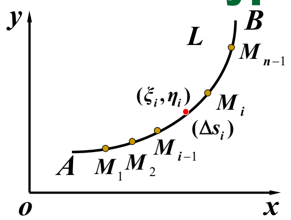
北京邮电大学
BEIJING UNIVERSITY OF POSTS AND TELECOMMUNICATIONS

Lecture 12

Line Integrals

Line Integrals of the First Type



Example 1.The mass of a plane curve body

1. Partition

$$A, M_1, M_2, \cdots, M_{n-1}, B$$

2. Homogenization

Take $(\xi_i,\eta_i)\in(\Delta s_i)$,

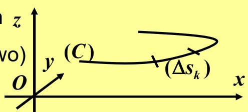
$$\Delta M_i \approx \rho(\xi_i,\eta_i)\cdot \Delta s_i .$$

3. Summation $M \approx \sum_{i=1}^{n}\rho(\xi_i,\eta_i)\cdot \Delta s_i .$ **Approximation**

4. Limit $M = \lim_{d\to 0}\sum_{i=1}^{n}\rho(\xi_i,\eta_i)\cdot \Delta s_i .$ **Precise value**

Line Integrals of the First Type

Definition Line integral with respect to arc length



Suppose (C) is a measureable smooth curve in space (or plane), f is a function of three (or two) variables defined on (C), which can be divided into $(\Delta s_k), k=1,\cdots,n$. Δs_k is the arc length of the subarc (Δs_k) and M_k is any point (ξ_k,η_k,ζ_k) on the subarc. If the limit

$$\lim_{d\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\Delta s_k, \left(\lim_{d\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k)\Delta s_k\right) \quad d=\max_{1\le k\le n}\Delta s_k .$$

exists uniquely, then f is integrable over the curve (C), and the limit is called **the line integral of f along (C) with respect to arc length, the line integral of the first type**, which is denoted by

$$\int_{(C)} f(x,y,z)ds = \lim_{d\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\boxed{\Delta s_k}, \left(\int_{(C)} f(x,y)ds = \lim_{d\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k)\boxed{\Delta s_k}\right).$$

Element of arc

Line Integrals of the First Type

Notes:

1. If (C) is a **simple closed** curve, the line integral is often denoted by

$$\oint_{(C)} f(x,y)ds \text{ or } \oint_{(C)} f(x,y,z)ds.$$

$\vec{r}\,'(t)$ is continuous and never 0.

2. If f is continuous on the smooth curve (C), then the line integral of f along (C) with respect to arc length exists.

3. Suppose that (C) is a **piecewise-smooth curve**, that is, (C) is a union of a finite number of smooth curves $(C_1),(C_2),\cdots,(C_n)$, then we define the integral of f along (C) as the sum of the integrals of f along each of the smooth pieces of (C):

$$\int_{(C)} f(x,y)ds = \int_{(C_1)} f(x,y)ds + \int_{(C_2)} f(x,y)ds + \cdots + \int_{(C_n)} f(x,y)ds$$

Line Integrals of the First Type

Notes:

4. **Geometric meaning of line integrals with respect to arc length**

Just as for an ordinary single integral, we can interpret the line integral of a positive function as an area. If $f(x,y)\ge 0$, $\int_{(C)} f(x,y)ds$ represents the area of one side of the "fence" or "curtain", whose base is (C) and whose height above the point (x,y) is $f(x,y)$.

5. $\int_{(C)} ds = L$, where L is the arc length of the curve of (C).

Line Integrals of the First Type

Properties:

1. Linearity Property

(1) $\int_{(C)} [f(x,y)\pm g(x,y)]ds = \int_{(C)} f(x,y)ds \pm \int_{(C)} g(x,y)ds.$

(2) $\int_{(C)} kf(x,y)ds = k\int_{(C)} f(x,y)ds.$

2. Additivity with respect to the domain of integration

Suppose that $(C)=(C_1)+(C_2)$ and $(C_1),(C_2)$ have no common part except for their boundaries. Then

$$\int_{(C)} f(x,y)ds = \int_{(C_1)} f(x,y)ds + \int_{(C_2)} f(x,y)ds.$$

Line Integrals of the First Type

3. Domination

(1) $\int_{(C)} f(x,y)ds \le \int_{(C)} g(x,y)ds, \quad (f(x,y) \le g(x,y));$

(2) $\left|\int_{(C)} f(x,y)ds\right| \le \int_{(C)} |f(x,y)|ds;$

(3) If $m \le f(x,y) \le M, \forall (x,y) \in (C),$ then

$$mL \le \int_{(C)} f(x,y)ds \le ML,$$

where L is the arc length of the curve of (C).

