Ch 3.1 Introduction

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Outline

- Introduction to Fourier Representations of Signals & LTI Systems
  - Clue of this chapter
  - Complex sinusoids and Frequency Response of LTI System
  - Fourier Representations for Four Classes of Signals

Clue of this chapter

- In chapter 2, by representing signals as linear combinations of **shifted impulses**, we analyzed LTI systems through the **convolution sum (integral).**
- In this chapter, we explore an alternative representation for signals and LTI systems.
- We will represent signals as linear combinations of a set of basic signals---**complex exponentials**. The resulting representations are known as the **continuous-time and discrete-time Fourier series and transform**
  - which convert time-domain signals into **frequency**-domain (or ***spectral***) representations.

Complex Sinusoids and Frequency Response of LTI Systems

- A discrete-time LTI system with impulse response ***h*[*n*]**
  - **Input:** $x[n] = e^{j\Omega n} = \cos \Omega n + j \sin \Omega n$
  - **Output:** $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\Omega(n-k)}$

$$= e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k} = \underline{H\left(e^{j\Omega}\right)}e^{j\Omega n}$$

Complex scaling factor

$$H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}$$

~ a function of frequency Ω.

$$e^{j\Omega n} \longrightarrow \boxed{h[n]} \longrightarrow H(e^{j\Omega})e^{j\Omega n}$$

  - **Frequency response $H(e^{j\Omega})$: the response of an *LTI* system to a sinusoidal input**

Complex Sinusoids and Frequency Response of LTI Systems

- A continuous-time LTI system with impulse response ***h(t)***
  - **Input:** $x(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t$
  - **Output:** $y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau = e^{j\omega t}\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau = \underline{H(j\omega)} e^{j\omega t}$
  - **Frequency response :**

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

~ a function of frequency ω.

★ **Polar form for $H(j\omega)$:** $H(j\omega) = A(\omega) + jB(\omega) = |H(j\omega)| e^{j\arg\{H(j\omega)\}}$

$$|H(j\omega)| = \sqrt{A^2(\omega) + B^2(\omega)}$$

~ Magnitude response

$$\arg\{H(j\omega)\} = \arctan\frac{B(\omega)}{A(\omega)}$$

~ Phase response

$$\Rightarrow \quad y(t) = |H(j\omega)| e^{j(\omega t + \arg\{H(j\omega)\})}$$

Complex Sinusoids and Frequency Response of LTI Systems

Example 3.1 ***RC Circuit: Frequency response***

The impulse response of the system relating to the input voltage to the voltage across the capacitor in Fig. 3.2 is derived in Example 1.21 as

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

Find an expression for the frequency response, and plot the magnitude and phase response.

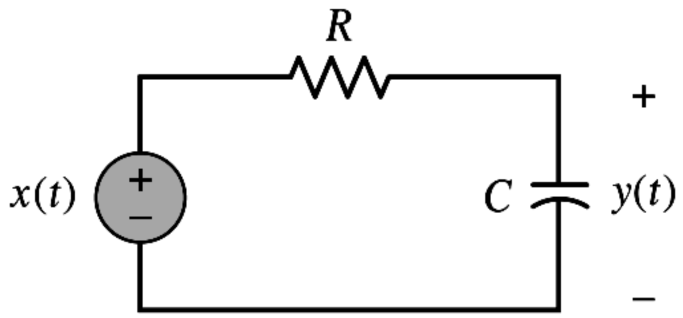


Figure 3.2 (p. 197)
RC circuit for Example 3.1.

<Sol.> Frequency response:

$$H(j\omega) = \frac{1}{RC} \int_{-\infty}^{\infty} e^{-\frac{\tau}{RC}} u(\tau) e^{-j\omega\tau} d\tau$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-\left(j\omega + \frac{1}{RC}\right)\tau} d\tau$$

$$= \frac{1}{RC} \frac{-1}{\left(j\omega + \frac{1}{RC}\right)} e^{-\left(j\omega + \frac{1}{RC}\right)\tau} \Bigg|_0^{\infty} = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$$

