Ch.2 *Time Domain Representations of Linear Time-Invariant Systems* (I)

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Outline

- Linear Time-invariant systems (LTI)
  - Introduction
  - Discrete time LTI systems: Convolution Sum
  - Convolution Sum Evaluation Procedure

Linear Time-invariant systems (LTI)

- LTI system: A system satisfying both the **linearity** and the **time-invariance** property
- LTI systems are mathematically easy to analyze and characterize, and consequently, easy to design.
    - They possess superposition theorem：If we represent the input to an LTI system in terms of linear combination of a set of basic signals, we can then use superposition to compute the output of the system in terms of responses to these basic signals.
- Highly useful signal processing algorithms have been developed utilizing this class of systems.

Representation of LTI systems

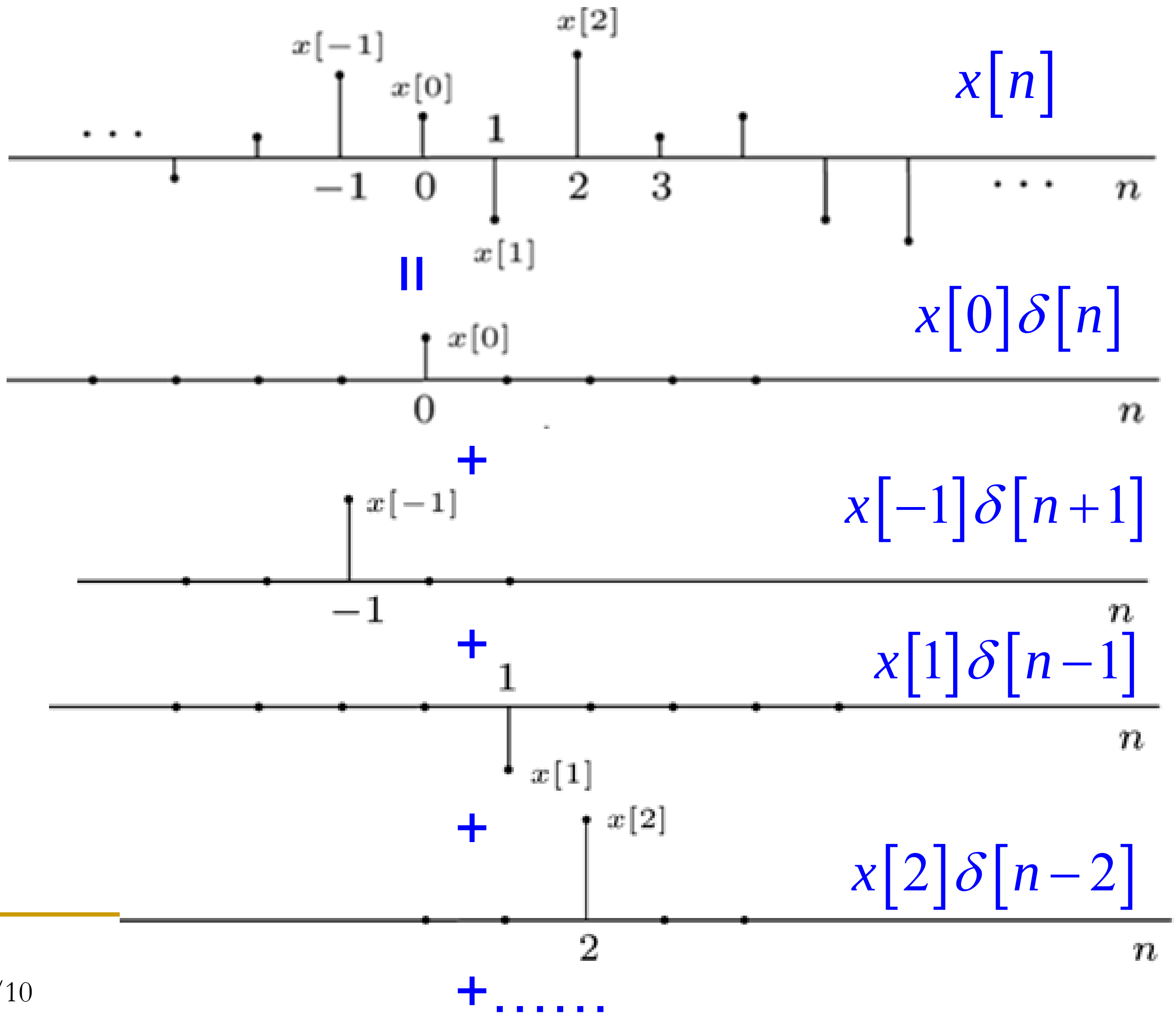
- Any linear time-invariant system (LTI) system, continuous-time or discrete-time, can be uniquely characterized by its
  - **Impulse response(冲激响应):** response of system to an impulse
  - **Frequency response(频率响应):** response of system to a complex exponential $e^{j2\pi ft}$ for all possible frequencies f.
  - **Transfer function(传递函数):** Laplace transform of impulse response
- Given one of the three, we can find other two provided that they exist.

Significance of unit impulse

- Every signal whether large or small can be represented in terms of linear combination of delayed impulses(延迟冲激).
- For DT or CT case, there are two natural choices for these two basic building blocks
  - For CT: shifted unit impulses $\boldsymbol{\delta(t)}$.
  - For DT: shifted unit samples(impulse sequences) $\boldsymbol{\delta[n]}$.

Superposition Sum for DT Systems

- Representing DT Signals with Sums of Unit Samples



Superposition Sum for DT Systems

- Representing DT Signals with a linear combination of delayed and advanced unit sample sequences

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n]$$
$$+ x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{Coefficients}} \underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

  - ***x*[*n*]**: entire signal;
  - ***x*[*k*]**: specific value of the signal ***x*[*n*]** at time *k*.

Superposition Sum for DT Systems

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

- The response of the LTI system to an input will be

input output

$$x[k]\delta[n-k] \quad \Rightarrow \quad x[k]H\{\delta[n-k]\}=x[k]h[n-k]$$

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k] \quad \Rightarrow \quad y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

For time-invariant system:

$$H\{\delta[n]\}=h[n] \quad \Rightarrow \quad H\{\delta[n-k]\}=h[n-k]$$

- The response of a linear system to x[n] will be the superposition of the scaled responses $\boldsymbol{h[n]}$ of the system to each of these shifted unit impulses.

Time-Domain Characterization of LTI DT System

- Knowing $\boldsymbol{h[n]}$, one can compute the output of the system for any arbitrary input
  - Ex: $x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + 0.75\delta[n-5]$
    - Since the system is time-invariant:

input output

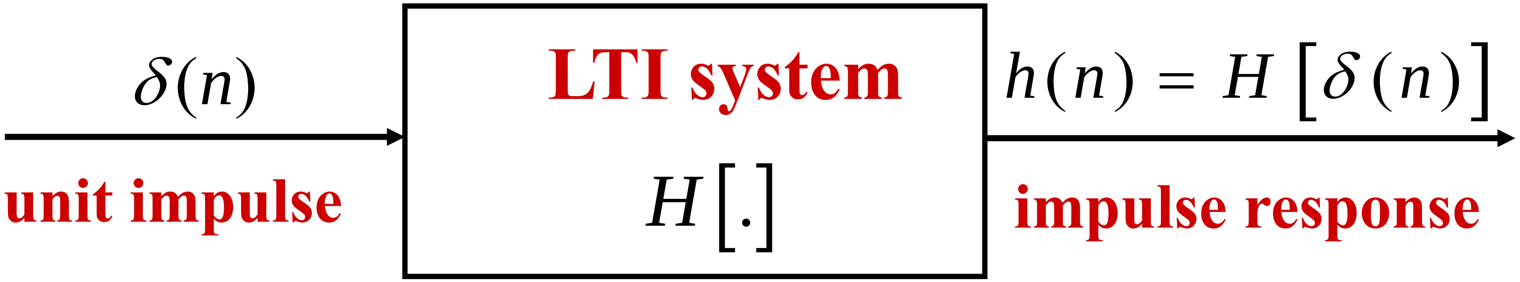
$$\boldsymbol{\delta[n] \to h[n]} \quad \Rightarrow \quad \begin{aligned} \delta[n+2] &\to h[n+2] \\ \delta[n-1] &\to h[n-1] \\ \delta[n-2] &\to h[n-2] \\ \delta[n-5] &\to h[n-5] \end{aligned}$$

- As the system is linear: $\boldsymbol{0.5\delta[n+2] \to 0.5h[n+2], \cdots}$

$$\boldsymbol{y[n] = 0.5h[n+2] + 1.5h[n-1] - h[n-2] + 0.75h[n-5]}$$

Impulse Response (冲激响应/脉冲响应)

- The response of a discrete-time system to a unit sample sequence $\delta[n]$ is called the **unit sample response** or simply, the **impulse response**, and is denoted by $\boldsymbol{h[n]}$.



