Lecture 13

Chapter 5 Linear Transformation

5.3 Matrix Representation

5.4 Similar Matrices

5.3 Matrix Representation of Linear Transformations

In **Section 5.1**, we have known that each $m \times n$ matrix A defines a linear transformation L_A from $\mathbf{R}^n$ to $\mathbf{R}^m$, where

$$L_A(\boldsymbol{x}) = A\boldsymbol{x}, \qquad \boldsymbol{x} \in \mathbf{R}^n.$$

Goal of this section:

- We will see that for each linear transformation L mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$, there is an $m \times n$ matrix A such that

$$L(\boldsymbol{x}) = A\boldsymbol{x}.$$

- We will also show that any linear transformation between **finite-dimensional** vector spaces can be represented by a matrix.

Linear Transformation $L: \mathbf{R}^n \to \mathbf{R}^m$

Theorem. If L is a linear transformation mapping $\mathbf{R}^n$ into $\mathbf{R}^m$, there is an $m \times n$ matrix A such that

$$L(\boldsymbol{x}) = A\boldsymbol{x}$$

for each $\boldsymbol{x} \in \mathbf{R}^n$. In fact, the jth column vector of A is given by $\boldsymbol{a}_j = L(\mathbf{e}_\mathrm{j}), j = 1,2, \dots, n$.

Proof. For $j = 1,2, \dots, n$, define $\boldsymbol{a}_j = L(\mathbf{e}_\mathrm{j})$ and let

$$A = (a_{ij}) = (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n).$$

If $\boldsymbol{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_\mathbf{n}$ is an arbitrary element of $\mathbf{R}^n$, then

$$\begin{aligned} L(\boldsymbol{x}) &= L(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_\mathbf{n}) \\ &= x_1L(\mathbf{e}_1) + x_2L(\mathbf{e}_2) + \cdots + x_nL(\mathbf{e}_\mathbf{n}) \\ &= x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A\boldsymbol{x}. \end{aligned}$$

A: the **standard matrix representation [标准矩阵表示]** of L.

Example. Define the linear transformation $L: \mathbf{R}^3 \to \mathbf{R}^2$ by

$$L(\boldsymbol{x}) = (x_1 + x_2, x_2 + x_3)^T,$$

for each $\boldsymbol{x} = (x_1, x_2, x_3)^T \in \mathbf{R}^3$. Compute the standard matrix representation of L.

Solution. We compute $L(\mathbf{e_1}), L(\mathbf{e_2}), L(\mathbf{e_3})$:

$$L(\mathbf{e_1}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad L(\mathbf{e_2}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad L(\mathbf{e_3}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we choose these vectors to be the columns of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

To check the result, we compute $A\boldsymbol{x}$:

$$A\boldsymbol{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix} = L(\boldsymbol{x}).$$

Linear Transformation $L: V \to W$

- Let $E = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ be a basis of vector space V;
  Let $F = \{\mathbf{w_1}, \mathbf{w_2}, \dots, \mathbf{w_m}\}$ be a basis of vector space W.

- For any vector $\boldsymbol{x} \in V$, we have

$$\boldsymbol{x} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_n\mathbf{v_n} = (\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n})\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= (\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n})[\boldsymbol{x}]_E,$$

$[\boldsymbol{x}]_E \in \mathbf{R}^n$: coordinate vector of $\boldsymbol{x}$ w.r.t. basis E

and

$$L(\boldsymbol{x}) = L(x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_n\mathbf{v_n})$$
$$= x_1L(\mathbf{v_1}) + x_2L(\mathbf{v_2}) + \cdots + x_nL(\mathbf{v_n}). \quad \textbf{(1)}$$

- $L(\mathbf{v_j}), j = 1,2, \dots, n$ are vectors in vector space W,

$$L(\mathbf{v_j}) = a_{1j}\mathbf{w_1} + a_{2j}\mathbf{w_2} + \cdots + a_{mj}\mathbf{w_m} = (\mathbf{w_1}, \mathbf{w_2}, \dots, \mathbf{w_m}) \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

$$= (\mathbf{w_1}, \mathbf{w_2}, \dots, \mathbf{w_m}) \left[L(\mathbf{v_j})\right]_F \quad \textbf{(2)}$$

where a_{ij} are scalars, $i = 1,2, \dots, m, j = 1,2, \dots, n$.

