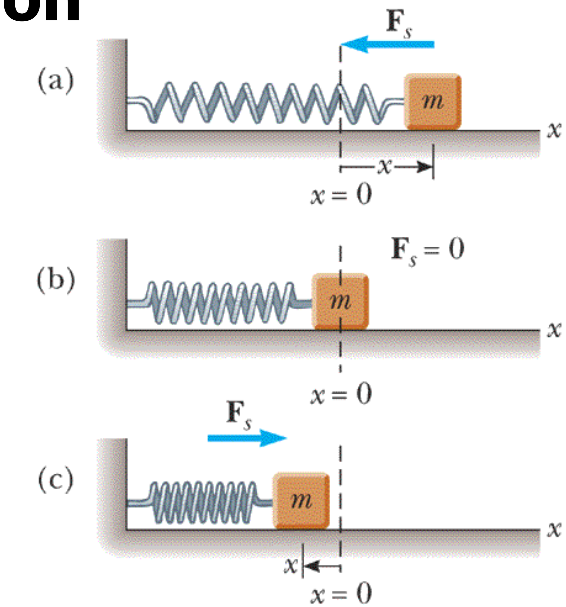
Chapter 12 Oscillations

§ 1 The Causes of Oscillation

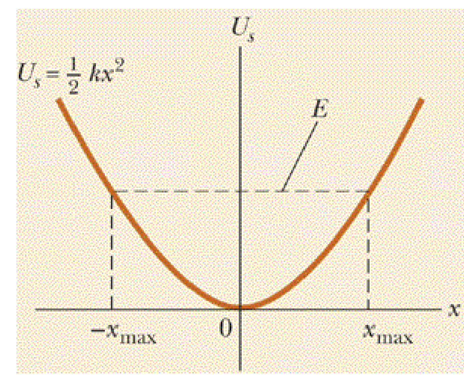


Existence of a restoring force

No matter what the direction of the displacement, the force always acts in a direction to restore the system to its equilibrium position.

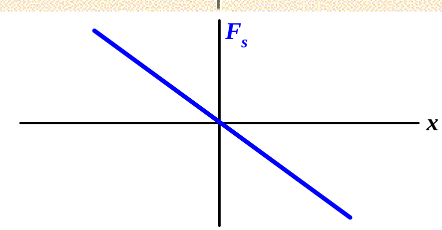
For a block-spring system $\boxed{F = -kx}$

Existence of a point of stable equilibrium



The systems tends to return to equilibrium when slightly displaced.

For a block-spring system $\boxed{U(x) = \dfrac{1}{2}kx^2}$

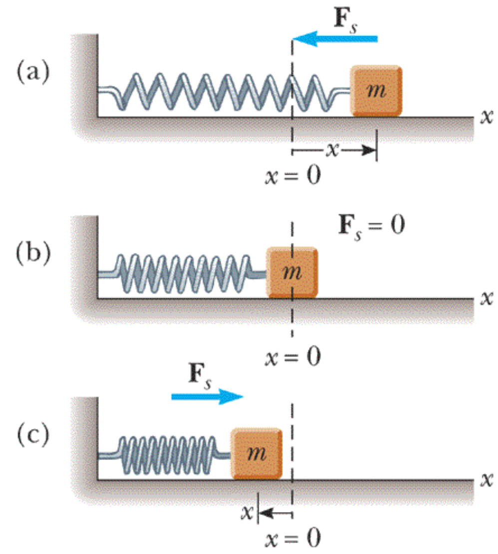


$$U(x) = -\int_0^x F_c dx = -\int_0^x (-kx)\,dx = \frac{1}{2}kx^2, \quad F_c = -\frac{dU}{dx} = -kx$$

§ 2 Simple Harmonic Motion (SHM)

- **The block-spring system** (P299)
  - **Newton's second law for block-spring system**

$$-kx = m\frac{d^2x}{dt^2}$$



- **Dynamics' equation**

Denote the ratio *k/m* with symbol ω^2

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

Dynamics' equation for SHM

Take a tentative solution,

$$\boxed{x = A\cos(\omega t + \phi)}$$

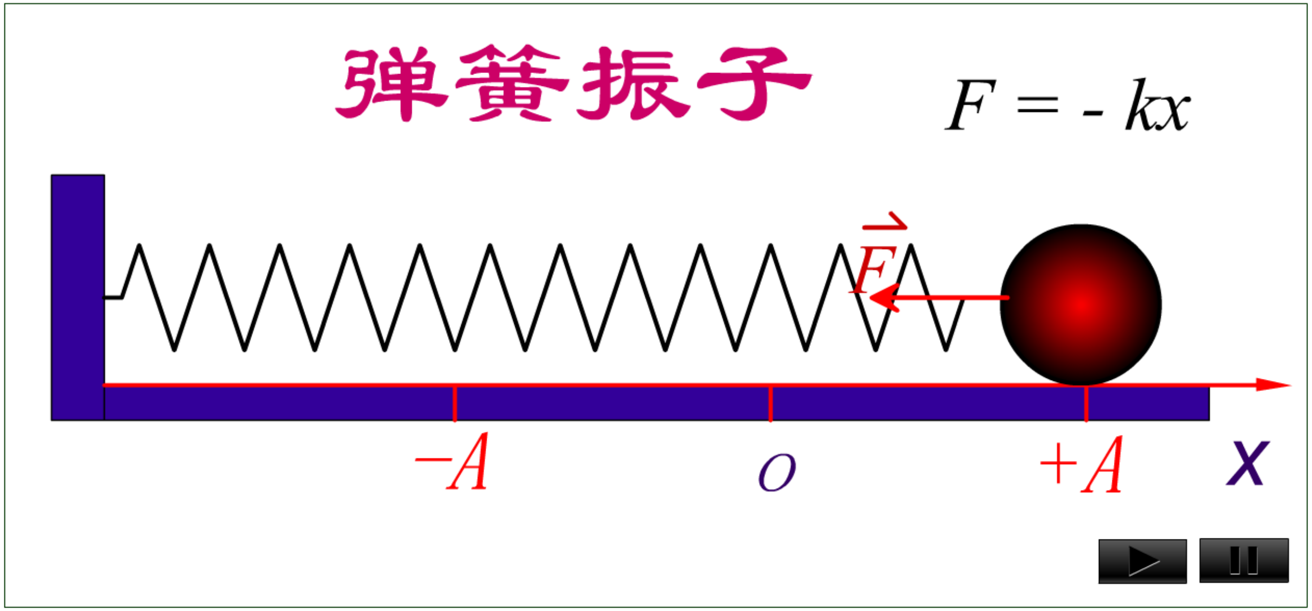
Kinematics' equation for SHM

***A* and ϕ arise from the integral constants.**

The block-spring system

- The simple harmonic motion
  - The motion action is governed by $$\frac{d^2x}{dt^2}+\omega^2 x=0$$
  - Be described in terms of sine and cosine function

$$x = A\cos(\omega t+\phi)$$



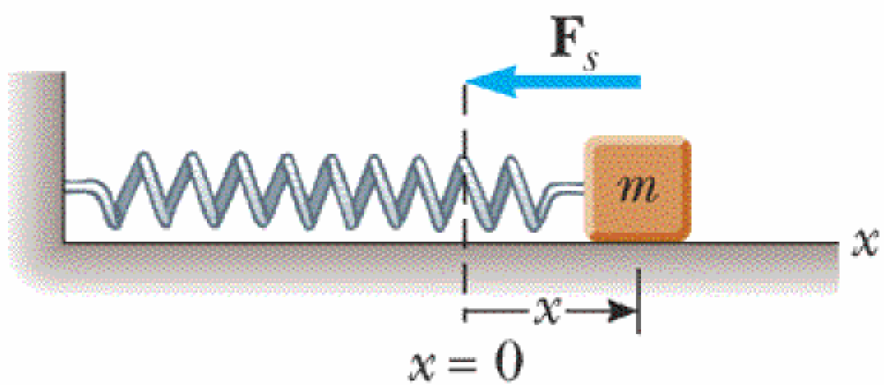
§ 3 The Characteristic Quantities for SHM

(P301)

$$\boxed{x = A\cos(\omega t + \phi)}$$

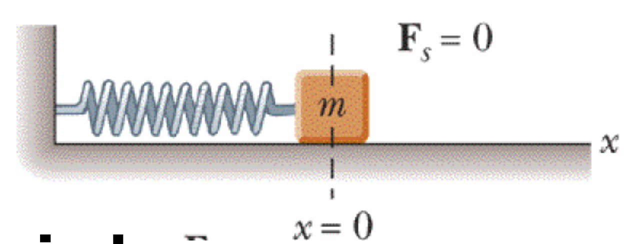
- **The amplitude *A***
  - **Maximum magnitude of displacement from equilibrium**

$$A = |x_{\max}|$$



The Characteristic Quantities for SHM

$$\boxed{x = A\cos(\omega t + \phi)}$$



- **Angular Frequency, Frequency, and Period**
  - **The period, T, is the time for oscillator to go through one full cycle of motion.**

$$[\omega(t+T)+\phi]-(\omega t+\phi)=2\pi, \quad T=\frac{2\pi}{\omega}$$

  - **The frequency, f, is the number of cycles in a unit of time. (SI unit: Hz)**

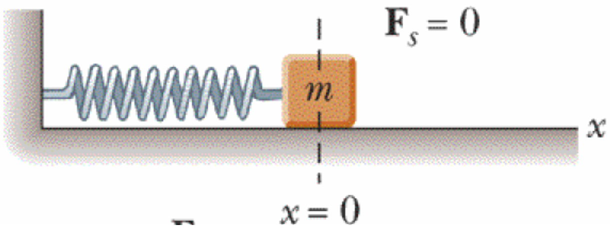
$$f=\frac{1}{T}=\frac{\omega}{2\pi}$$

  - **The angular frequency, ω, is 2π times the frequency. (SI unit: rad/s)**

$$\omega = 2\pi f = \frac{2\pi}{T}$$

The Characteristic Quantities for SHM

$$\boxed{x = A\cos(\omega t + \phi)}$$



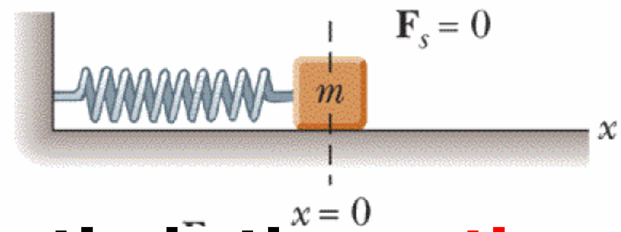
- T , f, ω **relate to the essential nature of an oscillator, which often called natural (intrinsic) period, natural frequency, and natural angular frequency.**

  - For a block-spring oscillator: $$\boxed{\omega = \sqrt{\frac{k}{m}}}$$
  - For a pendulum: $$\boxed{\omega = \sqrt{\frac{g}{L}}} \qquad \boxed{\omega = \sqrt{\frac{mgh}{I}}}$$

All determined by the essential natures of different oscillators.

