Lecture 08

Chapter 3. Vector Spaces

3.4. Basis and Dimension

3.5. Changing of Basis

3.4 Basis and Dimension

The elements of a minimal spanning set form the basic building blocks for the whole vector space.

Basis of Vector Space

Definition 1. The vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ form a **basis [基，基底]** for a vector space $\boldsymbol{V}$ **if and only if**

(i) $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ are linearly independent;

(ii) $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ span $\boldsymbol{V}$.

Example 1. In $\mathbf{R}^3$, the vectors $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ form a basis of the vector space $\mathbf{R}^3$, which are called the **standard basis [标准基]** for $\mathbf{R}^3$.

However, there are many bases for $\mathbf{R}^3$. For example,

$$\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right\} \text{ and } \left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$$ are both bases for $\mathbf{R}^3$.

Example 2. In $\mathbf{R}^{2\times 2}$, consider the set $\{E_{11}, E_{12}, E_{21}, E_{22}\}$, where

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(i) If

$$c_1E_{11} + c_2E_{12} + c_3E_{21} + c_4E_{22} = O,$$

then

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so $c_1 = c_2 = c_3 = c_4 = 0$. $E_{11}, E_{12}, E_{21}, E_{22}$ are linearly independent.

(ii) If A is in $\mathbf{R}^{2\times 2}$, then

$$A = a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}.$$

Thus, $E_{11}, E_{12}, E_{21}, E_{22}$ span $\mathbf{R}^{2\times 2}$ and hence form a basis for $\mathbf{R}^{2\times 2}$.

$\{E_{11}, E_{12}, E_{21}, E_{22}\}$: Standard basis for $\mathbf{R}^{2\times 2}$

Example 3. In the vector space P_3, the vector set $\{1, x, x^2\}$ is a basis, since all vectors in P_3 can be represented as a linear combination of them.

This basis is called the standard basis for P_3.

Theorem 1. If $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a **spanning set** for a vector space $\boldsymbol{V}$, then any collection of m vectors in $\boldsymbol{V}$, where $m > n$, is linearly dependent.

Proof. Let $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}$ be m vectors in $\boldsymbol{V}$ where $m > n$.

- Since $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ span $\boldsymbol{V}$, we have

$$\mathbf{u_i} = a_{i1}\mathbf{v_1} + a_{i2}\mathbf{v_2} + \cdots + a_{in}\mathbf{v_n} \qquad \text{for } i = 1,2, \ldots, m.$$

- A linear combination $c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_m\mathbf{u_m}$ can be written in the form

$$c_1 \sum_{j=1}^{n} a_{1j}\mathbf{v_j} + c_2 \sum_{j=1}^{n} a_{2j}\mathbf{v_j} + \cdots + c_m \sum_{j=1}^{n} a_{mj}\mathbf{v_j}\,.$$

Rearranging the terms, we have

Theorem 1. If $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ is a **spanning set** for a vector space $\boldsymbol{V}$, then any collection of m vectors in $\boldsymbol{V}$, where $m > n$, is linearly dependent.

Proof. **(continue)**

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_m\mathbf{u_m} = \sum_{i=1}^{m}\left[c_i\left(\sum_{j=1}^{n} a_{ij}\mathbf{v_j}\right)\right] = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}c_i\right)\mathbf{v_j}\,.$$

Consider the system of equations

$$\sum_{i=1}^{m} a_{ij}c_i = 0\,, \qquad j = 1,2,\dots,n,$$

where c_i, $i = 1,2,\dots,m$ are unknowns. This is a homogeneous system with more unknowns (m) than equations (n), then the system must have a nontrivial solution $(\hat{c}_1, \hat{c}_2, \dots, \hat{c}_m)$. Then

$$\hat{c}_1\mathbf{u_1} + \hat{c}_2\mathbf{u_2} + \cdots + \hat{c}_m\mathbf{u_m} = \sum_{j=1}^{n} 0\mathbf{v_j} = \mathbf{0}.$$

Corollary 1. If $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ and $\{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ are both bases for a vector space $\boldsymbol{V}$, then $n = m$.

