Lecture 10

Chapter 4 Analytic Geometry

4.1. Cartesian Coordinate System

4.2. Algebra in Euclidean Geometry

4.1 Analytic Geometry and Cartesian Coordinate System

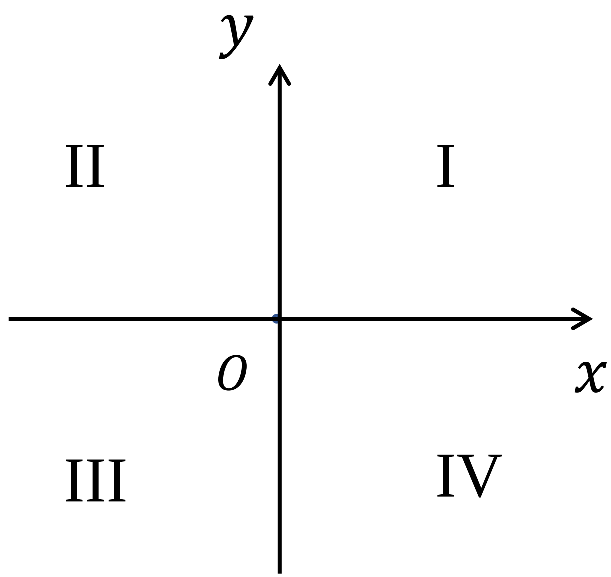
Geometry is a branch of mathematics concerned with questions of *shape*, *size*, *relative position of figures*, and the *properties of space*.

Benefiting from the introduction of **Cartesian coordinate system** [笛卡尔坐标系] (also referred to as **rectangular coordinate system** [直角坐标系]), we can define shapes in Euclidean geometry by a group of equations which are generally considered in algebra.

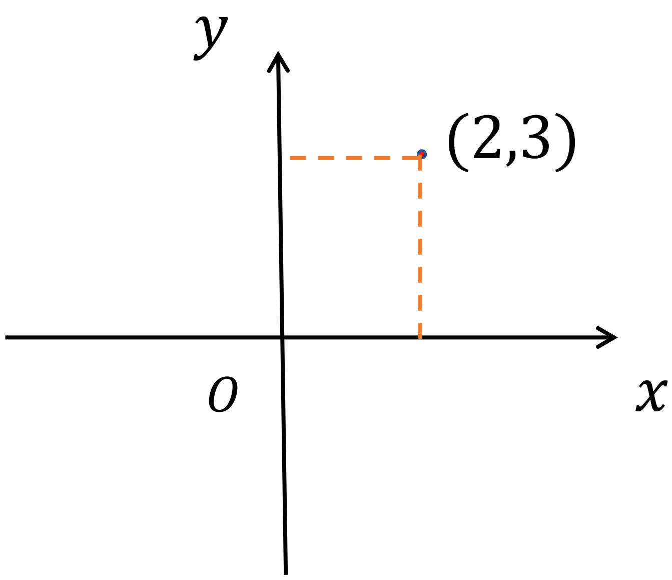
René Descartes (1596-1650) was a French philosopher, mathematician and physicist.

Cartesian Coordinate System on Plane

- Original point O [原点]
- 2 axes: x-axis [x轴], y-axis [y轴]
- 4 quadrants [象限]



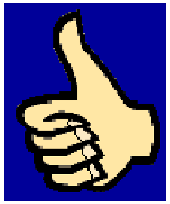
An ordered pair of numbers (x, y), is used to denote the point with x as its x-coordinate and y as its y-coordinate.

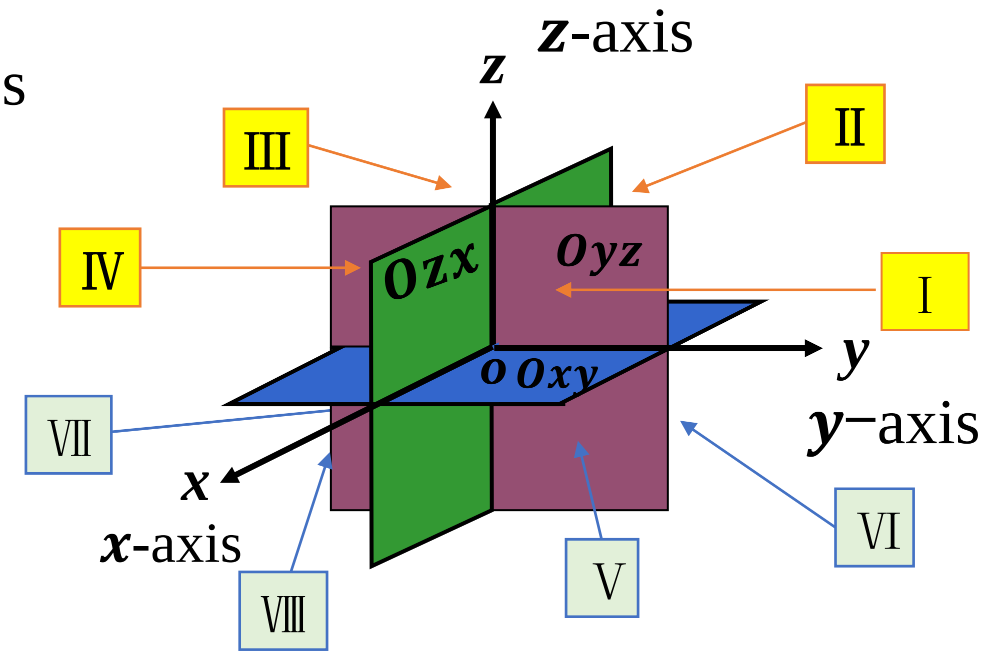


For instance, $(2,3)$ denotes a point settled at the position $x = 2$ and $y = 3$.

Cartesian Coordinate System in Space

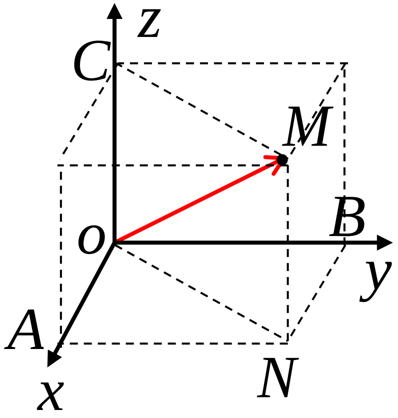
- origin O
- 3 axes
- 3 coordinate planes [坐标平面]
- 8 octants[卦限]





Any point M in the space can be represented by a tuple of three numbers, say (x, y, z).

For instance, the coordinate of O can be written as (0,0,0).



Vectors in Cartesian Coordinate System

Recall the definition of coordinate vector (**Definition 3.5.1**), for a given ordered basis E of a finite-dimensional vector space V, any vector x in V, has a **unique** **coordinate vector** associated with x.

Now let $E = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be the standard basis of vector space $\mathbf{R}^3$ and $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be a vector in $\mathbf{R}^3$.