The Characteristic Quantities for SHM

$$x = A\cos(\omega t + \phi)$$



- **The phase**
  - **The phase $(\omega t + \phi)$ can reflect entirely the motion state of an oscillator.**

$$\text{Phase} \text{ ——— } \omega t+\phi \Longleftrightarrow \begin{Bmatrix} x \\ v \end{Bmatrix} \text{ ——— } \text{State of motion}$$

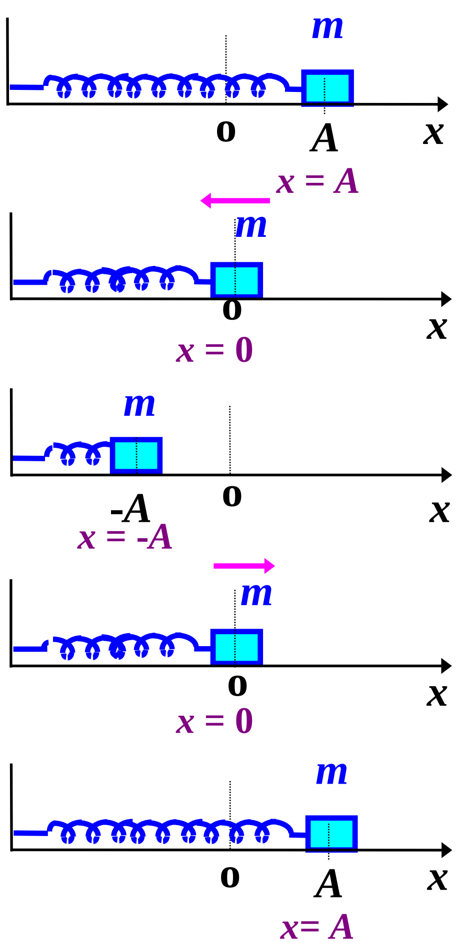
$$x = A\cos(\omega t + \phi), \qquad v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

  - **When $t = 0$, ϕ reflect the initial motion state of the oscillator.**

The relationship between motion state and phase

$$x(t) = A\cos(\omega t + \phi), \quad v = -\omega A\sin(\omega t + \phi)$$

Motion state (x, v) $\longleftrightarrow$ **Phase** $(\omega t + \phi)$



0

π/2

π

3π/2

2π

The Characteristic Quantities for SHM

$$x = A\cos(\omega t + \phi)$$

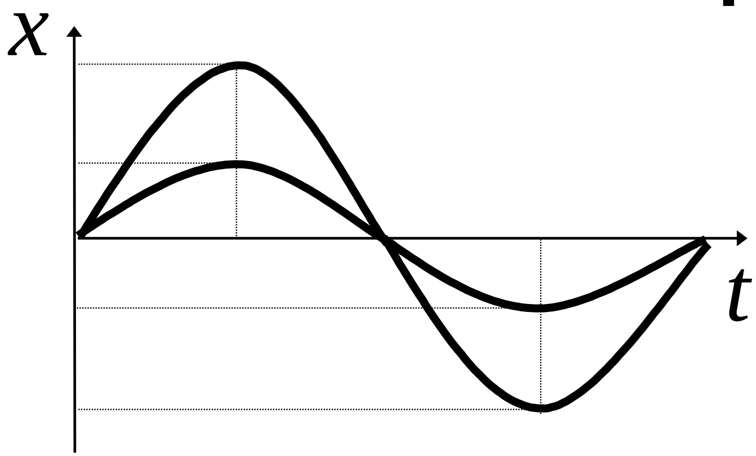
- ω relate to the essential nature of an oscillator, which often called natural (intrinsic) natural angular frequency.

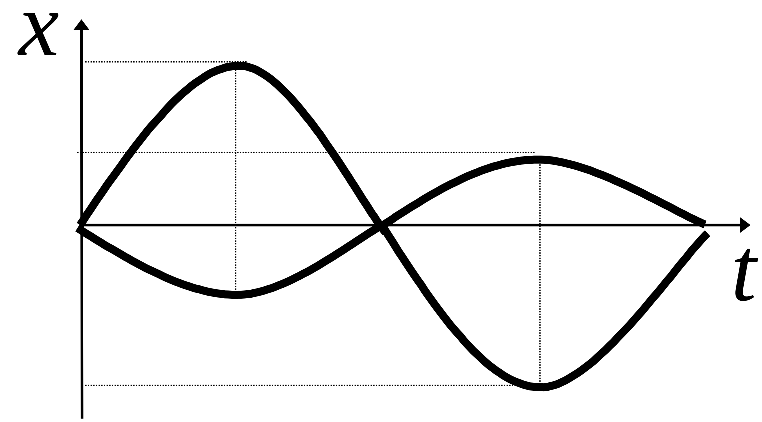
- A and ϕ are determined by initial conditions (How the motion starts)

When $t=0$, $x=x_0$, $v=v_0$

$$\begin{cases} x_0 = A\cos\phi \\ v_0 = -\omega A\sin\phi \end{cases} \Rightarrow \begin{cases} A = \sqrt{x_0^2 + \dfrac{v_0^2}{\omega^2}} \\ \phi = \arctan\left(-\dfrac{v_0}{\omega x_0}\right) \end{cases}$$

- **Phase difference play a an important role for oscillator**
  - **Two oscillators with phases:** $\theta_1=\omega t+\phi_1,\ \theta_2=\omega t+\phi_2$





$$\Delta\theta=\theta_2-\theta_1=2k\pi$$

$$k=0,\pm1,\pm2\cdots$$

In phase

$$\Delta\theta=\theta_2-\theta_1=(2k+1)\pi$$

$$k=0,\pm1,\pm2\cdots$$

Out of phase

$$\pi>\Delta\theta=\theta_2-\theta_1>0,$$

Ahead in phase

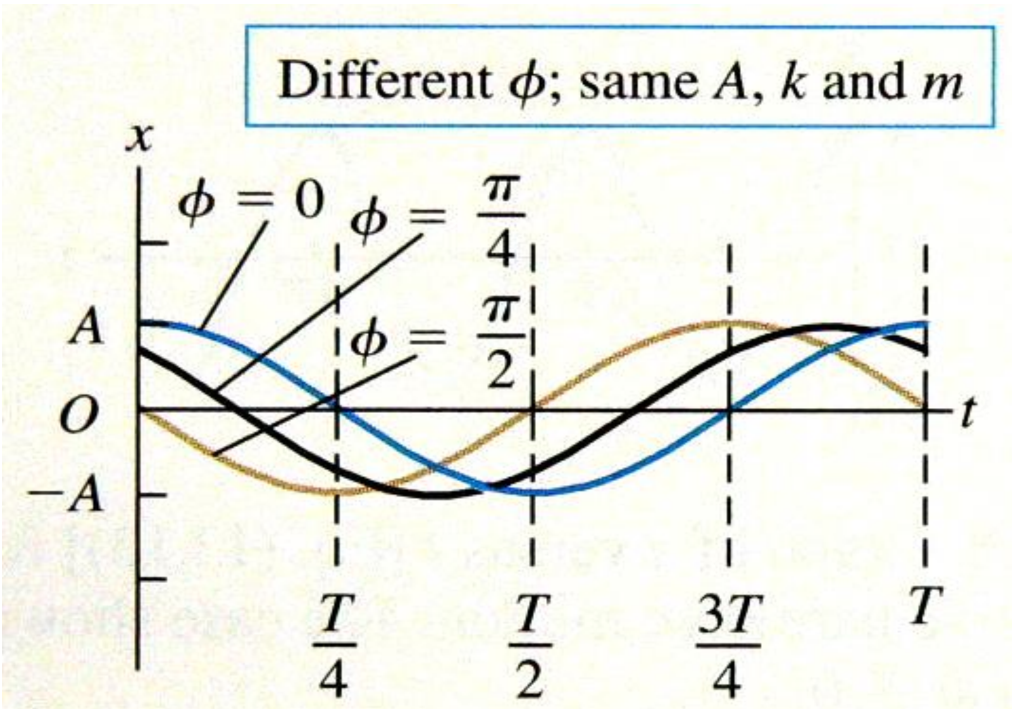
$$-\pi<\Delta\theta=\theta_2-\theta_1<0$$

