Stochastic Processes: Overview

Dong Yan
EIC. HUST

Overview

- OUTLINE

  Important concepts and relationships

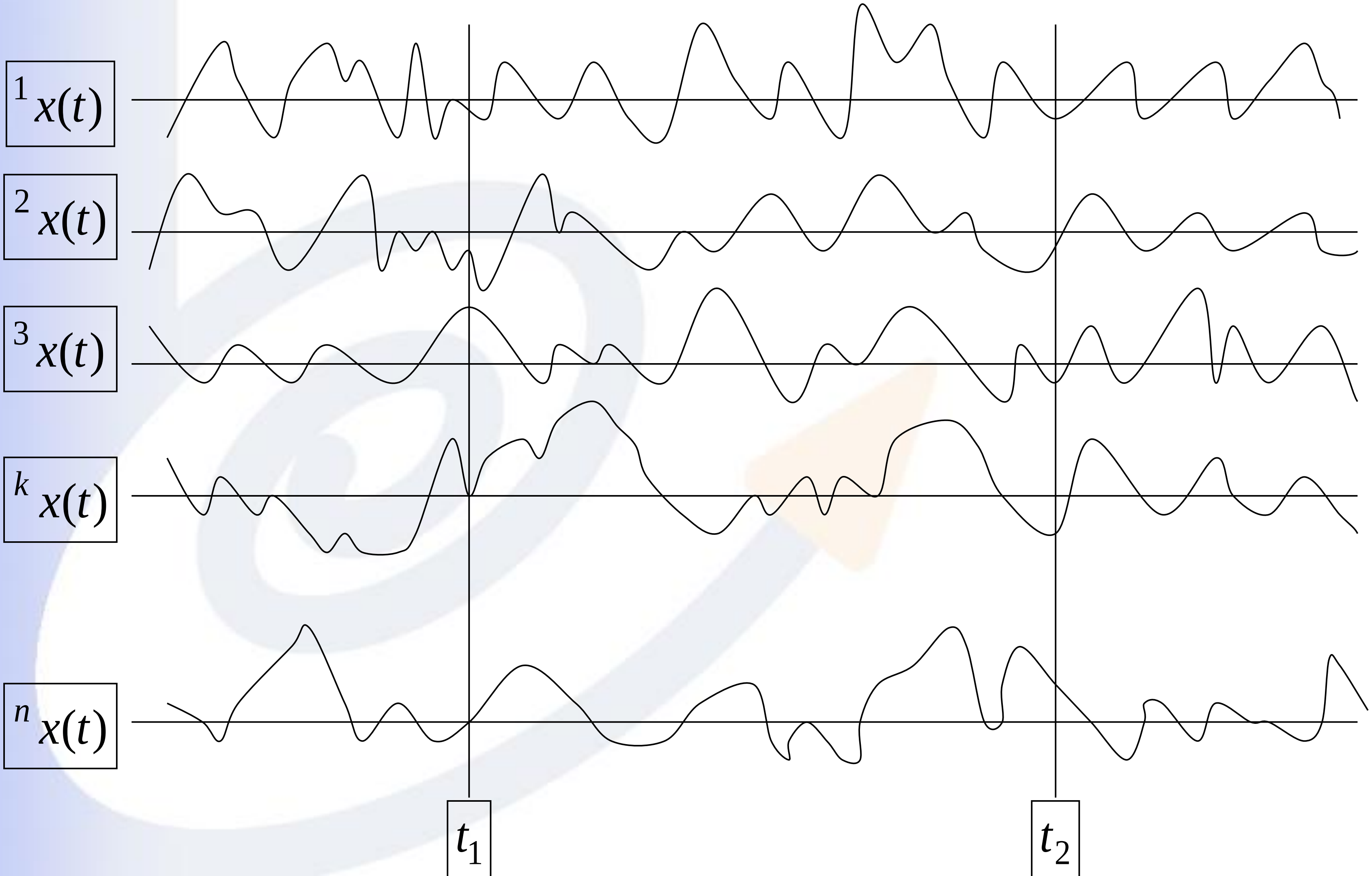
  Correlation functions and stationary processes

  Power spectrum and linear systems

  Markov chains

  Poisson processes

Important concepts and relationships



Important concepts and relationships

A stochastic process can be viewed as a function of 2 variables, time t and outcome ω.

$$\{x(t,\omega), t \in T, \omega \in \Omega\}$$

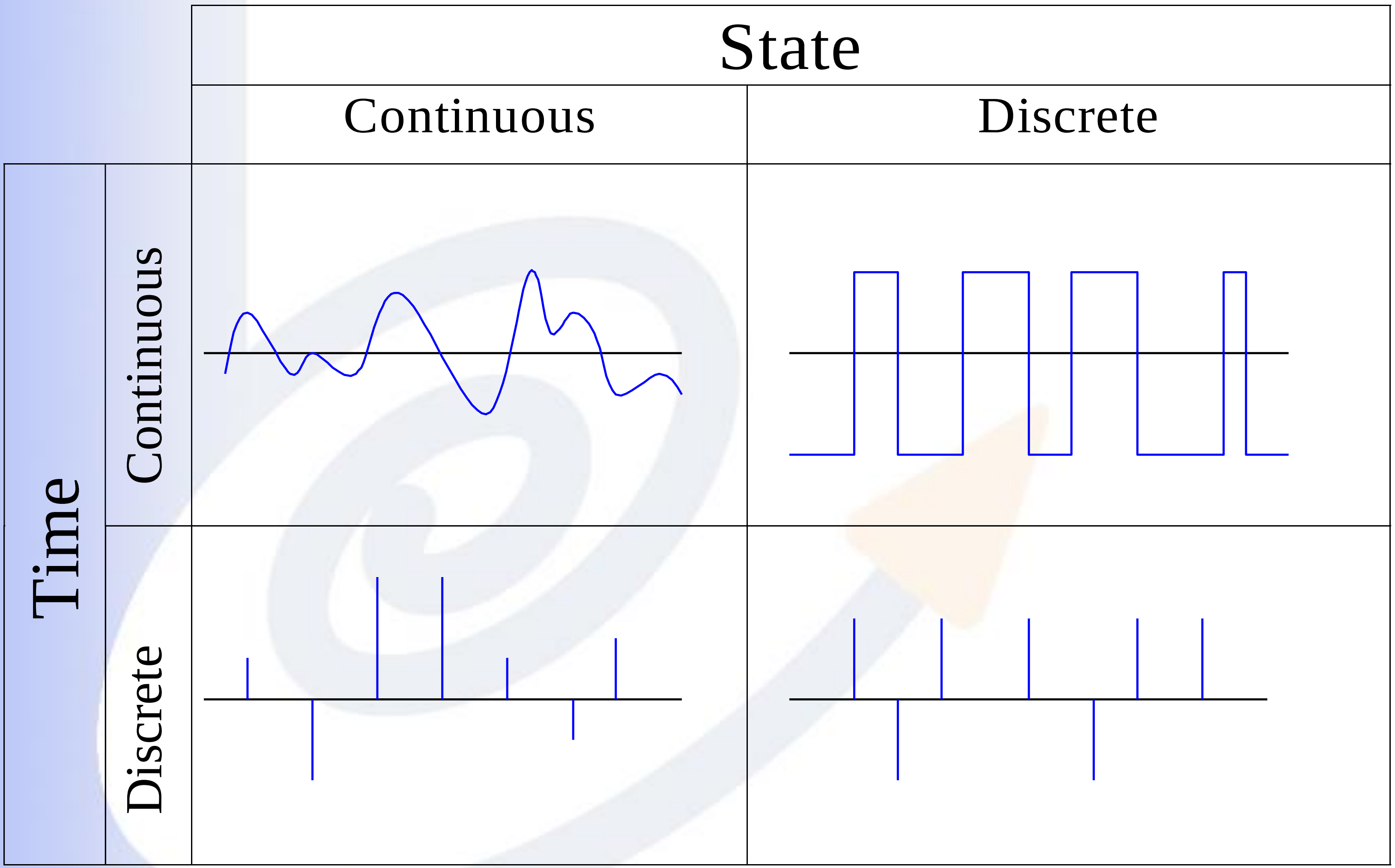
(i) For a fixed ω， x(t) is a function of time, i.e. sample function;

(ii) For fixed t, x(ω) is a family of random variables, i.e. ensemble;

(iii) x(t, ω), a family of functions with both t and variables ω;

(iv) for a fixed t_0 and a fixed ω_0 ,$x(t_0, \omega_0)$ is a single number.

