Lecture 13

Green's Formula and Its Applications

The Relation between Line Integral of Second Type and Double Integral

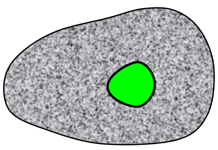
- Simple curve, closed curve
- Simply connected domain
- Multiply connected domain
- Positive direction of (C)

The Relation between Line Integral of Second Type and Double Integral

1. If a curve does not intersect itself anywhere between its endpoints, we call it a simple curve. A curve is called closed if its terminal point coincide with its initial point.

2. A simply connected domain D is a domain such that every simple closed contour within it enclose only points of D.

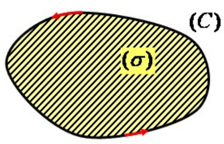
3. A domain that is not simply connected is said to be multiply connected.

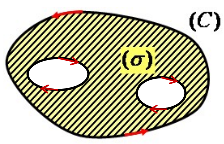


The Relation between Line Integral of Second Type and Double Integral

4. The positive orientation of a simple closed curve (C) refers to a single counterclockwise traversal of (C), we denoted by (+C).

If (C) is the boundary of multiply connected region (σ), the positive orientation of the (C) defined as follows: when one walks along (+C), the region always at his (her) left-side.



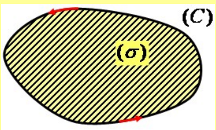


Green's Theorem

Theorem Suppose there is a closed bounded domain $(\sigma)\subset \mathbf{R}^2$ bounded by a piecewise smooth simple curve (C), and functions $P(x,y),Q(x,y)\in C^{(1)}((\sigma))$. Then the following relation holds:

$$\iint_{(\sigma)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma=\oint_{(+C)}\left[P(x,y)dx+Q(x,y)dy\right],$$

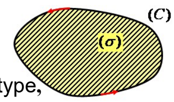
where $(+C)$ indicates that the integration is in the positive direction of (C).



The Proof of Green's Theorem

Proof

Case I. if $(\sigma)\subset \mathbf{R}^2$ is a simple connected region.

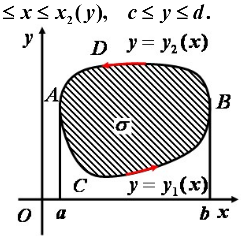


i) Suppose that the domain (σ) is both x-type and y-type, which can be expressed as

$$y_1(x)\le y\le y_2(x),\quad a\le x\le b \quad \text{or} \quad x_1(y)\le x\le x_2(y),\quad c\le y\le d.$$

Then $\oint_{(C)} P(x,y)dx=-\iint_{(\sigma)}\frac{\partial P}{\partial y}d\sigma$

$$\iint_{(\sigma)}\frac{\partial P}{\partial y}d\sigma=\int_a^b dx\int_{y_1(x)}^{y_2(x)}\frac{\partial P}{\partial y}dy$$
$$=\int_a^b\left[P\left(x,y_2(x)\right)-P\left(x,y_1(x)\right)\right]dx.$$



On the other hand,

$$\oint_{(+C)} P(x,y)dx=\int_{\overset{\frown}{ACB}} P(x,y)dx+\int_{\overset{\frown}{BDA}} P(x,y)dx,$$

$$=\int_a^b P\left(x,y_1(x)\right)dx+\int_b^a P\left(x,y_2(x)\right)dx=-\int_a^b\left[P\left(x,y_2(x)\right)-P\left(x,y_1(x)\right)\right]dx.$$

The Proof of Green's Theorem

Proof (continued)

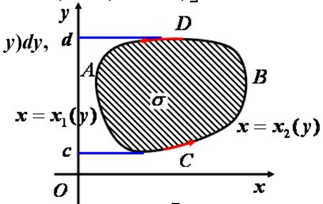
Similarly, we have

$$\iint_{(\sigma)} \frac{\partial Q}{\partial x} d\sigma = \int_c^d dy \int_{x_1(x)}^{x_2(x)} \frac{\partial Q}{\partial x} dx = \int_c^d \left[Q(x_2(y), y) - Q(x_1(y), y)\right] dx,$$

and $\oint_{(+C)} Q(x,y)dy = \int_{\overset{\frown}{DAC}} Q(x,y)dy + \int_{\overset{\frown}{CBD}} Q(x,y)dy,$

$$= \int_d^c Q(x_1(y), y)dy + \int_c^d Q(x_2(y), y)dy$$

$$= \int_c^d \left[Q(x_2(y), y) - Q(x_1(y), y)\right] dy.$$



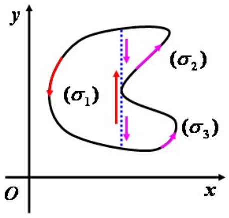
Therefore, $\iint_{(\sigma)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma = \oint_{(+C)} \left[P(x,y)dx + Q(x,y)dy\right].$

The Proof of Green's Theorem

Proof (continued)

ii) Suppose that the domain (σ) can be divided into some subdomains $(\sigma_i)(i=1,2,3)$ by a line parallels to y – axis or parallels to x – axis, each domain is type of domain in condition i).

