Chap 4 Spectral Analysis of Stochastic Processes

Dong Yan
EI. HUST.

- References:

1. Stochastic Processes and Their Applications, Frank E. Beichelt, O211. W138
   Chap 8, Spectral analysis of stationary processes
2. Probability, Random Variables and Random Signal Principles, 2^{nd} Edition, Peyton Z. Peebles, O211 W58/2
   Chap 7, Spectral characteristic of random processes

Chapter 4: Spectral Analysis

Content:

4.1 Spectral Density Functions

4.2 Spectral Analysis of Linear Systems

4.3 Spectrum of Amplitude-modulated Signals

4.4 Narrow-band Gaussian Processes

4.1 Spectral Density Functions

Content:

Average of the power

- If X(t) is ergodic, then

$$R_{XX}(0)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x^2(t)dt$$

- R(0) represents the time average of the power of a random process X(t).
- *Def.1* average of the power

$$\overline{P}_X=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x^2(t)dt$$

- Parseval theorem

$$\boxed{\int_{-\infty}^{\infty}x^2(t)dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}X^2(\omega)d\omega}$$

- The representation of the average energy in terms of time and frequency of a signal.
- Thus,

$$\overline{P}_X=\lim_{T\to\infty}\frac{1}{4\pi T}\int_{-T}^{T}X^2(\omega)d\omega$$

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1.1 Autospectral Density Functions

Def.2 Spectral Density Function (S.D.F.)

- The S.D.F. of a random process X(t) is

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} |X_T(\omega)|^2 \qquad S_{XX}(f) = \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2$$

- (autospectral density function) (Power Spectrum)
- From Def.1 and Def. 2,

$$\begin{aligned}\overline{P}_X &= \lim_{T\to\infty} \frac{1}{4\pi T} \int_{-T}^{T} X^2(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = \frac{1}{\pi} \int_{0}^{\infty} S_{XX}(\omega) d\omega \\ &= \int_{-\infty}^{\infty} S_{XX}(f) df = 2\int_{0}^{\infty} S_{XX}(f) df\end{aligned}$$

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

Definitions associated with the S.D.F.

1. The *k*th moment of the S.D.F.

$$m_k = \int_0^\infty \omega^k S_{XX}(\omega)d\omega$$

2. Modal frequency ω_m

   the frequency where the S.D.F. peaks.

3. Mean frequency

$$\overline{\omega}_X = \frac{\int_0^\infty \omega S_{XX}(\omega)d\omega}{\int_0^\infty S_{XX}(\omega)d\omega} = \frac{m_1}{m_0}$$

   the mean frequency is not equal to the average value of the various frequencies involved in a random process; it is the mathematical mean of the S.D.F.

4. The *k*th moment about the mean of the S.D.F.

$$\mu_k = \int_0^{\infty} (\omega - \overline{\omega})^k S_{XX}(\omega) d\omega$$

Thus,

$$\mu_0 = m_0 = \int_0^{\infty} S_{XX}(\omega) d\omega = \pi \overline{P}_X$$

$$\mu_1 = 0$$

$$\mu_2 = m_2 - \frac{m_1^2}{m_0}$$

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

Definitions associated with the S.D.F.

5. The bandwidth parameter of the S.D.F.

$$\varepsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}}$$

$\varepsilon = 0$, the random process has a narrow-band spectrum;

$\varepsilon = 1$, the random process has a wide-band spectrum;

Definitions associated with the S.D.F.

6. The spectral width parameter of the S.D.F.

$$\nu = \sqrt{\frac{\mu_2}{\mu_0}\frac{1}{\overline{\omega}}}$$

7. The spectral peakedness parameter

$$Q_P = \frac{\int_0^\infty \omega S^2{}_{XX}(\omega)d\omega}{(\int_0^\infty S_{XX}(\omega)d\omega)^2}$$

The value increases with increasing sharpness of the S.D.F.

