First Year Special Relativity – Lecture 5
Space-time diagrams and world lines

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1 In this lecture

- A way of visualising Relativisitic events - space-time diagrams;
- How to draw the trajectory of objects in such diagrams - world lines;
- Transforming between frames in such diagrams.

2 Introduction

The material in this lecture is not covered in Young and Freedman and in McCall only briefly in Sec. 6.1.

In the previous lecture, we say how the Lorentz transformations change a given position in space and time, called an ‘event’, from one inertial frame to another. This can be represented graphically in a ‘space-time’ diagram. The trajectory of an object can be considered as a sequence of events. These events form a ‘world line’ for the object, which is the graphical representation of the trajectory in a space-time diagram.

3 Space-time diagrams

A space-time diagram is simply a visual illustration of the ct and x values of events. By convention, the axes are drawn with the ct axis vertical and x axis horizontal. An event is a single point in such a diagram. A space-time diagram was already included in Lecture 4, which showed the lines along which events will move when they are Lorentz transformed.

The Lorentz transformations defined in Lecture 4 are for passive transformations, where the axes are changing, not the events themselves. This is equivalent to a passive rotation; all we are doing is changing the coordinate system, not the physical object. We saw under a passive rotation that the x and y axes rotate by the same angle. This is not the same for a Lorentz transformation, but we can find how the ct and x axes change under a Lorentz transformation in a space-time diagram quite easily. Specifically, any event on the x' axis by definition has $t' = 0$. This means all such events must satisfy

$$ct' = 0 = \gamma(ct - \beta x) \qquad \text{so} \qquad ct = \beta x$$

This is a straight line in the space-time diagram with gradient β, i.e. it makes an angle α to the original x axis, where $\tan\alpha = \beta$. Similarly, any point on the ct' axis has $x' = 0$ so now all such events must satisfy

$$x' = 0 = \gamma(x - \beta ct) \qquad \text{so} \qquad ct = \left(\frac{1}{\beta}\right) x$$

which is a straight line with a gradient of $1/\beta$. Since $1/\tan\alpha = \tan(\pi/2 - \alpha)$, this is an angle of $\pi/2 - \alpha$ to the x axis, which corresponds to an angle of α to the original ct axis. Note that the gradient in both cases is positive so this does *not* look like the rotation case and the two axes are not perpendicular to each other. Fig. 1 compares the two types of transformation.

You are probably not familiar with how to resolve a vector in non-perpendicular coordinates into its components. The trick is to always move parallel to the ‘other’ axis. Hence, you can see

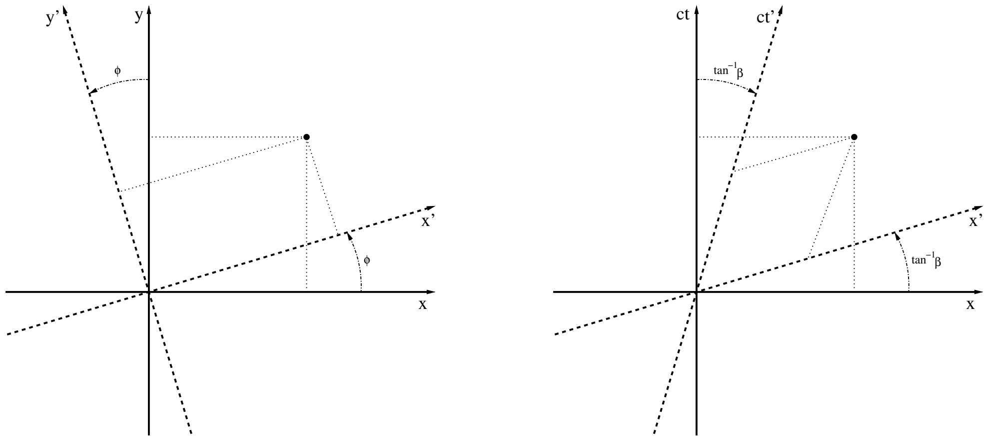


Figure 1: Left: Passive rotation by ϕ of the coordinate system with a given space position. Right: Passive Lorentz transformation by β of the coordinate system with a given space-time event.

in the figure how the new components for the given event correspond to those the second observer will measure. That this is required to resolve components is straightforward to understand. As stated above, any event on the x' axis has $t' = 0$, i.e. the x' axis is a line of constant t', in this case $t' = 0$. Lines of constant t' with values other than 0 will also be parallel to the x' axis but higher or lower, depending on the constant value, as shown in Fig. 2. Any line of constant t' (or t) is referred to as a 'line of simultaneity', because all events on that line are simultaneous in the primed (or unprimed) frame.

In a similar way, lines of constant x' are all parallel to the ct' axis, with the line $x' = 0$ actually corresponding to the ct' axis itself, also shown in Fig. 2. Note that any object at rest in the primed frame will by definition have the same x' value for all times. Hence lines of constant x' correspond to objects at rest in the transformed frame.

