Lecture 15

Chapter 6 Matrix Diagonalization

6.4 Eigenvalues and Eigenvectors

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Problem:

apply a linear operator repeatedly to a given vector, i.e. determine the vector $\boldsymbol{w} = A^m \boldsymbol{x_1}, \boldsymbol{x_1} \in V$,

$$\boldsymbol{w} = A^m \boldsymbol{x_1} = \underbrace{A \cdot A \cdots A}_{m \text{ times}} \boldsymbol{x_1}, \qquad \boldsymbol{x_1} \in V.$$

This equation can be calculated as follows

$$\boldsymbol{y_1} = A\boldsymbol{x_1},$$

$$\boldsymbol{y_2} = A\boldsymbol{y_1} = A(A\boldsymbol{x_1}) = A^2\boldsymbol{x_1},$$

$$\cdots$$

$$\boldsymbol{w} = \boldsymbol{y_m} = A\boldsymbol{y_{m-1}} = A(A^{m-1}\boldsymbol{x_1}) = A^m\boldsymbol{x_1}.$$

If the vector $\boldsymbol{x_1}$ satisfies $A\boldsymbol{x_1} = \lambda_1 \boldsymbol{x_1}$

where λ_1 is a scalar, the previous process cannot be simpler

$$\boldsymbol{y_1} = A\boldsymbol{x_1} = \lambda_1 \boldsymbol{x_1},$$

$$\boldsymbol{y_2} = A\boldsymbol{y_1} = A(\lambda_1 \boldsymbol{x_1}) = \lambda_1 A\boldsymbol{x_1} = \lambda_1^2 \boldsymbol{x_1},$$

$$\dots$$

$$\boldsymbol{w} = \boldsymbol{y_m} = A\boldsymbol{y_{m-1}} = A(\lambda_1^{m-1} \boldsymbol{x_1}) = \lambda_1^{m-1} A\boldsymbol{x_1} = \lambda_1^{m} \boldsymbol{x_1}.$$

If we choose $\boldsymbol{x_1}$ as a basis of the vector space V, the linear transformation A from V to itself may be particularly simple.

Since for any vector $\boldsymbol{v} \in V$ and $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}\}$ is a basis of V, we must have

$$\boldsymbol{v} = c_1 \boldsymbol{x_1} + c_2 \boldsymbol{x_2} + \cdots + c_n \boldsymbol{x_n}$$

$$
\begin{aligned}
A\boldsymbol{v} &= A(c_1\boldsymbol{x_1} + c_2\boldsymbol{x_2} + \cdots + c_n\boldsymbol{x_n}) \\
&= c_1A\boldsymbol{x_1} + c_2A\boldsymbol{x_2} + \cdots + c_nA\boldsymbol{x_n} \\
&= c_1\lambda_1\boldsymbol{x_1} + c_2A\boldsymbol{x_2} + \cdots + c_nA\boldsymbol{x_n}.
\end{aligned}
$$

We may expect that $\boldsymbol{x_2}, \dots, \boldsymbol{x_n}$ share the same property as that for $\boldsymbol{x_1}$, that is

$$A\boldsymbol{x_j} = \lambda_j\boldsymbol{x_j}, \qquad j = 2, \dots, n.$$

Then

$$
\begin{aligned}
A\boldsymbol{v} &= c_1A\boldsymbol{x_1} + c_2A\boldsymbol{x_2} + \cdots + c_nA\boldsymbol{x_n} \\
&= c_1\lambda_1\boldsymbol{x_1} + c_2\lambda_2\boldsymbol{x_2} + \cdots + c_n\lambda_n\boldsymbol{x_n}.
\end{aligned}
$$

Furthermore, it is easy to calculate that

$$A^m\boldsymbol{v} = c_1\lambda_1^m\boldsymbol{x_1} + c_2\lambda_2^m\boldsymbol{x_2} + \cdots + c_n\lambda_n^m\boldsymbol{x_n}.$$

Questions:

(1) Can we find the scalar λ and vector $\boldsymbol{x}$ so that $A\boldsymbol{x} = \lambda\boldsymbol{x}$?

(2) How to find them?

Concepts and Examples

Definition 1. Let A be an $n \times n$ matrix. The **eigenvalue [本征值]** or a **characteristic value [特征值]** of A is a scalar, say λ, such that

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

holds for a **nonzero** vector $\boldsymbol{x}$. The vector $\boldsymbol{x}$ is said to be an **eigenvector [本征向量]** or a **characteristic vector [特征向量]** belonging to the eigenvalue λ.

Example 1. Consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- We have $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Then $\lambda_1 = 1$ is an eigenvalue of A, and $(-1,1)^T$ is the eigenvector of A belonging to $\lambda_1 = 1$.

