Lecture 05

Chapter 2. Matrix Algebra

2.5 Determinant of a Matrix

2.6 Properties of Determinants

2.7 Cramer's Rule

Overview

With each **square matrix**, it is possible to associate a **number** called the **determinant of the matrix [矩阵的行列式]**. This value provides important information when the matrix is that of the coefficients of a system of a linear equations, or when it corresponds to a linear transformation of a vector space.

In this lecture, we will introduce the idea of determinant of matrices and show some properties of determinants. After that, we will show how the determinant of a matrix play an important role in some applications and give a theorem used to find the solution of a linear equations system.

2.5 Determinant of a Matrix

With each $n \times n$ matrix A, it is possible to associate a scalar, $\det(A)$, whose value will tell us whether the matrix is nonsingular.

Case I: 1×1 matrices

Let $A = (a)$ be a 1×1 matrix. Then A has a multiplicative inverse if and only if $a \neq 0$. Thus, if we define

$$\det(A) = a \ \text{ or } |A| = a,$$

the singularity of A can be determined by $\det(A)$.

A is nonsingular **if and only if** $\det(A) \neq 0$.

Case II: 2×2 matrices

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. We know that A is nonsingular **if and only if** A is row equivalent to I.

(i) If $a_{11} \neq 0$, we can test whether A is row equivalent to I by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}.$$

The resulting matrix is row equivalent to I **if and only if**

$$a_{11}a_{22} - a_{12}a_{21} \neq 0.$$

Case II: 2×2 matrices

(ii) If $a_{11} = 0$, interchange the two rows of A

$$A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{21} & a_{22} \\ 0 & a_{12} \end{pmatrix}$$

The resulting matrix is row equivalent to I **if and only** if $a_{12}a_{21} \neq 0$.

Define $$\det(A) = a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

for a 2×2 matrix A.

A is nonsingular **if and only if** $\det(A) \neq 0$.

Case III: 3×3 matrices

A 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is nonsingular **if and only if** A is row equivalent to I.

(i) If $a_{11} \neq 0$, then

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ 0 & \dfrac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{pmatrix}$$

The resulting matrix will be row equivalent to I if and only if the 2×2 matrix

$$\begin{pmatrix} \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ \dfrac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{pmatrix}$$

is nonsingular and $a_{11} \neq 0$. These requests can be equivalently written as

$$a_{11} \cdot \begin{vmatrix} \frac{a_{11}a_{22}-a_{21}a_{12}}{a_{11}} & \frac{a_{11}a_{23}-a_{21}a_{13}}{a_{11}} \\ \frac{a_{11}a_{32}-a_{31}a_{12}}{a_{11}} & \frac{a_{11}a_{33}-a_{31}a_{13}}{a_{11}} \end{vmatrix} \neq 0.$$

This condition can be simplified to

$$a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \neq 0.$$

The determinant of a 3×3 matrix A can be defined by

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} \\ + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

A is nonsingular **if and only if** $\det(A) \neq 0$.

Discuss the cases

(ii) $a_{11} = 0, a_{21} \neq 0$;

(iii) $a_{11} = a_{21} = 0, a_{31} \neq 0$;

(iv) $a_{11} = a_{21} = a_{31} = 0$.

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

To see how to calculate the determinant of a 3×3 matrix:

Sarrus' scheme

$$D = \left| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$$D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Or in other point of view,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Question: Define determinant for 4×4 matrices?

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

A is nonsingular **if and only if** A is row equivalent to I.

(i) $a_{11} \neq 0$

(ii) $a_{11} = 0,\ a_{21} \neq 0$

(iii) $a_{11} = a_{21} = 0,\ a_{31} \neq 0$

(iv) $a_{11} = a_{21} = a_{31} = 0,\ a_{41} \neq 0$

(v) $a_{11} = a_{21} = a_{31} = a_{41} = 0$

No Sarrus' rule!