Important concepts and relationships

- 

Important concepts and relationships

- A stochastic process $\{X(t), t \in T\}$ is only completely determined if for all integers n=1,2,… and for all n-tuples $\{t_1, t_2, \ldots, t_n\}$ with $t_i \in T$, the joint distribution functions of the random vectors $(X(t_1), X(t_2), \ldots, X(t_n))$ are know:

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n)$$
$$= P\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\}$$

- Joint density functions are found from appropriate derivatives of the distribution functions.

$$f_X(x_1; t_1) = dF_X(x_1; t_1) / dx_1$$
$$f_X(x_1, \ldots, x_N; t_1, \ldots, t_N)$$
$$= \partial^N F_X(x_1, \ldots, x_N; t_1, \ldots, t_N) / (\partial x_1 \ldots \partial x_N)$$

Important concepts and relationships

- Mean value function E[X(t)]

$$\bar{x}(t) = E[x(t)] = \int_{-\infty}^{\infty} x(t) f(x) dx$$

$$\bar{x}(t_i) = E[x(t_i)] = \frac{1}{n} \sum_{k=1}^{n} {}^{k} x(t_i), \qquad i = 1,2,.....$$

- Variance function

$$Var[X(t)] = E[X(t) - \bar{x}(t)]^2 = E[X^2(t)] - \bar{x}^2(t)$$

- Covariance(autocovariance) function

$$C_{XX}(t_1, t_2) = Cov[X(t_1), X(t_2)]$$

$$= E\big[[X(t_1) - \bar{x}(t_1)][X(t_2) - \bar{x}(t_2)]\big]$$

$$= E\big[X(t_1) X(t_2)\big] - \bar{x}(t_1)\bar{x}(t_2)$$

Important concepts and relationships

- Correlation(autocorrelation) function

$$R_{XX}(t_1,t_2)=E\left[X(t_1)X(t_2)\right]$$

$$C_{XX}(t_1,t_2)=R_{XX}(t_1,t_2)-\overline{x}(t_1)\overline{x}(t_2)$$

- Correlation coefficients

  (normalized autocovariance function)

$$\rho_{XX}(t_1,t_2)=\frac{C_{XX}(t_1,t_2)}{\sqrt{Var[X(t_1)]Var[X(t_2)]}}$$

Important concepts and relationships

$C_{XX}(t_1,t_2)$、$R_{XX}(t_1,t_2)$、$\rho_{XX}(t_1,t_2)$

are symmetric in t_1 and t_2

$$C_{XX}(t_1,t_2) = C_{XX}(t_2,t_1)$$

$$R_{XX}(t_1,t_2) = R_{XX}(t_2,t_1)$$

$$\rho_{XX}(t_1,t_2) = \rho_{XX}(t_2,t_1)$$

Important concepts and relationships

Cross-covariance function

$$C_{XY}(t_1,t_2) = Cov[X(t_1),Y(t_2)]$$
$$= E\left[[X(t_1)-\bar{x}(t_1)][Y(t_2)-\bar{y}(t_2)]\right]$$
$$= E\left[X(t_1)Y(t_2)\right]-\bar{x}(t_1)\bar{y}(t_2)$$

Cross-correlation function

$$R_{XY}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right]$$
$$C_{XY}(t_1,t_2) = R_{XY}(t_1,t_2)-\bar{x}(t_1)\bar{y}(t_2)$$

Mutual correlation coefficients

$$\rho_{XY}[t_1,t_2] = \frac{C_{XY}(t_1,t_2)}{\sqrt{Var[X(t_1)]Var[Y(t_2)]}}$$

Important concepts and relationships

X(t) and Y(t) are mutually uncorrelated

$$C_{XY}[t_1,t_2]=0 \quad t_1,t_2 \in T$$

Overview

- OUTLINE

  Important concepts and relationships

  Correlation functions and stationary processes

  Power spectrum and linear systems

  Markov chains

  Poisson processes

Correlation functions and stationary processes

Weakly stationary

$$E[X(t)] = m$$

$$R_{XX}(t_1, t_2) = R_{XX}(\tau)$$

$$R_{XX}(0) = E[X^2(t)]$$

Ergodic

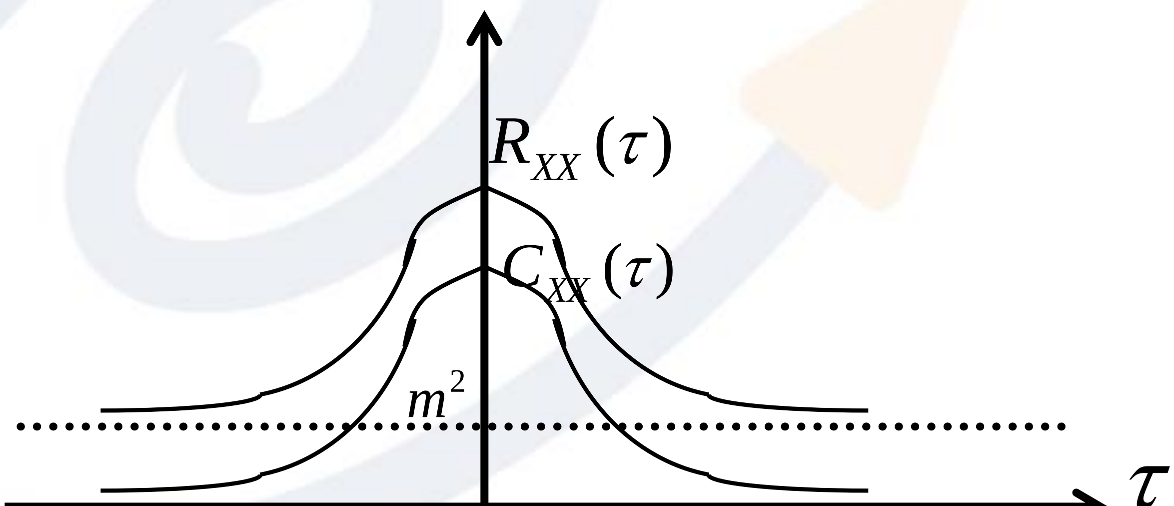
Correlation functions and stationary processes

Correlation function of weakly stationary S.P.

$$R_{XX}(t_1,t_2)=R_{XX}(\tau)$$

$$R_{XX}(0)=E[X^2(t)]$$

$$C_{XX}(\tau)=R_{XX}(\tau)-m^2$$



Correlation functions and stationary processes

Properties of $R_{XX}(\tau)$

i) An even function: $\boxed{R_{XX}(\tau) = R_{XX}(-\tau)}$

ii) $R_{XX}(0) \geq |R_{XX}(\tau)|$

iii) $R_{XX}(0) = E[X^2(t)] \geq 0$

If X(t) is ergodic, then

$$R_{XX}(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

Correlation functions and stationary processes

iv) If $X(t)=X(t+T)$, then

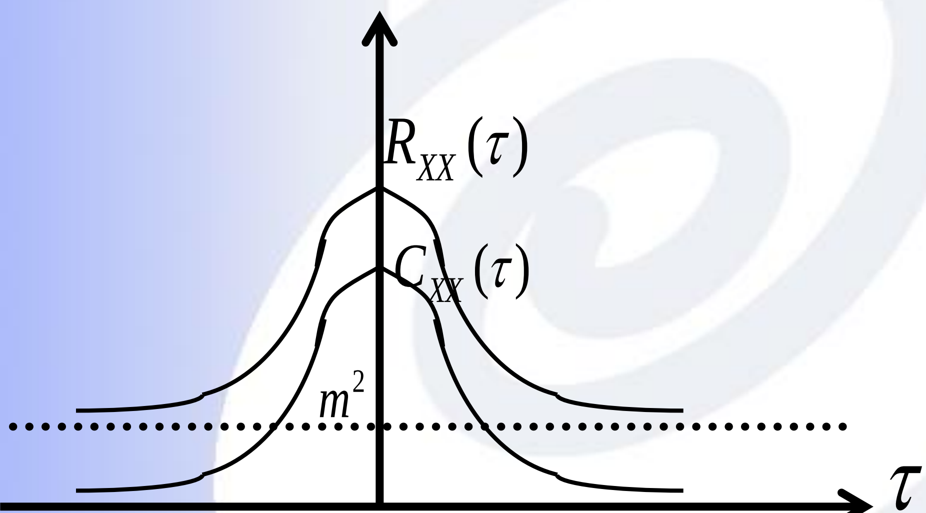
$$R_{XX}(\tau) = R_{XX}(\tau + T)$$

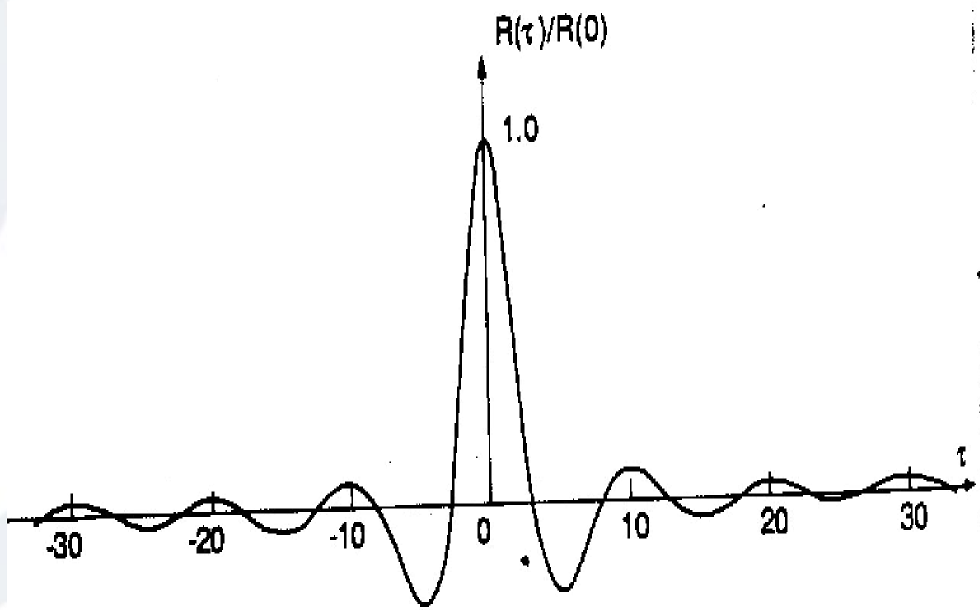
Correlation functions and stationary processes

iv) If X(t) is a non-period process, as $|\tau| \to \infty$

$$\lim_{|\tau \to \infty|} R_{XX}(\tau) = m^2$$

$X(t)$ and $X(t+\tau)$ are uncorrelation when *m*=0





Correlation functions and stationary processes

Jointly Stationary Processes

$$R_{XY}(t_1,t_2)=E\left[X(t_1)Y(t_2)\right]=R_{XY}(\tau)$$

$$R_{YX}(t_1,t_2)=E\left[Y(t_1)X(t_2)\right]=R_{YX}(\tau)$$

Properties of $R_{XY}(\tau)$

i) $R_{YX}(\tau)=R_{XY}(-\tau)$

ii) $|R_{XY}(\tau)|\le\sqrt{R_X(0)R_Y(0)}\le\frac{1}{2}[R_{XX}(0)+R_{YY}(0)]$

$|R_{YX}(\tau)|\le\sqrt{R_X(0)R_Y(0)}\le\frac{1}{2}[R_{XX}(0)+R_{YY}(0)]$

iii) $R_{XY}(\tau)$ is not always maximum at $\tau=0$

Stochastic Convergence

Convergence in Mean Square $\boxed{X_n \xrightarrow{m.s} X}$

$$\lim_{n\to\infty} E[(X_n - X)^2] = 0$$

Limit in mean (square):

$$\underset{n\to\infty}{\mathrm{l.i.m}}\, X_n = X$$

$\{X_n\}$ is also called expected square convergence.

