Chapter 2
Stochastic Processes

Chapter 2: Stochastic Processes

2.4 Some Important Stochastic Processes

- *2.4.1 Gaussian(Normal) Processes*
- 2.4.2 Independent Increment Processes
- 2.4.3 Wiener(-Levy) Processes
- *2.4.4 Markov Processes*
- 2.4.5 Counting Processes
- *2.4.6 Poisson Processes*
- 2.4.7 Bernoulli Processes
- *2.4.8 Narrow band Processes*

2.4.1 Gaussian(Normal) Processes

(in 2.2.2 of textbook)

Def.1 Gaussian process

A stochastic process $\{x(t), t \in T\}$ is a Gaussian process if the random vectors

$$(X(t_1), X(t_2), \ldots, X(t_n))$$

have a joint Gaussian (Normal) distribution for all n-tuples $(t_1, t_2, \ldots, t_n)$ with $t_i \in T$ and

$$t_1 < t_2 < \ldots < t_n; \quad n = 1, 2, \ldots$$

2.4.1 Gaussian(Normal) Processes

Def.1 Gaussian process based on a single sample wave

Precondition: the stochastic process satisfies the ergodic property.

A stochastic process $\{x(t), t \in T\}$ is a Gaussian process if for any given time t the random variable $x(t)$ is normally distributed.

Gaussian process is a second-order moment process, as the existence of first-order moment and second-order moments.

2.1.3 Distribution and Density Functions

e.g.1.

- Given: The stochastic process X(t) is given by $X(t)=Y_1+Y_2 t$, $t>0$, whereas Y_1 and Y_2 are independent Gaussian random variables, with zero mean and variance σ^2

Obtain: one-dimensional distribution and two-dimensional distribution of X(t).

Sln:

$$\mathrm{Var}\{Y_1\} + \mathrm{Var}\{tY_2\} = \sigma^2 + t^2\sigma^2 = \sigma^2(1 + t^2)$$

$$f_{X,1}(x;t) = [2\pi\sigma^2(1 + t^2)]^{-1/2}\exp\{-x^2/[2\sigma^2(1 + t^2)]\}$$

$$f_{X,Y}(u,v) = \frac{\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{u-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{u-\mu_1}{\sigma_1}\right)\left(\frac{v-\mu_2}{\sigma_2}\right) + \left(\frac{v-\mu_2}{\sigma_2}\right)^2\right]\right\}}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}}$$

2.1.3 Distribution and Density Functions

$$\mathrm{Var}\{X(t_k)\} = \sigma^2(1 + t_k^2)$$

$$\begin{aligned}\mathrm{Cov}\{X(t_1), X(t_2)\} &= E\{X(t_1)X(t_2)\} \\ &= E\{(Y_1 + t_1Y_2)(Y_1 + t_2Y_2)\} \\ &= \sigma^2 + t_1t_2\sigma^2 = \sigma^2(1 + t_1t_2)\end{aligned}$$

$$\rho(t_1, t_2) = \frac{\mathrm{Cov}\{X(t_1), X(t_2)\}}{\sqrt{\mathrm{Var}\{X(t_1)\}\,\mathrm{Var}\{X(t_2)\}}} = \frac{1 + t_1t_2}{[(1 + t_1^2)(1 + t_2^2)]^{1/2}}$$

$$f_{X,2}(x_1, x_2; t_1, t_2) = \frac{\exp\{-[(1 + t_2^2)x_1^2 - 2(1 + t_1t_2)x_1x_2 + (1 + t_1^2)x_2^2]/[2\sigma^2(t_1 - t_2)^2]\}}{2\pi\sigma^2|t_1 - t_2|}$$

2.4.1 Gaussian(Normal) Processes

- One/two-dimensional distributions:
  - The distribution can be decided by mean values $E[X(t_1)]$ and $E[X(t_2)]$ and covariance function $C_{XX}(t_1,t_2)$ or correlation function $R_{XX}(t_1,t_2)$
  - n-dimensional distribution:

    $$E[X(t_i)], \quad 1 \le i \le n$$

    $$Cov_{XX}(t_i,t_j) \quad or \quad R_{XX}(t_i,t_j), \quad 1 \le i \le n, 1 \le j \le n$$

  - A Gaussian random process is completely specified by its mean and correlation functions.

