Year 1 – Relativity
Lecture 6

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Overview of lectures

- Lecture 1: Introduction, concepts and classical results
- Lecture 2: The postulates of Relativity
- Lecture 3: Length contraction and simultaneity
- Lecture 4: The Lorentz transformations
- Lecture 5: Space-time diagrams and world lines
- Lecture 6: Four-vectors and causality
- Lecture 7: Energy and momentum
- Lecture 8: Rest mass energy and particle decays
- Lecture 9: Particle reactions
- Lecture 10: The relativistic Doppler effect

Previously on Relativity

- Looked at space-time diagrams
  - How events and axes move under Lorentz transformations
- Saw the concept of world lines
  - "History" of an object in space-time
- Saw how world lines change under Lorentz transformations
  - World lines change in position and gradient

What we will do today

- Introduce four-vectors
  - Similar concept to "normal" vectors but with four components
  - See that the (equivalent of) length-squared of all four-vectors is constant under Lorentz transformations
- Discuss the separation of two events
  - Separation = the length-squared of the four-vector which gives the difference of the two events
  - The sign of the separation is a critical value
  - Discuss how this relates to causality

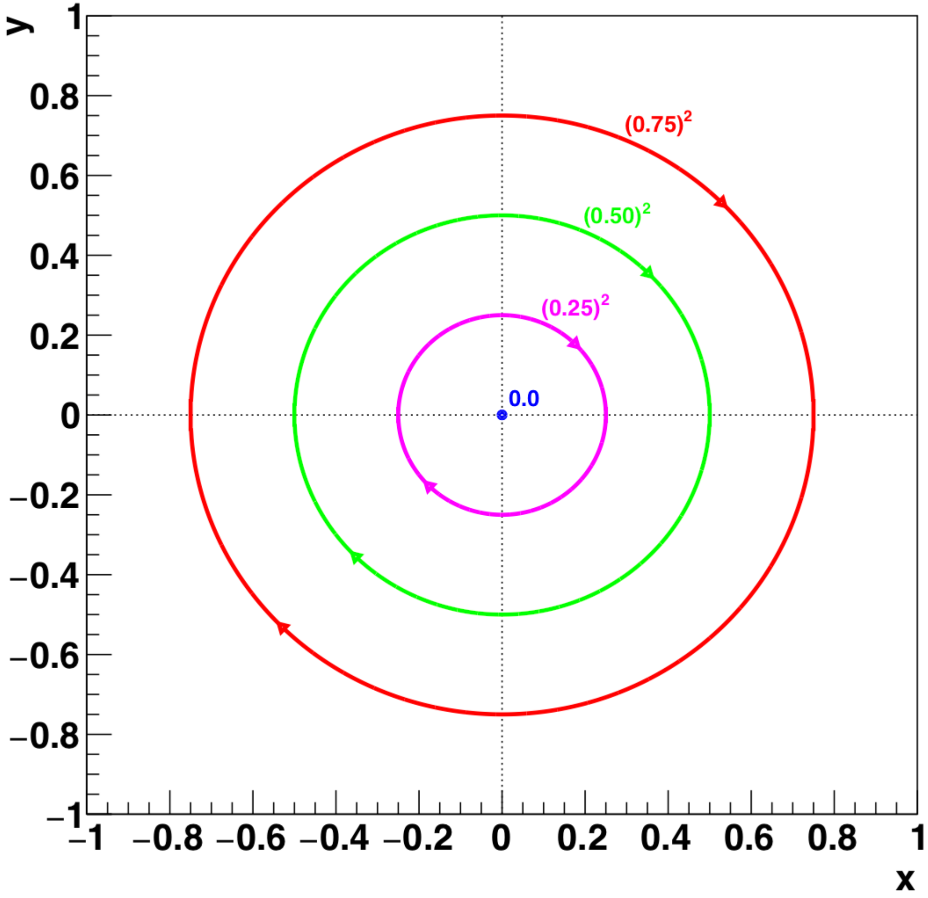
Four-vector notations

- The space-time four-vector (“four-position”)
  - I will write as (ct,r) = (ct,x,y,z)
  - Similar to writing the three-vector r = (x,y,z)
  - Some books use uppercase bold/underlined but this is ambiguous with some three-vectors; e.g. E
- The components can also be numbered
  - For any three-vector a, can write a = (a_1,a_2,a_3)
  - The “extra” component in four-vectors is numbered 0
  - Hence writing (ct,r) = (x^0,x^1,x^2,x^3) is very standard notation but a bit confusing when you first see it, so not used in this course