- We also have $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

So $\lambda_2 = 3$ is also an eigenvalue of A, and $(1,1)^T$ is the corresponding eigenvector.

How to find eigenvalues and eigenvectors?

$$A\boldsymbol{x} = \lambda \boldsymbol{x} \;\Leftrightarrow\; (A - \lambda I)\boldsymbol{x} = \boldsymbol{0}.$$

λ is an eigenvalue of A

$\Leftrightarrow$ the homogeneous system $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ has nontrivial solutions.

$\Leftrightarrow$ the coefficient matrix $A - \lambda I$ is singular.

Calculate eigenvalues of a matrix:

solve the equation $\det(A - \lambda I) = 0$.

Calculate the corresponding eigenvectors:

solve the linear system $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$; find the nullspace $N(A - \lambda I)$ (**eigenspace [本征空间]** corresponding to the eigenvalue λ)

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

$p(\lambda)$ is a polynomial of the unknown value λ, which is called the **characteristic polynomial [特征多项式].**

The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation [特征方程]** of the matrix A.

Question 1: How many eigenvalues does an $n \times n$ matrix have?

A polynomial of degree n has exactly n roots in the field of complex numbers, if we count the multiplicity of these roots. (代数学基本定理)

As a result, an $n \times n$ matrix A has exactly n eigenvalues, some of which may be repeated and some of which may be complex numbers.

Question 2: How many eigenvectors are there corresponding to an eigenvalue λ?

Infinite many.

Let A be an $n \times n$ matrix and λ be a scalar. The following statements are **equivalent**:

(a) λ is an eigenvalue of A.

(b) $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution.

(c) $N(A - \lambda I) \neq \{\mathbf{0}\}$.

(d) $A - \lambda I$ is singular.

(e) $\det(A - \lambda I) = 0$.

Example 2. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Find the eigenvalues of A and the corresponding eigenspace.

Solution. The characteristic polynomial of A is

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (\lambda - 1)(\lambda - 3).$$

The characteristic polynomial has two solutions $\lambda_1 = 1$ and $\lambda_2 = 3$.

As $\lambda_1 = 1$, the corresponding eigenvector can be found by solving the linear system $\quad (A - \lambda_1 I)\boldsymbol{x} = (A - I)\boldsymbol{x} = \mathbf{0}$.

$$(A - I|\mathbf{0}) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The solution to the system is $\boldsymbol{x} = (-\alpha, \alpha)^T, \alpha \in \mathbf{R}$.

Eigenspace: $\quad N(A - \lambda_1 I) = \text{Span}\{(-1,1)^T\}$.

Eigenvectors: $\quad \alpha(-1,1)^T, \ \ \alpha \neq 0$.

Example 2. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Find the eigenvalues of A and the corresponding eigenspace.

Solution. (continue)

As $\lambda_2 = 3$, we have to solve the following linear system

$$(A - \lambda_2 I)\boldsymbol{x} = (A - 3I)\boldsymbol{x} = \mathbf{0}.$$

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Solutions: $\boldsymbol{x} = (\alpha, \alpha)^T, \alpha \in \mathbf{R}$.

Eigenspace: $N(A - \lambda_2 I) = \mathrm{Span}\{(1,1)^T\}$.

Eigenvectors: $\alpha(1,1)^T, \alpha \neq 0$.

Example 3. Let $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$. Find the eigenvalues of A and the corresponding eigenspace.

Solution. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = -\lambda(\lambda - 1)^2.$$

Thus the eigenvalues of A are $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$.

For $\lambda_1 = 0$, the eigenspace is $N(A)$,

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution to $A\boldsymbol{x} = 0$: $\boldsymbol{x} = (\alpha, \alpha, \alpha)^T, \alpha \in \mathbf{R}$.

The eigenspace corresponding to $\lambda_1 = 0$ consists of all vectors of the form $\alpha(1,1,1)^T$.

Example 3. Let $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$. Find the eigenvalues of A and the corresponding eigenspace.

Solution. (continue)

For $\lambda_2 = \lambda_3 = 1$, we solve the system $(A - I)\boldsymbol{x} = \mathbf{0}$:

$$A - I = \begin{pmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Setting $x_2 = \alpha$ and $x_3 = \beta$, we get $x_1 = 3\alpha - \beta$ and solution.

$$x = (3\alpha - \beta, \alpha, \beta), \quad \alpha, \beta \in \mathbf{R}.$$

Thus the eigenspace corresponding to $\lambda_2 = \lambda_3 = 1$ consists of all vectors of form

$$\begin{pmatrix} 3\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \alpha, \beta \in \mathbf{R}.$$

Complex Eigenvalues

If A is an $n \times n$ matrix with **real** entries, then the coefficients of the characteristic polynomial of A are real. Hence, all its complex roots must occur in **conjugate pairs**. That is, we have the following

Property. If $\lambda = a + bi$ $(b \neq 0)$ is an eigenvalue of a real matrix A, then $\bar{\lambda} = a - bi$ must also be an eigenvalue of A.

