北京郵電大學
BEIJING UNIVERSITY OF POSTS AND TELECOMMUNICATIONS

Lecture 14

Surface Integrals

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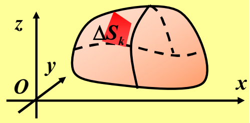
Surface Integrals of the First Type

Definition Surface integral with respect to surface area

Suppose $\{(\Delta S_k)\}$ is a portion of a smooth surface (S), $f(x,y,z)$ is a function of three variables defined on (S), ΔS_k is the area of the subsurface (ΔS_k), and (ξ_k,η_k,ζ_k) is any point on the subsurface.



If the limit of the sum

$$\lim_{d\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\Delta S_k,$$

exists uniquely, then we say the function f is integrable over the surface (S), and the limit is called the surface integral of function f over (S) with respect to surface area or surface integral of the first type, that is

$$\iint_{(S)} f(x,y,z)dS = \lim_{d\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\boxed{\Delta S_k}.$$

Element of area of the surface

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Surface Integrals of the First Type

Just as the arc length can be found by evaluate the line integral of the first type, the surface area in space also can be found by evaluate

$$\text{Area} = \iint_{(S)} dS.$$

Existence: If the function $f(x,y,z)$ is continuous on the smooth surface (S), then $\iint_{(S)} f(x,y,z)dS$ exists.

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Surface Integrals of the First Type

Properties:

1. **Linearity Property**

$$(1)\iint_{(S)}[f(x,y,z)+g(x,y,z)]dS = \iint_{(S)} f(x,y,z)dS + \iint_{(S)} g(x,y,z)dS;$$

$$(2)\iint_{(S)} kf(x,y,z)dS = k\iint_{(S)} f(x,y,z)dS.$$

2. **Additivity with respect to the domain of integration**

Suppose that $(S)=(S_1)+(S_2)$ and $(S_1),(S_2)$ have no common part except for their boundaries. Then

$$\iint_{(S)} f(x,y,z)dS = \iint_{(S_1)} f(x,y,z)dS + \iint_{(S_2)} f(x,y,z)dS$$

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Surface Integrals of the First Type

3. **Domination**

(1) $\iint_{(S)} f(x,y,z)dS \le \iint_{(S)} g(x,y,z)dS$, $\quad (f(x,y,z)\le g(x,y,z));$

(2) $\left|\iint_{(S)} f(x,y,z)dS\right| \le \iint_{(S)} |f(x,y,z)|\,dS;$

(3) If $m\le f(x,y,z)\le M, \forall (x,y,z)\in(S)$, then

$$mS \le \iint_{(S)} f(x,y,z)dS \le MS,$$

where S is are of the surface of (S).

4. **Mean Value Theorem**

Suppose that $f\in C((S))$, then there exists at least one point $(\xi,\eta,\zeta)\in(S)$, such that

$$\iint_{(S)} f(x,y,z)dS = f(\xi,\eta,\zeta)S,$$

where S is area of the surface of (S).

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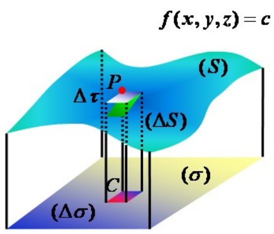
Application of Multiple Integrals

Computation of the surface area

The right figure shows a surface (S) lying above its "shadow" region (σ) in a plane beneath it. The surface is defined by an equation $f(x,y,z)=c$. If the surface is smooth, we can define and calculate its area as a double integral over (σ).



The first step is to partition the region (σ) into small rectangles $(\Delta\sigma)$. Directly above $(\Delta\sigma)$ lies a patch of surface (ΔS), we may approximate it with a portion $(\Delta\tau)$ of a tangent plane.

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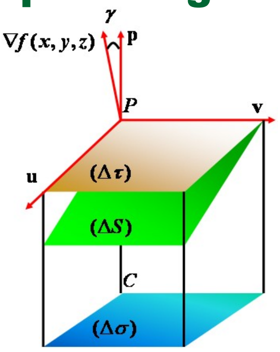
Application of Multiple Integrals

In the right figure gives a magnified view of $(\Delta\sigma)$ and $(\Delta\tau)$, showing the gradient vector $\nabla f(x,y,z)$ at P and a unit vector $\mathbf{p}$ that is normal to xOy – plane. The angle between ∇f and $\mathbf{p}$ is γ. Then

$\Delta\tau\,|\cos\gamma| = \Delta\sigma$ or $\Delta\tau = \dfrac{\Delta\sigma}{|\cos\gamma|}$,

provided $\cos\gamma \neq 0$. Thus the area is

$$\lim_{d\to 0}\sum \Delta\tau = \lim_{d\to 0}\sum \frac{\Delta\sigma}{|\cos\gamma|} = \iint_{(\sigma)} \frac{1}{|\cos\gamma|}\,d\sigma.$$



Application of Multiple Integrals

Moreover, if the surface can be expressed as a equation

$$z = f(x,y), \quad (x,y) \in (\sigma_{xy}) \subset \mathbf{R}^2,$$

Then $$S = \iint_{(\sigma_{xy})} \frac{1}{|\cos\gamma|}\,d\sigma = \iint_{(\sigma_{xy})} \sqrt{z_x^2 + z_y^2 + 1}\,d\sigma = \iint_{(\sigma_{xy})} \sqrt{z_x^2 + z_y^2 + 1}\,dxdy.$$

In our discussion of surface area, the element of surface area

$$dS = \sqrt{z_x^2 + z_y^2 + 1}\,dxdy.$$

If the surface can be expressed as a parametric equation

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in (\sigma) \subseteq \mathbf{R}^2$$

then
$$S = \iint_{(\sigma)} \|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)\|\,dudv,$$

and the element of surface area

$$dS = \|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)\|\,dudv.$$

Surface Integrals of the First Type

Computation of the Surface Integrals of the First Type

If (S) is a surface defined by the equation $z = z(x,y)$, and $f(x,y,z)$ is a continuous function defined on (S), then we have the following computation formula.

$$\iint_{(S)} f(x,y,z)\,dS = \iint_{(\sigma_{xy})} f(x,y,z(x,y))\sqrt{z_x^2 + z_y^2 + 1}\,d\sigma.$$

