Chap 4　Spectral Analysis of Stochastic Processes

Dong Yan

EI. HUST.

Chapter 4: Spectral Analysis

- References:

1. Stochastic Processes and Their Applications, Frank E. Beichelt, O211. W138
   Chap 8, Spectral analysis of stationary processes
2. Probability, Random Variables and Random Signal Principles, 2nd Edition, Peyton Z. Peebles, O211 W58/2
   Chap 7, Spectral characteristic of random processes

Chapter 4: Spectral Analysis

Content:

4.1 Spectral Density Functions

Content:

4.1.1 Autospectral Density Functions

4.1.2 Wiener-Khintchine Theorem

4.1.3 Crossspectral Density Functions

4.1.4 S.D.F. of Derived Random Processes

Average of the power

- If X(t) is ergodic, then

$$R_{XX}(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt$$

- R(0) represents the time average of the power of a random process X(t).
- *Def.1* average of the power

$$\overline{P}_X = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt$$

- Parseval theorem

$$\boxed{\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^2(\omega)d\omega}$$

- The representation of the average energy in terms of time and frequency of a signal.
- Thus,

$$\overline{P}_X = \lim_{T\to\infty} \frac{1}{4\pi T}\int_{-T}^{T} X^2(\omega)d\omega$$

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1.1 Autospectral Density Functions

Def.2 Spectral Density Function (S.D.F.)

- The S.D.F. of a random process X(t) is

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} |X_T(\omega)|^2 \qquad S_{XX}(f) = \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2$$

- (autospectral density function) (Power Spectrum)
- From Def.1 and Def. 2,

$$\begin{aligned}\overline{P}_X &= \lim_{T\to\infty} \frac{1}{4\pi T} \int_{-T}^{T} X^2(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = \frac{1}{\pi} \int_{0}^{\infty} S_{XX}(\omega) d\omega \\ &= \int_{-\infty}^{\infty} S_{XX}(f) df = 2\int_{0}^{\infty} S_{XX}(f) df\end{aligned}$$

Definitions associated with the S.D.F.

1. The *k*th moment of the S.D.F.

$$m_k = \int_0^{\infty} \omega^k S_{XX}(\omega) d\omega$$

2. Modal frequency ω_m

   the frequency where the S.D.F. peaks.

3. Mean frequency

$$\overline{\omega}_X = \frac{\int_0^{\infty} \omega S_{XX}(\omega) d\omega}{\int_0^{\infty} S_{XX}(\omega) d\omega} = \frac{m_1}{m_0}$$

   the mean frequency is not equal to the average value of the various frequencies involved in a random process; it is the mathematical mean of the S.D.F.

Definitions associated with the S.D.F.

4. The kth moment about the mean of the S.D.F.

$$\mu_k = \int_0^{\infty} (\omega - \overline{\omega})^k S_{XX}(\omega) d\omega$$

Thus,

$$\mu_0 = m_0 = \int_0^{\infty} S_{XX}(\omega) d\omega = \pi \overline{P}_X$$

$$\mu_1 = 0$$

$$\mu_2 = m_2 - \frac{m_1^2}{m_0}$$

Definitions associated with the S.D.F.

5. The bandwidth parameter of the S.D.F.

$$\varepsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}}$$

$\varepsilon = 0$, the random process has a narrow-band spectrum;

$\varepsilon = 1$, the random process has a wide-band spectrum;

Definitions associated with the S.D.F.

6. The spectral width parameter of the S.D.F.

$$\nu = \sqrt{\frac{\mu_2}{\mu_0}\frac{1}{\overline{\omega}}}$$

7. The spectral peakedness parameter

$$Q_P = \frac{\int_0^{\infty} \omega S^2{}_{XX}(\omega) d\omega}{\left(\int_0^{\infty} S_{XX}(\omega) d\omega\right)^2}$$

The value increases with increasing sharpness of the S.D.F.

