Chap 4 Spectral Analysis of Stochastic Processes

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Content:

4.3 Spectrum of Amplitude-modulated Signals

Random Amplitude Processes

Consider a signal of the form

$$Y(t) = \sqrt{2}A(t)\cos(w_c t + \theta), \qquad t > 0$$

whereas A(t) is a random process representing the amplitude and θ is a random variable uniformly distributed between 0 and 2π , and w_c is constant. A(t) and θ are *independent*.

Obtain: power spectrum of Y(t)

$\sqrt{2}\cos(w_c t + \theta)$ is seen as the *carrier signal*.

A(t) is the *modulation signal*. For analog communications, A(t) may represent a speech signal. In digital communications, A(t) is a continuous-time wave form that represents a sequence of data pulses.

Random Amplitude Processes

$$E[Y(t)] = \sqrt{2}E[A(t)\cos(w_c t + \theta)]$$
$$= \sqrt{2}E[A(t)]E[\cos(w_c t + \theta)] = 0$$

$$R_{YY}(t, t+\tau) = E[Y(t)Y(t+\tau)]$$
$$= 2E[A(t)\cos(w_c t + \theta)A(t+\tau)\cos(w_c t + w_c \tau + \theta)]$$
$$= 2E[A(t)A(t+\tau)]E[\cos(w_c t + \theta)\cos(w_c t + w_c \tau + \theta)]$$
$$= E[A(t)A(t+\tau)]\cos(w_c \tau)$$
$$= R_{AA}(t, t+\tau)\cos(w_c \tau)$$

If A(t) is a stationary process, then Y(t) is also stationary with the correlation function

$$R_{YY}(\tau) = R_{AA}(\tau)\cos(w_c \tau)$$

Random Amplitude Processes

Modulation theorem of Fourier transforms:

- If v(t) has Fourier transform V(f), then the Fourier transform of the signal

$$w(t) = v(t)\cos(w_c t)$$

is

$$W(f) = \frac{1}{2}[V(f - f_c) + V(f + f_c)]$$

where $w_c = 2\pi f_c$

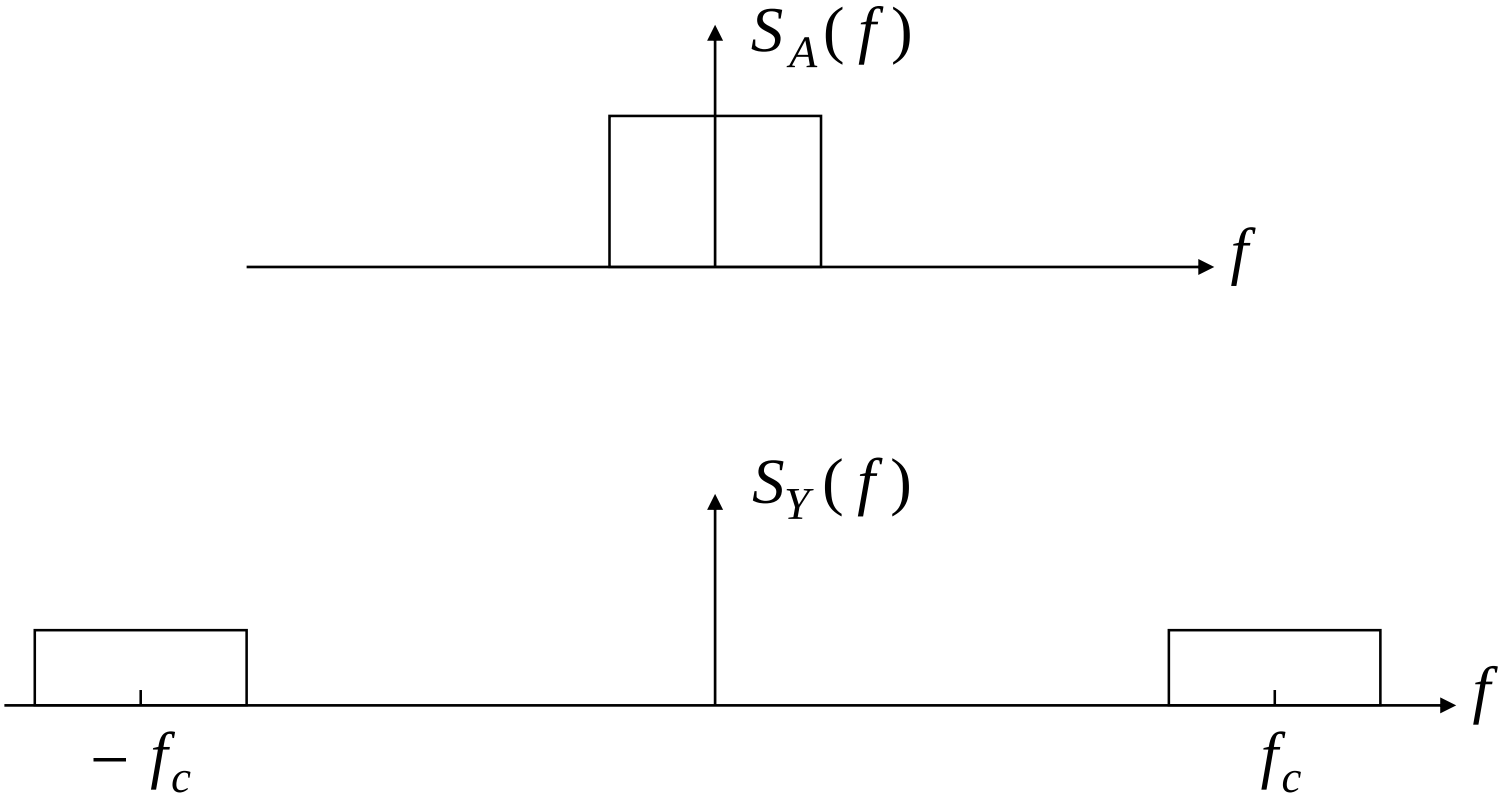
The spectrum of Y(t) is $S_Y(f) = \frac{1}{2}[S_A(f - f_c) + S_A(f + f_c)]$

$$Y(t) = \sqrt{2}A(t)\cos(w_c t + \theta), \qquad t > 0$$

$$R_{YY}(\tau) = R_{AA}(\tau)\cos(w_c \tau)$$

Random Amplitude Processes

- Spectrum of amplitude-modulated signal Y(t)



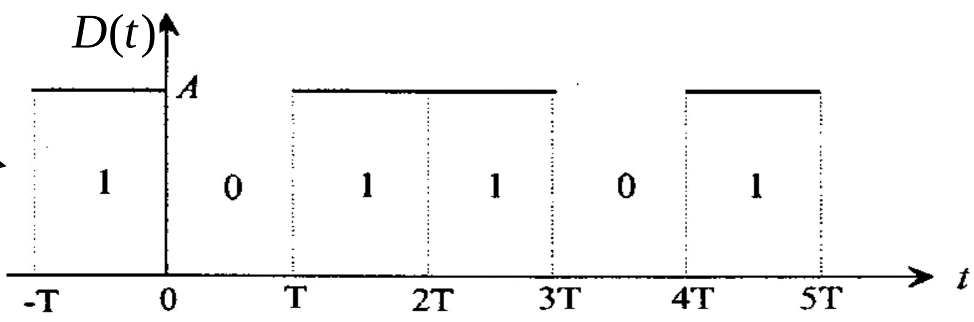
A random data signal with rectangular pulses

- Rectangular pulse: $$p_T(t) = \begin{cases} 1 & \text{for } 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$
- Define a random process as

$$D(t) = \sum_{n=-\infty}^{\infty} A_n p_T(t-nT), \;\; nT \le t < (n+1)T$$

- where *An* is a random variable representing the amplitude of the *n*th pulse. *An*, $-\infty < n < \infty$, are independent and identically distributed random variables.

A sample wave of D(t), where *An* is 0 or 1 (binary case)



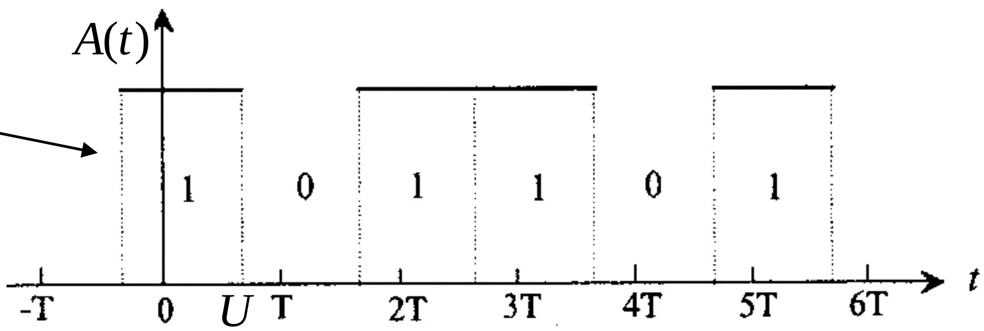
Pulse Code Modulation

- D(t) is *not wide-sense stationary* because it's correlation function depends on the absolute value of time.

A random data signal with rectangular pulses

- Random delay of D(t): $A(t) = D(t - U), \qquad -\infty < t < \infty$
- where U is uniformly distributed on the interval [0,T].

