Chapter 6: Markov Processes and Discrete-Time Markov Chains

OUTLINE

6.1 Markov Processes

6.2 Chapman-Kolmogorov Equation

6.3 Basic Concepts of Markov Chains(MC)

6.4 Classification of States

6.5 Ergodic MC and Stationary Distribution

6.3 Markov Chain

- Definition
- One-step transition probabilities
- Homogeneity
- One-step transition probability matrix
- M-step transition probabilities
- M-step transition probability matrix
- Properties of transition probabilities
- Initial distribution
- Absolute distribution

6.3 Markov Chain

- State Space : Finite number of states, {S=(0, 1,….,i,….,j…)}
- Index set: Discrete Time, $\{\mathrm{T} = (0, 1, 2, \ldots)\}$
- Markovian property

$$
\begin{aligned}
&P\{x(t_n) = x_n \mid x(t_0) = t_0, x(t_1) = t_1, \ldots, x(t_{n-1}) = t_{n-1}\} \\
&= P\{x(t_n) = x_n \mid x(t_{n-1}) = t_{n-1}\} \\
&= P\{X_n = i_n \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\} \\
&= P\{X_n = i_n \mid X_{n-1} = i_{n-1}\}
\end{aligned}
$$

$$\text{for any } n = 1,2,\ldots \text{ and any } i_0, i_1, i_2, \ldots, i_n \in S$$

6.3 Markov Chain

- One-step transition probabilities:

$$p_{ij}(n) = P\{X_{n+1} = j \mid X_n = i\}; \;\; n = 0,1,\ldots$$

- <u>Homogeneity:</u>

$$p_{ij}(n) = P\{X_{n+1} = j \mid X_n = i\} = p_{ij} \quad \text{for all } \; n = 0,1,\ldots$$

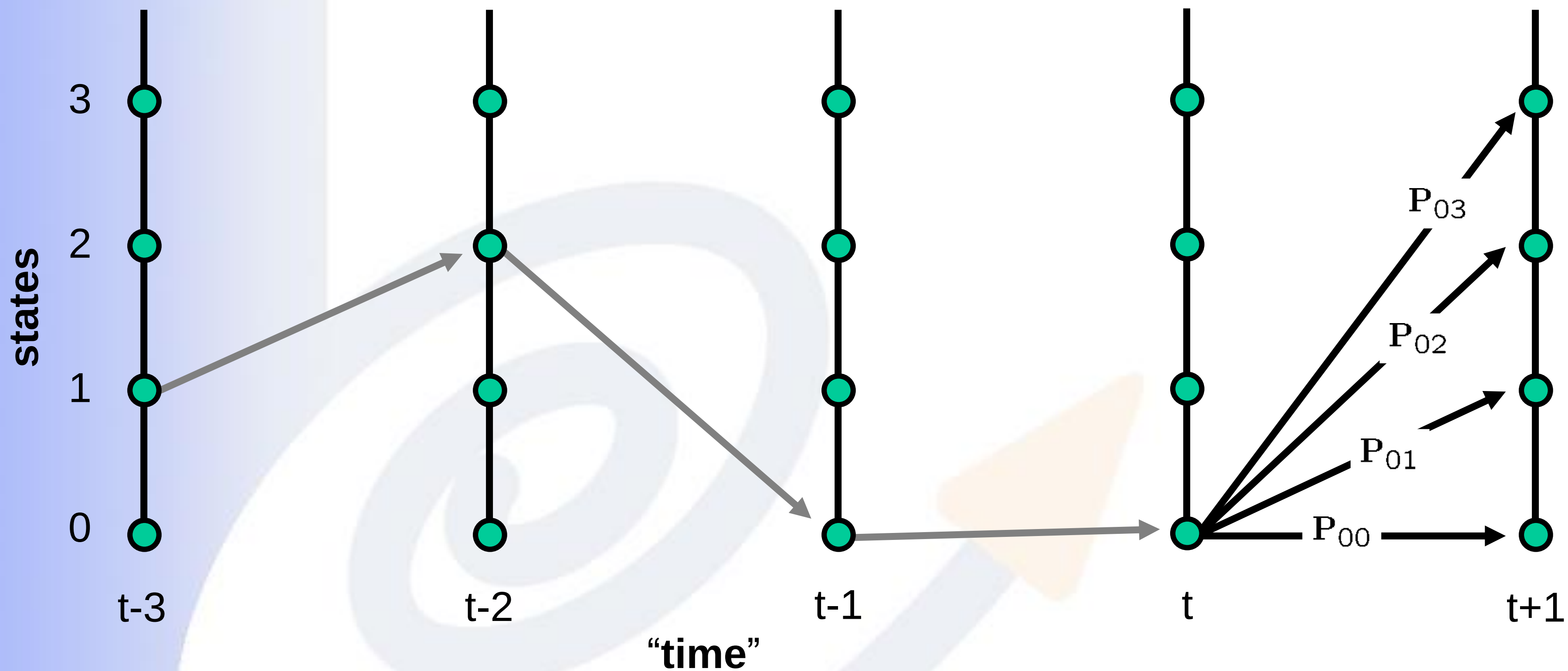
Transition probabilities is independent with time.

Markov chain only depends on going <u>ONE</u> step.

