SOLUTIONS

Module:	Engineering Mathematics		
Module Code	BBC4111	Paper	A
Time allowed	2hrs	Filename	Solutions_2324_BBC4111_A
Rubric	ANSWER EIGHT QUESTIONS ONLY		
Examiners	Dr Ting Mei		

Question1. [30 marks total, 3 marks for each blank]
Write down answers of the following questions in the blank part after questions.

a) The principal root of $(-2+2\sqrt{3}\mathrm{i})^{\frac{1}{4}}$ is ($\sqrt{2}\exp(\frac{\pi}{6}\mathrm{i})$ or $\frac{\sqrt{6}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}$).

b) $\lim\limits_{z\to -\mathrm{i}}\dfrac{z+\mathrm{i}}{2z(z^2+1)}=$ ($-\frac{1}{4}$).

c) Let x, y be real numbers. If the function $f(z)=(x^3-3xy^2)+\mathrm{i}v(x,y)$ is analytic, then $f'(z)=$($3z^2$ or $3x^2+6\mathrm{i}xy-3y^2$).

d) $\left[\frac{1}{\sqrt{2}}(1-i)\right]^{-i}=$ ($\exp(-\frac{1}{4}+2n)\pi, n\in Z$).

e) Suppose $f(z)=\displaystyle\int_{|s|=2}\frac{s^3-2s^2-1}{(s-z)^3}\mathrm{d}s$, then $f'(1)=$ ($6\pi\mathrm{i}$).

f) $\operatorname*{Res}\limits_{z=0} z^2\sin\frac{1}{z}=$ ($-\frac{1}{3!}$).

g) The solution of the initial problem $\begin{cases} u_{tt}-4u_{xx}=0, & -\infty<x<+\infty,\ t>0, \\ u(x,0)=x^2-x, & -\infty<x<+\infty, \\ u_t(x,0)=\sin x, & -\infty<x<+\infty. \end{cases}$ is

($x^2+4t^2-x+\frac{1}{2}\sin x\sin 2t$).

h) The eigenvalues of the eigenvalue problem $\begin{cases} u''(x)+\lambda u(x)=0, 0<x<1, \\ u(0)=u'(1)=0, \end{cases}$ are

($\left(n\pi-\frac{\pi}{2}\right)^2, n=1,2,\cdots$).

i) Let $P_n(x)$ be the Legendre polynomial of degree n, then $\int_{-1}^{1}(x^4-2x^3+x)P_3(x)\mathrm{d}x=$ ($-\frac{8}{35}$).

j) Suppose that $\mathcal{F}[f(x)]=F(\lambda)$, where $\mathcal{F}[f(x)]$ is the Fourier integral transformation of $f(x)$, then for any constant $a,b\in R, a>0$, $\mathcal{F}[f(ax+b)]=$ ($\frac{1}{a}\mathrm{e}^{-\mathrm{i}\frac{b}{a}\lambda}F(\frac{\lambda}{a})$).

Question 2. [10 marks total, 2 marks for each blank]

Please determine whether the following statements are true. Put "T" if the statement is true or "F" if it's wrong.

a) The function $f(z)=\mathrm{Log}(z-2\mathrm{i})$ is analytic in the domain $\{(x,y): x>0, y=2\}$. (T)

b) If $f(z)$ is analytic at the point z_0, then $f(z)$ is analytic in some neighborhood of z_0. (T)

c) Suppose $\sum\limits_{n=1}^{\infty}c_n$ converges and $\sum\limits_{n=1}^{\infty}|c_n|$ diverges, then the radius of convergence of the power series $\sum\limits_{n=1}^{\infty}c_n z^n$ is $R=1$. (T)

d) The general solution of the Legendre equation $(1-x^2)y''(x)-2xy'(x)+6y(x)=0$ is $y(x)=C_1P_2(x)+C_2Q_2(x)$. (T)

e) Let $J_v(x)$ be the first kind of Bessel function of order v. Then for all v, $J_v(x)$ have finite values at $x=0$. (F)

Question 3. [12 marks]

Find the Laurent series expansions for the function $f(z)=\dfrac{2z-1}{z(z+1)}$ in the following annular domains

a) $1<|z|<\infty$;

b) $1<|z-1|<2$.