4. Mean Value Theorem

Suppose that $f \in C((C))$, and (C) is a continuous curve. Then there exists at least one point $(\xi,\eta) \in (C)$, such that

$$\int_{(C)} f(x,y)ds = f(\xi,\eta)L,$$

where L is the arc length of the curve of (C).

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Arc Length Along a Curve

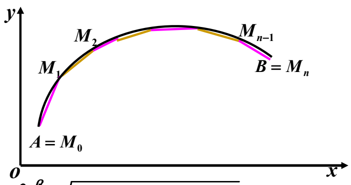
1. Arc length of a plane curve

(1) Parametric equations

$x = x(t), y = y(t), \quad (\alpha \le t \le \beta),$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}dt$$



The arc length of the curve is $s = \int_\alpha^\beta \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}dt$

(2) General equation in rectangular coordinates

$$y = y(x), \quad (a \le x \le b)$$

The arc length of the curve is $s = \int_a^b \sqrt{1 + [y'(x)]^2}dx$

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Arc Length Along a Curve

1. Arc length of a plane curve

(3) General equation in polar coordinates

$$\rho = \rho(\theta), \quad (\alpha \le \theta \le \beta)$$

The arc length of the curve is $s = \int_\alpha^\beta \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2}d\theta$

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Arc Length Along a Curve

2. Arc length of a space curve

Parametric equations $x = x(t), y = y(t), z = z(t), \quad (\alpha \le t \le \beta),$

The length of the curve is $s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt.$

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Computation of Line Integrals of the First Type

Proposition If the parametric equation of simple smooth space curve is

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (\alpha \le t \le \beta)$$

and the function f is continuous on the curve (C), then

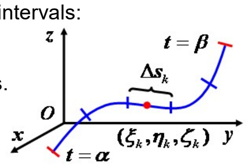
$$\int_{(C)} f(x,y,z)ds = \int_\alpha^\beta f[x(t),y(t),z(t)]\sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}dt.$$

Proof Partition the interval $[\alpha,\beta]$ into subintervals:

$$\alpha = t_0 < t_1 < t_2 < \cdots < t_n = \beta.$$

The curve (C) is divided into n segment arcs.

Let the segmental arc (Δs_k) correspond to $[t_{k-1},t_k]$ and the arc length is Δs_k.



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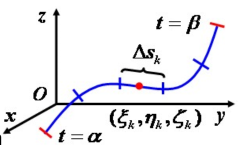
Computation of Line Integrals of the First Type

It is easy to see that

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}dt.$$

Since (C) is smooth and the integrand $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ is continuous, then by the mean value theorem for the integral, we have



$$\Delta s_k = \sqrt{\dot{x}^2(\tau_k) + \dot{y}^2(\tau_k) + \dot{z}^2(\tau_k)}\Delta t_k, \quad t_{k-1} \le \tau_k \le t_k.$$

Let $x(\tau_k) = \xi_k, \quad y(\tau_k) = \eta_k, \quad z(\tau_k) = \zeta_k.$

Obviously, the point should lie on the segmental arc (Δs_k).

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Computation of Line Integrals of the First Type

Form the sum $\Delta s_k = \sqrt{\dot{x}^2(\tau_k)+\dot{y}^2(\tau_k)+\dot{z}^2(\tau_k)}\Delta t_k$

$$\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\Delta s_k = \sum_{k=1}^{n} f\left[x(\tau_k),y(\tau_k),z(\tau_k)\right]\cdot\sqrt{\dot{x}^2(\tau_k)+\dot{y}^2(\tau_k)+\dot{z}^2(\tau_k)}\Delta t_k .$$

Since the integrand f is continuous on the curve (C), the line integral $\int_{(C)} f(x,y,z)ds$ exists. Then,

$$\int_{(C)} f(x,y,z)ds = \lim_{d\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\Delta s_k$$

$$= \lim_{d'\to 0}\sum_{k=1}^{n} f\left[x(\tau_k),y(\tau_k),z(\tau_k)\right]\cdot\sqrt{\dot{x}^2(\tau_k)+\dot{y}^2(\tau_k)+\dot{z}^2(\tau_k)}\Delta t_k$$

$$= \int_{\alpha}^{\beta} f\left[x(t),y(t),z(t)\right]\sqrt{\dot{x}^2(t)+\dot{y}^2(t)+\dot{z}^2(t)}dt,$$

where $d=\max\limits_{1\le k\le n}\Delta s_k, d'=\max\limits_{1\le k\le n}\Delta t_k$.

