EE120 - Fall'19 - Lecture 21 Notes[1]

Murat Arcak

19 November 2019

[1] Licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

The z-Transform

Chapter 10 *in* Oppenheim & Willsky

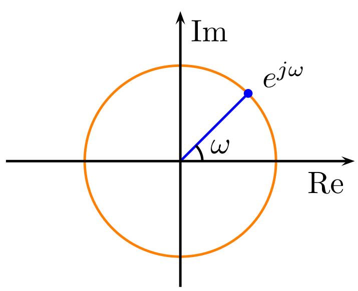
The z-Transform is defined as:

$$\boxed{X(z) := \sum_{n=-\infty}^{\infty} x[n] z^{-n}} \tag{1}$$

where $z \in \mathbb{C}$. It recovers the DTFT when z is on the unit circle, *i.e.*, when $z = e^{j\omega}$ for some $\omega \in [0, 2\pi)$:

$$\boxed{X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega}).} \tag{2}$$

Thus, the DTFT converges if the region of convergence (ROC) for the z-transform includes the unit circle.



<u>Example 1:</u> $x[n] = \delta[n] \;\rightarrow\; X(z) = 1$, ROC: entire complex plane

<u>Example 2:</u> $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} \quad \text{if } \underbrace{|az^{-1}| < 1}_{ROC:\ |z|>|a|}$$

Poles and zeros: $X(z) = \frac{z}{z-a} \rightarrow$ pole at $z = a$, zero at $z = 0$.

DTFT:

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad \text{if } |a| < 1$$

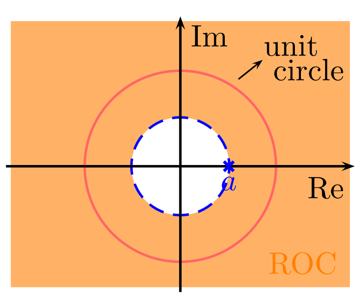
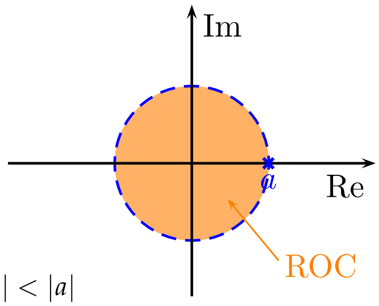


Figure 1: Former Berkeley EECS professors Lotfi Zadeh (above) and Eliahu Jury (below) were among those who developed the theory of z transforms in the 1950s. Research in sampling was partly motivated by radar which came to prominence during World War II.

Example 3: $x[n] = -a^n u[-n-1]$

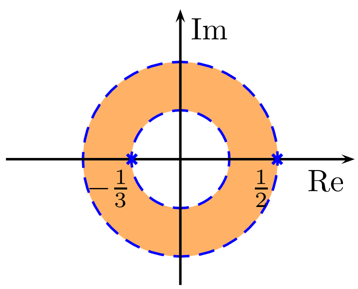
$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{n=1}^{\infty} -a^{-n} z^n \\ &= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n \\ &= 1 - \frac{1}{1-a^{-1}z} \quad \text{if } |a^{-1}z| < 1 \\ &= \frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}} \quad ROC: \; |z| < |a| \end{aligned}$$



DTFT converges is $|a| > 1$.

Example 4: $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(\frac{-1}{3}\right)^n u[n]$

$$X(z) = \underbrace{\underset{|z|<\frac{1}{2}}{\frac{1}{1-\frac{1}{2}z^{-1}}} + \underset{|z|>\frac{1}{3}}{\frac{1}{1+\frac{1}{3}z^{-1}}}}_{ROC:\ \frac{1}{3}<|z|<\frac{1}{2}} = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$



Example 5: $x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{-1}{3}\right)^n u[-n-1]$

$$ROC = \{z: \; |z| > \tfrac{1}{2}\} \cap \{z: \; |z| < \tfrac{1}{3}\} = \emptyset$$

Example 6: $x[n] = a^n$, $a \neq 0$.

$$x[n] = a^n u[n] + a^n u[-n-1]$$

$$ROC = \{z: \; |z| > a\} \cap \{z: \; |z| < a\} = \emptyset$$

Properties of the ROC

Section 10.2 *in* Oppenheim & Willsky

As seen in the examples above, the ROC is a ring or disk in the z-plane, centered at the origin. Note that it does not contain any poles. If $x[n]$ is right-sided (*e.g.*, Example 1) then the ROC extends from the outermost pole to ∞.

