

# Chap 4 Spectral Analysis of Stochastic Processes

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# Chapter 4: Spectral Analysis



## Content:

4.1 Spectral Density Functions

4.2 Spectral Analysis of Linear Systems

4.3 Spectrum of Amplitude-modulated  
Signals

4.4 Narrow-band Gaussian Processes

# 4.3 Spectrum of Amplitude-modulated Signals

## Random Amplitude Processes

Consider a signal of the form

$$Y(t) = \sqrt{2}A(t)\cos(\omega_c t + \theta), \quad t > 0$$

whereas  $A(t)$  is a random process representing the amplitude and  $\theta$  is a random variable uniformly distributed between 0 and  $2\pi$ , and  $\omega_c$  is constant.  $A(t)$  and  $\theta$  are *independent*.

Obtain: power spectrum of  $Y(t)$

- ✓  $\sqrt{2}\cos(\omega_c t + \theta)$  is seen as the *carrier signal*.
- ✓  $A(t)$  is the *modulation signal*.
- ✓ For analog communications,  $A(t)$  may represent a speech signal. In digital communications,  $A(t)$  is a continuous-time wave form that represents a sequence of data pulses.

# Random Amplitude Processes



$$\begin{aligned} E[Y(t)] &= \sqrt{2}E[A(t)\cos(w_c t + \theta)] \\ &= \sqrt{2}E[A(t)]E[\cos(w_c t + \theta)] = 0 \end{aligned}$$

$$\begin{aligned} R_{YY}(t, t + \tau) &= E[Y(t)Y(t + \tau)] \\ &= 2E[A(t)\cos(w_c t + \theta)A(t + \tau)\cos(w_c t + w_c \tau + \theta)] \\ &= 2E[A(t)A(t + \tau)]E[\cos(w_c t + \theta)\cos(w_c t + w_c \tau + \theta)] \\ &= E[A(t)A(t + \tau)]\cos(w_c \tau) \\ &= R_{AA}(t, t + \tau)\cos(w_c \tau) \end{aligned}$$

If  $A(t)$  is a stationary process, then  $Y(t)$  is also stationary with the correlation function

$$R_{YY}(\tau) = R_{AA}(\tau)\cos(w_c \tau)$$

# Random Amplitude Processes



## Modulation theorem of Fourier transforms

- If  $x(t)$  has Fourier transform  $X(f)$ , then the Fourier transform of the signal  $y(t) = x(t) \cos(\omega_c t)$

is

$$Y(\omega) = \pi[X(\omega + \omega_c) + X(\omega - \omega_c)]$$

$$Y(f) = \frac{1}{2}[X(f - f_c) + X(f + f_c)]$$

where  $\omega_c = 2\pi f_c$

$$Y(t) = \sqrt{2}A(t) \cos(\omega_c t + \theta), \quad t > 0$$

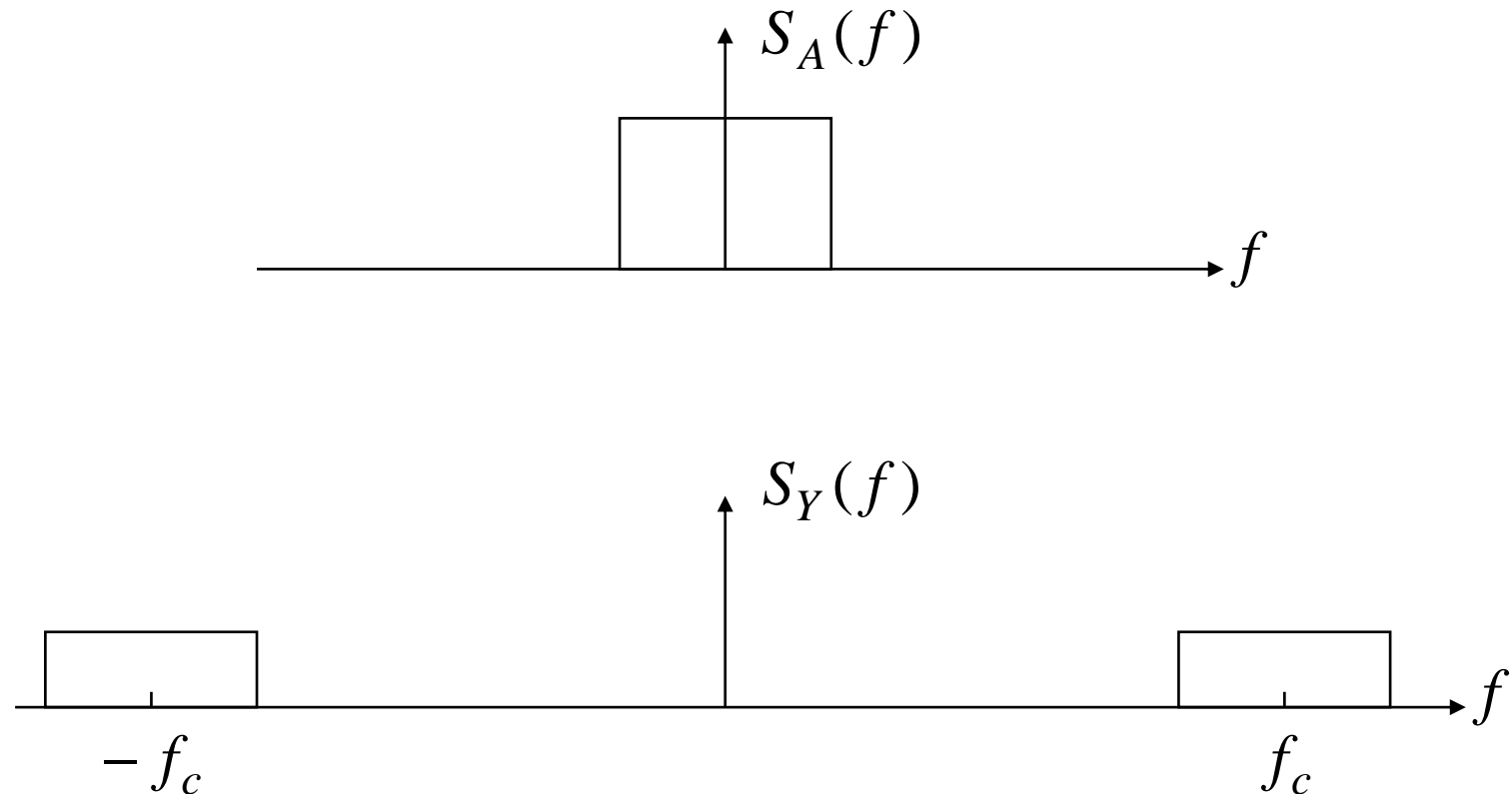
$$R_{YY}(\tau) = R_{AA}(\tau) \cos(\omega_c \tau)$$

The spectrum of  $Y(t)$  is  $S_Y(f) = \frac{1}{2}[S_A(f - f_c) + S_A(f + f_c)]$

# Random Amplitude Processes



- Spectrum of amplitude-modulated signal  $Y(t)$



$$S_Y(f) = \frac{1}{2} [S_A(f - f_c) + S_A(f + f_c)]$$

# A random data signal with rectangular pulses



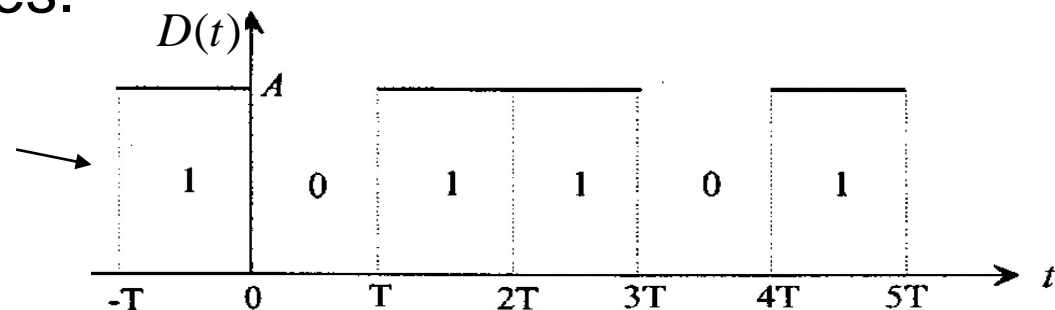
- Rectangular pulse:  $p_T(t) = \begin{cases} 1 & \text{for } 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}$

- Define a random process as

$$D(t) = \sum_{n=-\infty}^{\infty} A_n p_T(t - nT), \quad nT \leq t < (n+1)T$$

- where  $A_n$  is a random variable representing the amplitude of the  $n$ th pulse.  $A_n, -\infty < n < \infty$ , are independent and identically distributed random variables.

A sample wave of  $D(t)$   
*Pulse Code Modulation*

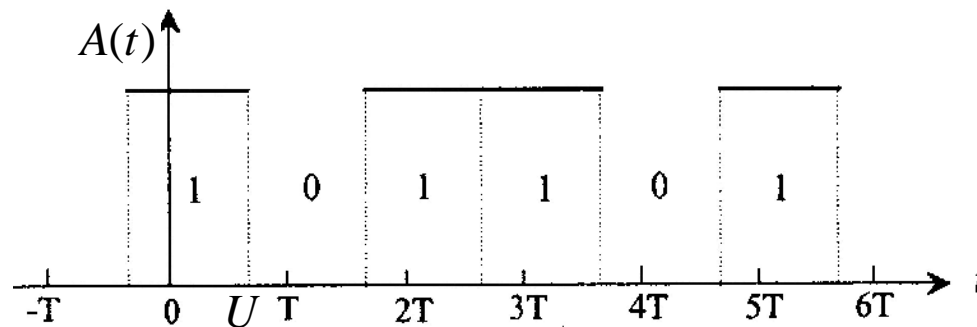


- $D(t)$  is *not wide-sense stationary* because its correlation function depends on the absolute value of time.

# A random data signal with rectangular pulses



- Random delay of  $D(t)$ :  $A(t) = D(t - U)$ ,  $-\infty < t < \infty$
- where  $U$  is uniformly distributed on the interval  $[0, T]$ .