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1.1 Autospectral Density Functions

- Properties of the S.D.F.

i) Nonnegative $S_{XX}(\omega) = \lim_{T \to \infty} \frac{1}{2T} |X_T(\omega)|^2 \geq 0$

ii) Real

iii) Even, if X(t) is real, $S_{XX}(\omega) = S_{XX}(-\omega)$

iv) If $\int_{-\infty}^{\infty} |R_{XX}(\tau)| d\tau < \infty$, then $S_{XX}(\omega)$ is a continuous function of ω

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1 Spectral Density Functions

Content:

4.1.2 Wiener-Khintchine Theorem

- The Fourier transform of the correlation function of an ergodic process:

$$\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau = \lim_{T\to\infty}\frac{1}{2T}|X_T(\omega)|^2$$

- And the definition of S.D.F.,

$$S_{XX}(\omega) = \lim_{T\to\infty}\frac{1}{2T}|X_T(\omega)|^2$$

- Obtain,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega$$

4.1.2 Wiener-Khintchine Theorem

Theorem: Wiener-Khintchine Theorem

For a weakly stationary random process X(t), its correlation function and the power spectrum are Fourier transform pair.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$

Standard Fourier Transform

$$F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwt} dw$$

4.1.2 Wiener-Khintchine Theorem

- Both correlation function and power spectrum are real and even functions.

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau$$

$$= 2\int_{0}^{\infty} R_{xx}(\tau) \cos\omega t \, d\tau$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} S_{xx}(\omega) \cos\omega\tau \, d\omega$$

4.1.2 Wiener-Khintchine Theorem

- The average power and correlation function

$$\frac{1}{\pi}\int_0^{\infty} S_{XX}(\omega)d\omega = \overline{P}_X = R_{XX}(0) = E[X^2(t)]$$

- If E[X(t)]=0,

$$Var[X(t)] = \frac{1}{\pi}\int_0^{\infty} S_{XX}(\omega)d\omega = \overline{P}_X = R_{XX}(0)$$

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

Example: White Noise

The process with autocorrelation function,

$$R(\tau) = A\delta(\tau)$$

and power spectrum,

$$S(\omega) = A = \text{constant}$$

The continuous white noise is a real, stationary, continuous-time stochastic process with constant spectral density.