(one time unit, one jump)

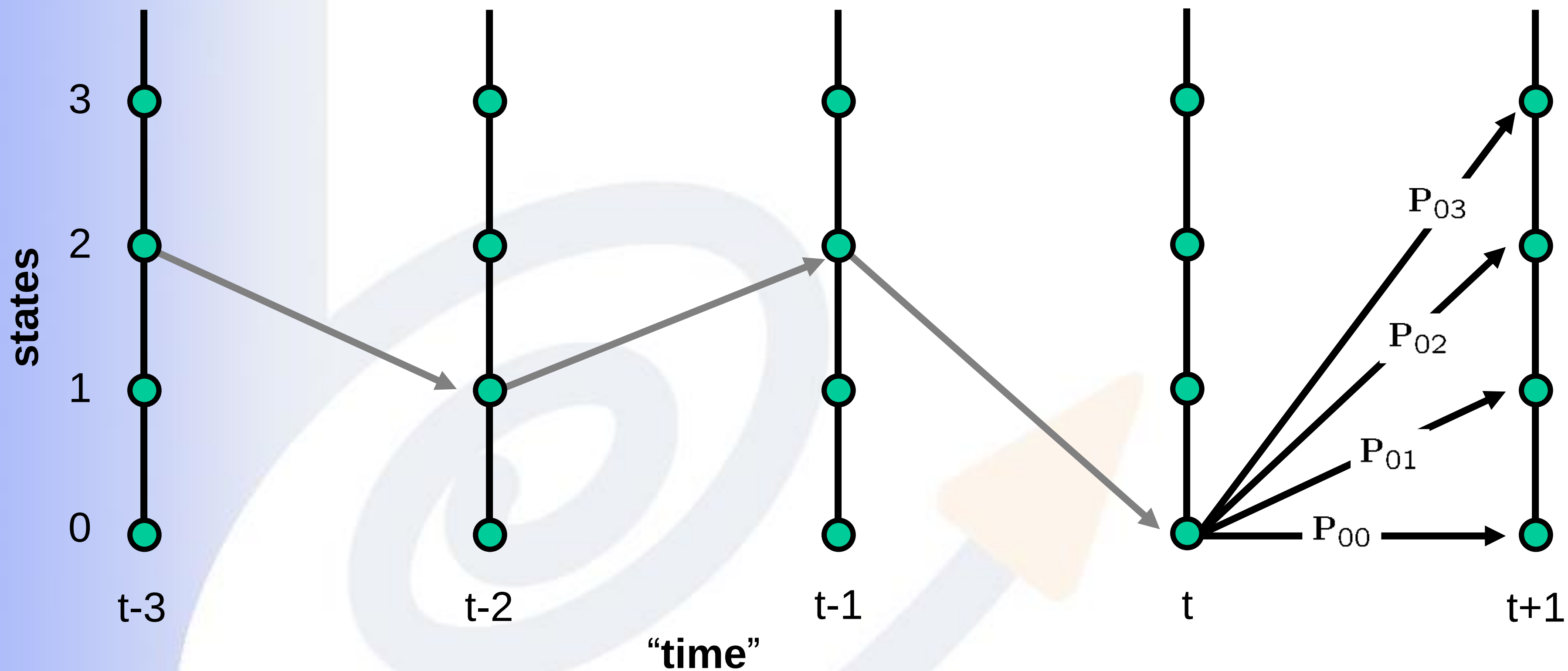
6.3 Markov Chain

- Sample path (1):