Computation of Line Integrals of the First Type

Special case

1. If (C) is a simple smooth plane curve given by the parametric equation

$$x=x(t),\quad y=y(t),\qquad (\alpha\le t\le\beta)$$

and the function f is continuous on the curve (C), then

$$\int_{(C)} f(x,y)ds = \int_{\alpha}^{\beta} f\left[x(t),y(t)\right]\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}dt;$$

2. If the parametric equation of simple smooth plane curve is

$$y=y(x),\qquad (a\le x\le b)$$

and the function f is continuous on the curve (C), then

$$\int_{(C)} f(x,y)ds = \int_{a}^{b} f\left[x,y(x)\right]\sqrt{1+\left(y'(x)\right)^2}dx;$$

Computation of Line Integrals of the First Type

Special case

3. If the parametric equation of simple smooth plane curve is

$$x=x(y),\qquad (c\le y\le d)$$

and the function f is continuous on the curve (C), then

$$\int_{(C)} f(x,y)ds = \int_{c}^{d} f\left[x(y),y\right]\sqrt{\left(x'(y)\right)^2+1}dy.$$

4. If the parametric equation of simple smooth plane curve is

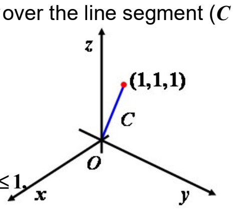
$$\rho=\rho(\theta)\ \ (\alpha\le\theta\le\beta),$$

and the function f is continuous on the curve (C), then

$$\int_{(C)} f(x,y)ds = \int_{\alpha}^{\beta} f(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)\sqrt{\rho^2(\theta)+\rho'^2(\theta)}\, d\theta.$$

Evaluating a Line Integral

Example 1. Integrate $f(x,y,z)=x-3y^2+z$ over the line segment (C) joining the origin and the point $(1,1,1)$



Solution The parametric equation of the line segment (C) is

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1}\triangleq t,\quad \Leftrightarrow x=t,\quad y=t,\quad z=t,\quad 0\le t\le 1,$$

then $\sqrt{\dot{x}^2(t)+\dot{y}^2(t)+\dot{z}^2(t)}=\sqrt{1+1+1}=\sqrt{3}.$

Therefore,

$$\int_{(C)} f(x,y,z)ds = \int_0^1 f(t,t,t)\sqrt{3}dt = \int_0^1 (t-3t^2+t)\sqrt{3}dt = 0.$$

Evaluating a Line Integral

Example 2. Find $I=\int_{(C)} y\,ds,$ where (C) is the segment of the parabola $y^2=2x$ between the two points $(2,-2)$ and $(2,2)$.

Solution Choose y as the variable of integration and regard the equation of the path $y^2=2x$ as a parametric equation with the parameter y: $x=\dfrac{y^2}{2}, y=y(-2\le y\le 2).$

Hence,

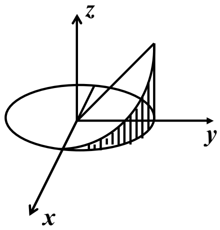
$$I=\int_{(C)} y\,ds = \int_{-2}^{2} y\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy = \int_{-2}^{2} y\sqrt{1+y^2}dy = 0.$$

Note: If the path of integration (C) is symmetric about the x-axis (or y-axis) and the integrand is an odd function respect to y-variable (or x-variable), then the line integral is zero.

Evaluating a Line Integral

Example 3. (**The lateral area of a cylinder**)

Find the lateral area of the part of the elliptic cylinder $\dfrac{x^2}{5}+\dfrac{y^2}{9}=1$ cut by the plane $z=y$ and $z=0$ located in the first octant.



Solution $A=\int_{(C)} f(x,y)ds = \int_{(C)} z\ ds = \int_{(C)} y\ ds$

(C): $x=\sqrt{5}\cos t,\ y=3\sin t,\ 0\le t\le\dfrac{\pi}{2}$

$$A=\int_{(C)} yds = \int_0^{\frac{\pi}{2}} 3\sin t\sqrt{5\sin^2 t+9\cos^2 t}dt$$

$$=\frac{9}{2}+\frac{15}{8}\ln 5$$

Evaluating a Line Integral

Example 4. Find $\oint_{(C)} y^2 ds$, where (C) is $\begin{cases} x^2+y^2+z^2=4, \\ x+y+z=0. \end{cases}$

Solution By symmetric,

$$\oint_{(C)} x^2 ds = \oint_{(C)} y^2 ds = \oint_{(C)} z^2 ds.$$

Then

$$\oint_{(C)} y^2 ds = \frac{1}{3}\oint_{(C)} (x^2+y^2+z^2) ds = \frac{1}{3}\oint_{(C)} 4 ds = \frac{4}{3}\cdot 4\pi = \frac{16}{3}\pi.$$

