First Year Special Relativity – Lecture 7
Energy and momentum

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1 In this lecture

- Four-vector for E,p and its dependence on an objects speed;
- Check low speed approximations;
- Look at invariant length-squared;
- Look at the special case of zero mass.

2 Introduction

The material in this lecture is covered in Young and Freedman, Secs. 37.7 and 37.8 and in McCall in Secs. 6.1 and 6.2.

There are many three-vectors in physics; position, velocity, momentum, electric and magnetic fields, etc. There would be no point in generalising the concept to four-vectors if the only one which existed was $(ct, \vec{r})$. Hence, as you probably already guessed, there are many other four-vectors. In the same way that all three-vectors have the same rotation transformation, then all four-vectors undergo the same Lorentz transformation under a given boost. We shall look at one of the other most important four-vectors over the next few lectures, namely the combination of energy and momentum. As we have seen, $\vec{r}$ (a three-vector) is combined with t (which is a scalar under rotations) and these mix together under a Lorentz transformation; i.e. t goes with $\vec{r}$. Similarly E (a scalar) goes with $\vec{p}$ (a three-vector).

However, it is a common mistake to think every three-vector must have another variable to go with it; there are many three-vectors which do not. One is velocity; we already saw this does *not* transform using the Lorentz transformations but in a more complicated way, without any other scalar variable being involved. Another example is the electric and magnetic fields; in this case they together form a structure called a 'tensor' and under boosts, the two vectors mix into each other. The most obvious example of this is that a stationary point charge has a electric field but no magnetic field. However, in the frame of an observer moving relative to the stationary charge, the charge appears to be moving and you should know from the E&M course that a moving charge, i.e. a current, has a magnetic field associated with it.

3 Energy and momentum

The energy E and momentum $\vec{p}$ of any object form a four-vector $(E, \vec{p}c)$, called the 'four-momentum'. As for the time component in the $(ct, \vec{r})$ case, the c ensures the dimensions of all the components are the same. (Warning: some books define four-momentum with the c in other positions, e.g. $(E/c, \vec{p})$.) Since all four-vectors change in the same way, then when changing frames p_y and p_z are unchanged, while

$$\begin{pmatrix} E' \\ p'_x c \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_x c \end{pmatrix}$$

Clearly in the rest frame $\vec{p} = 0$ but E cannot also be zero or the above equations would give zero energy and momentum in all frames. Let the rest frame energy be E_0. Consider a passive transformation of the object at rest (i.e. $u = 0$) in the x direction by v, so that the object we are considering is moving in the opposite direction with $u' = -v$. We can define $\beta'_u = u'/c = -\beta$ and $\gamma'_u = 1/\sqrt{1-\beta'^2_u} = \gamma$. Dropping the subscript of p_x for clarity, we get

$$E' = \gamma E_0 = \gamma'_u E_0, \qquad p'c = -\gamma\beta E_0 = \gamma'_u \beta'_u E_0$$

We can find the value of E_0 by requiring consistency with the classical momentum. For small β'_u, then $\gamma'_u \approx 1$ so $p' = \gamma'_u \beta'_u E_0/c \approx u' E_0/c^2$. But we know non-relativistically that for a particle moving at speed u in the x direction $p = mu$ so we conclude $E_0/c^2 = m$, i.e.

$$E_0 = mc^2$$

which is an equation you possibly might have seen before. This energy is often called the 'rest energy' or 'rest mass energy'. Its implications are profound and affect all kinematics, as well as resulting in nuclear energy and weapons; this will be discussed in the next few lectures.

Now we have the energy in the rest frame, we can find the exact formulæ for energy and momentum in any other frame. Substituting into the above gives

$$E' = \gamma'_u mc^2, \qquad p'c = \gamma'_u \beta'_u mc^2 \quad \text{so} \quad p' = \gamma'_u mu'$$

Since these equations only involve quantities all measured in the same frame, we will drop the primes from now on. The change in energy when moving, compared to being at rest, is by definition what we mean by kinetic energy K. Hence for a moving object, generally $K = E - mc^2$ which can also be written as $K = (\gamma_u - 1)mc^2$.

For a velocity $\vec{u}$ in any direction, the momentum term above obviously generalises to $\vec{p} = \gamma_u m\vec{u}$. These equations give E and $\vec{p}$ in terms of $\vec{u}$, but it is useful to also have the inverse relations. Since

$$\vec{u} = \frac{\gamma_u mc^2 \vec{u}}{\gamma_u mc^2} = \frac{\vec{p}c^2}{E} \qquad \text{then} \qquad \vec{\beta}_u = \frac{\vec{u}}{c} = \frac{\vec{p}c}{E}$$

It is also sometimes convenient to have γ_u and $\gamma_u \vec{\beta}_u$ which are

$$\gamma_u = \frac{\gamma_u mc^2}{mc^2} = \frac{E}{mc^2} \qquad \gamma_u \vec{\beta}_u = \frac{\vec{p}c}{E}\frac{E}{mc^2} = \frac{\vec{p}c}{mc^2}$$

One comment on the expression for $E = \gamma_u mc^2$. As $u \to c$, then $\gamma_u \to \infty$ and hence $E \to \infty$ too. Therefore, this illustrates yet another reason why we consider nothing can go faster than light; it takes infinite energy to speed an object up to light speed, so achieving $u = c$ is not possible, let alone $u > c$.

