EE 120: Signals and Systems
Bonus Review: Root Locus
December 12, 2019

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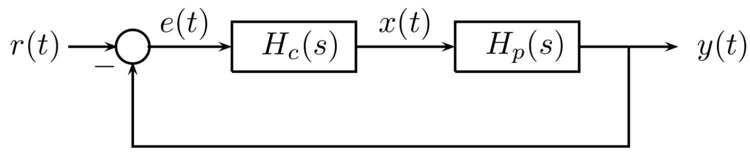
Suppose we have a feedback system like the one shown below:



in which we have taken a "plant" $H_p(s)$ and closed a feedback look around it with a controller $H_c(s)$. The transfer function of this "closed-loop" system is

$$H_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}.$$

We will choose a controller of the form $H_c(s) = KG(s)$, where K is the *controller gain*. One we have chosen a $G(s)$, the act of "designing" the controller will just be choosing K. We would like to know how our choice of K we can affect the closed-loop poles. This is equivalent to asking: What values can the roots of the equation

$$1 + KG(s)H_p(s) = 0$$

have for different values of K? The *root locus*, which is the set of all possible closed-loop poles for $K \in [0, \infty)$, answers this question.

Sketching the Root Locus

The root locus turns out to be a collection of line segments on the plane, called *branches*. There's one branch for each closed-loop pole. We can make a rough sketch of the branches with a few simple rules:

1. Each branch *starts* on a pole of $G(s)H_p(s)$. Each branch *ends* in one of two ways: on a zero of $G(s)H_p(s)$, or going to infinity.

2. The branches that go to infinity tend to straight-line asymptotes. The asymptotes all meet at the point $(\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{m} \beta_k)/(n-m)$, where α_k and β_k are the poles and zeros of $G(s)H_p(s)$, and leave this point at angles

   $(\pi + 2(k-1)\pi)/n - m$, $k = 1, \ldots, n-m$.

3. Parts of the real line to the left of an odd number of real poles and zeros of $H_p(s)$ must be on one of the branches.

4. Branches between two real poles must leave the real axis. These "breakaway points" occur when the branches meet, which will be at the roots of $\frac{d}{ds}(G(s)H_p(s)) = 0$. This is the only way that two branches can intersect.

5. The root locus is *conjugate symmetric*: if s is on a branch, so is s^*.

These rules all come from the properties of the equation $G(s)H_p(s) = -\frac{1}{K}$ when it is split into its magnitude and phase parts:

$$|G(s)H_p(s)| = \frac{\prod_{k=1}^{m} |s-\beta_k|}{\prod_{k=1}^{n} |s-\alpha_k|} = \frac{1}{K}$$
$$\angle G(s) + \angle H_p(s) = \sum_{k=1}^{m} \angle(s-\beta_i) - \sum_{k=1}^{n} \angle(s-\alpha_k) = -\pi + 2\pi w, \quad \text{for integer } w.$$

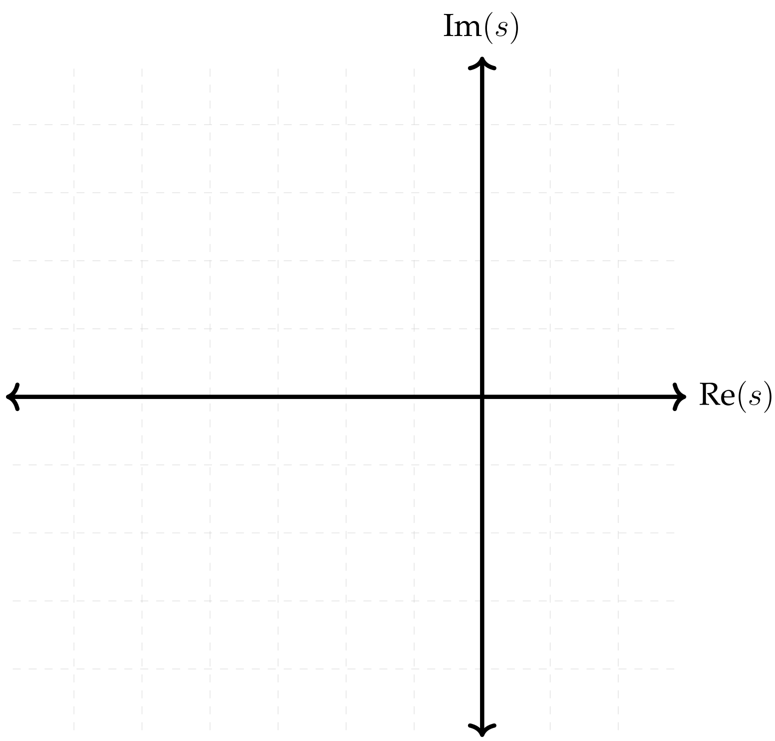
In addition, the "break-away point" condition in Rule 4 comes from the observation that if $G(s)H_p(s)$ has a repeated root, then $\frac{d}{ds}(G(s)H_p(s))$ has the same root. Proofs for the other rules are available in Lecture Note 19. More details and examples are available in Oppenheim & Willsky, Section 11.3.

