OUTLINE

6.1 Markov Processes

6.2 Chapman-Kolmogorov Equation

6.3 Basic Concepts of Markov Chains(MCs)

6.4 Classification of States

6.5 Ergodic MC and Stationary Distribution

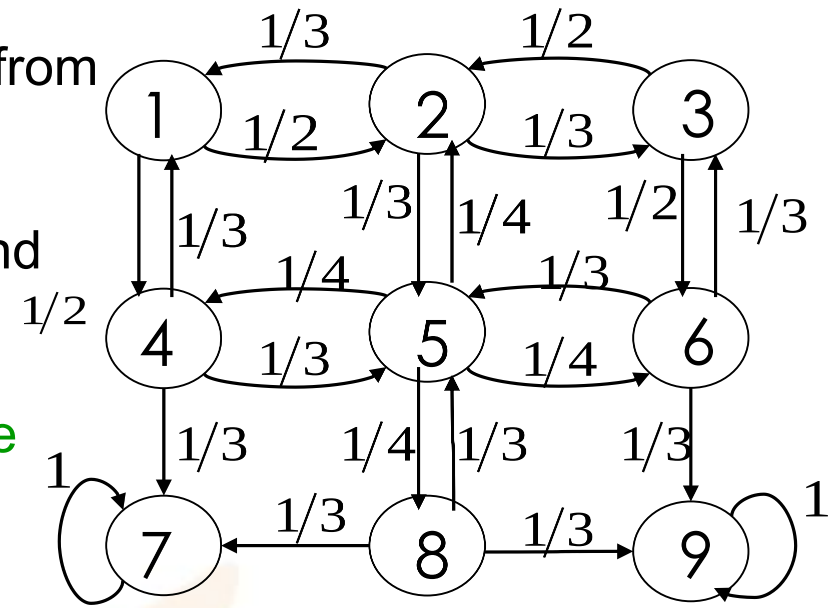
6.4 Classification of States

- The behavior of a Markov chain depends on the structure of the transition matrix *P*.
- We can classify a state space into different subsets according to the character of states.

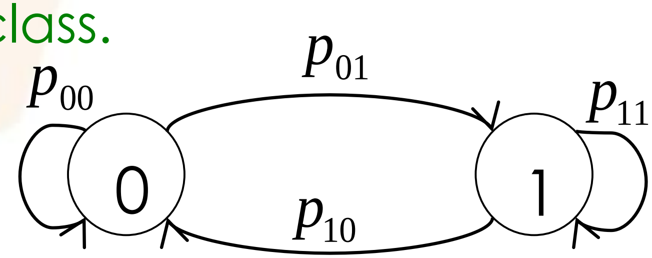
6.4 Classification of States

- *Accessible*: state j is said to be accessible from i if for some $n \geq 0, p_{ij}^{(n)} > 0$. “i—>j”
- *Communication*: if i—>j, and j—>i, then i and j communicate. “ i<—>j ”
- If i and j communicate, they are in the same *equivalence class.*

  Two classes are either disjoint or the same.
- *Closed class*: if no states outside of the class can be reached from any state within the class.
- The max closed class is the Markov chain itself.
- Closed class is *irreducible* if its states are communicate.
- *Irreducible Markov chain*: its only irreducible closed class is the set of states in its state space S.



{7},{9},{7,9}are closed class.



6.4 Classification of States

e.g.1: Consider a Markov chain with transition probability matrix:

A random walk with two absorbing barriers.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \cdot \\ \cdot \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdot & \cdot & N-1 & N \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array},$$

where $p + q = 1$ and $p > 0$ and $q > 0$.

The chain has three classes {0}, {*N*}, and {1,2,…N-1}.

Closed classes: {0}, {*N*}, {0,*N*}

6.4 Classification of States

First passage time T_{ij} is the first passage time from *i* to *j*.

First entrance probability $f_{ij}^{(n)}$ is the probability mass function for T_{ij}

$$f_{ij}^{(n)} = P\left\{T_{ij} = n\right\} = P\left\{X_n = j,\ X_{n-1} \neq j,\ \ldots,\ X_1 \neq j \middle| X_0 = i\right\}$$

$$n = 1,\ 2,\ \ldots.$$

Recurrence time of state j : T_{jj}

Reaching probability from i to j f_{ij} is the probability that *j* is ever reached from *i*.

$$f_{ij} = P\{T_{ij} < \infty\} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

6.4 Classification of States

If $f_{jj} = 1$ state *j* is called *recurrent* and $P\{T_{jj} < \infty\} = 1$

$\mu_{jj} = E[T_{jj}]$ denote *the mean recurrence time* of state *j*.