$E = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ standard basis of $\mathbf{R}^3$ and $\boldsymbol{x} = (x_1, x_2, x_3)^T$

The coordinate vector of $\boldsymbol{x}$ w.r.t. the basis E can be found by solving

$$c_1\mathbf{e_1} + c_2\mathbf{e_2} + c_3\mathbf{e_3} = \boldsymbol{x}, \qquad \textbf{(*)}$$

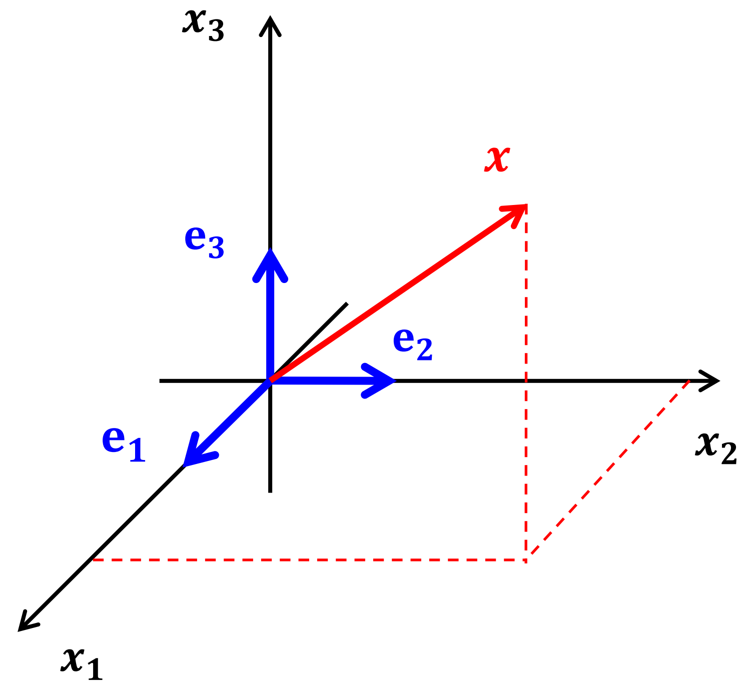
where c_i, $i = 1,2,3$ are scalars. If we denote $[\boldsymbol{x}]_E = (c_1, c_2, c_3)^T$, (*) can be written as

$$(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})[\boldsymbol{x}]_E = I[\boldsymbol{x}]_E = [\boldsymbol{x}]_E = \boldsymbol{x}.$$

This implies that three coordinates of vector $\boldsymbol{x} \in \mathbf{R}^3$ w.r.t. the standard basis E are exactly three components of vector $\boldsymbol{x}$.

Remark. In general, the coordinate vector $[\boldsymbol{x}]_E$ of vector $\boldsymbol{x}$ w.r.t. a given ordered basis E is **different** from the vector $\boldsymbol{x}$.

We can associate the coordinate vector $[\boldsymbol{x}]_E$ with a directed line segment from point (0,0,0) to (x_1, x_2, x_3), then each vector $\boldsymbol{x}$ can be represented by the **directed line segment**.



Remark. The expression of a point (a, b, c) is different from the row vector (a, b, c) discussed in the previous chapters.

4.2 Algebra in Euclidean Geometry

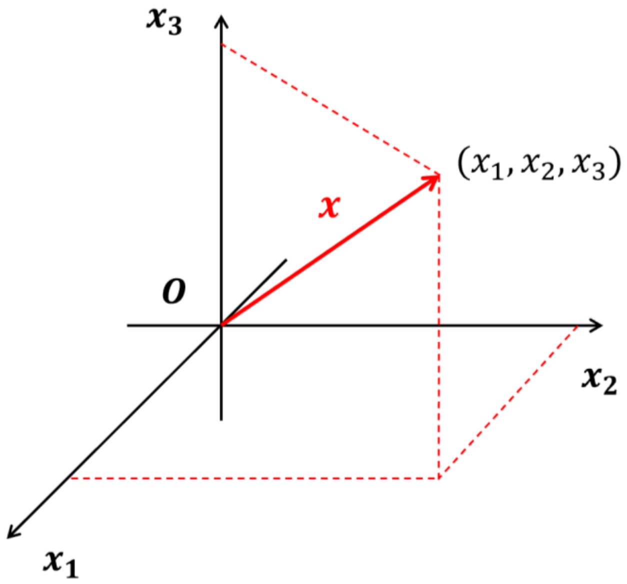
We shall restrict our study in $\mathbf{R}^2$ or $\mathbf{R}^3$, but most of these concepts and conclusions can be extended to all vector spaces.

Euclidean Length

Definition 1. Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T$, $n = 2,3$ be a vector in $\mathbf{R}^n$, then its **Euclidean length [欧几里得长度]** is defined by

$$l(\boldsymbol{x}) = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2},$$

and we use $\|\boldsymbol{x}\|$ to denote this number. If $\|\boldsymbol{x}\| = 1$, we say x is a **unit vector [单位向量].**



Example. Let $\boldsymbol{x} = (1,0,1)^T$, $\boldsymbol{y} = (0,1,-1)^T$. Find $\|\boldsymbol{x}+\boldsymbol{y}\|$, $\|\boldsymbol{x}-\boldsymbol{y}\|$, $\|-\boldsymbol{x}\|$ and $\|\boldsymbol{2y}\|$.

Theorem 1. Let $\boldsymbol{u}, \boldsymbol{v}$ be two vectors in $\mathbf{R}^n$, $n = 2,3$ and λ be a scalar. We have

(1) **positive homogeneity** [正齐次性]: $\|\lambda \boldsymbol{u}\| = |\lambda|\|\boldsymbol{u}\|$;

(2) **triangle inequality** [三角不等式]: $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$;

(3) **positivity** [非负性]: $\|\boldsymbol{u}\| \geq 0$ and $\|\boldsymbol{u}\| = 0$ if and only if $\boldsymbol{u} = \mathbf{0}$.

Example. Let $\boldsymbol{x} = (x_1, \dots, x_n)^T$ be a nonzero vector in $\mathbf{R}^{\boldsymbol{n}}$, $n = 2,3$. Show that $\boldsymbol{x}^0 = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ is a unit vector.

Proof. By positive homogeneity, we have

$$\|\boldsymbol{x}^0\| = \left\|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right\| = \frac{1}{\|\boldsymbol{x}\|}\|\boldsymbol{x}\| = 1.$$

Remark. $\boldsymbol{x}^{\mathbf{0}}$ is called **direction vector** [方向向量] of $\boldsymbol{x}$, since it always heads in the same direction as $\boldsymbol{x}$ and we have

$$\boldsymbol{x} = \|\boldsymbol{x}\|\boldsymbol{x}^0.$$

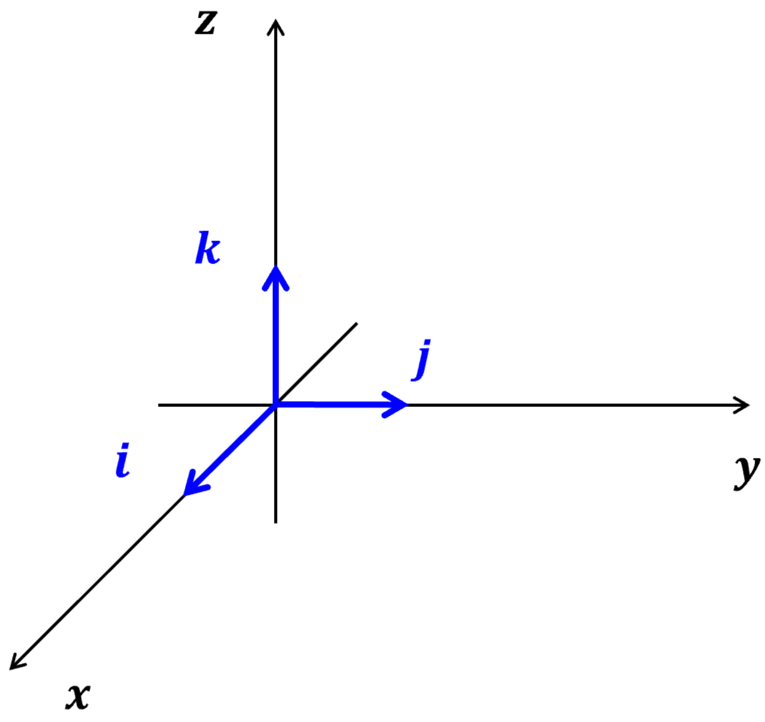
Moreover, since $\mathbf{0} = 0 \cdot \boldsymbol{u}^0$, where $\boldsymbol{u}^0$ is arbitrary unit vector, we can think that zero vector $\mathbf{0}$ can have any direction.

