Chapter 2
Stochastic Processes

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EI HUST

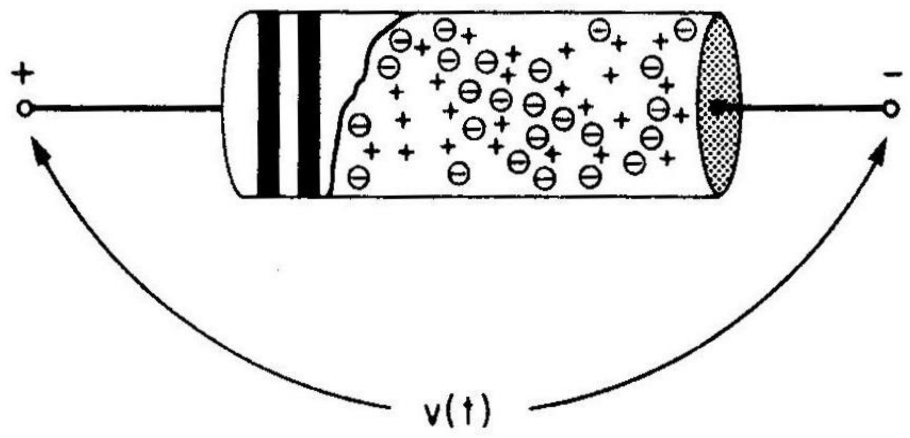
Chapter 2: Stochastic Processes

2.1 Basic Concepts

- 2.1.1 Definition and examples
- 2.1.2 Types of Stochastic Process
- 2.1.3 Distribution and Density Functions
- 2.1.4 Moments (Functions)

2.1.1 Definition and examples

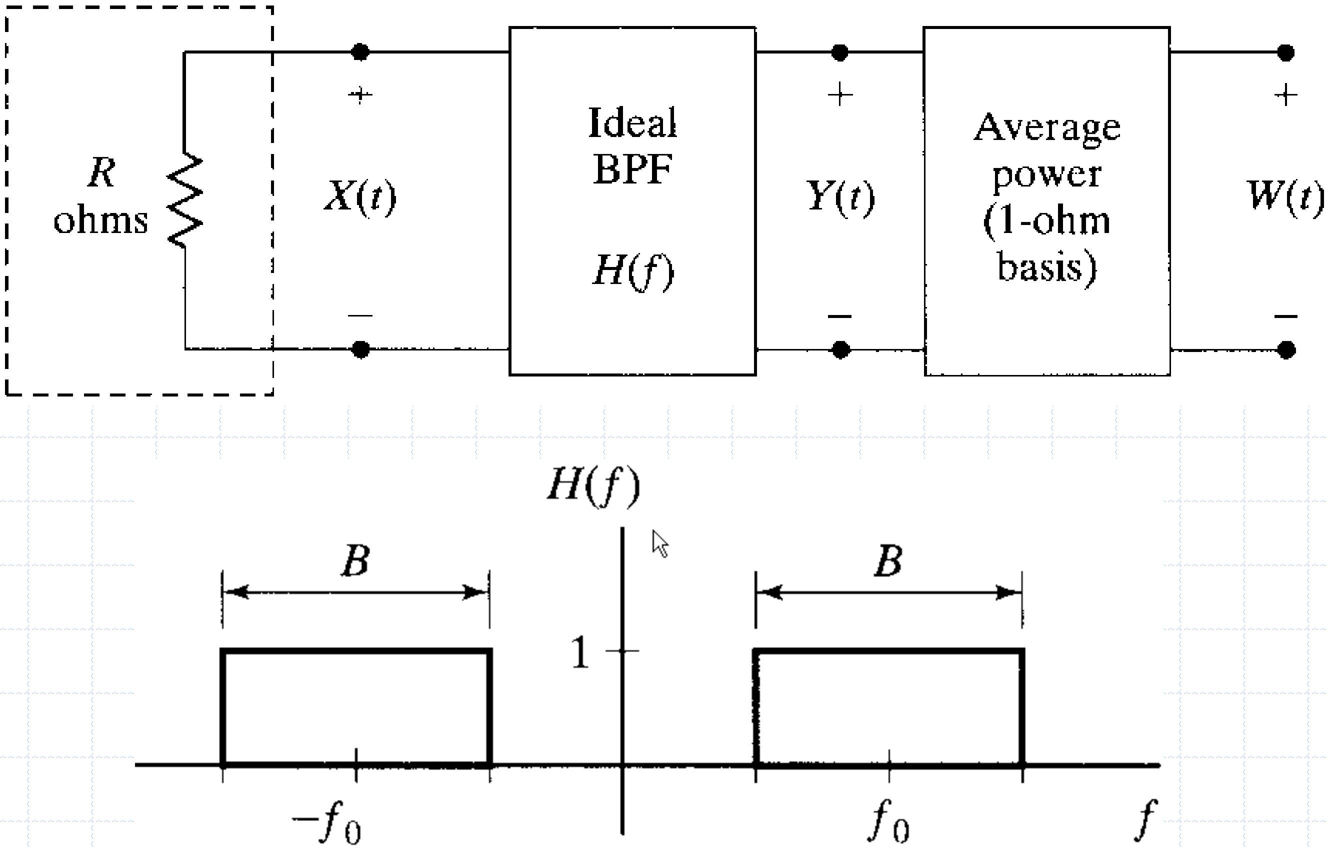
- Random Experiments: Measure thermal noise voltage at a resistor for a period of time.



2.1.1 Definition and examples

A possible measure system

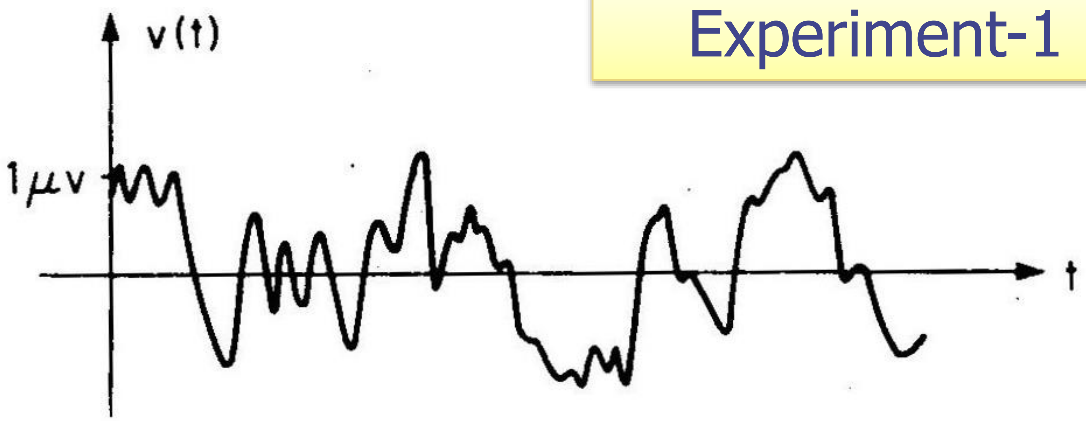
Temperature = $T°$ K



(b) Band-pass filter transfer function

2.1.1 Definition and examples

Experiment-1



Thermal Noise Voltage at a Resistor

the power W(t) settles down to a value close to

$$w_0 = 4kTRB,$$

k=1.38X10^{-23}joule/K (Boltzmann's constant);

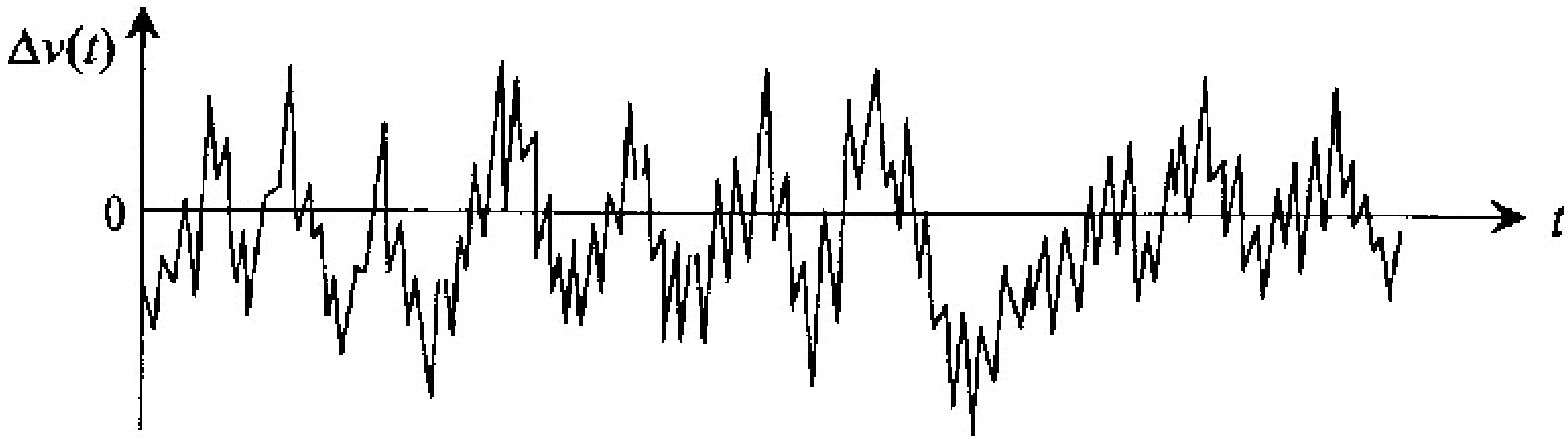
T, temperature; R, resistance;

B, bandwidth of the circuit.