4.1.1 Autospectral Density Functions

- Properties of the S.D.F.

i) Nonnegative $S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} |X_T(\omega)|^2 \geq 0$

ii) Real

iii) Even, if X(t) is real, $S_{XX}(\omega) = S_{XX}(-\omega)$

iv) If $\int_{-\infty}^{\infty} |R_{XX}(\tau)| \, d\tau < \infty$, then $S_{XX}(\omega)$ is a continuous function of ω

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1 Spectral Density Functions

Content:

4.1.1 Autospectral Density Functions

4.1.2 Wiener-Khintchine Theorem

4.1.3 Crossspectral Density Functions

4.1.4 S.D.F. of Derived Random Processes

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1.2 Wiener-Khintchine Theorem

- The Fourier transform of the correlation function of an ergodic process:

$$\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau = \lim_{T\to\infty}\frac{1}{2T}|X_T(\omega)|^2$$

- And the definition of S.D.F.,

$$S_{XX}(\omega) = \lim_{T\to\infty}\frac{1}{2T}|X_T(\omega)|^2$$

- Obtain,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega$$

Theorem: Wiener-Khintchine Theorem

For a weakly stationary random process X(t), its correlation function and the power spectrum are Fourier transform pair.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$

Standard Fourier Transform

$$F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwt} dw$$

4.1.2 Wiener-Khintchine Theorem

- Both correlation function and power spectrum are real and even functions.

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau$$

$$= 2\int_{0}^{\infty} R_{xx}(\tau) \cos \omega t \, d\tau$$

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{\pi} \int_{0}^{\infty} S_{xx}(\omega) \cos \omega\tau \, d\omega$$

4.1.2 Wiener-Khintchine Theorem

- The average power and correlation function

$$\frac{1}{\pi}\int_0^{\infty} S_{XX}(\omega)d\omega = \overline{P}_X = R_{XX}(0) = E[X^2(t)]$$

- If E[X(t)]=0,

$$Var[X(t)] = \frac{1}{\pi}\int_0^{\infty} S_{XX}(\omega)d\omega = \overline{P}_X = R_{XX}(0)$$

4.1.2 Wiener-Khintchine Theorem

Example: White Noise

The process with autocorrelation function,

$$R(\tau) = A\delta(\tau)$$

and power spectrum,

$$S(\omega) = A = \text{constant}$$

The continuous white noise is a real, stationary, continuous-time stochastic process with constant spectral density.

the "most random" stochastic process

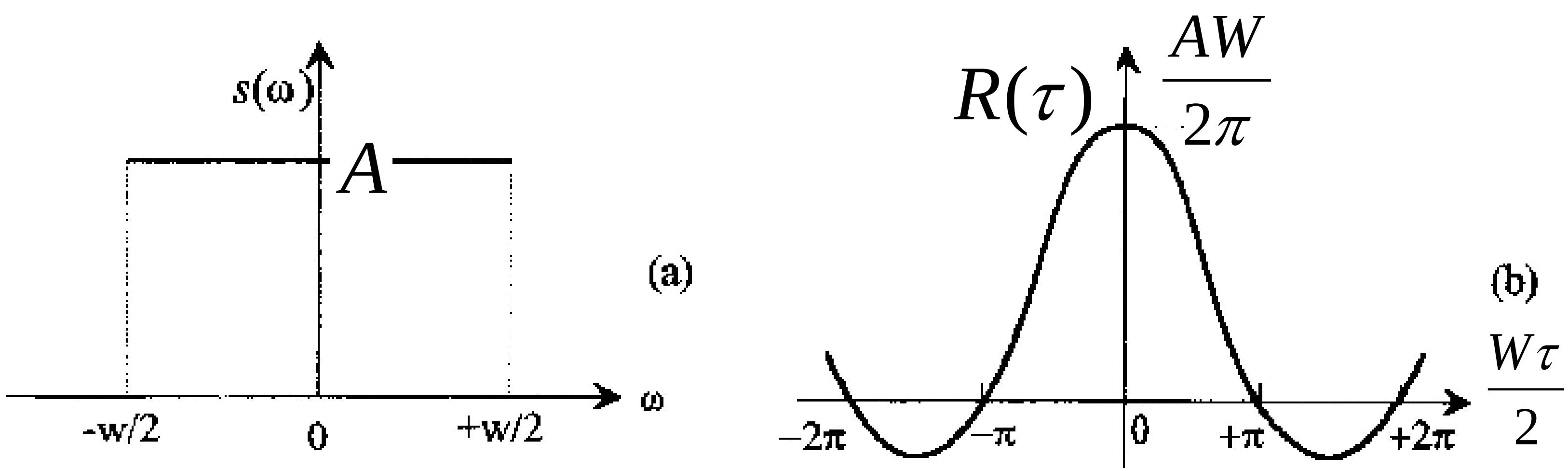
4.1.2 Wiener-Khintchine Theorem

Example 2. Band-limited White Noise

$$S(\omega) = A, \text{ where } -\frac{W}{2} < \omega < \frac{W}{2}$$

Thus,

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-W/2}^{W/2} Ae^{i\omega\tau} d\omega = \frac{A}{\pi\tau}\sin\frac{1}{2}W\tau$$