Case IV: $n \times n$ matrices

Gottfried Wilhelm Leibniz (1646-1716), German mathematician and philosopher

Permutation group

Pierre-Simon, marquis de Laplace (1749-1827), French mathematician and astronomer

Submatrices

Observation

For a 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, we can rewrite its determinant as follows

$$\begin{aligned} \det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13}) \end{aligned}$$

where $M_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$, $M_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$, $M_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$.

Definition. Let $A = (a_{ij})$ be an $n \times n$ matrix and let M_{ij} be the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column. The determinant of M_{ij} is called the **minor [子式]** of entry a_{ij}. The **cofactor [代数余子式]** A_{ij} of a_{ij} is defined by

$$A_{ij} = (-1)^{i+j} \det\left(M_{ij}\right).$$

Example. $(n=2)$ Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then M_{ij}'s are given by

$$M_{11} = (a_{22}), \quad M_{12} = (a_{21}), \quad M_{21} = (a_{12}), \quad M_{22} = (a_{11}),$$

and the cofactors are given by

$$A_{11} = (-1)^{1+1}a_{22},\ A_{12} = (-1)^{1+2}a_{21},\ A_{21} = (-1)^{2+1}a_{12},\ A_{22} = (-1)^{2+2}a_{11}.$$

We may rewrite the determinant in the form

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} = a_{11}A_{11} + a_{12}A_{12},$$

or

$$\det(A) = a_{21}(-a_{12}) + a_{22}a_{11} = a_{21}A_{21} + a_{22}A_{22}.$$

Example. $(n = 3)$ Given a 3×3 matrix A, we have

$$M_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}, \quad M_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}, \quad M_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

and $$A_{11} = (-1)^{1+1}\det(M_{11}), \qquad A_{12} = (-1)^{1+2}\det(M_{12}),$$

$$A_{13} = (-1)^{1+3}\det(M_{13}).$$

The determinant of A can be rewritten as

$$\begin{aligned} \det(A) &= a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13}) \\ &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}. \end{aligned}$$

We can also write as

$$\det(A) = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23},$$

$$\det(A) = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}.$$

Example. If $A = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$, then

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 2 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} + 5 \cdot (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 4 \cdot (-1)^{1+3} \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix}$$

$$= 2 \cdot (6 - 8) - 5 \cdot (18 - 10) + 4 \cdot (12 - 5)$$

$$= -16.$$

Note: The determinant can be represented as a cofactor expansion using any row or column.

Definition 1. The **determinant [行列式]** of an $n \times n$ matrix A, denoted by $\det(A)$, is a scalar associated with the matrix A, that is defined inductively as follows

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

where

$$A_{1j} = (-1)^{1+j} \det\left(M_{1j}\right), \qquad j = 1,2,\cdots,n$$

are the cofactors associated with the entries in the first row of A.

Theorem. If A is an $n \times n$ matrix with $n \geq 2$, then $\det(A)$ can be expressed as a cofactor expansion using any row or column of A:

$$\begin{aligned} \det(A) &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \end{aligned}$$

for $i = 1,2,\cdots,n$ and $j = 1,2,\cdots,n$.

Example 1. Evaluate the determinant of $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 2 \\ 2 & 1 & 4 & 4 \\ 1 & 5 & 3 & 1 \end{pmatrix}$.

Solution. Expand along the first row, we get

$$\det(A) = 1 \cdot (-1)^{1+1} \begin{vmatrix} 3 & 1 & 2 \\ 1 & 4 & 4 \\ 5 & 3 & 1 \end{vmatrix} + 2 \cdot (-1)^{1+2} \begin{vmatrix} 2 & 1 & 2 \\ 2 & 4 & 4 \\ 1 & 3 & 1 \end{vmatrix} + 1 \cdot (-1)^{1+3} \begin{vmatrix} 2 & 3 & 2 \\ 2 & 1 & 4 \\ 1 & 5 & 1 \end{vmatrix} + 2 \cdot (-1)^{1+4} \begin{vmatrix} 2 & 3 & 1 \\ 2 & 1 & 4 \\ 1 & 5 & 3 \end{vmatrix}$$

