Chapter 2
Stochastic Processes

Chapter 2: Stochastic Processes

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2.2.1 Stationary Processes

$$F_X(x_1,\ldots,x_n;t_1,\ldots,t_n)$$
$$= P\{X(t_1)\le x_1,\ldots,X(t_n)\le x_n\}$$

$$F_X(x_1,\ldots,x_n;t_1,t_2,\ldots,t_n)$$
$$= F_X(x_1,\ldots,x_n;t_1+\tau,t_2+\tau,\ldots,t_n+\tau)$$

2.2.1 Stationary Processes

Def.1 strict stationary

The joint distribution functions of the random vectors

$\left(X(t_1), X(t_2), \ldots, X(t_n)\right)$ and $\left(X(t_1+\tau), X(t_2+\tau), \ldots, X(t_n+\tau)\right)$

is the same for all τ, then the stochastic process X(t) is said to be strictly stationary.

$$F_X(x_1, \ldots, x_n; t_1, t_2, \ldots, t_n)$$
$$= F_X(x_1, \ldots, x_n; t_1+\tau, t_2+\tau, \ldots, t_n+\tau) \quad n=1,2,\ldots$$

The probability distribution of a strict stationary stochastic process is invariant against absolute time shifts.

2.2.1 Stationary Processes

- For a strictly stationary process:

  The one dimensional distribution functions do not depend on *t*.

$$F_X(x_1;t_1) = F_X(x_1;t_1+\tau) = F_X(x)$$

Thus,

$$\bar{x}(t) = E[x(t)] = m = \text{constant}$$

$$Var[X(t)] = \text{constant}$$

2.2.1 Stationary Processes

- For a strictly stationary process:

  The two dimensional distribution functions only depend on the time difference,

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; 0, t_2 - t_1) = F_X(x_1, x_2; \tau)$$

Thus,

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$= E[X(0)X(t_2 - t_1)] = R_{XX}(\tau)$$

$$C_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] - \overline{x}(t_1)\overline{x}(t_2)$$
$$= R_{XX}(\tau) - m^2 = C_{XX}(\tau)$$

2.2.1 Stationary Processes

- weakly stationary

Weakly stationary only defined for second-order moment processes.

If $E[X^2(t)]$ exists for $t \in T$, X(t) is called a second-order moment processes.

Usually the existence of the second moments of all X(t) is assumed.

2.2.1 Stationary Processes

Def.2 weakly stationary

a second-order process is said to be weakly stationary if it has properties:

$$\bar{x}(t) = E[x(t)] = m$$

$$C_{XX}(t_1, t_2) = C_{XX}(\tau) \qquad R_{XX}(t_1, t_2) = R_{XX}(\tau)$$

Weakly stationary

=wide-sense stationary

(stationary in wide sense)

=second-order stationary

Weakly stationary is a more relaxed condition for the stationarity of a stochastic process.

2.2.1 Stationary Processes

strictly stationary V.S. weakly stationary

- A strictly stationary process is not necessarily a weakly stationary process, since there are strictly stationary processes which are not second-order moment processes.

- But, if a second-order moment process is strictly stationary, then it is also weakly stationary.

2.2.1 Stationary Processes

- weakly stationary

Example 1. Random phase processes

X(t)=Acos(wt+ε)， t>0, whereas A and w are constants and ε is random variable uniformly distributed between –πandπ.

$$E[x(t)] = 0$$

$$C_{XX}(t_1, t_2) = \frac{A^2}{2}\cos w_0(t_2 - t_1)$$

$$= \frac{A^2}{2}\cos w_0\tau = C_{XX}(\tau)$$

2.2.1 Stationary Processes

- weakly stationary

Example 2. Random amplitude processes

X(t)=Ycos(wt)+Zsin(wt), t>0, whereas Y and Z are independent random variables, and EY=EZ=0, VarY=VarZ=σ^2.

$$E[x(t)] = 0$$

$$C_{XX}(t_1, t_2) = \sigma^2 \cos w(t_2 - t_1)$$

$$= \sigma^2 \cos w\tau = C_{XX}(\tau)$$

2.2.1 Stationary Processes

- weakly stationary

Example 3. Random amplitude processes

$$X(t)=\sum_{i=1}^{n}(A_i \cos w_i t + B_i \sin w_i t)$$

where A_i and B_i are all independent random variables, and $E(A_i)=E(B_i)=0$, $Var(A_i)=Var(B_i)=\sigma_i^2$.

2.1.4 Moments

Review

e.g.4.

Given: $X(t)$ and $Y(t)$ are two second-order moment processes.
$W(t)=X(t)+Y(t)$.