Review 2.2.1 Stationary Processes

- weakly stationary

Example 1. Random phase processes

X(t)=Acos(wt+ε)， t>0, whereas A and w are constants and ε is random variable uniformly distributed between –π and π.

$$E[x(t)] = 0$$

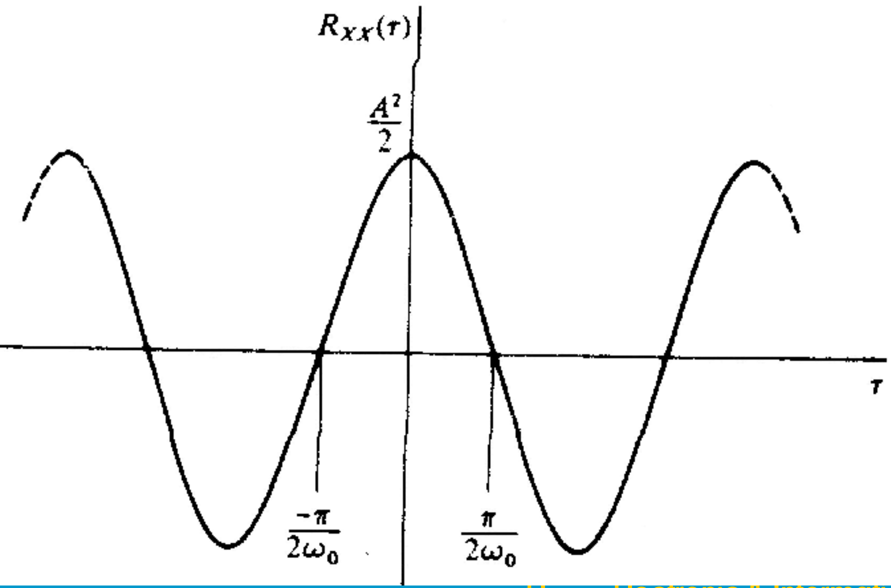
$$C_{XX}(t_1,t_2) = \frac{A^2}{2}\cos w_0(t_2 - t_1)$$

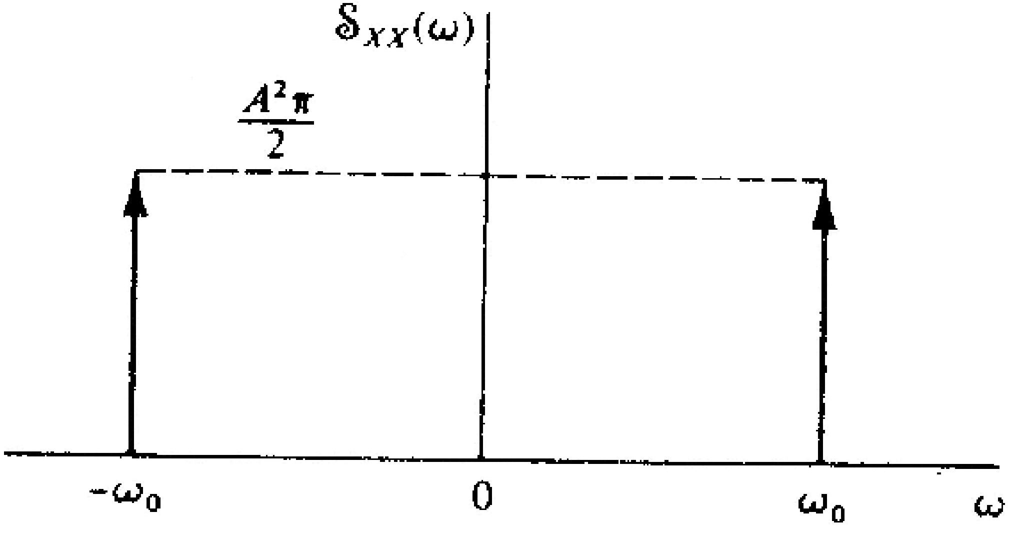
$$= \frac{A^2}{2}\cos w_0\tau = C_{XX}(\tau)$$