2.4.1 Gaussian(Normal) Processes

- If a Gaussian process is weakly stationary, it is also strictly stationary.

- The significance of Gaussian process

  The Gaussian process plays a significant role in stochastic analysis of random phenomena observed in natural sciences, since many random phenomena can be approximately represented by a normal process.

2.4 Some Important Stochastic Processes

2.4.1 Gaussian(Normal) Processes

2.4.2 Independent Increment Processes

2.4.3 Wiener(-Levy) Processes

2.4.4 Markov Processes

2.4.5 Counting Processes

2.4.6 Poisson Processes

2.4.7 Bernoulli Processes

2.4.8 Narrow band Processes

2.2.3 Independent Increment Processes

- Increments

  the increment of a stochastic process $\{X(t), t \in T\}$ with respect to the interval $[t_1, t_2]$ is the difference $X(t_2) - X(t_1)$.

2.2.3 Independent Increment Processes

Def.1 Independent Increment Process

A stochastic process X(t) is said to be an independent increment process if $X(t_{i+1}) - X(t_i)$, where i=0,1,2,…, is statistically independent .

(and thereby statistically uncorrelated).

2.2.3 Independent Increment Processes

Def.2 Stationary Increment Process

A stochastic process X(t) is said to be a **stationary increment process** if its increments $X(t_2+\tau)-X(t_1+\tau)$ have the same probability distribution for all τ with $t_1+\tau\in T, t_2+\tau\in T$; t_1, t_2 fixed, but arbitrary.

2.2.3 Independent Increment Processes

Def.2 Stationary Increment Process

an equivalent definition:

A stochastic process X(t) is said to be a stationary increment process if the probability distribution of $X(t+\tau)-X(t)$ does not depend on t for any fixed τ ; $t, t+\tau \in T$

2.2.3 Independent Increment Processes

- A stochastic process with stationary increments need not be stationary in any sense.(strictly or weakly)

2.2.3 Independent Increment Processes

Def.3

Stationary Independent Increment Process

A stochastic process X(t) which possess both stationary as well as independent increments properties is called a stationary independent increment process.

2.2.3 Independent Increment Processes

e.g.1.

Given: X(t) is an independent increment process, $X(t_0)=0$.

Prove: Its autocovariance function is equal to variance function.

Sln:

$$C_{XX}(t_1,t_2)=Var(X(t_1))$$

$$C_{XX}(t_1,t_2)=Var(X(t_2)) \quad ?$$

2.4 Some Important Stochastic Processes

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2.4.3 Wiener(-Levy) Processes

(Brownian motion process)

- A special Gaussian process.

Def.1 Wiener-Levy process

a stochastic process X(t) is said to be a Wiener-Levy process if

(i) X(t) has stationary independent increment.

(ii) Every independent increment is normally distributed.

(iii) E[X(t)]=0 for all time.

(iv) X(0)=0.

2.4.3 Wiener(-Levy) Processes

- Increments distribution of a Wiener process

$$X(t_2) - X(t_1) \sim N(0, \sigma^2 |t_2 - t_1|)$$

- The distribution of a Wiener process

$$X(t) \sim N(0, \sigma^2 |t|)$$

2.4 Some Important Stochastic Processes

2.4.4 Markov Processes

Def.1 Markov Process

A stochastic process X(t) is said to be a Markov process if it satisfies the following conditional probability:

$$Pr\{x(t_n) \leq x_n | x(t_1) = x_1, x(t_2) = x_2, \ldots, x(t_{n-1}) = x_{n-1}\}$$

$$= Pr\{x(t_n) = x_n | x(t_{n-1}) = x_{n-1}\}, \quad \text{where } t_1 < t_2 < \cdots\ t_{n-1} < t_n$$