Since the sum of line integral of second type over the common edges of these subdomains is zero, we can obtain



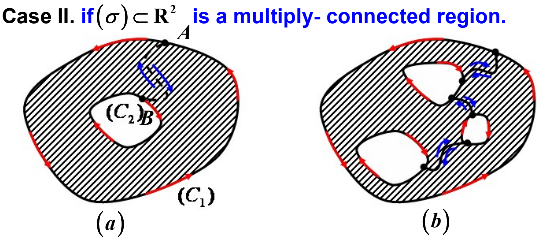
$$\iint_{(\sigma)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma = \sum_{i=1}^{3} \iint_{(\sigma_i)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma$$

$$= \sum_{i=1}^{3} \oint_{(+C_i)} \left[P(x,y)dx + Q(x,y)dy\right]$$

$$= \oint_{(+C)} \left[P(x,y)dx + Q(x,y)dy\right].$$

The Proof of Green's Theorem

Case II. if $(\sigma) \subset \mathbf{R}^2$ **is a multiply- connected region.**



We can cut (σ) by adding one or several secants to reduce (σ) into a simply-connected region. For example, in Figure (a) above,

$$(+C) = (+C_1) \cup (-C_2)$$

Add a secant $\overset{\frown}{AB}$ to cut the region (σ), then (σ) becomes a simply-connected region and its boundary $(\overline{C})$ is

The Proof of Green's Theorem

$$(\overline{C}) = (+C_1) \cup \overset{\frown}{AB} \cup (-C_2) \cup \overset{\frown}{BA}.$$

Then $\iint_{(\sigma)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma = \oint_{(+\overline{C})} [Pdx + Qdy]$

$$= \oint_{(+C_1)} [Pdx + Qdy] + \oint_{(\overset{\frown}{AB})} [Pdx + Qdy] + \oint_{(-C_2)} [Pdx + Qdy] + \oint_{(\overset{\frown}{BA})} [Pdx + Qdy]$$

$$= \oint_{(+C_1)} [Pdx + Qdy] + \oint_{(-C_2)} [Pdx + Qdy]$$

$$= \oint_{(+C)} [Pdx + Qdy].$$

If the multiply-connected region (σ) has more than one hole (see Figure (b)), we can handle it similarly. Therefore, Green's formula holds for multiply-connected regions as well.

The Relation between Line Integral of Second Type and Double Integral

Note: $\oint_{(+C)} \left[P(x,y)dx + Q(x,y)dy\right] = \iint_{(\sigma)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma$

(1) $(+C)$ denotes the positively oriented (C).

$$\oint_{(-C)} \left[P(x,y)dx + Q(x,y)dy\right] = -\iint_{(\sigma)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma$$

(2) $(+C)$ is a closed curve.

(3) $P(x,y), Q(x,y) \in C^{(1)}((\sigma)).$

Use Green's Theorem to Find the Line Integral of the Second Type

$$\oint_{(+C)} \left[P(x,y)dx + Q(x,y)dy\right] = \iint_{(\sigma)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma$$

Example 1. Find $I = \oint_{(+C)} (x+y)dx - (x-y)dy,$ where (C) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Solution Since $P = x + y$ and $Q = y - x,$ then, by Green's Theorem, we have $I = \iint_{(\sigma)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = -2\iint_{(\sigma)} dxdy = -2\pi ab.$

Use Green's Theorem to Find the Line Integral of the Second Type

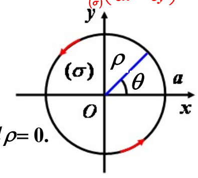
Example 2. Find $I=\oint_{(+C)} xy^2dx-x^2ydy$, where (C) is $x^2+y^2=a^2$.

Solution Since $P=xy^2$ and $Q=-x^2y$, $\oint_{(+C)}\left[P(x,y)dx+Q(x,y)dy\right]=\iint_{(\sigma)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma$

then, by Green's Theorem, we have

$$I=\oint_{(C)} xy^2dx-x^2ydy=\iint_{(\sigma)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma$$

$$=\iint_{(\sigma)}(-4xy)\,dxdy \underset{0\le\rho\le a,0\le\theta\le2\pi}{\overset{x=\rho\cos\theta,y=\rho\sin\theta}{=}} -4\int_0^{2\pi}\cos\theta\sin\theta d\theta\int_0^a\rho^2\rho d\rho=0.$$



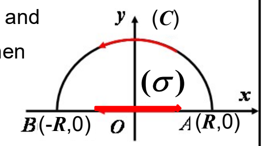
DIY Find $I=\oint_{(+C)} xy^2dx-x^2ydy$, where (C) is the upper semicircle $y=\sqrt{a^2-x^2}$.

