Lecture 12

Chapter 5 Linear Transformation

5.1 Definition and Examples

5.2 Image and Kernel

5.1 Definition and Examples

Linear mappings from one vector space to another play an important role in mathematics.

Linear Transformations

Definition 1. Let V and W be two vector space, and $L: V \to W$ be a mapping from V to W. If

$$L(\alpha \boldsymbol{v_1} + \beta \boldsymbol{v_2}) = \alpha L(\boldsymbol{v_1}) + \beta L(\boldsymbol{v_2})$$

holds for all $\boldsymbol{v_1}, \boldsymbol{v_2} \in V$, where α, β are scalars, we say that L is a **linear transformation [线性变换].**

In case that $V = W$, L is also called **linear operator [线性算子]** on V.

Theorem 1. Let $L: V \to W$ be a mapping. Then L is a linear transformation if and only if it satisfies

$$L(\boldsymbol{v_1} + \boldsymbol{v_2}) = L(\boldsymbol{v}_1) + L(\boldsymbol{v_2}) \tag{1}$$

$$L(\alpha \boldsymbol{v}) = \alpha L(\boldsymbol{v}), \tag{2}$$

for any $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v} \in V$ and α scalar.

Proof. The necessary part is clear. Assume that L satisfies (1) and (2).

For any $\boldsymbol{v_1}, \boldsymbol{v_2} \in V$, α, β scalars, $\alpha\boldsymbol{v_1}$ and $\beta\boldsymbol{v_2}$ are both vectors in V.

By (1), we have $\quad L(\alpha\boldsymbol{v_1} + \beta\boldsymbol{v_2}) = L(\alpha\boldsymbol{v_1}) + L(\beta\boldsymbol{v_2}).$

By (2), we have $\quad L(\alpha\boldsymbol{v_1}) = \alpha L(\boldsymbol{v_1}), \qquad L(\beta\boldsymbol{v_2}) = \beta L(\boldsymbol{v_2})$

showing that $\quad L(\alpha\boldsymbol{v_1} + \beta\boldsymbol{v_2}) = \alpha L(\boldsymbol{v_1}) + \beta L(\boldsymbol{v_2}).$

In other words, L is a linear transformation.

Linear Operators on $\mathbf{R}^2$

Example 1. Let $L: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$L(\boldsymbol{x}) = x_1\mathbf{e_1},$$

for any $\boldsymbol{x} = (x_1, x_2)^T \in \mathbf{R}^2$.

- $\boldsymbol{x} = (x_1, x_2)^T$, $\boldsymbol{y} = (y_1, y_2)^T \in \mathbf{R}^2$, α, β scalars,

$$\alpha\boldsymbol{x} + \beta\boldsymbol{y} = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)^T,$$

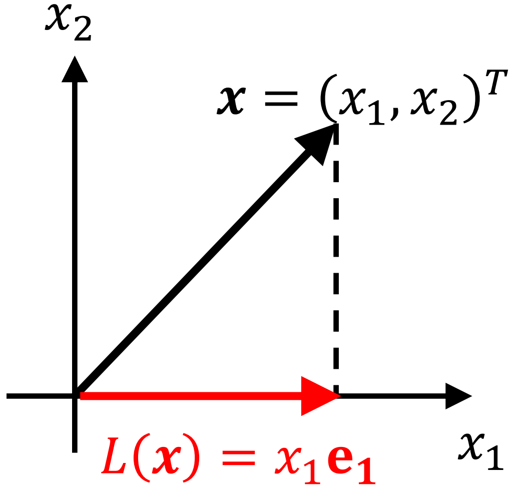
- $L(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = (\alpha x_1 + \beta y_1)\mathbf{e_1}$,

- $\alpha L(\boldsymbol{x}) + \beta L(\boldsymbol{y}) = \alpha(x_1\mathbf{e_1}) + \beta(y_1\mathbf{e_1}) = (\alpha x_1 + \beta y_1)\mathbf{e_1}$,

$$\Longrightarrow \qquad L(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \alpha L(\boldsymbol{x}) + \beta L(\boldsymbol{y}).$$

L is a linear transformation on $\mathbf{R}^2$.

