems" (1948). This is the first published use of root locus analysis.

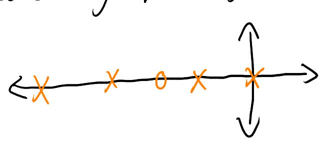


The plant's transfer function is $H_P(s) = \frac{1}{s(1+\frac{1}{4}s)(1+s)} = \frac{1}{4s(s+4)(s+1)}$,

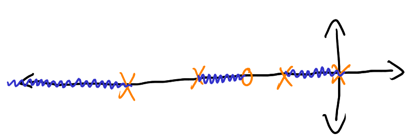
and the controller's is $G(s) = \frac{1+\frac{1}{2}s}{3(1+\frac{1}{6}s)} = \frac{2(s+2)}{18(s+6)}$,

so the overall open-loop transfer function is $H_p(s)\,G(s) = \frac{1}{36} \times \frac{s+2}{s(s+1)(s+4)(s+6)}$.

- Start the locus sketch by plotting the open-loop poles and zeros.



- Then, use rule 3 to draw the parts of the locus on the real axis.



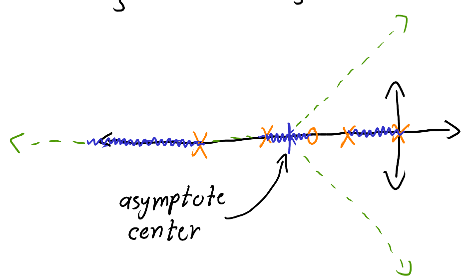
Solution to problem 2, continued:

- Since there are four poles and one zero, there will be three asymptotes. By rule 2, they will have center

$$\frac{(0+(-1)+(-4)+(-6))-(-2)}{4-1} = \frac{-11+2}{3} = -3,$$

and depart from this center with angles

$$\frac{\pi+(0)2\pi}{3} = \frac{\pi}{3}, \quad \frac{\pi+(1)2\pi}{3} = \pi, \quad \frac{\pi+(2)2\pi}{3} = \frac{5\pi}{3}.$$



Since two of the asymptotes point in the direction of the right half of the plane, we know that high gains can destabilize this system.

Problem 3: [2] Consider a feedback system with

$$G(s)H_p(s) = \frac{(s-a)}{s(s+2)}.$$

Sketch the root locus for the following values of a. For each value of a, answer: (i) Can the system be stabilized by feedback? (ii) Can certain feedback gains cause the system to exhibit oscillatory behavior?

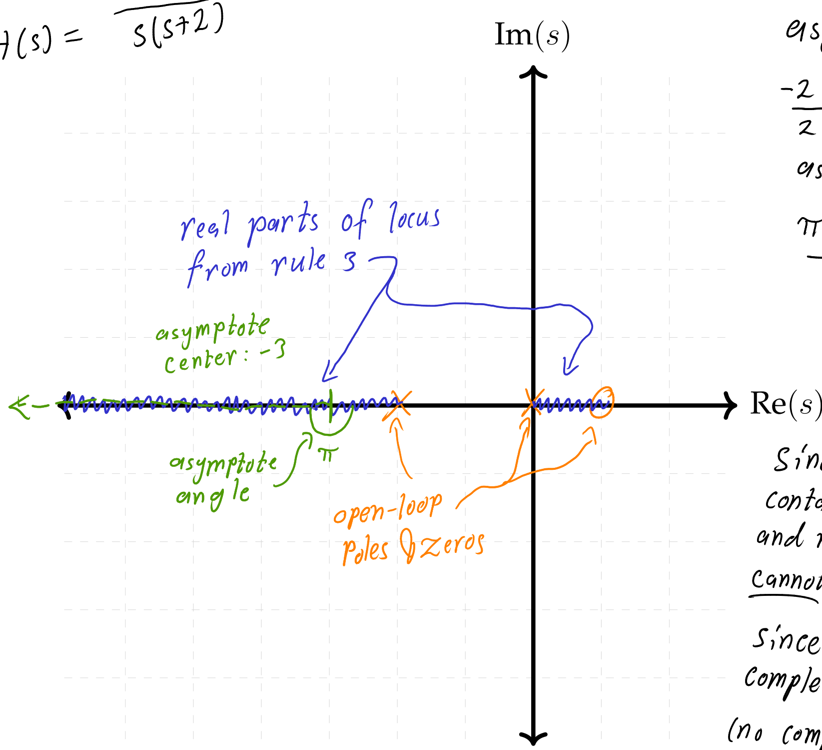
a) $a = 1$;

b) $a = -1$;

c) $a = -2$;

d) $a = -3$.