Now, these 6 rules will allow you to make a pretty good sketch of the root locus. Specifically, you can get the starting points and the parts of the locus on the real line exactly right, and you can estimate where the branches will go for large values of K with the help of the asymptotes. You can also get the "break-away points" exactly right as well, but these rules don't tell you what happens after the branches leave the real axis.

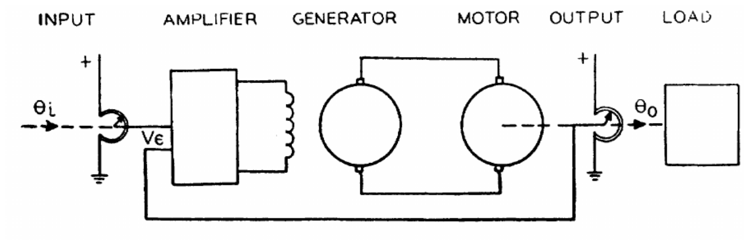
Problem 1: Let

$$H_p(s) = \frac{1}{s^2+2s-3}, \quad H_c(s) = K.$$

Sketch the root locus of the closed-loop system. Then, using the locus, find a value of K that stabilizes the system.



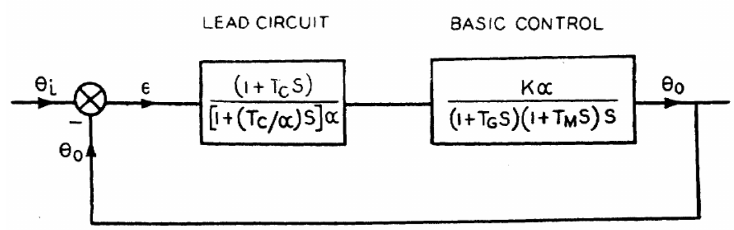
Problem 2: [1] Consider the motor servomechanism shown below:



This system's transfer function is

$$H_p(s) = \frac{1}{s(1+T_g s)(1+T_M s)},$$

with $T_G = \frac{1}{4}$, $T_m = 1$. The pole at zero models a *time delay* in the system, meaning that the servo will take some time to respond to a command. To get rid of the time delay, we will put the system in the following feedback loop:



where we've used a *lead controller* of the form

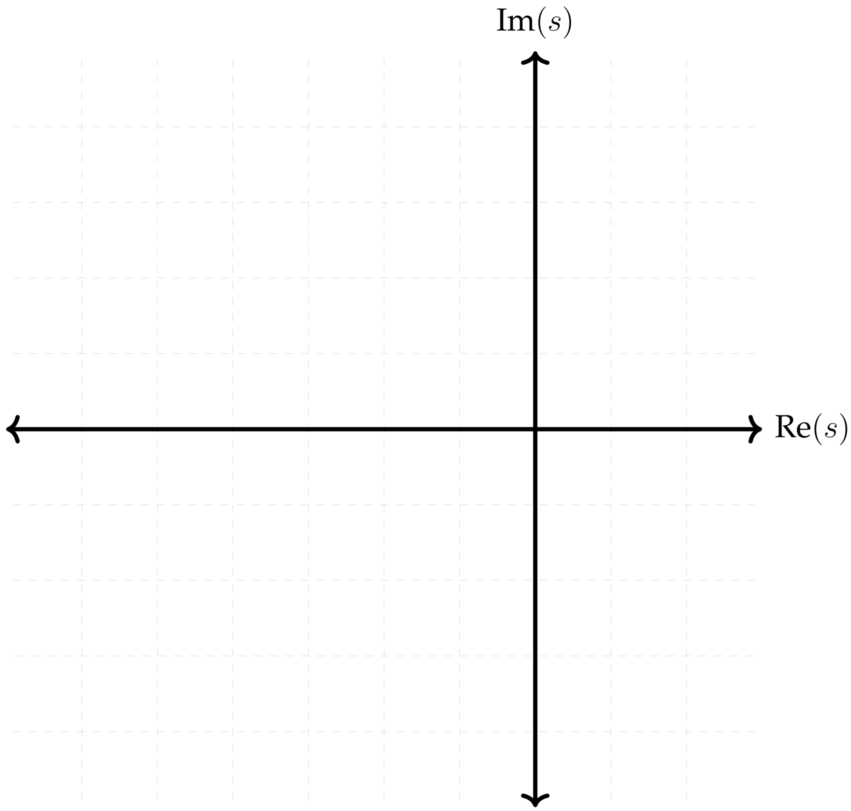
$$H_c(s) = K\frac{1+T_c s}{(1+(T_c/\alpha)s)\alpha}, \tag{1}$$

with $T_c = \frac{1}{2}$, $\alpha = 3$.

