Lecture 2

Chapter 1 Equation Systems and Matrices

1.3 Reduced Row Echelon Form

1.4 Consistency of Linear Systems

Review of last lecture

Example. Consider the linear system

$$\begin{aligned} 2x_1 + 4x_2 + \ \ x_3 &= 1, \\ x_1 + 2x_2 + 2x_3 &= 2, \\ 3x_1 + 5x_2 - \ \ x_3 &= 3. \end{aligned}$$

Solution. The augmented matrix is

$$\textbf{pivot } \boldsymbol{a_{11} = 2} \quad \left(\begin{array}{ccc|c} \boxed{2} & 4 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 3 & 5 & -1 & 3 \end{array}\right) \leftarrow \textbf{pivotal row}$$

After eliminating the entries below the pivot, we have

$$\cdots \xrightarrow{r_2+\left(-\frac{1}{2}\right)r_1,\, r_3+\left(-\frac{3}{2}\right)r_1} \left(\begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -1 & -\frac{5}{2} & \frac{3}{2} \end{array}\right)$$

←the second row cannot be taken as pivot row since $\boldsymbol{a_{22} = 0}$

Review of last lecture

Example. Consider the linear system

$$\begin{aligned} 2x_1 + 4x_2 + x_3 &= 1, \\ x_1 + 2x_2 + 2x_3 &= 2, \\ 3x_1 + 5x_2 - x_3 &= 3. \end{aligned}$$

Solution. (continue) We interchange the second and third row of the matrix and get

$$\left(\begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -1 & -\frac{5}{2} & \frac{3}{2} \end{array}\right) \xrightarrow{r_2 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 0 & \boxed{-1} & -\frac{5}{2} & \frac{3}{2} \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \end{array}\right)$$

The matrix represents a strict triangle linear system, so we can stop. The solution is $(8, -4, 1)$ by back substitution.

Review of last lecture

In the last lecture, we had seen that an $n \times n$ system may be reduced to the triangular form, by using the three **elementary row operations** applied on the augmented matrix.

But this progress may be broken if, at any step, all the possible choices for a pivot element are 0.

Since the progress of reducing a system to the triangular form is a progress of eliminating variables, at the stage of the reduction breaks down, it seems natural to move over the next column and eliminate the rest variables. By doing this, it is clear that the equation system can not be reduced to triangular form.

1.3 Reduced Row Echelon Form

The process of solving a linear system can be greatly simplified by using the notations of matrices and the solving process is equivalent to the process of reducing the augmented matrix of a linear system into another form of an equivalent system.

Row Echelon Form

Example 1. Consider the system represented by the augmented matrix

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right). \quad \begin{array}{l} \leftarrow \textbf{pivotal row} \\ \\ \\ \\ \\ \end{array}$$

Solution. Using row operation III to eliminate the nonzero entries in the last four rows of the first column, the resulting matrix will be

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right) \quad \begin{array}{l} \\ \leftarrow \textbf{pivotal row} \\ \\ \\ \\ \end{array}$$

All possible choices of pivot are 0 in the second column!

Row Echelon Form

Solution. (continue) It is nature to continue our reduction process from the next column.

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right) \begin{array}{l} \\ \leftarrow \textbf{pivotal row} \\ \\ \\ \\ \end{array}$$

Continue the elimination process, we have

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} \\ \\ \leftarrow \textbf{pivotal row} \\ \\ \\ \end{array}$$

Row Echelon Form

Solution. (continue) We end up with

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right).$$

The last two rows mean that

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = -4,$$
$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = -3.$$

Since there are no 5-tupes that could possibly satisfy these equations, the system has no solution.

Row Echelon Form

Solution'. (continue) If we change the right hand of the equations, so as to obtain a system has a solution set, such as

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right),$$

then the process will yield the augmented matrix

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Will be satisfied for any 5-tuple.

Row Echelon Form

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solution'. (continue) and the solution set will be the solution set of all 5-tuples satisfying the first three equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + \ \ x_5 &= 1, \\ x_3 + x_4 + 2x_5 &= 0, \\ x_5 &= 3. \end{aligned}$$

Definition 1. (Lead and Free Variables) The variables corresponding the first nonzero elements in each row of the augmented matrix are referred to as the **lead variables [首变量],** and the remaining variables corresponding to the columns skipped in the reduction process are referred to as the **free variables [自由变量].**

In the above system, the lead variables are x_1, x_3 and x_5, and the free variables are x_2 and x_4.