Complex Sinusoids and Frequency Response of LTI Systems

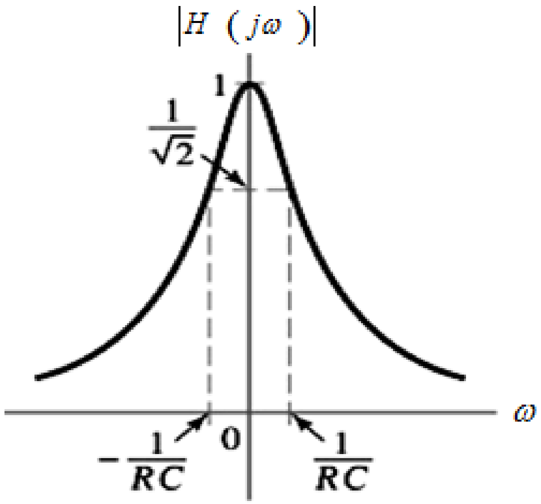
$$H(j\omega)=\frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}$$

- **Magnitude response :**

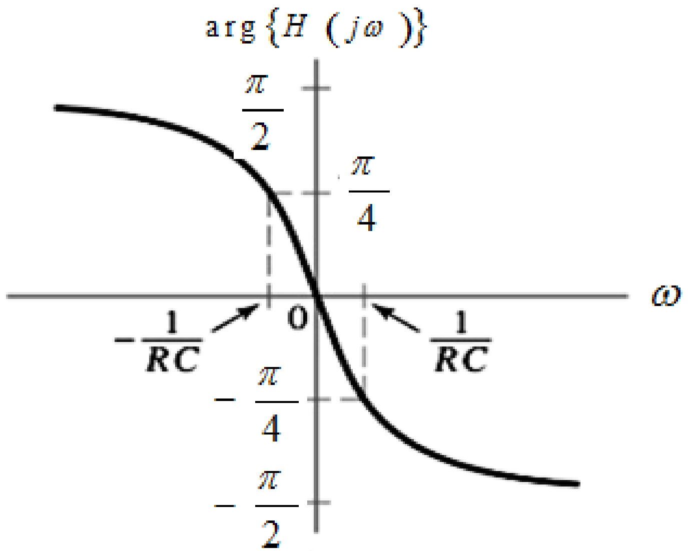
$$|H(j\omega)|=\frac{1}{RC}\Big/\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}$$

- **Phase response :**

$$\arg\{H(j\omega)\}=-\arctan(\omega RC)$$



(a)



(b)

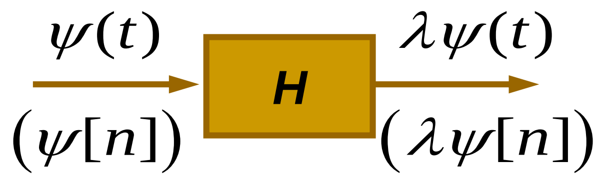
Eigenvalue and eigenfunction of LTI system

- **Matrix eigenproblem:** If $\mathbf{e}_\mathrm{k}$ is an eigenvector of a matrix $\mathbf{A}$ with eigenvalue $\boldsymbol{\lambda_k}$, then

$$\mathbf{A}\mathbf{e}_k = \lambda_k \mathbf{e}_k$$

- **Eigenrepresentation for LTI system:**

$$H\{\psi(t)\} = \lambda\psi(t)$$

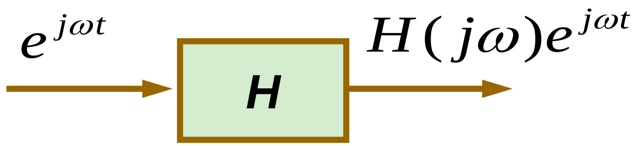


Eigenfunction: $\psi(t)$

Eigenvalue: λ

Eigenvalue and eigenfunction of LTI system

- **Continuous-time case:**



Eigenfunction: $\psi(t) = e^{j\omega t}$

Eigenvalue: $\lambda = H(j\omega)$

- **For Arbitrary input =** **weighted superpositions of eigenfunctions**

Ex. Input: $x(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t}$

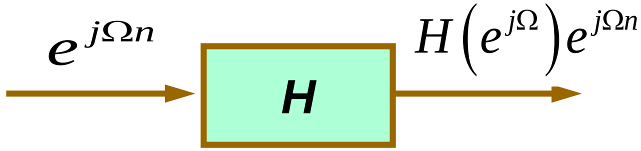
Output: $y(t) = x(t) * h(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t} * h(t) = \sum_{k=1}^{M} a_k H(j\omega_k) e^{j\omega_k t}$

$$\text{where} \quad H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau.$$

By representing arbitrary signals as weighted superpositions of eigenfunctions, we **transform the operation of convolution to multiplication.**

