Chapter 5 Poisson Processes

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OUTLINE

5.1 Poisson Processes (2.1,2.2)

5.2 Generalization of the Poisson Processes (2.3, 2.4)

(2.5, 2.6, 2.7 Canceled)

5.1 Poisson Processes

Simeon Denis Poisson, 1781-1840

OUTLINE:

5.1.1 Fundamentals of the Poisson Processes

5.1.2 Some Properties of the Poisson Processes

5.1.3 Interarrival Times and Waiting Times

5.1.4 Generating a Poisson Process

5.1.1 Fundamentals of the Poisson Processes

Applications of Poisson processes:

- Teletraffic management: Model of data packages arriving
- Web search: Model of Web pages' refreshing
- Reliability engineering: Model of software reliability
- ……

1. Three Definitions:

Definition 1

- *Definition.1* Poisson process

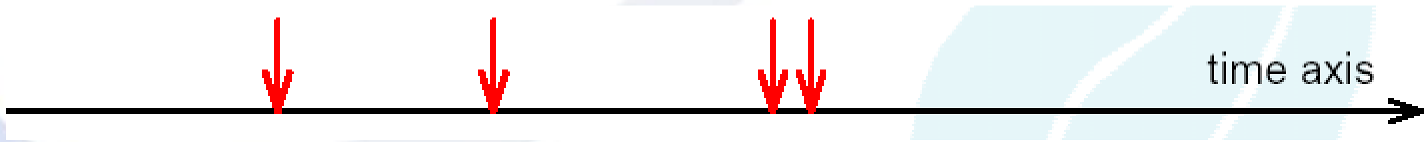
A counting process N(t) is said to be a Poisson process with mean rate (or intensity) $\nu\lambda$ (or) if

(i) N(t) has stationary independent increment.

(ii) N(0)=0.

(iii) The number in any time interval of length τ is Poisson distributed with mean $\lambda\tau$, That is,

$$P\{N(t+\tau)-N(t)=k\}=\frac{(\lambda\tau)^k}{k!}e^{-\lambda\tau}$$



- $N(t+\tau)-N(t)$ is called a Poisson increment process.

$$X(t)=N(t+\tau)-N(t)$$

$$C_{XX}(t_1,t_2)=\begin{cases}\lambda(t_1+\tau-t_2) & for\ 0<t_2-t_1<\tau \\ 0 & otherwise\end{cases}$$

The Poisson increment process is covariance stationary.

5.1.1 Fundamentals of the Poisson Processes

Poisson process implies:

- Independence between events in nonoverlapping intervals.
- The number of events in any interval of length λ is Poisson distributed with mean $\lambda\tau$
- Average number of packets generated in the interval of length 1 is λ

Definition 2:

recall counting processes:

If the interarrival times (are independent, identically distributed random variables) obey an exponential distribution, the process is called a Poisson process.

5.1.1 Fundamentals of the Poisson Processes

- *Definition 3:*

  A counting process $\{ N(t) \mid t \geq 0 \}$ is said to be a Poisson Process with **rate** $\lambda > 0$ if,

  i. $N(0) = 0$
  ii. The process has stationary and independent increments.
  iii. $N(t)$ satisfies

$$P\{X(t+h) - X(t) = 1\} = \lambda h + o(h)$$

$$P\{X(t+h) - X(t) \geq 2\} = o(h)$$

A function f( ) is said to be o(h) if $\lim_{h \to 0} \frac{f(h)}{h} = 0$

5.1.1 Fundamentals of the Poisson Processes

- Definitions (1), (2) and (3) are equivalent.
- Proof definition(1) and (3) are equivalent:

definition(1) and (3) are equivalent

The derivation of differential equations for $P_n(t)$, $n = 0, 1, 2, \ldots$.

the interval $(0, t + h) = (0, t) \cup [t, t + h)$

$P_n(t + h), n \geq 1,$ can be computed as:

a) the probability of n arrivals during $(0, t)$ and no arrivals during $[t, t + h)$;

b) the probability of $n - 1$ arrivals during $(0, t)$ and one arrival during $[t, t + h)$;

c) the probability of $x \geq 2$ arrivals during $[t, t + h)$ and $n - x$ arrivals during $(0, t)$.

These are three mutually exclusive and exhaustive possibilities. They give:

$$P_n(t + h) = P_n(t)(1 - \lambda h - o(h)) + P_{n-1}(t)\, \lambda h + o(h)$$

$$= P_n(t)(1 - \lambda h) + P_{n-1}(t)\, \lambda h + o(h)$$

Rearranging and dividing by h gives:

definition(1) and (3) are equivalent

$$\frac{P_n(t+h)-P_n(t)}{h}=-\lambda P_n(t)+\lambda P_{n-1}(t)+\frac{o(h)}{h},\ n\geq 1,\ t\geq 0$$

Taking the limit as $h \to 0^+$, gives the differential equations (actually claim the two-sided limit is justified):

$$P_n'(t)=-\lambda P_n(t)+\lambda P_{n-1}(t),\ n\geq 1,\ t\geq 0.$$

Applying similar reasoning to the case $n = 0$, we have

$$P_0'(t)=-\lambda P_0(t),\quad for\ t \geq 0.$$

$P_0(0) = 1$ (there have been no arrivals at all), so the last equation comes to us complete with initial condition.

Its unique solution is $P_0(t) = e^{-\lambda t}$.

$P_n(0) = 0$ for all $n \geq 1$. Using the just computed expression for $P_0(t)$, we obtain:

$$P_1'(t) = -\lambda P_1(t) + \lambda e^{-\lambda t},\ P_1(0) = 0.$$

definition(1) and (3) are equivalent

This is a non-homogeneous equation.

first find the general solution of the corresponding homogeneous equation:

$$P_1'(t) = -\lambda P_1(t) \quad -> \quad P_1(t) = C_1\, \mathrm{e}^{-\lambda t}.$$

Replace the constant by a function of t: $z(t)$.

To determine $z(t)$ by inserting into the **original** equation:

$$z'(t)\, e^{-\lambda t} - \lambda z(t)\, e^{-\lambda t} = - \lambda z(t)\, e^{-\lambda t} + \lambda\, e^{-\lambda t}$$

$$z'(t) = \lambda \quad --> \quad z(t) = C_2 + \lambda t$$

$$P_1(t) = (C_2 + \lambda t)\, e^{-\lambda t},\;\; P_1(0) = 0.$$

Finally: $$P_1(t) = \lambda t\, e^{-\lambda t}.$$

Repeat the construction for $P_2(t)$, $P_3(t)$, … etc., and a final induction proof lets us conclude that:

definition(1) and (3) are equivalent

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

the probability of exactly n arrivals during $(0, t)$.

It is easy to verify that

$$\sum_{n=0}^{\infty} P_n(t) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \equiv 1$$

From this, we can also compute the probabilities of at least N arrivals, or at most N arrivals, etc.

λ is an expected arrival rate.