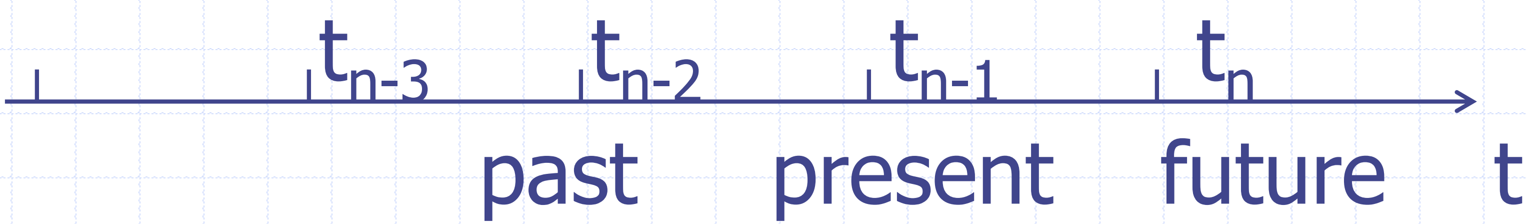
(Markovian property)

2.4.4 Markov Processes

- The Markovian property has the following implication:

  If t_n is a time point in the future, t_{n-1} the present time, and correspondingly t_1, t_2,…, t_{n-2}, time points in the past,

  the future development of a process does not depend on its evolvement in the past, but only on its present state.



2.4.4 Markov Processes

Independent increment process

V.S.

Markov process

Every independent increment process is a Markov process, although there are many Markov processes that do not have independent increments.

2.4 Some Important Stochastic Processes

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2.4.5 Counting Processes

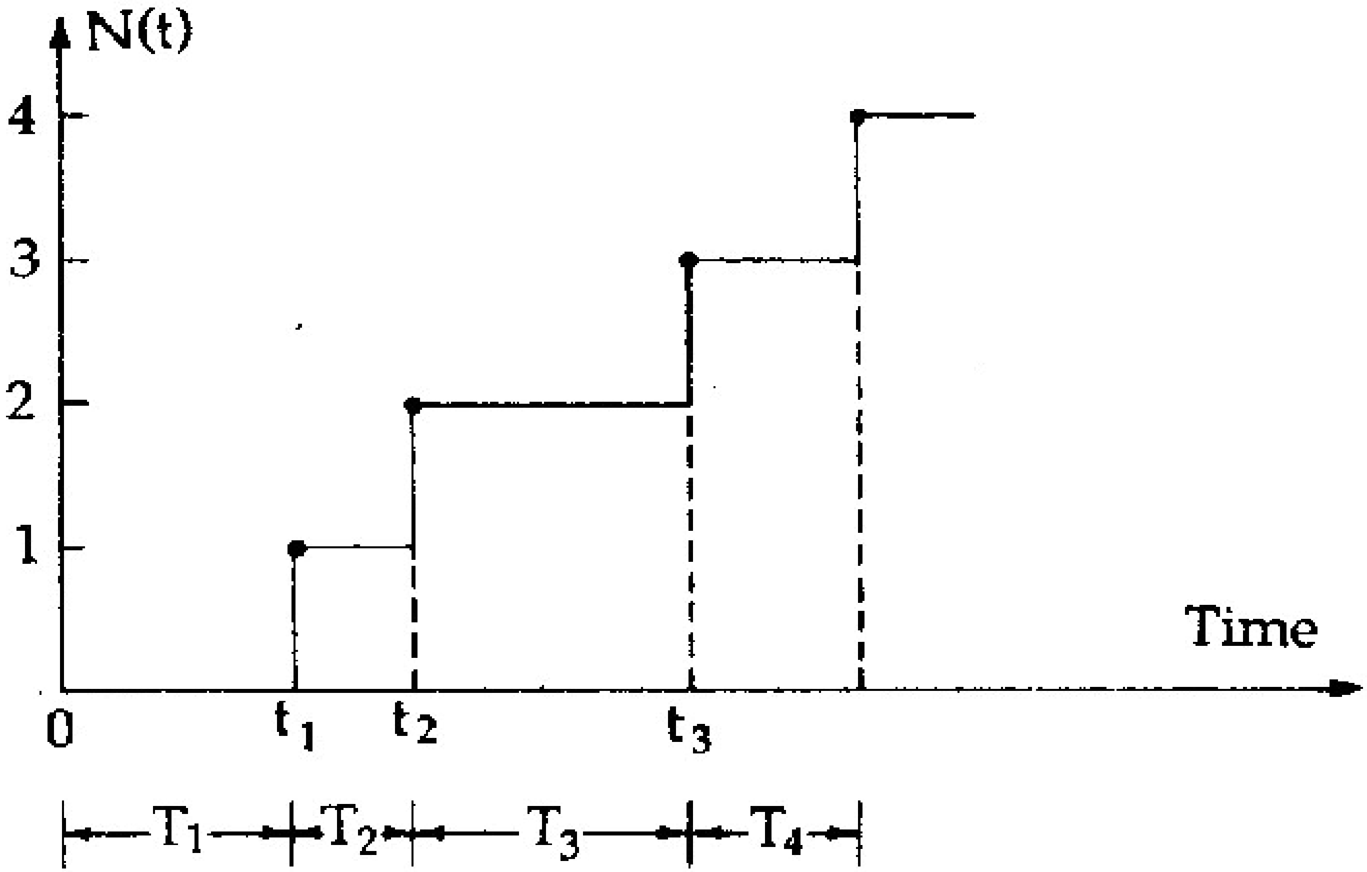
- Counting processes deal with the frequency of occurrence of random events.

