Ch 3.4 Properties of Fourier Representations

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Outline

- Properties of Fourier Representations
  - Periodicity
  - Linearity and Symmetry
  - Convolution
  - Differentiation and integration
  - Time-shift and frequency-shift
  - Finding inverse FT by using partial-fraction expansions
  - Multiplication
  - Scaling
  - Parseval relationships
  - Time-bandwidth product
  - Duality

The Four Fourier Representation

Time Domain	Periodic (*t*, *n*)	Non-periodic (*t*, *n*)	
Continuous (*t*)	**Fourier Series** $x(t)=\sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$ $X[k]=\frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$ $x(t)$ has period *T*, $\omega_o = 2\pi / T$.	**Fourier Transform** $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ $X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	Nonperiodic $(k,\ \omega)$
Discrete [*n*]	**Discrete-Time Fourier Series** $x[n]=\sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n}$ $X[k]=\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}$ *x*[*n*] and *X*[*k*] have period *N*, $\Omega_o = 2\pi / N$.	**Discrete-Time Fourier Transform** $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$ $X(e^{j\Omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ $X(e^{j\Omega})$ has period 2π .	Periodic $(k,\ \Omega)$
	Discrete [*k*]	Continuous $(\omega,\ \Omega)$	Frequency Domain

Periodicity Property

- Signals that are periodic in time have discrete frequency-domain representations, while nonperiodic time signals have continuous frequency-domain representations

Time-Domain Property	Frequency-Domain Property
Continuous	*Nonperiodic*
Discrete	*Periodic*
Periodic	*Discrete*
Nonperiodic	*Continuous*

Linearity Property

$$z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(j\omega) = aX(j\omega) + bY(j\omega)$$

$$z(t) = ax(t) + by(t) \xleftrightarrow{FS;\omega_o} Z[k] = aX[k] + bY[k]$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFT} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFS;\Omega_o} Z[k] = aX[k] + bY[k]$$

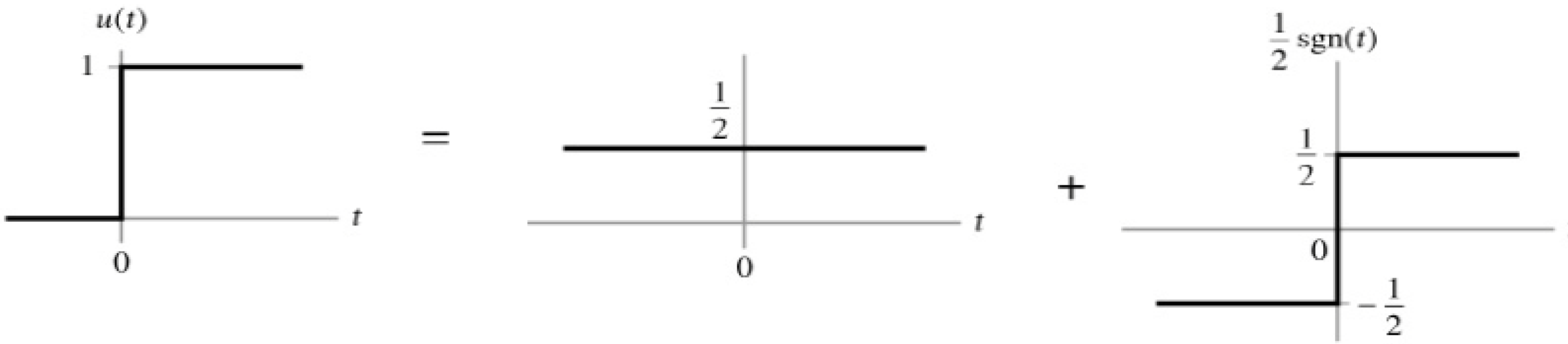
Note: in the case of the FS and DTFS, the signals being summed are assumed to have the same fundamental period.

- The linearity property is used to find Fourier representations of signals that are constructed as sums of signals whose representations are already known.

Linearity Property

Example. **Representation of a step function as the sum of a constant and a signum function.**

<Sol.>



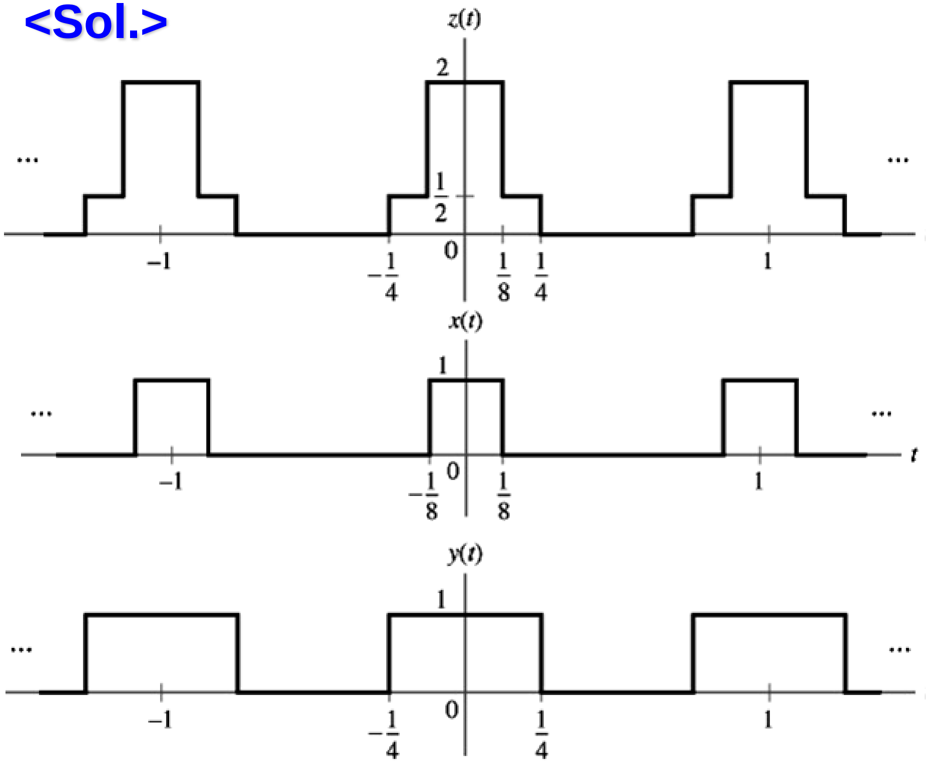
$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t) \xleftrightarrow{FT} \pi\delta(\omega) + \frac{1}{j\omega}$$

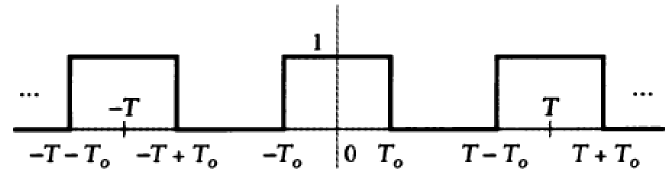
Linearity in the FS

Example 3.30 *Linearity in The FS*

Suppose $z(t)$ is the periodic signal. Use the linearity and the results of Example 3.13 to determine the FS coefficients $Z[k]$.

<Sol.>





$$X[k] = \sin\left(2\pi k T_0 / T\right) / \pi k$$

$$x(t) \xleftrightarrow{FS;2\pi} X[k] = (1/(k\pi))\sin(k\pi/4)$$

$$y(t) \xleftrightarrow{FS;2\pi} Y[k] = (1/(k\pi))\sin(k\pi/2)$$

$$z(t) = \frac{3}{2}x(t) + \frac{1}{2}y(t)$$

$$z(t) \xleftrightarrow{FS;2\pi} Z[k] = \frac{3}{2k\pi}\sin(k\pi/4) + \frac{1}{2k\pi}\sin(k\pi/2)$$

Symmetry Properties: Real and Imaginary Signals

- Symmetry property for real-valued signal $x(t)$:

$$X^*(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]^* = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt = X(-j\omega)$$

  - **Complex-conjugate symmetry**

$$\boxed{X^*(j\omega) = X(-j\omega)}$$

$$\mathrm{Re}\{X(j\omega)\} = \mathrm{Re}\{X(-j\omega)\}$$

$$\mathrm{Im}\{X(j\omega)\} = -\mathrm{Im}\{X(-j\omega)\}$$

  - **Complex-conjugate for DTFS:** $X^*[k] = X[-k] = X[N-k]$

Conclusions: If $x(t)$ is real valued, then the real part of the transform is an even function of frequency, while the imaginary part is an odd function of frequency. This also implies that the magnitude spectrum is an even function while the phase spectrum is an odd function.

