Question 1. Write down the correct answer on the following blanks.

a) Let $\mathbf{v}_1=(1,2,1)^T$ and $\mathbf{v}_2=(1,0,-1)^T$, then $\mathbf{v}_1\cdot\mathbf{v}_2=$ ①, $\mathbf{v}_1\times\mathbf{v}_2=$ ② and $\cos\theta=$ ③, where θ is the include angle of $\mathbf{v}_1$ and $\mathbf{v}_2$.

b) Suppose A be an $n\times n$ matrix and $A^2-2A-4E=O$, then A^{-1} is ④, where E is an $n\times n$ identity matrix and O is an $n\times n$ zero matrix.

c) Suppose that α_1 and α_2 are linear dependent, then does α_1, α_2, β_1 and β_2 ⑤ (are/are not) linear dependent? Where $\alpha_1,\alpha_2,\beta_1,\beta_2$ are all vectors in vector space V.

d) Suppose a base of $\mathbf{R}^3$ contents three vectors, $\xi_1=(1,1,0)^T,\xi_2=(0,1,1)^T,\xi_3=(-1,2,1)^T$. If the coordinate vector of $\mathbf{v}=(t,0,0)^T$ with respect to this base is $(1,1,-1)^T$, then $t=$ ⑥.

e) Suppose that $A=\begin{pmatrix}1&1&1\\1&1&-1\\1&-1&1\end{pmatrix}$, then $\det(A)=$ ⑦.

f) Suppose $A=\begin{pmatrix}1&-1&3&4\\2&1&-2&0\\4&-1&4&8\\0&-3&8&8\end{pmatrix}$, then $\operatorname{rank}(A)=$ ⑧.

g) Suppose L_1 and L_2 are two lines, $\mathbf{s}_1$ and $\mathbf{s}_2$ are their direction vectors, respectively. If L_1 is collinear with L_2, then $\mathbf{s}_1\times\mathbf{s}_2=$ ⑨.

h) Let A be an $n\times n$ matrix. If for all $\mathbf{x}\in\mathbf{R}^n$, we have $\mathbf{x}^TA\mathbf{x}>0$, then A is said to be ⑩.

Solution.

① 0, ② $(-2,2,-2)^T$, ③ $\frac{\pi}{2}$, ④ $\frac{1}{4}(A-2E)$, ⑤ are,

⑥ 2, ⑦ −4, ⑧ 2, ⑨ 0, ⑩ positive defined.

Question 2. Suppose that linear system

$$\begin{array}{rcrcrcl} x_1 & & & + & x_3 & = & k \\ 4x_1 & + & x_2 & + & 2x_3 & = & k+2 \\ 6x_1 & + & x_2 & + & 4x_3 & = & 2k+3 \end{array}$$

is consistent, determine k and give all solutions of the linear system.

Solution. The augmented matrix of linear system is

$$\left(\begin{array}{ccc|c}1&0&1&k\\4&1&2&k+2\\6&1&4&2k+3\end{array}\right)$$

By primary row operations, we have

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & k \\ 4 & 1 & 2 & k+2 \\ 6 & 1 & 4 & 2k+3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & k \\ 0 & 1 & -2 & -3k+2 \\ 0 & 1 & -2 & -4k+4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & k \\ 0 & 1 & -2 & -3k+2 \\ 0 & 0 & 0 & -k+2 \end{array}\right)$$

Since the linear system is consistent, we have $k=2$.

The corresponding augmented matrix of the equivalent linear system is

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since there is a free variable x_3, so if we take $x_2 = \alpha$, then the solutions can be written as

$$(2-\alpha, 2\alpha-4, \alpha)^T = \alpha(-1,2,1)^T + (2,-4,0)^T.$$

Question 3. Suppose that $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A^* BA = 2BA - 8E$, where A^* is the adjoint matrix of A, E is 3×3 identity matrix. Find B.

Solution. Since A is an inverse matrix, so we can multiply both sides of the equation by A on the left and A^{-1} on the right, that is

$$A(A^* BA)A^{-1} = A(2BA - 8E)A^{-1}$$

Notice that $AA^* = \det(A)E$, $\det(A) = -2$ and $AA^{-1} = E$, then

$$B(A+E) = 4E$$

So,

$$B = 4(A+E)^{-1} = 4\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note. Student may calculate B directly, it is also adoptable.