A sample wave of A(t), where A_n is 0 or 1 (binary case)



- A(t) is *wide-sense stationary*.
- Prove:
- Let $\boxed{E[A_n] = 0}$, $\boxed{E[A_n^2] = \alpha^2}$, then

$E[A(t)] = E[D(t)] = 0$, and $E[A_n A_k] = 0$ for $n \neq k$

A random data signal with rectangular pulses

- Correlation function of A(t)

$$E[A(t)A(t+\tau)] = E\left\{ \sum_{n=-\infty}^{\infty} A_n p_T(t-nT-U) \sum_{k=-\infty}^{\infty} A_k p_T(t+\tau-kT-U) \right\}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} E[A_n A_k] E[p_T(t-nT-U) p_T(t+\tau-kT-U)]$$

$$= \sum_{n=-\infty}^{\infty} \alpha^2 E[p_T(t-nT-U) p_T(t+\tau-nT-U)]$$

$$E[p_T(t-nT-U) p_T(t+\tau-nT-U)]$$

$$= \frac{1}{T} \int_0^T p_T(t-nT-u) p_T(t+\tau-nT-u) du \quad , \qquad v = u + nT$$

$$= \frac{1}{T} \int_{nT}^{(n+1)T} p_T(t-v) p_T(t+\tau-v) dv$$

A random data signal with rectangular pulses

$$E[A(t)A(t+\tau)] = \sum_{n=-\infty}^{\infty} \alpha^2 \frac{1}{T} \int_{nT}^{(n+1)T} p_T(t-v)\,p_T(t+\tau-v)\,dv$$

$$= \alpha^2 \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{nT}^{(n+1)T} p_T(t-v)\,p_T(t+\tau-v)\,dv$$

$$= \alpha^2 \frac{1}{T} \int_{-\infty}^{\infty} p_T(t-v)\,p_T(t+\tau-v)\,dv$$

Let $u = t + \tau - v$

$$E[A(t)A(t+\tau)] = \alpha^2 \frac{1}{T} \int_{-\infty}^{\infty} p_T(u-\tau)\,p_T(u)\,du$$

$$R_A(\tau) = \alpha^2 \frac{1}{T} p_T(\tau) * p_T(-\tau)$$

A random data signal with rectangular pulses

- Spectrum of A(t)

$$R_A(\tau) = \alpha^2 \frac{1}{T} p_T(\tau) * p_T(-\tau)$$

$$S_A(f) = \alpha^2 \frac{1}{T} |Z(f)|^2$$

- where $Z(f)$ is the Fourier transform of $p_T(\tau)$
- If $Y(t) = \sqrt{2}A(t)\cos(w_c t + \theta), \qquad t > 0$

then $S_Y(f) = \frac{\alpha^2}{2T}\left[\ |Z(f - f_c)|^2 + |Z(f + f_c)|^2\right]$

Sample wave of Y(t):

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4.4 Narrow-band Gaussian Processes

4.4.1 The Definition of Band

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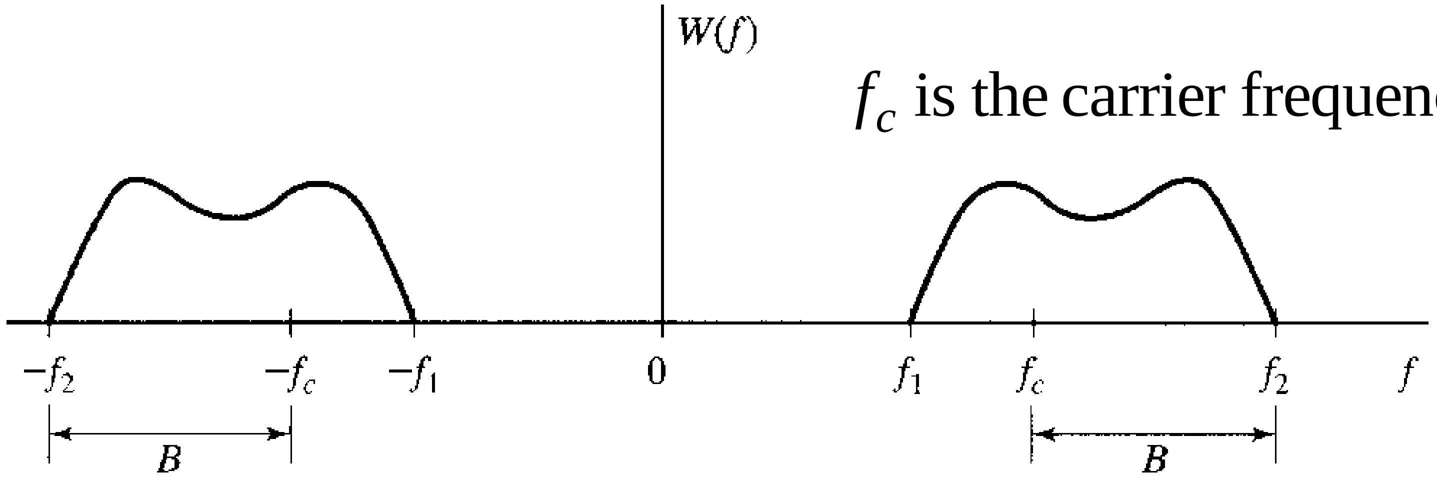
4.4.1. The Definition of Band

- In most cases, the signal we deal with is always band limited
- Band limited means that the band of a signal is finite.
- How can we define the bandwidth of a signal?
- Here, we just list some criterions for defining bandwidth

Case1: Ideal Band Pass Signals

I) Nonzero in a finite interval: $\text{bandwidth} = f_2 - f_1 \le 2B$

(Absolute bandwidth, cutoff bandwidth)



f_c is the carrier frequency

FIGURE 4–3 An ideal band-pass frequency function.

II) Half-power bandwidth (3dB bandwidth)

Let Wm be maximum value of |W(f)|,

$|W(f_3)| = |W(f_4)| = W_m/\sqrt{2}, \quad |W(f)| > W_m/\sqrt{2} \text{ for } f_3 < f < f_4$

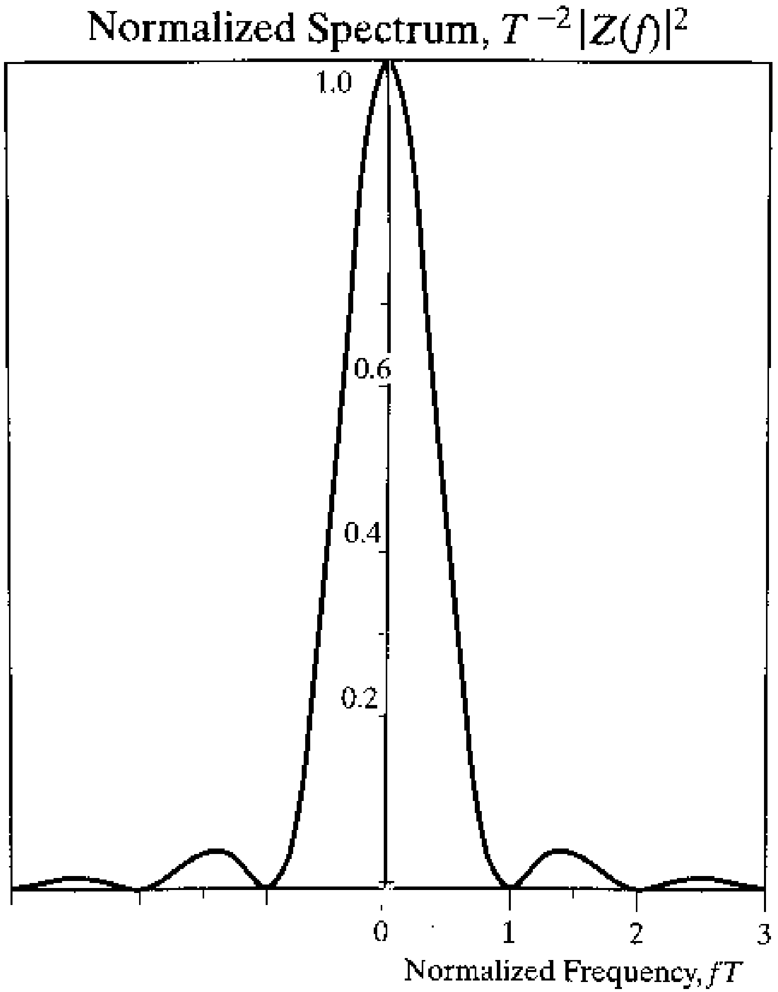
and $\quad |W(f)| < W_m/\sqrt{2} \text{ for } 0 < f < f_3 \text{ and } f_4 < f < \infty$

Case 2: Nonideal Band Pass Signals

- Null-to-null bandwidth
  - Nonzero in infinite interval
  - But the energy concentrates in a small interval

- Normalized bandwidth

1-(-1)=2



4.4 Narrow-band Gaussian Processes

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通信系统中的
数字信号正交变换理论

- 我们知道，现实中产生的物理可实现的信号是实信号
- 但通信系统却提出要将实信号正交分解为复信号，为什么要进行正交分解？直接利用现实中的信号不行吗？

- 设有一个实信号x(t)，其正交分解后的复信号为 z(t)，该信号的极坐标表示为：

$$z(t)=x(t)+j\hat{x}(t)=a(t)e^{j\varphi(t)}$$

从这个表达式中，我们很容易得到信号的：

瞬时包络 $a(t)$

瞬时相位 $\varphi(t)$

瞬时频率 $\dfrac{d(\varphi(t))}{dt}$

这三个参数，恰好是信号分析，参数测量和识别调制的基础。这就是对实信号进行解析表示的意义所在。

- 通过上面的介绍，我们知道了为什么要将信号进行正交解析表示。可是，怎样对信号进行正交表示呢?
- 一个实信号的频谱具有共轭对称性。
- 所以，对于一个实信号，只要取其正频域部分或者负频域部分就能完全加以描述，而不会丢失任何信息！并且，所得的新信号是一个复信号!

