# Chap. 3
# Time-Domain Analysis of Second-Order Stochastic Processes in Linear Systems


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# Chap 3: Stochastic Processes in Linear System

OUTLINE:

## 3.1 Linear System and Unit Impulse Response

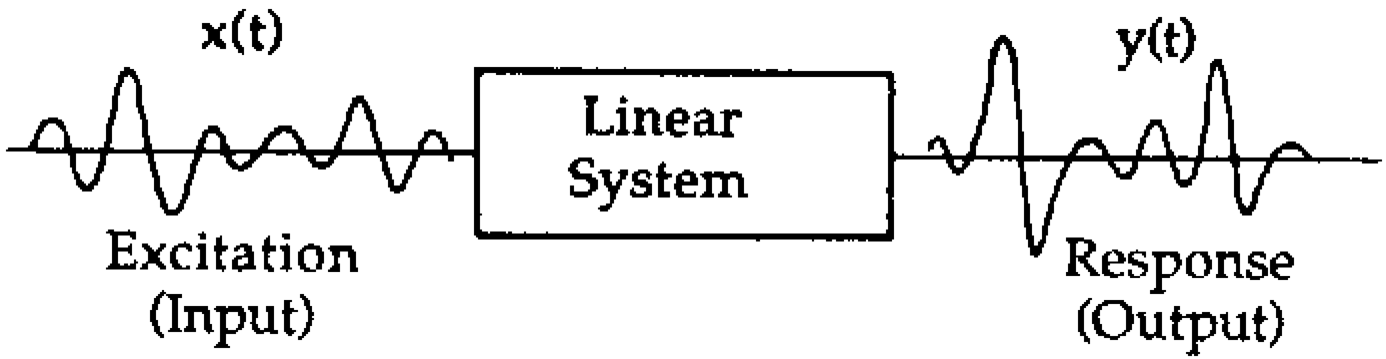
3.2 Linear operations and convergence of random processes

3.3 Input and Output Mean Levels

3.4 Input and Output Correlation Functions

# 3.1 Linear System and Unit Impulse Response

(1) What is a Linear System?




$L[\bullet]$ : a linear operator. $L[x(t)] = y(t)$

Linear System:

$$L[ax(t)] = ay(t)$$

$$L[x_1(t) + x_2(t)] = y_1(t) + y_2(t)$$

Time Invariant:

$$L[x(t+\tau)] = y(t+\tau)$$

Time-Invariant Linear System (Linear Time-Invariant System:LTI)

# 3.1 Linear System and Unit Impulse Response

Linear operations:

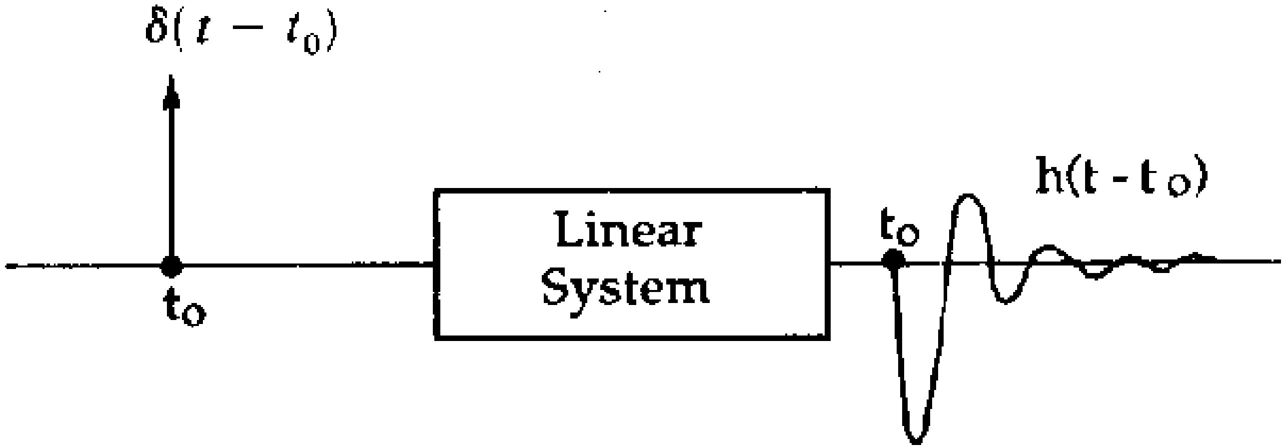
integration, differentiation, multiplication, summing and so on.

Discrete-time LTI

Continuous-time LTI

# 3.1 Linear System and Unit Impulse Response

(2) Impulse Response Function




$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = x(t) * h(t)$$

$$Y(\omega) = X(\omega)H(\omega)$$

Frequency response function: $H(\omega)$

Frequency response amplitude operator: $|H(\omega)|^2$

Stability of LTI: $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

Bounded-input means bounded output.

# 3.1 Linear System and Unit Impulse Response

*e.g.* The input of a linear system is given by $x(t) = ke^{i\omega t}$
Where k is a constant. The output is

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} ke^{i\omega(t-\tau)}h(\tau)d\tau$$

$$= ke^{i\omega t}\int_{-\infty}^{\infty} e^{-i\omega\tau}h(\tau)d\tau = ke^{i\omega t}H(\omega)$$

Letting $\omega = 0$ , then input: $x(t) = k$

Output: $y(t) = kH(0)$

Letting $\omega = \omega_i$, then input: $x(t) = ke^{i\omega_i t}$

Output: $y(t) = ke^{i\omega_i t}H(\omega_i)$

# (3) Evaluation of Impulse and Frequency Response Functions

Discrete system: what is the h(k)?

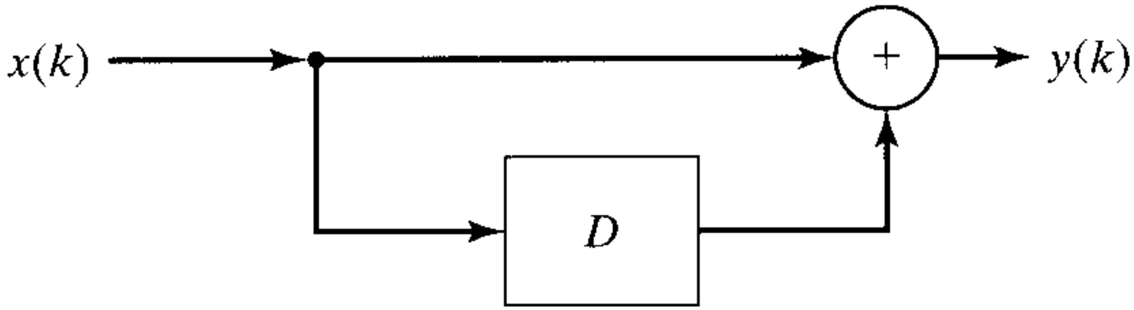



FIGURE 3–1 A simple discrete-time, linear, time-invariant filter.

$$y(k) = \sum_{-\infty}^{\infty} h(n)x(k-n)$$

$$y(k) = x(k) + x(k-1)$$

•

$n$=0,$n$=1, $h(n)$=1

$$\Rightarrow h(k) = \begin{cases} 1, & k = 0, k = 1 \\ 0, & otherwise \end{cases}$$