Use Green's Theorem to Find the Line Integral of the Second Type

Example 3 Evaluate $I=\int_{(C)} xy^2dy-x^2ydx$, where (C) is the half circle $y=\sqrt{R^2-x^2}$ form $A(R,0)$ to $B(-R,0)$.

Solution Since (C) is not closed, so we can not use Green's formula directly. We supplement the path of integration with the directed line segment $\overrightarrow{BA}$, and denote $(C^*)=(C)\cup\overrightarrow{BA}$, then (C^*) is closed. Then

$$I=\oint_{(C^*)} xy^2dy-x^2ydx-\int_{\overline{BA}} xy^2dy-x^2ydx\triangleq I_1+I_2.$$



Applying Green's Theorem for I_1, we have

$$I_1=\oint_{(C^*)} xy^2dy-x^2ydx=\iint_{(\sigma)}(y^2+x^2)dxdy=\int_0^{\pi}d\theta\int_0^R\rho^2\rho d\rho=\frac{\pi}{4}R^4.$$

$$I_2=\int_{\overline{BA}} xy^2dy-x^2ydx=\int_{-R}^{R}x^2\cdot 0dx=0.\quad\Rightarrow I=\frac{\pi}{4}R^4.$$

Find Area of a Region by Line Integrals of Second type

Note If $P(x,y)=-y$ and $Q(x,y)=x$, we can calculate the area of a region (σ), which is bounded by (C), by

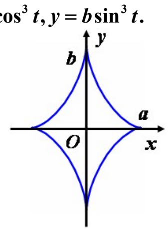
$$\sigma=\iint_{(\sigma)}dxdy=\frac{1}{2}\oint_{(C)}xdy-ydx.$$

Example Find the area of the starlike shape $x=a\cos^3 t, y=b\sin^3 t$.

Solution Area $=\frac{1}{2}\oint_{(C)}xdy-ydx$

$$=\frac{1}{2}\int_0^{2\pi}(3ab\cos^4 t\sin^2 t dt+3ab\sin^4 t\cos^2 t)dt$$

$$=\frac{1}{2}\int_0^{2\pi}3ab\cos^2 t\sin^2 t dt=\frac{3}{8}\pi ab.$$



Use Green's Theorem to Find the Double Integral

Example 3. Find $I=\iint_D e^{-y^2}dxdy$, where D is the triangle whose vertices are $O(0,0)$, $A(1,1)$ and $B(0,1)$.

Solution **Method I.**

Take $P=0$ and $Q=xe^{-y^2}$, then, by Green's Theorem, we have

$$I=\iint_{(\sigma)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\oint_{(+C)}xe^{-y^2}dy=\int_{\overline{OA}}xe^{-y^2}dy=\int_0^1ye^{-y^2}dy=\frac{1}{2}\left(1-e^{-1}\right).$$

Method II. $I=\int_0^1dy\int_0^y e^{-y^2}dx=\int_0^1ye^{-y^2}dy=\frac{1}{2}\left(1-e^{-1}\right).$

Use Green's Theorem to Find the Double Integral

Example 4. Find $I=\oint_{(C)}\frac{xdy-ydx}{x^2+y^2}$ where (C) is the counter-clockwise unit circle.

Note: The condition $P(x,y),Q(x,y)\in C^{(1)}((\sigma))$ must be satisfied in the theorem. So, we can't use Green's formula to solve the problem, since $Q(x,y)=\frac{x}{x^2+y^2},P(x,y)=\frac{-y}{x^2+y^2}\notin C^{(1)}((\sigma))$.

Solution The parametric equation of (C) is $\begin{cases}x=\cos t\\ y=\sin t\end{cases}, t:0\to2\pi.$

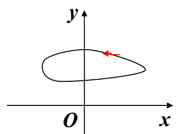
$$I=\oint_{(C)}\frac{xdy-ydx}{x^2+y^2}=\int_0^{2\pi}\frac{\cos^2 t+\sin^2 t}{\cos^2 t+\sin^2 t}dt=2\pi.$$

Use Green's Theorem to Find the Line Integral of the Second Type

Example 4. Find $I=\oint_{(+C)}\frac{xdy-ydx}{x^2+y^2}$, where (C) is any piecewise smooth simple closed curve with positive direction which does not pass through the origin.

