Chapter 4: Spectral Analysis

Content:

4.1 Spectral Density Functions

4.2 Spectral Analysis of Linear Systems

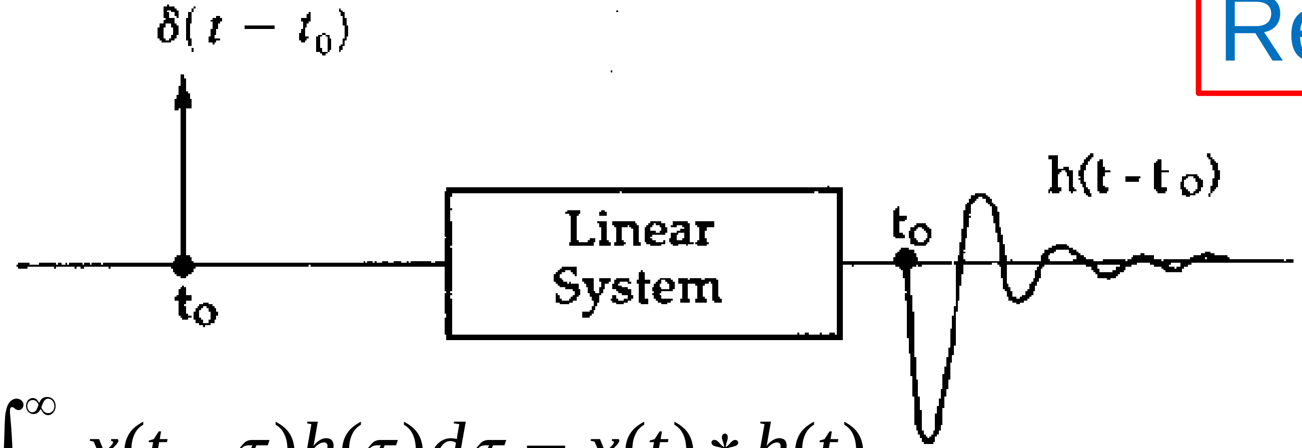
4.3 Spectrum of Amplitude-modulated Signals

4.4 Narrow-band Gaussian Processes

Linear System and Unit Impulse Response

Impulse Response Function

Review



$$y(t)=\int_{-\infty}^{\infty}x(t-\tau)h(\tau)d\tau=x(t)*h(t)$$

$$Y(\omega)=X(\omega)H(\omega)$$

Frequency response function: $H(\omega)$

Frequency response amplitude operator: $|H(\omega)|^2$

Stability of LTI: $\int_{-\infty}^{\infty}|h(t)|\,dt<\infty$

Bounded-input means bounded output.

3.3.1 Input and Output Mean Levels

If X(t) is a mean-square integrable random process, then

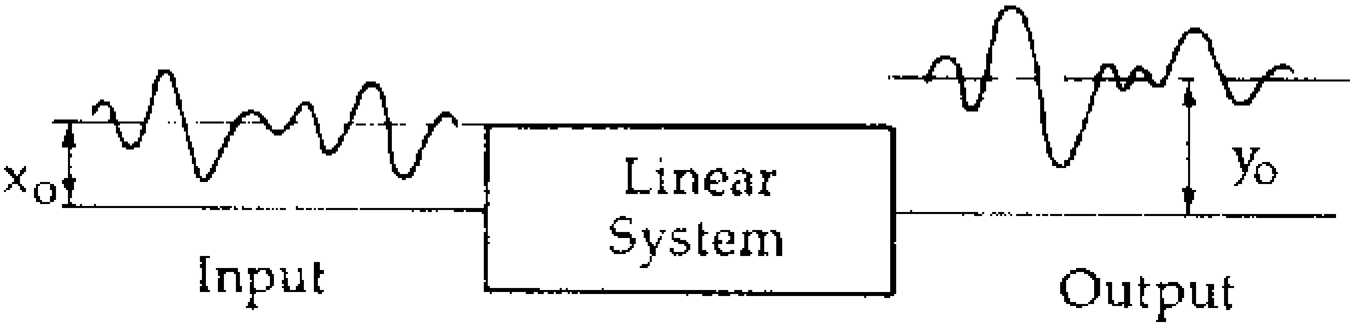
Review

$$E[Y(t)] = E[\int_{-\infty}^{\infty} X(t-\tau)h(\tau)d\tau]$$

$$= \int_{-\infty}^{\infty} E[X(t-\tau)]h(\tau)d\tau$$

If X(t) is a stationary process, then

$$\boxed{E[Y(t)] = m_X \int_{-\infty}^{\infty} h(\tau)d\tau = m_X H(0) = m_Y}$$



3.3.2 Input and Output Correlation Functions Relationship

Review

$$R_{XY}(t_1,t_2) \quad R_{YX}(t_1,t_2) \quad \boxed{\tau = t_1 - t_2}$$

$$\boxed{R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau)} \quad \boxed{R_{YX}(\tau) = R_{XX}(\tau) * h(\tau)}$$