$\left[L(\mathbf{v_j})\right]_F \in \mathbf{R}^m$: coordinate vector of $L(\mathbf{v_j}) \in \boldsymbol{W}$ w.r.t. basis F

- Substituting **(2)** into **(1),** we obtain

$$L(\boldsymbol{x}) = x_1(\mathbf{w_1}, \dots, \mathbf{w_m})[L(\mathbf{v_1})]_F + \cdots + x_n(\mathbf{w_1}, \dots, \mathbf{w_m})[L(\mathbf{v_n})]_F.$$

$$L(\boldsymbol{x}) = x_1 L(\mathbf{v}_1) + x_2 L(\mathbf{v_2}) + \cdots + x_n L(\mathbf{v_n}). \qquad \textbf{(1)}$$

- By using operation rules of partition matrices, we can rewrite

$$L(\boldsymbol{x}) = \left((\mathbf{w_1}, \dots, \mathbf{w_m})[L(\mathbf{v_1})]_F, \dots, (\mathbf{w_1}, \dots, \mathbf{w_m})[L(\mathbf{v_n})]_F\right)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= (\mathbf{w_1}, \dots, \mathbf{w_m})([L(\mathbf{v_1})]_F, \dots, [L(\mathbf{v_n})]_F)[\boldsymbol{x}]_E. \qquad \textbf{(3)}$$

- Meanwhile, $L(\boldsymbol{x})$ is a vector in the vector space W, so we can write

$$L(\boldsymbol{x}) = (\mathbf{w_1}, \dots, \mathbf{w_m})[L(\boldsymbol{x})]_F. \qquad \textbf{(4)}$$

- Comparing **(3)** and **(4),** we obtain the following conclusion

$$[L(\boldsymbol{x})]_F = ([L(\mathbf{v_1})]_F, \dots, [L(\mathbf{v_n})]_F)[\boldsymbol{x}]_E. \qquad \textbf{(5)}$$

$$L(\boldsymbol{x}) = x_1(\mathbf{w_1}, \dots, \mathbf{w_m})[L(\mathbf{v_1})]_F + \cdots + x_n(\mathbf{w_1}, \dots, \mathbf{w_m})[L(\mathbf{v_n})]_F.$$

$$[L(\boldsymbol{x})]_F = ([L(\mathbf{v_1})]_F, \dots, [L(\mathbf{v_n})]_F)[\boldsymbol{x}]_E. \qquad \textbf{(5)}$$

- If we take

$$A = ([L(\mathbf{v_1})]_F, \dots, [L(\mathbf{v_n})]_F) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

then **(5)** can also be written as

$$[L(\boldsymbol{x})]_F = A[\boldsymbol{x}]_E. \qquad \textbf{(6)}$$

The matrix A is called the **matrix representing L relative to bases E and F [线性变换L关于基E和F的表示矩阵].**

Matrix Representation Theorem

Theorem 1. Let $L: V \to W$ be a linear transformation, $E = \{\mathbf{v_1}, \dots, \mathbf{v_n}\}$ and $F = \{\mathbf{w_1}, \dots, \mathbf{w_m}\}$ be bases of V and W, respectively. There exists an $m \times n$ matrix A such that

$$[L(\boldsymbol{x})]_F = A[\boldsymbol{x}]_E,$$

where $[\cdot]_E, [\cdot]_F$ are coordinate vectors relative to basis E and basis F, and

$$\boldsymbol{a_j} = \left[L\left(\mathbf{v_j}\right)\right]_F, \qquad j = 1, \dots, n,$$

is the jth column of the matrix A.