- 假设有一个信号x(t)，取其正频域部分的频谱分量，这部分频谱可以用一个复函数z(t)来表示。则：

$$Z(w)=\left\{\begin{matrix} 2X(w), & w>0 \\ X(w), & w=0 \\ 0, & w<0 \end{matrix}\right\}$$

w > 0 的分量加倍是为了使 z(t) 与原信号能量相等）。

- 再引入一个阶跃滤波器：

$$H(w)=\begin{cases}-j, & w>0\\ 0, & w=0\\ j, & w<0\end{cases}$$

这样，我们可以得到：

$$Z(w)=X(w)[1+jH(w)]$$

$$z(t)=x(t)+jx(t)*h(t)$$ *为卷积符号

- 易于求出 $h(t) = \dfrac{1}{\pi t}$

我们把 $x(t) * h(t)$ 叫做 $x(t)$ 的 *Hilbert* 变换。

我们可以发现，一个实数的 *Hilbert* 变换同原信号正交。所以，一个实信号要进行正交分解，只需要：

4.4 Narrow-band Gaussian Processes

- References

[1] Analysis of stochastic signals, 朱华，北京理工大学, *Chapter 5*

[2] Random signal analysis, 李晓风，电子工业出版社, *Chapter 6*

4.4.2 Hilbert Transform

- Def. of Hilbert Transform

$$
\begin{cases}
\hat{g}(t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{+\infty}\dfrac{g(\tau)}{t-\tau}d\tau = \dfrac{1}{\pi t} * g(t) \\[2ex]
g(t) = -\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{+\infty}\dfrac{\hat{g}(\tau)}{t-\tau}d\tau = -\dfrac{1}{\pi t} * \hat{g}(t)
\end{cases}
$$

- The functions above are Hilbert Transform pairs
- Hilbert Transform is an operation denoted by H[*]
- Relationship in frequency domain:

$$
\boxed{H(w) = \begin{cases} -j & w \geq 0 \\ j & w < 0 \end{cases}}
\qquad
\boxed{\Rightarrow \quad |H(w)| = 1, \quad \varphi(w) = \begin{cases} -\pi/2 & w \geq 0 \\ \pi/2 & w < 0 \end{cases}}
$$

$$
\boxed{\hat{G}(w) = -j\,\mathrm{sgn}(w) G(w) = H(w) G(w)}
$$

Example for Hilbert Transform

e.g. Find the H.T. for $f(x) = \cos \omega t$ and its analytical signal.

Solution: $$\hat{f}(t) = \frac{1}{\pi t} * f(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\cos \omega\tau}{t-\tau} d\tau = \sin \omega t$$

$$x(t) = \cos \omega_c t \qquad X(j\omega) = \pi(\delta(\omega+\omega_c) + \delta(\omega-\omega_c))$$

$$x(t) = \sin \omega_c t \qquad X(j\omega) = \pi j(\delta(\omega+\omega_c) - \delta(\omega-\omega_c))$$

- Hilbert Transform is like an ideal phase shifter of 90 degree
- Let $\boxed{\tilde{f}(t) = f(t) + j\hat{f}(t)}$

$\tilde{f}(t)$ is called *the analytical signal* or pre-envelope of *f*(*t*)

$$\boxed{\tilde{f}(t) = \cos \omega t + j \sin \omega t = e^{j\omega t}}$$

- The compare between the frequency spectral of $f(t)$ and $\tilde{f}(t)$
- The spectral of $\tilde{f}(t)$ is the positive parts of the spectral of *f*(*t*)

The use of Hilbert Transform

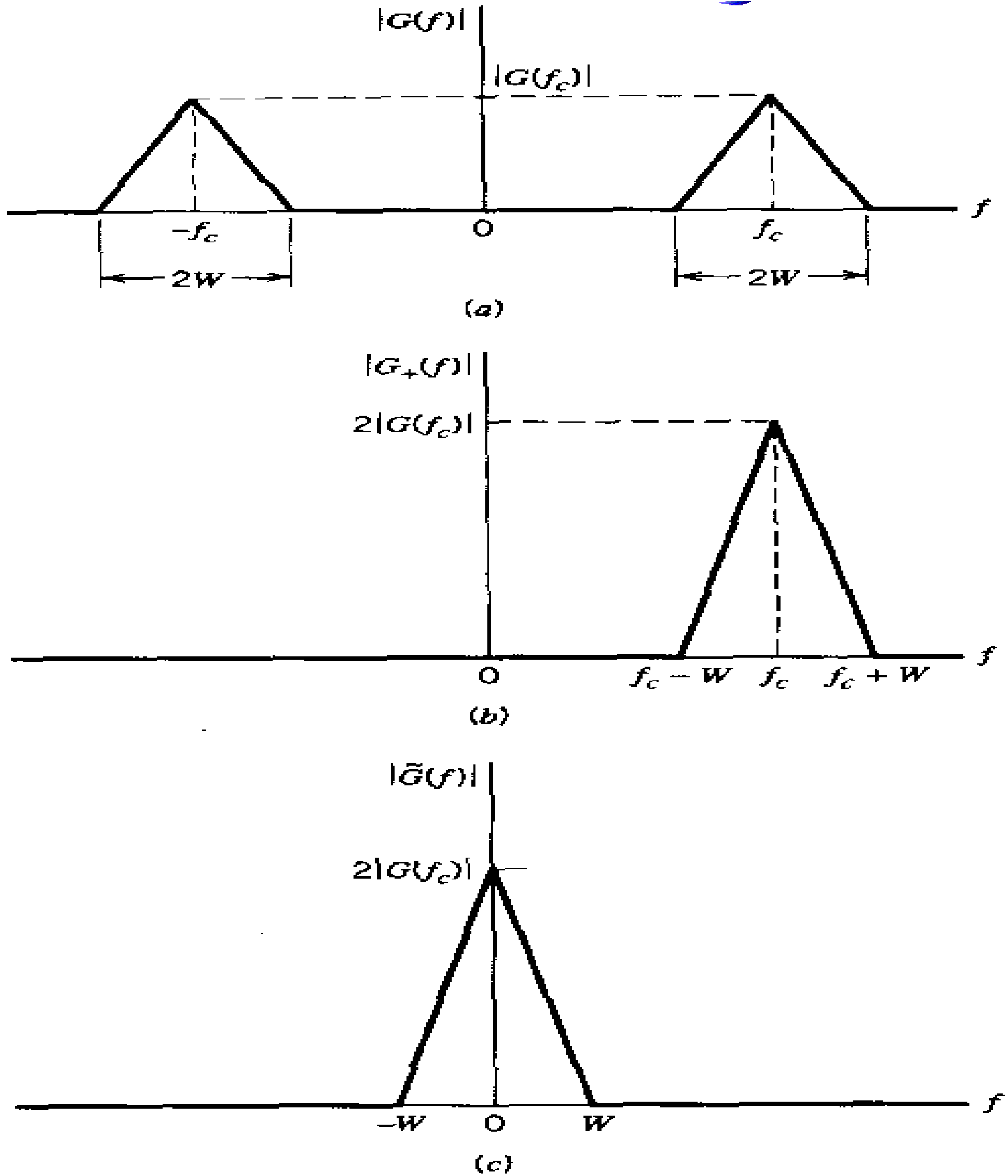
- Get the one-sided spectral: Let

$$\tilde{g}(t) = g(t) + j\hat{g}(t)$$

- Then:

$$\begin{aligned}
\tilde{G}(\omega) &= G(\omega) + j\hat{G}(\omega) \\
&= G(\omega) + j\left[-j\,\mathrm{sgn}\,\omega\, G(\omega)\right] \\
&= G(\omega)[1 + \mathrm{sgn}\,\omega] \\
&= \begin{cases} 2G(\omega), & \omega \geq 0 \\ 0, & \omega < 0 \end{cases} \\
&\boxed{= 2G_{+}(\omega)}
\end{aligned}$$

- $\tilde{g}(t)$ is called *the analytical signal* or preenvelope of g(t)



|G(f)|
|G(fc)|
−fc
0
fc
f
2W
2W
(a)
|G+(f)|
2|G(fc)|
0
fc − W
fc
fc + W
f
(b)
|G̃(f)|
2|G(fc)|
−W
0
W
f
(c)

Properties of Hilbert transform (1)

- Linearity: $H\{ax(t)+by(t)\}=a\hat{x}(t)+b\hat{y}(t)$
- Fourier Transform:

$$\boxed{\hat{X}(w)=F\{\hat{x}(t)\}=-j\,\mathrm{sgn}(w)X(w)=\begin{cases}-jX(w), & w\geq 0\\ 0, & w=0\\ jX(w), & w<0\end{cases}}$$

- Orthogonality:

$$\int_{-\infty}^{\infty}x(t)\hat{x}(t)dt=0$$

- Successive Hilbert transforms:

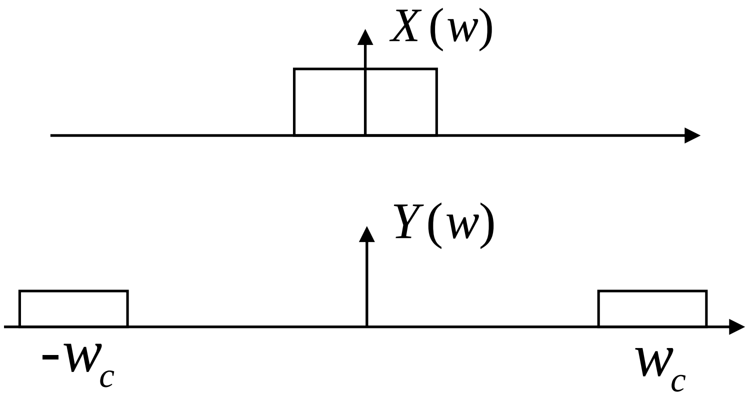
$$H\{\hat{x}(t)\}=-x(t)$$

- Convolution:

$$\boxed{H\{x(t)*y(t)\}=\hat{x}(t)*y(t)=x(t)*\hat{y}(t)}$$

Properties of Hilbert Transform (2)