$$= 1 \cdot \left(3 \cdot \begin{vmatrix} 4 & 4 \\ 3 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 5 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 4 \\ 5 & 3 \end{vmatrix}\right) - 2 \cdot \left(2 \cdot \begin{vmatrix} 4 & 4 \\ 3 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}\right)$$

$$+1 \cdot \left(2 \cdot \begin{vmatrix} 1 & 4 \\ 5 & 1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix}\right) - 2 \cdot \left(2 \cdot \begin{vmatrix} 1 & 4 \\ 5 & 3 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix}\right)$$

$$= -24 + 19 - 34 + 32 - 4 - 8 - 38 + 6 + 18 + 68 + 12 - 18 = 29.$$

Example 2. Evaluate $\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix}$.

Solution. We would expand along the first column. The first three will drop out, leaving

$$2(-1)^{4+1}\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -2 \cdot 3(-1)^{3+3}\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 12.$$

Finish.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Remark. By Laplace's definition of determinant, each term in the determinant is a product of n entries in different rows and different columns, the number of terms in the expansion is $n!$.

Example 3. Find the coefficient of x^3 in

$$\begin{vmatrix} 1 & x & 2 & 3 \\ x & 1 & 0 & 4 \\ 0 & 2 & 3 & 1 \\ 2 & 1 & x & 2 \end{vmatrix},$$

where x is a real variable.

2.6 Properties of Determinants

Exploring the properties of determinants may decrease the complexity of calculating a given determinant.

Determinant of Transposed Matrix

Theorem 1. For an $n \times n$ matrix A, we have $\det(A) = \det(A^T)$.

Proof. The proof follows by induction on n. Clearly the result holds if $n = 1$.

Assume that the result holds for all $k \times k$ matrices and that A is $(k+1) \times (k+1)$.

Expanding $\det(A)$ along the first row of A, we get

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \cdots + (-1)^{2+k} a_{1,k+1}\det\left(M_{1,k+1}\right).$$

Since the M_{1j}'s are all $k \times k$ matrices, it follows by induction hypothesis that

$$\det(A) = a_{11}\det\left(M_{11}^T\right) - a_{12}\det\left(M_{12}^T\right) + \cdots + (-1)^{2+k} a_{1,k+1}\det\left(M_{1,k+1}^T\right).$$

The right hand side is just the expansion by minors of $\det(A^T)$, along the first column of A^T. Therefore

$$\det(A) = \det\left(A^T\right).$$

Determinant of Triangular Matrices

Theorem 2. If A is an $n \times n$ triangular matrix, the determinant of A equals the product of all diagonal entries of A.

Proof. In view of $\det(A) = \det(A^T)$, it suffices to prove the theorem for lower triangular matrices.

The result follows easily by using a cofactor expansion and induction on n.

Determinant of Matrices with all zeros in a row or column

Theorem 3. Let A be an $n \times n$ matrix. If A has a row or column consisting entirely of zeros, then $\det(A) = 0$.

Proof. The statement can be proved by cofactor expansion along the row or column with all zero entries.

Determinant of Matrices with identical rows or columns

Theorem 4. Let A be an $n \times n$ matrix. If A has two identical rows or two identical columns, then $\det(A) = 0$.

Proof. In case $n = 2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{pmatrix}$, then $\det(A) = a_{11}a_{12} - a_{12}a_{11} = 0$.

The statement follows by induction on n.

Theorem 5. Let A be an $n \times n$ matrix and A_{jk} be the cofactor of a_{jk} for $j, k = 1,2,\cdots,n$. Then

$$\sum_{k=1}^{n} a_{ik} A_{jk} = a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A), & i = j, \\ 0, & i \neq j. \end{cases}$$

Proof. If $i = j$, this equation is nothing but the cofactor expansion of $\det(A)$ along the ith row of A.

