Chapter 5 Poisson Processes

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Chapter 5: Poisson Processes

OUTLINE

5.1 Poisson Processes (2.1,2.2)

5.2 Generalization of the Poisson Processes (2.3, 2.4)

5.3 Filtered Poisson Processes (2.5)

5.4 Two-Dimensional and Marked Poisson Processes (2.6)

5.5 Poisson Arrival See Time Averages (2.7)

5.3 Filtered Poisson Processes

- Compound Poisson Processes: $X(t)=\sum_{n=1}^{N(t)} Y_n$

Sometimes {Yn } cannot be used directly but the function of Yn.

Definition of filtered Poisson Processes:

A stochastic process $\{X(t), t \leq 0\}$ is called a filtered Poisson process if

$$X(t)=\sum_{n=1}^{N(t)} w(t, S_n, Y_n)$$

where N(t) is a Poisson process with intensity ν and {Yn} are identical independent distribution random variables. The process N(t) and the sequence {Yn} are independent. {Sn} are arrival times. w() is the response function defined by the three arguments.

5.3 Filtered Poisson Processes

- If $w(t, S_n, Y_n) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ $\qquad X(t) = \sum_{n=1}^{N(t)} w(t, S_n, Y_n)$

  then X(t)=N(t), X(t) is a Poisson process.

- If $w(t, S_n, Y_n) = \begin{cases} Y_n & t \geq 0 \\ 0 & t < 0 \end{cases}$

  then $X(t) = \sum_{n=1}^{N(t)} Y_n$, X(t) is a compound Poisson process.

- A frequently used function assumes the following form:

$$w(t, \tau, y) = w(t - \tau, y)$$

5.3 Filtered Poisson Processes

Example 1: (example 2.2.3 in textbook)

A cable TV company collects $1/unit time from each subscriber. Subscribers sign up in accordance with a Poisson process with rate λ. What is the expected total revenue received in (0,t]?

Let *Si* denote the arrival time of the *i*th customer.

The revenue generated by this customer in (0,t] is *t-Si*.

The total revenue received in (0,t] is $X(t)=\sum_{i=1}^{N(t)}(t-S_i)$

Y(t) is a filtered Poisson process, if define

$$w(t,S_i,Y_i)=\begin{cases} t-S_i & t\ge S_i \\ 0 & t<S_i \end{cases}$$

5.3 Filtered Poisson Processes

Probability generating function of X(t):

Let {Ui} be i.i.d. random variables with uniform distribution on (0,t).

$$E[z^{X(t)} \mid N(t)=n] = E[z^{\sum_{i=1}^{N(t)} w(t,S_i,Y_i)} \mid N(t)=n]$$

$$= E[z^{\sum_{i=1}^{n} w(t,U_i,Y_i)}] \qquad U_i, Y_i \text{ are i.i.d. random variables respectly}$$

$$= (E[z^{w(t,U,Y)}])^n$$

$$E[z^{X(t)}] = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} (E[z^{w(t,U,Y)}])^n$$

$$= \exp(-\lambda t) \exp\{\lambda t E[z^{w(t,U,Y)}]\}$$

$$= \exp\{\lambda t \{E[z^{w(t,U,Y)}] - 1\}\}$$

5.3 Filtered Poisson Processes

Mean function of X(t):

$$E[X(t)] = \frac{dE[z^{X(t)}]}{dz}\Big|_{z=1} = \frac{d\{\lambda t\{E[z^{w(t,U,Y)}]-1\}\}}{dz}\Big|_{z=1}$$

$$= \lambda t \frac{dE[z^{w(t,U,Y)}]}{dz}\Big|_{z=1} = \lambda t E[\frac{dz^{w(t,U,Y)}}{dz}\Big|_{z=1}]$$

$$= \lambda t E[w(t,U,Y) z^{w(t,U,Y)-1}\Big|_{z=1}]$$

$$E[X(t)] = \lambda t E[w(t,U,Y)]$$

Variance function of X(t): $Var[X(t)] = \lambda t E[w^2(t,U,Y)]$

Mean of compound Poisson process: $E[X(t)] = \lambda t E[Y]$

Variance of compound Poisson process: $Var[X(t)] = \lambda t E[Y^2]$

Note: w(t,U,Y) includes two random variables: U, Y

5.3 Filtered Poisson Processes

Example 1: (example 2.2.3 in textbook)

A cable TV company collects $1/unit time from each subscriber. Subscribers sign up in accordance with a Poisson process with rate l. What is the expected total revenue received in (0,t]?

$$X(t) = \sum_{i=1}^{N(t)} (t - S_i)$$

X(t) is a filtered Poisson process, if define $w(t, S_i, Y_i) = \begin{cases} t - S_i & t \geq S_i \\ 0 & t < S_i \end{cases}$

Obtain: $E[X(t)] = \lambda t E[w(t, U, Y)]$

$$= \lambda t E[w(t, U)] = \lambda t \int_0^t (t-u) \frac{1}{t} du$$

$$= \lambda [tu |_0^t - \frac{1}{2} u^2 |_0^t] = \lambda [t^2 - \frac{1}{2} t^2]$$

$$E[X(t)] = \frac{1}{2} \lambda t^2$$

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5.4 Two-Dimensional and Marked Poisson Processes

1. Two-Dimensional Poisson Processes

S denotes a two-dimensional plane. Let A be a subset of plane S. Points is scattered randomly over S.