We can think of this linear transformation as a **projection [投影]** onto x_1-axis.



$$L(\boldsymbol{x}) = x_1\boldsymbol{e}_{\mathbf{1}}$$

Example 2. Let $L: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

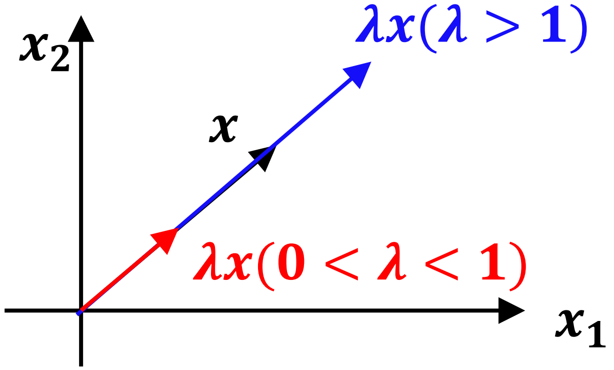
$$L(\boldsymbol{x}) = \lambda \boldsymbol{x},$$

where λ is a real number.

- $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^2$, α, β scalars,

$$L(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \lambda(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \alpha\lambda\boldsymbol{x} + \beta\lambda\boldsymbol{y} = \alpha L(\boldsymbol{x}) + \beta L(\boldsymbol{y}),$$

$\Longrightarrow$ L is a linear transformation (also a linear operator).



This linear transformation can be thought of as a **stretching** or **shrinking** by a factor λ.

Example 3. Let the mapping $L: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$L(\boldsymbol{x}) = (x_1, -x_2)^T,$$

for each $\boldsymbol{x} = (x_1, x_2)^T \in \mathbf{R}^2$.

- $\boldsymbol{x} = (x_1, x_2)^T$, $\boldsymbol{y} = (y_1, y_2)^T \in \mathbf{R}^2$, α, β scalars,

$$\alpha\boldsymbol{x} + \beta\boldsymbol{y} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{pmatrix},$$

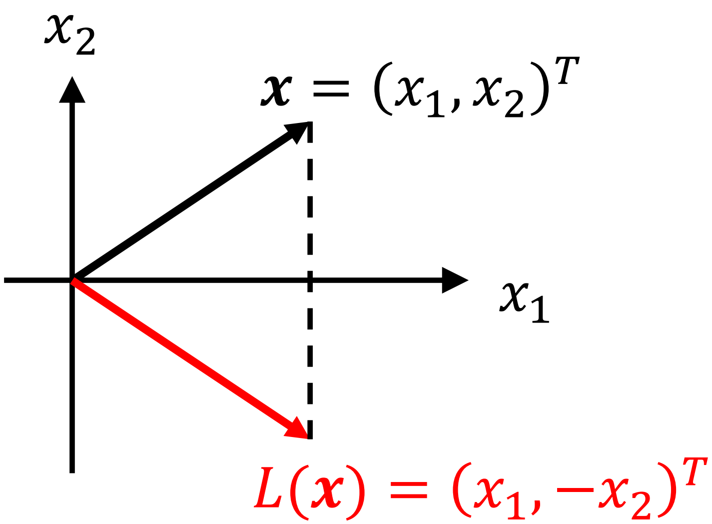
$$L(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ -(\alpha x_2 + \beta y_2) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x_1 \\ -\alpha x_2 \end{pmatrix} + \begin{pmatrix} \beta y_1 \\ -\beta y_2 \end{pmatrix}$$

$$= \alpha L(\boldsymbol{x}) + \beta L(\boldsymbol{y})$$

Thus L is a linear transformation on $\mathbf{R}^2$.

The linear transformation L has the effect of **reflecting vectors** about the x_1-axis.