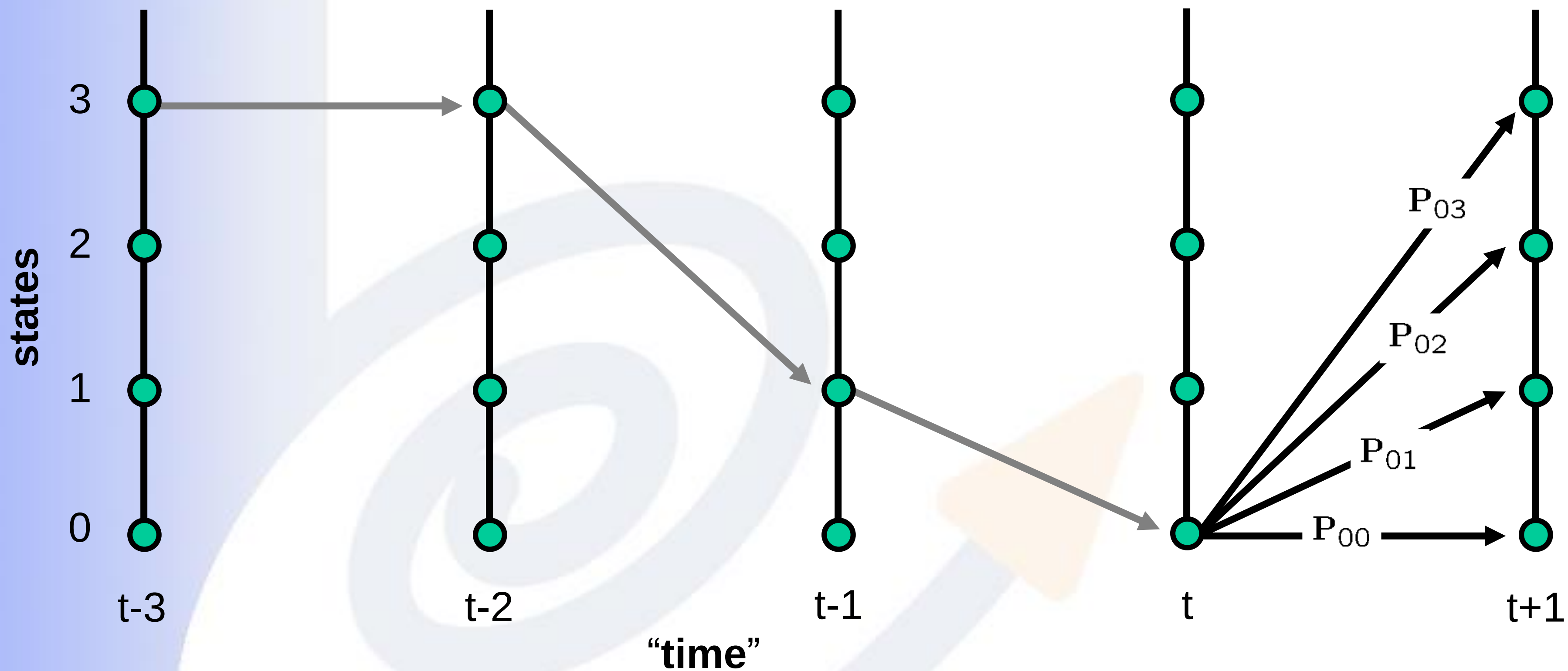
6.3 Markov Chain

- Sample path (2):



6.3 Markov Chain

- Sample path (3):



6.3 Markov Processes

<u>Markov Chains</u>:

State Space {0, 1, ...}

Discrete Time

$\{T = (0, 1, 2, \ldots)\}$

Continuous Time

$\{T = [0, \infty)\}$

Finite number of states

The Markovian property

Stationary transition probabilities (<u>Homogeneity</u>)

A set of initial probabilities $P\{X_0 = i\}$ for i

6.3 Markov Chains

Stage (t) $\qquad\qquad$ Stage (t + 1)

State i $\xrightarrow{\hspace{3cm}}$ State j

(with prob. p_{ij})

These are conditional probabilities!
Note that given $X_t = i$, must enter some state at stage t + 1

state space

$X_t = i \longrightarrow X_{t+1} = ?$ $\quad$ 0 $\quad$ 1 $\quad$ 2……j……m

with probabilities: $\quad p_{i0} \quad p_{i1} \quad p_{i2}\ldots\ldots p_{ij}\ldots\ldots p_{im}$

$$p_{ij} \geq 0, \quad \sum_{j=0}^{m} p_{ij} = 1, \quad i, j \in S$$

6.3 Markov Chains

One-step transition probability matrix:

Convenient to give transition probabilities in matrix form

$$\mathbf{P} = \mathbf{P}_{(m+1)(m+1)}$$

go to jth state next stage

$$\mathbf{P} = \mathbf{P}_{(m+1)(m+1)} = \begin{array}{cc} & \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & j & \cdots & m \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{array} & \begin{bmatrix} P_{00} & \cdots & & P_{0j} & & P_{0m} \\ P_{10} & \cdots & & P_{1j} & & P_{1m} \\ P_{20} & \cdots & & P_{2j} & & P_{2m} \\ \cdots & \cdots & \cdots & \cdots & & \\ P_{i0} & \cdots & & P_{ij} & & P_{1m} \\ \cdots & \cdots & \cdots & \cdots & & \\ P_{m0} & \cdots & & P_{mj} & \cdots & P_{mm} \end{bmatrix} \end{array}$$

Rows are given in this stage

Rows sum to 1

6.3 Markov Chains

Example 1:

t = day index 0, 1, 2, ...

$X_t = 0$ <u>rainy</u> on t^{th} day

$= 1$ <u>sunny</u> on t^{th} day

two states ===> S=(0, 1)

$0 \to 0 \quad p_{00} = P(X_{t+1} = 0 \mid X_t = 0) = 1/4$

$0 \to 1 \quad p_{01} = P(X_{t+1} = 1 \mid X_t = 0) = 3/4$

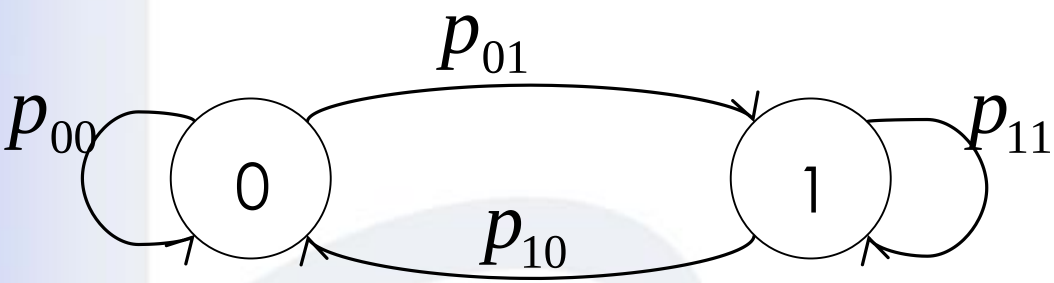
$1 \to 0 \quad p_{10} = P(X_{t+1} = 0 \mid X_t = 1) = 1/2$

