Homework 2

Problem 1: the original probability is $\frac{1}{3}$ for each one, we choose No.1, we have $P(A_1=1)=\frac{1}{3}$ for winning, $\frac{2}{3}$ for losting. But when we know No.3 is goat, then if we choice changes to No.2 then we have $\frac{2}{3}$ for winning, because the No.3 ~~probabili~~ uncertainty reduces to zero. All in all, we'd better pick door No.2

problem 2

(a) there are $24 \times 4 = \cancel{48}\ 96$ kinds of outcomes, which has $\log_2 \cancel{48}\ 96 = \cancel{5.6}\ 6.58$ Bits.
For a single weight operation, we have three outcomes, which decrease $\log_2 3 = 1.58$ Bits
So all we have to make the weighting operations is $\left\lceil \frac{\log_2 96}{\log_2 3} \right\rceil = 5$

(b) ① fetch 8 coins to the left, 8 coins to the right. if balance, the fake
2 coin is in the left coins, or we have test the left, the right each time
② For the 8 coins, fetch 3 to the left, 3 to the right. if balance, the fake
1 coin is in the other 2 coins. check 2 coins each time. totally $1+2+2=5$ ~~times~~
③ if not balance, mix all the 6 coins and fetch 2 to left side, 2 to right side
if balance, the fake is in the other 2, if not balance, put the left side 2 coins
on the two sides of balance. if balance the fake is in the two coins. ~~fa~~ if not
1 the fake is in the right side two coins.
④ For the 2 coins, put a true coin to one side of balance, other side put
1 any one of the 2 coins. if balance, the other is fake, if not this one ~~is~~ is fake
Totally, we have five times' operation

Problem 3

(a) $H(X) = \frac{1}{3}\log 3 + \frac{2}{3}(\log 3 - \log 2) = \log 3 - \frac{2}{3}\log 2 = 0.918$ bit.

$H(Y) = \frac{1}{3}\log 3 + \frac{2}{3}(\log 3 - \log 2) = \log 3 - \frac{2}{3}\log 2 = 0.918$ bit

1b) $H(X|Y) = \frac{1}{3}\cdot\log 2 + \frac{1}{3}\log 2 + 1\cdot\log 1 = \frac{2}{3} = 0.67$ bits

$H(Y|X) = 1\times\log 1 + \frac{1}{3}\log 2 + \frac{1}{3}\log 2 = \frac{2}{3} = 0.67$ bits

1c) $H(X,Y) = \frac{1}{3}\log 3 + \frac{1}{3}\log 3 + \frac{1}{3}\log 3 = 1.5849$ bits

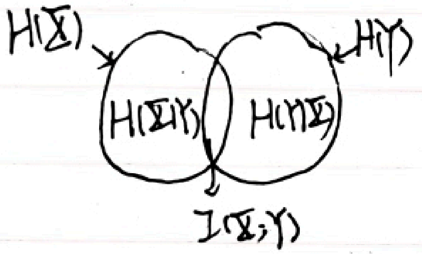
1d) $H(Y) - H(Y|X) = 0.918 - 0.67 = 0.252$ bits

1e) $I(X;Y) = \frac{1}{3}\cdot\log(\frac{1}{3}/\frac{2}{3}\cdot\frac{1}{3}) + \frac{1}{3}\cdot\log(\frac{1}{3}/\frac{2}{3}\cdot\frac{2}{3}) + \frac{1}{3}\log(\frac{1}{3}/\frac{2}{3}\times\frac{1}{3})$

$= \frac{1}{3}\log(\frac{3}{2}) + \frac{1}{3}\log\frac{3}{4} + \frac{1}{3}\log\frac{3}{2}$

$= \log_2 3 - \frac{4}{3}\log_2 2 = 0.252$ bits

f)



problem 4

sol: $H(p(x)) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4 = \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$ bits

$H(q(x)) = \frac{1}{3}\log_2 3 + \frac{1}{3}\log 3 + \frac{1}{3}\log 3 = \log_2 3 = 1.585$ bits

