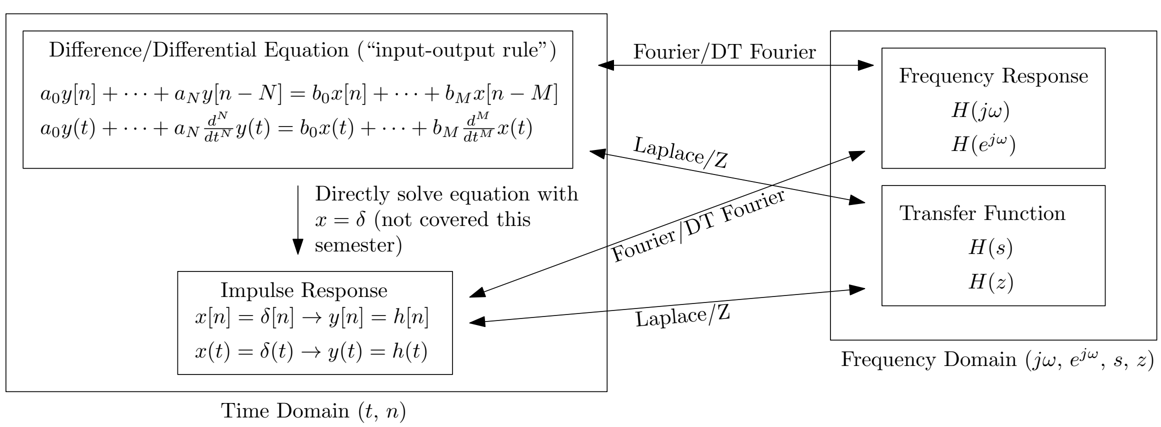
In this class, we've encountered a lot of ways to represent an LTI System, such as:

1. An *input-output rule* in the form of a linear differential equation (in continuous time) or difference equation (in discrete time) with constant coefficients;
2. An *impulse response*, a function of time which yields the output of the system when convolved with the input;
3. A *frequency response*, a rational function of an imaginary variable $j\omega$ (in continuous time) or $e^{j\omega}$ (in discrete time), which gives the output of the system when the input is a complex exponential;
4. A *transfer function*, a rational function of a complex variable s (in continuous time) or z (in discrete time) whose *poles* characterize the system behavior, especially its stability.

A system given in one form may be expressed in any other others using the transforms we have learned. This diagram shows how the representations and the transforms are related:



To go from the difference/differential equation representation to either of the frequency-domain transformations, you can just use the appropriate transform.

Going from the difference/differential equation to the impulse response is a little more tricky. There are two approaches. First, you can directly solve the equation with $x = \delta$ and using the method of homogeneous and particular solutions. On the other hand, You could first find the transfer function of the system using the Laplace transform or Z transform, solve for Y when $X = 1$, and take the inverse Laplace transform of Y to find h. Since we didn't really cover the direct method this semester, we recommend the Laplace/Z method.

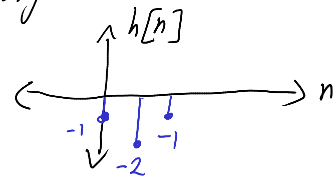
Problem 1: [1] Consider an LTI system defined by the difference equation

$$y[n] = -x[n] - 2x[n-1] - x[n-2].$$

a) Determine the impulse response of this system.

Since there's only one y term (no $y[n] + y[n-1]$ or anything like that), we can substitute $x = \delta$ to get h directly.

$$\boxed{h[n] = -\delta[n] - 2\delta[n-1] - \delta[n-2]}$$



b) Determine if the system is causal and/or stable.

$h[n] = 0$ for all $n < 0$, so $\boxed{\text{it's causal.}}$

The impulse response is finite in length, meaning

$\sum_{i=-\infty}^{\infty} |h[i]| < \infty$, so $\boxed{\text{it's stable.}}$

c) Find the frequency response $H(e^{j\omega})$.