Proof. By **Theorem 1**,

- since $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ span $\boldsymbol{V}$ and $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}$ are linearly independent, we have $m \leq n$;
- by the same reason, $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}$ span $\boldsymbol{V}$, and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ are linearly independent, so $n \leq m$.

Therefore, $n = m$.

Dimension of Vector Space

Definition 2. Let $\boldsymbol{V}$ be a vector space. If $\boldsymbol{V}$ has a basis consisting of n vectors, we say that $\boldsymbol{V}$ has **dimension [维数]** n, denoted by $\dim \boldsymbol{V} = n$.

-The subspace $\{\mathbf{0}\}$ of $\boldsymbol{V}$ is said to have dimension 0;

-$\boldsymbol{V}$ is said to be **finite-dimensional [有限维]** if there is a finite set of vectors that spans $\boldsymbol{V}$; otherwise, we say that $\boldsymbol{V}$ is **infinite-dimensional [无限维]**.

Example.

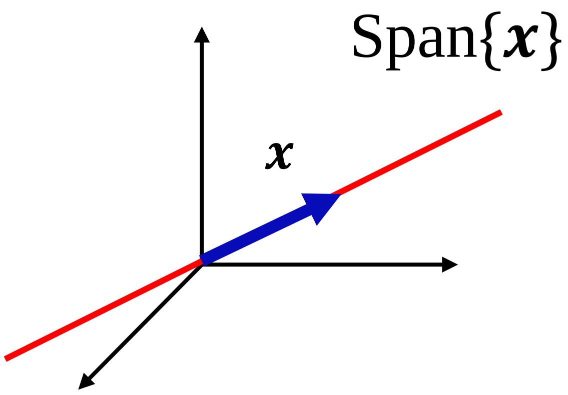
- $\mathbf{R}^3$: standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $\dim \mathbf{R}^3 = 3$.
- $\mathbf{R}^{2\times 2}$: standard basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$, $\dim \mathbf{R}^{2\times 2} = 4$.
- P_n: standard basis $\left\{1, x, \dots, x^{n-1}\right\}$, $\dim P_n = n$.

Example 4.

- Let $\boldsymbol{x} = (x_1, x_2, x_3)^T$ be a **nonzero** vector in $\mathbf{R^3}$, then

$$\text{Span}\{\boldsymbol{x}\} = \{\alpha\boldsymbol{x} | \alpha \in \mathbf{R}\}$$

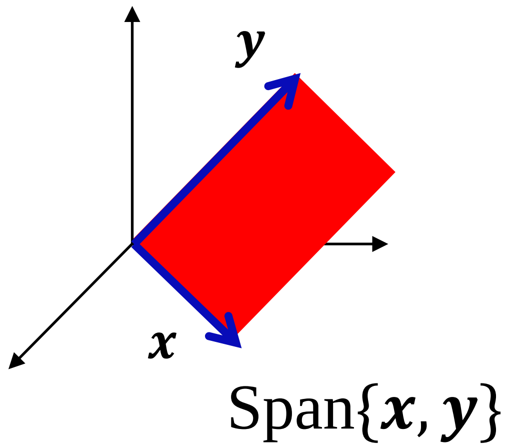
is a one-dimensional vector space.



- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two **linearly independent** vectors in $\mathbf{R^3}$, then

$$\text{Span}\{\boldsymbol{x}, \boldsymbol{y}\} = \{\alpha\boldsymbol{x} + \beta\boldsymbol{y} | \alpha, \beta \in \mathbf{R}\}$$

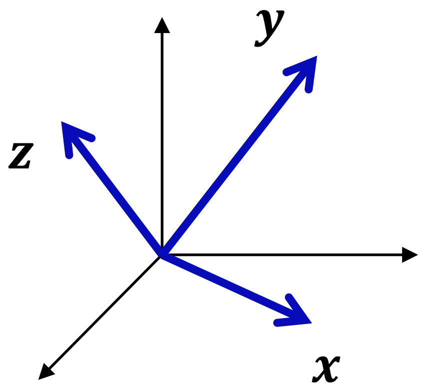
is a two-dimensional vector space.