Evaluating a Line Integral

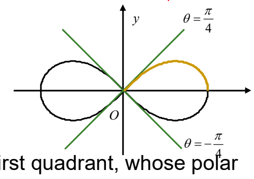
Example 5. Evaluate $\int_{(C)} |y|\, ds$ in polar coordinates, where (C) is

$$(x^2+y^2)^2 = a^2(x^2-y^2), a>0.$$

$$\int_{(C)} f(x,y)ds = \int_\alpha^\beta f(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)\sqrt{\rho^2(\theta)+\rho'^2(\theta)}\, d\theta.$$

Solution The polar coordinate of (C) is

$$\rho^2 = a^2\cos 2\theta, a>0.$$



By symmetric, $\int_{(C)} |y|\, ds = 4\int_{(C_1)} y\, ds,$

where (C_1) is the lemniscate located in the first quadrant, whose polar coordinate is

$$\rho = a\sqrt{\cos 2\theta}, 0<\theta<\frac{\pi}{4}.$$

Then

$$\int_{(C)} |y|\, ds = 4\int_0^{\frac{\pi}{4}} \rho(\theta)\sin\theta\sqrt{\rho^2(\theta)+\rho'^2(\theta)}\, d\theta = 4\int_0^{\frac{\pi}{4}} a^2\sin\theta\, d\theta = 2(2-\sqrt{2})a^2.$$

Properties of line integral of the first type

Properties:

(1) $\int_{(C)} ds = L$ (L denotes the length of the curve (C));

(2) 对称性: $\int_{(C)} f(x,y,z)ds = \begin{cases} 0, & f(x,y,-z) = -f(x,y,z) \\ 2\int_{(C_1)} f(x,y,z)ds, & f(x,y,-z) = f(x,y,z) \end{cases}$

where (C) is symmetry about xoy plane.

(3) 轮换对称性: $\int_{(C)} x^2 ds = \int_{(C)} y^2 ds = \int_{(C)} z^2 ds = \frac{1}{3}\int_{(C)} (x^2+y^2+z^2)ds = \frac{1}{3}\int_{(C)} ds = \frac{2\pi}{3}$

where $(C): \begin{cases} x^2+y^2+z^2=1 \\ x+y+z=0 \end{cases}$

(4) 带入性

(5) 无向性

Find Mass, Center of Mass

Example 4. A coil spring lies along the helix

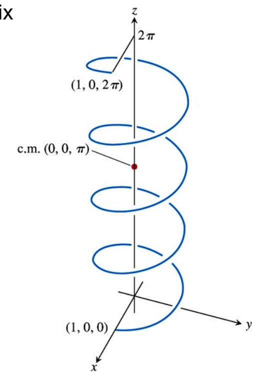
$x=\cos 4t, \quad y=\sin 4t, \quad z=t, \quad 0\le t\le 2\pi.$

The spring's density is a constant, $\delta=1$.

Find the spring's mass.

Solution Find mass

$$M = \int_{\text{Helix}} \delta ds = \int_0^{2\pi} (1)\sqrt{17}dt = 2\pi\sqrt{17}$$



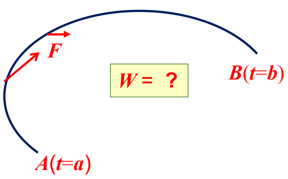
Work Done by a Force Over a Curve in Space

Suppose that the vector field $\vec{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ represents a force throughout a region in space (it might be the force of gravity or an electromagnetic force of some kind) and that

$$(C): \quad \vec{\mathbf{r}}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \quad t\in[a,b]([b,a]),$$

is an oriented smooth curve in space.

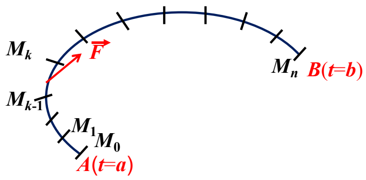
Then the work done by $\vec{F}$ over the curve (C) from A to B can be also obtain by the integral.



Work Done by a Force Over a Curve in Space

Find W

1) Partition

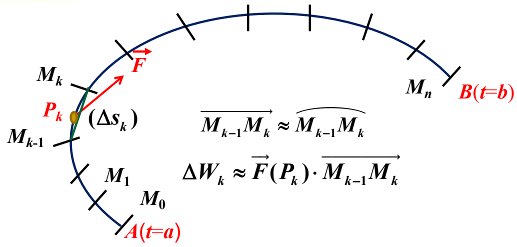


$W = \sum_{k=1}^{n} \Delta W_k$ $\quad\Delta W_k$: the work done by $\vec{F}$ along the small arc $\overset{\frown}{M_{k-1}M_k}$