Example 3. Random Phase Processes

$$R_{XX}(\tau) = \frac{A^2}{2}\cos w_0 \tau$$

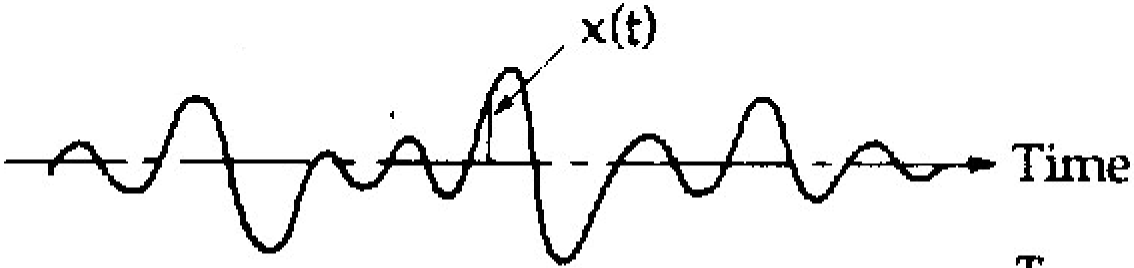
Obtain: $$S(\omega) = \frac{A^2 \pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$





4.1.2 Wiener-Khintchine Theorem

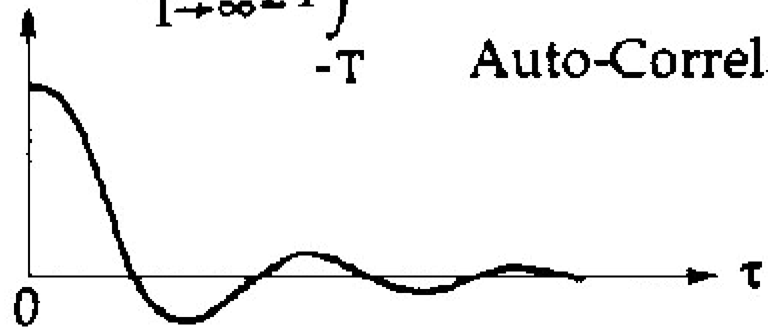
- Summary



$$\text{Ave. Energy } P_x = \frac{1}{2T}\int_{-T}^{T}\{x(t)\}^2\,dt$$

$$R_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau)\,dt$$

Auto-Correlation Function

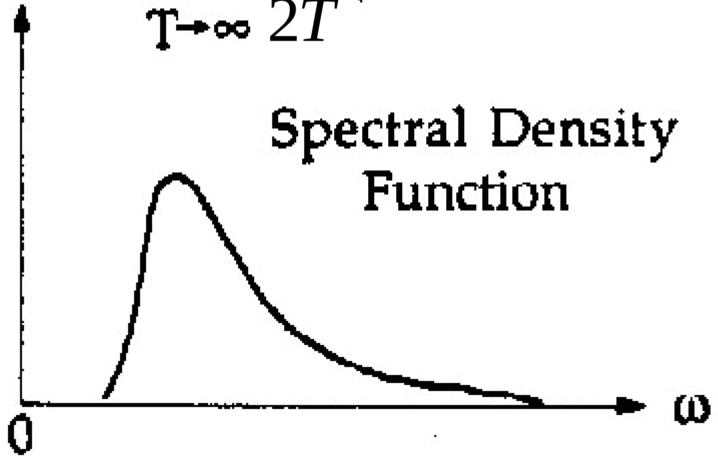


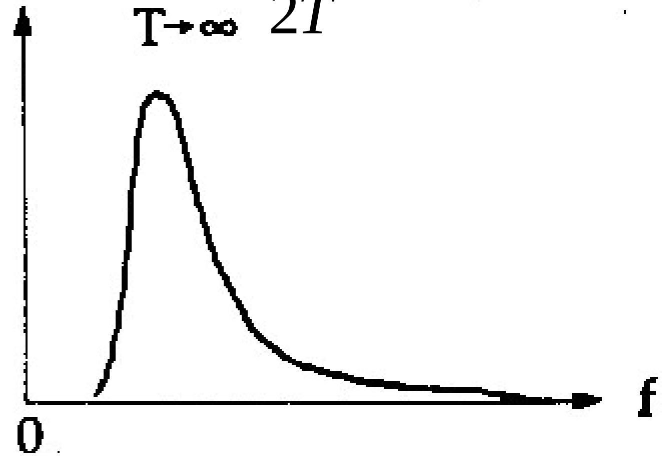
(Fourier Transform)