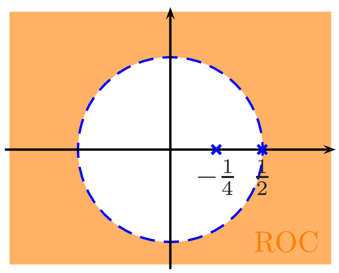
Inverse z-Transform by Partial Fraction Expansion

Example 7:

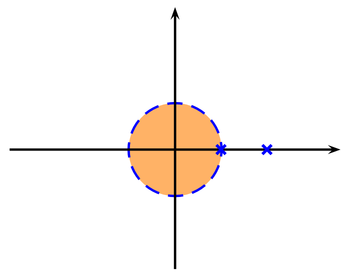
$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} = \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}$$

$$\begin{aligned} A_1 &= \left(1 - \frac{1}{4}z^{-1}\right) X(z)\big|_{z=\frac{1}{4}} = -1 \\ A_2 &= \left(1 - \frac{1}{2}z^{-1}\right) X(z)\big|_{z=\frac{1}{2}} = 2 \end{aligned}$$

1)

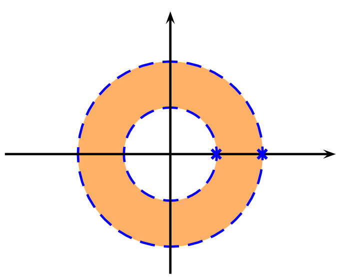


2)



$$x[n] = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u[n] \qquad x[n] = -\left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u[-n-1]$$

3)



$$x[n] = -2\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{1}{4}\right)^n u[n]$$

How to perform a PFE in general?

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}, \quad a_0 \neq 0$$

Suppose unrepeated poles: $d_1, d_2, ..., d_N$.

If $M < N$,

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

If $M \geq N$,

$$\begin{aligned} X(z) &= \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \\ &\Downarrow \\ x[n] &= \sum_{r=0}^{M-N} B_r \delta[n-r] + \underbrace{\sum_{k=1}^{N} A_k d_k^n u[n]}_{\text{if right-sided}} \end{aligned}$$

Example 8:

$$\begin{aligned} X(z) &= \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} \quad M=N=2 \\ &= B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}} \end{aligned}$$

Matching coefficients: $A_1 = -9,\ A_2 = 8, B_0 = 2$.

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

Differentiation (in z-domain) Property:

$$\begin{aligned} x[n] &\overset{\mathcal{Z}}{\longleftrightarrow} X(z) && ROC = R \\ nx[n] &\overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz} && ROC = R \end{aligned}$$

Proof: $$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[z]z^{-n} \\ \frac{dX(z)}{dz} &= \sum_{n=-\infty}^{\infty} -nx[n]z^{-(n+1)} = -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n} \\ &\sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -z\frac{dX(z)}{dz} \end{aligned}$$

Example 9:

$$\begin{aligned} a^n u[n] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}} \\ na^n u[n] &\overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}\left\{\frac{1}{1-az^{-1}}\right\} = z\frac{az^{-2}}{(1-az^{-1})^2} = \frac{az^{-1}}{(1-az^{-1})^2} \end{aligned}$$

Back to Partial Fraction Expansions: If d_k is a pole of multiplicity two, include two terms:

$$A_{k_1}\frac{1}{1-d_k z^{-1}} + A_{k_2}\frac{d_k z^{-1}}{(1-d_k z^{-1})^2}$$
$$\updownarrow$$
$$\left(A_{k_1} + A_{k_2} n\right) d_k^n u[n]$$