Matrix Representation Theorem

$$
\begin{tikzcd}
\boldsymbol{x} \in V \arrow[rr, "L = L_A"] \arrow[d, leftrightarrow] & & L(\boldsymbol{x}) \in W \arrow[d, leftrightarrow] \\
{[\boldsymbol{x}]_E \in \mathbf{R}^n} \arrow[rr, "A"] & & {[L(\boldsymbol{x})]_F \in \mathbf{R}^m}
\end{tikzcd}
$$

$$= A[\boldsymbol{x}]_E$$

Example 1. Let $L: \mathbf{R}^3 \to \mathbf{R}^3$ be a linear transformation defined by

$$L(\mathbf{e_1}) = \mathbf{e_2},\ L(\mathbf{e_2}) = \mathbf{e_3},\ L(\mathbf{e_3}) = \mathbf{e_1}.$$

Find the standard matrix representation of L.

Solution. Since $E = F = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, we can calculate

$$[L(\mathbf{e_1})]_F = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad [L(\mathbf{e_2})]_F = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad [L(\mathbf{e_3})]_F = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, the standard matrix representation of L is

$$A = ([L(\mathbf{e_1})]_F, [L(\mathbf{e_2})]_F, [L(\mathbf{e_3})]_F) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 2. Let $D: P_3 \to P_2$ be the linear transformation of differential. Find the standard matrix representation of D. Using the representation matrix to find $D(1 - x + 3x^2)$.

Solution. The standard basis of P_3 is $E = \{1, x, x^2\}$ and the standard basis of P_2 is $F = \{1, x\}$. It is easy to calculate that

$$D(1) = \frac{\mathrm{d}(1)}{\mathrm{d}x} = 0 \cdot 1 + 0 \cdot x = (1, x)\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$D(x) = \frac{\mathrm{d}(x)}{\mathrm{d}x} = 1 \cdot 1 + 0 \cdot x = (1, x)\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$D(x^2) = \frac{\mathrm{d}(x^2)}{\mathrm{d}x} = 0 \cdot 1 + 2 \cdot x = (1, x)\begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

then

$$[D(1)]_F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad [D(x)]_F = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [D(x^2)]_F = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Example 2. Let $D: P_3 \to P_2$ be the linear transformation of differential. Find the standard matrix representation of D. Using the representation matrix to find $D(1 - x + 3x^2)$.

Solution. (continue) The standard matrix representation is

$$A = \left([D(1)]_F, [D(x)]_F, \left[D\left(x^2\right)\right]_F\right) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The coordinate vector of $p(x) = 1 - x + 3x^2$ w.r.t. the standard basis E is

$$[p(x)]_E = (1, -1, 3)^T.$$

Therefore, we have

$$\left[D\left(p(x)\right)\right]_F = A[p(x)]_E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}.$$

The corresponding polynomial can be written as

$$D\left(p(x)\right) = (-1) \cdot 1 + 6 \cdot x = -1 + 6x.$$

5.4 Similar Matrices

Introduction

- If L is a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W, the matrix representation of L will depend on the **ordered bases** chosen for V and W.

- By using **different** bases, it is possible to represent L by **different** $m \times n$ matrices.

- In this section, we consider different matrix representations of linear transformations and characterize the relationship between matrices representing the same linear transformation.

Example 1. Let $L: \mathbf{R}^3 \to \mathbf{R}^2$ be a linear transformation defined by

$$L(\boldsymbol{x}) = (x_1 + x_2)\boldsymbol{\varepsilon}_1 + (x_2 - x_3)\boldsymbol{\varepsilon}_2.$$

Let $E_1 = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, $E_2 = \{\mathbf{v_1} = (1,1,1)^T, \mathbf{v_2} = (1,1,0)^T, \mathbf{v_3} = (1,0,0)^T\}$ be two bases of $\mathbf{R}^3$ and $F = \{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2\}$ be the standard basis of $\mathbf{R}^2$. Find the matrix A_1 representing L relative to bases E_1 and F, and matrix A_2 representing L relative to bases E_2 and F.