We apply the DTFT to the difference equation, and use the time-shift property ($x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$):

$$Y(e^{j\omega}) = -X(e^{j\omega}) - 2e^{-j\omega} X(e^{j\omega}) - e^{-j2\omega} X(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = -1 - 2e^{-j\omega} - e^{-2j\omega} = -e^{-j\omega}(e^{j\omega} + 2 + e^{-j\omega})$$

$$= -e^{-j\omega}(e^{j\omega/2} + e^{-j\omega/2})^2 = -e^{-j\omega}(2\cos(\tfrac{\omega}{2}))^2 = \boxed{-4e^{-j\omega}\cos^2(\omega/2)}.$$

d) Determine the response of this system when the input is $x[n] = 1 + (-1)^n$. Your answer should be of the form $y[n] = a + b(-1)^n$.

$1 = e^{j0n}$; $(-1)^n = e^{j\pi n}$; $x[n] = e^{j0n} + e^{j\pi n}$, so

$$y[n] = H(e^{j0})e^{j0n} + H(e^{j\pi})e^{j\pi n} = (-4)e^{j0n} + (-4(0))e^{j\pi n}$$

$$= -4$$

$$\boxed{= -4 - 0 \times (-1)^n, \text{ i.e. } a = 4, b = 0}.$$

[1] Adapted from EE120 Midterm 1, Spring 2013

Problem 2: Consider an LTI system defined by the differential equation

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t). \tag{1}$$

a) Find the impulse response and transfer function of this system.

Applying bilateral Laplace transform yields

$$s^2Y(s) + 3sY(s) + 2Y(s) = X(s) = (s^2+3s+2)Y(s)$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \boxed{H(s) = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)}}$$

To find impulse response, we'll find the partial fraction expansion and take the inverse Laplace transform:

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{1}{s+1} - \frac{1}{s+2};$$

To find A & B:

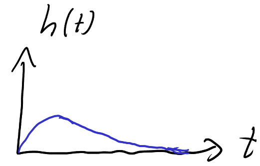
$A(s+2) + B(s+1) = 1$

$s=-1 \to A=1$

$s=-2 \to -B=1$

Inverse Laplace gives

$$\boxed{h(t) = (e^{-t} - e^{-2t})u(t)}.$$



b) Find the response of this system to $x(t) = e^t u(t)$.

Laplace again: $s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s-1}$

$$\Rightarrow Y(s) = \frac{1}{(s-1)(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-1} = \frac{-1/2}{s+1} + \frac{1/3}{s+2} + \frac{1/6}{s-1}$$

To find $A, B,$ & C:

$A(s+2)(s-1) + B(s+1)(s-1) + C(s+1)(s+2) = 1$

$s=1 \to C(2)(3) = 6C = 1 \to C = 1/6$

$s=-2 \to B(-1)(-3) = 3B = 1 \to B = 1/3$

$s=-1 \to A(1)(-2) = -2A = 1 \to A = -\frac{1}{2}$

Inverse Laplace yields

$$\boxed{y(t) = \left(-\frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t} + \frac{1}{6}e^{t}\right)u(t)}.$$

First, $\frac{2s+3}{s^2+6s+9} = \frac{2s+3}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2} = \frac{2}{s+3} - \frac{3}{(s+3)^2}$

To find A & B:

$A(s+3)+B = 2s+3$

matching terms: $A=2$

$3A+B=3$

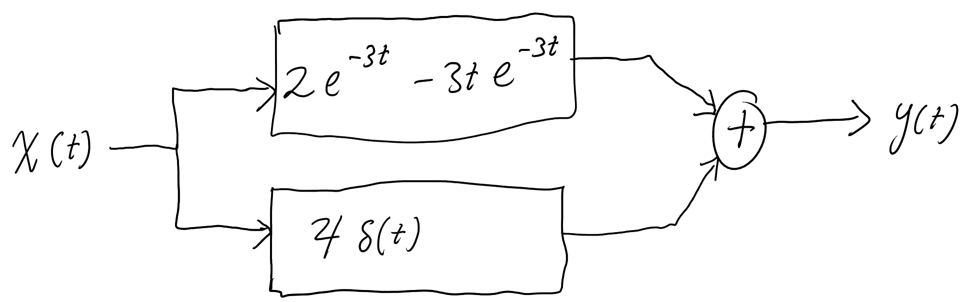
$\rightarrow B=-3$

$\Rightarrow H(s) = 4 + \frac{2}{s+3} - \frac{3}{(s+3)^2}$.