Example 10:
$$\begin{aligned} X(z) &= \frac{-\frac{1}{2}+z^{-1}}{(1-\frac{1}{2}z^{-1})^2} \qquad \begin{matrix} M=1 \\ N=2 \end{matrix} \qquad |z|>\frac{1}{2} \\ &= A_{11}\frac{1}{1-\frac{1}{2}z^{-1}} + A_{12}\frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \\ &= \frac{A_{11}+\frac{1}{2}\left(A_{12}-A_{11}\right)z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \end{aligned}$$

$$\left.\begin{aligned} A_{11} &= -\tfrac{1}{2} \\ \tfrac{1}{2}(A_{12}-A_{11}) &= 1 \end{aligned}\right\} A_{12} = \frac{3}{2}$$

$$x[n] = \left(-\frac{1}{2}+\frac{3}{2}n\right)\left(\frac{1}{2}\right)^n u[n]$$

Signal	Transform	ROC
$\delta[n]$	1	all z
$\delta[n-m]$	z^{-m}	all z except $z=0$ if $m>0$, all z except $z=\infty$ if $m<0$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert>1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert<1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert>a$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert<a$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert>a$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert<a$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>r$
$r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>r$

Table 1: z transforms of several functions.

Properties of the z-Transform

1) Linearity: $ax_1[n]+bx_2[n] \longleftrightarrow aX_1(z)+bX_2(z)$

ROC contains $R_1\cap R_2$ where R_i is the ROC of $x_i[n]$, $i=1,2$.

2) Time Shifting: $x[n-n_0] \longleftrightarrow z^{-n_0}X(z)$

ROC unchanged, except for possible addition/deletion of 0 and ∞.

Example 11: Find the inverse z-Transform (right-sided) of

$$\begin{aligned} X(z) &= \frac{1}{z^{-1}\left(1-\frac{1}{2}z^{-1}\right)} = z\frac{1}{1-\frac{1}{2}z^{-1}} \\ x[n] &= \left(\frac{1}{2}\right)^{n+1} u[n+1] \end{aligned}$$

3) Scaling in the z-domain:

$$z_0^n x[n] \xleftrightarrow{\mathcal{Z}} X\left(\frac{z}{z_0}\right) \quad \text{ROC} = |z_0| \cdot R$$

where R is the ROC of $x[n]$. Compare to:

$$\begin{aligned} e^{j\omega_0 n}x[n] &\xleftrightarrow{DTFT} X\left(e^{j(\omega-\omega_0)}\right) \\ e^{s_0 t}x(t) &\xleftrightarrow{\mathcal{L}} X(s-s_0) \quad \text{ROC} = R + Re\{s_0\} \end{aligned}$$

4) Time Reversal:

$$x[-n] \xleftrightarrow{\mathcal{Z}} X\left(\frac{1}{z}\right) \quad \text{ROC} = 1/R$$

Example 12:

$$\begin{aligned} x[n] = \left(\frac{1}{2}\right)^n u[n] &\quad \leftrightarrow \quad X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \\ x[-n] = 2^n u[-n] &\quad \leftrightarrow \quad X\left(\frac{1}{z}\right) = \frac{1}{1-\frac{1}{2}z} = \frac{-2z^{-1}}{1-2z^{-1}} \quad |z| < 2 \end{aligned}$$

5) Convolution Property:

$$x_1[n] * x_2[n] \xleftrightarrow{\mathcal{Z}} X_1(z)X_2(z) \quad \text{ROC contains } R_1 \cap R_2$$

6) Differentiation in z-domain:

$$nx[n] \xleftrightarrow{\mathcal{Z}} -z\frac{dX(z)}{dz} \quad \text{ROC unchanged}$$

Proof and example on page 4.

7) Initial Value Theorem: If $x[n] = 0$ for $n < 0$, then

$$\boxed{x[0] = \lim_{z\to\infty} X(z)}$$

Proof: $X(z) = x[0] + \underbrace{x[1]z^{-1} + x[2]z^{-2} + ...}_{\to 0 \text{ as } z\to\infty}$