Lecture 09

Chapter 3 Vector Spaces

3.6 Row Space and Column Space

3.6 Row Space and Column Space of Matrices

$$A = \begin{array}{cc} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{l} \boldsymbol{a}(1,:) \\ \boldsymbol{a}(2,:) \\ \\ \boldsymbol{a}(m,:) \end{array} \\ \begin{array}{cccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \quad & \boldsymbol{a}_n \end{array} & \end{array}$$

$\boldsymbol{a}(i,:), i = 1,2,\ldots,m$ are vectors in $\mathbf{R}^{1\times n}$, which are referred to as the **row vectors [行向量]** of A;

$\boldsymbol{a}_j, j = 1,2,\ldots n$ are vectors in $\mathbf{R}^m$, which are referred to as the **column vectors [列向量]** of A.

Spaces spanned by row vectors and column vectors lead to two **subspaces**.

Concepts and Examples

Definition 1. Let A be an $m \times n$ matrix.

- The subspace spanned by the **row vectors** of A is called the **row space [行空间]** of A.
- The subspace spanned by the **column vectors** of A is called the **column space [列空间]** of A.

Remark. The row space of A is a subspace of $\mathbf{R}^{1\times n}$ and the column space of A is a subspace of $\mathbf{R}^m$.

Example 1. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Find its row space and column space.

Row space. The two row vectors of A are $(1,0,0)$, $(0,1,0)$.

So the row space of A is the set of all 3-tuples of the form

$$\alpha(1,0,0) + \beta(0,1,0) = (\alpha, \beta, 0),$$

which is a two-dimensional subspace of $\mathbf{R}^{1\times 3}$.

Column space. The column vectors of A are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix},$

so the column space of A is the set of all vectors of the form

$$\alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

which is the whole space $\mathbf{R}^2$.

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$. Find its row space and column space.

Row space. The three row vectors of A are

$$\boldsymbol{a}(1,:) = (1,1,0), \qquad \boldsymbol{a}(2,:) = (1,0,1), \qquad \boldsymbol{a}(3,:) = (2,2,0)$$

The row space of A is $\mathrm{Span}\{\boldsymbol{a}(1,:), \boldsymbol{a}(2,:), \boldsymbol{a}(3,:)\}$.

Notice that the vectors $\boldsymbol{a}(1,:)$ and $\boldsymbol{a}(3,:)$ are linearly dependent, then we can remove one of them from the spanning set. Therefore, the row space of A is equal to $\mathrm{Span}\{\boldsymbol{a}(1,:), \boldsymbol{a}(2,:)\}$.

Since $\boldsymbol{a}(1,:)$ and $\boldsymbol{a}(2,:)$ are linearly independent, the vector set $E = \{\boldsymbol{a}(1,:), \boldsymbol{a}(2,:)\}$ is a basis of the row space of A.

Any vector $\boldsymbol{x}$ in the row space of A can be written as

$$\boldsymbol{x} = \alpha\boldsymbol{a}(1,:) + \beta\boldsymbol{a}(2,:) = (\alpha + \beta, \alpha, \beta).$$

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$. Find its row space and column space.

Column Space. The three column vectors of A are

$$\boldsymbol{a_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \qquad \boldsymbol{a_2} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad \boldsymbol{a_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

It is easy to see that the vector set

$$F = \{\boldsymbol{a_2}, \boldsymbol{a_3}\} = \left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

is a basis of the column space of A. So the column space of A can be written as span$\{\boldsymbol{a_2}, \boldsymbol{a_3}\}$. Any vector $\boldsymbol{y}$ in the column space of A can be written as

$$\boldsymbol{y} = \lambda \boldsymbol{a_2} + \mu \boldsymbol{a_3} = \begin{pmatrix} \lambda \\ \mu \\ 2\lambda \end{pmatrix}.$$

Linear System

Let us review the linear system $\quad A\boldsymbol{x} = \boldsymbol{b},$

where $\quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$

If we expand the left hand side of the linear system, we have

$$A\boldsymbol{x} = x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n}.$$

$A\boldsymbol{x}$ is a linear combination of column vectors of the matrix A.

Theorem 1. (Consistency Theorem for Linear System) A linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent **if and only if** $\boldsymbol{b}$ is in the column space of A.

Proof. The necessary condition is clear.

If $\boldsymbol{b}$ is in the column space of A, there exist scalars $x_1, x_2, \ldots, x_n$ such that

$$\boldsymbol{b} = x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n},$$

which implies that $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ is a solution of the linear system $A\boldsymbol{x} = \boldsymbol{b}$.

Remark. If $\boldsymbol{b}$ is replaced by $\boldsymbol{0}$, then $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$, **if and only if** the column vectors of A are linearly independent.

Rank of a Matrix

Definition 2. The dimension of **row space** of the $m \times n$ matrix A is called **rank [秩]**, which is denoted by $\operatorname{rank}(A)$ or $r(A)$.

- The rank of the matrix O is defined as 0.

Example. The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ has rank 2.

The matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$ has rank 2.

Theorem 2. Two row equivalent matrices have the same row space.

Recall: A is **row equivalent** to B if one can obtain B from A by a finite sequence of elementary row operations.

Elementary row operations: (I) $r_i \leftrightarrow r_j$; (II) λr_i; (III) $r_i + kr_j$.

Proof of the theorem. If B is row equivalent to A, then B can be formed from A by a finite sequence of row operations. Thus the row vector of B must be linear combinations of the row vectors of A. Consequently, the row space of B must be a subspace of the row space of A. Since A is row equivalent to B, by the same reasoning, the row space of A is a subspace of the row space B.

To determine the rank of a matrix, we can reduce the matrix to its **row echelon form** by using the Gauss elimination.

Example 2. The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$ can be reduced to

$U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ by performing elementary row operations.

It is easy to see that the rank of U is 2, since $(1,1,0), (0,1,-1)$ form a basis of the row space.

Since U and A are row equivalent, by **Theorem 2** they have the same row space. So the rank of A is 2.

Example 3. $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix} \longrightarrow U = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

We see that $\operatorname{rank}(A) = \operatorname{rank}(U) = 2$.

Remark. For a matrix U **in row echelon form**, the nonzero rows form a basis for the row space.

$$\operatorname{rank}(U) = \text{the number of its nonzero rows}.$$

Theorem 3. If A is an $n \times n$ invertible matrix, then

$$\operatorname{rank}(A) = n.$$

Proof. If A is an invertible matrix, it must be row equivalent to the $n \times n$ identity matrix. The rank of an $n \times n$ identity matrix is n. By **Theorem 2**, the rank of A is also n.

Remark. The rank defined in **Definition 2** is precisely called **rank of rows [行秩]**.

We can also define the **rank of columns [列秩]**, as the dimension of **column space** of the matrix A.

Question: What is the relation between the rank of rows and the rank of columns?

Column Space

Question: Will elementary row operations change the linear independency of two (column) vectors?

Precisely, let $\mathbf{w}, \mathbf{v}$ be two (column) vectors in $\mathbf{R}^m$. Let E be an $m \times m$ elementary matrix.

(1) If $\mathbf{w}$ and $\mathbf{v}$ are linearly dependent, will $E\mathbf{w}$ and $E\mathbf{v}$ be linearly dependent, too?

(2) If $\mathbf{w}$ and $\mathbf{v}$ are linearly independent, will $E\mathbf{w}$ and $E\mathbf{v}$ be linearly independent, too?

Answer to (1):

If $\mathbf{w}$ and $\mathbf{v}$ are linearly dependent, there exist scalars α, β not all zero such that

$$\alpha\mathbf{w} + \beta\mathbf{v} = \mathbf{0}.$$

Multiplying E to both side of the equality, we have

$$\alpha E\mathbf{w} + \beta E\mathbf{v} = E(\alpha\mathbf{w} + \beta\mathbf{v}) = E\mathbf{0} = \mathbf{0}$$

This implies that $E\mathbf{w}$ and $E\mathbf{v}$ are still linearly dependent.

Answer to (2):

Assume that **w** and **v** are linearly independent.

If there exist two scalars λ, μ such that

$$\lambda(E\mathbf{w}) + \mu(E\mathbf{v}) = \mathbf{0},$$

we then multiply E^{-1} to both side to obtain that

$$\lambda\mathbf{w} + \mu\mathbf{v} = E^{-1}\big(\lambda(E\mathbf{w}) + \mu(E\mathbf{v})\big) = E^{-1}\mathbf{0} = \mathbf{0}$$

Since **w** and **v** are linearly independent, then the only possible solution to the previous equation is $\lambda = \mu = 0,$

which implies that $E\mathbf{w}$ and $E\mathbf{v}$ are linearly independent, too.

Lemma 1. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ and $T = \{E\mathbf{v_1}, E\mathbf{v_2}, \ldots, E\mathbf{v_n}\}$, where E is an invertible matrix. Then we have

(1) If vectors in S are linearly dependent, then vectors in T are linearly dependent;

(2) If vectors in S are linearly independent, then vectors in T are linearly independent.

Elementary row operations do **NOT** change the linear (in)dependency of a set of vectors.

Theorem 4. Elementary row operations do not change the **dimension** of the column space of a matrix A.

Proof. Let E be the elementary matrix corresponding to an elementary operation. Let $A = (\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_n})$ where $\boldsymbol{a_j}$, $j = 1,2, \dots, n$, is the jth column of A. The column space of A is defined by

$$V = \mathrm{Span}\{\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_n}\}.$$

By multiplying E on the left side of A, we have

$$EA = (E\boldsymbol{a_1}, E\boldsymbol{a_2}, \dots, E\boldsymbol{a_n}),$$

the column space of the result matrix is

$$W = \mathrm{Span}\{E\boldsymbol{a_1}, E\boldsymbol{a_2}, \dots, E\boldsymbol{a_n}\}.$$

Theorem 4. Let A be an $m \times n$ matrix. Elementary row operations do not change the **dimension** of its column space.