Eigenvalue and eigenfunction of LTI system

- **Discrete-time case:**



Eigenfunction: $\psi[n] = e^{j\Omega n}$

Eigenvalue: $\lambda = H(e^{j\Omega})$

Ex. Input: $x[n] = \sum_{k=1}^{M} a_k e^{j\Omega_k n}$ ⟹ **Output:** $y[n] = \sum_{k=1}^{M} a_k H\left(e^{j\Omega_k}\right) e^{j\Omega_k n}$

- **Summary**
  - The response of an LTI system to a complex sinusoidal input lead to a characterization of system behavior that is termed the frequency response of the LTI system.
  - Rather than describing system's behavior as a function of time, frequency response describe it as a function of frequency.

How to represent signals as weighted superpositions of complex *sinusoidals?*

Fourier Representations for Four classes of Signals

Table 3.1 Relationship between Time Properties of a Signal and the Approximate Fourier Representation

Time Property	Periodic	Nonperiodic
Continuous (t)	Fourier Series (FS)	Fourier Transform (FT)
Discrete [n]	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)

- **FS *for* $x(t)$ = continuous-time signal with fundamental period T.**

$$\hat{x}(t) = \sum_{k} A[k] e^{jk\omega_0 t},$$

where "^" denotes approximate value.

$\omega_0 = 2\pi/T$: Fundamental frequency of $x(t)$

$e^{jk\omega_0 t}$: the k-th harmonic of $e^{j\omega_0 t}$.

$A[k]$: the weight applied to the k^{th} harmonic.

Periodic Signals: Fourier Series Representations

- **DTFS for $x[n]$ = discrete-time signal with fundamental period N.**

$$\hat{x}[n] = \sum_{k} A[k] e^{jk\Omega_0 n},$$ **$\Omega_o = 2\pi/N$: Fundamental frequency of $x[n]$**

- $\exp(jk\Omega_o n)$ are N-periodics in the frequency index k.

$$e^{j(N+k)\Omega_0 n} = e^{jN\Omega_0 n} e^{jk\Omega_0 n} = e^{j2\pi n} e^{jk\Omega_0 n} = e^{jk\Omega_0 n}$$

There are only *N* distinct complex sinusoids of the form $\exp(jk\Omega_0 n)$ should be used in above equation.

$$\hat{x}[n] = \sum_{k} A[k] e^{jk\Omega_0 n} = \sum_{k=0}^{N-1} A[k] e^{jk\Omega_0 n}$$

while $$\hat{x}(t) = \sum_{k} A[k] e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} A[k] e^{jk\omega_0 t}$$

Periodic Signals: Fourier Series Representations

- **Mean-square error (MSE) between the signal and its series representation**
  - Continuous-time case:

$$MSE = \frac{1}{T}\int_0^T \left| x(t) - \hat{x}(t) \right|^2 dt$$

  - Discrete-time case:

$$MSE = \frac{1}{N}\sum_{n=0}^{N-1} \left| x[n] - \hat{x}[n] \right|^2 dt$$

The optimum weights or coefficients $A[k]$ are obtained by minimizing the MSE between the signal and its series representation.

Nonperiodic Signals: Fourier-Transform Representations

- **FT of continuous-time signal:**

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$X(j\omega)d\omega/(2\pi)$: the weight applied to the sinusoid $e^{j\omega t}$

- **DTFT of discrete-time signal:**

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega$$

$X(e^{j\Omega})d\Omega/(2\pi)$: the weight applied to the sinusoid $e^{j\Omega n}$

- $\exp(j\Omega n)$ are periodical with period 2π.

$$e^{j(\Omega+2\pi)n} = e^{j\Omega n}e^{j2\pi n} = e^{j\Omega n}$$

Nonperiodic Signals: Fourier-Transform Representations

► **Problem 3.1** Identify the appropriate Fourier representation for each of the following signals:

(a) $x[n] = (1/2)^n u[n]$
(b) $x(t) = 1 - \cos(2\pi t) + \sin(3\pi t)$
(c) $x(t) = e^{-t}\cos(2\pi t)u(t)$
(d) $x[n] = \sum_{m=-\infty}^{\infty} \delta[n - 20m] - 2\delta[n - 2 - 20m]$

Answers:

(a) DTFT
(b) FS
(c) FT
(d) DTFS