- **Input-Output Relationship** - A consequence of the linear, time-invariance property is that an LTI discrete-time system is **completely characterized** by its impulse response.

Discrete time LTI systems: Convolution Sum

- Input: $x[n]$ as a weighted sum of time-shifted impulses

$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- Impulse response of LTI system: $H(\bullet)$ or $h[n]$
- Output:

$$y[n]=H\{x[n]\}=H\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}=\sum_{k=-\infty}^{\infty} H\{x[k]\delta[n-k]\}$$

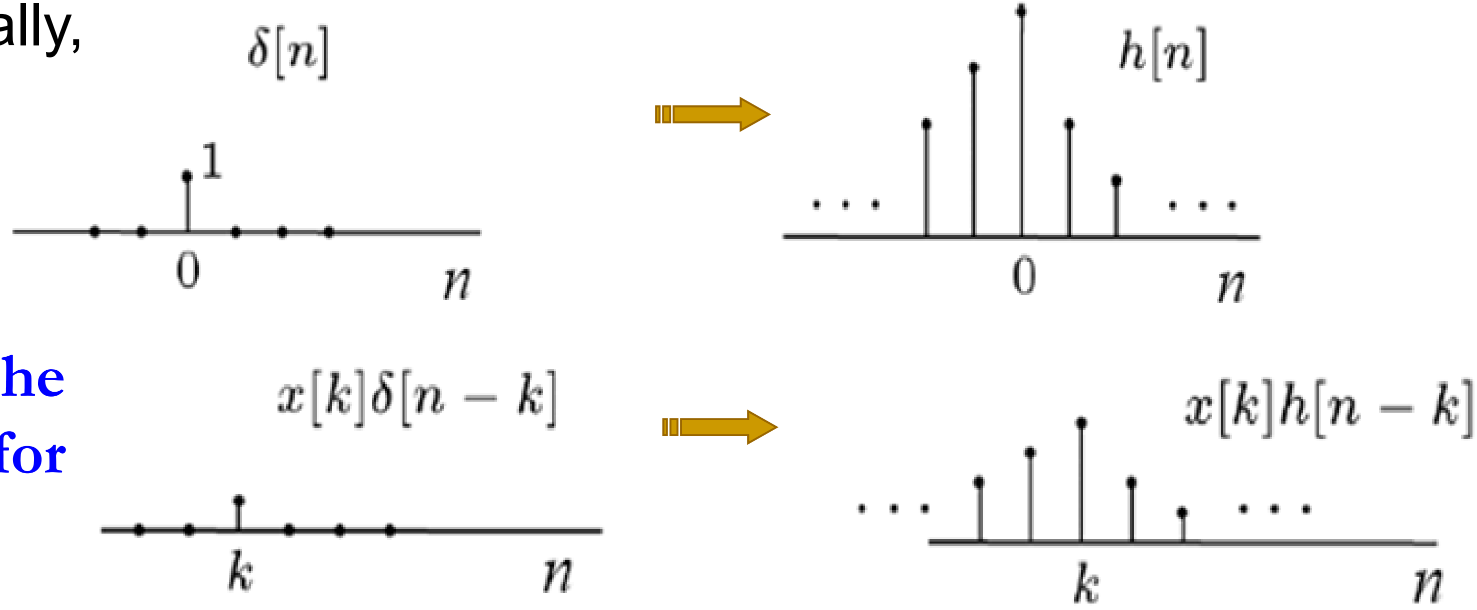
$$=\sum_{k=-\infty}^{\infty} x[k]H\{\delta[n-k]\}=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution Sum (卷积和)

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=x[n]*h[n]$$

- **Convolution sum**: the output of an LTI system is a weighted sum of the response of the system to time-shifted impulses. The sum is denoted by the symbol *.
- Graphically,

Sum up all the responses for all k′s.



