First Year Special Relativity – Lecture 9
Particle reactions

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1 In this lecture

- The centre-of-mass frame;
- Particle reactions.

2 Introduction

The material in this lecture is covered in Young and Freedman, Secs. 37.7 and 37.8 and in McCall in Secs. 6.3, 6.4 and 6.6.

We saw that conservation of four-momentum allowed us to work out the energy and momentum of the particles produced in the decay of a heavier particle. Here, we will look at particle reactions rather than decays; these are characterised by having two initial particles rather than just one.

3 Constant, invariant and conserved quantities

It is important to distinguish between the three concepts of a quantity being constant, invariant or conserved. These are different things.

1. A *constant* quantity has a fixed value. Examples are the speed of light c, Planck's constant h and the electron mass m_e.

2. An *invariant* is a quantity that is unchanged under a transformation. Hence a Lorentz (or four) invariant is one that is unchanged under a Lorentz transformation. However, within an inertial frame, it can change with time. Examples are any four-momentum length-squared, so $c^2\tau^2$ for the $(ct, \vec{r})$ four-position and m^2c^4 for the $(E, \vec{p}c)$ four-momentum. In fact, the equivalent of the three-vector dot-product for two four-vectors is also a Lorentz invariant (in the same way as the three-vector dot-product is a scalar under rotations). For two four-vectors $(A, \vec{a})$ and $(B, \vec{b})$, the equivalent of the dot-product is $AB - \vec{a}.\vec{b}$. Hence, the length-squared is actually the dot-product of the four-vector with itself. One example is the dot-product of the four-position and four-momentum, which is $Et - \vec{p}.\vec{r}$. You may recognise this as (the negative of) the wave phase times $\hbar$ in quantum mechanics; the phase of a wave (such as the amplitude peak) must be the same in all frames since all observers will agree on the peak.

3. A *conserved* quantity does not change with time for a given physical system. It can have a different value in another frame but its value in that frame should also be conserved. Examples are the total energy E_T and the total momentum $\vec{p}_T$ of an isolated system; these change under Lorentz transformations but whatever their value in each frame, it does not change with time.

Some quantities satisfy more than one of these labels. The speed of light c is not only constant but is also invariant (from the second postulate) and conserved. While E_T and $\vec{p}_T$ are conserved, the total mass m_T is both conserved and invariant, which is why it is a very useful quantity to work with.

4 Centre-of-mass

In the previous lecture, we calculated the particle decays in the frame where the decaying particle was at rest, as this was a convenient choice. We could then boost the result to any other frame we wanted, e.g. if the decaying particle happened to be moving in our frame.

This idea that calculations can be easier in the rest frame can be generalised to more complicated systems. Consider two particles bouncing off each other, which is the relativistic equivalent of a classical billiard ball problem. Because we now know that the total mass m_T is an important quantity, we can define the 'centre-of-mass' (CM) frame to be the one where the total momentum is zero. (This is alternatively called the 'centre-of-momentum' frame.) The total energy E_T in this frame, often labelled as E_{CM}, is given by $E_{\mathrm{CM}} = E_T = m_T c^2$, even if there is no particle of mass m_T. Hence, E_{CM}, despite being called an energy, is actually invariant and conserved as it is directly related to m_T.

The CM energy is important as it determines what reactions can happen, as explained later in the lecture. With $E_T^2 = p_T^2 c^2 + m_T^2 c^4$ and m_T the same in all frames, then since the momentum is non-zero in any other frame than the CM frame, E_T is higher in those frames. This higher incoming energy does not change what reactions can happen; it is used to provide the kinetic energy for the whole system which has non-zero momentum in this frame. Hence, some energy in every other frame than the CM frame is 'wasted' and not available for the reaction.

5 Particle reactions

We will consider particle reactions where two incoming particles collide and can produce some number (at least one) of outgoing particles.