Linear Transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$

Example 4. Let $L_A: \mathbf{R}^n \to \mathbf{R}^m$ be a mapping defined by

$$L_A(\boldsymbol{x}) = A\boldsymbol{x},$$

where A is an $m \times n$ matrix. Show that L_A is a linear transformation.

Proof. $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbf{R}^n$, α a real number,

$$L_A(\boldsymbol{x} + \boldsymbol{y}) = A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = L_A(\boldsymbol{x}) + L_A(\boldsymbol{y})$$

$$L_A(\alpha\boldsymbol{x}) = A(\alpha\boldsymbol{x}) = \alpha(A\boldsymbol{x}) = \alpha L_A(\boldsymbol{x}).$$

By **Theorem 1**, the mapping L_A is a linear transformation.

Exercise. Let $A = (1,1)$, find L_A.

Example 5. Let $L: \mathbf{R}^n \to \mathbf{R}$ be the mapping defined by

$$L(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{x} \rangle.$$

Determine if L is a linear transformation.

Solution. Since $L(2\boldsymbol{x}) = 2^2 L(\boldsymbol{x})$, L is **NOT** a linear transformation.

Linear Transformations from V to W

If L is a linear transformation from a vector space V into a vector space W, then

(1) $L(\mathbf{0}_V) = \mathbf{0}_W$, where $\mathbf{0}_V$, $\mathbf{0}_W$ are the zero vectors in V, W.

(2) $L(\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \cdots + \alpha_n\boldsymbol{v}_n) = \alpha_1 L(\boldsymbol{v}_1) + \alpha_2 L(\boldsymbol{v}_2) + \cdots + \alpha_n L(\boldsymbol{v}_n)$;

(3) $L(-\boldsymbol{v}) = -L(\boldsymbol{v})$, for all $\boldsymbol{v} \in V$.

Example 6. If V is any vector space, then the **identity operator** I is

defined by $\qquad I(\boldsymbol{v}) = \boldsymbol{v} \qquad$ for all $\boldsymbol{v} \in V$.

Clearly, I is a linear transformation (operator) from V to itself.

Example 7. Let $D\colon C^{(1)}(a,b) \to C(a,b)$ be the derivative of a continuous derivable function defined in the interval (a,b).

- $f, g \in C^{(1)}(a,b)$, α, β scalars,

$$D(\alpha f + \beta g) = \frac{\mathrm{d}}{\mathrm{d}x}(\alpha f + \beta g) = \alpha \frac{\mathrm{d}f}{\mathrm{d}x} + \beta \frac{\mathrm{d}g}{\mathrm{d}x}$$

$$= \alpha D(f) + \beta D(g).$$

Then D is a linear transformation.

Example 8. Let $I: C[a, b] \to R$ be the definite integral of a continuous function defined on the interval $[a, b]$.

- $f, g \in C[a, b]$, α, β scalars,

$$I(\alpha f + \beta g) = \int_a^b (\alpha f + \beta g)\mathrm{d}x = \alpha \int_a^b f\,\mathrm{d}x + \beta \int_a^b g\,\mathrm{d}x$$

$$= \alpha I(f) + \beta I(g).$$

So I is a linear transformation.

5.2 Image and Kernel

Let $L: V \to W$ be a linear transformation. Consider the effect that L has on subspaces of V.

Definition 1. Let $L: V \rightarrow W$ be a linear transformation. The vector set

$$\ker(L) = \{\boldsymbol{v} \in V | L(\boldsymbol{v}) = \mathbf{0}_W\}$$

is called **kernel [核]** of L, where $\mathbf{0}_W$ is the zero vector in the vector space W.

Definition 2. Let $L: V \rightarrow W$ be a linear transformation and let S be a subspace of V. The set $L(S)$ defined by

$$L(S) = \{\boldsymbol{w} \in W | \boldsymbol{w} = L(\boldsymbol{v}), \boldsymbol{v} \in S\}$$

is called **image [像]** of S. The image of the entire space, $L(V)$, is called the **range [值域]** of L.

