UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 01
Spring 2023

1. **Properties of Probability Measures**

   Use the axioms of probability to show the following facts. Clearly identify which axioms are used, and where. You may also use facts from lecture, but identify where you use them.

   a. (Subadditivity, also known as the "union bound") If $A_1, A_2, \dots \in \mathcal{F}$, then

   $$P\left(\cup_{i\geq 1} A_i\right) \leq \sum_{i\geq 1} P(A_i).$$

   b. (Continuity from below) If $A_1 \subset A_2 \subset \dots \in \mathcal{F}$, then

   $$P\left(\cup_{i\geq 1} A_i\right) = \lim_{i\to\infty} P(A_i).$$

   Here and throughout the course, the notation $A \subset B$ means that A is a subset of B (not necessarily a proper subset).
   (Hint: If $a_i \geq 0$ for each $i \geq 1$, then we have *monotone convergence*: $\sum_{i\geq 1} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$.)

Solution:

a. First, note that the set $A = \cup_{i\geq 1} A_i$ is an event, since it is a countable union of events (this is the σ-algebra property of $\mathcal{F}$). Now, decompose A into the disjoint union $A = \sqcup_{n\geq 1}(A_n \setminus B_{n-1})$, where $B_n := \cup_{i=1}^{n} A_i$ (with convention $B_0 = \emptyset$). Then, by σ-additivity and monotonicity (since $(A_n \setminus B_{n-1}) \subset A_n$):

$$P(A) = P\left(\sqcup_{n\geq 1}(A_n \setminus B_{n-1})\right) = \sum_{n\geq 1} P(A_n \setminus B_{n-1}) \leq \sum_{n\geq 1} P(A_n).$$

b. Using the same decomposition as in the previous part, we have $A = \sqcup_{i\geq 1}(A_i \setminus A_{i-1})$, and therefore by σ-additivity (twice), and non-negativity of probabilities with monotone convergence,

$$P(A) = \sum_{i\geq 1} P(A_i \setminus A_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i \setminus A_{i-1}) = \lim_{n\to\infty} P(A_n).$$

2. **Independence**

   Events $A, B \in \mathcal{F}$ are said to be **independent** if $P(A \cap B) = P(A)P(B)$.

   a. Show that if events A, B are independent, then the probability exactly one of the events occurs is
   $$P(A) + P(B) - 2P(A)P(B).$$

   b. Show that if event A is independent of itself, then $P(A) = 0$ or 1.

   Solution:

   a. The probability of the event that exactly one of A and B occur is
   $$\begin{aligned} &P(A \cap B^{\mathsf{c}}) + P(A^{\mathsf{c}} \cap B) \\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - 2P(A \cap B) \\ &= P(A) + P(B) - 2P(A)P(B). \end{aligned}$$

   b. $P(A \cap A) = P(A)P(A)$, so $P(A) = P(A)^2$. This implies that $P(A) \in \{0, 1\}$.

3. **Balls and Bins**

Suppose n bins are arranged from left to right. You sequentially throw the n balls, and each ball lands in a bin chosen uniformly at random, independent of all other balls.

a. Formulate an appropriate probability space for modeling the outcome of this experiment.

b. Let A_i denote the event that exactly i bins are empty, for $0 \le i \le n$. Compute the probability of the event:

$$\{\text{all empty bins sit to the left of all bins containing at least one ball}\}$$

in terms of the $P(A_i)$'s.

c. Practice your CS70 skills by computing $P(A_1)$.

Solution:

a. There is some flexibility in how we do this. For example, we can take the order of throws into account in our model, or just the final configuration. We'll do the former, since it makes the third part a bit easier due to the uniformity that P enjoys.

Hence, any outcome of our experiment is the sequence of bins where the balls land. I.e., we can take $\Omega = \{1, \dots, n\}^n$. In this manner, if $\omega = (\omega_1, \dots, \omega_n) \in \Omega$, then ω_i would denote the bin that the ith toss lands in.

Since everything is discrete, it is natural to choose $\mathcal{F} = 2^{\Omega}$, and since each ball is equally likely to land in any bin, we have P uniform over the sample space. I.e.,

$$P(\{\omega\}) = n^{-n} \quad \forall \omega \in \Omega.$$

b. Call our event of interest B. By the law of total probability, we have

$$P(B) = \sum_{i=0}^{n} P(B \cap A_i) = \sum_{i=0}^{n} \frac{1}{\binom{n}{i}} P(A_i).$$

The last identity follows because if we have exactly i empty bins, there are $\binom{n}{i}$ ways to choose which bins are empty, and only one of those ways will put all empty bins on the left. Since all such configurations of the empty bins are equally likely, the identity follows.

c. There are n choices for the empty bin, then $n-1$ choices for the bin that will have two balls, followed by $\binom{n}{2}$ choices of said two balls, and lastly $(n-2)!$ ways to throw the remaining $n-2$ balls into the $n-2$ remaining bins, so that each bin has exactly one ball. Therefore the desired probability is

$$\frac{n(n-1)(n-2)!\binom{n}{2}}{n^n} = \frac{n!\binom{n}{2}}{n^n}.$$