- $A(t)$  is *wide-sense stationary*.
- Prove:
- For simple, let  $p_T(t) = \begin{cases} 1 & \text{for } 0 \leq t < T \\ -1 & \text{otherwise} \end{cases}$
- Let  $E[A_n] = 0$ ,  $E[A_n^2] = \alpha^2$ , then  
 $E[A(t)] = E[D(t)] = 0$  , and  $E[A_n A_k] = 0$  for  $n \neq k$



# A random data signal with rectangular pulses



- Correlation function of  $A(t)$

$$\begin{aligned} E[A(t)A(t+\tau)] &= E\left\{ \sum_{n=-\infty}^{\infty} A_n p_T(t-nT-U) \sum_{k=-\infty}^{\infty} A_k p_T(t+\tau-kT-U) \right\} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} E[A_n A_k] E[p_T(t-nT-U) p_T(t+\tau-kT-U)] \\ &= \sum_{n=-\infty}^{\infty} \alpha^2 E[p_T(t-nT-U) p_T(t+\tau-nT-U)] \end{aligned}$$

$$\begin{aligned} &E[p_T(t-nT-U) p_T(t+\tau-nT-U)] \\ &= \frac{1}{T} \int_0^T p_T(t-nT-u) p_T(t+\tau-nT-u) du \quad , \quad v = u + nT \\ &= \frac{1}{T} \int_{nT}^{(n+1)T} p_T(t-v) p_T(t+\tau-v) dv \end{aligned}$$

# A random data signal with rectangular pulses

$$\begin{aligned} E[A(t)A(t+\tau)] &= \sum_{n=-\infty}^{\infty} \alpha^2 \frac{1}{T} \int_{nT}^{(n+1)T} p_T(t-v) p_T(t+\tau-v) dv \\ &= \alpha^2 \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{nT}^{(n+1)T} p_T(t-v) p_T(t+\tau-v) dv \\ &= \alpha^2 \frac{1}{T} \int_{-\infty}^{\infty} p_T(t-v) p_T(t+\tau-v) dv \end{aligned}$$

Let  $u = t + \tau - v$

$$E[A(t)A(t+\tau)] = \alpha^2 \frac{1}{T} \int_{-\infty}^{\infty} p_T(u-\tau) p_T(u) du$$

$$R_A(\tau) = \alpha^2 \frac{1}{T} p_T(\tau) * p_T(-\tau)$$

# A random data signal with rectangular pulses



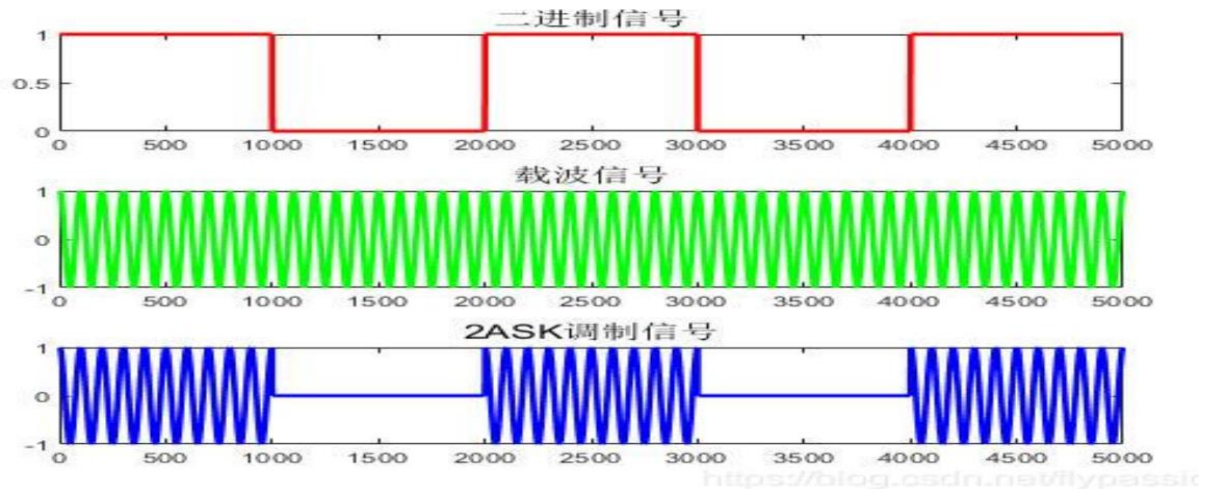
- Spectrum of  $A(t)$

$$R_A(\tau) = \alpha^2 \frac{1}{T} p_T(\tau) * p_T(-\tau) \quad S_A(f) = \alpha^2 \frac{1}{T} |P_T(f)|^2$$

- If  $Y(t) = \sqrt{2}A(t) \cos(w_c t + \theta), \quad t > 0$

then 
$$S_Y(f) = \frac{\alpha^2}{2T} [|P_T(f - f_c)|^2 + |P_T(f + f_c)|^2]$$

Sample wave of  $Y(t)$ :



# Chapter 4: Spectral Analysis



## Content:

4.1 Spectral Density Function

4.2 Spectral Analysis of Linear System

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4.4 Narrow-band Gaussian Processes

## 4.4 Narrow-band Gaussian Processes



### 4.4.1 The Definition of BandWidth (BW)

4.4.2 Hilbert Transform and analytical signal

4.4.3 Representation of Narrow-Band Signals

4.4.4 Narrow Band Random Processes

4.4.5 Gaussian Narrow-Band Random Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

## 4.4.1. The Definition of Band



- In most cases, the signal we deal with is always band limited
- Band limited means that the band of a signal is finite.
- How can we define the bandwidth of a signal?
- Here, we just list some criterions for defining bandwidth

# Case1: Ideal Band Pass Signals



I) **Nonzero in a finite interval:**  $BW = f_2 - f_1 \leq 2B$

(Absolute bandwidth, cutoff bandwidth)

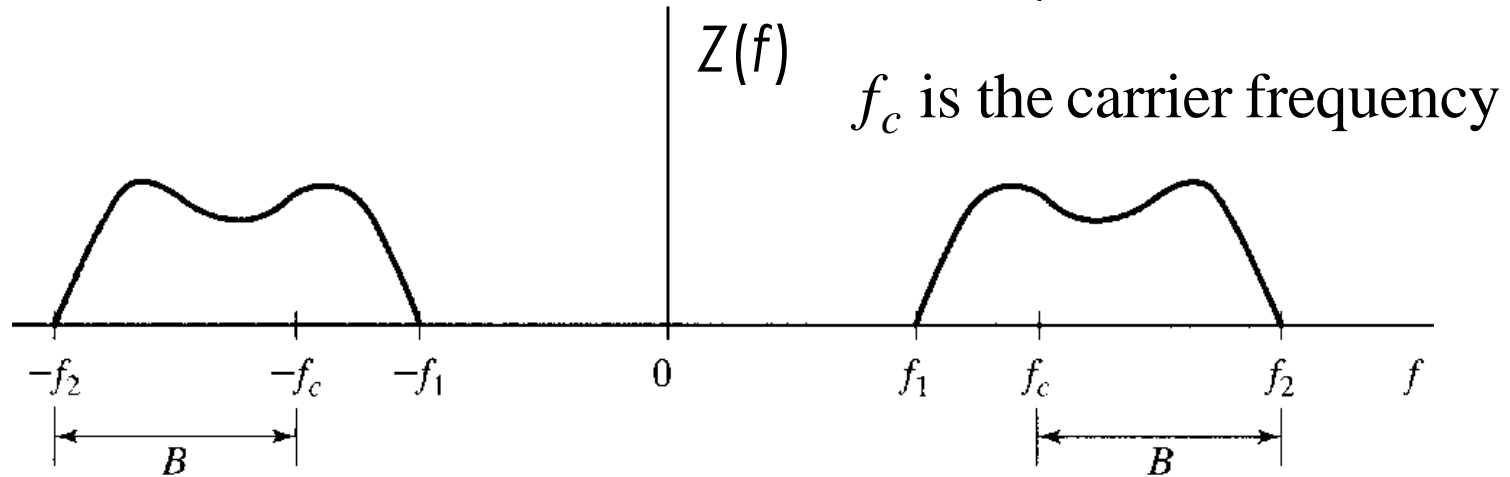


FIGURE 4-3 An ideal band-pass frequency function.

II) **Half-power bandwidth (3dB bandwidth)**

Let  $Z_m$  be maximum value of  $|Z(f)|$ ,

$$|Z(f_3)| = |Z(f_4)| = Z_m/\sqrt{2}, \quad |Z(f)| > Z_m/\sqrt{2} \text{ for } f_3 < f < f_4$$

and  $|Z(f)| < Z_m/\sqrt{2}$  for  $0 < f < f_3$  and  $f_4 < f < \infty$

# Case 2: Nonideal Band Pass Signals

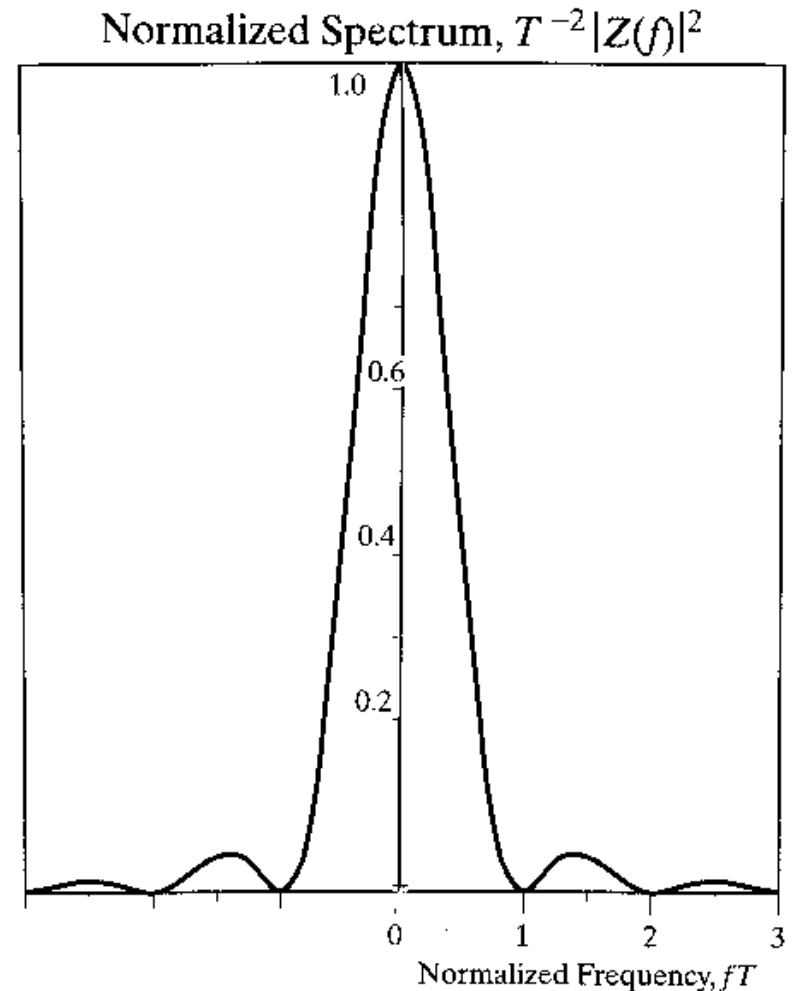


## III. Null-to-null bandwidth

- ✓ Nonzero in infinite interval
- ✓ But the energy concentrates in a small interval