In $\mathbf{R}^3$, we also denote unit direction vectors of the x-, y- and z-axis as $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, respectively. That is

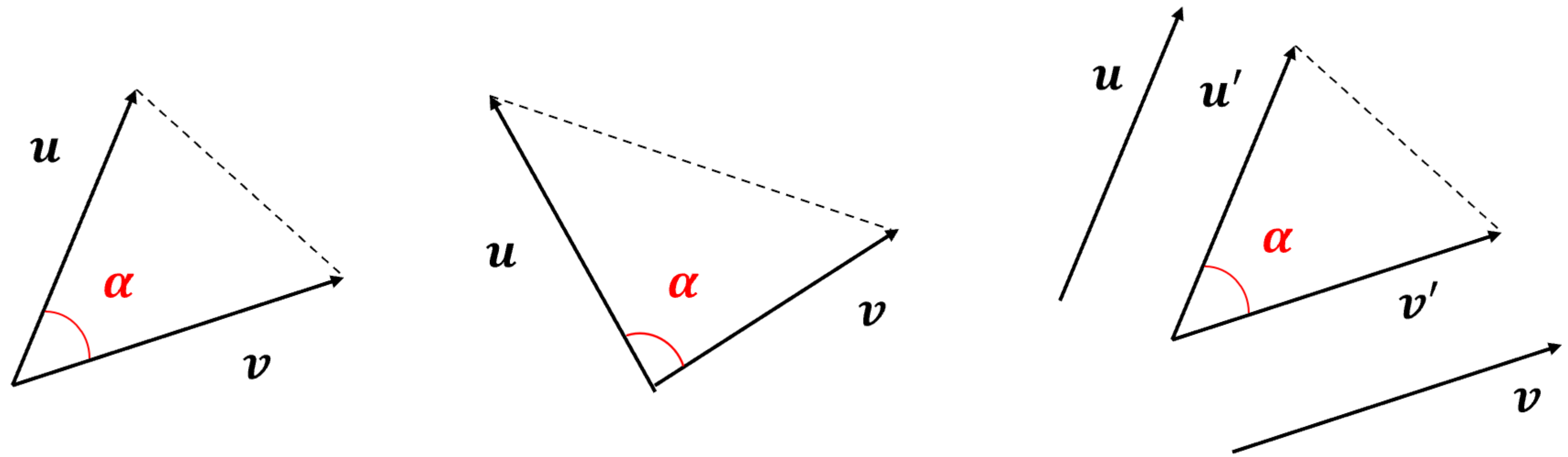
$$\boldsymbol{i} = \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\boldsymbol{j} = \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\boldsymbol{k} = \mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$



Included Angle of Two Vectors



Included angle [夹角] of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

Definition 2. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in $\mathbf{R}^n$, $n = 2,3$.

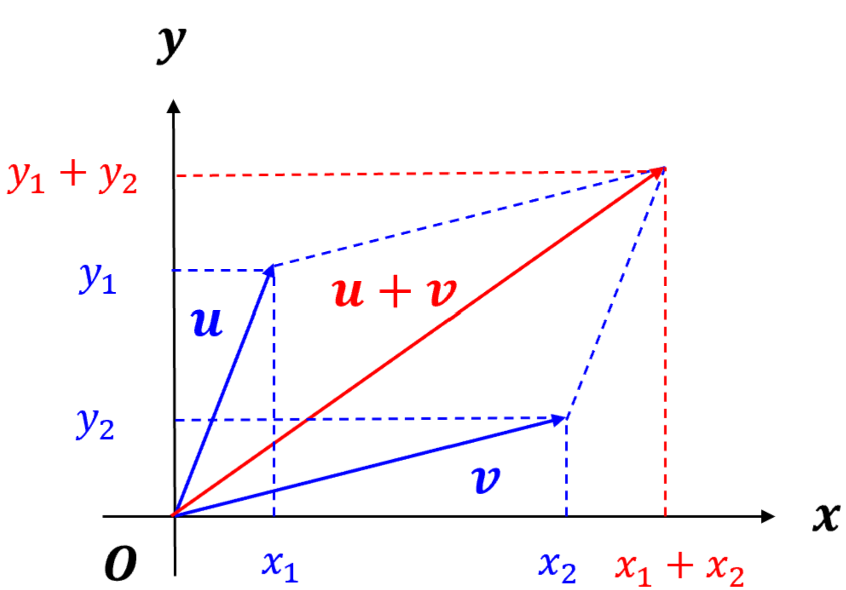
- They are called **collinear [共线]** (or **parallel [平行]**), denoted by $\boldsymbol{u} \parallel \boldsymbol{v}$, if their included angle is 0 or π;
- They are called **orthogonal [正交]** (or **perpendicular [垂直]**), denoted by $\boldsymbol{u} \perp \boldsymbol{v}$, if their included angle is $\frac{\pi}{2}$.

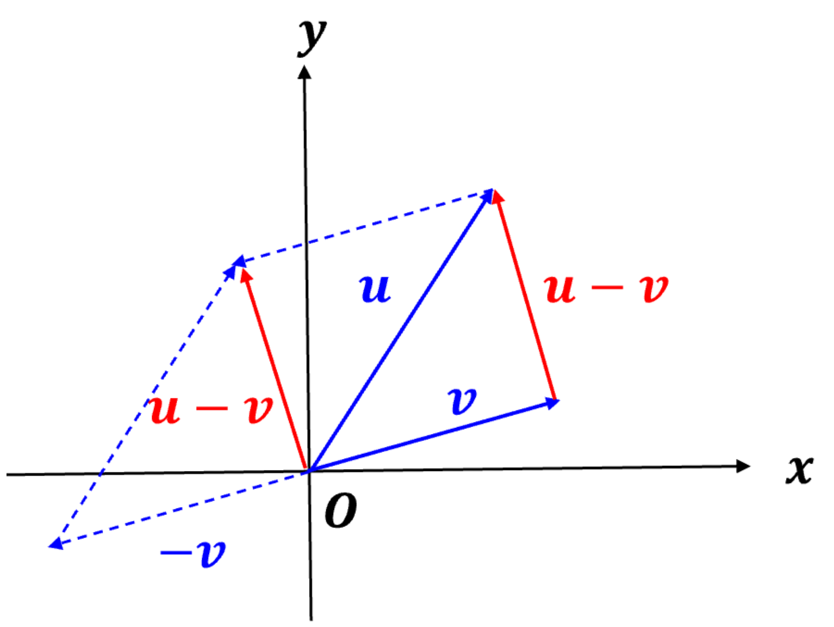
Geometric Interpretations of Operations on Vectors

Take $\mathbf{R}^2$ as an example.