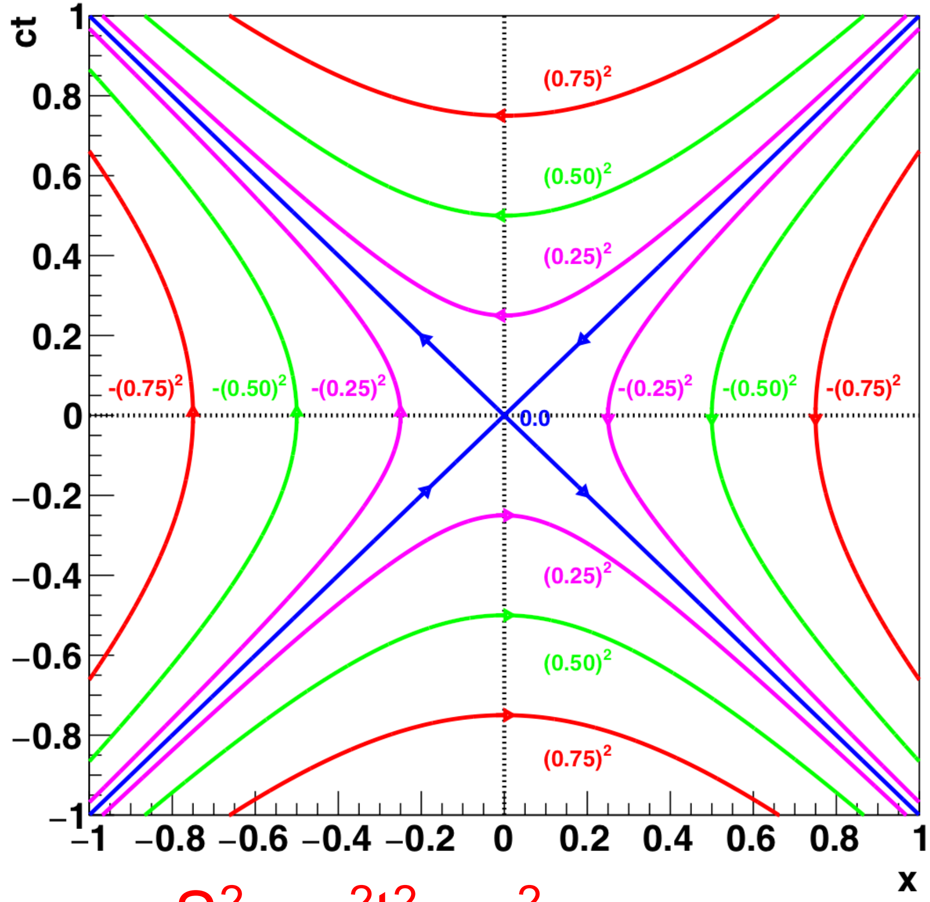
Full equation for LT

Lines of constant length-squared

Rotations



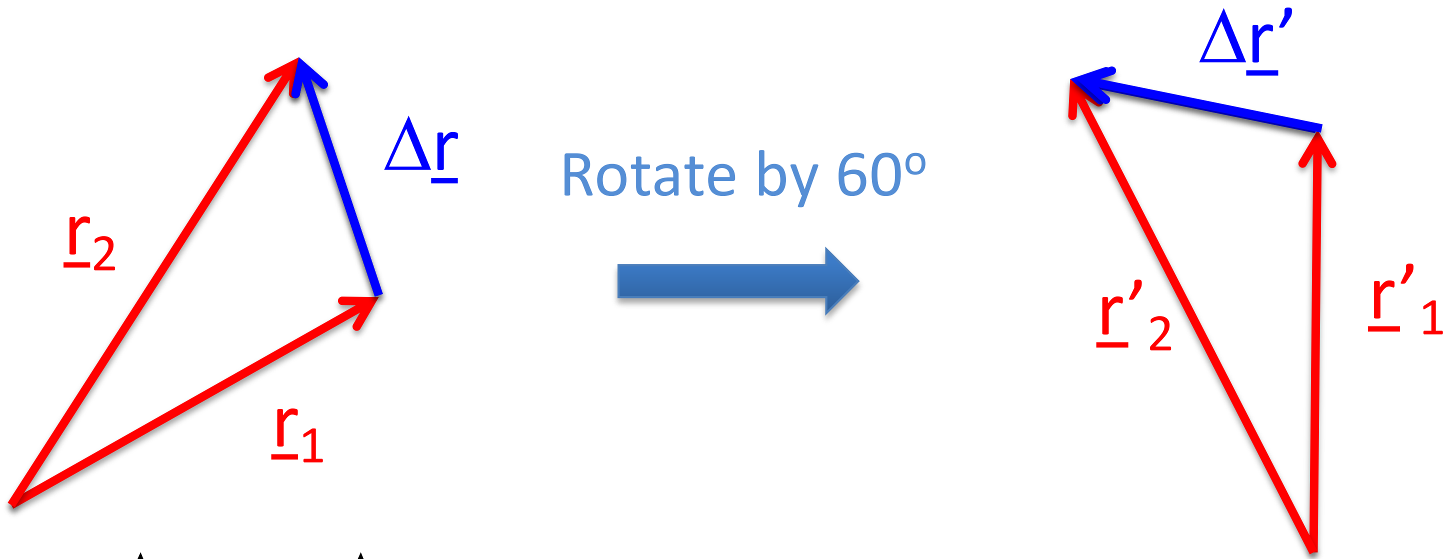
Lorentz transformations



$S^2 = c^2t^2 - r^2$

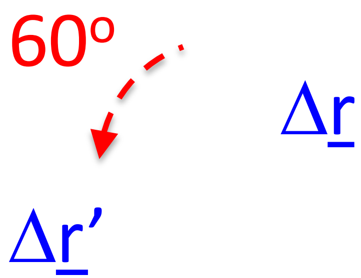
S^2 can be >0, =0 or <0

Distance between space points



$\underline{r}_2 = \underline{r}_1 + \Delta\underline{r}$ so $\Delta\underline{r} = \underline{r}_2 - \underline{r}_1$