## IV. Normalized bandwidth

$$1 - (-1) = 2$$





## 4.4 Narrow-band Gaussian Processes



4.4.1 The Definition of Band

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4.4.6 Sine Wave Plus Narrow-Band Noise

# 通信系统中的 数字信号正交变换理论



- 现实中产生的物理可实现的信号是实信号
- 但通信系统却提出要将实信号正交分解为复信号
- 为什么要进行正交分解？直接利用现实中的信号不行吗？

$$x(t) \text{ 与 } \hat{x}(t) \text{ 相正交, } \int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$$



- 设有一个实信号  $x(t)$ ，其正交分解后的复信号为  $z(t)$ ，
- $z(t)$  称为  $x(t)$  的解析表示

$$z(t) = x(t) + j\hat{x}(t) = a(t)e^{j\varphi(t)}$$

从这个表达式中，很容易得到信号的：

瞬时包络  $a(t)$

瞬时相位  $\varphi(t)$

瞬时频率  $\frac{d(\varphi(t))}{dt}$

这三个参数，恰好是信号分析，参数测量和识别调制的基础

这就是对实信号进行解析表示的意义所在。



- 怎样对信号进行解析表示呢？
- 一个实信号的频谱具有共轭对称性。
- 所以，对于一个实信号，只要取其正频域部分或者负频域部分就能完全加以描述，而不会丢失任何信息！并且，所得的新信号是一个复信号！

- 假设有一个信号  $x(t)$ ，取其正频域部分的频谱分量，这部分频谱可以用一个复函数  $z(t)$  来表示。则：

$$Z(\omega) = \begin{cases} 2X(\omega), & \omega > 0 \\ X(\omega), & \omega = 0 \\ 0, & \omega < 0 \end{cases}$$

$\omega > 0$  的分量加倍是为了使  $z(t)$  与原信号能量相等。



- 再引入一个阶跃滤波器：

$$H(w) = \begin{cases} -j, & w > 0 \\ 0, & w = 0 \\ j, & w < 0 \end{cases}$$

这样，我们可以得到：

$$Z(w) = X(w) [1 + jH(w)]$$

$$z(t) = x(t) + jx(t) * h(t) \quad * \text{为卷积符号}$$

- 易于求出  $h(t) = \frac{1}{\pi t}$

$x(t) * h(t)$  叫做  $x(t)$  的 *Hilbert* 变换

- 可以发现，一个实数的 *Hilbert* 变换同原信号正交。  
所以，一个实信号要进行正交分解，只需要：



## 4.4.2 Hilbert Transform



- Def. of Hilbert Transform  $h(t) = \frac{1}{\pi t}$

$$\begin{cases} \hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x(\tau)}{t - \tau} d\tau = \frac{1}{\pi t} * x(t) \\ x(t) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\hat{x}(\tau)}{t - \tau} d\tau = -\frac{1}{\pi t} * \hat{x}(t) \end{cases}$$

- The functions above are Hilbert Transform pairs
- Hilbert Transform is an operation denoted by  $H[*]$
- Relationship in frequency domain:

$$H(w) = \begin{cases} -j & w \geq 0 \\ j & w < 0 \end{cases}$$

$$\Rightarrow |H(w)| = 1, \quad \phi(w) = \begin{cases} -\pi/2 & w \geq 0 \\ \pi/2 & w < 0 \end{cases}$$

$$\hat{G}(w) = H(w)G(w) = -j \operatorname{sgn}(w) G(w)$$



## 4.4 Narrow-band Gaussian Processes



- References

- [1] Analysis of stochastic signals, 朱华, 北京理工大学, *Chapter 5*
- [2] Random signal analysis, 李晓风, 电子工业出版社, *Chapter 6*

# The use of Hilbert Transform

- Get the one-sided spectral: Let

$$\tilde{x}(t) = x(t) + j\hat{x}(t)$$

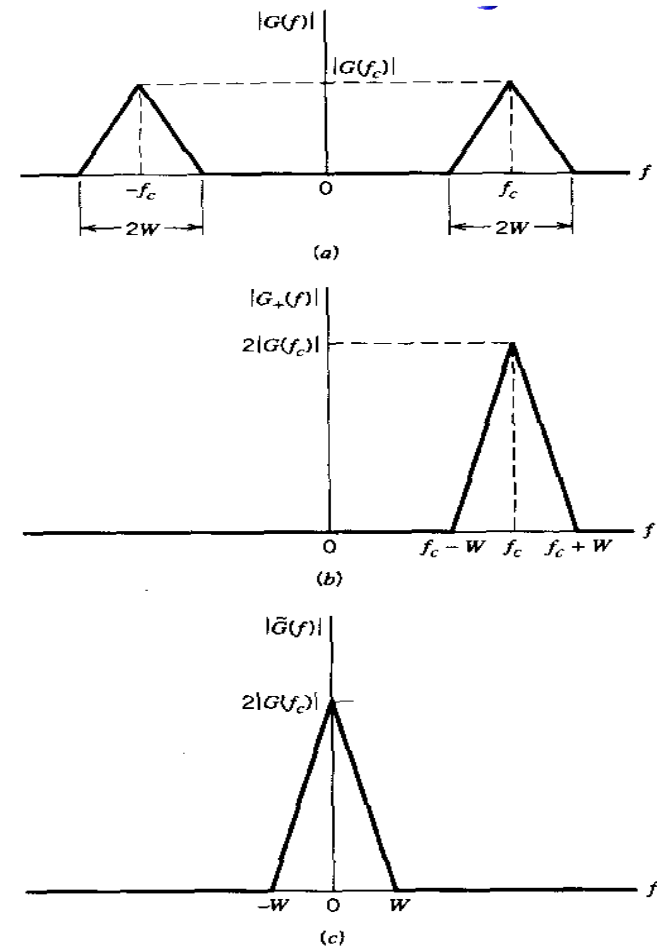
- Then:

$$\begin{aligned}\tilde{X}(\omega) &= X(\omega) + j\hat{X}(\omega) \\ &= X(\omega) + j[-j\operatorname{sgn}\omega X(\omega)] \\ &= X(\omega)[1 + \operatorname{sgn}\omega]\end{aligned}$$

$$= \begin{cases} 2X(\omega), & \omega \geq 0 \\ 0, & \omega < 0 \end{cases}$$

$$= 2X_+(\omega)$$

- $\tilde{x}(t)$  is called *the analytical signal* or pre-envelope of  $x(t)$



# Example for Hilbert Transform



*e.g.* Find the H.T. for  $x(t) = \cos \omega t$  and its analytical signal.

Solution:

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\cos \omega \tau}{t - \tau} d\tau = \sin \omega t$$

- Hilbert Transform is like an ideal phase shifter of 90 degree

- Let  $\tilde{x}(t) = x(t) + j\hat{x}(t)$

$\tilde{x}(t)$  *is the analytical signal* or pre-envelope of  $x(t)$

$$\tilde{x}(t) = \cos \omega t + j \sin \omega t = e^{j\omega t}$$

- The compare between the frequency spectral of  $x(t)$  and  $\tilde{x}(t)$
- *The spectral of  $\tilde{x}(t)$  is the positive parts of the spectral of  $x(t)$*

$$x(t) = \cos \omega_c t \quad X(\omega) = \pi(\delta(\omega + \omega_c) + \delta(\omega - \omega_c))$$

$$x(t) = \sin \omega_c t \quad X(\omega) = j\pi(\delta(\omega + \omega_c) - \delta(\omega - \omega_c))$$

# Properties of Hilbert transform (1)



$$\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x(\tau)}{t - \tau} d\tau = \frac{1}{\pi t} * x(t)$$

- Fourier Transform:  
$$\hat{X}(w) = F\{\hat{x}(t)\} = -j \operatorname{sgn}(w) X(w) = \begin{cases} -jX(w), & w \geq 0 \\ 0, & w = 0 \\ jX(w), & w < 0 \end{cases}$$
- Orthogonality:  $\int_{-\infty}^{\infty} x(t) \hat{x}(t) dt = 0$
- Linearity:  $H\{ax(t) + by(t)\} = a\hat{x}(t) + b\hat{y}(t)$
- Successive Hilbert transforms:  $H\{\hat{x}(t)\} = -x(t)$
- Convolution:  $H\{x(t) * y(t)\} = \hat{x}(t) * y(t) = x(t) * \hat{y}(t)$

# HT for Random Processes (1)



- If  $X(t)$  is a **WSS (wide-sense stationary) R.P.**, then  $\hat{X}(t)$  is also a WSS R.P. . And they are jointly WSS.

$$R_{\hat{X}}(\tau) = R_X(\tau), S_{\hat{X}}(\omega) = S_X(\omega)$$

$$R_{\hat{X}X}(\tau) = \hat{R}_X(\tau), \quad R_{XX}(\tau) = -\hat{R}_X(\tau), \quad \tau = t_1 - t_2$$

$$\text{Then, } R_{XX}(\tau) = -R_{\hat{X}X}(\tau) \quad (1)$$

- From the property of cross-correlation function:

$$R_{XX}(\tau) = R_{\hat{X}X}(-\tau) \quad (2)$$

$R_{XX}(\tau)$  and  $R_{\hat{X}X}(\tau)$  are **odd functions**:

$$R_{\hat{X}X}(-\tau) = -R_{\hat{X}X}(\tau)$$

$$R_{\hat{X}X}(0) = 0$$

# H.T. for Random Process (2)



The spectrum of the *analytical signal* of R.P.  $X(t)$

- Let  $Z(t) = X(t) + j\hat{X}(t)$

- Then,

$$\begin{aligned} R_Z(\tau) &= E[Z(t)Z^*(t+\tau)] \\ &= E[(X(t) + j\hat{X}(t))(X(t+\tau) - j\hat{X}(t+\tau))] \\ &= R_X(\tau) + R_{\hat{X}}(\tau) + jR_{X\hat{X}}(\tau) - jR_{\hat{X}X}(\tau) \\ &= 2R_X(\tau) + j2\hat{R}_X(\tau) \end{aligned}$$

$$R_{\hat{X}X}(\tau) = \hat{R}_X(\tau) \quad \Rightarrow \quad S_{\hat{X}X}(\omega) = \begin{cases} -jS_X(\omega), & \omega \geq 0 \\ jS_X(\omega), & \omega < 0 \end{cases}$$

$$R_{XX}(\tau) = -\hat{R}_X(\tau) \quad \Rightarrow \quad S_{XX}(\omega) = \begin{cases} jS_X(\omega), & \omega \geq 0 \\ -jS_X(\omega), & \omega < 0 \end{cases}$$

$$S_Z(\omega) = \begin{cases} 4S_X(\omega) & \omega \geq 0 \\ 0 & \omega < 0 \end{cases}$$