the “most random” stochastic process

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

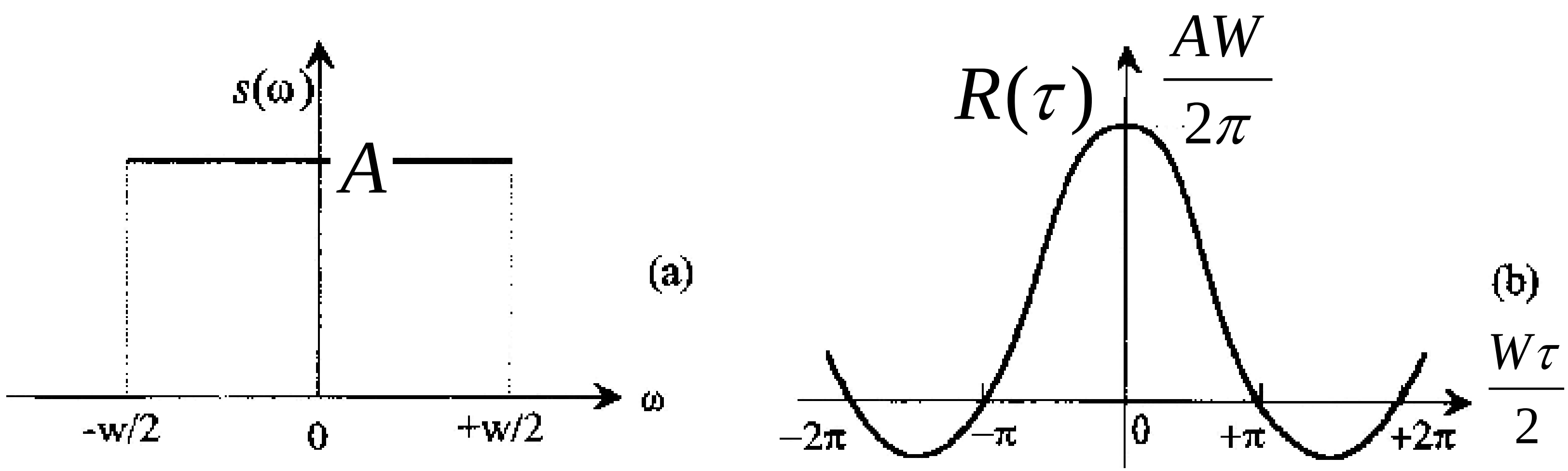
4.1.2 Wiener-Khintchine Theorem

Example 2. Band-limited White Noise

$$S(\omega) = A, \text{ where } -\frac{W}{2} < \omega < \frac{W}{2}$$

Thus,

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-W/2}^{W/2} Ae^{i\omega\tau}\,d\omega = \frac{A}{\pi\tau}\sin\frac{1}{2}W\tau$$



Review 2.2.1 Stationary Processes

- weakly stationary

Example 1. Random phase processes

X(t)=Acos(wt+ε)， t>0, whereas A and w are constants and ε is random variable uniformly distributed between –π and π.

$$E[x(t)] = 0$$

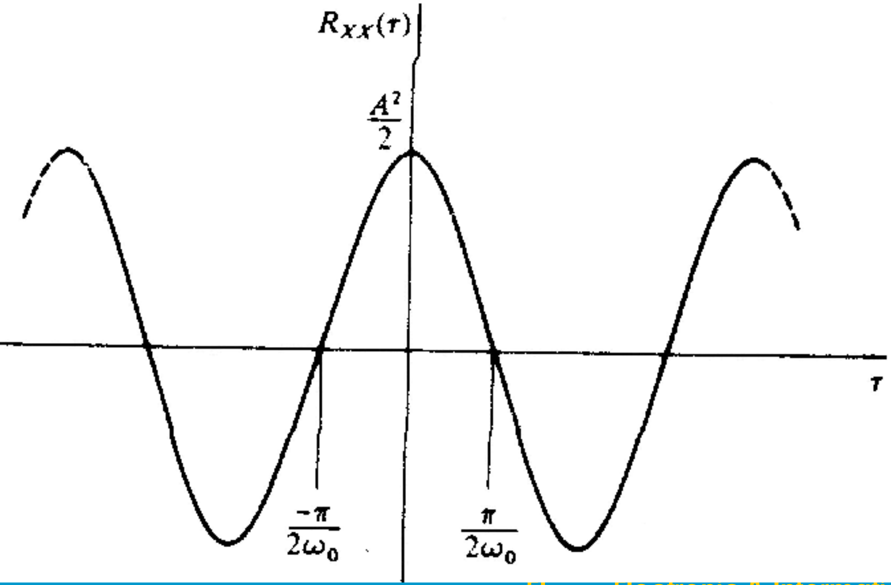
$$C_{XX}(t_1, t_2) = \frac{A^2}{2}\cos w_0(t_2 - t_1)$$

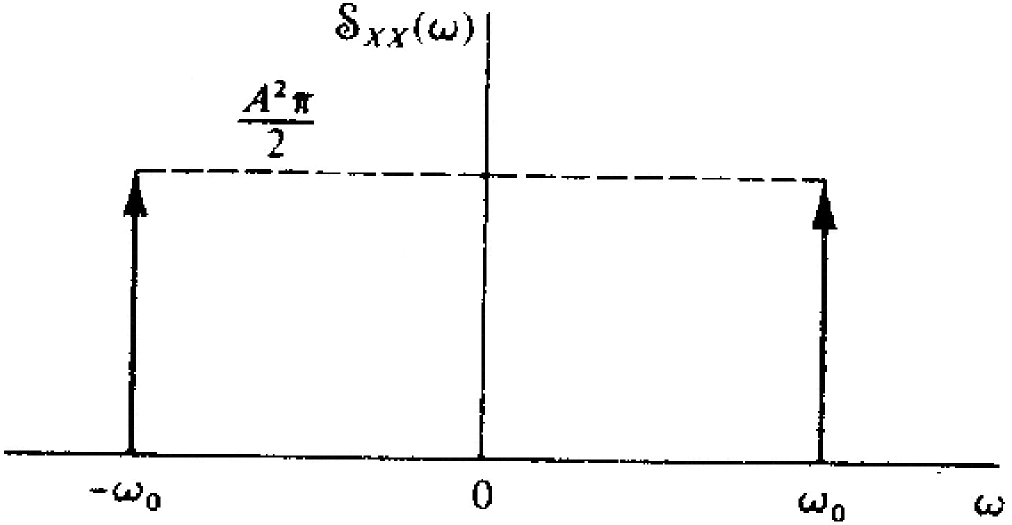
$$= \frac{A^2}{2}\cos w_0 \tau = C_{XX}(\tau)$$