Redefine origin to be at $\underline{r}_1$



$\Delta\underline{r}$ rotates like any other vector, so its length is invariant

Event separation

- Same holds for four-vectors and events
  - 'separation' between two events is defined to be the length-squared of the four-vector resulting from subtracting the four-vectors of the two events
  - Sometimes written,

$$(c\Delta t, \Delta \underline{r}) = (ct_2 - ct_1, \underline{r}_2 - \underline{r}_1)$$

  - This difference is also a four-vector and obeys the LT
  - Separation given,

$$\Delta S^2 = c^2 \Delta t^2 - |\Delta \underline{r}|^2$$

  - As we will see, events with different signs of ΔS^2 have very different properties wrt each other

Length-squared of a four-vector

Length-squared notations

- Four-position length-squared $S^2 = c^2t^2 - r^2$
  - We will treat the negative sign as simply something we have to remember
  - Some (usually older) books define it as $r^2 - c^2t^2$
  - Some define four-vectors to include an imaginary time component $(ict,\underline{r})$; squaring and adding all four components gives $r^2 - c^2t^2$ automatically
- General Relativity generalises this
  - Components multiplied by a "metric" (+1,−1,−1,−1)
  - These values change in gravitation fields so the ict idea is less used now

Menti question

- Go to www.menti.com

- Question 1: A proper time between two events can be defined for
  A. Any two events
  B. Only two events with $\Delta S^2 > 0$
  C. Only two events with $\Delta S^2 = 0$
  D. Only two events with $\Delta S^2 < 0$

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Can one event affect another?

