Ch 3.3 Fourier Transform

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Outline

- Fourier transform
  - Continuous-Time Nonperiodic Signals: The Fourier Transform (FT)
  - Discrete-Time Nonperiodic Signals: The Discrete-Time Fourier Transform (DTFT)

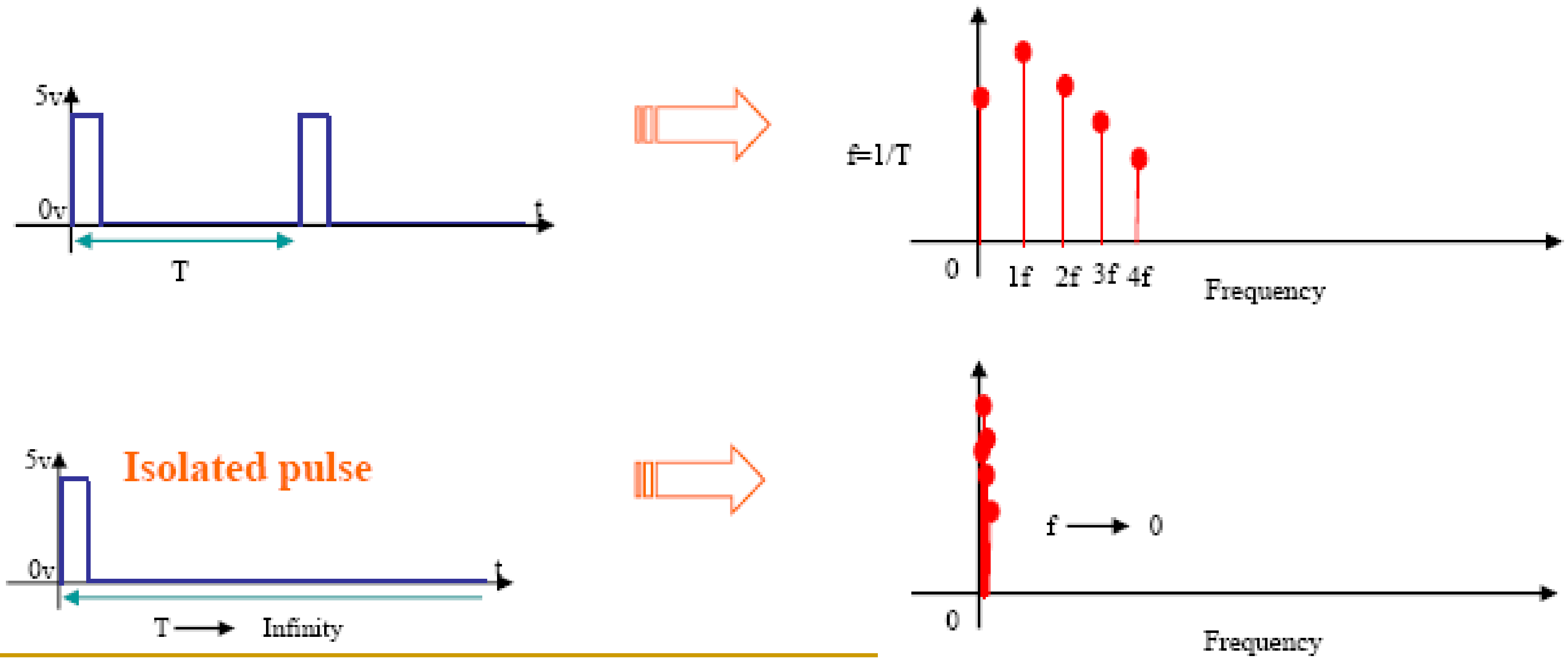
CT Nonperiodic Signals: The Fourier Transform

- The Fourier Series can only be applied to the periodic signals. However, the periodic signals are non-informational.
- Non-periodic signals cannot be analyzed using the Fourier series, the Fourier Transform (FT) is required.
- This is where the second way of representing signal in frequency-domain – as the sum of an uncountable infinity of sinusoids – becomes important and useful.

CT Nonperiodic Signals: The Fourier Transform

- We get to the Fourier Transform from the Fourier Series when we no longer have a periodic signal, e.g. when the period → infinite …

$$x(t)=\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, \quad X_n=\frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$



CT Nonperiodic Signals: The Fourier Transform

$$x(t)=\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, \quad X_n=\frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

when $T\to\infty,\ X_n\to 0,$ define $X(jn\omega_0)=TX_n$

$$\lim_{T\to\infty} X(jn\omega_0)=\lim_{T\to\infty} TX_n=\lim_{T\to\infty} T\cdot\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}dt$$

$T\to\infty$ ⇨ $\omega_0\to d\omega,\quad n\omega_0\to\omega$

$$\therefore \quad X(j\omega)=\lim_{T\to\infty} X(jn\omega_0)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t)=\lim_{T\to\infty}\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}=\lim_{T\to\infty}\sum_{n=-\infty}^{\infty}\frac{X(jn\omega_0)}{T}e^{jn\omega_0 t}$$

$$=\frac{1}{2\pi}\lim_{T\to\infty}\sum_{n=-\infty}^{\infty} X(jn\omega_0)e^{jn\omega_0 t}\omega_0=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

CT Nonperiodic Signals: The Fourier Transform

- **FT representation of time-domain signal $x(t)$:**

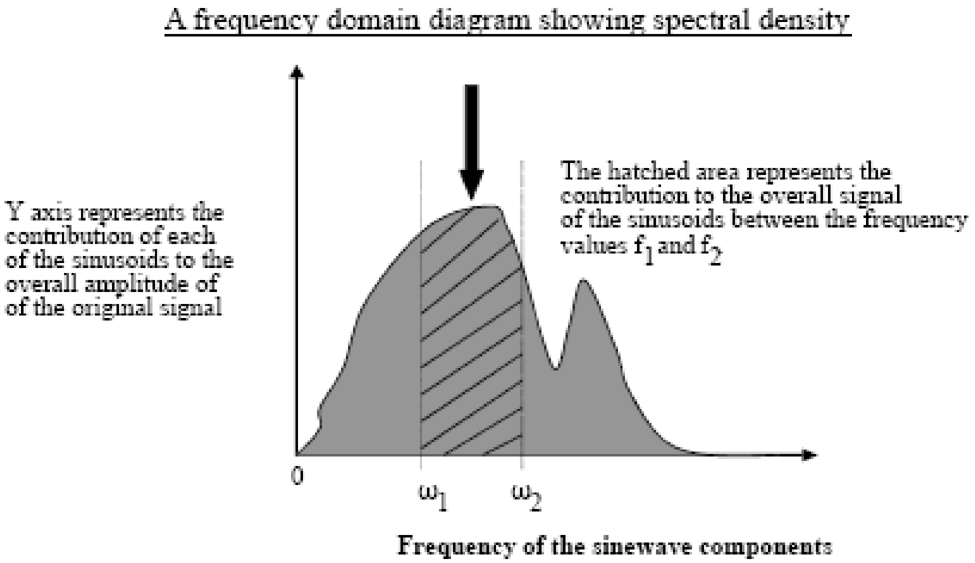
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$ **~ inverse Fourier Transform**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$ **~ Fourier Transform**

