First Year Special Relativity – Lecture 8
Rest mass energy and particle decays

Mitesh Patel, 30th May 2023

1 In this lecture

- A word on units;
- The meaning of mass in Relativity;
- Look at energy and momentum conservation and the implications for mass conservation;
- Look at the kinematics of a particle decays.

2 Introduction

The material in this lecture is covered in Young and Freedman, Sec. 37.8 and in McCall in Secs. 6.3 to 6.5.

In the previous lecture, we saw that energy and momentum form a four-vector, like space and time, and hence they also undergo Lorentz transformations. In classical mechanics, energy and momentum are conserved and this is essential for calculating what happens when objects collide. The same is true relativistically. However, mass needs to be handled quite differently.

3 Units

In practice, it is very hard to get macroscopic objects to move at any significant fraction of c. Hence, the experimental tests of four-momentum in Special Relativity effectively all involve elementary particles. The particle energies involved are usually far smaller than those we experience in our everyday life. We will use units which are better suited to these scales than SI units. Energies will be measured in electron-volts (eV) which is the energy acquired by an electron (or any particle with a charge e) in accelerating through $1\,\mathrm{V}$. There are also keV ($10^3\,\mathrm{eV}$), MeV ($10^6\,\mathrm{eV}$), GeV ($10^9\,\mathrm{eV}$), etc.

The combination pc also has dimensions of energy and so can be given in the same units. Hence, p itself is energy over c, so momentum can be written in units of eV/c (or keV/c, etc.). Similarly, mc^2 can be given in eV and hence m can be given in eV/c^2. For example, in SI units the electron mass is $m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$ so $m_e c^2 = 8.2 \times 10^{-14}\,\mathrm{J}$. Since $1\,\mathrm{eV} = 1.60 \times 10^{-19}\,\mathrm{J}$, then $m_e c^2 = 5.11 \times 10^5\,\mathrm{eV}$ or $0.511\,\mathrm{MeV}$. Therefore $m_e = 0.511\,\mathrm{MeV}/c^2$.

4 Implications of the rest mass energy

The existence of the rest mass energy $E_0 = mc^2$ has profound physical consequences. This equation can be considered in both directions, i.e. that energy implies mass ($m = E_0/c^2$), but also that mass implies energy ($E_0 = mc^2$). Let's consider both these in turn.

Consider a box containing some material of any type, where the total momentum of the box and material is zero, so the total system is at rest, even if not all the parts are stationary. Classically, its mass would be simply the sum of the masses $\sum_i m_i$ of the box and its contents, and so would be a constant. However, $m = E_0/c^2$ means that its mass is the total energy of everything divided by c^2, where the energy includes all the rest mass energies $m_i c^2$, but can also have contributions from any other types of energy of the constituents; kinetic, potential, thermal, binding, etc. E.g. if we heat up the box, then the material inside has more thermal energy (or equivalently, kinetic energy of the molecules it is made of) and so the total energy increases. Hence, the heated box has a higher mass than the cold box. Similarly, your phone weighs more when it is fully charged. Another example: two objects with the same sign electrostatic charge will have a higher potential energy if they are placed close together. Their total energy will be $E = m_1c^2 + m_2c^2 + V$ and so the total mass will be higher than the sum of the two masses by V/c^2. Conversely, if they have opposite sign charges, the total mass will be lower. A physical example of this is the hydrogen atom, which is made by bringing an electron and a proton together. If these two particles are initially at rest, they will bind together with a binding energy (in the lowest energy configuration) of around 13 eV. This energy is emitted as a photon when the atom is formed. Classically we would say the hydrogen atom has a mass of $m_e + m_p$. In Relativity, as both the electron and proton are initially at rest, the original system has a total energy of $(m_e + m_p)c^2$. The hydrogen atom is formed by emitting 13 eV, so the remaining energy of the hydrogen atom is $(m_e + m_p)c^2 - 13\,\mathrm{eV}$ in its rest frame. This means its mass must be $m_e + m_p - 13\,\mathrm{eV}/c^2$; it is actually less than the classical expectation by 13 eV. However, $m_p = 938.3\,\mathrm{MeV}$ and $m_e = 0.511\,\mathrm{MeV}$ so this is a correction of 13 eV in 938.8 MeV, or approximately a change of 10^{-8} of the mass. This is typical of mass changes in classical physics, which is why the assumption of the sum of masses being constant is a very good approximation for low speeds.

