Appendix B Basic Probability

① Borel-Cantelli Theorem: let A_n be events such that $\sum_{n=1}^{\infty} P(A_n) < \infty$
Then $P(A_n) = 0$

② mutual independence: we say that the events $\{A_j, j \in J\}$ are mutually independent if

$$P(\cap_{j \in K} A_j) = \prod_{j \in K} P(A_j), \ \forall \text{ finite } K \subset J$$

while pairwise independent $\not\Rightarrow$ mutually independent.

③ Conditional Probability: $P[A|B] = \frac{P(A \cap B)}{P(B)} \Leftrightarrow P(A \cap B) = P(B) P(A|B)$

④ Discrete RV

$E(X) = \sum_{n=1}^{N} x_n p_n$, $E(h(X)) = \sum_{n=1}^{N} h(x_n) p_n$ $\quad \begin{cases} \text{if } X \geq 0 \text{ and } E(X)=0 \text{ then } P(X=0)=1 \\ \text{if } X \geq 0 \text{ and } E(X)<\infty \text{ then } P(X<\infty)=1 \end{cases}$

$E(h_1(X) + h_2(X)) = E(h_1(X)) + E(h_2(X))$. $X \leq Y$ implies that $E(X) \leq E(Y)$

$Var(X) = E(X^2) - E^2(X)$ and $\sigma_X = \sqrt{Var(X)}$, $Var(aX) = a^2 Var(X)$

⑤ Bernoulli: $X = \{(0, 1-p), (1, p)\}$ (parameter $p \in [0,1]$) $E(X) = p$, $Var(X) = p(1-p)$

Geometric: $X = G(p)$ (parameter $p \in [0,1]$) $P(X=n) = (1-p)^{n-1} p$ $n \geq 1$ $E(X) = \frac{1}{p}$ $Var(X) = \frac{1-p}{p^2}$

Binomial: $X = B(N, p)$ if $P(X=n) = \binom{N}{n} p^n (1-p)^{N-n}$ $(n=0, \dots, N)$ $E(X) = Np$ $Var(X) = Np(1-p)$

poisson: $X = P(\lambda)$ if $P(X=n) = \frac{\lambda^n}{n!} e^{-\lambda}$, $n \geq 0$ $E(X) = \lambda$ $Var(X) = \lambda$

⑥ Multiple DRV:

$P_{i,j} = P(X = x_i, Y = y_j)$ $\forall (i,j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ $\quad P(X = x_i) = \sum_{j=1}^{n} P_{i,j}$

$E(h(X,Y)) = \sum_{i=1}^{m} \sum_{j=1}^{n} h(x_i, y_j) P_{i,j}$ and $E(h(X,Y)) = E(h_1(X,Y)) + E(h_2(X,Y))$

$cov(X,Y) = E(XY) - E(X)E(Y)$ (if $cov(X,Y)=0$, X and Y are uncorrelated)

Independent RVs are uncorrelated, the converse is not true.

Conditional Expectation $E[Y|X=x_i] = \sum_j y_j P[Y=y_j|X=x_i]$ while $E[Y|X]$ is a function $g(X)$ of X with $g(x_i) = E[Y|X=x_i]$; $E(E[Y|X]) = E(Y)$, $E[h(X)Y|X] = h(X)E[Y|X]$

$E[Y|X] = E[Y]$ if X and Y are independent.

⑦ General RVs

cdf: $F_X(x) = P(X \le x)$, pdf: $f_X(x) = \frac{d}{dx}F_X(x)$ $\quad P(a \le X \le b) = \int_a^b f_X(x)\,dx = F_X(b) - F_X(a)$

Uniformly distributed in $[a,b]$: $X = U[a,b] \Rightarrow f_X(x) = \frac{1}{b-a}$ $\quad F_X(x) = \max\{0, \min\{1, \frac{x-a}{b-a}\}\}$

Exponentially distributed with rate λ: $X = Exp(\lambda) \Rightarrow f_X(x) = \lambda e^{-\lambda x}$ $(x \ge 0)$ $\quad F_X(x) = 1 - e^{-\lambda x}$ $(x \ge 0)$

$E(h(X)) = \int_{-\infty}^{+\infty} h(x)\,dF_X(x) \simeq \int_{-\infty}^{\infty} h(x) f_X(x)\,dx$ $\quad E(Exp(\lambda)) = \frac{1}{\lambda}$ $\quad var(Exp(\lambda)) = \frac{1}{\lambda^2}$

while we have "Functions of Independent RVs are Independent"

⑧ X, Y are independent RVs. let $V = \min\{X, Y\}$, $W = \max\{X, Y\}$ then

$P(V > v) = P(X > v, Y > v) = P(X > v)P(Y > v)$ $\quad P(W \le w) = P(X \le w, Y \le w) = P(X \le w)P(Y \le w)$

let $Z = X + Y$. then we have $f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\,dx = f_X * f_Y(z)$

⑨ assume that X has a known pdf $f_X(x)$ let $Y = g(X)$.

1) $Y = aX + b \Rightarrow f_Y(y) = \frac{1}{|a|} f_X(x)$ / $\vec{Y} = \vec{A}\vec{X} + \vec{b} \Rightarrow f_Y(y) = \frac{1}{|A|} f_X(x)$

2) $Y = X^2 \Rightarrow f_Y(y) = \frac{1}{g'(x_i)} f_X(x_i)$ where $g(x_i) = y$