$1 \to 1 \quad p_{11} = P(X_{t+1} = 1 \mid X_t = 1) = 1/2$

$$\therefore \ \mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$$

6.3 Markov Chains

- State transition graph



Note:

$$p_{00} = P(X_1 = 0 \mid X_0 = 0) = 1/4$$

$$= P(X_{36} = 0 \mid X_{35} = 0)$$

Also,

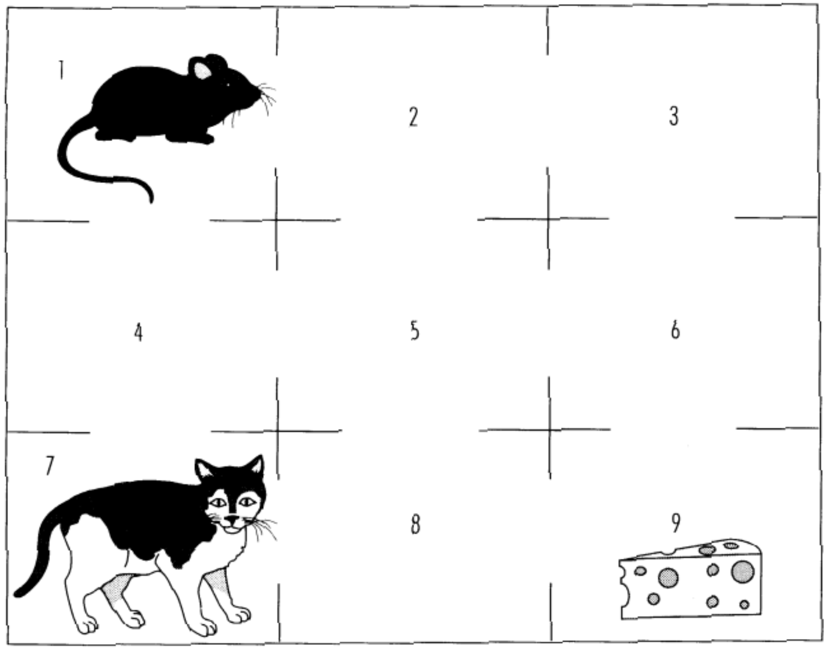
$$= P(X_2 = 0 \mid X_1 = 0, X_0 = 1)$$

$$= P(X_2 = 0 \mid X_1 = 0) = p_{00}$$

6.3 Markov Chains

Example 2: A mouse in a maze

When the mouse is in a given cell, it will choose the next cell to visit with probability 1/k, where k is the number of adjoining cells. Assume that once the mouse finds either the piece of cheese or the cat, it will stay there forever.



n = step of mouse, 0, 1, 2, ...

X_n = the position of the mouse after n changes of cells.

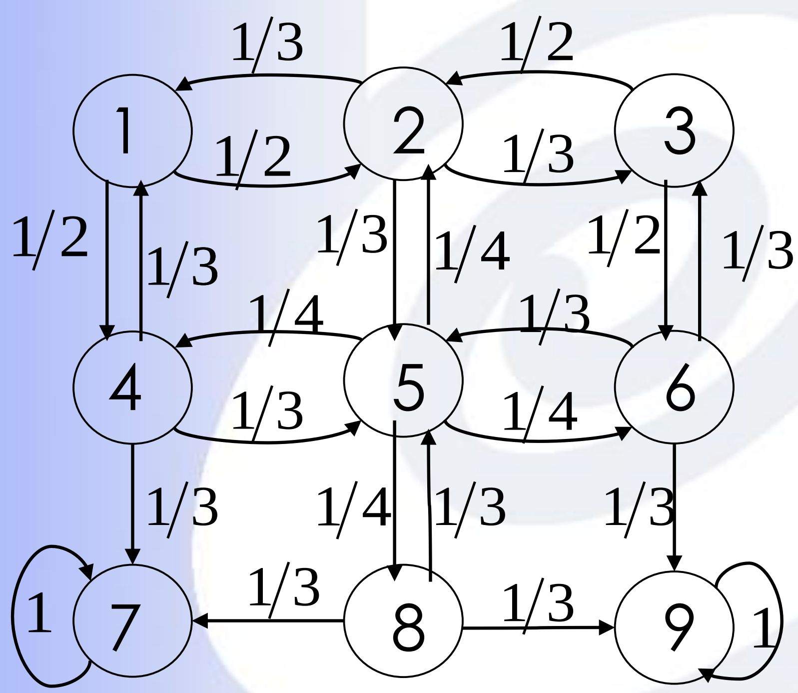
Nine states ===>S= (1,2,…9)

$\{X_n n=0,1,…\}$ is a Markov chain

6.3 Markov Chains

- Starting state probability vector: (1, 0, 0, 0, 0, 0, 0, 0, 0)

- State transition graph:



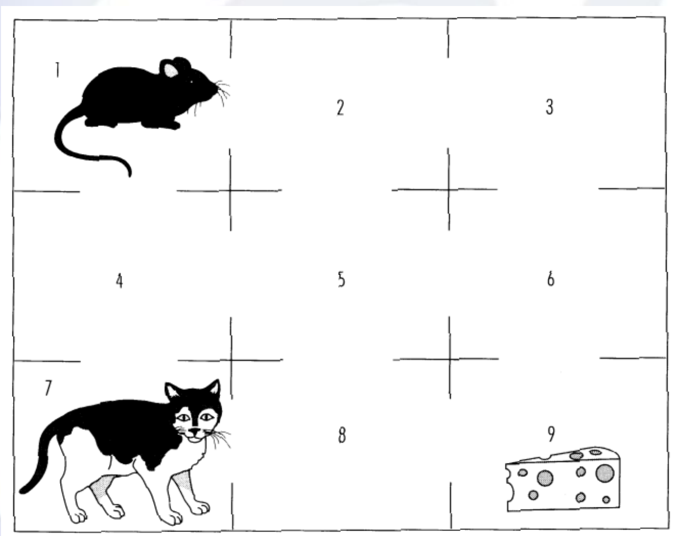
Transition probability matrix:

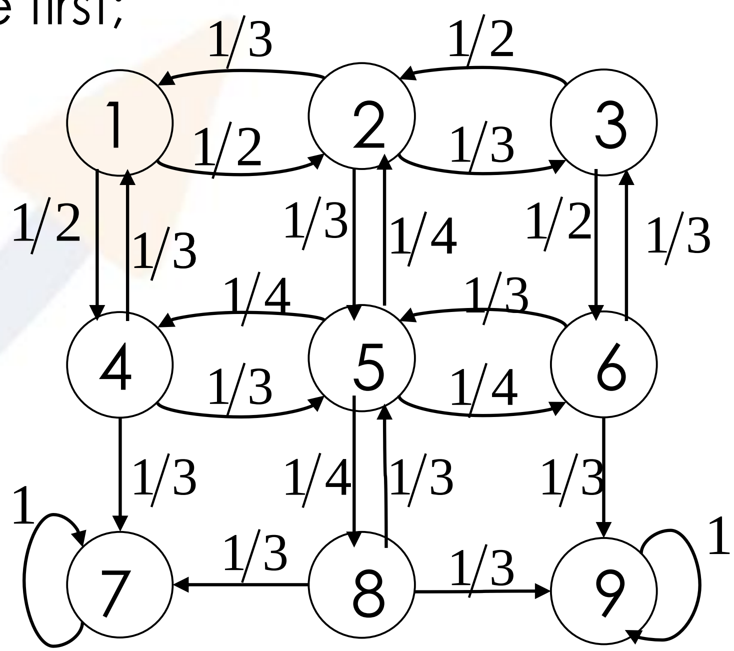
$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

6.3 Markov Chains

- Things of interest to the mouse:

1) The probability distribution for the time (the number of cell changes) in reaching the cheese before reaching the cat;
2) The probability distribution for the time in reaching the cat before reaching the cheese ;
3) The probability of reaching the cheese first;





6.3 Markov Chains

Example 3: A single-server queue with a finite number of waiting rooms and a constant service time

A shoeshine boy operates in a city. There are four chairs, three of which are for waiting customers. When all chairs are occupied, arriving customers will seek service elsewhere. Each shoeshine takes ten minutes.

X_n denote the number of waiting customers immediately after the completion of the *n*th shoeshine. (A departure and an arrival do not occur at the same time.)

A_n denote the number of arrivals during the *n*th shoeshine.

$P\{A_n=i\}=a_i$, for i=0,1,…, and n=1,2,…

At t

$$X_{n+1}=\begin{cases}\min(3,\ A_{n+1}) & \text{if } X_n=0\\ \min(3,\ X_n-1+A_{n+1}) & \text{if } X_n=1,\ 2,\ 3.\end{cases}$$

6.3 Markov Chains

- X_{n+1} is determined probabilistically if X_n is known.
- The process $\{X_n \mid n=1,2,\ldots\}$ is a Markov chain with state space $S=\{0,1,2,3\}$.
- The transition probability matrix is:

$$P = \begin{bmatrix} a_0 & a_1 & a_2 & {}^{\geq}a_3 \\ a_0 & a_1 & a_2 & {}^{\geq}a_3 \\ 0 & a_0 & a_1 & {}^{\geq}a_2 \\ 0 & 0 & a_0 & {}^{\geq}a_1 \end{bmatrix}.$$

- Problems:

  The long-run expected number of lost customers per hour;

  The average occupancy of the queue;

  The average length of an idle period.

6.3 Markov Chains

Example 4: a periodic-review (s,S) inventory system

Weekly demands for a given aircraft spare part at a maintenance depot are i.i.d. random variables with

$$P\{D_n = i\} = a_i, \qquad i,\ n = 0,\ 1,\ 2,\ \ldots, \qquad \text{and} \qquad \sum_{i=0}^{\infty} a_i = 1.$$

where *Dn* denotes the demand in week *n*.

Xn denote the inventory position at the start of week *n* before the receipt of replenishment, if any.

Assume that inventory replenishment is instantaneous and unfilled demands are lost.

The inventory policy used is of (*s*,*S*) type:

- If at the beginning of a week n, the store level *Xn* is *s* or higher, we do not order;
- If it is lower than *s*, an order is made to bring the inventory level to S.

6.3 Markov Chains

- The stochastic process {Xn,n=0,1,…} forms a Markov chain with state space S={0,1,…,S}

$$X_{n+1} = \begin{cases} \max\{X_n - D_n,\ 0\} & \text{if } X_n \geq s \\ \max\{S - D_n,\ 0\} & \text{if } X_n < s \end{cases}$$

- Transition probabilities: (s,S) = (2,5)

$$p_{ij} = \begin{cases} r_S & \text{if } 0 \leq i < s \text{ and } j = 0 \\ a_{S-j} & \text{if } 0 \leq i < s \text{ and } 1 \leq j \leq S \\ r_i & \text{if } s \leq i \leq S \text{ and } j = 0 \\ a_{i-j} & \text{if } s \leq i \leq S \text{ and } 1 \leq j \leq i \\ 0 & \text{otherwise,} \end{cases}$$

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & r_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 1 & r_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 2 & r_2 & a_1 & a_0 & 0 & 0 & 0 \\ 3 & r_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 4 & r_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 5 & r_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{array}$$

6.3 Markov Chains

- Problems:

  The expected amount of lost sales;

  The average level of inventory per week;

  The expected number of weeks between successive replenishments.