Mean squared error is the most commonly used measure of the difference between two random variables.

In practice, the mean squared error is usually obtained by averaging over time, not over a set of a statistical samples.

Stochastic Convergence

Theorem1 $\{X_n\}$ and $\{Y_n\}$ are second order moment sequence, U is a second order moment random variable, $\{c_n\}$ is a constant sequence, a,b,c are constants.

then, $\underset{n\to\infty}{\mathrm{l.i.m}}\, X_n = X \quad \underset{n\to\infty}{\mathrm{l.i.m}}\, Y_n = Y \quad \lim_{n\to\infty} c_n = c$

$$\underset{n\to\infty}{\mathrm{l.i.m}}\, c_n = \lim_{n\to\infty} c_n = c$$

$$\mathrm{l.i.m}\, U = U \qquad \underset{n\to\infty}{\mathrm{l.i.m}}(aX_n + bY_n) = aX + bY$$

$$\underset{n\to\infty}{\mathrm{l.i.m}}\, c_n U = cU \qquad \lim_{n\to\infty} E[X_n] = E[X] = E[\underset{n\to\infty}{\mathrm{l.i.m}}\, X_n]$$

$$\lim_{n,m\to\infty} E[X_n Y_m] = E[XY] = E[(\underset{n\to\infty}{\mathrm{l.i.m}}\, X_n)(\underset{m\to\infty}{\mathrm{l.i.m}}\, Y_m)]$$

Stochastic Convergence

Mean-Square Continuity, Differentiability and Integrability

- Mean-Square Continuity

$$\lim_{\varepsilon \to 0} E[X(t-\varepsilon)] = E[X(t)] = E\{\underset{\varepsilon \to 0}{\text{l.i.m.}}\, X(t+\varepsilon)\}$$

For a stationary process, if and only if Rxx() is continuous at $\tau = 0$,that is $\lim_{\varepsilon \to 0}[R_{XX}(\varepsilon) - R_{XX}(0)] = 0$

- Mean-Square Differentiability

$$\dot{X}(t) = \frac{dX(t)}{dt} = \underset{\varepsilon \to 0}{\text{l.i.m}} \frac{X(t) - X(t-\varepsilon)}{\varepsilon}$$

For a stationary process, if and only if Rxx() is twice differentiable at $\tau = 0$

$$\left.\frac{\partial^2 R_{XX}(\tau)}{\partial \tau^2}\right|_{\tau \to 0} \equiv \lim_{\varepsilon^2 \to 0} \frac{1}{\varepsilon^2}[R_{XX}(\varepsilon) - 2R_{XX}(0) + R_{XX}(-\varepsilon)]$$

Stochastic Convergence

Mean-Square Differentiability

If X(t) is mean-square differentiable at t

$$\frac{dE[X(t)]}{dt} = E[\dot{X}(t)]$$

$$\frac{\partial R_{XX}(t_1,t_2)}{\partial t_1} = \frac{\partial E[X(t_1)X(t_2)]}{\partial t_1} = E[\dot{X}(t_1)X(t_2)]$$

$$\frac{\partial R_{XX}(t_1,t_2)}{\partial t_2} = \frac{\partial E[X(t_1)X(t_2)]}{\partial t_2} = E[X(t_1)\dot{X}(t_2)]$$

$$\frac{\partial^2 R_{XX}(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial^2 E[X(t_1)X(t_2)]}{\partial t_1 \partial t_2} = E[\dot{X}(t_1)\dot{X}(t_2)]$$

Stochastic Convergence

Mean-Square Integrability

$$X^{(-1)}(t) = \int_0^t X(u)du = \underset{\varepsilon \to 0}{\mathrm{l.i.m}} \sum_{i=1}^{t/\varepsilon} \varepsilon X(i\varepsilon)$$

For a stationary process, if and only if Rxx() is Riemann-integrable on (0,t),

$$\int_0^t R_{XX}(u)du \equiv \lim_{\varepsilon \to 0}[\varepsilon \sum_{i=1}^{t/\varepsilon} R_{XX}(i\varepsilon)]$$

Stochastic Convergence

Mean-Square Integrability

$$E[\int_0^t X(u)du] = \int_0^t E[X(u)]du$$

$$E[\int_0^t X(t_1)dt_1 \int_0^t X(t_2)dt_2] = \int_0^t \int_0^t E[X(t_1)X(t_2)]dt_1 dt_2$$

$$= \int_0^t \int_0^t R_{XX}(t_1, t_2)]dt_1 dt_2$$

Theorem 3 If X(t) is mean-square continuous on the interval (a, b), the integral exists

$$Z(t) = \int_a^t X(u)du \quad (a \le t \le b)$$

and Z(t) is mean-square differentiable, $\dot{Z}(t) = X(t)$

Mean-Square Integrability

If a Gaussian process X(t) is mean-square differentiable its derivative $\dot{X}(t)$ is a Gaussian process.

If a Gaussian process X(t) is mean-square continuous on the interval (a, b), its integration

$$Z(t) = \int_a^t X(u)du \quad (a \le t \le b)$$

is a Gaussian process.

Overview

- OUTLINE

  Important concepts and relationships

  Correlation functions and stationary processes

  Power spectrum and linear systems

  Markov chains

  Poisson processes

Power spectrum and linear systems

Def.1 average of the power

$$\overline{P}_X = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

$$\overline{P}_X = \lim_{T\to\infty} \frac{1}{4\pi T} \int_{-T}^{T} X^2(\omega)d\omega$$

Def.2 Spectral Density Function (S.D.F.)

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} | X_T(\omega) |^2$$

$$\overline{P}_X = \lim_{T\to\infty} \frac{1}{4\pi T} \int_{-T}^{T} X^2(\omega)d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = \frac{1}{\pi} \int_{0}^{\infty} S_{XX}(\omega)d\omega$$

Wiener-Khintchine Theorem

Theorem: Wiener-Khintchine Theorem

For a weakly stationary random process X(t), its correlation function and the power spectrum are Fourier transform pair.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$

$$F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$$

Standard Fourier Transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwt} dt$$

Power spectrum and linear systems

Properties of power spectrum:

real, nonnegative, even.

The average power and correlation function

$$\frac{1}{\pi}\int_0^\infty S_{XX}(\omega)d\omega = \overline{P}_X = R_{XX}(0) = E[X^2(t)]$$

If E[X(t)]=0,

$$Var[X(t)] = \int_0^\infty S_{XX}(\omega)d\omega = \overline{P}_X = R_{XX}(0)$$

Power spectrum and linear systems

Cross-Spectral Density Function (CS.D.F)

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} X^*(\omega)Y(\omega)$$

$$S_{YX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} X(\omega)Y^*(\omega)$$

$$S_{XY}(\omega) = C_{XY}(\omega) + i\mathrm{Q}_{XY}(\omega)$$

$$S_{YX}(\omega) = C_{YX}(\omega) + i\mathrm{Q}_{YX}(\omega)$$

Wiener-Khintchine Theorem

Theorem: Wiener-Khintchine Theorem

For two jointly stationary random processes X(t) and Y(t), their cross-correlation function and cross-power density function are Fourier transform pair.

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau \qquad S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-i\omega\tau} d\tau$$

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\omega\tau} d\omega \qquad R_{YX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega) e^{i\omega\tau} d\omega$$

Standard Fourier Transform

$$F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwt} dt$$

Power spectrum and linear systems

Properties of Cross-Spectral Density Function

i) $C_{XY}(\omega)$ and $C_{YX}(\omega)$ are even functions

ii) $Q_{XY}(\omega)$ and $Q_{YX}(\omega)$ are odd functions

iii) $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}^{*}(\omega)$

iv) If X(t) and Y(t) are jointly stationary random processes and uncorrelated,

$$S_{XY}(\omega) = S_{YX}(\omega) = 2m_X m_Y \delta(\omega)$$

Power spectrum and linear systems

Coherency function

$$\gamma_{XY}(\omega) = \frac{|S_{XY}(\omega)|^2}{S_{XX}(\omega)S_{YY}(\omega)} = \frac{C_{XY}^2(\omega) + Q_{XY}^2(\omega)}{S_{XX}(\omega)S_{YY}(\omega)}$$

Mutual correlation coefficients

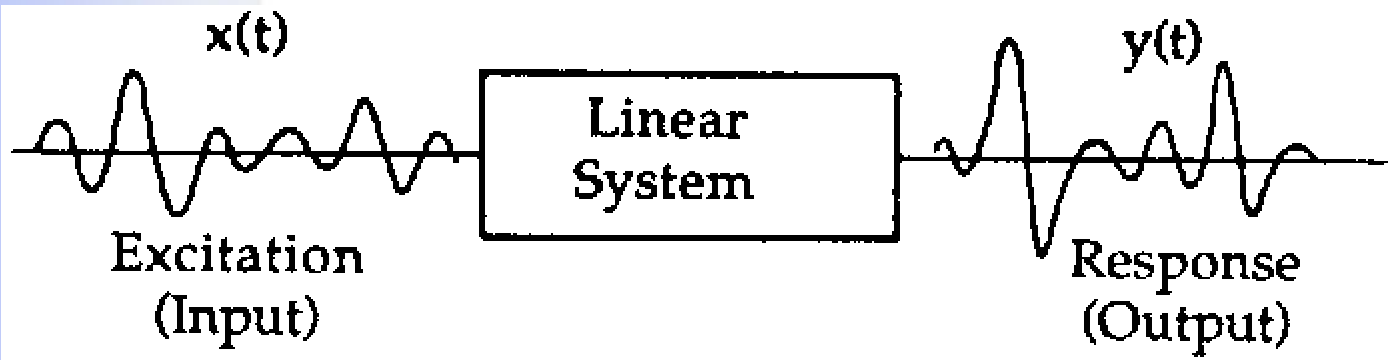
$$\rho_{XY}[t_1, t_2] = \frac{C_{XY}(t_1, t_2)}{\sqrt{Var[X(t_1)]Var[Y(t_2)]}}$$

$$0 \leq \gamma_{XY}(\omega) \leq 1$$

$\gamma_{XY}(\omega) \equiv 1$ if and only if X(t) and Y(t) are exactly linearly related.