Solution Let $Q(x,y)=\frac{x}{x^2+y^2},P(x,y)=\frac{-y}{x^2+y^2}$.

Since (C) does not pass through the origin, there are two cases.

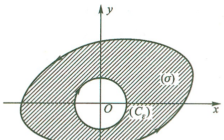


Case I. if the interior of (C) does not contain the origin, it is easy to see that $P(x,y),Q(x,y)\in C^{(1)}((\sigma))$, and $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, and so by Green's formula

$$\oint_{(C)}\frac{xdy-ydx}{x^2+y^2}=\iint_{(\sigma)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=0.$$

Use Green's Theorem to Find the Line Integral of the Second Type

Example 4. Find $I=\oint_{(+C)}\frac{xdy-ydx}{x^2+y^2}$, where (C) is any piecewise smooth simple closed curve with positive direction which does not pass through the origin.



Case II. If the interior of (C) contain the origin
Then, taking a small circle (C_ε) with positive direction, then by Green's formula over $(C)\cup(-C_\varepsilon)$,

$$\oint_{(C)\cup(-C_\varepsilon)}\frac{xdy-ydx}{x^2+y^2}=\iint_{(\sigma)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=0.$$

Thus $$\oint_{(C)}\frac{xdy-ydx}{x^2+y^2}=-\oint_{(-C_\varepsilon)}\frac{xdy-ydx}{x^2+y^2}=\oint_{(+C_\varepsilon)}\frac{xdy-ydx}{x^2+y^2}$$

$$\overset{x=\varepsilon\cos t}{\underset{y=\varepsilon\sin t}{=}}\int_0^{2\pi}\frac{\varepsilon\cos t\cdot\varepsilon\cos t+\varepsilon\sin t\cdot\varepsilon\sin t}{\varepsilon^2\cos^2 t+\varepsilon^2\sin^2 t}dt=2\pi.$$

Path Independence

If A and B are two points in a region D in space, the work $\int\vec{F}\cdot\vec{dr}$ done in moving a particle from A to B by a field $\vec{F}$ defined on D usually depends on the path taken.

Example 1. Evaluate $I=\int_L 2xydx-(3x+y)dy$, where the initial and terminal points of the path of integration are $O(0,0)$ and $A(1,1)$.
And the path is (1) the parabola $y=x^2$ (2) the parabola $x=y^2$
(3) the broken line OBA with $B=(0,1)$.

Solution $(1)\int_L 2xydx-(3x+y)dy=\int_0^1[2x\cdot x^2-(3x+x^2)\cdot 2x]dx=-2$

$(2)\int_L 2xydx-(3x+y)dy=\int_0^1[2y^2\cdot y\cdot 2y-(3y^2+y)]dy=-\frac{7}{10}$

$(3)\int_L 2xydx-(3x+y)dy$

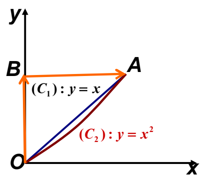
$=\int_{OB}2xydx-(3x+y)dy+\int_{BA}2xydx-(3x+y)dy=\int_0^1(-y)dy+\int_0^1 2xdx=\frac{1}{2}$

Path Independence

For some kinds of line integrals of the second type, the value of the integral depends only on the initial point A and the terminal point B and is independence of the path of integration.

Example 2 Evaluate $I=\int_L 2yx^3 d$ ... points of the path of integration ...

(1) the parabola $y=x^2$ (2) the straight line $y=x$
(3) the broken line OBA with $B=(0,1)$.



If the path from O to A is (C_1), then

$$\int_{(C)}2yx^3dy+3x^2y^2dx=\int_0^1 7x^6dx=1$$

If the path from O to A is (C_2), then

$$\int_{(C)}2yx^3dy+3x^2y^2dx=\int_0^1 5x^4dx=1$$

If the path from O to A is (C_3), then

$$\int_{(C)}2yx^3dy+3x^2y^2dx=0+\int_0^1 3x^2dx=1$$

Question When will the line integral of the second type independent on the path taken?

Path Independence

Definition Path Independence and Conservative Field

Let $\vec{F}$ be a field defined on a region D in space and suppose that the line integral $\int_{(C)}\vec{F}\cdot\vec{dr}$ is independent of the path (C) for any path (C) from A to B contained in D. Then the integral $\int_{(C)}\vec{F}\cdot\vec{dr}$ is **independent of the path of integration in D** and the vector field $\vec{F}$ is **a conservative field(保守场) in D**. In this case, the line integral can be written as $\int_A^B\vec{F}\cdot\vec{dr}$.