6.3 Markov Chains

2-step transition probabilities

In example 1: What is $p_{00}^{(2)}$ = P(X_2 = 0 | X_0 = 0)?

This is a two-step transition probability:

$$\begin{array}{ccc} \text{stage} & & \text{stage} \\ 0 & \longrightarrow & 2 \\ \text{or } t & \longrightarrow & t+2 \end{array}$$

What is $p_{01}^{(2)}$ $p_{10}^{(2)}$ $p_{11}^{(2)}$?

What is P(X_m = 0 | X_0 = 0)?

This is a m-step transition probability: $p_{00}^{(m)}$

6.3 Markov Chains

- m-step transition probabilities

$$p_{ij}^{(m)} = P\{X_{n+m} = j \mid X_n = i\}, \quad m = 0,1,\ldots$$

- m-step transition probability matrix

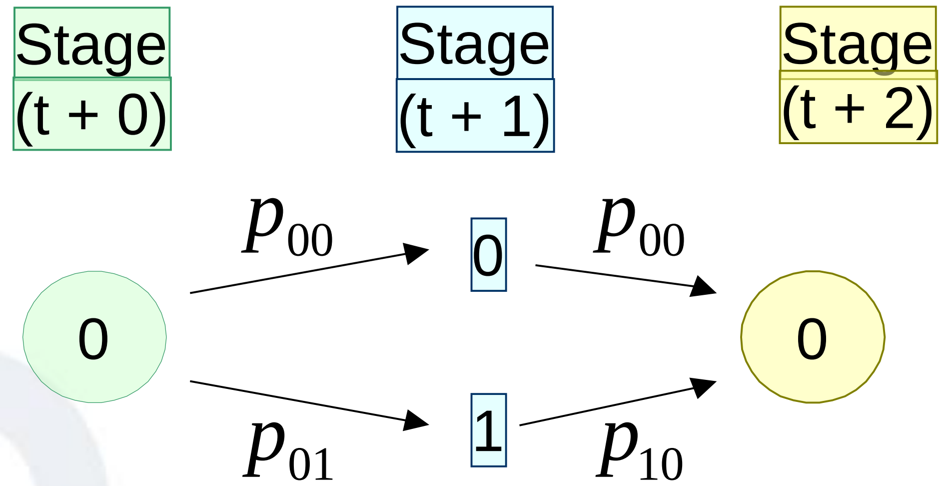
$$\mathbf{P}^{(m)} = \left[p_{ij}^{(m)} \right], \quad m = 0,1,\ldots$$

- Convenient to define

$$p_{ij}^{(0)} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

6.3 Markov Chains

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$$



$$P\{X_2 = 0, X_1 = 0 \mid X_0 = 0\}$$
$$= P\{X_1 = 0 \mid X_0 = 0\} P\{X_2 = 0 \mid X_1 = 0\} = p_{00} p_{00}$$

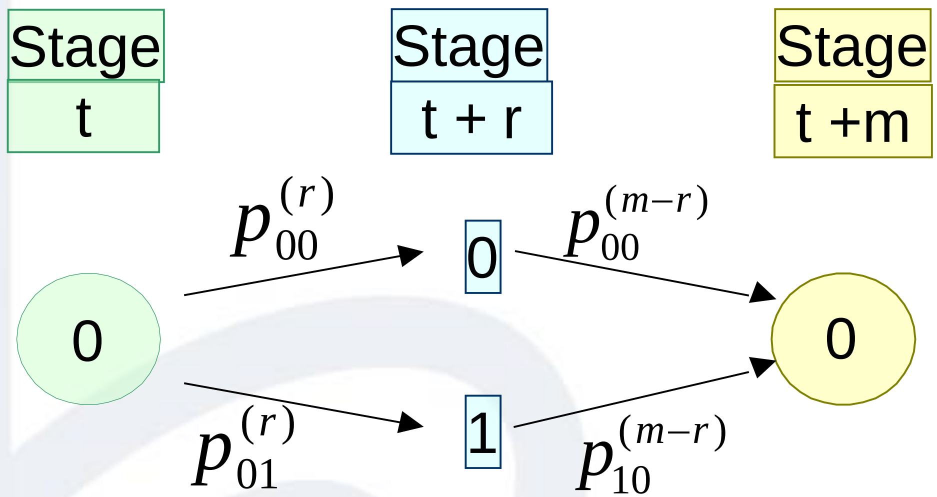
$$P\{X_2 = 0, X_1 = 1 \mid X_0 = 0\}$$
$$= P\{X_1 = 1 \mid X_0 = 0\} P\{X_2 = 0 \mid X_1 = 1\} = p_{01} p_{10}$$

$$P\{X_2 = 0 \mid X_0 = 0\} = p_{00}^{(2)} = p_{00} p_{00} + p_{01} p_{10}$$
$$= \frac{1}{4}\frac{1}{4} + \frac{3}{4}\frac{1}{2} = \frac{7}{16}$$

$$p_{00}^{(2)} = \sum_{k \in S} p_{0k} p_{k0}$$

$$p_{ij}^{(2)} = \sum_{k \in S} p_{ik} p_{kj}$$

6.3 Markov Chains



$$p_{00}^{(2)} = \sum_{k \in S} p_{0k}\, p_{k0}$$

$$p_{00}^{(m)} = \sum_{k \in S} p_{0k}^{(r)}\, p_{k0}^{(m-r)}$$

$$r = 0,1,\ldots m$$

$$p_{ij}^{(2)} = \sum_{k \in S} p_{ik}\, p_{kj}$$

$$p_{ij}^{(m)} = \sum_{k \in S} p_{ik}^{(r)}\, p_{kj}^{(m-r)}$$

- Properties of transition probabilities:

1) Chapman-Kolmogorov equations

$$p_{ij}^{(m)} = \sum_{k \in S} p_{ik}^{(r)} p_{kj}^{(m-r)}, \quad r = 0,1,\ldots,m; \quad i, j \in S$$

or

$$\mathbf{P}^{(m)} = \mathbf{P}^{(r)}\mathbf{P}^{(m-r)}, \; r = 0,1,\ldots,m$$

2) $\mathbf{P}^{(m)} = \mathbf{P}\mathbf{P}^{(m-1)}$

3) $\mathbf{P}^{(m-1)} = \mathbf{P}\mathbf{P}^{(m-2)}, \cdots, \quad \mathbf{P}^{(m)} = \mathbf{P}^{m}$

4) $p_{ij}^{(m)} = \sum_{k_1 \in S} \cdots \sum_{k_{m-1} \in S} p_{ik_1} p_{k_1 k_2} \cdots p_{k_{m-1} j}$

6.3 Markov Chains

Example 1:

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\mathbf{P}^{(2)} = \mathbf{P}^2 = \begin{bmatrix} p_{00}^{(2)} & p_{01}^{(2)} \\ p_{10}^{(2)} & p_{11}^{(2)} \end{bmatrix} = \begin{bmatrix} 7/16 & 9/16 \\ 3/8 & 5/8 \end{bmatrix}$$

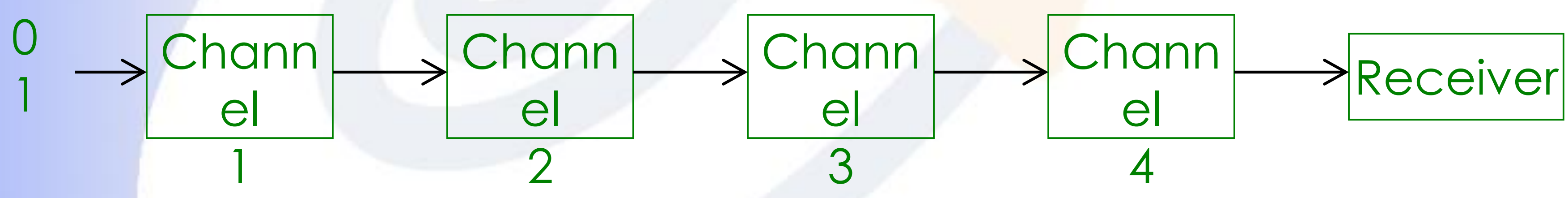
6.3 Markov Chains

Example 2:

Consider a communication system which transmits the digits 0 and 1 through several stages.

At each stage the probability that the same digit will be received by the next stage, as transmitted, is 0.75.

What is the probability that a 0 that is entered at the first stage is received as a 0 by the 4th stage?



Solution:

We want to find $p_{00}^{(4)}$. The state transition matrix P is given by

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

Hence

$$P^2 = \begin{bmatrix} 0.625 & 0.375 \\ 0.375 & 0.625 \end{bmatrix} \quad \text{and} \quad P^4 = P^2P^2 = \begin{bmatrix} 0.53125 & 0.46875 \\ 0.46875 & 0.53125 \end{bmatrix}$$

Therefore the probability that a zero will be transmitted through four stages as a zero is $p_{00}^{(4)} = 0.53125$

6.3 Markov Chains

- Initial distribution $\mathbf{s}(0)$
- (Starting state probability vector):

A probability distribution of $X(t_0)$

$$s(0) = \{s_i(0) = P(X_0 = i),\ i \in S,\ \sum_{i \in S} p_i = 1\}$$

$$P\{x(t_0), x(t_1), \cdots, x(t_n)\} = P\{x(t_0)\} \prod_{r=1}^{n} P\{x(t_r) \mid x(t_{r-1})\}$$

$$\begin{aligned}
&P\{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\} \\
&= P\{X_n = i_n \mid X_{n-1} = i_{n-1}\} \cdot P\{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\} \\
&= p_{i_{n-1} i_n} \cdot P\{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\} \\
&= s_{i_0}(0) p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}
\end{aligned}$$

6.3 Markov Chains

- Absolute distribution $\mathbf{s}(m)$:

One-dimensional state probabilities vector of the Markov chain after m steps.

$$\boxed{\mathbf{s}(m) = \{s_j(m) = P(X_m = j), j \in S, \ \sum_{j \in S} s_j(m) = 1\}}$$

- Properties of absolute distribution:

1) $$s_j(m) = \sum_{i \in S} s_i(0) p_{ij}^{(m)}, \quad m = 1,2,\ldots; j \in S$$

$$\boxed{\mathbf{s}(m) = \mathbf{s}(0)\mathbf{P}^{(m)} = \mathbf{s}(0)\mathbf{P}^{m}}$$