Power spectrum and linear systems

- LTI: Linear Time-invariant system



- If input is a random signal, what are the relationships between input and output ?
- Mean functions, correlation functions, stationarities,
- Power spectrums

Power spectrum and linear systems

Input and Output Mean Levels:

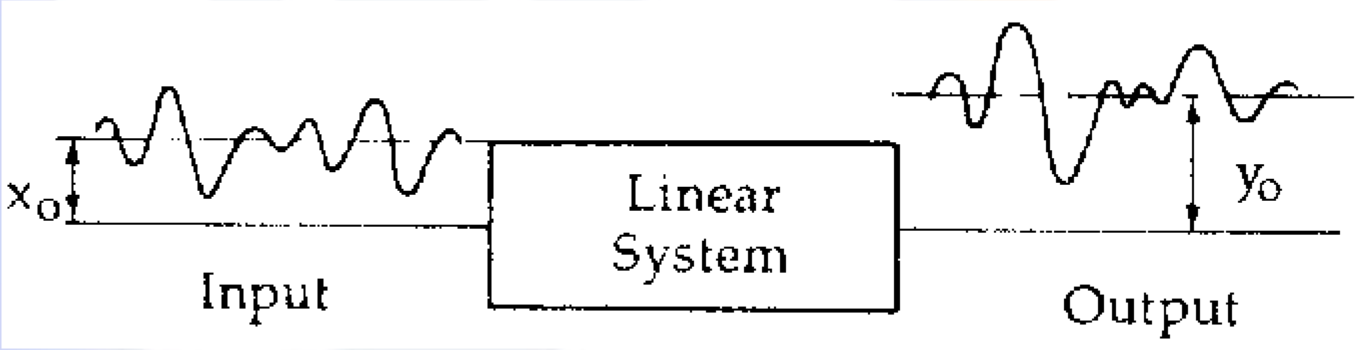
If X(t) is a mean-square integrable random process, then

$$E[Y(t)] = E[\int_{-\infty}^{\infty} X(t-\tau)h(\tau)d\tau]$$

$$= \int_{-\infty}^{\infty} E[X(t-\tau)]h(\tau)d\tau$$

If X(t) is a stationary process, then

$$\boxed{E[Y(t)] = m_X \int_{-\infty}^{\infty} h(\tau)d\tau = m_X H(0) = m_Y}$$



Power spectrum and linear systems

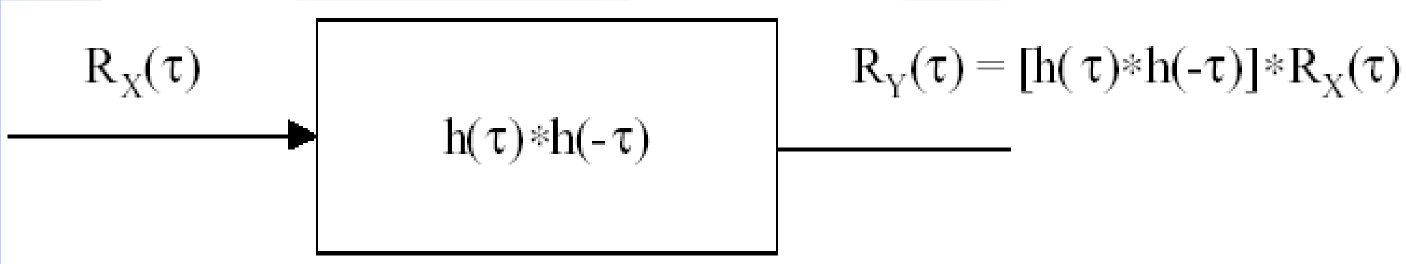
Input and Output Correlation Functions Relationship

1. $R_{XY}(t_1,t_2)$ $R_{YX}(t_1,t_2)$ $\boxed{\tau = t_1 - t_2}$

$$\boxed{R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau)} \quad \boxed{R_{YX}(\tau) = R_{XX}(\tau) * h(\tau)}$$

2. $R_{YY}(\tau)$

$$\boxed{\begin{aligned} R_{YY}(\tau) &= R_{XY}(\tau) * h(\tau) = R_{YX}(\tau) * h(-\tau) \\ &= h(\tau) * h(-\tau) * R_{XX}(\tau) \end{aligned}}$$



Power spectrum and linear systems

Input and Output Spectral Relationship

(a) $R_Y(\tau) = R_{YX}(\tau) * h(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$

$$\boxed{\begin{aligned} S_{YY}(\omega) &= H(\omega)H^*(\omega)S_{XX}(\omega) \\ &= |H(\omega)|^2 S_{XX}(\omega) \end{aligned}}$$

(b) From definition of power spectrum

$$S_{YY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} |Y(\omega)|^2$$

$$= \lim_{T\to\infty} \frac{1}{2T} |H(\omega)|^2 |X(\omega)|^2 = |H(\omega)|^2 S_{XX}(\omega)$$

Power spectrum and linear systems

- If the input to a LTI is stationary, so does the output.
- The input and output processes are jointly stationary.

- If the input is a Gaussian process, so does the output.
- The input and output processes are jointly Gaussian.

Overview

- OUTLINE
  - Important concepts and relationships
  - Correlation functions and stationary processes
  - Power spectrum and linear systems
  - Markov chains
  - Poisson processes

Markov Processes

- Markovian property

$$Pr\{x(t_n) \leqslant x_n | x(t_1) = x_1, x(t_2) = x_2, \ldots, x(t_{n-1}) = x_{n-1}\}$$

$$= Pr\{x(t_n) = x_n | x(t_{n-1}) = x_{n-1}\}, \quad \text{where } t_1 < t_2 < \cdots t_{n-1} < t_n$$

Transition probability density:

$$f\{x(t_r) | x(t_{r-1})\}$$

Transition probability:

$$P\{x(t_r) | x(t_{r-1})\}$$

Markov Processes

Homogeneity:

The transition probability density is invariant with time τ

$$f\{x(t_r+\tau) \mid x(t_{r-1}+\tau)\} = f\{x(t_r) \mid x(t_{r-1})\}$$

or $$P\{x(t_r+\tau) \mid x(t_{r-1}+\tau)\} = P\{x(t_r) \mid x(t_{r-1})\}$$

Markov Processes

$$f\{x(t_1), x(t_2), \ldots, x(t_{n-1})\}$$
$$= f\{x(t_{n-1})|x(t_{n-2})\} \cdot f\{x(t_1), x(t_2), \ldots, x(t_{n-2})\}$$

$$f\{x(t_1), x(t_2), \ldots, x(t_n)\} = f\{x(t_1)\} \prod_{r=2}^{n} f\{x(t_r)|x(t_{r-1})\}$$

$$P\{x(t_1), x(t_2), \cdots, x(t_n)\} = P\{x(t_1)\} \prod_{r=2}^{n} P\{x(t_r) \mid x(t_{r-1})\}$$

The statistical characteristic of Markov process decided by the initial condition and conditional probability density function or conditional probability .