Property. If $\lambda = a + bi$ $(b \neq 0)$ is an eigenvalue of a real matrix A, then $\bar{\lambda} = a - bi$ must also be an eigenvalue of A.

Proof. If $A = (a_{ij})$ is a matrix with complex entries, then $\bar{A} = (\bar{a}_{ij})$ is the matrix formed from A by conjugating each of its entries. Therefore, A is a real matrix if and only if $\bar{A} = A$.

If λ is a complex eigenvalue of a real $n \times n$ matrix A and $\boldsymbol{z}$ is an eigenvector belonging to λ, then

$$A\bar{\boldsymbol{z}} = \bar{A}\bar{\boldsymbol{z}} = \overline{A\boldsymbol{z}} = \overline{\lambda\boldsymbol{z}} = \bar{\lambda}\bar{\boldsymbol{z}},$$

which means that $\bar{\lambda}$ is an eigenvalue of A and $\bar{\boldsymbol{z}}$ is an eigenvector corresponding to $\bar{\lambda}$.

Product and Sum of Eigenvalues

Characteristic polynomial of A

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} \quad \textbf{(1)}$$

- Expanding the determinant in **(1)** along the first column, we have

$$\det(A - \lambda I) = (a_{11} - \lambda)\det(M_{11}) + \sum_{i=2}^{n} a_{i1}(-1)^{i+1}\det(M_{i1}),$$

where the minors $M_{i1}, i = 2, \dots, n$, do not contain the two diagonal elements $(a_{11} - \lambda)$ and $(a_{ii} - \lambda)$. Expanding $\det(M_{11})$ in the same way, we conclude that

$$(a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda) \quad \textbf{(2)}$$

is the only term involving a product of more than $n - 2$ of the diagonal elements.

Characteristic polynomial of A

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} \qquad \textbf{(1)}$$

- In **(2)**, the coefficient of λ^n is $(-1)^n$. Hence if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, then

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n). \qquad \textbf{(3)}$$

Comparing the coefficient of λ^{n-1} of **(1)** and **(3)**, we find that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}\,.$$

Trace [迹] of the matrix A

- Take $\lambda = 0$ in the equalities **(1)** and **(3)**, we have

$$\prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n = \det(A)\,.$$

Theorem 1. Let A be an $n \times n$ matrix and λ_i, $i = 1,2, \ldots, n$ be eigenvalues of A, then

(1) $$\prod_{i=1}^{n} \lambda_i = \det(A)$$

(2) $$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = \mathrm{tr}(A).$$

Eigenvalues and Eigenvectors of Similar Matrices

Theorem 2. Let $A \sim B$, then they have the same characteristic polynomial and eigenvalues.

Proof. A and B are similar matrices, so there exists an invertible matrix S such that $B = S^{-1}AS$. Thus

$$\begin{aligned} p_B(\lambda) = \det(B - \lambda I) &= \det(S^{-1}AS - \lambda S^{-1}IS) \\ &= \det(S^{-1}(A - \lambda I)S) \\ &= \det\left(S^{-1}\right)\det(A - \lambda I)\det S \\ &= p_A(\lambda). \end{aligned}$$

Example. $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}.$

It is easy to see that the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 3.$

Let $S = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ and set $B = S^{-1}AS$, then the eigenvalues of B should be the same as those of A by **Theorem 2**.

$$B = S^{-1}AS = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 6 & 6 \end{pmatrix}.$$

Check directly that the eigenvalues of B are $\lambda_1 = 2$ and $\lambda_2 = 3$.

6.5 Diagonalization

We consider the possibility of a matrix that is similar to a diagonal matrix.

Let us get back to the problem of computing the series

$$\boldsymbol{x}, A\boldsymbol{x}, A^2\boldsymbol{x}, \ldots$$

If the matrix A is similar to a diagonal matrix Λ, say

$$A = S^{-1}\Lambda S,$$

where S is an invertible matrix, then

$$A^m\boldsymbol{x} = \left(S^{-1}\Lambda S\right)^m\boldsymbol{x} = S^{-1}\Lambda^m S\boldsymbol{x},$$

which is much easier to compute.

Suppose that there exists an invertible matrix X such that

$$X^{-1}AX = \Lambda.$$

This equation implies that

$$AX = X\Lambda. \qquad \textbf{(1)}$$

If we denote the ith column of X by $\boldsymbol{x_i}$, and

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

Then **(1)** implies that

$$A(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}) = (\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

$$A(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}) = (\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

$$(A\boldsymbol{x_1}, A\boldsymbol{x_2}, \dots, A\boldsymbol{x_n}) = (\lambda_1\boldsymbol{x_1}, \lambda_2\boldsymbol{x_2}, \dots, \lambda_n\boldsymbol{x_n})$$

For any $i = 1,2, \dots, n$, we have $A\boldsymbol{x_i} = \lambda_i\boldsymbol{x_i}$.