$X(j\omega)$: **spectrum or spectral density of $x(t)$**

- **Notation for FT pair:**

$$x(t) \xleftrightarrow{FT} X(j\omega).$$



CT Nonperiodic Signals: The Fourier Transform

- **Convergence condition for FT:**

Pointwise convergence of $x(t) = \hat{x}(t)$ **is guaranteed at all values of** t **except those corresponding to discontinuities if** $x(t)$ **satisfies the Dirichlet conditions for nonperiodic signals:**

- $x(t)$ **is absolutely integrable:**

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty.$$

- $x(t)$ **has a finite number of maximum, minima, and discontinuities in any finite interval.**
- **The size of each discontinuity is finite.**

CT Nonperiodic Signals: The Fourier Transform

Example 3.24 ***FT of A Real Decaying Exponential***

Find the FT of $x(t) = e^{-at}u(t)$, shown in Fig. 3.39(a).

<Sol.>

- For $a \le 0$, since $x(t)$ is not absolutely integrable, i.e.,

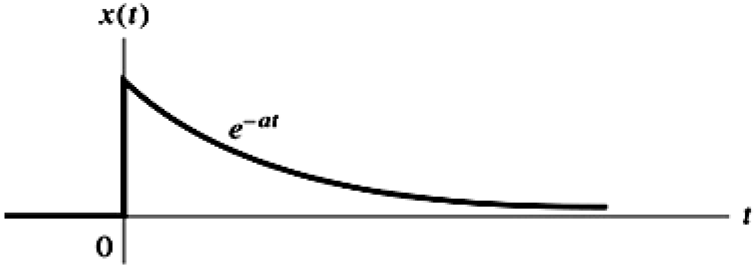
$$\int_0^{\infty} e^{-at}\,dt = \infty, \quad a \le 0 \quad \Rightarrow$$

The FT of $x(t)$ does not converge for $a \le 0$.

- For $a > 0$, the FT of $x(t)$ is

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_0^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= -\frac{1}{a+j\omega} e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$

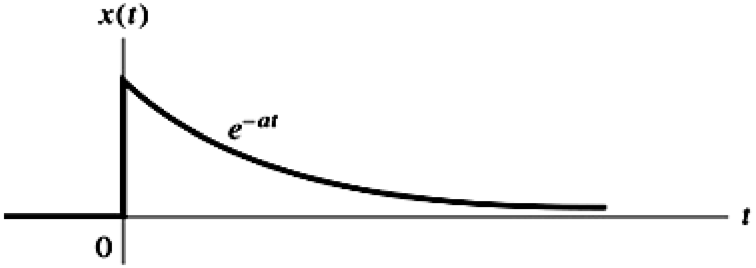


- Magnitude and phase of $X(j\omega)$:

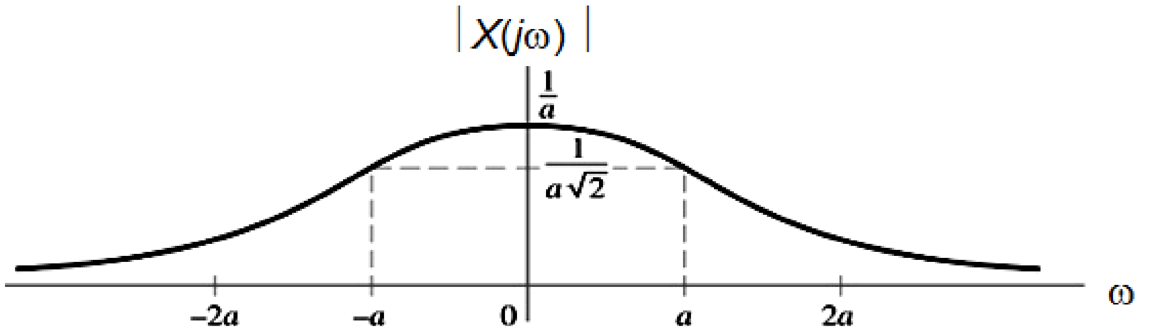
$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \qquad \arg\{X(j\omega)\} = -\arctan(\omega/a).$$

CT Nonperiodic Signals: The Fourier Transform

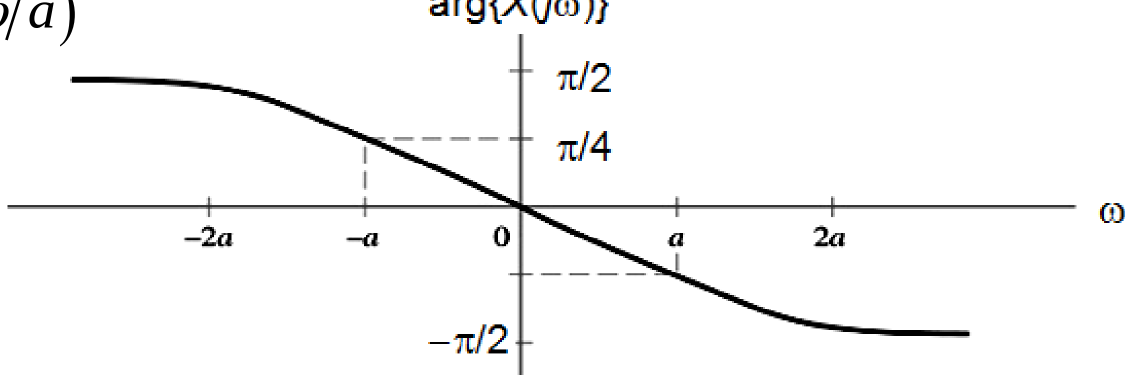
$$\mathbf{x(t) = e^{-a\,t}u(t)} \xleftrightarrow{FT} X(j\omega) = \frac{1}{a + j\omega}$$



$$\left|X(j\omega)\right| = \frac{1}{\sqrt{a^2 + \omega^2}}$$



$$\arg\left\{X(j\omega)\right\} = -\arctan(\omega/a)$$

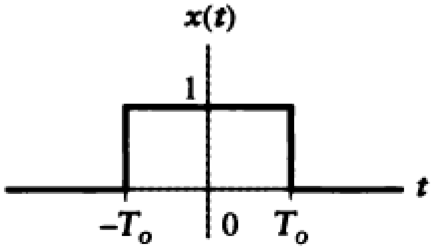


CT Nonperiodic Signals: The Fourier Transform

Example 3.25 ***FT of A Rectangular Pulse***

Consider the rectangular pulse depicted in Fig. 3.40 (a) and defined as

$$x(t)=\begin{cases}1, & -T_0<t<T_0\\ 0, & |t|>T_0\end{cases}$$



Find the FT of $x(t)$.