Chapman-Kolmogorov Equation

- **In terms of transition probability density function or transition probability**

$$t_0 < t_j < t$$

$$f\{x(t), x_0(t_0)\} = \int f\{x(t), x_0(t_0), x_j(t_j)\}\, dx_j$$

$$= \int f\{x(t) | x_0(t_0), x_j(t_j)\} f\{x_0(t_0), x_j(t_j)\}\, dx_j$$

$$f\{x(t), x_0(t_0)\} = \int f\{x(t) | x_j(t_j)\} f\{x_0(t_0), x_j(t_j)\}\, dx_j$$

$$f\{x(t) | x_0(t_0)\} = \int f\{x(t) | x_j(t_j)\} f\{x_j(t_j) | x_0(t_0)\}\, dx_j$$

$$p\{x(t) | x_0(t_0)\} = \sum_j p\{x(t) | x_j(t_j)\} p\{x_j(t_j) | x_0(t_0)\}$$

Markov chains

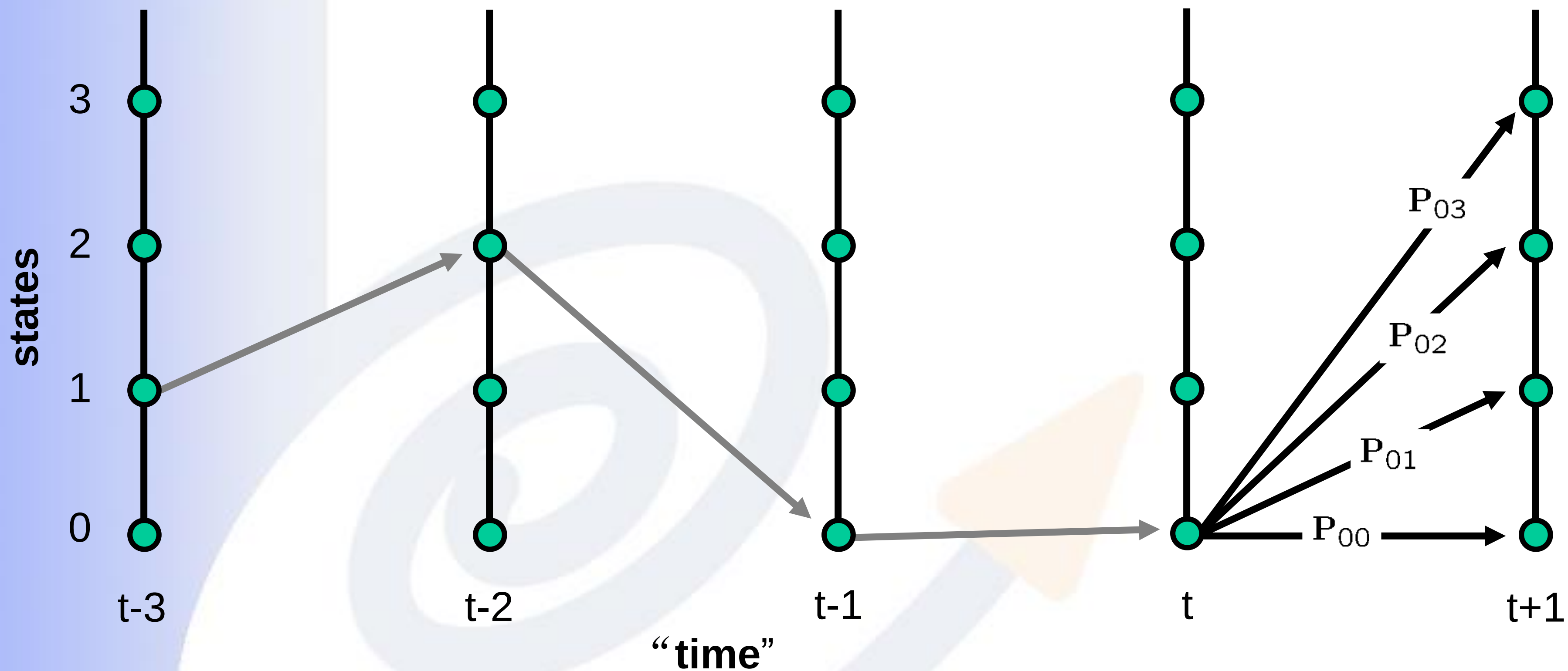
- Markov chains
- State Space : Finite number of states, {Z=(0, 1,…,i,…,j…)}
- Index set: Discrete Time, {T = (0, 1, 2, ...)}
- Markovian Property

$$P\{x(t_n) = x_n \mid x(t_0) = t_0, x(t_1) = t_1, \ldots, x(t_{n-1}) = t_{n-1}\}$$
$$= P\{X_n = i_n \mid X_{n-1} = i_{n-1}\}$$

for any $n = 1,2,\ldots$ and any $i_0, i_1, i_2, \ldots, i_n \in Z$

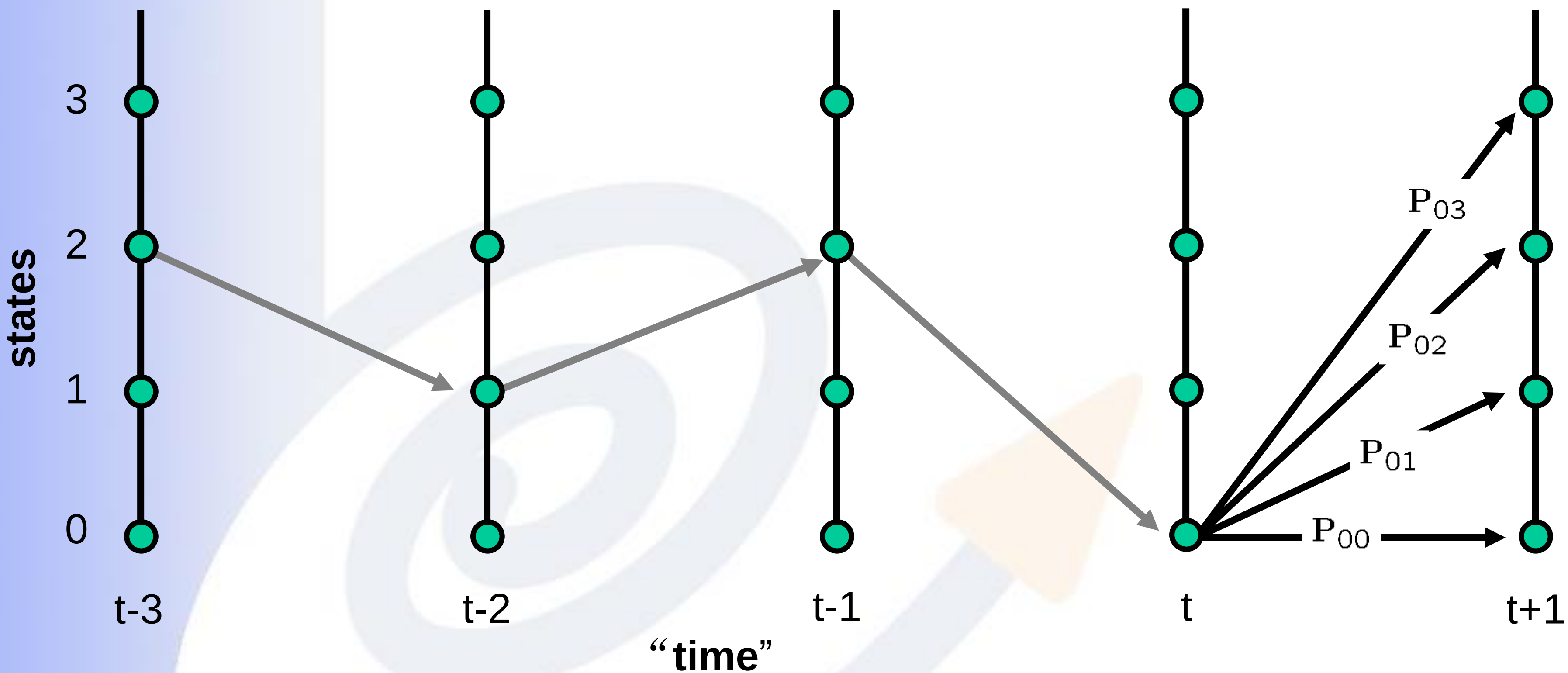
Markov chains

- Sample path (1):



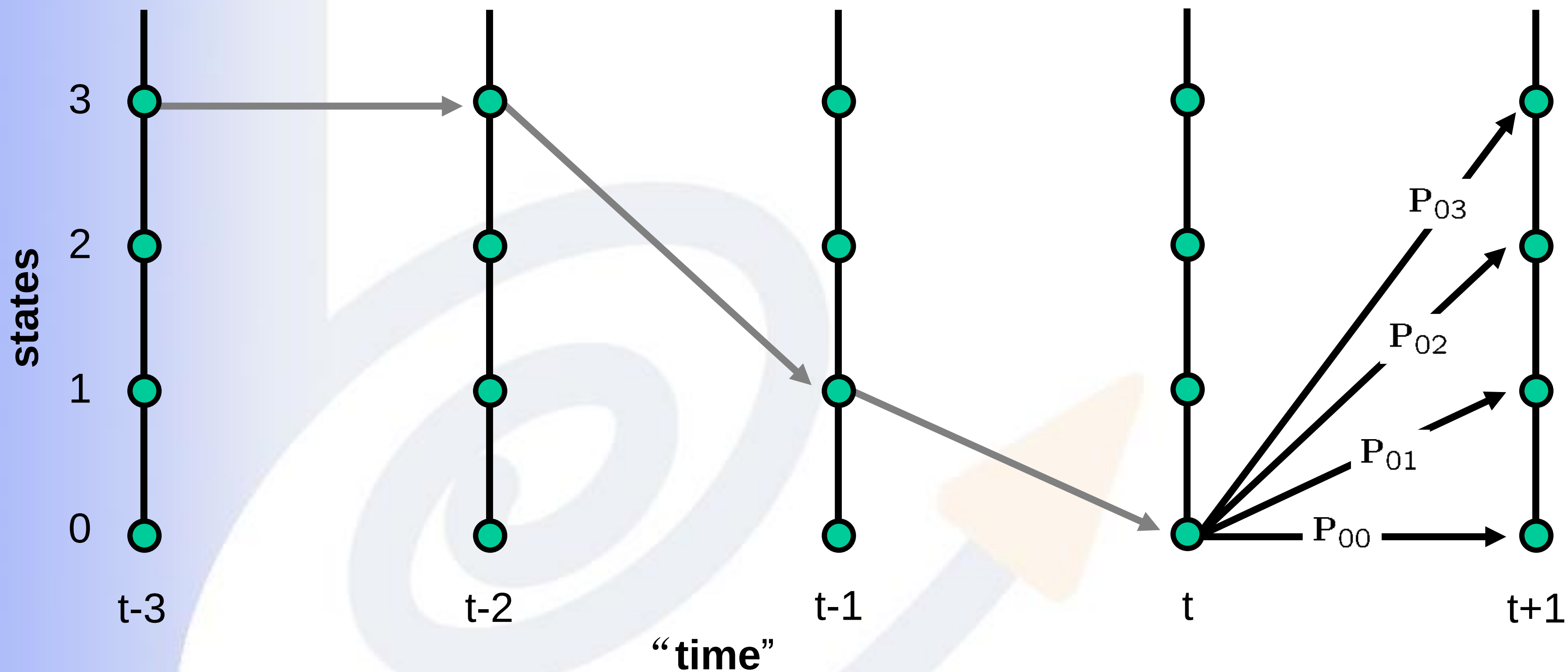
Markov chains

- Sample path (2):



