First Year Special Relativity – Lecture 3
Length contraction and simultaneity

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1 In this lecture

- Length contraction;
- Non simultaneity.

2 Introduction

The material in this lecture is covered in Young and Freedman, Secs. 37.2 and 37.4 and in McCall in Secs. 5.9 and 5.10.

We have seen that time runs slower for an object when it is moving relative to an external observer, the so-called 'time-dilation' effect. This lecture discusses some related effects. The first is called 'Lorentz contraction' (or 'length contraction'), in which an object also get shorter along its direction of motion. The second is that occurrences which happen at the same time (i.e. are simultaneous) for one observer are not necessarily simultaneous for other observers.

3 Lorentz contraction

We considered the light clock in Lecture 2, for which the period T is lengthened to $T' = \gamma T$ when it is moving. In fact, *any* clock will have exactly the same dilation factor. We can now ask how far does the clock move between each tick, i.e. within the time of the period.

Consider two inertial frames; one where the clock is at rest and another where the clock is moving with speed v. An observer in the moving frame of the clock has a long wooden rod and cuts this so that, as the clock moves past it, the rod is just the right length so the clock is at one end when it first ticks and reaches the other end when it ticks a second time; see Fig. 1. Note, this observer is in the rest frame of the wooden rod. Since the clock has speed v and the period in the moving frame is T' the observer will find the stationary wooden rod has a length $l = vT'$.

Now consider how this appears to an observer in the clock rest frame. This observer sees the clock as stationary but sees the wooden rod moving with speed v (but in the opposite direction, of course). The clock ticks must still occur just as the start and end of the rod pass the clock as it is the same system being observed. For this observer, the length of the moving wooden rod must be $l' = vT = vT'/\gamma$ and so

$$l' = \frac{l}{\gamma}$$

This is the Lorentz contraction equation. Note, the rod is stationary in the first frame considered, i.e. the moving frame of the clock, not the frame where the clock is at rest. Hence, the length in the rest frame of the rod is l, sometimes called the 'proper length', and in a moving frame is $l' = l/\gamma$, which is shorter. We conclude that lengths get shorter when an object is moving. This is called 'length contraction', or 'Lorentz contraction', as the objects appear to shrink compared to their length at rest. Although not shown by this argument, it is only the length in the direction of motion which is shortened.

You might think that this happens purely due to the particular molecular forces of this particular rod. However, the argument above is very general so it does apply to all objects. In

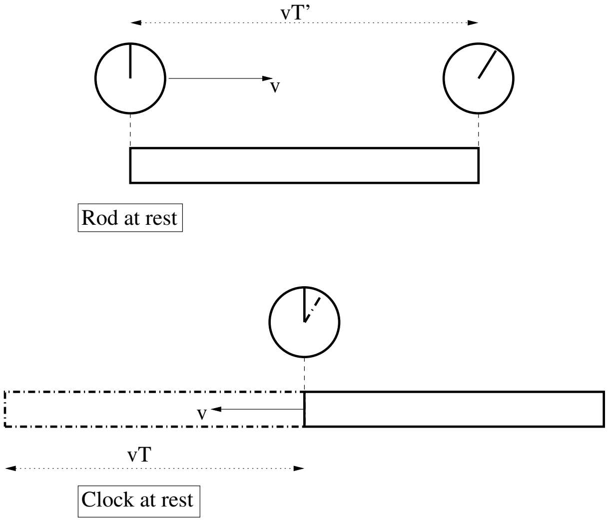

Figure 1: Top: A wooden rod cut to be the same length as the distance the clock moves in one period, shown in the rest frame of the rod. Bottom: The same system in the rest frame of the clock.

fact, it is even more general. Instead of trimming a rod to the right length, we could instead hold flags at the positions where the clock ticks. We would see the same contraction of the distance between the flags but in this case, there is no physical object between the flags. Hence, we have to conclude that it is space itself, and hence any object or distance in that space, which has contracted. This sometimes leads to confusion; what is the 'rest frame' for a distance if there is no object? The distance is defined by its beginning and end (marked by the flags in the case above) and by definition these have to have the same velocity or the distance would be changing with time. The 'rest frame' for the distance is then the inertial frame in which the beginning and end positions of the distance are not moving.

To summarise: time dilation means time slows down for an object in a moving frame compared with its rest frame, while Lorentz contraction means lengths get shorter for an object in a moving frame compared with its rest frame.

As should be clear from the above derivation, length contraction and time dilation are of course inextricably linked. They are two aspects of the same effect and a system when viewed in different frames can use one or the other to explain what is happening.

4 The light clock revisited

Now we know about Lorentz contraction, we can now look at the light clock again but with a different orientation. We previously studied it in a frame moving perpendicular to the direction of motion of the light. Now consider it in a frame moving parallel to the motion of the light, as shown in Fig. 2.