In case that $i \neq j$, let A^* be the matrix obtained by replacing the jth row of A by the ith row of A

$$A^* = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ \color{red}{a_{i1}} & \color{red}{a_{i2}} & \color{red}{\cdots} & \color{red}{a_{in}} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{array}{l} \\ \\ \\ \\ \leftarrow \textbf{the } \boldsymbol{j}\textbf{th row} \\ \\ \\ \end{array}$$

Theorem 5. Let A be an $n \times n$ matrix and A_{jk} be the cofactor of a_{jk} for $j, k = 1,2,\cdots,n$. Then

$$\sum_{k=1}^{n} a_{ik} A_{jk} = a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A), & i = j, \\ 0, & i \neq j. \end{cases}$$

Proof. (continued)

$$A^* = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ \color{red}{a_{i1}} & \color{red}{a_{i2}} & \color{red}{\cdots} & \color{red}{a_{in}} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{array}{l} \\ \\ \\ \\ \leftarrow \textbf{the } \boldsymbol{j}\textbf{th row} \\ \\ \\ \end{array}$$

Since the ith row and the jth row are identically equal in matrix A^*, we can expand $\det(A^*)$ along the jth row and obtain

$$0 = \det(A^*) = \sum_{k=1}^{n} a_{ik} A_{jk}^* = \sum_{k=1}^{n} a_{ik} A_{jk}.$$

Algebraic Rules of Determinants

Theorem 7.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_{i1}+c_{i1} & b_{i2}+c_{i2} & \cdots & b_{in}+c_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Proof. Expand the determinant along the ith row.

Question: If E is an elementary matrix, then EA is the matrix obtained from A by doing the same elementary row operation.

$$\det(E) = ? \qquad \det(EA) = ?$$

(1) If E_α is the elementary matrix of **type II**, then $\det(E_\alpha) = \alpha$ and

$$E_\alpha A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

By expanding the determinant along the ith row, it follows that

$$\det(E_\alpha A) = \alpha \det(A).$$

(2) If E_m is the elementary matrix of **type III**, then $\det(E_m) = 1$,

$$E_mA = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} + ma_{j1} & a_{i2} + ma_{j2} & \cdots & a_{in} + ma_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Expanding the determinant along the ith row, using (1) and **Theorem 4**, it follows that

$$\det(E_mA) = \det(A).$$

(3) If E is the elementary matrix of **type I**, then

$$EA = \left(\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array}\right) \begin{array}{l} \\ \\ \leftarrow \textbf{the } \boldsymbol{i}\textbf{th row} \\ \\ \leftarrow \textbf{the } \boldsymbol{j}\textbf{th row} \\ \\ \\ \end{array}$$

It follows from **Theorem 7** and Properties (1),(2) that

$$\det(EA) = -\det(A).$$

In particular, $\det(E) = -1$.

Summary: If E is an elementary matrix, then

$$\det(EA) = \det(E)\det(A)$$

where

$$\det(E) = \begin{cases} -1, & \text{if } E \text{ is of } \textbf{type I} \\ \alpha \neq 0, & \text{if } E \text{ is of } \textbf{type II} \\ 1, & \text{if } E \text{ is of } \textbf{type III} \end{cases}$$

Thus we have

(i) Interchanging two rows of a matrix changes the sign of the determinant (**Theorem 2.6.10**);

(ii) Multiplying a single row of a matrix by a nonzero scalar has the effect of multiplying the value of determinant by the scalar (**Theorem 2.6.8**);

(iii) Adding a multiple of one row to another does not change the value of the determinant (**Theorem 2.6.9**).

Remark. The above properties that hold for rows also hold for columns.

Example 1. Evaluate $D = \begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix}$.

Solution.

$$D = \begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & -5 & 3 & -3 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 3 & 1 & -1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -5 & 3 & -3 \\ 0 & -24 & 18 & -19 \\ 0 & 10 & -5 & 5 \\ 0 & 16 & -10 & 11 \end{vmatrix}$$

$$= -\begin{vmatrix} -24 & 18 & -19 \\ 10 & -5 & 5 \\ 16 & -10 & 11 \end{vmatrix} = -5\begin{vmatrix} -24 & 18 & -19 \\ 2 & -1 & 1 \\ 16 & -10 & 11 \end{vmatrix} = -5\begin{vmatrix} 14 & -1 & -19 \\ 0 & 0 & 1 \\ -6 & 1 & 11 \end{vmatrix}$$

$$= 5\begin{vmatrix} 14 & -1 \\ -6 & 1 \end{vmatrix} = 40.$$

Example 2. Evaluate $D = \begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$.

