Lecture 04

Chapter 2 Matrix Algebra

2.3. Partitioned Matrices

2.4. Elementary Matrices

2.3. Partitioned Matrices

It is useful to think of a matrix as being composed of a number of sub-matrices. These smaller sub-matrices are called as blocks.

Partitioned Matrices

For a given matrix, it is useful to rewrite it into **sub-matrices [子矩阵]**.

For example, let

$$C = \begin{pmatrix} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{pmatrix}$$

can be rewrite as

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \left(\begin{array}{cc|ccc} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ \hline 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{array}\right).$$

The smaller matrices are often referred to as **blocks [块].**

Partitioned Matrices

Definition 1. (Partitioned Matrix) A **partitioned matrix [分块矩阵]** is a matrix broken into blocks.

$$C=\begin{pmatrix}1 & -2 & 4 & 1 & 3\\ 2 & 1 & 1 & 1 & 1\\ 3 & 3 & 2 & -1 & 2\\ 4 & 6 & 2 & 2 & 4\end{pmatrix}$$

$$\begin{pmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}=\left(\begin{array}{cc|ccc}1 & -2 & 4 & 1 & 3\\ 2 & 1 & 1 & 1 & 1\\ \hline 3 & 3 & 2 & -1 & 2\\ 4 & 6 & 2 & 2 & 4\end{array}\right)$$

$$\begin{pmatrix}C_{11}\\ C_{21}\\ C_{31}\\ C_{41}\end{pmatrix}=\left(\begin{array}{ccccc}1 & -2 & 4 & 1 & 3\\ \hline 2 & 1 & 1 & 1 & 1\\ \hline 3 & 3 & 2 & -1 & 2\\ \hline 4 & 6 & 2 & 2 & 4\end{array}\right)$$

$$\left(\begin{array}{c|c|c|c|c}1 & -2 & 4 & 1 & 3\\ 2 & 1 & 1 & 1 & 1\\ 3 & 3 & 2 & -1 & 2\\ 4 & 6 & 2 & 2 & 4\end{array}\right)=(C_{11},C_{12},C_{13},C_{14},C_{15})$$

Block addition and scalar multiplication

Suppose that matrices A and B are two $m \times n$ matrices and are partitioned **in the same way**. That is,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{sr} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ \vdots & \vdots & & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{sr} \end{pmatrix}$$

where A_{ij}, B_{ij} are submatrices of order $p_i \times q_j$, $i = 1,2, \dots, s, j = 1,2, \dots, r.$

Then the block addition of these two matrices is given by

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1r} + B_{1r} \\ A_{21} + B_{21} & A_{22} + B_{22} & \cdots & A_{2r} + B_{2r} \\ \vdots & \vdots & & \cdots \\ A_{s1} + B_{s1} & A_{s2} + B_{s2} & \cdots & A_{sr} + B_{sr} \end{pmatrix}, \ \alpha A = \begin{pmatrix} \alpha A_{11} & \alpha A_{12} & \cdots & \alpha A_{1r} \\ \alpha A_{21} & \alpha A_{22} & \cdots & \alpha A_{2r} \\ \vdots & \vdots & & \cdots \\ \alpha A_{s1} & \alpha A_{s2} & \cdots & \alpha A_{sr} \end{pmatrix}.$$

Block Addition and Scalar Multiplication

Example. If $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 2 & 1 \end{array}\right)$

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 3 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ \hline 2 & 2 & 2 & 3 \end{array}\right)$$

perform block addition for $A + B$.

Solution.

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 5 & 4 & 2 \\ 5 & 1 & 3 & 3 \\ 1 & 3 & 1 & 1 \\ \hline 3 & 3 & 4 & 4 \end{array}\right).$$

Block Multiplication

One useful way of **partitioning a matrix is into columns**. For example, let

$$B = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{pmatrix}$$

We can partition B into three column submatrices:

$$B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3) = \left(\begin{array}{c|c|c} -1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{array}\right)$$

Then the multiplication of matrix can be represented as

$$AB = A(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3) = (A\boldsymbol{b}_1, A\boldsymbol{b}_2, A\boldsymbol{b}_3).$$

Example 1. Check the above equality for $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & -1 \end{pmatrix}$.