- If $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are **linearly independent**, then

$$\mathrm{Span}\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\} = \{\alpha\boldsymbol{x} + \beta\boldsymbol{y} + \gamma\boldsymbol{z} | \alpha, \beta, \gamma \in \mathbf{R}\}$$

is just $\mathbf{R}^3$.



$$\mathrm{Span}\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\} = \mathbf{R}^3$$

Example 5. The vector space of all polynomials P is infinite-dimensional.

Proof. In fact, if P is finite-dimensional, say of dimension n, any set of $n+1$ vectors would be linearly dependent. However, we can prove that $1, x, x^2, \dots, x^n$ are linearly independent. This means that P cannot be of dimension n. Since n is arbitrary, P must be infinite-dimensional.

Note: The same argument shows that the vector space $C[a,b]$ is infinite-dimensional.

Theorem. If $\boldsymbol{V}$ is a vector space of dimension $n > 0$:
(i) Any set of n linearly independent vectors span $\boldsymbol{V}$;
(ii) Any n vectors that span $\boldsymbol{V}$ are linearly independent.

Proof. (i) Suppose that $\mathbf{v_1}, \dots, \mathbf{v_n}$ are linearly independent and $\mathbf{v}$ is any other vector in $\boldsymbol{V}$. Since $\boldsymbol{V}$ has dimension n, it has a basis consisting of n vectors and these vectors span $\boldsymbol{V}$. Then $\mathbf{v_1}, \dots, \mathbf{v_n}, \mathbf{v}$ must be linearly dependent. Therefore, any $\mathbf{v} \in \boldsymbol{V}$ can be written in form of linear combination of $\mathbf{v_1}, \dots, \mathbf{v_n}$. This means that $\mathbf{v_1}, \dots, \mathbf{v_n}$ span $\boldsymbol{V}$.

Theorem. If $\boldsymbol{V}$ is a vector space of dimension $n > 0$:

(i) Any set of n linearly independent vectors span $\boldsymbol{V}$;

(ii) Any n vectors that span $\boldsymbol{V}$ are linearly independent.

Proof. (ii) Suppose that $\mathbf{v_1}, \dots, \mathbf{v_n}$ span $\boldsymbol{V}$. If $\mathbf{v_1}, \dots, \mathbf{v_n}$ are linearly dependent, then one of the $\mathbf{v_i}$'s, say $\mathbf{v_n}$, can be written as a linear combination of the others. It follows that $\mathbf{v_1}, \dots, \mathbf{v_{n-1}}$ will still span $\boldsymbol{V}$ and this means that the dimension of $\boldsymbol{V}$ must be smaller than n. This contradicts with $\dim \boldsymbol{V} = n$.

Example 6. Show that $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbf{R}^3$.

Proof. Since $\dim \mathbf{R}^3 = 3$, it is enough to show that the three vectors are linearly independent. This follows since

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = 2 \neq 0.$$

Theorem. If $\boldsymbol{V}$ is a vector space of dimension $n \geq 0$, then

(i) No set of less than n vectors can span $\boldsymbol{V}$.

(ii) Any subset of less than n linearly independent vectors can be extended to form a basis for $\boldsymbol{V}$.

(iii) Any spanning set containing more than n vectors can be pared down to form a basis for $\boldsymbol{V}$.

Note: This theorem can be easily proved and this is left to the reader.

Some Vector Spaces and their Standard Basis

(1) $\mathbf{R}^{n}$: the set $\{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_n}\}$.

(2) $\mathbf{R}^{m\times n}$: the set $\{E_{ij} | i = 1,2, \dots m; j = 1,2, \dots, n\}$, where E_{ij} is the $m \times n$ matrix with all zero entries except the (i, j)th entry.

(3) P_n: the set $\{1, x, x^2, \dots, x^{n-1}\}$.

3.5 Changing of basis

Coordinate of Vector

If $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ form a **basis** of a finite-dimensional vector space $\boldsymbol{V}$, then they are linearly independent and they span the whole space $\boldsymbol{V}$.