What about considering how mass implies energy? Given that we now know that mass can be changed, this means an increase in mass requires energy input, or conversely a decrease in mass will emit energy. In particular, if we have a reaction where the final objects have less mass than the original objects, e.g. the binding energy is increased, then the difference of the rest mass energy will be liberated. The hydrogen atom is a case in point, where the energy is emitted as a photon. However, the protons and neutrons in nuclei are held together by the strong force which has much higher binding energies. Hence, nuclear reactions can result in much larger mass changes and hence much larger energy releases. This is the basic physics underlying atomic weapons and power stations.

One final point on this topic: we talk about the electron having a mass with a very specific value, as given above. Why isn't this able to vary as well? The electron is one of what we consider to be the fundamental particles which have no internal structure. There is no way to put energy into an electron at rest; you cannot "heat" an electron. Hence, fundamental particles do have well-defined fixed values for mass.

5 Mass conservation

Energy and momentum are conserved in all processes for an isolated system; this is due to fundamental properties of time and space and holds both classically and relativistically. However, as discussed above, the mass of an object can change so the sum of the masses will not in general be conserved in reactions. This only appears to be true in classical physics because we can only rearrange the electrons and nuclei and don't change the nuclei themselves, or the numbers or types of the fundamental particles. The changes in mass are then very small and so negligible; again classical physics is an approximation to the exact relativistic theory.

However, in what initially sounds like a contradiction to the above, there *is* a mass quantity which is conserved. This is the total mass m_T defined through the equation

$$E_T^2 = p_T^2c^2 + m_T^2c^4 \qquad \text{so} \qquad m_T = \frac{\sqrt{E_T^2 - p_T^2c^2}}{c^2}$$

where E_T and $\vec{p}_T$ are the total energy and momentum. Since E_T and $\vec{p}_T$ are conserved, then clearly m_T will be as well. Note that m_T is defined using the total energy and momentum and is not the sum of the individual particle masses; $m_T \neq \sum_i m_i$. It is often the case that m_T does not correspond to the mass of any physical particle in the system.

6 Particle decay

We will consider a simple example of the above which is also an extreme case in terms of changing mass. The Higgs boson was discovered in 2012 at CERN and one way it was observed was when it decayed to two photons; $H \to \gamma\gamma$. (Note, γ is the standard symbol for a very high energy photon; do not confuse this with the Lorentz transformation parameter!) Hence, before the decay occurs, there is a Higgs boson which has a mass of $m_H = 125.2\,\mathrm{GeV}/c^2$ (which is around 130 times the proton mass). After the decay, we have two photons which both have zero mass. It is clear the initial particle mass is certainly not equal to the sum of the two final particle masses. The Higgs mass has been converted (using $E = mc^2$) into energy and this energy is carried by the photons. Consider this decay in the Higgs rest frame as shown in Fig. 1.

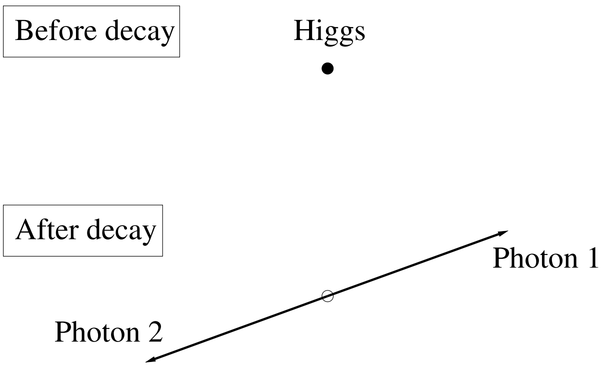

Figure 1: Particles both before and after the decay of a Higgs to two photons $H \to \gamma\gamma$ in the rest frame of the Higgs.

The total four-momentum is then just the Higgs four-momentum and so $(E_T, \vec{p}_T) = (E_H, \vec{p}_H) = (m_H c^2, \vec{0})$. To keep the total momentum zero, the two photons must be moving back-to-back with the same magnitude of momentum and hence, since $E = |\vec{p}|c$, they have the same energy E_γ. Therefore $E_T = m_H c^2 = 2E_\gamma$ so $E_\gamma = m_H c^2/2 = 62.6\,\mathrm{GeV}$. The photons equally share the energy created from the Higgs mass in this frame. The total energy ($E_T = m_H c^2$) and momentum ($\vec{p}_T = \vec{0}$) are unchanged so

$$m_T = \frac{\sqrt{E_T^2 - p_T^2 c^2}}{c^2} = m_H$$

is true both before and after the decay.