2.1.1 Definition and examples

Experiment-2

Thermal noise with higher temperature

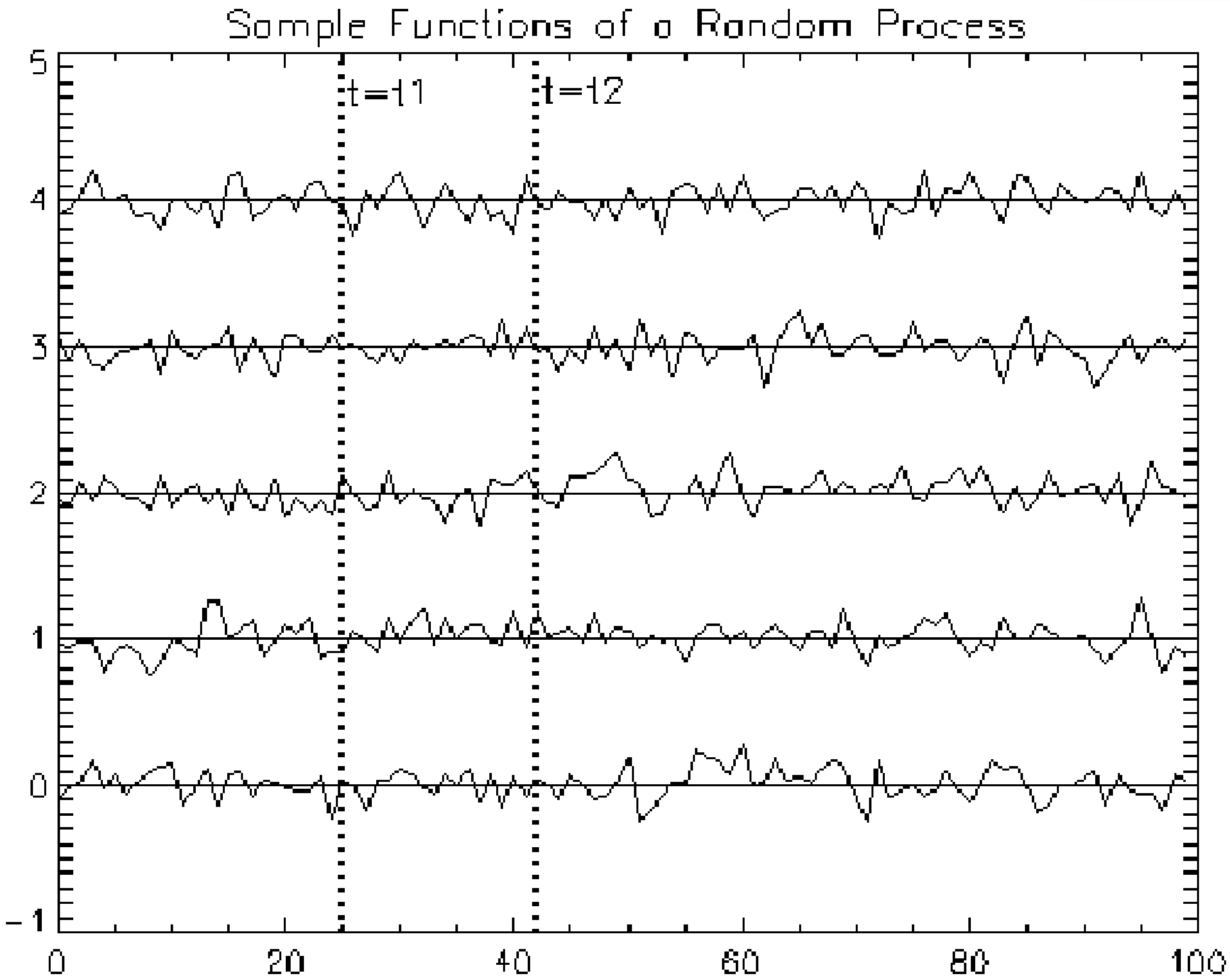


the voltage fluctuations in an electrical circuit around a nominal value caused by thermal noise under high temperature

2.1.1 Definition and examples

Sample Waves = Sample functions:

In experiment-1



2.1.1 Definition and examples

- *Def.* A family of random variables *X(t)* on probability space (Ω ,F, P) where *t* is a parameter belonging to an index set T is called a stochastic process, and is denoted by $\{X(t), t \in T\}$

Sample function X(t) : a time wave about the observed random phenomenon (a realization of the process).

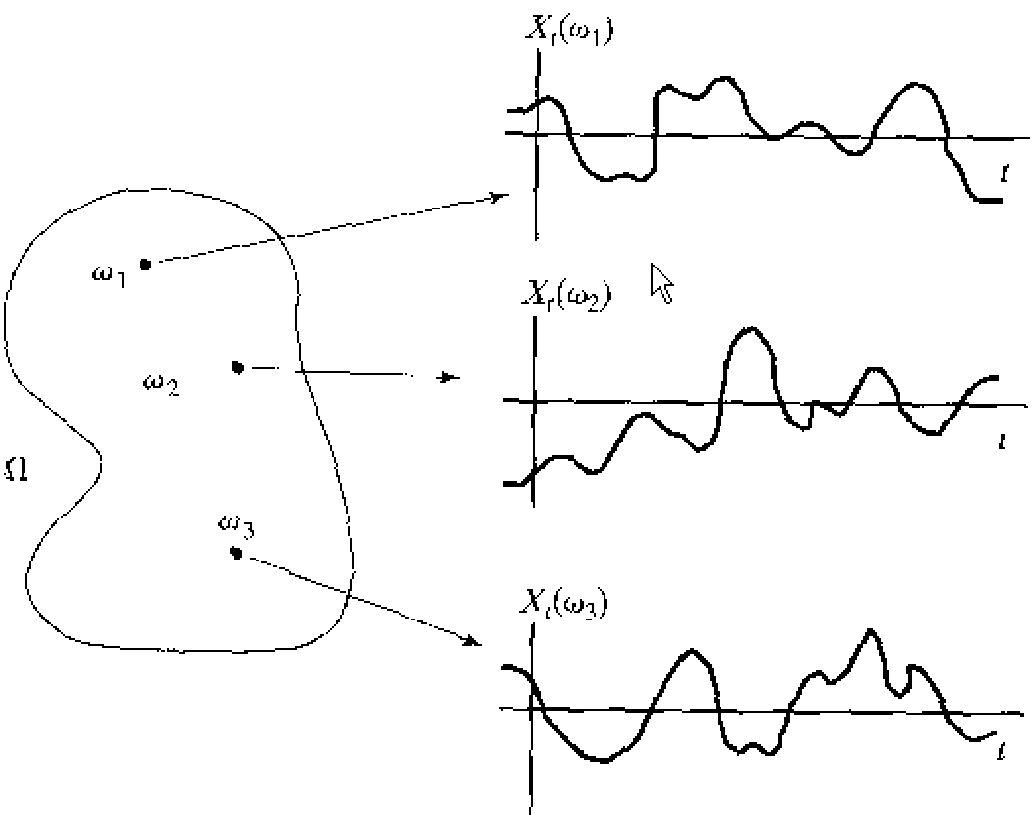
Sample space Ω : the set of all possible waves in any given random phenomenon.

State space E : the set of all possible values of *X(t)* .