Any vector $\boldsymbol{x}$ can be written **uniquely** as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$

$$\boldsymbol{x} = \alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \cdots + \alpha_n \mathbf{v_n}.$$

By using this basis, we find a simple way to represent all vectors in the space $\boldsymbol{V}$. (**coordinate system [坐标系]**)

Definition 1. Suppose $E = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ is an **ordered basis** of vector space $\boldsymbol{V}$. Then any vector $\boldsymbol{x}$ in $\boldsymbol{V}$ can be **uniquely** represented as

$$\boldsymbol{x} = \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n}$$

where $\alpha_i, i = 1,2, \dots, n$ are scalars and are called **coordinates [坐标]** of the vector $\boldsymbol{x}$ in $\boldsymbol{V}$ with respect to the ordered basis E.

The vector $(\alpha_1, \alpha_2, \dots, \alpha_n)_E^T$ is called **coordinate vector [坐标向量]** of vector $\boldsymbol{x}$ in $\boldsymbol{V}$ with respect to basis E, denoted by $[\boldsymbol{x}]_E$.

$$\boldsymbol{x} = \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n} = (\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n})\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}$$

$$= (\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n})[\boldsymbol{x}]_E.$$

Example 1. Suppose $E=\{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$, $F=\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ are two vector sets of $\mathbf{R^3}$. Show that E and F are both bases of $\mathbf{R^3}$ and find the coordinate vectors of $\boldsymbol{x}=(1,2,3)^T$ w.r.t. the ordered bases E and F.

Solution. E is the standard basis of $\mathbf{R^3}$. To show that F forms a basis of $\mathbf{R^3}$ it is enough to show that the three vectors are linearly independent. This can be done by check

$$\begin{vmatrix}1&1&0\\1&1&1\\1&0&1\end{vmatrix}=1\neq 0.$$

Solution. (continue) $\boldsymbol{x} = (1,2,3)^T$

- Suppose that $[\boldsymbol{x}]_E = (x_1, x_2, x_3)^T$, then we have

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = \boldsymbol{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3} = x_1\begin{pmatrix}1\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\end{pmatrix} + x_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

which implies that $[\boldsymbol{x}]_E = (1,2,3)^T$.

- Suppose that $[\boldsymbol{x}]_F = (y_1, y_2, y_3)^T$, then we have

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = \boldsymbol{x} = y_1\begin{pmatrix}1\\1\\1\end{pmatrix} + y_2\begin{pmatrix}1\\1\\0\end{pmatrix} + y_3\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}y_1 + y_2\\ y_1 + y_2 + y_3\\ y_1 + y_3\end{pmatrix}$$

This leads to the linear system

$$\begin{aligned} y_1 + y_2 \phantom{{}+y_3} &= 1\\ y_1 + y_2 + y_3 &= 2\\ y_1 \phantom{{}+y_2} + y_3 &= 3 \end{aligned}$$

By solving the system, we obtain $[\boldsymbol{x}]_F = (2, -1, 1)^T$.

Remark.

- The coordinate vectors of $\boldsymbol{x}$ w.r.t. different bases will be generally different from each other, so when we refer to a coordinate vector of a vector, we must **make sure** which basis is used.
- When $\boldsymbol{V} = \mathbf{R}^n$ and $E = \{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_n}\}$ is the standard basis, for any vector $x \in \mathbf{R}^n$ we have $[\boldsymbol{x}]_E = \boldsymbol{x}$.

Example 2. Show that

$$E = \{1, x, x^2\} \text{ and } F = \{1 + x, x(1 + x), x^2\}$$

are both bases of P_3. Find the coordinate vectors of polynomial $p(x) = (1 + x)^2$ w.r.t. the ordered bases E and F.

Solution. E is the standard basis of P_3. To show F is a basis of P_3, it is enough to show that the three polynomials in F are linearly independent. For this purpose, assume that

$$\alpha_1(1 + x) + \alpha_2 x(1 + x) + \alpha_3 x^2 = 0.$$

We then have

$$\begin{aligned} \alpha_1 \qquad\qquad &= 0 \\ \alpha_1 + \alpha_2 \qquad &= 0 \\ \alpha_2 + \alpha_3 &= 0 \end{aligned} \qquad \Longrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Thus the three vectors in the set F are linearly independent and F forms a basis of P_3.

