Chapter 2
Stochastic Processes

Dong Yan
EI. HUST.

Chapter 2: Stochastic Processes

2.1 Basic Concepts

2.2 Stationary Stochastic Processes

2.3 Properties of Correlation Functions

2.4 Some Important Stochastic Processes

2.3 Properties of Correlation Functions

OUTLINE:

2.3.1 Properties of $R_{XX}(t_1, t_2)$

2.3.2 Properties of $R_{XY}(t_1, t_2)$

2.3.1 Properties of $R_{XX}(t_1,t_2)$

1.Definition of correlation function for a stochastic process

recall $\{X(t), t \in T\}$

2.1.4 Moments

Review

4. Correlation(autocorrelation) functions

- For a second-order moment process, the correlation function is

$$R_{XX}(t_1,t_2)=E\left[X(t_1)X(t_2)\right]$$

or

$$R_{XX}(t,t+\tau)=E\left[X(t)X(t+\tau)\right]$$

- Calculate correlation function for discrete stochastic processes and continuous stochastic processes respectively:

$$R_{XX}(t_1,t_2)=E\left[X(t_1)X(t_2)\right]$$

$$=\sum_{k=1}^{n}\sum_{j=1}^{n}kjP(x(t_1)=k,x(t_2)=j)$$

$$R_{XX}(t_1,t_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(t_1)x(t_2)f_X(x_1,x_2;t_1,t_2)dx_1dx_2$$

2.3.1 Properties of $R_{XX}(t_1,t_2)$

Relationship with other moments

$$R_{XX}(t,t)=E[X^2(t)]$$

$$C_{XX}(t_1,t_2)=R_{XX}(t_1,t_2)-\bar{x}(t_1)\bar{x}(t_2)$$

$$|R_X(t_1,t_2)| \leq \frac{E[X(t_1)]^2+E[X(t_2)]^2}{2}$$

$$|R_{XX}(t_1,t_2)| \leq \frac{R_{XX}(t_1,t_1)+R_{XX}(t_2,t_2)}{2}$$

2.3.1 Properties of $R_{XX}(t_1, t_2)$

2. Properties of $R_{XX}(t_1, t_2)$

i) $R_{XX}(t,t) \geq 0$

ii) $R_{XX}(t_1,t_2) = R_{XX}(t_2,t_1)$

iii) $|R_X(t_1,t_2)| \leq \sqrt{R_X(t_1,t_1)R_X(t_2,t_2)}$

iv) Nonnegative definite (positive semi-definite).

$$\sum_{i=1}^{n}\sum_{k=1}^{n} \alpha_i \alpha_k^* R_X(t_i,t_k) \geq 0$$

Nonnegative definiteness implies i), ii), iii).

If a function R(*,*) is nonnegative definite, it can be a correlation function for some second-order random processes.

