Chapter 29 Maxwell's Equations

§ 1 Displacement Current and the Extended Ampére's Law

$$\oint_L \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{encl}}$$

Electric current → Magnetic field

$$\oint_L \vec{E} \cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

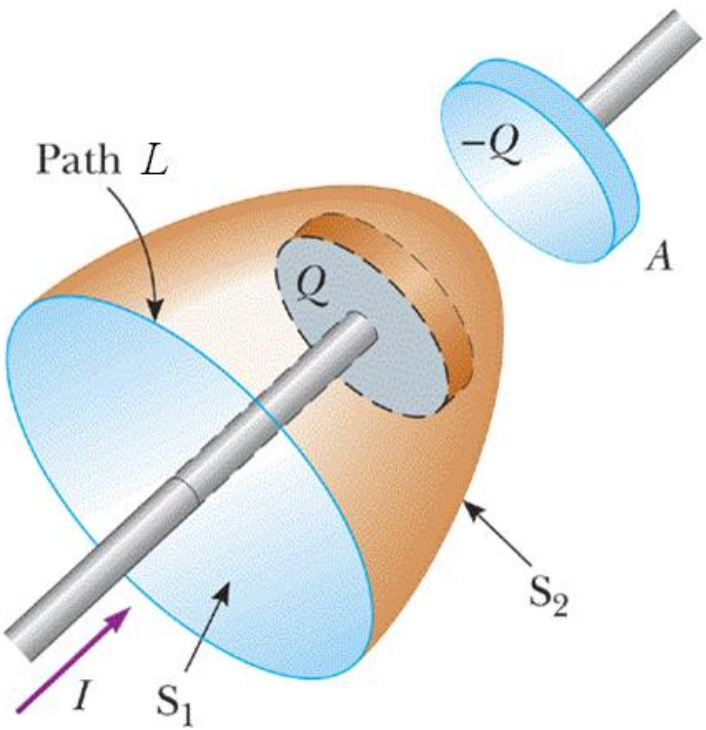
Changing magnetic field → Electric field

Changing electric field $\xrightarrow{?}$ Magnetic field

The Contradiction of Ampére's Law

- **The contradiction in applying Ampére's law to a charging capacitor**
  - **Apply Ampére's law to a circular loop L.**

    Ampère's law states that $\oint \vec{B}\cdot d\vec{s} = \mu_0 I$, where I is the total current through *any* surface bounded by the path L.



Surface S_1: the circular area in which the conduction current I penetrates.

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 \iint_{S_1} \vec{j}\cdot d\vec{A} = \mu_0 I$$

Surface S_2: the paraboloid passing between the capacitor's plates.

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 \iint_{S_2} \vec{j}\cdot d\vec{A} = 0$$

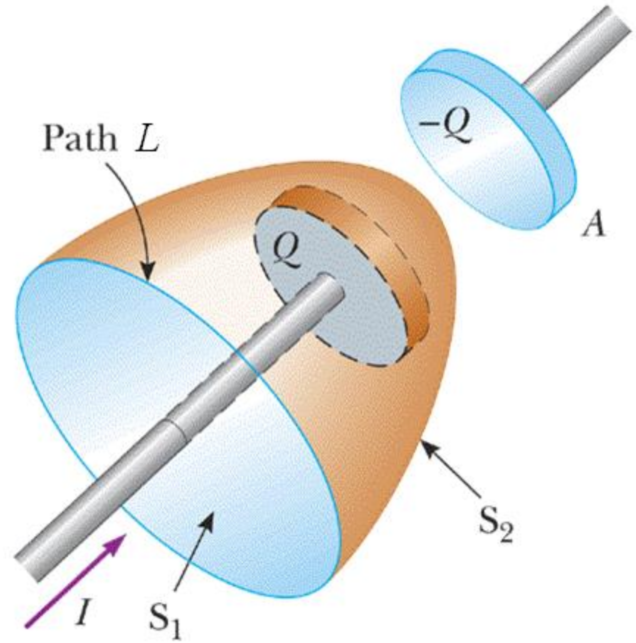
The Contradiction of Ampére's Law

Surface S$_1$:

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 \iint_{S_1} \vec{j}\cdot d\vec{A} = \mu_0 I$$

Surface S$_2$:

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 \iint_{S_2} \vec{j}\cdot d\vec{A} = 0$$



- **Question: Does Ampére's law need to be modified?**
  - **What is wrong with the Ampére's law?**
  - **How to treat the discontinuity of the current?**

  Ampére's law is valid only if the conduction current is continuous in space.