$\mathrm{E}(n, t)$ denote the expected number of arrivals during $(0, t)$.

$$E(n,t) = \sum_{n=0}^{\infty} n \cdot P_n(t) = \sum_{n=0}^{\infty} n \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \lambda t$$

2. Moments of Poisson processes

- Mean value function: $E[N(t)] = \lambda t$

  arrival rate $\lambda = E[N(t)]/t$

  λ is the expected number of arrivals in unit time.
- Variance function: $Var[N(t)] = \lambda t$
- Correlation functio $R(t,t+\tau) = E[N(t)N(t+\tau)]$

$$= E\left[N(t)\left\{\overline{N(t+\tau) - N(t)} + N(t)\right\}\right]$$

$$= E[N(t)]E[N(t+\tau) - N(t)] + E[N^2(t)]$$

$$= \lambda^2 t\tau + (\lambda t)^2 + \lambda t$$

Method 2:

$$R(t_1,t_2) = Cov(t_1,t_2) + E[N(t_1)]E[N(t_2)]$$

$$= Var[N(t_1)] + \lambda^2 t_1 t_2 = \lambda^2 t_1 t_2 + \lambda t_1, \qquad t_1 < t_2$$

5.1.1 Fundamentals of the Poisson Processes

- Covariance function:

$$Cov_N[t_1,t_2] = \lambda \min(t_1,t_2)$$

$$Cov(t_1,t_2) = R(t_1,t_2) - m(t_1)m(t_2) = \lambda t_1,\ t_1 < t_2$$

Poisson process is not a stationary process itself.

- Characteristic function:

$$\phi(u) = E[e^{iuN(t)}] = \exp\{\lambda t(e^{iu} - 1)\}$$

5.1 Poisson Processes

OUTLINE:

5.1.1 Fundamentals of the Poisson Process

5.1.2 Some Properties of the Poisson Processes

5.1.3 Interarrival Times and Waiting Times

5.1.4 Generating a Poisson Process

5.1.2 Some Properties of the Poisson Processes

1. $N_1(t), N_2(t), \ldots N_n(t)$ are independent Poisson processes, with mean values $\lambda_1 t, \lambda_2 t, \ldots \lambda_n t,$ respectively.
   $\boxed{N(t) = N_1(t) + N_2(t) + \ldots + N_n(t)}$ is also a Poisson process with mean $\boxed{(\lambda_1 + \lambda_2 + \ldots + \lambda_n)t.}$
   (p54, Decomposition of Poisson Process)
2. $N_1(t), N_2(t)$ are two independent Poisson processes with mean $\lambda_1 t$ and $\lambda_2 t$ respectively.
   $\boxed{N(t) = N_1(t) - N_2(t)}$ is not a Poisson process; instead, it has the probability distribution,

$$P\{N_1(t) - N_2(t) = n\} = e^{-(\lambda_1+\lambda_2)t} \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}} I_n(2\sqrt{\lambda_1 \lambda_2}\, t)$$

where $I_n()$ is a modified Bessel function of order *n*.

- Proof:

$$\Pr\{N(t)=n\}=\sum_{k=0}^{\infty}\Pr\{N_1(t)=n+k\}\Pr\{N_2(t)=k\}$$

$$=\sum_{k=0}^{\infty}\frac{e^{-\lambda_1 t}(\lambda_1 t)^{n+k}}{(n+k)!}\frac{e^{-\lambda_2 t}(\lambda_2 t)^{k}}{k!}$$

$$=e^{-(\lambda_1+\lambda_2)t}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}}\sum_{k=0}^{\infty}\frac{\left(\sqrt{\lambda_1\lambda_2}\,t\right)^{2k+n}}{k!(n+k)!}$$

$$=e^{-(\lambda_1+\lambda_2)t}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}}I_n(2\sqrt{\lambda_1\lambda_2}\,t)$$

$$I_n(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k+n}}{k!\,\Gamma(n+k+1)}$$

$$\boxed{\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}dx}$$

$$\boxed{\Gamma(n+1)=n\Gamma(n)=n!}$$

5.1.2 Some Properties of the Poisson Processes

3. If the Poisson process N(t) with mean νt is filtered such that every occurrence of the event is not counted, the process has a constant probability p of being counted. Then the resulting counting process is also a Poisson process with mean $p\nu t$.

Proof: (p54, Decomposition of Poisson Process)

$$Pr\{M(t) = n | N(t) = n + r\} = \binom{n+r}{n} p^n q^r,$$

where $p + q = 1$

$$Pr\{N(t) = n + r\} = e^{-\nu t} \frac{(\nu t)^{n+r}}{(n+r)!}$$

$$Pr\{M(t) = n\} = \sum_{r=0}^{\infty} \binom{n+r}{n} p^n q^r \cdot e^{-\nu t} \frac{(\nu t)^{n+r}}{(n+r)!}$$

5.1.2 Some Properties of the Poisson Processes

$$Pr\{M(t)=n\} = e^{-\nu t}\frac{(p\nu t)^n}{n!}\sum_{r=0}^{\infty}\frac{(q\nu t)^r}{r!}$$

$$= e^{-\nu t}\frac{(p\nu t)^n}{n!}e^{q\nu t} = e^{-p\nu t}\frac{(p\nu t)^n}{n!}$$

4. Let X be the number of occurrences of an event that takes place in accordance with a Poisson process with intensity *v*. Find the number X that has the largest probability in a specified time *t*.

$$Pr\{X=0\} < Pr\{X=1\} < \cdots < Pr\{X=r-1\}$$
$$\leqslant Pr\{X=r\} > Pr\{X=r+1\} > \cdots$$

$$\frac{Pr\{X=r+1\}}{Pr\{X=r\}} = \frac{e^{-\nu t}(\nu t)^{r+1}/(r+1)!}{e^{-\nu t}(\nu t)^{r}/r!} = \frac{\nu t}{r+1}$$

$$r \geq \nu t - 1 \qquad r = [\nu t]$$

5.1.2 Some Properties of the Poisson Processes

Example 1:

Analysis of records obtained in the Gulf of Mexico indicates that tropical storms come to the Gulf in accordance with a Poisson process with intensity 0.68 per year. Obtain the number of storms having the highest probability in a 5-year period.

$$\nu = 0.68$$

$$t = 5$$

$$r = [\nu t] = [0.68 \times 5] = 3$$

5.1 Poisson Process

OUTLINE:

5.1.1 Fundamentals of the Poisson Process

5.1.2 Some Properties of the Poisson Process

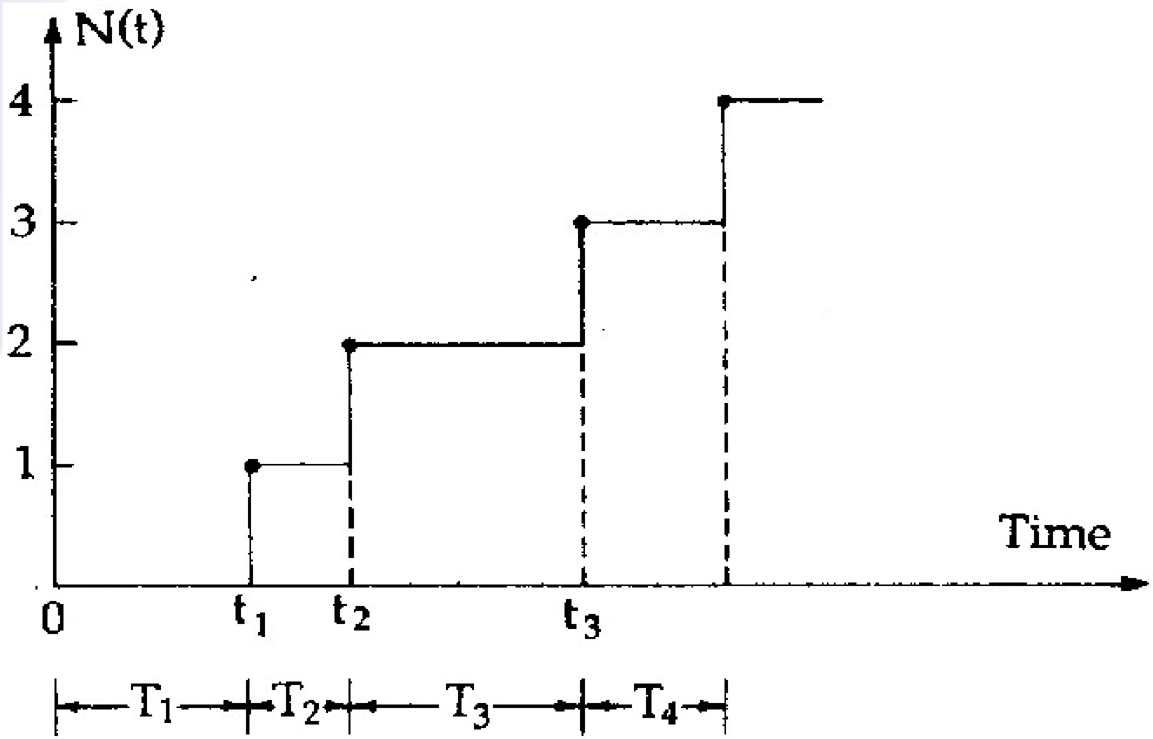
5.1.3 Interarrival Times and Waiting Times

5.1.4 Generating a Poisson Process

5.1.3 Interarrival Times and Waiting Times

1. Interarrival Times T_n : the time intervals between two successive occurrences of random events. (p51)

- $T_1, T_2, T_3, \ldots$ a random variables sequence.



$T_1=t_1, T_2=t_2-t_1, T_3=t_3-t_2, \ldots$ Interarrival times

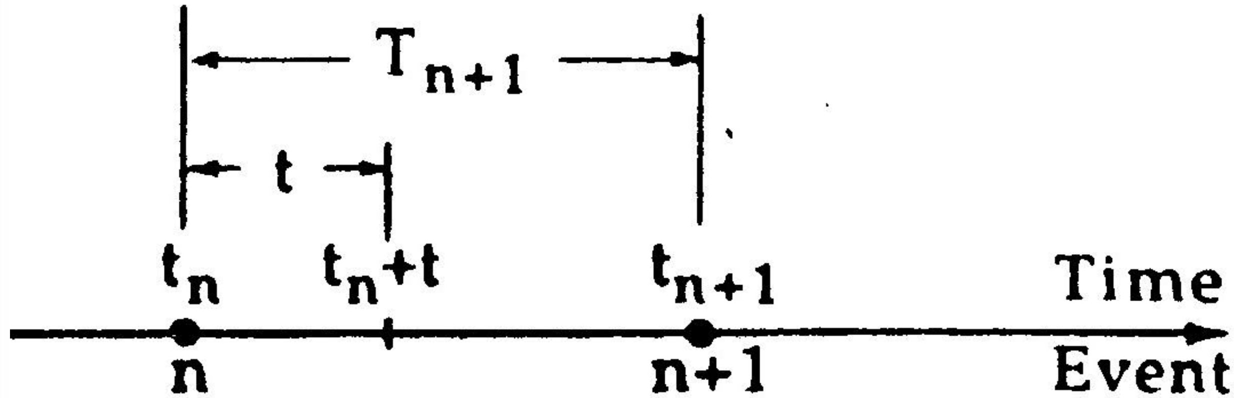
5.1.3 Interarrival Times and Waiting Times

So that λ is the expected number of arrivals in unit time, or the **arrival rate**.

What is the relationship between λ and inter-arrival times ?

Theorem: the interarrival times of a Poisson process with intensity λ are independent, identically distributed exponential random variables with mean $1/\lambda$

Proof:



$$P\{N(T_{n+1} > t)\} = P\{N(t_n + t) = n \mid N(t_n) = n\}$$

$$= P\{N(t_n + t) - N(t_n) = 0 \mid N(t_n) = n\}$$

$$= P\{N(t_n + t) - N(t_n) = 0\} \quad \text{(independent increments)}$$

$$= P\{N(t) = 0\} \quad \text{(stationary increments)} \boxed{= e^{-\lambda t}}$$

$$\boxed{\therefore F_{T_i}(t) = P\{T_i \le t\} = 1 - e^{-\lambda t},\ i = 1,2,\ldots}$$

$$\boxed{\therefore f_{T_i}(t) = \lambda e^{-\lambda t},\ i = 1,2,\ldots}$$

5.1.3 Interarrival Times and Waiting Times

- Independent and stationary increments
  $\Longrightarrow$ independent interarrival times

5.1.3 Interarrival Times and Waiting Times

Example:

- How often do Web pages change?

  If changes to a page follow a Poisson process of rate λ,

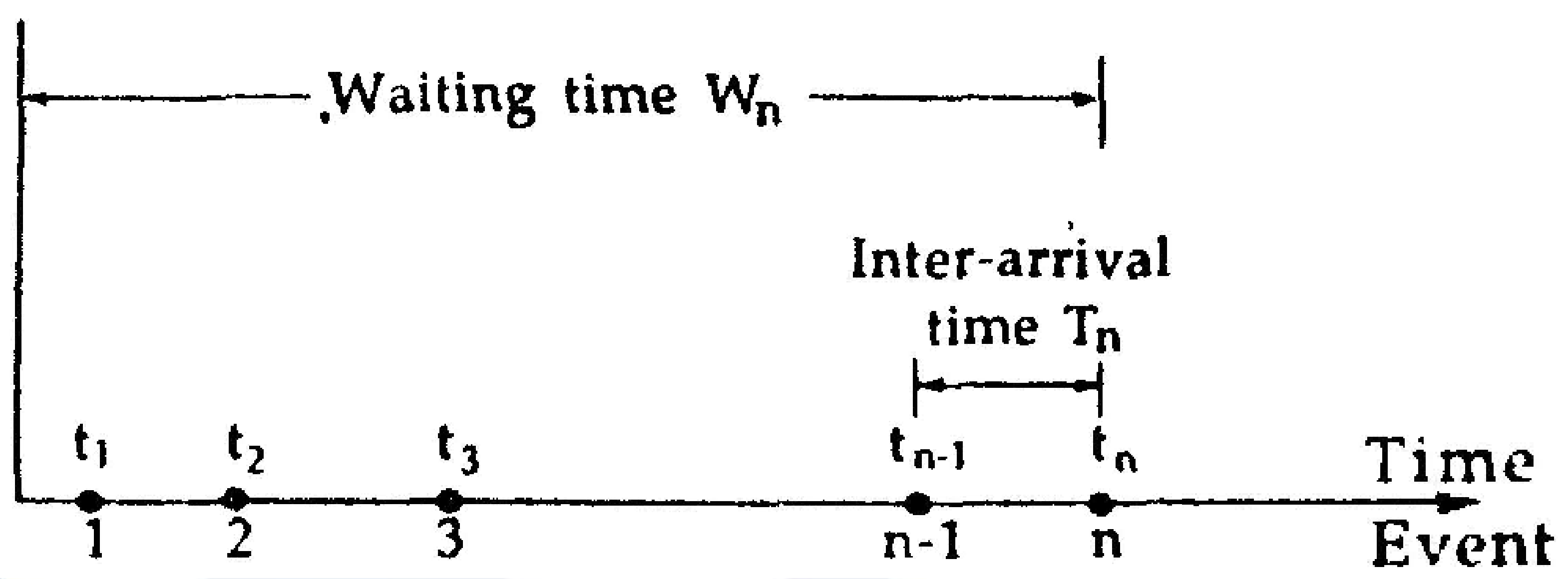
  Its change intervals follow the distribution $\lambda e^{-\lambda t}$

  Example:

  $\lambda = 0.1$ (once every 10 days on average)

  Optimal refresh strategy for crawling the Web.