Symmetry Properties: Real and Imaginary Signals

- Symmetry property for imaginary-valued signal $x(t)$

$$X^*(j\omega)=\left[\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\right]^*=\int_{-\infty}^{\infty}x^*(t)e^{j\omega t}dt=-\int_{-\infty}^{\infty}x(t)e^{-j(-\omega)t}dt$$

$$\Rightarrow \boxed{X^*(j\omega)=-X(-j\omega)} \Rightarrow \begin{aligned}\mathrm{Re}\{X(j\omega)\}&=-\mathrm{Re}\{X(-j\omega)\}\\ \mathrm{Im}\{X(j\omega)\}&=\mathrm{Im}\{X(-j\omega)\}\end{aligned}$$

Table 3.4 ***Symmetry Properties for Fourier Representation of Real- and Imaginary-Valued Signals***

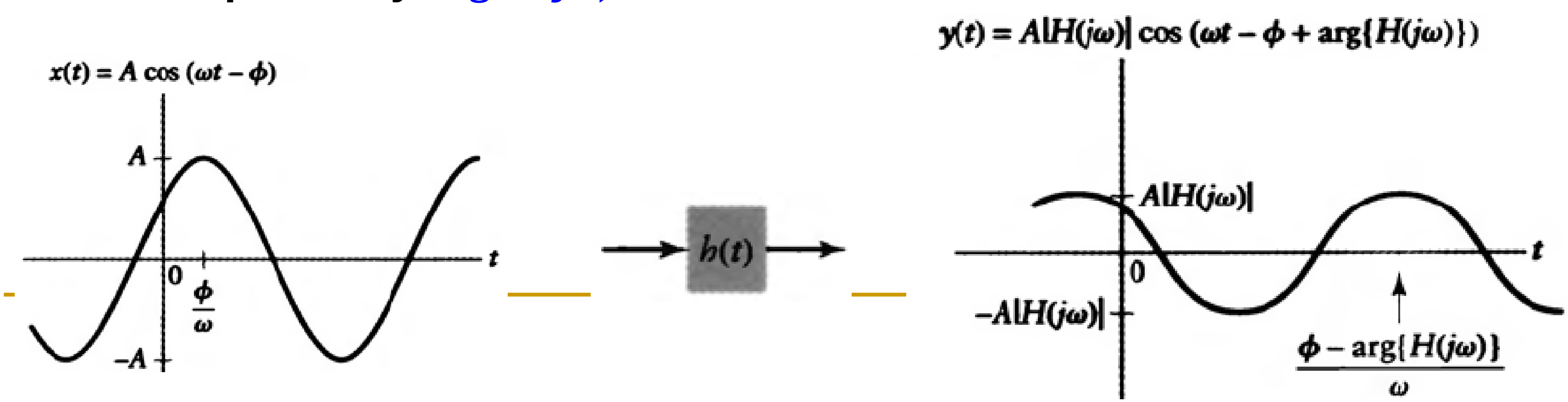
Representation	Real-Valued Time Signals	Imaginary-Valued Time Signals
FT	$X^*(j\omega)=X(-j\omega)$	$X^*(j\omega)=-X(-j\omega)$
FS	$X^*[k]=X[-k]$	$X^*[k]=-X[-k]$
DTFT	$X^*(e^{j\Omega})=X(e^{-j\Omega})$	$X^*(e^{j\Omega})=-X(e^{-j\Omega})$
DTFS	$X^*[k]=X[-k]$	$X^*[k]=-X[-k]$

Complex-conjugate Symmetry in FT

- A simple characterization of the output of an LTI system with a real-valued impulse response when the input is a real-valued sinusoid.
  - Input signal: $x(t) = A\cos(\omega t - \phi) = (A/2)e^{j(\omega t-\phi)} + (A/2)e^{-j(\omega t-\phi)}$
  - Real-valued impulse response of LTI system: $h(t) \xleftrightarrow{FT} H(j\omega)$
  - Output signal:

$$
\begin{aligned}
y(t) &= |H(j\omega)|(A/2)e^{j(\omega t-\phi+\arg\{H(j\omega)\})} + |H(-j\omega)|(A/2)e^{-j(\omega t-\phi-\arg\{H(-j\omega)\})} \\
&= |H(j\omega)|A\cos(\omega t - \phi + \arg\{H(j\omega)\})
\end{aligned}
$$

♣ **The LTI system modifies the amplitude of the input sinusoid by $|H(j\omega)|$ and the phase by $\arg\{H(j\omega)\}$.**



Symmetry Properties: Even and Odd Signals

- **$x(t)$ is real valued and has even symmetry.**

$$X^*(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j(-\omega)t} dt \quad = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \quad = X(j\omega)$$

$\Rightarrow$ **$\text{Im}\{X(j\omega)\} = 0$**

If $x(t)$ is real and even, then $X(j\omega)$ is also real and even.

- **If $x(t)$ is real and odd, then $X^*(j\omega) = -X(j\omega)$ and $X(j\omega)$ is purely imaginary and odd.**

Convolution Property

- **Convolution of Nonperiodic Signals — Continuous-time case**

$$y(t)=h(t)*x(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$

since $x(t-\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega(t-\tau)}d\omega$

$$y(t)=\int_{-\infty}^{\infty}h(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}e^{-j\omega\tau}d\omega\right]d\tau$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau\right]X(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

$$y(t)=h(t)*x(t)\xleftrightarrow{FT}Y(j\omega)=X(j\omega)H(j\omega)$$

We conclude that convolution of $h(t)$ and $x(t)$ in the time domain corresponds to multiplication of their Fourier transforms, $H(j\omega)$ and $X(j\omega)$ in the frequency domain;

Convolution Property

Example 3.31 ***Solving A Convolution Problem in The Frequency Domain***

Let $x(t) = (1/(\pi t))\sin(\pi t)$ be the input to a system with impulse response $h(t) = (1/(\pi t))\sin(2\pi t)$. Find the output $y(t) = x(t)*h(t)$.

<Sol.>

$$X(j\omega)=\begin{cases}1, & -W<\omega<W\\ 0, & |\omega|>W\end{cases} \Rightarrow x(t)=\frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega=-\frac{1}{j2\pi t}e^{j\omega t}\Big|_{-W}^{W}=\frac{1}{\pi t}\sin(Wt)$$

$$\Rightarrow \quad x(t)\xleftrightarrow{FT}X(j\omega)=\begin{cases}1, & |\omega|<\pi\\ 0, & |\omega|>\pi\end{cases} \qquad h(t)\xleftrightarrow{FT}H(j\omega)=\begin{cases}1, & |\omega|<2\pi\\ 0, & |\omega|>2\pi\end{cases}$$

$$\therefore \quad Y(j\omega)=X(j\omega)H(j\omega)=\begin{cases}1, & |\omega|<\pi\\ 0, & |\omega|>\pi\end{cases}$$

$$\Rightarrow \quad \mathbf{y(t) = (1/\pi t)\sin(\pi t)}$$

Convolution Property

Example 3.32 ***Finding Inverse FT's by Means of The Convolution Property***

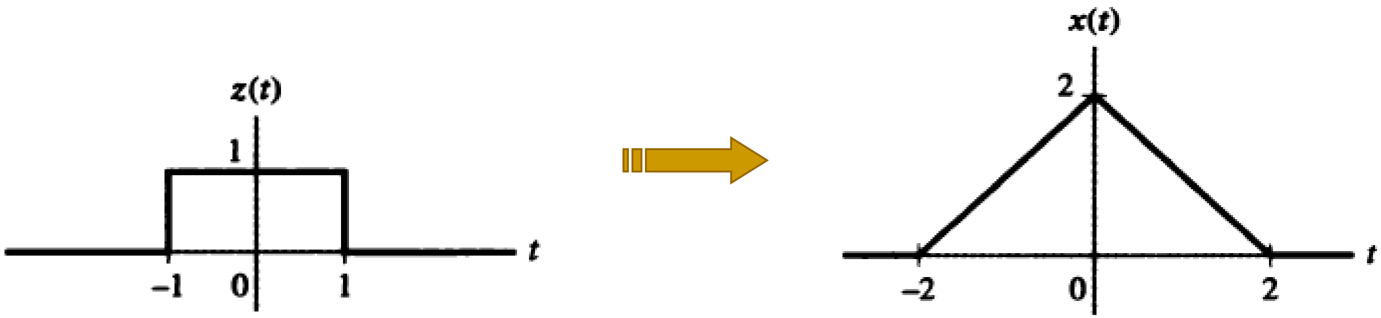
Use the convolution property to find *x*(*t*), where

$$x(t) \xleftrightarrow{FT} X(j\omega) = \frac{4}{\omega^2}\sin^2(\omega)$$

<Sol.> **Write $X(j\omega) = Z(j\omega)\,Z(j\omega)$, where** $Z(j\omega) = \frac{2}{\omega}\sin(\omega)$

$$x(t) = z(t) * z(t) \xleftrightarrow{FT} Z(j\omega)Z(j\omega)$$

Using the result of Example 3.25: $z(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases} \xleftrightarrow{FT} Z(j\omega) = \frac{2}{\omega}\sin(\omega)$



Convolution Property

- **Convolution of Nonperiodic Signals — Discrete-time case**

$$x[n] \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right)$$

$$h[n] \xleftrightarrow{DTFT} H\left(e^{j\Omega}\right)$$

$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega}) H(e^{j\Omega})$$

Convolution Property

- **Convolution of Periodic Signals**

Unstable system

Define the periodic convolution of two CT signals $x(t)$ and $z(t)$, each having period T, as

$$y(t) = x(t) \otimes z(t) = \int_0^T x(\tau) z(t-\tau)\, d\tau$$

$$\Rightarrow \quad y(t) = x(t) \circledast z(t) \xleftrightarrow{FS;\frac{2\pi}{T}} Y[k] = TX[k]Z[k]$$

