



Chapter 6 : Markov Processes and Discrete-Time Markov Chains

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Chapter 6: Markov Processes and Discrete-Time Markov Chains



OUTLINE

6.1 Markov Processes

6.2 Chapman-Kolmogorov Equation

6.3 Basic Concepts of Markov Chains(MC)

6.4 Classification of States

6.5 Ergodic MC and Stationary Distribution

6.1 Markov Processes



◆ Recall

Definition of the Markov process



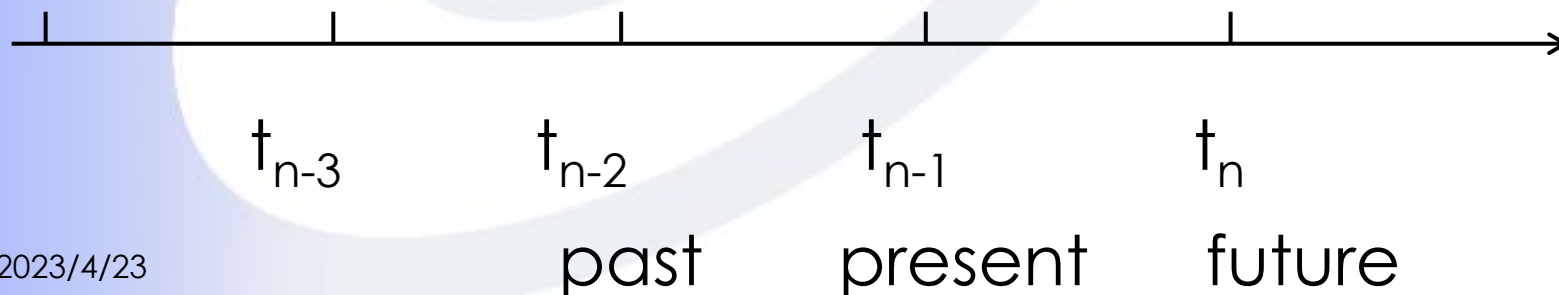


2.2.4 Markov Processes

- ◆ A Markov process possess the Markovian property.
- ◆ The Markovian property has the following implication:

If t_n is a time point in the **future**, t_{n-1} the **present time**, and correspondingly t_1, t_2, \dots, t_{n-2} , time points in the **past**,

the future development of a process does not depend on its evolvment in the past, but **only on its present state**.





2.2.4 Markov Process

Def.1 Markov Process

A stochastic process $X(t)$ is said to be a **Markov process** if it satisfies the following conditional probability:

$$\begin{aligned} Pr\{x(t_n) \leq x_n | x(t_1) = x_1, x(t_2) = x_2, \dots, x(t_{n-1}) = x_{n-1}\} \\ = Pr\{x(t_n) \leq x_n | x(t_{n-1}) = x_{n-1}\}, \quad \text{where } t_1 < t_2 < \dots < t_{n-1} < t_n \end{aligned}$$

(Markovian property)

$$\begin{aligned} Pr\{x(t_n) = x_n | x(t_1) = x_1, x(t_2) = x_2, \dots, x(t_{n-1}) = x_{n-1}\} \\ = Pr\{x(t_n) = x_n | x(t_{n-1}) = x_{n-1}\}, \quad \text{where } t_1 < t_2 < \dots < t_{n-1} < t_n \end{aligned}$$



Independent increment processes V.S. Markov processes

Every independent increment process is a Markov process, although there are many Markov processes that do not have independent increments.

- ◆ Wiener and Poisson process are Markov process.



6.1 Markov Processes

- ◆ Markov processes have enormous practical implications mainly for three reasons:
 1. Many practical phenomena can be modeled by Markov processes;
 2. The input necessary for their practical application is generally much more easily to provide than that for other classes of stochastic processes;
 3. Computer algorithms are available for numerical evaluations.



6.1 Markov Processes

- ◆ The state and time of a Markov process can be discrete and continuous.
- ◆ Classification of Markov processes
 - A Markov process with a discrete state and discrete time is called a **Markov chain**.
 - A Markov process with a discrete state and continuous time is called a **continuous-time Markov chain**.
 - A Markov process with a continuous state is called a **diffusion process**.