Solution. Sum all other rows to the first row

$$D = \begin{vmatrix} 6 & 6 & 6 & 6 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 6 \times (1 \times 2 \times 2 \times 2) = 48.$$

Example 3. Evaluate $D = \begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}$

Solution.

$$D = \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & a & 2a+b & 3a+2b+c \\ 0 & a & 3a+b & 6a+3b+c \end{vmatrix} = \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & a & 3a+b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & 0 & a \end{vmatrix} = a^4.$$

Example 4. Evaluate $D_{2n} = \begin{vmatrix} a & & & & & b \\ & \ddots & & & \iddots & \\ & & a & b & & \\ & & c & d & & \\ & \iddots & & & \ddots & \\ c & & & & & d \end{vmatrix}$, where the matrix is $2n \times 2n$

and all entries except entries on the two diagonals are zeros.

$$D_2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Solution. Expand along the first row

$$D_{2n} = a \begin{vmatrix} a & & & & & b & 0 \\ & \ddots & & & \iddots & & \\ & & a & b & & & \\ & & c & d & & & \\ & \iddots & & & \ddots & & \\ c & & & & & d & \\ 0 & & & & & & d \end{vmatrix} + b(-1)^{1+2n} \begin{vmatrix} 0 & a & & & & & b \\ & & \ddots & & & \iddots & \\ & & & a & b & & \\ & & & c & d & & \\ & & \iddots & & & \ddots & \\ & c & & & & & d \\ c & & & & & & 0 \end{vmatrix},$$

$$= ad\, D_{2n-2} - bc(-1)^{2n-1+1} D_{2n-2} = (ad - bc)^1 D_{2(n-1)} = (ad - bc)^2 D_{2(n-2)}$$

$$= \cdots = (ad - bc)^{n-1} D_2 = (ad - bc)^n.$$

Example 5. (Vandermonde Matrix) Show that

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le j < i \le n} (x_i - x_j).$$

Solution. By mathematical induction on n. We have $D_2 = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$.

Assume that the statement holds for $n = k \geq 2$.

$$D_{k+1} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_{k+1} \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_{k+1}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^k & x_2^k & x_3^k & \cdots & x_{k+1}^k \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_{k+1} - x_1 \\ 0 & x_2^2 - x_1 x_2 & x_3^2 - x_1 x_3 & \cdots & x_{k+1}^2 - x_1 x_{k+1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_2^k - x_1 x_2^{k-1} & x_3^k - x_1 x_3^{k-1} & \cdots & x_{k+1}^k - x_1 x_{k+1}^{k-1} \end{vmatrix}$$

$$D_{k+1} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_{k+1} - x_1 \\ 0 & x_2^2 - x_1 x_2 & x_3^2 - x_1 x_3 & \cdots & x_{k+1}^2 - x_1 x_{k+1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_2^k - x_1 x_2^{k-1} & x_3^k - x_1 x_3^{k-1} & \cdots & x_{k+1}^k - x_1 x_{k+1}^{k-1} \end{vmatrix}$$

$$= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & \cdots & x_{k+1} - x_1 \\ x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_{k+1}(x_{k+1} - x_1) \\ \vdots & \vdots & & \vdots \\ x_2^{k-1}(x_2 - x_1) & x_3^{k-1}(x_3 - x_1) & \cdots & x_{k+1}^{k-1}(x_{k+1} - x_1) \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1) \cdots (x_{k+1} - x_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_{k+1} \\ \vdots & \vdots & & \vdots \\ x_2^{k-1} & x_3^{k-1} & \cdots & x_{k+1}^{k-1} \end{vmatrix}$$

