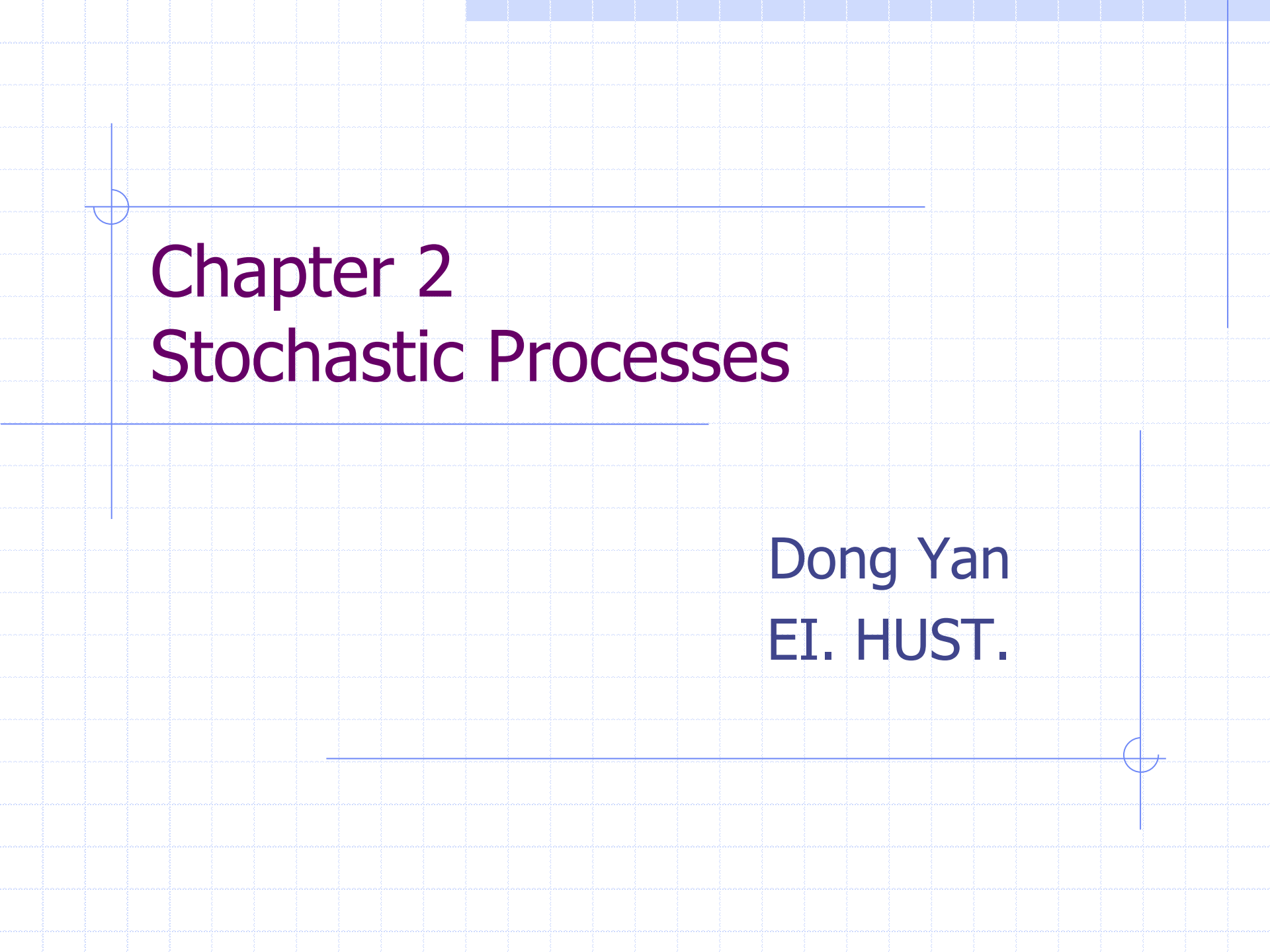




Chapter 2

Stochastic Processes

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Chapter 2: Stochastic Processes

2.1 Basic Concepts

2.2 Stationary Stochastic Processes

2.3 Properties of Correlation Functions

2.4 Some Important Stochastic Processes

2.3 Properties of Correlation Functions

OUTLINE:

2.3.1 Properties of $R_{XX}(t_1, t_2)$

2.3.2 Properties of $R_{XY}(t_1, t_2)$

2.3.1 Properties of $R_{XX}(t_1, t_2)$

1. Definition of correlation function for a stochastic process $\{X(t), t \in T\}$

recall

2.1.4 Moments

Review

4. Correlation(autocorrelation) functions

- ◆ For a second-order moment process, the correlation function is $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$ or