Def.1 Counting Process

- An integer-valued continuous-time stochastic process N(t) is called a counting process of the series of events if N(t) represents the total number of occurrences of the event in the time interval t=0 to t.

2.4.5 Counting Processes

A sample wave of a counting process N(t)



$T_1=t_1, T_2=t_2-t_1, T_3=t_3-t_2, \ldots$ Interarrival times

2.4.5 Counting Processes

- If the interarrival times are independent, identically distributed random variables, then the process is called a renewal process.
- If the interarrival times(are independent, identically distributed random variables) obey an exponential distribution, the process is called a Poisson process.

2.4 Some Important Stochastic Processes

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1.2 An Example of Random Processes

1.2.1 Poisson Distribution

"Occurrence of events"

λ = average rate of occurrence per second;

N = the number of occurrences of an event in an arbitrary period.

$$\Pr\left(N = k\right) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0,1,2,\ldots$$

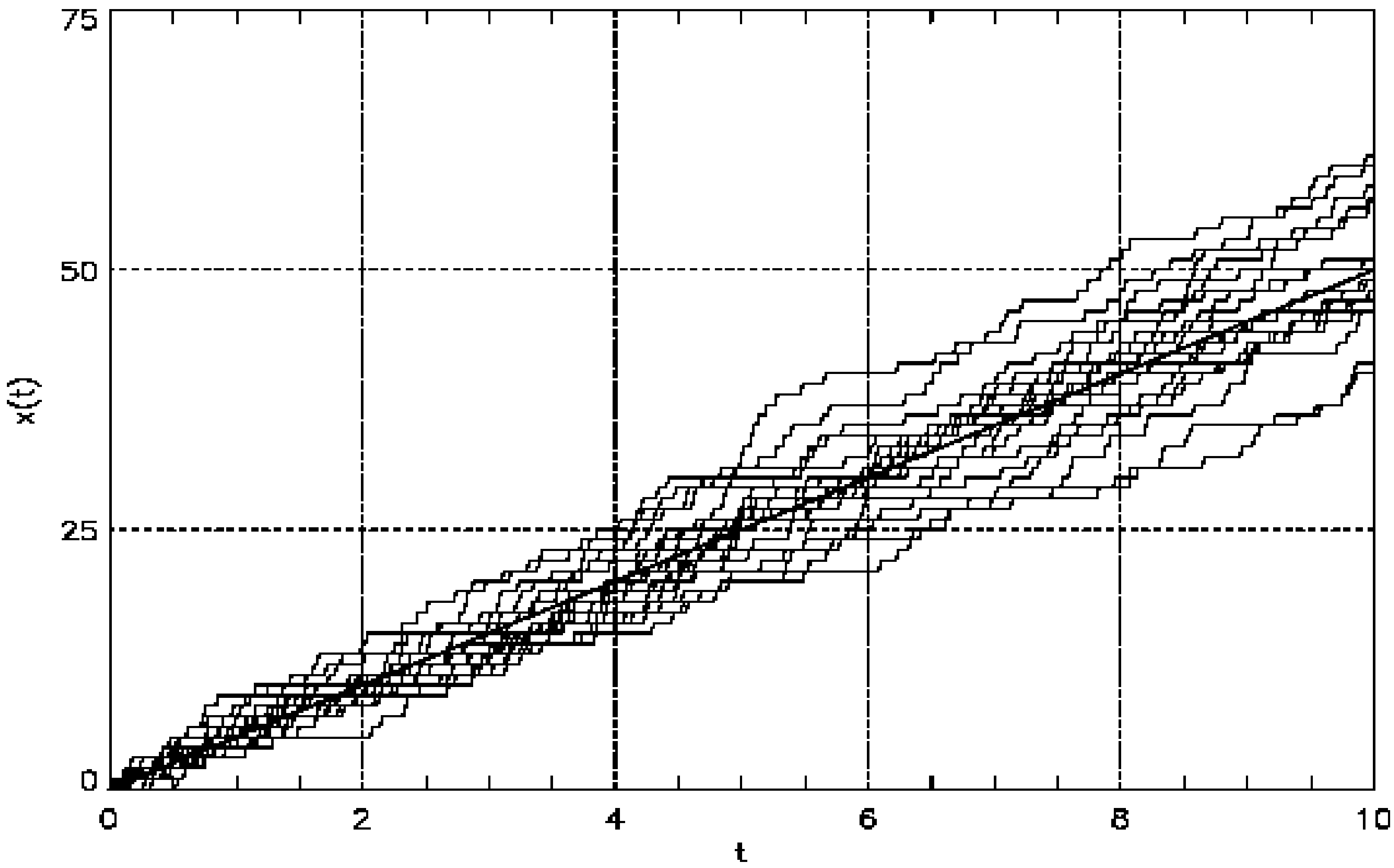
$$E\left(N\right) = \lambda$$

$$\mathrm{Var}\left(N\right) = \lambda$$

$$Cv_N^2 = 1/\lambda$$

2.4.6 Poisson Processes

- Sample waves of a Poisson process



2.4.6 Poisson Processes

- *Def.1* Poisson process

A counting process N(t) is said to be a Poisson process with mean rate (or intensity) λ if

(i) N(0)=0.