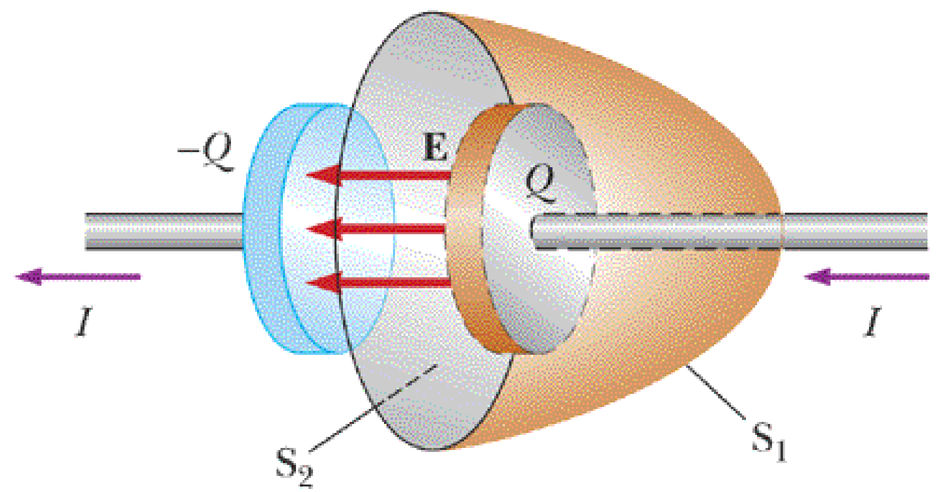
- **How to save Ampére's law from the contradiction?**
  - **The contradiction comes from the discontinuity of the conduction current.**

    The conduction current I is interrupted in the region between capacitor's two plates, there is also a changing electric field $\vec{E}$ or a changing electric flux Φ_E in this region.

$$I = \frac{dQ}{dt} = \frac{d(\sigma A)}{dt} = \frac{d\sigma}{dt} A$$

$$= \frac{d}{dt}(\varepsilon_0 E) A = \varepsilon_0 \frac{d}{dt}(EA)$$

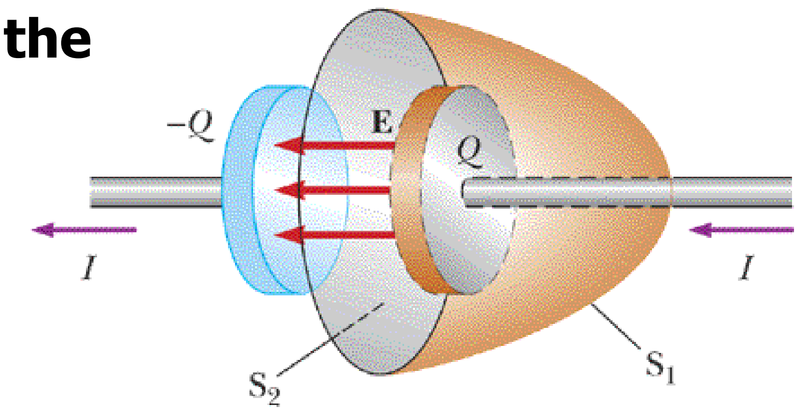
$$= \varepsilon_0 \frac{d\Phi_E}{dt}$$



The Displacement Current

- **How to save Ampére's law from the contradiction?**

$$I = \frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt}$$



- **To keep the continuity of the current, Maxwell made a postulation that there exists a fictitious current in the region between the plates, called the displacement current I_d.**

$$\boxed{I_d = \varepsilon_0 \frac{d\Phi_E}{dt}} = \varepsilon_0 \frac{d}{dt} \iint_S \vec{E} \cdot d\vec{A} = \iint_S \left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