Solution. (Find A_1) By **Theorem 1**, if we take E_1 as the basis of $\mathbf{R}^3$ and F as the basis of $\mathbf{R^2}$, then we have

$$[L(\mathbf{e_1})]_F = (1,0)^T, \qquad [L(\mathbf{e_2})]_F = (1,1)^T, \qquad [L(\mathbf{e_3})]_F = (0,-1)^T.$$

Therefore,

$$A_1 = ([L(\mathbf{e_1})]_F, [L(\mathbf{e_2})]_F, [L(\mathbf{e_3})]_F) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Example 1. Let $L: \mathbf{R}^3 \to \mathbf{R}^2$ be a linear transformation defined by

$$L(\boldsymbol{x}) = (x_1 + x_2)\boldsymbol{\varepsilon}_1 + (x_2 - x_3)\boldsymbol{\varepsilon}_2.$$

Let $E_1 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $E_2 = \{\mathbf{v}_1 = (1,1,1)^T, \mathbf{v}_2 = (1,1,0)^T, \mathbf{v}_3 = (1,0,0)^T\}$ be two bases of $\mathbf{R}^3$ and $F = \{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2\}$ be the standard basis of $\mathbf{R}^2$. Find the matrix A_1 representing L relative to bases E_1 and F, and matrix A_2 representing L relative to bases E_2 and F.

Solution. (Find A_2) If we take E_2 as the basis of $\mathbf{R}^3$ and F as the basis of $\mathbf{R}^2$, then we have

$$L(\mathbf{v}_1) = 2\boldsymbol{\varepsilon}_1 + 0\boldsymbol{\varepsilon}_2, \quad L(\mathbf{v}_2) = 2\boldsymbol{\varepsilon}_1 + 1\boldsymbol{\varepsilon}_2, \quad L(\mathbf{v}_3) = \boldsymbol{\varepsilon}_1 + 0\boldsymbol{\varepsilon}_2.$$

Therefore,

$$A_2 = ([L(\mathbf{v}_1)]_F, [L(\mathbf{v}_2)]_F, [L(\mathbf{v}_3)]_F) = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In general, suppose that

- $L: V \to W$ is a linear transformation;
- $E_1 = \left\{\mathbf{v}_1^{(1)}, \dots, \mathbf{v}_n^{(1)}\right\}$ and $E_2 = \{\mathbf{v}_1^{(2)}, \dots, \mathbf{v}_n^{(2)}\}$ are two bases of V;
- $F = \{\mathbf{w_1}, \dots, \mathbf{w_m}\}$ is a basis of vector space W.

The matrix representations of linear transformation relative to bases E_1 and F, and relative to bases E_2 and F, are different.

By using the idea of changing of basis, we know

$$\left(\mathbf{v}_1^{(2)}, \dots, \mathbf{v}_n^{(2)}\right) = \left(\mathbf{v}_1^{(1)}, \dots, \mathbf{v}_n^{(1)}\right) S,$$

and

$$[\boldsymbol{x}]_{E_1} = S[\boldsymbol{x}]_{E_2}.$$

Meanwhile, since A_1 and A_2 are two matrices representing linear transformation L, then

$$[L(\boldsymbol{x})]_F = A_1[\boldsymbol{x}]_{E_1} = A_2[\boldsymbol{x}]_{E_2}$$

Therefore,

$$A_1[\boldsymbol{x}]_{E_1} = A_1 S[\boldsymbol{x}]_{E_2} = A_2[\boldsymbol{x}]_{E_2},$$

which implies that $$A_2 = A_1 S. \qquad \textbf{(1)}$$

As in **Example 1**, the transition matrix from E_2 to E_1 is

$$S = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By relation **(1)**, we know that A_2 can be calculated by

$$A_2 = A_1 S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 2. Let $L: \mathbf{R}^3 \to \mathbf{R}^2$ be a linear transformation defined by

$$L(\boldsymbol{x}) = (x_1 + x_2)\boldsymbol{\varepsilon_1} + (x_2 - x_3)\boldsymbol{\varepsilon_2}.$$

Let $E = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be the standard basis of $\mathbf{R}^3$ and $F_1 = \{\boldsymbol{\varepsilon_1}, \boldsymbol{\varepsilon_2}\}$, $F_2 = \{\mathbf{u_1} = (1,2)^T, \mathbf{u_2} = (1,1)^T\}$ be two bases of $\mathbf{R}^2$. Find matrix A_1 representing L relative to bases E and F_1, and A_2 representing L relative to bases E and F_2.