Question 4. Let $\mathbf{v}_1=(1,0,1)^T$, $\mathbf{v}_2=(0,1,1)^T$ and $\mathbf{v}_3=(0,0,1)^T$. Show that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is a basis of $\mathbf{R}^3$, then use Gram-Schmidt orthogonalization process to deduce an orthonormal basis of $\mathbf{R}^3$.

Solution. Consider linear system

$$\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Since the determinant of the coefficient matrix is $\det(A)=\det\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}=1\neq 0$

So, $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ are linear independent, then $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is a basis of $\mathbf{R}^3$.

If we take $\boldsymbol{\beta}_1=\dfrac{\mathbf{v}_3}{\|\mathbf{v}_3\|}=(0,0,1)^T$, then

$$\mathbf{p}_1=\langle\mathbf{v}_1,\boldsymbol{\beta}_1\rangle\boldsymbol{\beta}_1=\left(\begin{pmatrix}1&0&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\boldsymbol{\beta}_2=\frac{\mathbf{v}_1-\mathbf{p}_1}{\|\mathbf{v}_1-\mathbf{p}_1\|}=\frac{(1,0,1)^T-(0,0,1)^T}{\|(1,0,1)^T-(0,0,1)^T\|}=(1,0,0)^T$$

$$\mathbf{p}_2=\langle\mathbf{v}_2,\boldsymbol{\beta}_1\rangle\boldsymbol{\beta}_1+\langle\mathbf{v}_2,\boldsymbol{\beta}_2\rangle\boldsymbol{\beta}_2=\left((0,1,1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}+\left((0,1,1)\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\boldsymbol{\beta}_3=\frac{\mathbf{v}_2-\mathbf{p}_2}{\|\mathbf{v}_2-\mathbf{p}_2\|}=\frac{(0,1,1)^T-(0,0,1)^T}{\|(0,1,1)^T-(0,0,1)^T\|}=(0,1,0)^T.$$

So, the orthonormal set is $\left\{\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$.

Question 5. Find the equation of plane which is perpendicular to line $L:\begin{cases}x_1+x_3=1\\x_2-x_3=0\end{cases}$ and pass through the original point.

Solution. Notice that the direction vector of the line or the normal vector of the plane can be calculated by

$$(1,0,1)^T \times (0,1,-1)^T = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -\mathbf{i}+\mathbf{j}+\mathbf{k} = (-1,1,1)^T$$

and notice the plane passes through the original point, so the equation of the plane is

$$-x_1 + x_2 + x_3 = 0 \text{ or } x_1 - x_2 - x_3 = 0\,.$$

Question 6. Change quadratic equation $x_1 x_2 = 1$ into its standard form at standard position.

Solution. This quadratic equation can be written as

$$\mathbf{x}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = (x_1, x_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\,.$$

Since the eigenvalues of its coefficient matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is $\lambda_{1,2} = \pm 1$ and the corresponding unit eigenvectors are $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, respectively. Then, if we take

$$Q = (\mathbf{v}_1, \mathbf{v}_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

we have

$$Q^T A Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So, if we take

$$\mathbf{x}' = Q\mathbf{x}$$

then, we have

$$\mathbf{x}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = (\mathbf{x}')^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\mathbf{x}') = 2$$

or

$$(x_1')^2 - (x_2')^2 = 2\,.$$

Question 1. Write down the correct answer on the following blanks.

a) Let $\mathbf{v}_1=(1,2,1)^T$ and $\mathbf{v}_2=(1,0,-1)^T$, then $\mathbf{v}_1\cdot\mathbf{v}_2=$ __①__, $\mathbf{v}_1\times\mathbf{v}_2=$ __②__ and $\cos\theta=$ __③__, where θ is the include angle of $\mathbf{v}_1$ and $\mathbf{v}_2$.

b) Suppose A be an $n\times n$ matrix and $A^2-2A-4E=O$, then A^{-1} is __④__, where E is an $n\times n$ identity matrix and O is an $n\times n$ zero matrix.

c) Suppose that α_1 and α_2 are linear dependent, then does $\alpha_1,\alpha_2,\beta_1$ and β_2 __⑤__ (are/are not) linear dependent? Where $\alpha_1,\alpha_2,\beta_1,\beta_2$ are all vectors in vector space V.

d) Suppose a base of $\mathbf{R}^3$ contents three vectors, $\xi_1=(1,1,0)^T,\xi_2=(0,1,1)^T,\xi_3=(-1,2,1)^T$. If the coordinate vector of $\mathbf{v}=(t,0,0)^T$ with respect to this base is $(1,1,-1)^T$, then $t=$ __⑥__.