<Sol.>

- The rectangular pulse $x(t)$ is absolutely integrable, provided that $T_0<\infty$.

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-T_0}^{T_0}e^{-j\omega t}\,dt=-\frac{1}{j\omega}e^{-j\omega t}\Big|_{-T_0}^{T_0}=\frac{2}{\omega}\sin(\omega T_0),\quad \omega\neq 0$$

$$X(j0)=\int_{-T_0}^{T_0}e^{-j0t}\,dt=\int_{-T_0}^{T_0}1dt=2T_0$$

$$\lim_{\omega\to 0}\frac{2}{\omega}\sin(\omega T_0)=2T_0 \quad\Longrightarrow\quad X(j\omega)=\frac{2}{\omega}\sin(\omega T_0)=2T_0\,\mathrm{sinc}(\omega T_0/\pi)$$

CT Nonperiodic Signals: The Fourier Transform

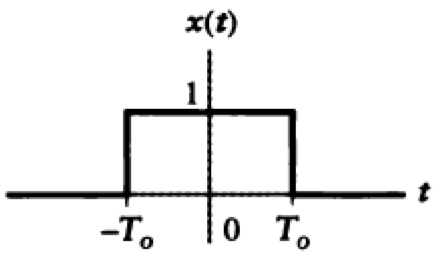
$$X(j\omega) = \frac{2}{\omega}\sin(\omega T_0) = 2T_0\,\mathrm{sinc}(\omega T_0/\pi)$$

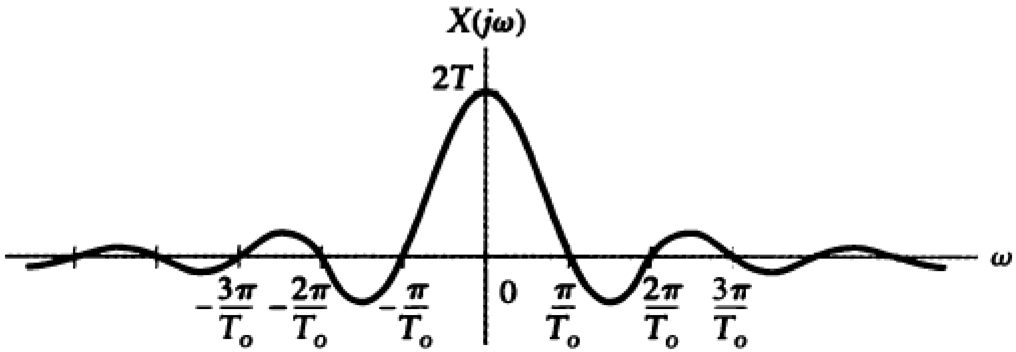
- Magnitude spectrum:

$$|X(j\omega)| = 2\left|\frac{\sin(\omega T_0)}{\omega}\right|$$

- Phase spectrum:

$$\arg\{X(j\omega)\} = \begin{cases} 0, & \sin(\omega T_0)/\omega > 0 \\ \pi, & \sin(\omega T_0)/\omega < 0 \end{cases}$$





As T_o increases, the nonzero time extent of $x(t)$ increases, while $X(j\omega)$ becomes more concentrated about the frequency origin.

CT Nonperiodic Signals: The Fourier Transform

Example 3.26 ***Inverse FT of A Rectangular Pulse***

Find the inverse FT of the rectangular spectrum depicted in Fig. 3. 42 (a) and given by

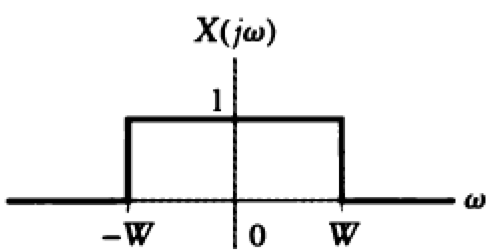
$$X(j\omega) = \begin{cases} 1, & -W < \omega < W \\ 0, & |\omega| > W \end{cases}$$

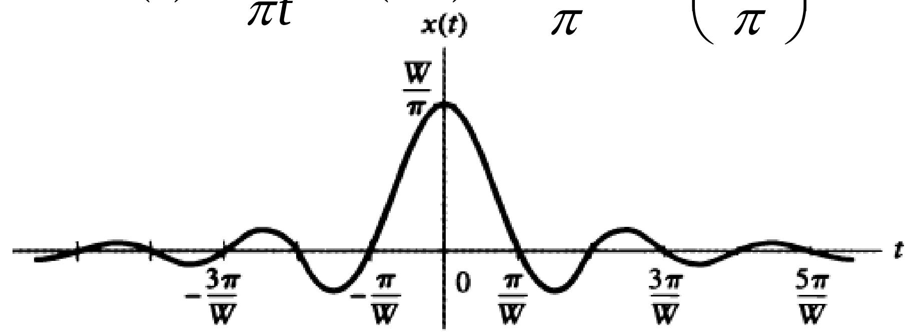
<Sol.>

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = -\frac{1}{j2\pi t}e^{j\omega t}\Big|_{-W}^{W} = \frac{1}{\pi t}\sin(Wt), \quad t \neq 0$$

$$x(0) = \frac{1}{2\pi}\int_{-W}^{W} e^{j0t}\,dt = \frac{1}{2\pi}\int_{-W}^{W} 1\,dt = \frac{W}{\pi}$$

$$\lim_{t\to 0}\frac{1}{\pi t}\sin(Wt) = W/\pi \quad \Longrightarrow \quad x(t) = \frac{1}{\pi t}\sin(Wt) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$





As W increases, $X(j\omega)$ becomes less concentrated about $\omega = 0$, while the time-domain representation $\mathbf{x(t)}$ becomes more concentrated about $t = 0$.

♣ Duality between Example 3.25 and 3.26.

CT Nonperiodic Signals: The Fourier Transform

Example 3.27 ***FT of The Unit Impulse***

Find the FT of $x(t) = \delta(t)$.

<Sol.>

- $x(t)$ does not satisfy the Dirichlet conditions, since the discontinuity at the origin is infinite.

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = e^{-j\omega 0} = 1 \Rightarrow \delta(t) \xleftrightarrow{FT} 1$$

Example 3.28 ***Inverse FT of An Impulse Spectrum***

Find the inverse FT of $X(j\omega) = 2\pi\,\delta(\omega)$.

<Sol.>

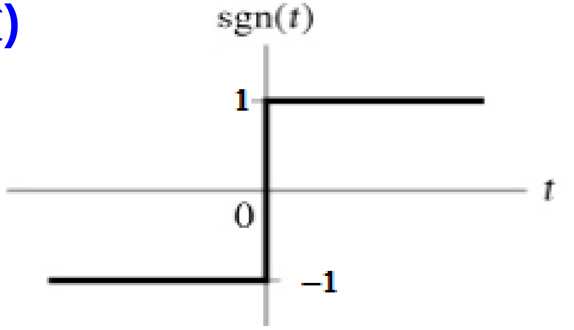
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t}\, d\omega = e^{-j0t} = 1 \Rightarrow 1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

♣ Duality between Example 3.27 and 3.28.