Inverse Laplace yields

$$\boxed{h(t) = 4\delta(t) + \left(2e^{-3t} - 3te^{-3t}\right)u(t)}.$$

(Recall $\frac{1}{(s+a)^2} \longleftrightarrow te^{-at}u(t)$)



We can rewrite $H(s)$ as

$$H(s) = 4 + \frac{2s+3}{(s+3)^2} + \frac{4(s+3)^2+2s+3}{(s+3)^2},$$

so both of the system's poles are at $s=-3$. $\boxed{\text{The system is stable.}}$

The effect of the constant term just turned out to be adding a zero.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1-1.25e^{-j\omega}}{1-0.8e^{-j\omega}}.$$

cross-multiplying yields

$$Y(e^{j\omega})(1-0.8e^{-j\omega}) = X(e^{j\omega})(1-1.25e^{-j\omega})$$

$$\rightarrow \quad Y(e^{j\omega}) - 0.8e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) - 1.25e^{-j\omega}X(e^{j\omega}).$$

Inverse DTFT (using the time-shift property) yields

$$\boxed{y[n] - 0.8y[n-1] = x[n] - 1.25x[n-1]}.$$

(Recall $x[n-n_0] \longleftrightarrow e^{-j\omega n_0}X(e^{j\omega})$)

$$H(e^{j\omega}) = \frac{1-1.25e^{-j\omega}}{1-0.8e^{-j\omega}} = \frac{1}{1-0.8e^{-j\omega}} - 1.25e^{-j\omega}\frac{1}{1-0.8e^{-j\omega}}.$$

Inverse DTFT yields

$$\boxed{h[n] = (0.8)^n u[n] - 1.25(0.8)^{n-1}u[n-1]}.$$

(Recall $\frac{1}{1-ae^{-j\omega}} \longleftrightarrow a^n u[n]$)

Laplace transform:

$$S(s) = \frac{1}{s} - \frac{1/2}{s+1} + \frac{1/2}{s+2} = \frac{(s+1)(s+2) - \frac{1}{2}(s+2) + \frac{1}{2}(s+1)}{s(s+1)(s+2)}$$

$$= \frac{s^2 + \frac{5}{2}s + 2}{s(s+1)(s+2)} ;$$

The transfer function is just the ratio of the output to the input. The input was $u(t)$, whose Laplace transform is $\frac{1}{s}$, so

$$\boxed{H(s) = \frac{S(s)}{\left(\frac{1}{s}\right)} = \frac{s^2 + \frac{5}{2}s + 2}{(s+1)(s+2)}} .$$

(the zeros are complex, so I left them unfactored.)

To find the impulse response, we start with partial fractions:

$$\frac{s^2 + \frac{5}{2}s + 2}{(s+1)(s+2)} = A + \frac{B}{s+1} + \frac{C}{s+2} = 1 + \frac{1/2}{s+1} - \frac{1}{s+2} .$$

I knew to add this because num. and den. are of the same degree

To find A, B, & C:

$A(s+1)(s+2) + B(s+2) + C(s+1) = s^2 + \frac{5}{2}s + 2.$

matching s^2 terms gives $A = 1$.

$s = -2 \rightarrow -C = 4 - 5 + 2 = 1 \rightarrow C = -1$

$s = -1 \rightarrow B = 1 - \frac{5}{2} + 2 = \frac{1}{2}$

Inverse Laplace yields

$$\boxed{h(t) = \delta(t) + \left(\frac{1}{2}e^{-t} - e^{-2t}\right)u(t)} .$$