Work Done by a Force Over a Curve in Space

Find W

2) Homogenization



$$\overrightarrow{M_{k-1}M_k} \approx \overset{\frown}{M_{k-1}M_k}$$

$$\Delta W_k \approx \vec{F}(P_k)\cdot\overrightarrow{M_{k-1}M_k}$$

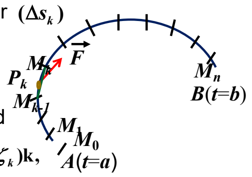
3) Summation $\quad W=\sum_{k=1}^{n}\Delta W_k \approx \sum_{k=1}^{n}\vec{F}(P_k)\cdot\overrightarrow{M_{k-1}M_k}$

4) Precision $\quad W=\lim_{d\to 0}\sum_{k=1}^{n}\vec{F}(P_k)\cdot\overrightarrow{M_{k-1}M_k}, \quad d=\max_{1\le k\le n}\Delta s_k.$

Work Done by a Force Over a Curve in Space

If the coordinate of M_{k-1} and M_k are $(x_{k-1},y_{k-1},z_{k-1})$ and (x_k,y_k,z_k). (ξ_k,η_k,ζ_k) is a point in (Δs_k). Then the work over (Δs_k) can be calculated by

$$\Delta W_k \approx \vec{F}(\xi_k,\eta_k,\zeta_k)\cdot\overrightarrow{M_{k-1}M_k}.$$



If $\overrightarrow{M_{k-1}M_k}$ is expressed as $(\Delta x_k,\Delta y_k,\Delta z_k)$, and

$\vec{F}(\xi_k,\eta_k,\zeta_k)=P(\xi_k,\eta_k,\zeta_k)\mathbf{i}+Q(\xi_k,\eta_k,\zeta_k)\mathbf{j}+R(\xi_k,\eta_k,\zeta_k)\mathbf{k}$,

then

$$\Delta W_k \approx (P(\xi_k,\eta_k,\zeta_k),Q(\xi_k,\eta_k,\zeta_k),R(\xi_k,\eta_k,\zeta_k))\cdot(\Delta x_k,\Delta y_k,\Delta z_k)$$

Since $W=\lim_{d\to 0}\sum_{k=1}^{n}\Delta W_k$, we have

$$W=\lim_{d\to 0}\sum_{k=1}^{n}\vec{F}(P_k)\cdot\overrightarrow{M_{k-1}M_k}$$

$$=\lim_{d\to 0}\sum_{k=1}^{n}\{P(\xi_k,\eta_k,\zeta_k)\Delta x_k+Q(\xi_k,\eta_k,\zeta_k)\Delta y_k+R(\xi_k,\eta_k,\zeta_k)\Delta z_k\}.$$

Definition of Line integrals of the second type

Definition Line integral with respect to coordinates

Suppose (C) is an oriented smooth space curve from one point A to to another point B and $\vec{F}=P(x,y,z)\mathbf{i}+Q(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$ is a vector function bounded on (C). (C) is divided into n small oriented subarcs $\overset{\frown}{M_{k-1}M_k}$ with the arc length Δs_k. Let (ξ_k,η_k,ζ_k) be any point on the subarc $\overset{\frown}{M_{k-1}M_k}$. If the limit

$$\lim_{d\to 0}\sum_{k=1}^{n}\vec{F}(\xi_k,\eta_k,\zeta_k)\cdot\overrightarrow{M_{k-1}M_k}$$

$$=\lim_{d\to 0}\sum_{k=1}^{n}\{P(\xi_k,\eta_k,\zeta_k)\Delta x_k+Q(\xi_k,\eta_k,\zeta_k)\Delta y_k+R(\xi_k,\eta_k,\zeta_k)\Delta z_k\}$$

exists uniquely, then the limit is called **the line integral of $\vec{F}$ along the oriented smooth curve (C) with respect to coordinates** or **the line integral of the second type**, which is denoted by

Definition of Line integrals of the second type

$$\int_{(C)}\vec{F}\cdot d\vec{r}=\int_{(C)}[P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz]$$

$$=\lim_{d\to 0}\sum_{k=1}^{n}\vec{F}(\xi_k,\eta_k,\zeta_k)\cdot\overrightarrow{M_{k-1}M_k}$$

$$=\lim_{d\to 0}\sum_{k=1}^{n}\{P(\xi_k,\eta_k,\zeta_k)\Delta x_k+Q(\xi_k,\eta_k,\zeta_k)\Delta y_k+R(\xi_k,\eta_k,\zeta_k)\Delta z_k\}$$

where $d\vec{r}=(dx,dy,dz)$.