(a) $H(s) = \dfrac{s-1}{s(s+2)}$



asymptote center:

$$\frac{-2-1}{2-1} = -3$$

asymptote angle:

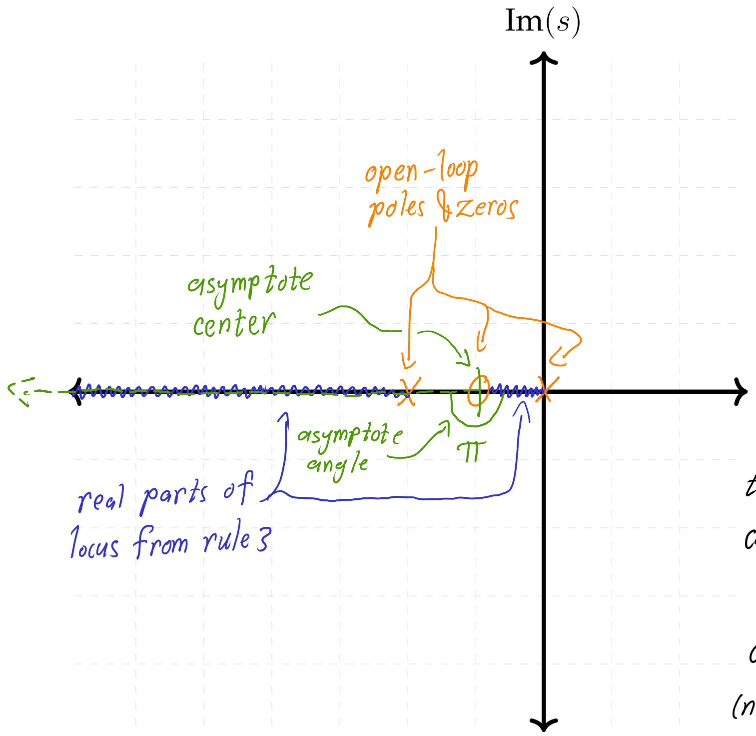
$$\frac{\pi + (0)2\pi}{2-1} = \pi$$

Since one branch is entirely contained in the imaginary axis and right half-plane, this system cannot be stabilized by feedback.

Since both branches lie completely on the real axis (no complex poles), this system will never oscillate.

[2] Adapted from Oppenheim & Willsky, problem 11.26.

(b) $H(s) = \dfrac{s+1}{s(s+2)}$



asymptote center:

$\dfrac{-2+1}{2-1} = -1$

asymptote angle:

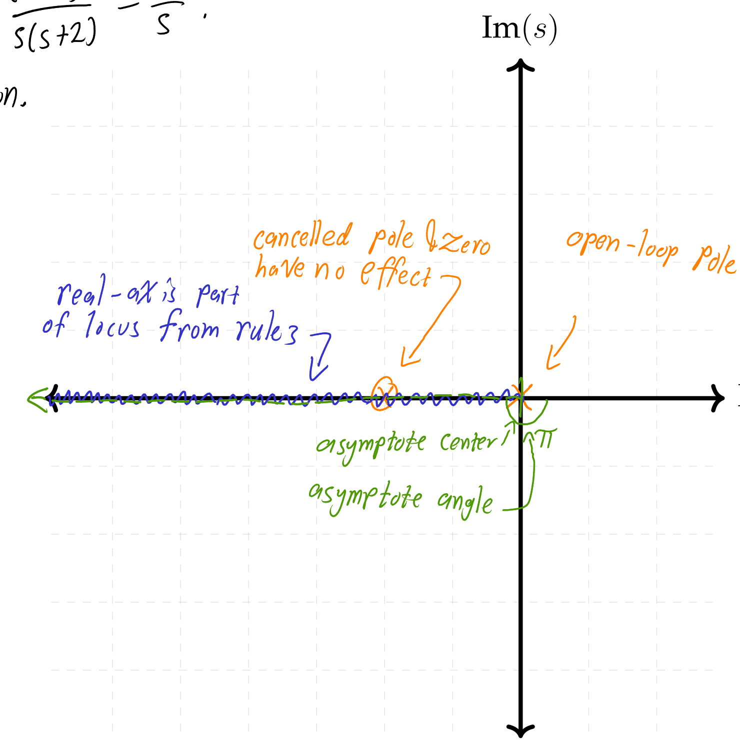
$\dfrac{\pi + (0)2\pi}{2} = \pi$

high-gain feedback sends poles to -1 and $-\infty$, so this system can be stabilized.

