Ch.2 *Time Domain Representations of Linear Time-Invariant Systems* (III)

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Outline

- Linear Time-invariant systems (LTI)
  - Differential and Difference Equation Representations of LTI systems
  - Block Diagram Representations

Differential and Difference Equation Representations of LTI systems

- **Linear constant-coefficient** difference and differential equations provide **another representation** for the input-output characteristics of LTI systems.
- Difference equations are used to represent discrete-time systems, while differential equations represent continuous-time systems.
- Linear constant-coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Input = $x(t)$, output = $y(t)$

- Linear constant-coefficient difference equation:

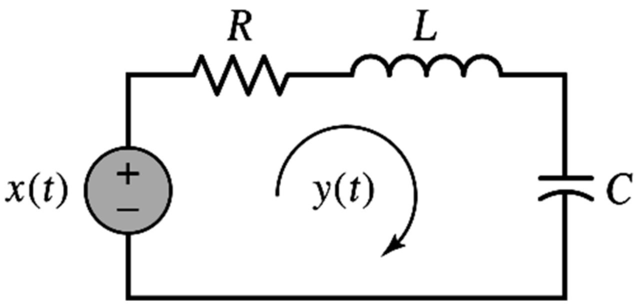
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Input = $x[n]$, output = $y[n]$

The *order* of the differential or difference equation is (N, M).
Often, $N \geq M$, and the order is described using only N.

Ex. RLC circuit depicted in Fig. 2.26.

Input $x(t)$ = voltage source, output $y(t)$ = loop current



$$Ry(t)+L\frac{d}{dt}y(t)+\frac{1}{C}\int_{-\infty}^{t}y(\tau)\,d\tau=x(t)$$

$$\frac{1}{C}y(t)+R\frac{d}{dt}y(t)+L\frac{d^2}{dt^2}y(t)=\frac{d}{dt}x(t)$$

$\Rightarrow$ ***N* = 2**

Ex. Second-order difference equation:

$$y[n]+y[n-1]+\frac{1}{4}y[n-2]=x[n]+2x[n-1]$$

$\Rightarrow$ ***N* = 2**

Differential and Difference Equation Representations of LTI systems

- Difference equations are easily rearranged to obtain recursive formulas for computing the current output of the system from the input signal and the past outputs.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \Longrightarrow y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k] - \frac{1}{a_0}\sum_{k=1}^{N} a_k y[n-k]$$

Ex. $y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$

$$\Longrightarrow y[n] = x[n] + 2x[n-1] - y[n-1] - \frac{1}{4}y[n-2]$$

$$y[0] = x[0] + 2x[-1] - y[-1] - \frac{1}{4}y[-2]$$

$$y[1] = x[1] + 2x[0] - y[0] - \frac{1}{4}y[-1]$$

$$y[2] = x[2] + 2x[1] - y[1] - \frac{1}{4}y[0] \quad \cdots\cdots$$

- Initial conditions: $y[-1]$ and $y[-2]$.

Differential and Difference Equation Representations of LTI systems

- **The initial conditions represent the "memory" of an LTI system.**

$$y[n]=\frac{1}{a_0}\sum_{k=0}^{M}b_k x[n-k]-\frac{1}{a_0}\sum_{k=1}^{N}a_k y[n-k]$$

$$\sum_{k=0}^{N}a_k\frac{d^k}{dt^k}y(t)=\sum_{k=0}^{M}b_k\frac{d^k}{dt^k}x(t)$$

  - The initial conditions for Nth-order difference equation are the ***N*** values.