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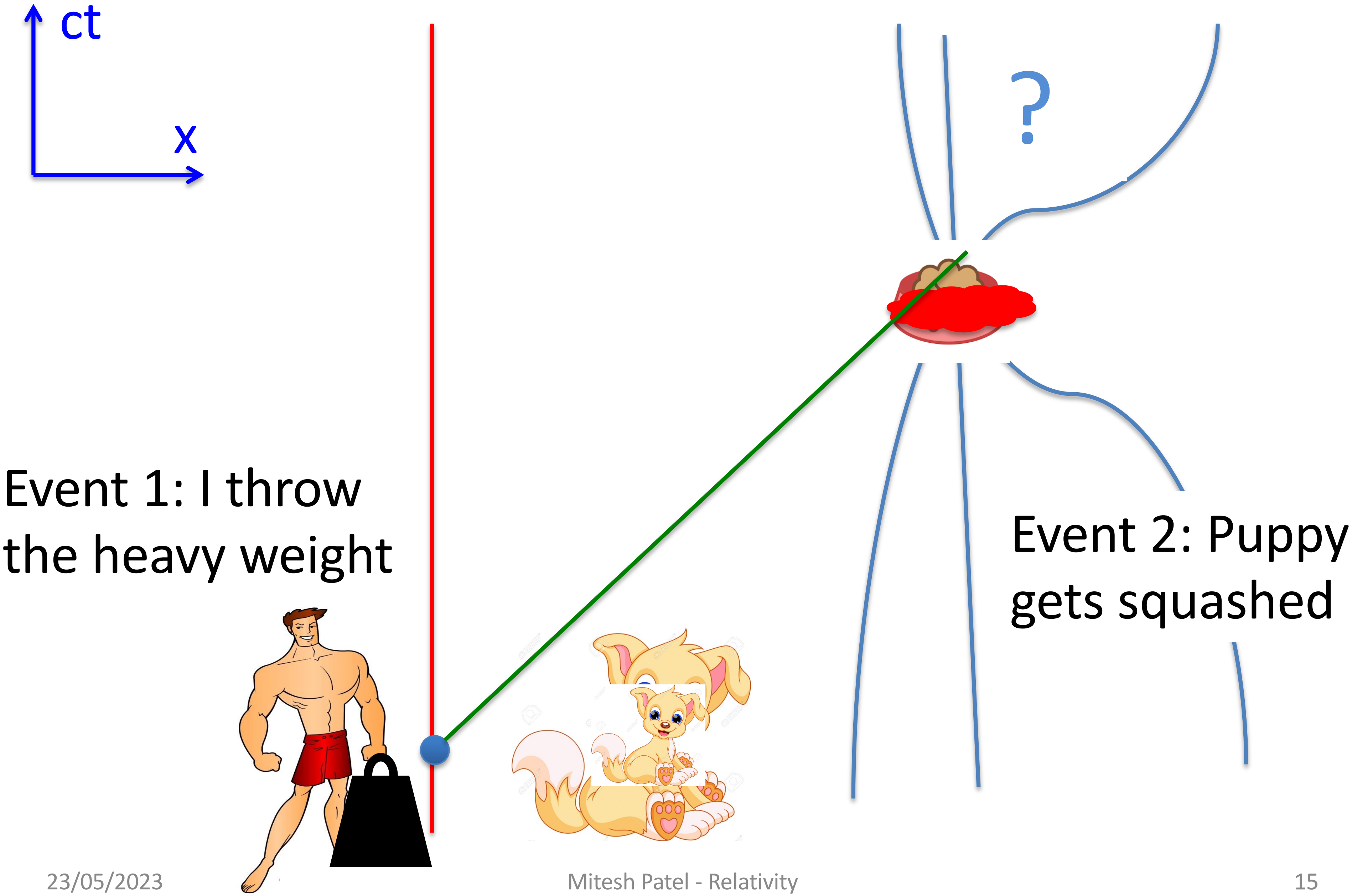