Line integrals of the second type

Notes:

1. The differences between line integrals of first type and second type:

	The integrand	The path
First Type	A scalar function	Has no direction
Second Type	A dot product of two vectors	Has direction

2. If the vector function $\vec{F}$ is continuous on an oriented smooth curve (C), then the line integral of $\vec{F}$ along (C) with respect to coordinates exists.

Line integrals of the second type

Notes:

3. Suppose that (C) is an oriented piecewise-smooth curve, that is, (C) is a union of a finite number of oriented smooth curves$(C_1),(C_2),\cdots,(C_n)$, then we define the integral of $\vec{F}$ along (C) as the sum of the integrals of F along each of the smooth pieces of (C):

$$\int_{(C)}\vec{F}\cdot d\vec{r}=\int_{(C_1)}\vec{F}\cdot d\vec{r}+\int_{(C_2)}\vec{F}\cdot d\vec{r}+\cdots+\int_{(C_n)}\vec{F}\cdot d\vec{r}$$

4. Definite integral can be regarded as a special line integral of the second type. For example,

$$\int_a^b f(x)dx=\int_{(C)}P(x,y)dx+Q(x,y)dy$$

where (C) is the segment from $A(a,0)$ to $B(b,0)$, and

$$P(x,y)=f(x),Q(x,y)=0.$$

Properties of Line Integrals of the Second Type

Properties

1. Linearity Property

$$\int_{(C)} \left(k_1\vec{F_1}+k_2\vec{F_2}\right)\cdot\vec{dr} = k_1\int_{(C)}\vec{F_1}\cdot\vec{dr}+k_2\int_{(C)}\vec{F_2}\cdot\vec{dr}.$$

2. Additivity with respect to the domain of integration

Suppose that $(C)=(C_1)+(C_2)$ and $(C_1),(C_2)$ have no common part except for their boundaries. Then

$$\int_{(C)}\vec{F}\cdot\vec{dr}=\int_{(C_1)}\vec{F}\cdot\vec{dr}+\int_{(C_2)}\vec{F}\cdot\vec{dr}.$$

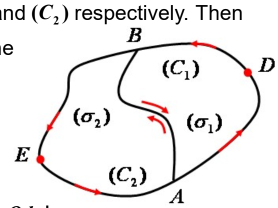
Properties of Line Integrals of the Second Type

3. If the direction of the path of integration (C) is reversed (denoted by $(-C)$), then the value of the line integral will change sign, namely

$$\int_{(C)}\vec{F}\cdot\vec{dr}=-\int_{(-C)}\vec{F}\cdot\vec{dr}.$$

Properties of Line Integrals of the Second Type

4. Suppose a plane domain bounded by a closed curve (C) is divided into two domains (σ_1) and (σ_2) with no common inner points, and their boundaries are denoted by (C_1) and (C_2) respectively. Then the line integral of the second type along the closed curve (C) is equal to the sum of the line integral of the second type along the closed curves (C_1) and (C_2) with the same direction as (C),



$$\oint_{(C)}[Pdx+Qdy]=\oint_{(C_1)}[Pdx+Qdy]+\oint_{(C_2)}[Pdx+Qdy].$$

Computation of Line Integral of the Second Type

Proposition Suppose that the parametric equation of a smooth oriented curve (C) is

$$\vec{\mathbf{r}}=\vec{\mathbf{r}}(t)=(x(t),y(t),z(t)) \quad (t\in[\alpha,\beta] \text{ or } [\beta,\alpha])$$

with $t=\alpha$ and $t=\beta$ corresponding to the initial and terminal points of the curve respectively, and that the vector valued function

$$\vec{F}(x,y,z)=(P(x,y,z),Q(x,y,z),R(x,y,z))$$

is continuous on the curve (C). Then

$$\int_{(C)}\vec{F}(x,y,z)\cdot\vec{dr}=\int_{(C)}[P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz].$$

$$=\int_\alpha^\beta\vec{F}(x(t),y(t),z(t))\cdot\vec{\mathbf{r}}'(t)dt$$

$$=\int_\alpha^\beta[P(x(t),y(t),z(t))\dot{x}(t)+Q(x(t),y(t),z(t))\dot{y}(t)+R(x(t),y(t),z(t))\dot{z}(t)]dt$$

Computation of Line Integral of the Second Type

Special case

Suppose the vector valued function

$$\vec{F}(x,y)=(P(x,y),Q(x,y))$$

is continuous on the plane curve (C). And the parametric equation of the simple smooth oriented plane curve (C) is

(1) $\vec{\mathbf{r}}=\vec{\mathbf{r}}(t)=(x(t),y(t)) \quad (t:\alpha\to\beta)$,

then $\int_{(C)}[P(x,y)dx+Q(x,y)dy]=\int_\alpha^\beta[P(x(t),y(t))\dot{x}(t)+Q(x(t),y(t))\dot{y}(t)]dt.$