$$\boxed{\begin{aligned} R_{YY}(\tau) &= R_{XY}(\tau) * h(\tau) = R_{YX}(\tau) * h(-\tau) \\ &= h(\tau) * h(-\tau) * R_{XX}(\tau) \end{aligned}}$$

If the input to a LTI is stationary, so does the output.

The input and output processes are jointly stationary.

If the input is a Gaussian process, so does the output.

The input and output processes are jointly Gaussian.

4.2 Spectral Analysis of Linear System

Input and Output Spectral Relationship

(a)
$$\boxed{S_{XY}(\omega) = H^*(\omega) S_{XX}(\omega)} \qquad \boxed{S_{YX}(\omega) = H(\omega) S_{XX}(\omega)}$$

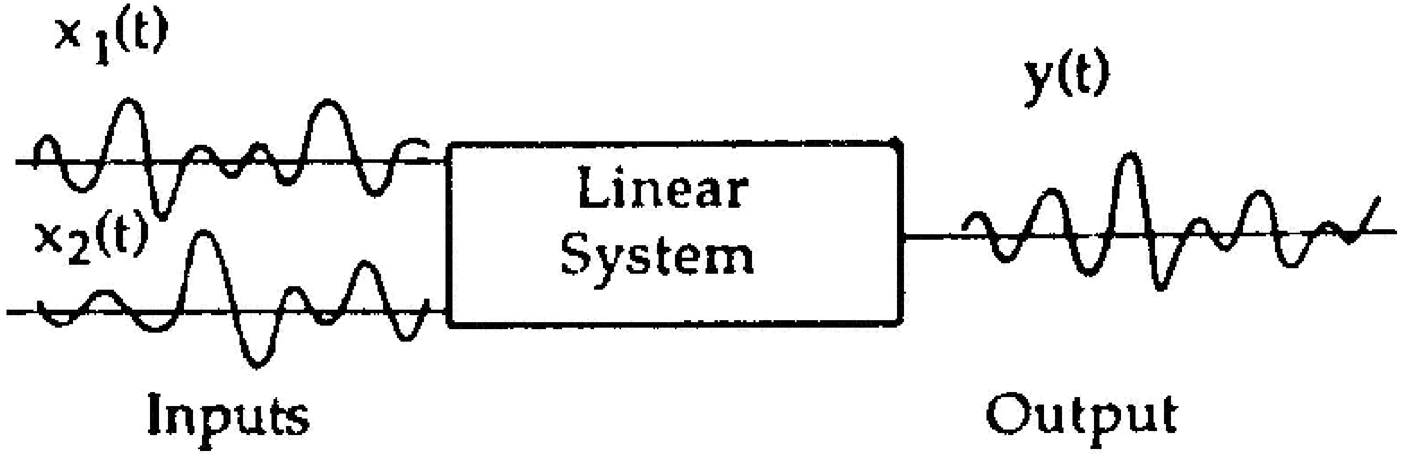
$$\boxed{\begin{aligned} S_{YY}(\omega) &= H(\omega) H^*(\omega) S_{XX}(\omega) \\ &= |H(\omega)|^2 S_{XX}(\omega) \end{aligned}}$$

(b) From definition of power spectrum

$$\begin{aligned} S_{YY}(\omega) &= \lim_{T\to\infty} \frac{1}{2T} |Y(\omega)|^2 \\ &= \lim_{T\to\infty} \frac{1}{2T} |H(\omega)|^2 |X(\omega)|^2 = |H(\omega)|^2 S_{XX}(\omega) \end{aligned}$$

