UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 02
Spring 2023

1. **The Probabilistic Method**

   We introduce a proof technique — the *probabilistic method*. If we wish to show that there exists an element with property A in a set $\mathcal{X}$, it suffices to show that there exists a probability distribution p over $\mathcal{X}$ such that under p, the probability assigned to elements with property A is greater than 0.

   (Why does this work? If there is no element with property A, then there cannot possibly exist a p that assigns a positive probability to elements with property A, because we require that $p(\varnothing) = 0$.) Such a proof method is nonconstructive, meaning that it doesn't provide a method for finding such an element, yet it demonstrates the element exists.

   Consider a sphere that has $\frac{1}{10}$ of its surface colored blue, and the rest colored red. Show that no matter how the colors are distributed, it is possible to inscribe a cube in the sphere with all of its vertices red.

   Hint: If we sample an inscribed cube uniformly at random among all possible inscribed cubes, what is (an upper bound on) the probability that the sampled cube has at least one blue vertex?

2. **Suspicious Game**

   You are playing a card game with your friend in which you take turns picking a card from a deck. (Assume that you never run out of cards.) If you draw one of the special *bullet* cards, then you lose the game. Unfortunately, you do not know the contents of the deck. Your friend claims that $\frac{1}{3}$ of the deck is filled with bullet cards. However, you don't fully trust your friend: you believe he is lying with probability $\frac{1}{4}$. Assume that if your friend is lying, then the opposite is true: $\frac{2}{3}$ of the deck is filled with bullet cards!

   What is the probability that you win the game if you go first?

3. **Law of the Unconscious Statistician**

a. Prove the *Law of the Unconscious Statistician* (LOTUS): Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X \colon \Omega \to \mathbb{Z}$ and $F \colon \mathbb{Z} \to \mathbb{Z}$ be random variables. Note that the composition $Y = F(X) \colon \Omega \to \mathbb{Z}$ is another random variable. If $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$, and $\mathbb{E}_{\mathcal{L}_X}$ is expectation with respect to the *law* of X on $\mathbb{Z}$, then

$$\mathbb{E}(F(X)) = \mathbb{E}_{\mathcal{L}_X}(F).$$

You should assume that Ω is **discrete** for the sake of simplicity, although LOTUS holds more generally.

b. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the space of all sequences of independent fair coin tosses. Formulate N, the minimum number of tosses needed until we see heads, as a random variable on Ω.

c. Find $\mathbb{E}(N^2)$.

Hint: By the linearity of expectation, $\mathbb{E}(N^2) = \mathbb{E}(N(N-1)) + \mathbb{E}(N)$. You may use the Law of the Unconscious Statistician from part a, and the following identity:

$$\sum_{k=1}^{\infty} k(k-1)p^{k-2} = -\frac{d}{dp} \sum_{k=1}^{\infty} kp^{k-1}.$$