Example 3. Random Phase Processes

$$R_{XX}(\tau) = \frac{A^2}{2}\cos w_0 \tau$$

Obtain: $S(\omega) = \dfrac{A^2 \pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

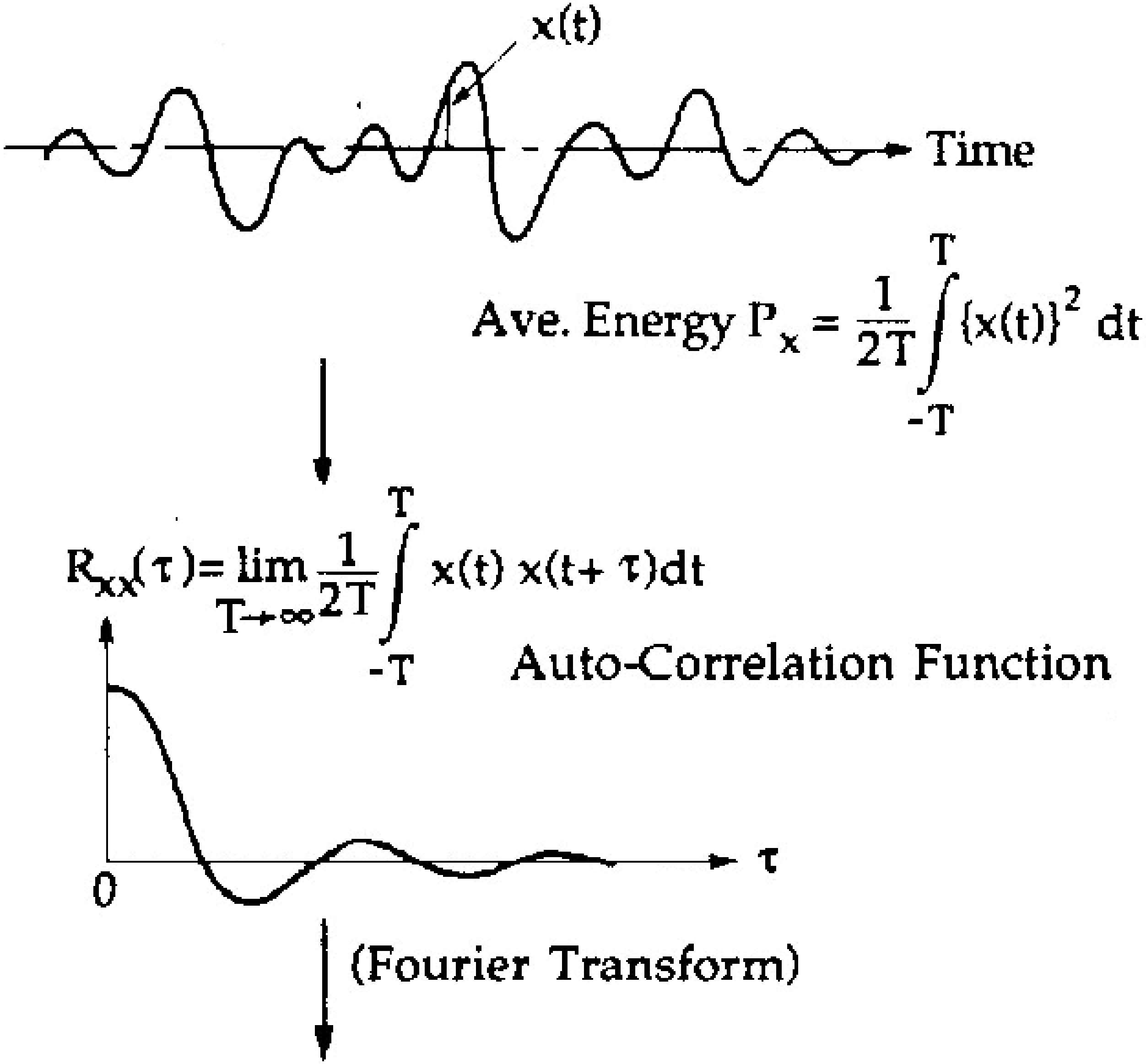




EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1.2 Wiener-Khintchine Theorem

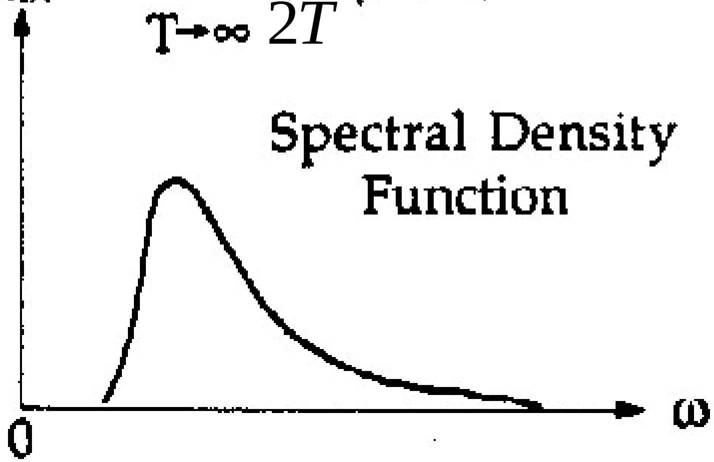