4.2 Spectral Analysis of Linear System

e.g. Response of a system of dual inputs



$y_1(t)$: response of $x_1(t)$ $x(t) = x_1(t) + x_2(t)$

$y_2(t)$: response of $x_2(t)$ $y(t) = y_1(t) + y_2(t)$

$$R_{XX}(t_1, t_2) = R_{X_1X_1}(t_1, t_2) + R_{X_2X_2}(t_1, t_2) + R_{X_1X_2}(t_1, t_2) + R_{X_2X_1}(t_1, t_2)$$

$$R_{YY}(t_1, t_2) = R_{Y_1Y_1}(t_1, t_2) + R_{Y_2Y_2}(t_1, t_2) + R_{Y_1Y_2}(t_1, t_2) + R_{Y_2Y_1}(t_1, t_2)$$

4.2 Spectral Analysis of Linear System

If $x_1(t)$ and $x_2(t)$ are joint stationary processes, then

$$R_{XX}(\tau) = R_{X_1X_1}(\tau) + R_{X_2X_2}(\tau) + R_{X_1X_2}(\tau) + R_{X_2X_1}(\tau)$$

$$R_{YY}(\tau) = R_{Y_1Y_1}(\tau) + R_{Y_2Y_2}(\tau) + R_{Y_1Y_2}(\tau) + R_{Y_2Y_1}(\tau)$$

$$\boxed{S_{XX}(\omega) = S_{X_1X_1}(\omega) + S_{X_2X_2}(\omega) + S_{X_1X_2}(\omega) + S_{X_2X_1}(\omega)}$$

$$\boxed{\begin{aligned} S_{YY}(\omega) &= |H(\omega)|^2 S_{XX}(\omega) \\ &= S_{Y_1Y_1}(\omega) + S_{Y_2Y_2}(\omega) + S_{Y_1Y_2}(\omega) + S_{Y_2Y_1}(\omega) \end{aligned}}$$

Moreover, If $x_1(t)$ and $x_2(t)$ are uncorrelated and zero-mean, then

$$R_{XX}(\tau) = R_{X_1X_1}(\tau) + R_{X_2X_2}(\tau) \qquad \boxed{S_{XX}(\omega) = S_{X_1X_1}(\omega) + S_{X_2X_2}(\omega)}$$

$$R_{YY}(\tau) = R_{Y_1Y_1}(\tau) + R_{Y_2Y_2}(\tau) \qquad \boxed{S_{YY}(\omega) = S_{Y_1Y_1}(\omega) + S_{Y_2Y_2}(\omega)}$$

Coherency Function of Input and Output

$$S_{YX}(\omega) = H(\omega) S_{XX}(\omega)$$

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$|H(\omega)|^2 = \frac{|S_{YX}(\omega)|^2}{|S_{XX}(\omega)|^2} = \frac{|S_{YX}(\omega)|^2}{S_{XX}^2(\omega)}$$

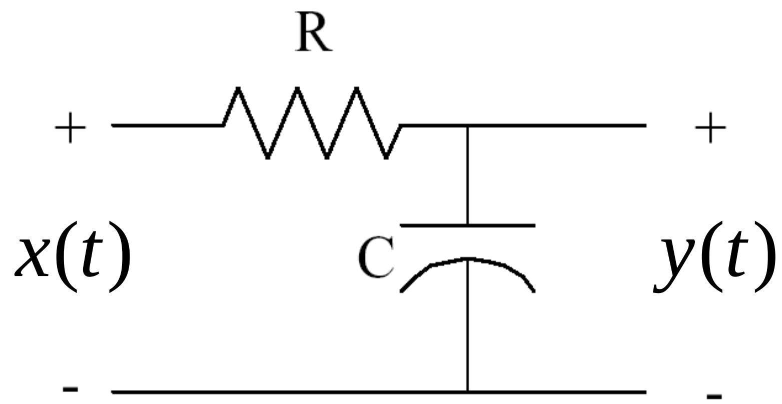
$$|H(\omega)|^2 = \frac{S_{YY}(\omega)}{S_{XX}(\omega)}$$

$$|S_{XY}(\omega)|^2 = S_{XX}(\omega) S_{YY}(\omega)$$

$$\gamma_{XY}(\omega) = \frac{|S_{XY}(\omega)|^2}{S_{XX}(\omega) S_{YY}(\omega)} = 1$$