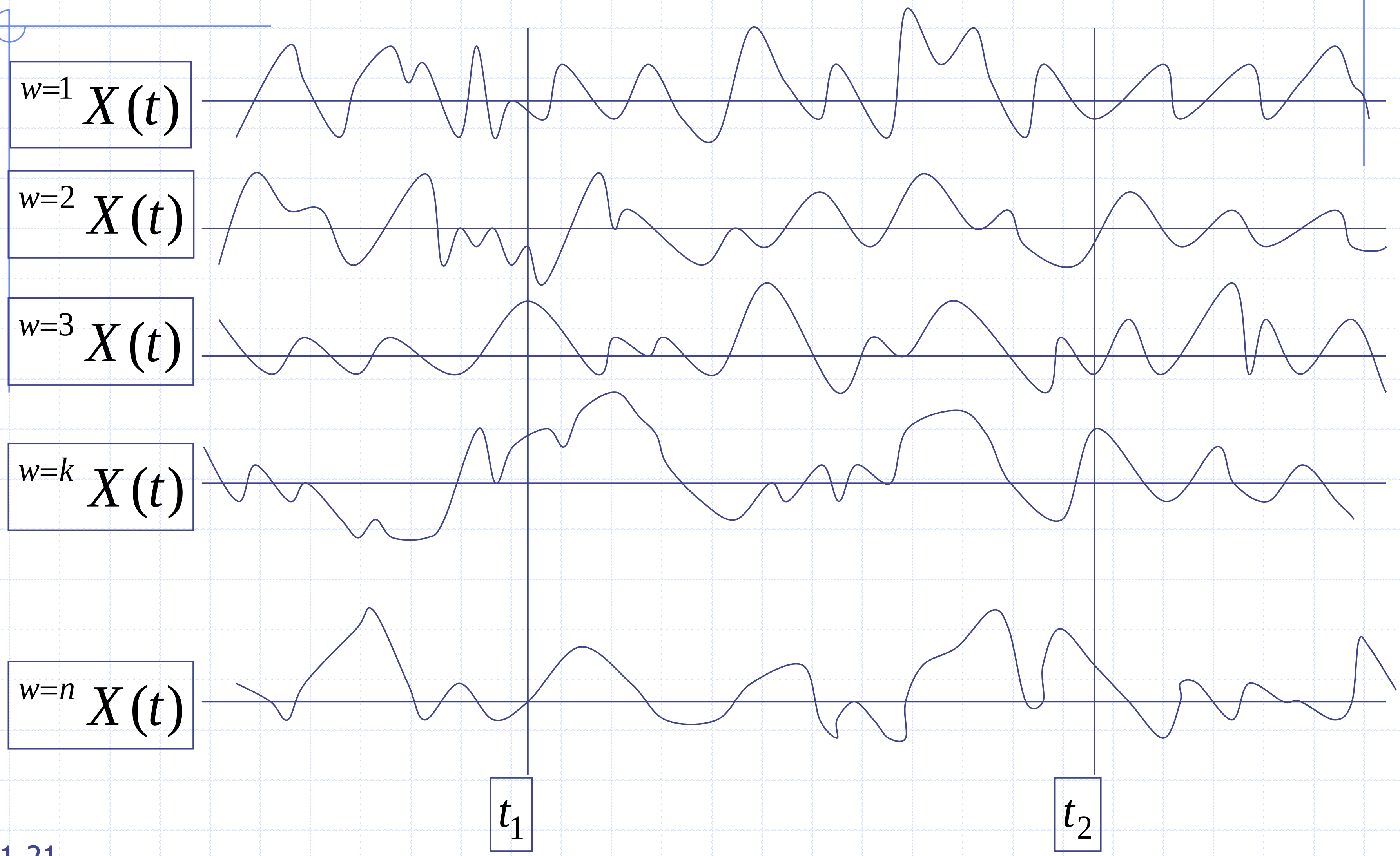
2.1.1 Definition and examples

A stochastic process can be viewed as a function of 2 variables, time t and outcome ω, $\{X(t,\omega), t \in T, \omega \in \Omega\}$

a) For a fixed ω, X(t) is a function of time, i.e. a sample function;



2.1.1 Definition and examples



2.1.1 Definition and examples

b) For fixed t, X(ω) is a random variable on the probability space (Ω ,F, P). (or the state at time t);

c) For a fixed t_0 and a fixed ω_0 , $X(t_0, \omega_0)$ is a single number;

d) X(t, ω), a family of functions with both t and variables ω.

2.1.1 Definition and examples

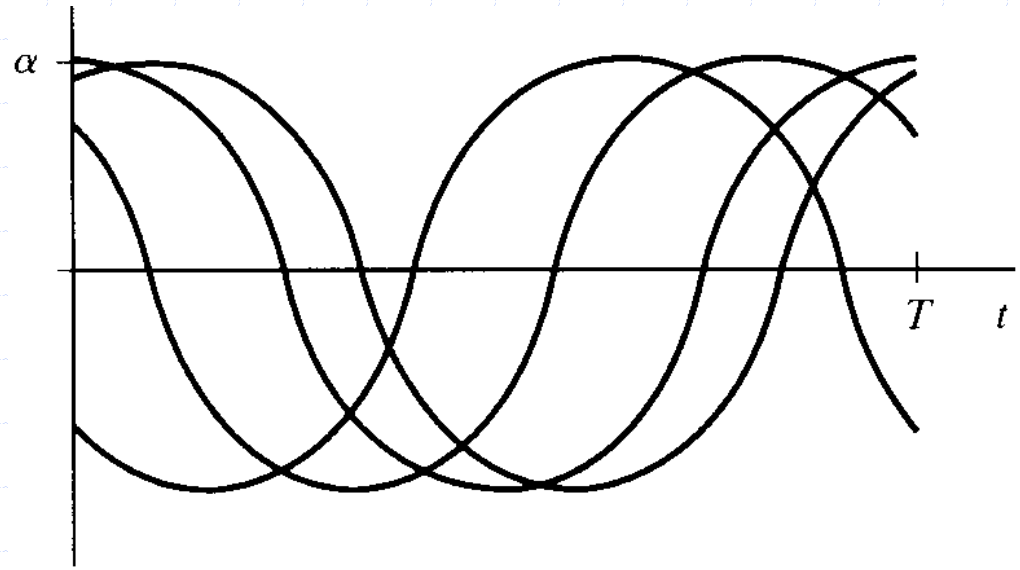
Example 1: An oscillator with a random phase

The stochastic process X(t) is given by

$$X(t) = A\cos(wt + \varepsilon),\quad t>0,$$

whereas A and w are constants and ε is a random variable uniformly distributed between 0 and 2π.

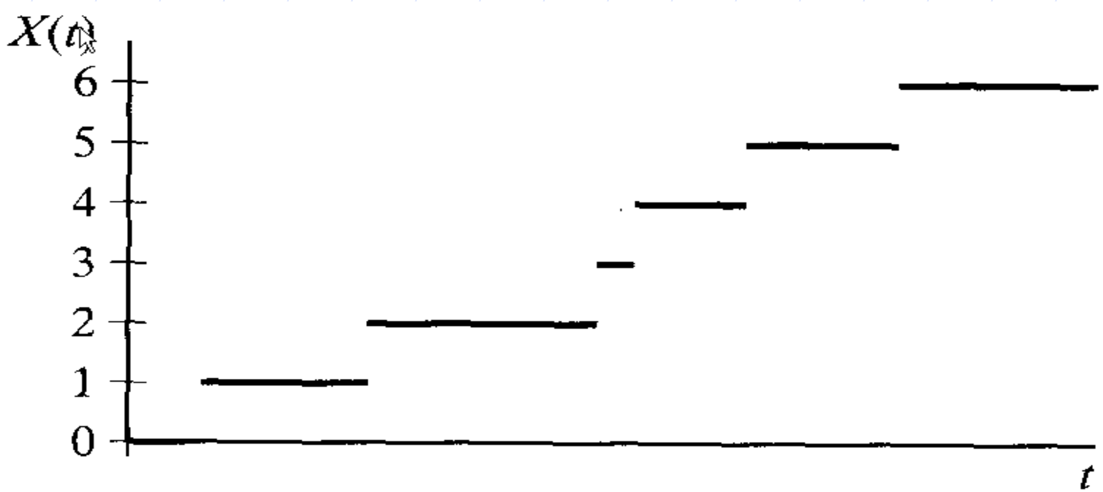
The randomness can be expressed via one variable.



