《随机过程》试卷

一、$X(t)=a\cos(\omega t+\theta)$, a,ω is constant and $\theta \sim U[0,2\pi]$

So we have $m_X(t)=E[X(t)]=\int_0^{2\pi} a\cos(\omega t+\theta)\frac{1}{2\pi}d\theta=\frac{a}{2\pi}\sin(\omega t+\theta)\Big|_0^{2\pi}=0$

$R_X(t,t+\tau)=E[X(t)X(t+\tau)]=E[a\cos(\omega t+\theta)a\cos(\omega(t+\tau)+\theta)]$

$=E[\frac{a^2}{2}[\cos(\omega(2t+\tau)+2\theta)+\cos\omega\tau]]$ where $E[\cos(\omega(2t+\tau)+2\theta)]=0$

So we will have $R_X(t,t+\tau)=\frac{a^2}{2}\cos\omega\tau$

$D_X(t)=R_X(0)=\frac{a^2}{2}$

二. $X(t)=A\cos\omega t+B\sin\omega t$ while A, B are independent and $A,B \overset{iid}{\sim} N(0,\sigma^2)$

(1) $m_X(t)=E(A\cos\omega t+B\sin\omega t)=E(A)\cos\omega t+E(B)\sin\omega t=0$

$D_X(t)=var(A\cos\omega t+B\sin\omega t)=\sigma^2(\cos^2\omega t+\sin^2\omega t)=\sigma^2$

$R_X(t,t+\tau)=E[(A\cos\omega t+B\sin\omega t)(A\cos\omega(t+\tau)+B\sin\omega(t+\tau))]$

$=E(A^2)E[\cos\omega t\cos\omega(t+\tau)]+E[B^2]E[\sin\omega t\sin\omega(t+\tau)]$

$=\sigma^2[\frac{1}{2}(\cos\omega(2t+\tau)+\cos\omega\tau)-\frac{1}{2}[\cos(\omega(2t+\tau))-\frac{1}{2}\cos\omega\tau]$

$=\sigma^2\cos\omega\tau$

(2) for $m_X(t)=0$. $D_X(t)=\sigma^2$ so we have

$f_{X,1}(x,t)=\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{1}{2}(\frac{x}{\sigma})^2)$

(3) $B_X(\tau)=R_X(\tau)-m_X(t)m_X(t+\tau)=\sigma^2\cos\omega\tau$

so we have the covariance matrix is

$$B=\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}=\begin{bmatrix} \sigma^2 & \sigma^2\cos\omega & \sigma^2\cos 2\omega \\ \sigma^2\cos\omega & \sigma^2 & \sigma^2\cos\omega \\ \sigma^2\cos 2\omega & \sigma^2\cos\omega & \sigma^2 \end{bmatrix}$$

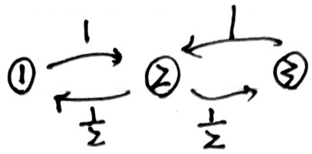
三. (1) $m_X(t)=\lambda t$. $D_X(t)=\lambda t$. $R_X(t,t+\tau)=\lambda t(1+\lambda\tau)$

(2) the first even arrival can be seen as the first time interval

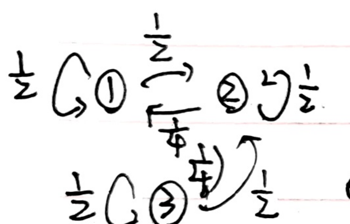
So it follows the expotiental distribution. So we can have

$\Rightarrow P\{W_X^{(1)} < W_Y^{(1)}\} = \int_0^\infty (1-\exp\{-\lambda_1 k\})\lambda_2 \exp\{-\lambda_2 k\}\, dk$

simplifying it, and we have $P\{W_X^{(1)} < W_Y^{(1)}\} = 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}$

四. (1) We can have the plot  we can find the closed set $\{1,2,3\}$ is irreducible. and for i we have $d(i)=2\ \forall i \in \{1,2,3\}$ so the mc is periodic, which means the markov chain is not ergodic

(2) the same method, we have that $\{1,2,3\}$ is irreducible

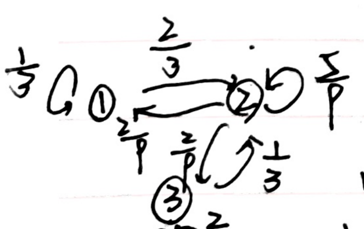


and $d(i)=1$ for $\forall i \in \{1,2,3\}$, the states are finite. So the markov chain is ergodic, the limit distribution exists. we let $\vec{\pi} = (\pi_0, \pi_1, \pi_2)$ denote, and we have

$$\vec{\pi} = \vec{\pi}\cdot\vec{P} \Rightarrow \begin{cases} \pi_0 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1 \\ \pi_1 = \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 \\ \pi_2 = \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 \end{cases}$$

and $\pi_1 + \pi_2 + \pi_3 = 1$ finally, we have $\pi_0 = \frac{1}{4}$ $\pi_1 = \frac{1}{2}$ $\pi_2 = \frac{1}{4}$