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$$

- ◆ Calculate correlation function for discrete stochastic processes and continuous stochastic processes respectively:

$$\begin{aligned} R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \sum_{k=1}^n \sum_{j=1}^n kjP(x(t_1) = k, x(t_2) = j) \end{aligned}$$

$$R_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1)x(t_2)f_X(x_1, x_2; t_1, t_2)dx_1dx_2$$

2.3.1 Properties of $R_{XX}(t_1, t_2)$

Relationship with other moments

$$R_{XX}(t, t) = E[X^2(t)]$$

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \bar{x}(t_1)\bar{x}(t_2)$$

$$|R_X(t_1, t_2)| \leq \frac{E[X(t_1)]^2 + E[X(t_2)]^2}{2}$$

$$|R_{XX}(t_1, t_2)| \leq \frac{R_{XX}(t_1, t_1) + R_{XX}(t_2, t_2)}{2}$$

2.3.1 Properties of $R_{XX}(t_1, t_2)$

2. Properties of $R_{XX}(t_1, t_2)$

i) $R_{XX}(t, t) \geq 0$

ii) $R_{XX}(t_1, t_2) = R_{XX}(t_2, t_1)$

iii) $|R_X(t_1, t_2)| \leq \sqrt{R_X(t_1, t_1)R_X(t_2, t_2)}$

iv) Nonnegative definite (positive semi-definite).

$$\sum_{i=1}^n \sum_{k=1}^n \alpha_i \alpha_k^* R_X(t_i, t_k) \geq 0$$

Nonnegative definiteness implies i), ii), iii).

If a function $R(*, *)$ is nonnegative definite, it can be a correlation function for some second-order random processes.