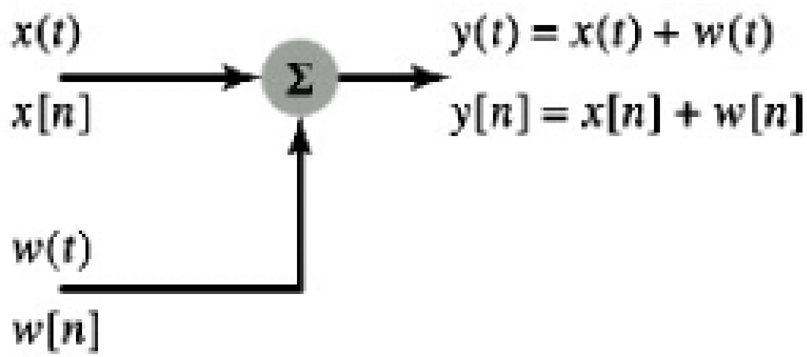
$$y[-N],y[-N+1],\ldots,y[-1],$$

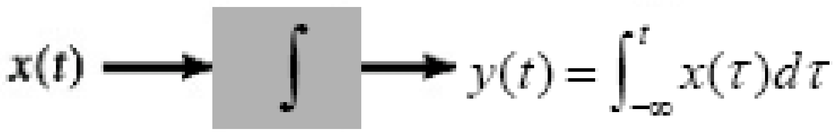
  - The initial conditions for Nth-order differential equation are the ***N*** values.

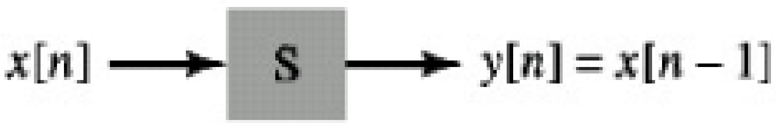
$$y(t)\Big|_{t=0-},\quad \frac{d}{dt}y(t)\Big|_{t=0-},\quad \frac{d^2}{dt^2}y(t)\Big|_{t=0-},\quad \ldots,\quad \frac{d^{N-1}}{dt^{N-1}}y(t)\Big|_{t=0-}$$

Block Diagram Representations

- **A block diagram is an interconnection of the elementary operations that act on the input signal.**
- **Four elementary operations for block diagram:**
  - **Scalar multiplication:** $y(t) = cx(t)$ or $y[n] = cx[n]$, where c is a scalar
  - **Addition:** $y(t) = x(t) + w(t)$ or $y[n] = x[n] + w[n]$
  - **Integration for continuous-time *LTI* system:** $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
  - **A time shift for discrete-time LTI system:** $y[n] = x[n - 1]$

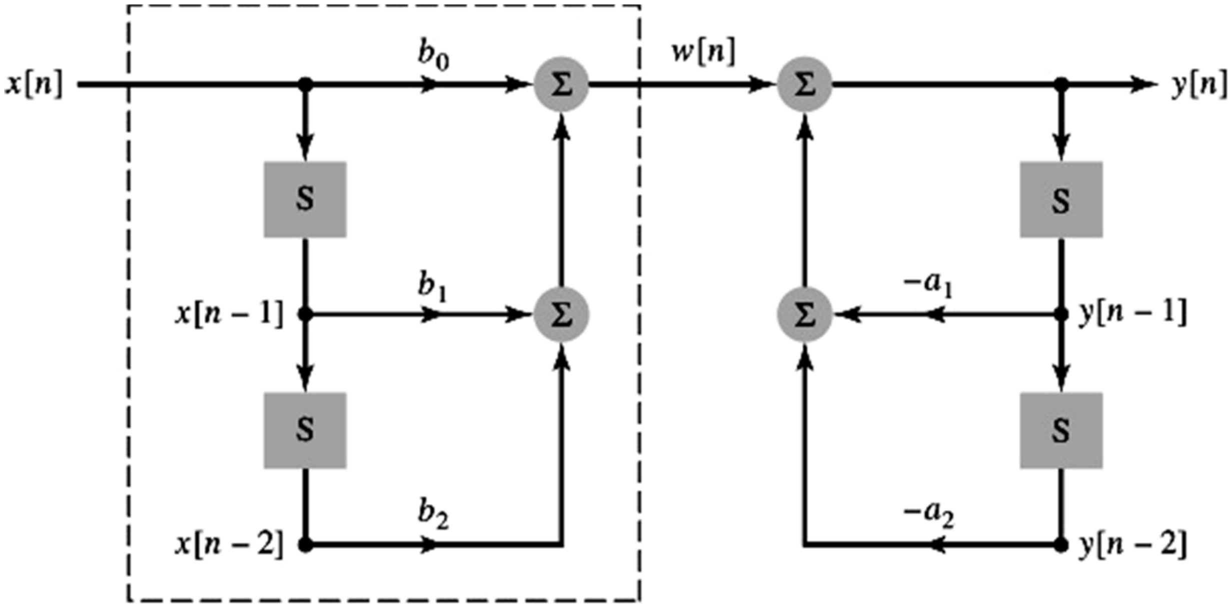






Block Diagram Representations

- **Ex. A discrete-time LTI system: Direct Form I (Cascade Form)**



$$w[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$y[n] = w[n] - a_1 y[n-1] - a_2 y[n-2]$$