Convolution in Time-Domain ↔ Multiplication in Frequency-Domain

$$y[n] = x[n] \circledast z[n] = \sum_{k=0}^{N-1} x[k] z[n-k]$$

$$\Rightarrow \quad y[n] = x[n] \circledast z[n] \xleftrightarrow{DTFS;\frac{2\pi}{N}} Y[k] = NX[k]Z[k]$$

Convolution Property

- **Summary on convolution properties**

$$x(t) * z(t) \xleftrightarrow{FT} X(j\omega)Z(j\omega)$$

$$x(t) \circledast z(t) \xleftrightarrow{FS;\omega_o} TX[k]Z[k]$$

$$x[n] * z[n] \xleftrightarrow{DTFT} X(e^{j\Omega})Z(e^{j\Omega})$$

$$x[n] \circledast z[n] \xleftrightarrow{DTFS;\Omega_o} NX[k]Z[k]$$

Differentiation and Integration Properties

- **Differentiation and integration are operations that apply to continuous functions.**
- **Differentiation in Time**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega \xrightarrow[\text{sides with respect to } t]{\text{Differentiating both}} \frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X(j\omega)j\omega e^{j\omega t}}\,d\omega$$

$$\Rightarrow \quad \boxed{\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)}$$

$$\mathcal{F}\left[\frac{d}{dt}x(t)\right]\Bigg|_{\omega=0} = (j\omega)\times X(j\omega)\big|_{\omega=0} = 0$$

$$\boxed{\frac{d^n}{dt}x(t) \xleftrightarrow{FT} (j\omega)^n X(j\omega)}$$

Differentiation and Integration Properties

Example 3.37 ***Verifying the Differentiation Property***

The differentiation property implies that

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega} \quad \Rightarrow \quad \frac{d}{dt}\left(e^{-at}u(t)\right) \xleftrightarrow{FT} \frac{j\omega}{a+j\omega}$$

Verify this result by differentiating and taking the FT of the result.

<Sol.>

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= -ae^{-at}u(t) + \delta(t)$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{-at}u(t)\right) \xleftrightarrow{FT} \frac{-a}{a+j\omega} + 1 = \frac{j\omega}{a+j\omega}$$

Differentiation and Integration Properties

- **Differentiation in Frequency**

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt \xrightarrow[\text{sides with respect to } \omega]{\text{Differentiating both}} \frac{d}{d\omega}X(j\omega)=\int_{-\infty}^{\infty}-jtx(t)e^{-j\omega t}dt$$

$$\Rightarrow \boxed{-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)}$$

$$\boxed{-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\Omega}X\left(e^{j\Omega}\right)}$$

Problem 3.25 Use the frequency-differentiation property to find the FT of ***$te^{-at}u(t)$***.

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega} \Rightarrow te^{-at}u(t) \xleftrightarrow{FT} j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)=\frac{1}{(a+j\omega)^2}$$

Differentiation and Integration Properties

- **Integration**
  - ♣ The operation of integration applies only to continuous dependent variables.

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

$$\Rightarrow Y(j\omega) = X(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] = \pi X(j\omega)\delta(\omega) + \frac{1}{j\omega}X(j\omega)$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)}$$

- **Time-average value or dc component:**

$$X(j\mathbf{0}) = \int_{-\infty}^{\infty} x(t)\, e^{-j\mathbf{0}t} dt = \int_{-\infty}^{\infty} x(t)dt$$

Differentiation and Integration Properties

- **Commonly used differentiation and integration properties**

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FS;\quad \omega_0} jk\omega_0 X[k]$$

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$$

$$-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\Omega}X\left(e^{j\Omega}\right)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega)+\pi X(j0)\delta(\omega)$$

Time-shift Property

If $x(t) \xleftrightarrow{FT} X(j\omega)$,

$$\int_{-\infty}^{\infty} x(t-t_o)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_o)}\,d\tau = e^{-j\omega t_o}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau$$

$$= e^{-j\omega t_o}X(j\omega) = \left|X(j\omega)\right| e^{j\left[\arg\{X(j\omega)\} - \omega t_0\right]}$$

Hence, a shift in time leaves the magnitude spectrum unchanged and introduces a phase shift that is a linear function of frequency.

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0}X(j\omega)$$

$$x(t-t_0) \xleftrightarrow{FS;\ \omega_0} e^{-jk\omega_0 t_0}X(k)$$

$$x[n-n_0] \xleftrightarrow{DTFT} e^{-j\Omega n_0}X(e^{j\Omega})$$

$$x[n-n_0] \xleftrightarrow{DTFS;\Omega_0} e^{-jk\Omega_0 n_0}X[k]$$

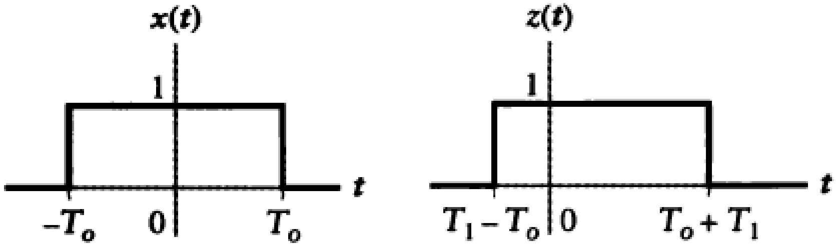
Time-shift Property

Example 3.41 ***Finding an FT Using the Time-Shift Property***

Use the FT of the rectangular pulse $x(t)$ depicted in Fig. 3.62 (a) to determine the FT of the time-shift rectangular pulse $z(t)$ depicted in Fig. 3. 62 (b).

<Sol.> $z(t) = x(t - T_1)$

$$X(j\omega) = \frac{2}{\omega}\sin(\omega T_0)$$



$$Z(j\omega) = e^{-j\omega T_1} X(j\omega) = e^{-j\omega T_1}\frac{2}{\omega}\sin(\omega T_0)$$

Example **Find the spectrum of signal:** $y(t) = \int_{-2}^{t} e^{-2\tau} e^{-5(t-\tau)} d\tau$

<Sol.> $y(t) = e^{-2t}u(t+2) * e^{-5t}u(t) \;\; = e^{4}e^{-2(t+2)}u(t+2) * e^{-5t}u(t)$

$$Y(j\omega) = \frac{e^{4}e^{j2\omega}}{j\omega+2}\cdot\frac{1}{j\omega+5} = \frac{e^{j2\omega+4}}{(j\omega)^2 + 7j\omega + 10}$$

Frequency-shift Property

Suppose that: $x(t) \xleftrightarrow{FT} X(j\omega)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X\big(j(\omega-\gamma)\big)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\eta)e^{j(\eta+\gamma)t}d\eta$$

$$= e^{j\gamma t}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\eta)e^{j\eta t}d\eta = e^{j\gamma t}x(t)$$

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega-\gamma))$$

$$e^{jk_o\omega_o t}x(t) \xleftrightarrow{FS;\omega_o} X[k-k_o]$$

$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\Omega-\Gamma)})$$

$$e^{jk_o\Omega_o n}x[n] \xleftrightarrow{DTFS;\Omega_o} X[k-k_o]$$

Frequency-shift Property

$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

$$\Rightarrow \quad e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\delta(\omega-\omega_0)$$

$$\cos(\omega_0 t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] \xleftrightarrow{FT} \pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]$$

$$\sin(\omega_0 t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] \xleftrightarrow{FT} j\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]$$

Finding Inverse FT by Using Partial-fraction Expansions (部分分式展开)

- **Inverse Fourier Transform**

$$X(j\omega)=\frac{b_M(j\omega)^M+\cdots+b_1(j\omega)+b_0}{(j\omega)^N+a_{N-1}(j\omega)^{N-1}+\cdots a_1(j\omega)+a_0}\qquad (M\geq N)$$

$$=c_0+c_1(j\omega)+c_2(j\omega)^2+\cdots+c_{M-N}(j\omega)^{M-N}+\frac{B(j\omega)}{A(j\omega)}$$

$$x(t)=c_0\delta(t)+c_1\delta'(t)+c_2\delta''(t)+\cdots+c_{M-N}\delta^{(M-N)}(t)+F^{-1}\left[\frac{B(j\omega)}{A(j\omega)}\right]$$

$$\frac{B(j\omega)}{A(j\omega)}=\frac{A_1}{j\omega-d_1}+\frac{A_2}{j\omega-d_2}+\cdots+\frac{A_N}{j\omega-d_N}$$

$$\text{where}\quad A_k=(j\omega-d_k)\left.\frac{B(j\omega)}{A(j\omega)}\right|_{j\omega=d_k},\quad k=1,2,\cdots,N$$

Finding Inverse FT by Using Partial-fraction Expansions

if $$\frac{B(j\omega)}{A(j\omega)} = \frac{A_1}{j\omega - d_1} + \frac{A_2}{j\omega - d_2} + \cdots + \frac{A_N}{j\omega - d_N}, \quad A_k = (j\omega - d_k)\left.\frac{B(j\omega)}{A(j\omega)}\right|_{j\omega = d_k}.$$

$$A_k e^{d_k t} u(t) \xleftrightarrow{FT} \frac{A_k}{j\omega - d_k}$$

$$\Rightarrow \quad F^{-1}\left[\frac{B(j\omega)}{A(j\omega)}\right] = \left(A_1 e^{d_1 t} + A_2 e^{d_2 t} + \cdots + A_N e^{d_N t}\right) u(t)$$

