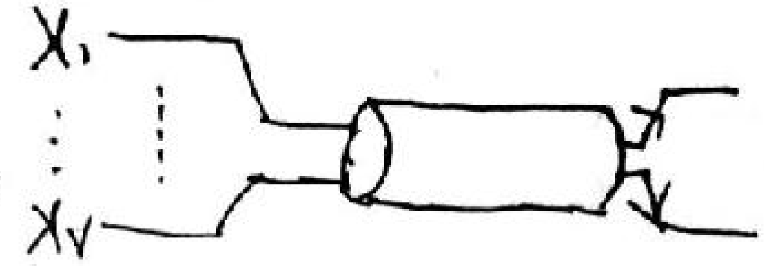
Multiplexing: A / B

Consider the model as follows:



the nontrivial aspect of the problem is that ν is a random variable.

Assume the users are active independently, then we have ν is Binomial (N, p) / $B(N, p)$

① Gaussian Random variable (the binomial distribution is well approximated by Gaussian)

1) standard normal RV: $f_W(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}$ $x \in \mathbb{R} \Leftrightarrow W =_D N(0,1)$

2) $X =_D N(\mu, \sigma^2)$ we have $X = \mu + \sigma W$ so we have pdf: $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$

② CLT (Central Limit Theory) Let $\{X(n), n \geq 1\}$ be RV with $E(X(n)) = \mu$ and $var(X(n)) = \sigma^2$

then as $n \to \infty$. $\frac{X(1) + \cdots + X(n) - n\mu}{\sigma\sqrt{n}} \Rightarrow N(0,1)$ ($\Rightarrow$ means convergence in distribution)

So we have $X(n) \xrightarrow{a.s.} X$ implies $X(n) \xrightarrow{P} X$ implies $X(n) \Rightarrow X$

③ Confidence interval

$[\mu_n - 1.65 \frac{\sigma_n}{\sqrt{n}}, \mu_n + 1.65 \frac{\sigma_n}{\sqrt{n}}] = 90\%$ $[\mu_n - 2\frac{\sigma_n}{\sqrt{n}}, \mu_n + 2\frac{\sigma_n}{\sqrt{n}}] = 95\%$

where $\sigma_n^2 = \frac{n}{n-1}\{\frac{\sum_{m=1}^n X(m)^2}{n} - \mu_n^2\}$ $\mu_n = \frac{\sum_{m=1}^n X(m)}{n}$

④ Buffer: at any time n, a transmission completes with probability μ and a new packet arrives with probability λ, independently of the past. X_n is a Markov chain

$p_2 = \lambda(1-\mu)$ $(X(n) \to X(n+1))$, $p_0 = \mu(1-\lambda)$ $(X(n+1) \to X(n))$ $p_1 = 1 - p_0 - p_2$ $(X(n) \to X(n))$

So we have balance equations.

$$\begin{cases} \pi(0) = (1-p_2)\pi(0) + p_0\pi(1) \\ \pi(n) = p_2\pi(n-1) + p_1\pi(n) + p_0\pi(n+1) & 1 \leq n \leq N-1 \\ \pi(N) = p_2\pi(N-1) + (1-p_0)\pi(N) \end{cases} \Rightarrow \begin{array}{l} \pi(i) = \pi(0)\rho^i, i = 0, 1, \cdots, N \text{ where } \rho := \frac{p_2}{p_0} \\ \pi(0) = [\sum_{i=0}^N \rho^i]^{-1} = \frac{1-\rho}{1-\rho^{N+1}} \end{array}$$

$E(X) = \sum_{i=0}^N i\pi(i) = \pi(0)\sum_{i=0}^N i\rho^i \approx \frac{\rho}{1-\rho} = \frac{p_2}{p_0 - p_2} = \frac{\lambda(1-\mu)}{\mu - \lambda}$ (average Packet number)

(Average Delay) $W = \sum_{k \geq 0} \frac{k+1}{\mu}\phi(k)$ where $\phi(k) = \pi(k)(1-\mu) + \pi(k+1)\mu$ Substitute it, we have

$W = \frac{1-\mu}{\mu-\lambda} = \frac{1}{\lambda}E(X)$

It is tempting to conclude that the probability that a packet finds k packet in the buffer upon its arrival is $P[X_{n+1}=k+1 \mid A_n=1] = P[X_n=k \mid A_n=1] = P[X_n=k] = \pi(k)$

Little's Law) $L = \lambda W$

L is the average number of customers in a system, λ is the average arrival rate of customer. and W is the average time that a customer spend in the system.

⑤ Multiple Access

the fraction of time that exactly one device transmits is $P(X(n)=1) = Np(1-p)^{N-1}$

the maximum over p of this success rate occurs when $p = 1/N \Rightarrow \lambda^* = (1-\frac{1}{N})^{N-1} \simeq \frac{1}{e}$

⑥ characteristic Functions

$\phi_X(u) = E(e^{iuX})$ $u \in \mathbb{R}$ $\Leftrightarrow$ $\phi_X(u) = \int_{-\infty}^{+\infty} e^{iux} f_X(x)\,dx$

let $X =_D N(0,1)$ Then $\phi_X(u) = e^{-\frac{u^2}{2}}$ (proof: P60)

⑦ Moments of $N(0,1)$

$\phi_X(u) = E(e^{iuX}) = E(\sum_{n=0}^{\infty} \frac{1}{n!}(iuX)^n) = \sum_{n=0}^{\infty} \frac{1}{n!}(iu)^n E(X^n)$

while $\phi_X(u) = e^{-\frac{u^2}{2}} = \sum_{m=0}^{\infty} \frac{1}{m!}(-\frac{u^2}{2})^m = \sum_{m=0}^{\infty} \frac{1}{m!}(-\frac{1}{2})^m u^{2m}$

match the coefficients of u^{2m}, and we can find

$\frac{1}{(2m)!} i^{2m} E(X^{2m}) = \frac{1}{m!}(-\frac{1}{2})^m \Rightarrow E(X^{2m}) = \frac{(2m)!}{m!\,2^m}$, $E(X^{2m+1}) = 0$

⑧ 1) Poission as a limit of Binomial: $P(\lambda) \sim B(n, \lambda/n)$

2) Exponential as limit of Geometric: $Exp(\lambda) \sim G(\lambda/n)$

⑨ Error Function $Q(x) := P(X > x)$ where $X =_D N(0,1)$

Bounds on EF: $\frac{x}{1+x^2}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\} \le Q(x) \le \frac{1}{x\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\}$ $\forall x > 0$

⑩ Adaptive Multiple Access

$$p(n+1) = \begin{cases} p(n) & \text{if } X(n)=1 \\ a p(n) & \text{if } X(n)>1 \\ \min\{b p(n), 1\}. & \text{if } X(n)=0 \end{cases}$$