## 4.4 Narrow-band Gaussian Processes



4.4.1 The Definition of Band

4.4.2 Hilbert Transform and analytical signal

**4.4.3 Representation of Narrow-Band Signals**

4.4.4 Narrow Band Random Processes

4.4.5 Gaussian Narrow-Band Random  
Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

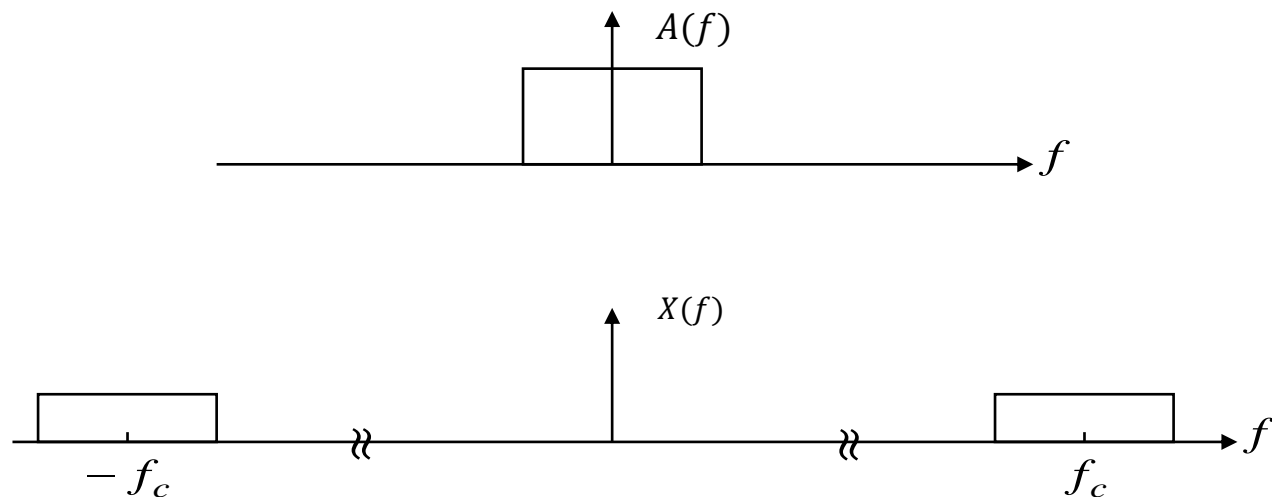


## 4.4.3 Representation of Narrow-Band Signal (Canonic form)

- The narrow-band signal with following form:

$$X(t) = A(t) \cos[2\pi f_c t + \phi(t)]$$

- $A(t)$  is the *envelop* and  $f_c$  is the *carrier frequency*.
- $A(t)$  and  $\phi(t)$  are low-frequency bandlimited signals, their max frequencies are much lower than  $f_c$
- $A(t)$  and  $\phi(t)$  can be seen as the *baseband signals*.





# Hilbert Transform of a narrow-band signal



- For a **narrow-band signal**  $x(t)$

Prove can be done easily in frequency domain

$$H[A(t) \cos 2\pi f_c t] = A(t) \sin 2\pi f_c t$$

$$H[A(t) \sin 2\pi f_c t] = -A(t) \cos 2\pi f_c t$$

- If  $A(t)$  and  $\phi(t)$  are low-frequency signal, then:

$$H[A(t) \cos(\omega_c t + \phi(t))] = A(t) \sin[\omega_c t + \phi(t)]$$

$$H[A(t) \sin(\omega_c t + \phi(t))] = -A(t) \cos[\omega_c t + \phi(t)]$$



## 4.4.3 Representation of Narrow-Band Signal (Canonic form)

- The narrow-band signal with following form:

$$X(t) = A(t) \cos[2\pi f_c t + \phi(t)]$$

- Then:  $X(t) = A(t) \cos \phi(t) \cos 2\pi f_c t - A(t) \sin \phi(t) \sin 2\pi f_c t$