(a) Typical sample functions if Θ is uniform on $[0, 2\pi]$

Example 2: A counting process

1. Let X(t) be the total number of phone calls received before time t. X(0)=0.
2. The outcome of one experiment will be a increasing step function.
3. If we record every call, all of the steps in the recorded graphs will be size of one.
4. The randomness is in the location of the steps which are the arrival times of the phone calls.
5. For a fixed time interval [0,T],both the total number of steps and the location of the steps are random.



2.1 Basic Concepts

2.1.1 Definition and examples

2.1.2 Types of Stochastic Process

2.1.3 Distribution and Density Functions

2.1.4 Moments (functions)

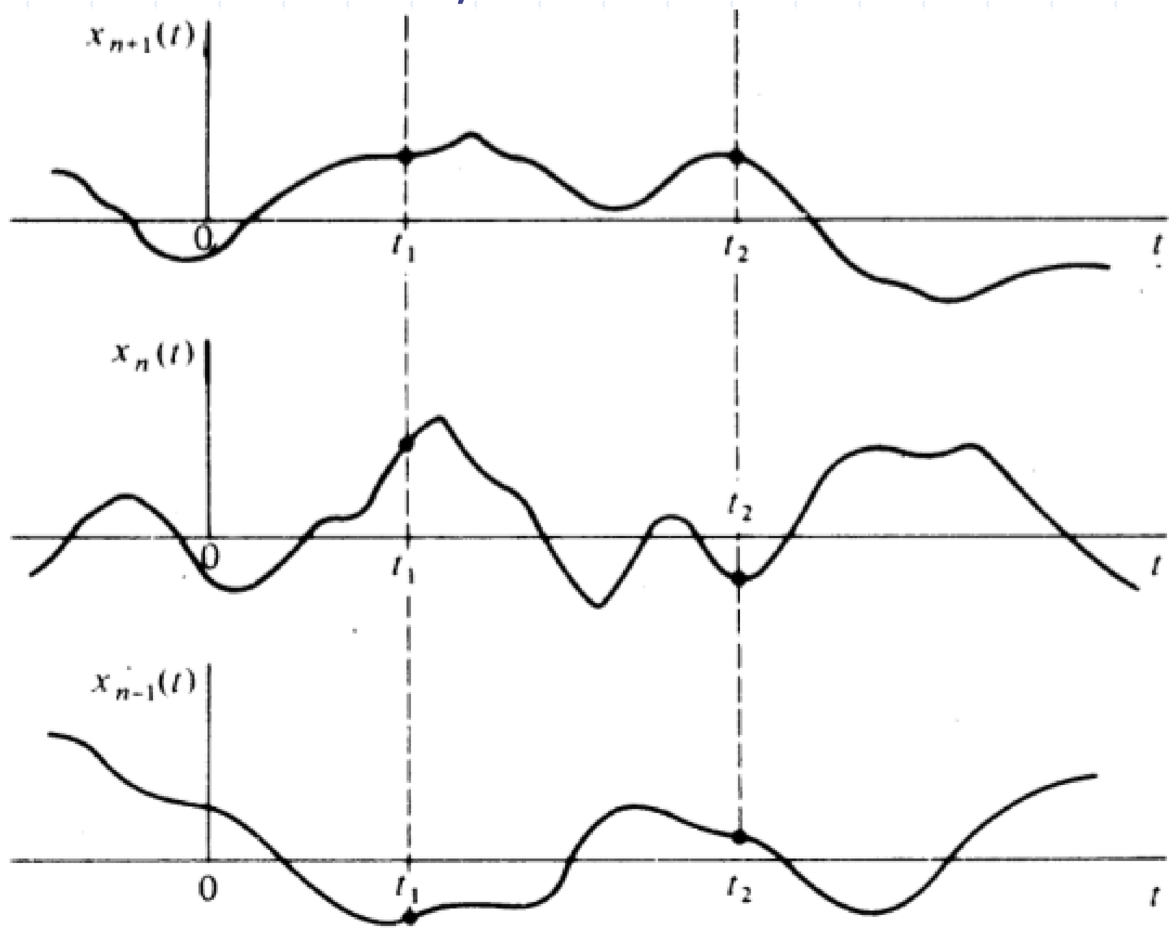
2.1.2 Types of Stochastic Process

- According to the characteristics of t and the random variable $x(t)$ at time t (or the state at time t), random processes can be classified to four cases.

2.1.2 Types of Stochastic Process

1. Continuous stochastic process

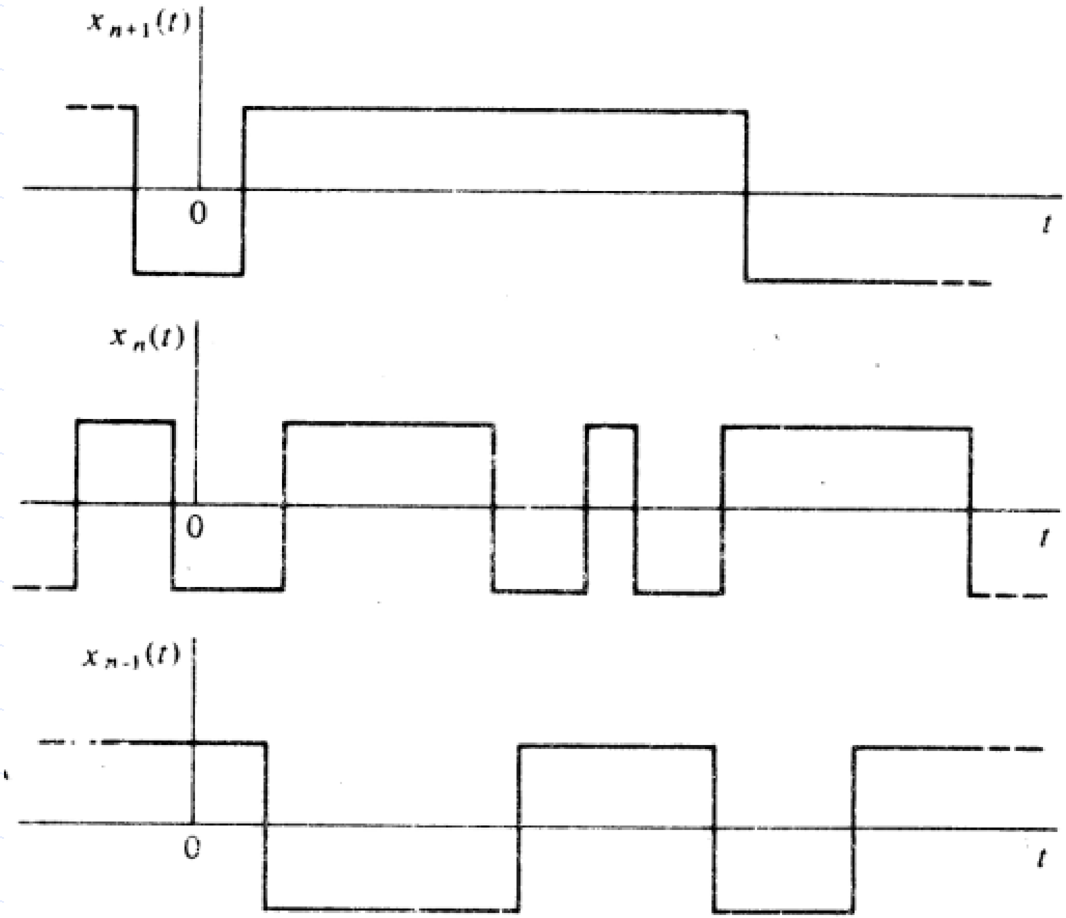
Both t and state are continuous. Such as thermal noise wave, wave profile in the ocean, and so on.



