First Year Special Relativity – Lecture 6
Four-vectors and causality

Mitesh Patel, 23rd May 2023

1 In this lecture

- Four vectors;
- The separation between two events;
- Implications for causality.

2 Introduction

The material in this lecture is not covered in Young and Freedman. and partially in McCall in Secs. 5.8 and 5.13.

We have seen that ct and x mix into each under a Lorentz transformation along the x axis, quite like x and y do under a rotation around the z axis. We know that vectors have well-defined properties when rotated. We can take the analogy of Lorentz transformations and rotations further and combine time and space into a 'four-vector'.

3 Four-vectors

We form the 'space-time' four-vector from ct and $\vec{r}$ and we will write this vector as $(ct, \vec{r})$ or (ct, x, y, z). This is an equivalent notation to writing the normal position vector as (x, y, z). To be absolutely clear which vectors I mean, I will use 'three-vector' for the standard vectors you have met before Relativity. The four-vector $(ct, \vec{r})$ indicates a 'position' in the 4D space-time (sometimes called 'Minkowski space'). Clearly, a specific four-position is what we mean by an event, so another way to think of a Lorentz transformation is that it changes a four-vector in a well-defined way, just like a rotation changes a three-vector. In fact, the full equation for a Lorentz transformation along the x axis in terms of all four components of a four-vector is

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

while the second equation is the full rotation case, for comparison.

When we refer to the coordinates of a three-vector, we often write $\vec{v} = (v_x, v_y, v_z)$ or (v_1, v_2, v_3). For four-vectors, the time coordinate is written before the space coordinates and to preserve the numbering of the space coordinates, ct gets the label of '0', called the 'zeroth' coordinate. In fact, the notation used by researchers is mainly $(ct, \vec{r}) = (x^0, x^1, x^2, x^3)$ although I will not use this here; it can be a little confusing as x is being used for all four components.

As you know, the length of a three-vector $|\vec{v}|$ does not change under a rotation. Obviously therefore neither does the square of the length $|\vec{v}|^2 = v_x^2 + v_y^2 + v_z^2$. Similarly there is a length-squared for a four-vector but it is not quite what you might initially think. For the space-time four-vector, it is given by $S^2 = (ct)^2 - |\vec{r}|^2 = c^2t^2 - x^2 - y^2 - z^2$. Hence it does not simply sum all four of the squared components but you have to remember to subtract the space coordinate terms from the time coordinate term. Unlike a three-vector, the length-squared of a four-vector can therefore be positive, zero or negative. The reason for the S^2 definition is, as for a three-vector under rotations, the length-squared of a four-vector is unchanged by a Lorentz transformation. Since y and z don't change, the calculation to show this reduces to

$$\begin{aligned}(ct')^2 - x'^2 &= \gamma^2(ct-\beta x)^2 - \gamma^2(x-\beta ct)^2 = \gamma^2(c^2t^2 - 2\beta ctx + \beta^2x^2 - x^2 + 2\beta ctx - \beta^2c^2t^2) \\ &= \gamma^2[c^2t^2(1-\beta^2) - x^2(1-\beta^2)] = (ct)^2 - x^2\end{aligned}$$

As we discussed for rotations, values which do not change are called scalars or invariants and so the length-squared S^2 of a four-vector is an invariant, or more specifically a 'Lorentz invariant'.

We saw the curves along which events move under Lorentz transformations in Lecture 3; this is reproduced below, together with the rotation case for comparison. We now understand that those correspond to lines of constant length-squared, i.e. constant $c^2t^2 - x^2$ (given that y and z don't change). The diagram with these curves labelled by their Lorentz invariant values shows that the different parts of the figure have different signs for the length-squared.

