

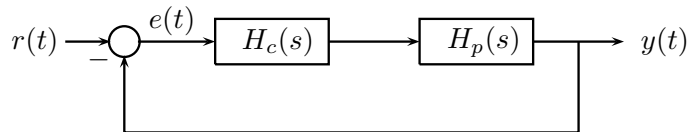
# EE120 - Fall'19 - Lecture 20 Notes<sup>1</sup>

Murat Arcak

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## Steady State Tracking Accuracy



$$e(t) = r(t) - y(t)$$

$$E(s) = R(s) - Y(s)$$

$$= R(s) - \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}R(s) = \frac{1}{1 + H_c(s)H_p(s)}R(s)$$

Suppose  $r(t)$  is a unit step. How do we guarantee  $e(t)$  converges to zero instead of a different constant?

$$R(s) = \frac{1}{s} \implies E(s) = \frac{1}{1 + H_c(s)H_p(s)} \cdot \frac{1}{s}$$

Final Value Theorem<sup>2</sup>:

$$e_{ss} := \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \frac{1}{1 + H_c(0)H_p(0)}$$

To ensure  $e_{ss} = 0$ , we need  $\lim_{s \rightarrow 0} H_c(s)H_p(s) = \infty$ , i.e.,

$H_c(s)H_p(s)$  must have at least one pole at  $s = 0$ .

<sup>2</sup> From Lecture 15:

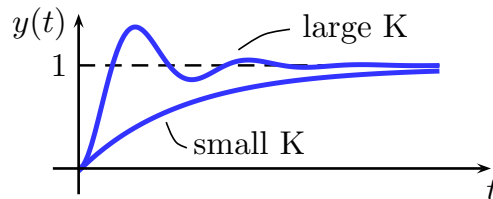
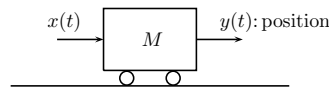
$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$$

if  $x(t)$  has a finite limit as  $t \rightarrow \infty$  and  $x(t) = 0$  for  $t < 0$ .

### Example 1: Position control

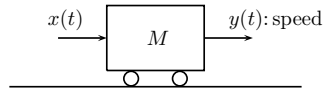
$$H_p(s) = \frac{1}{Ms^2 + bs} \quad H_c(s) = K$$

$$H_c(s)H_p(s) = \frac{K}{s(Ms + b)} \rightarrow e_{ss} = 0$$



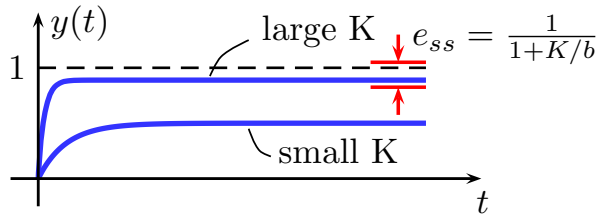
Example 2: Speed control

$$H_p(s) = \frac{1}{Ms + b} \quad H_c(s) = K$$

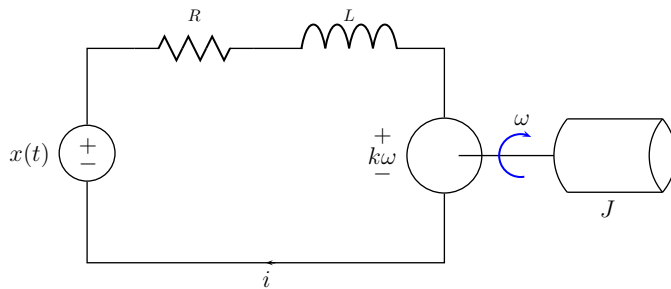


$$e_{ss} = \frac{1}{1 + H_c(0)H_p(0)} = \frac{1}{1 + K/b} \neq 0 \text{ if } b \neq 0$$

$$y_{ss} = 1 - e_{ss} = 1 - \frac{1}{1 + K/b} = \frac{K/b}{1 + K/b}$$



Steady-state error decreases with increasing  $K$ , but increasing  $K$  is not always a viable approach (poor damping if  $\# \text{poles} - \# \text{zeros} = 2$ , instability if  $\# \text{poles} - \# \text{zeros} \geq 3$ ).

Example 3: Speed control of a DC motor

Suppose we want to control  $y(t) = \omega$  (angular velocity).