Row Echelon Form

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solution'. (continue) Take the free variables $x_2 = \alpha$, $x_4 = \beta$, then

$$\begin{aligned} x_1 + x_3 + \ x_5 &= 1 - \alpha - \beta, \\ x_3 + 2x_5 &= -\beta, \\ x_5 &= 3. \end{aligned}$$

By back substitution, the solution to the system is $(4 - \alpha, \alpha, -\beta - 6, \beta, 3)$, where α, β can be taken as any real numbers.

Row Echelon Form

Definition 2. (Row Echelon Form) A matrix is said to be in **row echelon form [行阶梯形]** if

(i) in every nonzero row, the first nonzero entry (counting from left to right) is 1;

(ii) if row k is a nonzero row, the number of leading zero entries in row $k+1$ must be greater than the number of leading zero entries in row k;

(iii) if there are rows whose entries are all zeros, they must be below all nonzero rows.

Examples. Row Echelon form

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Row Echelon Form

Examples. Matrices not in row echelon form

$$\begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{pmatrix}$$ does not satisfy condition (i);

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$ fails to satisfy condition (iii);

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ fails to satisfy condition (ii).

Gauss Elimination

Definition 3. (Gauss Elimination) The process of using **elementary row operations I, II, III** to transform a linear system into anther one whose augmented matrix is in **row echelon form** is called **Gaussian elimination [高斯消元法].**

Remark. By Gauss elimination, we can transform a linear system equivalently into a system of row echelon form, and the linear system is consistent if and only if there is no row in the row echelon form like

$$(0 \ \ 0 \ \ \ldots \ \ 0 \,|\, 1),$$

since the corresponding equation of this row is

$$0 = 1,$$

and this equation never holds.

Reduced Row Echelon Form

Example 1'. Let us continue the discussion on

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Row Echelon Form

We can do more steps to change the row echelon form into a simpler one.

$$\cdots \xrightarrow{r_2-2r_3, r_1-r_3} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-r_2} \left(\begin{array}{ccccc|c} \boxed{1} & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & \boxed{1} & 1 & 0 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced Row Echelon Form

Example 1'. (Continue)

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & \boxed{1} & 1 & 0 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The final augmented matrix represents the following linear system

$$\begin{aligned} x_1 + x_2 \qquad\qquad\qquad &= 4, \\ x_3 + x_4 \qquad &= -6, \\ x_5 &= 3. \end{aligned}$$

This linear system is much better since there is no need to use back substitution to find out the solution.

Reduced Row Echelon Form

Definition 4. (Reduced Row Echelon Form) A matrix is said to be in **reduced row echelon form [最简行阶梯形]** if it satisfies the following conditions:

(1) The matrix is in row echelon form;

(2) The first nonzero entry in each row is the only nonzero entry in its column.

Examples. Which of the following matrices are in reduced row echelon form?

$$(1)\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 2\end{pmatrix},\ (2)\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix},\ (3)\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\ (4)\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 0\end{pmatrix},\ (5)\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Reduced Row Echelon Form

Definition 5. (Gauss-Jordan Elimination and Reduction) The process of using elementary row operations to transform a linear system into a linear system with augmented matrix in **reduced row echelon form** is called **Gauss-Jordan Elimination [高斯-若尔当消元法].**

1.4 Consistency of Linear Systems

All linear systems can be classified as *overdetermined system* or *underdetermined system*.

Overdetermined Systems

Definition 1. (Overdetermined System) A linear system is called **overdetermined [超定的]** if there are more equations than unknowns.

Example.

$$
\begin{aligned}
x_1 + x_2 &= 2, \\
x_1 - x_2 &= 1, \\
x_1 \phantom{{}+x_2} &= 4.
\end{aligned}
$$

Remark. Overdetermined systems are usually (but not always) inconsistent.

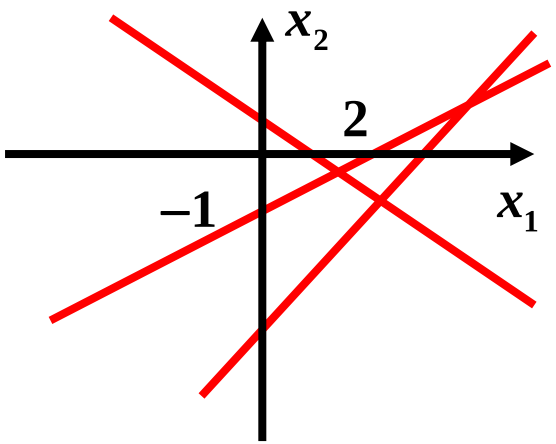
Overdetermined Systems

Example 1. Find solution of

$$\begin{aligned} x_1 + \quad x_2 &= 1 \\ x_1 - \quad x_2 &= 3 \\ -x_1 + 2x_2 &= -2 \end{aligned}$$

Solution. By Gauss Elimination, the augmented matrix of this system can be transformed to echelon form

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right)$$



The last row means that the system is inconsistent.