Method: 1. substitution 2. projection

Surface Integrals of the First Type

If $f(x,y,z)$ is continuous on a smooth surface (S) whose equation is

$$y = y(x,z), \quad (x,z) \in (\sigma_{xz}) \subset \mathbf{R}^2,$$

then the surface integral of first type of f on (S) can be reduce to double integral $\iint_{(S)} f(x,y,z)\,dS = \iint_{(\sigma_{xz})} f(x, y(x,z), z)\sqrt{y_x^2 + y_z^2 + 1}\,d\sigma.$

If $f(x,y,z)$ is continuous on a smooth surface (S) whose equation is

$$x = x(y,z), \quad (y,z) \in (\sigma_{yz}) \subset \mathbf{R}^2,$$

then the surface integral of first type of f on (S) can be reduce to double integral $\iint_{(S)} f(x,y,z)\,dS = \iint_{(\sigma_{yz})} f(x(y,z), y, z)\sqrt{x_y^2 + x_z^2 + 1}\,d\sigma.$

If the surface can be expressed as a parametric equation

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in (\sigma) \subseteq \mathbf{R}^2$$

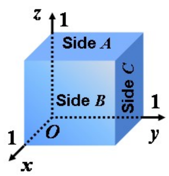
then

$$\iint_{(S)} f(x,y,z)\,dS = \iint_{(\sigma)} f(x(u,v), y(u,v), z(u,v))\,\|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)\|\,dudv.$$

Integrating Over a Surface

Example 1. Integrate $f(x,y,z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1, y = 1$ and $z = 1$.

Solution We integrate xyz over each of the six sides and add the results. Since $xyz = 0$ on the sides that lie in the coordinate planes, the integral over the surface of the cube reduces to

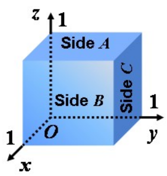


$$\iint_{\text{Cube surface}} xyz\,dS = \iint_{\text{Side A}} xyz\,dS + \iint_{\text{Side B}} xyz\,dS + \iint_{\text{Side C}} xyz\,dS.$$

Integrating Over a Surface

Solution (continued)

Side A is the surface $z = 1$ over the square region $(\sigma_{xy}): 0 \le x \le 1, 0 \le y \le 1$, in the xy – plane.



$$\iint_{\text{Side A}} xyz\,dS \overset{\text{Side}A:z=1}{=} \iint_{(\sigma_{xy})} xy\cdot z(x,y)\sqrt{z_x^2 + z_y^2 + 1}\,dxdy$$

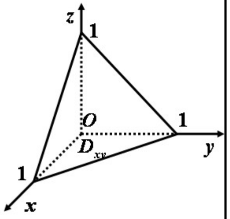
$$= \iint_{(\sigma_{xy})} xy\,dxdy = \int_0^1\int_0^1 xy\,dxdy = \int_0^1 \frac{y}{2}\,dy = \frac{1}{4}$$

Symmetry tell us that the integrals of xyz over sides B and C are also $\frac{1}{4}$.

Hence, $$\iint_{\text{Cube surface}} xyz\,d\sigma = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

Integrating Over a Surface

Example 2. Evaluate $\oint\!\!\oint_{\Sigma} xyzdS$, where Σ is bounded by $x=0, y=0, z=0$ and $x+y+z=1$.



Solution The surface can be divided into four parts. We denote the part on $x=0, y=0, z=0$ and $x+y+z=1$ by $\Sigma_1, \Sigma_2, \Sigma_3$ and Σ_4, respectively.

Then, $\oint\!\!\oint_{\Sigma} xyzdS$

$$=\iint_{\Sigma_1} xyzdS+\iint_{\Sigma_2} xyzdS+\iint_{\Sigma_3} xyzdS+\iint_{\Sigma_4} xyzdS.$$

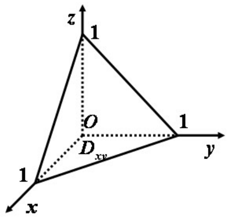
Since the integrand on the surface Σ_1, Σ_2 and Σ_3 are both zero, then

$$\oint\!\!\oint_{\Sigma} xyzdS=\iint_{\Sigma_4} xyzdS=\iint_{D_{xy}} \sqrt{3}xy(1-x-y)dxdy,$$

Integrating Over a Surface

Solution (continued)

where D_{xy} is bounded by the lines $x=0, y=0$ and $x+y=1$, then



$$\oint\!\!\oint_{\Sigma} xyzdS=\sqrt{3}\iint_{D_{xy}} xy(1-x-y)dxdy$$

$$=\sqrt{3}\int_0^1 xdx\int_0^{1-x} y(1-x-y)dy$$

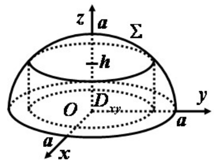
$$=\sqrt{3}\int_0^1 x\left[(1-x)\frac{y^2}{2}-\frac{y^3}{3}\right]_0^{1-x} dx$$

$$=\sqrt{3}\int_0^1 x\cdot\frac{(1-x)^3}{6}dx$$

$$=\frac{\sqrt{3}}{6}\int_0^1 (x-3x^2+3x^3-x^4)dx=\frac{\sqrt{3}}{120}.$$

Integrating Over a Surface

Example 3. Evaluate $\iint_{\Sigma}\frac{dS}{z}$, where Σ is a sphere $x^2+y^2+z^2=a^2$ cut from the top by a plane $z=h(0<h<a)$.



