# Lecture 06

**Chapter 3. Vector spaces**

**3.1 Definitions and Examples**

# Overview

The operations of **addition** and **scalar multiplication** are used in many diverse contexts in mathematics. Regardless of the context, however, these operations usually obey the same set of **algebraic rules**. Thus, a general theory of mathematical systems involving addition and scalar multiplication will be applicable to many areas in mathematics. Mathematical systems of this form are called **vector spaces [向量空间]** or **linear spaces [线性空间]**.

In this lecture, we will give the definition of a vector space.

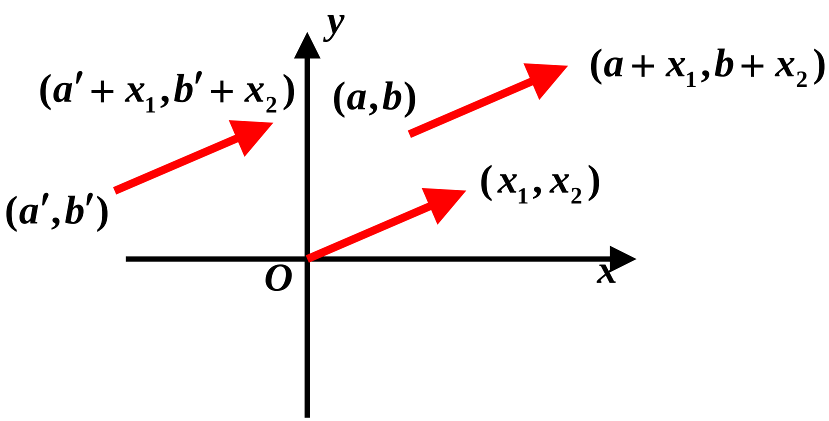
# Euclidean $n$-Spaces

The most elementary vector spaces are Euclidean $n$-spaces [$n$维欧几里得空间，欧式空间] $\mathbf{R}^n$, $n = 1,2, \ldots$. For simplicity, let us first consider $\mathbf{R}^2$.

Nonzero **vectors** in $\mathbf{R}^2$ can be represented geometrically by **directed line segment**. This geometric representation will help us visualize how the operations of addition and scalar multiplication work in $\mathbf{R}^2$.

Given a nonzero vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we can associate it with the line segment in the plane from (0,0) to $(x_1, x_2)$. If we equate line segments that have the same length and direction, $\boldsymbol{x}$ can be represented by any line segment from $(a, b)$ to $(a + x_1, b + x_2)$.

# Euclidean $n$-Spaces




The length of any vector can be thought as the length of the line segment, therefore, the **length** of vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is $\sqrt{x_1^2 + x_2^2}$.

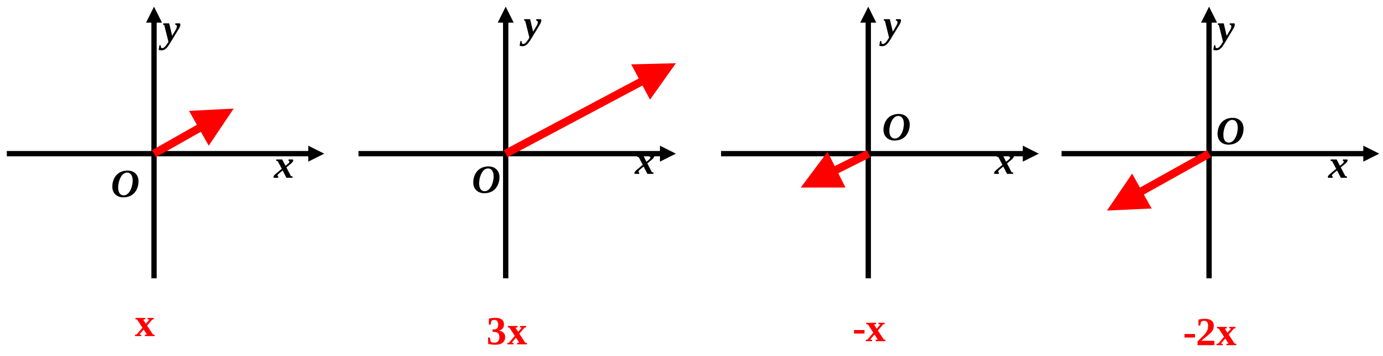
We define the **scalar product** of the scalar $\alpha$ and the vector $\boldsymbol{x}$ as

$$\alpha\boldsymbol{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}.$$

# Euclidean $n$-Spaces

For example, if $\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then

$$3\boldsymbol{x} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \qquad -\boldsymbol{x} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \qquad -2\boldsymbol{x} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}.$$