Discrete time LTI systems: Convolution Sum

- Illustration of the convolution sum

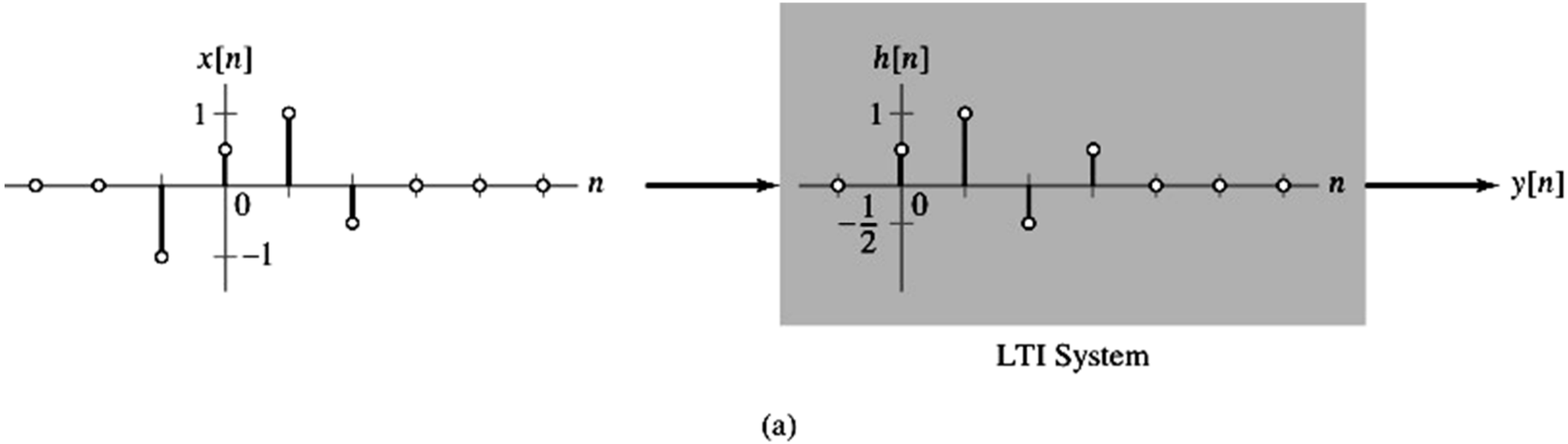
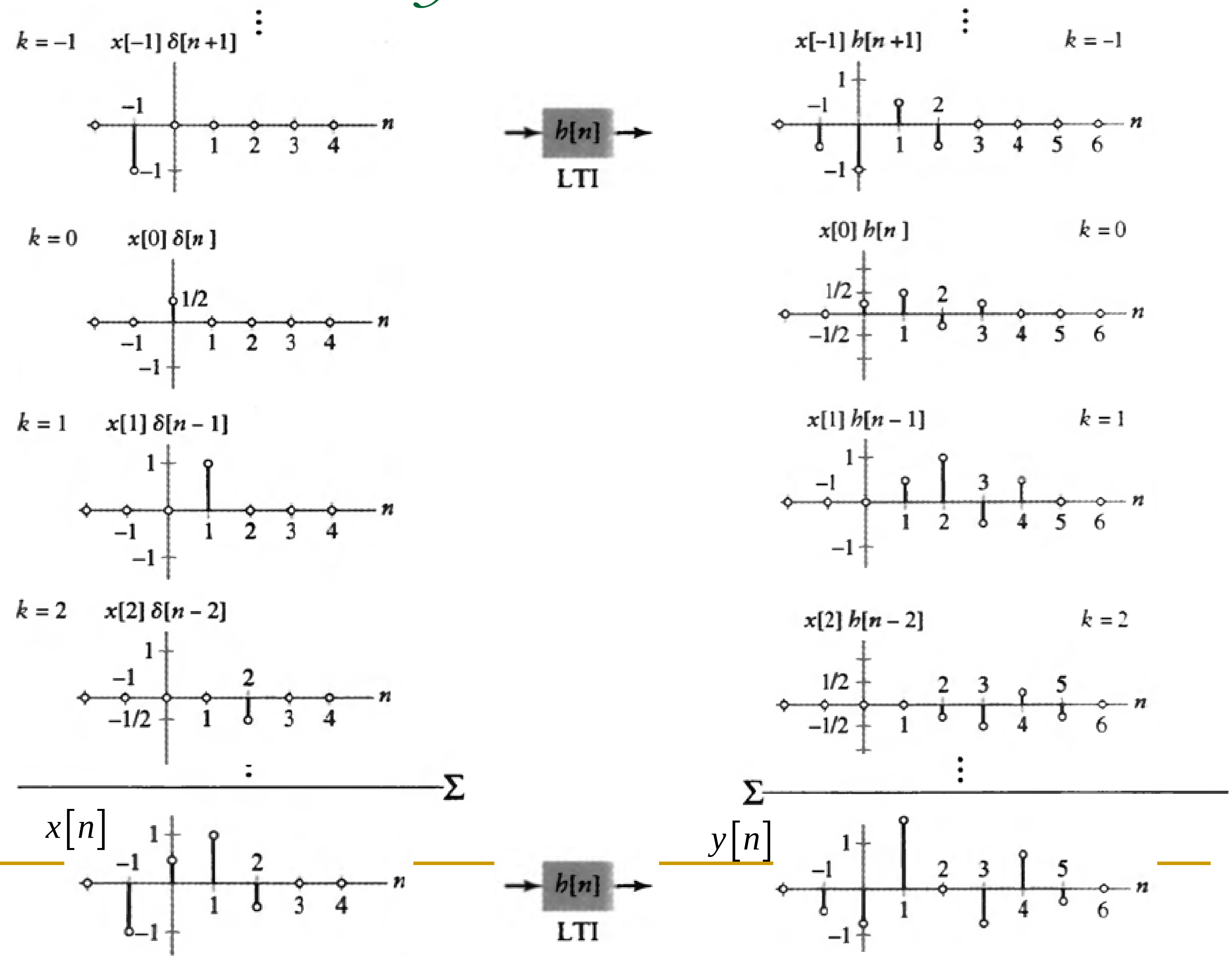


Figure 2.2a Illustration of the convolution sum. (a) LTI system with impulse response *h*[*n*] and input *x*[*n*].

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-1}^{2} x[k]h[n-k]$$

Discrete time LTI systems: Convolution Sum



Discrete time LTI systems: Convolution Sum

Example 2.1 ***Multipath Communication Channel: Direct Evaluation of the Convolution Sum***

Consider the discrete-time LTI system model representing a two-path propagation channel. If the strength of the indirect path is a = ½, then

$$y[n] = x[n] + \frac{1}{2}x[n-1]$$

Letting $x[n] = \delta[n]$, we find the impulse response is

$$h[n] = \begin{cases} 1, & n = 0 \\ \frac{1}{2}, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the output of this system in response to the input

$$x[n] = \begin{cases} 2, & n = 0 \\ 4, & n = 1 \\ -2, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Discrete time LTI systems: Convolution Sum

<Sol.>

- input: $x[n]=\begin{cases}2, & n=0\\ 4, & n=1\\ -2, & n=2\\ 0, & \text{otherwise}\end{cases}$
  $=2\delta[n]+4\delta[n-1]-2\delta[n-2]$
- system: $h[n]=\begin{cases}1, & n=0\\ \frac{1}{2}, & n=1\\ 0, & \text{otherwise}\end{cases}$
  $=\delta[n]+\frac{1}{2}\delta[n-1]$
- output: $y[n]=2h[n]+4h[n-1]-2h[n-2]$

$$=2\left\{\delta[n]+\frac{1}{2}\delta[n-1]\right\}+4\left\{\delta[n-1]+\frac{1}{2}\delta[n-2]\right\}-2\left\{\delta[n-2]+\frac{1}{2}\delta[n-3]\right\}$$

$$=2\delta[n]+5\delta[n-1]-\delta[n-3]=\begin{cases}0, & n<0\\ 2, & n=0\\ 5, & n=1\\ 0, & n=2\\ -1, & n=3\\ 0, & n\geq 4\end{cases}$$

Convolution Sum Evaluation Procedure

- An alternative approach to evaluate the convolution sum

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  - Define the intermediate signal:

$$w_n[k]=x[k]h[n-k]$$

where k = independent variable
n = constant.

where $h[n-k]=h[-(k-n)]$

~ a reflected and time-shifted version of $h[k]$.

$$\Rightarrow \quad y[n]=\sum_{k=-\infty}^{\infty} w_n[k]$$

The time shift n determines the time at which we evaluate the output of the system.

Convolution Sum Evaluation Procedure

Example 2.2 ***Convolution Sum Evaluation by using Intermediate Signal***

Consider a system with impulse response $h[n] = \left(\frac{3}{4}\right)^n u[n]$

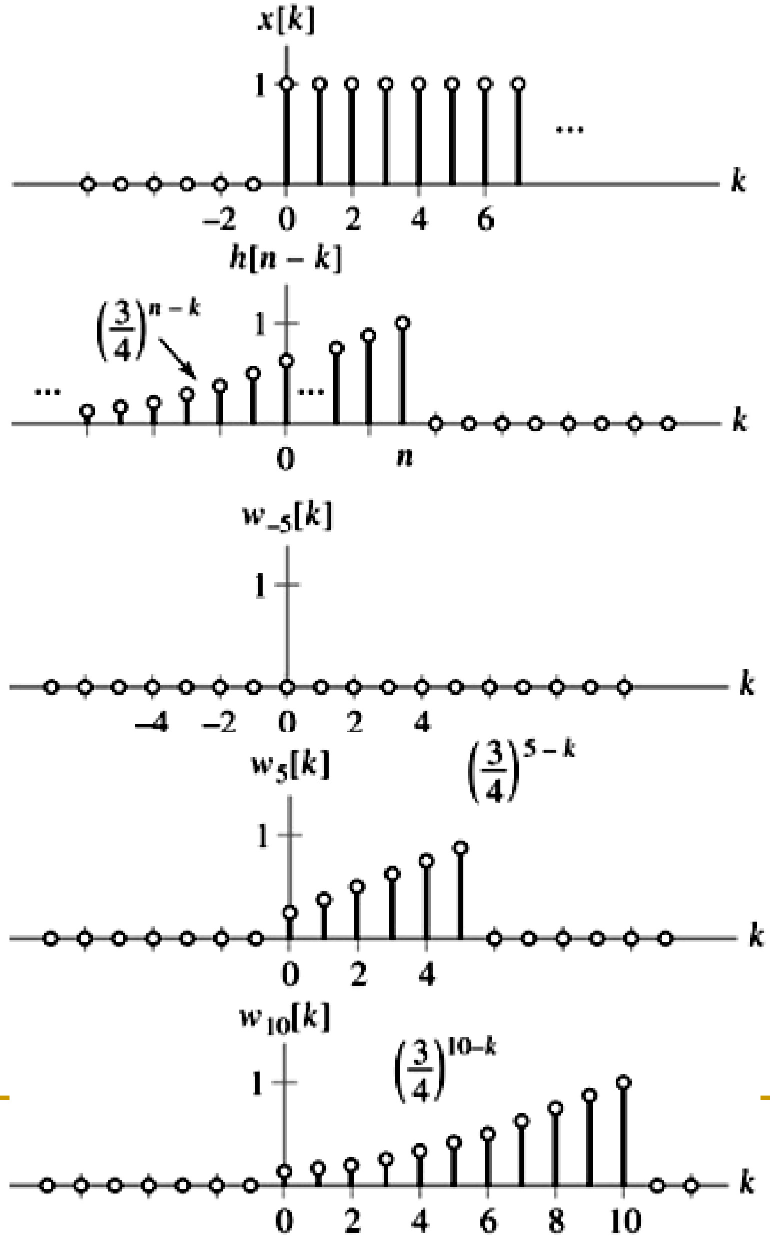
determine the output at time n = -5, 5, 10 when the input is $x[n] = u[n]$.