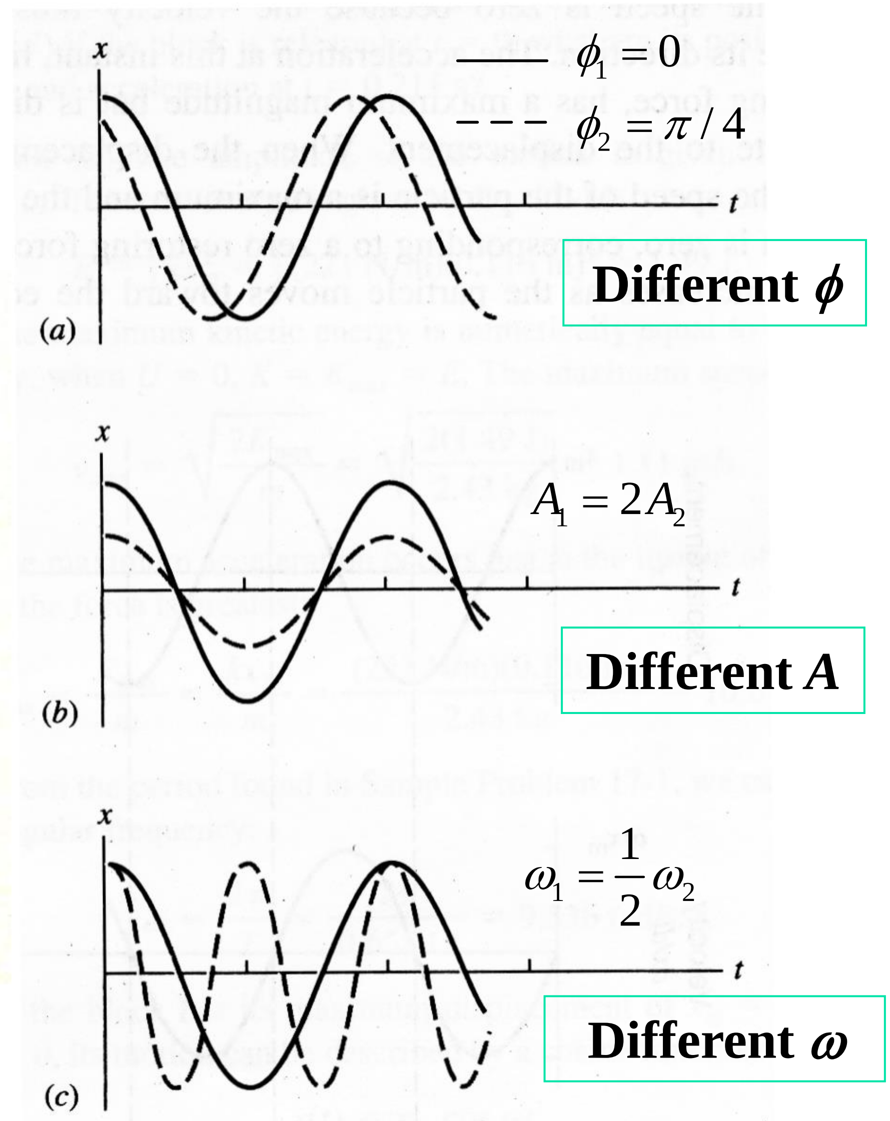
Lag in phase

The Roles Characteristic Quantities

$$x = A\cos(\omega t + \phi)$$

Several SHM with different characteristic quantities





The relations among the position, velocity, and acceleration

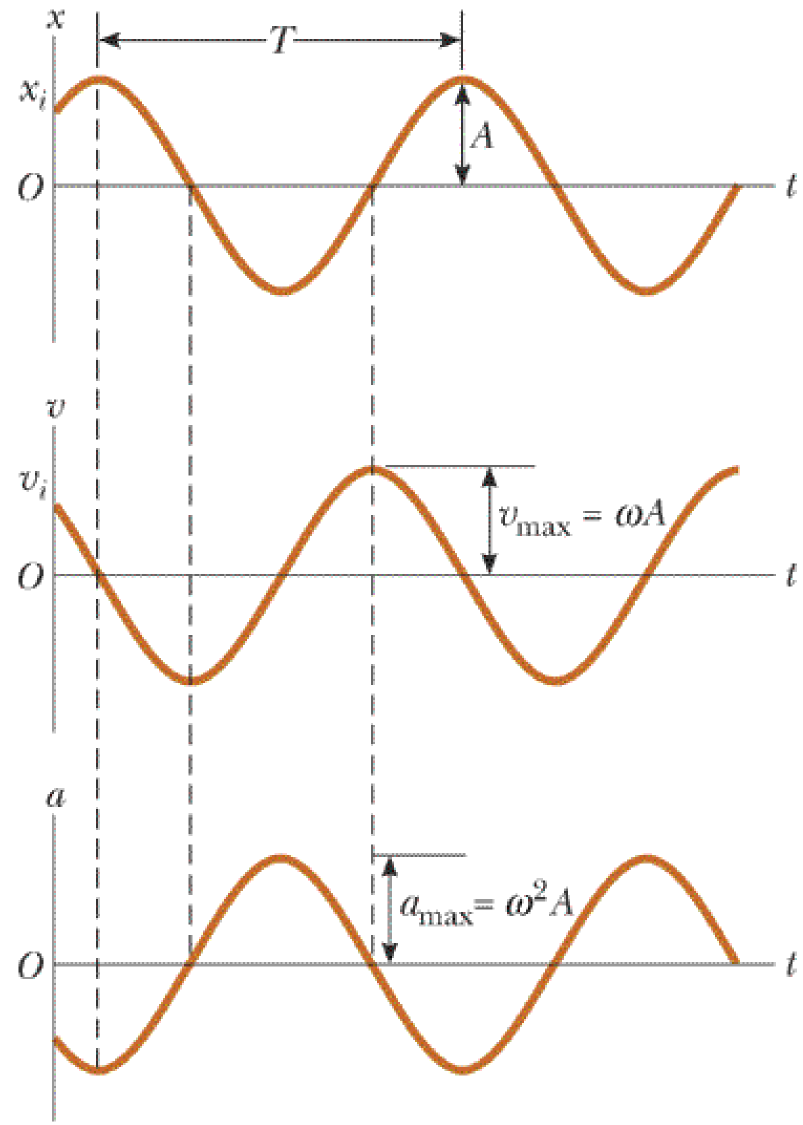
$$x = A\cos(\omega t + \phi)$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

$$= \omega A\cos(\omega t + \phi + \frac{\pi}{2})$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi)$$

$$= \omega^2 A\cos(\omega t + \phi + \pi)$$



- **The velocity is π/2 ahead in phase of the position.**
- **The acceleration is π out of phase with the position.**

Example

An object of mass 4 kg is attached to a spring of k =100 N/m. The object is given an initial velocity of v_0 = –5 m/s and an initial displacement of x_0=1 m. Find the kinematics equation.

Solution:

$$x = A\cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{4}} = 5\ \mathrm{s}^{-1}, \quad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{2}\ \mathrm{m}$$

$$t = 0,\ \ 1 = \sqrt{2}\cos\phi, \qquad \phi = \frac{\pi}{4}\ \text{ or }\ -\frac{\pi}{4}$$

$$\text{with}\ \ v_0 = -\omega A\sin\phi < 0, \qquad \sin\phi > 0, \qquad \therefore \phi = \frac{\pi}{4}$$

$$\therefore x = \sqrt{2}\cos(5t + \frac{\pi}{4})\ \ \mathrm{m}$$

Example

A particle undergoes SHM with $A = 4$ cm, $f = 0.5$ Hz. The displacement $x = -2$ cm when $t = 1$ s, and is moving in the positive x-axis. Write the kinematics equation.

Solution: $A = 0.04\ \text{m}, \quad f = 0.5\ \text{Hz}, \quad \omega = 2\pi f = \pi\ \text{rad/s},$

$$x = 0.04\cos(\pi t + \phi)\ \text{m}, \qquad \phi = ?$$

When $t = 1\text{s}, \quad -0.02 = 0.04\cos(\pi + \phi) = -0.04\cos\phi$

$$\cos\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3} \text{ or } -\frac{\pi}{3}$$

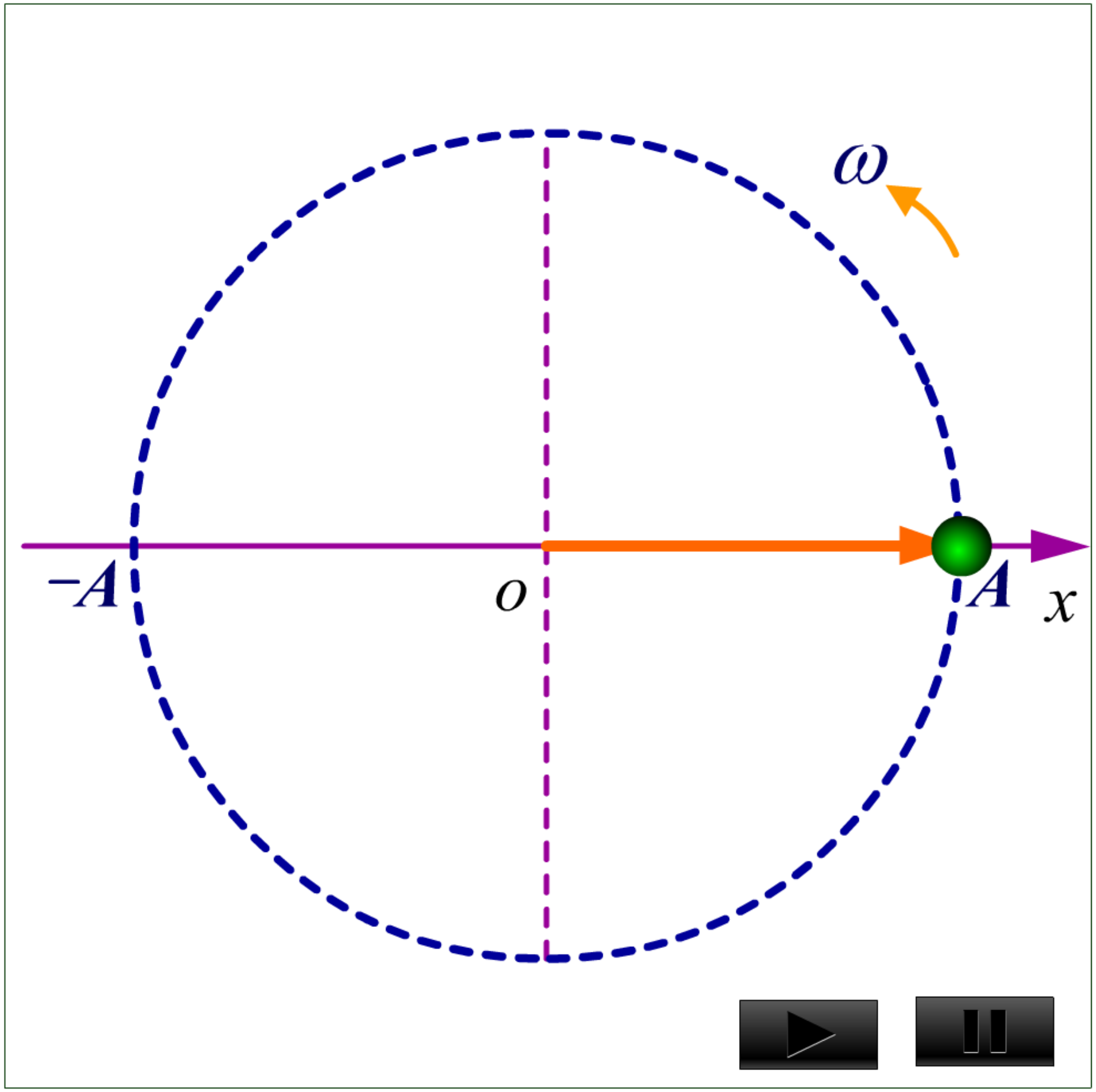
$$v = -\omega A\sin(\omega t + \phi)$$

$$= -0.04\pi\sin(\pi + \phi) = 0.04\pi\sin\phi > 0, \quad \phi = \frac{\pi}{3}$$

$$x = 0.04\cos(\pi t + \frac{\pi}{3})\ \text{m}$$

Too complicated !

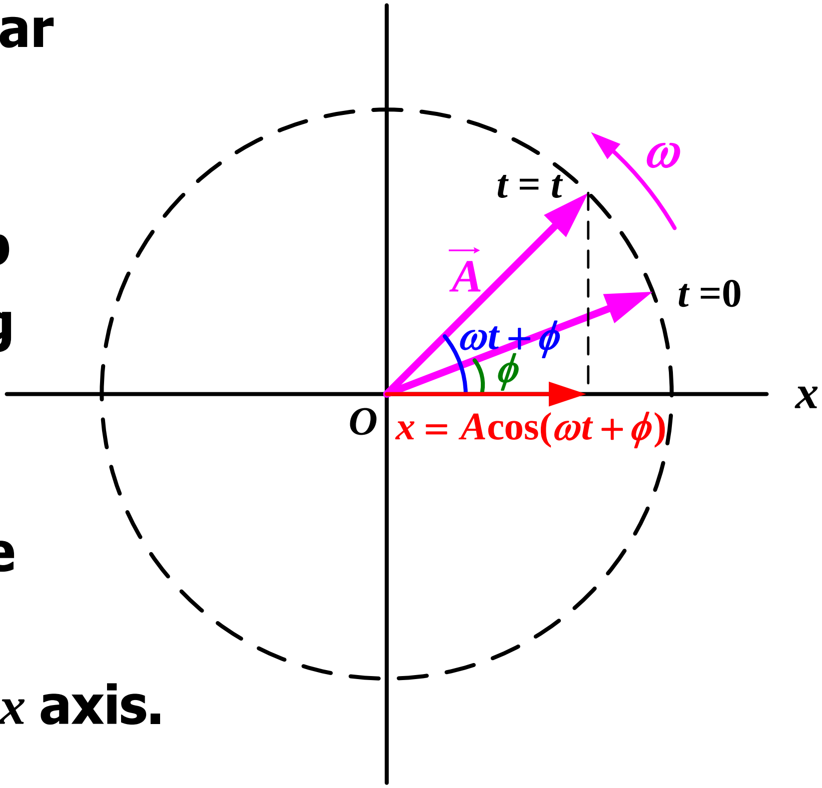
§ 4 The Circle of Reference (P306)



Circle of Reference or Phasor

- **The corresponding relation between SHM and uniform circular motion —— Circle of Reference (参考圆) or Phasor (旋转矢量)**