2. Waiting Time W_n (S_n in textbook, arrival sequence $\{S_n\}$): the time up to a specific number of occurrences of the event from $t = 0$. (p51)

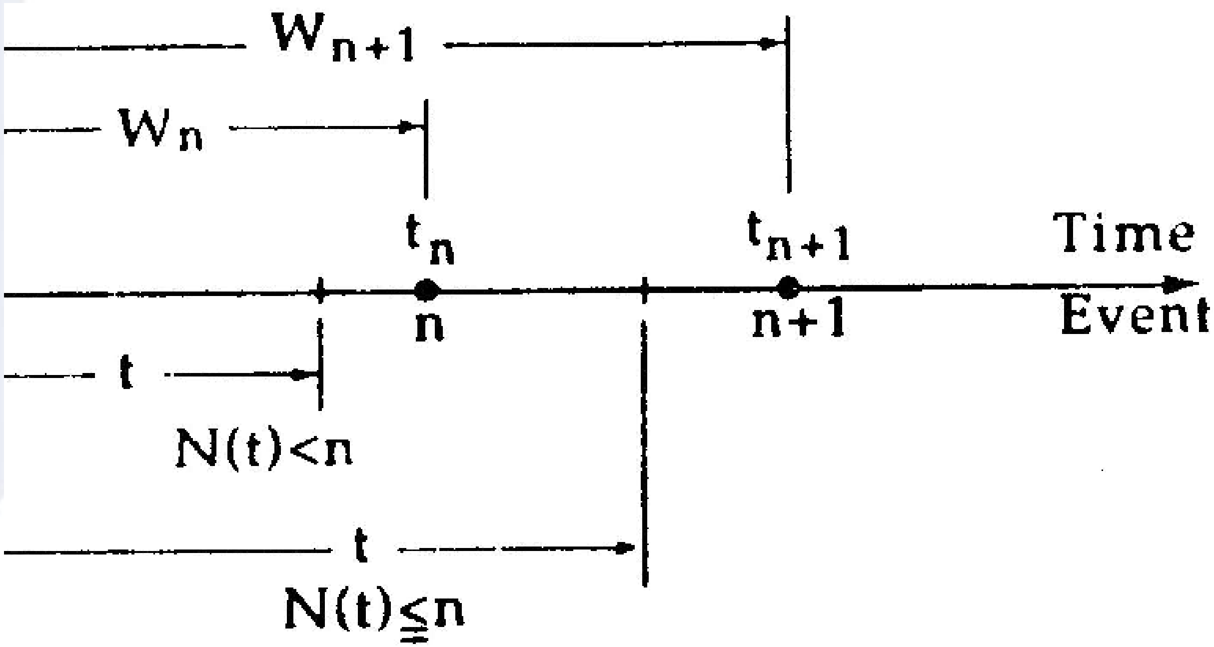


$$W_n = T_1 + T_2 + \cdots T_n$$

$$T_n = W_n - W_{n-1}$$

5.1.3 Interarrival Times and Waiting Times

Relationship of waiting time and events:



$$Pr\{N(t) < n\} = Pr\{W_n > t\}$$

$$Pr\{N(t) \leq n\} = Pr\{W_{n+1} > t\}, n = 0, 1, 2, \ldots$$

5.1.3 Interarrival Times and Waiting Times

Relationship of waiting time and event:

$$Pr\{N(t) \geq n\} = Pr\{W_n \leq t\} = F_{W_n}(t)$$

$$Pr\{N(t) > n\} = Pr\{W_{n+1} \leq t\} = F_{W_{n+1}}(t)$$

$$Pr\{N(t) = n\} = Pr\{N(t) \geq n\} - Pr\{N(t) > n\}$$

$$= F_{W_n}(t) - F_{W_{n+1}}(t), \qquad n = 1, 2, 3, \ldots$$

$$\boxed{Pr\{N(t) = 0\} = Pr\{W_1 > t\} = 1 - F_{W_1}(t)}$$

5.1.3 Interarrival Times and Waiting Times

Distribution of Waiting Time:

$$Pr\{W_n \le t\} = Pr\{N(t) \ge n\} = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

$$f_{W_n}(t) = -\sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} + \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!}$$

$$= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Gamma or Erlang distribution with parameters

- n and λ

3. The conditional distribution of arrival time

Problem: What is the probability that exactly m events occur in the interval $[0,t]$ given that exactly n events occur in the interval $[0, t+\tau]$; $m=0,1,\ldots,n$? (p52, Past Arrival Times)

$$
\begin{aligned}
&P\{N(t)=m \mid N(t+\tau)=n\} \\
&= P\{N(t)=m, N(t+\tau)=n\} \,/\, P\{N(t+\tau)=n\} \\
&= P\{N(t)=m, N(t+\tau)-N(t)=n-m\} \,/\, P\{N(t+\tau)=n\} \\
&= P\{N(t)=m\}P\{N(\tau)=n-m\} \,/\, P\{N(t+\tau)=n\} \text{ (independent increments)} \\
&= \frac{(\lambda t)^m}{m!} e^{-\lambda t} \frac{(\lambda\tau)^{n-m}}{(n-m)!} e^{-\lambda\tau} \,/\, \frac{(\lambda(t+\tau))^n}{n!} e^{-\lambda(t+\tau)} \text{(stationary increments)} \\
&= \frac{n!}{m!\,(n-m)!} \frac{t^m \tau^{n-m}}{(t+\tau)^n} = \binom{n}{m} \left(\frac{t}{t+\tau}\right)^m \left(\frac{\tau}{t+\tau}\right)^{n-m}
\end{aligned}
$$

It is a binomial distribution with parameters $p=\dfrac{\tau}{t+\tau}$ and n.

5.1.3 Interarrival Times and Waiting Times

For n=m=1,

$$P\{N(t)=1 \mid N(t+\tau)=1\} = \frac{t}{t+\tau}$$

The random time to the Poisson event occurring in [0,t] is uniformly distributed over this interval.

Past Arrival Times Given $N(t)=n$:

that is joint density function of $W_1, W_2, \ldots W_n$ given $N(t)=n$.

Order statistic:

Let $Y_1, Y_2, \ldots Y_n$ be i.i.d. random variables with common density *f*, and $Y_{(1)}, Y_{(2)}, \ldots Y_{(n)}$ are the corresponding *n* order statistics ($Y_{(i)}$ is the *i*th smallest of $\{Y_i\}$).