- Let:  $X_I(t) = A(t) \cos \phi(t)$  ,  $X_Q(t) = -A(t) \sin \phi(t)$

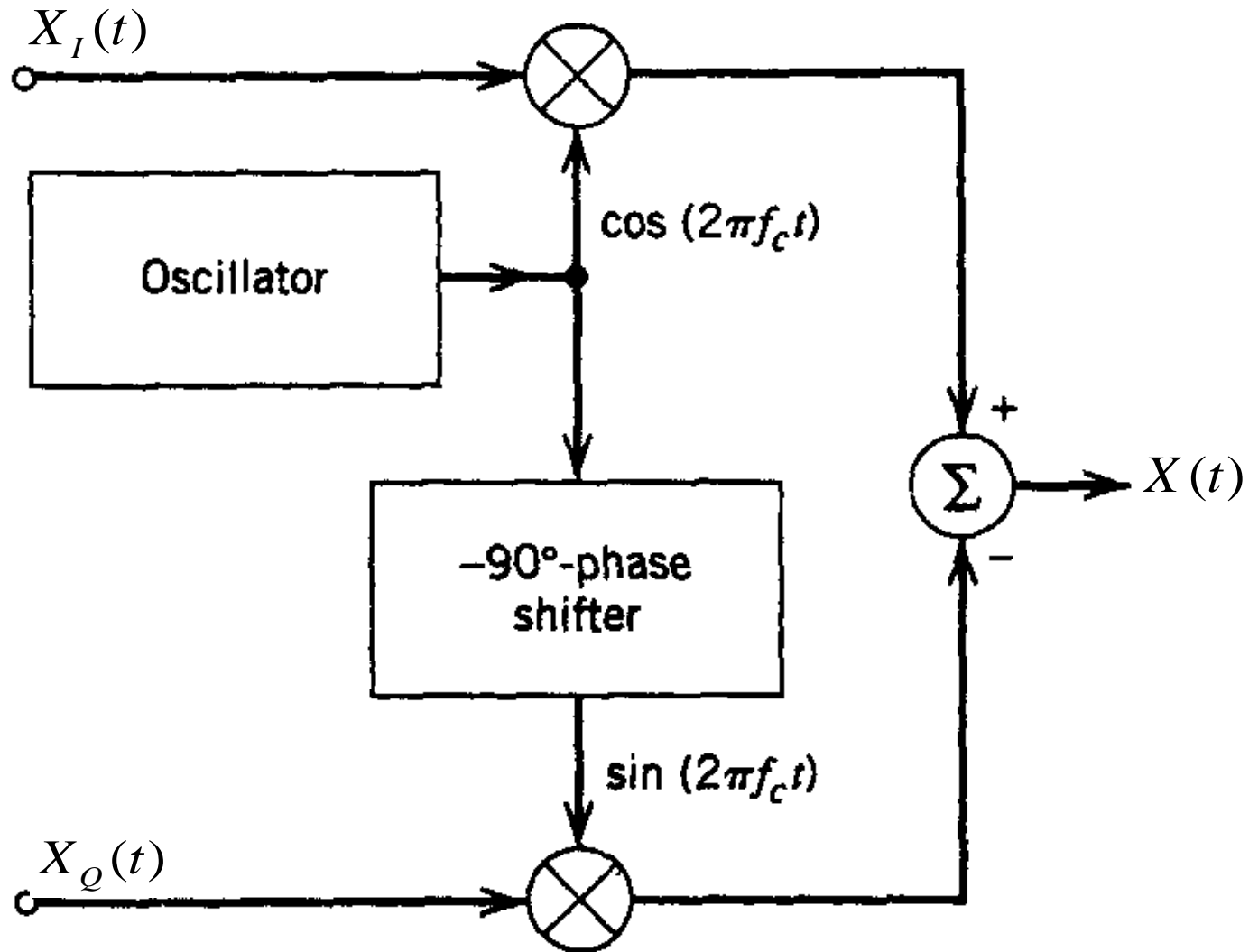
- Then

- $X(t) = X_I(t) \cos 2\pi f_c t + X_Q(t) \sin 2\pi f_c t$

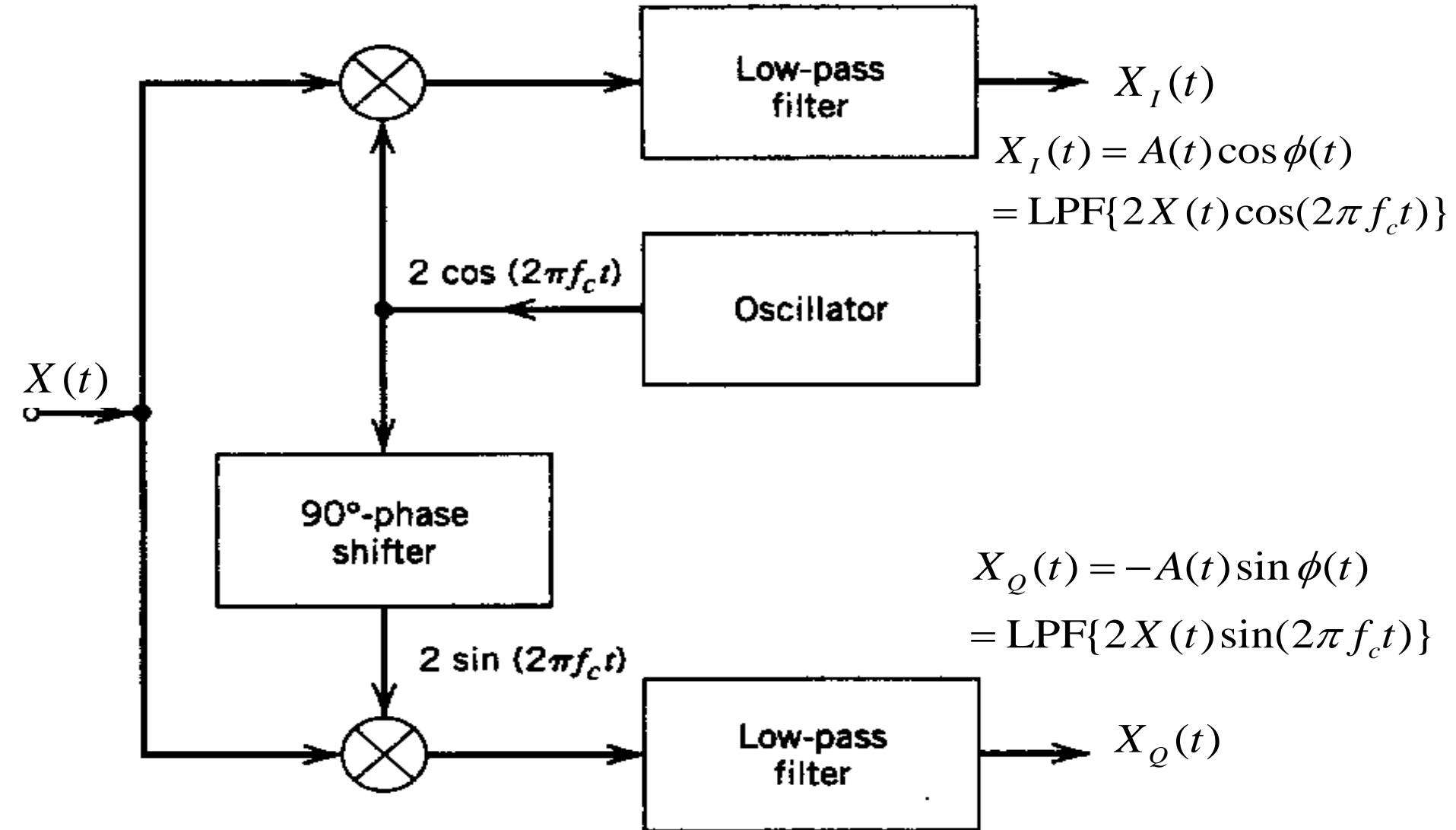
$$A(t) = \sqrt{X_I^2(t) + X_Q^2(t)}$$
$$\phi(t) = \arctan \frac{X_Q(t)}{X_I(t)}$$

- $X_I(t)$  and  $X_Q(t)$  are low-frequency bandlimited signals, their max frequencies are lower than  $f_c$
- $X_I(t)$  and  $X_Q(t)$  can be seen as the *baseband signals*.
- Define  $a(t) = X_I(t) + jX_Q(t) = A(t)e^{-j\phi(t)}$
- is the *complex envelop* of the baseband signal.

# Block Diagram for recover the narrow-band signal (*Modulation*)



# Block Diagram for decompose a narrow-band signal (*Demodulation*)





## 4.4.3 Representation of Narrow-Band Signal (Canonic form)

$$X(t) = A(t) \cos[2\pi f_c t + \phi(t)]$$

$$X(t) = X_I(t) \cos 2\pi f_c t + X_Q(t) \sin 2\pi f_c t$$

- Make Hilbert Transform on both side, we get:

$$\hat{X}(t) = X_I(t) \sin 2\pi f_c t - X_Q(t) \cos 2\pi f_c t$$

- Then:

$$X_I(t) = X(t) \cos 2\pi f_c t + \hat{X}(t) \sin 2\pi f_c t$$

$$X_Q(t) = X(t) \sin 2\pi f_c t - \hat{X}(t) \cos 2\pi f_c t$$

- The analytical signal  $\tilde{X}(t)$  :  $\tilde{X}(t) = X(t) + j\hat{X}(t)$
- The relationship between **complex envelop**  $a(t)$  and  $\tilde{X}(t)$

$$\tilde{X}(t) = a(t)e^{-j2\pi f_c t}$$

$$a(t) = X_I(t) + jX_Q(t) = A(t)e^{-j\phi(t)}$$

$$X(t) = \text{Re}[a(t)e^{-j2\pi f_c t}] = \text{Re}[A(t)e^{-j[2\pi f_c t + \phi(t)]}]$$

$$\hat{X}(t) = \text{Im}[a(t)e^{-j2\pi f_c t}] = \text{Im}[A(t)e^{-j[2\pi f_c t + \phi(t)]}]$$

## 4.4 Narrow-band Gaussian Processes



4.4.1 The Definition of Band

4.4.2 Hilbert Transform

4.4.3 Representation of Narrow-Band Signals

**4.4.4 Narrow Band Random Processes**

4.4.5 Gaussian Narrow-Band Random Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

# Facts about Narrow-band R.P.(1)



$$X(t) = A(t) \cos[2\pi f_c t + \phi(t)] = X_I(t) \cos 2\pi f_c t + X_Q(t) \sin 2\pi f_c t$$

- If  $X(t)$  is a **WSS R.P. with zero mean**, then the following facts are established:
- **$X_I(t)$  and  $X_Q(t)$**  are also WSS with zero mean, and  $\sigma_{X_I}^2 = \sigma_{X_Q}^2 = \sigma_X^2$ , and they are jointly stationary.
- $X_I(t)$  and  $X_Q(t)$  has the same autocorrelation functions,

$$R_{X_I}(\tau) = R_{X_Q}(\tau) \quad R_{X_I X_Q}(\tau) = R_{X_Q X_I}(-\tau) = -R_{X_Q X_I}(\tau)$$

- $X_I(t)$  and  $X_Q(t)$  are **orthogonal at the same time  $t$** :

$$R_{X_I X_Q}(0) = 0$$

$$R_{XX}(\tau) = R_{X_I X_I}(\tau) \cos 2\pi f_c \tau + R_{X_Q X_I}(\tau) \sin 2\pi f_c \tau$$

# Facts about Narrow-band R.P



Prove:

a.  $X_I(t)$  and  $X_Q(t)$  are joint stationary processes

$$\begin{aligned} R_{XX}(\tau) &= E[X(t+\tau)X(t)] \\ &= E\{ [X_I(t+\tau)\cos\omega_c(t+\tau) + X_Q(t+\tau)\sin\omega_c(t+\tau)] \\ &\quad [X_I(t)\cos(\omega_c t) + X_Q(t)\sin(\omega_c t)] \} \\ &= R_{X_I X_I}(t+\tau, t)\cos\omega_c(t+\tau)\cos(\omega_c t) \\ &\quad + R_{X_Q X_Q}(t+\tau, t)\sin\omega_c(t+\tau)\sin(\omega_c t) \\ &\quad + R_{X_I X_Q}(t+\tau, t)\cos\omega_c(t+\tau)\sin(\omega_c t) \\ &\quad + R_{X_Q X_I}(t+\tau, t)\sin\omega_c(t+\tau)\cos(\omega_c t) \end{aligned}$$



# Facts about Narrow-band R.P



$$t_1 = 0,$$

$$R_{XX}(\tau) = R_{X_I X_I}(t_1 + \tau, t_1) \cos \omega_c \tau + R_{X_Q X_I}(t_1 + \tau, t_1) \sin \omega_c \tau$$

$$\therefore R_{X_I X_I}(t + \tau, t) = R_{X_I X_I}(\tau), \quad R_{X_Q X_I}(t + \tau, t) = R_{X_Q X_I}(\tau)$$

$$t_2 = \frac{\pi}{2\omega_c},$$

$$R_{XX}(\tau) = R_{X_Q X_Q}(t_2 + \tau, t_2) \cos \omega_c \tau - R_{X_I X_Q}(t_2 + \tau, t_2) \sin \omega_c \tau$$

$$\therefore R_{X_Q X_Q}(t + \tau, t) = R_{X_Q X_Q}(\tau), \quad R_{X_I X_Q}(t + \tau, t) = R_{X_I X_Q}(\tau)$$

So,  $X_I(t)$  and  $X_Q(t)$  are joint stationary processes.

$$\begin{aligned} \therefore R_{XX}(\tau) &= R_{X_I X_I}(\tau) \cos \omega_c \tau + R_{X_Q X_I}(\tau) \sin \omega_c \tau \\ &= R_{X_Q X_Q}(\tau) \cos \omega_c \tau - R_{X_I X_Q}(\tau) \sin \omega_c \tau \end{aligned}$$

# Facts about Narrow-band R.P



b. Mean values and variances of  $X_I(t)$  and  $X_Q(t)$

$$X(t) = X_I(t) \cos \omega_c t + X_Q(t) \sin \omega_c t$$

$$t_1 = 0, \quad E[X(t_1)] = E[X_I(t_1)] = 0$$

$$t_2 = \frac{\pi}{2\omega_c}, \quad E[X(t_2)] = E[X_Q(t_2)] = 0$$

$$\begin{aligned} R_{XX}(\tau) &= R_{X_I X_I}(\tau) \cos \omega_c \tau + R_{X_Q X_I}(\tau) \sin \omega_c \tau \\ &= R_{X_Q X_Q}(\tau) \cos \omega_c \tau - R_{X_I X_Q}(\tau) \sin \omega_c \tau \end{aligned}$$

$$\therefore R_{XX}(0) = R_{X_I X_I}(0) = R_{X_Q X_Q}(0) = \sigma_X^2$$

$$\sigma_{X_I}^2 = \sigma_{X_Q}^2 = \sigma_X^2$$

# Facts about Narrow-band R.P



c.  $X_I(t)$  and  $X_Q(t)$  are uncorrelated at same times t .