2.1.2 Types of Stochastic Process

2. Discrete stochastic process

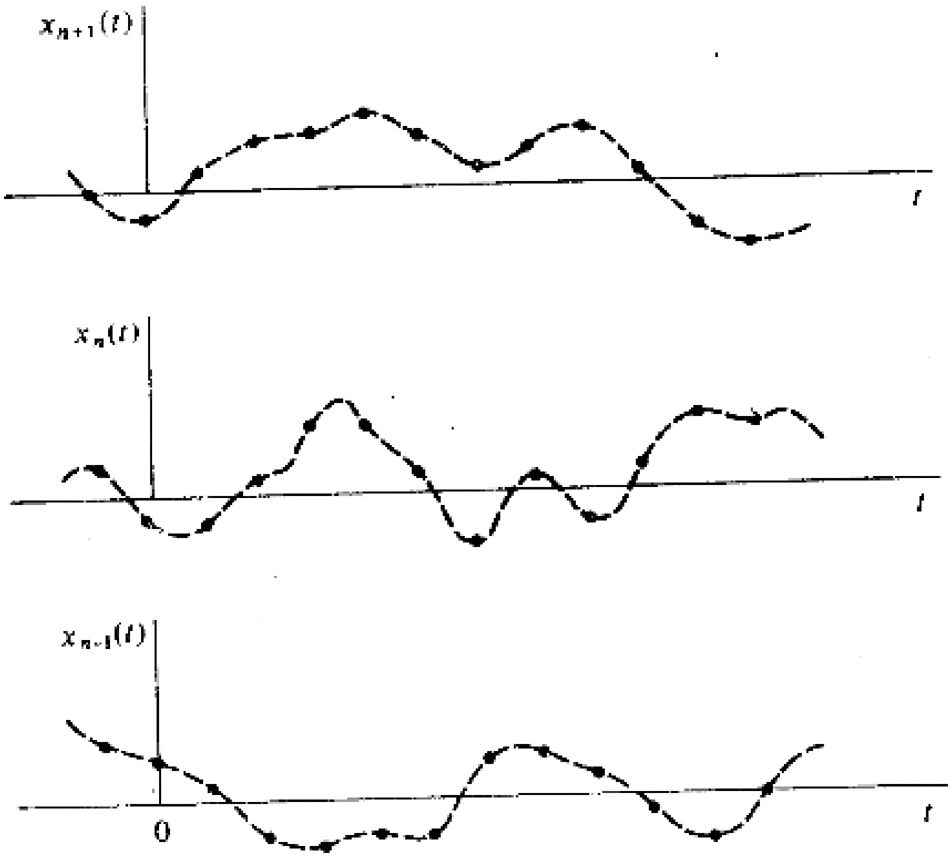
State is discrete, while t is continuous. Such as the counting process in example 2 or the random telegraph signal.



2.1.2 Types of Stochastic Process`1

3. Continuous random sequence

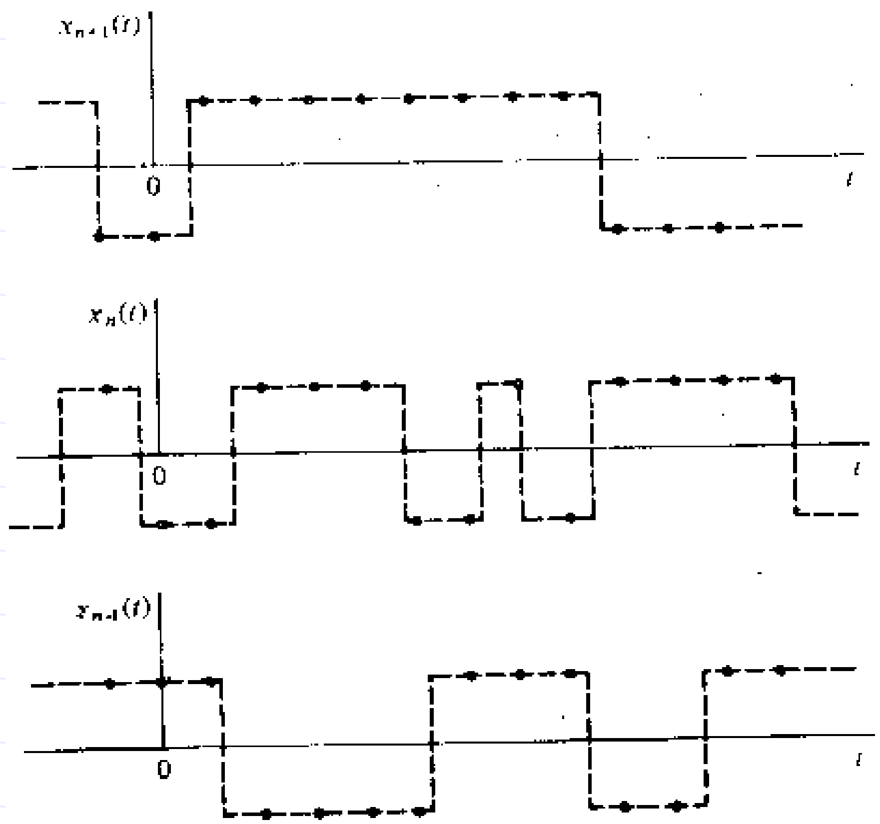
State is continuous, while t is discrete. Such a sequence can be formed by periodically sampling the ensemble members of a continuous random process.



2.1.2 Types of Stochastic Process

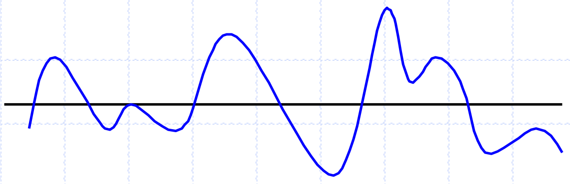
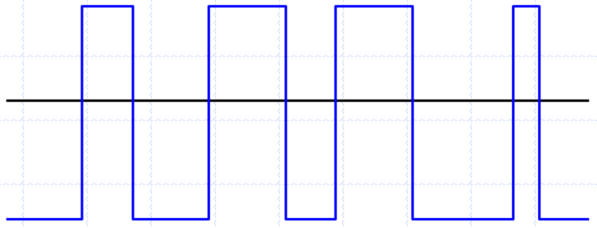
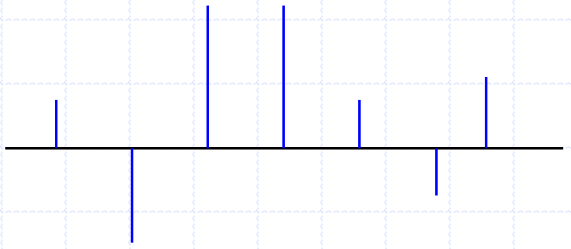
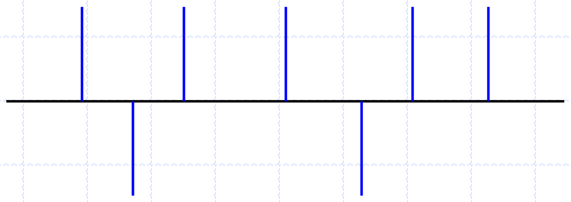
4. Discrete random sequence

Both t and state are discrete. A discrete random sequence developed by sampling the sample functions of the random telegraph signal.