Theorem 1. If $L: V \to W$ is a linear transformation and S is a subspace of V, then

(1) $\ker(L)$ is a subspace of V;

(2) $L(S)$ is a subspace of W.

Example 1. Let $L: \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation defined by

$$L(\boldsymbol{x}) = x_1\mathbf{e}_1.$$

Find the kernel and the range of L.

Solution.

Find $\ker(L)$**.** A vector $\boldsymbol{x} = (x_1, x_2)^T$ is in $\ker(L)$ if and only if $x_1 = 0$. Therefore, the kernel of L is

$$\ker(L) = \{\boldsymbol{x} | \boldsymbol{x} = (0, \alpha)^T, \alpha \in \mathbf{R}\},$$

which is a one-dimensional subspace of $\mathbf{R}^2$ spanned by $\mathbf{e}_2$.

Find $L(\mathbf{R}^2)$. A vector $\boldsymbol{y}$ is in the range of L if and only if y is a multiple of $\mathbf{e}_1$. Therefore,

$$L(\mathbf{R}^2) = \mathrm{Span}\{\mathbf{e}_1\}.$$

Example 2. Let $L: \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation defined by

$$L(\boldsymbol{x}) = \lambda \boldsymbol{x},$$

where $\lambda \neq 0, \lambda \in \mathbf{R}$. Find the kernel and the range of L.

Solution.

Find $\ker(L)$. Since $\lambda \neq 0$, the only vector satisfies $L(\boldsymbol{x}) = \lambda \boldsymbol{x} = \mathbf{0}$ is the zero vector, $\mathbf{0}$. Therefore,

$$\ker(L) = \{\mathbf{0}\}.$$

Find $L(\mathbf{R}^2)$. Since $\lambda \neq 0$, then

$$L\left(\mathbf{R}^2\right) = \mathbf{R}^2.$$

Example 3. Let $D: P_{n+1} \to P_n$ be the transformation of differentiation. Find its kernel and range.

Solution. Find $\ker(D)$. Let $p(x) \in P_m$, where $x \in \mathbf{R}$, then

$$D\big(p(x)\big) = p'(x), \qquad x \in \mathbf{R}.$$

Consider the differential equation

$$p'(x) = 0, \qquad x \in \mathbf{R}.$$

The solution is $p(x) = C$, where C is an arbitrary constant.

Therefore

$$\ker(D) = \{p(x) | p(x) = C, x \in \mathbf{R} \text{ and } C \text{ is constant}\}.$$

Recall: P_n is the vector space of all polynomials of degree less than n.

Example 3. Let $D: P_{n+1} \to P_n$ be the transformation of differentiation. Find its kernel and range.

Solution. Find $D(P_n)$.

By rules of differentiation, if $p(x)$ is a polynomial of degree n, then $p'(x)$ is a polynomial of degree $n - 1$, at most (for $n \geq 1$). Therefore, $D(P_{n+1}) = P_n$ for $n \geq 1$.

P_1 is the set of all polynomials of constant value, and therefore $p'(x) = 0$ for $p \in P_1$. We then have $D(P_1) = \{\mathbf{0}\}$.

Example 4. Let $I: C[a,b] \to \mathbf{R}$ be the linear transformation of definite integral defined on the interval $[a,b]$ where $a < b$. Find its kernel and image.

Solution. Find $\ker(I)$.

$$\ker(I) = \left\{ f \in C[a,b] \,\middle|\, \int_a^b f(x)\,dx = 0 \right\}.$$

Find $I(C[a,b])$. Notice that $f(x) = C$, where C is a constant, is one of the functions in $C[a,b]$ and it is easy to calculate that

$$\int_a^b f(x)\,dx = C\int_a^b dx = C(b-a).$$

Since $b - a$ is a nonzero constant and $C \in \mathbf{R}$, we have

$$I(C[a,b]) = \mathbf{R}.$$

Review

- Definition of Linear Transformations
- Kernel and Image

Preview

- Matrix Representation of Linear Transformations
- Similar Matrices