$$\begin{aligned} R_{XX}(\tau) &= R_{X_I X_I}(\tau) \cos \omega_c \tau + R_{X_Q X_I}(\tau) \sin \omega_c \tau \\ &= R_{X_Q X_Q}(\tau) \cos \omega_c \tau - R_{X_I X_Q}(\tau) \sin \omega_c \tau \end{aligned}$$

$$\therefore R_{X_I X_I}(\tau) = R_{X_Q X_Q}(\tau)$$

$$\left. \begin{aligned} R_{X_I X_Q}(\tau) &= -R_{X_Q X_I}(\tau) & (1) \\ R_{X_I X_Q}(\tau) &= R_{X_Q X_I}(-\tau) & (2) \end{aligned} \right\} \Rightarrow \begin{aligned} R_{X_Q X_I}(-\tau) &= -R_{X_Q X_I}(\tau) \\ R_{X_I X_Q}(-\tau) &= -R_{X_I X_Q}(\tau) \end{aligned}$$

$$\therefore R_{X_I X_Q}(0) = R_{X_Q X_I}(0) = 0$$

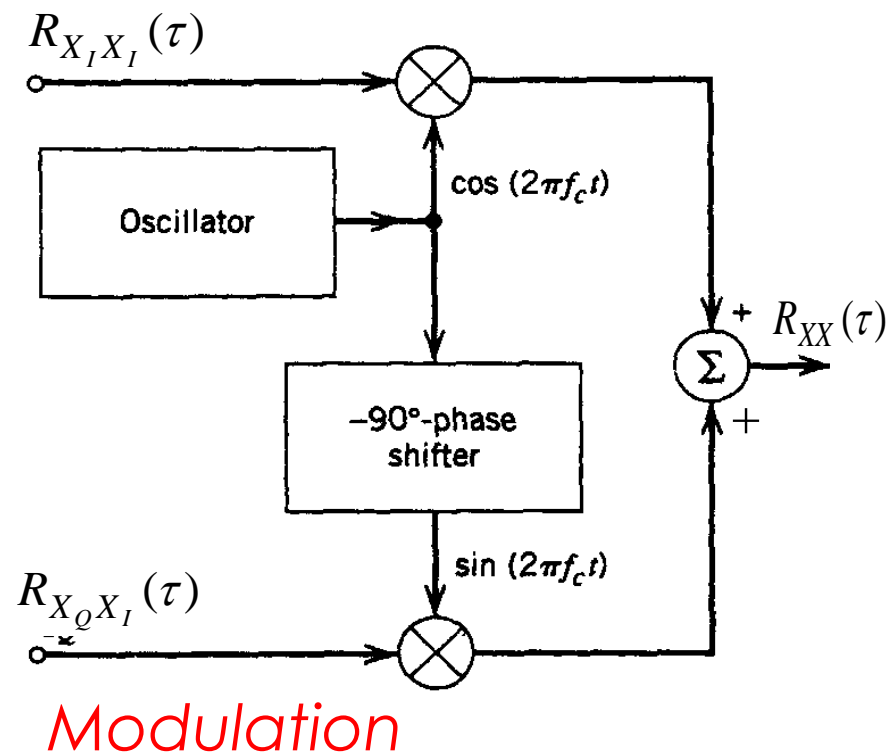
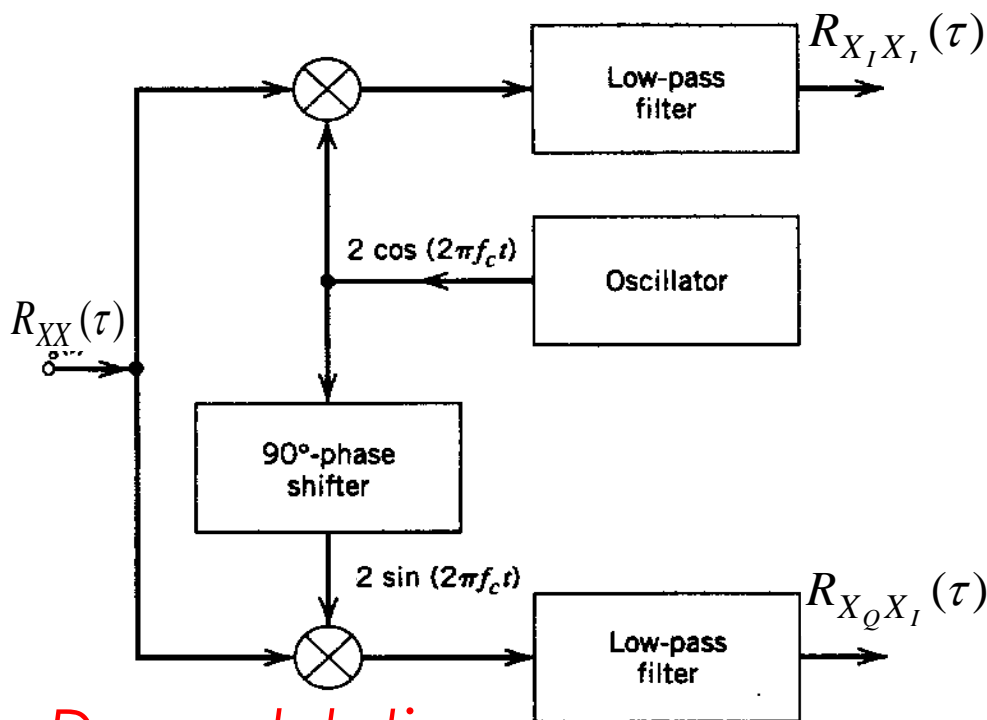
# Facts about Narrow-band R.P.(2)



$$X(t) = X_I(t) \cos \omega_c t + X_Q(t) \sin \omega_c t$$

$$R_{XX}(\tau) = R_{X_I X_I}(\tau) \cos \omega_c \tau + R_{X_Q X_I}(\tau) \sin \omega_c \tau$$

- The relationship between correlation functions is similar to that of between  $X(t)$ ,  $X_I(t)$  and  $X_Q(t)$ .



# Facts about Narrow-band R.P.(3)



- Relationship between power spectrum:

$$R_{X_I}(\tau) = \text{LPF}\{2R_{XX}(\tau) \cos(\omega_c \tau)\}$$

$$R_{X_Q X_I}(\tau) = \text{LPF}\{2R_{XX}(\tau) \sin(\omega_c \tau)\}$$

- $S_{X_I}(\omega), S_{X_Q}(\omega)$  concentrates in  $|\omega| < \Delta\omega/2$ , so they are low-frequency R.P., then:

$$S_{X_I}(\omega) = S_{X_Q}(\omega) = S_X(\omega + \omega_c) + S_X(\omega - \omega_c) \quad |\omega| < \Delta\omega/2$$

$$S_{X_Q X_I}(\omega) = -S_{X_I X_Q}(\omega) = j[S_X(\omega + \omega_c) - S_X(\omega - \omega_c)] \quad |\omega| < \Delta\omega/2$$

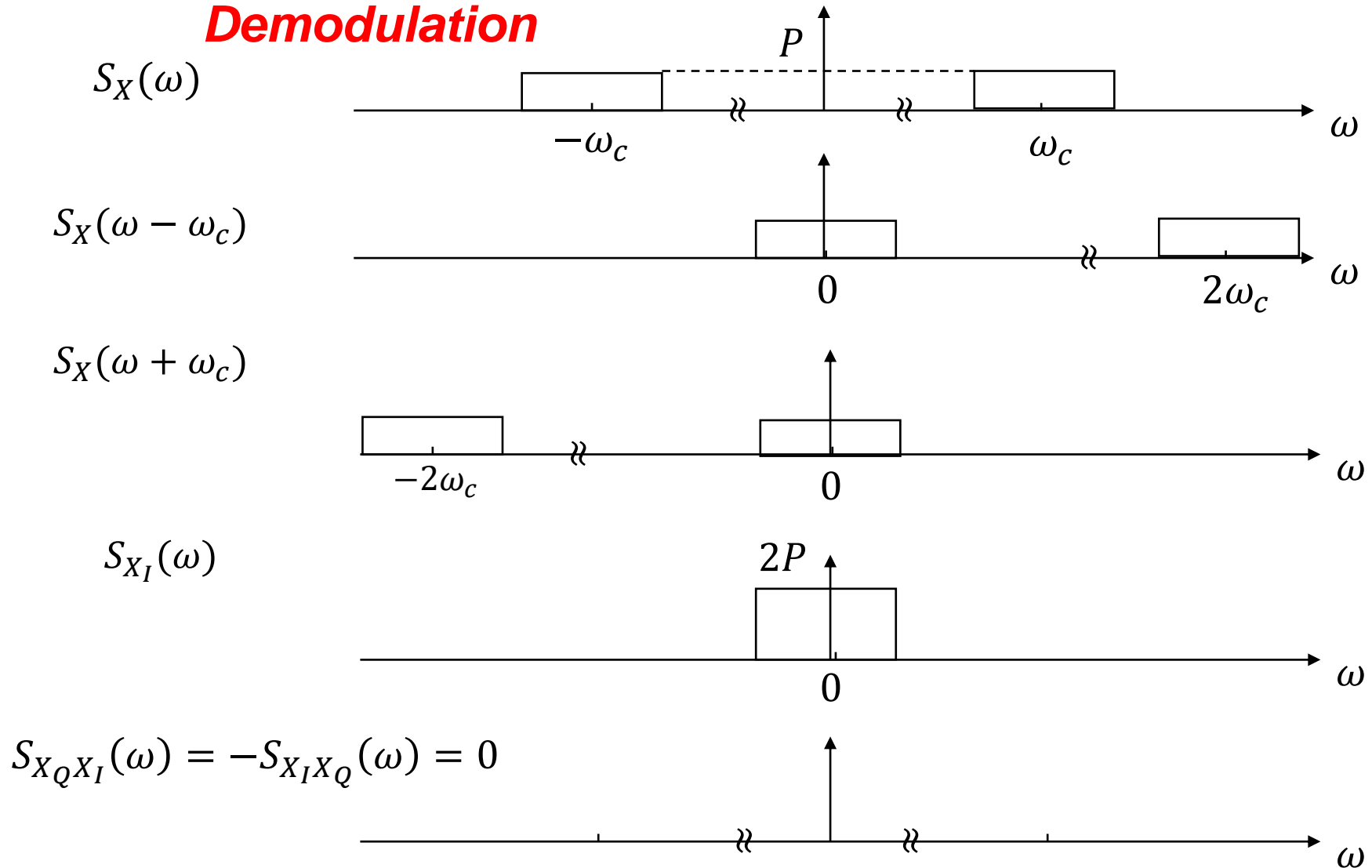
- If the S.D.F of  $X(t)$  is *symmetry* about  $\omega_c$ , then:

$$S_{X_I X_Q}(\omega) = S_{X_Q X_I}(\omega) = 0$$

# Relationship between power spectrum:

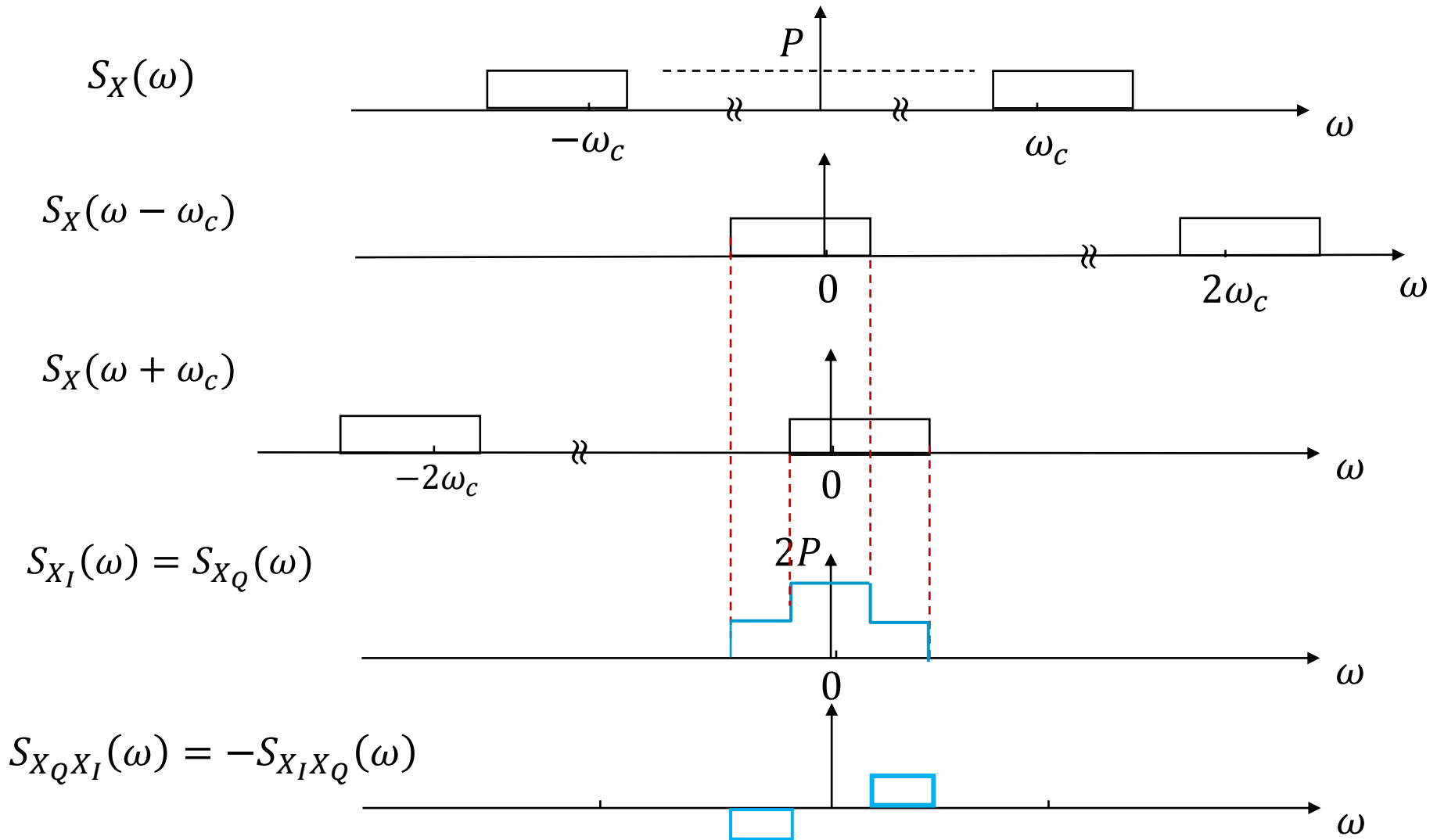


## Demodulation



The S.D.F of  $X(t)$  is symmetry about  $\omega_c$

# Relationship between power spectrum:



The S.D.F of  $X(t)$  is not symmetry about  $\omega_c$

# Facts About Narrow-band R.P.(4)



- If the S.D.F of  $X(t)$  is *symmetry* about  $\omega_c$ , then:

$$S_{X_I X_Q}(\omega) = S_{X_Q X_I}(\omega) = 0$$

$$R_{X_I X_Q}(\tau) = R_{X_Q X_I}(\tau) = 0$$

- As  $X_I(t)$  and  $X_Q(t)$  are WSS with zero mean,  $X_I(t)$  and  $X_Q(t)$  are *uncorrelated* at any time and *orthogonal* at any time.
- And:

$$R_X(\tau) = R_{X_I}(\tau) \cos \omega_c \tau = R_{X_Q}(\tau) \sin \omega_c \tau$$



## 4.4 Narrow-band Gaussian Processes



- 4.4.1 The Definition of Band
- 4.4.2 An introduction of Hilbert Transform
- 4.4.3 Representation of Narrow-Band Signals
- 4.4.4 Narrow Band Random Processes
- 4.4.5 Gaussian Narrow-Band Random Processes
- 4.4.6 Sine Wave Plus Narrow-Band Noise



1. Distribution of Amplitudes and Phases for Narrow-Band Processes
  - a. Peyton Z. Peebles, Probability, Random variables, and Random Signal Principles, O211 W58/2, Section 8.6, Bandpass, Band-limited and Narrowband Processes
  - b. 樊昌信, 通信原理, TN91 20/4, Section 2.6 窄带随机过程
2. Distribution of a Sinusoidal Signal Plus Noise
  - a. Section 10.6, Envelop and Phase of a sinusoidal signal plus noise
  - b. Section 2.7 正弦波加窄带高斯过程

## 4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes



**Suppose:** 
$$X(t) = A(t) \cos[\omega_0 t + \varphi(t)]$$
$$= X_I(t) \cos \omega_0 t + X_Q(t) \sin \omega_c t$$

- (1)  $X(t)$  is a narrow-band Gaussian Random Process with zero mean;
- (2) The variance of  $X(t)$  is  $\sigma_X^2$ ;
- (3) The P.D.F of  $X(t)$  is symmetry about  $\omega_c$ .

**Obtain:** Distribution of Envelope and Phase of  $X(t)$

$$A(t) = \sqrt{X_I^2(t) + X_Q^2(t)}$$
$$\phi(t) = \arctan \frac{X_Q(t)}{X_I(t)}$$

$$X_I(t) = X(t) \cos \omega_0 t + \hat{X}(t) \sin \omega_0 t$$
$$X_Q(t) = X(t) \sin \omega_0 t - \hat{X}(t) \cos \omega_0 t$$

## 4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes



- $X_I(t)$  and  $X_Q(t)$  are all linear combination of  $X(t)$ , so, if  $X(t)$  is a Gaussian random variable, then ,  $X_I(t)$  and  $X_Q(t)$  are also *Gaussian* random variable with zero mean and variance  $\sigma_X^2$  .

**Step1:** Find the distribution of  $X_I(t)$  and  $X_Q(t)$

**Step2:** Using Jacobian transformation to find the distribution of  $A(t)$  and  $\phi(t)$

Step1:

- (1) Since the P.D.F of  $X(t)$  is symmetry about  $\omega_c$  ,  $X_I(t)$  and  $X_Q(t)$  are *uncorrelated*.  $X_I(t)$  and  $X_Q(t)$  are Gaussian random variable, then they are independent;

- (2) 
$$f_{X_I X_Q}(x_i, x_q) = f_{X_I}(x_i) f_{X_Q}(x_q) = \frac{1}{2\pi\sigma_X^2} \exp\left[-\frac{x_{it}^2 + x_{qt}^2}{2\sigma_X^2}\right]$$

## 4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes



- Let  $A_t, \phi_t$  denote  $A(t), \phi(t)$  respectively, then:

$$f_{A\phi}(a_t, \phi_t) = |J| f_{X_I X_Q}(x_{it}, x_{qt})$$

- And: 
$$X_{it} = A_t \cos \phi_t, \quad X_{qt} = A_t \sin \phi_t$$
$$0 \leq A_t < \infty, \quad 0 \leq \phi_t < 2\pi$$

- So: 
$$f_{A\phi}(a_t, \phi_t) = |J| f_{X_I X_Q}(x_{it}, x_{qt})$$
$$= a_t f_{X_I X_Q}(x_{it}, x_{qt})$$
$$= \frac{a_t}{2\pi\sigma_X^2} \exp\left[-\frac{a_t^2}{2\sigma_X^2}\right] \quad a_t \geq 0, 0 \leq \phi_t < 2\pi$$

## 4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes



- Thus:

$$f_A(a_t) = \int_0^{2\pi} f_{A\phi}(a_t, \varphi_t) d\varphi_t = \frac{a_t}{\sigma_X^2} \exp\left(-\frac{a_t^2}{2\sigma_X^2}\right) \quad a_t \geq 0$$

$$f_\phi(\varphi_t) = \int_0^\infty f_{A\phi}(a_t, \varphi_t) da_t = \frac{1}{2\pi} \quad 0 \leq \phi_t < 2\pi$$

- That is: The envelope follows a *Reyleigh distribution*; the phase follows a uniform distribution.
- And:  $f_{A\phi}(a_t, \varphi_t) = f_A(a_t) \cdot f_\phi(\varphi_t)$  illustrates that the *envelope and phase are independent*.

**The mean of Reyleigh:**

$$\sigma_X \sqrt{\frac{\pi}{2}} \approx 1.253\sigma_X$$

**The variance of Reyleigh:**

$$\sigma_X^2 \frac{4 - \pi}{2} \approx 0.429\sigma_X^2$$

## 4.4 Narrow-band Gaussian Processes



4.4.1 The Definition of Band

4.4.2 An introduction of Hilbert Transform

4.4.3 Representation of Narrow-Band Signals

4.4.4 Narrow Band Random Processes

4.4.5 Gaussian Narrow-Band Random Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

## 4.4.6 Distribution of a Sinusoidal Signal Plus Noise



*Given:*  $r(t)$  is a sinusoidal signal with random phase plus a zero-mean, stationary Gaussian random process with a narrow-band spectrum,

$$r(t) = s(t) + n(t)$$

$$s(t) = A \cos[\omega_0 t + \theta]$$

$$n(t) = n_c(t) \cos(\omega_0 t) - n_s(t) \sin(\omega_0 t)$$

where  $A$  is constant,  $\theta$  is uniformly distributed on  $[0, 2\pi]$ ,  $n(t)$  is the narrow-band Gaussian noise with zero mean and variance  $\sigma_n^2$ .

*Obtain:* Probability density function of amplitude and phase of  $r(t)$ .