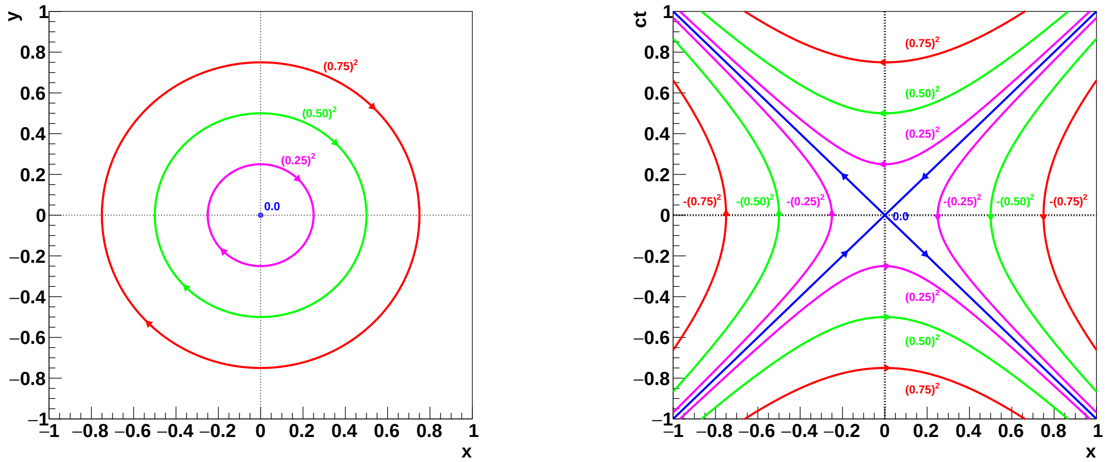

Figure 1: Lines of constant length-squared for rotations (left) and Lorentz transformations (right). The values of the length-squared in both cases are indicated.

Beware: there are several other notations and conventions. The above is closest to the modern notation used within General Relativity and particle physics. However, there are others. One is to call ct the fourth component and then the invariant interval is $x^2+y^2+z^2-c^2t^2$ which is changed in sign compared to the previous definition. There is even a notation where the fourth component is made imaginary and written as ict, so that the negative sign in the invariant interval is 'automatically' taken into account by simply summing the squares. This sounds attractive, but General Relativity generalises the constants $(+1, -1, -1, -1)$ to be variables. Hence, using ict only handles the Special Relativity case.

4 Event separation

Just as for three-vectors, we can do arithmetic with four-vectors. Two four-vectors can be added or subtracted by adding the coordinates separately. If we want the distance between two points $\vec{r}_1$ and $\vec{r}_2$ in three-vector space, then we know this distance squared is $(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$. We can consider this as taking the vector difference of the two vectors $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ and then taking the length-squared $|\Delta\vec{r}|^2$. It should be clear that under rotations, $\Delta\vec{r}$ will behave just like all other three-vectors.

Again, the same holds for four-vectors and events. The 'separation' between two events is defined to be the length-squared of the four-vector resulting from subtracting the four-vectors of the two events. This four-vector is sometimes written as $(c\Delta t, \Delta\vec{r}) = (ct_2 - ct_1, \vec{r}_2 - \vec{r}_1)$. This difference is also a four-vector and so obeys the Lorentz transformations. The separation is $\Delta S^2 = c^2\Delta t^2 - |\Delta\vec{r}|^2$ and is also clearly a Lorentz invariant. If we consider two events, one at the origin $t = 0, \vec{r} = 0$ and the other at ct and $\vec{r}$ then the separation is $\Delta S^2 = c^2t^2 - |\vec{r}|^2$ and so corresponds to what we discussed earlier in the lecture. Events with different signs of the separation have very different properties with respect to each other.

5 Causality

The sign of the invariant interval between two events is crucial for understanding one of the most important concepts in Relativity, namely 'causality'. The base of this word is 'cause' and it concerns whether one event can cause, or more generally affect, another. If they can, they are said to be 'causally connected'.