# (3) Evaluation of Impulse and Frequency Response Functions

Method 1: Theoretical Approach

Property of the Fourier transform

Time Differentiation:

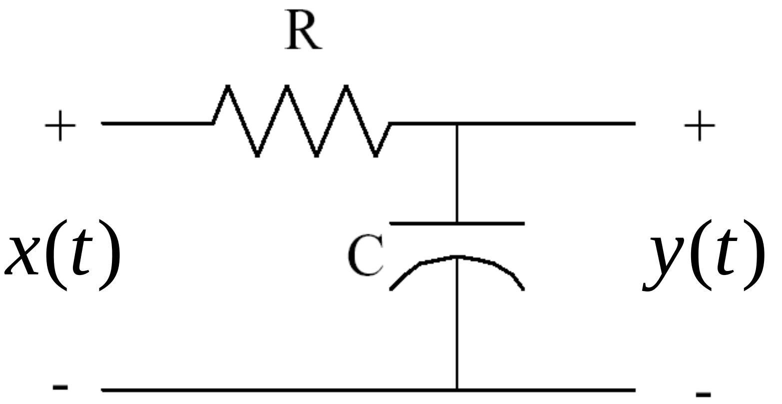
If $$f(t) \leftrightarrow F(\omega)$$

then $$\frac{d^n f(t)}{dt^n} \leftrightarrow (i\omega)^n F(\omega)$$

# (3) Evaluation of Impulse and Frequency Response Functions

*e.g.1* A first-order RC low-pass filter

(a) $$RC\frac{dy(t)}{dt}+y(t)=x(t)$$




$$RCi\omega Y(\omega)+Y(\omega)=X(\omega)$$

$$\boxed{H(\omega)=\frac{1}{1+i\omega RC}=\frac{\alpha}{\alpha+i\omega}}$$

where $\alpha = 1/RC$

(b) Letting $x(t)=e^{i\omega t}$ $\quad y(t)=e^{i\omega t}H(\omega)$

$$RC\frac{d[e^{i\omega t}H(\omega)]}{dt}+e^{i\omega t}H(\omega)=e^{i\omega t}$$

# (3) Evaluation of Impulse and Frequency Response Functions

*e.g.2* A second-order system

$$m\frac{d^2y(t)}{dt^2}+r\frac{dy(t)}{dt}+ky(t)=x(t)$$

$$(i\omega)^2 mY(\omega)+i\omega rY(\omega)+kY(\omega)=X(\omega)$$

$$\boxed{H(\omega)=\frac{1}{k-m\omega^2+i\omega r}}$$

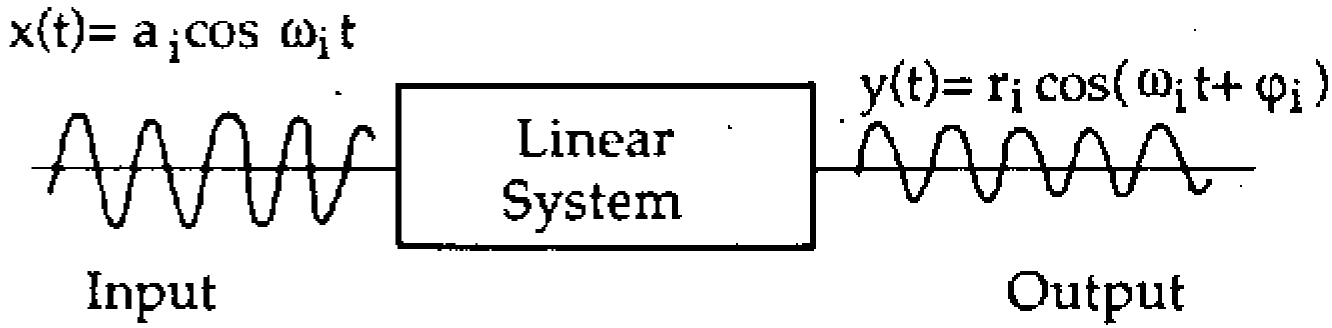
*e.g.3*

$$y(t)=\frac{1}{2T}\int_{t-T}^{t+T}x(u)du$$

What is the h(t) ?

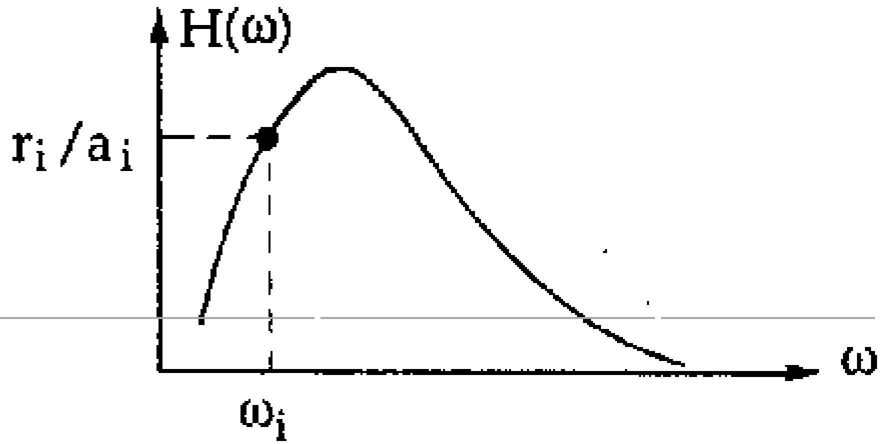
## Method 2: Experimental Approach

frequency response function $H(\omega)$




$$x(t) = ke^{i\omega t}$$

$$y(t) = ke^{i\omega_i t} H(\omega_i)$$



OUTLINE:

3.1 Linear System and Unit Impulse Response

3.2 Linear operations and convergence of random processes

3.3 Input and Output Mean Levels

3.4 Input and Output Correlation Functions

# 3.2 Integrated and Differentiated Random Processes

OUTLINE:

3.2.1 Stochastic Convergence

3.2.2 Mean-Square Continuity, Differentiability and Integrability

3.2.3 Mean,Variance, and Covariance

3.2.4 Autocorrelation Function of Derived Random Processes

# 3.2 Integrated and Differentiated Random Processes

Reference:

1. William A. Gardner
   - Introduction to Random Processes with Applications to Signals and Systems,
   - Index Entry: O211.6 W77
   - Section 2.5, Convergence
   - Chap 7, Stochastic Calculus
2. 刘次华
   - 随机过程
   - 6.3 随机分析

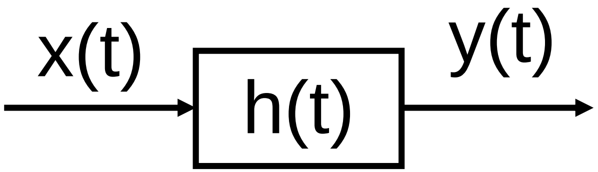
# 3.2 Integrated and Differentiated Random Processes

Introduction:

Linear operations of a signal:

summing , multiplication, integration, differentiation and so on.

e.g. a linear system




$$y(t) = \int_{-\infty}^{\infty} x(u)h(t-u)du$$

# 3.2 Integrated and Differentiated Random Processes

## Introduction:

Recall (for nonrandom functions) :