For this closed-loop system, sketch the part of the root locus *on the real axis*, and sketch the asymptotes.

Can a high K destabilize this system?

[1]Adapted from an example in Evans, "Graphical Analysis of Control Systems" (1948). This is the first published use of root locus analysis.


Im(s)
Re(s)

Problem 3: [2] Consider a feedback system with

$$G(s)H_p(s) = \frac{(s-a)}{s(s+2)}.$$

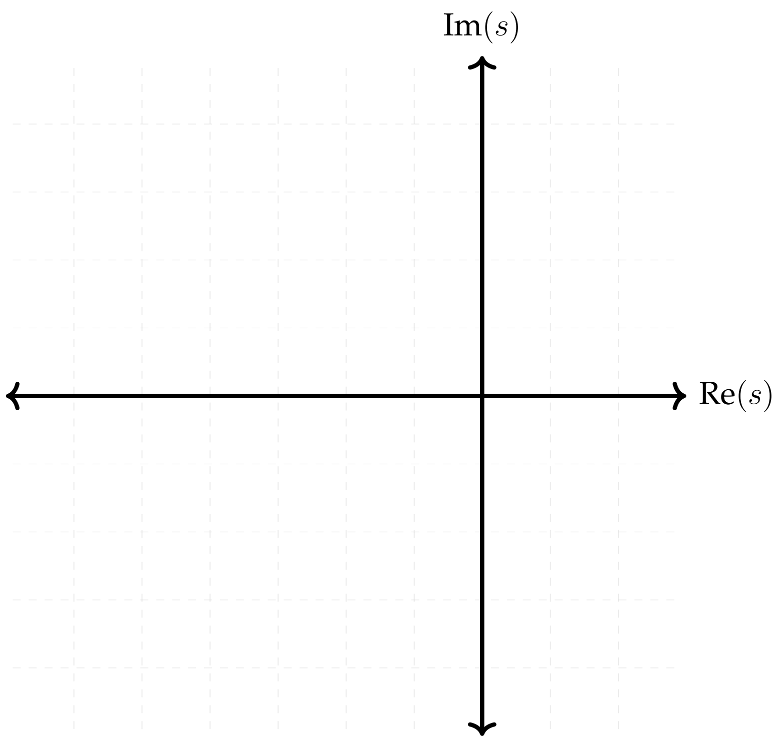
Sketch the root locus for the following values of a. For each value of a, answer: (i) Can the system be stabilized by feedback? (ii) Can certain feedback gains cause the system to exhibit oscillatory behavior?

a) $a = 1$;

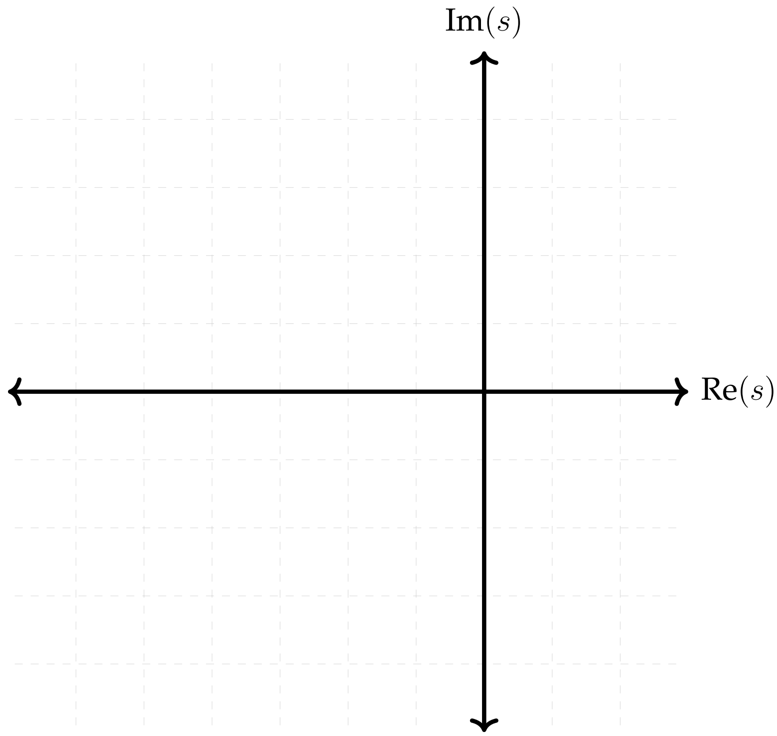
b) $a = -1$;

c) $a = -2$;