Solution The equation of Σ is

$$z=\sqrt{a^2-x^2-y^2}.$$

The projection of Σ on the xOy – plane is D_{xy},

$$D_{xy}=\{(x,y)\,|\,x^2+y^2\le a^2-h^2\}.$$

Since $\sqrt{z_x^2+z_y^2+1}=\frac{a}{\sqrt{a^2-x^2-y^2}}$,

we have

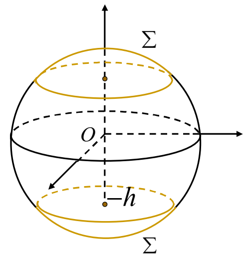
$$\iint_{\Sigma}\frac{dS}{z}=\iint_{D_{xy}}\frac{adxdy}{a^2-x^2-y^2}=a\int_0^{2\pi}d\theta\int_0^{\sqrt{a^2-h^2}}\frac{\rho d\rho}{a^2-\rho^2}=2\pi a\ln\frac{a}{h}.$$

Integrating Over a Surface

Exercise If Σ is a sphere $x^2+y^2+z^2=a^2$ cut from the top by a plane $z=\pm h(0<h<a)$, find

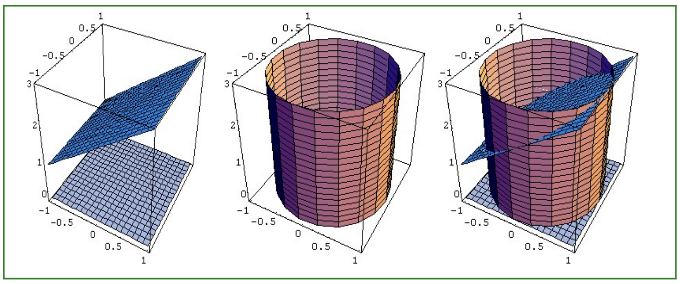
$$\iint_{\Sigma}\frac{dS}{z}=(\quad 0 \quad)$$

$$\iint_{\Sigma}\frac{dS}{|z|}=(\;4\pi a\ln\frac{a}{h}\;)$$



Integrating Over a Surface

Example 4 Evaluate $\oint\!\!\oint_{\Sigma} xdS$, where Σ is the boundary surface of the domain enclosed by $x^2+y^2=1, z=x+2$ and $z=0$.



Integrating Over a Surface

Example 4 Evaluate $\oint\!\!\oint_{\Sigma} xdS$, where Σ is the boundary surface of the domain enclosed by $x^2+y^2=1, z=x+2$ and $z=0$.

Solution The surface can be divided into four parts. We denote the part on $z=0, z=x+2$ and $x^2+y^2=1$ by Σ_1, Σ_2 and Σ_3, respectively.

Then, $\oint\!\!\oint_{\Sigma} xdS=\iint_{\Sigma_1} xdS+\iint_{\Sigma_2} xdS+\iint_{\Sigma_3} xdS$

$$\Sigma_1,\Sigma_2 \xrightarrow{\text{projection}} (\sigma_{xy})$$

$$\iint_{\Sigma_1} xdS=\iint_{(\sigma_{xy})} xdxdy=0$$

$$\iint_{\Sigma_2} xdS=\iint_{(\sigma_{xy})} x\sqrt{2}dxdy=0$$

Integrating Over a Surface

Example 4 Evaluate $\oiint\limits_{\Sigma} xdS$, where Σ is the boundary surface of the domain enclosed by $x^2+y^2=1, z=x+2$ and $z=0$.

Solution(Cont.) $\Sigma_3=\Sigma_{31}(y=\sqrt{1-x^2})+\Sigma_{32}(y=-\sqrt{1-x^2})$

$$\Sigma_{31},\Sigma_{32}\xrightarrow{\text{projection}}(\sigma_{xz})=\{(x,z):-1\le x\le 1, 0\le z\le x+2\}$$

$$\iint\limits_{\Sigma_3} xdS=\iint\limits_{\Sigma_{31}} xdS+\iint\limits_{\Sigma_{32}} xdS$$

$$=2\iint\limits_{(\sigma_{xz})} x\sqrt{1+y_x^2+y_z^2}\,dxdz=2\iint\limits_{(\sigma_{xz})} x\sqrt{1+\frac{x^2}{1-x^2}}\,dxdz$$

$$=2\int_{-1}^{1}dx\int_0^{x+2}\frac{x}{\sqrt{1-x^2}}dz \quad =\pi.$$

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Integrating Over a Surface

Exercise Evaluate $\oiint\limits_{\Sigma} x^2dS$, where Σ is the boundary surface of the sphere $x^2+y^2+z^2=1.$

Solution Method I

$$\oiint\limits_{\Sigma} x^2dS=\frac13\oiint\limits_{\Sigma}(x^2+y^2+z^2)dS=\frac13\oiint\limits_{\Sigma}dS=\frac43\pi.$$

Method II $\oiint\limits_{\Sigma} x^2dS=\iint\limits_{\Sigma_{up}} x^2dS+\iint\limits_{\Sigma_{down}} x^2dS$

$\Sigma_{up}: z=\sqrt{1-x^2-y^2}$, $\Sigma_{down}: z=-\sqrt{1-x^2-y^2}$

$\Sigma_{up},\Sigma_{down}\xrightarrow{\text{projection}}(\sigma_{xy})\qquad(\sigma_{xy}): x^2+y^2\le 1$

$$\oiint\limits_{\Sigma} x^2dS=2\iint\limits_{(\sigma_{xy})} x^2\frac{1}{\sqrt{1-x^2-y^2}}d\sigma$$

$$=8\int_0^{\pi/2}\int_0^1\frac{\rho^2}{\sqrt{1-\rho^2}}\cos^2\theta\rho\, d\rho\, d\theta=\frac43\pi.$$

Integrating Over a Surface

Exercise Evaluate $\iint\limits_{(S)}(xy+yz+zx)dS$, where (S) is the part of the cone $z=\sqrt{x^2+y^2}$ cut by the surface $x^2+y^2=2x.$

Solution

$$\iint\limits_{(S)}(xy+yz+zx)dS=\iint\limits_{(\sigma)}(xy+y\sqrt{x^2+y^2}+x\sqrt{x^2+y^2})\sqrt{1+z_x{}^2+z_y{}^2}\,d\sigma$$

where (σ) is the projection of (S) on xoy plane $x^2+y^2\le 2x.$

Let $x=\rho\cos\theta, y=\rho\sin\theta,$ Then the boundary curve is

$$\rho=2\cos\theta,\theta\in[-\frac{\pi}{2},\frac{\pi}{2}].$$

$$\iint\limits_{(S)}(xy+yz+zx)dS=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}(\rho\cos\theta\cdot\rho\sin\theta+\rho\sin\theta\cdot\rho+\rho\cos\theta\cdot\rho)\sqrt2\rho\, d\rho\, d\theta$$

$$=\frac{\sqrt2}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2\cos\theta)^4(\cos\theta\sin\theta+\sin\theta+\cos\theta)d\theta=8\sqrt2\int_0^{\frac{\pi}{2}}\cos^5\theta d\theta=\frac{64\sqrt2}{15}.$$

Properties of surface integral of the first type

Properties:

(1). $\iint\limits_{(S)}dS=S$

(2). 对称性：

$$\iint\limits_{(S)}f(x,y,z)dS=\begin{cases}0, & f(x,y,-z)=-f(x,y,z)\\ 2\iint\limits_{(S_1)}f(x,y,z)dS, & f(x,y,-z)=f(x,y,z)\end{cases}$$

where (S) is symmetry about xoy plane.