Conventional integral (Riemann integral),

The limit of the sequence of areas of rectangles:

$$\int_{-\infty}^{t} x(u)du \equiv \lim_{\varepsilon \to 0} \varepsilon \sum_{i}^{t/\varepsilon} x(i\varepsilon)$$

Conventional derivative

The limit of the sequence of differences:

$$\frac{dx(t)}{dt} \equiv \lim_{\varepsilon \to 0} \frac{x(t) - x(t-\varepsilon)}{\varepsilon}$$

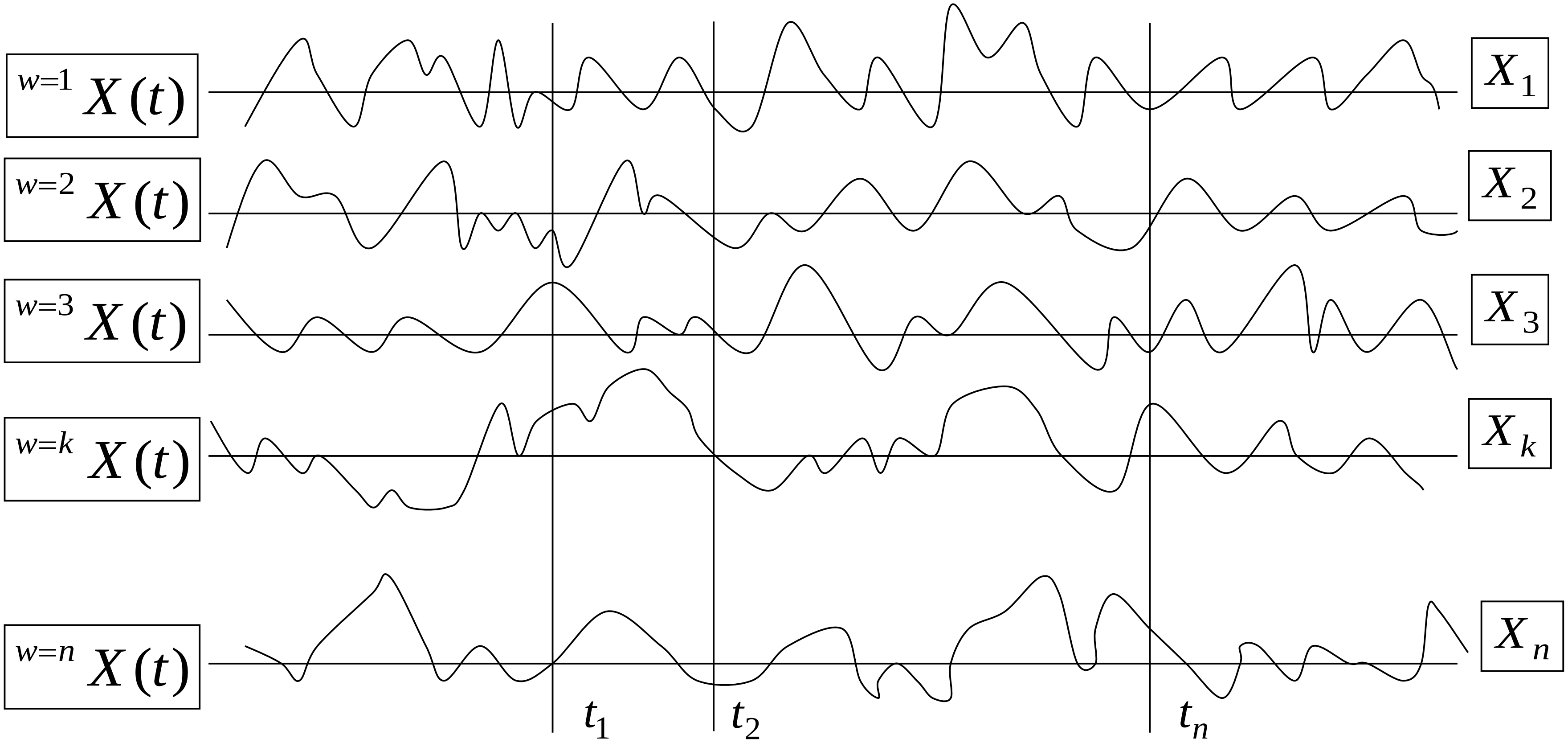
## Introduction:

For random processes:

If we define the derivative and the integral of a process X(t) in terms of the conventional derivative and the integral of each sample function x(t),

then we must require the existence of limits of samples of sequences of random variables for every sample point $\omega$ in the sample space.

# 2.1.1 Definition and examples




- Are X1, X2, ..., Xn the same value?
- After sampling, a random variable sequence X(n) can be obtained from X(t). What is the limit of X(n)?

# 3.2 Integrated and Differentiated Random Processes

## 3.2.1 Stochastic Convergence

3.2.2 Mean-Square Continuity, Differentiability and Integrability

3.2.3 Mean,Variance, and Covariance

3.2.4 Autocorrelation Function of Derived Random Processes

# 3.2.1 Stochastic Convergence

A sequence of random variables $\{X_n\}$:

$\{X_n\}$ is a family of sequences of real numbers,

$$\{X_n(\omega) : \omega \in \Omega\}$$

together with a sequence of joint probability distributions,

$$\{F(x_1, x_2, \ldots, x_n)\}$$

The second order moments of $\{X_n\}$ exist.

Second-order moment sequence

# 3.2.1 Stochastic Convergence

1. Convergence with Probability 1

(Convergence Almost Surely)

$$X_n \xrightarrow{a.e} X$$

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \text{for all } \omega \in \tilde{\Omega} \subseteq \Omega$$

where $P(\tilde{\Omega}) = 1$

that is,

$$P\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = 1$$

For a sequence that converges with probability one, there can be particular sample sequences $\{X_n(\omega) : \omega \notin \tilde{\Omega}\}$ that do not converge.

However, the probability of the event that the sequence does not converge is zero: $P\{\omega \in \Omega : \omega \notin \tilde{\Omega}\} = 0$

# 3.2.1 Stochastic Convergence

## 2. Convergence in Mean Square

$$X_n \xrightarrow{m.s} X$$

$$\lim_{n\to\infty} E[(X_n - X)^2] = 0$$

Limit in mean square: $\underset{n\to\infty}{\mathrm{l.i.m}}\, X_n = X$

$\{X_n\}$ is also called expected square convergence.

Mean squared error is the most commonly used measure of the difference between two random variables.

In practice, the mean squared error is usually obtained by averaging over time, not over a set of a statistical samples.