<u>Addition of vectors.</u> $\boldsymbol{u} = (x_1, y_1)^T$, $\boldsymbol{v} = (x_2, y_2)^T$.

$$\boldsymbol{u} + \boldsymbol{v} = (x_1 + x_2, y_1 + y_2)^T, \quad \boldsymbol{u} - \boldsymbol{v} = (x_1 - x_2, y_1 - y_2)^T.$$

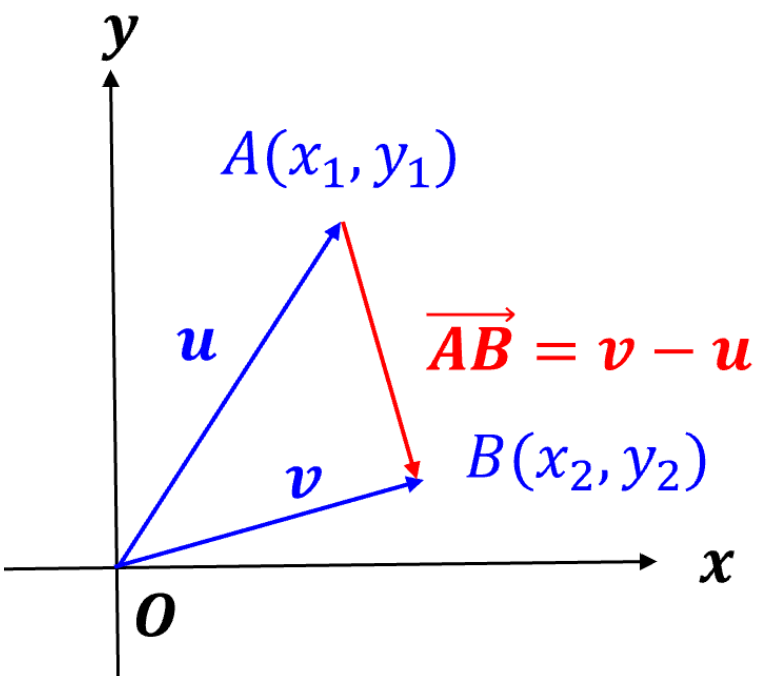




Theorem 2. (Distance between two points) Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in two dimensional Euclidean space. The distance between them can be calculated by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

In three dimensional Euclidean space, the distance of two points, say $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ can be calculated by

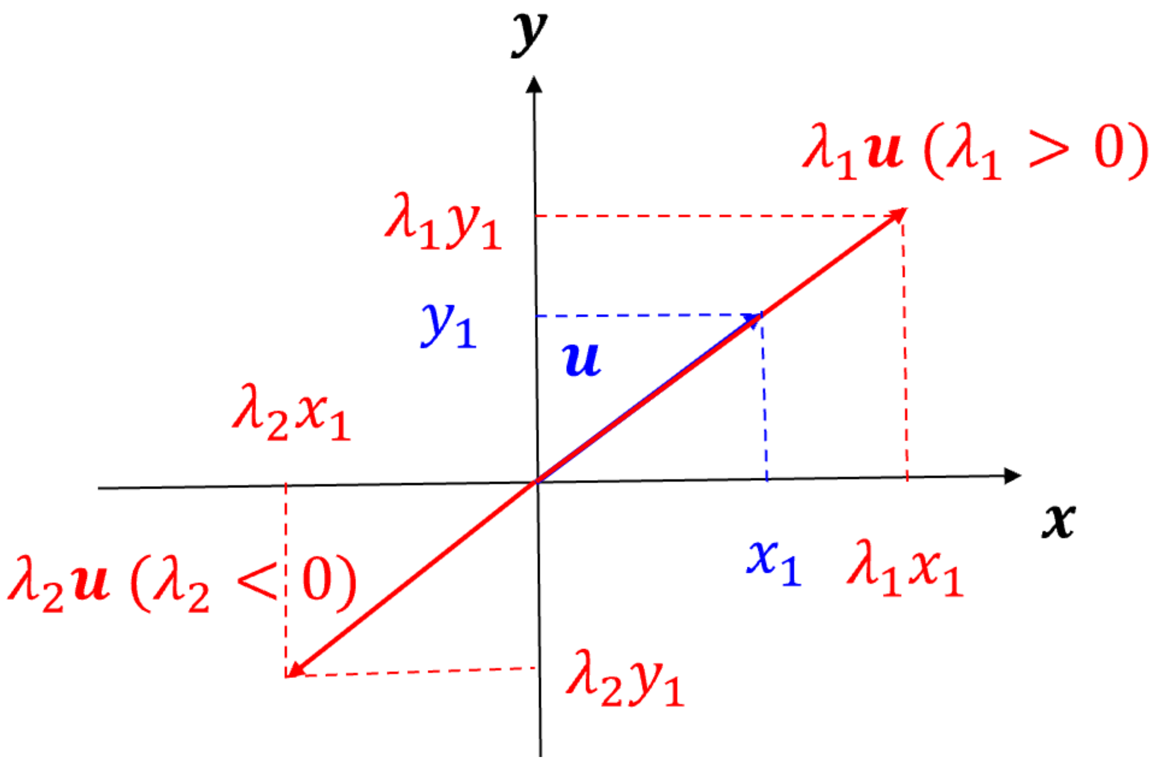


$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

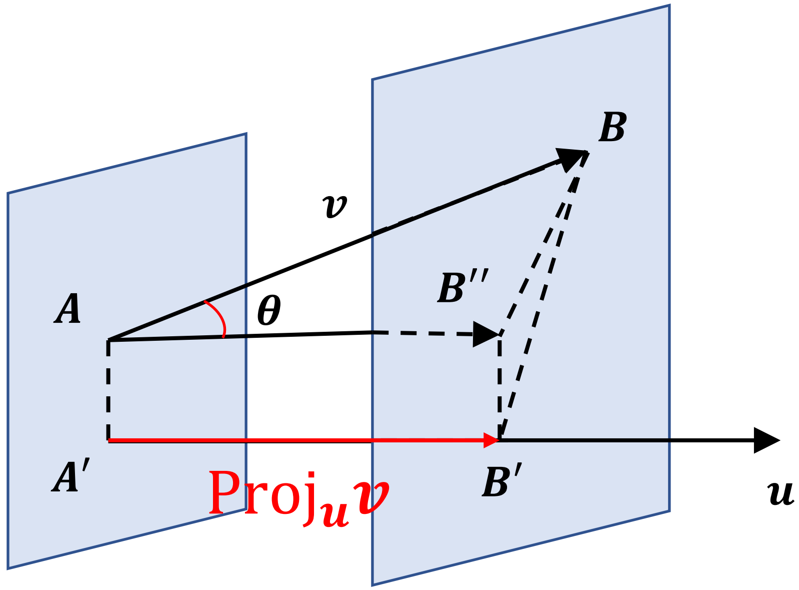
Scalar multiplication.

Let $\boldsymbol{u} = (x_1, y_1)^T$ and λ be a scalar. By the definition of scalar multiplication, we have

$$\lambda\boldsymbol{u} = \lambda(x_1, y_1)^T = (\lambda x_1, \lambda y_1)^T.$$



Projection of Vectors



The vector $\overrightarrow{A'B'}$ is called the **projection vector** of $\boldsymbol{v}$ onto $\boldsymbol{u}$, and is denoted by $\text{proj}_{\boldsymbol{u}}\boldsymbol{v}$.

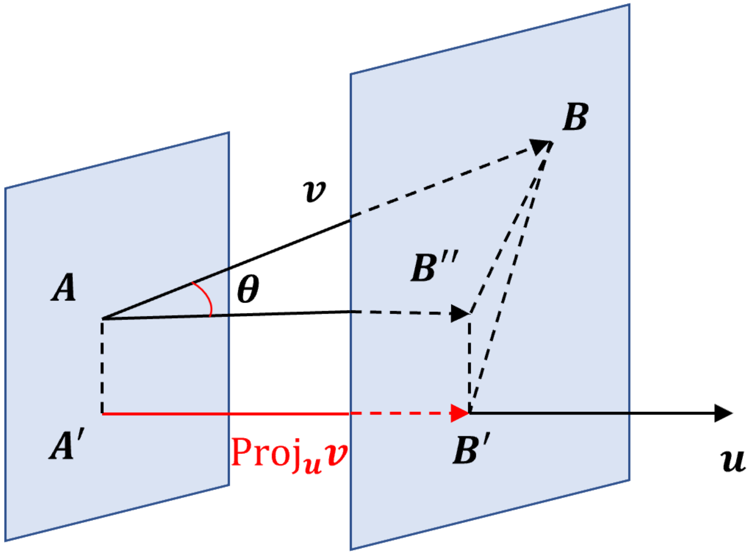
Definition 3. Let the included angle of vector $\boldsymbol{u}$ and $\boldsymbol{v}$ be θ. The **orthogonal projection vector [正交投影向量]** of $\boldsymbol{v}$ onto $\boldsymbol{u}$, or the **projection vector [投影向量],** is

$$\mathrm{proj}_{\boldsymbol{u}}\, \boldsymbol{v} = \|\boldsymbol{v}\| \cos\theta \cdot \boldsymbol{u}^0.$$