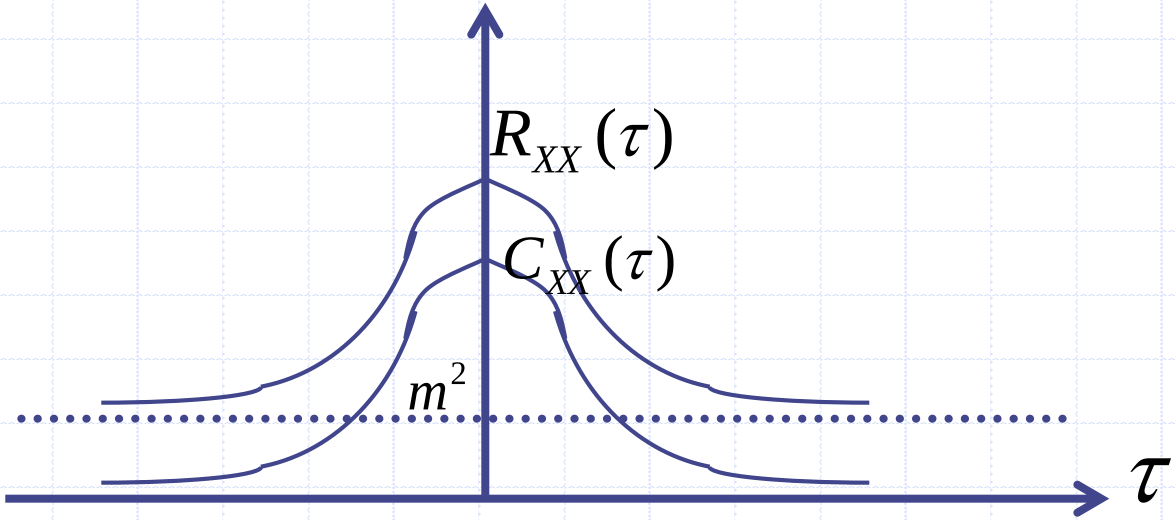
2.3.1 Properties of $R_{XX}(t_1, t_2)$

3. Correlation function of weakly stationary S.P.

$$R_{XX}(t_1, t_2) = R_{XX}(\tau)$$

$$R_{XX}(0) = E[X^2(t)]$$

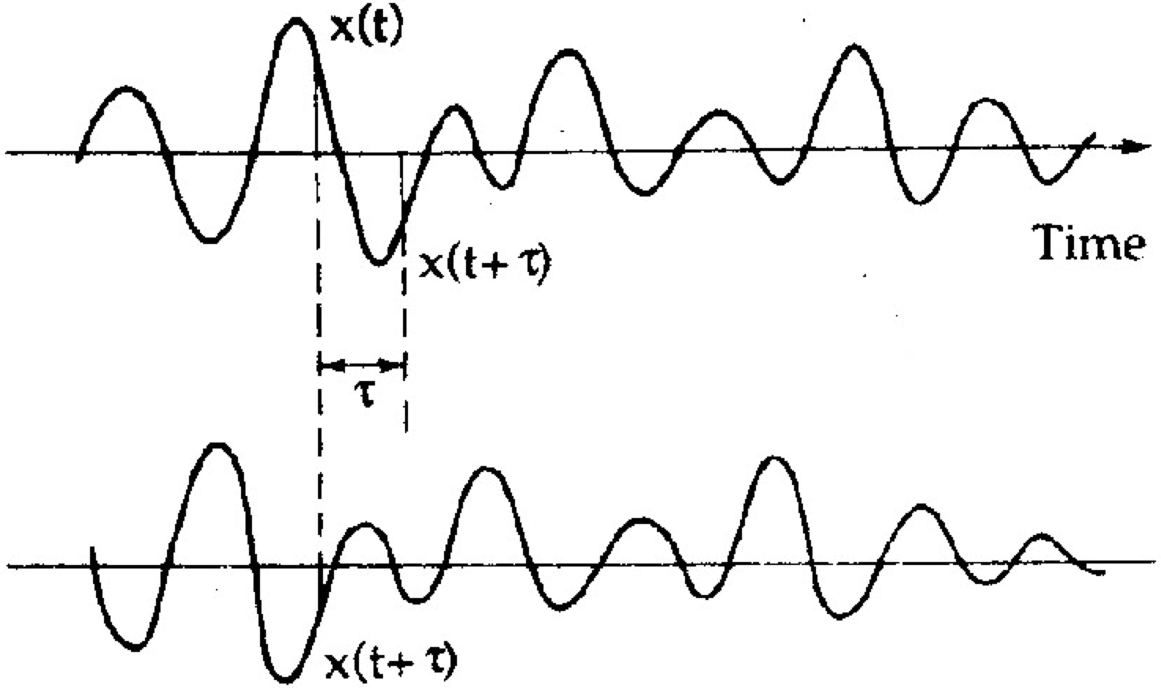
$$C_{XX}(\tau) = R_{XX}(\tau) - m^2$$



2.3.1 Properties of $R_{XX}(t_1, t_2)$

- If X(t) is ergodic, then

$$R_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt$$



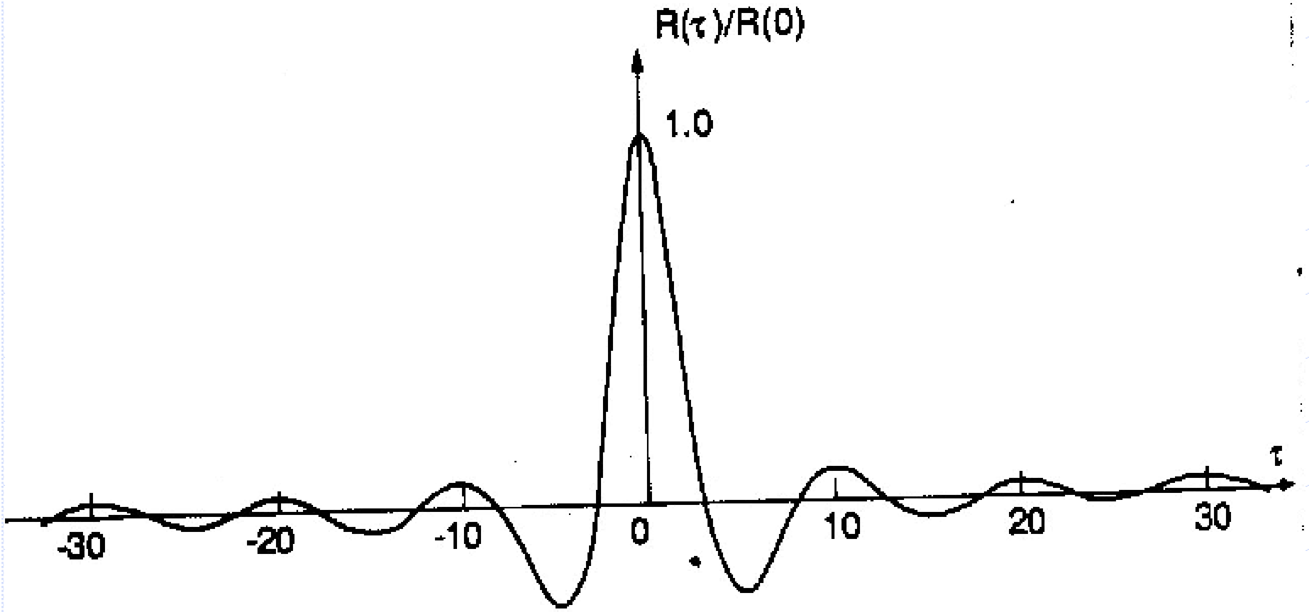
4. Properties of $R_{XX}(\tau)$

i) An even function: $R_{XX}(\tau) = R_{XX}(-\tau)$

ii) $R_{XX}(0) \geq | R_{XX}(\tau) |$

iii) $R_{XX}(0) = E[X^2(t)] \geq 0$

iv) If X(t) is ergodic, then $R_{XX}(0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$