4 Small speed approximation

We already used $p \approx mu$ for small speeds, but we can also ask what is the energy in this approximation? We need to improve on simply $\gamma_u \approx 1$ or else we will just get $E \approx mc^2$ again. We can get a better approximation for γ_u by considering it as a binomial expansion, for which

$$(1+a)^b \approx 1 + ab$$

Since $\gamma_u = (1-\beta_u^2)^{-1/2}$, then with $a = -\beta_u^2$ and $b = -1/2$

$$\gamma_u = (1-\beta_u^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)\left(-\beta_u^2\right) \approx 1 + \frac{1}{2}\beta_u^2$$

Using this

$$E = \gamma_u mc^2 \approx mc^2 + \frac{1}{2}mc^2\beta_u^2 \approx mc^2 + \frac{1}{2}mu^2$$

Hence for low speeds, the kinetic energy is $K \approx mu^2/2$. This is of course just the non-relativistic kinetic energy, which is now seen to be an approximation to a more complicated expression. The dependence of E and pc on β_u is shown in the left of Fig. 1.

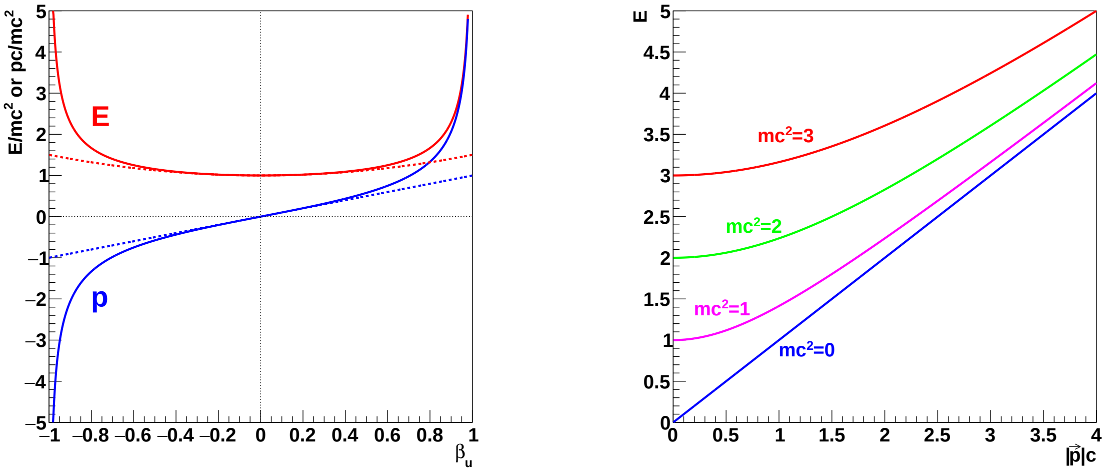

Figure 1: Left: Energy and momentum as a function of β_u. The dashed lines show the classical approximations. The energy and momentum are plotted in units of mc^2. Right: Energy as a function of momentum magnitude for various values of mass.

5 Energy-momentum invariant

Since all four-vectors transform in the same way, the length-squared for every four-vector must be an invariant. For any four-vector $(a, \vec{b})$, the length-squared is $a^2 - |\vec{b}|^2$. What is this for the energy-momentum four-vector? We can find this easily

$$E^2 - (pc)^2 = \gamma_u^2 m^2 c^4 - \gamma_u^2 \beta_u^2 m^2 c^4 = \gamma_u^2 (1 - \beta_u^2) m^2 c^4 = m^2 c^4 = (mc^2)^2$$

which is the rest energy squared. This is in fact obvious from considering the invariant in the rest frame, where $\vec{p} = 0$ and $E = E_0 = mc^2$. As mass is a real value, then $(mc^2)^2$ is always positive and you may hear the four-momentum referred to as a 'time-like' four-vector, although it has no implications for causality as it does not refer to time-ordering of events.

The above equation is often rearranged to give

$$E^2 = (pc)^2 + (mc^2)^2 = p^2 c^2 + m^2 c^4$$

In this form, it is clear that E is always greater than both pc and mc^2. A plot of how energy varies with momentum is shown in Fig. 1. It is clear that $E \to mc^2$ as the momentum goes to zero (i.e. the energy is just the rest energy) and it asymptotes to $E \to |\vec{p}|c$ for high momentum (i.e. the mass becomes negligible compared with the momentum). This is expanded to show the whole four-vector Lorentz transformation space in Fig. 2, which illustrates that the four-momentum is physical only in the upper central region.

