UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 02
Spring 2023

1. **The Probabilistic Method**

We introduce a proof technique — the *probabilistic method*. If we wish to show that there exists an element with property A in a set $\mathcal{X}$, it suffices to show that there exists a probability distribution p over $\mathcal{X}$ such that under p, the probability assigned to elements with property A is greater than 0.

(Why does this work? If there is no element with property A, then there cannot possibly exist a p that assigns a positive probability to elements with property A, because we require that $p(\varnothing) = 0$.) Such a proof method is nonconstructive, meaning that it doesn't provide a method for finding such an element, yet it demonstrates the element exists.

Consider a sphere that has $\frac{1}{10}$ of its surface colored blue, and the rest colored red. Show that no matter how the colors are distributed, it is possible to inscribe a cube in the sphere with all of its vertices red.

Hint: If we sample an inscribed cube uniformly at random among all possible inscribed cubes, what is (an upper bound on) the probability that the sampled cube has at least one blue vertex?

Solution: Pick an inscribed cube uniformly at random, enumerate its vertices $1, \ldots, 8$, and let B_i be the event that vertex i is blue. Note that

$$\mathbb{P}\left(\bigcup_{i=1}^{8} B_i\right) \leq \sum_{i=1}^{8} \mathbb{P}(B_i) = \sum_{i=1}^{8} \frac{1}{10} = \frac{8}{10} < 1.$$

In other words, the probability of at least one vertex being blue is strictly less than 1, so the probability that no vertex is blue (every vertex is red) is strictly greater than 0. Because a *randomly sampled* inscribed cube has nonzero probability of having all vertices red, there must exist at least one inscribed cube with all vertices red by the probabilistic method.

2. **Suspicious Game**

You are playing a card game with your friend in which you take turns picking a card from a deck. (Assume that you never run out of cards.) If you draw one of the special *bullet* cards, then you lose the game. Unfortunately, you do not know the contents of the deck. Your friend claims that $\frac{1}{3}$ of the deck is filled with bullet cards. However, you don't fully trust your friend: you believe he is lying with probability $\frac{1}{4}$. Assume that if your friend is lying, then the opposite is true: $\frac{2}{3}$ of the deck is filled with bullet cards!

What is the probability that you win the game if you go first?

Solution: Let p denote the probability of randomly selecting a bullet card; p stays the same since you never run out of cards. Because the game ends when the first bullet card is drawn, the number of turns N before the game ends is a Geometric random variable with parameter p. The probability that you win is the probability that N is even, so we have

$$\begin{aligned}\mathbb{P}(\text{win} \mid p) = \mathbb{P}(N \text{ is even} \mid p) &= \sum_{\substack{k=1 \\ k \text{ is even}}}^{\infty} p(1-p)^{k-1} = p(1-p)\sum_{i=0}^{\infty}(1-p)^{2i} \\ &= \frac{p(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}.\end{aligned}$$

Now, we don't know whether $p = \frac{1}{3}$ or $\frac{2}{3}$, so we can use the law of total probability:

$$\begin{aligned}\mathbb{P}(\text{win}) &= \mathbb{P}\left(\text{win} \mid p = \frac{1}{3}\right) \cdot \mathbb{P}\left(p = \frac{1}{3}\right) + \mathbb{P}\left(\text{win} \mid p = \frac{2}{3}\right) \cdot \mathbb{P}\left(p = \frac{2}{3}\right) \\ &= \frac{2}{5} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} \\ &= 0.3625.\end{aligned}$$

Note. It may be tempting to first calculate p as

$$\mathbb{P}(B) = \mathbb{P}(B \mid L) \cdot \mathbb{P}(L) + \mathbb{P}(B \mid L^c) \cdot \mathbb{P}(L^c) = \frac{5}{12},$$

where B is the event of drawing a bullet card and L is the event that your friend is lying. Then, one would plug in $p = \frac{5}{12}$ to find the probability of winning as $\frac{7}{19} \approx 0.3684$. However, this does not work as it is not the case that N is a Geometric random variable with parameter $\frac{5}{12}$: the order of conditioning is reversed in this case.

Remark. The reason why the game is suspicious is because $\frac{1-p}{2-p} \leq \frac{1}{2}$ for $p \in [0, 1]$, so your chances of winning are always unfavorable!