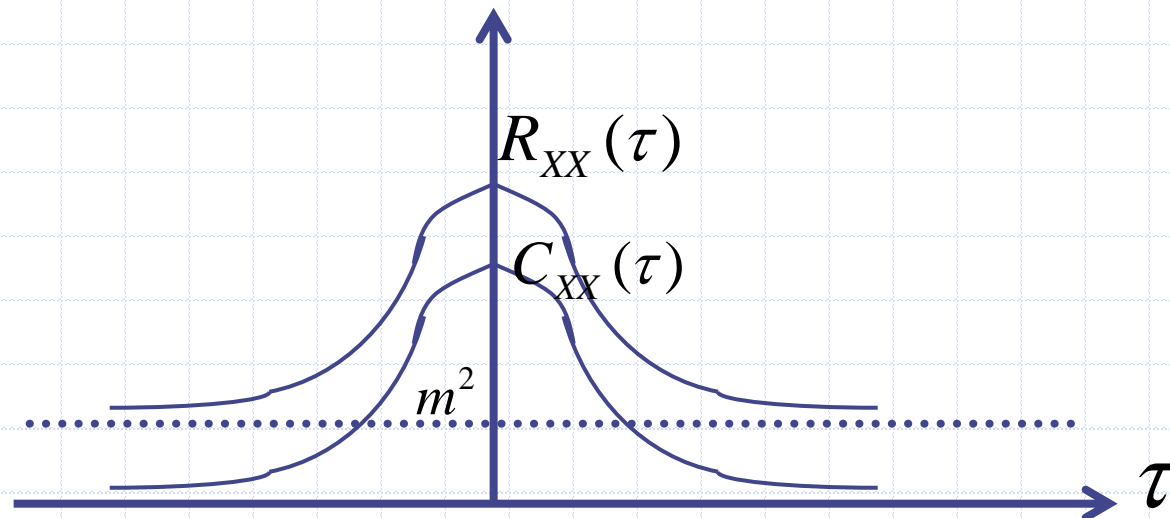
2.3.1 Properties of $R_{XX}(t_1, t_2)$

3. Correlation function of weakly stationary S.P.

$$R_{XX}(t_1, t_2) = R_{XX}(\tau)$$

$$R_{XX}(0) = E[X^2(t)]$$

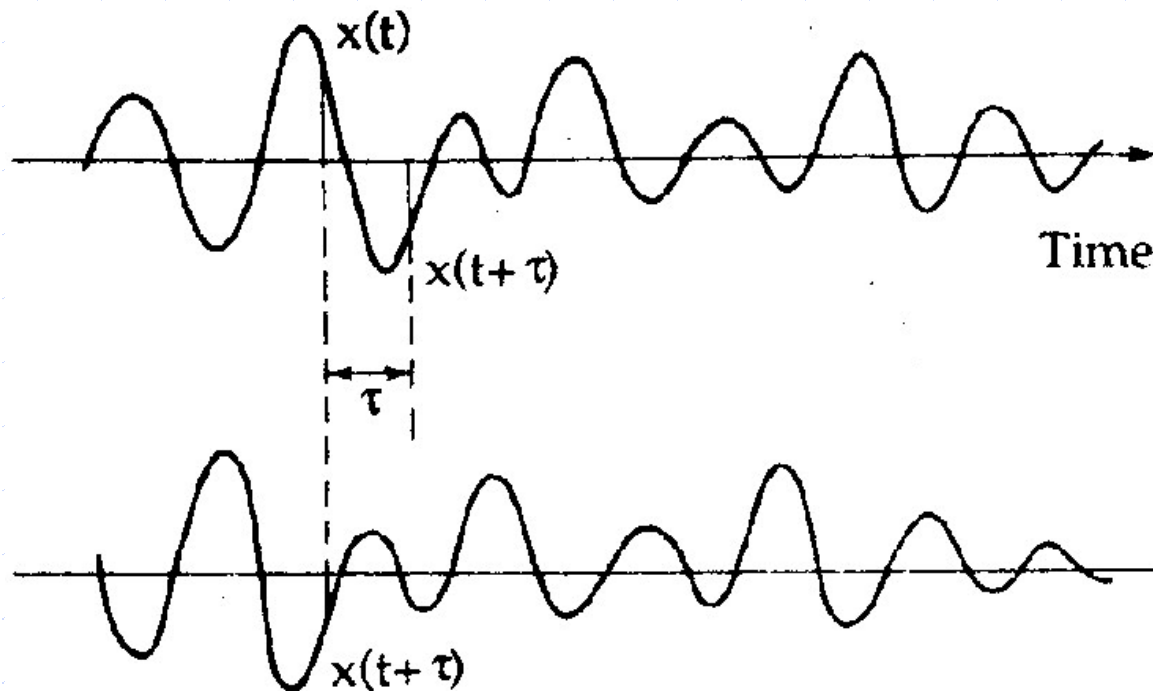
$$C_{XX}(\tau) = R_{XX}(\tau) - m^2$$



2.3.1 Properties of $R_{XX}(t_1, t_2)$

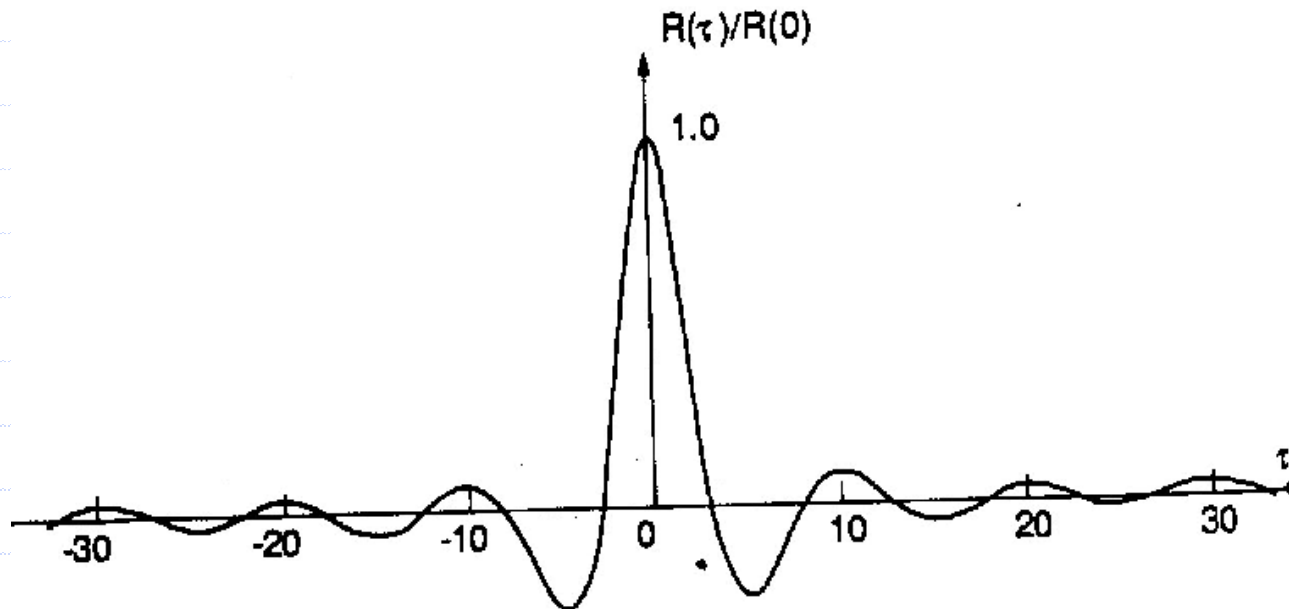
If $X(t)$ is ergodic, then

$$R_{XX}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau)dt$$



4. Properties of $R_{XX}(\tau)$

- i) An even function: $R_{XX}(\tau) = R_{XX}(-\tau)$
- ii) $R_{XX}(0) \geq |R_{XX}(\tau)|$
- iii) $R_{XX}(0) = E[X^2(t)] \geq 0$
- iv) If $X(t)$ is ergodic, then $R_{XX}(0) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x^2(t) dt$