Vandermonde determinant of order k

By induction hypothesis, we have

$$D_{k+1} = (x_2 - x_1)(x_3 - x_1) \cdots (x_{k+1} - x_1) \prod_{2 \le j < i \le k+1} (x_i - x_j) = \prod_{1 \le j < i \le k+1} (x_i - x_j) .$$

Example 6. Evaluate $D = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \\ 1 & 2^4 & 3^4 & 4^4 \end{vmatrix}.$

Solution.

$$D = 2 \cdot 3 \cdot 4 \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{vmatrix}$$

$$= 2 \cdot 3 \cdot 4 \cdot (2-1)(3-1)(4-1)(3-2)(4-2)(4-3)$$

$$= 288.$$

Determinant and Singularity of a Matrix

Theorem 12. An $n \times n$ matrix A is singular if and only if $\det(A) = 0$.

Proof. The matrix A can be reduced to row echelon form with a finite number of row operations, which means

$$U = E_k E_{k-1} \cdots E_1 A$$

where U is in row echelon form and E_i's are elementary matrices. Since

$$\det(U) = \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A)$$

and the determinants of E_i's are all nonzero, it follows that $\det(A) = 0$ if and only if $\det(U) = 0$.

If A is singular, then U has a row consisting entirely of zeros and hence $\det(U) = 0$.

If A is nonsingular, then U is triangular with 1's along the diagonal and hence $\det(U) = 1$.

Theorem 11. If A and B are $n \times n$ matrices, then
$$\det(AB) = \det(A)\det(B).$$

Proof. If B is singular, then AB is also singular and therefore,
$$\det(AB) = 0 = \det(A)\det(B).$$

If B is nonsingular, then B can be written as a product of elementary matrices. We have already seen that the result holds for elementary matrices. Thus

$$\begin{aligned}\det(AB) &= \det(AE_kE_{k-1}\cdots E_1)\\ &= \det(A)\det(E_k)\det(E_{k-1})\cdots\det(E_1)\\ &= \det(A)\det(E_kE_{k-1}\cdots E_1)\\ &= \det(A)\det(B)\end{aligned}$$

Remark. For any $n \times n$ matrices A, B, in general $AB \neq BA$, but we always have $\det(AB) = \det(BA)$.

2.7 Cramer’s Rule

Cramer’s rule allows us to list explicitly the solution of an $n \times n$ linear system.

Adjoint of a Matrix

Definition 1. Let A be an $n \times n$ matrix and A_{ij} be the cofactor of a_{ij}, where $i = 1,2,\cdots,n,\ j = 1,2,\cdots,n$. The **adjoint matrix [伴随矩阵]** of A is defined by

$$\mathrm{adj}A = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

To form the adjoint matrix of A, we have to replace each term by its cofactor and then transpose the resulting matrix.

$$A(\mathrm{adj}A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \det(A) \end{pmatrix}$$

$$\sum_{k=1}^{n} a_{ik} A_{jk} = \begin{cases} \det(A), & i = j, \\ 0, & i \neq j. \end{cases}$$

$$(\mathrm{adj}A)A = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \det(A) \end{pmatrix}$$

$$\sum_{k=1}^{n} a_{ki} A_{kj} = \begin{cases} \det(A), & i = j, \\ 0, & i \neq j. \end{cases}$$

Theorem 1. If A is an $n \times n$ nonsingular matrix and adj$A$ is the adjoint matrix of A, then we have

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A.$$

Example 1. For a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the adjoint matrix is

$$\operatorname{adj}(A) = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

and $\det(A) = a_{11}a_{22} - a_{12}a_{21}$. By **Theorem 1**, in case that $a_{11}a_{22} - a_{12}a_{21} \neq 0$,

we have

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \\ 3 & 0 & 1 \end{pmatrix}$. Find A^{-1}.