Obtain: $E[W(t)]$ and $R_{WW}(t_1,t_2)$

Sln: $E[W(t)] = E[X(t)]+E[Y(t)]$

$$R_{WW}(t_1,t_2)= R_{XX}(t_1,t_2)+ R_{YY}(t_1,t_2) + R_{XY}(t_1,t_2)+ R_{YX}(t_1,t_2)$$

If $E[X(t)]=E[Y(t)]=0$, $C_{XY}(t_1,t_2)=0$,
$X(t)$ and $Y(t)$ are uncorrelated,

then, $E[W(t)]=0$,

$$R_{WW}(t_1,t_2)= R_{XX}(t_1,t_2)+ R_{YY}(t_1,t_2)$$

$$C_{WW}(t_1,t_2)= R_{WW}(t_1,t_2)=C_{XX}(t_1,t_2)+ C_{YY}(t_1,t_2)$$

2.2.1 Stationary Processes

Def.3 covariance stationary

A second-order process is said to be covariance stationary if it has properties:

$$C_{XX}(t_1, t_2) = C_{XX}(\tau)$$

A wide-sense stationary process is also covariance stationary.

A covariance stationary process need not be wide-sense stationary .

2.2.1 Stationary Processes

e.g.5-1. A signal plus a noise (signal is a function of time)

Given: $v(t)$ is a deterministic signal, $X(t)$ is wide-sense stationary process. $Y(t)=v(t)+X(t)$.

Obtain: $E[Y(t)]$ and $Cov_{YY}(t_1,t_2)$

Mean of Y(t) is not a constant.

Sln: $E[Y(t)] = E[v(t)] + E[X(t)]=v(t) + E[X(t)]$

$C_{YY}(t_1,t_2)= R_{YY}(t_1,t_2) - E[Y(t_1)]E[Y(t_2)]$

$R_{YY}(t_1,t_2)= R_{XX}(t_1,t_2)+ R_{vv}(t_1,t_2)+R_{Xv}(t_1,t_2)+ R_{vX} (t_1,t_2)$

$= R_{XX}(t_1,t_2)+ v(t_1)v(t_2)+v(t_1)E[X(t_2)]+ v(t_2)E[X(t_1)]$

$E[Y(t_1)]E[Y(t_2)]$

$= E[X(t_1)]E[X(t_2)]+v(t_1)v(t_2)+v(t_1)E[X(t_2)]+ v(t_2)E[X(t_1)]$

$C_{YY}(t_1,t_2)= R_{XX}(t_1,t_2)- E[X(t_1)]E[X(t_2)]=$

2.2.1 Stationary Processes

e.g.5-2. A signal plus a noise (signal is a function of time)

$Y(t)=v(t)+X(t)$. Let $W(t)=Y(t)-v(t)$.

Obtain: $E[W(t)]$ and $Cov_{ww}(t_1,t_2)$

Sln:

$E[W(t)]=E[Y(t)]-v(t)= E[X(t)] =m$

Mean of W(t) is a constant.

$R_{WW}(t_1,t_2)=R_{XX}(t_1,t_2)=R_{XX}(\tau)$

W(t) is a wide-sense stationary process.

2.2.1 Stationary Processes

nonstationary processes

Example 1. Random amplitude processes (1)

X(t)=Acos(wt)， t>0, where A is a nonnegative random variable with $E(A) < \infty$

$$E[X(t)] = E(A)\cos\omega t$$

$$\begin{aligned} C_{XX}(t_1, t_2) &= E[(A\cos wt_1)(A\cos wt_2)] - E[X(t_1)]E[X(t_2)] \\ &= [E(A^2) - E^2(A)]\cos wt_1 \cos wt_2 \\ &= Var(A)\cos wt_1 \cos wt_2 \end{aligned}$$

2.2.1 Stationary Processes

- nonstationary processes

Example 1. Random amplitude processes (2)