Example **Find the inverse FT of** $X(j\omega) = \dfrac{j\omega + 1}{(j\omega)^2 + 5j\omega + 6}$

<Sol.> $X(j\omega) = \dfrac{j\omega + 1}{(j\omega + 2)(j\omega + 3)} = \dfrac{-1}{j\omega + 2} + \dfrac{2}{j\omega + 3}$

$$\Rightarrow \quad x(t) = -e^{-2t}u(t) + 2e^{-3t}u(t)$$

Finding Inverse FT by Using Partial-fraction Expansions

- **Inverse Discrete-Time Fourier Transform**

$$X\left(e^{j\Omega}\right)=\frac{b_M e^{-j\Omega M}+\cdots+b_1 e^{-j\Omega}+b_0}{e^{-j\Omega N}+a_{N-1}e^{-j\Omega(N-1)}+\cdots a_1 e^{-j\Omega}+a_0}\qquad (M\geq N)$$

$$=c_0+c_1 e^{-j\Omega}+c_2 e^{-j2\Omega}+\cdots+c_{M-N}e^{-j(M-N)\Omega}+\frac{B\left(e^{j\Omega}\right)}{A\left(e^{j\Omega}\right)}$$

$$\Rightarrow\quad x[n]=c_0\delta[n]+c_1\delta[n-1]+\cdots+c_{M-N}\delta[n-(M-N)]+F^{-1}\left[\frac{B\left(e^{j\Omega}\right)}{A\left(e^{j\Omega}\right)}\right]$$

$$\frac{B\left(e^{j\Omega}\right)}{A\left(e^{j\Omega}\right)}=\frac{A_1}{1-d_1 e^{-j\Omega}}+\frac{A_2}{1-d_2 e^{-j\Omega}}+\cdots+\frac{A_N}{1-d_N e^{-j\Omega}}$$

$$\text{where}\quad A_k=\left(1-d_k e^{-j\Omega}\right)\left.\frac{B\left(e^{j\Omega}\right)}{A\left(e^{j\Omega}\right)}\right|_{e^{-j\Omega}=1/d_k},\quad k=1,2,\ldots,N$$

$$A_k\left(d_k\right)^n u[n]\xleftrightarrow{DTFT}\frac{A_k}{1-d_k e^{-j\Omega}}\quad\Rightarrow\quad F^{-1}\left[\frac{B\left(e^{j\Omega}\right)}{A\left(e^{j\Omega}\right)}\right]=\sum_{k=1}^{N}A_k\left(d_k\right)^n u[n]$$

Finding Inverse FT by Using Partial-fraction Expansions

Example 3.45 **Find the inverse DTFT of** $X\left(e^{j\Omega}\right)=\dfrac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}$

<Sol.> $$X\left(e^{j\Omega}\right)=\frac{-\frac{5}{6}e^{-j\Omega}+5}{\left(1+\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{3}e^{-j\Omega}\right)}=\frac{A_1}{1+\frac{1}{2}e^{-j\Omega}}+\frac{A_2}{1-\frac{1}{3}e^{-j\Omega}}$$

where

$$A_1=\left(1+\frac{1}{2}e^{-j\Omega}\right)\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}\Bigg|_{e^{-j\Omega}=-2}=\frac{-\frac{5}{6}e^{-j\Omega}+5}{1-\frac{1}{3}e^{-j\Omega}}\Bigg|_{e^{-j\Omega}=-2}=4$$

$$A_2=\left(1-\frac{1}{3}e^{-j\Omega}\right)\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}\Bigg|_{e^{-j\Omega}=3}=\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{2}e^{-j\Omega}}\Bigg|_{e^{-j\Omega}=3}=1$$

$$\Rightarrow\quad x[n]=4(-1/2)^n u[n]+(1/3)^n u[n]$$

Multiplication Property

- **Non-periodic continuous-time Signals**

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(jv)e^{jvt}dv,\quad z(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Z(j\eta)e^{j\eta t}d\eta$$

$$y(t)=x(t)z(t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X(jv)Z(j\eta)e^{j(\eta+v)t}d\eta dv$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(jv)Z(j(\omega-v))dv\right]e^{j\omega t}d\omega$$

$$\boxed{y(t)=x(t)z(t)\xleftrightarrow{FT}Y(j\omega)=\frac{1}{2\pi}X(j\omega)*Z(j\omega)}$$

where $X(j\omega)*Z(j\omega)=\int_{-\infty}^{\infty}X(jv)Z(j(\omega-v))dv$

Multiplication in Time-Domain ↔ Convolution in Frequency-Domain×(1/2π)

Multiplication Property

- **Non-periodic discrete-time Signals**

$$y[n] = x[n]z[n] \quad \xleftrightarrow{DTFT} \quad Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \otimes Z(e^{j\Omega})$$

where the symbol Ä denotes periodic convolution.

$$X\left(e^{j\Omega}\right) \otimes Z\left(e^{j\Omega}\right) = \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Z\left(e^{j(\Omega-\theta)}\right) d\theta$$

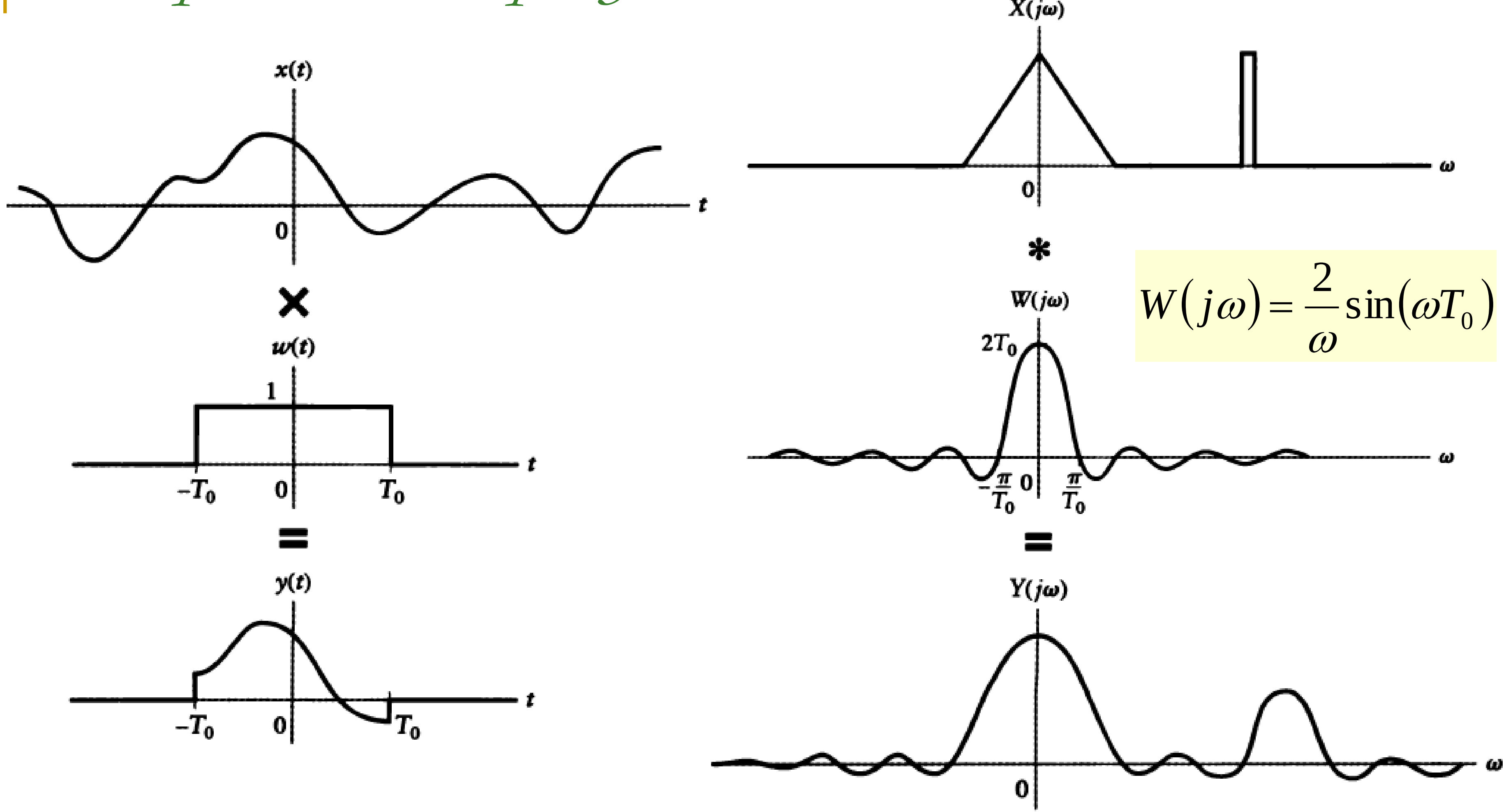
♣ Multiplication property can be used to study the effects of truncating a time-domain signal on its frequency-domain.

Windowing !