Path Independence

Theorem Suppose a region $G\subset R^2, P(x,y),Q(x,y)\in C(G)$.
Then the following three propositions are equivalent:

1. Along any piecewise smooth simple closed curve (C) inside G, the line integral $\oint_{(C)}P(x,y)dx+Q(x,y)dy=0.$
2. The value of the line integral $\int_A^B P(x,y)dx+Q(x,y)dy$ is independent of the path of integration in G.
3. The integrand representation $P(x,y)dx+Q(x,y)dy$ is a total differential of some function of two variables $u(x,y)$ in G, that is $du=P(x,y)dx+Q(x,y)dy$.

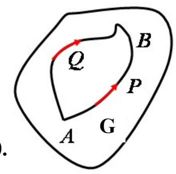
u is called a potential function of $\vec{F}=(P,Q)$ on D.

Proof We will prove these statements in the order

$$1^\circ\Rightarrow 2^\circ\Rightarrow 3^\circ\Rightarrow 1^\circ.$$

Path Independence

Proof (continued) $1^\circ\Rightarrow 2^\circ$ Suppose that A and B are two points in G,

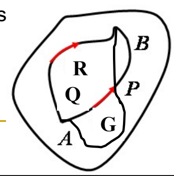


We travel from A to B by taking two arbitrary curves inside (σ), denoted by $\overset{\frown}{APB}$ and $\overset{\frown}{AQB}$. If these two curves do not intersect, then by proposition 1°

$$\int_{APB}[Pdx+Qdy]+\int_{BQA}[Pdx+Qdy]=\oint_{APBQA}[Pdx+Qdy]=0.$$

Therefore $\int_{APB}[Pdx+Qdy]=-\int_{BQA}[Pdx+Qdy]=\int_{AQB}[Pdx+Qdy].$

If these two curves have a intersection other than A and B, we can draw another curve $\overset{\frown}{ARB}\subset G$ from A and B such that it does intersect with both $\overset{\frown}{APB}$ and $\overset{\frown}{AQB}$ except for A and B.



$$\int_{APB}[Pdx+Qdy]=\int_{ARB}[Pdx+Qdy]=\int_{AQB}[Pdx+Qdy].$$

Path Independence

Proof (continued) $2^\circ \Rightarrow 3^\circ$ We take any point $A(x_0,y_0)\in G$ and form the line integral with variable upper limit $\int_{(x_0,y_0)}^{(x,y)}[Pdx+Qdy]$, under proposition 2°, we denote this integral by

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)}[Pdx+Qdy].$$

We will prove that $du=Pdx+Qdy$ or $\frac{\partial u}{\partial x}=P,\frac{\partial u}{\partial y}=Q.$ In fact, by the definition of the partial derivative of u, we have

$$\frac{\partial u}{\partial x}=\lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x},$$

while $u(x+\Delta x,y)-u(x,y)=\int_{(x_0,y_0)}^{(x+\Delta x,y)}[Pdx+Qdy]-\int_{(x_0,y_0)}^{(x,y)}[Pdx+Qdy].$

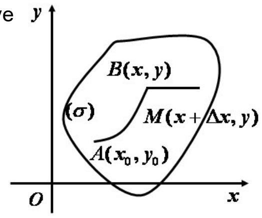
Path Independence

Proof (continued) By proposition 2°, we have

$$u(x+\Delta x,y)-u(x,y)=\int_{(x,y)}^{(x+\Delta x,y)}[Pdx+Qdy].$$

Then $\int_{(x,y)}^{(x+\Delta x,y)}[Pdx+Qdy]=\int_{x}^{x+\Delta x}Pdx$

$$=P(x+\theta\Delta x,y)\Delta x, 0\le\theta\le 1.$$



Thus, by the continuity of $P(x,y)$, we have

$$\frac{\partial u}{\partial x}=\lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}=\lim_{\Delta x\to 0}P(x+\theta\Delta x,y)=P(x,y).$$

Similarly, we have $\frac{\partial u}{\partial y}=Q(x,y).$ Then the proposition has been proved.