Solution. (continue) $p(x) = 1 + 2x + x^2$

- Suppose $[p(x)]_E = (c_1, c_2, c_3)^T$. Then

$$p(x) = c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2$$

By comparing the coefficients, we see that $[p(x)]_E = (1,2,1)^T$.

- Assume that $[p(x)]_F = (a_1, a_2, a_3)^T$ and then

$$p(x) = a_1(1 + x) + a_2 x(1 + x) + a_3 x^2$$

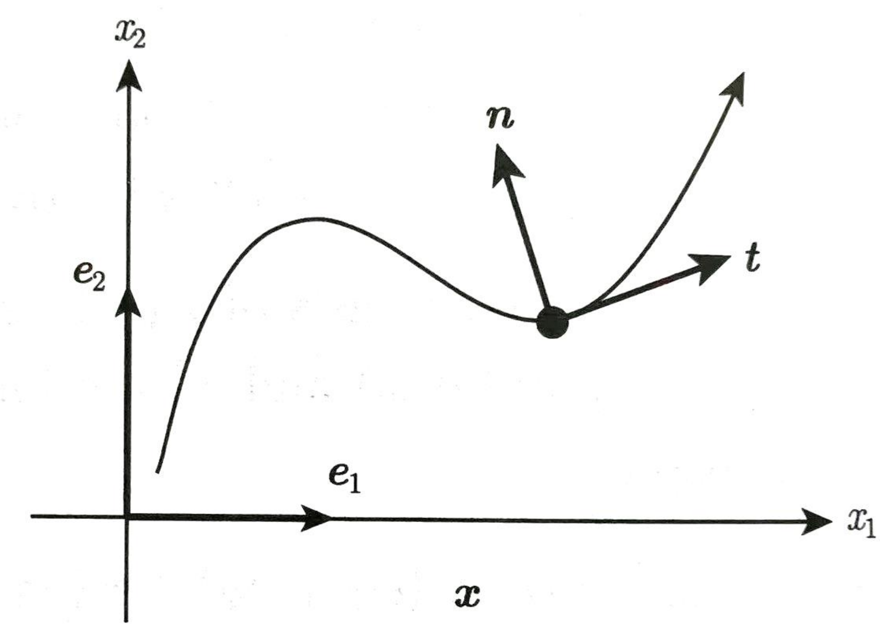
$$1 + 2x + x^2 = a_1 + (a_1 + a_2)x + (a_2 + a_3)x^2.$$

By comparing the coefficients, we see that

$$\begin{aligned} a_1 \qquad\qquad &= 1 \\ a_1 + a_2 \qquad &= 2 \\ a_2 + a_3 &= 1 \end{aligned}$$

Therefore $[p(x)]_F = (1,1,0)^T$.

Changing of basis in R^2



Method of changing basis [基变换]

E: the original basis of $\boldsymbol{V}$

F: the new basis of $\boldsymbol{V}$

Question: How does the coordinate vector of $\boldsymbol{x} \in \boldsymbol{V}$ change with the changing of basis?

(1) Given $[\boldsymbol{x}]_E$, find $[\boldsymbol{x}]_F$;

(2) Given $[\boldsymbol{x}]_F$, find $[\boldsymbol{x}]_E$.

Let $E = \{\mathbf{e_1}, \mathbf{e_2}\}$, $F = \left\{\mathbf{u_1} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$ be bases of $\mathbf{R}^2$.