4.2 Spectral Analysis of Linear System

e.g. A first-order RC low-pass filter



Let X(t) be a white noise,

$$R_{XX}(\tau) = \frac{N_0}{2}\delta(\tau)$$

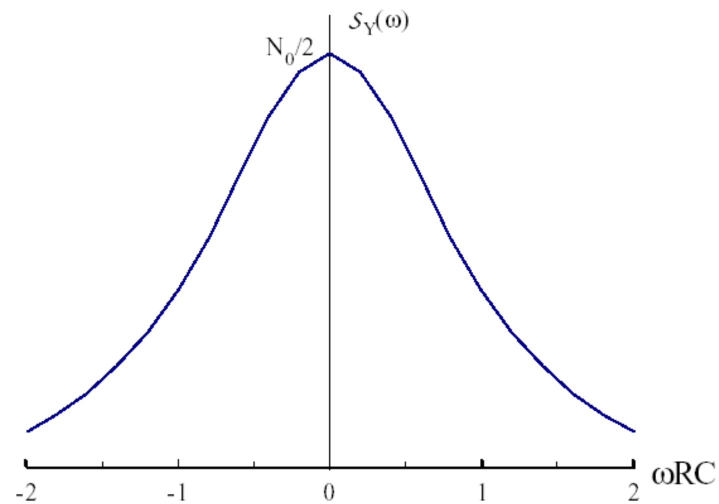
Obtain $S_{YY}(\omega)$

Sln:

$$S_{XX}(\omega) = \frac{N_0}{2} \qquad H(\omega) = \frac{1}{1+i\omega RC}$$

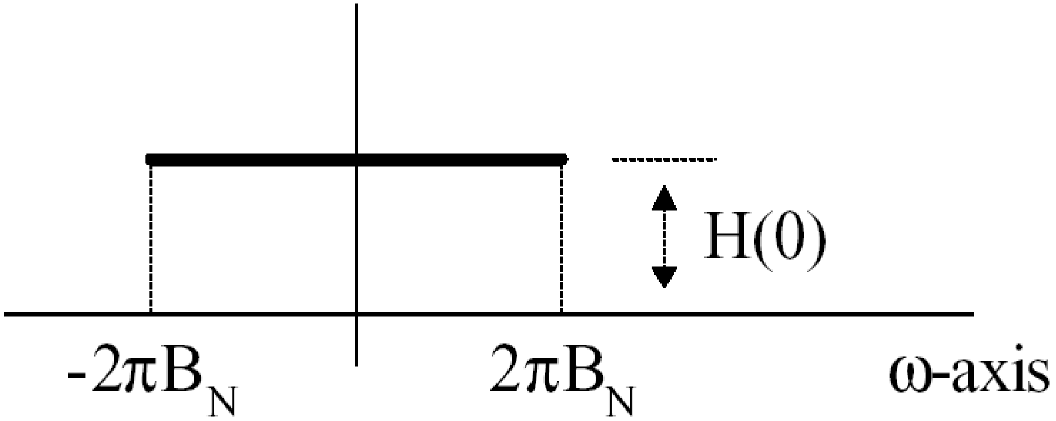
$$S_{YY}(\omega) = \frac{1}{1+i\omega RC}\frac{1}{1-i\omega RC}\frac{N_0}{2}$$