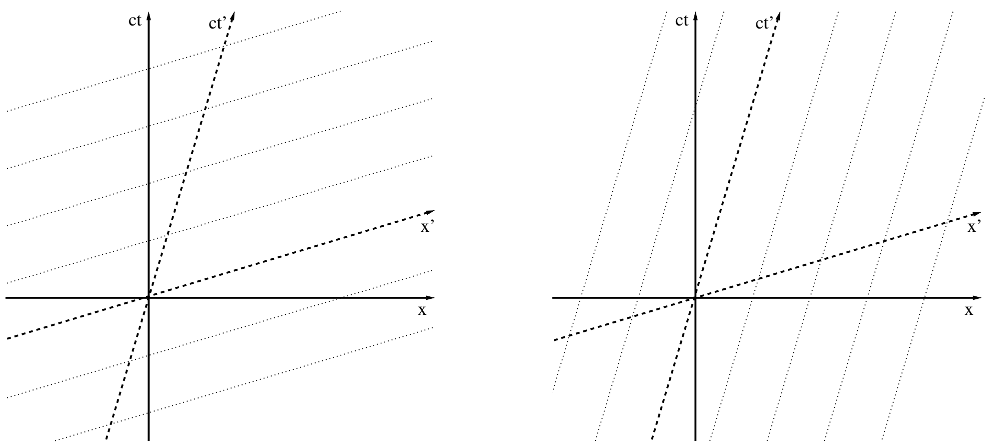


Figure 2: Left: Lines of constant t', also called lines of simultaneity. Right: Lines of constant x', corresponding to world lines of objects at rest in the transformed frame.

Of course, the second observer always considers their axes to be perpendicular and so will view the situation as shown in the right-hand diagram of Fig. 3. From the perspective of the second observer, the original frame coordinates both have negative gradients, of $-\beta$ and $-1/\beta$, as this figure shows. This view is of course equivalent to an inverse Lorentz transformation, just as it was for rotations.

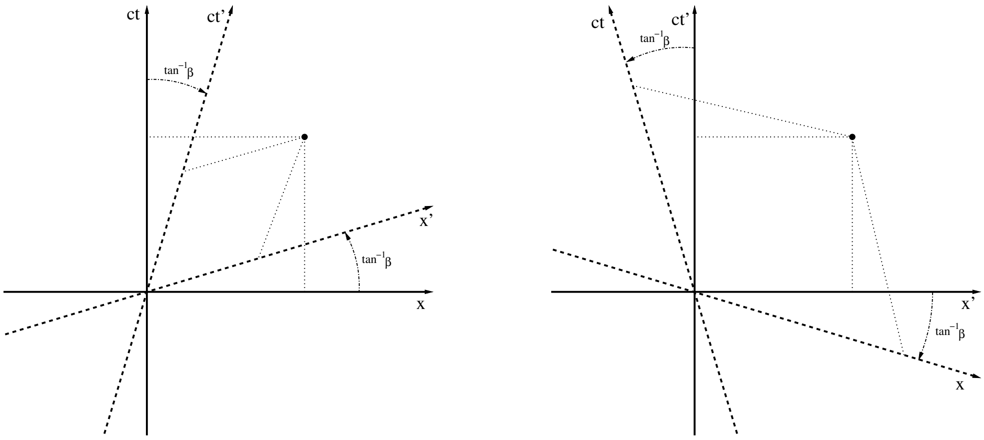


Figure 3: Left: Passive Lorentz transformation by β of the coordinate system with a given space-time event (as previously shown in Fig. 1). Right: View of same transformation from the perspective of an observer in the transformed frame.

4 World lines

You will have solved mechanics problems involving Newton's laws for the position of an object as a function of time, i.e. $x(t)$. It is common to plot the x as a function of t, for example as shown in the left of Fig. 4. For any point on the curve, the derivative of the function gives the velocity, while the second derivative gives the acceleration. Hence, an object without any acceleration has a straight line function (so the second derivative is zero) while a stationary object has a horizontal line (so the first derivative is also zero). The same type of graph is very useful in Relativity also. However, as we saw above, in Relativity space-time diagrams are standardly drawn with the axes swapped over; i.e. the ct axis is vertical and the x axis is horizontal, also shown in Fig. 4.

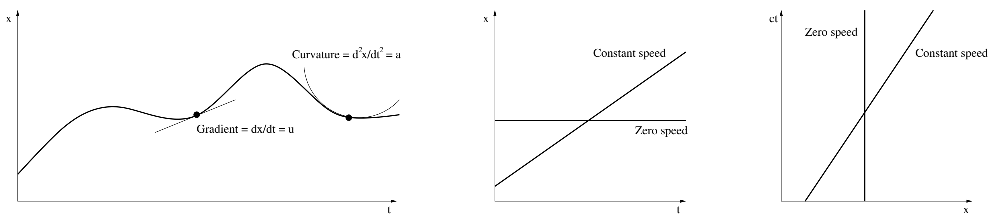


Figure 4: Left: Example of solution $x(t)$ for a mechanics problem, showing velocity and acceleration. Middle: Lines for zero and non-zero constant velocity. Right: Lines for zero and non-zero constant velocity in a space-time diagram, which has the axes exchanged.

