Discussion 1

1. properties of probability Measures

a. we have the inequality: $p(A \cup B) \le p(A) + p(B)$

for $i=2$, we have $p(A_1 \cup A_2) \le p(A_1) + p(A_2)$ and assume when $i=k$ is right,

$p(\bigcup_{i=1}^{k} A_i) \le \sum_{i=1}^{k} p(A_i)$ then let $A = \bigcup_{i=1}^{k} A_i$, $B = A_{k+1}$, we can easily have

$p(\bigcup_{i=1}^{k} A_i \cup A_{k+1}) = p(\bigcup_{i=1}^{k+1} A_i) \le \sum_{i=1}^{k} p(A_i) + p(A_{k+1}) = \sum_{i=1}^{k+1} p(A_i)$

so for $k+1$, the assumption is also right, generally. $p(\bigcup_{i\ge1} A_i) \le \sum_{i\ge1} p(A_i)$

b. for the above analysis, we have $p(\bigcup_{i\ge1} A_i) \le \sum_{i\ge1} p(A_i)$ and $p(A_k) \ge 0$ for $k \in N$

so we have the $\sum_{i\ge1} p(A_i) = \lim_{n\to\infty} \sum_{i=1}^{n} p(A_i)$, Since $A_1 \subset A_2 \subset \cdots \subset F$, we have

$\bigcup_{i\ge1} A_i = A_n (n\to\infty)$. Finally, we have $p(\bigcup_{i\ge1} A_i) = \lim_{n\to\infty} p(A_n)$

2. Independence

a) let event C is one of: even A and even B occurs. then we have

$p(C) = p(A)(1-p(B)) + p(B)(1-p(A)) = p(A) + p(B) - 2p(A)p(B)$

b) if event A is independent of itself, we have $p(A) = p(A)p(A)$

① assume $p(A) \ne 0$ then we have $p(A) = 1$

② assume $p(A) = 0$

In General, $p(A) = 1$ or 0

3. Balls and Bins

a. we can define the probability space (S, F, p) as follows

(1) The sample space $S = \{(X_1, X_2, \cdots X_n)\}$ for each X_k $(k=1, \cdots, n)$, X_k is an integer which ranges in $[1, n]$.

(2) the event space is all the subsets of S

(3) The probability measure p is defined as $p(E) = \frac{\Omega(E)}{\Omega(S)} = \frac{1}{n^n}$.

b. A_z denotes the event that exactly z bins are empty, for $0 \le z \le n$ and E denotes that all empty bins sit to the left of all bins containing at least one ball.

we have $p(E) = \sum_{z=0}^{n} p(E|A_z) = \sum_{z=0}^{n} \frac{1}{\binom{n}{z}} p(A_z)$

c. when $z=1$, we have the probability of one empty bin is

$$p(A_1) = \frac{n \cdot (n-1)(n-2)! \binom{n}{2}}{n^n} = \frac{n! \binom{n}{2}}{n^n}$$

From the bins' perspective, select one as empty we have n choices; select one as the bin containing 2 balls, we have $n-1$ choices. and arrange the left $(n-2)$ bins, we have $(n-2)!$ and in the $(n-2)+2$ intervals, insert 2 bins. $\binom{n}{2}$ choices