4.1.2 Wiener-Khintchine Theorem

- Summary

$$S_{xx}(\omega)=\lim_{T\to\infty}\frac{1}{2T}|X(\omega)|^2 \qquad S_{xx}(f)=\lim_{T\to\infty}\frac{1}{2T}|X(f)|^2$$





$$\frac{1}{\pi}\int_0^{\infty} S_{xx}(\omega)\,d\omega = 2\int_0^{\infty} S_{xx}(f)\,df = R_{xx}(0) = Var[x(t)] = \bar{P}_x$$

4.1.2 Wiener-Khintchine Theorem

- Summary

$R_X(\tau)$	$S_X(\omega)$
$\begin{cases} (T-\lvert\tau\rvert)/T, & \lvert\tau\rvert < T, \\ 0, & \text{otherwise} \end{cases}$	$T\{\sin(\omega T/2)/(\omega T/2)\}^2 = T\,\mathrm{sinc}^2(fT)$
1	$2\pi\delta(\omega)$
$\delta(\tau)$	1
$\exp(-\alpha\lvert\tau\rvert)$	$2\alpha/(\alpha^2+\omega^2)$
$\cos(\omega_0\tau)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\exp(-\alpha\lvert\tau\rvert)\cos(\omega_0\tau)$	$\alpha/[\alpha^2+(\omega-\omega_0)^2]\}+\{\alpha/[\alpha^2+(\omega+\omega_0)^2]\}$
$2W\,\mathrm{sinc}(2W\tau)=\sin(2\pi W\tau)/\pi\tau$	$\begin{cases} 1, & \lvert\omega\rvert \le 2\pi W \\ 0, & \text{otherwise} \end{cases}$

$$\mathrm{sinc}(u)=\sin(\pi u)/\pi u, -\infty < u < \infty$$

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1.2 Wiener-Khintchine Theorem

e.g. Suppose the continuous-time random process X(t) has autocorrelation function

$$R_X(\tau) = 1 + e^{-\alpha|\tau|}, \quad -\infty < \tau < \infty, \qquad \alpha > 0$$

Find the spectral density function for X(t).

$$S_X(\omega) = 2\pi\delta(\omega) + \frac{2\alpha}{\alpha^2 + \omega^2}$$

4.1 Spectral Density Functions

Content:

1. Definition of Cross-Spectral Density Function (CS.D.F)

Cross-Spectral Density Function of two random processes X(t) and Y(t) is

$$\boxed{\tau = t_1 - t_2}$$

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} X(\omega)Y^*(\omega)$$

$$S_{YX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} X^*(\omega)Y(\omega)$$

4.1.3 Cross-Spectral Density Functions

Theorem: Wiener-Khintchine Theorem

For two jointly stationary random processes X(t) and Y(t), their cross-correlation function and cross-power density function are Fourier transform pair.

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau \qquad S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-i\omega\tau} d\tau$$

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\omega\tau} d\omega \qquad R_{YX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega) e^{i\omega\tau} d\omega$$

Standard Fourier Transform

$$F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwt} dt$$

4.1.3 Cross-Spectral Density Functions

2. Real part and imaginary part

$$S_{XY}(\omega)=\int_{-\infty}^{\infty}R_{XY}(\tau)e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty}R_{XY}(\tau)\cos\omega\tau d\tau-i\int_{-\infty}^{\infty}R_{XY}(\tau)\sin\omega\tau d\tau$$

$$=C_{XY}(\omega)+iQ_{XY}(\omega)$$

$$C_{XY}(\omega)=\int_{-\infty}^{\infty}R_{XY}(\tau)\cos\omega\tau d\tau$$

$$=\int_{-\infty}^{0}R_{XY}(\tau)\cos\omega\tau d\tau+\int_{0}^{\infty}R_{XY}(\tau)\cos\omega\tau d\tau$$

$$=\int_{0}^{\infty}R_{XY}(-\tau)\cos\omega\tau d\tau+\int_{0}^{\infty}R_{XY}(\tau)\cos\omega\tau d\tau$$