In the clock rest frame, the total time for the light pulse to return is $T = 2d/c$. The distance between the light source and the mirror will be shortened to d/γ but the mirror is moving at speed v. Hence, the time for the light to reach the mirror t_1' is given by

$$ct_1' = \frac{d}{\gamma} + vt_1', \qquad \text{so} \qquad t_1' = \frac{d}{\gamma(c-v)} = \frac{d}{\gamma(1-\beta)c}$$

Since the source and detector are also moving, the time t_2' for the light to return from the mirror

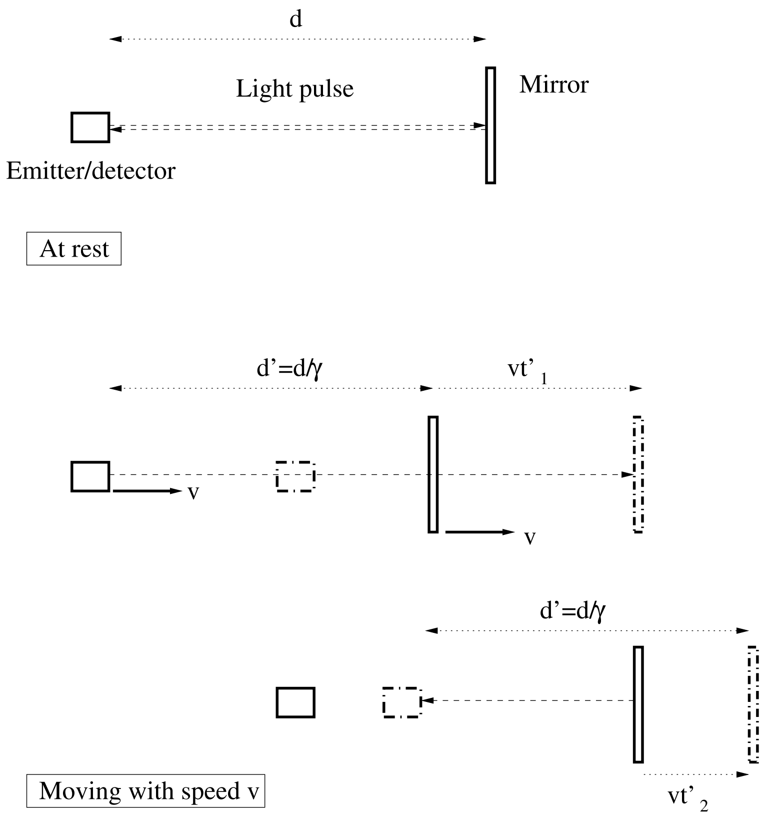

Figure 2: The lightclock at rest (top) and moving to the right with speed v (bottom), oriented so that the light pulse is along the direction of motion.

to the source is

$$ct'_2 = \frac{d}{\gamma} - vt'_2, \qquad \text{so} \qquad t'_2 = \frac{d}{\gamma(c+v)} = \frac{d}{\gamma(1+\beta)c}$$

Hence, the total time for the light to return, i.e. the clock period, is

$$T' = t'_1 + t'_2 = \frac{d}{\gamma(1-\beta)c} + \frac{d}{\gamma(1+\beta)c} = \frac{d(1+\beta)+d(1-\beta)}{\gamma(1-\beta)(1+\beta)c} = \frac{2d}{\gamma(1-\beta^2)c} = \frac{2d\gamma}{c} = \gamma T$$

Hence, the time dilation factor does not depend on which way round the clock is oriented, as must be the case for consistency; time must not depend on whether we do a rotation of the clock in its rest frame.

However, one interesting feature is that in the perpendicular frame, the time from the source to the mirror and from the mirror to the detector were both equal to $d\gamma/c$, but in this orientation, the two times t'_1 and t'_2 are no longer equal. You might have thought each leg of the light path should simply be time dilated by γ but this is not the case. This is related to the fact that the two ends of the light path are (obviously) not in the same position, i.e. that the observed time depends on position as well. We will see how this works in general in Lecture 4.

5 Simultaneity

Consider a slightly more complicated light clock in its rest frame, where an emitter sends light in two opposite directions to bounce off two mirrors and return to be detected. If the two

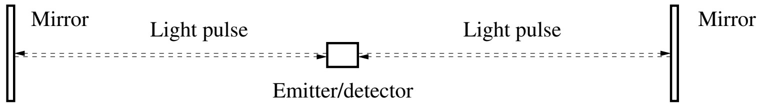

Figure 3: A double light clock, with both arms having an identical length.

mirrors are the same distance from the source then the two light pulses will arrive at the mirrors simultaneously and similarly return to the detector at the same time; see Fig. 3.

How does this look in a moving frame? As the light has speed c, then with one mirror moving towards and the other away from the source, then the light will no longer hit the two mirrors at the same time. By effectively an identical calculation to the above, you can show the times to reach the front and back mirrors are

$$\frac{d}{\gamma(1-\beta)c} \qquad \text{and} \qquad \frac{d}{\gamma(1+\beta)c}$$

respectively. However, having bounced off the mirrors, the light then reaches the detector after

$$\frac{d}{\gamma(1+\beta)c} \qquad \text{and} \qquad \frac{d}{\gamma(1-\beta)c}$$

respectively, which clearly sum to the same total so the light pulse does arrive back at the detector at the same time.

The critical thing here is that the arrival times of the light pulses at the mirrors are simultaneous in the rest frame of the clock, but are not simultaneous in any other inertial frame. However, the arrival at the detector is simultaneous in all inertial frames. The difference is that the light arriving at the mirrors is at two different locations, while the arrival at the detector is at the same location. The general principle is that for two simultaneous occurrences which are not at the same x position, then in any other frame moving along x, they are no longer simultaneous. In contrast, if they *are* simultaneous and at the same x position, they will be simultaneous in all frames moving along x. Hence, to be simultaneous in any frame moving in any direction, the two occurrences must be at exactly the same position.

The breaking of simultaneity raises significant issues. In particular, can the first occurrence affect the second, given that they may occur in the opposite order in some frames to others? We will discuss this in Lecture 6.

6 Revision questions

- Why do we get length contraction?
- Are two simultaneous occurrences that occur at different x positions in some frame also simultaneous in a frame moving along x?
- What about two simultaneous occurrences at the same x position?
- How does the relativity of simultaniety help understand e.g. length contraction?