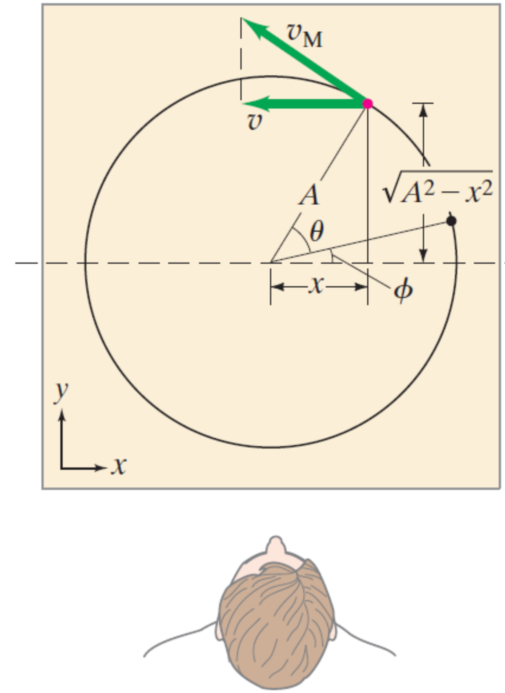
- **SHM is the projection of uniform circular motion of phasor $\vec{A}$ onto x axis.**
- **The circle in which the phasor moves so that the projection of phasor's top matches the motion of the oscillating body is called the circle of reference.**
- **The phasor $\vec{A}$ rotates with constant angular speed ω, and makes an angle $\omega t+\phi$ with the x axis. When $t=0$, the phasor $\vec{A}$ makes an angle ϕ with the x axis.**



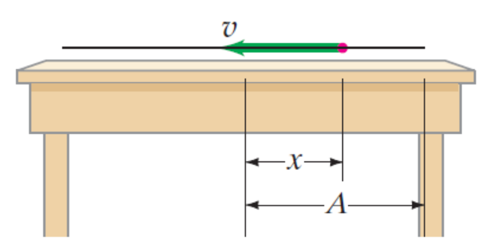
Corresponding Relation Between SHM and UCM

	For Simple Harmonic Motion	For Uniform Circular Motion
A	Amplitude	Radius
x	Displacement	Projection
ω	Angular Frequency	Angular Velocity
$\theta = \omega t + \phi$	Phase	Angle between Phasor and x axis

The simple harmonic motion is the side view of circular motion.

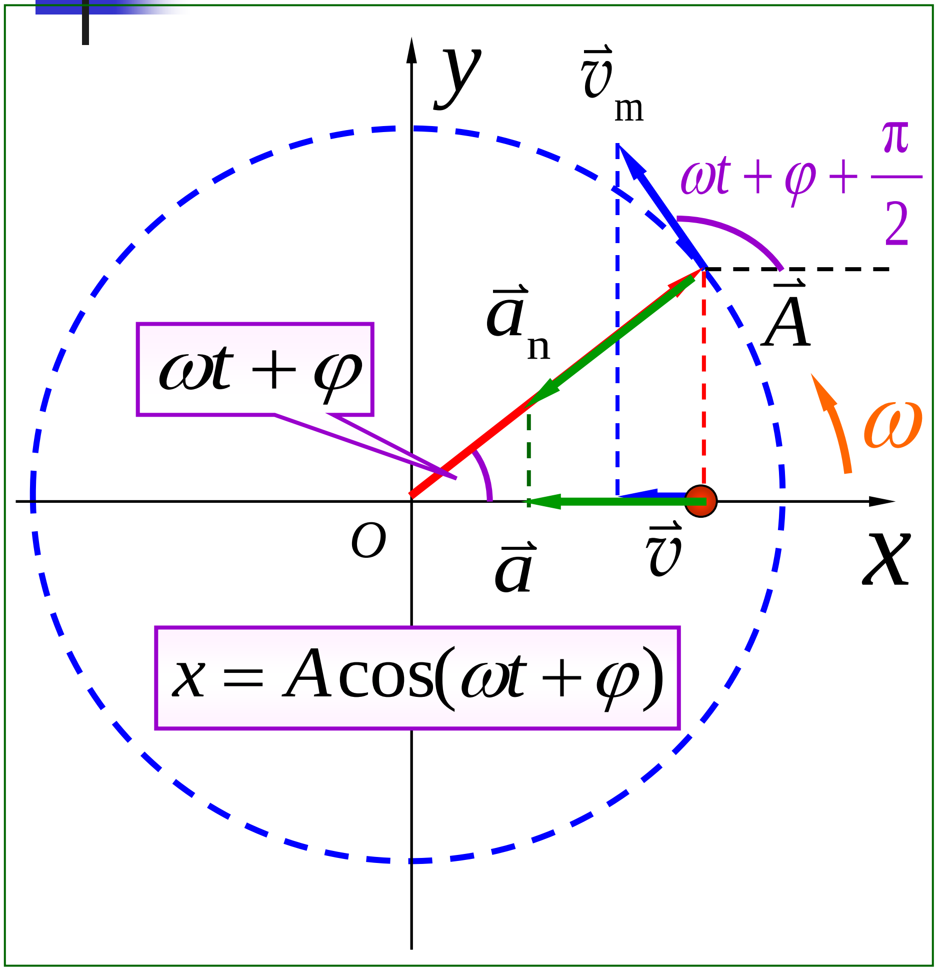


(a)



(b)

Corresponding Relation Between SHM and UCM



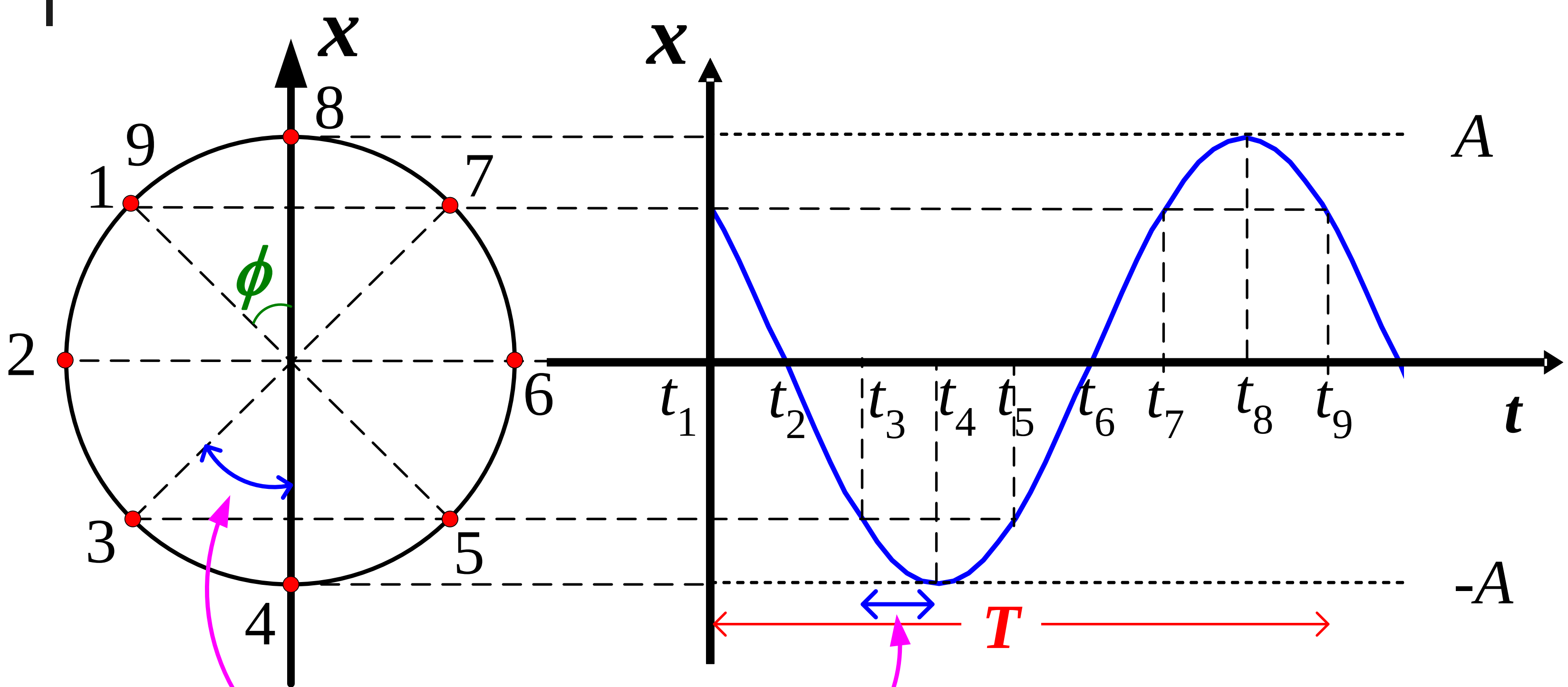
$$v_{\mathrm{m}} = A\omega$$

$$\boxed{v = -A\omega\sin(\omega t+\varphi)}$$

$$a_{\mathrm{n}} = A\omega^2$$

$$\boxed{a = -A\omega^2\cos(\omega t+\varphi)}$$

Draw *x*-*t* Graph Using Circle of Reference



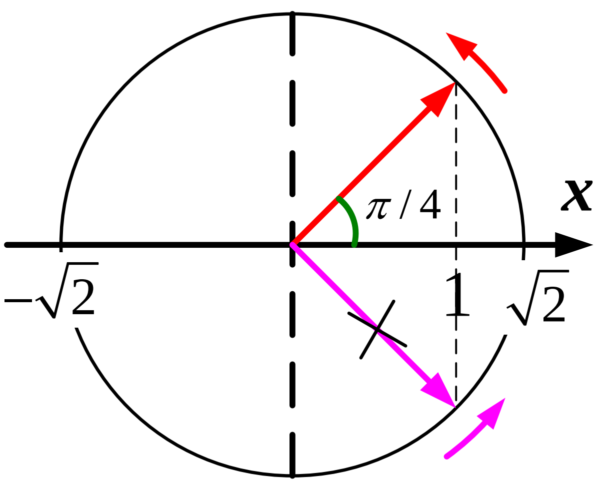
$$\frac{|\Delta\phi|}{2\pi}T = \Delta t$$

An object of mass 4 kg is attached to a spring of k =100 N/m. The object is given an initial velocity of v_0 = –5 m/s and an initial displacement of x_0=1 m. Find the kinematics equation.

Solution (II): $$x = A\cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{4}} = 5\ \mathrm{s}^{-1}, \qquad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{2}\ \mathrm{m}$$

Using the phasor $\quad x_0 = 1,\ v_0 < 0$



$$\therefore x = \sqrt{2}\cos(5t + \frac{\pi}{4}) \quad \mathrm{m}$$

Example

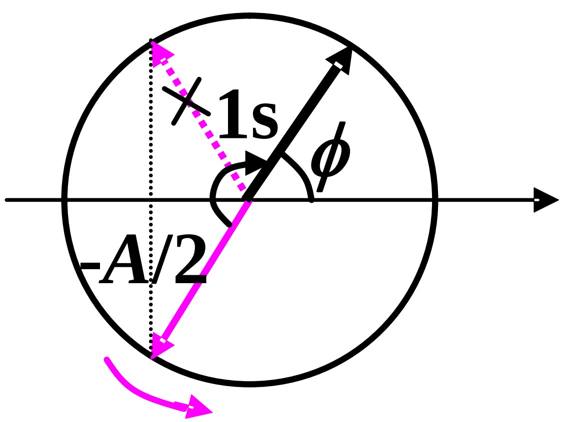
A particle undergoes SHM with A=4cm， f = 0.5Hz. The displacement x = –2cm when t = 1s, and is moving in the positive x-axis. Write the kinematics equation.