6 Velocity transformation

We can now easily rederive the result on the velocity transformation which we saw in Lecture 4 using the four-momentum. For motion along the x axis, $u = pc^2/E$ and under a Lorentz transformation also along the x axis

$$E' = \gamma(E - \beta pc), \qquad p'c = \gamma(pc - \beta E)$$

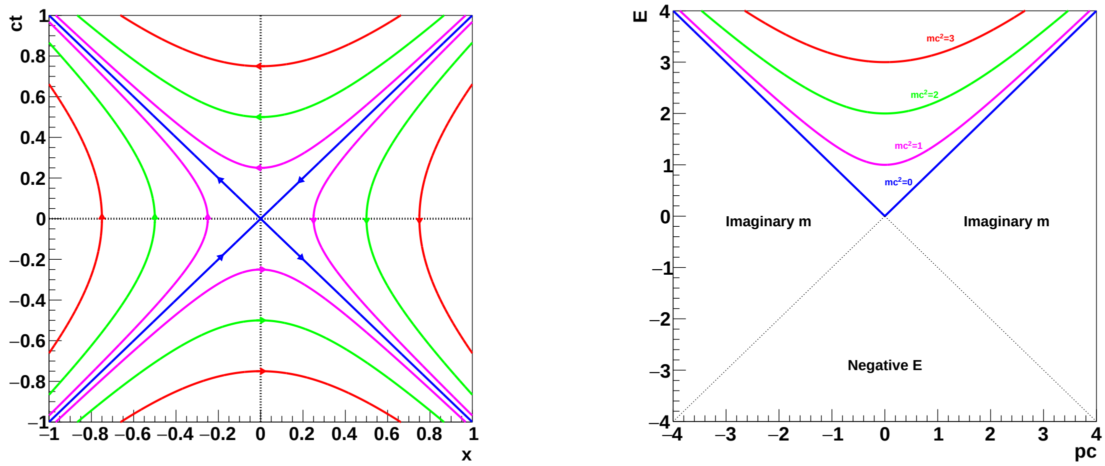

Figure 2: Left: The Lorentz transformations for ct and x. Right: Energy as a function of momentum for various values of mass expanded to show the whole four-vector Lorentz transformation space, showing why the four-momentum is unphysical in the other regions.

Hence

$$u' = \frac{p'c^2}{E'} = \frac{\gamma(pc^2 - \beta cE)}{\gamma(E - \beta pc)} = \frac{pc^2 - vE}{E - vp} = \frac{pc^2/E - v}{1 - vp/E} = \frac{u - v}{1 - vu/c^2}$$

which is, of course, the same as we got previously.

7 Photons

In the case of photons, the E and p equations need to be used carefully. We have seen no object can actually be accelerated to the speed of light because it would take infinite energy. However, photons (being light itself) do go at the speed of light. This is only possible because they are a special case in two ways; they have zero mass and always have speed c.

Putting $m = 0$ into the equations for E and p in terms of β_u seems to imply E and p would be zero. However, $|\vec{u}| = c$ and hence $|\vec{\beta}_u| = 1$, so γ_u becomes infinite. Putting this into the equations implies E and p would be infinite. The trick is to realise that m and γ_u only appear multiplied together as $\gamma_u m$. Although infinity times zero is not well-defined, you can roughly think of the product as being finite. The correct way to handle this is to eliminate this product from the equations. In fact, we already did this in writing $\vec{\beta}_u = \vec{p}c/E$. Since $|\vec{\beta}_u| = 1$ for photons, this means

$$E = |\vec{p}|c$$

which does indeed hold for a photon and is also shown in Fig. 1. In fact, this can also be found from the one equation which has m *without* a factor of γ_u, namely

$$E^2 = (pc)^2 + (mc^2)^2 = (pc)^2 \quad \text{for } m = 0$$

Note that $|\vec{u}| = c$ and $m = 0$ must *both* be true for photons because if only one were true, E and $\vec{p}$ would be zero (if $m = 0$) or infinite (if $|\vec{u}| = c$).

The above equation raises another issue. As shown in the last lecture (and is also true classically), for a particle with a given mass and velocity, the energy and momentum are fixed. We have said for all photons that $m = 0$ and speed $= c$ so do all photons have the same E and p?

No; they also vary in frequency f (or equivalently wavelength λ). This actually seen most clearly if we consider Quantum Mechanics. The Planck-Einstein relation states $E = hf$ while the de Broglie relation states $|\vec{p}| = h/\lambda$. Hence, the photon energy and momentum are determined by the light frequency (or equivalently wavelength). However, we might be worried because Quantum Mechanics is not a relativistic theory. For the case of the photon, since $E = |\vec{p}|c$, then in Quantum Mechanics this gives $hf = hc/\lambda$, which means $f\lambda = c$ as required. Hence, Relativity is consistent with Quantum Mechanics with regard to these relations.

8 Revision questions

- What is the form of the energy-momentum four-vector?
- Is it possible to accelerate an object to speed c?
- How do the properties of photons differ from the classical limit?
- What is relation between E and p for photons?