In a frame where the Higgs is moving, then the calculation can get very complicated as the photons can come off at arbitrary angles with respect to the direction of motion of the Higgs. The photon energies will not then be equal. However, the total four-momentum is still conserved, so

$$E_T = E_H = E_{\gamma 1} + E_{\gamma 2} \qquad \text{and} \qquad \vec{p}_T = \vec{p}_H = \vec{p}_{\gamma 1} + \vec{p}_{\gamma 2}$$

still holds. Clearly the invariant of the Higgs four-momentum, and hence also the total four-momentum, is $(m_T c^2)^2 = (m_H c^2)^2$ as before as invariants are the same in all frames. Critically, because the sum of two four-vectors is also a four-vector, then the invariant of the sum of the two photon four-momenta is also $(m_H c^2)^2$. This is in fact how the discovery of the Higgs was achieved. The Higgs lives for too short a time to be observed in a particle detector directly, but when it decays to two photons, these are easily detected and their energies and positions measured. By calculating the four-momentum of each photon and adding these to give the total four-momentum, then the total invariant, and hence m_T, can be found from this. This should be consistent with m_H if the two photons came from a Higgs, and indeed an excess at a mass around $125\,\mathrm{GeV}/c^2$ was found.

In the more general case, a particle can decay to two 'daughter' particles with unequal masses. For example this could be a large nucleus undergoing alpha decay to result in a smaller nucleus and the alpha particle. If the decaying particle has mass m_0 and the daughter particles have m_1 and m_2, then we need $m_0 \geq m_1 + m_2$ as otherwise there would not be enough energy to create the two daughter particles.

Again, consider the situation in the decaying particle rest frame. The initial energy is $E = m_0c^2$ and initial momentum $\vec{p} = 0$. After the decay, energy conservation requires

$$E_1 + E_2 = m_0c^2 \tag{1}$$

The two daughter particles must again be back-to-back to conserve momentum, i.e. $\vec{p}_2 = -\vec{p}_1$ so $p_2^2 = p_1^2$. Their momenta magnitudes are related to their energies by

$$p_1^2c^2 = E_1^2 - m_1^2c^4, \qquad p_2^2c^2 = E_2^2 - m_2^2c^4$$

so momentum conservation requires

$$E_1^2 - m_1^2c^4 = E_2^2 - m_2^2c^4$$

This looks hard to solve but there is a trick. Rearranging gives

$$E_1^2 - E_2^2 = (E_1 + E_2)(E_1 - E_2) = m_1^2c^4 - m_2^2c^4$$

Using Eqn. 1 this becomes

$$E_1 - E_2 = \frac{m_1^2c^4 - m_2^2c^4}{m_0c^2} \tag{2}$$

Taking the sum and difference of Eqns. 1 and 2 gives

$$E_1 = \frac{m_0c^2}{2} + \frac{m_1^2c^4 - m_2^2c^4}{2m_0c^2} = \frac{m_0^2c^4 + m_1^2c^4 - m_2^2c^4}{2m_0c^2}$$

and

$$E_2 = \frac{m_0c^2}{2} - \frac{m_1^2c^4 - m_2^2c^4}{2m_0c^2} = \frac{m_0^2c^4 + m_2^2c^4 - m_1^2c^4}{2m_0c^2}$$

Note, the equation is symmetric under interchange of $1 \leftrightarrow 2$, as would be expected. From these, the momenta can be calculated using the initial equations above but it is quite messy. Note that the particle with the higher mass gets more total energy than $m_0c^2/2$ while the lower mass particle gets less total energy. This is in direct constrast to the Galilean transformations, where the same *e.g.* momentum vector would be added each particle to boost from one frame to another. If the two decaying particles are the same type, so $m_1 = m_2$, then $E_1 = E_2 = m_0c^2/2$, as would be expected by symmetry.

The kinetic energy of the outgoing particles can in principle be accessed e.g. by letting them collide with some material and causing heating. The maximum amount of usable 'liberated' energy in the rest frame is clearly $K_1 + K_2 = (m_0 - m_1 - m_2)c^2$.

7 Revision questions

- What is the difference between the conservation of mass classically and in Relativity?
- Why can we talk about fundamental particles having a specific mass?
- What quantity is conserved in Relativistic particle decays?
- For a two-body decay such as $H \to \gamma\gamma$, can you derive the energy of the final state particles?