4. Properties of $R_{XX}(\tau)$

V) If $X(t)=X(t+T)$, then $R_{XX}(\tau) = R_{XX}(\tau + T)$

Example 1 Random phase processes

$X(t)=A\cos(w_0t+\varepsilon)$, $t>0$, whereas A and w_0 are constants and ε is random variable uniformly distributed between $-\pi$ and π .

$$E[x(t)] = 0$$

$$C_{XX}(\tau) = R_{XX}(\tau) = \frac{A^2}{2} \cos w_0\tau$$

4. Properties of $R_{XX}(\tau)$

Example 2. Random amplitude processes
 $X(t) = Y\cos(\omega t) + Z\sin(\omega t)$, $t > 0$, whereas Y and Z are independent random variables, and $EY = EZ = 0$, $\text{Var}Y = \text{Var}Z = \sigma^2$.

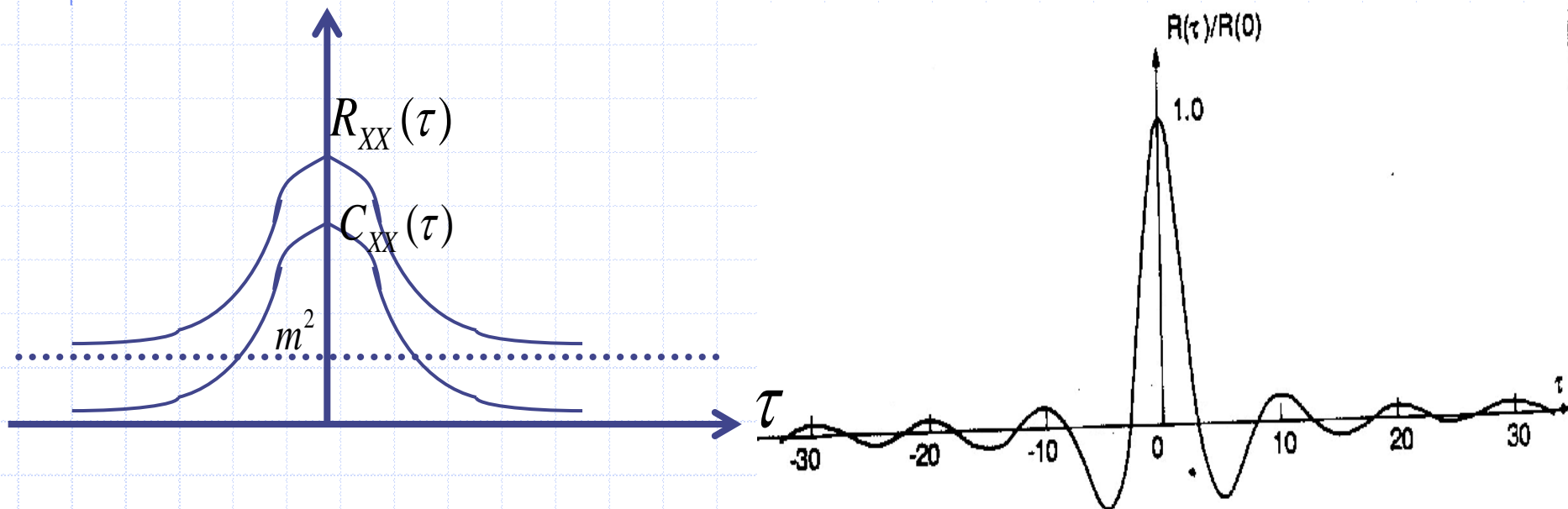
$$E[x(t)] = 0$$

$$C_{XX}(\tau) = R_{XX}(\tau) = \sigma^2 \cos \omega \tau$$

4. Properties of $R_{XX}(\tau)$

vi) If $X(t)$ is a non-period process, as $|\tau| \rightarrow \infty$
 $X(t)$ and $X(t + \tau)$ are independent, that is

$$\lim_{|\tau| \rightarrow \infty} R_{XX}(\tau) = m^2$$



2.3.1 Properties of $R_{XX}(t_1, t_2)$

Example 3. Pulse Code Modulation (binary case)

- ◆ A source generates the symbols **1** or **0** independently with probabilities **$1-p$** and **p** , respectively.
- ◆ The symbol **1** is transmitted by sending a pulse with constant **amplitude A** and **duration T** .
- ◆ The symbol **0** is transmitted by sending nothing during an interval of **length T** .

Example 3.