Path Independence

Proof (continued) $3^\circ \Rightarrow 1^\circ$ Suppose that (C) is an arbitrary piecewise smooth simple closed curve, whose equations are $x=x(t), y=y(t),$ $t:\alpha\to\beta$ and $x(\alpha)=x(\beta), y(\alpha)=y(\beta).$ Then

$$\oint_{(C)}[Pdx+Qdy]=\int_\alpha^\beta\{P[x(t),y(t)]\dot{x}(t)+Q[x(t),y(t)]\dot{y}(t)\}dt$$

$$=\int_\alpha^\beta\left[\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}\right]dt=\int_\alpha^\beta\left(\frac{d}{dt}u[x(t),y(t)]\right)dt=u[x(t),y(t)]\Big|_\alpha^\beta=0.$$

Path Independence

Note:

(1) If we regard the vector field $(P(x,y),Q(x,y))$ as a velocity field $v(x,y)$ of a fluid in the plane, namely $v=P\mathrm{i}+Q\mathrm{j}$, then

$$\oint_{(C)}[Pdx+Qdy]=\oint_{(C)}v\cdot d\vec{r}=\oint_{(C)}v\cdot\vec{T}ds$$

where $\vec{T}$ is the unit tangent vector of (C). The line integral expresses the quantity of the fluid flowing along the closed curve (C) per unit time. So we call the line integral

$$\oint_{(C)}\vec{F}\cdot d\vec{r}$$

the **circulation(环流量)** of the vector field $\vec{F}$ along the closed curve (C).

Path Independence

Note:

(2) **Physical interpretation of Theorem above**:

Proposition 1 $\oint_{(C)}P(x,y)dx+Q(x,y)dy=0.$

We call the field $\vec{F}$ an **irrotational vector field (无旋向量场)**.

Proposition 2 $\int_A^B P(x,y)dx+Q(x,y)dy$ is independent of the path of integration in G.

$\vec{F}=(P,Q)$ is a **conservative field(保守场).**

Proposition 3 There is a differentiable scalar field $u(x,y),(x,y)\in G$ such that $\nabla u=(P,Q).$

The field $\vec{F}=(P,Q)$ is called a **gradient field (梯度场).** $u(x,y)$ is called the **potential function** of $\vec{F}$, so $\vec{F}$ is also called a **potential field(势场).**

$\vec{F}$ is an irrotational vector field $\Longleftrightarrow$ $\vec{F}$ is a conservative field
$\Longleftrightarrow$ $\vec{F}$ is a gradient field

Path Independence

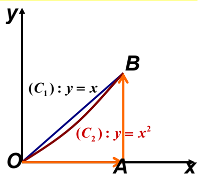
Theorem Let (σ) be a simply connected region in the plane, $P,Q,$ $\frac{\partial P}{\partial y},\frac{\partial Q}{\partial x}\in C((\sigma))$, then

The three propositions in last theorem are true $\Longleftrightarrow$ $\frac{\partial P}{\partial y}\equiv\frac{\partial Q}{\partial x},(x,y)\in(\sigma).$

Example $I=\int_L 2yx^3dy+3x^2y^2dx,$

$$P(x,y)=3x^2y^2, Q(x,y)=2yx^3,$$

$$\frac{\partial P(x,y)}{\partial y}=6x^2y=\frac{\partial Q(x,y)}{\partial x}.$$



说明: 根据上述定理，若在某区域G内$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$，则

1) 计算曲线积分时, 可选择方便的积分路径;

2) 求曲线积分时, 可利用格林公式简化计算,

若积分路径不是闭曲线, 可添加辅助线;

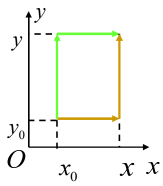
3) 可用积分法求$\mathrm{d}u=P\,\mathrm{d}x+Q\,\mathrm{d}y$在域 G 内的原函数:

取定点 $(x_0,y_0)\in G$ 及动点$(x,y)\in G$, 则其中一个原函数为

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)} P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y$$

$$=\int_{x_0}^{x} P(x,y_0)\,\mathrm{d}x+\int_{y_0}^{y} Q(x,y)\,\mathrm{d}y$$

或 $u(x,y)=\int_{y_0}^{y} Q(x_0,y)\,\mathrm{d}y+\int_{x_0}^{x} P(x,y)\,\mathrm{d}x$

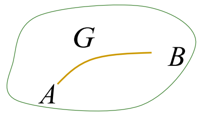


4) 若已知$\mathrm{d}u=P\,\mathrm{d}x+Q\,\mathrm{d}y$,则对$G$内任一分段光滑曲线$\overset{\frown}{AB}$,有

$$\int_{\overset{\frown}{AB}} P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y$$

$$=\int_A^B P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y$$

$$=\int_A^B \mathrm{d}u=u\Big|_A^B=u(B)-u(A)$$



注: 此式称为**曲线积分的基本公式**.它类似于微积分基本公式:

$$\int_a^b f(x)\,\mathrm{d}x=\int_a^b \mathrm{d}F(x) \quad (\text{其中 } F'(x)=f(x))$$

$$=F(x)\Big|_a^b=F(b)-F(a)$$

Path Independence

Example 3. Find the integral $\int_{(0,0)}^{(1,1)}(x-y)(dx-dy)$.