$$Var[X(t)] = C_{XX}(t,t) = Var(A)\cos^2 \omega\, t$$

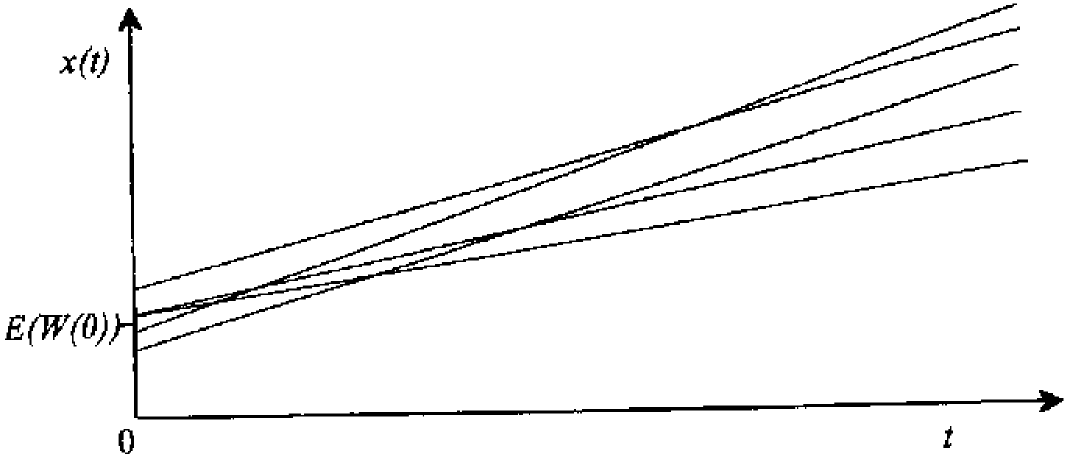
$$\rho_{XX}(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sqrt{Var[X(t_1)]Var[X(t_2)]}}$$

$$= \frac{Var(A)\cos w\, t_1 \cos w\, t_2}{\sqrt{Var(A)\cos^2 w\, t_1 Var(A)\cos^2 w\, t_2}} = \pm 1$$

nonstationary processes

Example 2. Process with linear sample waves (1)

- X(t)=Vt+W, V and W are assumed to be random variables with finite expected values and variances.



This model is used for describing the development of maintenance costs and of the degree of equipment wear over time.

$$E[X(t)] = E[V]t + E[W]$$

$$C_{XX}(t_1, t_2)$$
$$= E[(Vt_1 + W)(Vt_2 + W)] - E[X(t_1)E[X(t_2)]$$
$$= t_1 t_2 Var(V) + (t_1 + t_2) Cov(V, W) + Var(W)$$

2.2 Stationary Stochastic Processes

2.2.1 Stationary Processes

2.2.2 Ergodic Processes

2.1.1 Definition and examples

The statistical averages are calculated by the set of sample waves.

The time averages are calculated by a single sample wave.

2.2.2 Ergodic Processes

Def.1 Time average

(sample average, ensemble average)

The time average of a quantity is defined as

$$A[\cdot] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\cdot] dt$$

A[] is used to denote time average in a manner analogous to *E*[] for the statistical average.

Time average is the average of a single sample function.

2.2.2 Ergodic Processes

- Time average for a sample function x(t)

  (a lower case letter is used to imply a sample function)

$$\hat{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

- Time average for $x(t)x(t+\tau)$

  is called time autocorrelation function.

$$\hat{R}_{xx}(\tau) = A[x(t)x(t+\tau)]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt$$

2.2.2 Ergodic Processes

- For any one sample function of the process X(t), $\hat{x}$ is a number, and $\hat{R}_{xx}(\tau)$ is a deterministic function of τ .
- For all sample functions, $\hat{x}$ is a random variables, and $\hat{R}_{xx}(\tau)$ is a random wave.
- If X(t) is a stationary process, we obtain:

$$E[\hat{x}] = E[X(t)]$$

$$E[\hat{R}_{xx}(\tau)] = R_{XX}(\tau)$$

2.2.2 Ergodic Processes

- If $\hat{x}$ and $\hat{R}_{xx}(\tau)$ have zero variances, we obtain:

$$\hat{x} = E[X(t)]$$

$$\hat{R}_{xx}(\tau) = R_{XX}(\tau)$$

$E[X(t)]$ and $R_{XX}(\tau)$ are ensemble averages or statistical averages.

2.2.2 Ergodic Processes

Def.2 Ergodic Processes

A stationary process X(t) is said to be an ergodic process if the time average $\hat{x}$ and time correlation function $\hat{R}_{xx}(\tau)$ of a single sample wave is equal to the ensemble average.

$$\hat{x} = E[X(t)]$$

$$\hat{R}_{xx}(\tau) = R_{XX}(\tau)$$

2.2.2 Ergodic Processes

- Ergodicity is a very restrictive form of stationarity.
- Ergodicity property of a process means that a single sample wave can include all possible states in the set of sample wave if the observed time is sufficiently long.
- In the real world, we are usually forced to work with only one sample function of a process. Thereby statistical properties of a random process may be evaluated from analysis of a single sample waveform.
- We shall often assume a stationary process is ergodic to simplify problems.

Homework

- 2.9
- 2.14