Consider two events where the second is close in position to the first, but later in time. It is obvious that the first can affect the second. How far apart can the two events be such that the first can actually affect the second? We have assumed nothing can go faster than light so if an object travelling at a speed up to the speed of light from the first event can get to the second event, then clearly the first can affect the second. Hence, we require the distance to be less than or equal to the distance light can go in that time. The events must satisfy $|\Delta\vec{r}| \leq c\Delta t$ which means $|\Delta\vec{r}|^2 \leq c^2\Delta t^2$ or $\Delta S^2 = c^2\Delta t^2 - |\Delta\vec{r}|^2 \geq 0$, i.e. the invariant interval must not be negative. This is the condition for one event to be able cause the other.

Let's turn the argument around: when would two events *not* be able to affect each other? One obvious case is when they are separated in space but happen at exactly the same time. There is clearly no way any object or signal can move between two separated events instantaneously without going faster than c, which we assume is not allowed. The more general case is that a signal even at the speed of light would not get between them in the time available. This means the two events must satisfy $|\Delta\vec{r}|^2 > c^2\Delta t^2$ or $\Delta S^2 = c^2\Delta t^2 - |\Delta\vec{r}|^2 < 0$. Hence, as you might have guessed, the condition is that the invariant interval is negative. These events are called 'causally unconnected'.

One worry is that because events have different times in different inertial frames, could one appear to affect the other in one frame and not the other? The answer is no; since the condition depends on the invariant interval, which is the same in all frames, then the ability of one event to affect the other is the same in all frames. Clearly this must be absolutely required for physical laws to obey any logic. In fact, the speed limit of c applies not just to physical objects but in fact even to information. If the events can communicate in any way using faster-than-light signals, then there will be a frame where an earlier event was caused by a later event in that frame, which breaks all logic. Hence, the requirement of causality is very strong and limits all things which can influence events not to go faster than light.

Let's look at the Lorentz transformation diagram in Fig. 1 again. With one event at the origin, the other will move along the lines of constant separation. The whole of the central upper quadrant of the figure has a positive separation so all events there can be affected by an event at the origin. Note, the time of the second event is always positive, i.e. later than the time of the first event. You can always find a particular boost which moves the second event along the line until it is directly above the first event. This puts the second event at $x = 0$, i.e. so the two events only differ in time and not space. More generally, you can change the position order of the events, so if the first event is to the right of the second in one frame, it can become to the left of the second event in another frame. The events in the lower central quadrant also have a positive separation and here their times are always earlier than the origin; in this case they can cause the event at the origin but not vice versa. Any events which can influence, or be influenced by, a specific event are said to be within the 'light-cone' of that event, illustrated in Fig. 2. The name comes from considering the shape in two space dimensions (as it is hard to visualise in all three).