4. Properties of $R_{XX}(\tau)$

V) If X(t)=X(t+T), then $R_{XX}(\tau)=R_{XX}(\tau+T)$

Example 1 Random phase processes

$X(t)=A\cos(w_0 t+\varepsilon)$, $t>0$, whereas A and w_0 are constants and ε is random variable uniformly distributed between $-\pi$ and π.

$$E[x(t)]=0$$

$$C_{XX}(\tau)=R_{XX}(\tau)=\frac{A^2}{2}\cos w_0\tau$$

4. Properties of $R_{XX}(\tau)$

Example 2. Random amplitude processes

X(t)=Ycos(wt)+Zsin(wt)， t>0, whereas Y and Z are independent random variables, and EY=EZ=0, VarY=VarZ=σ^2.

$$E[x(t)] = 0$$

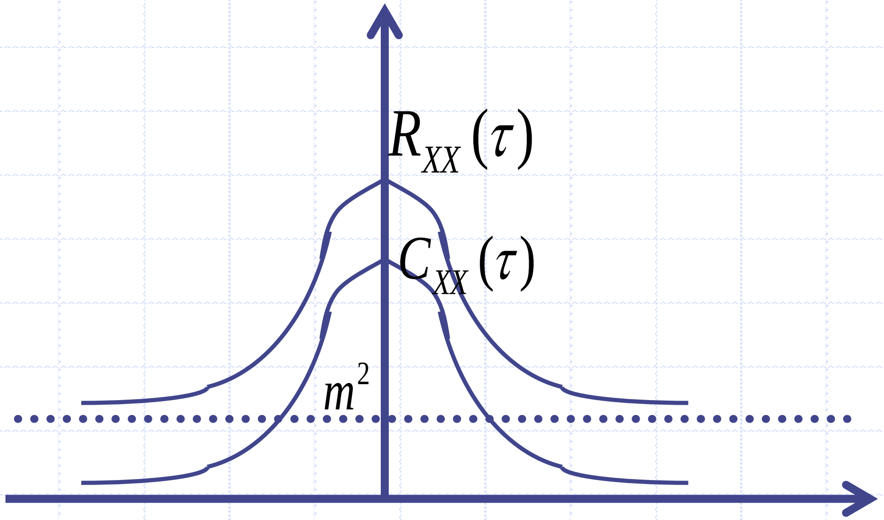
$$C_{XX}(\tau) = R_{XX}(\tau) = \sigma^2 \cos w\tau$$