- Summary

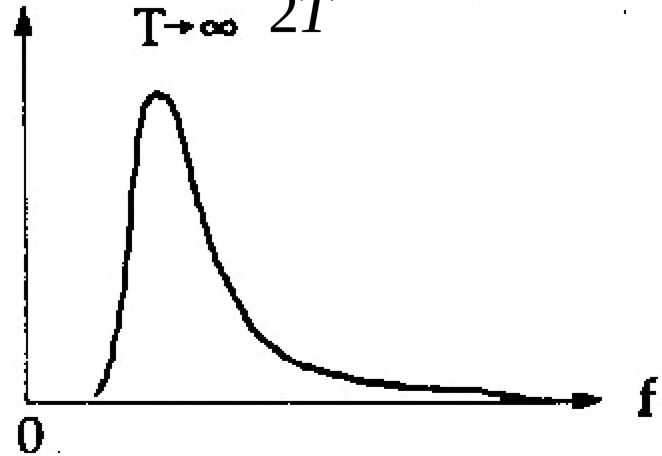


4.1.2 Wiener-Khintchine Theorem

- Summary

$$S_{xx}(\omega)=\lim_{T\to\infty}\frac{1}{2T}|X(\omega)|^2 \qquad S_{xx}(f)=\lim_{T\to\infty}\frac{1}{2T}|X(f)|^2$$





$$\frac{1}{\pi}\int_0^\infty S_{xx}(\omega)\,d\omega = 2\int_0^\infty S_{xx}(f)\,df = R_{xx}(0) = Var[x(t)] = \bar{P}_x$$