The scalar

$$(\boldsymbol{v})_{\boldsymbol{u}} = \|\boldsymbol{v}\| \cos\theta$$

is called the **orthogonal projection [正交投影]** of $\boldsymbol{v}$ onto $\boldsymbol{u}$, or simply the **projection [投影].**



Property 1. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbf{R^n}$, $n = 2,3$, λ be a scalar,

(1) $(\lambda\boldsymbol{v})_{\boldsymbol{u}} = \lambda(\boldsymbol{v})_{\boldsymbol{u}}$;

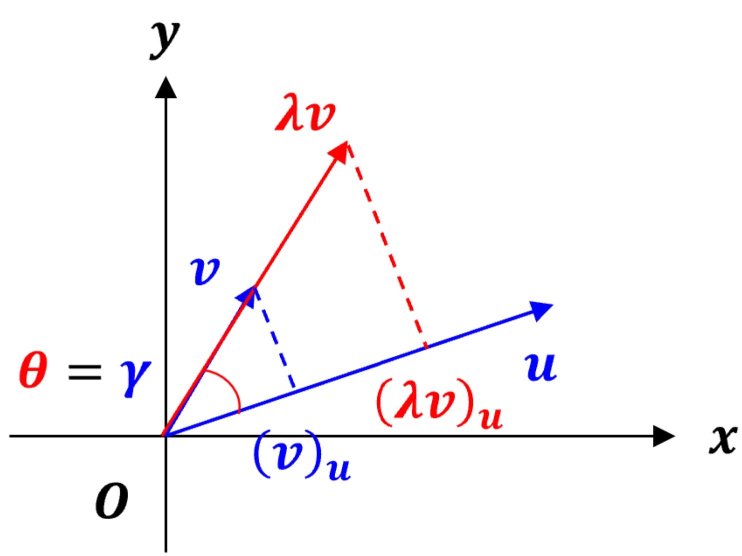
(2) $(\boldsymbol{v} + \boldsymbol{w})_{\boldsymbol{u}} = (\boldsymbol{v})_{\boldsymbol{u}} + (\boldsymbol{w})_{\boldsymbol{u}}$.

Proof of (1). We have

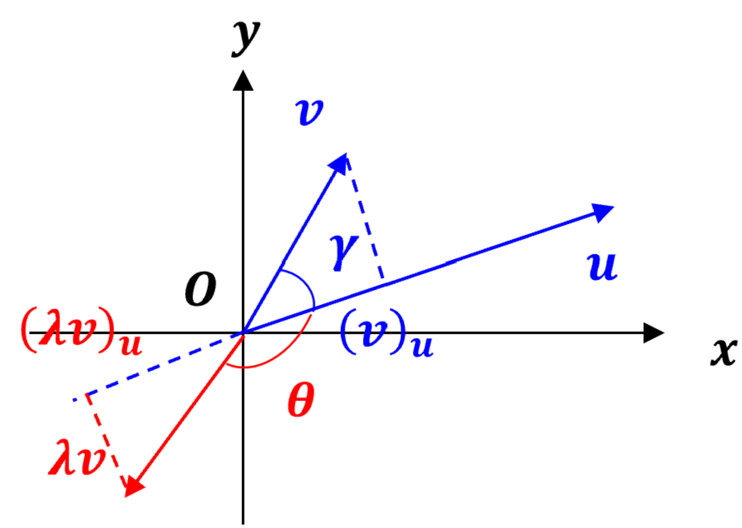
$$(\lambda\boldsymbol{v})_{\boldsymbol{u}} = \|\lambda\boldsymbol{v}\| \cos\theta = |\lambda|\|\boldsymbol{v}\| \cos\theta \,,$$

where θ is the included angle of $\lambda\boldsymbol{v}$ and $\boldsymbol{u}$. Let γ be the included angle of $\boldsymbol{v}$ and $\boldsymbol{u}$.

$\lambda > 0$:



$\lambda < 0$:

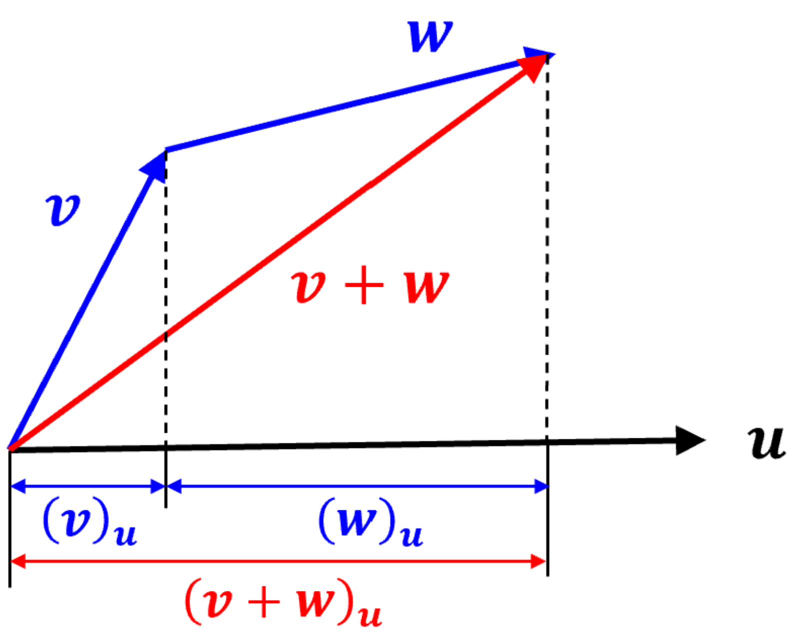


Property 1. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbf{R}^{\mathbf{n}}$, $n = 2,3$, λ be a scalar,

(1) $(\lambda \boldsymbol{v})_{\boldsymbol{u}} = \lambda(\boldsymbol{v})_{\boldsymbol{u}}$;

(2) $(\boldsymbol{v} + \boldsymbol{w})_{\boldsymbol{u}} = (\boldsymbol{v})_{\boldsymbol{u}} + (\boldsymbol{w})_{\boldsymbol{u}}$.

Proof of (2).



Definition 4. Let $\boldsymbol{u}$ be a vector in $\mathbf{R}^3$. If $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit direction vectors of x-, y-, z-axis, respectively and u_x, u_y, u_z are projections of $\boldsymbol{u}$ onto x-, y-, z-axis, then the **component representation [分量表示]** of $\boldsymbol{u}$ is

$$\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}.$$

Similarly, the component representation in $\mathbf{R}^2$ is

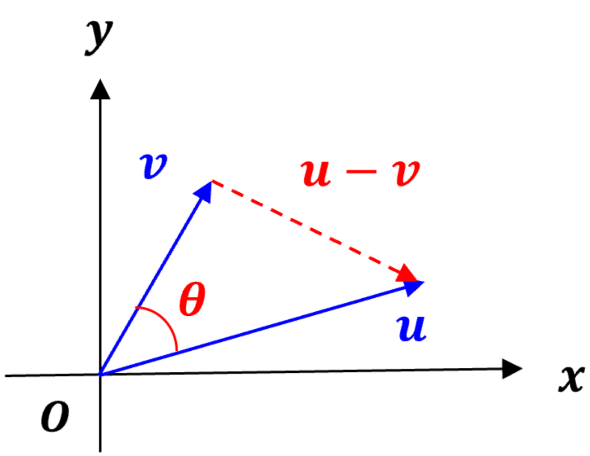
$$\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j}.$$

Inner Product

Let $\boldsymbol{u} = (x_1, y_1)^T$ and $\boldsymbol{v} = (x_2, y_2)^T$ be two vectors in $\mathbf{R}^2$.