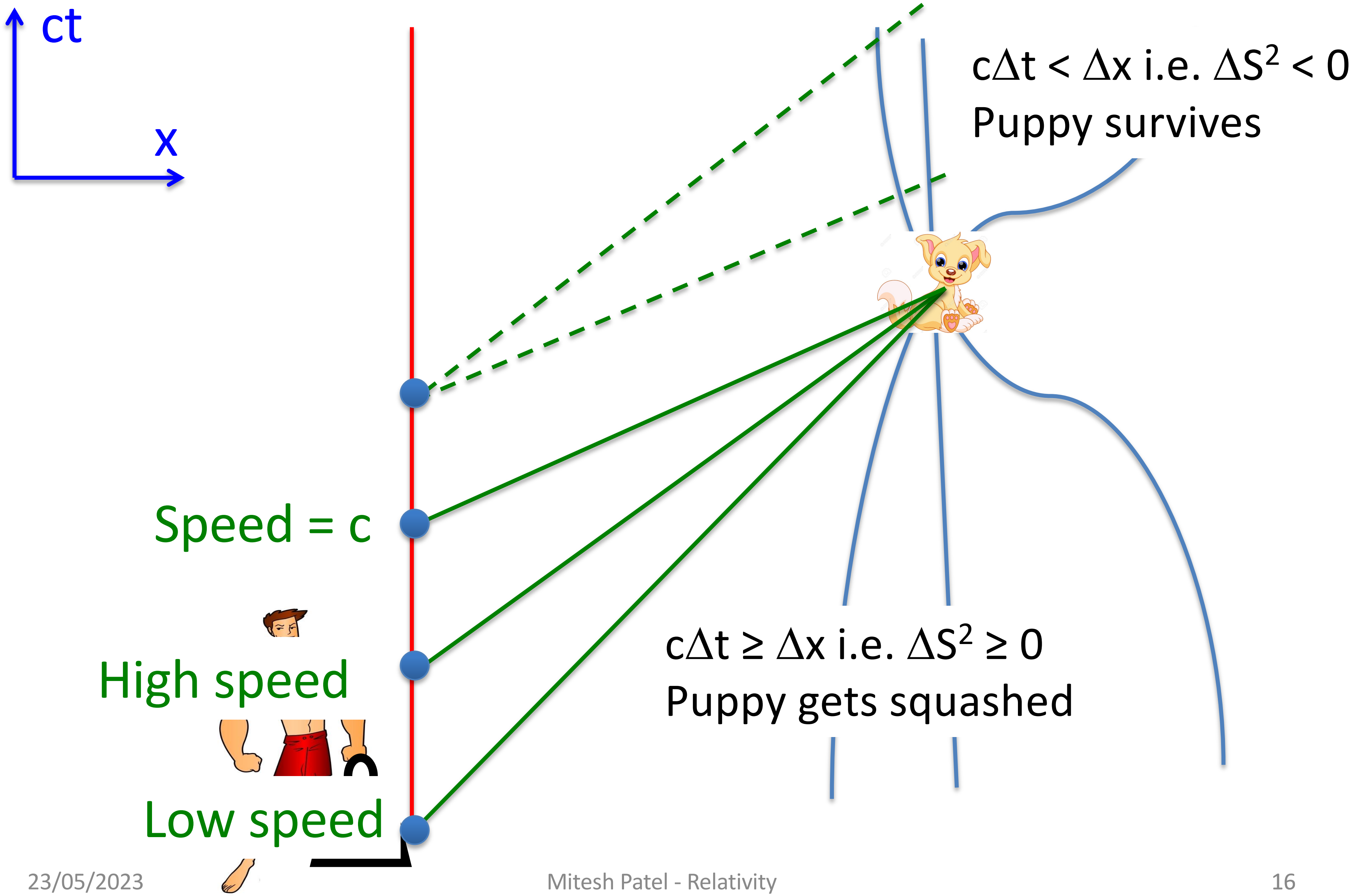
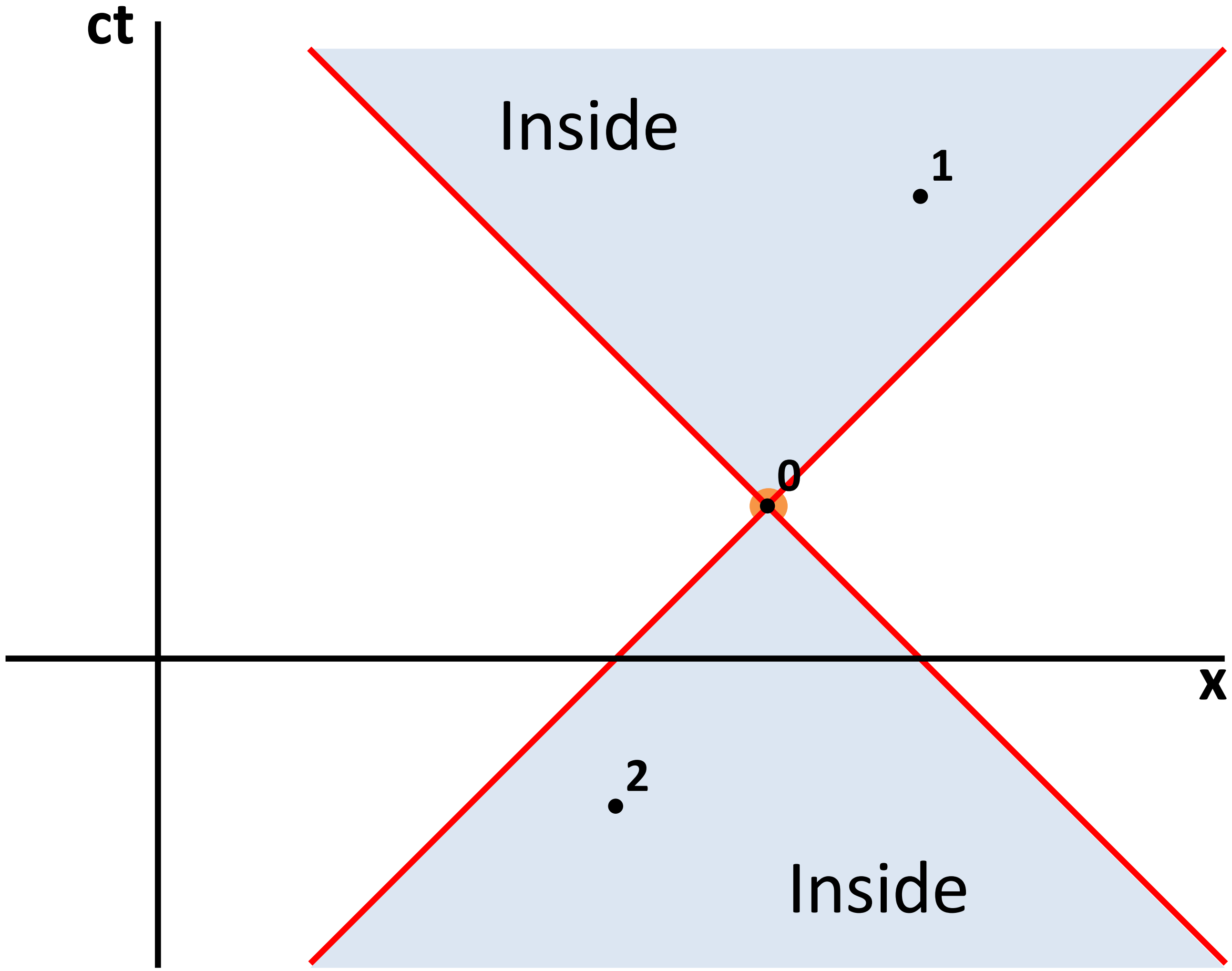