( P99~101, 3.5.2 )

# 3.2.1 Stochastic Convergence

## 2. Convergence in Mean Square $\boxed{X_n \xrightarrow{m.s} X}$

*Theorem1* $\{X_n\}$ and $\{Y_n\}$ are second order moment sequence, U is a second order moment random variable, $\{c_n\}$ is a constant sequence, a,b,c are constants,

$\underset{n\to\infty}{\text{l.i.m}}\, X_n = X$ , $\underset{n\to\infty}{\text{l.i.m}}\, Y_n = Y$ , $\lim_{n\to\infty} c_n = c$ , then, $\text{l.i.m}\, U = U$

$$\underset{n\to\infty}{\text{l.i.m}}\, c_n = \lim_{n\to\infty} c_n = c \qquad \underset{n\to\infty}{\text{l.i.m}}(aX_n + bY_n) = aX + bY$$

$$\underset{n\to\infty}{\text{l.i.m}}\, c_n U = cU \qquad \lim_{n\to\infty} E[X_n] = E[X] = E[\underset{n\to\infty}{\text{l.i.m}}\, X_n]$$

$$\lim_{n,m\to\infty} E[X_n Y_m] = E[XY] = E[(\underset{n\to\infty}{\text{l.i.m}}\, X_n)(\underset{m\to\infty}{\text{l.i.m}}\, Y_m)]$$

# 3.2.1 Stochastic Convergence

## 3. Convergence in Probability

$$X_n \xrightarrow{P} X$$

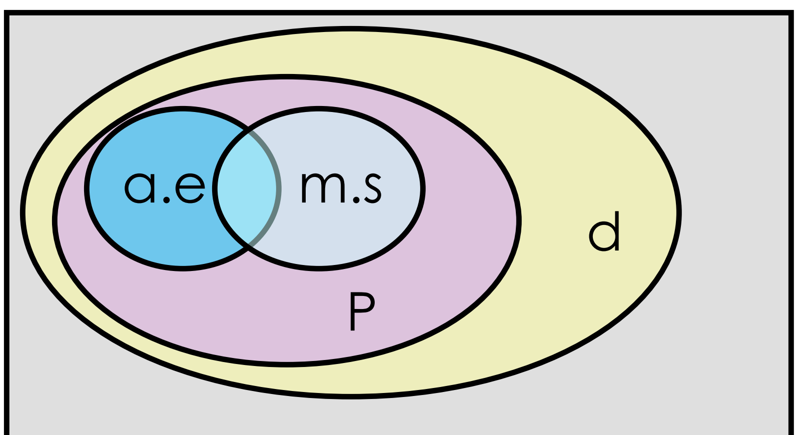
$$\lim_{n\to\infty} P\{|X_n(\omega) - X(\omega)| < \varepsilon\} = 1 \quad \text{for all } \varepsilon > 0$$

## 4. Convergence in Distribution

$$X_n \xrightarrow{d} X$$

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

for all continuity points x of $F_x(\cdot)$



3.2.1 Stochastic Convergence

3.2.2 Mean-Square Continuity, Differentiability and Integrability

3.2.3 Mean,Variance, and Covariance

3.2.4 Autocorrelation Function of Derived Random Processes

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

- Based on mean square convergence.
- Appropriate for the study of random processes in linear transformations.
- Inadequate for the study of nonlinear systems with random excitation.

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

*Def.1* Mean-Square Continuity

$$\underset{\varepsilon \to 0}{\mathrm{l.i.m}}\, X(t-\varepsilon) = X(t)$$

A process $\{x(t), t \in T\}$ is Mean-Square Continuous at ***t*** if and only if the following limit exists.

$$\lim_{\varepsilon \to 0} E[(X(t) - X(t-\varepsilon))^2] = 0$$

*Theorem1* X(t) is mean-square continuous at ***t*** if and only if $R_{XX}(t_1, t_2)$ is continuous at the moment $t_1 = t_2 = t$ , that is

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} [R_{XX}(t,t) - R_{XX}(t-\varepsilon_1, t-\varepsilon_2)] = 0$$

For a stationary process, if and only if Rxx() is continuous at $\tau = 0$ , that is $\lim_{\varepsilon \to 0}[R_{XX}(\varepsilon) - R_{XX}(0)] = 0$

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

## Mean-Square Continuity

If $\boldsymbol{X}(\boldsymbol{t})$ is mean-square continuous at $\boldsymbol{t}$, then the mean function is continuous in the ordinary sense at t.

$$\lim_{\varepsilon \to 0} E[X(t-\varepsilon)] = E[X(t)] = E\{\underset{\varepsilon \to 0}{\text{l.i.m.}}\, X(t+\varepsilon)\}$$

For a mean-square continuous process, the order of execution of the operations of expectation and limiting can be interchanged.

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

*Def.2* Mean-Square Differentiability

A process $\{x(t), t \in T\}$ is Mean-Square differentiable at *t* if and only if there exists a random variable, denoted by $\dot{X}(t)$ (or $dX(t)/dt$ ), such that the limit

$$\lim_{\varepsilon \to 0} E\left\{\left(\frac{X(t)-X(t-\varepsilon)}{\varepsilon} - \dot{X}(t)\right)^2\right\} = 0$$

exists.

$$\boxed{\dot{X}(t) = \frac{dX(t)}{dt} = \underset{\varepsilon \to 0}{\mathrm{l.i.m}} \frac{X(t)-X(t-\varepsilon)}{\varepsilon}}$$

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

## Mean-Square Differentiability

*Theorem 2* X(t) is mean-square differentiable at t if and only if $R_{XX}(t_1,t_2)$ is differentiable jointly at the moment $t_1=t_2=t$ , that is the limit

$$\frac{\partial^2 R_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}\Big|_{t_1=t_2=t} \equiv \lim_{\varepsilon_1,\varepsilon_2\to 0}\Big[\frac{R_{XX}(t,t)-R_{XX}(t-\varepsilon_1,t)}{\varepsilon_1\varepsilon_2} - \frac{R_{XX}(t,t-\varepsilon_2)-R_{XX}(t-\varepsilon_1,t-\varepsilon_2)}{\varepsilon_1\varepsilon_2}\Big]$$

must exists.

For a stationary process, if and only if Rxx() is twice differentiable at $\tau=0$

$$\frac{\partial^2 R_{XX}(\tau)}{\partial \tau^2}\Big|_{\tau\to 0} \equiv \lim_{\varepsilon^2\to 0}\frac{1}{\varepsilon^2}[R_{XX}(\varepsilon)-2R_{XX}(0)+R_{XX}(-\varepsilon)]$$

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

## Mean-Square Differentiability

If X(t) is mean-square differentiable at t

$$\frac{dE[X(t)]}{dt} = E[\dot{X}(t)]$$

$$\frac{\partial R_{XX}(t_1,t_2)}{\partial t_1} = \frac{\partial E[X(t_1)X(t_2)]}{\partial t_1} = E[\dot{X}(t_1)X(t_2)]$$

$$\frac{\partial R_{XX}(t_1,t_2)}{\partial t_2} = \frac{\partial E[X(t_1)X(t_2)]}{\partial t_2} = E[X(t_1)\dot{X}(t_2)]$$

$$\frac{\partial^2 R_{XX}(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial^2 E[X(t_1)X(t_2)]}{\partial t_1 \partial t_2} = E[\dot{X}(t_1)\dot{X}(t_2)]$$

- For a mean-square differentiable process, the order of execution of the operations of expectation and derivation can be interchanged.