Block Multiplication

In general, if A is an $m \times n$ matrix and B is an $n \times r$ matrix that has been partitioned into columns $(\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_r)$, then the block multiplication of A times B is given by

$$AB = (A\boldsymbol{b}_1, A\boldsymbol{b}_2, \dots, A\boldsymbol{b}_r).$$

In particular,

$$(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n) = A = AI = (A\mathbf{e}_1, A\mathbf{e}_2, \dots, A\mathbf{e}_n).$$

The columns of AB are linear combinations of columns of A.

Block Multiplication

A matrix can also be **partitioned into rows.**

For example, $A = \begin{pmatrix} \boldsymbol{a}(1,:) \\ \boldsymbol{a}(2,:) \\ \vdots \\ \boldsymbol{a}(m,:) \end{pmatrix}$, then $AB = \begin{pmatrix} \boldsymbol{a}(1,:)B \\ \boldsymbol{a}(2,:)B \\ \vdots \\ \boldsymbol{a}(m,:)B \end{pmatrix}$.

Example 2. Check the above equality for $A = \begin{pmatrix} 2 & 5 \\ 3 & 4 \\ 1 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 & -3 \\ -1 & 1 & 1 \end{pmatrix}$.

The rows of AB are linear combinations of rows of B.

Block Multiplication

Let A be an $m \times n$ matrix and B be an $n \times r$ matrix. Four steps.

t columns

$(r-t)$ columns

Case 1. $B = (B_1 \quad B_2)$, where B_1 is an $n \times t$ matrix and B_2 is an $n \times (r-t)$ matrix. Then $AB = (AB_1 \quad AB_2)$.

k rows

Case 2. $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, where A_1 is a $k \times n$ matrix and A_2 is an $(m-k) \times n$ matrix.

Then $AB = \begin{pmatrix} A_1 B \\ A_2 B \end{pmatrix}$.

$(m-k)$ rows

<u>Case 3.</u> If $A = (A_1 \quad A_2)$, where A_1 is a $m \times s$ matrix and A_2 is an $m \times (n-s)$ matrix, $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ where B_1 is an $s \times r$ matrix and B_2 is an $(n-s) \times r$ matrix, then

$$AB = \begin{array}{cc} \begin{pmatrix} A_1 & A_2 \end{pmatrix} & \\ s \text{ columns} \quad (n-s) \text{ columns} & \end{array} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{array}{l} s \text{ rows} \\ (n-s) \text{ rows} \end{array} = A_1B_1 + A_2B_2.$$

<u>Case 4</u>. If

$$A = \begin{array}{cc} \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) & \begin{array}{l} k \text{ rows} \\ (m-k) \text{ rows} \end{array} \\ s \text{ columns} \quad (n-s) \text{ columns} & \end{array}, \quad B = \begin{array}{cc} \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right) & \begin{array}{l} s \text{ rows} \\ (n-s) \text{ rows} \end{array} \\ t \text{ columns} \quad (r-t) \text{ columns} & \end{array}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}.$$

Block Multiplication

In general, if the blocks have the proper dimensions, the block multiplication can be carried out in the same manner as ordinary matrix multiplication. If

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ \vdots & \vdots & & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{st} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ \vdots & \vdots & & \vdots \\ B_{t1} & B_{t2} & \cdots & B_{tr} \end{pmatrix},$$

then

$$AB = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ C_{21} & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & & \vdots \\ C_{s1} & C_{s2} & \cdots & C_{sr} \end{pmatrix}, \qquad \text{where } C_{ij} = \sum_{k=1}^{t} A_{ik} B_{kj} .$$

Block Multiplication

Example 3. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right)$.

There are many ways to partition A into blocks and perform block multiplication.

(i) $$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 1 \\ 3 & 3 & 2 & 2 \end{array}\right)$$

Then

$$AB = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 1 \\ 3 & 3 & 2 & 2 \end{array}\right) \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right) = \left(\begin{array}{cc|cc} 8 & 6 & 4 & 5 \\ \hline 10 & 9 & 6 & 7 \\ 18 & 15 & 10 & 12 \end{array}\right).$$

Block Multiplication

Example 3. (continue)

(ii) $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 3 & 2 & 2 \end{array}\right)$

Then

$$AB = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 3 & 2 & 2 \end{array}\right) \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right) = \left(\begin{array}{cc|cc} 8 & 6 & 4 & 5 \\ 10 & 9 & 6 & 7 \\ \hline 18 & 15 & 10 & 12 \end{array}\right).$$

Remark. Although matrices can be partitioned in different ways, the products of block multiplication are the same.