(2) $y=y(x), \quad (x:a\to b)$,

then $\int_{(C)}[P(x,y)dx+Q(x,y)dy]=\int_a^b[P(x,y(x))+Q(x,y(x))y'(x)]dx.$

(3) $x=x(y), \quad (y:c\to d)$,

then $\int_{(C)}[P(x,y)dx+Q(x,y)dy]=\int_c^d[P(x(y),y)x'(y)+Q(x(y),y)]dy.$

Evaluating Line Integral of the Second Type

Example 1 Evaluate $\int_{(C)}(yzdx-xzdy+2z^2dz)$, where (C) is an oriented arc segment of the helix $x=a\cos t, y=a\sin t, z=kt$, from $t=0$ to $t=\pi$.

Solution

$$\int_{(C)}(yzdx-xzdy+2z^2dz)$$

$$=\int_0^\pi\left[a\sin t\cdot kt\cdot(a\cos t)'-a\cos t\cdot kt\cdot(a\sin t)'+2(kt)^2(kt)'\right]dt$$

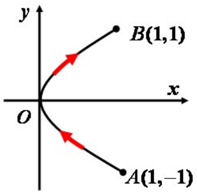
$$=k\pi^2\left(\frac{2}{3}k^2\pi-\frac{a^2}{2}\right).$$

Evaluating Line Integral of the Second Type

Example 2 Evaluate $\int_L xydx$, where L is a curve cut from a parabola $y^2 = x$ from point $A(1,-1)$ to point $B(1,1)$.

Solution I

$$AO:\begin{cases} x = x, \\ y = -\sqrt{x}, \end{cases} x:1 \to 0, \quad OB:\begin{cases} x = x, \\ y = \sqrt{x}, \end{cases} x:0 \to 1$$



$$\int_L xydx = \int_{AO} xydx + \int_{OB} xydx$$

$$= \int_1^0 x(-\sqrt{x})dx + \int_0^1 x\sqrt{x}dx$$

$$= 2\int_0^1 x^{\frac{3}{2}}dx = \frac{4}{5}.$$

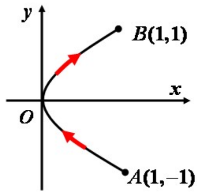
Evaluating Line Integral of the Second Type

Example 2 Evaluate $\int_L xydx$, where L is a curve cut from a parabola $y^2 = x$ from point $A(1,-1)$ to point $B(1,1)$.

Solution II

$$AB:\begin{cases} x = y^2, \\ y = y, \end{cases} -1 \le y \le 1$$

$$\int_L xydx = \int_{-1}^1 y^2 y(y^2)'dy = 2\int_{-1}^1 y^4 dy = \frac{4}{5}.$$



Evaluating Line Integral of the Second Type

Example 3. Evaluate $I = \int_L 2xydx - (3x + y)dy$, where the initial and terminal points of the path of integration are $O(0,0)$ and $A(1,1)$. And the path is

(1) the parabola $y = x^2$ (2) the parabola $x = y^2$

(3) the broken line OBA with $B = (0,1)$.

Solution

$$(1)\int_L 2xydx - (3x + y)dy = \int_0^1 [2x\cdot x^2 - (3x + x^2)\cdot 2x]dx = -2$$

$$(2)\int_L 2xydx - (3x + y)dy = \int_0^1 [2y^2\cdot y\cdot 2y - (3y^2 + y)]dy = -\frac{7}{10}$$

$$(3)\int_L 2xydx - (3x + y)dy$$

$$= \int_{OB} 2xydx - (3x + y)dy + \int_{BA} 2xydx - (3x + y)dy = \int_0^1 (-y)dy + \int_0^1 2xdx = \frac{1}{2}$$

Evaluating Line Integral of the Second Type

Example 4. Evaluate $I = \int_L 2yx^3dy + 3x^2y^2dx$, where the initial and terminal points of the path of integration are $O(0,0)$ and $A(1,1)$. And the path is

(1) the parabola $y = x^2$ (2) the straight line $y = x$

(3) the broken line OBA with $B = (0,1)$.

Solution $(1)\int_L 2yx^3dy + 3x^2y^2dx = \int_0^1 (2x^2\cdot x^3\cdot 2x + 3x^2\cdot x^4)dx = 1$

$$(2)\int_L 2yx^3dy + 3x^2y^2dx = \int_0^1 (2x\cdot x^3 + 3x^2\cdot x^2)dx = 1$$

$$(3)\int_L 2yx^3dy + 3x^2y^2dx = \int_0^1 2y\cdot 0^3 dy + \int_0^1 3x^2\cdot 1dx = 1$$

Note: For some kinds of line integrals of the second type, the value of an integral depends only the initial and terminal points and is independent of the path of integration.