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

## *Def.3* Mean-Square Integrability

A process $\{x(t), t \in T\}$ is mean-square integrable on the interval (0, t) if and only if there exists a random variable, denoted by $X^{(-1)}(t)$ (or $\int_0^t X(u)du$ ), such that the limit

$$\lim_{\varepsilon \to 0} E\left\{\left(\varepsilon \sum_{i=1}^{t/\varepsilon} X(i\varepsilon) - X^{(-1)}(t)\right)^2\right\} = 0$$

exists.

$$\boxed{X^{(-1)}(t) = \int_0^t X(u)du = \underset{\varepsilon \to 0}{\mathrm{l.i.m}} \sum_{i=1}^{t/\varepsilon} \varepsilon X(i\varepsilon)}$$

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

## Mean-Square Integrability

*Theorem 3* X(t) is mean-square integrable on the interval (0, t) if and only if $R_{XX}(t_1,t_2)$ is Riemann-integrable on the square (0,t)×(0,t); that is the limit

$$\int_0^t \int_0^t R_{XX}(t_1,t_2)dt_1 dt_2 \equiv \lim_{\varepsilon \to 0}[\varepsilon^2 \sum_{i,j=1}^{t/\varepsilon} R_{XX}(i\varepsilon, j\varepsilon)]$$

must exists.

For a stationary process, if and only if Rxx() is Riemann-integrable on (0,t),

$$\int_0^t R_{XX}(u)du \equiv \lim_{\varepsilon \to 0}[\varepsilon \sum_{i=1}^{t/\varepsilon} R_{XX}(i\varepsilon)]$$

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

Mean-Square Integrability

$$E[\int_0^t X(u)du] = \int_0^t E[X(u)]du$$

$$E[\int_0^t X(t_1)dt_1 \int_0^t X(t_2)dt_2] = \int_0^t \int_0^t E[X(t_1)X(t_2)]dt_1 dt_2$$

$$= \int_0^t \int_0^t R_{XX}(t_1, t_2)]dt_1 dt_2$$

- The operations of integration and expectation can be interchanged.

*Theorem 3* If X(t) is mean-square continuous on the interval (a, b), and the integral exists

$$Z(t) = \int_a^t X(u)du \quad (a \le t \le b)$$

then Z(t) is mean-square differentiable, $\dot{Z}(t) = X(t)$

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

## Mean-Square Integrability

- If a Gaussian process X(t) is mean-square differentiable, its derivative $\dot{X}(t)$ is a Gaussian process.

- If a Gaussian process X(t) is mean-square continuous on the interval (a, b), its integration

$$Z(t) = \int_a^t X(u)du \quad (a \leq t \leq b)$$

is a Gaussian process.

(P94, Example 3-5: the Wiener process is a nonstationary, second-order Gaussian process. It is the integral of another Gaussian process)

# 3.2.2 Mean-Square Continuity, Differentiability and Integrability

Stationarity:

- The derivative process of a stationary process is also stationary.
- The integral process of a stationary process need not to be stationary.

*Example:* Is a Poisson Process mean-square continuous, mean-square integrable, mean-square differentiable ?

Solution:

$$R(t_1,t_2)=Cov(t_1,t_2)+E[N(t_1)]E[N(t_2)]$$

$$=Var[N(t_1)]+\lambda^2 t_1 t_2=\lambda^2 t_1 t_2+\lambda t_1, \qquad t_1<t_2$$

The last is “not”. Sample waves are not continuous.

- Mean-square continuous does not mean sample wave’s continuity.

# 3.2 Integrated and Differentiated Random Processes

3.2.1 Stochastic Convergence

3.2.2 Mean-Square Continuity, Differentiability and Integrability

3.2.3 Mean,Variance, and Covariance

3.2.4 Autocorrelation Function of Derived Random Processes

# 3.2.3 Mean,Variance, and Covariance

1. The integrated random process

$$Z(t) = \int_0^t X(u)du$$

then,

$$E[Z(t)] = \int_0^t E[X(u)]du$$

$$Var[Z(t)] = E[Z(t)^2] - (E[Z(t)])^2$$

$$= \int_0^t \int_0^t E[X(u)X(v)]dudv - \int_0^t \int_0^t E[X(u)]E[X(v)]dudv$$

$$= \int_0^t \int_0^t C_{XX}(u,v)dudv$$

$$Cov_{ZZ}(t_1,t_2) = Cov[\int_0^{t_1} X(u)du, \int_0^{t_2} X(v)dv]$$

$$= \int_0^{t_2} \int_0^{t_1} Cov_{XX}(u,v)dudv$$

# 3.2.3 Mean,Variance, and Covariance

2. The differentiated random process

$$\dot{X}(t) = dX(t)/dt$$

then, $E[\dot{X}(t)] = dE[X(t)]/dt$

$$Var[\dot{X}(t)] = E[\dot{X}^2(t)] - (E[\dot{X}(t)])^2 = \frac{d^2 Var[X(t)]}{dt^2}$$

$$Cov_{\dot{X}\dot{X}}(t_1,t_2) = Cov[\frac{dX(t_1)}{dt_1}, \frac{dX(t_2)}{dt_2}] = \frac{\partial^2 Cov_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}$$

$$Cov_{X\dot{X}}(t_1,t_2) = Cov[X(t_1), \frac{dX(t_2)}{dt_2}] = \frac{dCov_{XX}(t_1,t_2)}{dt_2}$$

$$Cov_{\dot{X}X}(t_1,t_2) = Cov[\frac{dX(t_1)}{dt_1}, X(t_2)] = \frac{dCov_{XX}(t_1,t_2)}{dt_1}$$

# 3.2.3 Mean,Variance, and Covariance

2. The differentiated random process

*Example* The covariance function of the Wiener-Levy process is given by

$$Cov_{XX}(t_1,t_2)=\sigma^2\min(t_1,t_2)$$

or

$$Cov_{XX}(t_1,t_2)=\sigma^2 t_1 \quad \text{for } t_1<t_2$$

$$\frac{\partial^2 Cov_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}=\frac{\partial \sigma^2 U(t_2-t_1)}{\partial t_2}=\sigma^2\delta(t_2-t_1)$$

$$E[X(t)]=0, \qquad E[\dot{X}(t)]=0$$

$$R_{\dot{X}\dot{X}}(t_1,t_2)=\sigma^2\delta(t_2-t_1)=\sigma^2\delta(\tau)$$

Conclusion: the derivative of the Wiener-Levy process is a Gaussian white noise process, a special stationary process.