- intermediate signal: $w_n[k] = x[k]h[n-k]$

$$h[n-k] = \left(\frac{3}{4}\right)^{n-k} u(n-k) = \begin{cases} \left(\frac{3}{4}\right)^{n-k}, & k \le n \\ 0, & otherwise \end{cases}$$

$$w_n[k] = u(k)\left(\frac{3}{4}\right)^{n-k} u(n-k) = \begin{cases} \left(\frac{3}{4}\right)^{n-k}, & 0 \le k \le n \\ 0, & otherwise \end{cases}$$

Convolution Sum Evaluation Procedure



- n=-5: $w_{-5}[k] = 0$

$$y[-5] = \sum_{k=-\infty}^{\infty} w_{-5}[k] = 0$$

- n=5: $w_5[k] = \left(\frac{3}{4}\right)^{5-k}, \quad 0 \le k \le 5$

$$y[5] = \sum_{k=0}^{5} w_5[k] = \left(\frac{3}{4}\right)^5 \sum_{k=0}^{5} \left(\frac{4}{3}\right)^k$$

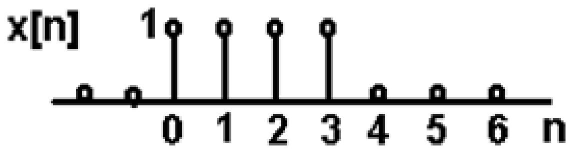
$$= \left(\frac{3}{4}\right)^5 \frac{1-(4/3)^6}{1-(4/3)} = 3.288$$

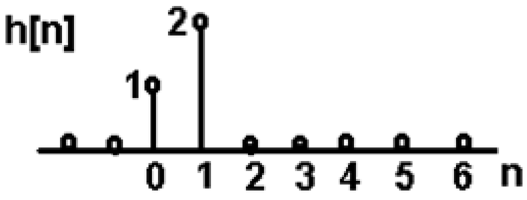
- n=10: $w_{10}[k] = \left(\frac{3}{4}\right)^{10-k}, 0 \le k \le 10$

$$y[10] = \sum_{k=0}^{10} w_{10}[k] = 3.831$$

Convolution Sum Evaluation Procedure

Ex. Develop the sequence $y[n]$ generated by the convolution of $x[n]$ and $h[n]$.

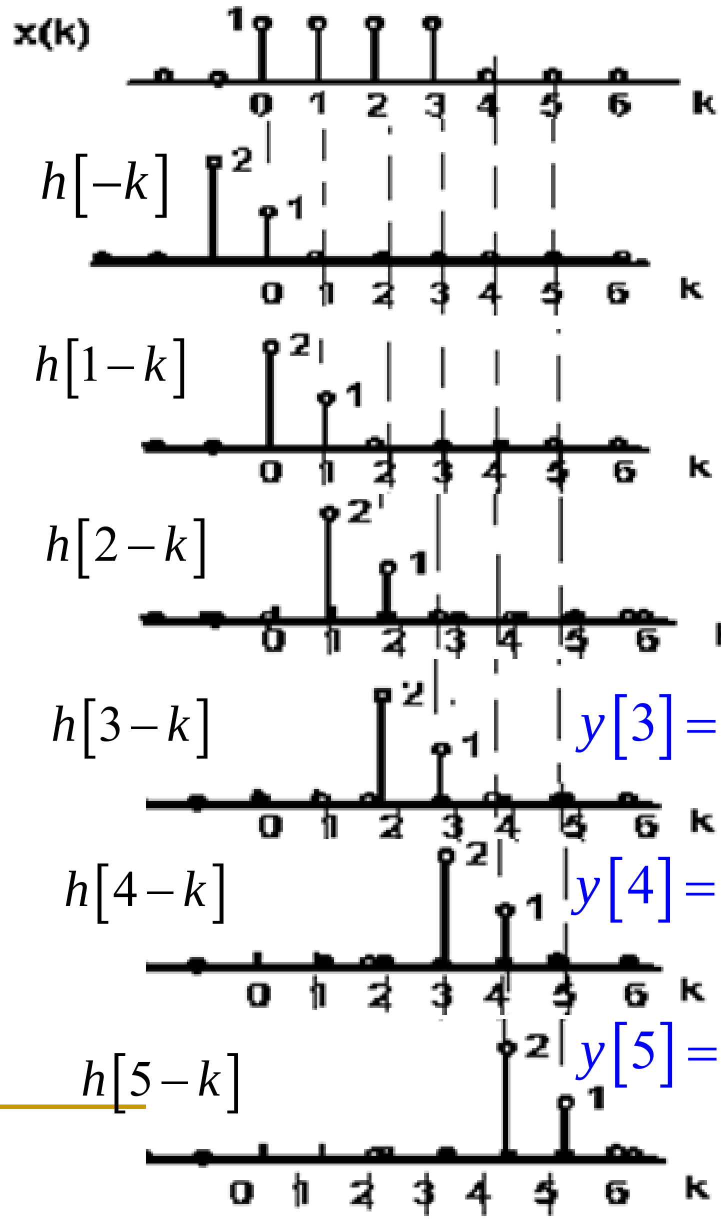




$$n=0:\ y[0]=\sum_{k=-\infty}^{\infty}x[k]h[0-k]=\sum_{k=-\infty}^{\infty}x[k]h[-k]$$
$$=x[0]h[0]\ =1$$