- Modulations:
- For a narrow-band signal *x*(*t*)



$$y(t) = x(t)\cos 2\pi f_c t$$

$$H[x(t)\cos 2\pi f_c t] = x(t)\sin 2\pi f_c t$$

$$H[x(t)\sin 2\pi f_c t] = -x(t)\cos 2\pi f_c t$$

Prove can be done easily in frequency domain

$$x(t) = \cos\omega_c t \qquad X(j\omega) = \pi(\delta(\omega+\omega_c) + \delta(\omega-\omega_c))$$

$$x(t) = \sin\omega_c t \qquad X(j\omega) = \pi j(\delta(\omega+\omega_c) - \delta(\omega-\omega_c))$$

- If *A*(*t*) and $\varphi(t)$ are low-frequency signal, then:

$$H[A(t)\cos(\omega_c t + \phi(t))] = A(t)\sin[\omega_c t + \phi(t)]$$

$$H[A(t)\sin(\omega_c t + \phi(t))] = -A(t)\cos[\omega_c t + \phi(t)]$$

HT for Random Processes (1)

- If $X(t)$ is a WSS (wide-sense stationary) R.P., then $\hat{X}(t)$ is also a WSS R.P. . And they are jointly WSS.

$$\boxed{R_{\hat{X}}(\tau) = R_X(\tau)\,,\, S_{\hat{X}}(\omega) = S_X(\omega)}$$

$$\boxed{R_{\hat{X}X}(\tau) = \hat{R}_X(\tau)\,, \qquad R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau), \qquad \tau = t_1 - t_2}$$

Then, $R_{X\hat{X}}(\tau) = -R_{\hat{X}X}(\tau)$ (1)

- From the property of cross-correlation function:

$$R_{X\hat{X}}(\tau) = R_{\hat{X}X}(-\tau) \qquad (2)$$

$R_{X\hat{X}}(\tau)$ and $R_{\hat{X}X}(\tau)$ are odd functions:

$$\boxed{R_{\hat{X}X}(-\tau) = -R_{\hat{X}X}(\tau)} \qquad \boxed{R_{\hat{X}X}(0) = 0}$$

H.T. for Random Process (2)

The spectrum of the *analytical signal* of R.P. *X*(*t*)

- Let $Z(t) = X(t) + j\hat{X}(t)$
- Then,

$$
\boxed{
\begin{aligned}
R_Z(\tau) &= E[Z(t)Z^*(t+\tau)] \\
&= E[(X(t)+j\hat{X}(t))(X(t+\tau)-j\hat{X}(t+\tau))] \\
&= R_X(\tau) + R_{\hat{X}}(\tau) + jR_{X\hat{X}}(\tau) - jR_{\hat{X}X}(\tau) \\
&= 2R_X(\tau) + j2\hat{R}_X(\tau)
\end{aligned}
}
$$

$$
R_{\hat{X}X}(\tau) = \hat{R}_X(\tau) \quad \Rightarrow \quad S_{\hat{X}X}(\omega) = \begin{cases} -jS_X(\omega), & \omega \geq 0 \\ jS_X(\omega), & \omega < 0 \end{cases}
$$

$$
R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau) \quad \Rightarrow \quad S_{X\hat{X}}(\omega) = \begin{cases} jS_X(\omega), & \omega \geq 0 \\ -jS_X(\omega), & \omega < 0 \end{cases}
$$

$$
\boxed{S_Z(\omega) = \begin{cases} 4S_X(\omega) & \omega \geq 0 \\ 0 & \omega < 0 \end{cases}}
$$

4.4 Narrow-band Gaussian Processes

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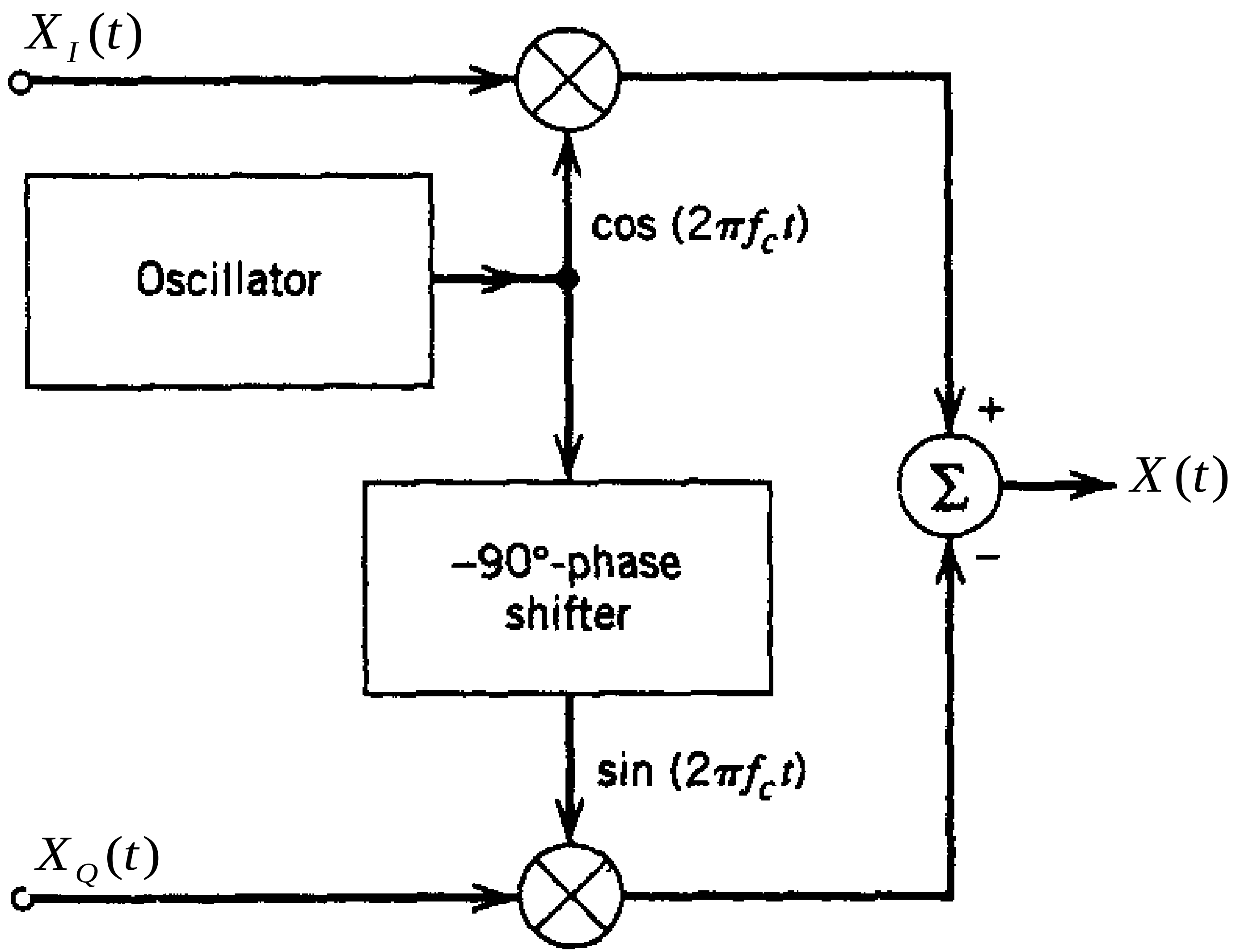
4.4.3 Representation of Narrow-Band Signal (Canonic form)

- The narrow-band signal with following form:

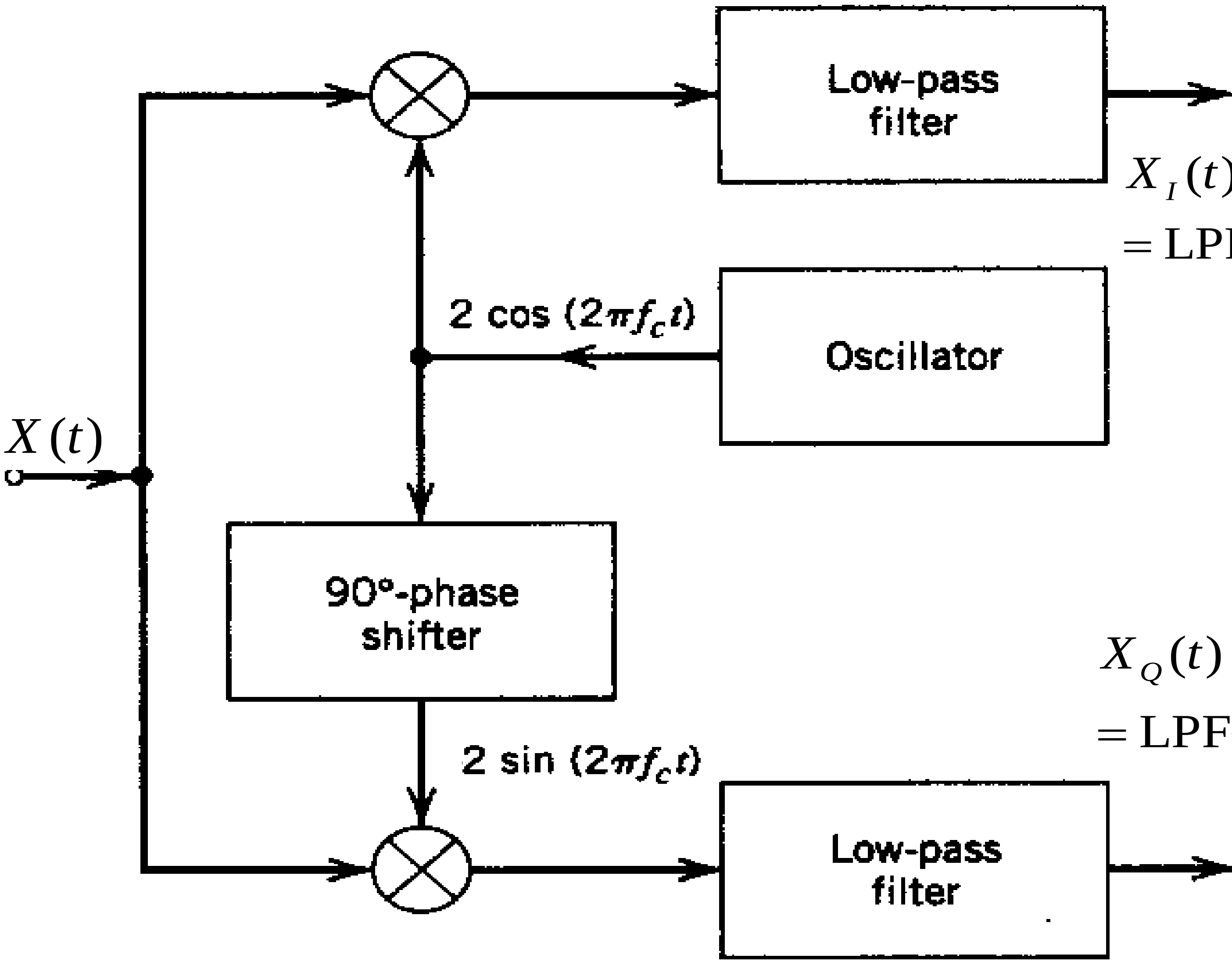
$$X(t) = A(t)\cos[2\pi f_c t + \phi(t)]$$

- Then: $\boxed{X(t) = A(t)\cos\phi(t)\cos 2\pi f_c t - A(t)\sin\phi(t)\sin 2\pi f_c t}$
- Let: $\boxed{X_I(t) = A(t)\cos\phi(t)\ ,\ \ X_Q(t) = -A(t)\sin\phi(t)}$
- Then $\boxed{X(t) = X_I(t)\cos 2\pi f_c t + X_Q(t)\sin 2\pi f_c t}$
- $\boxed{A(t) = \sqrt{X_I^2(t) + X_Q^2(t)} \qquad \phi(t) = \arctan\frac{X_Q(t)}{X_I(t)}}$
- $A(t)$ is the *envelop* and f_c is the *carrier frequency*.
- $X_I(t)$ and $X_Q(t)$ are low-frequency bandlimited signals, their max frequencies are lower than f_c
- $X_I(t)$ and $X_Q(t)$ can be seen as the *baseband signals*.
- Define: $\boxed{a(t) = X_I(t) + jX_Q(t) = A(t)e^{-j\phi(t)}}$

Block Diagram for recover the narrow-band signal (*Modulation*)



Block Diagraph for decompose a narrow-band signal (*Demodulation*)



$$X_I(t) = A(t)\cos\phi(t) = \mathrm{LPF}\{2X(t)\cos(2\pi f_c t)\}$$

$$X_Q(t) = -A(t)\sin\phi(t) = \mathrm{LPF}\{2X(t)\sin(2\pi f_c t)\}$$

4.4.3 Representation of Narrow-Band Signal (Canonic form)

$$a(t) = X_I(t) + jX_Q(t) = A(t)e^{-j\phi(t)}$$

- $a(t)$ is the *complex envelop* of the baseband signal.
- Make Hilbert Transform on both side of the canonic form, we get: $\boxed{\hat{X}(t) = X_I(t)\sin 2\pi f_c t - X_Q(t)\cos 2\pi f_c t}$
- Then:

$$\boxed{\begin{aligned} X_I(t) &= X(t)\cos 2\pi f_c t + \hat{X}(t)\sin 2\pi f_c t \\ X_Q(t) &= X(t)\sin 2\pi f_c t - \hat{X}(t)\cos 2\pi f_c t \end{aligned}}$$

- The relationship between complex envelop $a(t)$ and analytical signal $\tilde{X}(t)$ ：

$$\tilde{X}(t) = X(t) + j\hat{X}(t) = a(t)e^{-j2\pi f_c t}$$

$$X(t) = \mathrm{Re}[a(t)e^{-j2\pi f_c t}] = \mathrm{Re}[A(t)e^{-j[2\pi f_c t + \phi(t)]}]$$

$$\hat{X}(t) = \mathrm{Im}[a(t)e^{-j2\pi f_c t}] = \mathrm{Im}[A(t)e^{-j[2\pi f_c t + \phi(t)]}]$$

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Facts about Narrow-band R.P.(1)

$$X(t) = A(t)\cos[2\pi f_c t + \phi(t)]$$
$$= X_I(t)\cos 2\pi f_c t + X_Q(t)\sin 2\pi f_c t$$

- If X(t) is a *WSS R.P. with zero mean*, then $X_I(t)$ and $X_Q(t)$ are also WSS with zero mean, and $\sigma_{X_I}^2 = \sigma_{X_Q}^2 = \sigma_X^2$, and they are jointly stationary.
- $X_I(t)$ and $X_Q(t)$ has the same autocorrelation functions,

$$R_{X_I}(\tau) = R_{X_Q}(\tau) \qquad R_{X_I X_Q}(\tau) = R_{X_Q X_I}(-\tau) = -R_{X_Q X_I}(\tau)$$

- $X_I(t)$ and $X_Q(t)$ are *orthogonal* at the same time *t* :

$$\boxed{R_{X_I X_Q}(0) = 0}$$

$$R_{XX}(\tau) = R_{X_I X_I}(\tau)\cos 2\pi f_c \tau + R_{X_Q X_I}(\tau)\sin 2\pi f_c \tau$$

Facts about Narrow-band R.P

Prove:

a. $X_I(t)$ and $X_Q(t)$ are joint stationary processes

$$R_{XX}(\tau) = E[X(t+\tau)X(t)]$$

$$= E\{\ [X_I(t+\tau)\cos\omega_c(t+\tau) + X_Q(t+\tau)\sin\omega_c(t+\tau)]$$

$$[X_I(t)\cos(\omega_c t) + X_Q(t)\sin(\omega_c t)]\}$$

$$= R_{X_I X_I}(t+\tau, t)\cos\omega_c(t+\tau)\cos(\omega_c t)$$

$$+ R_{X_Q X_Q}(t+\tau, t)\sin\omega_c(t+\tau)\sin(\omega_c t)$$

$$+ R_{X_I X_Q}(t+\tau, t)\cos\omega_c(t+\tau)\sin(\omega_c t)$$

$$+ R_{X_Q X_I}(t+\tau, t)\sin\omega_c(t+\tau)\cos(\omega_c t)$$

Facts about Narrow-band R.P

$t_1 = 0,$

$$R_{XX}(\tau) = R_{X_I X_I}(t_1+\tau, t_1)\cos\omega_c\tau + R_{X_Q X_I}(t_1+\tau, t_1)\sin\omega_c\tau$$

$$\therefore \quad R_{X_I X_I}(t+\tau, t) = R_{X_I X_I}(\tau), \;\; R_{X_Q X_I}(t+\tau, t) = R_{X_Q X_I}(\tau)$$

$t_2 = \dfrac{\pi}{2\omega},$

$$R_{XX}(\tau) = R_{X_Q X_Q}(t_2+\tau, t_2)\cos\omega_c\tau - R_{X_I X_Q}(t_2+\tau, t_2)\sin\omega_c\tau$$

$$\therefore \quad R_{X_Q X_Q}(t+\tau, t) = R_{X_Q X_Q}(\tau), \;\; R_{X_I X_Q}(t+\tau, t) = R_{X_I X_Q}(\tau)$$

So, $X_I(t)$ and $X_Q(t)$ are joint stationary processes.

$\therefore$

$$\begin{aligned} R_{XX}(\tau) &= R_{X_I X_I}(\tau)\cos\omega_c\tau + R_{X_Q X_I}(\tau)\sin\omega_c\tau \\ &= R_{X_Q X_Q}(\tau)\cos\omega_c\tau - R_{X_I X_Q}(\tau)\sin\omega_c\tau \end{aligned}$$

Facts about Narrow-band R.P

b. Mean values and variances of $X_I(t)$ and $X_Q(t)$

$t_1 = 0, \qquad E[X(t_1)] = E[X_I(t_1)] = 0 \qquad \boxed{\therefore\ E[X_I(t)] = 0}$

$t_2 = \dfrac{\pi}{2\omega}, \qquad E[X(t_2)] = E[X_Q(t_2)] = 0 \qquad \boxed{\therefore\ E[X_Q(t)] = 0}$

$$\boxed{\begin{aligned} R_{XX}(\tau) &= R_{X_I X_I}(\tau)\cos\omega_c\tau + R_{X_Q X_I}(\tau)\sin\omega_c\tau \\ &= R_{X_Q X_Q}(\tau)\cos\omega_c\tau - R_{X_I X_Q}(\tau)\sin\omega_c\tau \end{aligned}}$$

$$\therefore \quad R_{XX}(0) = R_{X_I X_I}(0) = R_{X_Q X_Q}(0) = \sigma_X^2$$

$$\boxed{\therefore \quad \mathrm{Var}[X_I(t)] = \mathrm{Var}\left[X_Q(t)\right] = \sigma_X^2}$$