Since both branches lie completely on the real axis (no complex poles), this system will never oscillate.

(c) $H(s) = \dfrac{(s+2)}{s(s+2)} = \dfrac{1}{s}$.

Pole-zero cancellation.



asymptote center:

$\dfrac{(-2+0)-(-2)}{2-1} = 0$

asymptote angle:

$\dfrac{\pi + (0)2\pi}{2} = \pi$

feedback sends pole to $-\infty$, so this system can be stabilized.

Since both branches lie completely on the real axis (no complex poles), this syste will never oscillate.

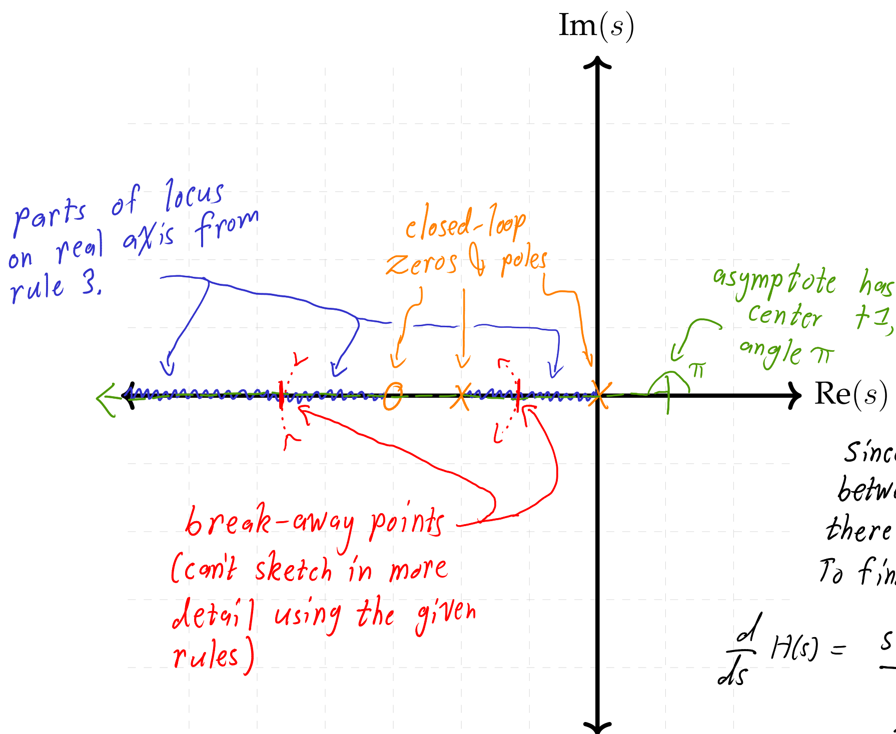
(d) $H(s) = \dfrac{s+3}{s(s+2)}$

asymptote center:

$$\frac{(-2+0)-(-3)}{2-1} = 1$$

asymptote angle:

$$\frac{\pi+(0)\pi}{2-1} = \pi$$



Since there's a region of locus between two real poles we know there are break-away point. To find them with rule 4:

$$\frac{d}{ds}H(s) = \frac{s(s+2)\frac{d}{ds}(s+3)-(s+3)\frac{d}{ds}(s(s+2))}{(s(s+2))^2}$$

$$= \frac{s^2+2s-(2s^2+8s+6)}{(s(s+2))^2}$$

$$= -\frac{s^2+6s+6}{(s(s+2))^2}$$

$\frac{d}{ds}H(s)=0$ at the roots of s^2+6s+6; quadratic formula yields $s=-3\pm\sqrt{3} \approx -4.7, -1.3$.

At high K the poles go to -3 and $-\infty$, so this system can be stabilized.

Since the locus breaks away into the complex plane, this system can have complex poles for certain K. Therefore, it can oscillate.