$$n=1:\ \ y[1]=\sum_{k=-\infty}^{\infty}x[k]h[1-k]$$
$$=x[0]h[1]+x[1]h[0]\ =3$$

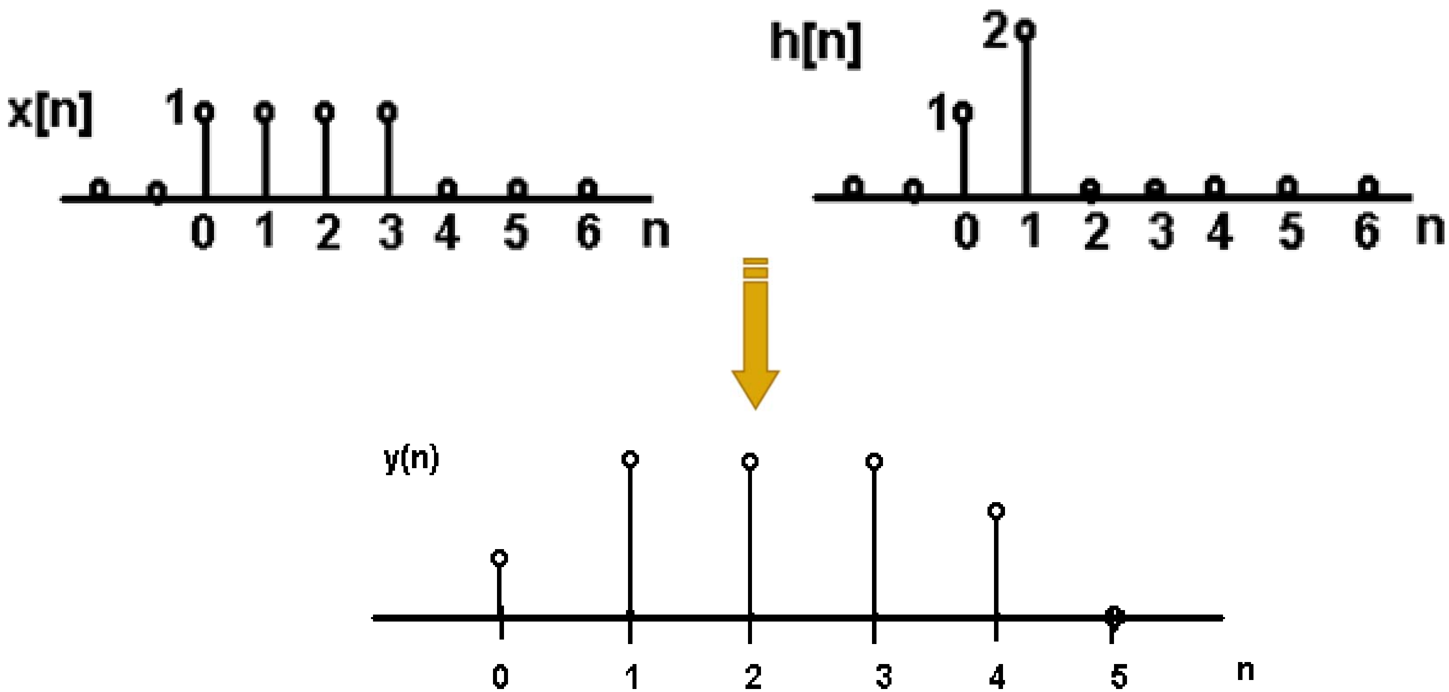
$$n=2:\ y[2]=\sum_{k=-\infty}^{\infty}x[k]h[2-k]$$
$$=x[1]h[1]+x[2]h[0]\ =3$$



$h[-k]$ $h[1-k]$ $h[2-k]$ $h[3-k]$ $y[3]=3$ $h[4-k]$ $y[4]=2$ $h[5-k]$ $y[5]=0$

Convolution Sum Evaluation Procedure

Ex. Develop the sequence $y[n]$ generated by the convolution of $x[n]$ and $h[n]$.



The length of $x[n]$: $\boldsymbol{M}$

The length of $h[n]$: $\boldsymbol{N}$

The length of $y[n]$: $\boldsymbol{M + N - 1}$

Convolution Sum Evaluation Procedure

- **Procedure 2.1: Reflect and Shift Convolution Sum Evaluation**
  - **Graph both $x[k]$ and $h[n-k]$ as a function of the independent variable k. To determine $h[n-k]$, first reflect $h[k]$ about $k=0$ to obtain $h[-k]$. Then shift by $-n$.**
  - **Begin with n large and negative. That is, shift $h[-k]$ to the far left on the time axis.**
  - **Write the mathematical representation for the intermediate signal $w_n[k]$.**
  - **Increase the shift n (i.e., move $h[n-k]$ toward the right) until the mathematical representation for $w_n[k]$ changes. The value of n at which the change occurs defines the end of the current interval and the beginning of a new interval**
  - **Let n be in the new interval. Repeat step 3 and 4 until all intervals of times shifts and the corresponding mathematical representations for $w_n[k]$ are identified. This usually implies increasing n to a very large positive number.**
  - **For each interval of time shifts, sum all the values of the corresponding $w_n[k]$ to obtain $y[n]$ on that interval.**

Convolution Sum Evaluation Procedure

Example 2.3 *Moving-Average System*

The output $y[n]$ of the four-point moving-average system is related to the input $x[n]$ according to the formula

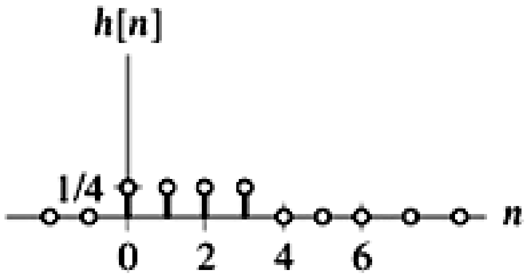
$$y[n] = \frac{1}{4}\sum_{k=0}^{3} x[n-k]$$

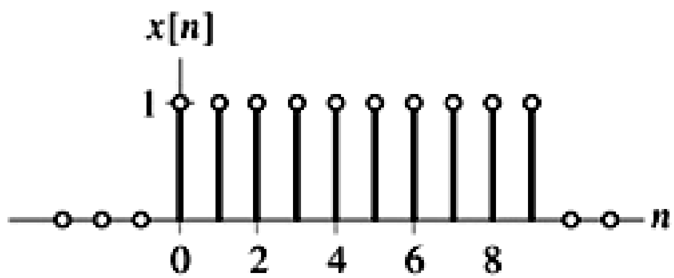
The impulse response $h[n]$ of this system is obtained by letting $x[n] = \delta[n]$, which yields

$$h[n] = \frac{1}{4}\left(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]\right) = \frac{1}{4}\left(u[n] - u[n-4]\right)$$

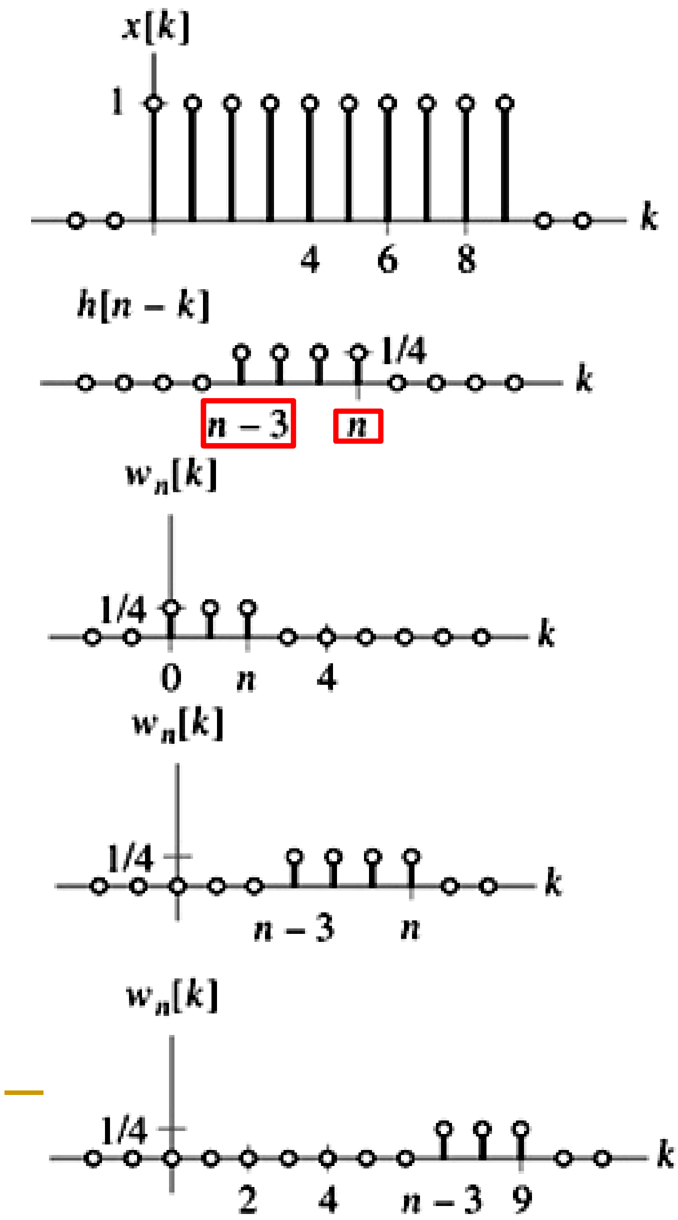
Determine the output of the system when the input is the rectangular pulse defined as

