Chapter 6 : Markov Processes and Discrete-Time Markov Chains

Dong Yan
EI. HUST.

OUTLINE

6. 1 Markov Processes

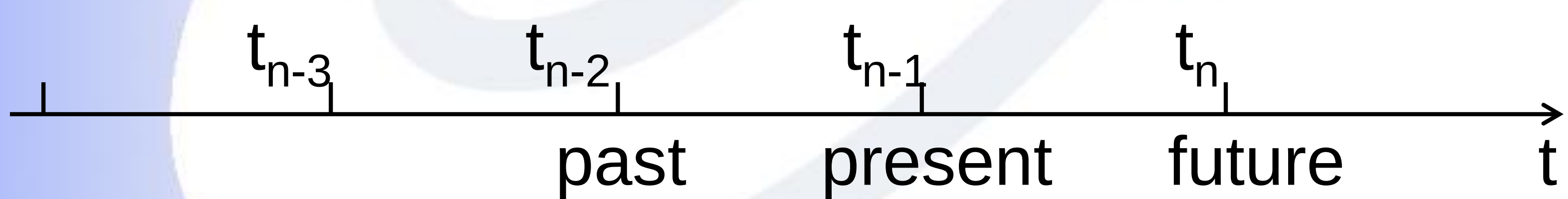
- Recall
  - Definition of the Markov process

2.2.4 Markov Processes

- A Markov process possess the Markovian property.
- The Markovian property has the following implication:

If t_n is a time point in the future, t_{n-1} the present time, and correspondingly t_1, t_2,…, t_{n-2}, time points in the past,

the future development of a process does not depend on its evolvement in the past, but only on its present state.



Def.1 Markov Process

A stochastic process X(t) is said to be a Markov process if it satisfies the following conditional probability:

$$
\begin{aligned}
&Pr\{x(t_n) \leqslant x_n | x(t_1) = x_1, x(t_2) = x_2, \ldots, x(t_{n-1}) = x_{n-1}\} \\
&\quad = Pr\{x(t_n) \leq x_n | x(t_{n-1}) = x_{n-1}\}, \qquad \text{where } t_1 < t_2 < \cdots \ t_{n-1} < t_n
\end{aligned}
$$

(Markovian property)

$$
\begin{aligned}
&Pr\{x(t_n) = x_n | x(t_1) = x_1, x(t_2) = x_2, \ldots, x(t_{n-1}) = x_{n-1}\} \\
&\quad = Pr\{x(t_n) = x_n | x(t_{n-1}) = x_{n-1}\}, \qquad \text{where } t_1 < t_2 < \cdots \ t_{n-1} < t_n
\end{aligned}
$$

Independent increment processes

V.S.

Markov processes

Every independent increment process is a Markov process, although there are many Markov processes that do not have independent increments.

- Wiener and Poisson process are Markov process.

6.1 Markov Processes

- Markov processes have enormous practical implications mainly for three reasons:

1. Many practical phenomena can be modeled by Markov processes;
2. The input necessary for their practical application is generally much more easily to provide than that for other classes of stochastic processes;
3. Computer algorithms are available for numerical evaluations.

6.1 Markov Processes

- The state and time of a Markov process can be discrete and continuous.

- Classification of Markov processes
  - A Markov process with a discrete state and discrete time is called a Markov chain.
  - A Markov process with a discrete state and continuous time is called a continuous-time Markov chain.
  - A Markov process with a continuous state is called a diffusion process.

6.2 Markov Processes

- Markovian Property writing in term of the probability density function:

$$f\{x(t_n)|x(t_1), x(t_2), \ldots, x(t_{n-1})\} = f\{x(t_n)|x(t_{n-1})\}$$

- Joint probability density function

$$\begin{aligned} f\{x(t_1), x(t_2), \ldots, x(t_n)\} &= f\{x(t_n)|x(t_1), x(t_2), \ldots, x(t_{n-1})\} \\ &\quad \times f\{x(t_1), x(t_2), \ldots, x(t_{n-1})\} \\ &= f\{x(t_n)|x(t_{n-1})\} \cdot f\{x(t_1), x(t_2), \ldots, x(t_{n-1})\} \end{aligned}$$

6.2 Markov Processes

- Similarly,

$$f\{x(t_1), x(t_2), \ldots, x(t_{n-1})\}$$

$$= f\{x(t_{n-1}) | x(t_{n-2})\} \cdot f\{x(t_1), x(t_2), \ldots, x(t_{n-2})\}$$

$$f\{x(t_1), x(t_2), \ldots, x(t_n)\} = f\{x(t_1)\} \prod_{r=2}^{n} f\{x(t_r) | x(t_{r-1})\}$$

$$P\{x(t_1), x(t_2), \cdots, x(t_n)\} = P\{x(t_1)\} \prod_{r=2}^{n} P\{x(t_r) | x(t_{r-1})\}$$

The statistical characteristic of Markov process decided by the initial condition and conditional probability density function or conditional probability .

6.2 Markov Processes

- Transition probability density:

$$f\{x(t_r)\mid x(t_{r-1})\}$$

- Transition probability:

$$P\{x(t_r)\mid x(t_{r-1})\}$$

- Homogeneity:

The transition probability density is invariant with time τ :

$$f\{x(t_r+\tau)\mid x(t_{r-1}+\tau)\}=f\{x(t_r)\mid x(t_{r-1})\}$$

or

$$P\{x(t_r+\tau)\mid x(t_{r-1}+\tau)\}=P\{x(t_r)\mid x(t_{r-1})\}$$

OUTLINE

6.2 Chapman-Kolmogorov Equation

- {X(t),t>=0} is a Markov process with density function *f*.
- In terms of transition probability density function

$t_0 < t_j < t$

$$f\{x(t), x_0(t_0)\} = \int f\{x(t), x_0(t_0), x_j(t_j)\}\, dx_j$$

$$= \int f\{x(t) | x_0(t_0), x_j(t_j)\} f\{x_0(t_0), x_j(t_j)\}\, dx_j$$

$$f\{x(t) | x_0(t_0), x_j(t_j)\} = f\{x(t) | x_j(t_j)\}$$

- Chapman-Kolmogorov Equation
- In terms of transition probability density function

$$f\{x(t), x_0(t_0)\} = \int f\{x(t)|x_j(t_j)\} f\{x_0(t_0), x_j(t_j)\}\, dx_j$$

$$f\{x(t)|x_0(t_0)\} = \int f\{x(t)|x_j(t_j)\} f\{x_j(t_j)|x_0(t_0)\}\, dx_j$$

6.2 Chapman-Kolmogorov Equation

- Chapman-Kolmogorov Equation
- In terms of transition probability

$$p\{x(t)|x_0(t_0)\} = \sum_j p\{x(t)|x_j(t_j)\}\, p\{x_j(t_j)|x_0(t_0)\}$$

OUTLINE

6.1 Markov Processes

6.2 Chapman-Kolmogorov Equation

6.3 Basic Concepts of Markov Chains(MC)

6.4 Classification of States

6.5 Ergodic MC and Stationary Distribution