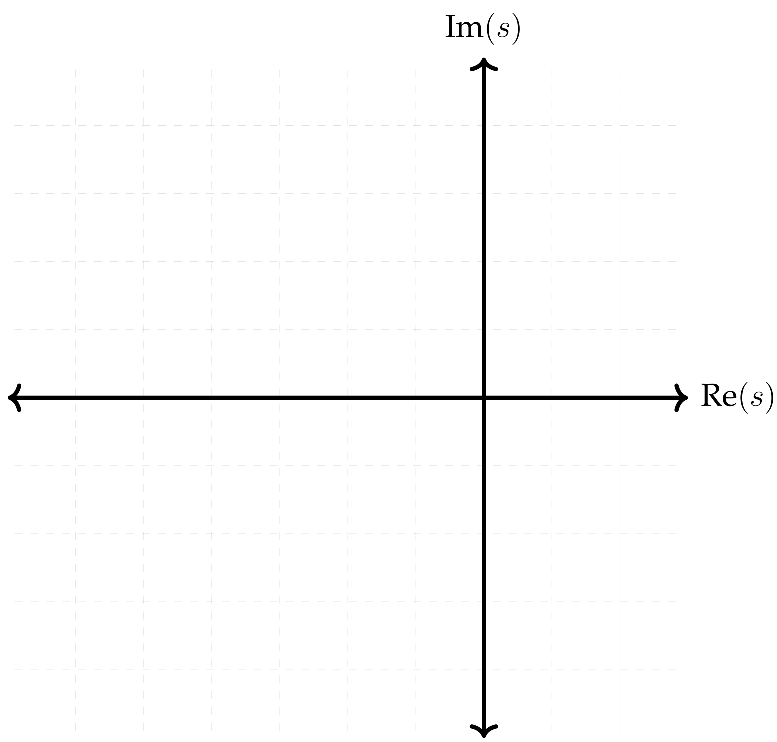
d) $a = -3$.



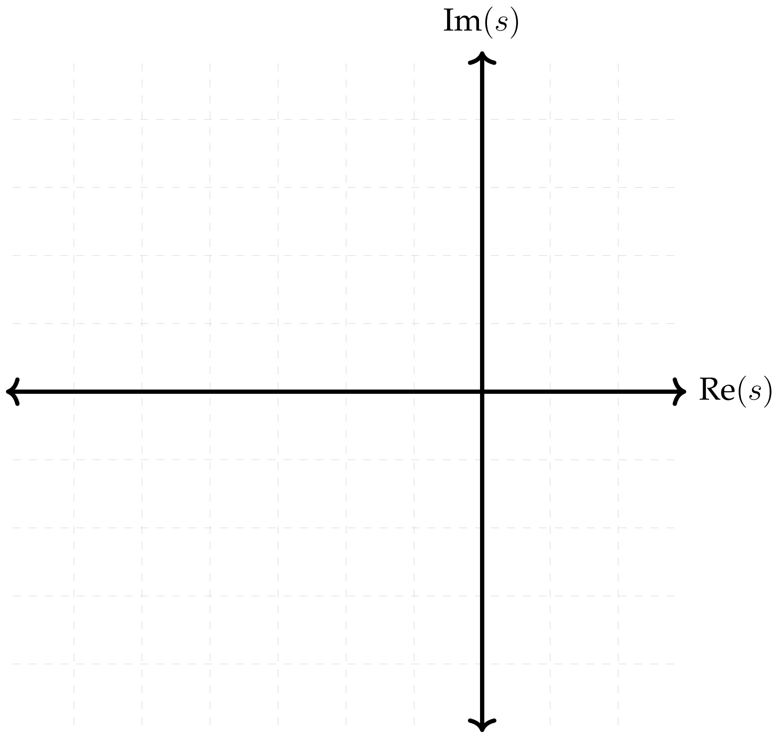
[2]Adapted from Oppenheim & Willsky, problem 11.26.



Im(s)
Re(s)



Im(s)
Re(s)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