Solution (II): $A = 0.04\ \text{m},\quad f = 0.5\ \text{Hz},\quad \omega = 2\pi f = \pi\ \text{rad/s},$

$$x = 0.04\cos(\pi t + \phi)\ \text{m},\qquad \phi = ?$$

Using the phasor:

Δt = 1 s corresponds to half a revolution.



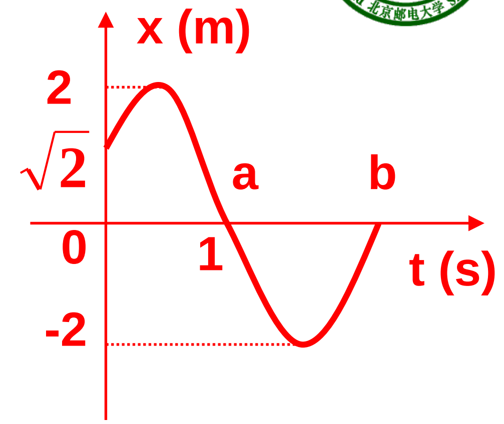
$$v > 0 \Rightarrow \Delta\phi + \phi = \omega\Delta t + \phi$$

$$= \pi + \phi = 4\pi/3$$

$$\Rightarrow \phi = \frac{\pi}{3}$$

Example

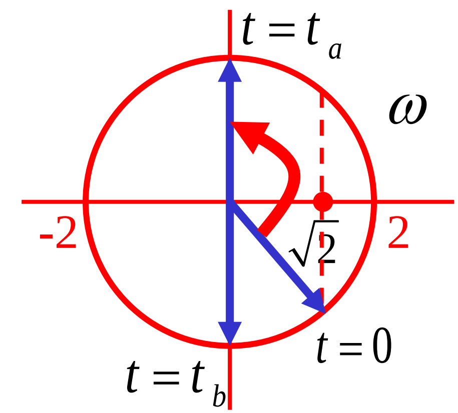
SHM: From given *x*-*t* graph, find θ_a, θ_b, ϕ, and the angular frequency ω.



Solution:

From circle of reference

$$\theta_a = \frac{\pi}{2}, \quad \theta_b = \frac{3\pi}{2}, \quad \phi = -\frac{\pi}{4}$$

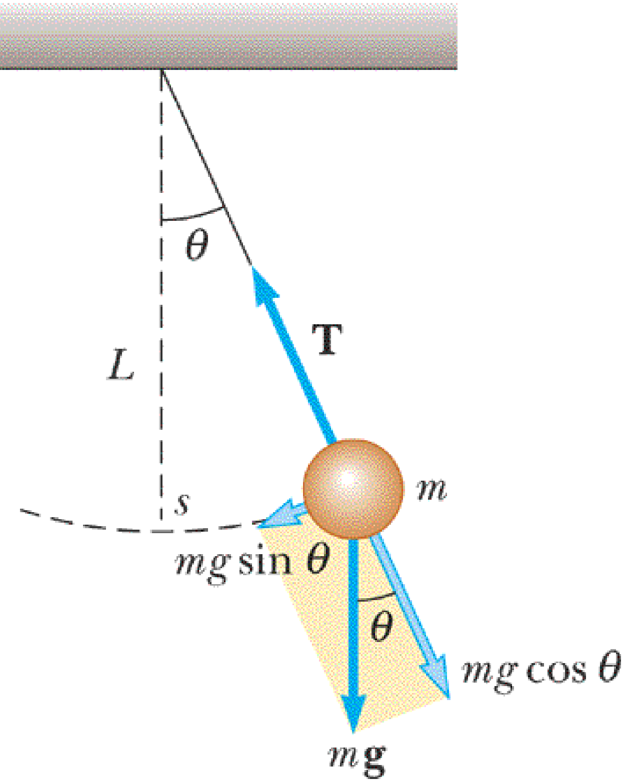


$$\therefore \omega = \frac{\Delta\theta}{\Delta t} = \frac{\theta_a - \phi}{\Delta t} = \frac{\frac{\pi}{2} - \left(-\frac{\pi}{4}\right)}{1} = \frac{3\pi}{4}\ \text{rad/s}$$

$$x = 2\cos\left(\frac{3\pi}{4}t - \frac{\pi}{4}\right)\ \text{m}$$

The simple pendulum (P307)

- **Newton's second law for the simple pendulum**



$$\tau_{\text{net-axis}} = I\alpha, \quad -mg\left(L\sin\theta\right) = \left(mL^2\right)\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Let $\omega = \sqrt{\frac{g}{L}}$ **,**

and for small angles $\sin\theta \approx \theta$

- **We get also a equation of motion of SHM**

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \qquad \theta = \theta_m \cos(\omega t + \phi)$$

The Physical Pendulum (复摆) (P308)

- **Newton's second law for rigid body:**

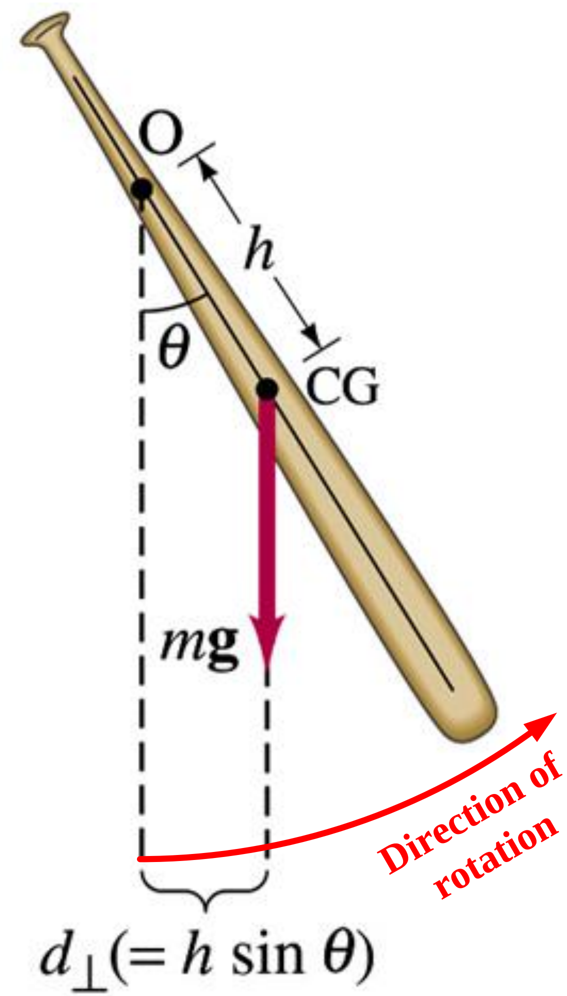
$$\tau_{\text{net-axis}} = I\alpha, \qquad -mgh\sin\theta = I\frac{d^2\theta}{dt^2}$$

It follows that:

$$\frac{d^2\theta}{dt^2} + \frac{mgh}{I}\sin\theta = 0, \qquad \sin\theta \approx \theta$$

$$\frac{d^2\theta}{dt^2} + \left(\frac{mgh}{I}\right)\theta = 0 \quad \Rightarrow$$

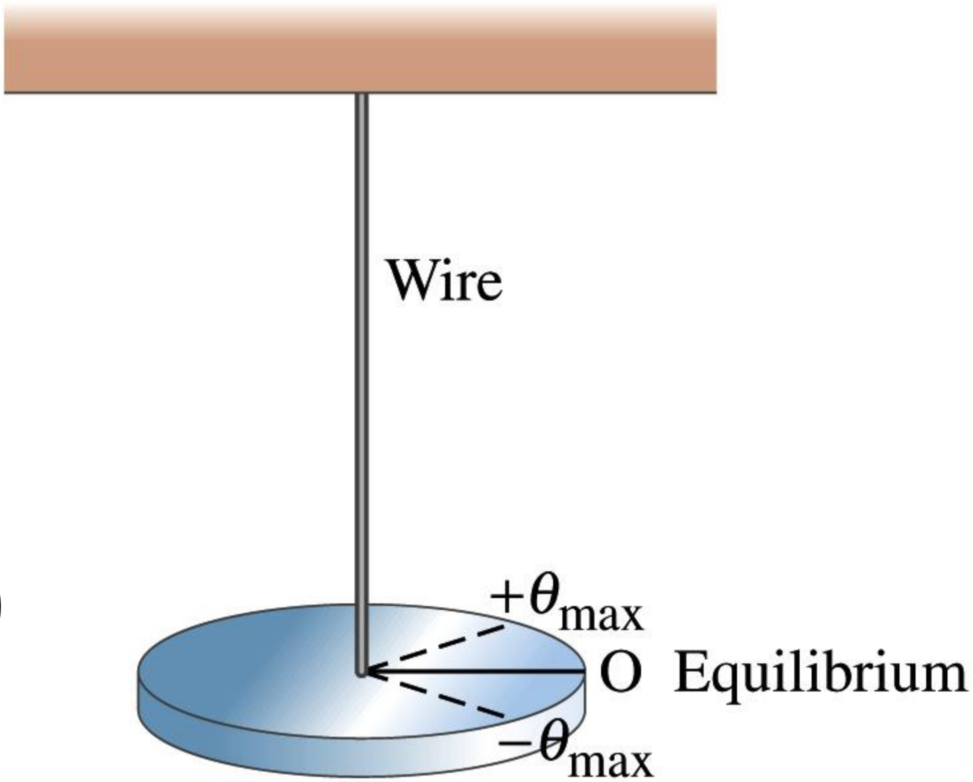
$$\theta = \theta_{\max}\cos(\omega t + \phi), \quad \omega = \sqrt{\frac{mgh}{I}}.$$



The Torsion Pendulum (扭摆) (P309)