(ii) N(t) has stationary independent increment.

(iii) The number in any time interval of length τ is Poisson distributed with mean $\lambda\tau$, That is,

$$P\{N(t+\tau)-N(t)=k\}=\frac{(\lambda\tau)^k}{k!}e^{-\lambda\tau}$$

- Both Wiener process and Poisson process have stationary independent increment.

2.4.6 Poisson Processes

- $N(t+\tau)-N(t)$ is called a Poisson increment process.
- Let $X(t)=N(t+\tau)-N(t)$

$$E[X(t)]=E[N(t+\tau)]-E[N(t)]=\lambda(t+\tau)-\lambda t=\lambda\tau$$

$$C_{XX}(t_1,t_2)=\begin{cases} Var_X(t_1+\tau-t_2) & for\ 0<t_2-t_1<\tau \\ 0 & otherwise \end{cases}$$

$$=\begin{cases} Var_X(\tau-u) & for\ 0<t_2-t_1<\tau \\ 0 & otherwise \end{cases}$$

where $u=t_2-t_1$

The Poisson increment process is covariance stationary.

2.4.6 Poisson Processes

- *Another definition of* Poisson process

A counting process N(t) is said to be a Poisson process with mean rate λ if

(i) N(0)=0.

(ii) N(t) has stationary independent increment.

(iii) N(t) satisfies

$$P\{X(t+h)-X(t)=1\}=\lambda h+o(h)$$

$$P\{X(t+h)-X(t)\geq 2\}=o(h)$$

A function f( ) is said to be o(h) if $\lim_{h\to 0}\frac{f(h)}{h}=0$

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2.4.7 Bernoulli Processes

- *Def.1* Bernoulli process

consider a series of independent repeated trials with two outcomes: success and failure, rain and no rain, and so on.

A counting process X_n is called a Bernoulli process if X_n represents the number of successes in n trials.