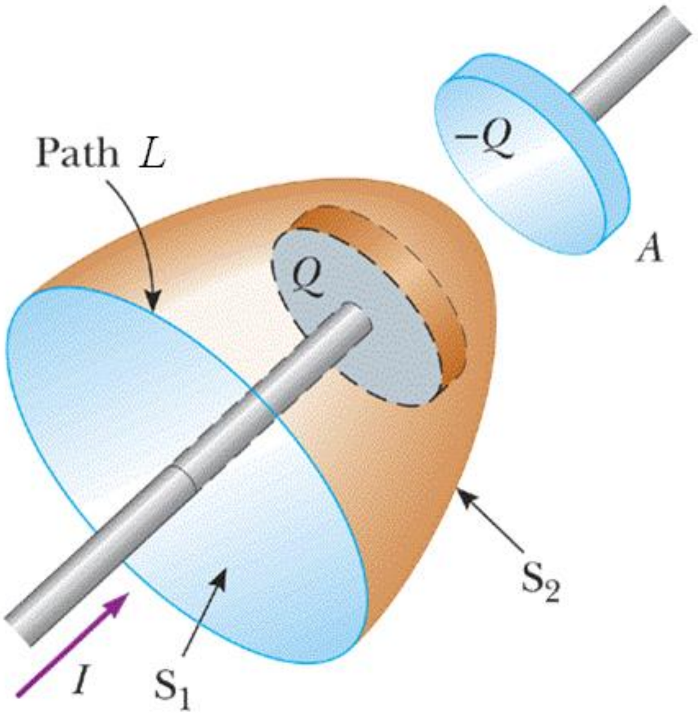
- **Displacement current density:**

$$\vec{j}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}, \quad I_d = \iint_S \vec{j}_d \cdot d\vec{A}$$

- **Extended Ampére's law or Ampére-Maxwell law:**
  - **The postulation of displacement current solved the discontinuity of the conduction current.**

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 (I_c + I_d)_{\text{encl}}$$

$$= \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$



Extended Ampére's law

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0(I_c + I_d)_{\text{encl}} = \mu_0 I_c + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

- The displacement current is also a source of magnetic field

Conduction current I_c

Displacement current I_d

or changing electric field

→ Magnetic field

- Magnetic field are produced both by conduction current and by changing electric field.

Extended Ampére's law

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}}$$

Electric current → **Magnetic field**

$$\oint_L \vec{E}\cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

Changing magnetic field → **Electric field**

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 (I_d)_{\text{encl}} = \mu_0 \iint_S \left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{A}$$

Changing electric field → **Magnetic field**

Example

Calculate the displacement current I_d between the square plates, 3.8 cm on a side, of a capacitor if the electric field is changing at a rate of 2.0×10^6 V/m·s. (The permittivity of free space is 8.85×10^{-12} $C^2/N \cdot m^2$.)

Solution:

The displacement current is

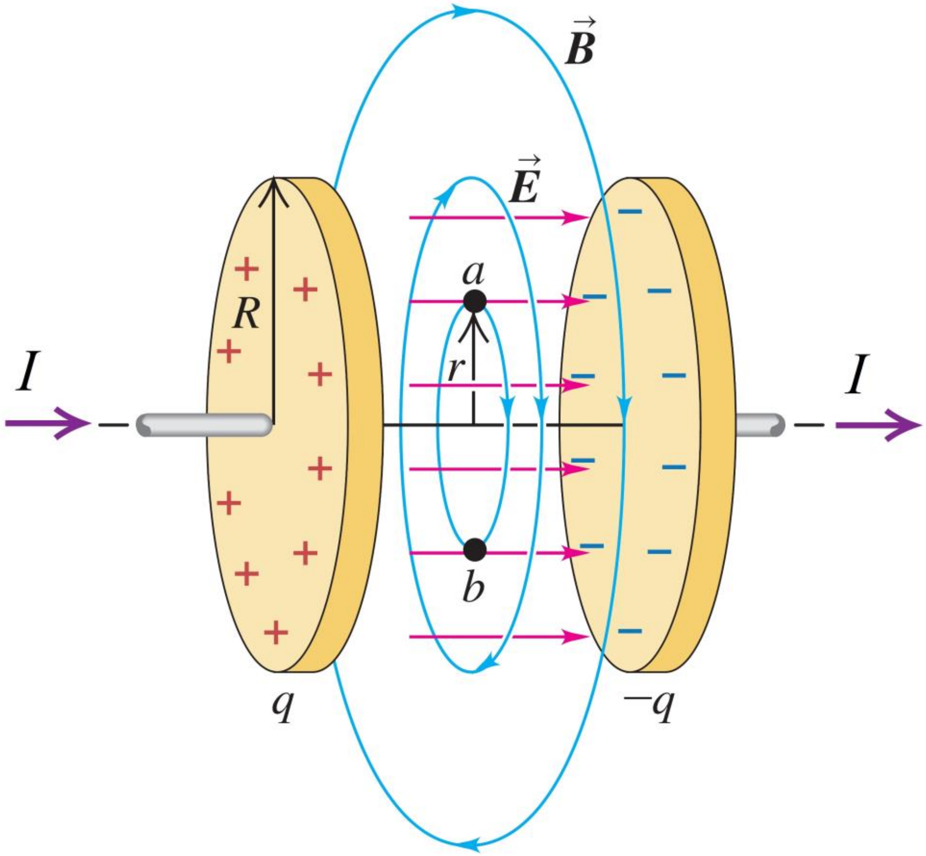
$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d(EA)}{dt} = \varepsilon_0 A \frac{dE}{dt}$$

$$= (8.85\times10^{-12})\times(0.038)^2\times(2.0\times10^6)$$

$$= 2.6\times10^{-8}\ \mathrm{A}$$

Example

Calculate the magnetic field in the region between the two capacitor's plates while the capacitor is charging with a increasing current *I*. The radius of plate is *R*.



Example

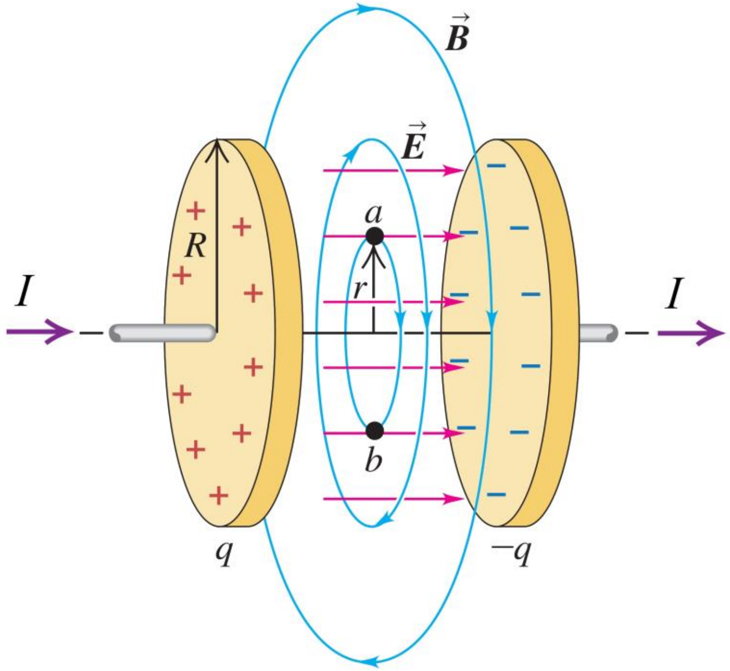
Solution: For a point a distance *r* from the center, we apply Ampére's law to a circular path of radius *r* passing through the point.