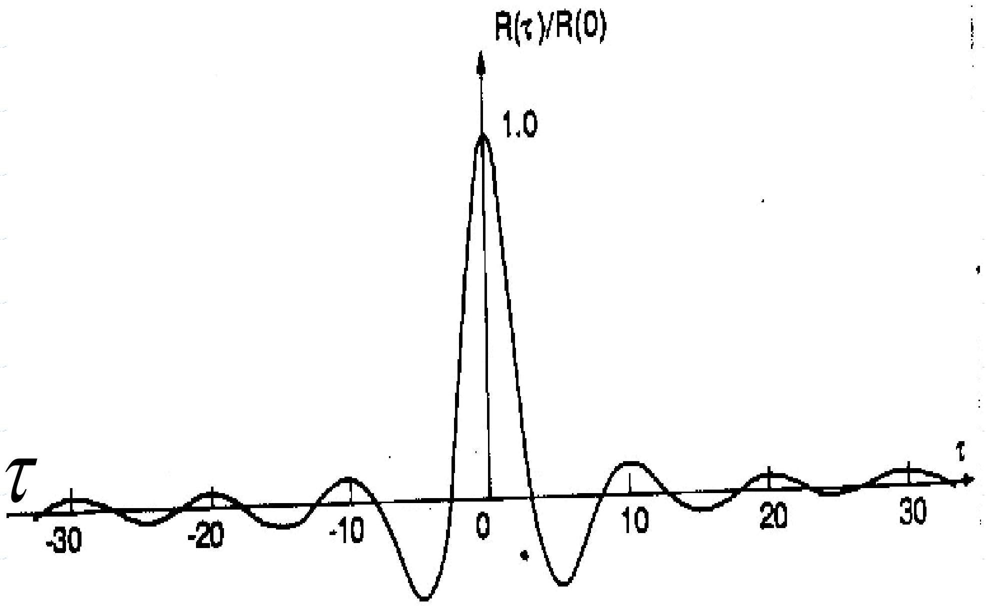
4. Properties of $R_{XX}(\tau)$

vi) If X(t) is a non-period process, as $|\tau| \to \infty$

$X(t)$ and $X(t+\tau)$ are independent, that is

$$\lim_{|\tau \to \infty|} R_{XX}(\tau) = m^2$$





2.3.1 Properties of $R_{XX}(t_1, t_2)$

Example 3. Pulse Code Modulation (binary case)

- A source generates the symbols 1 or 0 independently with probabilities 1-*p* and *p*, respectively.
- The symbol 1 is transmitted by sending a pulse with constant amplitude *A* and duration *T*.
- The symbol 0 is transmitted by sending nothing during an interval of length *T*.

Example 3.

A signal modulated in this way is represented by the stochastic process

$$X(t) = \sum_{n=-\infty}^{\infty} A_n h(t - nT), \;\; nT \le t < (n+1)T$$

Where the $A_n; n = 0, \pm 1, \pm 2, \ldots$ are independent binary random variables defined by

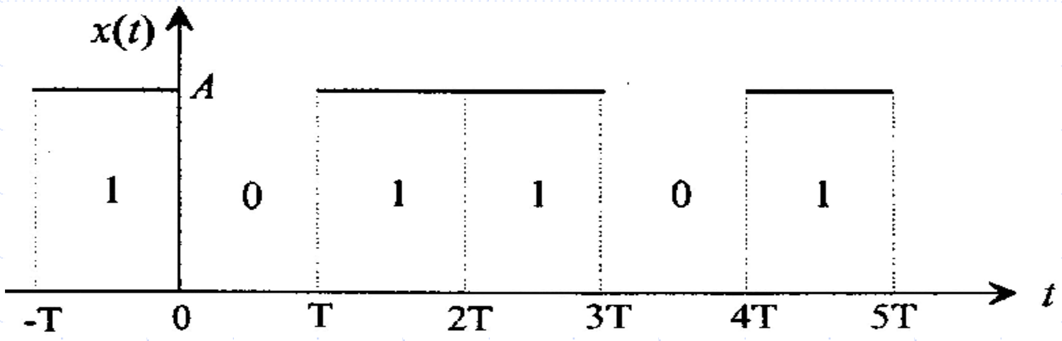
$$A_n = \begin{cases} 0 & \text{with probability } p \\ A & \text{with probability } 1-p \end{cases}$$

And h(t) is given by

$$h(t) = \begin{cases} 1 & \text{for } 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Example 3.

In particular, for any t, $X(t)=\begin{cases}0 & \text{with probability } p \\ A & \text{with probability } 1-p\end{cases}$



A sample wave x(t): …1 0 1 1 0 1…

Commonly, the time point $t=0$ has been chosen in such a way that is coincides with the beginning of a new transmission period.