2.4.7 Bernoulli Processes

$$\begin{bmatrix} outcomes \\ probability \end{bmatrix} = \begin{bmatrix} success & failure \\ p & q \end{bmatrix}$$

the probability of *k* successes in *n* trials is given by the following binomial distribution:

$$P(X_n = k) = \binom{n}{k} p^k q^{n-k}$$

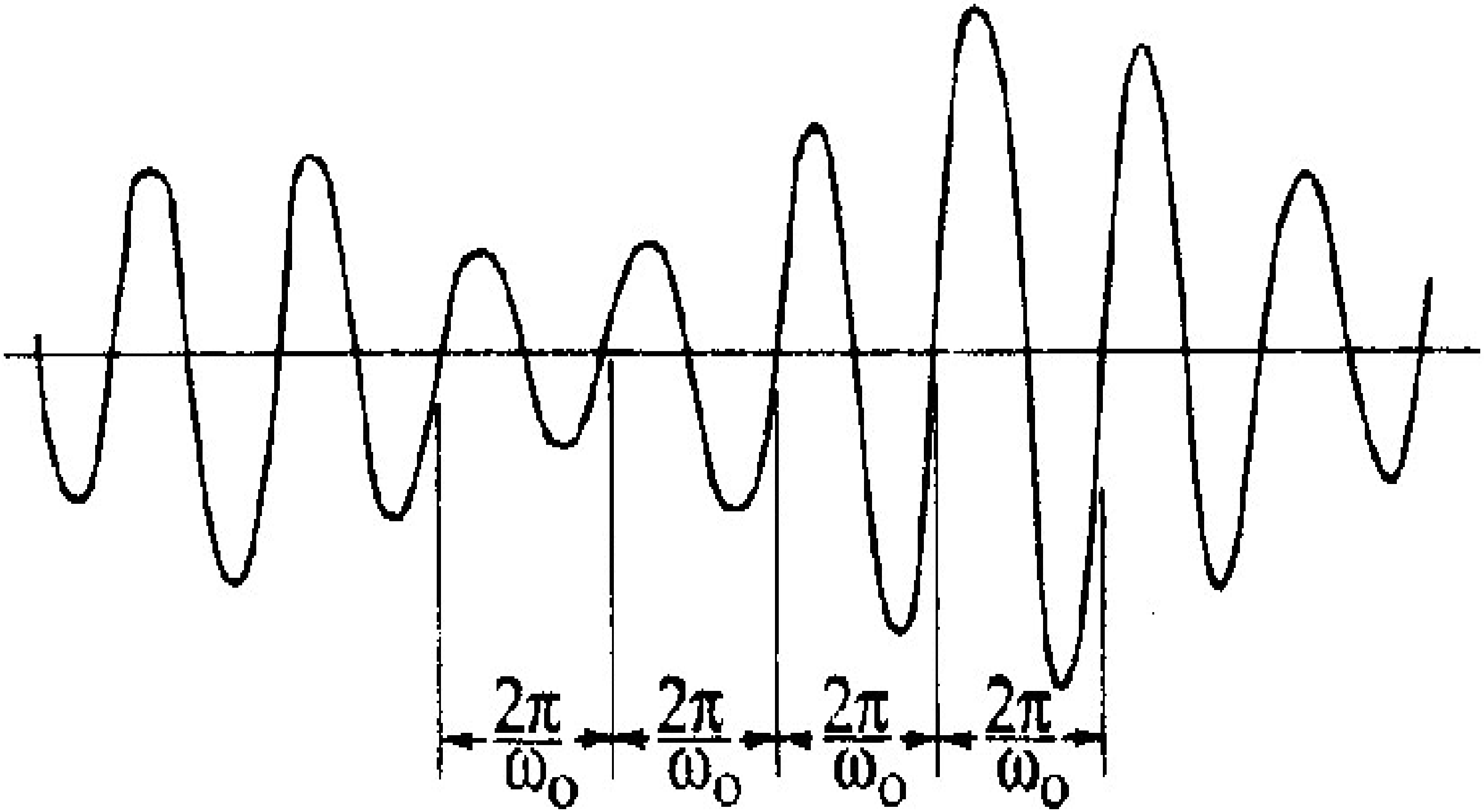
$$p + q = 1 \quad k = 1,2,\ldots,n; \quad n = 1,2,\ldots$$

2.4.7 Bernoulli Processes

- Poisson process and Bernoulli process are counting process.