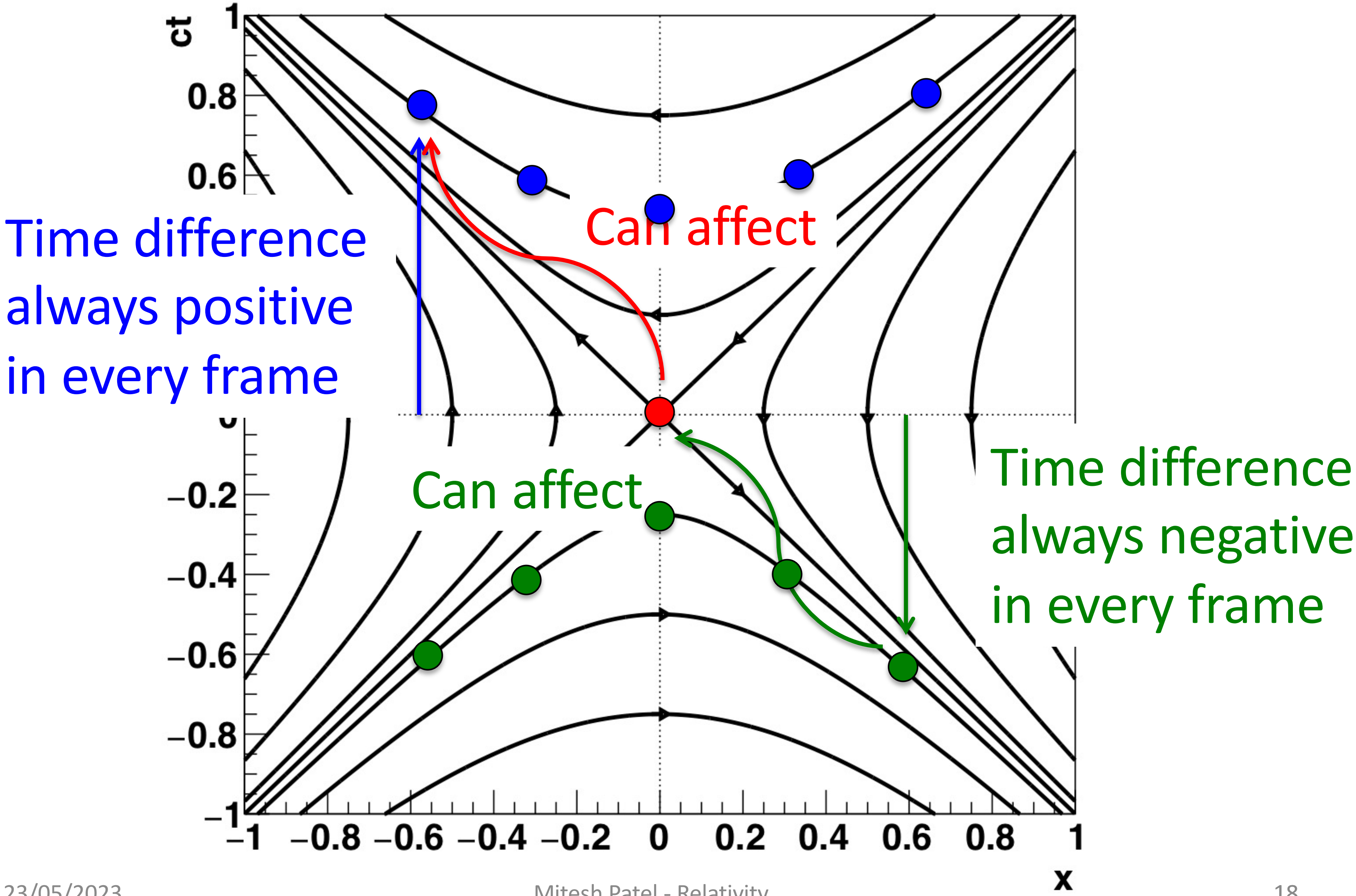
When can I throw the weight?