CT Nonperiodic Signals: The Fourier Transform

Example ***FT of The Signum Function* (符号函数)**

Find the FT of $\operatorname{sgn}(t) = \begin{cases} -1, & t<0 \\ 0, & t=0 \\ 1, & t>0 \end{cases}$



<Sol.> $$\operatorname{sgn}(t) = \lim_{\sigma\to 0}\left[e^{-\sigma t}u(t) - e^{\sigma t}u(-t)\right]$$

$$X(j\omega) = F\left[\operatorname{sgn}(t)\right] = \lim_{\sigma\to 0}\left[\int_0^{\infty} e^{-\sigma t}e^{-j\omega t}dt - \int_{-\infty}^{0} e^{\sigma t}e^{-j\omega t}dt\right]$$

$$= \lim_{\sigma\to 0}\left[-\frac{e^{-(\sigma+j\omega)t}}{\sigma+j\omega}\bigg|_0^{\infty} - \frac{e^{(\sigma-j\omega)t}}{\sigma-j\omega}\bigg|_{-\infty}^{0}\right]$$

$$= \lim_{\sigma\to 0}\left[\frac{1}{\sigma+j\omega} - \frac{1}{\sigma-j\omega}\right] = \frac{2}{j\omega}$$

DT Nonperiodic Signals: The DT Fourier Transform

- **The DTFT is used to represent a discrete-time nonperiodic signal as a superposition of complex sinusoids, which involve a continuum of frequencies on the interval $-\pi < \Omega \leq \pi$.**

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

~ inverse DTFT

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

~ Discrete-Time Fourier Transform (DTFT)

$X\left(e^{j\Omega}\right)$: **spectrum or spectral density of $x[n]$**

- **Notation for DTFT pair:**

$$x[n] \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right)$$

DT Nonperiodic Signals: The DT Fourier Transform

- **Condition for convergence of DTFT**
  - **If $x[n]$ is of finite duration and finite valued, then the DTFT converges.**
  - **If $x[n]$ is of infinite duration**
    - x[n] is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} \left| x\left[n\right] \right| < \infty \Rightarrow$$

The DTFT converges uniformly to a continuous function of Ω.

- x[n] is not absolutely summable, but has finite energy, i.e.,

$$\sum_{n=-\infty}^{\infty} \left| x\left[n\right] \right|^2 < \infty \Rightarrow$$

The DTFT converges in a mean-square error sense, but does not converge pointwise.

DT Nonperiodic Signals: The DT Fourier Transform

Example 3.17 ***DTFT of An Exponential Sequence***

Find the DTFT of the sequence $x[n] = \alpha^n u[n]$.

<Sol.> $$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n}$$

- **For $|\alpha| \geq 1$, the sum diverges.**
- **For $|\alpha| < 1$:** $$X(e^{j\Omega}) = \sum_{n=0}^{\infty}\left(\alpha e^{-j\Omega}\right)^n = \frac{1}{1-\alpha e^{-j\Omega}}$$

real valued α $\Longrightarrow$ $$X(e^{j\Omega}) = \frac{1}{1-\alpha\cos\Omega + j\alpha\sin\Omega}$$

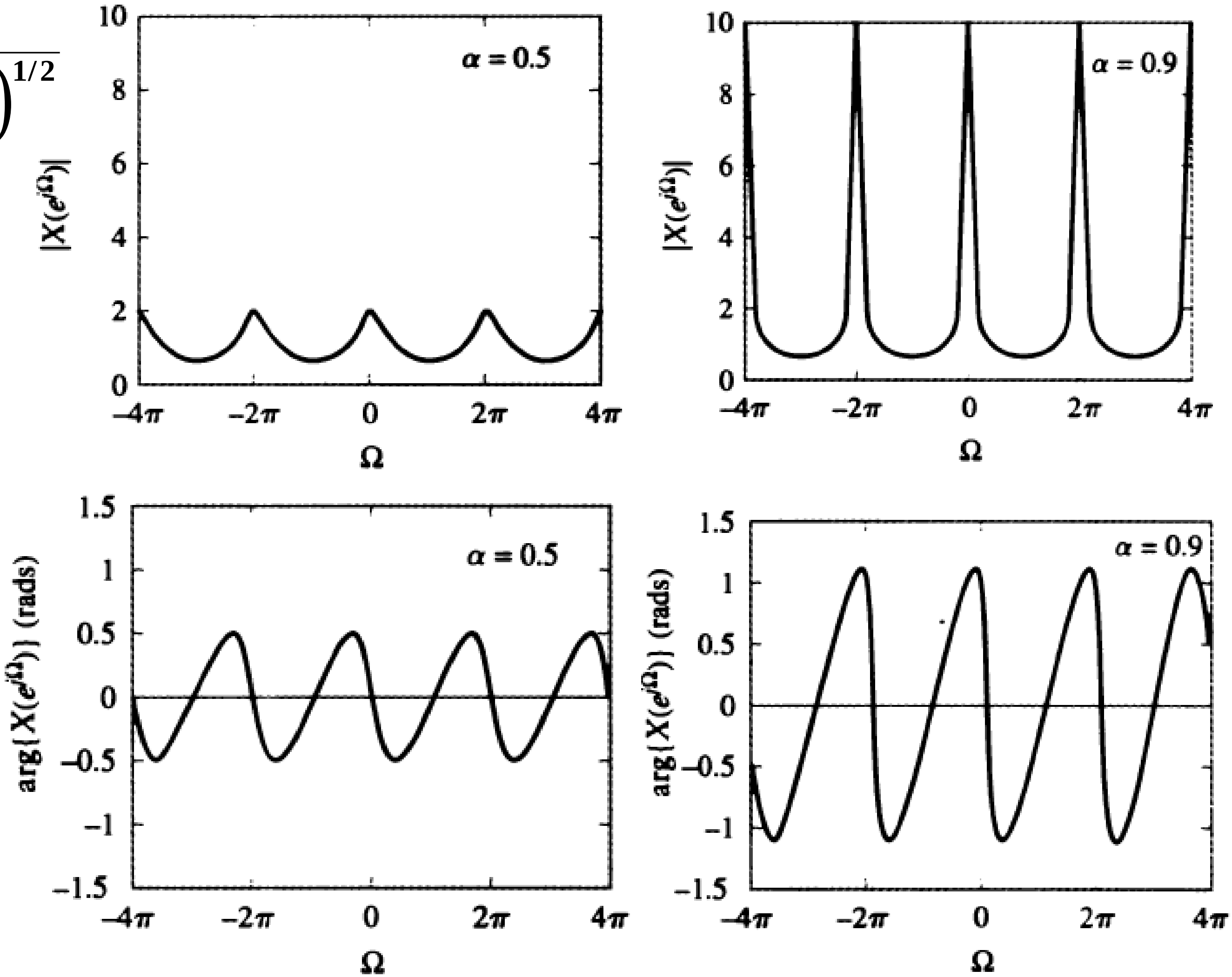
$$\left|X\left(e^{j\Omega}\right)\right| = \frac{1}{\left((1-\alpha\cos\Omega)^2 + \alpha^2\sin^2\Omega\right)^{1/2}} = \frac{1}{\left(\alpha^2 + 1 - 2\alpha\cos\Omega\right)^{1/2}}$$ **~ even function**

$$\arg\left\{X\left(e^{j\Omega}\right)\right\} = -\arctan\left(\frac{\alpha\sin\Omega}{1-\alpha\cos\Omega}\right)$$ **~ Odd function**