$$x[n] = u[n] - u[n-10]$$





Convolution Sum Evaluation Procedure



- **Five intervals**
  - $\boldsymbol{n<0}$**:** $w_n[k]=0 \quad y[n]=0$
  - $\boldsymbol{0\le n\le 3}$**:** $w_n[k]=1/4, \quad 0\le k\le n$

$$y[n]=\sum_{k=0}^{n}1/4 = \frac{n+1}{4}$$

  - $\boldsymbol{3<n\le 9}$**:** $w_n[k]=1/4, \quad n-3\le k\le n$

$$y[n]=\sum_{k=n-3}^{n}1/4 = 1$$

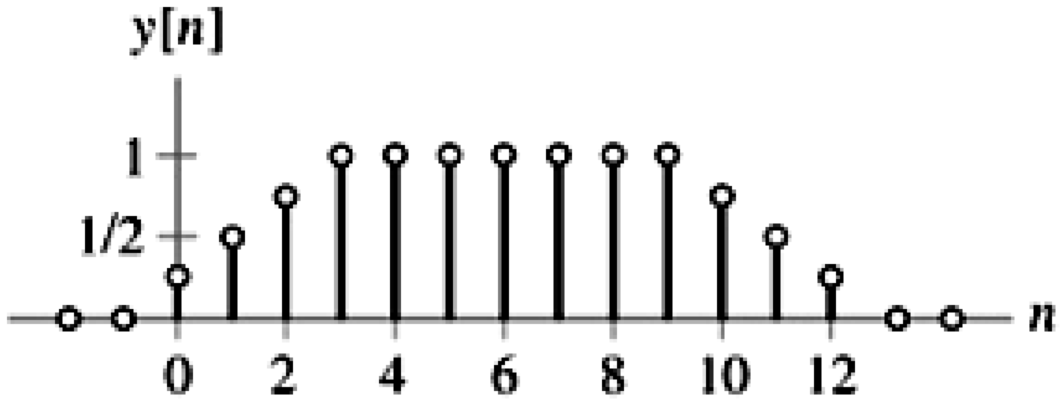
  - $\boldsymbol{9<n\le 12}$**:** $w_n[k]=1/4, \quad n-3\le k\le 9$

$$y[n]=\sum_{k=n-3}^{9}1/4 = \frac{13-n}{4}$$

  - $\boldsymbol{n>12}$**:** $w_n[k]=0 \quad y[n]=0$

Convolution Sum Evaluation Procedure

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{n+1}{4}, & 0 \le n \le 3 \\ 1, & 3 < n \le 9 \\ \dfrac{13-n}{4}, & 9 < n \le 12 \\ 0, & n > 12 \end{cases}$$



Convolution Sum Evaluation Procedure

Example 2.4 ***First-order Recursive System:***

The input-output relationship for the first-order recursive system is given by

$$y[n]-\rho y[n-1]=x[n]$$

Let the input be given by $x[n]=b^n u[n+4]$

We use convolution to find the output of this system, assuming that $b\neq\rho$ and that the system is causal.

<Sol.>

- **Impulse response:** $h[n]=\rho h[n-1]+\delta[n]$

  Since the system is causal, we have $h[n]=0$ for $n<0$.

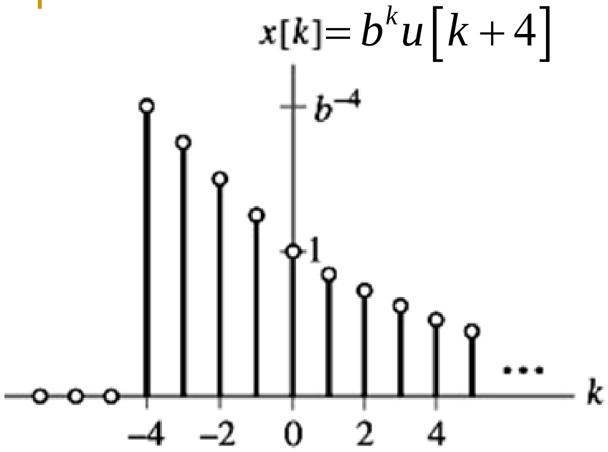
  $$h[0]=\delta[0]=1,\ \ h[1]=\rho,\ \ h[2]=\rho^2,\ \cdots\cdots,\ \Longrightarrow\ h[n]=\rho^n u[n]$$

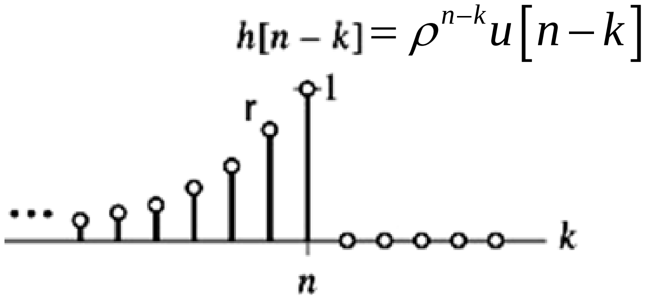
- **Graph both $x[k]$ and $h[n-k]$ as a function of k.**

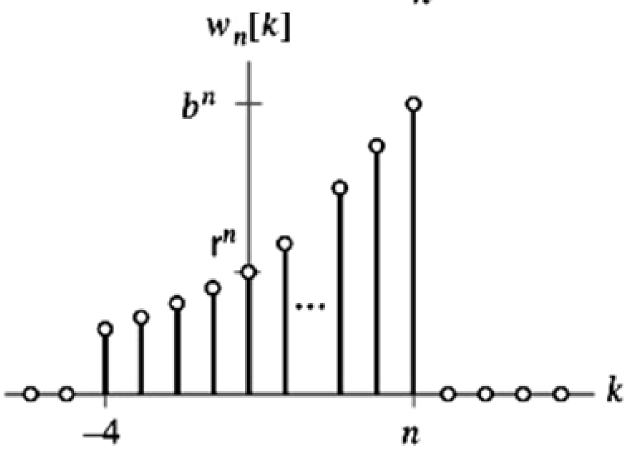
$$x[k]=b^k u[k+4]=\begin{cases} b^k, & k\geq -4 \\ 0, & otherwise \end{cases}$$

$$h[n-k]=\rho^{n-k}u[n-k]=\begin{cases} \rho^{n-k}, & k\leq n \\ 0, & otherwise \end{cases}$$

Convolution Sum Evaluation Procedure







- **Two intervals**
  - n **< -4 :** $w_n[k]=0 \quad y[n]=0$
  - n **≥-4:** $w_n[k]=\begin{cases} b^k\rho^{n-k}, & -4\le k\le n \\ 0, & \text{otherwise} \end{cases}$

$$y[n]=\sum_{k=-4}^{n} b^k\rho^{n-k}=\rho^n\sum_{k=-4}^{n}\left(\frac{b}{\rho}\right)^k$$

$$=\rho^n\left(\frac{\rho}{b}\right)^4\sum_{m=0}^{n+4}\left(\frac{b}{\rho}\right)^m$$