Solution. It is easy to see

$$f(z)=\frac{2z-1}{z(z+1)}=\frac{3}{z+1}-\frac{1}{z}.$$

a) In the annular domain $1<|z|<\infty$,

$$f(z)=\frac{1}{z}\frac{3}{1+\frac{1}{z}}-\frac{1}{z}=\frac{3}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n-\frac{1}{z}=\frac{2}{z}-3\sum_{n=2}^{\infty}\left(-\frac{1}{z}\right)^n.$$ **[6 marks]**

b) In the annular domain $1<|z-1|<2$,

$$f(z)=\frac{3}{2+z-1}-\frac{1}{1+z-1}=\frac{1}{2}\frac{3}{1+\frac{z-1}{2}}-\frac{1}{z-1}\frac{1}{1+\frac{1}{z-1}}$$

$$=\frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{z-1}{2}\right)^n-\frac{1}{z-1}\sum_{n=0}^{\infty}\left(-\frac{1}{z-1}\right)^n$$

$$=\frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{z-1}{2}\right)^n+\sum_{n=1}^{\infty}\left(-\frac{1}{z-1}\right)^n.$$ **[6 marks]**

Question 4. [12 marks]

Suppose the function $f(z)=\dfrac{e^z}{z^2(z^2+1)}$, then

a) find out all the singular points of $f(z)$, and indicate their types;

b) evaluate the residues of $f(z)$ at those singular points;

c) evaluate the integral $\displaystyle\oint_{|z-i|=\frac{3}{2}} f(z)\mathrm{d}z$.

Solution. a) $f(z)$ has three singular points $z_1=0,\ z_2=i,\ z_3=-i$. **[3 marks]**

For $z_1=0$, $f(z)=\dfrac{\frac{e^z}{z^2+1}}{z^2}$ with $\dfrac{e^z}{z^2+1}$ being analytic and nonzero at $z_1=0$. So $z_1=0$ is a pole of order two.

For $z_2=i$, $f(z)=\dfrac{\frac{e^z}{z^2(z+i)}}{z-i}$ with $\dfrac{e^z}{z^2(z+i)}$ being analytic and nonzero at $z_2=i$. So $z_2=i$ is a simple pole.

For $z_3=-\mathbf{i}$, $f(z)=\dfrac{\dfrac{e^z}{z^2(z-i)}}{z+i}$ with $\dfrac{e^z}{z^2(z-i)}$ being analytic and nonzero at $z_3=-\mathbf{i}$. So $z_3=-\mathbf{i}$ is a simple pole. **[3 marks]**

b) $\operatorname*{Res}_{z=0} f(z)=\left(\dfrac{e^z}{z^2+1}\right)'\Bigg|_{z=0}=1,$

$\operatorname*{Res}_{z=i} f(z)=\dfrac{e^z}{z^2(z+i)}\Bigg|_{z=i}=\dfrac{i}{2}e^{i},$

$\operatorname*{Res}_{z=-i} f(z)=\dfrac{e^z}{z^2(z-i)}\Bigg|_{z=-i}=-\dfrac{i}{2}e^{-i}.$ **[3 marks]**

c) Since only $z_1=0$ and $z_2=i$ are inside $|z-i|=\dfrac{3}{2}$, so

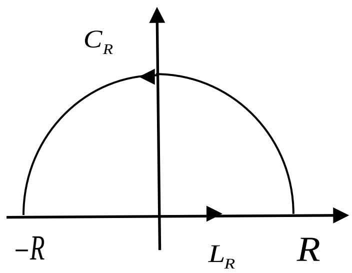
$$\int_{|z-i|=\frac{3}{2}} f(z)dz=2\pi i(\operatorname*{Res}_{z=0} f(z)+\operatorname*{Res}_{z=i} f(z))=2\pi i-\pi e^{i}.$$ **[3 marks]**