Overdetermined Systems

Example 2. Find solution(s) of

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 2x_1 - x_2 + 3x_3 &= 5 \end{aligned}$$

Solution. The augmented matrix of this system can be transformed into

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 0.2 & 0 \\ 0 & 0 & 1 & 1.5 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It is clear that the system is consistent with solution $(0.1, -0.3, 1.5)$.

Underdetermined Systems

Definition 2. (Underdetermined System) A linear system is called **underdetermined [不定的]** if there are fewer equations than unknowns.

Example.

$$\begin{aligned} x_1 - x_2 + x_3 &= 2, \\ 2x_1 + x_2 - x_3 &= 4. \end{aligned}$$

Remark. Underdetermined systems are usually (but not always) consistent with infinite solutions.

Underdetermined Systems

Example 3. Find solutions of

$$x_1 + 2x_2 + x_3 = 1$$

$$2x_1 + 4x_2 + 2x_3 = 3$$

Solution. The augmented matrix of this system can be transformed to

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

It is clear that the last row of the augmented matrix shows that this system is inconsistent or has no solution.

Underdetermined Systems

Example 4. Find solution(s) of

$$\begin{aligned} x_1 + x_2 + x_3 + \ \ x_4 + \ \ x_5 &= 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 2 \end{aligned}$$

Solution. The augmented matrix of this system can be transformed into

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

It is clear that this system is consistent with infinite many solutions, since there are two free variables.

Underdetermined Systems

Solution. (continue) We can choose the last row as pivotal row and the first nonzero entry as pivot element to eliminate the all entries above the pivot element, to get its **reduced row echelon form**

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right).$$

By the last matrix, it is clear that the solution of this system is

$$(1-\alpha-\beta, \alpha, \beta, 2, -1),$$

where α and β are any real numbers.

Homogeneous Systems

Definition 3. (Homogeneous Systems) A linear system is **homogeneous [齐次的]** if all its right hand sides are zeros.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0,$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0,$$
$$\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0,$$

where $a_{ij}, i = 1,2, \dots, m, j = 1,2, \dots, n$ are numbers.

Remark. Homogeneous systems are always consistent, since $(0,0, \dots, 0)$ is always a solution of these systems. This solution of all zero entries of a homogeneous system is called **trivial solution [平凡解];** otherwise, the solution is called **nontrivial solution [非平凡解].**

Homogeneous Systems

Example 5. Use Gauss-Jordan elimination to solve the system

$$-x_1 + x_2 - x_3 + 3x_4 = 0$$

$$3x_1 + x_2 - x_3 - x_4 = 0$$

$$2x_1 - x_2 - 2x_3 - x_4 = 0$$

Solution.

$$\left(\begin{array}{rrrr|r} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

Row echelon form

$$\to \left(\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

Reduced row echelon form

The solution to this system is $(\alpha, -\alpha, \alpha, \alpha)$, where α is any real number.

Homogeneous Systems

Theorem 1. (Nontrivial Solution of Underdetermined Homogeneous Systems) An $m \times n$ homogeneous system has a nontrivial solution if $n > m$.

Proof. A homogeneous system is always consistent. The row echelon form of the matrix can have at most m nonzero rows. Thus there are at most m lead variables.

Since $n > m$, there must be some free variables. The free variables can be assigned arbitrary values. For each assignment of values to the free variables, there is a solution to the system. In particular, we take one of these free variables as value 1 and the others as value 0, we will obtain a nonzero solution.