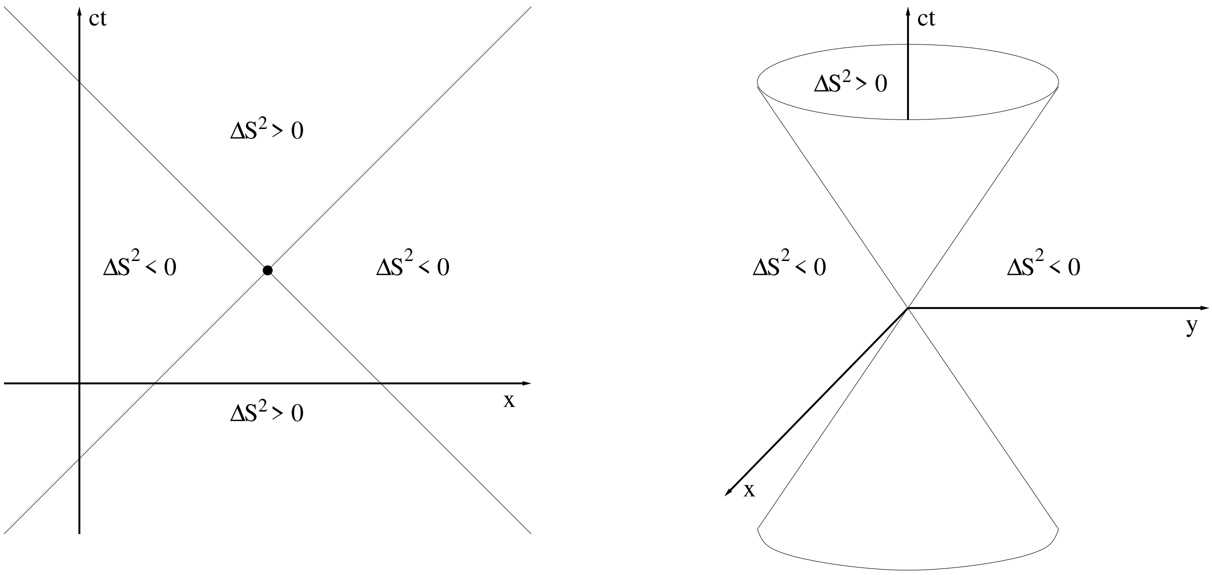

Figure 2: Left: The light-cone for the event marked by a circle and corresponding separations in ct, x. Right: 3D visualisation of light-cone in ct, x, y for an event at the origin.

What about the two quadrants to the left and right in Fig. 1? These have a negative separation from the origin and so the events cannot affect each other. Here, the time difference of the events can change between being positive and negative under boosts. Hence these are the ones for which the time order of the events changes in different frames. Indeed, there is always one particular boost which moves the event along the curve until it makes the time difference zero so that the two events are simultaneous in that frame. However, they will not be simultaneous in any other frame. The one exception is both events being at $x = 0, t = 0$ which are then simultaneous in all frames. Note that events cannot move from the left quadrant to the right quadrant, so the x order is always preserved. In summary:

1. For $\Delta S^2 > 0$, the time order of two events is the same in all frames and the first can affect the second. These events are called 'time-like' separated because there is always one frame in which they have no difference in position, but are separated in time. The separation is therefore $\Delta S^2 = c^2\tau^2$ where τ is the proper time between the events. The space order is different in different frames.

2. For $\Delta S^2 < 0$ the time order of two events is different in different frames and so they cannot logically affect each other. The space order is always preserved. These events are called 'space-like' separated because there is always one frame in which they have no difference in time but are separated in space.

3. There is a final, special case where ΔS^2 is exactly zero. The events are connected only by a light-speed signal and lie exactly on the diagonals of the light cone. They retain their time (and indeed space) order for any boost. In principle the first can affect the second; hence they are causally connected, as for time-like separated events. However, because c is the same in all frames, they always lie on the diagonals and there is no frame in which they

have either a zero time or a zero space difference. These are called 'light-like' separated, for obvious reasons.

Note, the time dilation formula can only be applied to time-like separated events because for space-like separated events, there is no frame in which the events are at the same position. Similarly, the length contraction formula can only be applied to space-like separated events, because time-like separated events cannot happen at the same time.

6 Tachyons

We have said it is essential for logical consistency that if one event causes the other, that this event happens first in every possible inertial frame. This seems to be true from the above so everything is fine.

However, imagine two events where a particle going *faster* that the speed of light moved from the first to the second. Such hypothetical particles have a general name of 'tachyons' (from the Greek 'tachy' meaning 'rapid'). Although we have assumed this doesn't happen, this does not sound so bad; the first event still happens before the second so it does sound logical. However, if the two events are connected by a particle going faster than light then they must have a negative separation. This means the order of the two events is different in some frames than in others. If you define several events along the tachyon world line and Lorentz transform them to such a frame, you will find the tachyon is apparently going in the reverse direction along the world line i.e. backwards in time compared to the original frame. This does lead to logical inconsistencies and so it is generally assumed tachyons cannot exist; they have certainly never been found experimentally.

7 Revision questions

- Write out a space-time four vector in 4 component terms and in abbreviated 2 component form;
- For a generic 4-vector, what quantity is invariant under Lorentz transformations?
- What does the sign of the separation, S, imply between two events?
- Sketch the ct, x plane for lines of constant S, for an event at the origin in this plane, label the different ΔS regions;
- Which regions are "time-like", "space-like" and "light-like"?