$$\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

The electric field between the plates:

$$E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{\pi R^2}$$

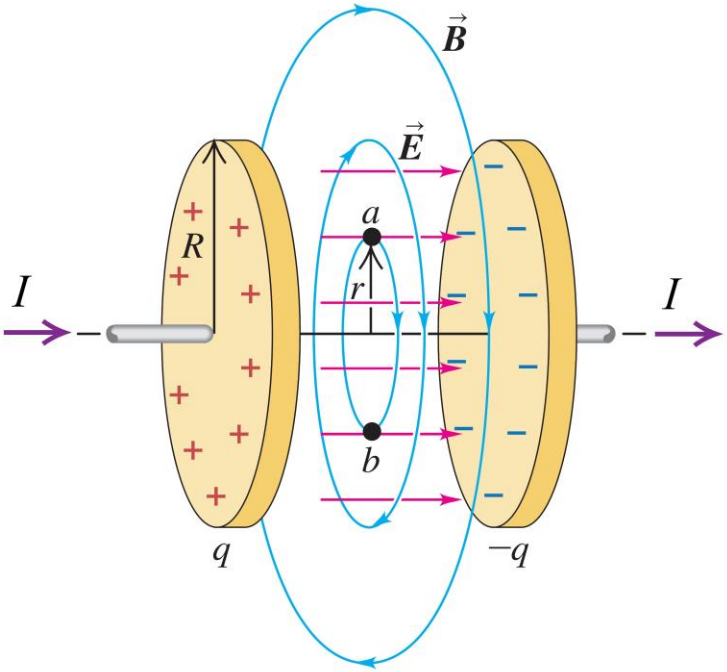
$$\Phi_E = \begin{cases} E(\pi r^2) = \dfrac{1}{\varepsilon_0}\dfrac{r^2}{R^2}Q, & \text{for } r < R \\ E(\pi R^2) = \dfrac{Q}{\varepsilon_0}, & \text{for } r > R \end{cases}$$



Example

$$\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\Phi_E = \begin{cases} \dfrac{1}{\varepsilon_0}\dfrac{r^2}{R^2}Q & \text{for } r < R \\ \dfrac{Q}{\varepsilon_0} & \text{for } r > R \end{cases}$$



For $r < R$:

$$\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0 \frac{r^2}{R^2}\frac{dQ}{dt} = \mu_0 \frac{r^2}{R^2} I, \qquad B = \frac{\mu_0 I}{2\pi}\frac{r}{R^2}$$

For $r > R$:

$$\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0 \frac{dQ}{dt} = \mu_0 I, \qquad B = \frac{\mu_0 I}{2\pi r}$$

§ 2 Maxwell's Equations

$$\oiint_S \vec{E}\cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0}$$

Gauss's law for electricity

$$\oiint_S \vec{B}\cdot d\vec{A} = 0$$

Gauss's law for magnetism

$$\oint_L \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

Faraday's law of induction

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}} + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 I_{\text{encl}} + \mu_0\varepsilon_0 \iint_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$

Ampére-Maxwell law

Maxwell's equations and Lorentz force give the fundamental relations of electromagnetism! They are fundamental in the sense that Newton's three laws are for mechanics.

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

Lorentz force

The Physical Meanings Embodied in Maxwell's Equations

$$\oiint_S \vec{E} \cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0}$$

Charged particles create an electric field (electrostatic).

$$\oint_L \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

An electric field (non-electrostatic) can also be created by a changing magnetic field.

$$\oiint_S \vec{B} \cdot d\vec{A} = 0$$

There are no magnetic monopoles.

$$\oint_L \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \iint_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}$$

A magnetic field can be created either by currents or by a changing electric field.