DT Nonperiodic Signals: The DT Fourier Transform

$$\left| X\left(e^{j\Omega}\right) \right| = \frac{1}{\left(\alpha^2 + 1 - 2\alpha\cos\Omega\right)^{1/2}}$$

$$\arg\left\{ X\left(e^{j\Omega}\right) \right\} = -\arctan\left(\frac{\alpha \sin\Omega}{1 - \alpha\cos\Omega} \right)$$



DT Nonperiodic Signals: The DT Fourier Transform

Example 3.18 ***DTFT of A Rectangular Pulse***

Let $x[n]=\begin{cases}1, & |n|\le M\\ 0, & |n|>M\end{cases}$ **as depicted in Fig. 3.30 (a). Find the DTFT of *x*[*n*].**

<Sol.> $$X\left(e^{j\Omega}\right)=\sum_{n=-M}^{M}1e^{-j\Omega n}=\sum_{m=0}^{2M}e^{-j\Omega(m-M)}=e^{j\Omega M}\sum_{m=0}^{2M}e^{-j\Omega m}$$

$$=e^{j\Omega M}\frac{1-e^{-j\Omega(2M+1)}}{1-e^{-j\Omega}},\qquad \Omega\neq 0,\ \pm 2\pi,\ \pm 4\pi,\ \ldots$$

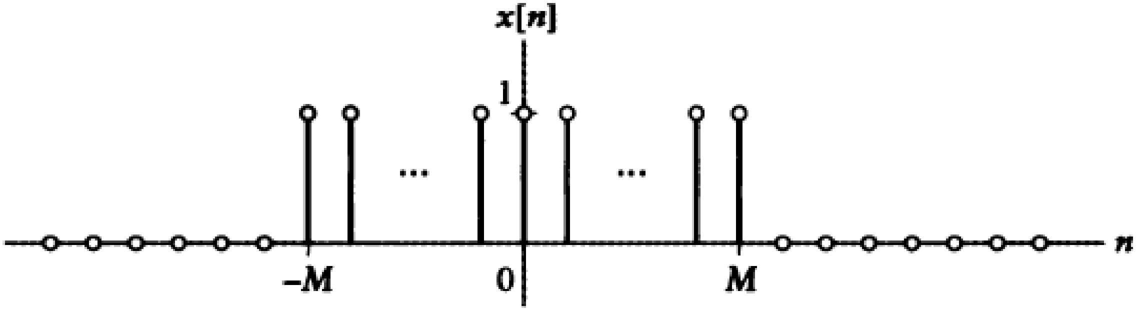
$$=e^{j\Omega M}\frac{e^{-j\Omega(2M+1)/2}\left(e^{j\Omega(2M+1)/2}-e^{-j\Omega(2M+1)/2}\right)}{e^{-j\Omega/2}\left(e^{j\Omega/2}-e^{-j\Omega/2}\right)}$$

$$=\frac{\left(e^{j\Omega(2M+1)/2}-e^{-j\Omega(2M+1)/2}\right)}{e^{j\Omega/2}-e^{-j\Omega/2}}=\frac{\sin\left(\Omega(2M+1)/2\right)}{\sin\left(\Omega/2\right)},\qquad \Omega\neq 0,\ \pm 2\pi,\ \ldots$$

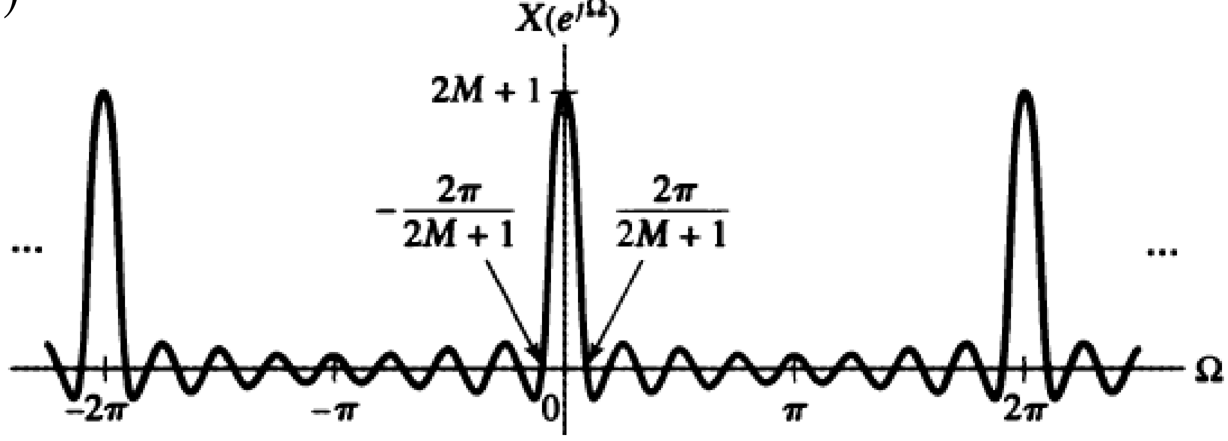
$$\lim_{\Omega\to 0,\pm 2\pi,\pm 4\pi,\ldots,}\frac{\sin\left(\Omega\left(2M+1\right)/2\right)}{\sin\left(\Omega/2\right)}=2M+1 \quad\Rightarrow\quad X\left(e^{j\Omega}\right)=\frac{\sin\left(\Omega\left(2M+1\right)/2\right)}{\sin\left(\Omega/2\right)}$$

DT Nonperiodic Signals: The DT Fourier Transform

$$x[n] = \begin{cases} 1, & |n| \le M \\ 0, & |n| > M \end{cases}$$



$$X(e^{j\Omega}) = \frac{\sin\left(\Omega(2M+1)/2\right)}{\sin\left(\Omega/2\right)}$$