Question 5. [8 marks]

Evaluate the integral $I=\displaystyle\int_0^{+\infty}\frac{1}{(x^2+1)(x^2+4)}dx.$

Solution. Suppose $f(z)=\dfrac{1}{(z^2+1)(z^2+4)}$, then the function $f(z)$ is analytic except $z_1=i, z_2=2i, z_3=-i, z_4=-2i$. Only $z_1=i, z_2=2i$ lie in the upper half plane. **[2 marks]**

We construct a contour in the upper half plane with $z_1=i, z_2=2i$ inside: a real segment L_R from $-R$ to R and a half circle $C_R:|z|=R$ from R to $-R$.



According to Cauchy's residue theorem, we have the following equation:

$$\int_{L_R} f(z)dz+\int_{C_R} f(z)dz=2\pi i\left[\operatorname*{Res}_{z=i} f(z)+\operatorname*{Res}_{z=2i} f(z)\right].$$

Notice that $\displaystyle\int_{L_R} f(z)dz=\int_{-R}^{R} f(x)dx$, and

$$\operatorname*{Res}_{z=i} f(z)=\frac{1}{(z+i)(z^2+4)}\Bigg|_{z=i}=-\frac{i}{6},\ \operatorname*{Res}_{z=2i} f(z)=\frac{1}{(z+2i)(z^2+1)}\Bigg|_{z=2i}=\frac{i}{12}.$$ **[2 marks]**

Then

$$\int_{-R}^{R} f(x)dx=2\pi i\left(\frac{i}{12}-\frac{i}{6}\right)-\int_{C_R} f(z)dz=\frac{\pi}{6}-\int_{C_R} f(z)dz.$$ **[2 marks]**

Next we claim that $\displaystyle\int_{C_R} f(z)dz$ tends to zero as R approaches the positive infinity. On C_R, $|z|=R$,

$$|f(z)|\le\frac{1}{(R^2-1)(R^2-4)}.$$

Then

$$\left|\int_{C_R} f(z)\mathrm{d}z\right| \le \frac{\pi R}{(R^2-1)(R^2-4)}, \qquad \text{[1 marks]}$$

which implies our claim.

Since $\int_{-R}^{R} f(x)\mathrm{d}x$ is an even function, we have

$$I=\int_0^{+\infty}\frac{1}{(x^2+1)(x^2+4)}\mathrm{d}x=\frac{\pi}{12}. \qquad \text{[1 marks]}$$

Question 6. [8 marks]

Determine the type of the linear partial differential equation $y^2u_{xx}+4xyu_{xy}+4x^2u_{yy}=x^2y$ and reduce it to the normal type.

Solution. The discriminate $\Delta=16x^2y^2-16x^2y^2=0.$ The equation is parabolic type, and the character equation is

$$y^2\mathrm{d}y^2-4xy\mathrm{d}x\mathrm{d}y+4x^2\mathrm{d}x^2=0 \qquad (xy\ne 0), \qquad \text{[4 marks]}$$

and its solution is $x^2-\frac{1}{2}y^2=C$.

Then take the transformation $\begin{cases}\xi=x^2-\frac{1}{2}y^2,\\ \eta=y,\end{cases}$ $J=\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}=2x\ne 0.$ **[2 marks]**

Therefore,

$$u_x=2xu_\xi,\ \ u_y=-yu_\xi+u_\eta,$$
$$u_{xx}=2u_\xi+4x^2u_{\xi\xi},\ \ u_{xy}=-2xyu_{\xi\xi}+2xu_{\xi\eta},$$
$$u_{yy}=-u_\xi+y^2u_{\xi\xi}-2yu_{\xi\eta}+u_{\eta\eta}.$$