e) Suppose that $A=\begin{pmatrix}1&1&1\\1&1&-1\\1&-1&1\end{pmatrix}$, then $\det(A)=$ __⑦__.

f) Suppose $A=\begin{pmatrix}1&-1&3&4\\2&1&-2&0\\4&-1&4&8\\0&-3&8&8\end{pmatrix}$, then $\operatorname{rank}(A)=$ __⑧__.

g) Suppose L_1 and L_2 are two lines, $\mathbf{s}_1$ and $\mathbf{s}_2$ are their direction vectors, respectively. If L_1 is collinear with L_2, then $\mathbf{s}_1\times\mathbf{s}_2=$ __⑨__.

h) Let A be an $n\times n$ matrix. If for all $\mathbf{x}\in\mathbf{R}^n$, we have $\mathbf{x}^TA\mathbf{x}>0$, then A is said to be __⑩__.

Solution.

① 0, ② $(-2,2,-2)^T$, ③ $\frac{\pi}{2}$, ④ $\frac{1}{4}(A-2E)$, ⑤ are,

⑥ 2, ⑦ -4, ⑧ 2, ⑨ 0, ⑩ positive defined.

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$$\begin{cases}x_1 + x_3 = k\\ 4x_1 + x_2 + 2x_3 = k+2\\ 6x_1 + x_2 + 4x_3 = 2k+3\end{cases}$$

is consistent, determine k and give all solutions of the linear system.

Solution. The augmented matrix of linear system is

$$\left(\begin{array}{ccc|c}1&0&1&k\\4&1&2&k+2\\6&1&4&2k+3\end{array}\right)$$

By primary row operations, we have

$$\left(\begin{array}{ccc|c} 1&0&1&k\\ 4&1&2&k+2\\ 6&1&4&2k+3 \end{array}\right)\to\left(\begin{array}{ccc|c} 1&0&1&k\\ 0&1&-2&-3k+2\\ 0&1&-2&-4k+4 \end{array}\right)\to\left(\begin{array}{ccc|c} 1&0&1&k\\ 0&1&-2&-3k+2\\ 0&0&0&-k+2 \end{array}\right)$$

Since the linear system is consistent, we have $k=2$.

The corresponding augmented matrix of the equivalent linear system is

$$\left(\begin{array}{ccc|c} 1&0&1&2\\ 0&1&-2&-4\\ 0&0&0&0 \end{array}\right)$$

$\left(\begin{array}{ccc|c} 1&0&1&1\\ 0&1&-2&-1\\ 0&0&0&0 \end{array}\right)$

$x_1+x_3=1$

$x_2-2x_3=-1$ $\quad x_3=\alpha.$

Since there is a free variable x_3, so if we take $x_2=\alpha$, then the solutions can be written as

$\therefore\ x_1=1-\alpha$

$x_2=-1+2\alpha$

$$(2-\alpha,2\alpha-4,\alpha)^T=\alpha(-1,2,1)^T+(2,-4,0)^T.$$

$\begin{pmatrix}1-\alpha\\2\alpha-1\\\alpha\end{pmatrix}$

$\alpha\begin{pmatrix}-1\\2\\1\end{pmatrix}+\begin{pmatrix}1\\-1\\0\end{pmatrix}$

Question 3. Suppose that $A=\begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix}$ and $A^*BA=2BA-8E$, where A^* is the adjoint matrix of A, E is 3×3 identity matrix. Find B.

$A\cdot A^*B\cdot E=(2B E-8E\cdot A^{-1})\cdot A$

$A^{-1}=\frac{1}{\det(A)}\cdot A^* \Rightarrow A\cdot A^{-1}=$

Solution. Since A is an inverse matrix, so we can multiply both sides of the equation by A on the left and A^{-1} on the right, that is

$A\cdot A^*=\det(A)\cdot E$

$$A\left(A^*BA\right)A^{-1}=A(2BA-8E)A^{-1}$$

$\det(A)\cdot E\cdot B=2AB-8E$

Notice that $AA^*=\det(A)E$, $\det(A)=-2$ and $AA^{-1}=E$, then

$\det(A)=-2$

$-2E\cdot B=2AB-8E$

$$B(A+E)=4E$$

$8E=2AB+2BE$

$B=\frac{8E}{2A+2E}$

$=\frac{4E}{A+E}$

So,

$B=4(A+E)^{-1}$

$$B=4(A+E)^{-1}=4\begin{pmatrix}\frac{1}{2}&0&0\\0&-1&0\\0&0&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}2&0&0\\0&-4&0\\0&0&2\end{pmatrix}.$$

$A+E=\begin{pmatrix}2&0&0\\0&-1&0\\0&0&2\end{pmatrix}$

$\begin{pmatrix}2&0&0\\0&-1&0\\0&0&2\end{pmatrix}\begin{pmatrix}\frac{1}{2}&0&0\\0&-1&0\\0&0&\frac{1}{2}\end{pmatrix}$

Note. Student may calculate B directly, it is also adoptable.