2.4 Some Important Stochastic Processes

2.4.8 Narrow-Band Processes



Example of narrow-band process

2.4.8 Narrow-Band Processes

- *Def.1* Narrow-Band Process

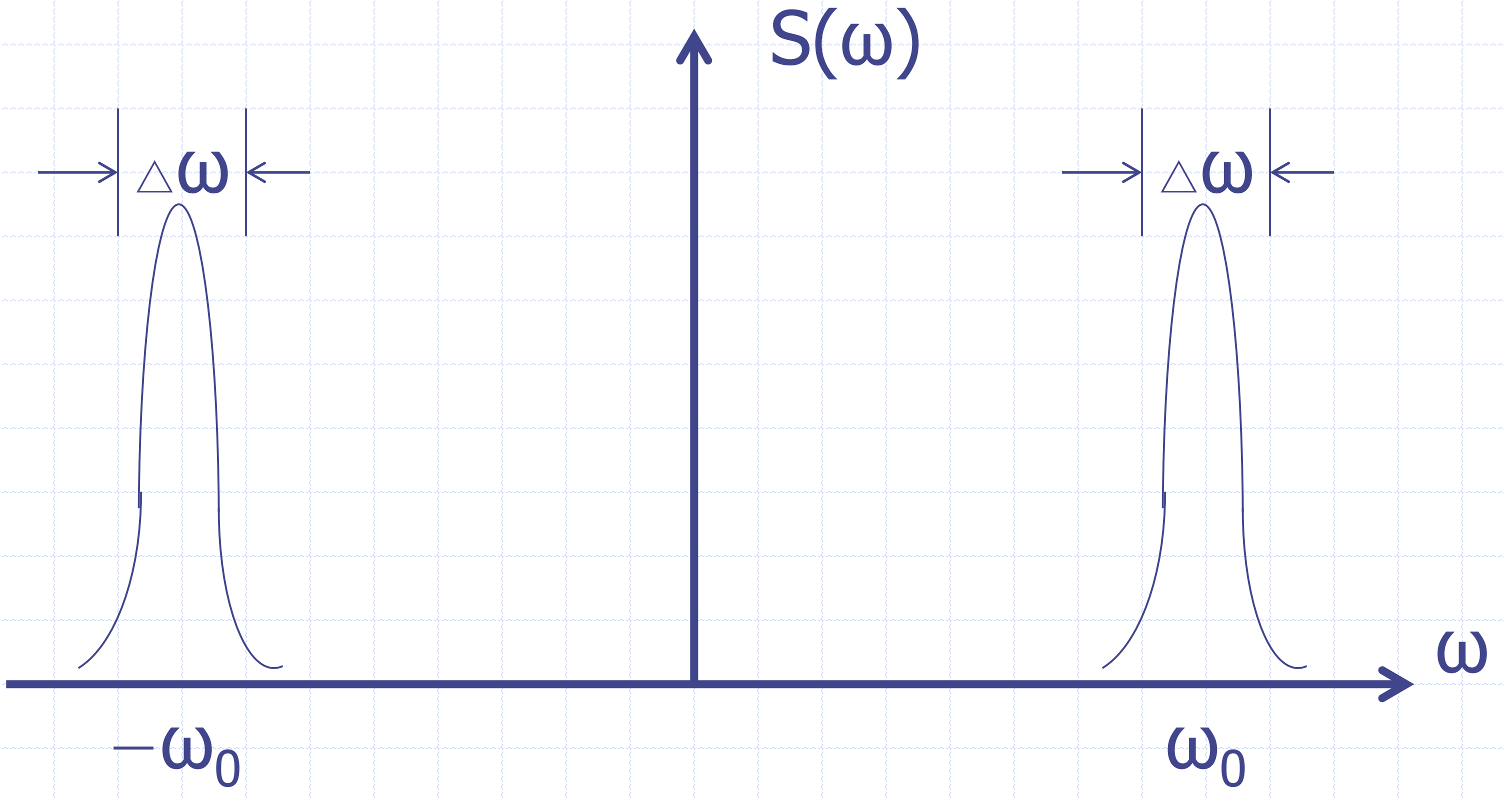
A continuous-state and continuous-time stationary stochastic process X(t) is called a narrow-band process if X(t) can be expressed by

$$X(t) = A(t)\cos\{\omega_0 t + \varepsilon(t)\}$$

Where ω_0=constant. The amplitude A(t) and the phase $\varepsilon(t)$ are random variables whose sample spaces are $0 \le A(t) < \infty$ and $0 \le \varepsilon(t) \le 2\pi$ respectively.