$$= \frac{N_0}{2}\frac{1}{1+(\omega RC)^2}$$



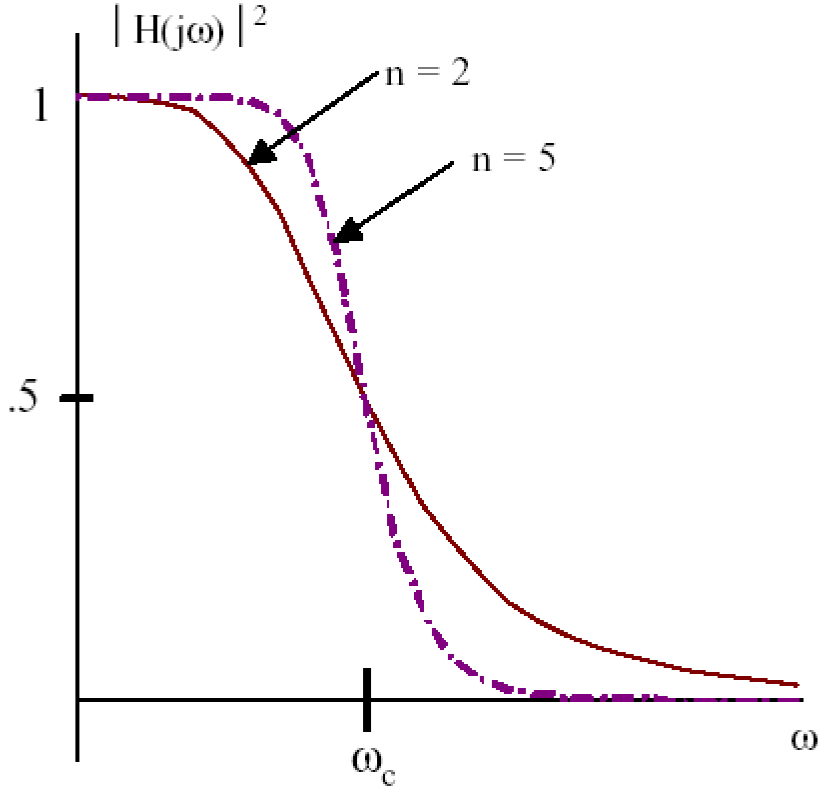
4.2 Spectral Analysis of Linear System

- Frequency response of an ideal low-pass filter



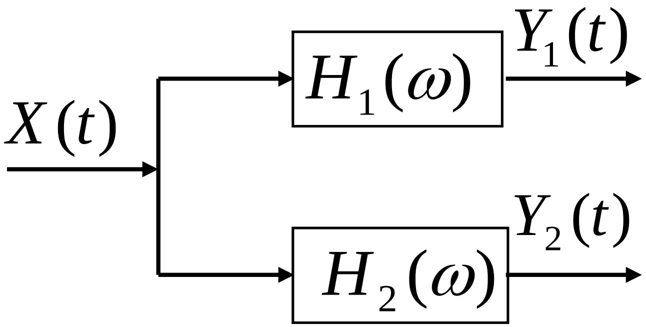
4.2 Spectral Analysis of Linear System

- Magnitude-squared response of a Butterworth filter



4.2 Spectral Analysis of Linear System

e.g. There are two LTI with frequency response $H_1(\omega)$ and $H_2(\omega)$ respectively. Let X(t) be stationary process with zero-mean. If Y1(t) and Y2(t) are mutually uncorrelated, what should $H_1(\omega)$ and $H_2(\omega)$ be?



Sln: $E[Y_1(t)] = m_X \int_{-\infty}^{\infty} h_1(\tau)d\tau = 0$

$$E[Y_2(t)] = m_X \int_{-\infty}^{\infty} h_2(\tau)d\tau = 0$$

$$R_{Y_1Y_2}(t_1, t_2) = E[Y_1(t_1)Y_2(t_2)]$$

$$= E[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(u)h_2(v)X(t_1 - u)X(t_2 - v)dudv]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(u)h_2(v)E[X(t_1 - u)X(t_2 - v)]dudv$$

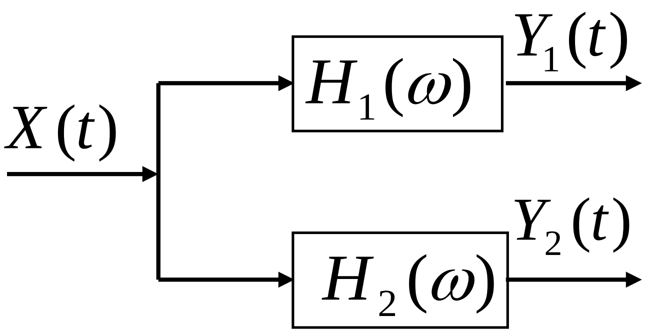
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(u)h_2(v)R_{XX}(t_1 - t_1 - u + v)]dudv = R_{Y_1Y_2}(\tau)$$

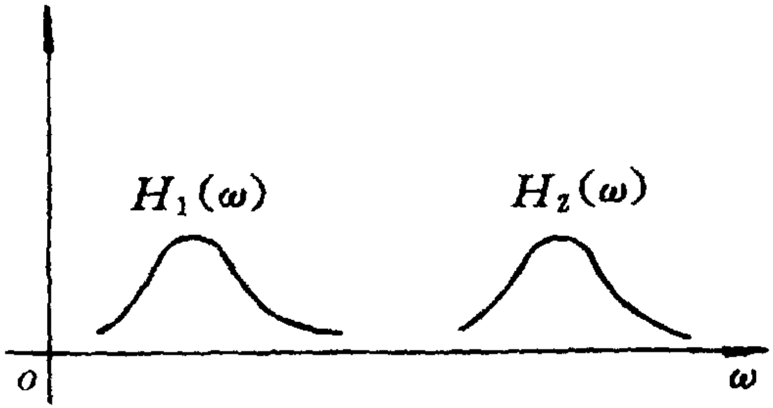
4.2 Spectral Analysis of Linear System

e.g.