Notice that
$$\mathbf{u_1} = 0\mathbf{e_1} + 2\mathbf{e_2},$$
$$\mathbf{u_2} = 3\mathbf{e_1} + 1\mathbf{e_2}.$$

If $[\boldsymbol{x}]_F = (c_1, c_2)^T$, we have

$$\boldsymbol{x} = c_1\mathbf{u_1} + c_2\mathbf{u_2} = (0c_1 + 3c_2)\mathbf{e_1} + (2c_1 + 1c_2)\mathbf{e_2}$$
$$= (\mathbf{e_1}, \mathbf{e_2})\begin{pmatrix} 0c_1 + 3c_2 \\ 2c_1 + 1c_2 \end{pmatrix} = (\mathbf{e_1}, \mathbf{e_2})\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Let
$$U = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} = (\mathbf{u_1}, \mathbf{u_2}).$$

Then
$$[\boldsymbol{x}]_E = U[\boldsymbol{x}]_F.$$

transition matrix from F to E

Let $E = \{\mathbf{e_1}, \mathbf{e_2}\}$, $F = \left\{\mathbf{u_1} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$ be bases of $\mathbf{R}^2$.

$$[\boldsymbol{x}]_E = U[\boldsymbol{x}]_F$$

- The matrix U is called the **transition matrix [过渡矩阵]** from the ordered basis F to the standard basis E.
- U is invertible, since each column of U is a vector of the basis F.
- If we know $[\boldsymbol{x}]_E$ and the transition matrix from basis F to basis E is U, then $[\boldsymbol{x}]_F = U^{-1}[\boldsymbol{x}]_E$.

  U^{-1} is the transition matrix from standard basis E to basis F.

Exercise. Let $\boldsymbol{x} = (1,1)^T$. Find the coordinates of $\boldsymbol{x}$ w.r.t. the basis F.

Let $\{\mathbf{u_1}, \mathbf{u_2}\}$, $\{\mathbf{v_1}, \mathbf{v_2}\}$ be two bases of $\mathbf{R^2}$.

Compute the **transition matrix** from basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ to $\{\mathbf{v_1}, \mathbf{v_2}\}$:

Step 1. Find the transition matrix U from basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ to the standard basis $\{\mathbf{e_1}, \mathbf{e_2}\}$,

$$U = (\mathbf{u_1}, \mathbf{u_2}).$$

Step 2. Find the transition matrix V from basis $\{\mathbf{v_1}, \mathbf{v_2}\}$ to the standard basis $\{\mathbf{e_1}, \mathbf{e_2}\}$,

$$V = (\mathbf{v_1}, \mathbf{v_2}).$$

Step 3. The transition matrix S from basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ to $\{\mathbf{v_1}, \mathbf{v_2}\}$ can be calculated by

$$S = V^{-1}U.$$

Let $\{\mathbf{u_1}, \mathbf{u_2}\}$, $\{\mathbf{v_1}, \mathbf{v_2}\}$ be two bases of $\mathbf{R^2}$.

$$
\begin{tikzcd}
\{\mathbf{u_1}, \mathbf{u_2}\} \arrow[r, "U"] \arrow[dr, "S = V^{-1}U"'] & \{\mathbf{e_1}, \mathbf{e_2}\} \arrow[d, "V^{-1}"] \\
 & \{\mathbf{v_1}, \mathbf{v_2}\}
\end{tikzcd}
$$

Example 3. Suppose

$$\mathbf{u_1} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}, \qquad \mathbf{v_1} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find the transition matrix corresponding to the change of basis from $\{\mathbf{u_1}, \mathbf{u_2}\}$ to $\{\mathbf{v_1}, \mathbf{v_2}\}$.

Solution.

- The transition matrix from basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ to the standard basis is $U = (\mathbf{u_1}, \mathbf{u_2}) = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$.
- The transition matrix from basis $\{\mathbf{v_1}, \mathbf{v_2}\}$ to the standard basis is $V = (\mathbf{v_1}, \mathbf{v_2}) = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$. $\qquad V^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$.
- The transition matrix S from $\{\mathbf{u_1}, \mathbf{u_2}\}$ to $\{\mathbf{v_1}, \mathbf{v_2}\}$ is

$$S = V^{-1}U = \begin{pmatrix} 3 & 4 \\ -4 & -5 \end{pmatrix}.$$

Changing of basis in an n-dim. vector space

Case $V = \mathbf{R}^n$.