Markov chains

- Sample path (3):



Markov chains

- Important concepts (1):

  One-step transition probabilities:

$$\boxed{p_{ij}(n) = P\{X_{n+1} = j \mid X_n = i\};\ \ n = 0,1,\ldots}$$

  Homogeneity:

$$\boxed{p_{ij}(n) = P\{X_1 = j \mid X_0 = i\} = p_{ij} \quad \text{for all } \ n = 0,1,\ldots}$$

  One-step transition probability matrix: $\mathbf{P}$

  M-step transition probabilities:

$$\boxed{p_{ij}^{(m)} = P\{X_{n+m} = j \mid X_n = i\},\ \ \ m = 0,1,\ldots}$$

  M-step transition probability matrix: $\mathbf{P}^{(m)}$

Markov chains

- Important concepts (2):

  Initial distribution $\mathbf{s}(0)$:

  A probability distribution of $X(t_0)$

  Absolute distribution $\mathbf{s}(m)$:

  One-dimensional state probabilities of the Markov chain after *m* steps

$$\mathbf{s}(m) = \{p_j^{(m)} = P(X_m = j), j \in S, \sum_{j \in S} p_j = 1\}$$

  Limiting distribution:

$$\lim_{m \to \infty} p_{ij}^{(m)} = \frac{1}{\mu_j}, \quad j \in S$$

Markov chains

- Important concepts (3):

  Ergodic Markov chain: Communicate irreducible aperiodic Markov chain with finite number of states is called an *Ergodic Markov chain*.

  For some $m \geq 1$, $p_{ij}^{(m)} \neq 0,\ i,j \in S$

  Stationary distribution: $\{\pi_j,\ j \in S\}$

  Stationary distribution exists when the Markov chain is ergodic.

$$\boxed{\begin{cases} \pi_j = \sum_{i \in Z} \pi_i p_{ij} \\ \sum_{j \in Z} \pi_j = 1,\ \pi_j \geq 0 \end{cases}}$$

Markov chains

- Important relationship (1)

Chapman-Kolmogorov equations

$$p_{ij}^{(m)} = \sum_{k \in Z} p_{ik}^{(r)} p_{kj}^{(m-r)}, \quad r = 0,1,\ldots,m \qquad \boxed{\mathbf{P}^{(m)} = \mathbf{P}\mathbf{P}^{(m-1)}}$$

or

$$\boxed{\mathbf{P}^{(m)} = \mathbf{P}^{(r)}\mathbf{P}^{(m-r)}, \ r = 0,1,\ldots,m} \qquad \boxed{\mathbf{P}^{(m)} = \mathbf{P}^{m}}$$

Joint distribution:

$$\boxed{P\{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\} = s_{i_0}(0)\, p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}}$$

Absolute distribution:

$$p_j(m) = \sum_{i \in S} s_i(0)\, p_{ij}^{(m)} = \sum_{i \in S} s_i(m-1)\, p_{ij}, \ m = 1,2,\ldots$$

$$\boxed{\mathbf{s}(m) = \mathbf{s}(0)\mathbf{P}^{(m)} = \mathbf{s}(m-1)\mathbf{P}}$$

Markov chains

- Important relationship (2)

  The unique stationary distribution and limit distribution exist if the Markov chain is ergodic.

  For any $i, j \in S$,

$$\pi_j = \lim_{m\to\infty} p_{ij}^{(m)} = \frac{1}{\mu_j}, \quad i, j \in S$$

$$m \to \infty, \quad \mathbf{P}^{(m)} = \begin{bmatrix} \pi_0 & \pi_1 & \cdots & \pi_j & \cdots \\ \cdots & \ddots & \ddots & \ddots & \cdots \\ \pi_0 & \pi_1 & \cdots & \pi_j & \cdots \end{bmatrix}$$

Overview

- OUTLINE

  Important concepts and relationships

  Correlation functions and stationary processes

  Power spectrum and linear systems

  Markov chains

  Poisson processes

Poisson processes

- Three equivalent definitions:

Def. 1 Poisson process

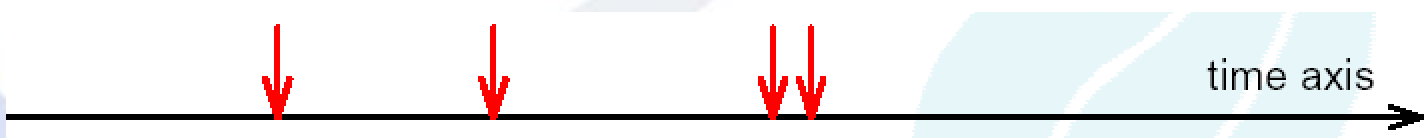
A counting process N(t) is said to be a Poisson process with mean rate (or intensity) ν (or λ) if

(i) N(t) has stationary independent increment.

(ii) N(0)=0.

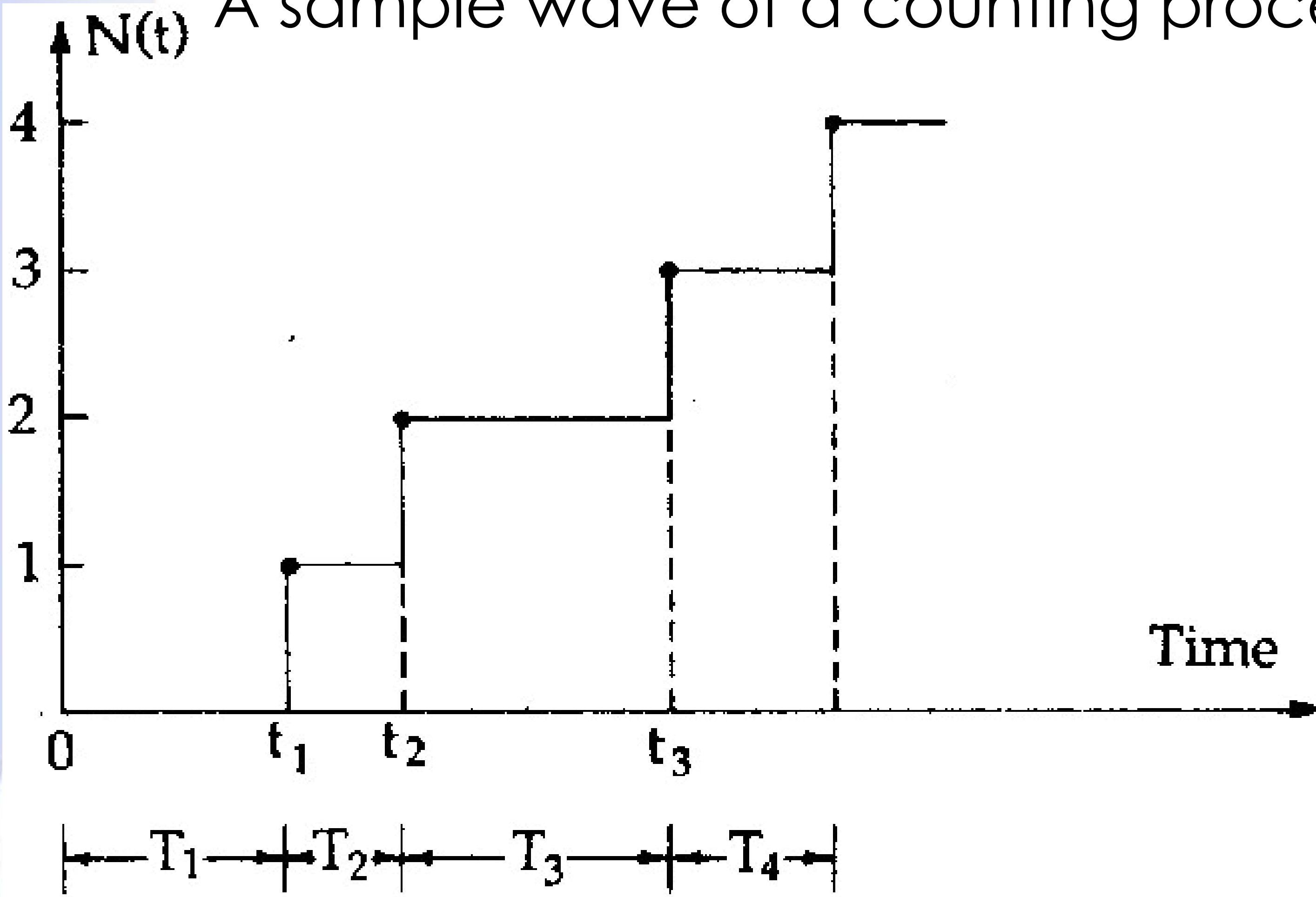
(iii) The number in any time interval of length τ is Poisson distributed with mean $\nu\tau$. That is,

$$P\{N(t+\tau)-N(t)=k\}=\frac{(\nu\tau)^k}{k!}e^{-\nu\tau}$$



Poisson processes

A sample wave of a counting process N(t)



$T_1=t_1, T_2=t_2-t_1, T_3=t_3-t_2, \ldots$ Interarrival times

Poisson processes

Def.2 Poisson process

If the interarrival times (are independent, identically distributed random variables) obey an exponential distribution, the process is called a Poisson process.