- **Truncate signal x(t) by a window function w(t) is represented by**

$$\boldsymbol{y(t) = x(t)w(t)}$$

Multiplication Property



$$W(j\omega)=\frac{2}{\omega}\sin(\omega T_0)$$

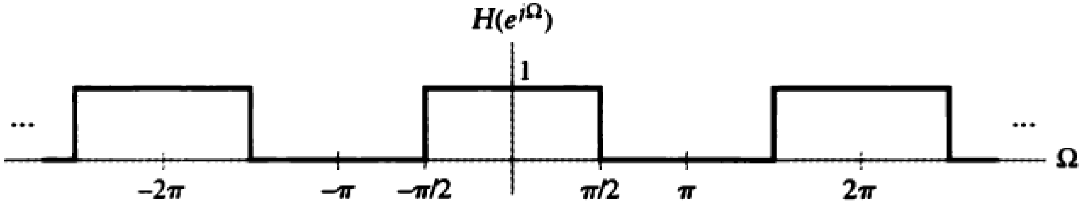
- **The general effect of the window is to smooth details in $X(j\omega)$ and introduce oscillations near discontinuities in $X(j\omega)$.**

Multiplication Property

Example 3.46 ***Truncating the Impulse Response***

The frequency response $H(e^{j\Omega})$ of an ideal discrete-time system is depicted in Fig. 3. 66(a). Describe the frequency response of a system whose impulse response is the ideal impulse response truncated to the interval $-M \le n \le M$.

<Sol.> $h[n] = \dfrac{1}{\pi n}\sin\left(\dfrac{\pi n}{2}\right)$



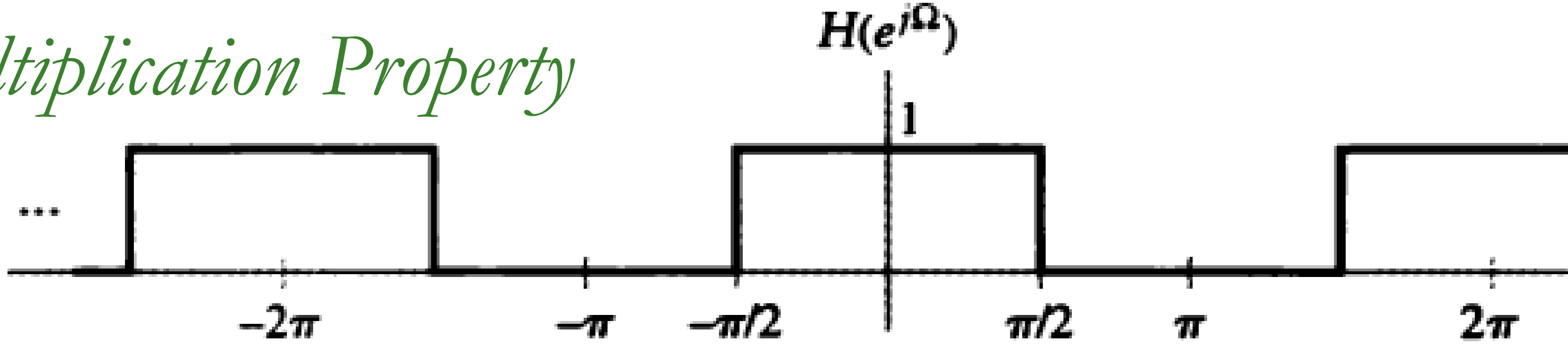
$$h_t[n] = \begin{cases} h[n], & |n| \le M \\ 0, & \text{otherwise} \end{cases} = h[n]w[n] \quad \textbf{where } w[n] = \begin{cases} 1, & |n| \le M \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad H_t\left(e^{j\Omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\theta}\right) W\left(e^{j(\Omega-\theta)}\right) d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} F_\Omega(\theta)\, d\theta$$

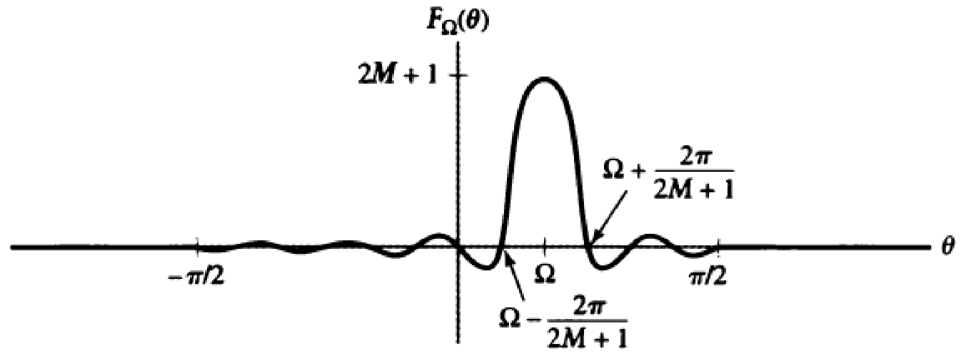
$$\textbf{where } W\left(e^{j(\Omega-\theta)}\right) = \frac{\sin\left((\Omega-\theta)(2M+1)/2\right)}{\sin\left((\Omega-\theta)/2\right)}$$

$$F_\Omega(\theta) = H\left(e^{j\theta}\right) W\left(e^{j(\Omega-\theta)}\right) = \begin{cases} W\left(e^{j(\Omega-\theta)}\right), & |\theta| < \pi/2 \\ 0, & |\theta| > \pi/2 \end{cases}$$

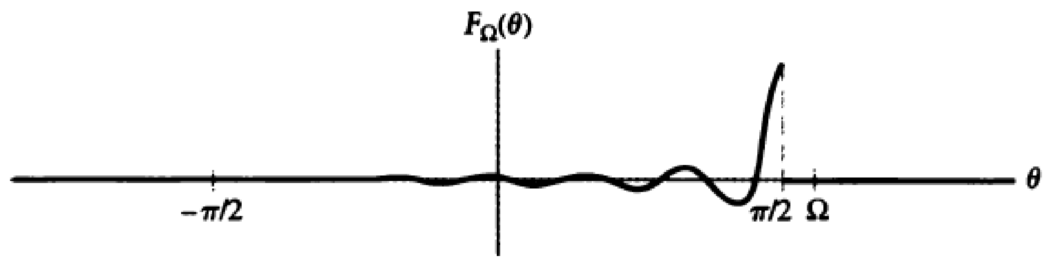
Multiplication Property

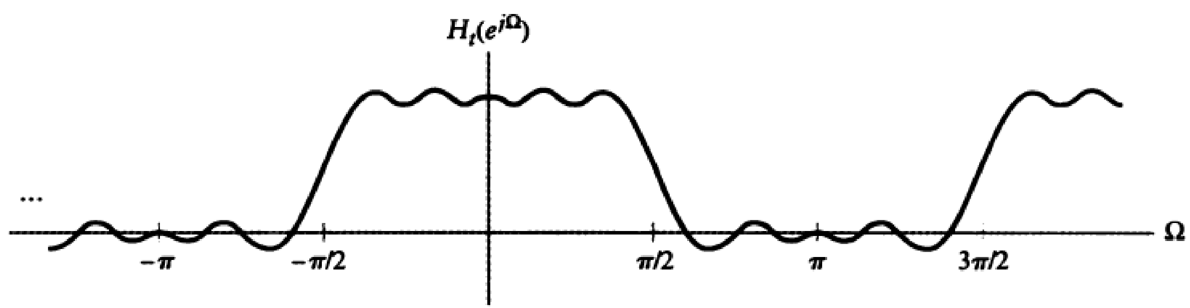


$$F_{\Omega}(\theta)=\begin{cases} W\left(e^{j(\Omega-\theta)}\right), & |\theta|<\pi/2 \\ 0, & |\theta|>\pi/2 \end{cases}$$



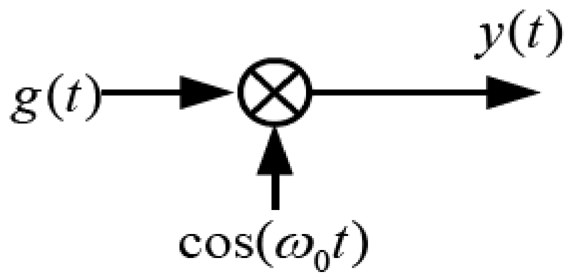
$$H_t\left(e^{j\Omega}\right)=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}F_{\Omega}(\theta)\,d\theta$$





