Year 1 – Relativity
Lecture 7

Mitesh Patel

Overview of lectures

- Lecture 1: Introduction, concepts and classical results
- Lecture 2: The postulates of Relativity
- Lecture 3: Length contraction and simultaneity
- Lecture 4: The Lorentz transformations
- Lecture 5: Space-time diagrams and world lines
- Lecture 6: Four-vectors and causality
- Lecture 7: Energy and momentum
- Lecture 8: Rest mass energy and particle decays
- Lecture 9: Particle reactions
- Lecture 10: The relativistic Doppler effect

Previously on Relativity

- Saw events and four-vectors
  - Events are points in ct,r space which can also be defined using four-vectors (ct,r)
  - Equivalent to three-vectors but with four components
- Four-vectors undergo Lorentz transformations
  - (All three-vectors rotate using the same equations)
  - Similarly, all four-vectors transform using the same LT equations
  - The length-squared $S^2 = c^2t^2 - r^2$ is invariant and so this holds for all other four-vectors also

What we will do today

- Introduce four-momentum for an object
  - Combining energy E and momentum p
  - See how these depend on the object's speed
- Check some of the four-momentum properties
  - In the low speed approximation, E and p should correspond to the classical quantities; $mu^2/2$ and $m\underline{u}$
  - Find the invariant length-squared for the four-momentum
  - Rederive the velocity transformation
  - See how it works for photons; special case as m=0 and $|\underline{u}|=c$

Four-vectors

- There are many three-vectors in physics
  - Position r, velocity v, momentum p, electric field E…
  - All rotate using the same transformation
  - There are also three-scalars = invariants under rotations; e.g. time t, energy E, mass m…
  - Equations which use these will be rotation covariant
- The "four-position" (ct,r) is not the only four-vector
  - There are many other four-vectors, all of which combine a three-vector with a three-scalar
  - There are also four-scalars = invariants under LTs
  - Fundamental physical laws use all these to be LT covariant

Menti question

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- Question 1: Every three-vector has an associated three-scalar. Together, these form a four-vector which undergoes Lorentz transformations when changing inertial frames
  - True
  - False

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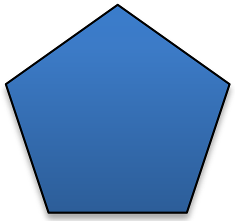
- Question 1: Every three-vector has an associated three-scalar. Together, these form a four-vector which undergoes Lorentz transformations when changing inertial frames
  – True
  – False ✓

Have already seen a three-vector which does **not** transform according to the LTs but in a more complicated way : the velocity;
E,B fields form a `tensor' structure

Transforming from rest

Object at rest



Observer frame
moving at negative v

Observer
frame at rest

Object moving
with positive $u = -v$

Four-momentum

- The energy E and momentum $\underline{p}$ of any object form a four-vector $(E, \underline{p}c)$, called the 'four-momentum'

- Since all four-vectors change in the same way, we already know how $(E, \underline{p}c)$ transforms...