Solution. The adjoint matrix of A is

$$\operatorname{adj}A = \begin{pmatrix} \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & -1 & -2 \\ 4 & -5 & 2 \\ -6 & 3 & 0 \end{pmatrix}$$

and

$$\det(A) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \\ 3 & 0 & 1 \end{vmatrix} = -6.$$

Therefore,

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}A = \frac{1}{-6}\begin{pmatrix} 2 & -1 & -2 \\ 4 & -5 & 2 \\ -6 & 3 & 0 \end{pmatrix}.$$

Cramer’s Rule

Theorem 2. Let A be an $n \times n$ nonsingular matrix and let $\boldsymbol{b} \in \mathbf{R}^n$. Let A_j be the matrix obtained by replacing the jth column of A by $\boldsymbol{b}$. If $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ is the unique solution of the linear system $A\boldsymbol{x} = \boldsymbol{b}$, then

$$x_j = \frac{\det(A_j)}{\det(A)}, \qquad j = 1,2,\cdots,n.$$

Proof. Since A is nonsingular, we know that

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \frac{1}{\det(A)}(\mathrm{adj}A)\boldsymbol{b}.$$

$$(\mathrm{adj}A)\boldsymbol{b} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} b_i A_{i1} \\ \sum_{i=1}^{n} b_i A_{i2} \\ \vdots \\ \sum_{i=1}^{n} b_i A_{in} \end{pmatrix}.$$

By Laplace's definition of determinant, we know that

$$\sum_{i=1}^{n} b_i A_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = \det(A_j)$$

which can be obtained by replacing the jth column of the $\det(A)$. It follows that

$$x_j = \frac{1}{\det(A)}\left(b_1 A_{1j} + b_2 A_{2j} + \cdots + b_n A_{nj}\right) = \frac{\det(A_j)}{\det(A)}.$$

Example 3. Solve the following linear system by using Cramer's rule.

$$\begin{aligned} 2x_1 + \ \ x_2 - 5x_3 + \ \ x_4 &= \ \ 8, \\ x_1 - 3x_2 \qquad\qquad - 6x_4 &= \ \ 9, \\ 2x_2 - \ \ x_3 + 2x_4 &= -5, \\ x_1 + 4x_2 - 7x_3 + 6x_4 &= \ \ 0. \end{aligned}$$

Solution. Let A be the coefficient matrix of the linear system and $\boldsymbol{b}$ be the right-hand side

$$A = \begin{pmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} \mathbf{8} \\ \mathbf{9} \\ \mathbf{-5} \\ \mathbf{0} \end{pmatrix}$$

It is clear that $\det(A) = 27 \neq 0$ and

$$\det(A_1) = \begin{vmatrix} \mathbf{8} & 1 & -5 & 1 \\ \mathbf{9} & -3 & 0 & -6 \\ \mathbf{-5} & 2 & -1 & 2 \\ \mathbf{0} & 4 & -7 & 6 \end{vmatrix} = 81, \quad \det(A_2) = \begin{vmatrix} 2 & \mathbf{8} & -5 & 1 \\ 1 & \mathbf{9} & 0 & -6 \\ 0 & \mathbf{-5} & -1 & 2 \\ 1 & \mathbf{0} & -7 & 6 \end{vmatrix} = -108$$

$$\det(A_3) = \begin{vmatrix} 2 & 1 & \mathbf{8} & 1 \\ 1 & -3 & \mathbf{9} & -6 \\ 0 & 2 & \mathbf{-5} & 2 \\ 1 & 4 & \mathbf{0} & 6 \end{vmatrix} = -27, \quad \det(A_4) = \begin{vmatrix} 2 & 1 & -5 & \mathbf{8} \\ 1 & -3 & 0 & \mathbf{9} \\ 0 & 2 & -1 & \mathbf{-5} \\ 1 & 4 & -7 & \mathbf{0} \end{vmatrix} = 27.$$

Then by Cramer's rule, we have $x_1 = 3,\ x_2 = -4,\ x_3 = -1,\ x_4 = 1.$

Review

- Definition of determinant
- Properties of determinant
- Definition of adjoint matrix
- Cramer's rule

Preview

The definition of vector spaces