Consider a photon being deflected by colliding with an electron in the CM frame; we could write the reaction as $\gamma + e \to \gamma + e$. Hence both the initial momenta $\vec{p}_{\gamma\, i} = -\vec{p}_{e\, i}$ and final momenta $\vec{p}_{\gamma\, f} = -\vec{p}_{e\, f}$ balance to give overall momentum conservation. Energy conservation requires $E_T = E_{\gamma\, i} + E_{e\, i} = E_{\gamma\, f} + E_{e\, f}$. It does not take long to convince yourself that both particles have the same initial and final momentum magnitude, and hence keep the same initial and final energy; $E_{\gamma\, i} = E_{\gamma\, f}$ and $E_{e\, i} = E_{e\, f}$. Hence all that happens is they bounce off at some different angle. Note, this is identical to classical billiard ball elastic collisions; it is basically purely set by the energy and momentum conservation. Again, the final four-momentum can be Lorentz transformed to see how this appears in other frames. This reaction is known as Compton scattering, after the person who first did the relativistic calculation and subsequent experiment to demonstrate it was correct.

We may want to calculate the actual energy and momentum needed to give a particular E_{CM}. Most of the work for this has already been done. The trick is to 'pretend' there is a particle of just the right mass, i.e. m_T, which is created when the incoming photon and electron collide and this pretend particle then decays to give the outgoing photon and electron. Considering the decay step, then this is identical to the situation in the last lecture, where we derived the energy of the two outgoing particles in terms of the decaying particle mass. This was done in the rest frame of the decaying particle, which is the same frame as what we now call the CM frame. Since the pretend decaying particle mass is now $m_0 = E_{\mathrm{CM}}/c^2$, then this gives

$$E_1 = \frac{E_{\mathrm{CM}}^2 + m_1^2 c^4 - m_2^2 c^4}{2E_{\mathrm{CM}}} \qquad \text{and} \qquad E_2 = \frac{E_{\mathrm{CM}}^2 + m_2^2 c^4 - m_1^2 c^4}{2E_{\mathrm{CM}}}$$

where, for the example of Compton scattering, 1 could refer to the photon and 2 the electron. What about the energy of the incoming particles? This is simple once you realise that the energy and momentum conservation argument does not rely on the time order. Hence, the process of the incoming particles forming the pretend particle is identical in terms of energy and momentum conservation to the pretend particle decaying to these particles. Hence the same equations as above also hold.

However, there are more complicated cases where the particles react and different particles come out. This can change the sum of the particle masses (as happens in decays) and this change can be in either direction, so the final state could have more kinetic energy from 'destroying' mass, or the initial state kinetic energy could be absorbed into 'creating' mass. You probably know matter and antimatter annihilate each other if they come into contact. An explicit example is an electron e^- and its antiparticle, called the 'positron' e^+. These have identical masses but opposite charges, as indicated by their symbols. When these are brought together, they react and often create a pair of photons; the reaction is written as

$$e^+ + e^- \to \gamma + \gamma$$

In the CM frame, since $m_{e^+} = m_{e^-}$, then $E_{e^+} = E_{e^-} = E_{\mathrm{CM}}/2$, as can also be found from the general equations with $m_1 = m_2$. Since the photons have no mass, then to have equal and opposite momenta they must each have the same energy, so $E_{e^+} = E_{e^-} = E_\gamma$ and these are all equal to $E_{\mathrm{CM}}/2$. This is therefore a very simple case, due to having equal mass particles in both cases.

If we think in terms of the masses, then the initial state has a sum of particle masses of $2m_e$, while the final state particle mass sum is clearly zero. Therefore, this is a case where the initial masses are destroyed and the rest mass energy has been released, here going into the energy of the photons. This means the reaction can proceed even if the electron and positron have negligible kinetic energy, i.e. are effectively at rest before they touch each other. In the case they are moving very slowly then $E_{\mathrm{CM}} \approx 2m_ec^2$ and hence $E_\gamma \approx m_ec^2 \approx 0.511\,\mathrm{MeV}$. This reaction would therefore result in a particular line in the photon spectrum at the value of the electron rest mass energy. Such a line has been detected by gamma ray astronomy observations from sources such as black holes, where positrons can be created through other reactions before then annihilating with electrons and giving photons. However, this spectral line has also been used to search for antimatter in other parts of the galaxy. If there are antimatter stars, then there must be some boundary in the galaxy between the matter and antimatter parts. Interstellar gas (which does not have relativistic speeds) along that boundary would be a mix of matter and antimatter and so would be undergoing the above reaction. No such signals of annihilation along boundaries have been seen, so we believe the galaxy is purely made of matter.