Solution Since $P(x,y)=x-y$ and $Q(x,y)=y-x$, then, we have

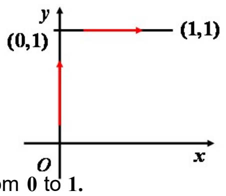
$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=-1.$$

Therefore, the integral does not depend on the path taken.

Method I. taking path $(0,0)\to(0,1)\to(1,1)$,

$$I=\int_0^1 Q(0,y)dy+\int_0^1 P(x,1)dx$$

$$=\int_0^1 ydy+\int_0^1 (x-1)dx=\frac{1}{2}+\frac{1}{2}-1=0.$$



Method II. taking path $(0,0)\to(1,1): y=x, x$ from 0 to 1.

$$I=\int_0^1[Q(x,x)+P(x,x)]dx=\int_0^1[(x-x)-(x-x)]dx=0.$$

Path Independence

Example 4. Is there a function $u(x,y)$, such that

$du=(3x^2+xy^2)dx+(3y^2+x^2y)dy$? If so, find it.

Solution Let $P(x,y)=3x^2+xy^2$ and $Q(x,y)=3y^2+x^2y$, then

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=2xy.$$

$\frac{\partial P}{\partial y},\frac{\partial Q}{\partial x}$ are both continuous and equal. Then there must exist a function $u(x,y)$, such that

$$du=(3x^2+xy^2)dx+(3y^2+x^2y)dy.$$

In the following, we will give three methods to find the potential function $u(x,y)$.

Path Independence

Example 4. Is there a function $u(x,y)$, such that

$du=(3x^2+xy^2)dx+(3y^2+x^2y)dy$? If so, find it.

Solution(cont.)

Method I. (by line integrals)

$$\Phi(x,y)=\int_{(0,0)}^{(x,y)}(3x^2+xy^2)dx+(3y^2+x^2y)dy$$

$$=\int_0^x(3x^2+x\cdot 0)dx+\int_0^y(3y^2+x^2y)dy=x^3+y^3+\frac{1}{2}x^2y^2.$$

Then the potential function

$$u(x,y)=x^3+y^3+\frac{1}{2}x^2y^2+C.$$

Method II. (by partial integrals)

We know $\frac{\partial u}{\partial x}=P(x,y)=3x^2+xy^2, \frac{\partial u}{\partial y}=Q(x,y)=3y^2+x^2y.$

Path Independence

Example 4. Is there a function $u(x,y)$, such that

$du=(3x^2+xy^2)dx+(3y^2+x^2y)dy$? If so, find it.

Solution(cont.) Integrating both sides of the equality $\frac{\partial u}{\partial x}=3x^2+xy^2$ w.r.t. x and regarding y as a constant, then

$$u(x,y)=x^3+\frac{1}{2}x^2y^2+\varphi(y).$$

Since $\frac{\partial u}{\partial y}=x^2y+\varphi'(y)=3y^2+x^2y$, then $\varphi'(y)=3y^2$, that is $\varphi(y)=y^3+C$.

So $u(x,y)=x^3+y^3+\frac{1}{2}x^2y^2+C.$

Method III. (combining terms into total differentials)

$$du=P(x,y)dx+Q(x,y)dy=(3x^2+xy^2)dx+(3y^2+x^2y)dy$$

$$=3x^2dx+3y^2dy+(xy^2dx+x^2ydy)$$

$$=dx^3+dy^3+\frac{1}{2}dx^2y^2=d\left(x^3+y^3+\frac{1}{2}x^2y^2\right),$$

So $u(x,y)=x^3+y^3+\frac{1}{2}x^2y^2+C.$

Path Independence

The method to find the potential function

(1) Line integral

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)} P(x,y)dx+Q(x,y)dy+C$$

$$=\int_{x_0}^{x} P(x,y_0)dx+\int_{y_0}^{y} Q(x,y)dy+C$$

(2) Partial integral

$$\frac{\partial u}{\partial x}=P(x,y)\Rightarrow u(x,y)=\int P(x,y)dx=f(x,y)+\varphi(y)$$

$$\frac{\partial u}{\partial y}=Q(x,y)\Rightarrow\frac{\partial}{\partial y}\left(f(x,y)+\varphi(y)\right)=Q(x,y)\Rightarrow\varphi(y)\Rightarrow u(x,y)$$

(3) Combining terms into total differentials

$$du(x,y)=P(x,y)dx+Q(x,y)dy=d(\cdots)$$

Path Independence

Example 5. Is there a function $u(x,y)$, such that

$du=\dfrac{(x+2y)dx+ydy}{(x+y)^2}$? If so, find it.