Proof. (continue)

Since the linear dependency does not change by multiplying an invertible matrix, the **numbers** of vectors in the minimal spanning set of V and W must be equal.

That is, elementary operations do not change the dimension of its column space.

Example 2. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix} \to U = \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$$\text{Column Space } (A) \neq \text{Column Space } (U),$$

$$\dim(\text{Column Space } (A)) = 2 = \dim(\text{Column Space } (U)).$$

Remark. Elementary row operations **do** change the column space, but do **NOT** change the **dimension** of the column space.

Example 3. $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix} \longrightarrow U = \begin{pmatrix} \boxed{1} & 2 & 0 & 3 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

$$\dim(\text{Column Space }(A)) = \dim(\text{Column Space }(U)) = 2.$$

Relation between rank of rows and rank of columns

Let us review the process of Gauss elimination, which changes an $m \times n$ matrix A into a row echelon form U. Let

$$A = (\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_n}),$$

$$U = (\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_n})$$

be the corresponding **row echelon form** of A and $\boldsymbol{u_j}$ be the jth column vector of U. By **Theorem 4**,

$$\dim(\text{Column Space } (A)) = \dim(\text{Column Space } (U)).$$

- U is in row echelon form, so

  dim(Column Space (U))= the number of leading 1s.

- Elementary row operations do not change row space,

  dim(Row Space (A))=dim(Row Space (U))

  =the number of nonzero rows of U.

Theorem 5. Let A be an $m \times n$ matrix. The dimension of its row space is **equal** to the dimension of its column space.

Corollary 1. If A is an $m \times n$ matrix, then $\text{rank}(A) \leq \min\{m, n\}$.

Corollary 2. If A is an $m \times n$ matrix, then $\text{rank}(A) = \text{rank}(A^T)$.

Corollary 3. If A is an $m \times n$ matrix, P is an $m \times m$ invertible matrix and Q is an $n \times n$ invertible matrix, then

$$\text{rank}(PAQ) = \text{rank}(A).$$

Rank and Nullity Theorem

$A_{m\times n}$

- When considering the linear system $A\boldsymbol{x} = \mathbf{0}$, the rank of A is the number of lead variables.

- Recall the definition of nullspace $N(A) = \ker(A) = \{\boldsymbol{x} | A\boldsymbol{x} = \mathbf{0}\}$.

  The dimension of nullspace of A is called **nullity [零化度].**

  The dimension of $N(A)$ is the number of free variables.

Theorem 6. (The Rank-Nullity Theorem) If A is an $m \times n$ matrix, then

$$\operatorname{rank}(A) + \dim N(A) = n\,.$$

Example 4. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$. Find $N(A)$ and $\dim N(A)$,

and verify that $$\operatorname{rank}(A) + \dim N(A) = 3\,.$$

Solution. Consider the linear system $A\boldsymbol{x} = \mathbf{0}$. The augmented matrix of the system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right)$$

whose reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Example 4. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$. Find $N(A)$ and $\dim N(A)$,

and verify that $\quad \operatorname{rank}(A) + \dim N(A) = 3$.

Solution. (continue) Therefore x_3 is a free variable. Let $x_3 = \alpha$, then the solution of the linear system is

$$x = \begin{pmatrix} -\alpha \\ \alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

This implies that $\quad N(A) = \operatorname{Span}\{(-1,1,1)^T\}$

and $\dim N(A) = 1$.

Recall that $\operatorname{rank}(A) = 2$, so we see that

$$\operatorname{rank}(A) + \dim N(A) = 3.$$

Example 5. Let $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix}$. Find a basis for the row space of A and a basis for $N(A)$. Verify that

$$\dim N(A) = n - r.$$

Solution. The reduced row echelon form of A is given by

$$U = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\{(1,2,0,3), (0,0,1,2)\}$ is a basis for the row space of A, and A has rank $r = 2$.

Since the system $A\boldsymbol{x} = \mathbf{0}$ and $U\boldsymbol{x} = \mathbf{0}$ are equivalent, it follows that $\boldsymbol{x}$ is in $N(A)$ is and only if

$$\begin{aligned} x_1 + 2x_2 \qquad + 3x_4 &= 0 \\ x_3 + 2x_4 &= 0. \end{aligned}$$

Solution. (continue)

$$(U|\mathbf{0}) = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We see that x_2 and x_4 are free variables. Let $x_2 = \alpha$ and $x_4 = \beta$.

Then $N(A)$ consists of all vectors of form

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2\alpha - 3\beta \\ \alpha \\ -2\beta \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

The vectors $(-2,1,0,0)^T$ and $(-3,0,-2,1)^T$ form a basis for $N(A)$. Therefore,

$$n - r = 4 - 2 = 2 = \dim N(A).$$

Review

- Row space and column space of a matrix
- Rank of a matrix

Preview

- Cartesian Coordinate System
- Algebra in Euclidean Geometry