Fundamentals of Information Theory

Homework 2

Problem 1 *(10 points) Suppose you're on a game show, and you've given the choice of three doors: Behind one door is a car; behind the others, goat. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice so that you have a higher chance to win the car instead of the goat?*

Problem 2 *(15 points) One is given* 24 *coins. It is known that precisely one coin is fake, which weights differently compared with genuine coins. It is not clear whether this fake coin is heavier or lighter, though. Now we use a balance to identify the fake coin, but we do not have the weights for this balance.*

(a) What is the minimum number of the weighting operations in order to identify this fake coin.

(b) Explain briefly about your weighting procedure to identify this fake coin.

Problem 3 *(10 points)*

Let $p(x, y)$ be given in Table 1, i.e., $p(X = 0, Y = 1) = 0$.

Table 1: $p(x, y)$

X\Y	0	1
0	1/3	0
1	1/3	1/3

Find.

(a) $H(X)$, $H(Y)$.

(b) $H(X|Y)$, $H(Y|X)$.

(c) $H(X, Y)$.

(d) $H(Y) - H(Y|X)$.

(e) $I(X; Y)$.

(f) Draw a Venn diagram for the quantities in (a) through (e).

Problem 4 *(10 points) Let the random variable X have three possible outcomes $\{a, b, c\}$. Consider two distributions on this random variable:*

Symbol	$p(x)$	$q(x)$
a	*1/2*	*1/3*
b	*1/4*	*1/3*
c	*1/4*	*1/3*

Calculate $H(p(x))$, $H(q(x))$, $D(p(x)||q(x))$, and $D(q(x)||p(x))$. Verify that in this case, $D(p(x)||q(x)) \neq D(q(x)||p(x))$.

Problem 5 *(10 points)*

Consider a non-uniform random variable X with $M = 8$ possible outcomes and probabilities (1/2, 1/4, 1/8, 1/16, 1/64, 1/64, 1/64, 1/64)*. We use one symbol to represent one outcome, so we have 8 types of symbols (A, B, C, D, E, F, G, H). Then we transmit 1024 symbols from point s to point d. In this 1024-symbol sequence, it turns out that the numbers of these 8 type symbols are $(516, 255, 126, 68, 17, 18, 10, 14)$, respectively*

(a) Compute the entropy $H(X)$ in bits.

(b) Consider a coding scheme c_1, $A \to 000$, $B \to 001$, $C \to 010$, $D \to 011$, $E \to 100$, $F \to 101$, $G \to 110$, $H \to 111$, to transmit these 1024 symbols. Compute the total bits using scheme c_1.

(c) Similarly, consider another coding scheme c_2, $A \to 0$, $B \to 10$, $C \to 110$, $D \to 1110$, $E \to 11110$, $F \to 111110$, $G \to 1111110$, $H \to 11111110$, to transmit these 1024 symbols. Compute the total bits using scheme c_2. On average, how many bits are used to transmit one symbol in this transmitted sequence?

Problem 6 *(10 points) Let X_1 and X_2 be identically distributed but not necessarily independent. Let*

$$\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)}.$$

(a) Show that $\rho = \frac{I(X_1;X_2)}{H(X_1)}$.

(b) Show that $0 \le \rho \le 1$.

(c) When is $\rho = 0$?

(d) When is $\rho = 1$?

Problem 7 *(10 points)*

Let X, Y and Z be joint random variables. Prove the following inequalities and find conditions for equality.

(a) $H(X, Y|Z) \ge H(X|Z)$.

(b) $I(X, Y; Z) \ge I(X; Z)$.

(c) $H(X, Y, Z) - H(X, Y) \le H(X, Z) - H(X)$.

(d) $I(X; Z|Y) \ge I(Z; Y|X) - I(Z; Y) + I(X; Z)$.