- **The restoring torque:** $\tau = -K\theta$

$$-K\theta = I\alpha = I\frac{d^2\theta}{dt^2}$$



$$\frac{d^2\theta}{dt^2} + \left(\frac{K}{I}\right)\theta = 0$$

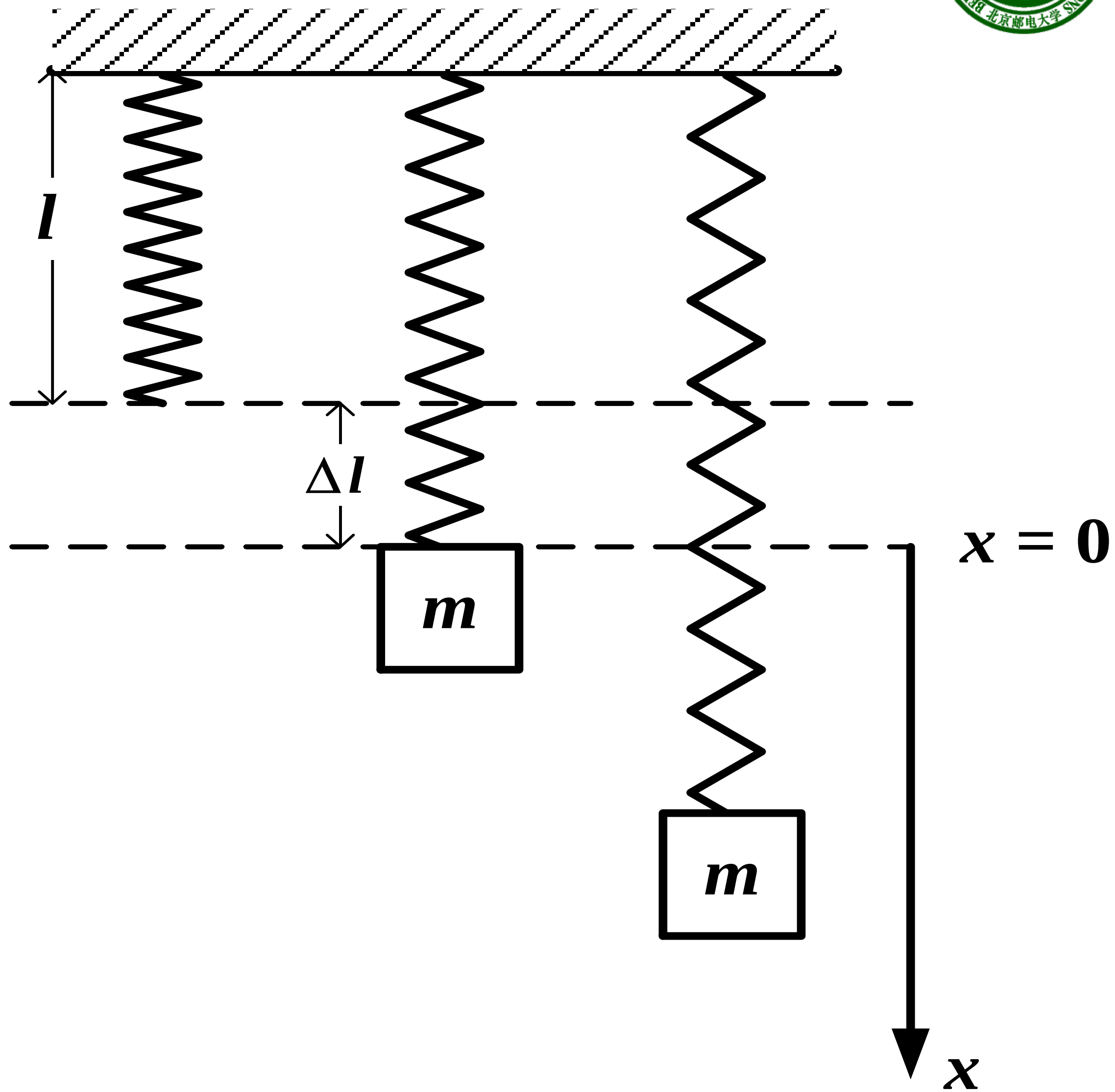
$$\theta = \theta_{max}\cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{K}{I}}$$

Example

Vertical SHM:

Suppose we hang a spring with force constant k and suspend from it a body with mass m. Oscillation will now be vertical. Will it still be SHM?

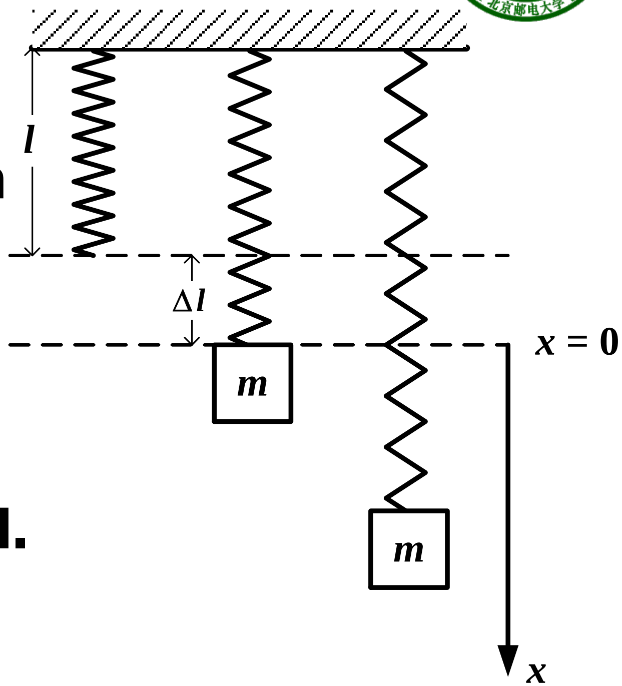


Example

Solution I: by Newton' second law

When the body hangs at rest, in equilibrium

$$k\Delta l = mg$$



Take $x = 0$ to be the equilibrium position, and take the positive x-direction to be downward.

$$F_{net} = -k(x+\Delta l) + mg = -kx - k\Delta l + mg$$

$$= -kx = m\frac{d^2x}{dt^2}$$

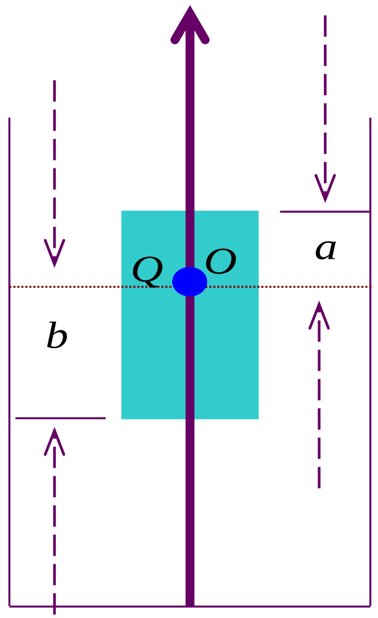
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0,$$

The body's motion is still SHM with the angular frequency:

$$\omega = \sqrt{\frac{k}{m}}$$

Example

A wooden block floats in water. We press it until its upper surface just under water, and release. Will the motion of the wooden block be **SHM**?



Example

Solution: **Take the point O at the surface of water to be the origin of x-axis. When the block is in equilibrium, the point Q of block coincides with origin point O.**

When block is in equilibrium

$$Sl\rho_{block}g = Sb\rho_{water}g$$

where S is the area of block's cross section, and $l=a+b$

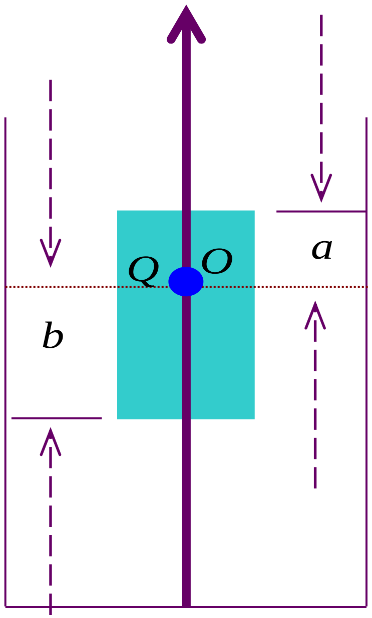
The net force: $$\sum F = S(b-x)\rho_{water}g - Sl\rho_{block}g$$

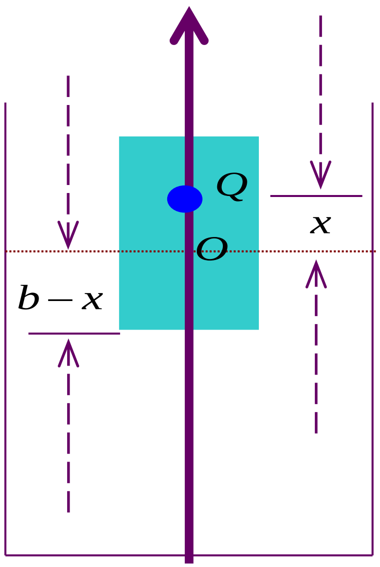
$$= S(b-x)\rho_{water}g - Sb\rho_{water}g$$

$$= -Sx\rho_{water}g$$

$$(Sl\rho_{block})\frac{d^2x}{dt^2} = -S\rho_{water}gx$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{g}{b}x = 0, \qquad x = A\cos\left(\sqrt{\frac{g}{b}}t + \phi\right)$$





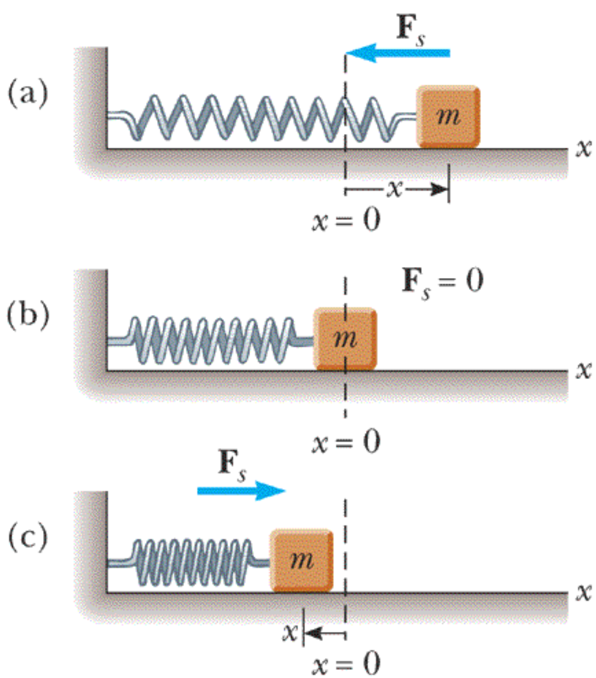
课后作业

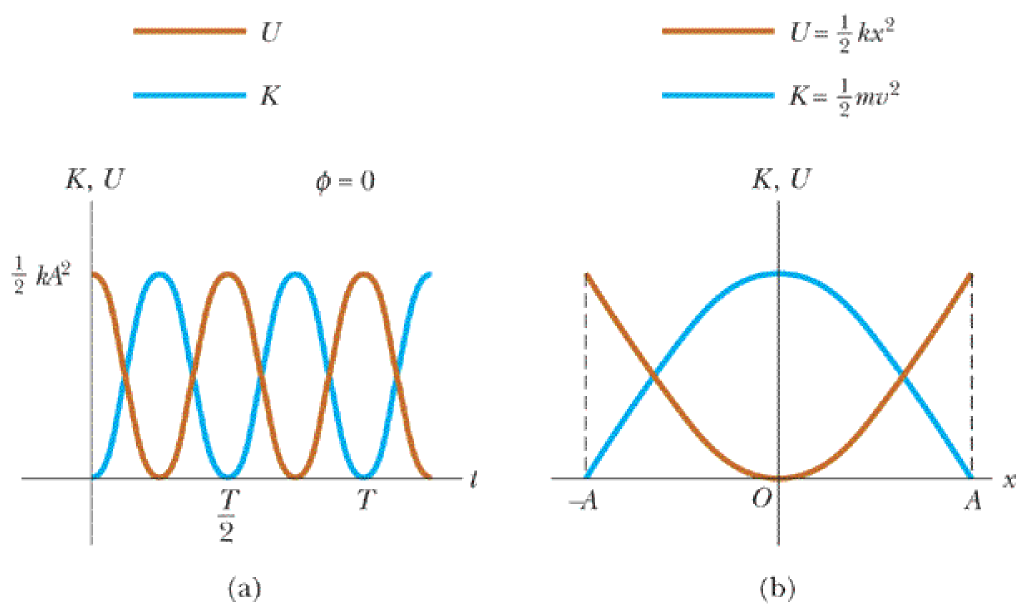
Ch12 (P318)