Evaluating Line Integral of the Second Type

Example 5. Evaluate $\oint_{(C)} [(z - y)dx + (x - z)dy + (x - y)dz]$, where the curve (C) is the circle $\begin{cases} x^2 + y^2 = 1, \\ z = 0, \end{cases}$ with anticlockwise direction looking from the origin (0,0).

Solution The parametric equation of the curve (C) is

$$\begin{cases} x = \cos t, \\ y = \sin t, 0 \le t \le 2\pi. \\ z = 0 \end{cases}$$

Then $\oint_{(C)} [(z - y)dx + (x - z)dy + (x - y)dz]$

$$= \oint_{(C)} [(-y)dx + xdy] = \int_0^{2\pi} [-\sin t(\cos t)' + \cos t(\sin t)']dt$$

$$= \int_0^{2\pi} dt = 2\pi.$$

Properties of line integral of the second type

Properties:

(1) 轮换对称性:

$$\oint_{(C)} xdy + ydz + zdx = 3\oint_{(C)} xdy, (C):\begin{cases} x = y = z = 0 \\ x + y + z = 1 \end{cases}$$

(2) 有向性

$$\oint_{(+C)} Pdx + Qdy + Rdz = -\oint_{(-C)} Pdx + Qdy + Rdz$$

The Relationship between Two Types of the Line Integrals

Suppose that there is a smooth curve defined by

$$\vec{r} = \vec{r}(t) = (x(t), y(t), z(t)) \quad (t \in [\alpha, \beta] \text{ or } [\beta, \alpha])$$

with $t = \alpha$ and $t = \beta$ corresponding to the initial and terminal points of the curve respectively. By definition of the line integral of the second type, we have

$$\int_{(C)} \vec{F} \cdot \vec{dr} = \int_{(C)} Pdx + Qdy + Rdz = \int_{\alpha}^{\beta} [P\dot{x}(t) + Q\dot{y}(t) + R\dot{z}(t)]dt = \int_{\alpha}^{\beta} \vec{F} \cdot \vec{r}\,'(t)dt.$$

By definition of the line integral of the first type, if we let $\cos\alpha, \cos\beta, \cos\gamma$ denote the direction cosines of the tangent line at point $(x(t), y(t), z(t))$

$$\int_{\alpha}^{\beta} [P\dot{x}(t) + Q\dot{y}(t) + R\dot{z}(t)]dt = \int_{\alpha}^{\beta} \pm \frac{[P\dot{x}(t) + Q\dot{y}(t) + R\dot{z}(t)]}{\sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}} \cdot \pm\sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}dt$$

$$= \int_{\alpha}^{\beta} \pm(P, Q, R) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \cdot \pm\sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}dt = \int_{(C)} [P\cos\alpha + Q\cos\beta + R\cos\gamma]ds.$$

The Relationship between Two Types of the Line Integrals

Theorem. The relationship between two types of line integrals

$$\int_{(C)} Pdx + Qdy + Rdz = \int_{(C)} [P\cos\alpha + Q\cos\beta + R\cos\gamma]ds$$

That is $\int_{(C)} \vec{F} \cdot \vec{dr} = \int_{(C)} \vec{F} \cdot \vec{T}ds$, where

$$\vec{T} = (\cos\alpha, \cos\beta, \cos\gamma) = \pm\frac{(\dot{x}(t), \dot{y}(t), \dot{z}(t))}{\sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}}.$$

The Relationship between Two Types of the Line Integrals

Example 6. Change the line integral of the second type into the first type

$$I = \int_L P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz,$$

where L is the arc $x = t, y = t^2, z = t^3$ from the point $(1,1,1)$ to $(0,0,0)$.

Solution

$$\vec{r}\,'(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t)) = (1, 2t, 3t^2) = (1, 2x, 3y).$$

$$\vec{T} = -\frac{(\dot{x}(t), \dot{y}(t), \dot{z}(t))}{\sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}} = -\frac{(1, 2x, 3y)}{\sqrt{1 + 4x^2 + 9y^2}}.$$

$$I = \int_L Pdx + dy + dz = \int_L [P\cos\alpha + Q\cos\beta + R\cos\gamma]ds = -\int_L \frac{P + 2xQ + 3yR}{\sqrt{1 + 4x^2 + 9y^2}}ds.$$

Review

- Line Integral of the First Type
  (with respect to arc)
- Line Integral of the Second Type
  (with respect to coordinate)
- Computation of the Line Integrals
  (definite integral)
- Relations between two types of Line Integrals