A signal modulated in this way is represented by the stochastic process

$$X(t) = \sum_{n=-\infty}^{\infty} A_n h(t - nT), \quad nT \leq t < (n+1)T$$

Where the $A_n; n = 0, \pm 1, \pm 2, \dots$ are independent binary random variables defined by

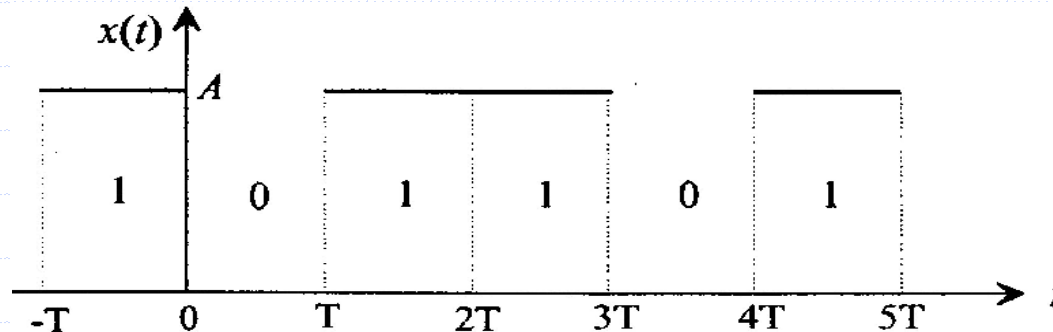
$$A_n = \begin{cases} 0 & \text{with probability } p \\ A & \text{with probability } 1 - p \end{cases}$$

And $h(t)$ is given by

$$h(t) = \begin{cases} 1 & \text{for } 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

Example 3.

In particular, for any t , $X(t) = \begin{cases} 0 & \text{with probability } p \\ A & \text{with probability } 1 - p \end{cases}$



A sample wave $x(t)$: ...1 0 1 1 0 1...

Commonly, the time point $t=0$ has been chosen in such a way that it coincides with the beginning of a new transmission period.

Obtain: $E[X(t)]$, $R_{xx}(t_1, t_2)$

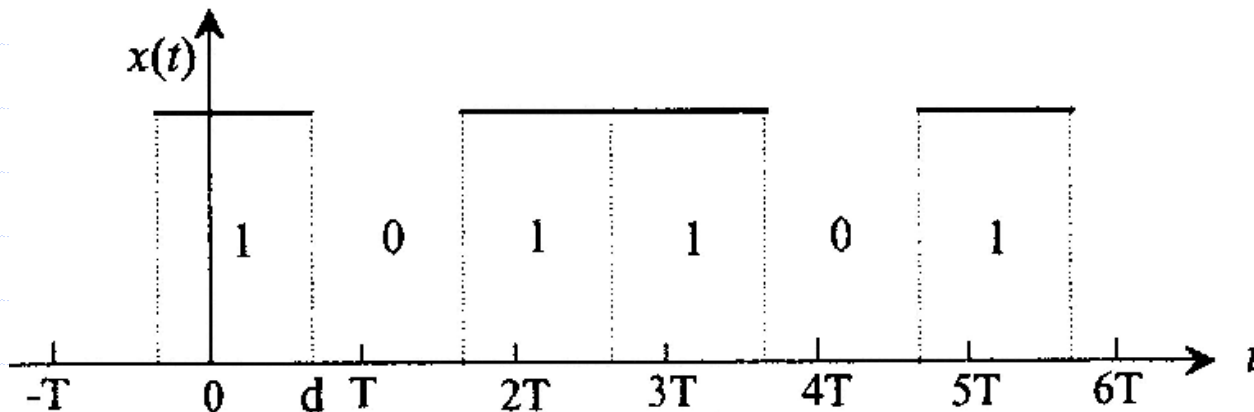
Discuss: Is $X(t)$ a stationary process?

Example 4. Random Delayed Pulse Code

Using the stochastic process $X(t)$ defined in example 3, define the process:

$$Y(t) = X(t - D)$$

where D is uniformly distributed over $[0, T]$.