Facts about Narrow-band R.P

c. $X_I(t)$ and $X_Q(t)$ are uncorrelated at same times t .

$$\boxed{\begin{aligned} R_{XX}(\tau) &= R_{X_I X_I}(\tau)\cos\omega_c\tau + R_{X_Q X_I}(\tau)\sin\omega_c\tau \\ &= R_{X_Q X_Q}(\tau)\cos\omega_c\tau - R_{X_I X_Q}(\tau)\sin\omega_c\tau \end{aligned}}$$

$$\therefore\ R_{X_I X_I}(\tau) = R_{X_Q X_Q}(\tau)$$

$$\left.\begin{aligned} R_{X_I X_Q}(\tau) &= -R_{X_Q X_I}(\tau) \qquad (1) \\ R_{X_I X_Q}(\tau) &= R_{X_Q X_I}(-\tau) \qquad (2) \end{aligned}\right\} \Rightarrow \quad \boxed{R_{X_Q X_I}(-\tau) = -R_{X_Q X_I}(\tau)} \quad \boxed{R_{X_I X_Q}(-\tau) = -R_{X_I X_Q}(\tau)}$$

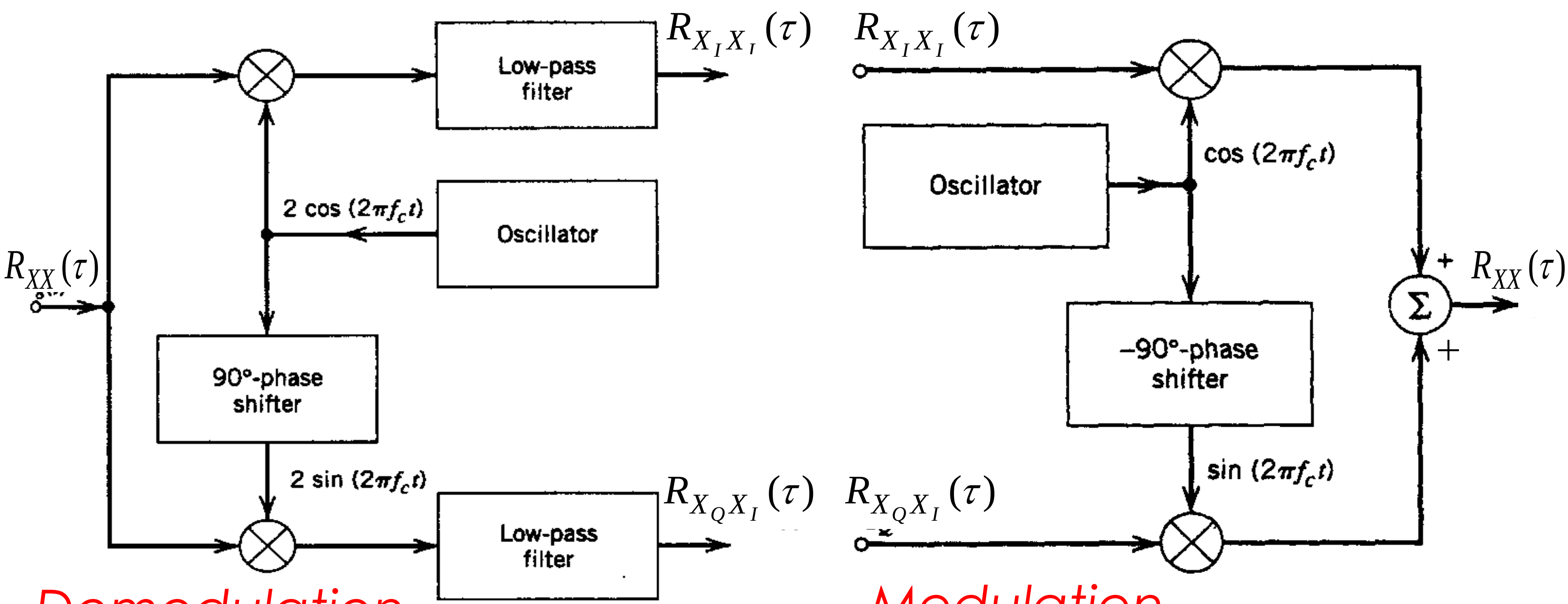
$$\boxed{\therefore\ R_{X_I X_Q}(0) = R_{X_Q X_I}(0) = 0}$$

Facts about Narrow-band R.P.(2)

$$X(t) = X_I(t)\cos 2\pi f_c t + X_Q(t)\sin 2\pi f_c t$$

$$\boxed{R_{XX}(\tau) = R_{X_I X_I}(\tau)\cos\omega_c\tau + R_{X_Q X_I}(\tau)\sin\omega_c\tau}$$

- The relationship between correlation functions is similar to that of between *X*(*t*), *X_I*(*t*) and *X_Q*(*t*).



Demodulation

Modulation

Facts about Narrow-band R.P.(3)

- Relationship between power spectrum:

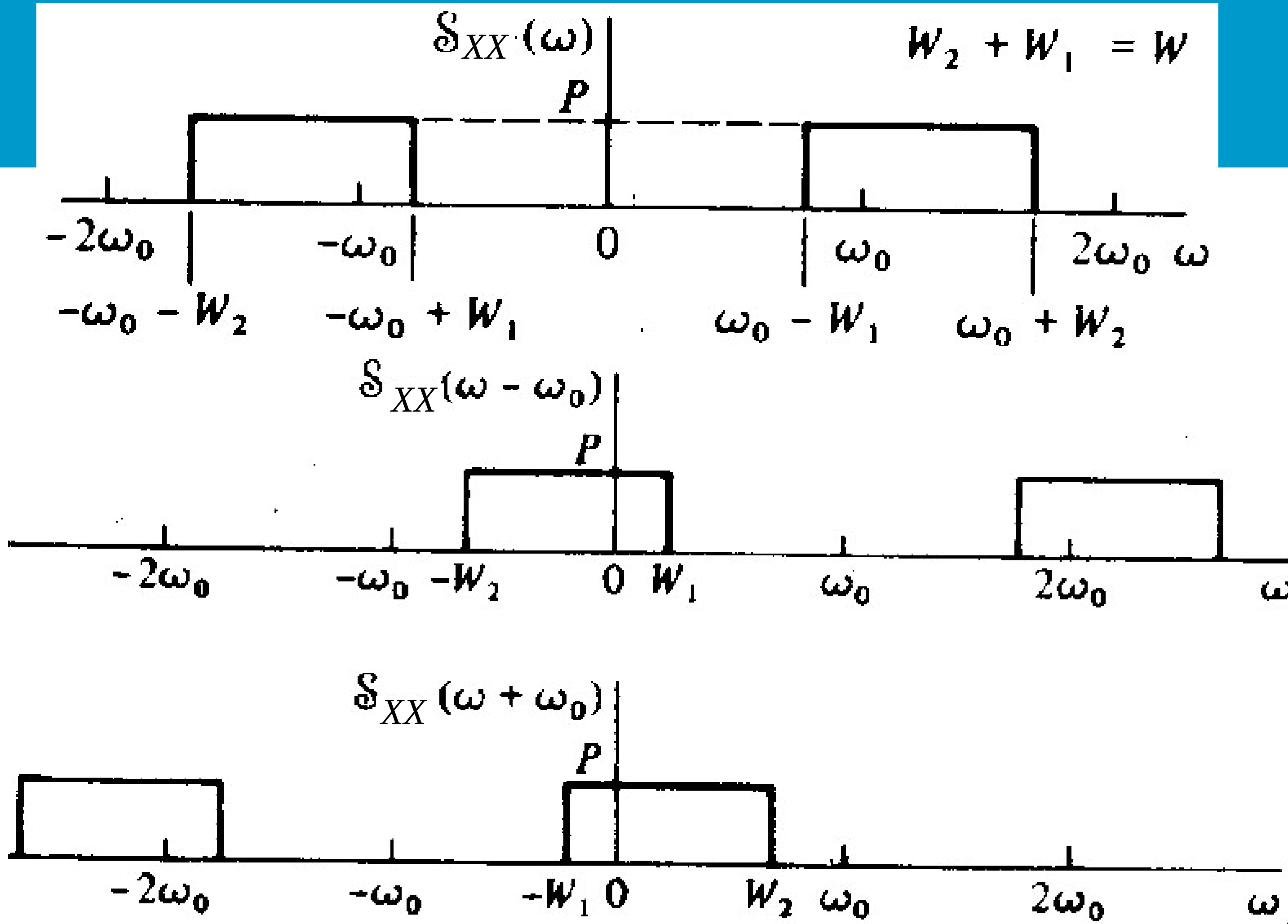
$$R_{X_I}(\tau) = \mathrm{LPF}\{2R_{XX}(\tau)\cos(\omega_c\tau)\}$$