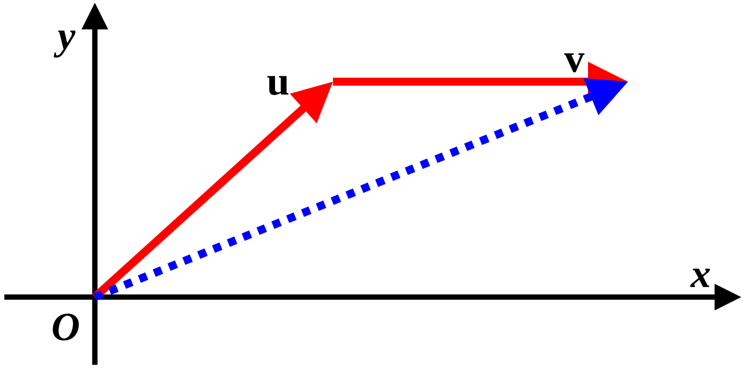
# Euclidean $n$-Spaces

The **sum** of two vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

is defined by

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$



# Euclidean $n$-Spaces

In general, **scalar multiplication** and **addition** in $\mathbf{R}^n$ are defined by

$$\alpha \boldsymbol{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}, \qquad \boldsymbol{x} + \boldsymbol{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$ and any scalar $\alpha$.

We can also view $\mathbf{R}^n$ as the set of all $n \times 1$ matrices with real entries. The addition and scalar multiplication of vectors in $\mathbf{R}^n$ are just the usual addition and scalar multiplication of matrices.

# The space $\mathbf{R}^{m \times n}$

More generally, let $\mathbf{R}^{m \times n}$ be the set of all $m \times n$ matrices with real entries.

- If $A = (a_{ij})$ and $B = (b_{ij})$, the sum $C = A + B$ is defined to be the $m \times n$ matrix $C = (c_{ij})$ where
$$c_{ij} = a_{ij} + b_{ij}.$$

- Given a scalar $\alpha$, we define $\alpha A$ to be the $m \times n$ matrix whose $(i, j)$-entry is $\alpha a_{ij}$.

By defining operations on $\mathbf{R}^{m \times n}$, we have created a mathematical system, and this system is called the **vector space $\mathbf{R}^{m \times n}$**.

# Vector space axioms

**Definition 1. (Vector Space)** Let $\boldsymbol{V}$ be a set on which the operations of **addition** and **scalar multiplication** are defined. By this, we mean that, with each pair of elements $\boldsymbol{x}, \boldsymbol{y}$ in $\boldsymbol{V}$, we can associate a unique element $\boldsymbol{x} + \boldsymbol{y}$ that is in $\boldsymbol{V}$; and with each element $\boldsymbol{x}$ in $\boldsymbol{V}$ and each scalar $\alpha$, we can associate a unique element $\alpha\boldsymbol{x}$ in $\boldsymbol{V}$.

The set $\boldsymbol{V}$ together with the operations of addition and scalar multiplication is said to form a **vector space [向量空间]** or a **linear space [线性空间]** if the following axioms are satisfied.

# Vector space axioms

**(A1) Associativity of addition:** $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$, for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\boldsymbol{V}$;

**(A2) Commutativity of addition:** $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$, for any $\boldsymbol{x}, \boldsymbol{y}$ in $\boldsymbol{V}$;

**(A3) Identity element of addition:** there exists an element $\boldsymbol{0} \in \boldsymbol{V}$, called the **zero vector**, such that $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \boldsymbol{V}$;

**(A4) Inverse elements of addition:** for any $\boldsymbol{x} \in \boldsymbol{V}$, there exists an element $-\boldsymbol{x} \in \boldsymbol{V}$, called the **additive inverse** of $\boldsymbol{x}$, such that $\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{0}$;

**(A5) Distributivity of scalar multiplication:** $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$, for any scalar $\alpha$ and for any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$;

**(A6) Distributivity of scalar multiplication:** $(\alpha + \beta)\boldsymbol{x} = \alpha\boldsymbol{x} + \beta\boldsymbol{x}$, for any scalars $\alpha, \beta$ and for any $\boldsymbol{x} \in \boldsymbol{V}$;

**(A7) Compatibility:** $\alpha(\beta\boldsymbol{x}) = (\alpha\beta)\boldsymbol{x}$, for any scalars $\alpha, \beta$ and for any $\boldsymbol{x} \in \boldsymbol{V}$;

**(A8) Identity element of scalar multiplication:** $1\boldsymbol{x} = \boldsymbol{x}$, for any $\boldsymbol{x} \in \boldsymbol{V}$.