§ 3 Electromagnetic Waves

- **The relationship between electric and magnetic field in empty space.**

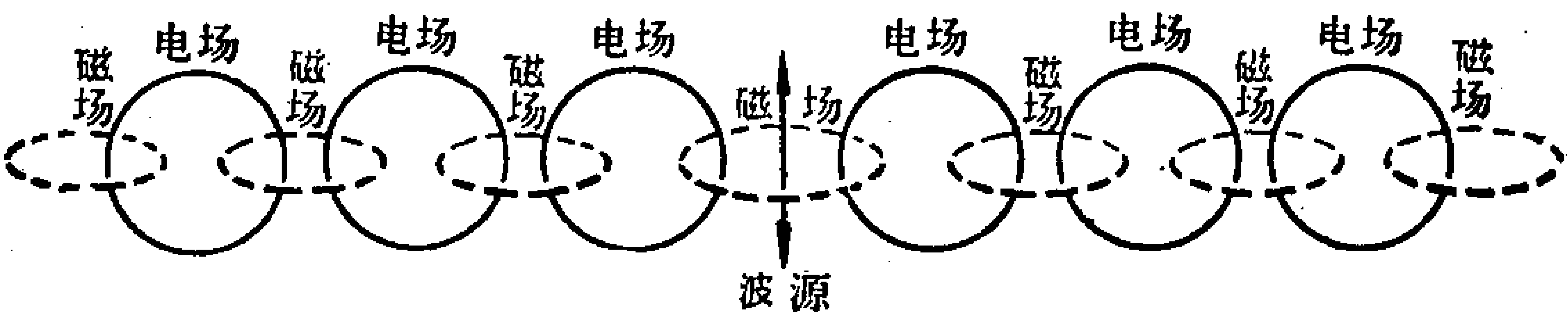
$$\oint_L \vec{E}\cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}, \qquad \oint_L \vec{B}\cdot d\vec{s} = \varepsilon_0\mu_0 \iint_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$

- **A time varying magnetic field induces a electric field in neighboring regions;**
- **A time varying electric field induces a magnetic field in neighboring regions.**

These relationships predicts the existence of electromagnetic waves consisting of time-varying electric and magnetic fields that travel from one region of space to another, even if no charge or current are present in space.

The Propagation of the Electromagnetic Wave

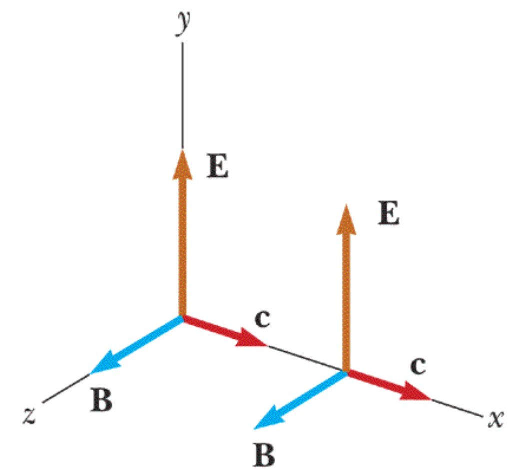
- **The mechanism for maintaining the propagation of the electromagnetic wave.**
  - **Unlike mechanical waves, which need a medium such as water or air to transit a wave, electromagnetic waves require no medium. The changing electric and magnetic fields create each other to maintain the propagation of the waves.**
  - **A exhibition map (not real) for propagation of electromagnetic waves**



- **The wave equation:**

From Maxwell's equations, we can obtain the wave equation for a wave which propagates in *x*-direction

$$\frac{\partial^2 E}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}, \qquad \frac{\partial^2 B}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2}$$

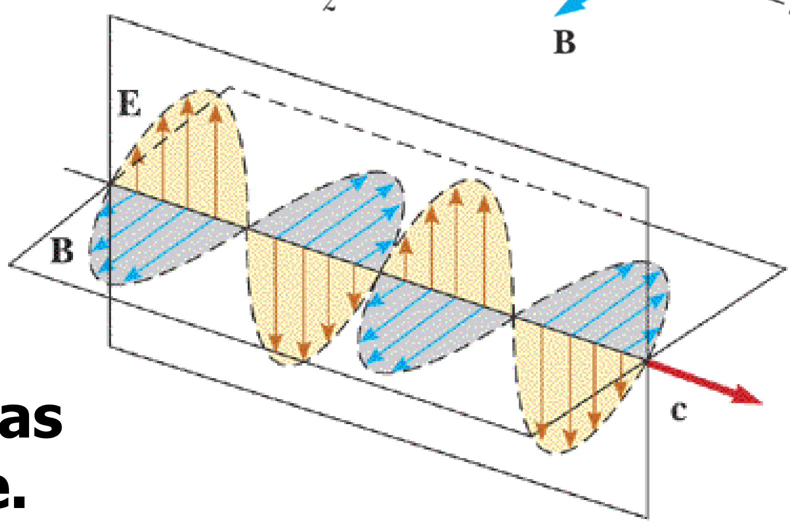


- **The wave speed:**

Generally, the wave equation

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi(x,t)}{\partial t^2}$$

$$\boxed{v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.997 \times 10^8 \text{ m/s} = c}$$