The joint density of $\{Y_{(i)}\}$ is given by

$$\boxed{f_{Y(1),\ldots,Y(n)}(y_1,\ldots,y_n) = n!\prod_{i=1}^{n} f(y_i) \qquad 0 < y_1 < \cdots < y_n}$$

If *f* follows the uniform density over (0,t), then

$$f_{Y(1),\ldots,Y(n)}(y_1,\ldots,y_n) = n!/t^n \qquad 0 < y_1 < \cdots < y_n$$

2.1.3 Interarrival Times and Waiting Times

Joint density function of $W_1, W_2, \ldots W_n$ given N(t)=n is

$$f_{W_1,\ldots,W_2|N(t)}(t_1,\ldots,t_n \mid n) = n!/t^n \qquad 0 < t_1 < \cdots < t_n < t.$$

Theorem:

A total of n random events occurs in time t in accord with a Poisson process with intensity ν. Then the waiting times $W_1, W_2, \ldots W_n$ are equivalent to the ordered sample of a random variable that has a uniform distribution between 0 and t.

5.1 Poisson Processes

OUTLINE:

5.1.1 Fundamentals of the Poisson Processes

5.1.2 Some Properties of the Poisson Processes

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5.1.4 Generating a Poisson Process

5.1.4 Generating a Poisson Process

Generating Interarrival Times of a Poisson by Computer Simulation: (p52, Example 2.2.1)

Generate the exponential variable X with parameter ν.

(i) generate $U \sim U(0,1)$, so $1-U \sim U(0,1)$

(ii) $X = -(1/\nu)\log(U)$, $X \sim F$, $F_X(x) = 1 - e^{-\nu x},\ x \geq 0$

随机样本生成法：

龚光鲁，钱敏平

应用随机过程教程－及在算法和智能计算中的随机模型

北京，清华大学出版社，2004

ISBN 7-302-06948-4 / O 313

第2章 随机样本生成法

5.1.4 Generating a Poisson Process

Poisson in Microsoft Excel:

- POISSON (x, mean, cumulative)

  x: 事件数。

  Mean: 期望值。

  Cumulative: 为一逻辑值，确定所返回的概率分布形式。如果 cumulative 为 TRUE，函数 POISSON 返回泊松累积分布概率，即随机事件发生的次数在 0 到 x 之间（包含 0 和 1）；如果为 FALSE，则返回泊松概率密度函数，即，随机事件发生的次数恰好为 x。

- 假设 cumulative = FALSE，POISSON= $\dfrac{\nu^{x}}{x!}e^{-\nu}$

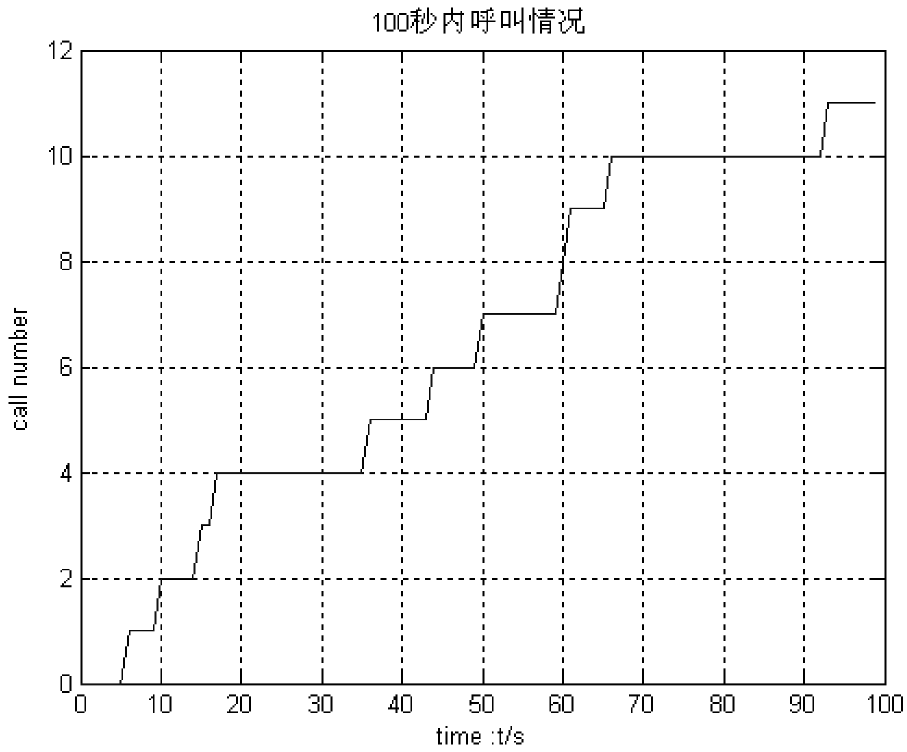
- 假设 cumulative = TRUE, POISSON= $\displaystyle\sum_{k=0}^{x}\frac{\nu^{k}}{k!}e^{-\nu}$

5.1.4 Generating a Poisson Process

4. Poisson in Matlab:

◆ 泊松整数序列发生器函数：x=poissrnd（lm）

lm=0.12



OUTLINE

5.1 Poisson Processes

5.2 Generalization of the Poisson Processes

5.2 Generalization of the Poisson Processes

OUTLINE:

5.2.1 Nonhomogeneous Poisson Processes

5.2.2 Compound Poisson Processes

5.2.1 Nonhomogeneous Poisson Processes

Definition1: A Poisson process with an intensity that is a nonnegative function of time, $\lambda(t)$, is defined as a nonhomogeneous Poisson process.

Definition 2: A counting process $\{N(t), t \geq 0\}$ is called an nonhomogeneous Poisson process with nonnegative intensity function $\lambda(t)$ if it has properties (p56)

i) $N(0)=0$,

ii) $\{N(t), t \geq 0\}$ has independent increments,

iii) $P\{X(t+h)-X(t)=1\}=\lambda(t)h+o(h)$

iv) $P\{X(t+h)-X(t)\geq 2\}=o(h)$

Definition 1 and definition 2 are equivalent.

5.2.1 Nonhomogeneous Poisson Processes

- $\lambda(t)$ is called the intensity function.

- Distribution:

$$P\{N(t)=n\} = \frac{\{\int_0^t \lambda(s)ds\}^n}{n!} \exp\{-\int_0^t \lambda(s)ds\}$$

$$E[N(t)] = Var[N(t)] = \int_0^t \lambda(s)ds = m_N(t)$$

$$\boxed{P\{N(t)=n\} = \frac{\{m_N(t)\}^n}{n!} \exp\{-m_N(t)\}}$$

5.2.1 Nonhomogeneous Poisson Processes

- Correlation function:

$$R(t,\tau)=E[N(t)N(t+\tau)]=E\{N(t)[N(t+\tau)-N(t)+N(t)]\}$$

$$R(t,\tau) = E[N(t)]E[N(t+\tau)-N(t)]+E[N^2(t)]$$

$$= \int_0^t \lambda(t)\,dt \int_0^{t+\tau} \lambda(t)\,dt + \int_0^t \lambda(t)\,dt$$

$$= \int_0^t \lambda(t)\,dt \left\{1 + \int_0^{t+\tau} \lambda(t)\,dt\right\}$$

- The increment process of a nonhomogeneous Poisson process is no longer stationary.