# Vector space axioms

**Remark:** The elements of $\boldsymbol{V}$ are called **vectors**, denoted by $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and etc. The term **real vector space** is used to indicate that the set of scalars is the set of real numbers.

Check that $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$ with operations of addition and scalar multiplication defined in the usual way are vector spaces.

# Vector space axioms

An important component of the definition is the **closure property [封闭性]** of the two operations. These properties can be summarized as follows:

**(C1)** If $\boldsymbol{x} \in \boldsymbol{V}$ and $\alpha$ is a scalar, then $\alpha\boldsymbol{x} \in \boldsymbol{V}$;

**(C2)** If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$, then $\boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{V}$.

The set $$\boldsymbol{W} = \left\{(a, 1)^T \mid a \quad \text{real}\right\}$$

with addition and scalar multiplication defined in the usual way

is NOT a vector space.

# The vector space $C[a, b]$

Let $C[a, b]$ be the set of all real-valued functions that are defined and continuous on the closed interval $[a, b]$.

- If we define the sum $f + g$ of two functions $f, g$ in $C[a, b]$ as

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in [a, b]$, then the new function $f + g$ is an element in $C[a, b]$.

- If $f$ is a function in $C[a, b]$ and $\alpha$ is a real number, define $\alpha f$ by

$$(\alpha f)(x) = \alpha f(x)$$

for all $x \in [a, b]$. Clearly $\alpha f$ is in $C[a, b]$.

Thus on $C[a, b]$, we have defined the operations of addition and scalar multiplication. It is easy to prove that all of the vector space axioms are satisfied. So $C[a, b]$ is a vector space.

**Question**: what is the zero vector in $C[a, b]$?

# The vector space $P_n$

Let $P_n$ be the set of all polynomials of degree less than $n$. Define $p + q$ and $\alpha p$ by

$$(p + q)(x) = p(x) + q(x)$$

and

$$(\alpha p)(x) = \alpha p(x)$$

for all real numbers $x$.

Then one can prove that $P_n$ with these addition and scalar multiplication is a vector space and

$$z(x) = 0x^{n-1} + 0x^{n-2} + \cdots + 0$$

is the zero polynomial.

# Additional properties of vector spaces

**Theorem.** If $\boldsymbol{V}$ is a vector space and $\boldsymbol{x}$ is any element of $\boldsymbol{V}$, then
(i) $0\boldsymbol{x} = \boldsymbol{0}$;
(ii) $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{0}$ implies that $\boldsymbol{y} = -\boldsymbol{x}$ (i.e. the additive inverse of $\boldsymbol{x}$ is unique);
(iii) $(-1)\boldsymbol{x} = -\boldsymbol{x}$.

# Inner Product and Outer Product Expansion of Vectors in $\mathbf{R}^n$

Let $\boldsymbol{x}, \boldsymbol{y}$ be two vectors in $\mathbf{R}^n$. The **inner product [内积]** of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$\boldsymbol{x}^T \boldsymbol{y} = (x_1, x_2, \dots, x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

The result of inner product is a $1 \times 1$ matrix or a scalar, it is also called **scalar product [标量积]** or **dot product [点积],** which is denoted by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ or $\boldsymbol{x} \cdot \boldsymbol{y}$.

**Property.** Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ be vectors in $\mathbf{R}^n$, $\lambda, \beta$ be real numbers.
**(1)** $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$;
**(2)** $\langle \boldsymbol{x}, \lambda \boldsymbol{y} + \beta \boldsymbol{z} \rangle = \lambda \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \beta \langle \boldsymbol{x}, \boldsymbol{z} \rangle$;

# Inner Product and Outer Product Expansion of Vectors in $\mathrm{R}^n$

The **outer product [外积]** of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$\boldsymbol{x}\boldsymbol{y}^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} (y_1, y_2, \dots, y_n) = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_n \end{pmatrix}$$

which is an $n \times n$ matrix.

**Remark.** The definitions of inner and outer product are just suitable for the vector space $\mathrm{R}^n$.

## Review

- Definition of vector space; examples

## Preview

- Subspaces
- Linear Independence