Recall the **inner product** of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T \boldsymbol{v} = x_1 x_2 + y_1 y_2.$$



By the law of cosine,

$$\|\boldsymbol{u} - \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta\,,$$

we can derive that

$$\boldsymbol{u} \cdot \boldsymbol{v} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\| \cos\theta\,,$$

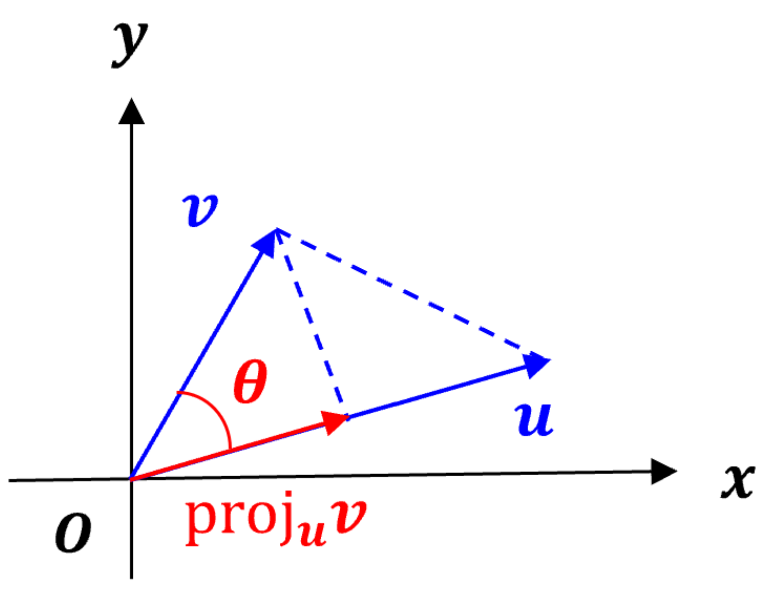
where θ is the included angle of $\boldsymbol{u}$ and $\boldsymbol{v}$.

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors, we have

$$\begin{aligned}\langle \boldsymbol{u}, \boldsymbol{v} \rangle &= \|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\| \cos\theta \\ &= \|\boldsymbol{u}\| \cdot \|\mathrm{proj}_{\boldsymbol{u}} \boldsymbol{v}\|.\end{aligned}$$

Similarly, we have

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \|\boldsymbol{v}\| \cdot \|\mathrm{proj}_{\boldsymbol{v}} \boldsymbol{u}\|.$$



The included angle of nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be calculated by

$$\theta = \arccos\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\cdot\|\boldsymbol{v}\|}\right) = \arccos\left(\boldsymbol{u}^{\boldsymbol{0}}\cdot\boldsymbol{v}^{\boldsymbol{0}}\right).$$

In case that $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular, we will have

$$\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\cdot\|\boldsymbol{v}\|\cos\frac{\pi}{2} = 0.$$

Conversely, if $\boldsymbol{u}\cdot\boldsymbol{v} = 0$, we have

$$\theta = \arccos\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\cdot\|\boldsymbol{v}\|}\right) = \arccos(0) = \frac{\pi}{2}.$$

Theorem 3. Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors in $\mathbf{R}^n$, $n = 2,3$. The necessary and sufficient condition of $\boldsymbol{u}\perp\boldsymbol{v}$ is $\boldsymbol{u}\cdot\boldsymbol{v} = 0$.

In case that $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, we have

$$\boldsymbol{u}^{\mathbf{0}} \cdot \boldsymbol{v}^{\mathbf{0}} = \|\boldsymbol{u}^{\mathbf{0}}\| \cdot \|\boldsymbol{v}^{\mathbf{0}}\| \cos 0 = 1\,,$$

or

$$\boldsymbol{u}^{\mathbf{0}} \cdot \boldsymbol{v}^{\mathbf{0}} = \|\boldsymbol{u}^{\mathbf{0}}\| \cdot \|\boldsymbol{v}^{\mathbf{0}}\| \cos(\pi) = -1\,.$$

If $\boldsymbol{u}^{\mathbf{0}} \cdot \boldsymbol{v}^{\mathbf{0}} = 1$ or -1, we have

$$\theta = \arccos\left(\boldsymbol{u}^{\mathbf{0}} \cdot \boldsymbol{v}^{\mathbf{0}}\right) = \arccos(1) = 0$$

or

$$\theta = \arccos\left(\boldsymbol{u}^{\mathbf{0}} \cdot \boldsymbol{v}^{\mathbf{0}}\right) = \arccos(-1) = \pi.$$

Theorem 4. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be nonzero vectors in $\mathbf{R}^n$, $n = 2,3$. The necessary and sufficient condition of $\boldsymbol{u} \parallel \boldsymbol{v}$ is $\left|\boldsymbol{u}^{\mathbf{0}} \cdot \boldsymbol{v}^{\mathbf{0}}\right| = 1$.

Example.

Find the included angle between the following vectors and determine their relative positions.

(1) $\boldsymbol{u} = (0,1,0)^T$ and $\boldsymbol{v} = (1,0,1)^T$;

(2) $\boldsymbol{u} = (1,-1,1)^T$ and $\boldsymbol{v} = (-1,-1,1)^T$;

(3) $\boldsymbol{u} = (1,0,-1)^T$ and $\boldsymbol{v} = (-1,0,1)^T$;

(4) $\boldsymbol{u} = (0,1,-1)^T$ and $\boldsymbol{v} = (0,2,-2)^T$.

Property 2. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbf{R}^n$, $n = 2,3$. Then

$$(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}.$$

Proof. It follows easily by the definition of inner product.

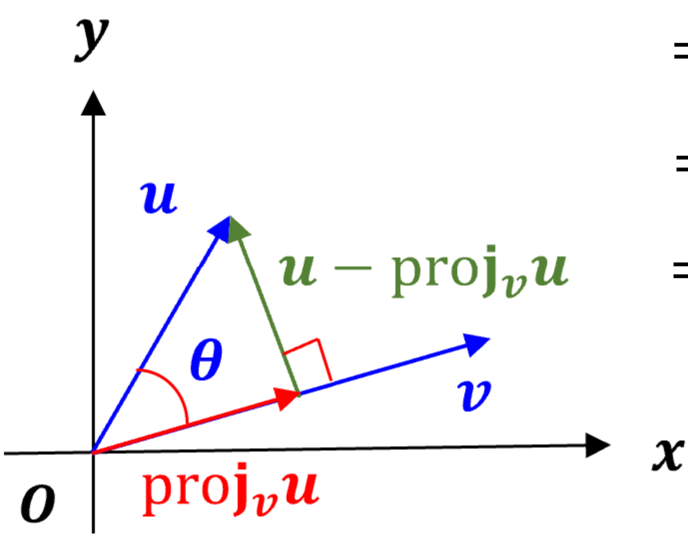
Example 5. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ are two nonzero vectors in $\mathbf{R}^n$.

Prove that $$(\boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\boldsymbol{u}\,) \perp \text{proj}_{\boldsymbol{v}}\boldsymbol{u}.$$

Proof. It is enough to show that $(\boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\boldsymbol{u}\,) \cdot \text{proj}_{\boldsymbol{v}}\boldsymbol{u} = 0$.