Solution. By **Example 1**, we know that the matrix A_1 representing L relative to bases E and F_1 is

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Now let us take bases E and F_2. Notice that

$$\mathbf{u_1} = \boldsymbol{\varepsilon_1} + 2\boldsymbol{\varepsilon_2}, \qquad \mathbf{u_2} = \boldsymbol{\varepsilon_1} + \boldsymbol{\varepsilon_2},$$

$$\Longrightarrow \qquad \boldsymbol{\varepsilon_1} = -\mathbf{u_1} + 2\mathbf{u_2}, \qquad \boldsymbol{\varepsilon_2} = \mathbf{u_1} - \mathbf{u_2}.$$

Example 2. Let $L: \mathbf{R}^3 \to \mathbf{R}^2$ be a linear transformation defined by

$$L(\boldsymbol{x}) = (x_1 + x_2)\boldsymbol{\varepsilon}_1 + (x_2 - x_3)\boldsymbol{\varepsilon}_2.$$

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbf{R}^3$ and $F_1 = \{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2\}$, $F_2 = \{\mathbf{u}_1 = (1,2)^T, \mathbf{u}_2 = (1,1)^T\}$ be two bases of $\mathbf{R}^2$. Find matrix A_1 representing L relative to bases E and F_1, and A_2 representing L relative to bases E and F_2.

Solution. (continue) then we have

$$L(\mathbf{e}_1) = 1\boldsymbol{\varepsilon}_1 = -\mathbf{u}_1 + 2\mathbf{u}_2$$

$$L(\mathbf{e}_2) = 1\boldsymbol{\varepsilon}_1 + 1\boldsymbol{\varepsilon}_2 = \mathbf{u}_2$$

$$L(\mathbf{e}_3) = -\boldsymbol{\varepsilon}_2 = -\mathbf{u}_1 + \mathbf{u}_2.$$

Therefore,

$$A_2 = \left([L(\mathbf{e}_1)]_{F_2}, [L(\mathbf{e}_2)]_{F_2}, [L(\mathbf{e}_3)]_{F_2}\right) = \begin{pmatrix} -1 & 0 & -1 \\ 2 & 1 & 1 \end{pmatrix}.$$

In general, suppose that

- $L: V \to W$ is a linear transformation;
- $E = \{\mathbf{v_1}, \dots, \mathbf{v_n}\}$ is a basis of V;
- $F_1 = \{\mathbf{w}_1^{(1)}, \dots, \mathbf{w}_m^{(1)}\}$, $F_2 = \{\mathbf{w}_1^{(2)}, \dots, \mathbf{w}_m^{(2)}\}$ are two bases of W.

Let U be the transition matrix from basis F_2 to F_1, that is

$$\left(\mathbf{w}_1^{(2)}, \dots, \mathbf{w}_m^{(2)}\right) = \left(\mathbf{w}_1^{(1)}, \dots, \mathbf{w}_m^{(1)}\right) U,$$

and $$[\boldsymbol{w}]_{F_1} = U[\boldsymbol{w}]_{F_2}, \qquad \forall \boldsymbol{w} \in W.$$

Then $$[L(\mathbf{v_i})]_{F_1} = U[L(\mathbf{v_i})]_{F_2}, \qquad i = 1, \dots, n.$$

and we have
$$\begin{aligned} A_1 &= \left([L(\mathbf{v_1})]_{F_1}, \dots, [L(\mathbf{v}_{\boldsymbol{n}})]_{F_1}\right) \\ &= \left(U[L(\mathbf{v_1})]_{F_2}, \dots, U[L(\mathbf{v}_{\boldsymbol{n}})]_{F_2}\right) \\ &= U\left([L(\mathbf{v_1})]_{F_2}, \dots, [L(\mathbf{v}_{\boldsymbol{n}})]_{F_2}\right) = UA_2, \end{aligned}$$

or $$A_2 = U^{-1}A_1. \qquad \textbf{(2)}$$

As in **Example 2**, the transition matrix from basis F_2 to F_1 is

$$U = (\mathbf{u_1}, \mathbf{u_2}) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Therefore,

$$U^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Since

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix},$$

then A_2 can be calculated by **(2)**

$$A_2 = U^{-1}A_1 = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Linear Operator $L: V \to V$