Block Multiplication

Review of **matrix multiplication**: $A_{m\times n}$, $B_{n\times r}$

(i) $A = \begin{pmatrix} \boldsymbol{a}(1,:) \\ \boldsymbol{a}(2,:) \\ \vdots \\ \boldsymbol{a}(m,:) \end{pmatrix}$, $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r)$,

$$AB = \begin{pmatrix} \boldsymbol{a}(1,:)\boldsymbol{b}_1 & \boldsymbol{a}(1,:)\boldsymbol{b}_2 & \cdots & \boldsymbol{a}(1,:)\boldsymbol{b}_r \\ \boldsymbol{a}(2,:)\boldsymbol{b}_1 & \boldsymbol{a}(2,:)\boldsymbol{b}_2 & \cdots & \boldsymbol{a}(2,:)\boldsymbol{b}_r \\ \vdots & \vdots & & \vdots \\ \boldsymbol{a}(m,:)\boldsymbol{b}_1 & \boldsymbol{a}(m,:)\boldsymbol{b}_2 & \cdots & \boldsymbol{a}(m,:)\boldsymbol{b}_r \end{pmatrix}$$

(ii) $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)$, $B = \begin{pmatrix} \boldsymbol{b}(1,:) \\ \boldsymbol{b}(2,:) \\ \vdots \\ \boldsymbol{b}(n,:) \end{pmatrix}$,

$$AB = \boldsymbol{a}_1\boldsymbol{b}(1,:) + \boldsymbol{a}_2\boldsymbol{b}(2,:) + \cdots + \boldsymbol{a}_n\boldsymbol{b}(n,:)$$

Quasi Triangular Matrices

Definition. (Quasi Triangular Matrices) A partitioned matrix has the form

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1r} \\ O & M_{22} & \cdots & M_{2r} \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & M_{rr} \end{pmatrix}$$

where M_{ij} $(i, j = 1,2, \dots, r)$ are blocks, called **quasi-upper triangular matrix [准上三角矩阵]**, and a partitioned matrix in the form of

$$M = \begin{pmatrix} M_{11} & O & \cdots & O \\ M_{21} & M_{22} & \cdots & O \\ \vdots & \vdots & & \vdots \\ M_{s1} & M_{s2} & \cdots & M_{ss} \end{pmatrix}$$

is called **quasi-lower triangular matrix [准下三角矩阵].** Both of them are called **quasi triangular matrix [准三角矩阵].**

2.4. Elementary Matrices

We will review the process of solving linear systems in terms of matrix multiplications, rather than row operations. These special matrices which will be used are called elementary matrices.

Equivalent Systems

Recall that two linear systems are equivalent, if they have the same solution set.

Given an $m \times n$ linear system $A\boldsymbol{x} = \boldsymbol{b}$, we can obtain an equivalent system by multiplying both sides of the equation by a nonsingular $m \times m$ matrix M

$$A\boldsymbol{x} = \boldsymbol{b} \quad \Leftrightarrow \quad MA\boldsymbol{x} = M\boldsymbol{b},$$

since these two systems have the same solution set.

Equivalent Systems

To obtain an equivalent system that is easy to solve, we can apply a sequence of nonsingular matrices $E_1, E_2, \dots, E_k$ to both sides of the equation $A\boldsymbol{x} = \boldsymbol{b}$ and obtain

$$U\boldsymbol{x} = \boldsymbol{c}$$

where $U = E_k \dots E_2 E_1 A$ and $\boldsymbol{c} = E_k \dots E_2 E_1 \boldsymbol{b}$. The new system will be equivalent to the original one provided that $M = E_k \dots E_2 E_1$ is nonsingular.

Theorem. If A and B are nonsingular $n \times n$ matrices, then AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$. Moreover, if $E_1, E_2, \dots, E_k$ are all nonsingular, then the product $E_1 E_2 \dots E_k$ is nonsingular and

$$(E_1 E_2 \dots E_k)^{-1} = E_k^{-1} \dots E_2^{-1} E_1^{-1}.$$

Elementary Matrices

Type I. **Type I** elementary matrices are obtained by interchanging two rows of I.

Example. $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an elementary matrix of **type I**.

Let A be a 3×3 matrix. Then

$$E_1 A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

$$AE_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{pmatrix}.$$

Elementary Matrices

Type I. **Type I** elementary matrices are obtained by interchanging two rows of I.

In general, we can interchange two **rows** of matrix A by multiplying a **type I** elementary matrix E on the **left** side of A.