By **Property 2**, we have

$$\begin{aligned}
(\boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\boldsymbol{u}\,) \cdot \text{proj}_{\boldsymbol{v}}\boldsymbol{u} &= \boldsymbol{u} \cdot \text{proj}_{\boldsymbol{v}}\boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\boldsymbol{u} \cdot \text{proj}_{\boldsymbol{v}}\boldsymbol{u} \\
&= \boldsymbol{u} \cdot \left(\|\boldsymbol{u}\| \cos\theta\, \boldsymbol{v}^0\right) - \|\text{proj}_{\boldsymbol{v}}\boldsymbol{u}\|^2 \\
&= (\|\boldsymbol{u}\| \cos\theta)\boldsymbol{u} \cdot \boldsymbol{v}^{\mathbf{0}} - \left\|\|\boldsymbol{u}\| \cos\theta\, \boldsymbol{v}^0\right\|^2 \\
&= (\|\boldsymbol{u}\| \cos\theta)(\|\boldsymbol{u}\| \cos\theta) - (\|\boldsymbol{u}\| \cos\theta)^2 \\
&= 0.
\end{aligned}$$



Theorem 5. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two nonzero vectors in $\mathbf{R}^n$, $n = 2,3$, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent if and only if $\boldsymbol{u} \parallel \boldsymbol{v}$.

Proof. Suppose that $\boldsymbol{u}$ and v are linearly dependent. Then there exist scalars c_1, c_2 not all zero such that

$$c_1\boldsymbol{u} + c_2\boldsymbol{v} = \mathbf{0}.$$

Assume without loss of generality $c_2 \neq 0$, then

$$\boldsymbol{v} = -\frac{c_1}{c_2}\boldsymbol{u} = \lambda\boldsymbol{u},$$

which means $\boldsymbol{u} \parallel \boldsymbol{v}$. Conversely, if $\boldsymbol{u} \parallel \boldsymbol{v}$, then there exists scalar λ such that $\boldsymbol{v} = \lambda\boldsymbol{u}$. This gives

$$\lambda\boldsymbol{u} - \boldsymbol{v} = 0,$$

meaning that $\boldsymbol{u}, \boldsymbol{v}$ are linearly dependent.

Cauchy-Schwarz Inequality

Theorem 6. (Cauchy-Schwarz inequality (1888)) If $x_1, x_2, \ldots, x_n \in \mathbf{R}$ and $y_1, y_2, \ldots, y_n \in \mathbf{R}$, then we have

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right). \tag{1}$$

If we take x_i and y_i, $i = 1,2, \ldots, n$ as components of two vectors in $\mathbf{R}^n$:

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T \quad \text{and} \quad \boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T,$$

then the inequality (1) can be rewritten as

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|. \tag{2}$$

The equality in (2) holds only when $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent.

Cauchy-Schwarz inequality $\quad |\boldsymbol{x}\cdot\boldsymbol{y}| \le \|\boldsymbol{x}\|\cdot\|\boldsymbol{y}\|.$

Proof. We have

$$|\boldsymbol{x}\cdot\boldsymbol{y}| = \big|\|\boldsymbol{x}\|\cdot\|\boldsymbol{y}\|\cos\theta\big| = \|\boldsymbol{x}\|\cdot\|\boldsymbol{y}\|\cdot|\cos\theta|,$$

where $\theta \in [0,\pi]$ is the included angle of $\boldsymbol{x},\boldsymbol{y}$. Since $|\cos\theta| \le 1$, we get

$$|\boldsymbol{x}\cdot\boldsymbol{y}| \le \|\boldsymbol{x}\|\cdot\|\boldsymbol{y}\|.$$

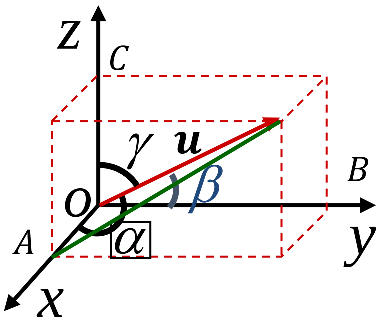
The equality holds if and only if $|\cos\theta| = 1$, which means $\theta = 0$ or π, $\boldsymbol{x},\boldsymbol{y}$ are parallel. By **Theorem 5**, we know that $\boldsymbol{x},\boldsymbol{y}$ are linearly dependent.

Definition 5. Let $\boldsymbol{u}$ be a vector in $\mathbf{R}^3$, u_x, u_y, u_z are projections of $\boldsymbol{u}$ onto x, y, z axes. Let α, β, γ be the included angles between $\boldsymbol{u}$ and x, y, z axes, respectively. These angles are called **direction angles [方向角]** and the cosines of these angles are called **direction cosine [方向余弦].**

$$\cos\alpha = \frac{u_x}{\|\boldsymbol{u}\|}$$

$$\cos\beta = \frac{u_y}{\|\boldsymbol{u}\|}$$

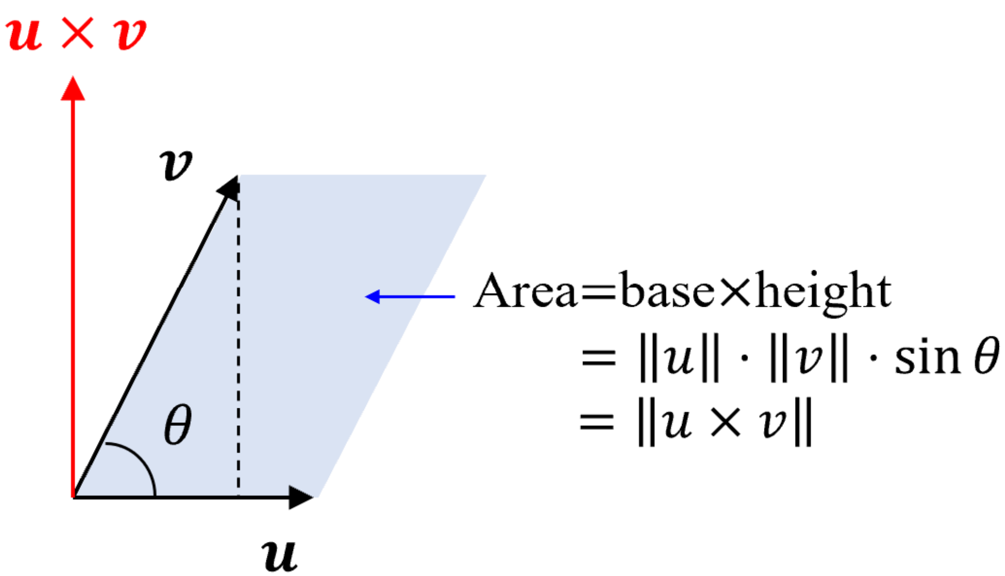
$$\cos\gamma = \frac{u_z}{\|\boldsymbol{u}\|}$$



$$\Rightarrow \quad (\cos\alpha, \cos\beta, \cos\gamma)^T = \frac{1}{\|\boldsymbol{u}\|}(u_x, u_y, u_z)^T = \boldsymbol{u}^0$$

Cross Product

Definition 6. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in $\mathbf{R}^3$, then the **cross product [叉积]** of them is a new vector that is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$, with a direction given by the right-hand rule, and a magnitude equal to the area of the parallelogram that the vectors span. The cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $\boldsymbol{u} \times \boldsymbol{v}$.



$\boldsymbol{u} \times \boldsymbol{v} = \|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\| \cdot \sin\theta \cdot \boldsymbol{n}$, $\boldsymbol{n}$ is a unit direction vector of $\boldsymbol{u} \times \boldsymbol{v}$, θ is the include angle of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Property 3. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be nonzero vectors in $\mathbf{R}^3$, λ, μ be scalars.