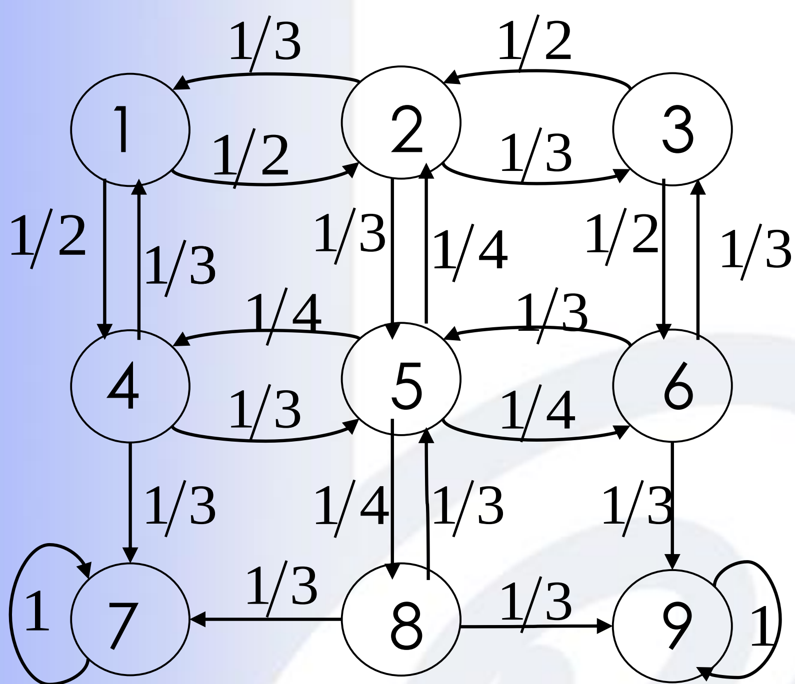
When the state space is infinite it is not necessary that $E[T_{jj}] < \infty$

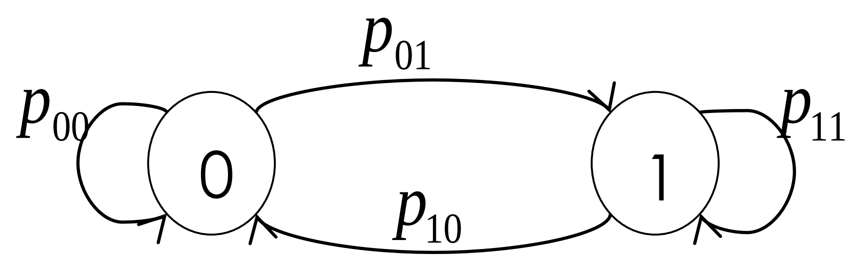
If $\mu_{jj} < \infty$ state *j* is called *positive recurrent*.

If $\mu_{jj} = \infty$ state *j* is called *null recurrent*.

If $f_{jj} < 1$ state *j* is called *transient*.

6.4 Classification of States





- 0,1 are positive recurrent
- 1,2,3,4,5,6,8 are transient
- 7,9 are positive recurrent

The states in an irreducible Markov Chain with finite state space are positive recurrent.

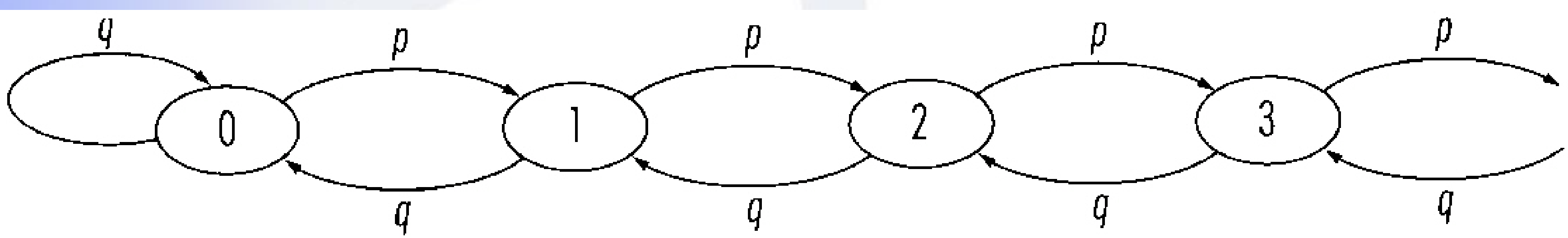
6.4 Classification of States

Example: (example 4.3.1)

A random walk with a reflecting barrier

where $p>0, q>0, p+q=1$

$$\begin{bmatrix} q & p & 0 & \cdot & \cdot & \cdot \\ q & 0 & p & 0 & \cdot & \cdot \\ 0 & q & 0 & p & 0 & \cdot \\ \cdot & 0 & q & 0 & p & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \end{bmatrix}$$



- All states are communicate, hence the chain is irreducible.
- $q>p$, the chain is positive recurrent. $q<p$, the chain is transient. $q=p$, the chain is null recurrent.

Chapter 6: Markov Processes and Discrete-Time Markov Chains

OUTLINE

6.1 Markov Processes

6.2 Chapman-Kolmogorov Equation

6.3 Basic Concepts of Markov Chains(MC)

6.4 Classification of States

6.5 Ergodic MC and Stationary Distribution

6.5 Ergodic MC and Stationary Distribution

1. Limiting distributions
2. Periodic Markov chains
3. Ergodic Markov chains
4. Stationary distributions

6.5 Ergodic MC and Stationary Distribution

1. Limiting distribution:

Example 1:

t = day index 0, 1, 2, ...