5.2.1 Nonhomogeneous Poisson Processes

Example: Based on a large statistical sample it is known that the number of cars which arrive for petrol week-days between 6:00 and 12:00 at a particular filling station can be described by an nonhomogeneous Poisson process, the intensity function is

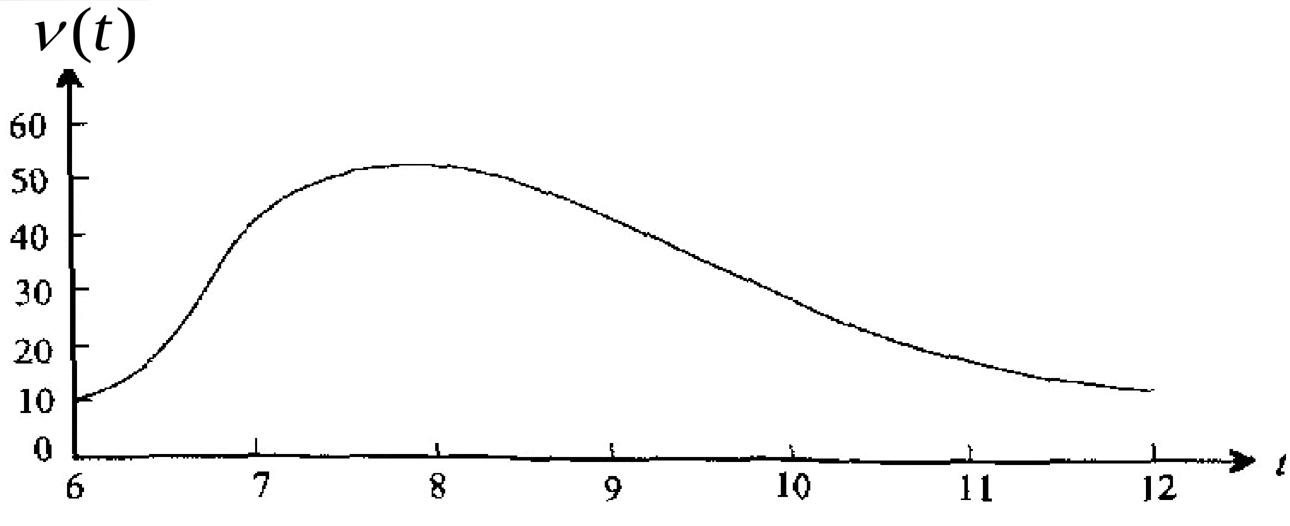
$$\boxed{v(t) = 10 + 35.4(t-6)e^{-\frac{1}{8}(t-6)^2}, 6 \le t \le 12}$$

1) How many cars on average arrive for petrol week-days between 6:00 and 12:00?

2) What is the probability that at least 90 cars arrive for petrol week-days between 7:00 and 9:00?

5.2.1 Nonhomogeneous Poisson Processes

Sln:



1) The average number is

$$E[N(t)] = \int_0^t \nu(s)ds = \int_6^{12} \nu(s)ds$$

$$= \int_0^6 [10 + 35.4te^{-\frac{1}{8}t^2}]dt$$

$$= [10t + 141.6(1 - e^{-\frac{1}{8}t^2})]\Big|_0^6 = 200$$

5.2.1 nonhomogeneous Poisson Processes

2) During the time interval [7:00, 9:00] the random number of arriving cars is Poisson distributed with parameter

$$\int_7^9 \nu(t)dt = [10t + 141.6(1 - e^{-\frac{1}{8}t^2})]\Big|_1^3 = 99$$

That is, on average 99 cars arrive for petrol between 7:00 and 9:00. The desired probability is

$$P\{N(9) - N(7) \geq 90\} = \sum_{n=90}^{\infty} \frac{99^n}{n!} e^{-99}$$

$$\approx 1 - \Phi(\frac{90 - 99}{\sqrt{99}})$$

$$\approx 1 - 0.1827 = 0.8173$$

Generating arrival times of a nonhomogeneous Poisson process with intensity function $\nu(t)$: (Example 2.3.1)

i) Generate a Poisson ($N_1(t)$) arrival sequence $\{T_i\}$ with intensity ν,

$$\nu \ge \nu(t), \text{ for all } t \ge 0$$

ii) The arrival at T_i will be counted as an arrival of $N_1(t)$ with probability $\nu(T_i)/\nu$.

- The counted process is a nonhomogeneous Poisson process

with intensity function $[\nu(T_i)/\nu]\nu = \nu(T_i)$.

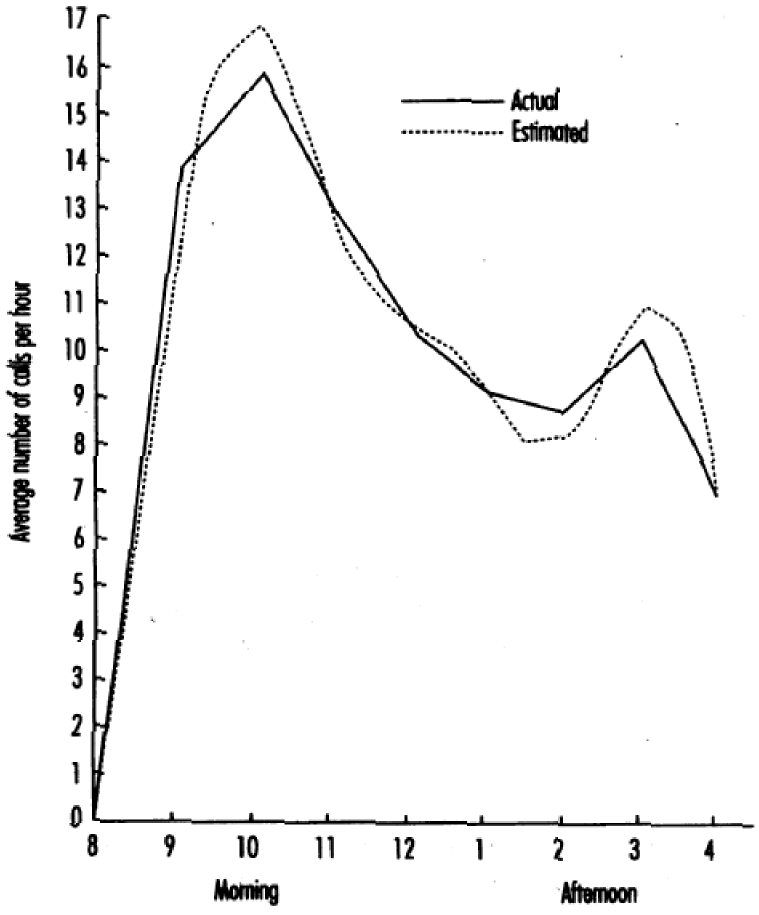
5.2.1 nonhomogeneous Poisson Processes

Modeling Arrivals to a Computer System (p63,Example 2.3.6)

$$\lambda(t) = 8.924 - 1.584\cos\frac{\pi t}{1.51} + 7.897\sin\frac{\pi t}{3.02} - 10.434\cos\frac{\pi t}{4.53} + 4.293\cos\frac{\pi t}{6.04}$$

The computer system is designed for online analysis of electrocardiograms.

Arrival data is analyzed for developing an input process for subsequence uses in computer simulation and analytical model building.