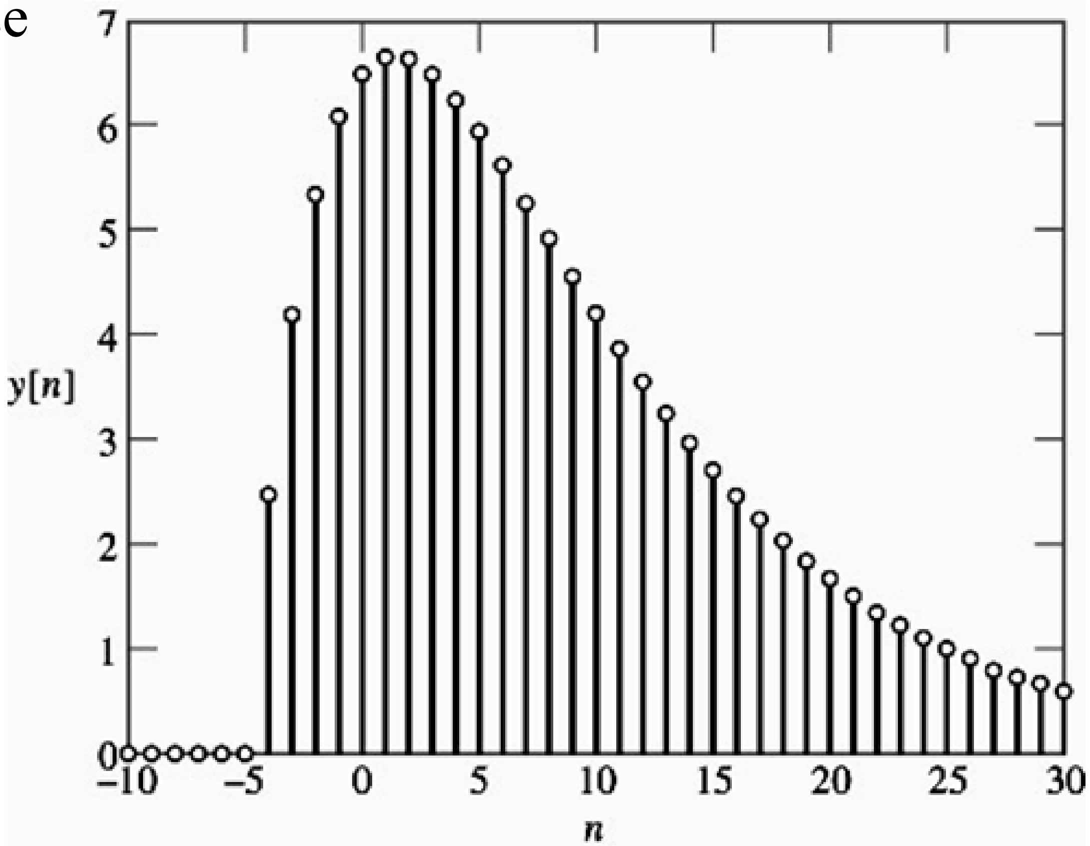
$$=\rho^n\left(\frac{\rho}{b}\right)^4\frac{1-(b/\rho)^{n+5}}{1-(b/\rho)}$$

$$=b^{-4}\left(\frac{\rho^{n+5}-b^{n+5}}{\rho-b}\right)$$

Convolution Sum Evaluation Procedure

$$y[n]=\begin{cases} b^{-4}\left(\dfrac{\rho^{n+5}-b^{n+5}}{\rho-b}\right), & n\geq -4 \\ 0, & \text{otherwise} \end{cases}$$

for $\rho=0.9, \quad b=0.8$



Convolution Sum Evaluation Procedure

Example 2.5 ***Investment Computation***

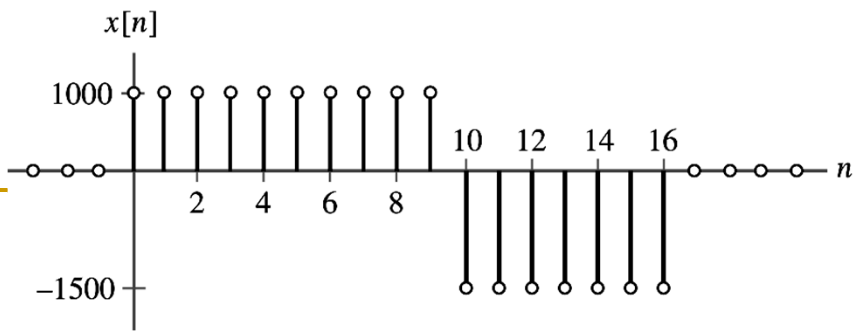
The first-order recursive system is used to describe the value of an investment earning compound interest at a fixed rate of r% per period if we set $\rho = 1+(r/100)$. Let $y[n]$ be the value of the investment at the start of period n. If there are no deposits or withdrawals, then the value at time n is expressed in terms of the value at the previous time as $y[n] = \rho y[n-1]$. Now, suppose $x[n]$ is the amount deposited ($x[n] > 0$) or withdrawn ($x[n] < 0$) at the start of period n. In this case, the value of the amount is expressed by the first-order recursive equation $y[n]=\rho y[n-1]+x[n]$

We use convolution to find the value of an investment earning 8% per year if \$1000 is deposited at the start of each year for 10 years and then \$1500 is withdrawn at the start each year for 7 years.

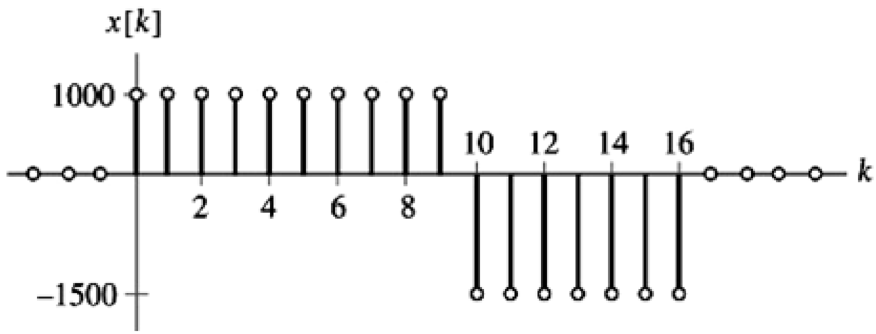
<Sol.>

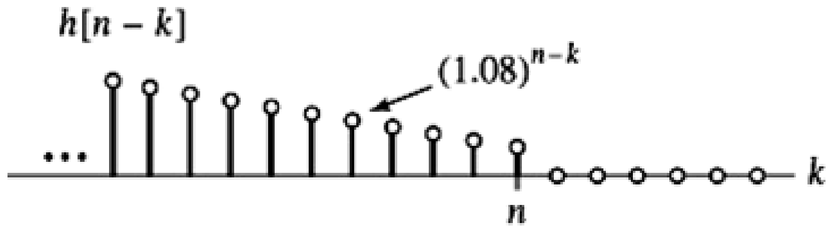
- **Impulse response :** $h[n]=\rho h[n-1]+\delta[n] \Rightarrow h[n]=\rho^n u[n],\ \rho=1.08$
- **input:**

$$x[n]=\begin{cases}1000, & 0\le n\le 9\\ -1500, & 10\le n\le 16\\ 0, & \text{otherwise}\end{cases}$$