(3). 轮换对称性：

(4). 带入性： $\iint\limits_{(S)}x^2dS=\iint\limits_{(S)}y^2dS=\iint\limits_{(S)}y^2dS=\frac13\iint\limits_{(S)}(x^2+y^2+z^2)dS=\frac13\iint\limits_{(S)}dS=\frac{4\pi}{3},$

$where (S): x^2+y^2+z^2=1$

(5). 无向性

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The Surface Integrals of the Second Type

If we want to find the rates at which fluids flow across a surface, we need to know the direction of the flow and it's obvious, that flows in different direction should have different effects.

The sides of a surface

Usually, a surface has two sides. A closed surface has an inside and an outside; a surface which is not closed may have front and back or up and down sides. This kind of surface is called two – side surface.

In fact, if we choose any point P on the surface and a normal vector at P, then, we move the point in the surface arbitrarily and return to the same position of P, the direction of normal vector may not change its direction for two – side surface.

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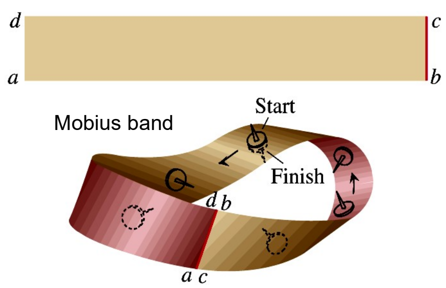
The Surface Integrals of the Second Type

If the normal vector changes direction, the surface is called one – side surface.

For example, we can turn a paper strip and stickup end to end as the right figure.



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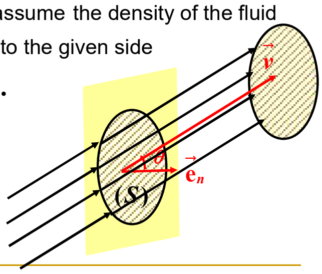
The Flux

Suppose that $\vec{v}(M)(M\in(G)\subseteq \mathbf{R}^3)$ is a field of velocity of an incompressible flow and (S) is a directed surface in (G). To find the flux of the fluid passing through the surface (S) to the given side.

(I). If the velocity of flow at every point of the field is same, that is, $\vec{v}$ is a constant vector, and (S) is a plane, and assume the density of the fluid $\mu = 1.$ Then the flux passing through (S) to the given side

$$Q = S\|\vec{v}\|\cos\theta = \vec{v}\cdot\vec{\mathrm{e}}_n S.$$



The Flux

(II). If the field of velocity of flow is not a constant vector field, and (S) is a piece of an oriented surface, then the computation of the flux requires the idea of integral.

1. Partition

$$(S)\longrightarrow(\Delta S_k),(k=1,2,\cdots,n)$$

2. Homogenization

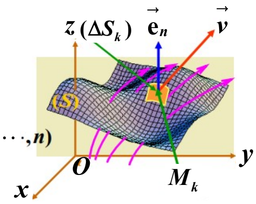
$$\Delta Q_k \approx \vec{v}(M_k)\cdot\vec{\mathrm{e}}_n(M_k)\Delta S_k,\quad (k=1,2,\cdots,n)$$

3. Summation

$$Q=\sum_{k=1}^{n}\Delta Q_k \approx \sum_{k=1}^{n}\vec{v}(M_k)\cdot\vec{\mathrm{e}}_n(M_k)\Delta S_k$$

4. Precision

$$Q=\lim_{d\to 0}\sum_{k=1}^{n}\vec{v}(M_k)\cdot\vec{\mathrm{e}}_n(M_k)\Delta S_k$$



The Surface Integrals of The Second Type

Definition Suppose that (S) is a rectifiable oriented surface and a side of (S) is given. Let the vector field $\vec{\mathrm{A}}(M)$ given. Arbitrarily partition (S) into n small parts and denote the k-th parts by $(\Delta S_k), k=1,2,\cdots,n.$ Select any point $M_k\in(\Delta S_k)$, and form the scalar product

$$\vec{\mathrm{A}}(M_k)\cdot\vec{\mathrm{e}}_n(M_k)\Delta S_k\quad (k=1,2,\cdots,n),$$

where $\vec{\mathrm{e}}_n(M_k)$ is a unit normal vector to the surface at the point M_k in the direction of the given sides. Form the sum:

$$\sum_{k=1}^{n}\vec{\mathrm{A}}(M_k)\cdot\vec{\mathrm{e}}_n(M_k)\Delta S_k.$$

If for any partition of the surface (S) and any selection of $M_k\in(\Delta S_k)$,

The Surface Integrals of The Second Type

Definition (continued)

the limit of the sum exists uniquely as $d\to 0$, where d is the maximum of the diameters of all $(\Delta S_k), k=1,2,\cdots,n,$ then this value of the limit is called the surface integral of the second type of the vector field $\vec{\mathrm{A}}(M)$ over the oirented surface (S) and is denoted by

$$\iint_{(S)}\vec{\mathrm{A}}(M)\cdot\vec{\mathrm{e}}_n dS=\lim_{d\to 0}\sum_{k=1}^{n}\vec{\mathrm{A}}(M_k)\cdot\vec{\mathrm{e}}_n(M_k)\Delta S_k.$$

If we denote $\vec{\mathrm{e}}_n dS=\overrightarrow{\mathrm{d}\mathrm{S}}$, then the integral can be written as

$$\iint_{(S)}\vec{\mathrm{A}}(M)\cdot\overrightarrow{\mathrm{d}\mathrm{S}}=\lim_{d\to 0}\sum_{k=1}^{n}\vec{\mathrm{A}}(M_k)\cdot\vec{\mathrm{e}}_n(M_k)\Delta S_k.$$