Question 4. Let $\mathbf{v}_1 = (1,0,1)^T$, $\mathbf{v}_2 = (0,1,1)^T$ and $\mathbf{v}_3 = (0,0,1)^T$. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbf{R}^3$, then use Gram-Schmidt orthogonalization process to deduce an orthonormal basis of $\mathbf{R}^3$.

(det

$u_3 = \frac{v_3}{\|v_3\|} = (0,0,1)^T$

$p_1 = \langle v_2, u_3 \rangle u_3 = (0,0,1)^T$

$q_2 = v_2 - p_1 = (0,1,0)^T$.

$u_2 = \frac{q_2}{\|q_2\|} = (0,1,0)^T$.

Solution. Consider linear system

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since the determinant of the coefficient matrix is $\det(A) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 1 \neq 0$

So, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linear independent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbf{R}^3$.

If we take $\boldsymbol{\beta}_1 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = (0,0,1)^T$, then

$$\mathbf{p}_1 = \langle \mathbf{v}_1, \boldsymbol{\beta}_1 \rangle \boldsymbol{\beta}_1 = \left(\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{\beta}_2 = \frac{\mathbf{v}_1 - \mathbf{p}_1}{\|\mathbf{v}_1 - \mathbf{p}_1\|} = \frac{(1,0,1)^T - (0,0,1)^T}{\|(1,0,1)^T - (0,0,1)^T\|} = (1,0,0)^T$$

$$\mathbf{p}_2 = \langle \mathbf{v}_2, \boldsymbol{\beta}_1 \rangle \boldsymbol{\beta}_1 + \langle \mathbf{v}_2, \boldsymbol{\beta}_2 \rangle \boldsymbol{\beta}_2 = \left((0,1,1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \left((0,1,1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{\beta}_3 = \frac{\mathbf{v}_2 - \mathbf{p}_2}{\|\mathbf{v}_2 - \mathbf{p}_2\|} = \frac{(0,1,1)^T - (0,0,1)^T}{\|(0,1,1)^T - (0,0,1)^T\|} = (0,1,0)^T.$$

So, the orthonormal set is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

1,1,1

$\frac{x_3}{1} = \frac{x_1 - 1}{-1}$

$x_1 = 1 - x_3$

$\frac{x_2}{1} = \frac{x_3}{1} = \frac{x_1 - 1}{-1} = \lambda$.

$x_1 = 1 + \lambda$

$x_2 = 0 + \lambda$.

$x_3 = 0 + \lambda$.

x_1

1 0 1

0 1 −1

0, 0, −1

Question 5. Find the equation of plane which is perpendicular to line $L: \begin{cases} x_1 + x_3 = 1 \\ x_2 - x_3 = 0 \end{cases}$ and pass through the original point.

$(1,1,1)$.

Solution. Notice that the direction vector of the line or the normal vector of the plane can be calculated by

$$(1,0,1)^T \times (0,1,-1)^T = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -\mathbf{i}+\mathbf{j}+\mathbf{k} = (-1,1,1)^T$$

and notice the plane passes through the original point, so the equation of the plane is

$$-x_1 + x_2 + x_3 = 0 \text{ or } x_1 - x_2 - x_3 = 0 .$$

Question 6. Change quadratic equation $x_1 x_2 = 1$ into its standard form at standard position.

Solution. This quadratic equation can be written as

$$\mathbf{x}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = (x_1, x_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2 .$$

Since the eigenvalues of its coefficient matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is $\lambda_{1,2} = \pm 1$ and the corresponding unit eigenvectors are $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, respectively. Then, if we take

$$Q = (\mathbf{v}_1, \mathbf{v}_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

we have

$$Q^T A Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

So, if we take

$$\mathbf{x}' = Q\mathbf{x}$$

then, we have

$$\mathbf{x}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = (\mathbf{x}')^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\mathbf{x}') = 2$$

or

$$(x_1')^2 - (x_2')^2 = 2 .$$