2) $$s_j(m) = \sum_{i \in S} s_i(m-1) p_{ij}, \quad m = 1,2,\ldots; j \in S$$

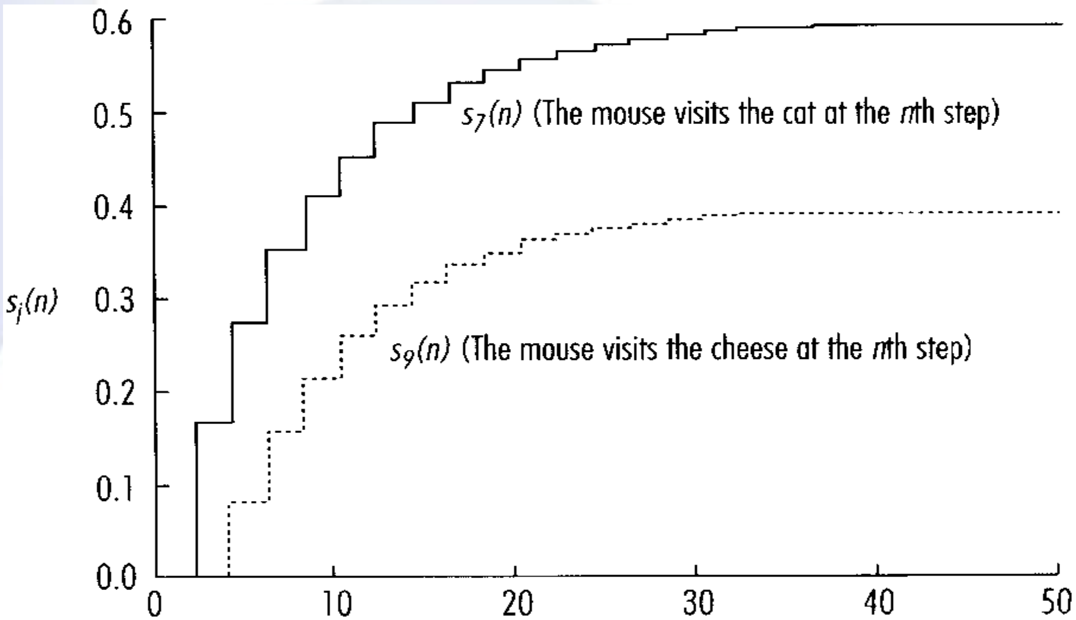
$$\boxed{\mathbf{s}(m) = \mathbf{s}(m-1)\mathbf{P}}$$

6.3 Markov Chains

- Absolute distributions $\mathbf{s}(m)$:

Example: A mouse in a maze

$s_7(n), s_9(n)$ the probabilities that the mouse will be visiting either the cat or the cheese at the *n*th cell change.



Stationary of absolute distribution

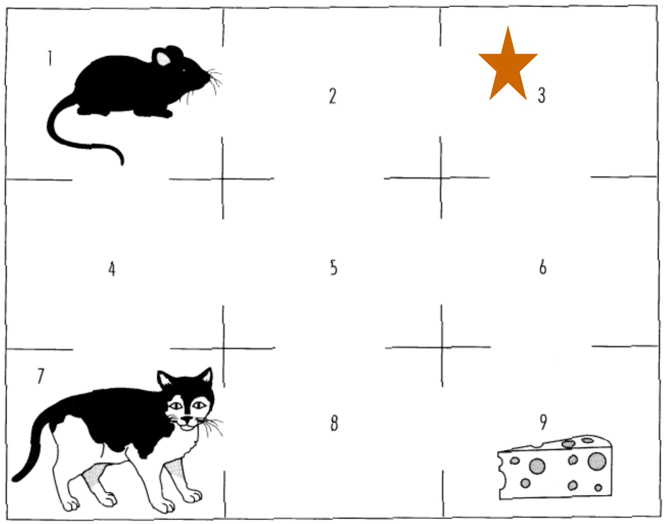
After a large number of steps, the mouse will either be with the cat or the cheese, with probabilities 0.6 and 0.4.

6.3 Markov Chains

- If the mouse starts from cell 3, s(0)=(0,0,1,0,…,0). By symmetry,

$$s_7(n) \qquad s_9(n)$$

After a large number of steps, the mouse will either be with the cat or the cheese, with probabilities 0.4 and 0.6.



6.3 Markov Chains

Example 1 revisited:

t = day index 0, 1, 2, ...

$X_t = 0$ rainy on t^{th} day

$\quad = 1$ sunny on t^{th} day

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\mathbf{P}^{(2)} = \begin{bmatrix} p_{00}^{(2)} & p_{01}^{(2)} \\ p_{10}^{(2)} & p_{11}^{(2)} \end{bmatrix} = \begin{bmatrix} 7/16 & 9/16 \\ 3/8 & 5/8 \end{bmatrix} = \begin{bmatrix} 0.4375 & 0.5625 \\ 0.375 & 0.625 \end{bmatrix}$$

$$\mathbf{P}^{(4)} = \begin{bmatrix} 0.40234375 & 0.59765625 \\ 0.3984375 & 0.6015625 \end{bmatrix}$$

$$\mathbf{P}^{(8)} = \begin{bmatrix} p_{00}^{(8)} & p_{01}^{(8)} \\ p_{10}^{(8)} & p_{11}^{(8)} \end{bmatrix} \approx \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix}$$

6.3 Markov Chains

- Homeworks
- 1, 2, 3, 5,
- 13(a), 23

- Experiments:
- 4, 8, 11, 16