2.1.2 Types of Stochastic Process

Summary

		State	
		Continuous	Discrete
Time	Continuous		
	Discrete		

2.1 Basic Concepts

Outline

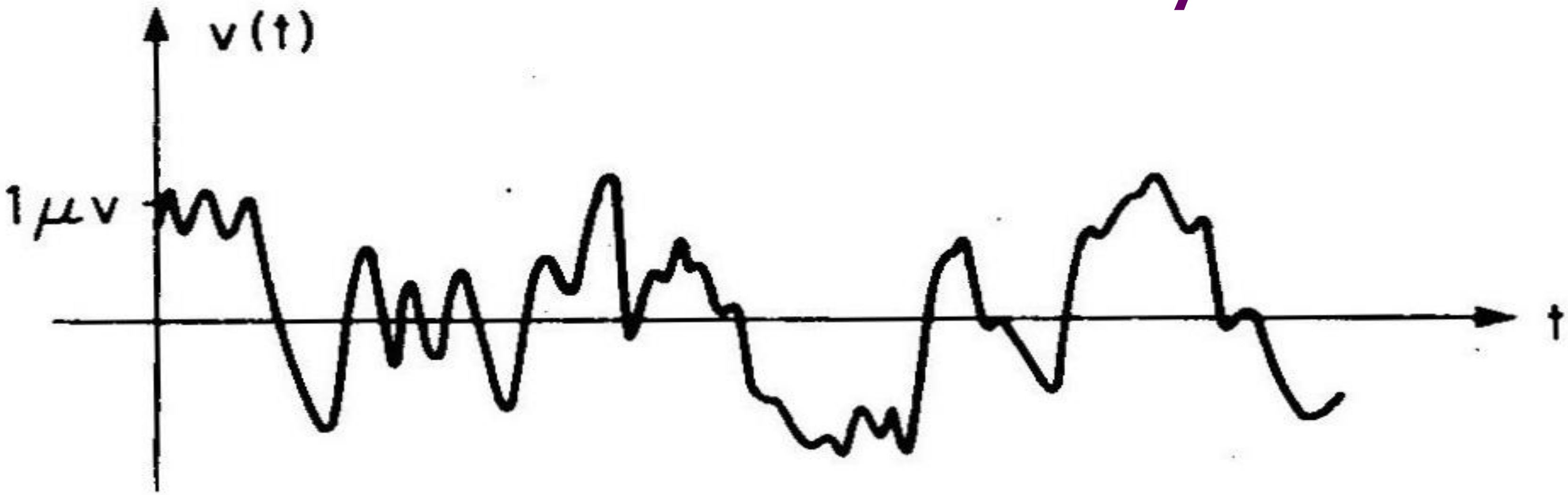
2.1.1 Definition and examples

2.1.2 Types of Stochastic Process

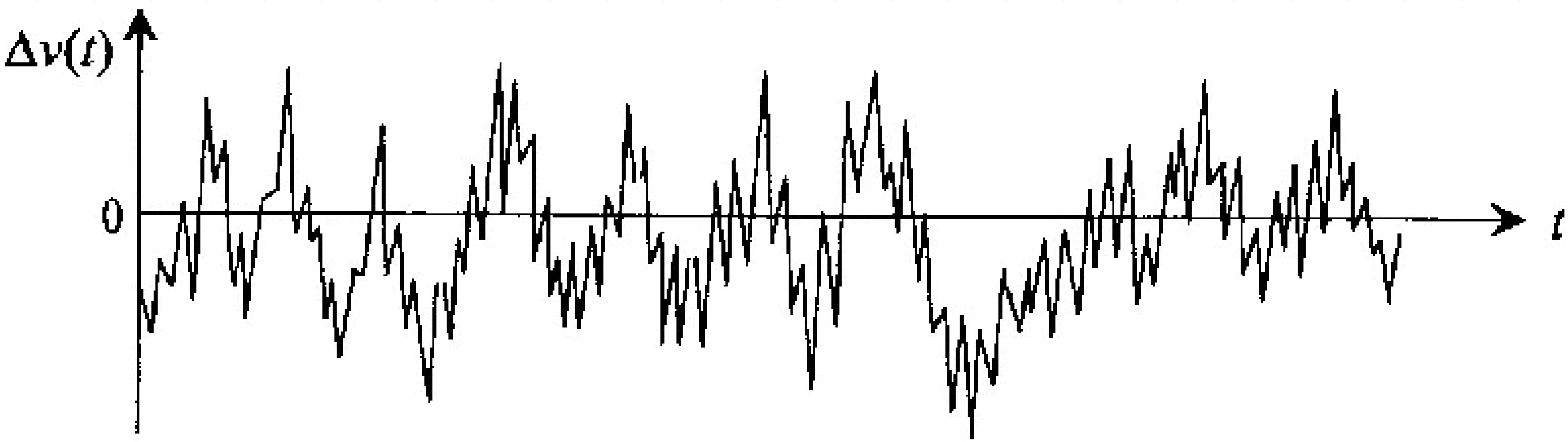
2.1.3 Distribution and Density Functions

2.1.4 Moments

2.1.3 Distribution and Density Functions

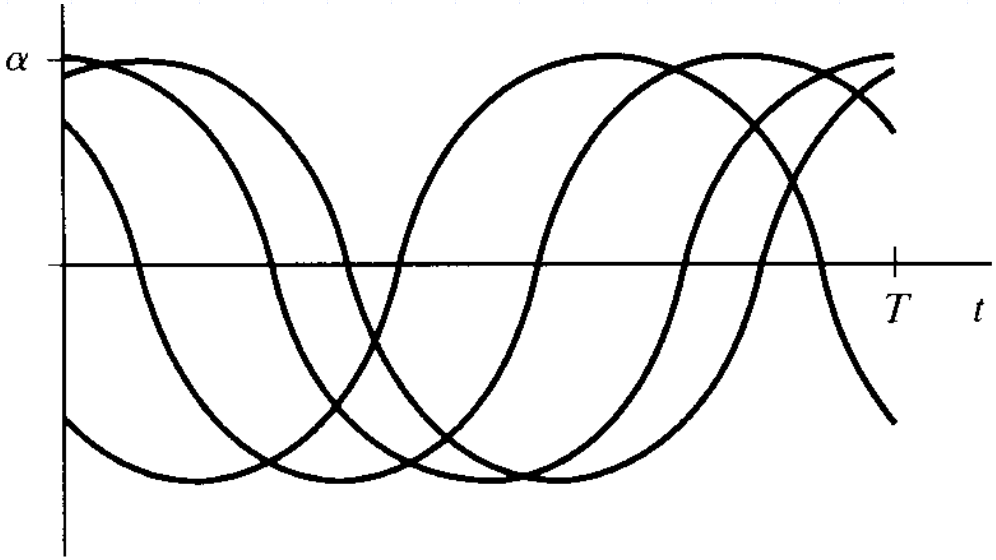


Thermal Noise Voltage at a Resistor under low temperature

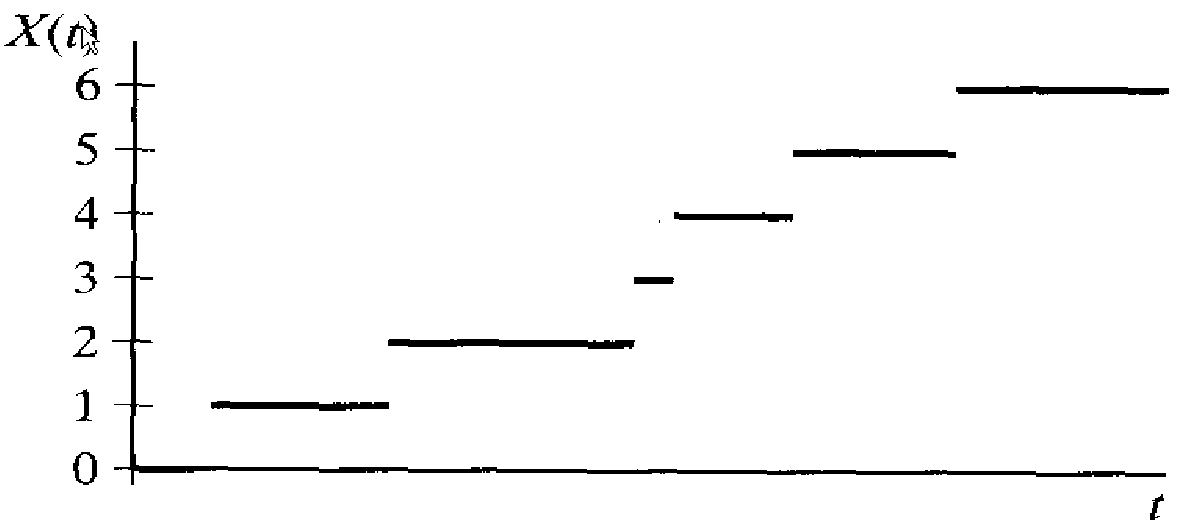


under high temperature

2.1.3 Distribution and Density Functions



(a) Typical sample functions if Θ is uniform on $[0, 2\pi]$



2.1.3 Distribution and Density Functions

Question:

How to describe a stochastic process?

How to compare the difference among different stochastic processes?

How to describe the speed that a stochastic process changes with respect to time(or the statistical dependence between different moments) ?

2.1.3 Distribution and Density Functions

- For a particular time t_1

  The distribution function associated with the random variables $X_1=X(t_1)$ will be denoted $F_x(x_1;t_1)$. It is defined as

$$F_X(x_1;t_1) = P\{X(t_1) \le x_1\}$$

  for any real number x_1

- The one-dimensional probability distribution of $\{X(t), t \in T\}$ is characterized by the family of one-dimensional probability distribution $\{F_X(x), t \in T\}$

2.1.3 Distribution and Density Functions

Because the statistical dependence generally exists among different time point t_i , the specification of $\{F_X(x), t \in T\}$ does not completely characterize a stochastic process.

The relationship among different time:

the joint distribution function.