First, find the transfer function  $H_p(s)$  from state-space model:

$$\begin{aligned} J \frac{d\omega(t)}{dt} &= k i(t) \\ L \frac{di(t)}{dt} &= -k\omega(t) - Ri(t) + x(t). \end{aligned}$$

Define the state vector  $\vec{z} := \begin{bmatrix} \omega \\ i \end{bmatrix}$  and rewrite the equations above as

$$\frac{d}{dt} \vec{z}(t) = A \vec{z}(t) + B x(t) \quad y(t) = C \vec{z}(t) \quad (1)$$

where

$$A = \begin{bmatrix} 0 & \frac{k}{J} \\ -\frac{k}{L} & -\frac{R}{L} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

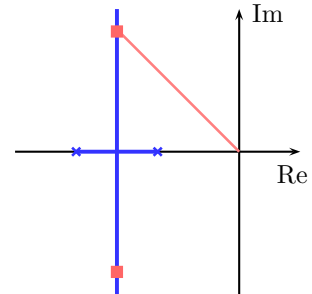
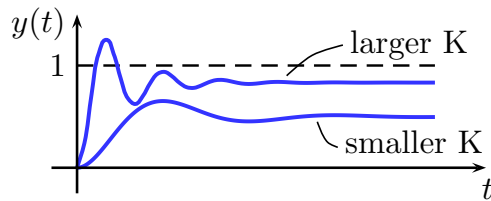
Then, using the formula from Lecture 17,

$$H_p(s) = C(sI - A)^{-1}B = \frac{Y(s)}{X(s)} = \frac{k}{JLs^2 + JRs + k^2}.$$

Constant gain control  $H_c(s) = K$  gives nonzero steady-state error:

$$e_{ss} = \frac{1}{1 + KH_p(0)} = \frac{1}{1 + \frac{K}{k}} \neq 0$$

Increasing the gain reduces  $e_{ss}$ , but leads to a poorly damped system:

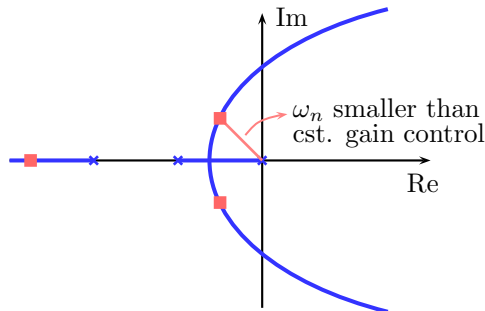


### Integral Control

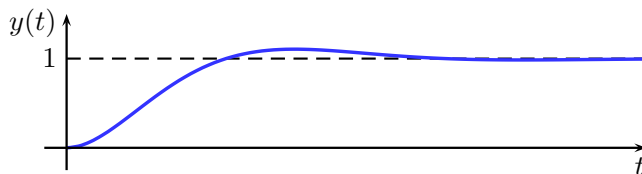
If  $H_p(s)$  does not contain a pole at  $s = 0$ , introduce one in  $H_c(s)$  to achieve zero steady-state tracking error.

The drawback, however, is that adding a pole at  $s = 0$  makes it harder to meet damping and natural frequency specifications.

For example, in the motor control problem above, integral control  $H_c(s) = \frac{K}{s}$  results in the following root locus with respect to gain  $K$ :



This means that  $e_{ss} = 0$  is achieved, but at the cost of slower response (smaller  $\omega_n$ ):

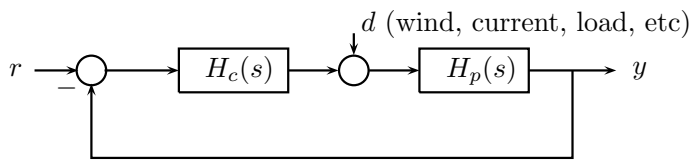


To achieve a faster response we can augment integral control with lead control:

$$H_c(s) = \frac{K}{s} \frac{s - \beta}{s - \alpha} \quad \alpha < \beta < 0.$$

The main features of this controller are similar to PID (proportional-integral-derivative) control which is very popular in industry.

### Disturbance Rejection

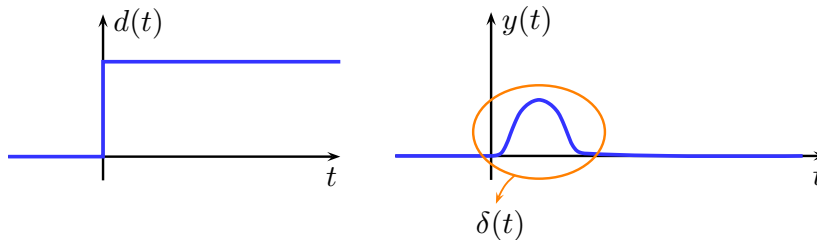


$$Y(s) = H_p(s) (H_c(s)(R(s) - Y(s)) + D(s))$$

$$(1 + H_c(s)H_p(s)) Y(s) = H_c(s)H_p(s)R(s) + H_p(s)D(s)$$

$$Y(s) = \underbrace{\frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}}_{\triangleq Y_{\text{nominal}}(s)} R(s) + \underbrace{\frac{H_p(s)}{1 + H_c(s)H_p(s)}}_{\triangleq \Delta(s)} D(s) \quad (2)$$

Suppose  $d(t) = u(t)$ : unit step. How do we guarantee  $y(t)$  recovers from this disturbance; that is,  $\delta(t) \triangleq y(t) - y_{\text{nominal}}(t) \rightarrow 0$ ?



$$\Delta(s) = \frac{H_p(s)}{1 + H_c(s)H_p(s)} \frac{1}{s}$$

$$\lim_{t \rightarrow \infty} \delta(t) = \lim_{s \rightarrow 0} s \Delta(s) = \lim_{s \rightarrow 0} \frac{H_p(s)}{1 + H_c(s)H_p(s)}$$

$$= 0 \text{ if } H_c(s) \text{ has a pole at } s = 0.$$

Thus, integral control achieves rejection of constant disturbances as well.

