Linear Algebra
线性代数

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课程要求

作业要求：每周日24:00前提交至**教学云平台**

每一页写清**班级**+**学号**+**姓名**，**题号；**逐页拍照上传

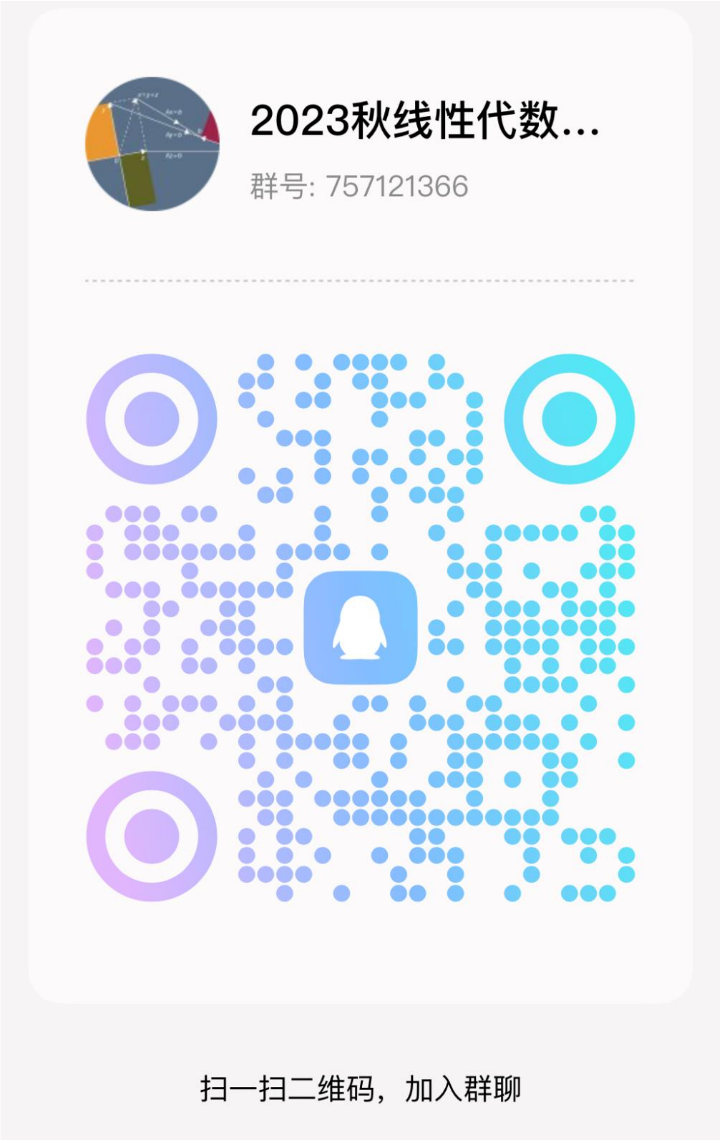
教材：张文博、杨娟等，Linear Algebra.

参考(非必需)

1. S.J. Leon, L. de Pillis, Linear Algebra with applications, 机械工业出版社;

2. G. Strang, Introduction to Linear Algebra, 清华大学出版社.

答疑：课程QQ群，邮件



Overview of the course

Linear algebra [线性代数] is a widely used tool to deal with many kinds of problems, such as business planning, engineering designing and so on.

Outline of the course:

- System of equations
- Matrices and determinants
- Vector spaces
- Linear transformations
- Orthogonality
- Eigenvalues

Lecture 1

Chapter 1. Equation Systems and Matrices

1.1 Systems of Linear Equations

1.2 Linear System in Matrix

Equations

Equations are like encrypted codes. You are given a certain amount of information about some unknow numbers, from which you have to deduce what the unknown numbers are.

Examples.

$$\begin{aligned} x + 3y &= 6, \\ 2x + \ \ y &= 7. \end{aligned}$$

$$x^2 = 2, \qquad \sin x = y$$

A **linear equation** is an algebraic equation in which each term is either a constant (real or complex) or the product of a constant and (the first power of) a single variable.

Systems of Linear Equations

Definition 1. A **linear equation in n unknowns [线性方程]** is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where a_i $(i = 1, 2, \ldots, n)$ and b are real numbers and x_i $(i = 1, 2, \ldots, n)$ are **variables (or unknowns) [变量，未知元]**.

Definition 2. A **linear system of m equations in n unknowns** is a system of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2,$$

$$\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m,$$

where a_{ij} $(i = 1, 2, \ldots, m, j = 1,2, \ldots, n)$ and b_i $(i = 1,2, \ldots, m)$ are real numbers. We will call this system **$m \times n$ linear system [线性代数方程组]**.

Systems of Linear Equations

Example 1. Some linear systems

(a)
$$\begin{aligned} 2x - y &= 0, \\ -x + 2y &= 3. \end{aligned}$$

(b)
$$\begin{aligned} x_1 - x_2 + x_3 &= 2, \\ 2x_1 + x_2 - x_3 &= 4. \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 &= 2, \\ x_1 - x_2 &= 1, \\ x_1 \phantom{{}+x_2} &= 4. \end{aligned}$$

Can you solve these systems?