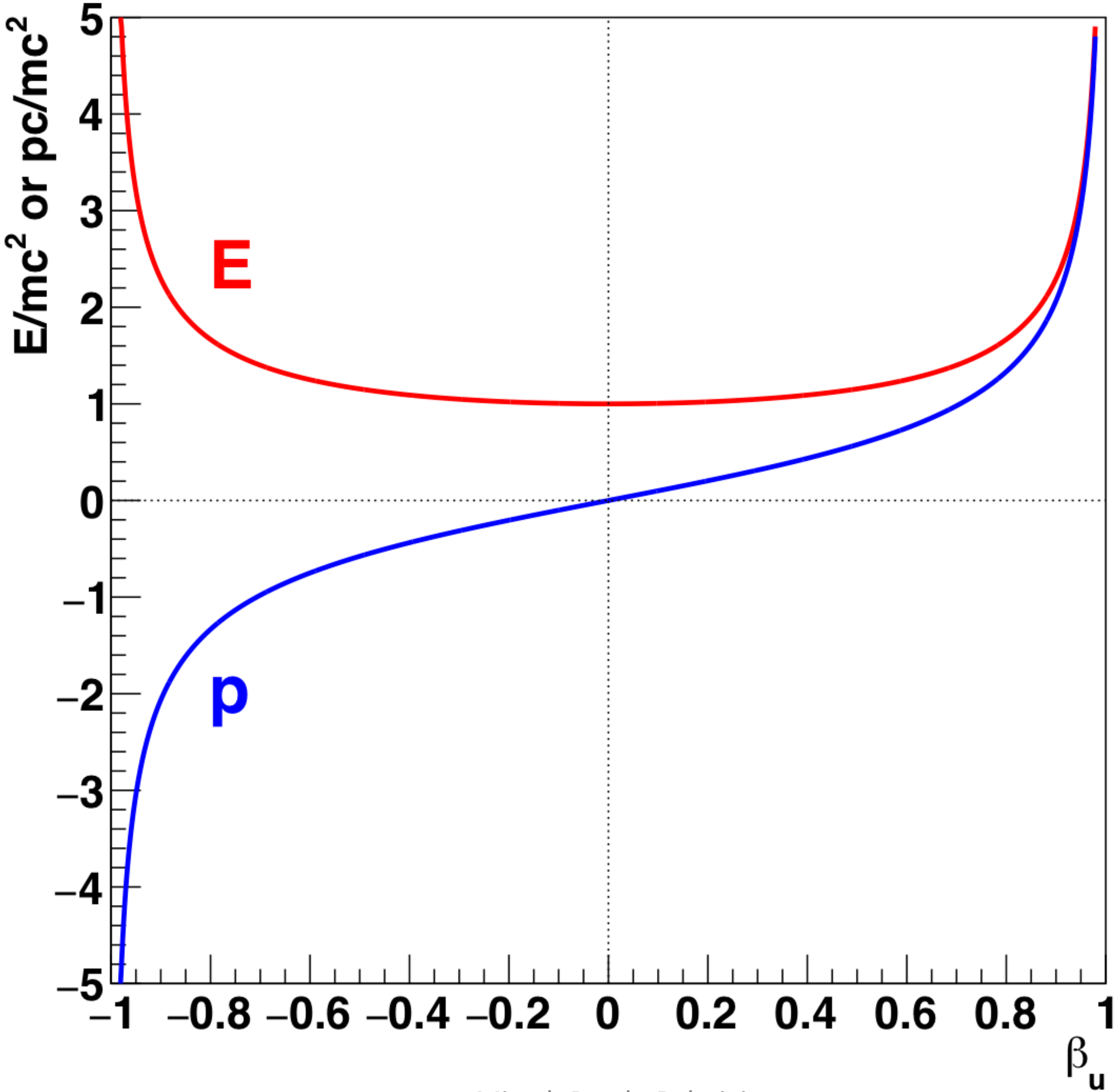
Four-momentum

- The energy E and momentum $\underline{p}$ of any object form a four-vector $(E, \underline{p}c)$, called the 'four-momentum'

- Can use what we know about how four-vectors transform to relate E and $\underline{p}$ to the rest-mass energy, $E_0=mc^2$, and velocity
  - $E = \gamma_u mc^2$,
  - $\underline{p} = \gamma_u m\underline{u}$