X_t = 0 rainy on t^{th} day

= 1 sunny on t^{th} day

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\mathbf{P}^{(2)} = \begin{bmatrix} p_{00}^{(2)} & p_{01}^{(2)} \\ p_{10}^{(2)} & p_{11}^{(2)} \end{bmatrix} = \begin{bmatrix} 7/16 & 9/16 \\ 3/8 & 5/8 \end{bmatrix} = \begin{bmatrix} 0.4375 & 0.5625 \\ 0.375 & 0.625 \end{bmatrix}$$

$$\mathbf{P}^{(4)} = \begin{bmatrix} 0.40234375 & 0.59765625 \\ 0.3984375 & 0.6015625 \end{bmatrix}$$

$$\mathbf{P}^{(8)} = \begin{bmatrix} p_{00}^{(8)} & p_{01}^{(8)} \\ p_{10}^{(8)} & p_{11}^{(8)} \end{bmatrix} \approx \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix}$$

$$\lim_{m\to\infty} \mathbf{P}^{(m)} = \begin{bmatrix} p_{00}^{(m)} & p_{01}^{(m)} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 1/\mu_0 & 1/\mu_1 \\ 1/\mu_0 & 1/\mu_1 \end{bmatrix}$$

Example 2:

0 1 → Channel 1 → Channel 2 → Channel 3 → Channel 4 → Receiver

$$\mathbf{P} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} \qquad \mathbf{P}^{(4)} = \begin{bmatrix} 0.53125 & 0.46875 \\ 0.46875 & 0.53125 \end{bmatrix}$$

$$\mathbf{P}^{(2)} = \begin{bmatrix} 0.625 & 0.375 \\ 0.375 & 0.625 \end{bmatrix} \qquad \mathbf{P}^{(8)} = \begin{bmatrix} 0.501953125 & 0.498046875 \\ 0.498046875 & 0.501953125 \end{bmatrix}$$

$$\lim_{m\to\infty} \mathbf{P}^{(m)} = \begin{bmatrix} p_{00}^{(m)} & p_{01}^{(m)} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

6.5 Ergodic MC and Stationary Distribution

- Limiting distribution (independent with i):

$$\lim_{m\to\infty} p_{ij}^{(m)} = \frac{1}{\mu_j}, \quad j \in S$$

Example 1:

$$\lim_{m\to\infty} p_{i0}^{(m)} = \frac{1}{\mu_0} = 0.4 \qquad \lim_{m\to\infty} p_{i1}^{(m)} = \frac{1}{\mu_1} = 0.6$$

Example 2:

$$\lim_{m\to\infty} p_{i0}^{(m)} = \frac{1}{\mu_0} = 0.5 \qquad \lim_{m\to\infty} p_{i1}^{(m)} = \frac{1}{\mu_1} = 0.5$$

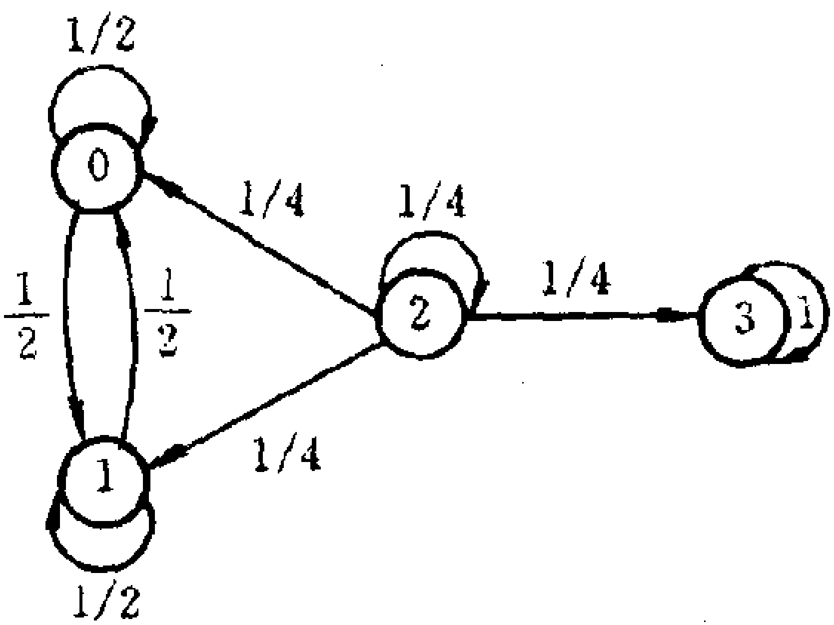
6.5 Ergodic MC and Stationary Distribution

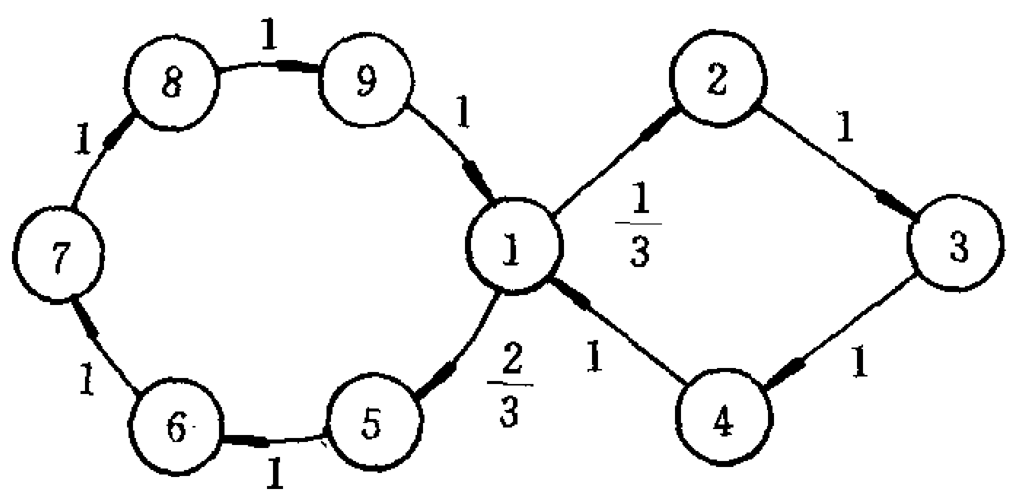
2. Periodic Markov chain: the period of state i:

d(i) denote the greatest common divisor of all integers $n \geq 1$ for which $p_{ii}^{(n)} > 0$.

The integer d(i) is called *the period of state i*.

- If d(i)=1, state i is called aperiodic.
- If $p_{ii}^{(1)} > 0$, then d(i)=1, state *i* is aperiodic.





6.5 Ergodic MC and Stationary Distribution

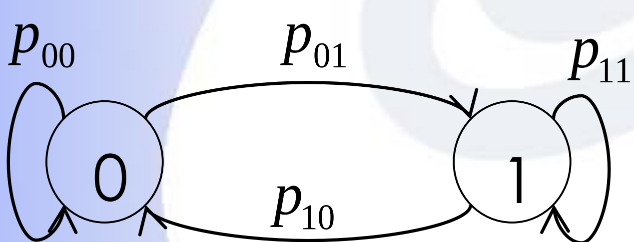
- If d is the period of state i, for all $n(\mathrm{mod})d \neq 0$

$$p_{ii}^{(n)} = 0$$

- If one state in a class is periodic, all states in the class are periodic.

- Aaperiodic Markov chain：

An irreducible Markov chain whose states have a period of 1 is called an aperiodic chain.



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.7 & 0.3 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix}$$

6.5 Ergodic MC and Stationary Distribution

3. Ergodic Markov chain

- An irreducible aperiodic Markov chain with finite number of states is called an *Ergodic Markov chain*.
- An ergodic Markov chain is irreducible.
- All states in an ergodic Markov chain are Communicate and Aperiodic.

Communicate: For some $m \ge 1,\ p_{ij}^{(m)} \neq 0,\ i,j \in S$

Example 1:

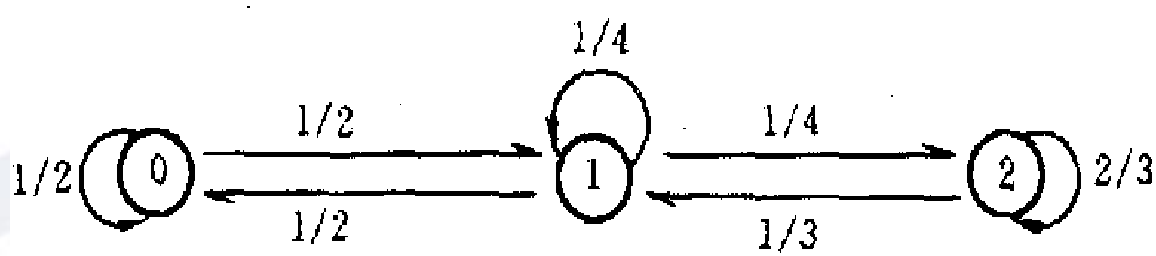
$$\mathbf{P} = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$$

Example 2:

$$\mathbf{P} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

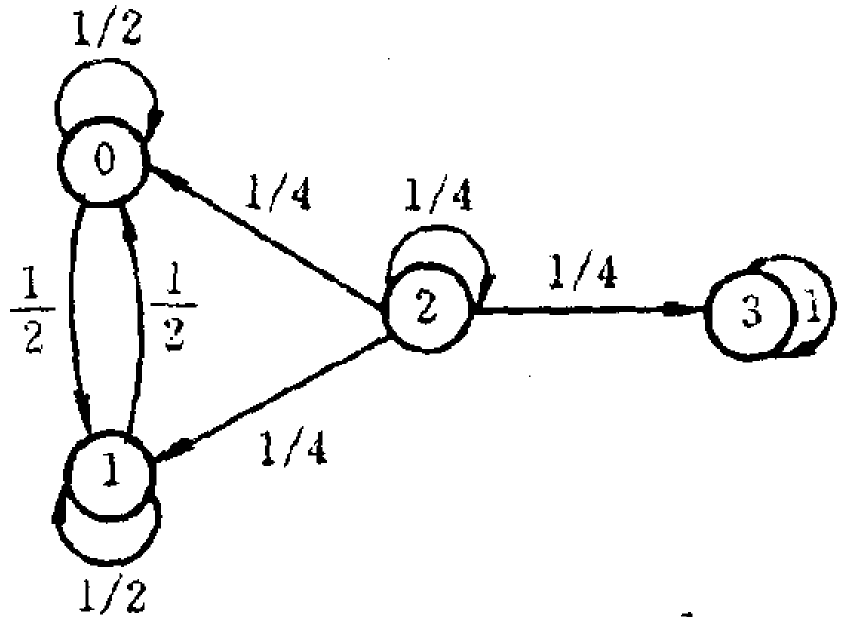
Example 3:

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$



6.5 Ergodic MC and Stationary Distribution

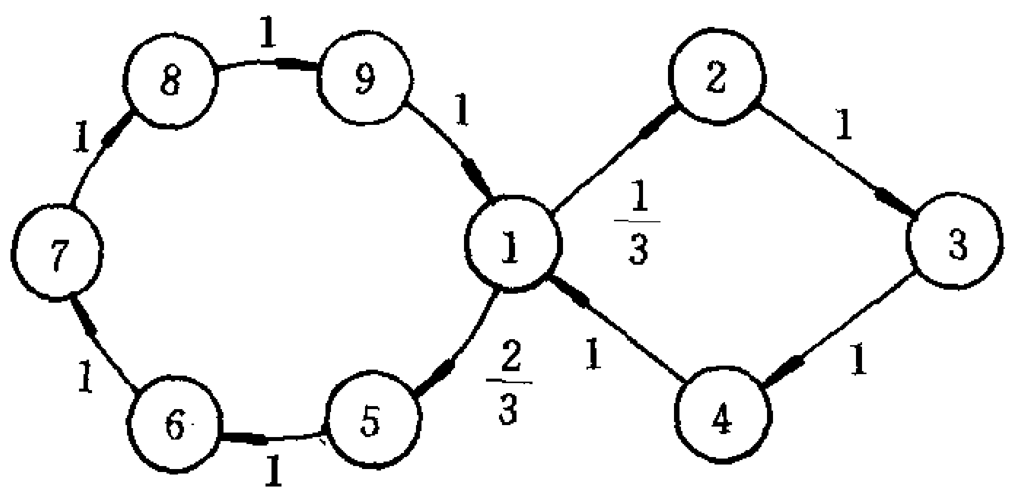
- Example 4:



{0,1},{3},{0,1,3} are closed class.

{0,1} are irreducible closed class.

- Example 5:



All states in state space are communicate.

A periodic irreducible Markov chain with d(i)=2

6.5 Ergodic MC and Stationary Distribution

Theorem:

The limiting distribution exists

if the Markov chain is ergodic.

6.5 Ergodic MC and Stationary Distribution

4. Stationary distribution:

Finding the limiting distribution:

If the limiting distribution of a Markov chain exists, it can be write as

$$\lim_{m\to\infty} P^{(m)} = \begin{bmatrix} \pi_0 & \cdots & \pi_i & \pi_N \\ \pi_0 & \cdots & \pi_i & \pi_N \\ \cdots & \cdots & \cdots & \cdots \\ \pi_0 & \cdots & \pi_i & \pi_N \end{bmatrix}$$

After a period of time the absolute distribution is not change with time:

$$\boxed{\begin{aligned} \mathbf{s}(m) &= \mathbf{s}(0)\mathbf{P}^{(m)} = \begin{bmatrix} \pi_0 & \cdots & \pi_i & \pi_N \end{bmatrix} \\ \mathbf{s}(m+1) &= \mathbf{s}(0)\mathbf{P}^{(m+1)} = \begin{bmatrix} \pi_0 & \cdots & \pi_i & \pi_N \end{bmatrix} \\ \mathbf{s}(m+1) &= \mathbf{s}(m)\mathbf{P} = s(m) \end{aligned}}$$

6.5 Ergodic MC and Stationary Distribution

4. Stationary distribution:

The probability distribution $\{\pi_j, j \in S\}$ is called stationary distribution of Markov process, if it satisfies:

$$\boxed{\begin{cases} \pi_j = \sum_{i \in S} \pi_i p_{ij} \\ \sum_{j \in S} \pi_j = 1, \ \pi_j \geq 0 \end{cases}} \quad \text{or} \quad \boxed{\begin{cases} \boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \\ \sum_{j \in S} \pi_j = 1, \ \pi_j \geq 0 \end{cases}}$$

Theorem:

The unique stationary distribution exists if the Markov chain is ergodic. For any $i, j \in S$, the stationary distribution is given by

$$\boxed{\pi_j = \lim_{m \to \infty} p_{ij}^{(m)} = \frac{1}{\mu_j}, \quad i,j \in S}$$

6.5 Ergodic MC and Stationary Distribution

First passage time T_{ij} is the first passage time from *i* to *j*.

First entrance probability $f_{ij}^{(n)}$ is the probability mass function for T_{ij}

$$f_{ij}^{(n)} = P\{T_{ij} = n\} = P\{X_n = j,\ X_{n-1} \neq j,\ \ldots,\ X_1 \neq j \mid X_0 = i\}$$

Mean of fist passage time:

$$\mu_{ij} = E[T_{ij}] = \sum_{n=1}^{\infty} n f_{ij}^{(n)}$$

Recurrence time of state j : T_{jj}

Mean of Recurrence time of state *j:*

$$\mu_j = \mu_{jj} = E[T_{jj}] = \sum_{n=1}^{\infty} n f_{jj}^{(n)}$$

6.5 Ergodic MC and Stationary Distribution

- Example 1:

$$\lim_{m\to\infty} p_{i0}^{(m)} = \frac{1}{\mu_0} = 0.4 \qquad \lim_{m\to\infty} p_{i1}^{(m)} = \frac{1}{\mu_1} = 0.6$$

$$\mathbf{P} = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix} \qquad \boxed{\begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \cdot \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}}$$