3. **Law of the Unconscious Statistician**

   a. Prove the *Law of the Unconscious Statistician* (LOTUS): Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X : \Omega \to \mathbb{Z}$ and $F : \mathbb{Z} \to \mathbb{Z}$ be random variables. Note that the composition $Y = F(X) : \Omega \to \mathbb{Z}$ is another random variable. If $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$, and $\mathbb{E}_{\mathcal{L}_X}$ is expectation with respect to the *law* of X on $\mathbb{Z}$, then

$$\mathbb{E}(F(X)) = \mathbb{E}_{\mathcal{L}_X}(F).$$

   You should assume that Ω is **discrete** for the sake of simplicity, although LOTUS holds more generally.

   b. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the space of all sequences of independent fair coin tosses. Formulate N, the minimum number of tosses needed until we see heads, as a random variable on Ω.

   c. Find $\mathbb{E}(N^2)$.
   Hint: By the linearity of expectation, $\mathbb{E}(N^2) = \mathbb{E}(N(N-1)) + \mathbb{E}(N)$. You may use the Law of the Unconscious Statistician from part a, and the following identity:

$$\sum_{k=1}^{\infty} k(k-1)x^{k-2} = \frac{d}{dx} \sum_{k=1}^{\infty} kx^{k-1}.$$

Solution:

   a. By the definition of expectation, the left-hand side is equal to

$$\begin{aligned}
\mathbb{E}(F(X)) &= \sum_{y \in \mathbb{Z}} y \, \mathbb{P}(F(X) = y) \\
&= \sum_{y \in \mathbb{Z}} y \, \mathbb{P}(X \in \{x : F(x) = y\}).
\end{aligned}$$

   The $\mathbb{Z}$ above refers to the second $\mathbb{Z}$ in $\Omega \xrightarrow{X} \mathbb{Z} \xrightarrow{f} \mathbb{Z}$. Now, the law $\mathcal{L}_X$ of X is a probability measure on (the first) $\mathbb{Z}$, such that

$$\mathcal{L}_X(B) = \mathbb{P}(X \in B) \quad \text{for } B \subset \mathbb{Z}.$$

   So, the above is precisely equal to

$$\begin{aligned}
&= \sum_{y \in \mathbb{Z}} y \mathcal{L}_X(\{x : F(x) = y\}) \\
&= \sum_{y \in \mathbb{Z}} y \mathcal{L}_X(F = y) = \mathbb{E}_{\mathcal{L}_X}(F).
\end{aligned}$$

   b. We write each outcome ω as $(\omega_1, \omega_2, \omega_3, \ldots)$, where $\omega_n \in \{H, T\}$ is the result of the nth toss. Then, we define N by

$$N(\omega) = \min\{n \geq 1 : \omega_n = H\}.$$

c. Per the hint, $\mathbb{E}(N^2) = \mathbb{E}(N(N-1)) + \mathbb{E}(N)$. We observe that N is a Geometric random variable with parameter $p = \frac{1}{2}$, which has expected value $\mathbb{E}(N) = 2$. Now, by the Law of the Unconscious Statistician,

$$\begin{aligned}
\mathbb{E}(N(N-1)) &= \sum_{k=1}^{\infty} k(k-1)\,\mathbb{P}(N=k) \\
&= \sum_{k=1}^{\infty} k(k-1)p(1-p)^{k-1}
\end{aligned}$$

Pulling out a factor of $p(1-p)$, we can apply the final hint.

$$\begin{aligned}
&= p(1-p)\sum_{k=1}^{\infty} k(k-1)(1-p)^{k-2} \\
&= -p(1-p)\frac{d}{dp}\sum_{k=1}^{\infty} k(1-p)^{k-1}
\end{aligned}$$

Note that $\mathbb{E}(N) = \frac{1}{p}$ is, by definition, equal to $p\sum_{k=1}^{\infty} k(1-p)^{k-1}$. Then,

$$\begin{aligned}
&= -p(1-p)\frac{d}{dp}\frac{1}{p^2} \\
&= \frac{2(1-p)}{p^2}.
\end{aligned}$$

For $p = \frac{1}{2}$, we find that $\mathbb{E}(N^2) = 4 + 2 = 6$.

Alternatively, we can observe the following recurrence relation. We remark that the approach above of finding $\mathbb{E}(N(N-1))$ is also applicable when N is *Poisson*, but not the following approach.

$$\begin{aligned}
\mathbb{E}(N^2) = \sum_{k=1}^{\infty} k^2 p(1-p)^{k-1} &= 1^2 \cdot p + (1-p)\sum_{k=1}^{\infty}(k+1)^2 p(1-p)^{k-1} \\
&= p + (1-p)\,\mathbb{E}((N+1)^2) \\
&= p + (1-p)(\mathbb{E}(N^2) + 2\,\mathbb{E}(N) + 1).
\end{aligned}$$