$$=\int_{0}^{\infty}[R_{YX}(\tau)+R_{XY}(\tau)]\cos\omega\tau d\tau$$

4.1.3 Cross-Spectral Density Functions

Imaginary part

$$Q_{YX}(\omega) = -\int_{-\infty}^{\infty} R_{XY}(\tau)\sin\omega\tau d\tau$$

$$= \int_{0}^{\infty} [-R_{XY}(\tau) + R_{YX}(\tau)]\sin\omega\tau d\tau$$

Amplitude Spectrum:

$$|S_{XY}(\omega)| = \sqrt{C_{XY}^{2}(\omega) + Q_{XY}^{2}(\omega)}$$

Phase Spectrum:

$$\varepsilon_{XY}(\omega) = \tan^{-1}\frac{Q_{XY}(\omega)}{C_{XY}(\omega)}$$

4.1.3 Cross-Spectral Density Functions

3. Properties of Cross-Spectral Density Function

i) $C_{XY}(\omega)$ and $C_{YX}(\omega)$ are even functions

ii) $Q_{XY}(\omega)$ and $Q_{YX}(\omega)$ are odd functions

iii) $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}^{*}(\omega)$

iv) If X(t) and Y(t) are jointly stationary random processes and uncorrelated,

$$S_{XY}(\omega) = S_{YX}(\omega) = 2\pi m_X m_Y \delta(\omega)$$

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1.3 Cross-Spectral Density Functions

4. Coherency function

$$\gamma_{XY}(\omega) = \frac{|S_{XY}(\omega)|^2}{S_{XX}(\omega)S_{YY}(\omega)} = \frac{C_{XY}^2(\omega) + Q_{XY}^2(\omega)}{S_{XX}(\omega)S_{YY}(\omega)}$$

Mutual correlation coefficients

$$\rho_{XY}[t_1, t_2] = \frac{C_{XY}(t_1, t_2)}{\sqrt{Var[X(t_1)]Var[Y(t_2)]}}$$

$$0 \le \gamma_{XY}(\omega) \le 1$$

$\gamma_{XY}(\omega) \equiv 1$ if and only if X(t) and Y(t) are exactly linearly related.

Example 1.

Given: a cross-spectral density function

$$S_{XY}(\omega) = \begin{cases} a + i\dfrac{b\omega}{\omega_0} & -\omega_0 < \omega < \omega_0 \\ 0 & otherwise \end{cases}$$

where $\omega_0 > 0$, a and b are real constants.

Obtain: the cross-correlation function.

4.1 Spectral Density Functions

Content:

4.1.1 Autospectral Density Functions

4.1.2 Wiener-Khintchine Theorem

4.1.3 Crossspectral Density Functions

4.1.4 S.D.F. of Derived Random Processes

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1.4 S.D.F. of Derived Random Processes

1. Autocorrelation function of derived random processes

$$R_{x\dot{x}}(t_1, t_2) = \frac{\partial}{\partial t_2} R_{xx}(t_1, t_2) = -\frac{d}{d\tau} R_{xx}(\tau)$$

$$R_{\dot{x}\dot{x}}(\tau) = \frac{\partial^2}{\partial t_1 \, \partial t_2} R_{xx}(t_1, t_2) = -\frac{d^2}{d\tau^2} R_{xx}(\tau)$$

$$R_{\ddot{x}\ddot{x}}(\tau) = \frac{d^4}{d\tau^4} R_{xx}(\tau)$$

4.1.4 S.D.F. of Derived Random Processes

2. Spectral density function of derived random processes

$$\mathscr{F}\{x(t)\} = X(\omega)$$

$$\mathscr{F}\{\dot{x}(t)\} = \dot{X}(\omega) = i\omega X(\omega)$$

$$\mathscr{F}\{\ddot{x}(t)\} = \ddot{X}(\omega) = -\omega^2 X(\omega)$$

$$S_{\dot{x}\dot{x}}(\omega) = \lim_{T\to\infty} \frac{1}{2T} |\dot{X}(\omega)|^2$$

$$= \omega^2 \lim_{T\to\infty} \frac{1}{2T} |X(\omega)|^2 = \omega^2 S_{xx}(\omega)$$

4.1.4 S.D.F. of Derived Random Processes

2. Spectral density function of derived random processes

$$S_{\ddot{x}\ddot{x}}(\omega) = \lim_{T\to\infty} \frac{1}{2T} |\ddot{X}(\omega)|^2$$

$$= \omega^4 \lim_{T\to\infty} \frac{1}{2T} |X(\omega)|^2 = \omega^4 S_{xx}(\omega)$$

Homework

4.1

4.2

4.7