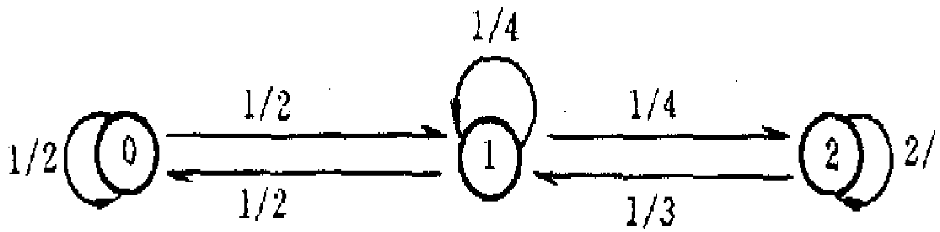
Example 2:

$$\lim_{m\to\infty} p_{i0}^{(m)} = \frac{1}{\mu_0} = 0.5 \qquad \lim_{m\to\infty} p_{i1}^{(m)} = \frac{1}{\mu_1} = 0.5$$

$$\mathbf{P} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} \qquad \boxed{\begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}}$$

- Example 3:

$$\mathbf{P}=\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$



$$\begin{cases} \begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix}=\begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix}\cdot\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix} \\ \sum_{j\in Z}\pi_j=1,\ \pi_j\geq 0 \end{cases}$$

$$\begin{cases} \pi_0=1/2\cdot\pi_0+1/2\cdot\pi_1 \\ \pi_1=1/2\cdot\pi_0+1/4\cdot\pi_1+1/3\cdot\pi_2 \\ \pi_2=1/4\cdot\pi_1+2/3\cdot\pi_2 \\ \pi_0+\pi_1+\pi_2=1,\ \pi_j\geq 0 \end{cases}$$

$$\pi_0=8/19$$
$$\pi_1=8/19$$
$$\pi_2=3/19$$

6.5 Ergodic MC and Stationary Distribution

- A Markov chain is strictly stationary if and only if its absolute state probabilities do not depend on time.

- An ergodic Markov chain is not a stationary process unless:

  A Markov chain is strictly stationary if and only if its initial distribution is stationary.

6.5 Ergodic MC and Stationary Distribution

- Example 4:

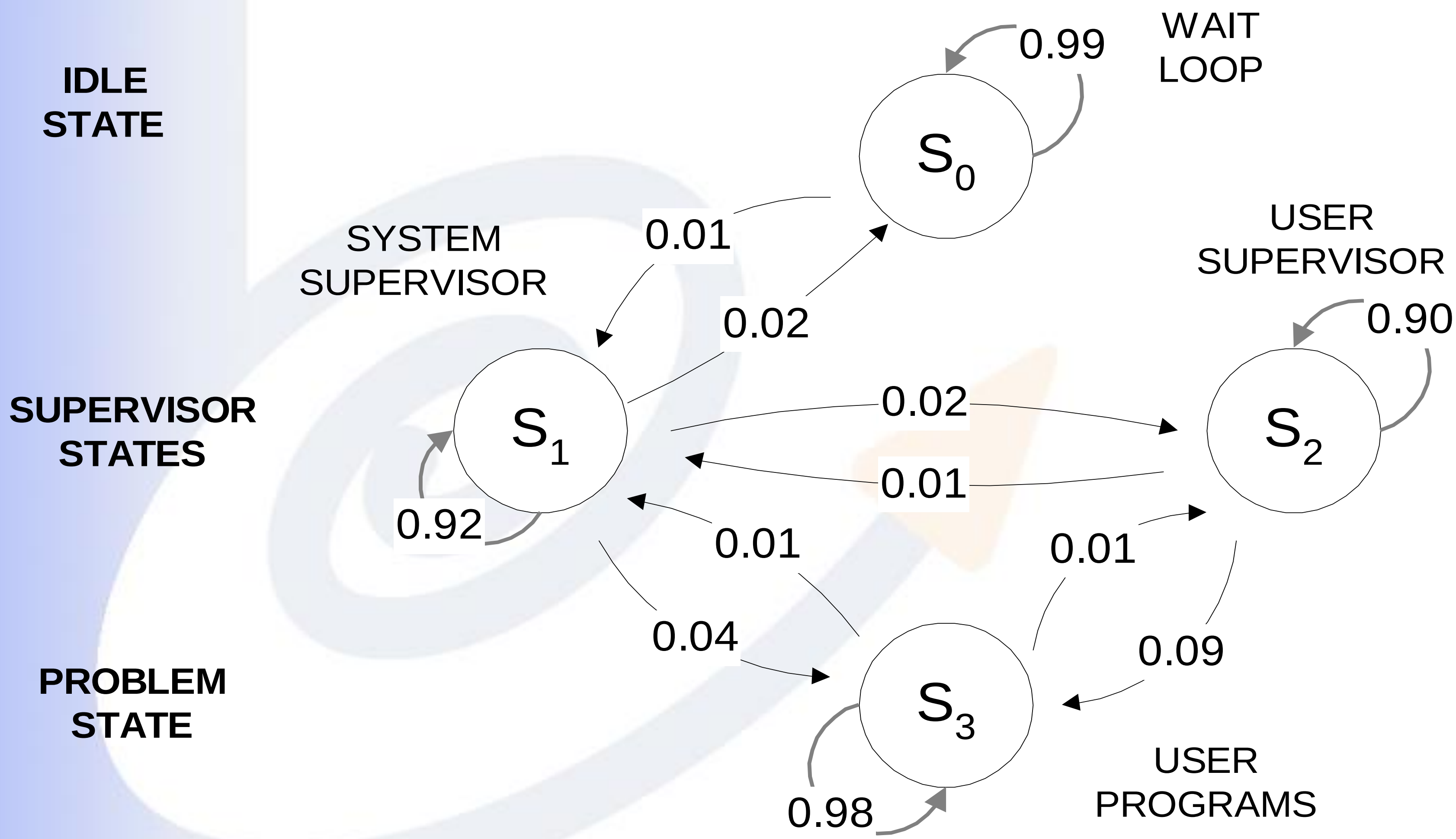
  CPU of a multiprogramming system is at any time executing instructions from:

  - User program ==> Problem State (S_3)
  - OS routine explicitly called by a user program (S_2)
  - OS routine performing system wide control task (S_1)

    ==> Supervisor State
  - Wait loop ==> Idle State (S_0)

6.5 Ergodic MC and Stationary Distribution

State Transition graph of discrete-time Markov of a CPU



6.5 Ergodic MC and Stationary Distribution

To State

From State		S_0	S_1	S_2	S_3
	S_0	0.99	0.01	0	0
From	S_1	0.02	0.92	0.02	0.04
State	S_2	0	0.01	0.90	0.09
	S_3	0	0.01	0.01	0.98

Transition Probability Matrix

6.5 Ergodic MC and Stationary Distribution

$$\pi_0 = 0.99\,\pi_0 + 0.02\,\pi_1$$

$$\pi_1 = 0.01\,\pi_0 + 0.92\,\pi_1 + 0.01\,\pi_2 + 0.01\,\pi_3$$

$$\pi_2 = 0.02\,\pi_1 + 0.90\,\pi_2 + 0.01\,\pi_3$$

$$\pi_3 = 0.04\,\pi_1 + 0.09\,\pi_2 + 0.98\,\pi_3$$

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3$$

Stationary state probabilities can be computed by solving system of equations. So we have:

$\pi_0 = 2/9$, $\pi_1 = 1/9$, $\pi_2 = 8/99$, $\pi_3 = 58/99$

6.5 Ergodic MC and Stationary Distribution

S_0 : waiting loop; $\pi_0 = 2/9$

S_1: system supervisor; $\pi_1 = 1/9$

S_2: user supervisor; $\pi_2 = 8/99$

S_3: user programs; $\pi_3 = 58/99$

- Utilization of CPU: $1 - \pi_0 = 77.7\%$
- 58.6% of total time spent for processing users programs (S_3).
- 19.1% (77.7 - 58.6) of time spent in supervisor state.

  11.1% in S_1

  8% in S_2

6.5 Ergodic MC and Stationary Distribution

- Finding the n-step transition probability matrix

$$|P - \lambda I| = 0$$

$$P = Q\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Q^{-1} \qquad P^n = Q\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} Q^{-1}$$

Example: Two-state Markov chain

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= p_{00} + p_{11} - 1 \end{aligned}$$

$$Q = \begin{pmatrix} 1 & p_{01} \\ 1 & -p_{10} \end{pmatrix} \qquad Q^{-1} = \frac{1}{p_{01} + p_{10}} \begin{pmatrix} p_{10} & p_{01} \\ 1 & -1 \end{pmatrix}$$

6.5 Ergodic MC and Stationary Distribution

◆

$$P(n) = P^n = \frac{1}{p_{01}+p_{10}}\begin{pmatrix}1 & p_{01}\\ 1 & -p_{10}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & (p_{00}+p_{11}-1)^n\end{pmatrix}\begin{pmatrix}p_{10} & p_{01}\\ 1 & 1\end{pmatrix}$$

$$= \frac{1}{p_{01}+p_{10}}\begin{pmatrix}p_{10} & p_{01}\\ p_{10} & p_{01}\end{pmatrix} + \frac{(p_{00}+p_{11}-1)^n}{p_{01}+p_{10}}\begin{pmatrix}p_{01} & -p_{01}\\ -p_{10} & p_{10}\end{pmatrix}$$

6.5 Ergodic MC and Stationary Distribution

- **Summary of Markov Chain**
- Important concepts (1):

One-step transition probabilities:

$$p_{ij}(n) = P\{X_{n+1} = j \mid X_n = i\};\ n = 0,1,\ldots$$

Homogeneity:

$$p_{ij}(n) = P\{X_1 = j \mid X_0 = i\} = p_{ij} \quad \text{for all } n = 0,1,\ldots$$

One-step transition probability matrix: $\mathbf{P}$

M-step transition probabilities:

$$p_{ij}^{(m)} = P\{X_{n+m} = j \mid X_n = i\},\quad m = 0,1,\ldots$$

M-step transition probability matrix: $\mathbf{P}^{(m)}$

6.5 Ergodic MC and Stationary Distribution

- Important concepts (2):