$$R_{Y_1Y_2}(\tau) = h_1(\tau) * h_2(-\tau) * R_{XX}(\tau) = 0$$

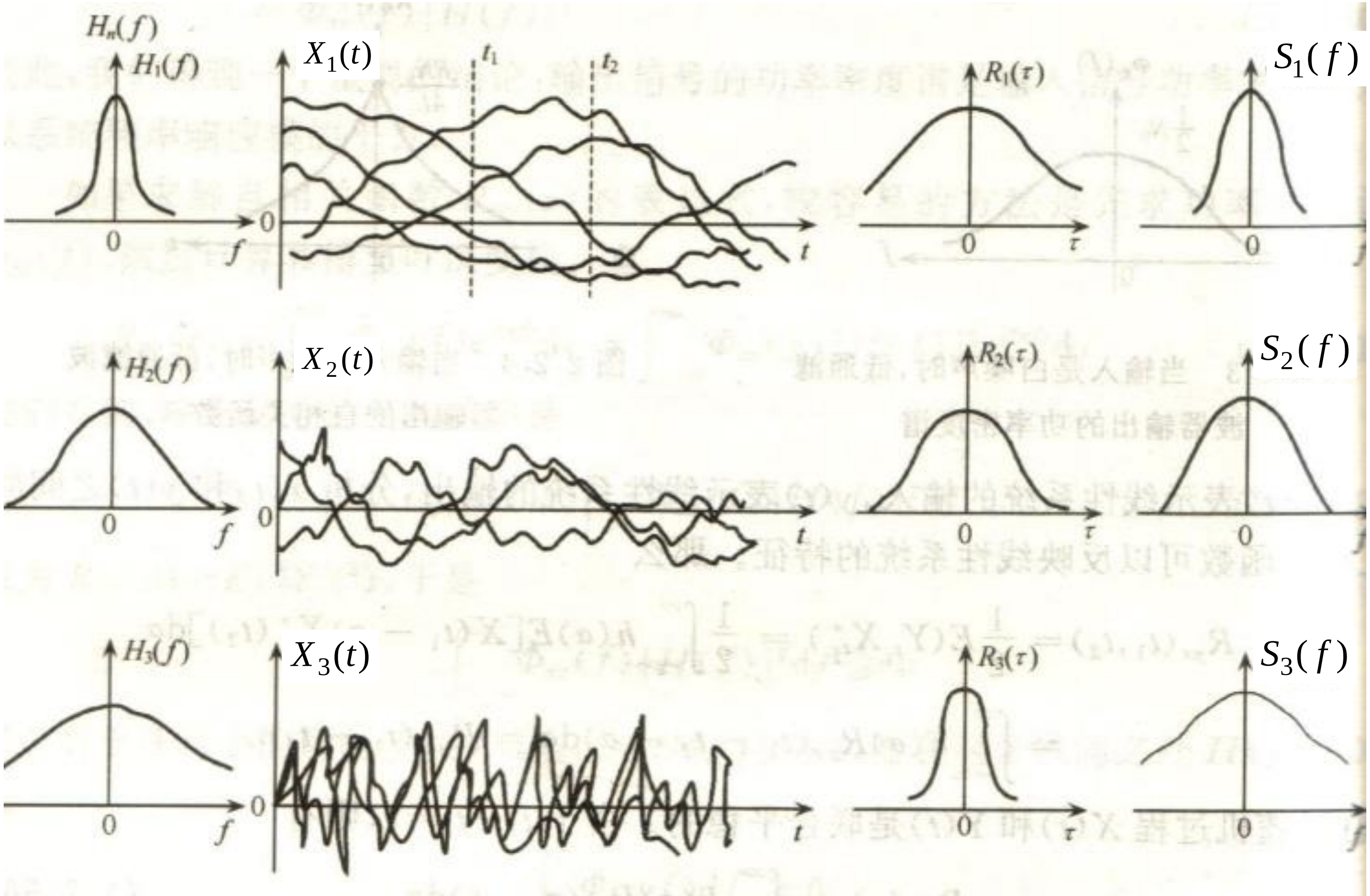
$$S_{Y_1Y_2}(\omega) = H_1(\omega)H_2^*(\omega)S_X(\omega) = 0$$





4.2 Spectral Analysis of Linear System

Relationships between LTI's frequency respor
waves of output, correlation functions and spectrums.



(a) $H_n(f)$ (b) $X_n(t)$ (c) $R_n(\tau)$ (d) $S_n(f)$