## 4.4.6 Distribution of a Sinusoidal Signal Plus Noise



Sln:

$$\begin{aligned} r(t) &= A \cos[\omega_0 t + \theta] + n_c(t) \cos(\omega_0 t) - n_s(t) \sin(\omega_0 t) \\ &= [A \cos \theta + n_c(t)] \cos(\omega_0 t) - [A \sin \theta + n_s(t)] \sin(\omega_0 t) \\ &= Z_c(t) \cos(\omega_0 t) - Z_s(t) \sin(\omega_0 t) \\ &= Z(t) \cos[\omega_0 t + \phi(t)] \end{aligned}$$

Where,

$$Z_c(t) = Z(t) \cos \phi(t)$$

$$Z_s(t) = Z(t) \sin \phi(t)$$

$$Z(t) = \sqrt{Z_c^2(t) + Z_s^2(t)}$$

$$\phi(t) = \arctan \frac{Z_s(t)}{Z_c(t)}$$

Step 1: find the joint distribution of  $Z_c(t)$  and  $Z_s(t)$

Step 2: Using Jacobian transformation to find the distribution of  $Z(t)$  and  $\phi(t)$

## 4.4.6 Distribution of a Sinusoidal Signal Plus Noise



- Modified Bessel function of order zero

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp[x \cos \theta] d\theta$$

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}} \quad x \gg 1$$

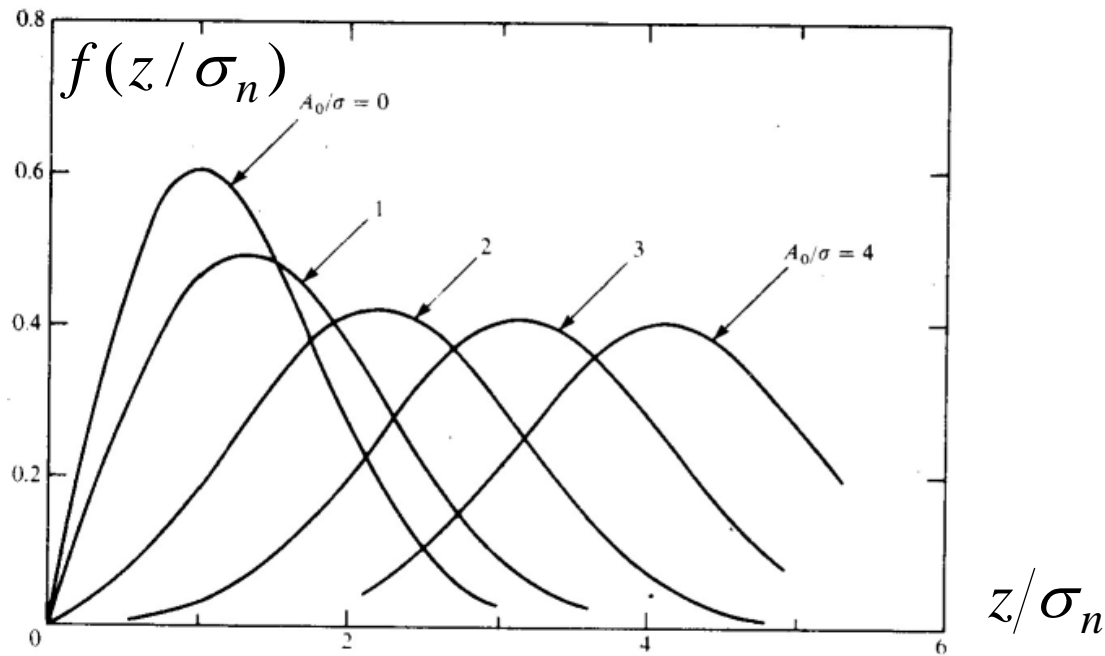
- Probability densities of the envelope follows Rice distribution:

$$f(z) = \frac{z}{\sigma_n^2} \exp\left[-\frac{1}{2\sigma_n^2}(z^2 + A^2)\right] I_0\left(\frac{Az}{\sigma_n^2}\right) \quad z \geq 0$$

## 4.4.6 Distribution of a Sinusoidal Signal Plus Noise



Probability densities of the envelope for various ratios  $\frac{A}{\sigma_n}$



$$\frac{A}{\sigma_n} \ll 1$$

,  $f(z/\sigma_n)$  is similar to Rayleigh distribution;

$$\frac{A}{\sigma_n} \gg 1$$

,  $f(z/\sigma_n)$  is similar to standard Gaussian distribution.

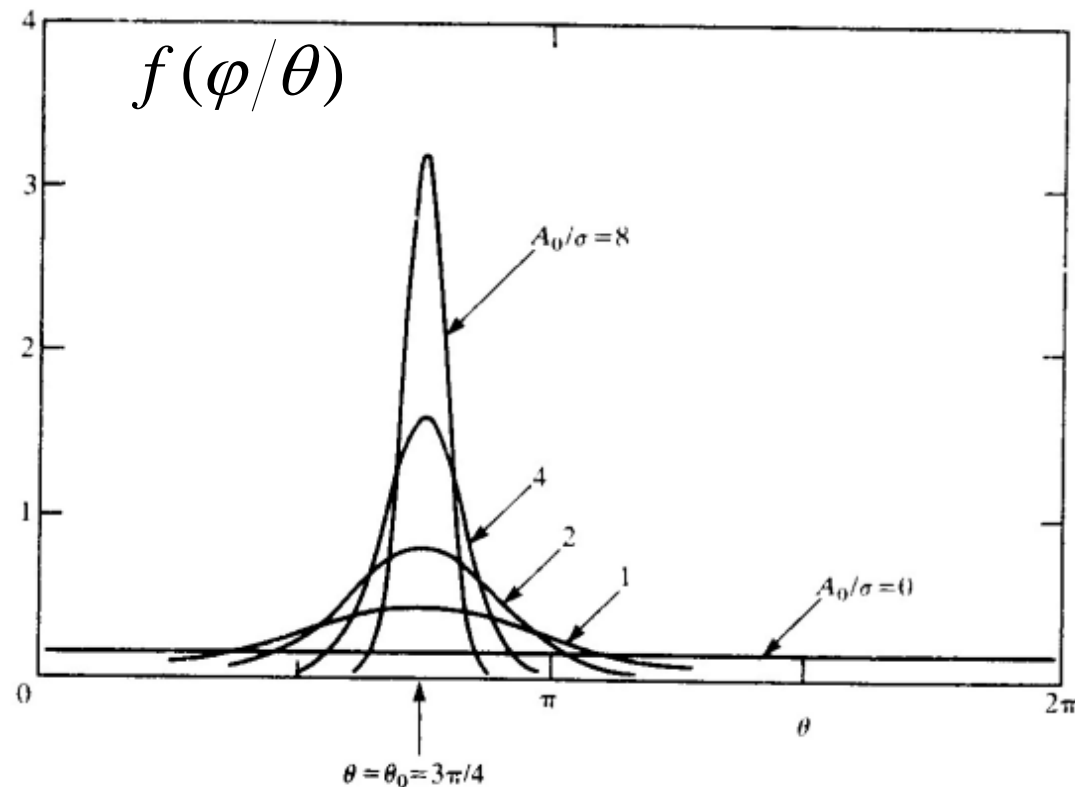
## 4.4.6 Distribution of a Sinusoidal Signal Plus Noise



Probability densities of the phase for various ratios  $\frac{A}{\sigma_n}$ .

$$f(\varphi/\theta) = \frac{1}{2\pi} e^{-\frac{\alpha^2}{2}} + \frac{\alpha \cos(\varphi - \theta)}{\sqrt{2\pi}} e^{-\frac{\alpha^2 \sin^2(\varphi - \theta)}{2}} \Phi \left[ \frac{\alpha \cos(\varphi - \theta)}{\sigma_n} \right]$$

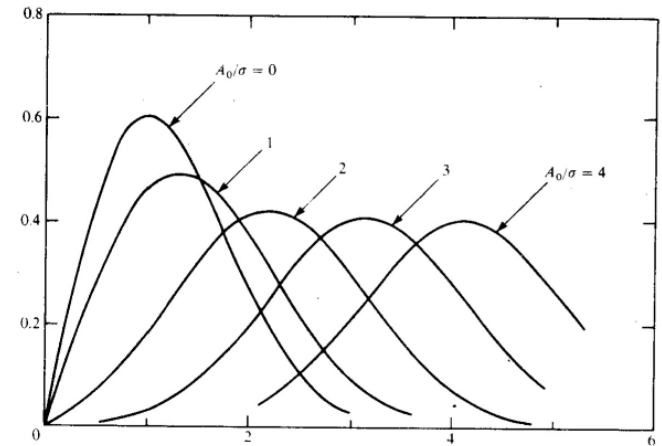
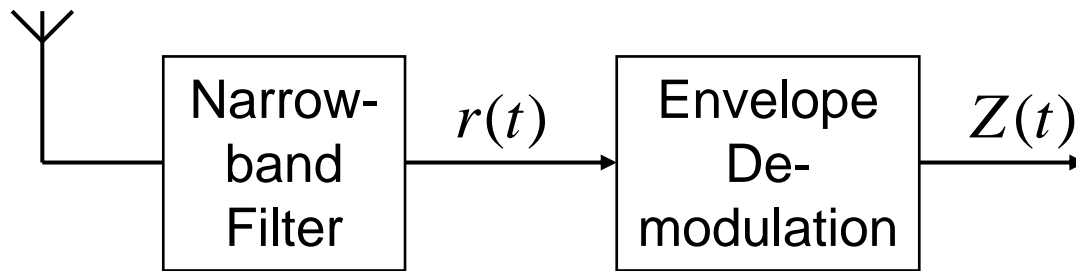
$$\alpha = \frac{A}{\sigma_n}$$



# Application



- From the amplitude  $Z(t)$  of the received signal  $r(t)$ , we may detect **whether the  $s(t)$  exists**.



- If  $s(t)$  exists**,  $Z(t)$  follows **Rice distribution**. More large of  $s(t)$  than noise, the peak wave is further from y-axis.
- If  $s(t)$  is not exist**,  $Z(t)$  follows **Reyleigh distribution** which is near to y-axis.

# Homework



4.14,

4.16



$$x(t) = \text{sgn}(t) = \begin{cases} +1 & (t > 0) \\ -1 & (t < 0) \end{cases} \quad X(j\omega) = \lim_{a \rightarrow 0} X_1(j\omega) = \lim_{a \rightarrow 0} \frac{-2j\omega}{a^2 + \omega^2} = \frac{2}{j\omega}$$

$$x(t) = \sin \omega_0 t \quad X(j\omega) = \pi j(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$$

$$x(t) = \cos \omega_0 t \quad X(j\omega) = \pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$$