Any object which exists for a finite time will form a line in the diagram, just like the one described above; this called the 'world line' of the object. In this case, because the axes are swapped, then a stationary object will give a vertical line while an object moving at constant velocity u will give a straight line with some finite gradient, now given by the inverse of $\beta_u = u/c$.

For the simple case of an object going through the origin, you can think of this explicitly as

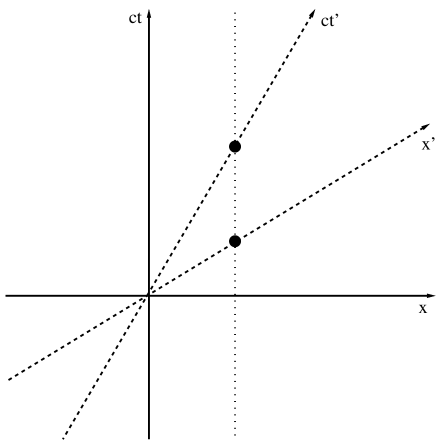
$$x = ut = \left(\frac{u}{c}\right) ct = \beta_u ct \qquad \text{so} \qquad ct = \left(\frac{1}{\beta_u}\right) x$$

Since we are limited to speeds $|u| \leq c$ for which $|\beta_u| \leq 1$ and hence $1/|\beta_u| \geq 1$, then the magnitude of the gradient must be always at least 1 and for light itself, it must always travel along lines with a gradient = 1, which have an angle of 45° in space-time diagrams. As stated at the beginning of the course, we will not consider acceleration so we only have to deal with cases like these.

5 Transforming world lines

An object which is stationary in one frame will not be stationary in a different frame but will be seen as moving with a constant velocity. Hence, a vertical world line will become tilted, but still be a straight line, in a different frame. More generally an object moving at a constant speed in one frame has a different constant speed in another frame so again it will remain a straight line but its gradient will have changed in the second frame. You can think of such a line as being a lot of events, each being the position of the object at a different time, with the times all very close to each other. As we can consider a world line as simply a lot of events, then applying the Lorentz transformations to these events will tell us directly how the world lines transform. In practise, since we are only dealing with straight lines, we only need to transform two events, as the new world line will be the straight line going through both of these. Alternatively we can use the velocity transformation formula presented in Lecture 4 to determine the new inverse gradient, although we still need to Lorentz transform one event on the world line to get the new intercept.

The rest frame case is particularly straightforward. An object at rest in a frame has a world line parallel to the ct axis in that frame. Under a Lorentz transformation, in the new frame, the ct axis will be tilted as shown in the right of Fig. 3. Hence, the world line must also have the same gradient as the ct axis in this frame. This is illustrated in Fig. 5.



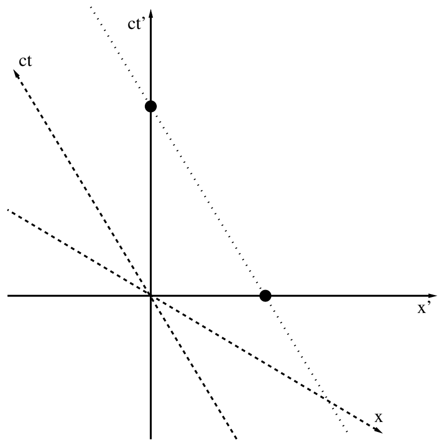


Figure 5: Left: A world line (dotted) for an object at rest in an initial frame. It passes through the ct' and x' axes at the two events marked by circles. Right: The transformed view of the world line, showing the transformed positions of the two events and the transformed world line going through these. Note this world line is still parallel to the original ct axis.

Overall, you should have got the impression that time and space are working in very similar ways within Relativity. In some ways, this is the big conceptual leap; time and space are mixed under Lorentz transformations and the inertial frame equivalence symmetry tells us there is a symmetry between space and time.

A final note. There is a health warning in using the non-perpendicular axes of the transformed frames; the overall γ factor in the Lorentz transformations means the axes are stretched compared with the original axes as well as tilted. This means, unlike for rotations, a distance of e.g. $1\,\mathrm{m}$ along the x axis is not the same length in a space-time diagram on the x' axis. Hence you often cannot use simple geometry to compare distances between inertial frames. You should always work in terms of events to be safe.

6 Revision questions

- On a space-time diagram draw:
  - a zero speed object;
  - an object travelling at c;
  - an unphysical (superluminal) line.
- Redraw your diagram from the point-of-view of a moving observer;
- Can you reproduce the world-line diagram to explain why both the tortoise and hare think they won their race?