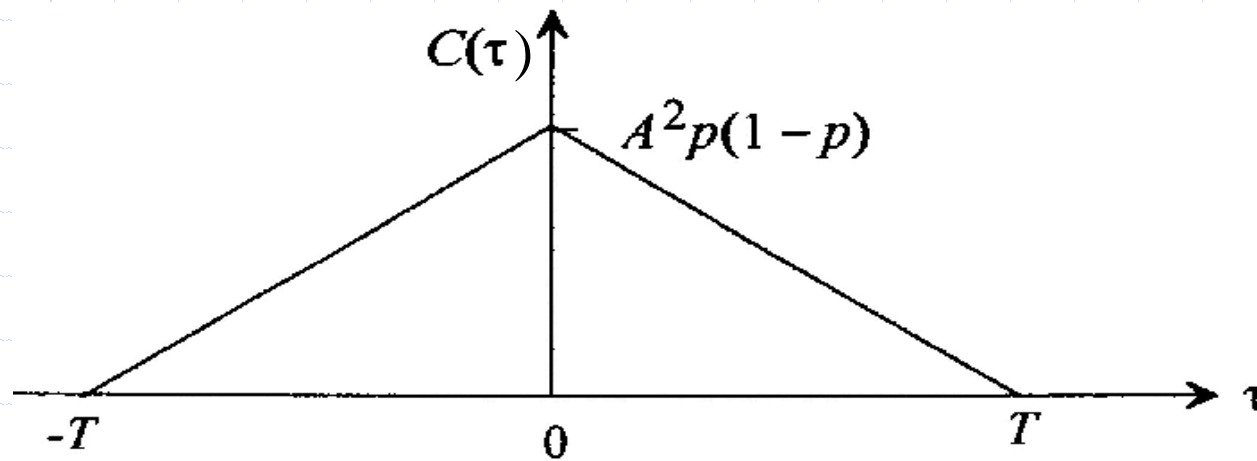
Obtain: $E[Y(t)], R_{YY}(t_1, t_2)$

Discuss: Is $Y(t)$ a stationary process?

Example 4. Random Delayed Pulse Code

Discuss: Is $Y(t)$ a stationary process?

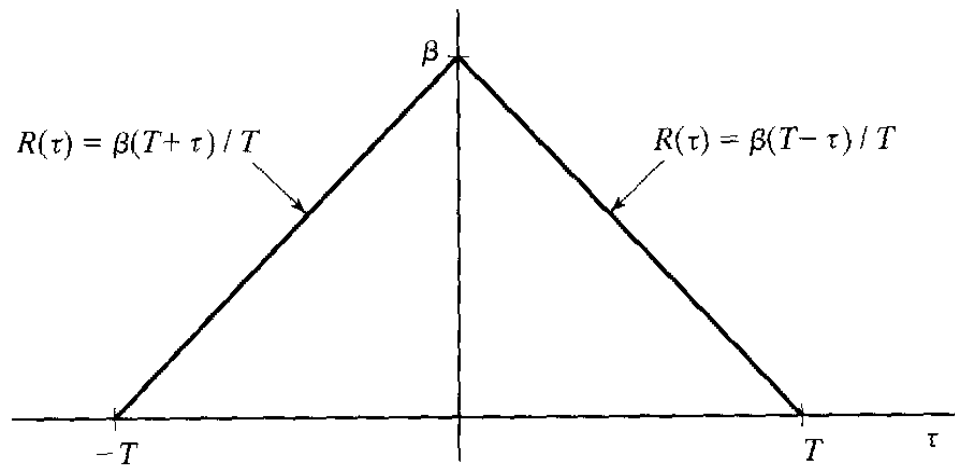
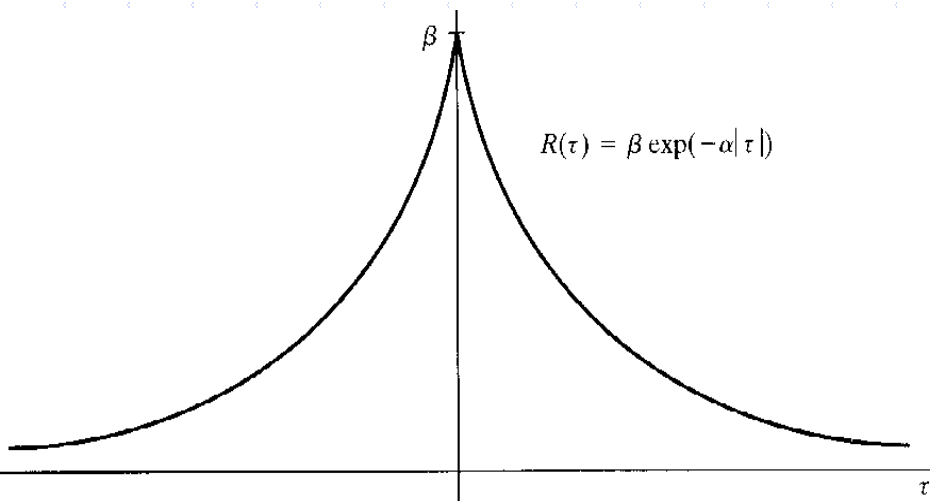
$$C_{YY}(\tau) = \begin{cases} A^2 p(1-p)(1 - \frac{|\tau|}{T}) & \text{for } |\tau| \leq T \\ 0 & \text{otherwise} \end{cases}$$



2.3.1 Properties of $R_{XX}(t_1, t_2)$

Examples of correlation functions for weakly stationary processes

1. $R(\tau) = \beta \exp(-\alpha|\tau|)$
2. $R(\tau) = \begin{cases} \beta(T - |\tau|)/T, & -T \leq \tau \leq T \\ 0, & \text{otherwise} \end{cases}$
3. $R(\tau) = \beta \exp(-\alpha|\tau|) \cos(\omega_0 \tau)$
4. $R(\tau) = 2W \{ \sin(2\pi W \tau) / 2\pi W \tau \}$



2.3 Properties of Correlation Function

OUTLINE:

2.3.1 Properties of $R_{XX}(t_1, t_2)$

2.3.2 Properties of $R_{XY}(t_1, t_2)$

2.3.2 Cross-Correlation Functions

1. Definition of cross-correlation function for two stochastic processes $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$

recall

2.1.4 Moments

Review

6. Cross-covariance functions

- ◆ For two second-order moment processes $X(t)$ and $Y(t)$, their cross-covariance function is

$$\begin{aligned}C_{XY}(t_1, t_2) &= \text{Cov}[X(t_1), Y(t_2)] \\&= E[(X(t_1) - \bar{x}(t_1))[Y(t_2) - \bar{y}(t_2)]] \\&= E[X(t_1)Y(t_2)] - \bar{x}(t_1)\bar{y}(t_2)\end{aligned}$$

2.1.4 Moments

Review

7. Cross-correlation functions

- ◆ For two second-order moment processes $X(t)$ and $Y(t)$, their cross-correlation function is

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \bar{x}(t_1)\bar{y}(t_2)$$

$$R_{XY}(t_1, t_2) = E[Y(t_2)X(t_1)] = R_{YX}(t_2, t_1)$$

$$|R_{X,Y}(t, s)| \leq \sqrt{R_X(t, t)R_Y(s, s)}$$

2.1.4 Moments

Review

Mutually uncorrelated

◆ For two second-order moment processes $X(t)$ and $Y(t)$, if

$$C_{XY}[t_1, t_2] = 0 \quad t_1, t_2 \in T$$

then $X(t)$ and $Y(t)$ are mutually uncorrelated

$$R_{XY}(t_1, t_2) = \bar{x}(t_1)\bar{y}(t_2)$$

2.3.2 Cross-Correlation Functions

Independent random processes:

The processes $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$ are said to be **independent random processes** if, for each positive integer n and each choice of (t_1, t_2, \dots, t_n) , $t_i \in T$, (s_1, s_2, \dots, s_n) , $s_i \in T$, the two random vectors $\mathbf{X} = (X(t_1), X(t_2), \dots, X(t_n))$ and $\mathbf{Y} = (Y(s_1), Y(s_2), \dots, Y(s_n))$ are independent.

Two random vectors \mathbf{X} and \mathbf{Y} are independent if and only if $F_{\mathbf{X}, \mathbf{Y}}(\mathbf{x}, \mathbf{y}) = F_{\mathbf{X}}(\mathbf{x})F_{\mathbf{Y}}(\mathbf{y})$.

*The independent random processes **are also** uncorrelated.
Uncorrelated random processes **need not be** independent.*

2.3.2 Cross-Correlation Function

jointly Gaussian random processes

The processes $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$ are said to be **jointly Gaussian random processes** if, for each positive integer n and each choice of (t_1, t_2, \dots, t_n) , $t_i \in T$, (s_1, s_2, \dots, s_n) , $s_i \in T$, the two random vectors $\mathbf{X} = (X(t_1), X(t_2), \dots, X(t_n))$ and $\mathbf{Y} = (Y(s_1), Y(s_2), \dots, Y(s_n))$ are **jointly Gaussian**.

The vector **(\mathbf{X}, \mathbf{Y})** has $2n$ -dimensional Gaussian density.

Jointly Gaussian random processes that are uncorrelated are also independent random processes.

2.3.2 Cross-Correlation Function

2. Jointly Stationary Processes

◆ For two stationary processes $X(t)$ and $Y(t)$, if

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = R_{XY}(\tau)$$

$$R_{YX}(t_1, t_2) = E[Y(t_1)X(t_2)] = R_{YX}(\tau)$$

then $X(t)$ and $Y(t)$ are called **Jointly Stationary Processes**.

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

2.1.4 Moments

Review

e.g. 4.

Given: $X(t)$ and $Y(t)$ are two second-order moment processes. $W(t) = X(t) + Y(t)$.

Obtain: $E[W(t)]$ and $R_{WW}(t_1, t_2)$

$$E[W(t)] = E[X(t)] + E[Y(t)]$$

$$R_{WW}(t_1, t_2) = R_{XX}(t_1, t_2) + R_{YY}(t_1, t_2) + R_{XY}(t_1, t_2) + R_{YX}(t_1, t_2)$$

If $X(t)$ and $Y(t)$ are **Jointly Stationary Processes**, then,

$$E[W(t)] = m_X + m_Y = m_W$$

$$R_{WW}(t_1, t_2) = R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) = R_{WW}(\tau)$$

2.3.2 Cross-Correlation Functions

e.g.4.

If $m_X = m_Y = 0$, and $X(t)$ and $Y(t)$ are mutually uncorrelated, then

$$m_W = 0$$

$$R_{WW}(\tau) = R_{XX}(\tau) + R_{YY}(\tau)$$

The sum of any two correlation functions is a valid correlation function.

2.3.2 Cross-Correlation Functions

e.g. The product of any two correlation functions is a valid correlation function.