2.1.3 Distribution and Density Functions

- For two random variables $X(t_1)$ and $X(t_2)$, the two-dimensional joint distribution function is

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

- In a similar manner,for N random variables $X(t_i)$, $i=1,2,\ldots,N$, the N-dimensional joint distribution function is

$$F_X(x_1,\ldots,x_N; t_1,\ldots,t_N)$$
$$= P\{X(t_1) \le x_1,\ldots, X(t_N) \le x_N\}$$

2.1.3 Distribution and Density Functions

- A stochastic process $\{X(t), t \in T\}$ is only then completely determined if for all integers n=1,2,... and for all n-tuples $\{t_1, t_2, \ldots, t_n\}$ with $t_i \in T$, the joint distribution functions of the random vectors$(X(t_1), X(t_2), \ldots, X(t_n))$ are know:

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) \\ = P\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\}$$

2.1.3 Distribution and Density Functions

- Joint density functions

  are found from appropriate derivatives of the distribution functions.

$$f_X(x_1;t_1) = dF_X(x_1;t_1)/dx_1$$

$$f_X(x_1,x_2;t_1,t_2) = \partial^2 F_X(x_1,x_2;t_1,t_2)/(\partial x_1,\partial x_2)$$

$$f_X(x_1,\ldots,x_N;t_1,\ldots,t_N)$$
$$= \partial^N F_X(x_1,\ldots,x_N;t_1,\ldots,t_N)/(\partial x_1 \ldots \partial x_N)$$

2.1.3 Distribution and Density Functions

e.g.1. A discrete-time filtering problem

Suppose that $X_0, X_1, X_2, \ldots$ is a sequence of independent random variables whose distribution is

$$P\{X_n = 0\} = P\{X_n = 1\} = 1/2$$

for each n. define a discrete-time random process X(t), t=0,1,2…, by setting $X(t) = X_t$

Let X(t) be the input to a discrete-time filter and the output is Y(t)=X(t)+X(t-1)

Obtain: the one- and two-dimensional distribution of the output process Y(t)

2.1.3 Distribution and Density Functions

e.g.2. Given: The stochastic process X(t) is given by $X(t)=Y_1+Y_2 t$, $t>0$, whereas Y_1 and Y_2 are independent Gaussian random variables, with zero mean and variance σ^2

Obtain: one-dimensional distribution and two-dimensional distribution of X(t).

Sln: $\mathrm{Var}\{Y_1\} + \mathrm{Var}\{tY_2\} = \sigma^2 + t^2\sigma^2 = \sigma^2(1 + t^2)$

$$E[X(t)] = E[Y_1] + E[tY_2] = E[Y_1] + tE[Y_2] = 0$$

$$f_{X,1}(x;t) = [2\pi\sigma^2(1 + t^2)]^{-1/2} \exp\{-x^2/[2\sigma^2(1 + t^2)]\}$$

$$f_{X,Y}(u,v) = \frac{\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{u-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{u-\mu_1}{\sigma_1}\right)\left(\frac{v-\mu_2}{\sigma_2}\right) + \left(\frac{v-\mu_2}{\sigma_2}\right)^2\right]\right\}}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}}$$

2.1.3 Distribution and Density Functions

$$\mathrm{Var}\{X(t_k)\} = \sigma^2(1 + t_k^2)$$

$$\begin{aligned}\mathrm{Cov}\{X(t_1), X(t_2)\} &= E\{X(t_1)X(t_2)\} \\ &= E\{(Y_1 + t_1Y_2)(Y_1 + t_2Y_2)\} \\ &= \sigma^2 + t_1t_2\sigma^2 = \sigma^2(1 + t_1t_2)\end{aligned}$$

$$\rho(t_1, t_2) = \frac{\mathrm{Cov}\{X(t_1), X(t_2)\}}{\sqrt{\mathrm{Var}\{X(t_1)\}\,\mathrm{Var}\{X(t_2)\}}} = \frac{1 + t_1t_2}{[(1 + t_1^2)(1 + t_2^2)]^{1/2}}$$

$$f_{X,2}(x_1, x_2; t_1, t_2) =$$

$$\frac{\exp\{-[(1 + t_2^2)x_1^2 - 2(1 + t_1t_2)x_1x_2 + (1 + t_1^2)x_2^2]/[2\sigma^2(t_1 - t_2)^2]\}}{2\pi\sigma^2|t_1 - t_2|}$$

Gaussian(Normal) Process

Def. Gaussian process

A stochastic process $\{x(t), t \in T\}$ is a Gaussian process if the random vectors $(X(t_1), X(t_2), \dots, X(t_n))$ have a joint Gaussian (Normal) distribution for all n-tuples $(t_1, t_2, \dots, t_n)$ with $t_i \in T$ and

$$t_1 < t_2 < \dots < t_n; \quad n = 1, 2, \dots$$

2.1.3 Distribution and Density Functions

- Conditional density functions

Ratios of joint density functions.

If for each $k\ (1 \le k \le n)$, the function $f_{X,k}$ is the k-dimensional density function for a continuous-amplitude random process X(t), then the conditional density function for $X(t_1)$, $X(t_2),\ldots,$ $X(t_m)$ given $X(t_{m+1})$, $X(t_{m+2}),\ldots,$ $X(t_n)$ is

$$f_{X,n}(x_1,\ldots,x_m;t_1,\ldots,t_m|x_{m+1},\ldots,x_n;t_{m+1},\ldots,t_n) = \frac{f_{X,n}(x_1,\ldots,x_n;t_1,\ldots,t_n)}{f_{X,n-m}(x_{m+1},\ldots,x_n;t_{m+1},\ldots,t_n)}$$

2.1 Basic Concepts

- 2.1.1 Definition and examples
- 2.1.2 Types of Stochastic Process
- 2.1.3 Distribution and Density Functions
- 2.1.4 Moments (functions)

2.1.4 Moments

E[X(t)] is a deterministic function.

1. Mean value function E[X(t)]

Mean value function is the expected value of X(t) as a function of t.

- For continuous stochastic processes

If the densities $f_X(x;t) = dF_X(x;t)/dx, t \in T$ exist, then

$$\bar{x}(t) = E[x(t)] = \int_{-\infty}^{\infty} x(t) f(x,t) dx$$

- For discrete stochastic processes

The mean value function is evaluated at every discrete time point.

$$\bar{x}(t_i) = E[x(t_i)] = \sum_{k=1}^{n} kP[x(t_i) = k], \quad i = 1, 2, \ldots..$$

2.1.4 Moments

- Second-order moment processes

If $E[X^2(t)]$ exists for $t \in T$, X(t) is called a second-order moment processes.

(mean square value function)

2.1.4 Moments

2. Variance function

- For a second-order moment process, the variance function is

$$Var[X(t)] = E[X(t) - \bar{x}(t)]^2$$
$$= E[X^2(t)] - \bar{x}^2(t)$$

2.1.4 Moments

3. Covariance(autocovariance) function

- For a second-order moment process, the covariance function is

$$C_{XX}(t_1,t_2) = Cov[X(t_1), X(t_2)]$$
$$= E\big[[X(t_1) - \bar{x}(t_1)][X(t_2) - \bar{x}(t_2)]\big]$$
$$= E\big[X(t_1)X(t_2)\big] - \bar{x}(t_1)\bar{x}(t_2)$$

Thus,

$$C_{XX}(t,t) = Var[X(t)]$$

2.1.4 Moments

4. Correlation(autocorrelation) function

- For a second-order moment process, the correlation function is

$$R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$

then,

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \overline{x}(t_1)\overline{x}(t_2)$$

2.1.4 Moments

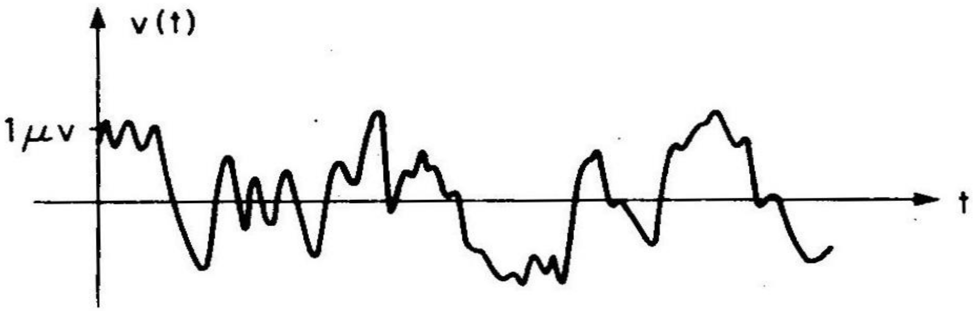
5. Correlation coefficients

(normalized autocovariance function)