The Surface Integrals of The Second Type

If we denote $\vec{\mathrm{A}}(M)=\left(P(x,y,z),Q(x,y,z),R(x,y,z)\right),$

$$\vec{\mathrm{e}}_n(M)=\left(\cos\alpha,\cos\beta,\cos\gamma\right),\quad M_k=\left(\xi_k,\eta_k,\zeta_k\right)$$

Then
$$\iint_{(S)}\vec{\mathrm{A}}(M)\cdot\overrightarrow{\mathrm{d}\mathrm{S}}=\lim_{d\to 0}\sum_{k=1}^{n}\vec{\mathrm{A}}(M_k)\cdot\vec{\mathrm{e}}_n(M_k)\Delta S_k$$
$$=\lim_{d\to 0}\sum_{k=1}^{n}\Big[P\left(\xi_k,\eta_k,\zeta_k\right)\cos\alpha_k+Q\left(\xi_k,\eta_k,\zeta_k\right)\cos\beta_k$$
$$+R\left(\xi_k,\eta_k,\zeta_k\right)\cos\gamma_k\Big]\Delta S_k \qquad =\left\|\boxed{\overrightarrow{\mathrm{d}\mathrm{S}}}\right\|$$

or
$$\iint_{(S)}\vec{\mathrm{A}}(M)\cdot\overrightarrow{\mathrm{d}\mathrm{S}}=\iint_{(S)}\left[P(x,y,z)\cos\alpha+Q(x,y,z)\cos\beta+R(x,y,z)\cos\gamma\right]\boxed{dS}.$$

This is the relationship between the two types of surface integrals.

The Surface Integrals of The Second Type

Note that $\cos\alpha dS$, $\cos\beta dS$, $\cos\gamma dS$ are the projections of dS onto the coordinate planes yOz, zOx and xOy, respectively.

Denote $\cos\alpha dS=dydz$, $\cos\beta dS=dzdx$, $\cos\gamma dS=dxdy$, or

$$\cos\alpha dS=dy\wedge dz,\ \cos\beta dS=dz\wedge dx,\ \cos\gamma dS=dx\wedge dy.$$

Then
$$\iint_{(S)}\vec{\mathrm{A}}(M)\cdot\overrightarrow{\mathrm{d}\mathrm{S}}=\iint_{(S)}\left[P(x,y,z)dydz+Q(x,y,z)dzdx+R(x,y,z)dxdy\right]$$
$$=\iint_{(S)}\left[P(x,y,z)dy\wedge dz+Q(x,y,z)dz\wedge dx+R(x,y,z)dx\wedge dy\right].$$

This is the coordinate form of the surface integral of the second type.

The surface integral of the second type is also called the surface integral with respect to coordinates.

The Surface Integrals of The Second Type

In the coordinate representation of the integral, $dy \wedge dz,\ dz \wedge dx,\ dx \wedge dy$ are the projections of dS onto the corresponding coordinate planes, whose signs depend on the direction of the normal vector.

For example: $\cos\gamma > 0,\quad dx \wedge dy = d\sigma_{xy}$

$\cos\gamma < 0,\quad dx \wedge dy = -d\sigma_{xy}$

where $d\sigma_{xy}$ is the projection of dS onto xOy plane

Properties of Surface Integrals of the Second Type

Let us suppose that all the integrals of the second type exist.

Property 1 If we change the side of the surface of integration then the sign of the integral will change, that is

$$\iint_{(S)} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{dS}} = -\iint_{(-S)} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{dS}}.$$

Property 2 If the surface (S) is divided into two pieces (S_1) and (S_2), and $(S) = (S_1) \cup (S_2)$, then

$$\iint_{(S)} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{dS}} = \iint_{(S_1)} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{dS}} + \iint_{(S_2)} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{dS}},$$

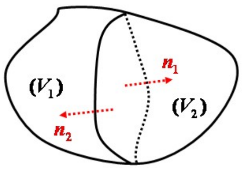
where all surfaces of integration are on the same side.

Properties of Surface Integrals of the Second Type

Property 3 Suppose the domain (V) in space bounded by the oriented closed surface (S) is divided into two domains (V_1) and (V_2) by another surface located inside the domain (V) and the boundaries of the domains (V_1) and (V_2) are denoted by (S_1) and (S_2) respectively. Then

$$\oiint_{(S)} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{dS}} = \oiint_{(S_1)} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{dS}} + \oiint_{(S_1)} \vec{\mathbf{A}} \cdot \overrightarrow{\mathbf{dS}},$$

where the surfaces in the above equality are both outside or both inside.



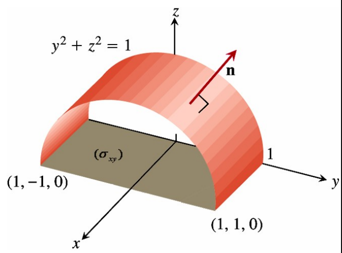
Finding Flux

Example 1. Find the flux of $\vec{\mathbf{F}} = yz\mathbf{j} + z^2\mathbf{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1, z \geq 0$, by the planes $x = 0$ and $x = 1$.

Solution The outward normal field on S may be calculated from the $G(x, y, z) = y^2 + z^2 - 1 = 0$ to be

$$\vec{\mathbf{e}}_n = +\frac{\vec{n}}{\|\vec{n}\|} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4y^2 + 4z^2}} = y\mathbf{j} + z\mathbf{k}.$$

then $\vec{\mathbf{F}} \cdot \vec{\mathbf{e}}_n = \left(yz\mathbf{j} + z^2\mathbf{k}\right) \cdot \left(y\mathbf{j} + z\mathbf{k}\right) = z.$



The flux can be calculated by the surface integral of first type.