DT Nonperiodic Signals: The DT Fourier Transform

Example 3.19 ***Inverse DTFT of A Rectangular Pulse***

Find the inverse DTFT of

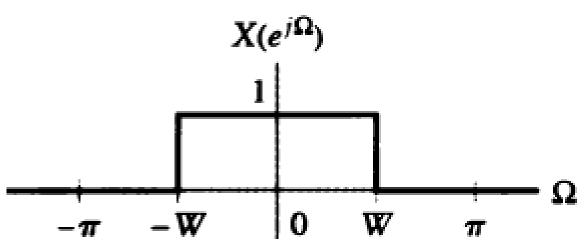
$$X\left(e^{j\Omega}\right)=\begin{cases}1, & |\Omega|\le W\\ 0, & W<|\Omega|<\pi\end{cases},$$

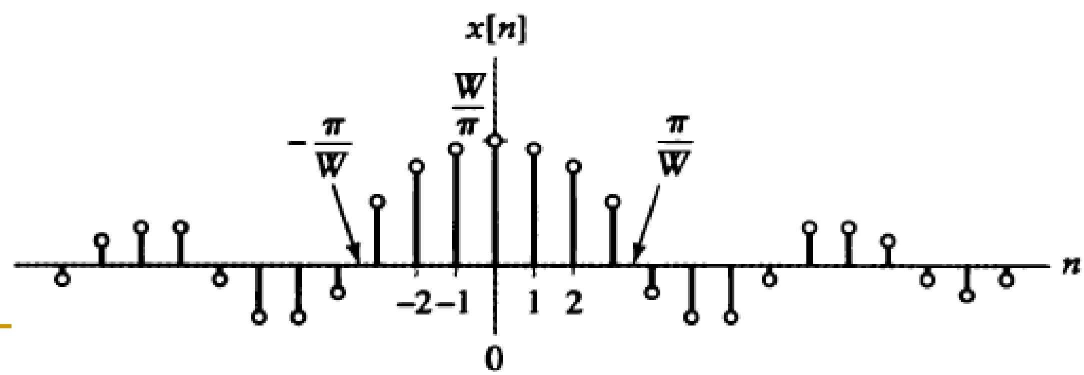
which is depicted in Fig. 3.31 (a).

<Sol.>

$$x[n]=\frac{1}{2\pi}\int_{-W}^{W}e^{j\Omega n}d\Omega=\frac{1}{2\pi nj}e^{j\Omega n}\Big|_{-W}^{W},\quad n\neq 0\quad=\frac{1}{\pi n}\sin(Wn),\quad n\neq 0.$$

$$\lim_{n\to 0}\frac{1}{n\pi}\sin(Wn)=\frac{W}{\pi}\quad\Rightarrow\quad x[n]=\frac{1}{\pi n}\sin(Wn)=\frac{W}{\pi}\operatorname{sinc}(Wn/\pi)$$



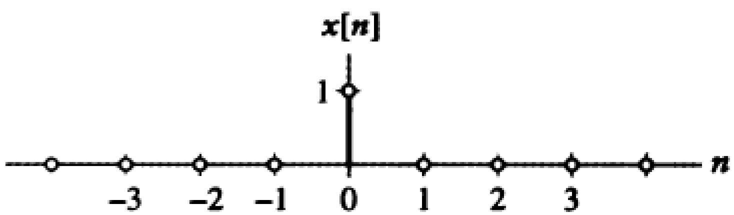


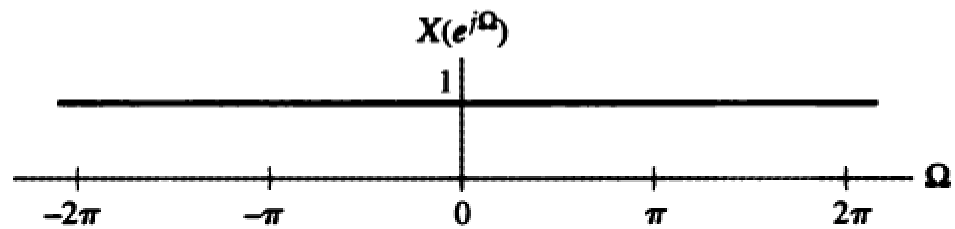
DT Nonperiodic Signals: The DT Fourier Transform

Example 3.20 ***DTFT of The Unit Impulse***

Find the DTFT of $x[n] = \delta[n]$.

<Sol.> $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = e^{-j\Omega 0} = 1$ $\Rightarrow$ $\delta[n] \xleftrightarrow{DTFT} 1.$





DT Nonperiodic Signals: The DT Fourier Transform

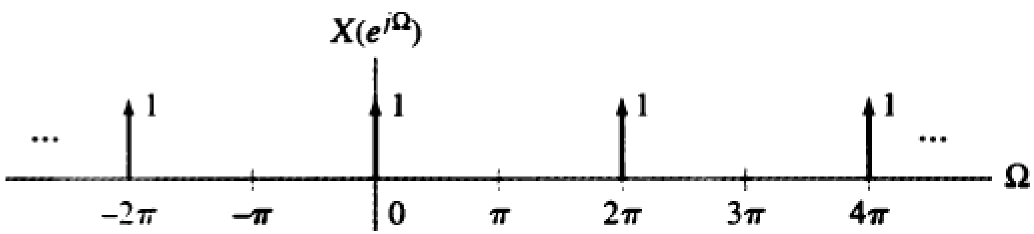
Example 3.21 ***Inverse DTFT of A Unit Impulse Spectrum***

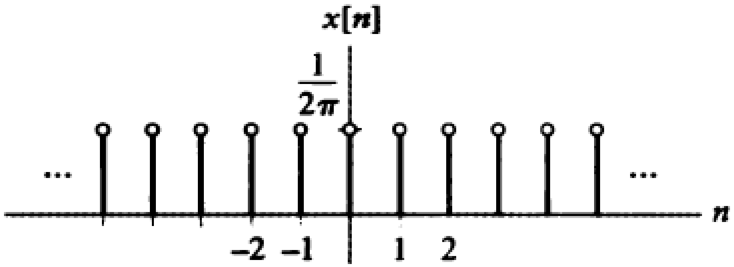
Find the inverse DTFT of $X(e^{j\Omega}) = \delta(\Omega),\ -\pi < \Omega \le \pi$.