5.2.1 nonhomogeneous Poisson Processes

A multiserver queue with nonhomogeneous Poisson arrivals and exponential service times: M(t)/M/s (p65,Example 2.3.8)

"Introduction to Queue"

5.2 Generalization of the Poisson Processes

OUTLINE:

5.2.1 Nonhomogeneous Poisson Processes

5.2.2 Compound Poisson Processes

5.2.2 Compound Poisson Processes

Definition: A stochastic process Y(t) is called a compound Poissson process if it is the sum of random variables X_n

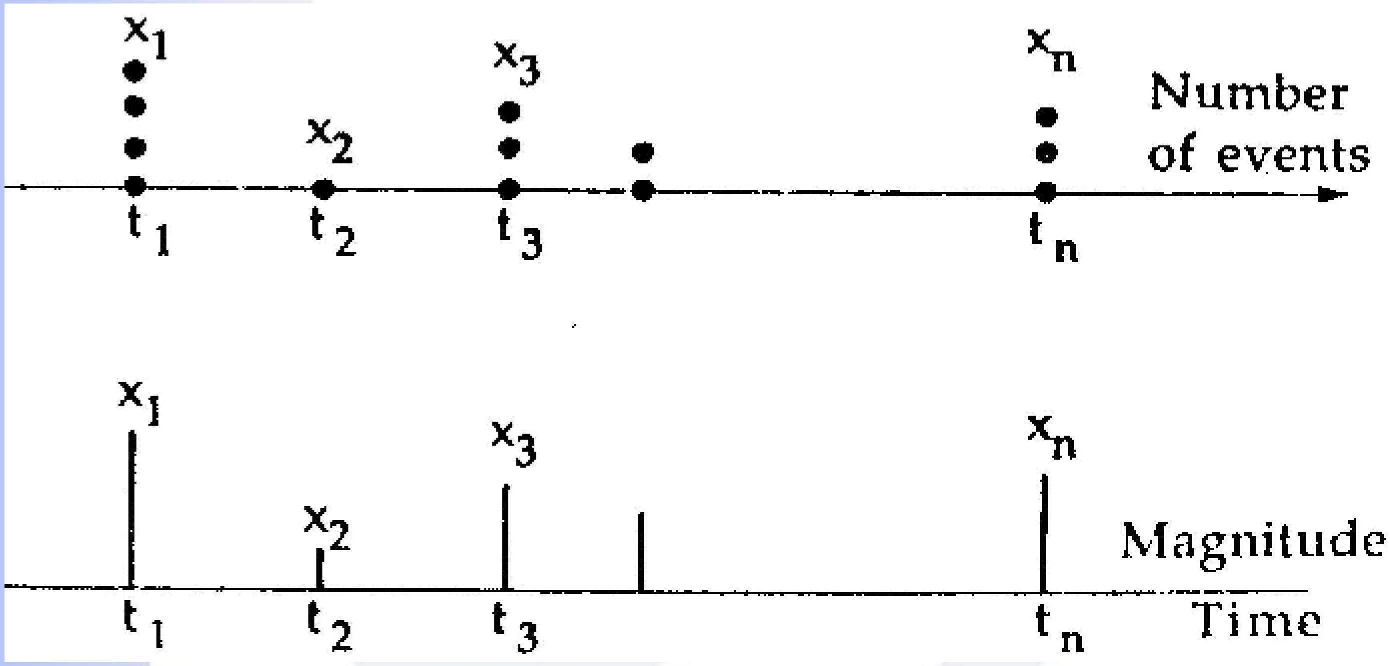
given by

$$Y(t) = \sum_{n=1}^{N(t)} X_n$$

where $N(t)$ is a Poisson process with intensity λ and X_n are independent random variables with identical distribution.

5.2.2 Compound Poisson Processes

X_n may be continuous random variables or discrete random variables.



$$Y(t) = \sum_{n=1}^{N(t)} X_n$$

Y(t) has independent increment.

If N(t) is a homogenous Poisson process, Y(t) has stationary increment. (p73)

5.2.2 Compound Poisson Processes

Characteristic function: $Y(t)=\sum_{n=1}^{N(t)} X_n$

$$\phi_Y(u)=E[e^{iuY(t)}]=E\{E[e^{iuY(t)}\mid N(t)]\}$$

$$=\sum_{n=0}^{\infty} E[e^{iuY(t)}\mid N(t)=n]P(N(t)=n)$$

$$=\sum_{n=0}^{\infty} E[\exp(iu\sum_{k=1}^{n} X_k)]P(N(t)=n)$$

$$=\sum_{n=0}^{\infty} \phi_X^n(u)\frac{(vt)^n}{n!}e^{-vt}=e^{-vt}e^{vt\phi_X(u)}$$

$$=e^{\{vt[\phi_X(u)-1]\}}$$

Conditional expectation:

$$E[Y\mid X=x]$$

$$=\int_{-\infty}^{\infty} y\,dF_{Y|X}(y\mid x)$$

$$=\int_{-\infty}^{\infty} y f(y\mid x)dy$$

$$=\sum_j y_j P(y_j\mid x_i)$$

$E[Y\mid X=x]$ is a random function of x

$$E\{E[Y\mid X]\}=E(Y)$$

5.2.2 Compound Poisson Processes

Moments of compound Poisson processes:

$$E[Y(t)] = \frac{1}{i}\left[\frac{d\phi_Y(u)}{du}\right]_{u=0} = \frac{1}{i}\nu t\phi_x'(0) = \nu t E[x]$$

$$E[Y^2(t)] = \frac{1}{i^2}\left[\frac{d^2\phi_Y(u)}{du}\right]_{u=0}$$

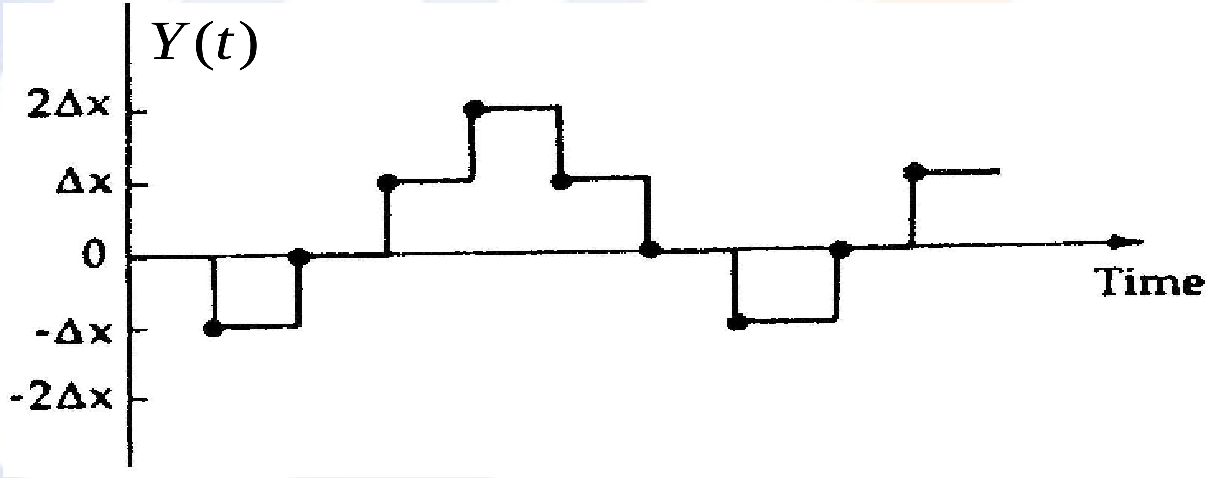
$$= \frac{1}{i^2}\left[\nu t\phi_x''(0) + (\nu t)^2\{\phi_x'(0)\}^2\right]$$

$$\mathrm{Var}[Y(t)] = \nu t E[x^2]$$

$$\mathrm{Cov}[Y(s), Y(t)] = \nu(\min s, t)E[X^2]$$

5.1.1 Random Walk

- A simple example about one-dimension random walk
- The particle moves a fixed distance $+\Delta x$ or $-\Delta x$ with equal probability of positive or negative direction at each step on a straight line path. And each step completes in Δt second.
- Time history of particle movement:



5.2.2 Compound Poisson Processes

- Brownian motion is a ceaseless random fluctuating motion of a microscopic particle suspended in a fluid or gas.
- Random walk —— a simplified random movement of a particle in one dimension.