Given: $X(t)$ and $Y(t)$ are two independent second-order moment processes. $W(t)=X(t)Y(t)$.

Obtain: $E[W(t)]$ and $R_{WW}(t_1, t_2)$

$$\begin{aligned} E[W(t)] &= E[X(t)Y(t)] = E[X(t)]E[Y(t)] \\ R_{WW}(t_1, t_2) &= E[X(t_1)Y(t_1) \bullet X(t_2)Y(t_2)] \\ &= E[X(t_1)X(t_2)] \bullet E[Y(t_1)Y(t_2)] \\ &= R_{XX}(t_1, t_2)R_{YY}(t_1, t_2) \end{aligned}$$

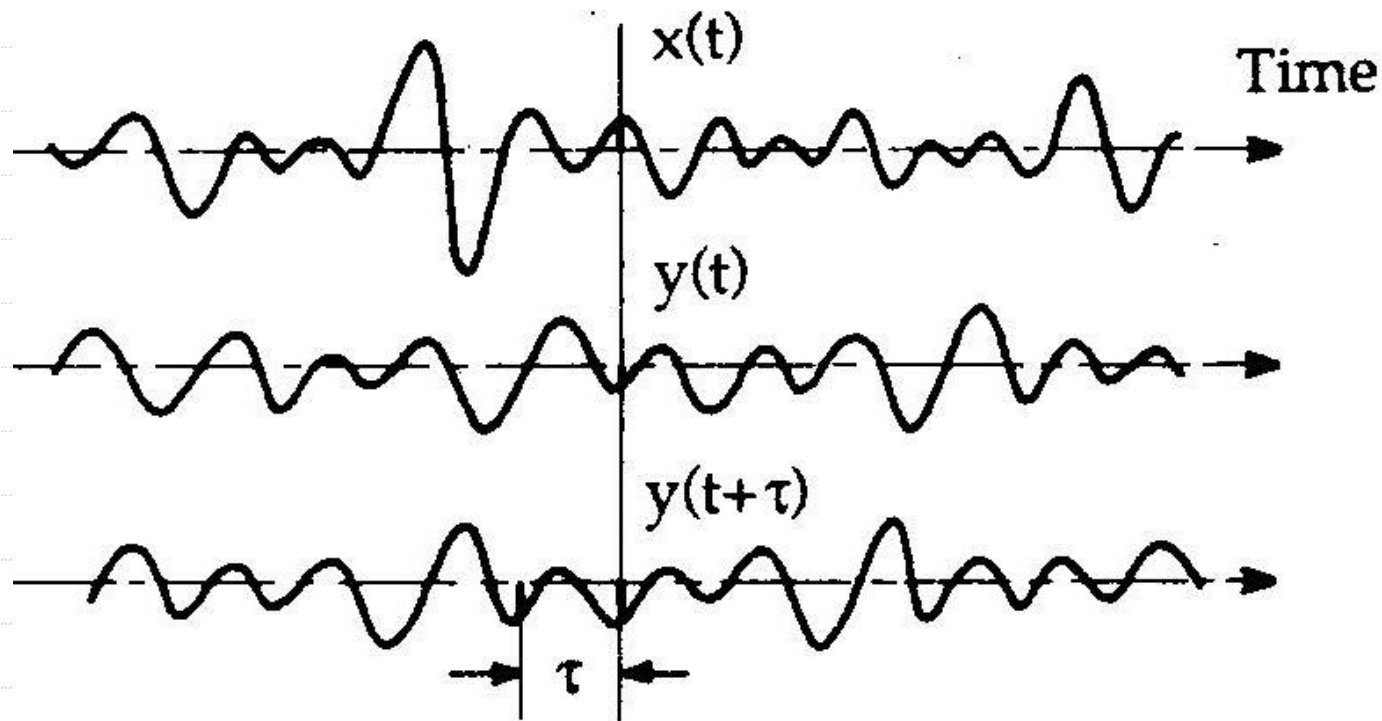
If $X(t)$ and $Y(t)$ are each Stationary Processes, then,

$$E[W(t)] = m_X m_Y = m_W \quad R_{WW}(t_1, t_2) = R_{WW}(\tau) = R_{XX}(\tau)R_{YY}(\tau)$$

2.3.2 Cross-Correlation Functions

If $X(t)$ and $Y(t)$ are Jointly Stationary Processes and $X(t), Y(t)$ are ergodic, then

$$R_{XY}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)y(t+\tau)dt$$



2.3.2 Cross-Correlation Functions

4. Properties of $R_{XY}(\tau)$

i) $R_{YX}(\tau) = R_{XY}(-\tau)$

ii) $|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)} \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$
 $|R_{YX}(\tau)| \leq \sqrt{R_X(0)R_Y(0)} \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

iii) $R_{XY}(\tau)$ is not always maximum at $\tau = 0$

Homework

2.8

2.16

2.17