Initial distribution $\mathbf{s}(0)$:

A probability distribution of $X(t_0)$

Absolute distribution $\mathbf{s}(m)$:

One-dimensional state probabilities of the Markov chain after m steps.

$$\mathbf{s}(m) = \{p_j^{(m)} = P(X_m = j), j \in S, \sum_{j \in S} p_j = 1\}$$

Limiting distribution:

$$\lim_{m \to \infty} p_{ij}^{(m)} = \frac{1}{\mu_j}, \ \ j \in S$$

6.5 Ergodic MC and Stationary Distribution

- Important concepts (3):

  Ergodic Markov chain: Communicate irreducible aperiodic Markov chain with finite number of states is called an *Ergodic Markov chain*.

  For some $m \geq 1$, $p_{ij}^{(m)} \neq 0,\ i,j \in S$

  Stationary distribution: $\{\pi_j, j \in S\}$

  Stationary distribution exists when the Markov chain is ergodic.

$$\begin{cases} \pi_j = \sum_{i \in Z} \pi_i p_{ij} \\ \sum_{j \in Z} \pi_j = 1,\ \pi_j \geq 0 \end{cases}$$

- Important relationship (1)

Chapman-Kolmogorov equations

$$p_{ij}^{(m)} = \sum_{k \in Z} p_{ik}^{(r)} p_{kj}^{(m-r)}, \quad r = 0,1,\ldots,m \qquad \mathbf{P}^{(m)} = \mathbf{P}\mathbf{P}^{(m-1)}$$

or

$$\mathbf{P}^{(m)} = \mathbf{P}^{(r)}\mathbf{P}^{(m-r)}, \; r = 0,1,\ldots,m \qquad \mathbf{P}^{(m)} = \mathbf{P}^{m}$$

Joint distribution:

$$P\{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\} = s_{i_0}(0)\, p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$$

Absolute distribution:

$$p_j(m) = \sum_{i \in S} s_i(0)\, p_{ij}^{(m)} = \sum_{i \in S} s_i(m-1)\, p_{ij}, \; m = 1,2,\ldots$$

$$\mathbf{s}(m) = \mathbf{s}(0)\mathbf{P}^{(m)} = \mathbf{s}(m-1)\mathbf{P}$$

6.5 Ergodic MC and Stationary Distribution

- Important relationship (2)

  The unique stationary distribution and limit distribution exist if the Markov chain is ergodic. For any $i, j \in S$,

$$\pi_j = \lim_{m \to \infty} p_{ij}^{(m)} = \frac{1}{\mu_j}, \quad i, j \in S$$

$$m \to \infty, \qquad \mathbf{P}^{(m)} = \begin{bmatrix} \pi_0 & \cdots & \pi_i & \cdots & \pi_N \\ \cdots & \ddots & \ddots & \ddots & \cdots \\ \pi_0 & \cdots & \pi_i & \cdots & \pi_N \end{bmatrix}$$

Ex. 1

Let $\{X_i;\ i = 0,1,\ldots\}$ be a sequence of independent, identically distributed binary random variables with

$$P(X_i = 1) = P(X_i = -1) = 1/2$$

Let moving averages Y_n be defined as follows

$$Y_n = \frac{1}{2}(X_n + X_{n-1});\quad n = 1,2,\ldots$$

Is $\{Y_1, Y_2, \ldots\}$ a Markov chain?

Ex. 2

- A wireless packet communications channel suffers from clustered errors. That is, whenever a packet has an error, the next packet will have an error with probability 0.9. When a packet is error-free, the next packet is error-free with probability 0.99.
- In steady state, what is the probability that a packet has an error?

Ex. 3

- The state of a proess changes daily according to a two-state Markov chain. If the process is in state *i* during one day, then it is in state *j* the following day with probability p_{ij} , where $p_{00} = 0.4, \quad p_{01} = 0.6, \quad p_{10} = 0.2, \quad p_{11} = 0.8$
- Every day a message is sent. If the state of the Markov chain that day is *i* then the message sent is "good" with probability p_i and is "bad" with probability $q_i = 1 - p_i, i = 0,1$
- If the process is in state 0 on Monday, what is the probability that a good message is sent on Tuesday?
- If the process is in state 0 on Monday, what is the probability that a good message is sent on Friday?
- In the long run, what proportion of messages is good?
- Let $Y_n = 1$ if a good message is sent on day *n* and let $Y_n = 2$ otherwise. Is $\{Y_n, \quad n \geq 1\}$ a Markov chain? If so, give its transition probability matrix. If not, briefly explain why not.

Ex.4

- Every week a professor gives exams to his students. He can give three possible types of exams, and his class is graded as either having done well or badly. Let $\boxed{p_i}$ denote the probability that the class does well on a type ***i*** exam, and suppose that $p_1 = 0.3, \quad p_2 = 0.6, \quad \text{and } p_3 = 0.9$.
- If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. Let $X_n, n \geq 1$ denote the exam type of the nth week.
- Find its transition probability matrix.
- If the exam type is 2 at first week, compute

$$P(X_1=2, X_2=1, X_3=3).$$

End of Chapter 6