Multiplication Property

- **Application: amplitude modulation (幅度调制)**



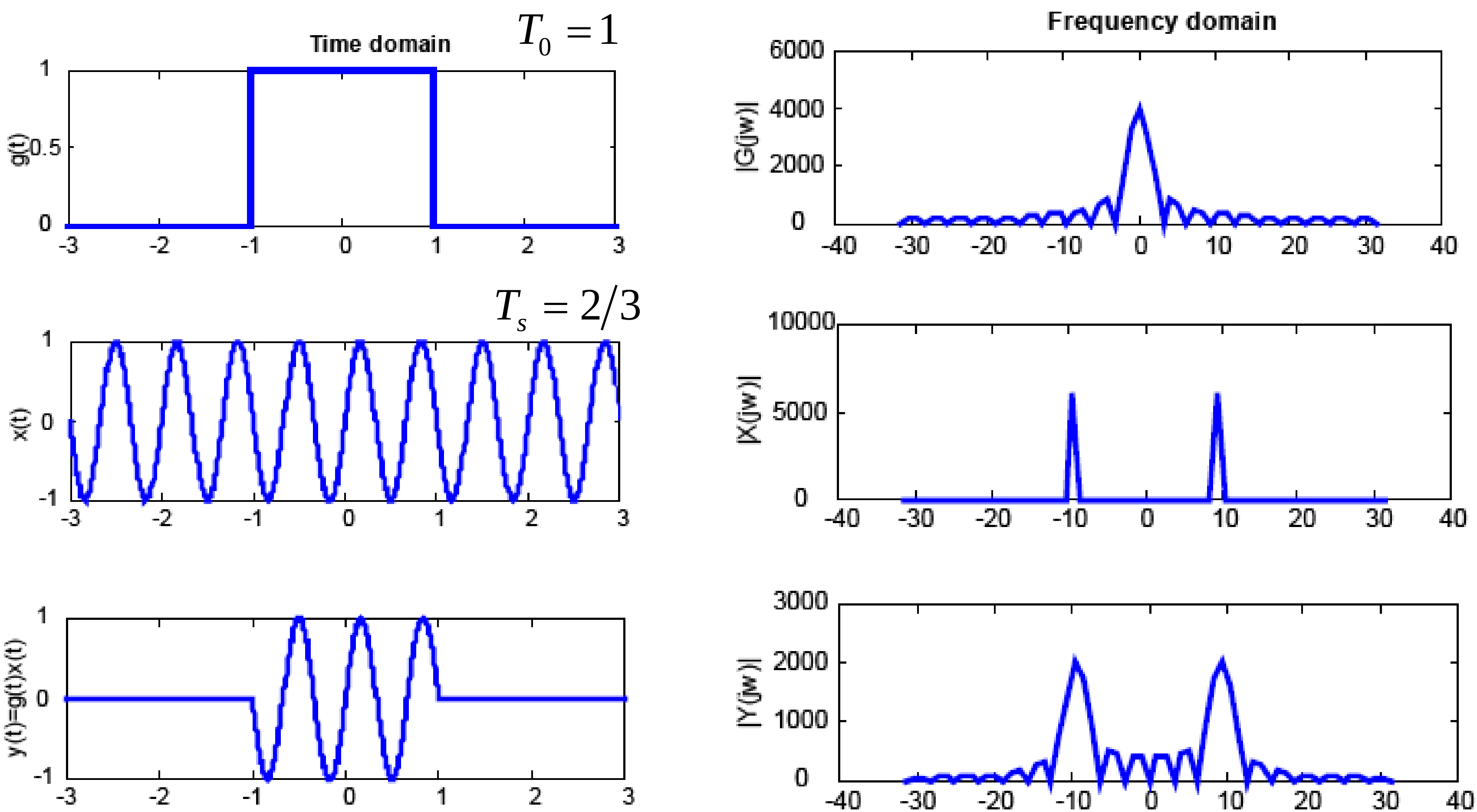
$$y(t) = g(t)\cos(\omega_0 t)$$

$$Y(j\omega) = \frac{1}{2\pi}\left[G(j\omega) * \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]\right]$$

$$= \frac{1}{2}G\left[j(\omega+\omega_0)\right] + \frac{1}{2}G\left[j(\omega-\omega_0)\right]$$

Multiplication Property

- **Application: amplitude modulation (幅度调制)**



Scaling Property

$$z(t) = x(at)$$

$$Z(j\omega) = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

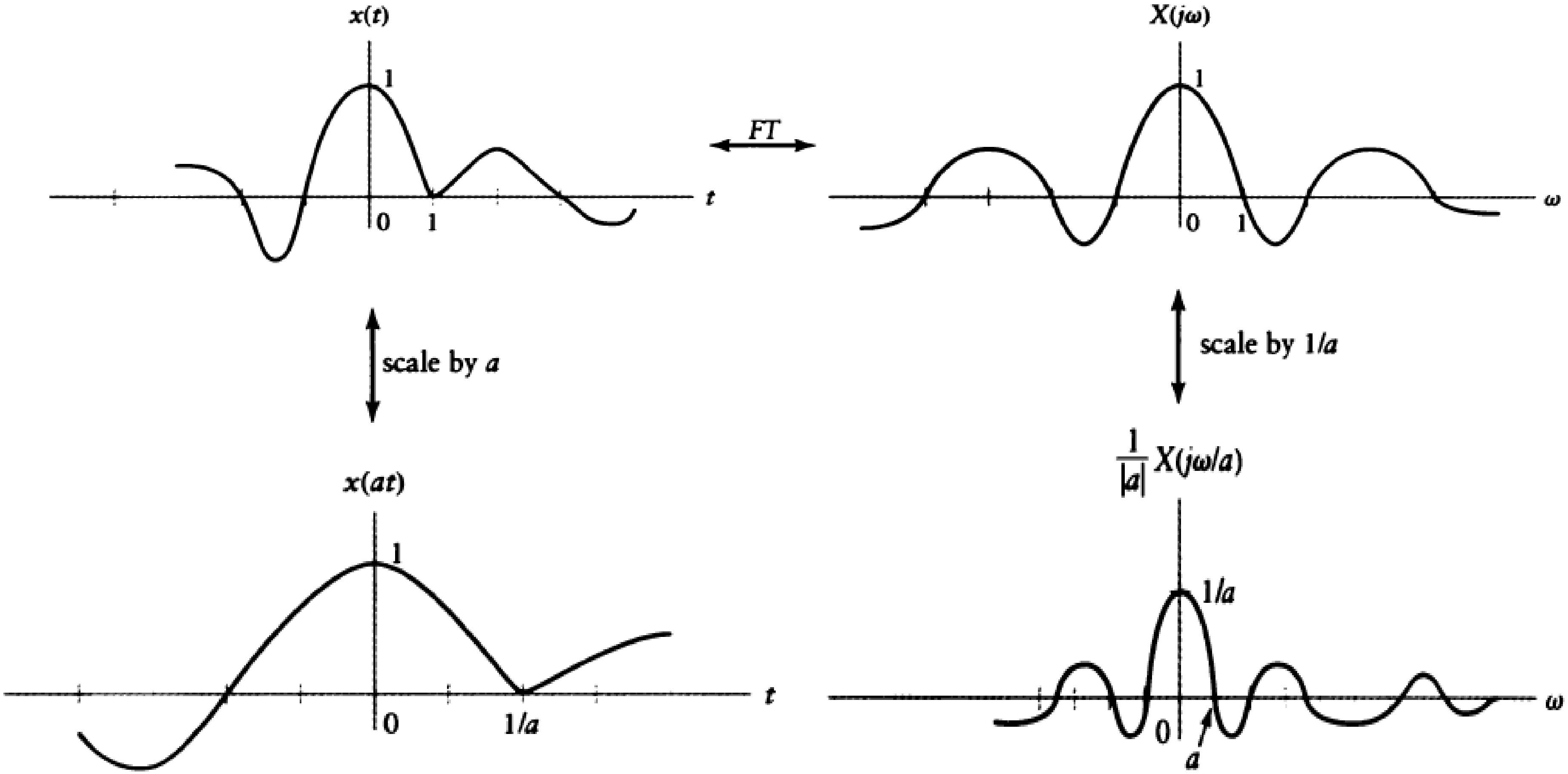
$$= \begin{cases} (1/a)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a > 0 \\ (1/a)\int_{\infty}^{-\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a < 0 \end{cases} = \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau$$

$$\boxed{x(at) \xleftrightarrow{FT} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)}$$

Scaling in Time-Domain ↔ Inverse Scaling in Frequency-Domain

- **Compressing a signal in time leads to expansion in the frequency domain and vice versa.**