五. (1) From the one-step transform probability matrix, we can have the plot.



for the set $\{1,2,3\}$ is irreducible and for state i $d(i)=1$. So we have the mc is ergonic.

(2) the initial distribution is $\vec{P}(0) = (1,0,0)$

then one-step, we have $\vec{P}(1) = \vec{P}(0)\cdot\vec{P} = (\frac{1}{3}, \frac{2}{3}, 0)$

then the distribution of $X(n=1) = \{\frac{1}{3}, \frac{2}{3}, 0\}$

(3) Since the chain is ergonic, so the limit distribution is the same as stationary distribution

$$\vec{\pi} = \vec{\pi}\cdot\vec{P} \Rightarrow \begin{cases} \pi_0 = \frac{1}{3}\pi_0 + \frac{2}{9}\pi_1 \\ \pi_1 = \frac{2}{3}\pi_0 + \frac{5}{9}\pi_1 + \frac{1}{3}\pi_2 \\ \pi_2 = \frac{2}{9}\pi_1 + \frac{2}{3}\pi_2 \end{cases}$$

Solve it we have $\pi_0 = \frac{1}{6}$ $\pi_1 = \frac{1}{2}$ $\pi_2 = \frac{1}{3}$

$\pi_0 + \pi_1 + \pi_2 = 1$

$\cdots \sin(2\pi\theta t)] = \int_0^1 \sin(2\pi\theta t)\,d\theta = \frac{-1}{2\pi t}\cos(2\pi\theta t)\Big|_0^1 = \frac{1-\cos 2\pi t}{2\pi t}$

only when $t=k$ $k\in Z$, $E[X(t)]=0$ so $E[X(t)]$ has the period $T=1$

$R_X(t,t+\tau) = E[X(t)X(t+\tau)] = E[\sin(2\pi\theta t)\sin(2\pi\theta(t+\tau))]$

$= E[-\frac{1}{2}(\cos(2\pi\theta(2t+\tau)) - \frac{1}{2}\cos(2\pi\theta\tau)]$

also $R_X(t,t+\tau)$ depends on the time t

Finally the $X(t) = \sin(2\pi\theta t)$ is a non-stationary process.

t. (1) $X(t)$ $\forall t$ $P(X(t)=1) = P(X(t)=-1) = \frac{1}{2}$ so we have $m_X(t) = 1\cdot\frac{1}{2} - 1\cdot\frac{1}{2} = 0$

$R_X(t,t+\tau) = E[X(t)X(t+\tau)]$ so the combinations of $X(t)$ is 4 cases.

① $X(t)=1, X(t+\tau)=-1$: $P_1 = P\{X(t)=1, X(t+\tau)=-1\} = P\{X(t+\tau)=-1 \mid X(t)=1\}P(X(t)=1)$

where $P\{X(t+\tau)=-1 \mid X(t)=1\} = \sum_{k=0}^{\infty} P\{\tau, 2k+1\} = \sum_{k=0}^{\infty} \exp\{-\lambda\tau\}\frac{1}{(2k+1)!}(\lambda\tau)^{2k+1}$

and $\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}(\lambda\tau)^{2k+1} = \frac{1}{2}(e^x - e^{-x})\Big|_{x=\tau\lambda} = \frac{1}{2}(\exp(\lambda\tau) - \exp(-\lambda\tau))$

$\Rightarrow P\{X(t+\tau)=-1 \mid X(t)=1\} = \frac{1}{2}(1-\exp(-2\lambda\tau))$

② $X(t)=1, X(t+\tau)=1$, using the same way, we have $P\{X(t+\tau)=1 \mid X(t)=1\}$

$= \exp(-\lambda\tau)\sum_{k=0}^{\infty}\frac{1}{(2k)!}(\lambda\tau)^{2k} = \exp(-\lambda\tau)\cdot\frac{1}{2}(\exp(\lambda\tau)+\exp(-\lambda\tau)) = \frac{1}{2}(1+\exp(-2\lambda\tau))$

so we can have $E[X(t)X(t+\tau)] = (-1)\times\frac{1}{4}(1-\exp(-2\lambda\tau)) + \frac{1}{4}(1+\exp(-2\lambda\tau))$