<Sol.> **Alternatively, define** $X\left(e^{j\Omega}\right) = \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k2\pi\right)$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} e^{j\Omega n}\Big|_{\Omega=0} = \frac{1}{2\pi}$$

$$\Rightarrow \quad \frac{1}{2\pi} \xleftrightarrow{DTFT} \delta(\Omega), \qquad -\pi < \Omega \le \pi.$$





♣ **Dilemma:** **The DTFT of $x[n] = 1/(2\pi)$ does not converge, since it is not a square summable signal, yet $x[n]$ is a valid inverse DTFT!**

DT Nonperiodic Signals: The DT Fourier Transform

Example 3.22 *Moving-Average Systems: Frequency Response*

Consider two different moving-average systems described by the input-output equations

$$y_1[n]=\frac{1}{2}\big(x[n]+x[n-1]\big) \qquad \textbf{and} \qquad y_2[n]=\frac{1}{2}\big(x[n]-x[n-1]\big)$$

The first system averages successive inputs, while the second forms the difference. The impulse responses are

$$h_1[n]=\frac{1}{2}\delta[n]+\frac{1}{2}\delta[n-1] \qquad \textbf{and} \qquad h_2[n]=\frac{1}{2}\delta[n]-\frac{1}{2}\delta[n-1]$$

Find the frequency response of each system and plot the magnitude responses.

<Sol.> **Frequency response is the DTFT of the impulse response.**

- **For $h_1[n]$:** $H_1(e^{j\Omega})=\sum_{n=-\infty}^{\infty}h_1[n]e^{-j\Omega n}=\frac{1}{2}\sum_{n=-\infty}^{\infty}\{\delta[n]+\delta[n-1]\}e^{-j\Omega n}$

$$H(e^{j\Omega})=\sum_{k=-\infty}^{\infty}h[k]e^{-j\Omega k}$$

$$=\frac{1}{2}+\frac{1}{2}e^{-j\Omega}=e^{-j\frac{\Omega}{2}}\frac{e^{j\frac{\Omega}{2}}+e^{-j\frac{\Omega}{2}}}{2}=e^{-j\frac{\Omega}{2}}\cos\left(\frac{\Omega}{2}\right).$$

$$\left|H_1(e^{j\Omega})\right|=\left|\cos\left(\frac{\Omega}{2}\right)\right|, \qquad \arg\{H_1(e^{j\Omega})\}=-\frac{\Omega}{2}.$$

DT Nonperiodic Signals: The DT Fourier Transform

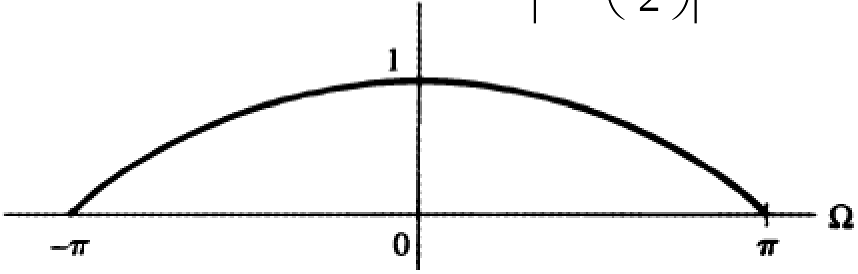
- **For $h_2[n]$:** $h_2[n]=\frac{1}{2}\delta[n]-\frac{1}{2}\delta[n-1]$

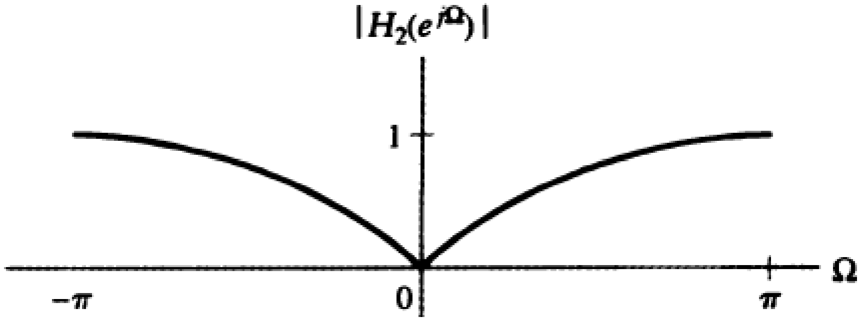
$$H_2(e^{j\Omega})=\frac{1}{2}\sum_{n=-\infty}^{\infty}\{\delta[n]-\delta[n-1]\}e^{-j\Omega n} = \frac{1}{2}-\frac{1}{2}e^{-j\Omega}$$

$$= je^{-j\frac{\Omega}{2}}\frac{e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}}{2j} = je^{-j\frac{\Omega}{2}}\sin\left(\frac{\Omega}{2}\right)$$

$$\left|H_2\left(e^{j\Omega}\right)\right|=\left|\sin\left(\frac{\Omega}{2}\right)\right|, \qquad \arg\{H_2(e^{j\Omega})\}=\begin{cases}\pi/2-\dfrac{\Omega}{2}, & \Omega>0\\ -\pi/2-\dfrac{\Omega}{2}, & \Omega<0\end{cases}$$

$$|H_1(e^{j\Omega})|=\left|\cos\left(\frac{\Omega}{2}\right)\right|$$





DT Nonperiodic Signals: The DT Fourier Transform

Example 3.23 ***Multipath Communication Channel: Frequency Response***

The input-output equation describing a discrete-time model of a two-path propagation channel is

$$y[n] = x[n] + ax[n-1]. \Rightarrow h[n] = \delta[n] + a\delta[n-1], \quad h^{inv}[n] = (-a)^n u[n]$$

The inverse system is stable provided that $|a| < 1$. Compare the magnitude responses of both systems for $a = 0.5\, e^{j\pi/3}$ and $a = 0.9\, e^{j2\pi/3}$.

<Sol.>

- **For $h[n]$:** $H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \{\delta[n] + a\delta[n-1]\} e^{-j\Omega n}$

$$= 1 + ae^{-j\Omega} = 1 + |a| e^{-j(\Omega - \arg\{a\})}$$

$$\because a = |a| e^{j\arg\{a\}}$$

$$= 1 + |a|\cos(\Omega - \arg\{a\}) - j|a|\sin(\Omega - \arg\{a\})$$

$$\Rightarrow \left|H(e^{j\Omega})\right| = \left(\left(1 + |a|\cos(\Omega - \arg\{a\})\right)^2 + |a|^2 \sin^2(\Omega - \arg\{a\})\right)^{1/2}$$

$$= \left(1 + |a|^2 + 2|a|\cos(\Omega - \arg\{a\})\right)^{1/2}$$

DT Nonperiodic Signals: The DT Fourier Transform

$$\left|H\left(e^{j\Omega}\right)\right| = \left(1 + |a|^2 + 2|a|\cos\left(\Omega - \arg\{a\}\right)\right)^{1/2}$$

$$\Rightarrow \quad \left|H\left(e^{j\Omega}\right)\right|_{\max} = 1 + |a|, \quad \text{when } \Omega = \arg\{a\}.$$

$$\left|H\left(e^{j\Omega}\right)\right|_{\min} = 1 - |a|, \quad \text{when } \Omega = \arg\{a\} - \pi.$$

♣ **If $|a| = 1$, then the multipath model applies zero gain to any sinusoid with frequency $\Omega = \arg\{a\} - \pi$.**