2.4.8 Narrow Band Processes

- Typical spectrum of a narrow band signal

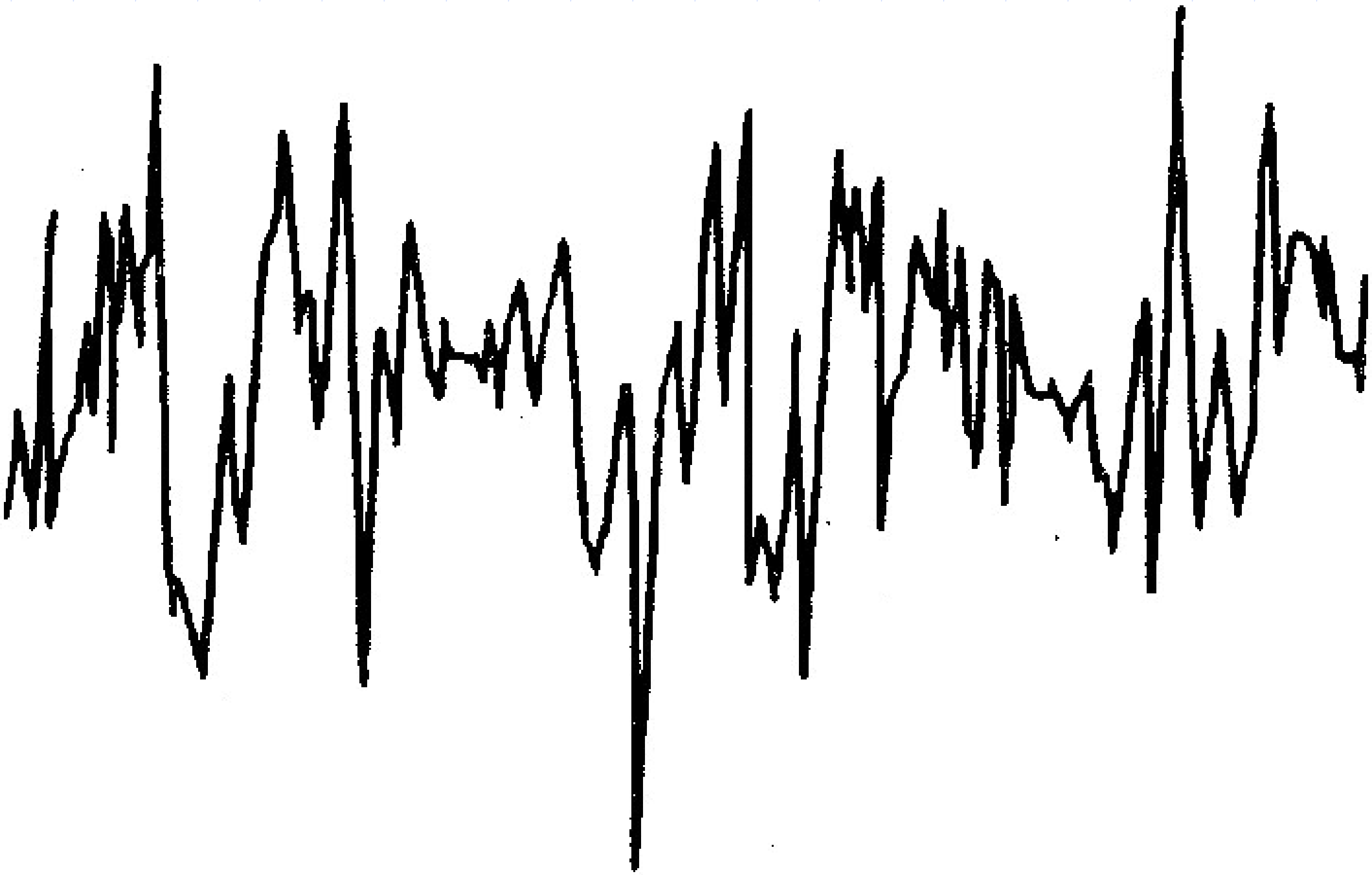


2.4.8 Narrow Band Processes

Wide-Band Processes

- A stochastic process that does not satisfy the condition of narrow-band process is called a non-narrow-band or wide-band process.

2.4.8 Narrow Band Processes



Example of a wide-band process

Homework

Self learning: Complex stochastic processes

Find

1. what is complex stochastic process?
2. What are it's moments?
3. What are the covariance function's properties of a complex stochastic process?