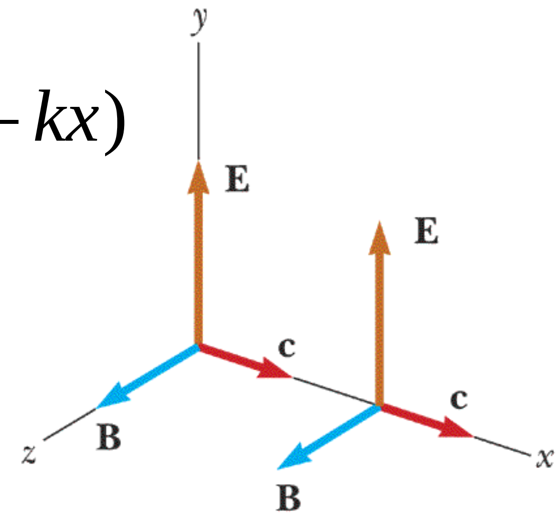


This speed is precisely the same as the speed of light in empty space.

The important features of electromagnetic waves

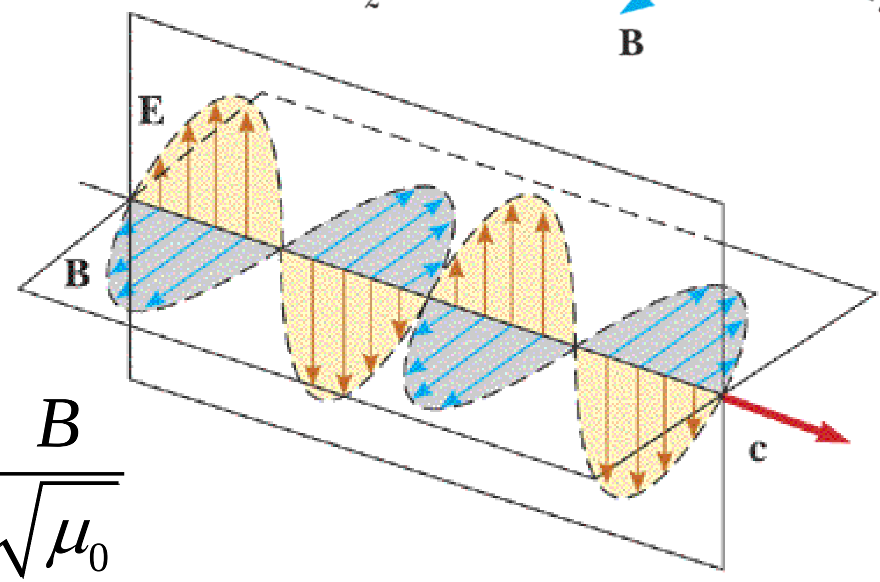
- **The sinusoidal plane wave is the simplest solution of the wave equations**

$$E = E_{\max}\cos(\omega t - kx), \quad B = B_{\max}\cos(\omega t - kx)$$



- **The wave is transverse.**

Both $\vec{E}$ and $\vec{B}$ are perpendicular to each other, and to the direction of propagation. The direction of propagation is $\vec{E}\times\vec{B}$



- **$\vec{E}$ and $\vec{B}$ are in phase, and has a definite ratio**

$$\frac{E}{B} = \frac{E_{\max}}{B_{\max}} = c, \qquad \sqrt{\varepsilon_0}E = \frac{B}{\sqrt{\mu_0}}$$

The important features of electromagnetic waves

- **Poynting vector: energy flow vector.**

The total energy density:

$$u = u_E + u_B = \frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0} = \frac{EB}{\mu_0 c}$$

The energy current density:

$$S = uc = \frac{EB}{\mu_0},$$

$$\boxed{\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}}$$

S **represents power per unit area perpendicular to the direction of wave propagation.**