Obtain: $E[X(t)]$, $R_{XX}(t_1,t_2)$

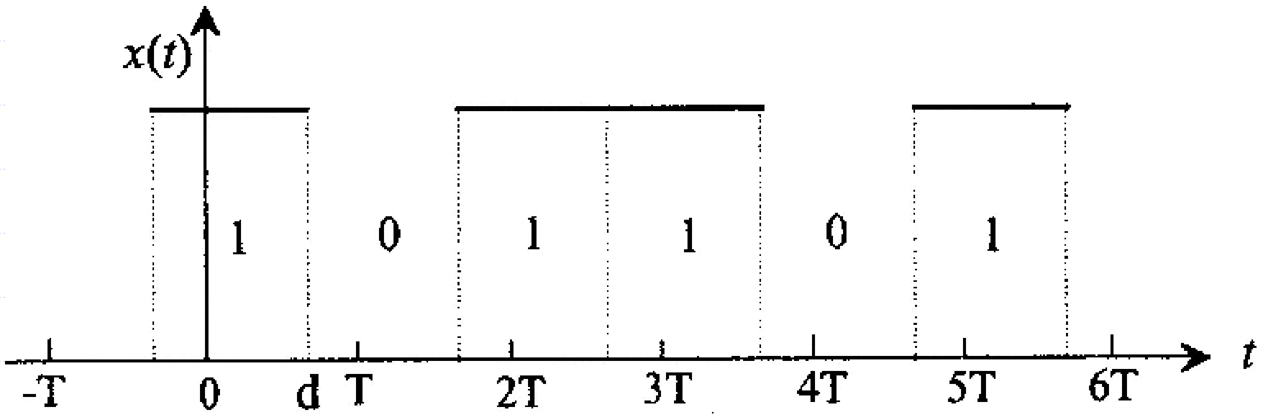
Discuss: Is X(t) a stationary process?

Example 4. Random Delayed Pulse Code

Using the stochastic process X(t) defined in example 3, define the process:

$$Y(t) = X(t - D)$$

where D is uniformly distributed over [0,T].



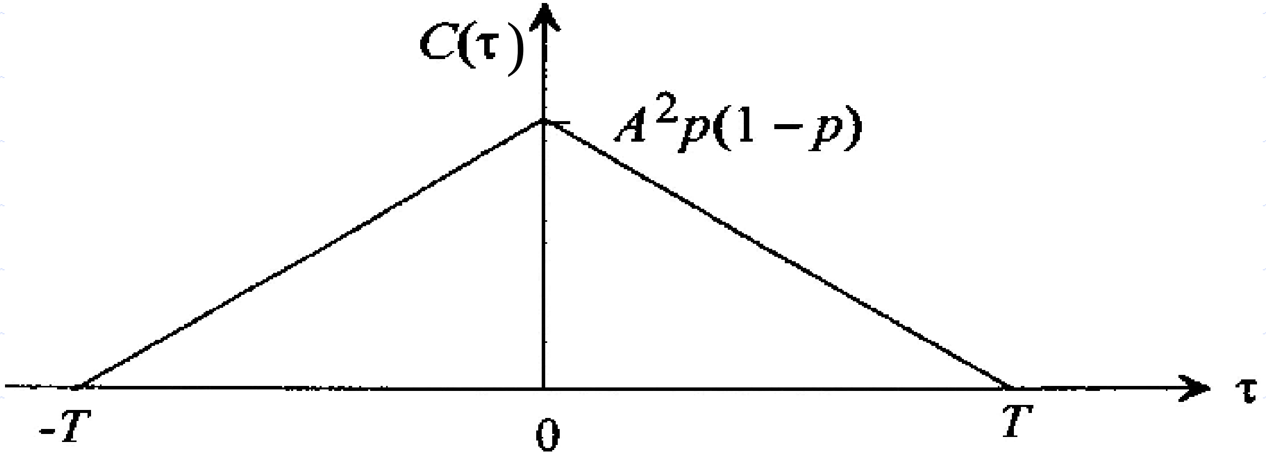
Obtain: E[Y(t)], $R_{YY}(t_1,t_2)$

Discuss: Is Y(t) a stationary process?