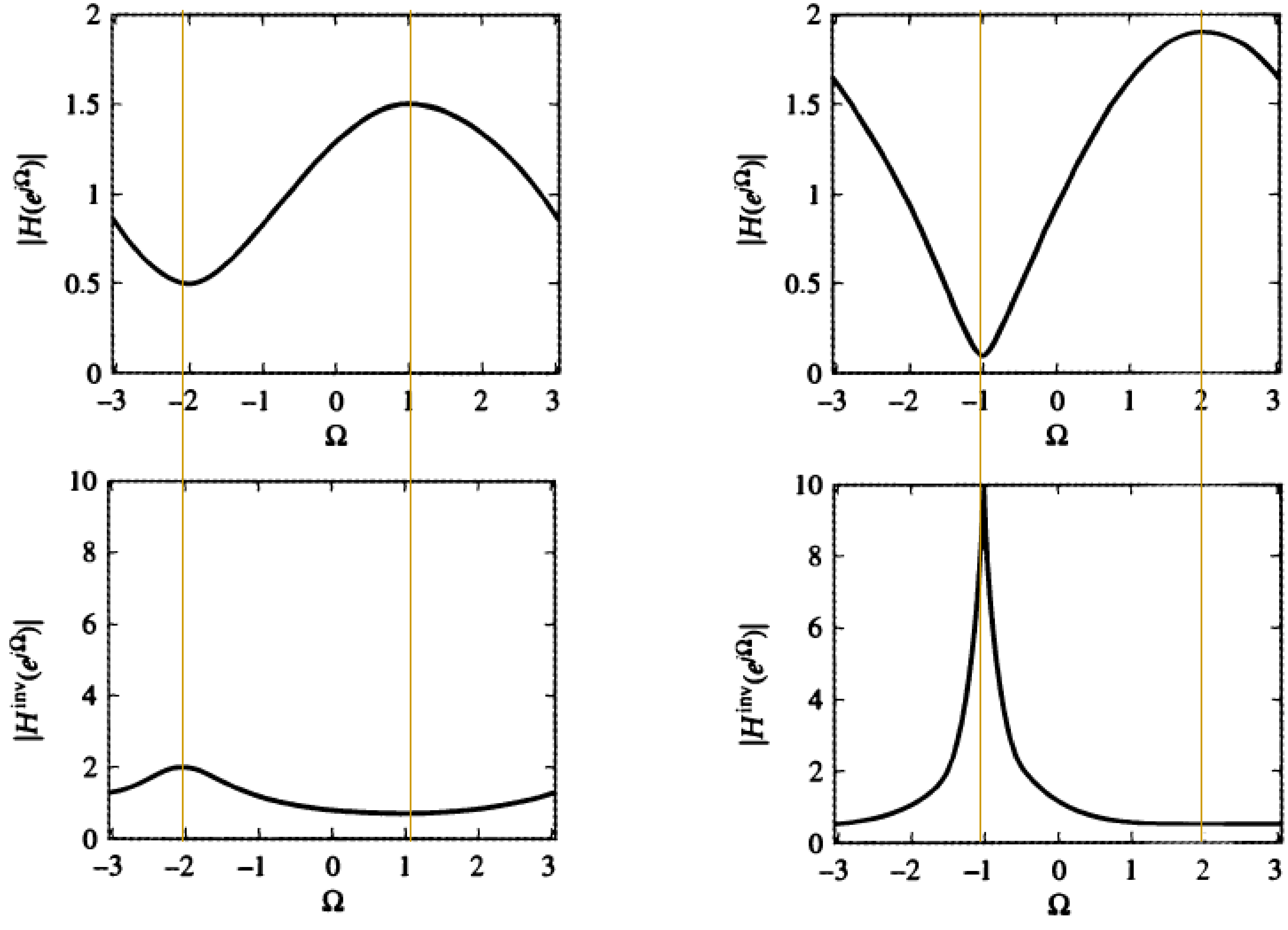
- **For** $h^{inv}[n] = (-a)^n u[n]$

$$a^n u[n] \xleftrightarrow{DTFT} \frac{1}{1 - ae^{-j\Omega}}, \quad |a| < 1 \quad \Rightarrow \quad H^{inv}\left(e^{j\Omega}\right) = \frac{1}{1 + ae^{-j\Omega}}, \quad |a| < 1$$

$$H^{inv}\left(e^{j\Omega}\right) = \frac{1}{H\left(e^{j\Omega}\right)} \quad \Rightarrow \quad \left|H^{inv}\left(e^{j\Omega}\right)\right| = \frac{1}{\left(1 + |a|^2 + 2|a|\cos\left(\Omega - \arg\{a\}\right)\right)^{1/2}}$$

♣ **The multipath system cannot be inverted when $|a| = 1$.**

DT Nonperiodic Signals: The DT Fourier Transform



(a) Echo coefficient $a = 0.5e^{j\pi/3}$

(b) Echo coefficient $a = 0.9e^{j2\pi/3}$

Summary

- **Fourier transform**
  - Continuous-Time Nonperiodic Signals: The Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega, \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  - Discrete-Time Nonperiodic Signals: The Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega, \quad X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

- Reference in textbook: 3.6, 3.7
- Homework: 3.54(a,d,f), 3.55(a,c,e); 3.52(b,c,e), 3.53(a,d,f)

Fourier Transform for Elementary Signals

$$\delta(t) \xleftrightarrow{FT} 1 \qquad 1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

$$\operatorname{sgn}(t) \xleftrightarrow{FT} \frac{2}{j\omega}$$

$$u(t) \xleftrightarrow{FT} \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\cos(\omega_0 t) \xleftrightarrow{FT} \pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]$$

$$\sin(\omega_0 t) \xleftrightarrow{FT} j\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]$$

$$x(t) = e^{-at}u(t) \xleftrightarrow{FT} X(j\omega) = \frac{1}{a+j\omega}$$

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & |t| > T_0 \end{cases} \xleftrightarrow{FT} X(j\omega) = \frac{2}{\omega}\sin(\omega T_0)$$

$$x(t) = \frac{1}{pt}\sin(Wt) \xleftrightarrow{FT} X(j\omega) = \begin{cases} 1, & -W < \omega < W \\ 0, & |\omega| > W \end{cases}$$

DTFT for Elementary Signals

$$\delta[n] \xleftrightarrow{DTFT} 1.$$

$$x[n] = \alpha^n u[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \frac{1}{1-\alpha e^{-j\Omega}}$$

$$x[n] = \begin{cases} 1, & |n| \le M \\ 0, & |n| > M \end{cases} \xleftrightarrow{DTFT} X(e^{j\Omega}) = \frac{\sin\left(\Omega(2M+1)/2\right)}{\sin\left(\Omega/2\right)}$$

$$x[n] = \frac{1}{\pi n}\sin(Wn) \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) = \begin{cases} 1, & |\Omega| \le W \\ 0, & W < |\Omega| < \pi \end{cases}$$