Example: Consider the Brownian motion of a particle on a line starting at

$Y = 0$ at time $t = 0$.

Assume that random impacts with other particles occur following a Poisson process with intensity ν.

Assume that a particle moves either $+a$ or $-a$ at each impact with equal probability.

5.2.2 Compound Poisson Processes

Y(t) denotes the location of the particle at time *t*, then

$$Y(t) = \sum_{n=1}^{N(t)} X_n$$

where X_n are independent identically distributed random variables with probability $P\{X_n=a\}=P\{X_n=-a\}=1/2$.

The characteristic function of X_n is

$$\phi_X(u) = E[e^{iuX}]$$

$$= P(X=a)e^{iua} + P(X=-a)e^{-iua}$$

$$= \cos au$$

5.2.2 Compound Poisson Processes

The characteristic function of Y(t) is

$$Y(t) = \sum_{n=1}^{N(t)} X_n$$

$$\phi_X(u) = \cos au$$

$$\phi_Y(u) = e^{\{vt[\phi_X(u)-1]\}} = \exp\{vt[\cos au - 1]\}$$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

$$a \to 0, \quad \boxed{\phi_Y(u) = \exp\{vt[-(au)^2/2]\} = \exp\{-\frac{1}{2}va^2tu^2]\}}$$

$$v \to \infty, \quad \boxed{\text{and } \; va^2 = \sigma^2 = \text{constant,}}$$

$$\boxed{\phi_Y(u) = \exp\{-\frac{1}{2}\sigma^2tu^2]\}}$$

5.2.2 Compound Poisson Processes

$Z(t) \sim N(0, \sigma^2 t)$, Find the characteristic function of $Z(t)$.

$$\exp\{-(\frac{z}{s})^2\} \quad z \overset{F}{\leftrightarrow} u \quad \sqrt{\pi}\, s \exp\{-(\frac{us}{2})^2\}, \quad s > 0$$

$$\phi_Z(u) = \int_{-\infty}^{\infty} e^{iuz} p(z)dz = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} \exp\{-\frac{z^2}{2\sigma^2 t}\} e^{iuz} dz$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} \exp\{-\frac{z^2}{2\sigma^2 t}\} e^{-i(-u)z} dz, \qquad s = \sqrt{2\sigma^2 t}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 t}} \sqrt{\pi} \sqrt{2\sigma^2 t} \exp\{-(\frac{-u\sqrt{2\sigma^2 t}}{2})^2\}$$

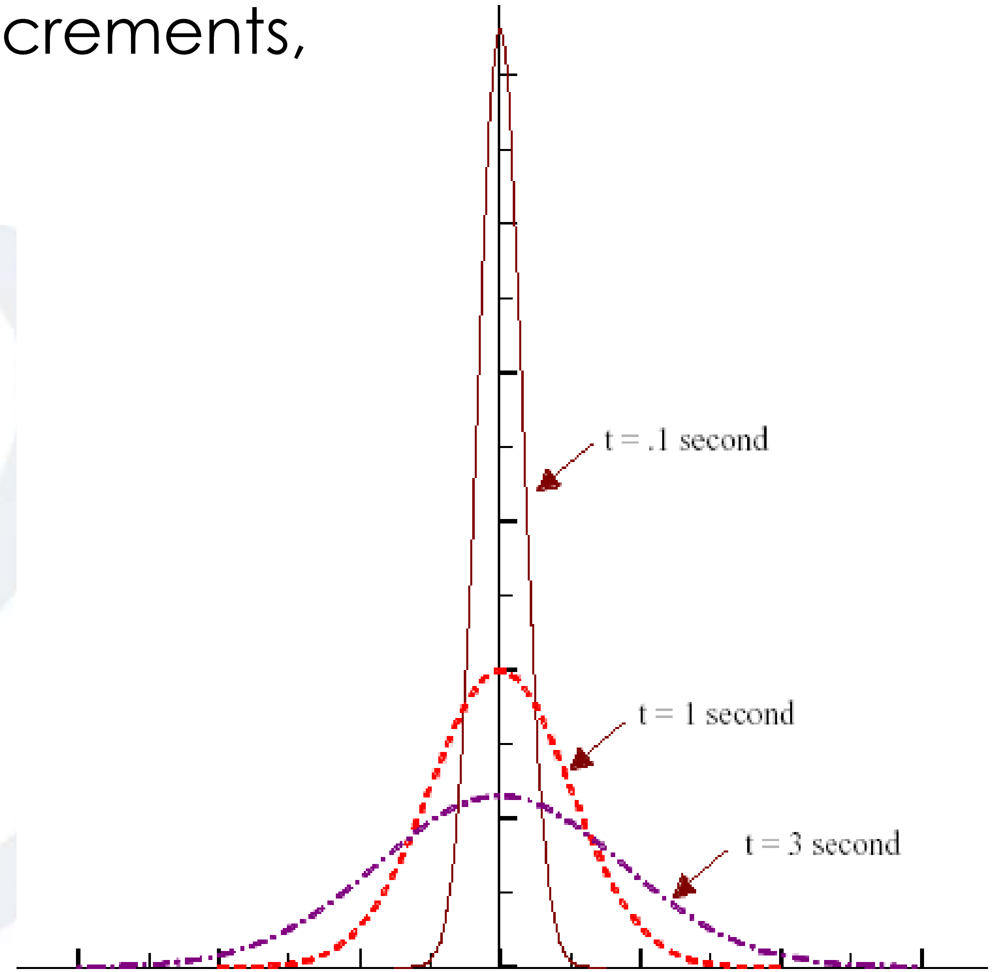
$$= \exp\{-\frac{1}{2}\sigma^2 t u^2\}$$

$$\boxed{\phi_Y(u) = \exp\{-\frac{1}{2}\sigma^2 t u^2]\}}$$

5.2.2 Compound Poisson Processes

Thus, $Y(t)$ is a normal distribution that has stationary independent increments,

$$Y(t) \sim N(0, \sigma^2 t)$$



Exercises

1. The number of cars which pass a certain intersection daily between 12:00 and 14:00 follows a homogeneous Poisson process with intensity 40 per hour. Among these there are 0.8% which disregard the STOP-sign. What is the probability that at least one car disregards the STOP-sign between 12:00 and 13:00?
2. An electronic system is subject to two types of shocks which arrive independently of each other according to homogeneous Poisson processes with intensities 0.002[per hour] and 0.01[per hour]. A shock of type 1 always causes a system failure, a shock of type 2 causes a system failure with probability 0.4. What is the probability that the

Exercises

- A nonhomogeneous Poisson process $N(t)$ has intensity function (mean arrival rate) $\lambda(t) = 1 + 2t$ for $t \geq 0$.

  Initially $N(0) = 0$.

  (a) Find the mean function.

  (b) Find the correlation function.

Home Work

Problems:

2,

4(example 2.3.2),

6,

30

For presentation:

Problems: 10, 11

P69, example2.3.11

A report on Applications of Stochastic Processes