This corresponds to the **first type** of elementary row operations.

(We can interchange two **columns** of matrix A by multiplying a **type I** elementary matrix E on the **right** side of A.)

Elementary Matrices

Type II: **Type II** elementary matrices are obtained by multiplying a row of I by a nonzero constant.

Example. $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is an elementary matrix of **type II**. Then

$$E_2 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{pmatrix},$$

$$A E_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 3a_{13} \\ a_{21} & a_{22} & 3a_{23} \\ a_{31} & a_{32} & 3a_{33} \end{pmatrix}.$$

Elementary Matrices

Type II: **Type II** elementary matrices are obtained by multiplying a row of I by a nonzero constant.

In general, we can multiply a **row** of matrix A by a nonzero constant, by multiplying a **type II** elementary matrix E_2 on the **left** side of matrix A.

This corresponds to the **second type** of elementary row operations.

(We can multiply a **column** of matrix A by a nonzero constant, by multiplying a **type II** elementary matrix E_2 on the **right** side of matrix A.)

Elementary Matrices

Type III: **Type III** elementary matrices are obtained from I by adding a multiple of one row to another row.

Example. $E_3 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an elementary matrix of **type III**. Then

$$E_3A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} + 3a_{31} & a_{12} + 3a_{32} & a_{13} + 3a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

$$AE_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 3a_{11} + a_{13} \\ a_{21} & a_{22} & 3a_{21} + a_{23} \\ a_{31} & a_{32} & 3a_{31} + a_{33} \end{pmatrix}$$

Elementary Matrices

Type III: **Type III** elementary matrices are obtained from I by adding a multiple of one row to another row.

In general, we can add a multiple of one **row** to another **row** of a matrix A by multiplying a **type III** elementary matrix E_3 on the **left** side of A.

This corresponds to the **third type** of elementary row operations.

(We can add a multiple of one **column** to another **column** of a matrix A by multiplying a **type III** elementary matrix E_3 on the **right** side of A.)

Elementary Matrices

In previous examples, we see that each type of these matrices can be associated with one elementary row operations. All row operations are invertible, it is natural to have the following theorem.

Theorem 1. (The Inverse Matrix of Elementary Matrices) Let E be an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

Elementary Matrices

Theorem 1. (The Inverse Matrix of Elementary Matrices) Let E be an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

Proof.

(1) If E is the elementary matrix of type I formed from I by interchanging the ith and jth rows, then E can be transformed back into I by interchanging the same rows again. Thus

$$EE = I$$

and hence E is its own inverse.

Elementary Matrices

Theorem 1. (The Inverse Matrix of Elementary Matrices) Let E be an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

Proof.

(2) If E_α is the elementary matrix of type II formed by multiplying the ith row of I by a nonzero scalar α, then E_α can be transformed into the identity by multiplying either its ith row or ith column by $1/\alpha$.

Elementary Matrices

Theorem 1. (The Inverse Matrix of Elementary Matrices) Let E be an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

Proof.

(3) If E is the elementary matrix of type III formed by I by adding m times of the ith row to the jth row, then E can be transformed back into I by subtracting m times the ith row from the jth row or by subtracting m times the jth column from the ith column.

Elementary Matrices

Definition 2. (Row Equivalent Matrices) Let A and B be two matrices. B is said to be **row equivalent [行等价]** to A if there exists a finite sequence $E_1, E_2, \dots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \dots E_1 A.$$

Property 1. (Row Equivalent Matrices) Let A, B, C be matrices, then

(1) If A is row equivalent to B, then B is row equivalent to A;

(2) If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.

Elementary Matrices

Theorem 2. (Equivalent Conditions for Nonsingularity) Let A be an $n \times n$ matrix. The following are equivalent:

(a) A is nonsingular;

(b) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$;

(c) A is row equivalent to I.

Proof. (a)⇒(b). If A is nonsingular and $\widehat{\boldsymbol{x}}$ is a solution to $A\boldsymbol{x} = \mathbf{0}$, then

$$\widehat{\boldsymbol{x}} = \boldsymbol{I}\widehat{\boldsymbol{x}} = \left(A^{-1}A\right)\widehat{\boldsymbol{x}} = A^{-1}(A\,\widehat{\boldsymbol{x}}) = A^{-1}\mathbf{0} = \mathbf{0}.$$

Thus $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

Elementary Matrices

Theorem 2. (Equivalent Conditions for Nonsingularity) Let A be an $n \times n$ matrix. The following are equivalent:

(a) A is nonsingular;

(b) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$;

(c) A is row equivalent to I.