- Kinetic energy then given $K=(\gamma_u-1)mc^2$

Energy and momentum vs β_u



Energy-momentum invariant

- We know the length-squared for every four-vector is an invariant

- For the four-momentum this invariant is

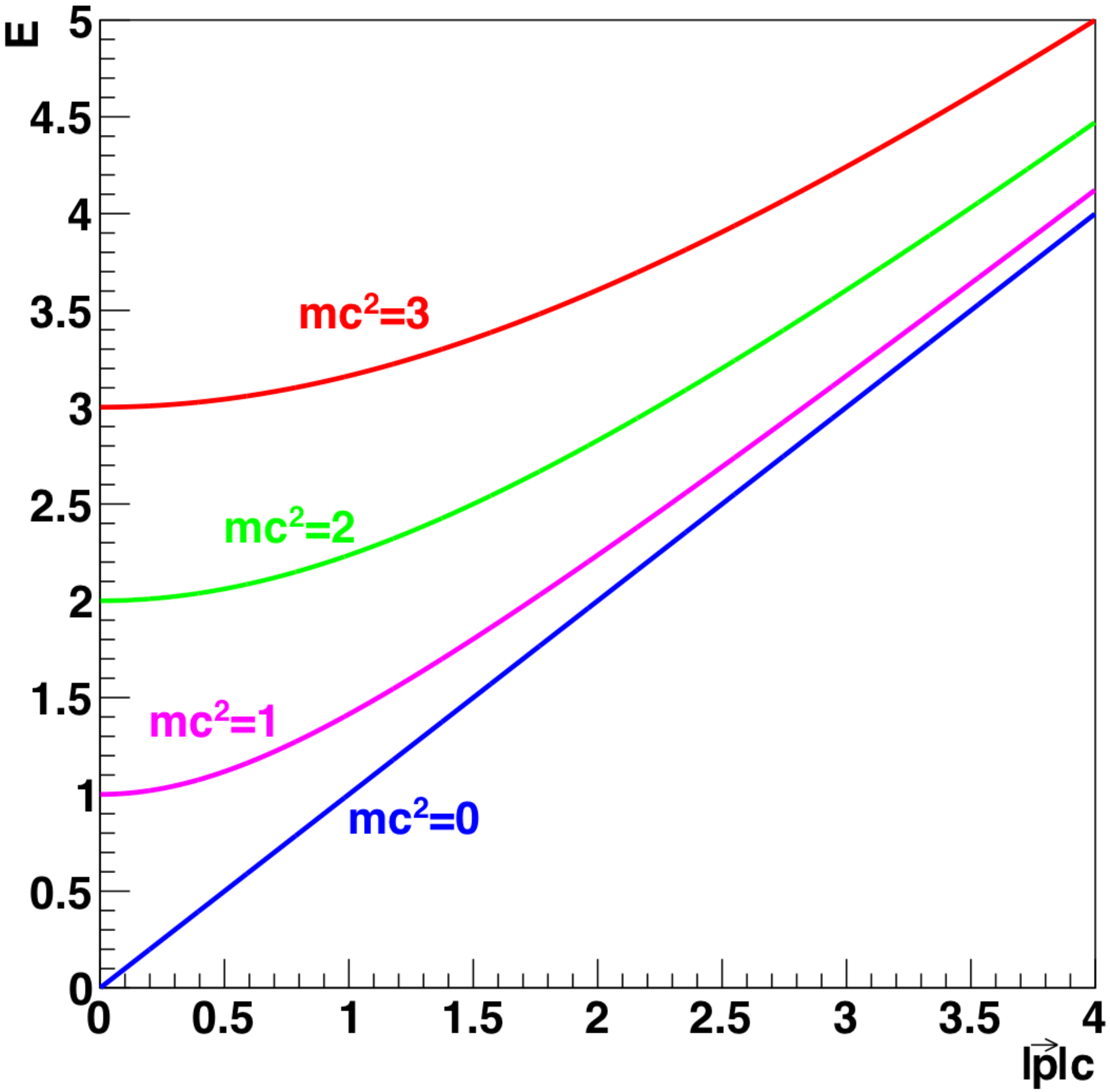
$$E^2 - (pc)^2 = (mc^2)^2$$

Often rearranged,

$$E^2 = p^2c^2 + m^2c^4$$

hence making it clear that E is always bigger than both pc and mc^2

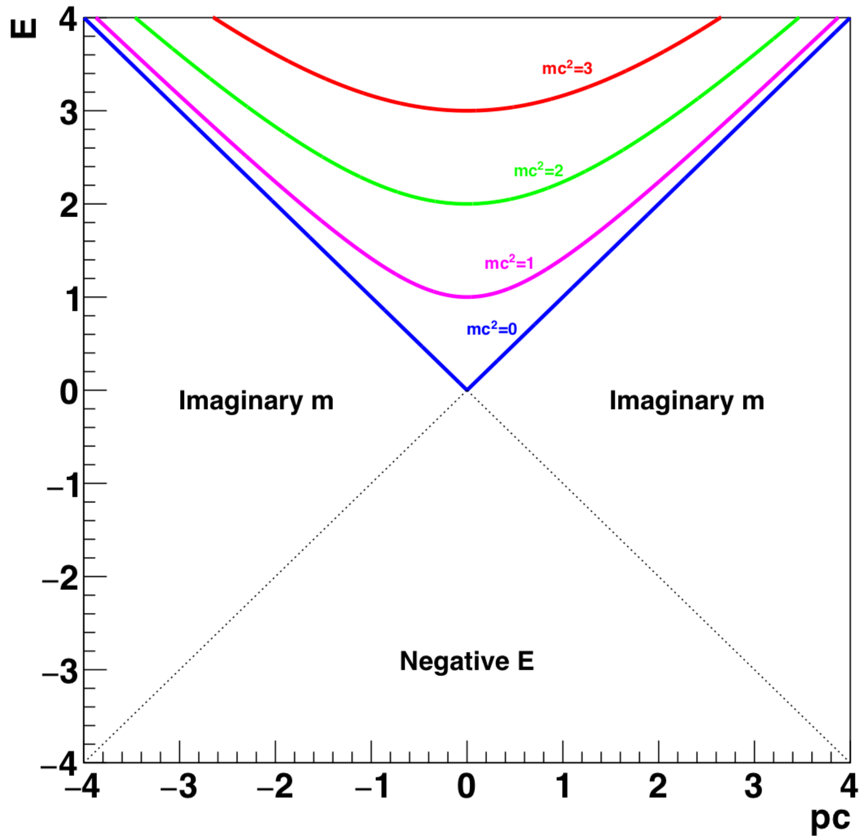
Energy vs momentum

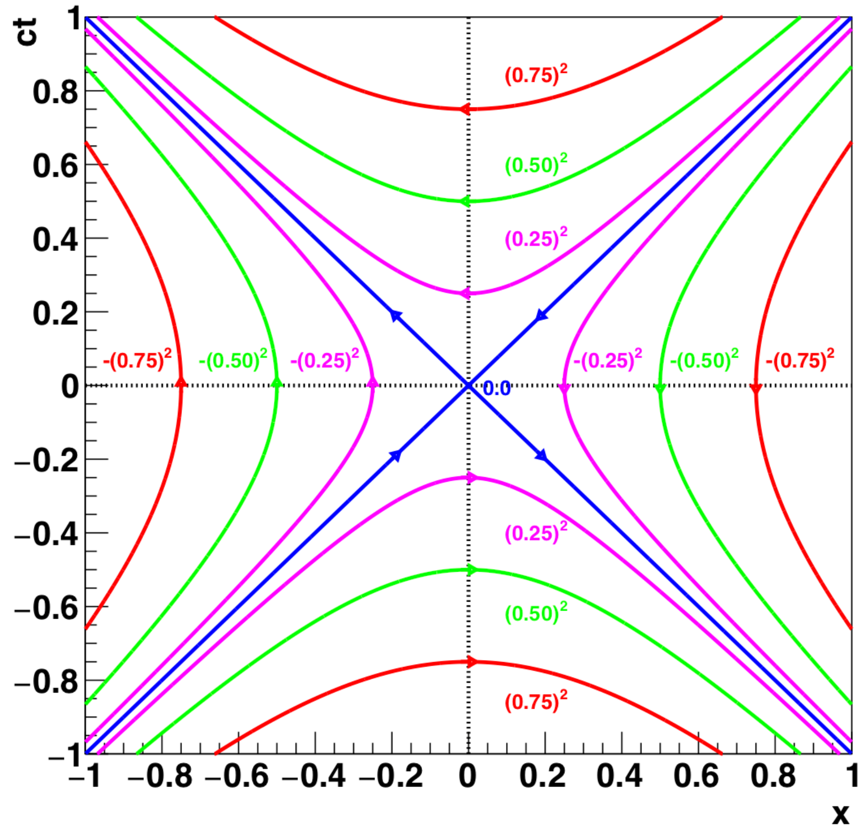


$E \rightarrow mc^2$ as the $p \rightarrow 0$ (i.e. the energy is just the rest energy)

$E \rightarrow |\underline{p}|c$ as p becomes large i.e. mass becomes negligible cf E

Energy vs momentum





Only top quadrant is relevant as

$E > 0$ and m is real

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- Question 2: In principle, it is possible to accelerate an object to the speed of light, i.e. to $u=c$
  - True
  - False

Menti question

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- Question 2: In principle, it is possible to accelerate an object to the speed of light, i.e. to $u=c$
  - True
  - False ✓
- As γ_u goes infinite and so this would need infinite energy

Menti question

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- Question 3: What are the first two terms in the binominal expansion of $(1 - x^2)^{-1/2}$?
  A. $1 + x$
  B. $1 - x$
  C. $1 + (x^2/2)$
  D. $1 - (x^2/2)$

Menti question

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- Question 3: What are the first two terms in the binominal expansion of $(1 - x^2)^{-1/2}$?