Example 4. Random Delayed Pulse Code

Discuss: Is Y(t) a stationary process?

$$C_{YY}(\tau) = \begin{cases} A^2 p(1-p)(1-\frac{|\tau|}{T}) & \text{for } |\tau| \le T \\ 0 & \textit{otherwise} \end{cases}$$



2.3.1 Properties of $R_{XX}(t_1, t_2)$

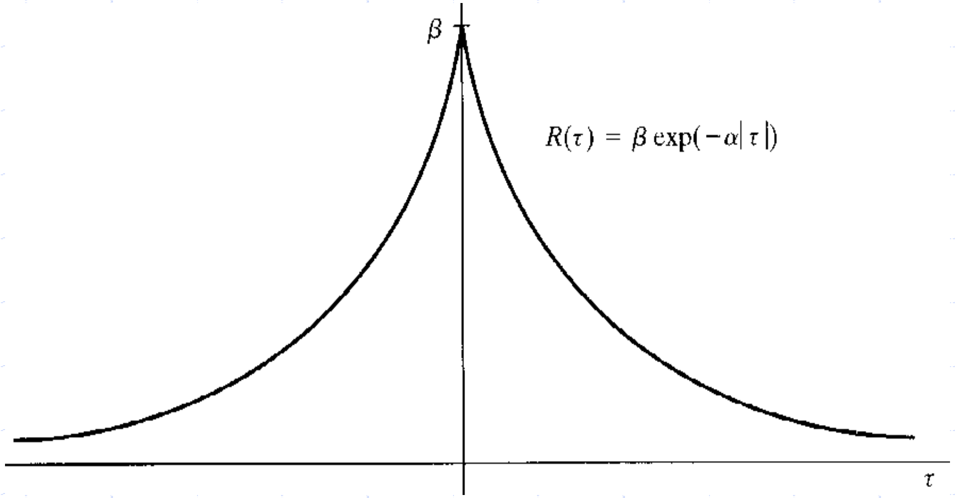
Examples of correlation functions for weakly stationary processes

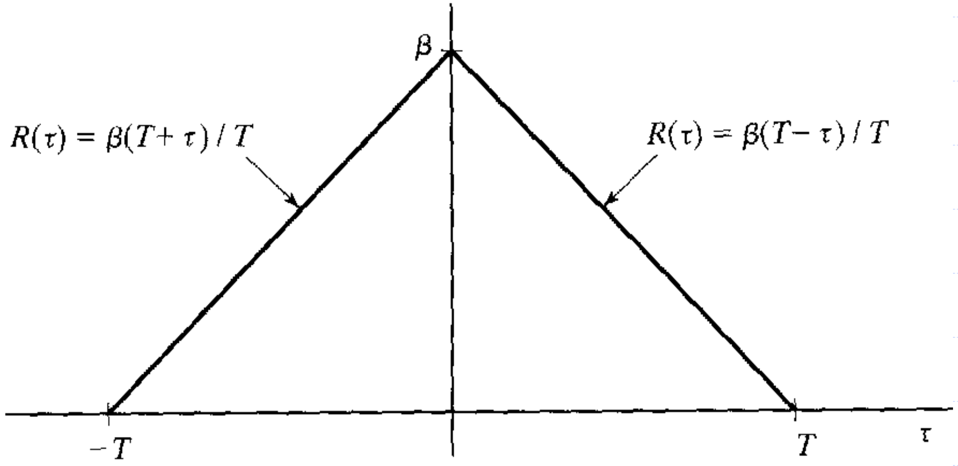
1. $R(\tau) = \beta \exp(-\alpha|\tau|)$

2. $R(\tau) = \begin{cases} \beta(T - |\tau|)/T, & -T \leq \tau \leq T \\ 0, & \text{otherwise} \end{cases}$

3. $R(\tau) = \beta \exp(-\alpha|\tau|)\cos(\omega_0 \tau)$

4. $R(\tau) = 2W\{\sin(2\pi W\tau)/2\pi W\tau\}$





2.3 Properties of Correlation Function

OUTLINE:

2.3.1 Properties of $R_{XX}(t_1, t_2)$

2.3.2 Properties of $R_{XY}(t_1, t_2)$

2.3.2 Cross-Correlation Functions

1. Definition of cross-correlation function for two stochastic processes and $\{X(t), t \in T\}$ $\{Y(t), t \in T\}$

recall

2.1.4 Moments

Review

6. Cross-covariance functions

- For two second-order moment processes X(t) and Y(t), their cross-covariance function is

$$C_{XY}(t_1,t_2)=Cov[X(t_1),Y(t_2)]$$
$$=E\big[[X(t_1)-\overline{x}(t_1)][Y(t_2)-\overline{y}(t_2)]\big]$$
$$=E[X(t_1)Y(t_2)]-\overline{x}(t_1)\overline{y}(t_2)$$