Prob. 21, 22, 23

§ 5 Energy in Simple Harmonic Motion (P304)

- **The total mechanical energy for an isolated simple harmonic oscillator**
  - **Kinetic energy:** $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi), \quad \omega^2 = \frac{k}{m}$
  - **Potential energy:** $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$
  - **Total mechanical energy:** $E_{\text{mech}} = K + U = \frac{1}{2}kA^2 = \text{constant}$





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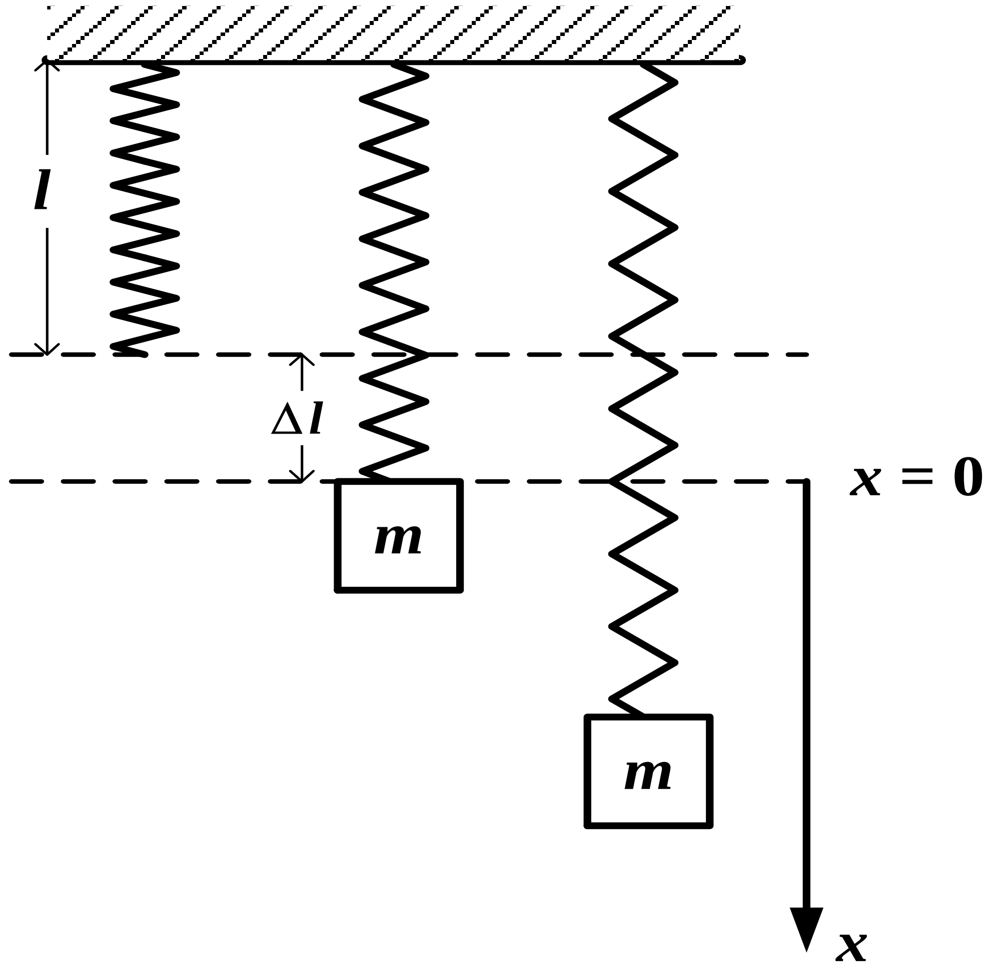
Vertical SHM:

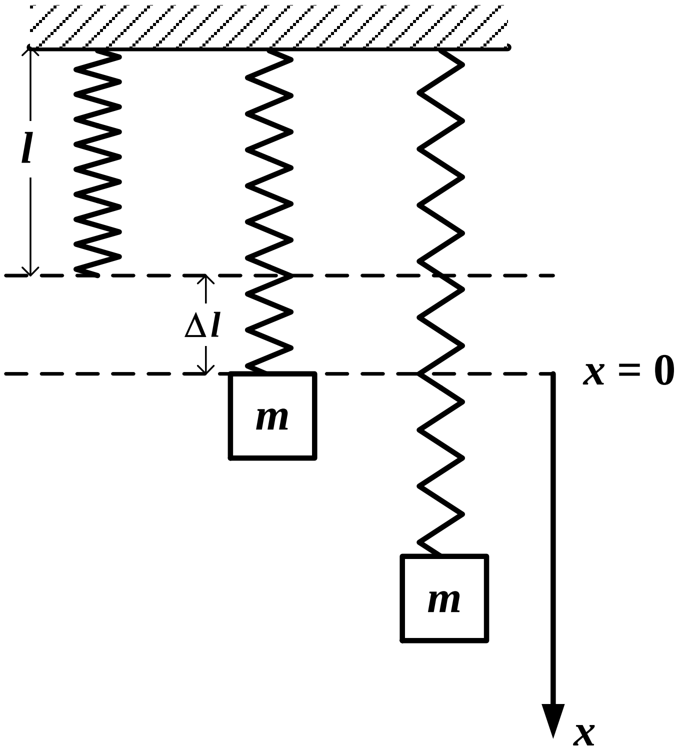
Suppose we hang a spring with force constant k and suspend from it a body with mass m. Oscillation will now be vertical. Will it still be SHM?



Example cont'd

Solution II: By energy analysis

When the body is at the position x, the total mechanical energy is



$$\frac{1}{2}mv^2 + \frac{1}{2}k(x+\Delta l)^2 - mgx = \text{constant}$$

by derivative on both sides

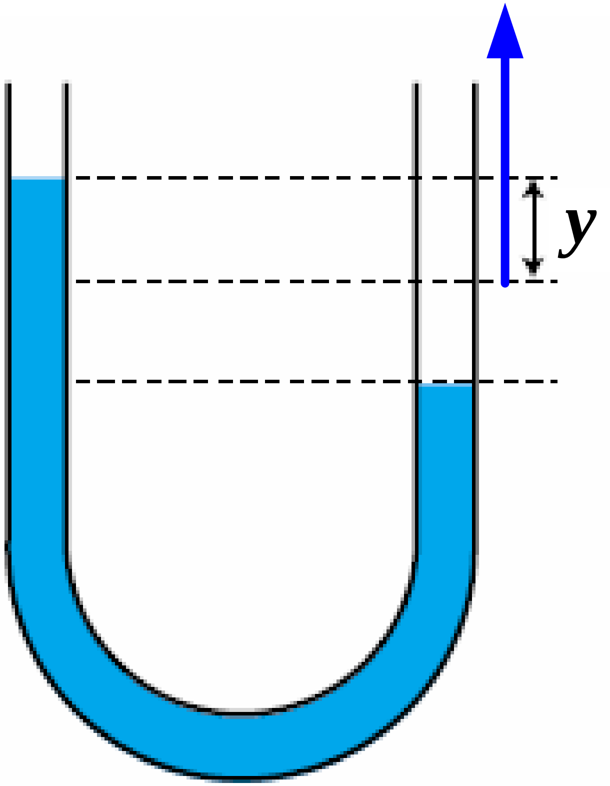
$$mv\frac{dv}{dt} + k(x+\Delta l)\frac{dx}{dt} - mg\frac{dx}{dt} = 0$$

$$\frac{dv}{dt} = \frac{d^2x}{dt^2}, \qquad \frac{dx}{dt} = v$$

$$m\frac{d^2x}{dt^2} + kx + (k\Delta l - mg) = 0, \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0, \qquad \omega = \sqrt{\frac{k}{m}}$$

Example

Liquid in a U-tube: A liquid of density ρ is poured into a U-shaped tube with a cross-section of S. The total mass of the liquid is m. The liquid in the U-tube can undergoes vibration about equilibrium. Find the vibration period of the liquid.



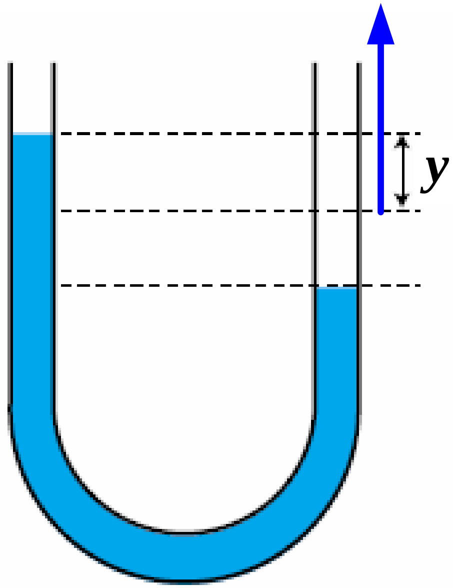
Example

Solution: The potential energy

$$U = (\rho S y g) y = \rho S g y^2$$

The kinetic energy:

$$K = \frac{1}{2} m \left(\frac{dy}{dt} \right)^2$$



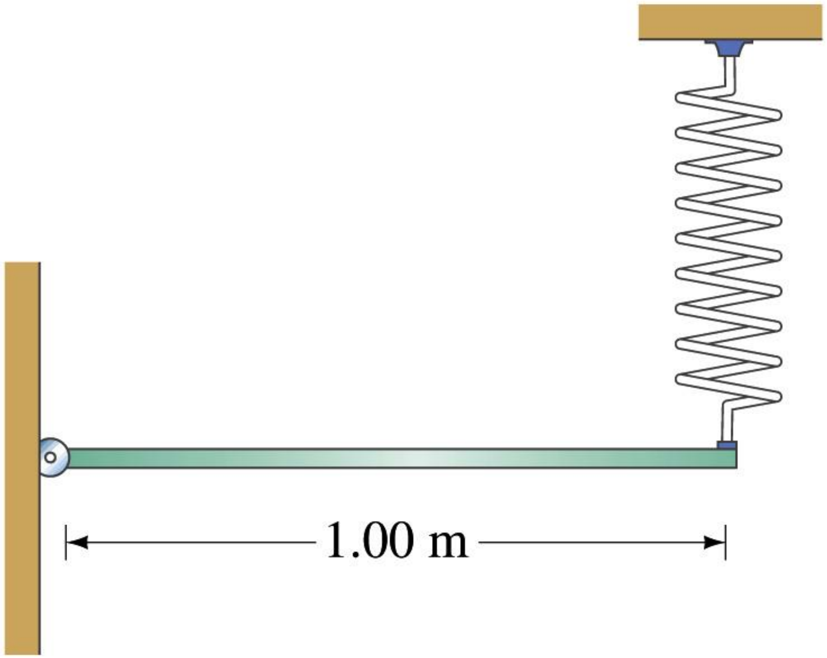
$$K + U = \frac{1}{2} m \left(\frac{dy}{dt} \right)^2 + \rho S g y^2 = \text{constant}$$

$$m \left(\frac{dy}{dt} \right) \left(\frac{d^2 y}{dt^2} \right) + 2\rho S g y \left(\frac{dy}{dt} \right) = 0, \qquad \frac{d^2 y}{dt^2} + \frac{2\rho S g}{m} y = 0,$$

$$m = \rho(SL), \qquad T = 2\pi \sqrt{\frac{m}{2\rho S g}} = 2\pi \sqrt{\frac{L}{2g}}$$

Example

A uniform meter stick of mass M is pivoted on a hinge at one end and held horizontal by a spring with spring constant k attached at the other end. The stick is displaced by a small angle from its horizontal equilibrium position and released. Find the angular frequency with which the stick moves with simple harmonic motion. (Using energy conservation.)