$D(p(x)||q(x)) = \frac{1}{2}\log\frac{3}{2} + \frac{1}{4}\log\frac{3}{4} + \frac{1}{4}\log\frac{3}{4} = \log 3 - \frac{3}{2}\log 2 = 0.085$ bits

$D(q(x)||p(x)) = \frac{1}{3}\log\frac{2}{3} + \frac{1}{3}\log\frac{4}{3} + \frac{1}{3}\log\frac{4}{3} = \frac{5}{3}\log_2 2 - \log_2 3 = 0.0817$ bits

So we can have $D(p(x)||q(x)) \neq D(q(x)||p(x))$

problem 5

1a) $H(X) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{16}\log 16 + \frac{4}{64}\log_2 64 = \frac{1}{2}+\frac{1}{2}+\frac{3}{8}+\frac{1}{4}$
$+ \frac{1}{16}\cdot 6 = 2$

1b) $1024\times 3 = 3072$ bits. the total bits using scheme C_1 is 3072

1c) $1\times 516 + 2\times 255 + 3\times 126 + 4\times 68 + 5\times 17 + 6\times 18 + 7\times 10 + 8\times 14 = 2051$ bits

$\frac{2051}{1024} = 2.003$ bits ≈ 2 bits, on average. one symbol needs 2 bits.

problem 6

(a) $\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)} = \frac{H(X_1)-H(X_2|X_1)}{H(X_1)} = \frac{H(X_2)-H(X_2|X_1)}{H(X_1)}$ identically distributed (X_1 and X_2 are

$\Rightarrow \rho = \frac{I(X_2;X_1)}{H(X_1)} = \frac{I(X_1;X_2)}{H(X_1)}$

(b) we can have $0 \le I(X_1;X_2) \le H(X_1) \Rightarrow 0 \le \rho \le 1$

(c) $\rho = 0 \Rightarrow I(X_1;X_2) = 0 \Rightarrow X_1$ and X_2 are independent

(d) $\rho = 1 \Rightarrow I(X_1;X_2) = H(X_1) \Rightarrow X_1$ and X_2 are the same

problem 7

(a) $H(X,Y|Z) = H(X|Z) + H(Y|X,Z)$ while $H(Y|X,Z) \geq 0$

So we have $H(X,Y|Z) \geq H(X|Z)$, where $H(Y|X,Z)=0$ for equality.

(b) $I(X,Y;Z) = I(X;Z) + I(Y;Z|X) \Rightarrow I(X,Y;Z) \geq I(X;Z)$

since $I(Y;Z|X) \geq 0$, where $I(Y;Z|X) = 0$ for equality.

(c) $H(X,Y,Z) = H(X,Z) + H(Y|X,Z)$, $H(X,Y) = H(Y|X) + H(X)$

So we have $H(X,Y,Z) - H(X,Y) = H(Y|X,Z) + H(X,Z) - H(X) - H(Y|X)$

while $H(Y|X,Z) - H(Y|X) \le 0 \Rightarrow H(X,Y,Z) - H(X,Y) \le H(X,Z) - H(X)$

where $H(X,Z) = H(X)$. for equality.

(d) $I(X;Z|Y) = H(X|Y) - H(X|Y,Z)$; $I(Z;Y|X) - I(Z;Y) + I(X;Z)$

$= H(Z|X) - H(Z|X,Y) - H(Z) + H(Z|Y) + H(X) - H(X|Z)$

$= H(Z|Y) - H(Z|X,Y)$

So we should prove $H(X|Y) - H(X|Y,Z) \geq H(Z|Y) - H(Z|X,Y)$

$\Rightarrow H(X|Y) + H(Z|X,Y) \geq H(Z|Y) + H(X|Y,Z)$

while $H(X,Y,Z) = H(Y) + H(X|Y) + H(Z|X,Y) = H(Y) + H(Z|Y) + H(X|Y,Z)$

So we have $H(X|Y) + H(Z|X,Y) = H(Z|Y) + H(X|Y,Z)$

$\Rightarrow I(X;Z|Y) = I(Z;Y|X) - I(Z;Y) + I(X;Z)$ for any time.