Poisson processes

Def.3 Poisson process

A counting process { N(t) | t ≥ 0 } is said to be a Poisson Process with **rate** ν > 0 if,

i. N(0) = 0
ii. The process has stationary and independent increments.
iii. N(t) satisfies

$$P\{X(t+h)-X(t)=1\}=\nu h+o(h)$$
$$P\{X(t+h)-X(t)\geq 2\}=o(h)$$

A function f( ) is said to be o(h) if $\lim_{h\to 0}\frac{f(h)}{h}=0$

Poisson processes

- Moments

(1) Mean value function: $E[N(t)] = \nu t$ $\boxed{\nu = E[N(t)]/t}$

(2) Variance function: $Var[N(t)] = \nu t$

(3) Correlation function:

$$R(\tau) = E[N(t)N(t+\tau)] = \nu^2 t\tau + (\nu t)^2 + \nu t$$

$$R(t_1, t_2) = \nu^2 t_1 t_2 + \nu t_1,\ t_1 < t_2$$

(4) Covariance function:

$$Cov_N[t_1, t_2] = \lambda \min(t_1, t_2)$$

(5) Characteristic function:

$$\phi(u) = E[e^{iuN(t)}] = \exp\{\lambda t(e^{iu} - 1)\}$$

Poisson processes

- Properties:

1. $N_1(t), N_2(t),...N_n(t)$ are independent Poisson processes, with mean values $\nu_1 t, \nu_2 t, ...\nu_n t,$ respectively.
   $N(t) = N_1(t) + N_2(t) + ... + N_n(t)$ is also a Poisson process with mean $\boxed{(\nu_1 + \nu_2 + ... + \nu_n)t.}$
2. $N_1(t), N_2(t)$ are two independent Poisson processes with mean $\nu_1 t$ and $\nu_2 t$ respectively.
   $\boxed{N(t) = N_1(t) - N_2(t)}$ is not a Poisson process; instead, it has the probability distribution,

$$P\{N_1(t) - N_2(t) = n\} = e^{-(\nu_1+\nu_2)t}\left(\frac{\nu_1}{\nu_2}\right)^{\frac{n}{2}} I_n(2\sqrt{\nu_1\nu_2}t)$$

where $I_n(\ )$ is a modified Bessel function of order n.

Poisson processes

3. If the Poisson process N(t) with mean νt is filtered such that every occurrence of the event is not counted, the process has a constant probability p of being counted. Then the resulting counting process is also a Poisson process with mean $\boxed{p\nu t}$.

$$P\{M(t)=n\}=e^{-p\nu t}\frac{(p\nu t)^n}{n!}$$

6.1.2 Some Properties of the Poisson Processes

4. Let X be the number of occurrences of an event that takes place in accordance with a Poisson process with intensity *v*. Find the number X that has the largest probability in a specified time *t*.

$$\frac{Pr\{X = r + 1\}}{Pr\{X = r\}} = \frac{e^{-\nu t}(\nu t)^{r+1}/(r+1)!}{e^{-\nu t}(\nu t)^{r}/r!} = \frac{\nu t}{r+1}$$

$$Pr\{X = 0\} < Pr\{X = 1\} < \cdots < Pr\{X = r - 1\}$$

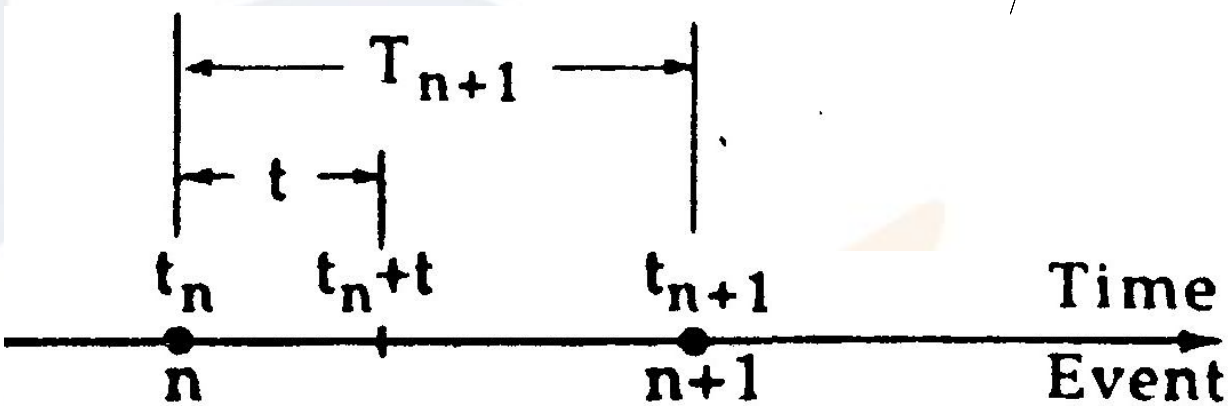
$$\leq Pr\{X = r\} > Pr\{X = r + 1\} > \cdots$$

$$r = \mathrm{int}[\nu t + 1] - 1$$

Poisson processes

- interarrival times

Theorem: the interarrival times of a Poisson process with intensity ν are independent, identically distributed exponential random variables with mean $1/\nu$



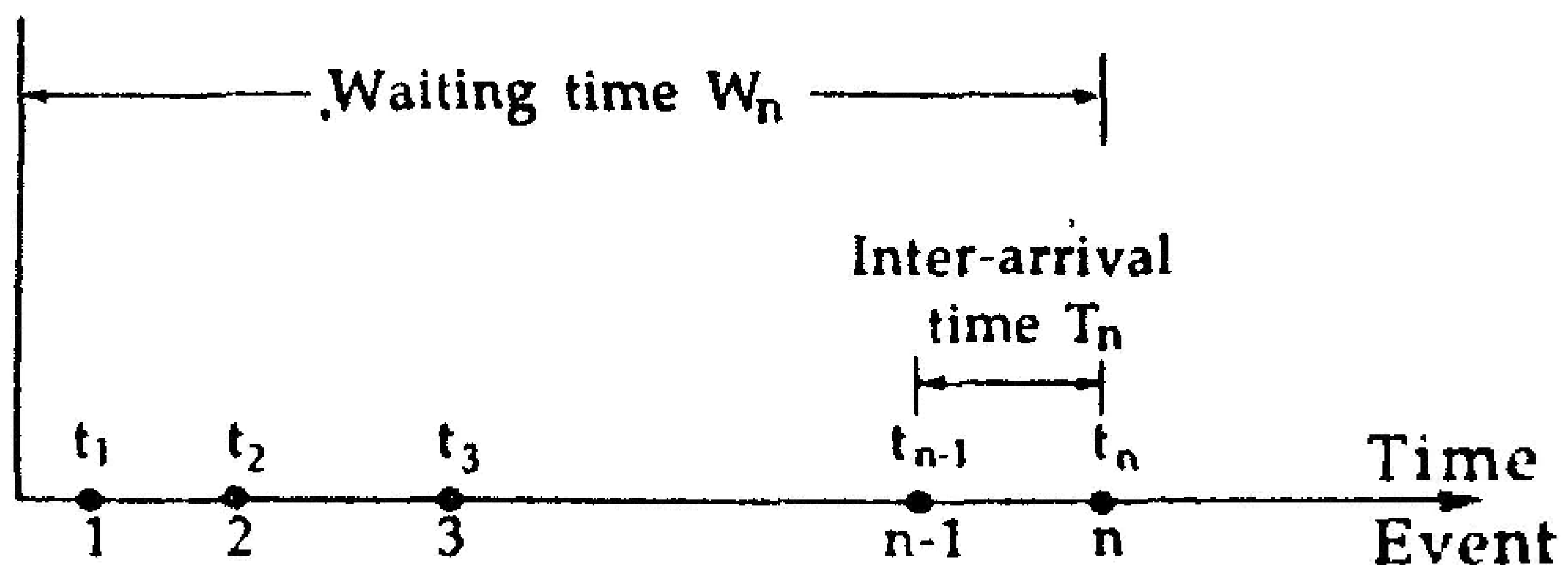
$$P\{T_{n+1} > t\} = P\{N(t) = 0\} = e^{-\nu t}$$

$$\therefore F_{T_i}(t) = P\{T_i \le t\} = 1 - e^{-\nu t},\ i = 1,2,\ldots$$

$$\therefore\ f_{T_i}(t) = \nu e^{-\nu t},\ i = 1,2,\ldots$$

Poisson processes

- Waiting Time W_n:
  the time up to a specific number of occurrences of the event from t=0.