4.1.2 Wiener-Khintchine Theorem

- Summary

$R_X(\tau)$	$S_X(\omega)$
$\begin{cases} (T - \lvert\tau\rvert)/T, & \lvert\tau\rvert < T, \\ 0, & \text{otherwise} \end{cases}$	$T\{\sin(\omega T/2)/(\omega T/2)\}^2 = T\,\mathrm{sinc}^2(fT)$
1	$2\pi\delta(\omega)$
$\delta(\tau)$	1
$\exp(-\alpha\lvert\tau\rvert)$	$2\alpha/(\alpha^2 + \omega^2)$
$\cos(\omega_0\tau)$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\exp(-\alpha\lvert\tau\rvert)\cos(\omega_0\tau)$	$\alpha/[\alpha^2 + (\omega - \omega_0)^2]\} + \{\alpha/[\alpha^2 + (\omega + \omega_0)^2]\}$
$2W\,\mathrm{sinc}(2W\tau) = \sin(2\pi W\tau)/\pi\tau$	$\begin{cases} 1, & \lvert\omega\rvert \le 2\pi W \\ 0, & \text{otherwise} \end{cases}$

$$\mathrm{sinc}(u) = \sin(\pi u)/\pi u, -\infty < u < \infty$$

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1.2 Wiener-Khintchine Theorem

e.g. Suppose the continuous-time random process X(t) has autocorrelation function

$$R_X(\tau) = 1 + e^{-\alpha|\tau|}, \quad -\infty < \tau < \infty, \qquad \alpha > 0$$

Find the spectral density function for X(t).

$$S_X(\omega) = 2\pi\delta(\omega) + \frac{2\alpha}{\alpha^2 + \omega^2}$$

4.1 Spectral Density Functions

Content:

4.1.1 Autospectral Density Functions

4.1.2 Wiener-Khintchine Theorem

4.1.3 Crossspectral Density Functions

4.1.4 S.D.F. of Derived Random Processes

4.1.3 Cross-Spectral Density Functions

1. Definition of Cross-Spectral Density Function (CS.D.F)

Cross-Spectral Density Function of two random processes X(t) and Y(t) is

$$S_{XY}(\omega) = \lim_{T\to\infty}\frac{1}{2T}X^*(\omega)Y(\omega)$$

$$S_{YX}(\omega) = \lim_{T\to\infty}\frac{1}{2T}X(\omega)Y^*(\omega)$$

4.1.3 Cross-Spectral Density Functions

Theorem: Wiener-Khintchine Theorem

For two jointly stationary random processes X(t) and Y(t), their cross-correlation function and cross-power density function are Fourier transform pair.

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau \qquad S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-i\omega\tau} d\tau$$

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\omega\tau} d\omega \qquad R_{YX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega) e^{i\omega\tau} d\omega$$

Standard Fourier Transform

$$F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwt} dt$$

4.1.3 Cross-Spectral Density Functions

2. Real part and imaginary part

$$S_{XY}(\omega)=\int_{-\infty}^{\infty}R_{XY}(\tau)e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty}R_{XY}(\tau)\cos\omega\tau d\tau-i\int_{-\infty}^{\infty}R_{XY}(\tau)\sin\omega\tau d\tau$$

$$=C_{XY}(\omega)+iQ_{XY}(\omega)$$

$$C_{XY}(\omega)=\int_{-\infty}^{\infty}R_{XY}(\tau)\cos\omega\tau d\tau$$

$$=\int_{-\infty}^{0}R_{XY}(\tau)\cos\omega\tau d\tau+\int_{0}^{\infty}R_{XY}(\tau)\cos\omega\tau d\tau$$

$$=\int_{0}^{\infty}R_{XY}(-\tau)\cos\omega\tau d\tau+\int_{0}^{\infty}R_{XY}(\tau)\cos\omega\tau d\tau$$

$$=\int_{0}^{\infty}[R_{YX}(\tau)+R_{XY}(\tau)]\cos\omega\tau d\tau$$

4.1.3 Cross-Spectral Density Functions

Real part and imaginary part

$$Q_{YX}(\omega) = -\int_{-\infty}^{\infty} R_{XY}(\tau)\sin\omega\tau d\tau$$

$$= \int_{0}^{\infty} [-R_{XY}(\tau) + R_{YX}(\tau)]\sin\omega\tau d\tau$$

Amplitude Spectrum:

$$|S_{XY}(\omega)| = \sqrt{C_{XY}^2(\omega) + Q_{XY}^2(\omega)}$$

Phase Spectrum:

$$\varepsilon_{XY}(\omega) = \tan^{-1}\frac{Q_{XY}(\omega)}{C_{XY}(\omega)}$$

4.1.3 Cross-Spectral Density Functions

3. Properties of Cross-Spectral Density Function

i) $C_{XY}(\omega)$ and $C_{YX}(\omega)$ are even functions

ii) $Q_{XY}(\omega)$ and $Q_{YX}(\omega)$ are odd functions

iii) $S_{XY}(\omega) = S_{YX}(-\omega) = S^*_{YX}(\omega)$

iv) If X(t) and Y(t) are jointly stationary random processes and uncorrelated,

$$S_{XY}(\omega) = S_{YX}(\omega) = 2\pi m_X m_Y \delta(\omega)$$

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

4.1.3 Cross-Spectral Density Functions

4. Coherency function

$$\gamma_{XY}(\omega) = \frac{|S_{XY}(\omega)|^2}{S_{XX}(\omega)S_{YY}(\omega)} = \frac{C_{XY}^2(\omega) + Q_{XY}^2(\omega)}{S_{XX}(\omega)S_{YY}(\omega)}$$

Mutual correlation coefficients

$$\rho_{XY}[t_1, t_2] = \frac{C_{XY}(t_1, t_2)}{\sqrt{Var[X(t_1)]Var[Y(t_2)]}}$$

$$0 \le \gamma_{XY}(\omega) \le 1$$

$\gamma_{XY}(\omega) \equiv 1$ if and only if X(t) and Y(t) are exactly linearly related.

4.1.3 Cross-Spectral Density Functions

Example 1.

Given: a cross-spectral density function

$$S_{XY}(\omega)=\begin{cases} a+i\dfrac{b\omega}{\omega_0} & -\omega_0<\omega<\omega_0 \\ 0 & otherwise \end{cases}$$

where $\omega_0>0$, a and b are real constants.

Obtain: the cross-correlation function.

4.1 Spectral Density Functions

Content:

4.1.1 Autospectral Density Functions

4.1.2 Wiener-Khintchine Theorem

4.1.3 Crossspectral Density Functions

4.1.4 S.D.F. of Derived Random Processes

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

1. Autocorrelation function of derived random processes

$$R_{x\dot{x}}(t_1, t_2) = \frac{\partial}{\partial t_2} R_{xx}(t_1, t_2) = -\frac{d}{d\tau} R_{xx}(\tau)$$

$$R_{\dot{x}\dot{x}}(\tau) = \frac{\partial^2}{\partial t_1 \, \partial t_2} R_{xx}(t_1, t_2) = -\frac{d^2}{d\tau^2} R_{xx}(\tau)$$

$$R_{\ddot{x}\ddot{x}}(\tau) = \frac{d^4}{d\tau^4} R_{xx}(\tau)$$

4.1.4 S.D.F. of Derived Random Processes

2. Spectral density function of derived random processes

$$\mathscr{F}\{x(t)\} = X(\omega)$$

$$\mathscr{F}\{\dot{x}(t)\} = \dot{X}(\omega) = i\omega X(\omega)$$

$$\mathscr{F}\{\ddot{x}(t)\} = \ddot{X}(\omega) = -\omega^2 X(\omega)$$

$$S_{\dot{x}\dot{x}}(\omega) = \lim_{T\to\infty} \frac{1}{2T} |\dot{X}(\omega)|^2$$

$$= \omega^2 \lim_{T\to\infty} \frac{1}{2T} |X(\omega)|^2 = \omega^2 S_{xx}(\omega)$$

4.1.4 S.D.F. of Derived Random Processes

2. Spectral density function of derived random processes

$$S_{\ddot{x}\ddot{x}}(\omega) = \lim_{T\to\infty} \frac{1}{2T} |\ddot{X}(\omega)|^2$$

$$= \omega^4 \lim_{T\to\infty} \frac{1}{2T} |X(\omega)|^2 = \omega^4 S_{xx}(\omega)$$

EI Electronic & Information Engineering Department
HUST HuaZhong University of Science & Technology

Homework

4.1

4.2

4.7