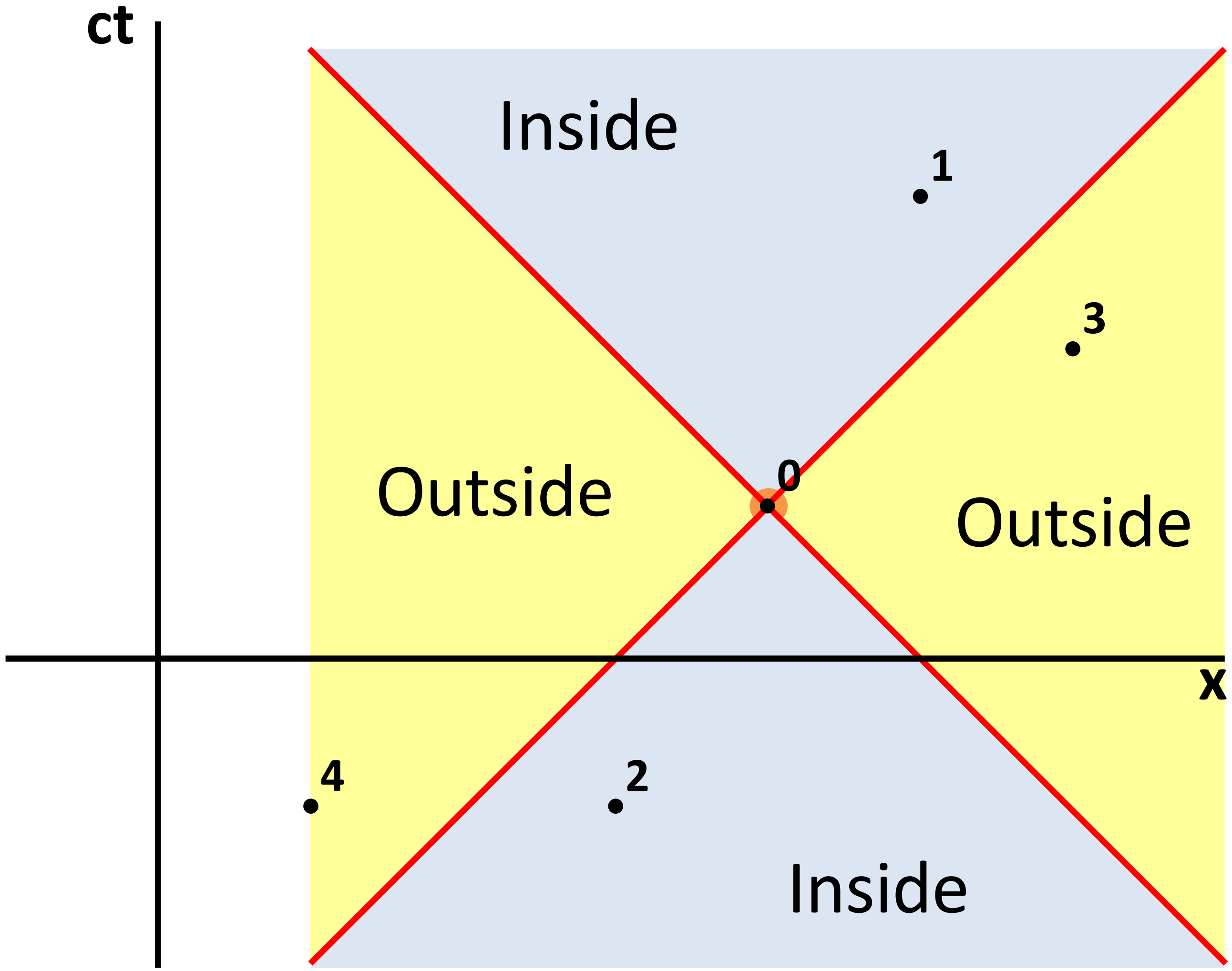
The light-cone



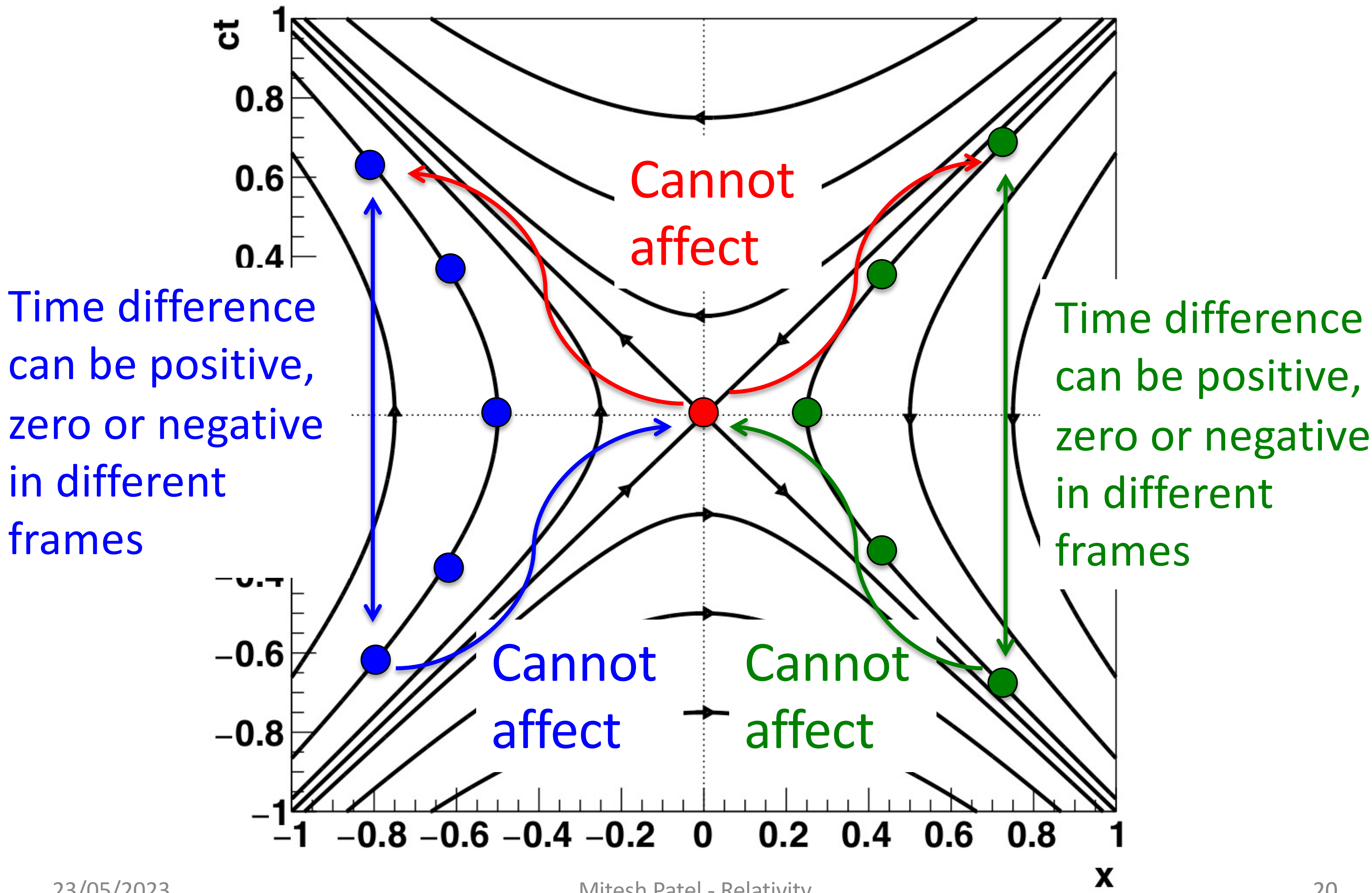
Events within light-cone



The light-cone



Events outside light-cone



Menti question

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- Question 2: Two events with exactly $\Delta S^2 = 0$ are causally connected
  A. True
  B. False

Menti question

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- Question 2: Two events with exactly $\Delta S^2 = 0$ are causally connected
  A. True
  B. False
- Answer: True, but only connected by something travelling at speed c

Menti question

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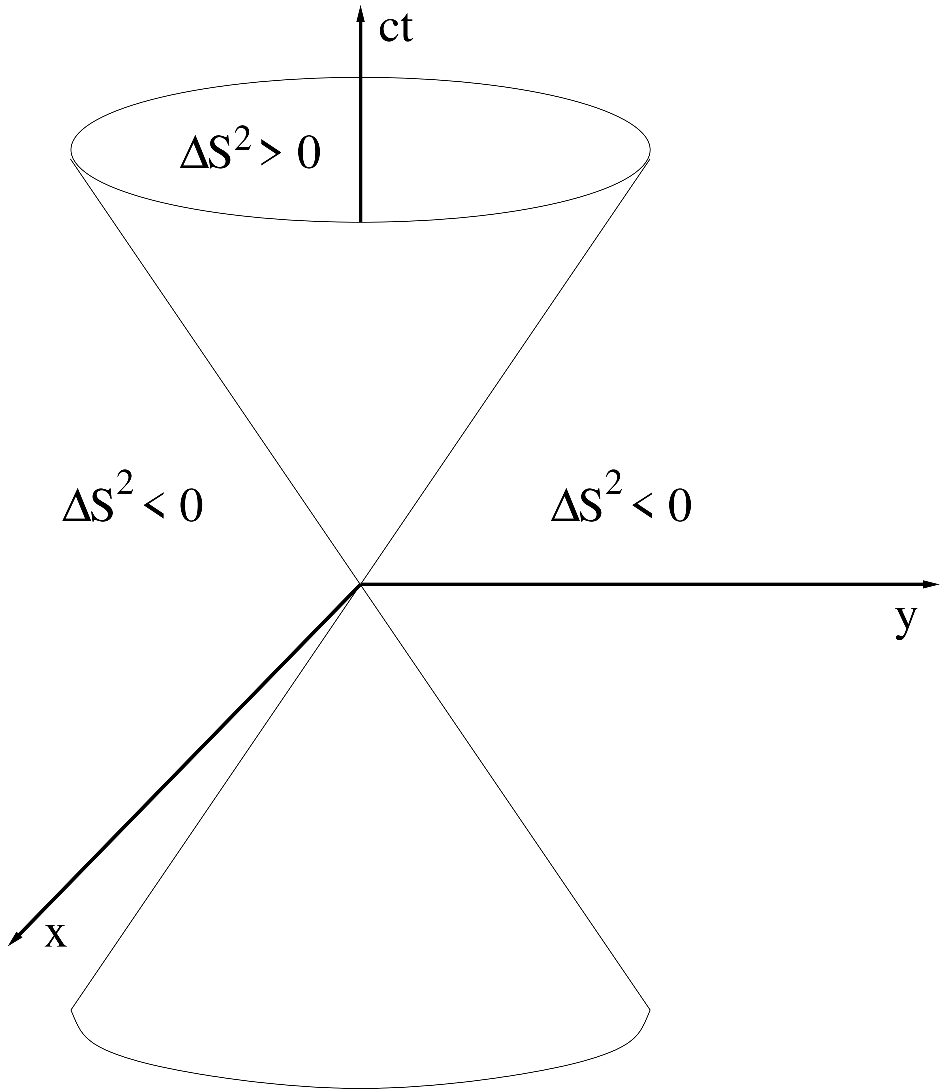
- Question 3: For any event, the space-time "volumes" inside and outside its light-cone are equal
  A. True
  B. False