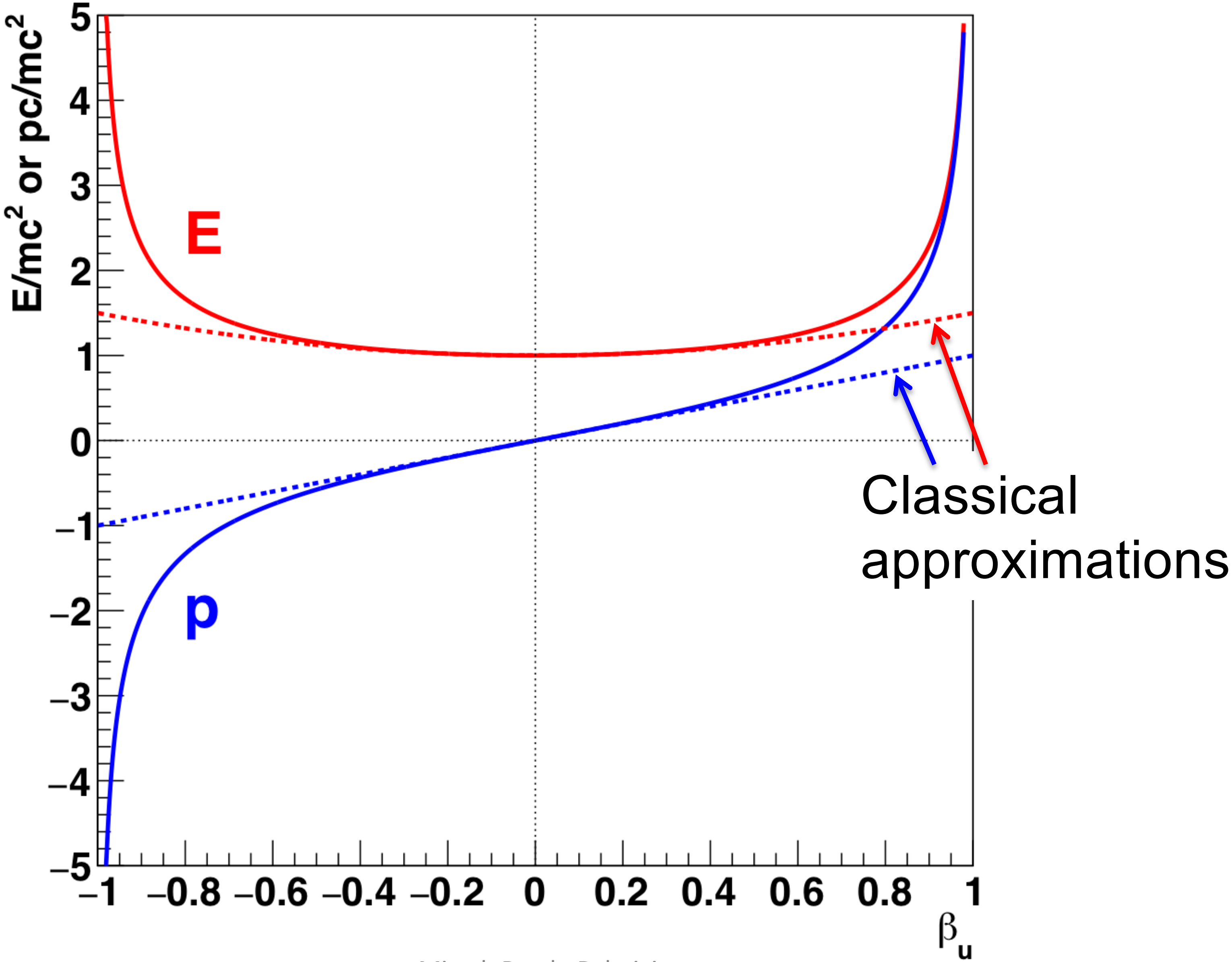
  A. $1 + x$

  B. $1 - x$

  C. $1 + (x^2/2)$ ✓

  D. $1 - (x^2/2)$

Energy and momentum vs β_u



Rederiving the velocity transform

Photons = Quantum Mechanics!

- Classically, fixed m and v give fixed E and p
  - But photons are not all the same
  - They vary in frequency f (and so also wavelength λ)
- The Planck-Einstein and de Broglie relations connect E,p to f,λ
  - Planck-Einstein: $E = hf$
  - de Broglie: $p = h/\lambda$
- $E = |\underline{p}|c$ therefore requires
  - $hf = hc/\lambda$ so $f\lambda = c$, which shows Relativity and Quantum Mechanics are consistent in this respect

Some notes on photons

- Photons always have speed c
  - There is no rest frame for photons
  - They also have no rest mass energy
- Are photons really a special case?
  - m=0 and $|\underline{u}|$=c require each other or the equations give zero or infinity
  - Hence any particle with m=0 must have the same properties; hence c should be called “the speed of massless particles”, not just “the speed of light”
  - Experimentally have seen the gluon, theoretically expect the graviton; both are m=0

What we did today

- Introduced the four-momentum
  - Combines E and $\underline{p}$ which are functions of velocity

    $$E = \gamma_u mc^2, \qquad \underline{p} = \gamma_u m\underline{u}$$

  - Classical expressions are approximations to exact formulae
  - Velocity is given by $\underline{u} = \underline{p}c^2/E$
  - Saw the four-momentum invariant is $(mc^2)^2$
- Looked at four-momentum for photons
  - Massless and $|\underline{u}| = c$ always
  - E, $\underline{p}$ and $\underline{u}$ still related by $\underline{u} = \underline{p}c^2/E$ so $E=|\underline{p}|c$