Solution Let $P(x,y)=\dfrac{x+2y}{(x+y)^2}$ and $Q(x,y)=\dfrac{y}{(x+y)^2}$, then

$$\frac{\partial P}{\partial y}=-\frac{2y}{(x+y)^3},\quad \frac{\partial Q}{\partial x}=-\frac{2y}{(x+y)^3}.$$

In the domain of $x+y>0$ or $x+y<0$, $\dfrac{\partial P}{\partial y},\dfrac{\partial Q}{\partial x}$ are both continuous and equal. Then there must exist a function $u(x,y)$, such that

$$du=\frac{(x+2y)dx+ydy}{(x+y)^2}.$$

Path Independence

Example 5. Is there a function $u(x,y)$, such that

$du=\dfrac{(x+2y)dx+ydy}{(x+y)^2}$? If so, find it.

Solution (continued) **Method I**.

$$\Phi(x,y)=\int_{(1,0)}^{(x,y)}\frac{(x+2y)dx+ydy}{(x+y)^2}=\int_1^x\frac{1}{x}dx+\int_0^y\frac{y}{(x+y)^2}dy$$

$$=\left[\ln|x|\right]_1^x+\left[\ln|x+y|+\frac{x}{x+y}\right]_0^y$$

$$=\ln|x+y|+\frac{x}{x+y}-1$$

$$=\ln|x+y|-\frac{y}{x+y}$$

and $u(x,y)=\ln|x+y|-\dfrac{y}{x+y}+C.$

Path Independence

Example 5. Is there a function $u(x,y)$, such that

$du=\dfrac{(x+2y)dx+ydy}{(x+y)^2}$? If so, find it.

Solution (continued) **Method II.**

We know $\dfrac{\partial u}{\partial x}=\dfrac{x+2y}{(x+y)^2}=\dfrac{1}{x+y}+\dfrac{y}{(x+y)^2}$

then $u(x,y)=\displaystyle\int\frac{\partial u}{\partial x}dx=\int\left(\frac{1}{x+y}+\frac{y}{(x+y)^2}\right)dx$

$$=\ln|x+y|-\frac{y}{x+y}+\varphi(y)$$

Since $\dfrac{\partial u}{\partial y}=\dfrac{1}{x+y}-\dfrac{x}{(x+y)^2}+\varphi'(y)=\dfrac{y}{(x+y)^2}$

then $\varphi'(y)=0$, that is $\varphi(y)=C.$

So $u(x,y)=\ln|x+y|-\dfrac{y}{x+y}+C$

Path Independence

Example 6. Find the integral $I_1=\displaystyle\int_{(0,1)}^{(1,0)}\frac{xdx+ydy}{\sqrt{x^2+y^2}}$ and $I_2=\displaystyle\oint_{(C)}\frac{xdx+ydy}{\sqrt{x^2+y^2}}$

where (C) is $x^2+y^2=1$.

Solution

$$\frac{xdx+ydy}{\sqrt{x^2+y^2}}=d\sqrt{x^2+y^2}$$

$$\Rightarrow I_1=\int_{(0,1)}^{(1,0)}\frac{xdx+ydy}{\sqrt{x^2+y^2}}=\sqrt{x^2+y^2}\Big|_{(0,1)}^{(1,0)}=0$$

$$I_2=\oint_{(C)}\frac{xdx+ydy}{\sqrt{x^2+y^2}}=\oint_{(C)}xdx+ydy=\int_0^{2\pi}[\cos t(-\sin t)+\sin t\cos t]dt=0$$

Path Independence

Example 7. Suppose that the integral $\int_L xy^2dx+y\varphi(x)dy$ is independent on the path taken, where φ has continuous derivative, and $\varphi(0)=0$,

Find $\displaystyle\int_{(0,0)}^{(1,1)}xy^2dx+y\varphi(x)dy.$

Solution Let $P(x,y)=xy^2$, $Q(x,y)=y\varphi(x)$, then

$$\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}(xy^2)=2xy,\quad \frac{\partial Q}{\partial x}=\frac{\partial}{\partial x}[y\varphi(x)]=y\varphi'(x),$$

Since the integral is independent of path, we have $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}.$

That is $y\varphi'(x)=2xy$ and then $\varphi(x)=x^2+C$. Since $\varphi(0)=0$, we have

$C=0$. Therefore, $\displaystyle\int_{(0,0)}^{(1,1)}xy^2dx+y\varphi(x)dy=\int_{(0,0)}^{(1,1)}xy^2dx+x^2ydy=\frac{1}{2}.$

Review

- Green’s formula
- Using Green’s formula to find the line integral of the second type
- Conditions for path independence of a line integral
- Potential function and potential field (methods for finding potential functions)