## Moving Beyond Step Inputs

So far we have discussed the case where the reference and the disturbance signals are constant, step inputs. How do we judge how well we track other reference signals and reject other disturbance signals? To answer this question we first rewrite (2) as

$$Y(s) = H_{r \rightarrow y}(s)R(s) + H_{d \rightarrow y}(s)D(s)$$

where

$$H_{r \rightarrow y}(s) := \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)} \quad H_{d \rightarrow y}(s) := \frac{H_p(s)}{1 + H_c(s)H_p(s)}.$$

Then, the Fourier transforms of reference, disturbance and output are related by:

$$Y(j\omega) = H_{r \rightarrow y}(j\omega)R(j\omega) + H_{d \rightarrow y}(j\omega)D(j\omega)$$

and the tracking and disturbance rejection objectives can be stated as:

**Tracking:** We want  $H_{r \rightarrow y}(j\omega) \approx 1$ , that is,

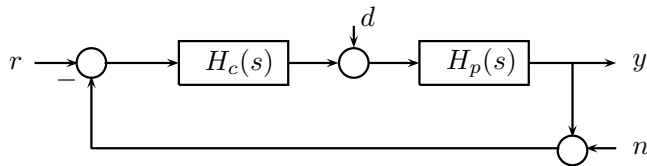
$$|H_c(j\omega)H_p(j\omega)| \gg 1. \quad (3)$$

**Disturbance rejection:** We want  $H_{d \rightarrow y}(j\omega) \approx 0$ . This is the case if, in addition to (3),

$$|H_c(j\omega)| \gg 1.$$

Note that in the previous sections we achieved these goals at  $\omega = 0$ : for tracking we demanded that  $H_c(s)H_p(s)$  have a pole at  $s = 0$  so  $H_{r \rightarrow y}(0) = 1$ , and for disturbance rejection we required that  $H_c(s)$  have a pole at  $s = 0$  so  $H_{d \rightarrow y}(0) = 0$ .

## Insensitivity to Noise



Now let's consider the effect of measurement noise,  $n(t)$ , as shown in the block diagram above. The transfer function from the noise to the output is (show this):

$$H_{n \rightarrow y}(s) := -\frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}.$$

Thus, to make the output  $y$  insensitive to the measurement noise  $n$ , we require  $H_{n \rightarrow y}(j\omega) \approx 0$ , that is

$$|H_c(j\omega)H_p(j\omega)| \ll 1. \quad (4)$$

Note that this requirement contradicts (3) which we demanded for tracking and disturbance rejection. Fortunately, references and disturbances are typically low-frequency signals whereas noise is dominant at high frequencies. Thus, a reasonable strategy is to design the controller  $H_c(s)$  such that (3) holds at low frequencies and (4) holds at high frequencies.

### *Some History: Black's Feedback Amplifier*

In its early days Bell Labs developed amplifiers that enabled long distance telephone communication. However, the amplifiers had significant variations in their gains and their nonlinearity caused interference between the channels. Addressing these problems Harold Black introduced a negative feedback around the amplifier that both reduced the variations in the gains and extended the linear range.

We illustrate these benefits on a static model of the amplifier in the figure below. Suppose the amplifier has gain  $\mu$  in its linear range and the output saturates at  $\pm 1$ . When a negative feedback with gain

$$\beta \gg \mu^{-1}$$

is applied, the relationship between the new input  $\tilde{x}$  and the output  $y$  is again a saturation nonlinearity (show this), but the new gain is

$$\frac{\mu}{1 + \beta\mu} \approx \beta^{-1}$$

which is robust to variations in  $\mu$ . In addition, the response is linear when  $|\tilde{x}| \leq \frac{1+\beta\mu}{\mu} \approx \beta$ , a significantly wider range than  $|x| \leq \mu^{-1}$ .

The drawback is that the amplifier gain is now significantly reduced, as Black explains in his 1934 paper in Bell System Technical Journal:

"... by building an amplifier whose gain is deliberately made say 40 decibels higher than necessary and then feeding the output back on the input in such a way as to throw away excess gain, it has been found possible to effect extraordinarily improvement in constancy of amplification and freedom from nonlinearity."