6 Reaction thresholds

Another reaction which can occur is when the electron and positron annihilation produces a muon and its antiparticle, called an antimuon.

$$e^+ + e^- \to \mu^+ + \mu^-$$

The muon is effectively identical in its properties to the electron but has a much larger mass; $m_\mu c^2 = 106\,\mathrm{MeV}$ so $m_\mu/m_e = 207$. Hence, this is a case where the initial sum of particle masses $2m_e$ is less than the final sum of particle masses $2m_\mu$. This means kinetic energy in the initial particles is needed to form the final particles. By an identical argument to the photon case, we have $E_{e^+} = E_{e^-} = E_{\mu^+} = E_{\mu^-}$. However, the minimum value of E_{μ^+} and E_{μ^-} must each be $m_\mu c^2$, even if the muons have no kinetic energy. This means unless $E_{\mathrm{CM}} \geq 2m_\mu c^2$, the reaction cannot occur. There is a 'threshold' for the electron energies required for this reaction to occur, in contrast to the previous photon reaction where the reaction can always occur for any E_{CM}. Specifically, if $E_{e^+} = E_{e^-} < m_\mu c^2$, then there is not enough initial energy to make the mass of the pair of muons at all.

If we set the electron energies to be just at the threshold, so $E_{\text{CM}} = 2m_\mu c^2$, then there is just enough energy to create the muons but they will not have any kinetic energy. Hence, the muons will be created at rest. If electrons with more energy than this minimum are used, the energy above threshold becomes muon kinetic energy.

It should be clear that observers in any inertial frame will agree on whether muons are produced or not. As discussed previously, all other inertial frames than the CM frame will have a higher total energy. However, no matter what the energy in any frame, if $E_{\text{CM}} < 2m_\mu c^2$ then despite the higher energy, the reaction can still not occur. This is why E_{CM} (or equivalently m_T) is the critical energy value for a reaction, not the total E in any arbitrary observer frame.

The idea of a threshold is very general and does not just apply to cases with two final particles. If a reaction creates N final particles, then if the sum of all the final particle masses is greater than the sum of the two initial particle masses, there will be a energy threshold for the reaction to occur. The threshold will be $E_{\text{CM}} = \sum_i^N m_i c^2$ and if exactly at this threshold, all the final particles will be at rest, while above this threshold, they will be moving and so have kinetic energy. There is one important special case, which is $N = 1$, i.e. creating a single particle. As before, if the mass M of this particle is more than the sum of the masses of the incoming particles, there will be a threshold of $E_{\text{CM}} = Mc^2$. However, the single particle must have no momentum in the CM frame, as momentum is conserved. Therefore, if the incoming energy is raised above the threshold, the particle created cannot simply move to gain kinetic energy. This means a single particle can only be created if $E_{\text{CM}} = Mc^2$ with the reaction not happening for lower *or* higher energies. Hence, unless the incoming particle energies are set precisely to the right values, production of the single particle cannot occur. This type of reaction is called a 'resonance' because, in quantum mechanics, it is effectively the same mathematics as for a resonance in a harmonic oscillator.

The above discussion explains why modern high energy particle accelerators like the Large Hadron Collider at CERN in Geneva are built the way they are. Firstly, they use two beams which are collided head-on, so all the energy goes into E_{CM} and there is no 'wasted' kinetic energy for the overall system. Secondly, the high beam energies give a high E_{CM} and so are more likely to be above threshold for creating previously undiscovered heavy new particles. This is precisely how the Higgs was able to be created and hence discovered.

7 Revision questions

- What is the centre-of-mass frame and how can you find such a frame?
- Why is this frame useful when trying to solve for kinematic quantities?
- What dictates whether reactions can occur in the case when the final state particles are heavier than the initial state particles?