Finding Flux

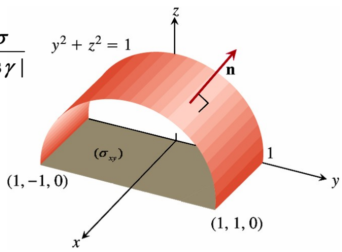
Solution (continued)

$$\text{Flux} = \iint_{(S)} \vec{\mathbf{F}} \cdot \vec{\mathbf{e}}_n dS = \iint_{(S)} z dS = \iint_{(\sigma_{xy})} z \frac{d\sigma}{|\cos\gamma|}$$

$$= \iint_{(\sigma_{xy})} z\sqrt{z_x^2 + z_y^2 + 1}\, d\sigma$$

$$= \iint_{(\sigma_{xy})} z \frac{1}{z} d\sigma = \iint_{(\sigma_{xy})} d\sigma$$

$$= 2.$$



Computation of Surface Integrals of the Second Type

Since $\iint_{(S)} \vec{\mathbf{A}}(M) \cdot \overrightarrow{\mathbf{dS}} = \iint_{(S)} \left[P(x,y,z)\, dy \wedge dz + Q(x,y,z)\, dz \wedge dx + R(x,y,z)\, dx \wedge dy\right]$

(1) If the surface (S) can be expressed by $z = z(x, y),\ (x, y) \in (\sigma_{xy})$, then the unit normal at any point on (S) is

$$\vec{\mathbf{e}}_n = \pm\frac{(-z_x, -z_y, 1)}{\sqrt{1 + {z_x}^2 + {z_y}^2}}.$$

and

$$\vec{\mathbf{A}}(M) \cdot \vec{\mathbf{e}}_n = \pm\frac{-Pz_x - Qz_y + R}{\sqrt{1 + {z_x}^2 + {z_y}^2}}.$$

Here the choice of the sign " ± " depends on the orientation of (S). If the direction of the surface (S) is upward, then we choose the sign "+". Otherwise, we choose the sign "-".

Computation of Surface Integrals of the Second Type

Applying the computation of surface integrals of the first type, we have

$$\iint_{(S)} \vec{A}(M)\cdot\vec{dS} = \iint_{(S)} \vec{A}(M)\cdot\vec{e}_n dS = \pm\iint_{(S)} \frac{-Pz_x - Qz_y + R}{\sqrt{1+z_x{}^2+z_y{}^2}} dS$$

$$= \pm\iint_{(\sigma_{xy})} \frac{-P(x,y,z(x,y))z_x(x,y) - Q(x,y,z(x,y))z_y(x,y) + R(x,y,z(x,y))}{\sqrt{1+z_x{}^2+z_y{}^2}}$$

$$\times\sqrt{1+z_x{}^2+z_y{}^2}\,dxdy$$

$$= \pm\iint_{(\sigma_{xy})} -P(x,y,z(x,y))z_x(x,y) - Q(x,y,z(x,y))z_y(x,y) + R(x,y,z(x,y))dxdy$$

where the choice of the sign "$\pm$ " depends on the orientation of (S).

Computation of Surface Integrals of the Second Type

In particular, we have

$$\iint_{(S)} R(x,y,z)dx\wedge dy = \iint_{(S)} R(x,y,z)\cos\gamma dS$$

$$= \iint_{(\sigma_{xy})} R(x,y,z(x,y))\cos\gamma\frac{d\sigma}{|\cos\gamma|} = \pm\iint_{(\sigma_{xy})} R(x,y,z(x,y))d\sigma.$$

The sign depends on whether the surface of integration is taken to be the upper or lower side.

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Computation of Surface Integrals of the Second Type

(2) Similarly, if the equation of (S) has the form $x = x(y,z)$, we have

$$\iint_{(S)} P(x,y,z)dy\wedge dz = \iint_{(S)} P(x,y,z)\cos\alpha dS$$

$$= \iint_{(\sigma_{yz})} P(x(y,z),y,z)\cos\alpha\frac{d\sigma}{|\cos\alpha|} = \pm\iint_{(\sigma_{yz})} P(x(y,z),y,z)d\sigma,$$

(3) If the equation of (S) has the form $y = y(x,z)$, we have

$$\iint_{(S)} Q(x,y,z)dz\wedge dx = \iint_{(S)} Q(x,y,z)\cos\beta dS$$

$$= \iint_{(\sigma_{xz})} Q(x,y(x,z),z)\cos\beta\frac{d\sigma}{|\cos\beta|} = \pm\iint_{(\sigma_{xz})} Q(x,y(x,z),z)d\sigma.$$

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Computation of Surface Integrals of the Second Type

If the equation of (S) has the form $z = z(x,y)$, $x = x(y,z)$, and $y = y(x,z)$, we have respectively

$$\iint_{(S)} R(x,y,z)dx\wedge dy = \pm\iint_{(\sigma_{xy})} R(x,y,z(x,y))d\sigma$$

$$\iint_{(S)} P(x,y,z)dy\wedge dz = \pm\iint_{(\sigma_{yz})} P(x(y,z),y,z)d\sigma,$$

$$\iint_{(S)} Q(x,y,z)dz\wedge dx = \pm\iint_{(\sigma_{xz})} Q(x,y(x,z),z)d\sigma.$$

where (σ_{yz}) and (σ_{xz}) are the projection of (S) onto the yOz – plane and zOx – plane. The sign of the integration depends on the angle between the normal vector and the positive direction of the axes.

Method: 1. Projection 2. substitution 3. Determining the sign

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Computation of Surface Integrals of the Second Type

Method I: $\iint_{(S)} \vec{A}(M)\cdot\vec{dS} = \iint_{(S)} \vec{A}(M)\cdot\vec{e}_n dS$

Method II: If the equation of (S) has the form $z = z(x,y)$,

$$\iint_{(S)} \vec{A}(M)\cdot\vec{dS}$$

$$= \pm\iint_{(\sigma_{xy})} -P(x,y,z(x,y))z_x(x,y) - Q(x,y,z(x,y))z_y(x,y) + R(x,y,z(x,y))dxdy$$

where the choice of the sign "$\pm$ " depends on γ.