$$R_{X_Q X_I}(\tau) = \mathrm{LPF}\{2R_{XX}(\tau)\sin(\omega_c\tau)\}$$

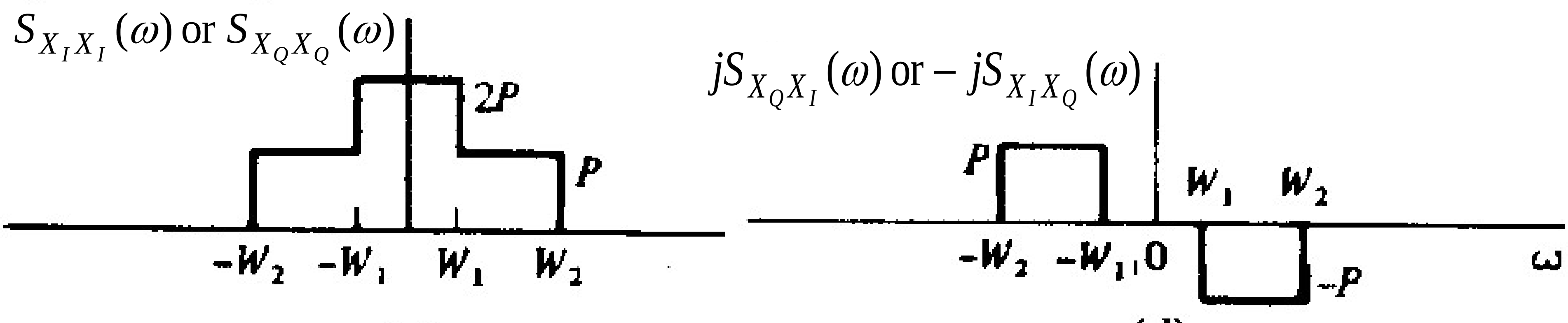
- $S_{X_I}(\omega), S_{X_Q}(\omega)$ concentrates in $|\omega| < \Delta\omega/2$, so they are low-frequency R.P. , then:

$$S_{X_I}(\omega) = S_{X_Q}(\omega) = S_X(\omega+\omega_c) + S_X(\omega-\omega_c) \quad |\omega| < \Delta\omega/2$$

$$S_{X_Q X_I}(\omega) = -S_{X_I X_Q}(\omega) = j[S_X(\omega+\omega_c) - S_X(\omega-\omega_c)] \quad |\omega| < \Delta\omega/2$$



$S_{XX}(\omega)$
$W_2 + W_1 = W$
P
$-2\omega_0$ $-\omega_0$ 0 ω_0 $2\omega_0$ ω
$-\omega_0 - W_2$ $-\omega_0 + W_1$ $\omega_0 - W_1$ $\omega_0 + W_2$
$S_{XX}(\omega - \omega_0)$
P
$-2\omega_0$ $-\omega_0$ $-W_2$ 0 W_1 ω_0 $2\omega_0$ ω
$S_{XX}(\omega + \omega_0)$
P
$-2\omega_0$ $-\omega_0$ $-W_1$ 0 W_2 ω_0 $2\omega_0$ ω



$S_{X_I X_I}(\omega)$ or $S_{X_Q X_Q}(\omega)$
$2P$
P
$-W_2$ $-W_1$ W_1 W_2
$jS_{X_Q X_I}(\omega)$ or $-jS_{X_I X_Q}(\omega)$
P
W_1 W_2
$-W_2$ $-W_1$ 0 ω
$-P$

Facts About Narrow-band R.P.(4)

- If the S.D.F of X(t) is *symmetry* about wc , then:

$$S_{X_I X_Q}(\omega) = S_{X_Q X_I}(\omega) = 0$$

$$R_{X_I X_Q}(\tau) = R_{X_Q X_I}(\tau) = 0$$

- As $X_I(t)$ and $X_Q(t)$ are WSS with zero mean, $X_I(t)$ and $X_Q(t)$ are *uncorrelated* at any time and *orthogonal* at any time.

- And: $R_X(\tau) = R_{X_I}(\tau)\cos\omega_c\tau = R_{X_Q}(\tau)\sin\omega_c\tau$

4.4 Narrow-band Gaussian Processes

4.4.1 The Definition of Band

4.4.2 An introduction of Hilbert Transform

4.4.3 Representation of Narrow-Band Signals

4.4.4 Narrow Band Random Processes

4.4.5 Gaussian Narrow-Band Random Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

References

1. Distribution of Amplitudes and Phases for Narrow-Band Processes
   a. Peyton Z.Peebles, Probability, Random variables, and Random Signal Principles, O211 W58/2，

      Section 8.6， Bandpass,Band-limited and Narrowband Processes
   b. 樊昌信，通信原理，TN91 20/4,

      Section 2.6 窄带随机过程

2. Distribution of a Sinusoidal Signal Plus Noise
   a. Section 10.6, Envelop and Phase of a sinusoidal signal plus noise
   b. Section 2.7 正弦波加窄带高斯过程

4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

Suppose:

(1) X(t) is a narrow-band Gaussian Random Process with zero mean;

(2) The variance of X(t) is σ_X^2 ;

(3) The P.D.F of X(t) is symmetry about ω_c .

Obtain: Distribution of Envelope and Phase of $X(t)$

$$X(t) = A(t)\cos[\omega_0 t + \varphi(t)]$$
$$= X_I(t)\cos\omega_0 t$$

- And: $X_I(t) = X(t)\cos\omega_0 t + \hat{X}(t)\sin\omega_0 t$

$$X_Q(t) = X(t)\sin\omega_0 t - \hat{X}(t)\cos\omega_0 t$$

4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

- $X_I(t)$ and $X_Q(t)$ are all linear combination of X(t), so, if X(t) is a Gaussian random variable, then , $X_I(t)$ and $X_Q(t)$ are also *Gaussian* random variable with zero mean and variance σ_X^2 .
- Step1: Find the distribution of $X_I(t)$ and $X_Q(t)$
- Step2: Using Jacobian transformation to find the distribution of $A(t)$ and $\phi(t)$

Step1:

(1) Since $R_{X_I X_Q}(\tau) = R_{X_Q X_I}(\tau) = 0$, $X_I(t)$ and $X_Q(t)$ are Gaussian random variable, they are independent;

(2) $$f_{X_I X_Q}(x_i, x_q) = f_{X_I}(x_i) f_{X_Q}(x_q) = \frac{1}{2\pi\sigma_X^2} \exp[-\frac{x_{it}^2 + x_{qt}^2}{2\sigma_X^2}]$$

4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

- Let A_t, ϕ_t denote $A(t)$ 、 $\phi(t)$ respectively, then:

$$f_{A\phi}(a_t, \varphi_t) = |J|\, f_{X_I X_Q}(x_{it}, x_{qt})$$

- And:

$$X_{it} = A_t \cos\phi_t, \qquad X_{qt} = A_t \sin\phi_t$$

$$0 \le A_t < \infty, \quad 0 \le \phi_t < 2\pi$$

- So:

$$\begin{aligned} f_{A\phi}(a_t, \varphi_t) &= |J|\, f_{X_I X_Q}(x_{it}, x_{qt}) \\ &= a_t\, f_{X_I X_Q}(x_{it}, x_{qt}) \\ &= \frac{a_t}{2\pi\sigma_X^2}\exp[-\frac{a_t^2}{2\sigma_X^2}] \quad a_t \ge 0, 0 \le \phi_t < 2\pi \end{aligned}$$

4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

- Thus:

$$f_A(a_t) = \int_0^{2\pi} f_{A\phi}(a_t, \varphi_t) d\varphi_t = \frac{a_t}{\sigma_X^2} \exp\left(-\frac{a_t^2}{2\sigma_X^2}\right) \quad a_t \geq 0$$

$$f_\phi(\varphi_t) = \int_0^{\infty} f_{A\varphi}(a_t, \varphi_t) da_t = \frac{1}{2\pi} \quad 0 \leq \phi_t < 2\pi$$

- That is: The envelope follows a *Reyleigh distribution*; the phase follows a uniform distribution.
- And: $f_{A\phi}(a_t, \varphi_t) = f_A(a_t) \cdot f_\phi(\varphi_t)$ illustrates that the envelope and phase are independent.

The mean of Reyleigh:

$$\sigma_X \sqrt{\frac{\pi}{2}} \approx 1.253\sigma_X$$

The variance of Reyleigh:

$$\sigma_X^2 \frac{4-\pi}{2} \approx 0.429\sigma_X^2$$

4.4 Narrow-band Gaussian Processes

4.4.1 The Definition of Band

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4.4.3 Representation of Narrow-Band Signals

4.4.4 Narrow Band Random Processes

4.4.5 Gaussian Narrow-Band Random Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

4.4.6 Distribution of a Sinusoidal Signal Plus Noise

Given: $r(t)$ is a sinusoidal signal with random phase plus a zero-mean, stationary Gaussian random process with a narrow-band spectrum,

$$r(t) = s(t) + n(t)$$

$$s(t) = A\cos[\omega_0 t + \theta]$$

$$n(t) = n_c(t)\cos(\omega_0 t) - n_s(t)\sin(\omega_0 t)$$

where A is constant, θ is uniformly distributed on $[0, 2\pi]$,

$n(t)$ is the narrow-band Gaussian noise with zero mean and variance σ_n^2 .

Obtain: Probability density function of amplitude and phase of $r(t)$.