- $E = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$, $F = \{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n}\}$: two bases of $\mathbf{R}^n$.
- $\boldsymbol{x} \in \mathbf{R}^n$

$$\boldsymbol{x} = \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n} = (\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n})\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$= (\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n})[\boldsymbol{x}]_E$$

$$\boldsymbol{x} = \beta_1\mathbf{u_1} + \beta_2\mathbf{u_2} + \cdots + \beta_n\mathbf{u_n} = (\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n})\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$= (\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n})[\boldsymbol{x}]_F.$$

Case $V = \mathbf{R}^n$.

- We have $(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})[\boldsymbol{x}]_E = (\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n})[\boldsymbol{x}]_F$.

- Let $V = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$ and $U = (\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n})$.

  Then U and V are both invertible matrices.

- If we denote $S = V^{-1}U,$

  then the relations between two coordinate vectors are

$$\boxed{[\boldsymbol{x}]_E = S[\boldsymbol{x}]_F} \quad \text{and} \quad \boxed{[\boldsymbol{x}]_F = S^{-1}[\boldsymbol{x}]_E.}$$

general n-dimensional vector space $\boldsymbol{V}$

- $E = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$, $F = \{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n}\}$: two ordered bases of $\boldsymbol{V}$.

$$\mathbf{u_j} = \sum_{i=1}^{n} a_{ij}\mathbf{v_i}, \qquad j = 1,2,\dots,n.$$

- $\boldsymbol{x} \in V$. $[\boldsymbol{x}]_E = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$, $[\boldsymbol{x}]_F = (\beta_1, \beta_2, \dots, \beta_n)^T$

$$\begin{aligned}
\boldsymbol{x} &= \beta_1\mathbf{u_1} + \beta_2\mathbf{u_2} + \cdots + \beta_n\mathbf{u_n} \\
&= \beta_1\sum_{i=1}^{n} a_{i1}\mathbf{v}_{\boldsymbol{i}} + \beta_2\sum_{i=1}^{n} a_{i2}\mathbf{v}_{\boldsymbol{i}} + \cdots + \beta_n\sum_{i=1}^{n} a_{in}\mathbf{v}_{\boldsymbol{i}} \\
&= \sum_{j=1}^{n}\beta_j\sum_{i=1}^{n} a_{ij}\mathbf{v}_{\boldsymbol{i}} = \sum_{i=1}^{n}\underbrace{\left(\sum_{j=1}^{n} a_{ij}\beta_j\right)}_{\alpha_i}\mathbf{v}_{\boldsymbol{i}}.
\end{aligned}$$

general n-dimensional vector space $\boldsymbol{V}$

Compared with $\boldsymbol{x} = \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n}$, we obtain

$$\alpha_i = \sum_{j=1}^{n} a_{ij}\beta_j\,.$$

This gives

$$[\boldsymbol{x}]_E = U[\boldsymbol{x}]_F$$

where $U = \left(a_{ij}\right)$ is the transition matrix.

The matrix U is invertible.

Example. Suppose that in P_3, we want to change from the ordered basis $\{1, x, x^2\}$ to the ordered basis $\{1, 2x, 4x^2 - 2\}$.

Solution. It is easier to find the transition matrix from $\{1, 2x, 4x^2 - 2\}$ to $\{1, x, x^2\}$, since

$$1 = 1 \cdot 1 + 0x + 0x^2$$
$$2x = 0 \cdot 1 + 2x + 0x^2$$
$$4x^2 - 2 = -2 \cdot 1 + 0x + 4x^2.$$

The transition matrix is $S = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$

Solution. (continue) The inverse of S will be the transition matrix from $\{1, x, x^2\}$ to $\{1, 2x, 4x^2 - 2\}$

$$S^{-1} = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}.$$

Given any $p(x) = a + bx + cx^2$ in P_3, to find the coordinates of $p(x)$ with respect to $\{1, 2x, 4x^2 - 2\}$, we simply multiply

$$\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a + \frac{1}{2}c \\ \frac{1}{2}b \\ \frac{1}{4}c \end{pmatrix}.$$

Thus, $\quad p(x) = \left(a + \frac{1}{2}c\right) \cdot 1 + \left(\frac{1}{2}b\right) \cdot 2x + \left(\frac{1}{4}c\right) \cdot (4x^2 - 2).$

Review

- Basis and dimension
- Changing of basis

Preview

- Row space and column space