6.2 Markov Processes

Markovian Property writing in term of the probability density function:

$$f\{x(t_n)|x(t_1), x(t_2), \dots, x(t_{n-1})\} = f\{x(t_n)|x(t_{n-1})\}$$

Joint probability density function

$$\begin{aligned} &f\{x(t_1), x(t_2), \dots, x(t_n)\} \\ &= f\{x(t_n)|x(t_1), x(t_2), \dots, x(t_{n-1})\} \\ &\quad \times f\{x(t_1), x(t_2), \dots, x(t_{n-1})\} \\ &= f\{x(t_n)|x(t_{n-1})\} \cdot f\{x(t_1), x(t_2), \dots, x(t_{n-1})\} \end{aligned}$$



6.2 Markov Processes

◆ Similarly,

$$\begin{aligned} f\{x(t_1), x(t_2), \dots, x(t_{n-1})\} \\ = f\{x(t_{n-1}) | x(t_{n-2})\} \cdot f\{x(t_1), x(t_2), \dots, x(t_{n-2})\} \end{aligned}$$

$$f\{x(t_1), x(t_2), \dots, x(t_n)\} = f\{x(t_1)\} \prod_{r=2}^n f\{x(t_r) | x(t_{r-1})\}$$

$$P\{x(t_1), x(t_2), \dots, x(t_n)\} = P\{x(t_1)\} \prod_{r=2}^n P\{x(t_r) | x(t_{r-1})\}$$

The statistical characteristic of Markov process decided by the initial condition and conditional probability density function or conditional probability .



6.2 Markov Processes

Transition probability density:

$$f\{x(t_r) | x(t_{r-1})\}$$

Transition probability:

$$P\{x(t_r) | x(t_{r-1})\}$$

◆ Homogeneity:

The transition probability density is invariant with time τ :

$$f\{x(t_r + \tau) | x(t_{r-1} + \tau)\} = f\{x(t_r) | x(t_{r-1})\}$$

or

$$P\{x(t_r + \tau) | x(t_{r-1} + \tau)\} = P\{x(t_r) | x(t_{r-1})\}$$

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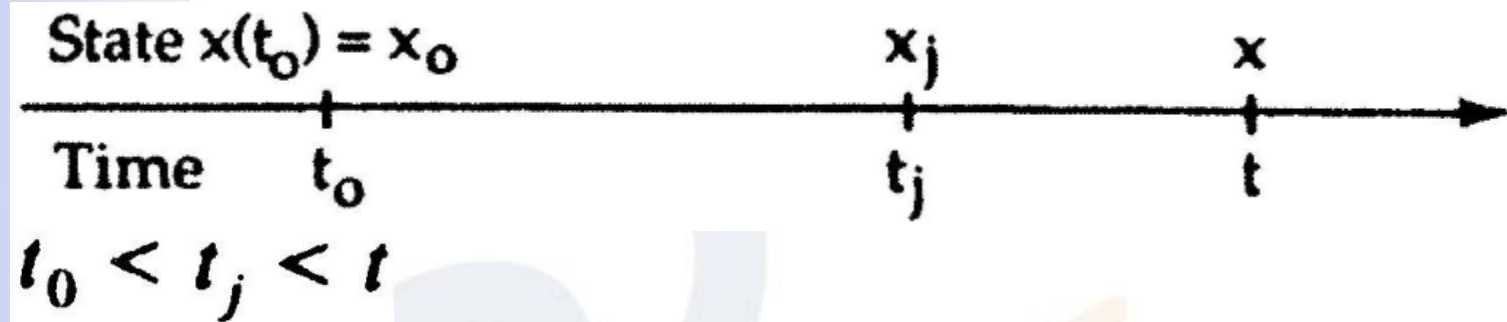
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6.5 Ergodic MC and Stationary Distribution



6.2 Chapman-Kolmogorov Equation

- ◆ $\{X(t), t \geq 0\}$ is a Markov process with density function f .
- ◆ In terms of transition probability density function



$$\begin{aligned} f\{x(t), x_0(t_0)\} &= \int f\{x(t), x_0(t_0), x_j(t_j)\} dx_j \\ &= \int f\{x(t) | x_0(t_0), x_j(t_j)\} f\{x_0(t_0), x_j(t_j)\} dx_j \\ f\{x(t) | x_0(t_0), x_j(t_j)\} &= f\{x(t) | x_j(t_j)\} \end{aligned}$$



6.2 Chapman-Kolmogorov Equation

- ◆ Chapman-Kolmogorov Equation
- ◆ In terms of transition probability density function

$$f\{x(t), x_0(t_0)\} = \int f\{x(t)|x_j(t_j)\} f\{x_0(t_0), x_j(t_j)\} dx_j$$

$$f\{x(t)|x_0(t_0)\} = \int f\{x(t)|x_j(t_j)\} f\{x_j(t_j)|x_0(t_0)\} dx_j$$



6.2 Chapman-Kolmogorov Equation

- ◆ Chapman-Kolmogorov Equation
- ◆ In terms of transition probability

$$p\{x(t)|x_0(t_0)\} = \sum_j p\{x(t)|x_j(t_j)\} p\{x_j(t_j)|x_0(t_0)\}$$

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