Theorem 1. Suppose that $L: V \to V$ is a linear operator and $E = \{\mathbf{v_1}, \dots, \mathbf{v_n}\}$, $F = \{\mathbf{w_1}, \dots, \mathbf{w_n}\}$ are two bases of vector space V. Let A and B be matrices representing L relative to bases E to E and F to F, respectively. There exists an invertible matrix S, such that

$$B = S^{-1}AS.$$

Proof. Suppose that the transition matrix from basis F to E is S, then S must be an invertible matrix and

$$[\boldsymbol{x}]_E = S[\boldsymbol{x}]_F.$$

If $\boldsymbol{x} \in V$, then $\qquad [L(\boldsymbol{x})]_E = A[\boldsymbol{x}]_E, \qquad [L(\boldsymbol{x})]_F = B[\boldsymbol{x}]_F.$

We then have

$$S^{-1}[L(\boldsymbol{x})]_E = [L(\boldsymbol{x})]_F = B[\boldsymbol{x}]_F = BS^{-1}[\boldsymbol{x}]_E,$$

Proof. **(continue)**

$$S^{-1}[L(\boldsymbol{x})]_E = [L(\boldsymbol{x})]_F = B[\boldsymbol{x}]_F = BS^{-1}[\boldsymbol{x}]_E,$$

or

$$[L(\boldsymbol{x})]_E = SBS^{-1}[\boldsymbol{x}]_E.$$

Comparing with $[L(\boldsymbol{x})]_E = A[\boldsymbol{x}]_E$, we have

$$B = S^{-1}AS.$$

$$\begin{tikzcd}
V\text{: Basis } E \arrow[r, "L: V \to V", "A"'] \arrow[d, shift left, "S^{-1}"] & V\text{: Basis } E \arrow[d, shift left, "S^{-1}"] \\
V\text{: Basis } F \arrow[r, "B", "L: V \to V"'] \arrow[u, shift left, "S"] & V\text{: Basis } F \arrow[u, shift left, "S"]
\end{tikzcd}$$

Definition 1. Let A and B be $n \times n$ matrices. B is said to be **similar [相似]** to A if there exists a **nonsingular** matrix S such that

$$B = S^{-1}AS.$$

Notation $B \sim A$ is used to denote B similar to A.

Remark. The matrices representing the same linear operator L on a vector space V relative to different bases E and F, are **similar**.

Example 3. Let $L: \mathbf{R}^3 \to \mathbf{R}^3$ be the linear operator defined by

$$L(\boldsymbol{x}) = (2x_1 - x_2, x_2, x_3)^T.$$

Find the matrix A representing L relative to basis

$$F = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\} = \left\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\right\}.$$

Solution. (I) By **Theorem 5.3.1**, we can calculate that

$$L(\mathbf{u_1}) = L\left((1,1,1)^T\right) = (1,1,1)^T = 1\mathbf{u_1} + 0\mathbf{u_2} + 0\mathbf{u_3},$$
$$L(\mathbf{u_2}) = L\left((1,1,0)^T\right) = (1,1,0)^T = 0\mathbf{u_1} + 1\mathbf{u_2} + 0\mathbf{u_3},$$
$$L(\mathbf{u_3}) = L\left((1,0,0)^T\right) = (2,0,0)^T = 0\mathbf{u_1} + 0\mathbf{u_2} + 2\mathbf{u_3}.$$

Therefore, the matrix A is

$$A = ([L(\mathbf{u_1})]_F, [L(\mathbf{u_2})]_F, [L(\mathbf{u_3})]_F) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solution. (II) If we take $E = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ as the standard basis of $\mathbf{R}^3$, then the matrix B representing L relative to basis E can be shown directly from the definition. That is,

$$B = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transition matrix S from basis F to E is $S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

The inverse matrix of S is $S^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$.

Then we have

$$A = S^{-1}BS = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Property 1. Let A, B and C be $n \times n$ matrices. Then

(1) $A \sim A$;

(2) If $A \sim B$, then $B \sim A$;

(3) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Review

- Matrix Representation of Linear Transformations
- Similar Matrices

Preview

- Inner product and Inner Product Space
- Orthogonal Sets and Orthogonal Subspaces
- Gram-Schmidt Orthogonalization