$$\Rightarrow \quad y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

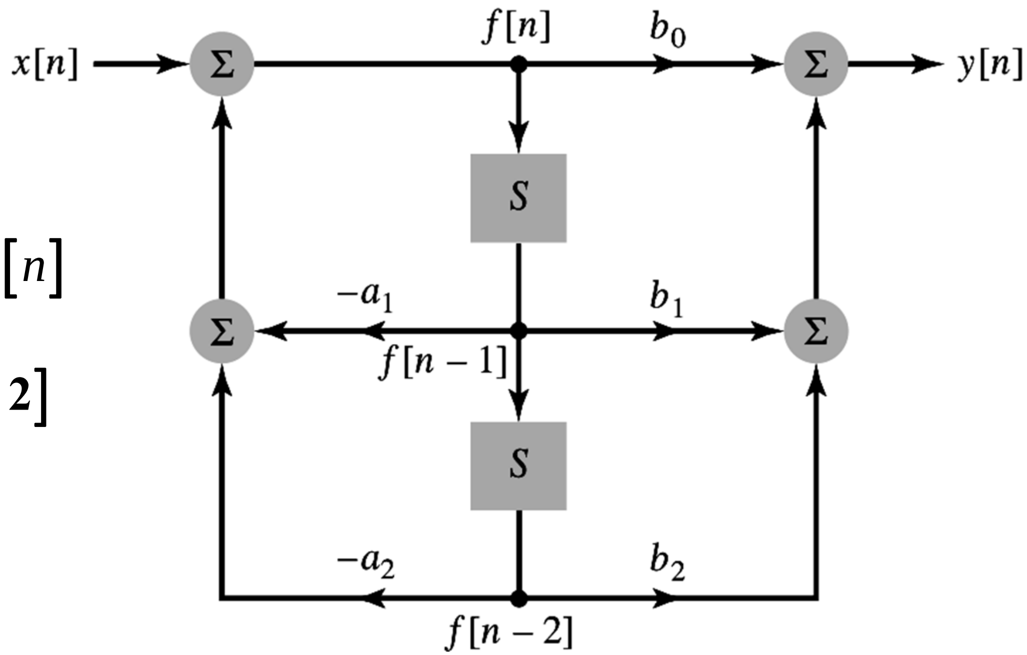
$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

Block Diagram Representations

- **Ex. A discrete-time LTI system: Direct Form II**
  - Interchange the order of Direct Form I: $h_1(t) * h_2(t) = h_2(t) * h_1(t)$
  - Denote the output of the new first system as $f[n]$.

$$f[n] = -a_1 f[n-1] - a_2 f[n-2] + x[n]$$

$$y[n] = b_0 f[n] + b_1 f[n-1] + b_2 f[n-2]$$



  - Direct Form II uses memory more efficiently.

Block Diagram Representations

- **Block diagram representation for continuous-time LTI system**

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \qquad \textbf{(2.54)}$$

  - Let $\boldsymbol{v^{(0)}(t)} = \boldsymbol{v(t)}$ be an arbitrary signal, and set

$$v^{(n)}(t) = \int_{-\infty}^{t} v^{(n-1)}(\tau)d\tau, \quad n = 1, \; 2, \; 3, \; \dots$$

  - Integrator with initial condition:

$$v^{(n)}(t) = \int_{0}^{t} v^{(n-1)}(\tau)d\tau + v^{(n)}(0), \quad n = 1, \; 2, \; 3, \; \dots$$

$$\Rightarrow \quad \frac{d}{dt} v^{(n)}(t) = v^{(n-1)}(t), \quad t > 0 \quad \text{and} \quad n = 1, \; 2, \; 3, \; \dots$$

  - Integrate N times to eq. (2.54)

$$\sum_{k=0}^{N} a_k y^{(N-k)}(t) = \sum_{k=0}^{M} b_k x^{(N-k)}(t)$$

Block Diagram Representations

- **Ex. Second-order system:**

$$y(t) = -a_1 y^{(1)}(t) - a_0 y^{(2)}(t) + b_2 x(t) + b_1 x^{(1)}(t) + b_0 x^{(2)}(t)$$