Computation of Surface Integrals of the Second Type

Method III: If the equation of (S) has the form $z = z(x,y)$, $x = x(y,z)$, and $y = y(x,z)$, we have respectively

$$\iint_{(S)} R(x,y,z)dxdy = \pm\iint_{(\sigma_{xy})} R(x,y,z(x,y))d\sigma_{xy}$$

$$\iint_{(S)} P(x,y,z)dydz = \pm\iint_{(\sigma_{yz})} P(x(y,z),y,z)d\sigma_{yz},$$

$$\iint_{(S)} Q(x,y,z)dzdx = \pm\iint_{(\sigma_{xz})} Q(x,y(x,z),z)d\sigma_{xz}.$$

where (σ_{yz}) and (σ_{xz}) are the projection of (S) onto the yOz – plane and zOx – plane. The sign of the integration depends on the angle between the normal vector and the positive direction of the axes.

Procedure: 1. Projection 2. substitution 3. Determining the sign

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Computation of Surface Integrals of the Second Type

Example 2 Calculate the integral $\iint\limits_{\Sigma} x^2dy\wedge dz+y^2dz\wedge dx+z^2dx\wedge dy$, where Σ is the outside surface of the rectangle $\{(x,y,z)|0\le x\le a,0\le y\le b,0\le z\le c\}$.

Solution We divide the surface into six parts:

$\Sigma_1: z=c(0\le x\le a,0\le y\le b)$ The upper

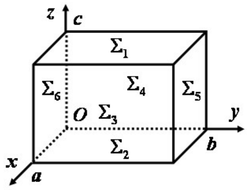
$\Sigma_2: z=0(0\le x\le a,0\le y\le b)$ The lower

$\Sigma_3: x=a(0\le y\le b,0\le y\le b)$ The front

$\Sigma_4: x=0(0\le y\le b,0\le y\le b)$ The back

$\Sigma_5: y=b(0\le x\le a,0\le z\le c)$ The right

$\Sigma_6: y=0(0\le x\le a,0\le z\le c)$ The left



Computation of Surface Integrals of the Second Type

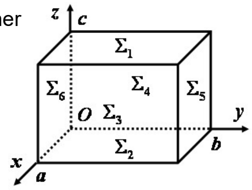
Solution (continued)

Except Σ_3,Σ_4, areas of the projection of the other parts onto yOz – plane are all zero, then

$$\iint\limits_{\Sigma} x^2dy\wedge dz=\iint\limits_{\Sigma_3} x^2dy\wedge dz+\iint\limits_{\Sigma_4} x^2dy\wedge dz,$$

and

$$\iint\limits_{\Sigma} x^2dy\wedge dz=\iint\limits_{(\sigma_{yz})} a^2dydz-\iint\limits_{(\sigma_{yz})} 0^2dydz=a^2bc.$$



Similarly, we have $\iint\limits_{\Sigma} y^2dz\wedge dx=b^2ac,$ and $\iint\limits_{\Sigma} z^2dx\wedge dy=c^2ab.$

Therefore $\iint\limits_{\Sigma} x^2dy\wedge dz+y^2dz\wedge dx+z^2dx\wedge dy=abc(a+b+c)$

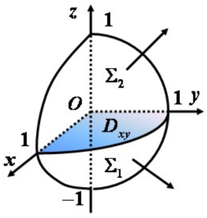
Computation of Surface Integrals of the Second Type

Example 3 Find $\iint\limits_{\Sigma} xyzdx\wedge dy$, where Σ is the outside of a sphere $x^2+y^2+z^2=1$ cut by $x\ge 0,y\ge 0.$

Solution Divide Σ into Σ_1 and Σ_2, where

$\Sigma_1: z_1=-\sqrt{1-x^2-y^2},\quad \Sigma_2: z_2=\sqrt{1-x^2-y^2},$

then $\iint\limits_{\Sigma} xyzdx\wedge dy=\iint\limits_{\Sigma_1} xyzdx\wedge dy+\iint\limits_{\Sigma_2} xyzdx\wedge dy,$ the unit normal vectors are shown as in right figure, then we have



Computation of Surface Integrals of the Second Type

Solution (continued)

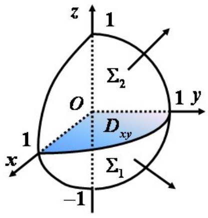
$$\iint\limits_{\Sigma} xyzdx\wedge dy=\iint\limits_{\Sigma_1} xyzdx\wedge dy+\iint\limits_{\Sigma_2} xyzdx\wedge dy$$

$$=-\iint\limits_{D_{xy}} xy\left(-\sqrt{1-x^2-y^2}\right)dxdy+\iint\limits_{D_{xy}} xy\sqrt{1-x^2-y^2}dxdy$$

$$=2\iint\limits_{D_{xy}} xy\sqrt{1-x^2-y^2}dxdy$$

$$=2\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta d\theta\int_0^1\rho^3\sqrt{1-\rho^2}d\rho$$

$$=1\cdot\frac{2}{15}=\frac{2}{15}.$$



Computation of Surface Integrals of the Second Type

Example 4 Evaluate $\oiint\limits_{(S)}\left[(x^2+y^2)dy\wedge dz+zdx\wedge dy\right]$, where (S) is the outside of the surface of the solid bounded by the cylinder $x^2+y^2=R^2$ and the planes $z=0,\ z=H\ (H>0).$

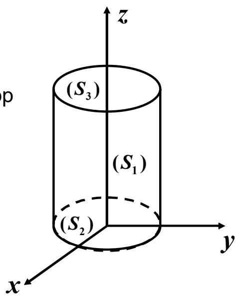
Solution Divide (S) into three parts:

(S_1): the cylinder (S_2): the base (S_3): the top

Firstly, we evaluate $\oiint\limits_{(S)}\left[(x^2+y^2)dy\wedge dz\right]$

Divide (S_1) into (S_{11}) and (S_{12}), where

$(S_{11}): x=\sqrt{R^2-y^2},\quad (S_{12}): x=-\sqrt{R^2-y^2},$



Computation of Surface Integrals of the Second Type

Solution (continued)