Scaling Property



$0 < a < 1$

Scaling Property

Example 3.48 *Scaling a Rectangular Pulse*

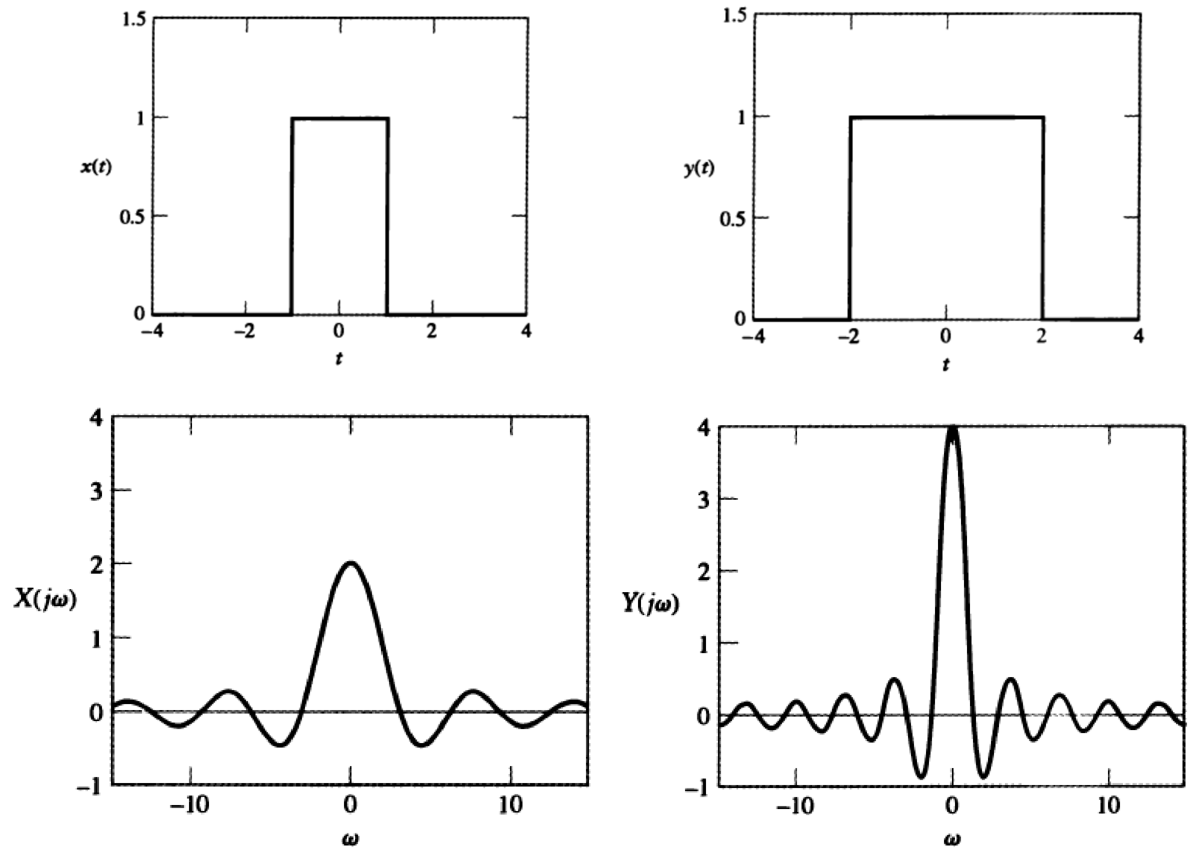
$$x(t)=\begin{cases}1, & |t|<1\\ 0, & |t|>1\end{cases} \xleftrightarrow{FT} X(j\omega)=\frac{2}{\omega}\sin(\omega T_0)=\frac{2}{\omega}\sin(\omega)$$

$$y(t)=\begin{cases}1, & |t|<2\\ 0, & |t|>2\end{cases}=x\left(\frac{t}{2}\right)$$

$$Y(j\omega)=2X(j2\omega)$$

$$=2\left(\frac{2}{2\omega}\right)\sin(2\omega)$$

$$=\frac{2}{\omega}\sin(2\omega)$$



Parseval Relationships

- **The energy or power in the time-domain representation of a signal is equal to the energy or power in the frequency-domain representation.**

FT:

$$W_x = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2 \mathrm{d}\omega$$

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$$W_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left\{\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt\right\} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega) X(j\omega)\, d\omega$$

Parseval Relationships

- **Parseval relationships for the Four Fourier representation.**

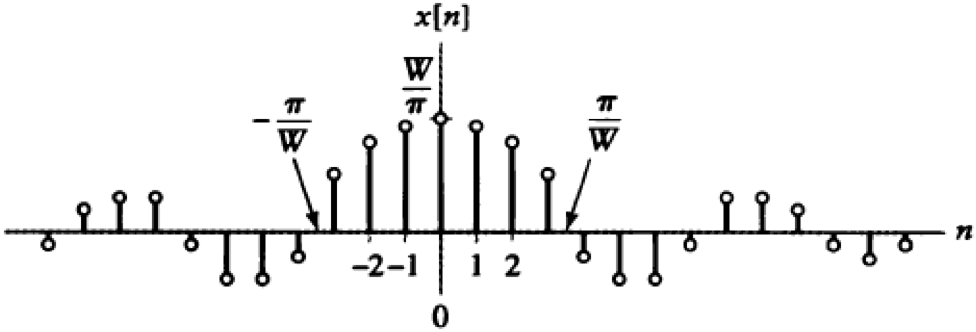
Representations	Parseval Relationships
FT	$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(j\omega) \rvert^2 d\omega$
FS	$\frac{1}{T} \int_{0}^{T} \lvert x(t) \rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert X[k] \rvert^2$
DTFT	$\sum_{n=-\infty}^{\infty} \lvert x[n] \rvert^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lvert X(e^{j\Omega}) \rvert^2 d\Omega$
DTFS	$\frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$

Energy is used for nonperiodic time-domain signals, while power applies to periodic time-domain signals.

Parseval Relationships

Example 3.50 ***Calculating Energy in a Signal***

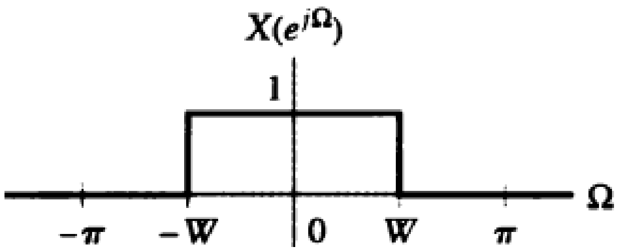
Let $x[n] = \dfrac{\sin(Wn)}{\pi n}$



Use Parseval's theorem to evaluate

$$\chi = \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \sum_{n=-\infty}^{\infty} \frac{\sin^2(Wn)}{\pi^2 n^2}$$

<Sol.> $$\chi = \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\Omega})\right|^2 d\Omega$$



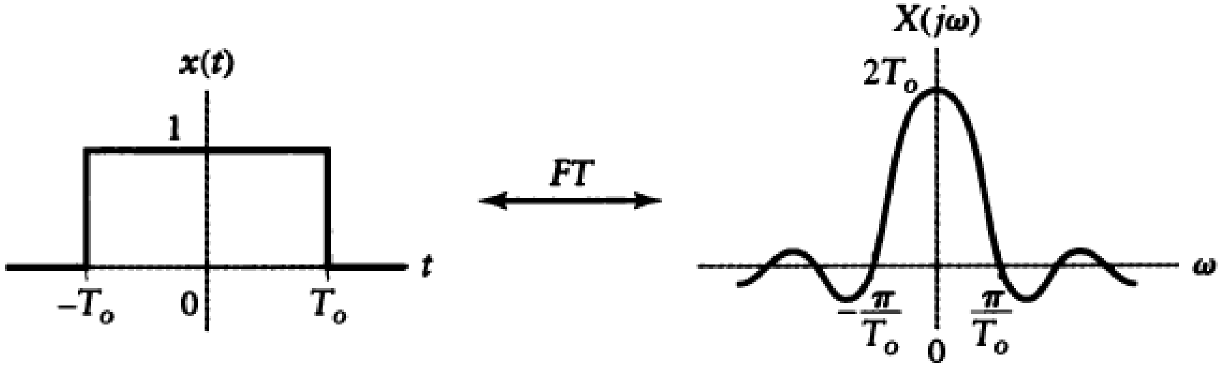
$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| \le W \\ 0, & W < |\Omega| < \pi \end{cases}$$

$$\chi = \frac{1}{2\pi}\int_{-W}^{W} 1 d\Omega = W / \pi$$

Time-Bandwidth Product (时间-带宽积)

- **Rectangular pulse illustrating the inverse relationship between the time and frequency extent of a signal.**

$$x(t)=\begin{cases}1, & |t|\le T_o\\ 0, & |t|>T_o\end{cases} \quad \xleftrightarrow{FT} \quad X(j\omega)=2\sin(\omega T_o)/\omega$$



- **The product of the time extent $2T_o$ and mainlobe width $2\pi/T_o$ is a constant.**

Signal's time-bandwidth product !

Time-Bandwidth Product

- **Effective duration:** $T_d = \left[\dfrac{\int_{-\infty}^{\infty} t^2 \left| x(t) \right|^2 dt}{\int_{-\infty}^{\infty} \left| x(t) \right|^2 dt} \right]^{1/2}$
  (有效时宽)

- **Effective bandwidth :** $B_w = \left[\dfrac{\int_{-\infty}^{\infty} \omega^2 \left| X(j\omega) \right|^2 d\omega}{\int_{-\infty}^{\infty} \left| X(j\omega) \right|^2 d\omega} \right]^{1/2}$
  (有效带宽)

- **The time-bandwidth product for any signal is lower bounded according to the relationship**

$$T_d B_w \geq 1/2$$

We cannot simultaneously decrease the duration and bandwidth of a signal.

Uncertainty principle !

Time-Bandwidth Product

Example 3.51 ***Bounding the Bandwidth of a Rectangular Pulse***

Let

$$x(t) = \begin{cases} 1, & |t| \le T_o \\ 0, & |t| > T_o \end{cases}$$

Use the uncertainty principle to place a lower bound on the effective bandwidth of $x(t)$.