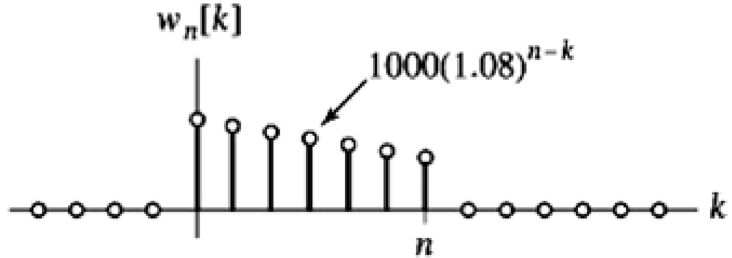
Convolution Sum Evaluation Procedure



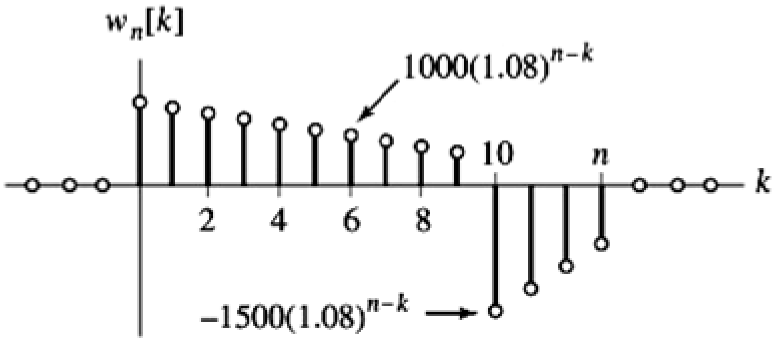


$$h[n-k]=1.08^{n-k}u[n-k]=\begin{cases}1.08^{n-k}, & k\le n\\ 0, & \text{otherwise}\end{cases}$$

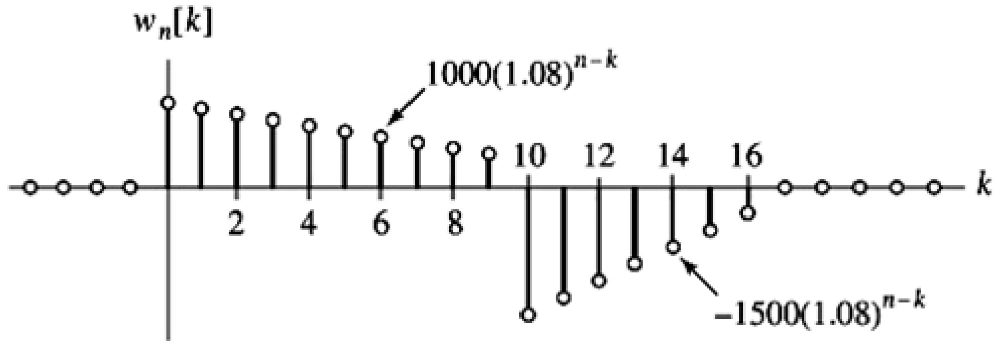
- **$0 \le$ n ≤ 9:**



- **$10 \le$ n ≤ 16:**



- **n ≥ 17:**



Convolution Sum Evaluation Procedure

- **Four intervals** $h[n-k]=\begin{cases}1.08^{n-k}, & k\le n\\ 0, & \text{otherwise}\end{cases}$ $x[n]=\begin{cases}1000, & 0\le n\le 9\\ -1500, & 10\le n\le 16\\ 0, & \text{otherwise}\end{cases}$

- $n<0$: $w_n[k]=0 \quad y[n]=0$

- $0\le n\le 9$: $w_n[k]=\begin{cases}1000*1.08^{n-k}, & 0\le k\le n\\ 0, & \text{otherwise}\end{cases} \Rightarrow y[n]=\sum_{k=0}^{n}1000*1.08^{n-k}$

- $10\le n\le 16$: $w_n[k]=\begin{cases}1000*1.08^{n-k}, & 0\le k\le 9\\ -1500*1.08^{n-k}, & 10\le k\le n\\ 0, & \text{otherwise}\end{cases}$

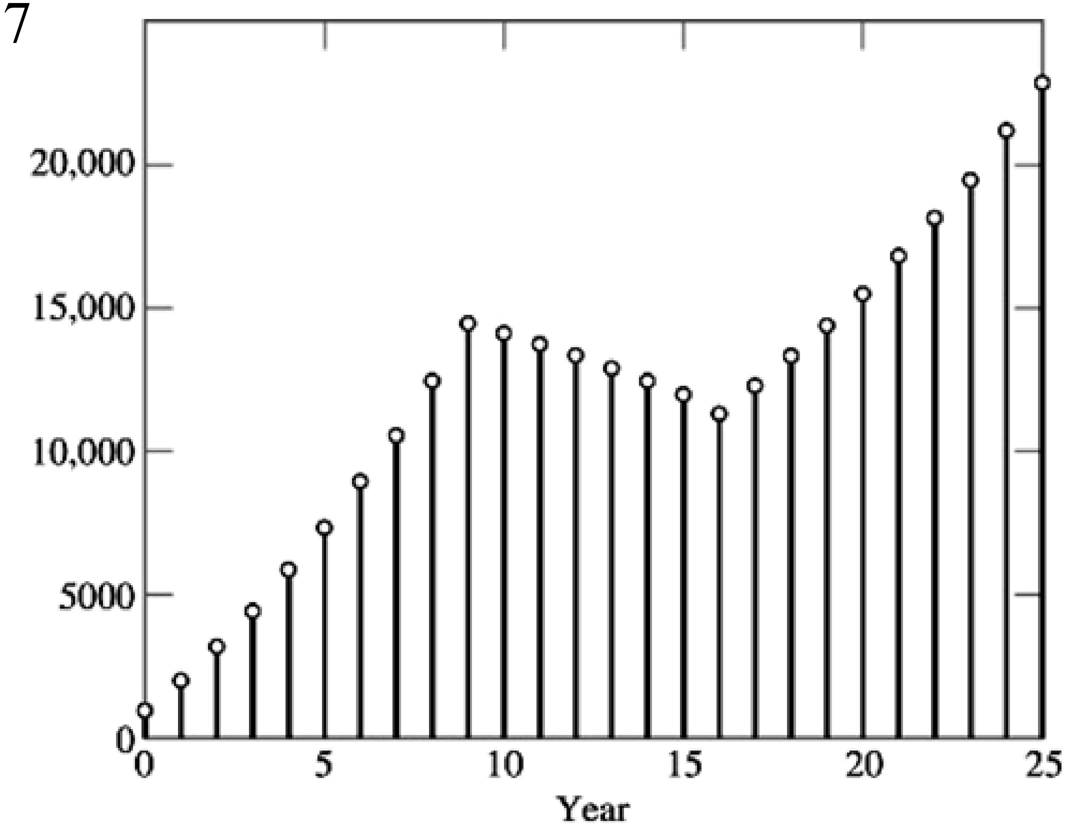
$$\Rightarrow y[n]=\sum_{k=0}^{9}1000*1.08^{n-k}-\sum_{k=10}^{n}1500*1.08^{n-k}$$

- $n\ge 17$: $w_n[k]=\begin{cases}1000*1.08^{n-k}, & 0\le k\le 9\\ -1500*1.08^{n-k}, & 10\le k\le 16\\ 0, & \text{otherwise}\end{cases}$

$$\Rightarrow y[n]=\sum_{k=0}^{9}1000*1.08^{n-k}-\sum_{k=10}^{16}1500*1.08^{n-k}$$

Convolution Sum Evaluation Procedure

$$y[n] = \begin{cases} 12500\left(1.08^{n+1} - 1\right), & 0 \le n \le 9 \\ 7246.89\left(1.08\right)^{n} - 18750\left(1.08^{n-9} - 1\right), & 10 \le n \le 16 \\ 3340.17\left(1.08\right)^{n}, & n \ge 17 \\ 0, & \text{otherwise} \end{cases}$$



Summary

- Linear Time-invariant systems (LTI)
  - Introduction
  - Discrete time LTI systems: Convolution Sum
  - Convolution Sum Evaluation Procedure

- Reference in textbook: 2.1, 2.2, 2.3
- Homework: 2.32, 2.34(a,c,d), 2.35