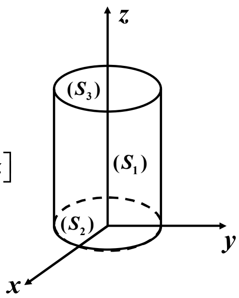
$$\oiint\limits_{(S)}\left[(x^2+y^2)dy\wedge dz\right]=\iint\limits_{(S_{11})}\left[(x^2+y^2)dy\wedge dz\right]$$

$$+\iint\limits_{(S_{12})}\left[(x^2+y^2)dy\wedge dz\right]+\iint\limits_{(S_2)}\left[(x^2+y^2)dy\wedge dz\right]$$

$$+\iint\limits_{(S_3)}\left[(x^2+y^2)dy\wedge dz\right]$$

$$=\iint\limits_{(\sigma_{yz})}\left[(R^2-y^2+y^2)dydz\right]-\iint\limits_{(\sigma_{yz})}\left[(R^2-y^2+y^2)dydz\right]$$

$$=0$$



Computation of Surface Integrals of the Second Type

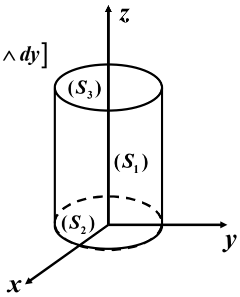
Solution (continued)

Secondly, we evaluate $\oiint\limits_{(S)}[zdx\wedge dy]$

$$\oiint\limits_{(S)}[zdx\wedge dy]=\iint\limits_{(S_1)}[zdx\wedge dy]+\iint\limits_{(S_2)}[zdx\wedge dy]+\iint\limits_{(S_3)}[zdx\wedge dy]$$

$$=-\iint\limits_{(\sigma_{xy})}[0dxdy]+\iint\limits_{(\sigma_{xy})}[Hdxdy]$$

$$=\pi R^2 H$$



Therefore

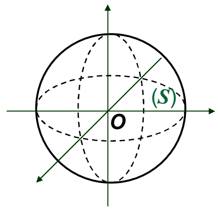
$$\oiint\limits_{(S)}\left[(x^2+y^2)dy\wedge dz+zdx\wedge dy\right]=\pi R^2 H$$

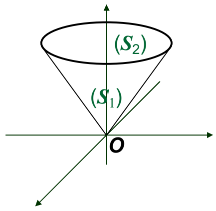
Finding Flux

Example 5. Find the flux of the vector field $\vec{F}=(x,y,z)$ at the directed surface (S), where

(1) (S) is the outside of the sphere $x^2+y^2+z^2=1$.

(2) (S) is the outside of the surface of the solid bounded by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=1$.





Finding Flux

Solution

(1) Method I: $\Phi=\iint\limits_{(S)}\vec{F}\cdot\vec{e}_n dS=\iint\limits_{(S)}(x,y,z)\cdot(x,y,z)dS=\iint\limits_{(S)}(x^2+y^2+z^2)dS=4\pi$

Method II:

$$(S^{up}):z=\sqrt{1-(x^2+y^2)},(S^{down}):z=-\sqrt{1-(x^2+y^2)},(\sigma_{xy}):x^2+y^2\le 1.$$

$$\Phi=\iint\limits_{(S)}xdy\wedge dz+ydz\wedge dx+zdx\wedge dy=\iint\limits_{(S^{up})}\cdots+\iint\limits_{(S^{down})}\cdots$$

$$=\iint\limits_{(\sigma_{xy})}\left(-x\frac{-x}{\sqrt{1-(x^2+y^2)}}-y\frac{-y}{\sqrt{1-(x^2+y^2)}}+\sqrt{1-(x^2+y^2)}\right)d\sigma$$

$$-\iint\limits_{(\sigma_{xy})}\left(-x\frac{x}{\sqrt{1-(x^2+y^2)}}-y\frac{y}{\sqrt{1-(x^2+y^2)}}-\sqrt{1-(x^2+y^2)}\right)d\sigma$$

Method III: $\Phi=\iint\limits_{(S)}xdy\wedge dz+\iint\limits_{(S)}ydz\wedge dx+\iint\limits_{(S)}zdx\wedge dy=I_1+I_2+I_3$

$$I_1=\iint\limits_{(S^{\text{front}})}\cdots+\iint\limits_{(S^{\text{back}})}\cdots=\iint\limits_{(\sigma_{yz})}\sqrt{1-(y^2+z^2)}d\sigma-\iint\limits_{(\sigma_{yz})}-\sqrt{1-(y^2+z^2)}d\sigma$$

Finding Flux

Solution(Cont.)

(2) Method I: $\Phi=\iint\limits_{(S)}\vec{F}\cdot\vec{e}_n dS=\iint\limits_{(S^{up})}\vec{F}\cdot\vec{e}_n dS+\iint\limits_{(S^{dowm})}\vec{F}\cdot\vec{e}_n dS$

$$=\iint\limits_{(S^{up})}(x,y,z)\cdot(0,0,1)dS+\iint\limits_{(S^{dowm})}(x,y,z)\cdot\frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}},-1\right)dS=\pi$$

Method II: $(S^{up}):z=1,(S^{down}):z=\sqrt{x^2+y^2},(\sigma_{xy}):x^2+y^2\le 1.$

$$\Phi=\iint\limits_{(S)}xdy\wedge dz+ydz\wedge dx+zdx\wedge dy$$

$$=\iint\limits_{(S^{up})}xdy\wedge dz+ydz\wedge dx+zdx\wedge dy+\iint\limits_{(S^{down})}xdy\wedge dz+ydz\wedge dx+zdx\wedge dy$$

$$=\iint\limits_{(\sigma_{xy})}(x\cdot 0+y\cdot 0+1)d\sigma-\iint\limits_{(\sigma_{xy})}\left(-x\frac{x}{\sqrt{x^2+y^2}}-y\frac{y}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}\right)d\sigma=\pi.$$

Method III: $\Phi=\iint\limits_{(S)}xdy\wedge dz+\iint\limits_{(S)}ydz\wedge dx+\iint\limits_{(S)}zdx\wedge dy=I_1+I_2+I_3$

Properties of surface integral of the second type

Properties:

(1). 轮换对称性: $\iint\limits_{(S)}zdxdy=\iint\limits_{(S)}ydzdx=\iint\limits_{(S)}xdydz,$

$(S):x+y+z=1$ cut by $x=0,y=0,z=0$

(2). 有向性

$$\iint\limits_{(+S)}\vec{A}(M)\cdot\vec{dS}=-\iint\limits_{(-S)}\vec{A}(M)\cdot\vec{dS}$$

Review

- Surface integrals with respect to surface area
- Surface integrals with respect to coordinates
- The relationship between two kinds of surface integrals