Let N(A) be the number of points in A.

N(A) is called a point process in S.

Stochastic process $\{N(A), A \subset S\}$ is a *two-dimensional Poisson process* if

i) N(A) follows a Poisson distribution with mean $\lambda |A|$.

ii) the numbers of points occurring in disjoint subsets of S are mutually independent.

where $|A|$ denote the size of the set A.

5.4 Two-Dimensional and Marked Poisson Processes

Two-dimensional nonhomogeneous Poisson processes:

Let $\lambda(x, y)$ be the intensity function of the Point process N(A). The process is a two-dimensional nonhomogeneous Poisson process if

i) N(A) follows a Poisson distribution with mean $\iint_A \lambda(x, y)dxdy$

ii) the numbers of points occurring in disjoint subsets of S are mutually independent.

n-dimensional Poisson processes

n-dimensional nonhomogeneous Poisson processes

Stars in space.

2. Marked Poisson processes

N(t) is a Poisson process with mean λ. {Sn} are the arrival times of N(t). Let {Yn} be i.i.d. random variables with a common distribution G(y) (or g(y)), where Yn is associated with the *n*th arrival of N(t). N(t) and {Yn} are independent.

The stochastic process (Sn, Yn) defined in the (t, y) plane is called *a marked Poisson process.*

$$\because \quad P\{Y \in (y, y+h)\} = g(y)h + o(h)$$

The process (Sn, Yn) (or N(A)) is a two-dimensional nonhomogeneous Poisson process with intensity function λ(t,y)=λg(y).

5.4 Two-Dimensional and Marked Poisson Processes

Mean of N(A):

$$E[N(A)] = \iint_A \lambda g(y)dydt$$

$$E[N(A)] = \iint_A \lambda dG(y)dt$$

Example:

G assumes the form of a multinomial distribution with

$$P\{Y = i\} = p_i, \quad i = 1,\ldots,n; \quad p_1 + p_2 + \cdots p_n = 1$$

Then the Poisson process N(A) is decomposed into n independent Poisson streams, each with density λp_i respectively.

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5.5 Poisson Arrival See Time Averages

- Important in queuing modeling
- In a service system, customers arrival process is Poisson.
- $X(s)$ represents the number of customers in the system at time $0<s<t$.
- So $X(t)$ is a stochastic process with a discrete state space $S=\{0,1,\dots\}$.
- B is a subset of S.

1. The ratio of the sojourn time $X(s)$ $(0<s<t)$ in B and the total time t.
2. The fraction of Poisson arrivals who find $X(s)(0<s<t)$ in B.

5.5 Poisson Arrival See Time Averages

- the *lack of anticipation assumption* (LAA) : for each $t \geq 0$, the arrival process $\{N(t+u)-N(t), u \geq 0\}$ is independent of $\{N(s), 0 \leq s \leq t\}$ and $\{X(s), 0 \leq s \leq t\}$

- Poisson processes satisfy LAA.

5.5 Poisson Arrival See Time Averages

1. The sojourn time X(s) (0<s<t) in B

- An indicator random variable U(t):

$$U(t) = \begin{cases} 1 & \text{if } X(t) \in B \\ 0 & \text{otherwise} \end{cases}$$

Assume that U(t) is left-continuous function.

$\int_0^t u(s)ds$ is the time the process X(s) is in the set B in [0,t].

- The fraction of time the process X(s) is in the set B in [0,t]:

$$V(t) = \frac{\int_0^t U(s)ds}{t}$$

5.5 Poisson Arrival See Time Averages

2. The fraction of Poisson arrivals who find $X(s)(0<s<t)$ in B.

   (who sees the system in state set B)

- The number of arrivals in (0,t] who find $X(s)(0<s<t)$ in B:

$$Y(t) = \int_0^t U(s)dN(s)$$

- The fraction of Poisson arrivals who find $X(s)(0<s<t)$ in B:

$$Z(t) = \frac{Y(t)}{N(t)}$$

- Under LAA, $P\{\lim_{n\to\infty} Z_n(t) = V(t)\} = 1$

The fraction of arrivals that sees the system in state set B is equal to the fraction of the time process is in that state set.

5.5 Poisson Arrival See Time Averages

$$E\{Y(t)\} = \lim_{n\to\infty} E\{Y_n(t)\} = \lambda t E\{V(t)\} = \lambda E\{\int_0^t U(s)ds\}$$

The expected number of arrivals who see the system in state set B in [0,t] is equal to the arrival rate multiplied by the length of time the system has been in B in [0,t].

End of Chapter 5