Substituting them into the original equation, we have

$$(2y^2-4x^2)u_\xi+4x^2u_{\eta\eta}=x^2y.$$

The standard form is

$$u_{\eta\eta}=\frac{\eta}{4}+\frac{2\xi}{2\xi+\eta^2}u_\xi. \qquad \text{[2 marks]}$$

Note: we also can take the other transformations, and obtain the standard form as follows:

(i) $\begin{cases}\xi=\frac{1}{2}y^2-x^2,\\ \eta=y,\end{cases}$ $u_{\eta\eta}=\frac{\eta}{4}+\frac{2\xi}{\eta^2-2\xi}u_\xi;$

(ii) $\begin{cases}\xi=y^2-2x^2,\\ \eta=y,\end{cases}$ $u_{\eta\eta}=\frac{\eta}{4}+\frac{2\xi}{\eta^2-\xi}u_\xi;$

(iii) $\begin{cases}\xi=2x^2-y^2,\\ \eta=y,\end{cases}$ $u_{\eta\eta}=\frac{\eta}{4}+\frac{2\xi}{\eta^2+\xi}u_\xi.$

Question 7. [12 marks]

Solve the following problem by means of separation of variables:

$$\begin{cases} u_t = 4u_{xx}, & 0 < x < \pi, t > 0, \\ u_x(0,t) = u_x(\pi,t) = 0, & t \geq 0, \\ u(x,0) = \cos 2x - 3\cos x, & 0 \leq x \leq \pi. \end{cases}$$

Solution. Let $u(x,t) = X(x)T(t)$ and substitute it into the equation, we have

$$T'(t)X(x) = 4X''(x)T(t).$$ **[2 marks]**

Dividing it by $4X(x)T(t)$, we have $\dfrac{T'(t)}{4T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$, that is,

$$X''(x) + \lambda X(x) = 0,$$
$$T'(t) + 4\lambda T(t) = 0.$$

And the boundary conditions become $X'(0) = X'(\pi) = 0$. **[3 marks]**

Solving the eigenvalue problem $\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = X'(\pi) = 0 \end{cases}$, we obtain the eigenvalues $\lambda_n = n^2$ and eigenfunctions $X_n(x) = \cos nx, \quad n = 0, 1, 2, \cdots$ **[2 marks]**

Solving the other problem about $T(t)$, we obtain

$$T_n(t) = a_n e^{-4n^2 t}, \quad n = 0, 1, 2, \cdots.$$

So $u_n(x,t) = a_n e^{-4n^2 t} \cos nx, \quad n = 0, 1, 2, \cdots$ **[2 marks]**

Assume $u(x,t) = \sum_{n=0}^{\infty} a_n e^{-4n^2 t} \cos nx$. According to the initial condition, we have

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cos nx = \cos 2x - 3\cos x.$$

We have $a_1 = -3, a_2 = 1, a_0 = a_3 = a_4 = \cdots = 0$. Hence the solution is

$$u(x,t) = -3e^{-4t} \cos x + e^{-16t} \cos 2x.$$ **[3 marks]**

Question 8. [8 marks]

Use the Laplace transformation to solve the ordinary differential equation

$$\begin{cases} x'''(t) + 2x''(t) - x'(t) - 2x(t) = 1, \\ x(0) = x'(0) = x''(0) = 0. \end{cases}$$

Solution. Let $L(p) = L[x(t)]$. Taking Laplace transform on the ODE, we have

$$p^3 L(p) + 2p^2 L(p) - pL(p) - 2L(p) = \frac{1}{p}.$$ **[4 marks]**

So $L(p) = \dfrac{1}{p(p+2)(p^2-1)} = \dfrac{1}{6(p-1)} - \dfrac{1}{2p} + \dfrac{1}{2(p+1)} - \dfrac{1}{6(p+2)}$. **[2 marks]**

Since $L(e^{at}) = \dfrac{1}{p-a}$, thus

$$x(t) = \frac{1}{6}e^t - \frac{1}{2} + \frac{1}{2}e^{-t} - \frac{1}{6}e^{-2t}.$$ **[2 marks]**