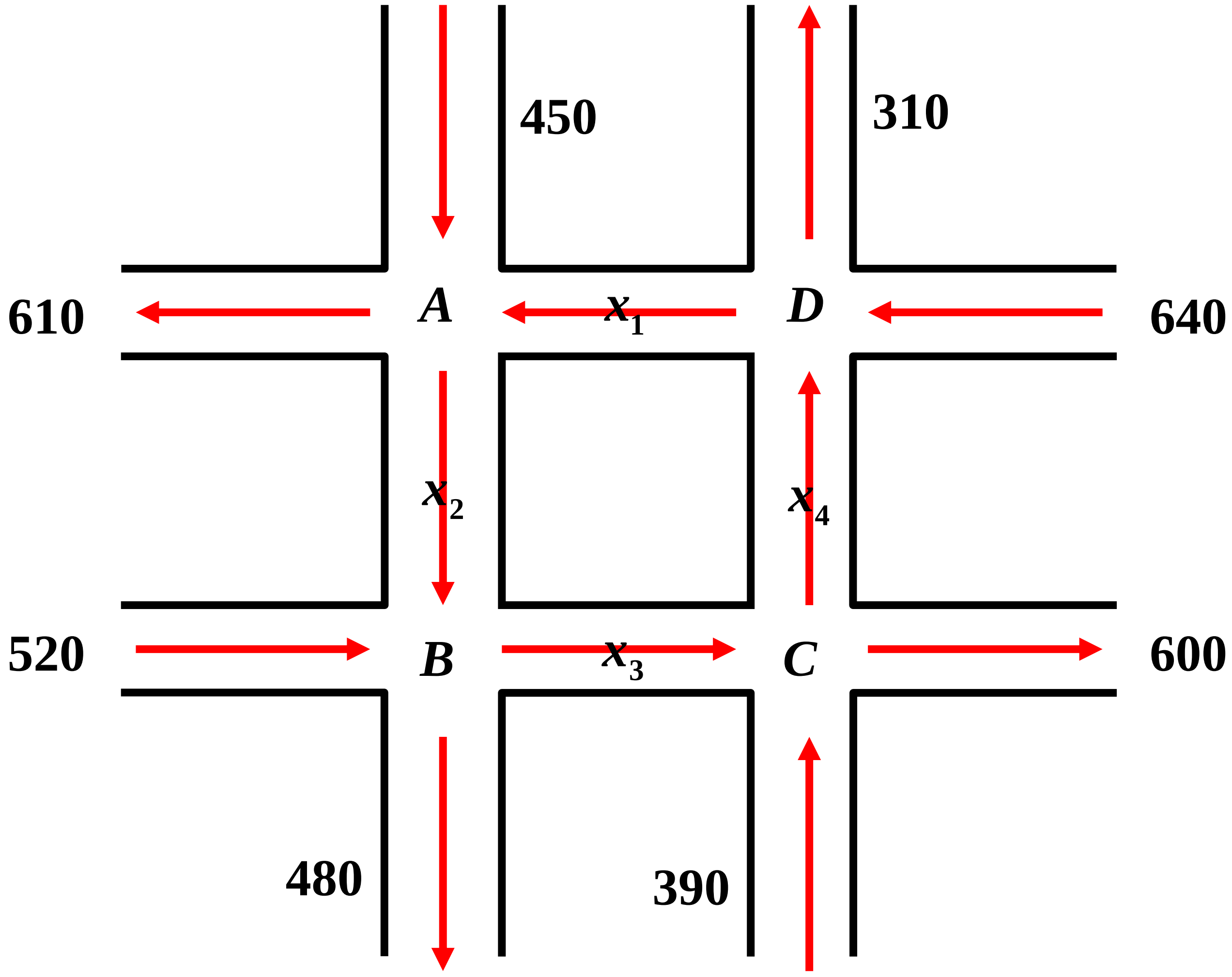
Homogeneous Systems

Remark. The key point in the proof is the fact that there exist **free variables** and all leading variables can be represented as a combination of free variables.

The assumption $n > m$ is in fact a sufficient condition but not necessary.

Application: Traffic Flow

Question:
Try to find the amount of traffic between each of the four intersections.



Application: Traffic Flow

Question: Try to find the amount of traffic between each of the four intersections.

Solution. At each intersection, the number of automobiles entering must be equal to the number leaving. So we have

$$\begin{aligned} x_1 + 450 &= x_2 + 610 \quad \text{(intersection A)} \\ x_2 + 520 &= x_3 + 480 \quad \text{(intersection B)} \\ x_3 + 390 &= x_4 + 600 \quad \text{(intersection C)} \\ x_4 + 640 &= x_1 + 310 \quad \text{(intersection D)} \end{aligned}$$

This is an equation system with four equations and four variables. The answer of the question is to find the solution of the linear system.

Application: Traffic Flow

Solution. (continue) The augmented matrix of this system is

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{array}\right)$$

and the reduced row echelon form is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 330 \\ 0 & 1 & 0 & -1 & 170 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

It is clear that this system has infinite many solutions. But if the amount of traffic were known between any pair of intersections, the traffic on the remaining arteries could be easily calculated.

Review

- Row Echelon Form, Reduced Row Echelon Form, lead variables, free variables
- Gauss Elimination, Gauss-Jordan Elimination

Preview

Matrix algebra

Exercises

P22: 3(b)(e), 4(d)(f),5;
P29: 3(b)(d),4(c).