Proof. (b)⇒(c). By elementary row operations, the system can be transformed into the form $U\boldsymbol{x} = \mathbf{0}$, where U is in row echelon form. If one of the diagonal elements of U were 0, the last row of U would consist entirely of 0's. But then $A\boldsymbol{x} = \mathbf{0}$ will have infinite nontrivial solutions. Thus U must be a triangular matrix with diagonal elements all equal to 1. Then I is the reduced row echelon form of A.

Elementary Matrices

Theorem 2. (Equivalent Conditions for Nonsingularity) Let A be an $n \times n$ matrix. The following are equivalent:

(a) A is nonsingular;

(b) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$;

(c) A is row equivalent to I.

Proof. (c)⇒(a). If A is row equivalent to I, there exist elementary matrices $E_1, E_2, \dots, E_k$ such that

$$A = E_k E_{k-1} \dots E_1 I = E_k E_{k-1} \dots E_1$$

Since $E_1, E_2 \dots, E_k$ are nonsingular, the product $E_k E_{k-1} \dots E_k$ is also invertible. Hence A is nonsingular. Finish.

Elementary Matrices

Theorem 2. (Equivalent Conditions for Nonsingularity) Let A be an $n \times n$ matrix. The following are equivalent:

(a) A is nonsingular;

(b) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$;

(c) A is row equivalent to I.

Remark. An $n \times n$ matrix A is singular **if and only if** $A\boldsymbol{x} = \mathbf{0}$ has nontrivial solutions.

Elementary Matrices

Corollary. The system of n linear equations in n unknowns $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution **if and only if** A is nonsingular.

Proof. If A is nonsingular, then $A^{-1}\boldsymbol{b}$ is the only solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Conversely, suppose that $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\widehat{\boldsymbol{x}}$. If A is singular, $A\boldsymbol{x} = \mathbf{0}$ has a solution $\boldsymbol{z} \neq \mathbf{0}$ by the Theorem. Let $\boldsymbol{y} = \widehat{\boldsymbol{x}} + \boldsymbol{z}$. Clearly, $\boldsymbol{y} \neq \widehat{\boldsymbol{x}}$ and

$$A\boldsymbol{y} = A(\widehat{\boldsymbol{x}} + \boldsymbol{z}) = A\,\widehat{\boldsymbol{x}} + A\boldsymbol{z} = \boldsymbol{b} + \mathbf{0} = \boldsymbol{b}.$$

Thus y is also a solution to $A\boldsymbol{x} = \boldsymbol{b}$, which is a contradiction.

Finding the Inverse Matrix

If A is **nonsingular**, then A is row equivalent to I, so there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \ldots E_1 A = I.$$

Multiplying both sides of this equation on the right by A^{-1}, we obtain

$$E_k E_{k-1} \ldots E_1 I = A^{-1}.$$

Thus the same series of elementary row operations that transform the nonsingular matrix A into I will transform I into A^{-1}.

This gives us a practical method for computing A^{-1}.

Finding the Inverse Matrix

$$E_k E_{k-1} \dots E_1 A = I.$$

$$E_k E_{k-1} \dots E_1 I = A^{-1}.$$

If we augment A by I, and perform the elementary row operations that transform A into I on the augmented matrix, then I will be transformed into A^{-1}.

That is, the reduced row echelon form of the augmented matrix $(A|I)$ will be $(I|A^{-1})$.

Finding the Inverse Matrix

Example. Compute A^{-1} for $A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$.

Solution.

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right).$$

Finding the Inverse Matrix

Example. Solve the system

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 12, \\ -x_1 - 2x_2 &= -12, \\ 2x_1 + 2x_2 + 3x_3 &= 8. \end{aligned}$$

Solution. The solution to the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -\frac{8}{3} \end{pmatrix}.$$

LU Factorization

If an $n \times n$ matrix A can be reduced to **strict upper triangular form** using **only** row operation III, then the reduction process can be represented in terms of a matrix factorization.

Example. $A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix}$. Using only row operation III, the matrix A can be transformed into $U = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}$.

Unit lower triangular

Strict upper triangular

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}.$$

***LU* factorization**

Review

- Partitioned matrices; block multiplication
- Elementary matrices; representing elementary row operations in terms of matrix multiplication; the method of finding inverse of a matrix

Preview

- The determinant of a matrix