(1) **anti-commutative [反交换律]:** $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$;

(2) **associative [结合律]:** $(\lambda \boldsymbol{u}) \times \boldsymbol{v} = \boldsymbol{u} \times (\lambda \boldsymbol{v}) = \lambda(\boldsymbol{u} \times \boldsymbol{v})$ and

$$(\lambda \boldsymbol{u}) \times (\mu \boldsymbol{v}) = (\lambda\mu)(\boldsymbol{u} \times \boldsymbol{v});$$

(3) **distributive [分配律]:** $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$ and

$$(\boldsymbol{v} + \boldsymbol{w}) \times \boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{u} + \boldsymbol{w} \times \boldsymbol{u}.$$

Example 6. (Cross Product of Standard Bases) Let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be the standard bases of $\mathbf{R}^3$. Find $\boldsymbol{i} \times \boldsymbol{i}$, $\boldsymbol{j} \times \boldsymbol{j}$, $\boldsymbol{k} \times \boldsymbol{k}$, $\boldsymbol{i} \times \boldsymbol{j}$, $\boldsymbol{j} \times \boldsymbol{k}$, $\boldsymbol{k} \times \boldsymbol{i}$, $\boldsymbol{j} \times \boldsymbol{i}$, $\boldsymbol{k} \times \boldsymbol{j}$ and $\boldsymbol{i} \times \boldsymbol{k}$.

Solution. By definition of cross product, we have

$$\|\boldsymbol{i} \times \boldsymbol{i}\| = \|\boldsymbol{j} \times \boldsymbol{j}\| = \|\boldsymbol{k} \times \boldsymbol{k}\| = 1 \cdot 1 \cdot |\sin 0| = 0,$$

so that $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \mathbf{0}$.

By the right-hand rule, we know that $\boldsymbol{i} \times \boldsymbol{j}$ is in the direction of $\boldsymbol{k}$, $\boldsymbol{j} \times \boldsymbol{k}$ is in the direction of $\boldsymbol{i}$, $\boldsymbol{k} \times \boldsymbol{i}$ is in the direction of $\boldsymbol{j}$; the included angle between $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are all $\pi/2$ so that $\|\boldsymbol{i} \times \boldsymbol{j}\| = \|\boldsymbol{j} \times \boldsymbol{k}\| = \|\boldsymbol{k} \times \boldsymbol{i}\| = 1$. As a result,

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}, \qquad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}, \qquad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}.$$

Finally by the anti-commutativity, we have

$$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{i} \times \boldsymbol{j} = -\boldsymbol{k}, \qquad \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}, \qquad \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}.$$

Theorem 7. (Component Representation of Cross Product) Let

$$\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k} \text{ and } \boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k},$$

Then

$$\boldsymbol{u} \times \boldsymbol{v} = \left(u_y v_z - u_z v_y\right)\boldsymbol{i} - (u_x v_z - u_z v_x)\boldsymbol{j} + \left(u_x v_y - u_y v_x\right)\boldsymbol{k}.$$

Proof. It suffices to use conclusions in **Example 6** to get

$$\boldsymbol{u} \times \boldsymbol{v} = \left(u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}\right) \times \left(v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}\right)$$

$$= u_x v_x \boldsymbol{i} \times \boldsymbol{i} + u_y v_y \boldsymbol{j} \times \boldsymbol{j} + u_z v_z \boldsymbol{k} \times \boldsymbol{k} + u_x v_y \boldsymbol{i} \times \boldsymbol{j} + u_y v_x \boldsymbol{j} \times \boldsymbol{i}$$
$$+ u_y v_z \boldsymbol{j} \times \boldsymbol{k} + u_z v_y \boldsymbol{k} \times \boldsymbol{j} + u_x v_z \boldsymbol{i} \times \boldsymbol{k} + u_z v_x \boldsymbol{k} \times \boldsymbol{i}$$

$$= \left(u_x v_y - u_y v_x\right)\boldsymbol{k} + \left(u_y v_z - u_z v_y\right)\boldsymbol{i} + (u_z v_x - u_x v_z)\boldsymbol{j}$$

$$= \left(u_y v_z - u_z v_y\right)\boldsymbol{i} - (u_x v_z - u_z v_x)\boldsymbol{j} + \left(u_x v_y - u_y v_x\right)\boldsymbol{k}.$$

$$\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}, \quad \boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k},$$

$$\boldsymbol{u} \times \boldsymbol{v} = \left(u_y v_z - u_z v_y\right)\boldsymbol{i} - (u_x v_z - u_z v_x)\boldsymbol{j} + \left(u_x v_y - u_y v_x\right)\boldsymbol{k}.$$

By using the matrix notation of determinant, we can also rewrite as

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}.$$

Exercise. Let $\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = -4\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$. Find $\boldsymbol{u} \times \boldsymbol{v}$ and $\boldsymbol{v} \times \boldsymbol{u}$.

Theorem 8. (Necessary and Sufficient Condition for Collinear Vectors) Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors in $\mathbf{R}^3$, then $\boldsymbol{u} \parallel \boldsymbol{v}$ if and only if $\boldsymbol{u} \times \boldsymbol{v} = \mathbf{0}$.

Proof. According to the definition of cross product, it is easy to see that $\boldsymbol{u} \times \boldsymbol{v} = \mathbf{0}$ if and only if $\theta = 0$ or π.

Triple Scalar or Box Product

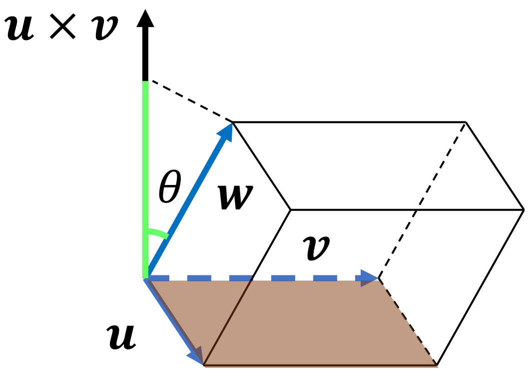
Definition 7. Suppose $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are three vectors in $\mathbf{R}^3$, then the product $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ is called the **triple scalar product [混合积]** of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$, denoted by $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$.

Volume of the parallelepiped

$= \text{Area of base} \times \text{Height}$

$= \|\boldsymbol{u} \times \boldsymbol{v}\| \cdot \|\boldsymbol{w}\| \cdot |\cos\theta|$

$= |(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}|$



Area of base = $\|\boldsymbol{u} \times \boldsymbol{v}\|$

Height = $\|\boldsymbol{w}\| \cdot |\cos\boldsymbol{\theta}|$

Theorem 9. (Component Representation of Triple Scalar Product)

$$\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k},$$

$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k},$$

$$\boldsymbol{w} = w_x\boldsymbol{i} + w_y\boldsymbol{j} + w_z\boldsymbol{k}.$$

Then
$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}.$$

Proof. Notice that $\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} u_x & u_y \\ v_x & v_z \end{vmatrix} \boldsymbol{k}$, so that

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} w_x - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} w_y + \begin{vmatrix} u_x & u_y \\ v_x & v_z \end{vmatrix} w_z = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}.$$

Property 4. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be three vectors in $\mathbf{R}^3$, then

(1) $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = (\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{u} = (\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v}$;

(2) $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = -(\boldsymbol{v} \times \boldsymbol{u}) \cdot \boldsymbol{w}$.

Theorem 10. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be three vectors in $\mathbf{R}^3$, then they are **coplanar [共面]** if and only if $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = 0$.

Review

- Cartesian coordinates on plane and in space;
- Projection of a vector onto another;
- Inner product and Cross product of vectors;
- Triple Scalar product of vectors

Preview

➢ Planes and Lines