2.1.4 Moments

Review

7. Cross-correlation functions

- For two second-order moment processes X(t) and Y(t), their cross-correlation function is

$$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)]$$

$$C_{XY}(t_1,t_2) = R_{XY}(t_1,t_2) - \overline{x}(t_1)\overline{y}(t_2)$$

$$R_{XY}(t_1,t_2) = E[Y(t_2)X(t_1)] = R_{YX}(t_2,t_1)$$

$$|R_{X,Y}(t,s)| \le \sqrt{R_X(t,t)R_Y(s,s)}$$

2.1.4 Moments

Review

Mutually uncorrelated

- For two second-order moment processes X(t) and Y(t), if

$$C_{XY}[t_1,t_2]=0 \qquad t_1,t_2 \in T$$

then X(t) and Y(t) are mutually uncorrelated

$$R_{XY}(t_1,t_2) = \overline{x}(t_1)\overline{y}(t_2)$$

2.3.2 Cross-Correlation Functions

Independent random processes:

The processes $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$ are said to be independent random processes if, for each positive integer n and each choice of $(t_1, t_2, \ldots, t_n)$, $t_i \in T$, $(s_1, s_2, \ldots, s_n)$, $s_i \in T$, the two random vectors $\mathbf{X} = (X(t_1), X(t_2), \ldots, X(t_n))$ and $\mathbf{Y} = (Y(s_1), Y(s_2), \ldots, Y(s_n))$ are independent.

Two random vectors **X** and **Y** are independent if and only if $F_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = F_{\mathbf{X}}(\mathbf{x}) F_{\mathbf{Y}}(\mathbf{y})$.

The independent random processes are also uncorrelated.

Uncorrelated random processes need not be independent.

2.3.2 Cross-Correlation Function

jointly Gaussian random processes

The processes $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$ are said to be jointly Gaussian random processes if, for each positive integer n and each choice of $(t_1, t_2, \ldots, t_n)$, $t_i \in T$, $(s_1, s_2, \ldots, s_n)$, $s_i \in T$, the two random vectors $\mathbf{X} = (X(t_1), X(t_2), \ldots, X(t_n))$ and $\mathbf{Y} = (Y(s_1), Y(s_2), \ldots, Y(s_n))$ are jointly Gaussian .

The vector **(X , Y)** has 2n-dimensional Gaussian density.

Jointly Gaussian random processes that are uncorrelated are also independent random processes .

2.3.2 Cross-Correlation Function

2. Jointly Stationary Processes

- For two stationary processes X(t) and Y(t), if

$$R_{XY}(t_1,t_2)=E\left[X(t_1)Y(t_2)\right]=R_{XY}(\tau)$$

$$R_{YX}(t_1,t_2)=E\left[Y(t_1)X(t_2)\right]=R_{YX}(\tau)$$

then X(t) and Y(t) are called Jointly Stationary Processes.

$$R_{XY}(\tau)=R_{YX}(-\tau)$$

2.1.4 Moments

Review

e.g.4.

Given: X(t) and Y(t) are two second-order moment processes. W(t)=X(t)+Y(t).

Obtain: E[W(t)] and $R_{WW}(t_1,t_2)$

$$E[W(t)] = E[X(t)] + E[Y(t)]$$

$$R_{WW}(t_1,t_2) = R_{XX}(t_1,t_2) + R_{YY}(t_1,t_2) + R_{XY}(t_1,t_2) + R_{YX}(t_1,t_2)$$

If X(t) and Y(t) are Jointly Stationary Processes, then,

$$E[W(t)] = m_X + m_Y = m_W$$

$$R_{WW}(t_1,t_2) = R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) = R_{WW}(\tau)$$

2.3.2 Cross-Correlation Functions

e.g.4.

If $m_X = m_Y = 0$, and X(t) and Y(t) are mutually uncorrelated, then

$$m_W = 0$$

$$R_{WW}(\tau) = R_{XX}(\tau) + R_{YY}(\tau)$$

The sum of any two correlation functions is a valid correlation function.

2.3.2 Cross-Correlation Functions

e.g. The product of any two correlation functions is a valid correlation function.

Given: X(t) and Y(t) are two independent second-order moment processes. W(t)=X(t)Y(t).