4.4.6 Distribution of a Sinusoidal Signal Plus Noise

Sln:
$$r(t) = A\cos[\omega_0 t + \theta] + n_c(t)\cos(\omega_0 t) - n_s(t)\sin(\omega_0 t)$$
$$= [A\cos\theta + n_c(t)]\cos(\omega_0 t) - [A\sin\theta + n_s(t)]\sin(\omega_0 t)$$
$$= Z_c(t)\cos(\omega_0 t) - Z_s(t)\sin(\omega_0 t)$$
$$= Z(t)\cos[\omega_0 t + \phi(t)]$$

Where,
$$Z_c(t) = Z(t)\cos\phi(t)$$
$$Z_s(t) = Z(t)\sin\phi(t)$$

$$Z(t) = \sqrt{Z_c^2(t) + Z_s^2(t)}$$
$$\phi(t) = ac\tan\frac{Z_s(t)}{Z_c(t)}$$

Step 1: find the joint distribution of $Z_c(t)$ and $Z_s(t)$

Step 2: Using Jacobian transformation to find the distribution of $Z(t)$ and $\phi(t)$

4.4.6 Distribution of a Sinusoidal Signal Plus Noise

- Modified Bessel function of order zero

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp[x\cos\theta]d\theta$$

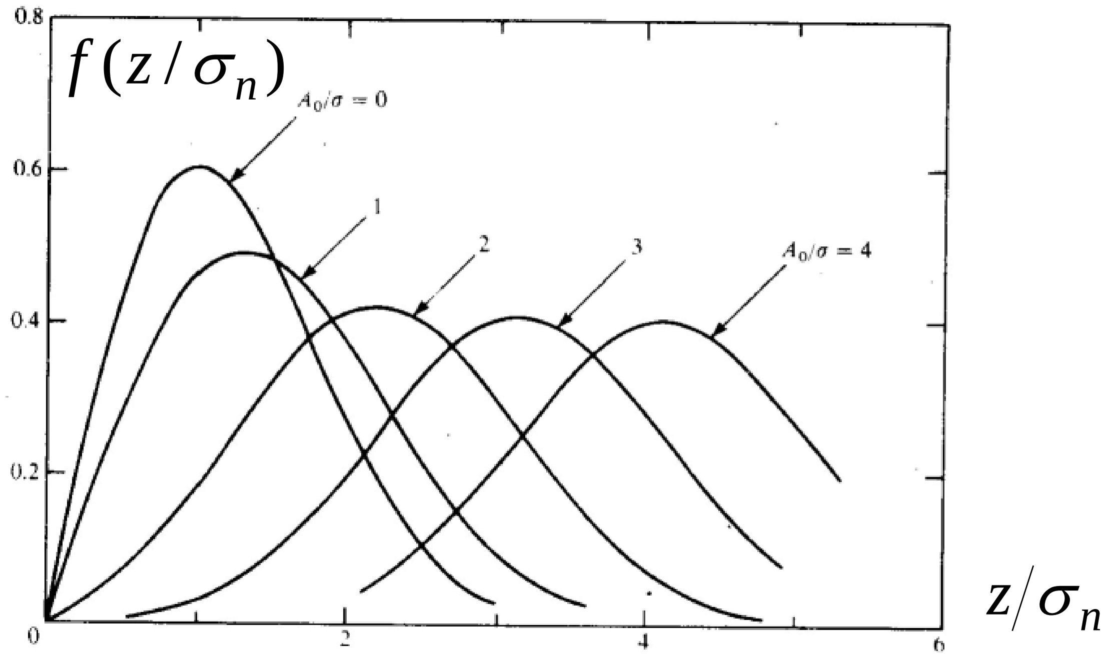
$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}} \qquad x >> 1$$

- Probability densities of the envelope follows Race distribution:

$$f(z) = \frac{z}{\sigma_n^2}\exp[-\frac{1}{2\sigma_n^2}(z^2 + A^2)]I_0(\frac{Az}{\sigma_n^2}) \qquad z \geq 0$$

4.4.6 Distribution of a Sinusoidal Signal Plus Noise

Probability densities of the envelope for various ratios $\dfrac{A}{\sigma_n}$



$\boxed{\dfrac{A}{\sigma_n} << 1}$, $f(z/\sigma_n)$ is similar to Reyleigh distribution;

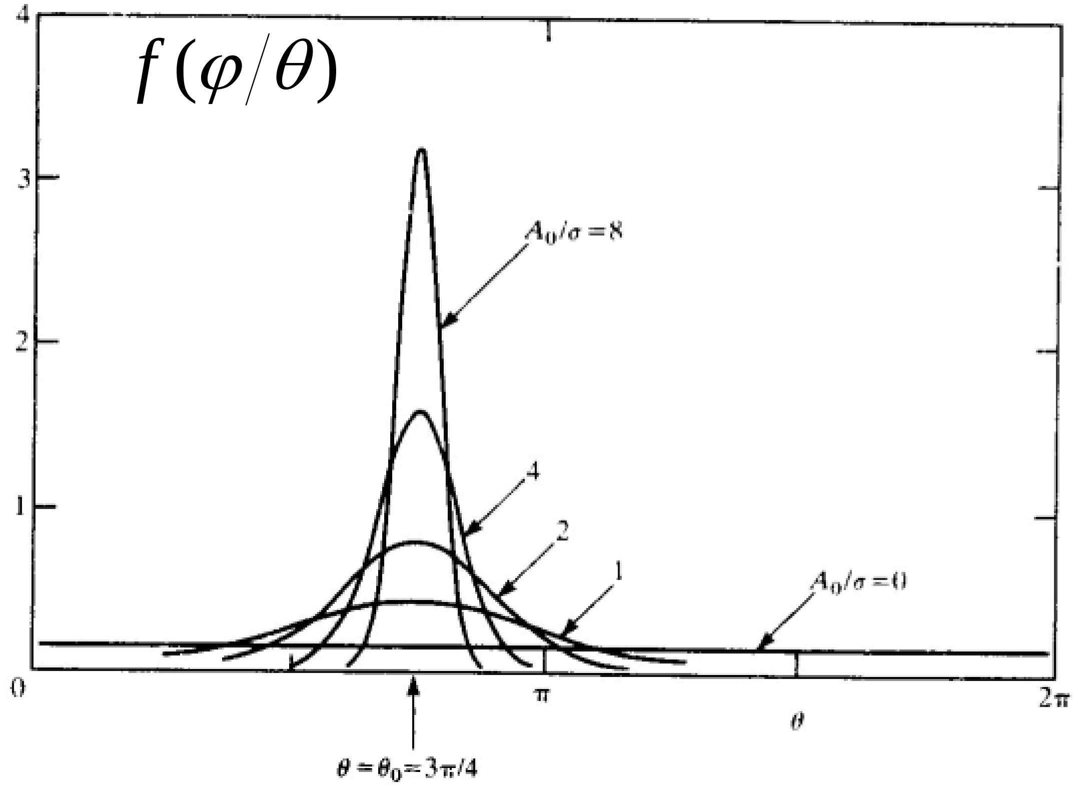
$\boxed{\dfrac{A}{\sigma_n} >> 1}$, $f(z/\sigma_n)$ is similar to standard Gaussian distribution.

4.4.6 Distribution of a Sinusoidal Signal Plus Noise

Probability densities of the phase for various ratios $\dfrac{A}{\sigma_n}$.

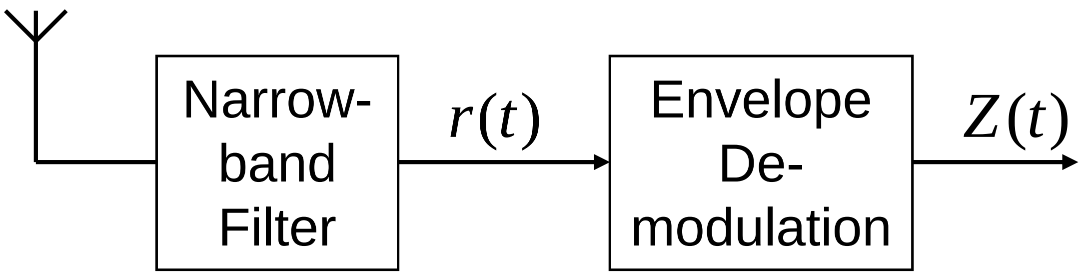
$$f(\varphi/\theta) = \frac{1}{2\pi} e^{-\frac{\alpha^2}{2}} + \frac{\alpha\cos(\varphi-\theta)}{\sqrt{2\pi}} e^{-\frac{\alpha^2 \sin^2(\varphi-\theta)}{2}} \Phi\left[\frac{\alpha\cos(\varphi-\theta)}{\sigma_n}\right]$$

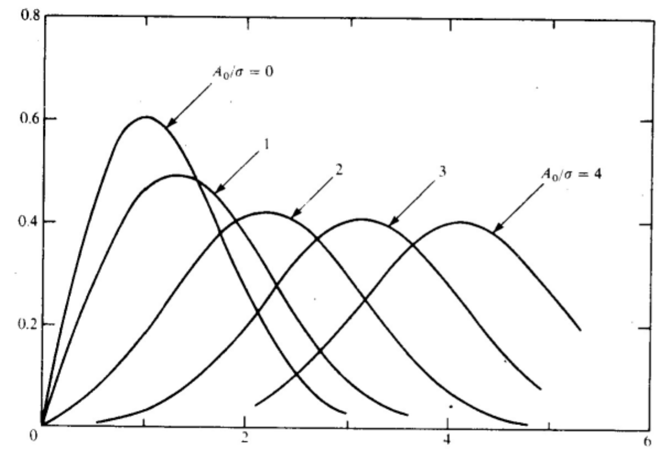
$$\alpha = \frac{A}{\sigma_n}$$



Application

- From the amplitude Z(t) of the received signal r(t), we may detect whether the s(t) exists.





- If s(t) exists, Z(t) follows Race distribution. More large of s(t) than noise, the peak wave is further from y-axis.
- If s(t) is not exist, Z(t) follows Reyleigh distribution which is near to y-axis.

Homework

4.14,

4.16

$$x(t) = \mathrm{sgn}(t) = \begin{cases} +1 & (t > 0) \\ -1 & (t < 0) \end{cases} \qquad X(j\omega) = \lim_{a \to 0} X_1(j\omega) = \lim_{a \to 0} \frac{-2j\omega}{a^2 + \omega^2} = \frac{2}{j\omega}$$

$$x(t) = \sin \omega_0 t \qquad X(j\omega) = \pi j(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$$

$$x(t) = \cos \omega_0 t \qquad X(j\omega) = \pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$$