Menti answer

- This is false; it looks like this in 1D but space is 3D so there is a lot more "volume" outside the light-cone than inside it

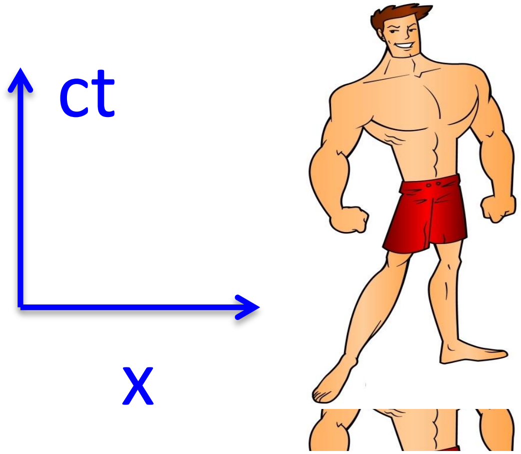


Summary of ΔS^2

- $\Delta S^2 > 0$ = `Time-like'
  - Time order always the same in all frames
  - Space order can change between frames
  - One frame has $\Delta x = 0$
  - $\Delta S^2 = c^2\tau^2$ where τ is the proper time
  - Causally connected
- $\Delta S^2 < 0$ = `Space-like'
  - Space order always the same in all frames
  - Time order can change between frames
  - One frame has $\Delta t = 0$
  - Causally unconnected
- $\Delta S^2 = 0$ = `Light-like'
  - Time and space order always the same in all frames
  - No frame has $\Delta x = 0$ or $\Delta t = 0$
  - Causally connected (by light-speed signal only)

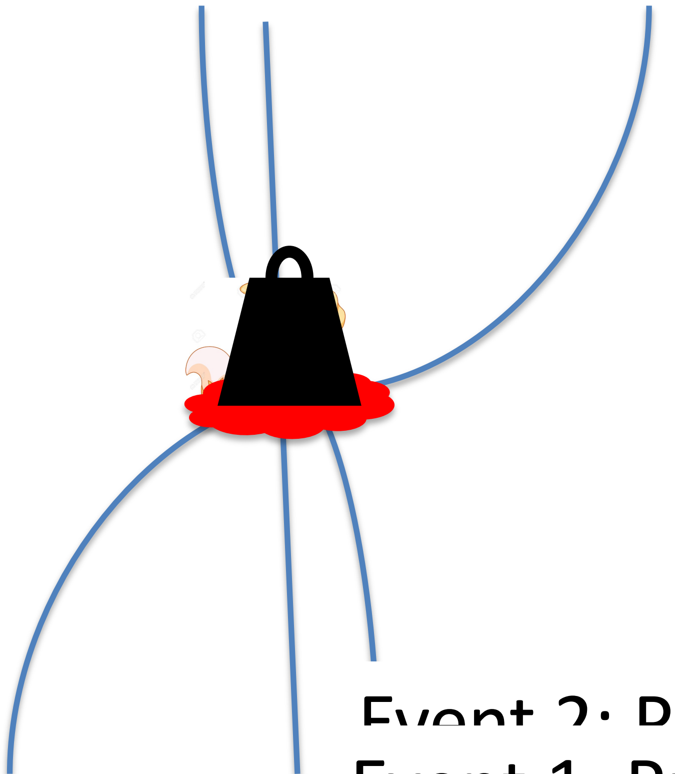
Tachyons

In a different frame



Event 2: I catch the tachyon weight

Event 1: I throw the tachyon weight with speed > c



Event 2: Puppy

Event 1: Puppy gets squashed

What we did today

- Introduced four-vectors
  - Four components and a lot of similarities to three-vectors
  - Saw that the length squared of all four-vectors is constant under Lorentz transformations
- Discussed the separation of two events
  - The length squared of the four-vector which is the difference of the two events
  - The sign of the separation tells us if events are causally connected or not
  - Works in all frames if nothing goes faster than c