The ith column vector $\boldsymbol{x_i}$ of X is in fact an eigenvector of A belonging to eigenvalue $\lambda_i, i = 1,2, \dots, n$.

Moreover, the eigenvectors $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$ are linearly independent.

As a result, A has n linearly independent eigenvectors.

Conversely, suppose that A has n linearly independent eigenvectors.

Suppose that $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$ are the n linearly independent eigenvectors of A, and

$$A\boldsymbol{x_i} = \lambda_i \boldsymbol{x_i}.$$

Let $X = (\boldsymbol{x_1}, \dots, \boldsymbol{x_n})$. Then X is invertible, and

$$AX = A(\boldsymbol{x_1}, \dots, \boldsymbol{x_n}) = (A\boldsymbol{x_1}, \dots, A\boldsymbol{x_n}) = (\lambda_1\boldsymbol{x_1}, \dots, \lambda_n\boldsymbol{x_n})$$

$$= (\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = X\Lambda.$$

Therefore, A is similar to a diagonal matrix.

Definition 1. Let A be an $n \times n$ matrix. If there exists an invertible matrix X and a diagonal matrix Λ, such that

$$X^{-1}AX = \Lambda,$$

we say A is **diagonalizable [可对角化]** and X **diagonalizes [对角化]** A.

Theorem 1. An $n \times n$ matrix A is diagonalizable **if and only if** A has n linearly independent eigenvectors.

Remark. Assume that A is diagonalizable.

- Columns of X: **eigenvectors** of A
- Diagonal elements of Λ: **eigenvalues** of A

Example 1. Let $B = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Show that B is diagonalizable.

Find the invertible matrix X and the diagonal matrix Λ such that $X^{-1}BX = \Lambda$.

Proof. Consider the characteristic equation of B,

$$\det(B - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)^2.$$

the eigenvalues of B are $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = 2.$

For $\lambda_1 = \lambda_2 = 1$: $\quad B - I = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

Eigenspace $\quad N(B - 1 \cdot I) = \mathrm{Span}\{\boldsymbol{x_1} = (1,1,0)^T, \boldsymbol{x_2} = (0,0,1)^T\}.$

The vectors $\boldsymbol{x_1}, \boldsymbol{x_2}$ are eigenvectors belonging to eigenvalue $\lambda_1 = \lambda_2 = 1$ and they are linearly independent.

Example 1. Let $B = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Show that B is diagonalizable.

Find the invertible matrix X and the diagonal matrix Λ such that $X^{-1}BX = \Lambda$.

Proof. (continue) For $\lambda_3 = 2$,

$$B - 2I = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

eigenspace $\quad N(B - 2 \cdot I) = \mathrm{Span}\{\boldsymbol{x_3} = (1,0,0)^T\}.$

We can take $\boldsymbol{x_3}$ as an eigenvector belong to eigenvalue $\lambda_3 = 2$.

As a result, we obtain $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ 3 eigenvectors of B, which are linearly independent. Therefore, the matrix B is diagonalizable, and it is similar to the diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Example 1. Let $B = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Show that B is diagonalizable.

Find the invertible matrix X and the diagonal matrix Λ such that $X^{-1}BX = \Lambda$.

Proof. (continue) Let $X = (\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Then X is invertible and $X^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

Finally one can verify that

$$X^{-1}BX = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \Lambda.$$

Remark. The choice of the invertible matrix X is not unique.

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Is it possible to diagonalize A?

Solution. Consider the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda).$$

The eigenvalues of A are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$.

For $\lambda_1 = \lambda_2 = 1$,

$$A - I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Eigenspace $\quad N(A - I) = \mathrm{Span}\{(1,0,0)^T\}$.

This vector space is one-dimensional, so that we can only choose **one** linearly independent eigenvector belonging to eigenvalue $\lambda_1 = \lambda_2 = 1$, such as $\quad \boldsymbol{x_1} = (1,0,0)^T$.

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Is it possible to diagonalize A?

Solution. (continue)

For $\lambda_3 = 2$,

$$A - 2I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

eigenspace $N(A - I) = \text{Span}\{(0,0,1)^T\}$.

The eigenvector belonging to eigenvalue λ_3 is

$$\boldsymbol{x}_3 = (0,0,1)^T.$$

Since the number of all linearly independent eigenvectors is **fewer** than the number of eigenvalues, A **cannot** be diagonalized.

Review

- Eigenvalues and eigenvectors of a matrix
- Diagonalization of a matrix

Preview

- Quadratic Form and its applications