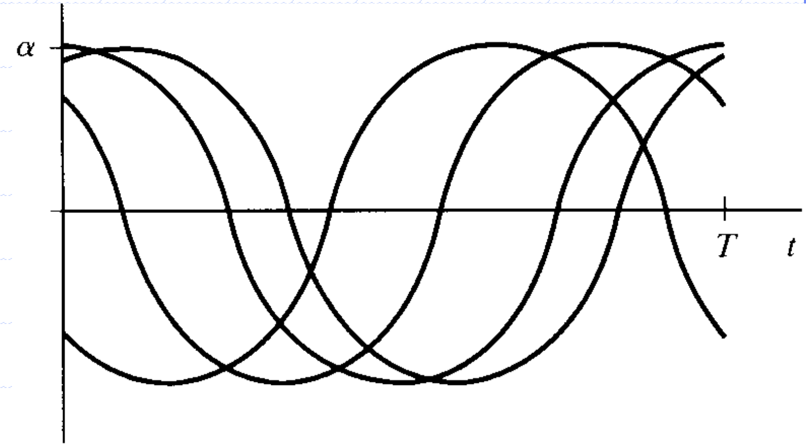
- For a second-order moment process, the correlation coefficients is

$$\rho_{XX}(t_1,t_2)=\frac{C_{XX}(t_1,t_2)}{\sqrt{Var[X(t_1)]Var[X(t_2)]}}$$

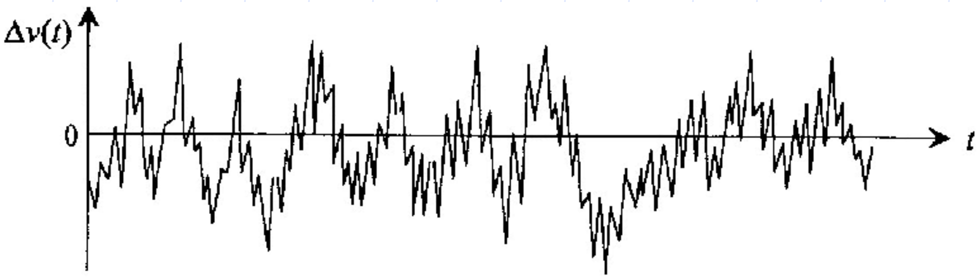
How to calculate ?



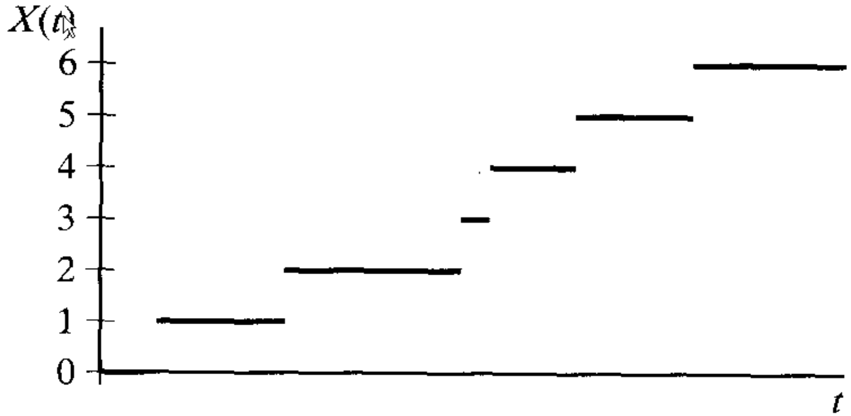
Thermal Noise Voltage at a Resistor under low temperature



(a) Typical sample functions if Θ is uniform on $[0, 2\pi]$



under high temperature



Poisson Processes

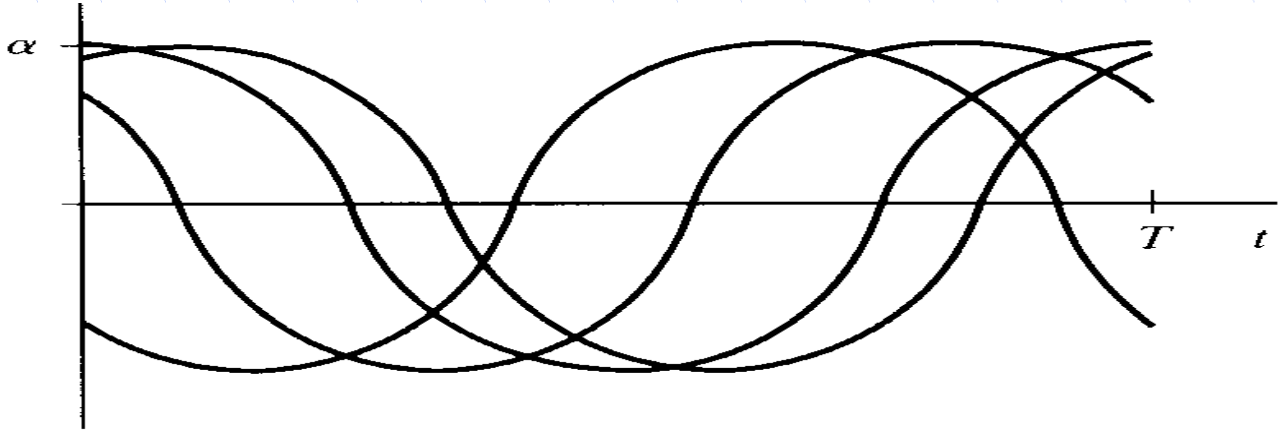
2.1.4 Moments

e.g.1.

Given: The stochastic process X(t) is given by $X(t)=A\cos(wt+\varepsilon)$, $t>0$, whereas A and w are constants and ε is random variable uniformly distributed between $-\pi$ and π.

Obtain: the digital characteristic of X(t).

Sln: $E[X(t)]$, $Var[X(t)]$, $C_{xx(}t_1,t_2)$, $R_{xx}(t_1,t_2)$



(a) Typical sample functions if Θ is uniform on [0, 2π]

2.1.4 Moments

e.g.2.

Given: The stochastic process $X(t)$ is given by $X(t)=Y\cos(wt)+Z\sin(wt)$, $t>0$, whereas Y and Z are independent random variables, and $EY=EZ=0$, $VarY=VarZ=\sigma^2$.

Obtain: $E[X(t)]$, $C_{xx}(t_1,t_2)$.

Sln:

2.1.4 Moments

$C_{XX}(t_1,t_2)$、$R_{XX}(t_1,t_2)$、$\rho_{XX}(t_1,t_2)$

are symmetric in t_1 and t_2

$$C_{XX}(t_1,t_2) = C_{XX}(t_2,t_1)$$

$$R_{XX}(t_1,t_2) = R_{XX}(t_2,t_1)$$

$$\rho_{XX}(t_1,t_2) = \rho_{XX}(t_2,t_1)$$

2.1.4 Moments

- $X(t_1)$ and $X(t_2)$ often are expected to be almost independent if the time difference $|t_2 - t_1|$ is sufficiently large, one anticipates that

$$\lim_{|t_2 - t_1| \to \infty} C_{XX}(t_1, t_2) = \lim_{|t_2 - t_1| \to \infty} \rho_{XX}(t_1, t_2) = 0$$

(E[X(t)]=m=0)

But the formula does not hold for all stochastic processes.

2.1.4 Moments

6. Cross-covariance function

- For two second-order moment processes X(t) and Y(t), their cross-covariance function is

$$
\begin{aligned}
C_{XY}(t_1,t_2) &= Cov[X(t_1),Y(t_2)] \\
&= E\big[[X(t_1)-\overline{x}(t_1)][Y(t_2)-\overline{y}(t_2)]\big] \\
&= E\big[X(t_1)Y(t_2)\big]-\overline{x}(t_1)\overline{y}(t_2)
\end{aligned}
$$

2.1.4 Moments

7. Cross-correlation function

- For two second-order moment processes X(t) and Y(t), their cross-correlation function is

$$R_{XY}(t_1,t_2)=E\left[X(t_1)Y(t_2)\right]$$

then,

$$C_{XY}(t_1,t_2)=R_{XY}(t_1,t_2)-\overline{x}(t_1)\overline{y}(t_2)$$

2.1.4 Moments

8. Mutual correlation coefficients

- For two second-order moment processes X(t) and Y(t), their normalized correlation coefficients is

$$\rho_{XY}[t_1,t_2]=\frac{C_{XY}(t_1,t_2)}{\sqrt{Var[X(t_1)]Var[Y(t_2)]}}$$

For two random variables,

$$\rho_{XY}=\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

2.1.4 Moments

Mutually uncorrelated

- For two second-order moment processes X(t) and Y(t), if

$$C_{XY}[t_1, t_2] = 0 \qquad t_1, t_2 \in T$$

then X(t) and Y(t) are mutually uncorrelated.

2.1.4 Moments

e.g.4.

Given: $X(t)$ and $Y(t)$ are two second-order moment processes. $W(t)=X(t)+Y(t)$.

Obtain: $E[W(t)]$ and $R_{WW}(t_1,t_2)$

Sln:

Homework

- 2.2
- 2.5
- 2.7