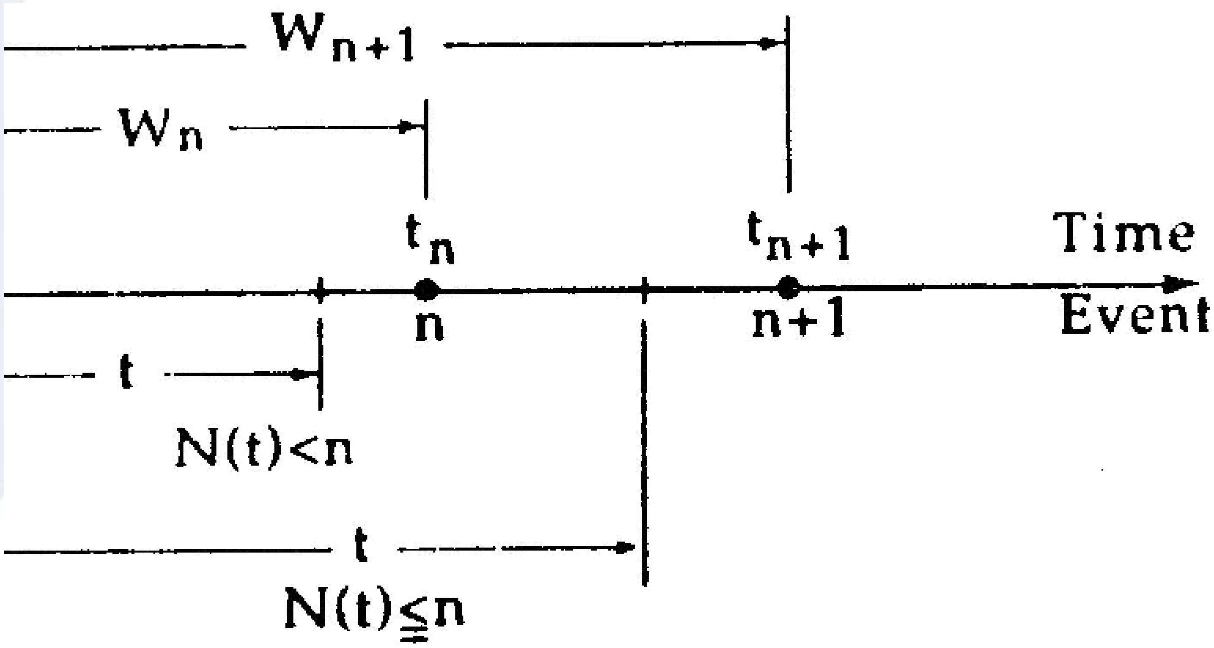


$$\boxed{W_n = T_1 + T_2 + \cdots T_n}$$

$$\boxed{T_n = W_n - W_{n-1}}$$

Poisson processes

Relationship of waiting time and event:



$$Pr\{N(t) < n\} = Pr\{W_n > t\}$$

$$Pr\{N(t) \leq n\} = Pr\{W_{n+1} > t\},\ n = 0, 1, 2, \ldots$$

Poisson processes

Relationship of waiting time and event:

$$Pr\{N(t) \geq n\} = Pr\{W_n \leq t\} = F_{W_n}(t)$$

$$Pr\{N(t) > n\} = Pr\{W_{n+1} \leq t\} = F_{W_{n+1}}(t)$$

$$Pr\{N(t) = n\} = Pr\{N(t) \geq n\} - Pr\{N(t) > n\}$$

$$= F_{W_n}(t) - F_{W_{n+1}}(t), \qquad n = 1, 2, 3, \ldots$$

$$Pr\{N(t) = 0\} = Pr\{W_1 > t\} = 1 - F_{W_1}(t)$$

Poisson processes

Distribution of Waiting Time:

$$Pr\{W_n \le t\} = Pr\{N(t) \ge n\} = \sum_{j=n}^{\infty} e^{-\nu t} \frac{(\nu t)^j}{j!}$$

$$f_{W_n}(t) = -\sum_{j=n}^{\infty} \nu e^{-\nu t} \frac{(\nu t)^j}{j!} + \sum_{j=n}^{\infty} \nu e^{-\nu t} \frac{(\nu t)^{j-1}}{(j-1)!}$$

$$= \nu e^{-\nu t} \frac{(\nu t)^{n-1}}{(n-1)!}$$

Gamma or Erlang distribution with parameters n and ν .

Poisson processes

3. The conditional distribution of arrival time:

Problem: What is the probability that exactly *m* events occur in the interval [0,*t*] given that exactly *n* events occur in the interval $[0, t+\tau]$; *m*=0,1,…,n?

$$Pr\{N(t)=m \mid N(t+\tau)=n\}$$

$$=\binom{n}{m}\left(\frac{t}{t+\tau}\right)^{m}\left(\frac{\tau}{t+\tau}\right)^{n-m}$$

It is a binomial distribution with parameters $p=\dfrac{\tau}{t+\tau}$ and n.

Inhomogeneous Poisson processes

Definition1: A Poisson process with an intensity that is a nonnegative function of time, $\nu(t)$, is defined as a inhomogeneous Poisson process.

Definition 2: A counting process $\{N(t), t \geq 0\}$ is called an inhomogeneous Poisson process with nonnegative intensity function $\nu(t)$ if it has properties

i) $N(0)=0$,

ii) $\{N(t), t \geq 0\}$ has independent increments,

iii) $P\{X(t+h) - X(t) = 1\} = \nu(t)h + o(h)$

iv) $P\{X(t+h) - X(t) \geq 2\} = o(h)$

Definition 1 and definition 2 are equivalent.

Inhomogeneous Poisson processes

- $\nu(t)$ is called the intensity function.

- Distribution:

$$P\{N(t)=n\}=\frac{\{\int_0^t \nu(s)ds\}^n}{n!}\exp\{-\int_0^t \nu(s)ds\}$$

$$E[N(t)]=Var[N(t)]=\int_0^t \nu(s)ds=m_N(t)$$

$$P\{N(t)=n\}=\frac{\{m_N(t)\}^n}{n!}\exp\{-m_N(t)\}$$

Inhomogeneous Poisson Processes

- Correlation function:

$$R(\tau) = E[N(t)]E[N(t+\tau) - N(t)] + E[N^2(t)]$$

$$= \int_0^t \nu(t)\,dt \int_0^{t+\tau} \nu(t)\,dt + \int_0^t \nu(t)\,dt$$

$$= \int_0^t \nu(t)\,dt \left\{1 + \int_0^{t+\tau} \nu(t)\,dt\right\}$$

- The increment process of a inhomogeneous Poisson process is no longer stationary.

Compound Poisson processes

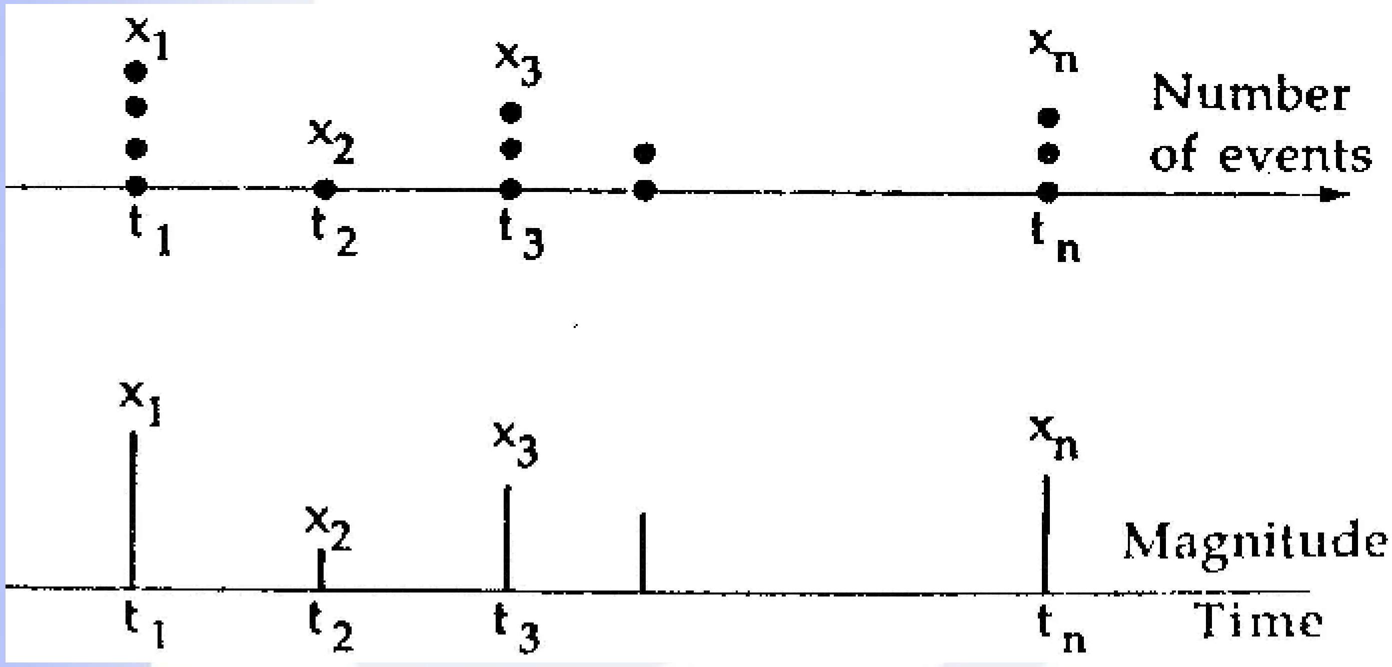
<u>Definition:</u> A stochastic process Y(t) is called a compound Poissson process if it is the sum of random variables X_n given by

$$Y(t) = \sum_{n=1}^{N(t)} X_n$$

where N(t) is a Poisson process with intensity ν and X_n are independent random variables with identical distribution.

Compound Poisson processes

X_n may be continuous random variables or discrete random variables.



$$Y(t) = \sum_{n=1}^{N(t)} X_n$$

Y(t) has independent increment.

Compound Poisson processes

Characteristic function:

$$\phi_Y(u) = E[e^{iuY(t)}] = e^{\{\lambda t[\phi_X(u)-1]\}}$$

Compound Poisson processes

Digital Characteristic :

$$E[Y(t)] = \frac{1}{i}\left[\frac{d\phi_Y(u)}{du}\right]_{u=0} = \frac{1}{i}\nu t\phi_x'(0) = \nu t E[x]$$

$$E[Y^2(t)] = \frac{1}{i^2}\left[\frac{d^2\phi_Y(u)}{du}\right]_{u=0}$$

$$= \frac{1}{i^2}\left[\nu t\phi_x''(0) + (\nu t)^2\{\phi_x'(0)\}^2\right]$$

$$\mathrm{Var}[Y(t)] = \nu t E[x^2]$$

$$\mathrm{Cov}[Y(s), Y(t)] = \nu(\min s, t)E[X^2]$$

End

Good luck!