*Example.* White Noise

The process with correlation function

$$R(\tau) = A\delta(\tau)$$

and the spectral density $S(\omega) = A = \text{constant}$

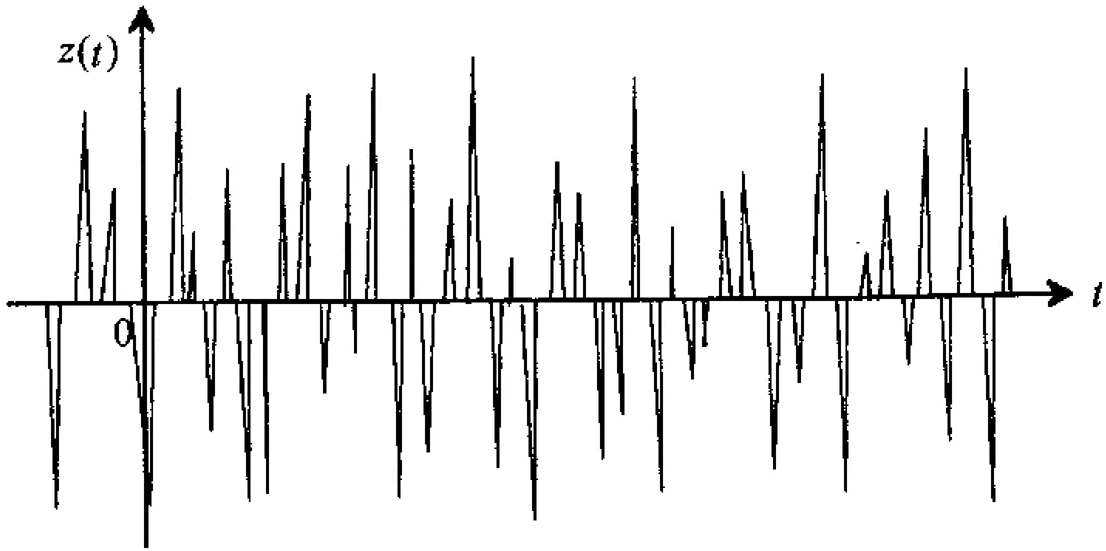
The continuous white noise is a real, stationary, continuous-time stochastic process with constant spectral density.

The "most random" stochastic process

# 3.2.3 Mean, Variance, and Covariance

*Example.* White Noise

White noise can be thought of as a sequence of extremely sharp pulses, which occur after extremely short time intervals, and which have independent, identically distributed amplitudes.




Gaussian white noise:

white noise with Gaussian distribution.

# 3.2.3 Mean,Variance, and Covariance

2. The differentiated random process

*Example* The autocorrelation of a random process X(t) is given by $R_{XX}(\tau) = \sigma^2 e^{-a\tau^2}$, and $E[X(t)] = 0$ , $t_1 - t_2 = \tau$

Obtain: $R_{\dot{X}\dot{X}}(\tau)$

# 3.2 Integrated and Differentiated Random Processes

3.2.1 Stochastic Convergence

3.2.2 Mean-Square Continuity, Differentiability and Integrability

3.2.3 Mean,Variance, and Covariance

3.2.4 Autocorrelation Function of Derived Random Processes

# 3.2.4 Autocorrelation Function of Derived Random Processes

Autocorrelation function of derived random processes

$$R_{x\dot{x}}(t_1, t_2) = E\left[x(t_1)\dot{x}(t_2)\right]$$

$$= E\left[x(t_1)\frac{x(t_2 + \Delta t) - x(t_2)}{\Delta t}\right]$$

$$= \frac{1}{\Delta t}\left\{R_{xx}(t_1, t_2 + \Delta t) - R_{xx}(t_1, t_2)\right\}$$

$$R_{x\dot{x}}(t_1, t_2) = \frac{\partial}{\partial t_2}R_{xx}(t_1, t_2) = -\frac{d}{d\tau}R_{xx}(\tau)$$

$$R_{\dot{x}\dot{x}}(\tau) = \frac{\partial^2}{\partial t_1\,\partial t_2}R_{xx}(t_1, t_2) = -\frac{d^2}{d\tau^2}R_{xx}(\tau)$$

$$R_{\ddot{x}\ddot{x}}(\tau) = \frac{d^4}{d\tau^4}R_{xx}(\tau)$$

OUTLINE:

3.1 Linear System and Unit Impulse Response

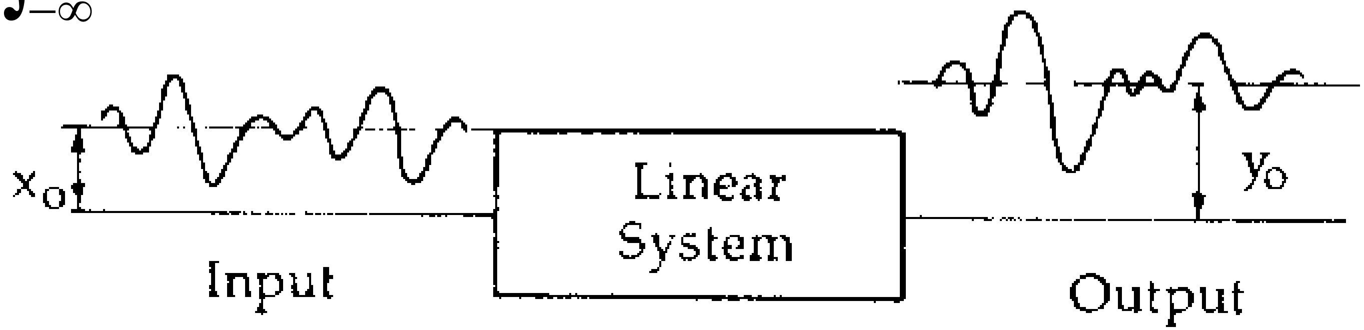
3.2 Linear operations and convergence of random processes

3.3 Input and Output Mean Levels

3.4 Input and Output Correlation Functions

# 3.3 Input and Output Mean Levels

$$Y(t) = \int_{-\infty}^{\infty} X(t-\tau)h(\tau)d\tau$$




For a stationary process, if and only if Rxx() is continuous at $\tau = 0$, that is $\lim_{\varepsilon \to 0}[R_{XX}(\varepsilon) - R_{XX}(0)] = 0$

***Theorem 3*** **If X(t) is mean-square continuous on the interval (a, b), its integral exists.**

**=> If Rxx() is continuous at $\tau = 0$, the second-order stationary process X(t) is integrable.**

# 3.3 Input and Output Mean Levels

$$E[Y(t)] = E[\int_{-\infty}^{\infty} X(t-\tau)h(\tau)d\tau]$$

$$= \int_{-\infty}^{\infty} E[X(t-\tau)]h(\tau)d\tau$$

If X(t) has constant mean or is a stationary process, then

$$\boxed{E[Y(t)] = m_X \int_{-\infty}^{\infty} h(\tau)d\tau = m_X H(0) = m_Y}$$

1. **The output mean can be determined from the input mean alone.**
2. **If the input mean is constant, so is the output mean.**
3. **If the input mean is constant and finite, and the system is stable, the output mean is constant and finite.**

# 3.3 Input and Output Mean Levels

**e.g. suppose a continuous-time linear system has impulse response**

$$h(t,\tau)=\begin{cases}\exp\{-\alpha(t-\tau)\}, & t\ge\tau\\ 0, & t<\tau\end{cases}\qquad \alpha>0$$

**If the input process X(t) has constant mean $\mu_X$ find the mean of the output process Y(t).**

**Solution: Since the system is a LTI system and the input mean is constant,**

$$\mu_Y=\mu_X\int_{-\infty}^{\infty}h(\tau)d\tau=\mu_X\int_0^{\infty}e^{-\alpha\tau}d\tau=\mu_X/\alpha$$

# 3.3 Input and Output Mean Levels

Calculate output's mean value in discrete-time LTI

$$Y(k)=\sum_{n=-\infty}^{\infty} h(k,n)X(n)$$

$$=\sum_{n=-\infty}^{\infty} h(k-n)X(n) \qquad \text{(for time-invariant system)}$$

$$E[Y(k)]=E\left[\sum_{n=-\infty}^{\infty} h(k-n)X(n)\right]=\sum_{n=-\infty}^{\infty} h(k-n)E[X(n)]$$

If X(n) has constant mean or is a stationary random process,

$$\mu_Y=\mu_X\sum_{k=-\infty}^{\infty} h(k)$$

# 3.3 Input and Output Mean Levels

*E.g.* A discrete-time linear system is described as follows: for any input x(k), the output is the signal

$$y(k) = \frac{1}{2} y(k-1) + x(k)$$

Suppose the input to this system is a random process X(t) with constant mean $\mu_X$. What is the mean of the output process Y(t)?