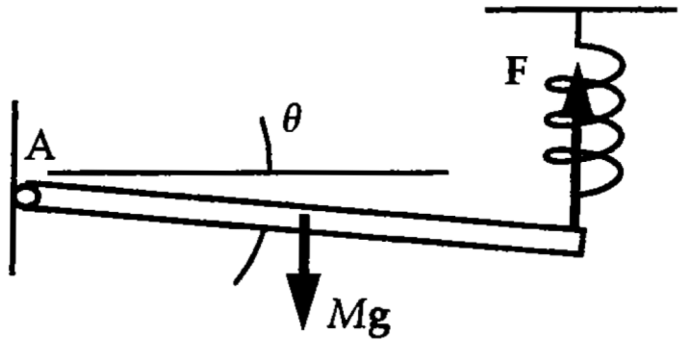
Example

Solution (II):

The total mechanical energy is

$$\frac{1}{2}I(\dot{\theta})^2+\frac{1}{2}k(x_0+l\theta)^2-Mg\frac{l\theta}{2}=\text{constant}$$

By derivative on both sides,



$$I\dot{\theta}(\ddot{\theta})+k(x_0+l\theta)l\dot{\theta}-\frac{1}{2}Mgl\dot{\theta}=0$$

$$I\ddot{\theta}+k(x_0+l\theta)l-\frac{1}{2}Mgl=0$$

$$kx_0=\frac{1}{2}Mg \quad \Rightarrow \quad I\ddot{\theta}+kl^2\theta=0$$

$$\ddot{\theta}+\frac{kl^2}{I}\theta=0, \qquad \omega=\sqrt{\frac{kl^2}{I}}=\sqrt{\frac{kl^2}{\frac{1}{3}Ml^2}}=\sqrt{\frac{3k}{M}}$$

§ 6 Superposition of SHM

- An object experiences two SHMs **simultaneously**.
  - Two SHMs

$$x_1 = A_1 \cos(\omega t + \phi_1)$$

$$x_2 = A_2 \cos(\omega t + \phi_2)$$

Resultant motion which is superposed by the two SHMs is also a SHM

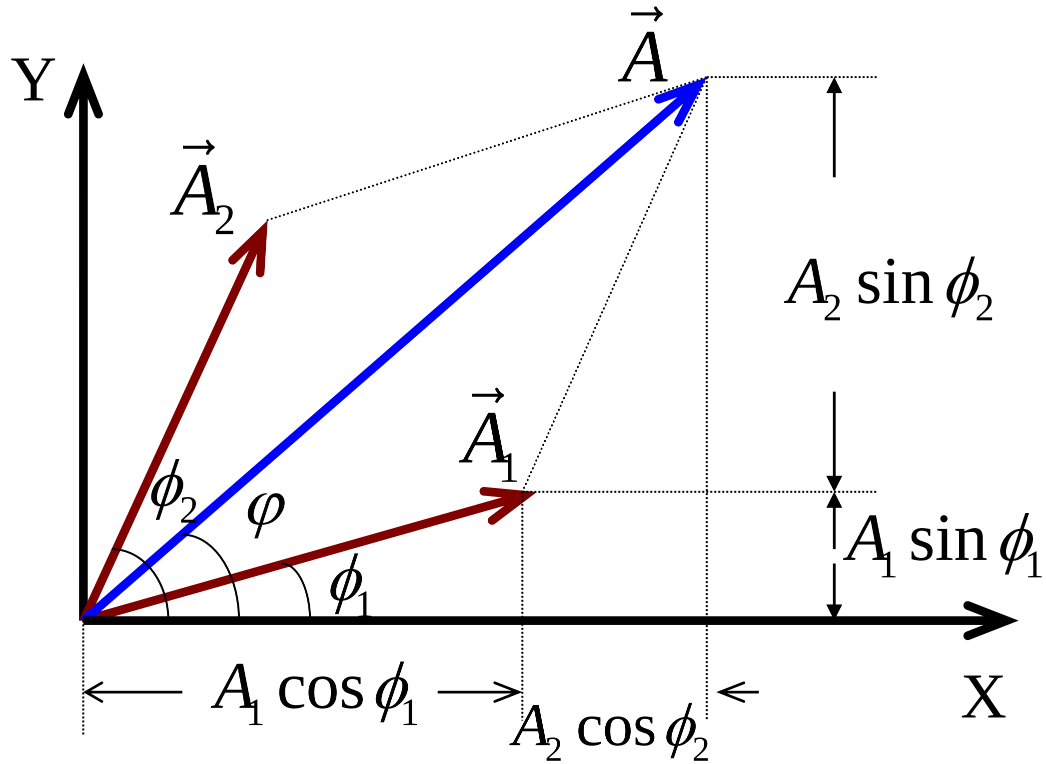
$$x = x_1 + x_2 = A \cos(\omega t + \phi)$$

Resultant Amplitude ?

Resultant Phase angle ?

Superposition of SHMs using phasor diagram

Using phasors,

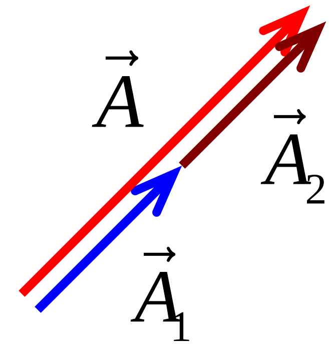


$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1)}$$

$$\varphi = \arctan\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}$$

Superposition of SHMs under different phase differences

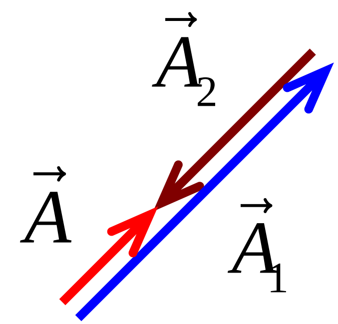
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\Delta\phi}$$



- **The phase difference $\Delta\phi=\phi_2-\phi_1$.**
  - **When $\Delta\phi=\phi_2-\phi_1=2k\pi$, $k=0, \pm1, \pm2, \ldots$**

    The two SHMs are in phase, the resultant amplitude take its maximum.

$$A = A_1 + A_2$$

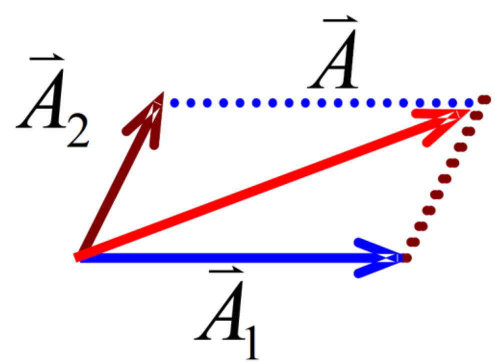


  - **When $\Delta\phi=\phi_2-\phi_1=(2k+1)\pi$, $k=0, \pm1, \pm2, \ldots$**

    The two SHMs are out of phase, the resultant amplitude take its minimum.

$$A = |A_1 - A_2|$$

  - **Generally, $\Delta\phi=\phi_2-\phi_1\neq k\pi$**



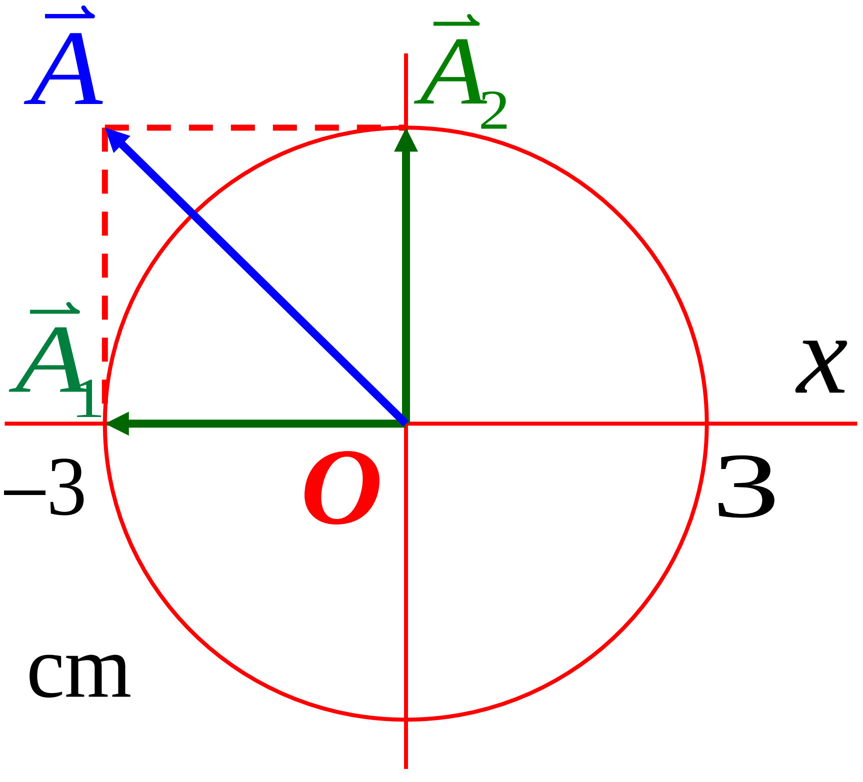
$$|A_1 - A_2| < A < A_1 + A_2$$

Example: x_1=3cos(2πt+π) cm, x_2=3cos(2πt+π/2) cm, find the superposition displacement of x_1 and x_2.

Solution (I):

Draw a circle of reference,

$$x = x_1 + x_2$$
$$= A\cos(\omega t + \phi)$$
$$= 3\sqrt{2}\cos(2\pi t + \frac{3\pi}{4}) \quad \text{cm}$$



Example

Example: x_1=3cos(2πt+π) cm, x_2=3cos(2πt+π/2) cm, find the superposition displacement of x_1 and x_2.

Solution (II):

$$A=\sqrt{A_1^2+A_2^2+2A_1A_2\cos(\phi_2-\phi_1)}$$

$$=\sqrt{3^2+3^2+2\times3\times3\cos\frac{\pi}{2}}=3\sqrt{2}\ \text{cm}$$

$$\tan\phi=\frac{A_1\sin\phi_1+A_2\sin\phi_2}{A_1\cos\phi_1+A_2\cos\phi_2}=\frac{3\sin\pi+3\sin\frac{\pi}{2}}{3\cos\pi+3\cos\frac{\pi}{2}}=-1,\quad \phi=\frac{3\pi}{4}$$

Solution (III):

$$x_1+x_2=3\left[\cos\left(2\pi t+\pi\right)+\cos\left(2\pi t+\frac{\pi}{2}\right)\right]=3\left[2\cos\left(2\pi t+\frac{3}{4}\pi\right)\cos\frac{\pi}{4}\right]$$

$$=3\sqrt{2}\cos(2\pi t+\frac{3\pi}{4})\quad \text{cm}$$

Problem

Ch12 (P319)

Prob. 34，35，36