Obtain: E[W(t)] and $R_{WW}(t_1,t_2)$

$$E[W(t)] = E[X(t)Y(t)] = E[X(t)]E[Y(t)]$$

$$\begin{aligned} R_{WW}(t_1,t_2) &= E[X(t_1)Y(t_1) \bullet X(t_2)Y(t_2)] \\ &= E[X(t_1)X(t_2)] \bullet E[Y(t_1)Y(t_2)] \\ &= R_{XX}(t_1,t_2)R_{YY}(t_1,t_2) \end{aligned}$$

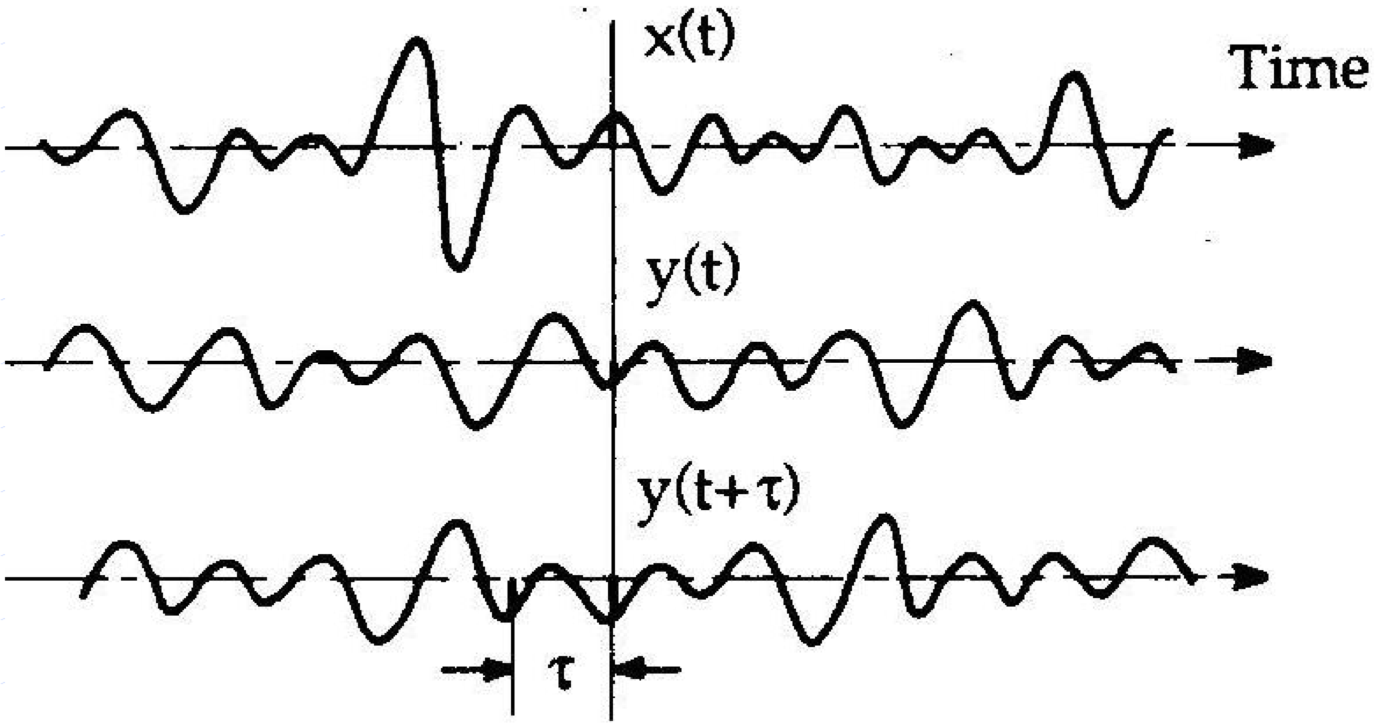
If X(t) and Y(t) are each Stationary Processes, then,

$$E[W(t)] = m_X m_Y = m_W \qquad R_{WW}(t_1,t_2) = R_{WW}(\tau) = R_{XX}(\tau)R_{YY}(\tau)$$

2.3.2 Cross-Correlation Functions

If X(t) and Y(t) are Jointly Stationary Processes and X(t),Y(t) are ergodic, then

$$R_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t+\tau)dt$$



2.3.2 Cross-Correlation Functions

4. Properties of $R_{XY}(\tau)$

i) $$R_{YX}(\tau) = R_{XY}(-\tau)$$

ii) $$|R_{XY}(\tau)| \le \sqrt{R_X(0)R_Y(0)} \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

$$|R_{YX}(\tau)| \le \sqrt{R_X(0)R_Y(0)} \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

iii) $R_{XY}(\tau)$ is not always maximum at $\tau = 0$

Homework

2.8
2.16
2.17