Solution:

Method 1: find the pulse response at first,

apply the proceeding formula

Method 2: the linear system is time-invariant, then

$Y(k) = \frac{1}{2} Y(k-1) + X(k)$, and the output mean is constant,

$$\mu_Y = \frac{1}{2} \mu_Y + \mu_X \qquad \mu_Y = 2\mu_X$$

OUTLINE:

3.1 Linear System and Unit Impulse Response

3.2 Linear operations and convergence of random processes

3.3 Input and Output Mean Levels

3.4 Input and Output Correlation Functions

(LTI system and the input is stationary process)

# 3.4 Input and Output Correlation Functions

1. $R_{XY}(t_1,t_2)$ $R_{YX}(t_1,t_2)$

$\tau = t_1 - t_2$

$R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau)$

$R_{YX}(\tau) = R_{XX}(\tau) * h(\tau)$

$\tau = t_2 - t_1$

$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$

$R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$

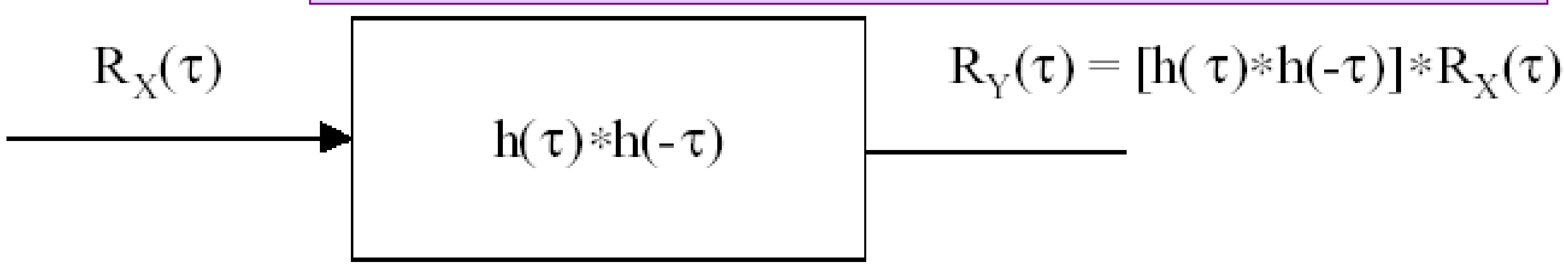
2. $R_{YY}(\tau)$

$\tau = t_1 - t_2$

$$R_{YY}(\tau) = R_{XY}(\tau) * h(\tau) = R_{YX}(\tau) * h(-\tau)$$
$$= h(\tau) * h(-\tau) * R_{XX}(\tau)$$

$\tau = t_2 - t_1$

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau) = R_{YX}(\tau) * h(\tau)$$
$$= h(\tau) * h(-\tau) * R_{XX}(\tau)$$



# 3.4 Input and Output Correlation Functions

$$R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau)$$

$$R_{YX}(\tau) = R_{XX}(\tau) * h(\tau)$$

$$\begin{aligned} R_{YY}(\tau) &= R_{XY}(\tau) * h(\tau) = R_{YX}(\tau) * h(-\tau) \\ &= h(\tau) * h(-\tau) * R_{XX}(\tau) \end{aligned}$$

$$\tau = t_1 - t_2$$

If the input to a LTI is stationary, so does the output.

The input and output processes are jointly stationary.

If the input is a Gaussian process, so does the output.

The input and output processes are jointly Gaussian.

# 3.4 Input and Output Correlation Functions

$$\boxed{\begin{aligned} R_{YY}(\tau) &= R_{XY}(\tau) * h(\tau) = R_{YX}(\tau) * h(-\tau) \\ &= h(\tau) * h(-\tau) * R_{XX}(\tau) \end{aligned}}$$

$$f(u) = \int_{-\infty}^{\infty} h(v-u)h(v)\,dv$$

$$f(u) = f(-u)$$

$$R_{YY}(\tau) = f(\tau) * R_{XX}(\tau) = \int_{-\infty}^{\infty} f(\tau - u) R_{XX}(u)du$$

$$= \int_{-\infty}^{\infty} f(u - \tau) R_{XX}(u)du$$

- Integrate the product of the autocorrelation function and a delayed version of the function *f*. No requirement to ‘flip” one of the functions about the origin, as is necessary in the general convolution procedure.

# 3.4 Input and Output Correlation Functions

*e.g.* the rectangular pulse of duration T>0 is defined as

$$p_T(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & ortherwise \end{cases}$$

Determine the function *f* for the linear filter with impulse response $h(t) = p_T(t)$

$$f(\tau) = \int_{-\infty}^{\infty} h(t)h(t-\tau)dt = \int_{\tau}^{T} p_T(t)\, p_T(t-\tau)dt$$

$$f(\tau) = \begin{cases} T - |\tau|, & |\tau| < T \\ 0, & ortherwise \end{cases}$$

# 3.4 Input and Output Correlation Functions

*e.g.* Consider the proceeding example. The input is a wide-sense stationary ransom process X(t) with autocorrelation function

$$R_X(\tau) = \sigma^2 \exp(-\gamma |\tau|), \qquad -\infty < \tau < \infty$$

where $\gamma$ is a positive constant. Find the autocorrelation function for the output process Y(t).

Solution:

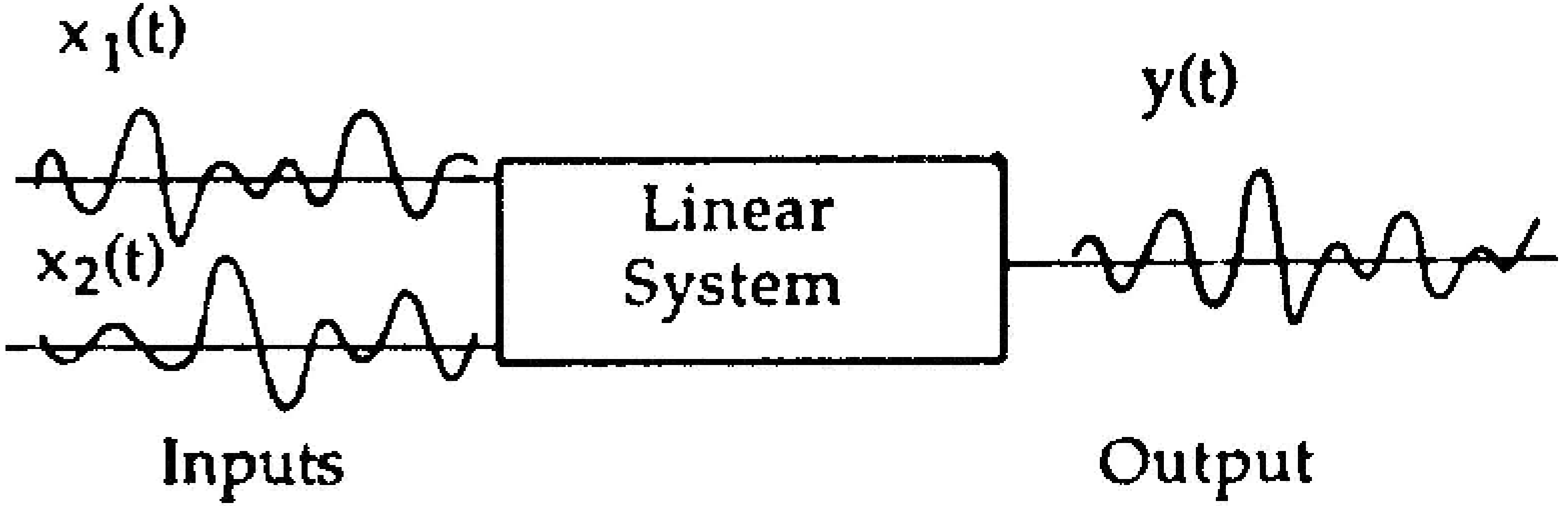
$$R_Y(\tau) = \int_{-\infty}^{\infty} \sigma^2 \exp(-\gamma |u|) f(u-\tau) du$$

$$= \int_{\tau-T}^{\tau+T} \sigma^2 \exp(-\gamma |u|) f(u-\tau) du$$

$$= \int_{\tau-T}^{\tau+T} \sigma^2 \exp(-\gamma |u|)[T - |u-\tau|] du$$

# 3.4 Input and Output Correlation Functions

*e.g.* Response of a system of dual inputs




$y_1(t)$ : response of $x_1(t)$ $\qquad x(t) = x_1(t) + x_2(t)$

$y_2(t)$ : response of $x_2(t)$ $\qquad y(t) = y_1(t) + y_2(t)$

$$R_{XX}(t_1,t_2) = R_{X_1X_1}(t_1,t_2) + R_{X_2X_2}(t_1,t_2) + R_{X_1X_2}(t_1,t_2) + R_{X_2X_1}(t_1,t_2)$$

$$R_{YY}(t_1,t_2) = R_{Y_1Y_1}(t_1,t_2) + R_{Y_2Y_2}(t_1,t_2) + R_{Y_1Y_2}(t_1,t_2) + R_{Y_2Y_1}(t_1,t_2)$$

# 3.4 Input and Output Correlation Functions

If $x_1(t)$ and $x_2(t)$ are joint stationary processes, then

$$R_{XX}(\tau) = R_{X_1X_1}(\tau) + R_{X_2X_2}(\tau) + R_{X_1X_2}(\tau) + R_{X_2X_1}(\tau)$$

$$R_{YY}(\tau) = R_{Y_1Y_1}(\tau) + R_{Y_2Y_2}(\tau) + R_{Y_1Y_2}(\tau) + R_{Y_2Y_1}(\tau)$$

Moreover, If $x_1(t)$ and $x_2(t)$ are uncorrelated and zero-mean, then

$$R_{XX}(\tau) = R_{X_1X_1}(\tau) + R_{X_2X_2}(\tau)$$

$$R_{YY}(\tau) = R_{Y_1Y_1}(\tau) + R_{Y_2Y_2}(\tau)$$

The output $y_1(t)$ and $y_2(t)$ are uncorrelated and zero-mean.

# 3.4 Input and Output Correlation Functions

The expected value of the instantaneous power in the process Y(t) is

$$E[Y^2(t)] = R_{YY}(0)$$

$$R_{YY}(0) = \int_{-\infty}^{\infty} f(u) R_{XX}(u)\,du = 2\int_{0}^{\infty} f(u) R_{XX}(u)\,du$$

- X(t) is White noise, $R_{XX}(\tau) = A\delta(\tau)$

$$R_{YY}(\tau) = A\int_{-\infty}^{\infty} f(u-\tau)\delta(u)\,du = Af(\tau)$$

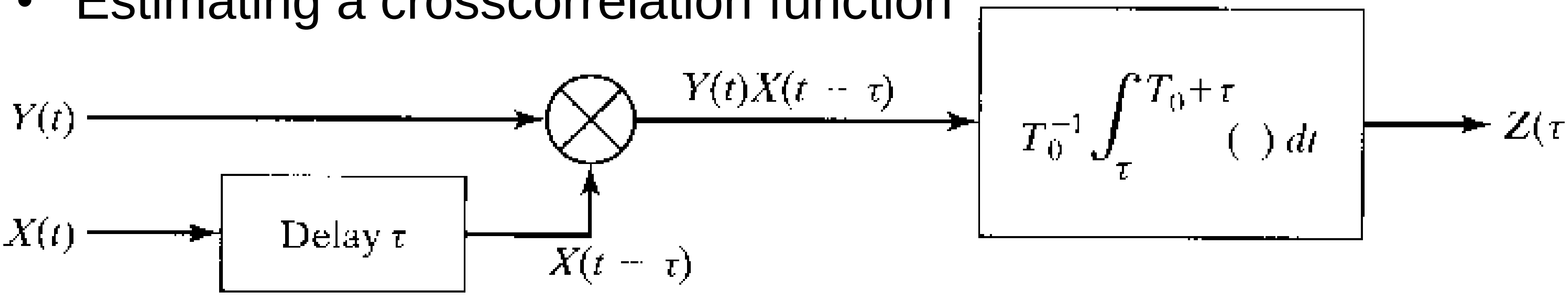
$$R_{YY}(0) = \int_{-\infty}^{\infty} Af(u)\delta(u)\,du = Af(0) = A\int_{-\infty}^{\infty} h^2(u)\,du$$

$$R_{YX}(\tau) = A\int_{-\infty}^{\infty} h(\tau-u)\delta(u)\,du = Ah(\tau)$$

Aapplication: System Identification

# 3.4 Input and Output Correlation Functions

- Estimating a crosscorrelation function




$$Z(\tau) = T_0^{-1}\int_{\tau}^{T_0+\tau} Y(t)X(t-\tau)\,dt = T_0^{-1}\int_0^{T_0} Y(t+\tau)X(t)\,dt$$

$$E\{Z(\tau)\} = T_0^{-1}\int_0^{T_0} E\{Y(t+\tau)X(t)\}\,dt = R_{Y,X}(\tau)$$

Accuracy of measurement: Mean-Square Error (MSE)

$$E\{[Z(\tau) - R_{Y,X}(\tau)]^2\}$$

Convergency limit of MSE whe$T_0 \to \infty$

# Chapter 3 Random processes in LTI system

- Homework

  3.4

  3.14

- Self-learning: Gaussian random processes

  1. What are the main conclusion about the Gaussian random processes?

  2. What are the main conclusion about the White Gaussian noise?

  3. Exercise 3-14 (Page 91)

  4. Exercise 3-15 (Page 94)

  5. Example 3-5 The Wiener-Levy process (Page 94)