<Sol.>

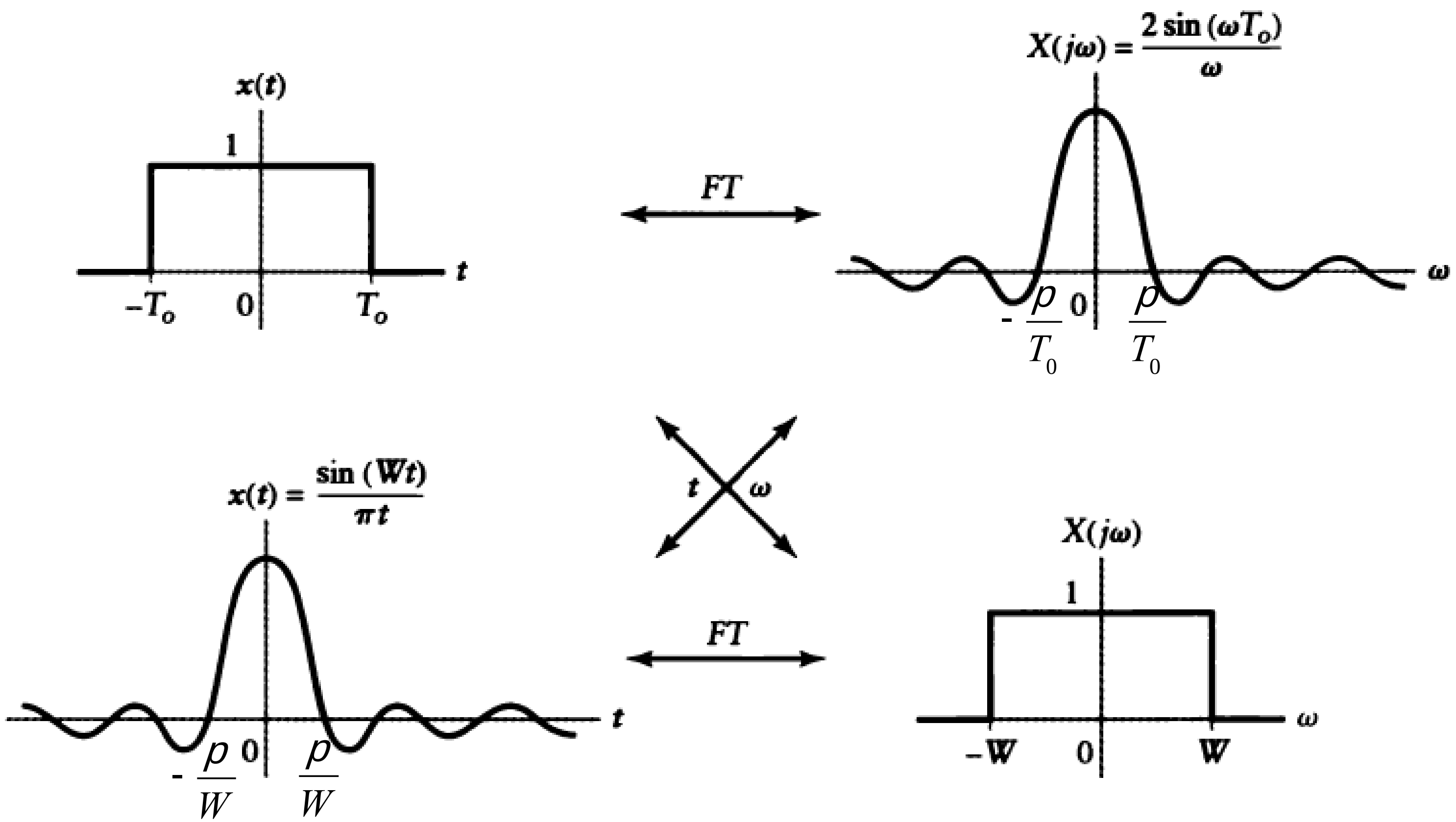
$$T_d = \left[\frac{\int_{-T_o}^{T_o} t^2 dt}{\int_{-T_o}^{T_o} dt} \right]^{1/2} = [(1/(2T_o))(1/3)t^3 \Big|_{-T_o}^{T_o}]^{1/2} = T_o / \sqrt{3}$$

The uncertainty principle states that

$$T_d B_w \ge 1/\mathbf{2} \quad \Rightarrow \quad B_w \ge \sqrt{3}/(2T_o)$$

Duality Property of FT (对偶特性)

- **Duality of rectangular pulses and sinc functions**



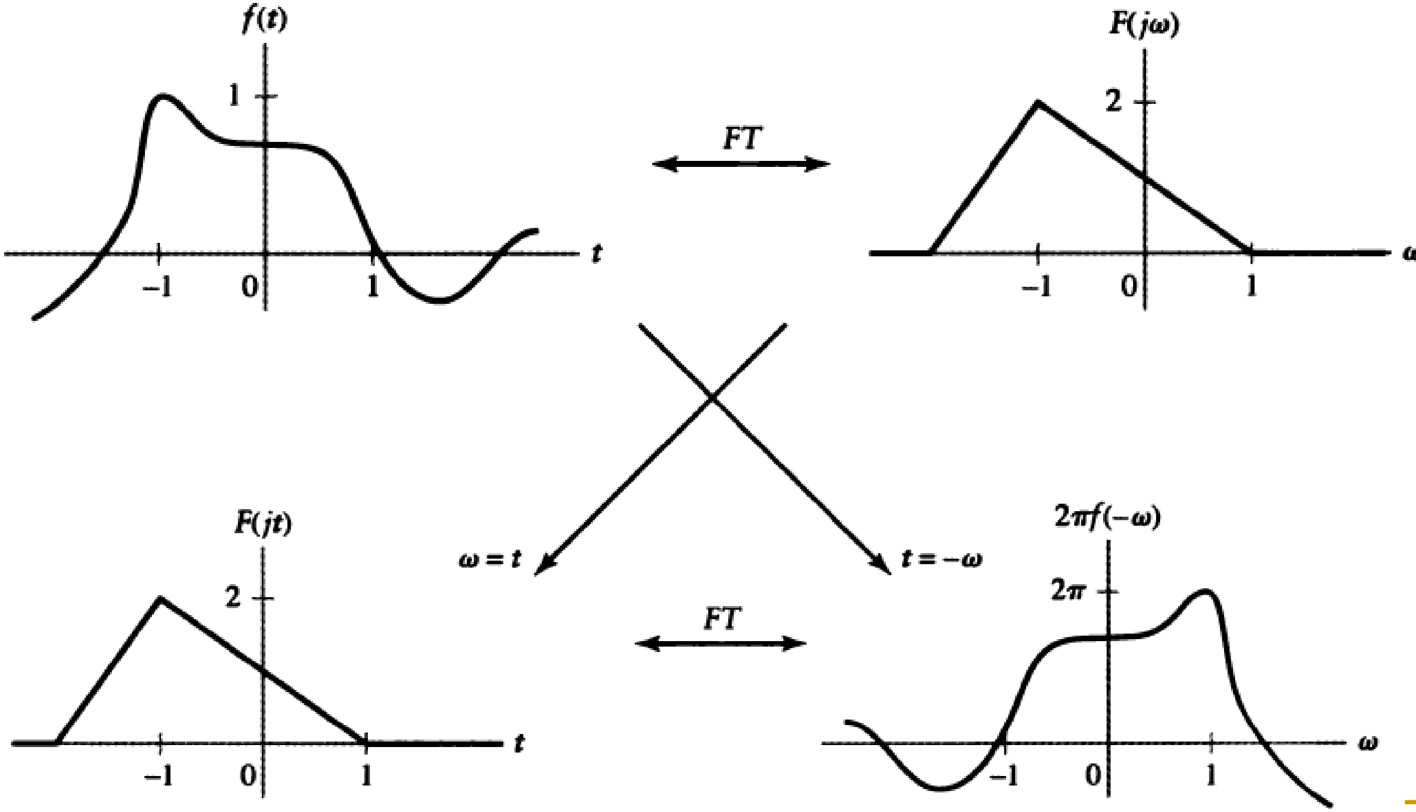
Duality Property of FT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

if $f(t) \xleftrightarrow{FT} F(j\omega)$, then

$$F(jt) \xleftrightarrow{FT} 2\pi f(-\omega)$$



Duality Property of FT

Example 3.52 ***Applying Duality***

Find the FT of $x(t) = \dfrac{1}{1+jt}$

<Sol.> $f(t) = e^{-t}u(t) \xleftrightarrow{FT} F(j\omega) = \dfrac{1}{1+j\omega}$

$$x(t) = F(jt) = \frac{1}{1+jt} \Rightarrow X(j\omega) = 2\pi f(-\omega) = 2\pi e^{\omega}u(-\omega)$$

- **Duality property of Fourier representations**

FT	$f(t) \xleftrightarrow{FT} F(j\omega)$		$F(jt) \xleftrightarrow{FT} 2\pi f(-\omega)$	
DTFS	$x[n] \xleftrightarrow{DTFS;\,2\pi/N} X[k]$		$X[n] \xleftrightarrow{DTFS;\,2\pi/N} (1/N)x[-k]$	
FS-DTFT	$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$		$X(e^{jt}) \xleftrightarrow{FS;\,1} x[-k]$	

Summary

- Properties of Fourier Representations
  - Periodicity
  - Linearity and Symmetry
  - Convolution
  - Differentiation and integration
  - Time-shift and frequency-shift
  - Finding inverse FT by using partial-fraction expansions
  - Multiplication
  - Scaling
  - Parseval relationships
  - Time-bandwidth product
  - Duality
- Reference in textbook: 3.8 ~ 3.18
- Homework: 3.58(a,d,f), 3.59(b,c,e), 3.60(a,b), 3.61(b,e); 3.73(b,c), 3.74(b,d)