$+ (-1)\cdot\frac{1}{4}(1-\exp(-2\lambda\tau)) + \frac{1}{4}(1+\exp(-2\lambda\tau)) = \exp(-2\lambda\tau)$ $(\tau>0)$

Finally, we have $R_X(t,t+\tau) = \exp(-2\lambda\tau)$ is a stationary process

(2) $S_X(\omega) = \mathcal{F}\{R_X(\tau)\} = \mathcal{F}\{\exp(-2\lambda\tau)\} = \int_0^{+\infty}\exp(-2\lambda\tau)\exp(-j\omega\tau)\,d\tau$ $(\tau>0)$

$= \frac{1}{2\lambda+j\omega}$

which is $S_X(\omega) = \frac{1}{2\lambda+j\omega}$

1. $S_X(\omega) = \frac{2}{\omega^4+5\omega^2+4} = \frac{2}{(\omega^2+1)(\omega^2+4)} = (\frac{1}{\omega^2+1} - \frac{1}{\omega^2+4})\frac{2}{3}$

$\Rightarrow S_X(\omega) = \frac{2}{3}\frac{1}{\omega^2+1} - \frac{2}{3}\frac{1}{\omega^2+4} \Rightarrow R_X(\tau) = \mathcal{F}^{-1}\{S_X(\omega)\} = \frac{1}{3}e^{-|\tau|} - \frac{1}{6}e^{-2|\tau|}$

the average power $\psi^2 = R_X(\tau)|_{\tau=0} = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

6. (1) $m_X(t) = 0$ and $S_X(w) = \frac{2}{w^2+1} \Rightarrow R_X(\tau) = \mathcal{F}^{-1}\{S_X(w)\} = e^{-|\tau|}$

(2) $var(X(t)) = R_X(0) = E[X^2(t)] = 1 \Rightarrow \sigma^2 = 1$

then we will have $X(t) \sim N(0,1)$

$\Rightarrow f_{X,1}(x,t) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)$

7. $R_X(\tau) = e^{-|\tau|} \Rightarrow S_X(w) = \frac{2}{w^2+1}$

Consider the LTI system, we have

$Y(t) = X(t) \cdot \frac{\frac{1}{jwc}}{\frac{1}{jwc} + R} \Rightarrow Y(t) = X(t) \cdot \frac{1}{1+jwRc} \Rightarrow H(w) = \frac{1}{1+jwRc}$

$= \beta \cdot \frac{1}{\beta + jw}$

For $X(t)$ is zero-mean, stationary signal, so does $Y(t)$

and $m_Y(t) = m_X(t) \cdot H(0) = 0$

$R_Y(t, t+\tau) = R_X(\tau) * h(\tau) * h(-\tau)$ where $h(\tau) = \beta \cdot e^{-\beta\tau}$ $\tau > 0$

$f(\tau) = h(\tau) * h(-\tau) = \int_0^\tau \beta \cdot e^{-\beta s} \beta \cdot e^{-\beta(\tau-s)} ds = \int_0^\tau \beta^2 \cdot e^{-\beta\tau} ds = \beta^2 \tau e^{-\beta\tau}$

$\Rightarrow R_Y(\tau) = R_X(\tau) * f(\tau) = \int_{-\infty}^{+\infty} \beta^2 s e^{-\beta s} \cdot e^{-|\tau-s|} ds$

$= \beta^2 \int_{-\infty}^{\tau} s e^{-\beta s} \cdot e^{-(\tau-s)} ds + \int_{\tau}^{+\infty} s e^{-\beta s} \cdot e^{\tau-s} ds$

$= \beta^2 \{ \int_{-\infty}^{\tau} e^{-\tau} s e^{(1-\beta)s} ds + \int_{\tau}^{+\infty} e^{\tau} s e^{-(\beta+1)s} ds$

$= \beta^2 [\frac{1}{(1-\beta)^2} e^{-\tau} ((1-\beta)\tau - 1) e^{(1-\beta)\tau} + \frac{1}{(\beta+1)^2} e^{\tau} \cdot (-(\beta+1)\tau + 1) e^{-(\beta+1)\tau} \}$

$= \beta^2 e^{-\beta\tau} [\frac{1}{1-\beta^2} 2\beta\tau - 4\beta \cdot \frac{1}{(\beta+1)^2 (1-\beta)^2}]$

$S_Y(w) = |H(w)|^2 S_X(w) = \frac{2}{(w^2+1)(1+w^2c^2R^2)}$