Systems of Linear Equations

Definition 3. The **solution [解]** of an $m \times n$ linear system is an ordered n-tuple of numbers

$$(x_1, x_2, \dots, x_n),$$

which satisfies all equations of the $m \times n$ linear system.

If there is at least one solution of an $m \times n$ linear system, we say the linear system is **consistent [相容]**. Otherwise, we say the linear system is **inconsistent [不相容]**.

If there is more than one solution, we call the set of solutions the **solution set [解集].**

Systems of Linear Equations

It is clear that the ordered pair $(1,2)$ is a solution to the system **(a)** in **Example 1**, since

$$\begin{aligned} 2\cdot(1)-1\cdot(2)&=0,\\ -1\cdot(1)+2\cdot(2)&=3. \end{aligned}$$

The ordered triple $(2,0,0)$ is a solution to the system **(b)**, since

$$1\cdot(2)-1\cdot(0)+1\cdot(0)=2,$$

$$2\cdot(2)+1\cdot(0)-1\cdot(0)=4.$$

Actually, let α be any real number, then the ordered triple $(2,\alpha,\alpha)$ is a solution.

However, the system **(c)** has no solution. No ordered pair will satisfy all the three equations in the system **(c)**.

2×2 Systems

Let us consider the following systems

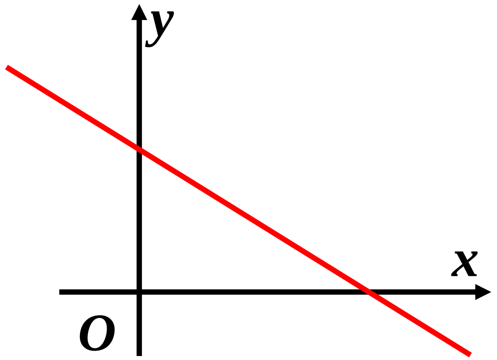
$$a_{11}x_1 + a_{12}x_2 = b_1,$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

Geometrically, a linear equation in 2 variables represents a line on the coordinate plane, and any line on the coordinate plane can be expressed as a linear equation in 2 variables.

The ordered pair (x_1, x_2) is a solution of the 2×2 system if and only if the point (x_1, x_2) lies on both two lines.

$$ax + by = c$$

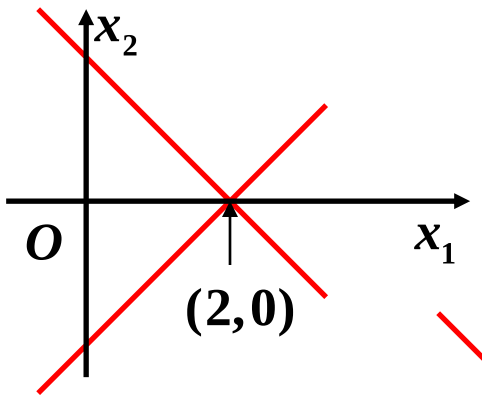
if and only if



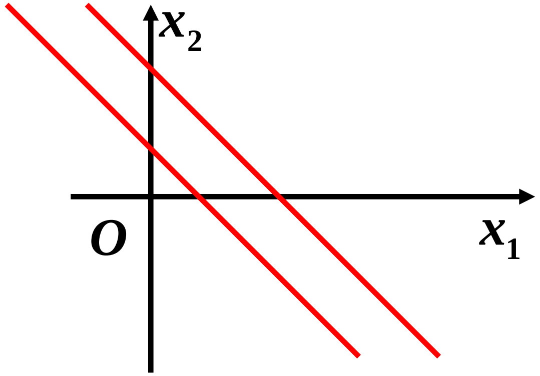
2×2 Systems

Example 2. 2×2 systems

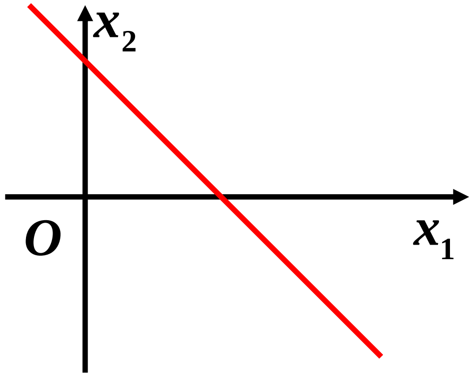
$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \end{aligned}$$



$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned}$$



$$\begin{aligned} x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}$$



2×2 Systems

There are only three possible relative positions for two lines on the xOy plane: **intersecting**, **parallel** or **coincident**.

The consistency of all 2×2 linear systems must be one of the following three cases:

(1) Consistent, with unique solution;

(2) Consistent, with infinite number of solutions;

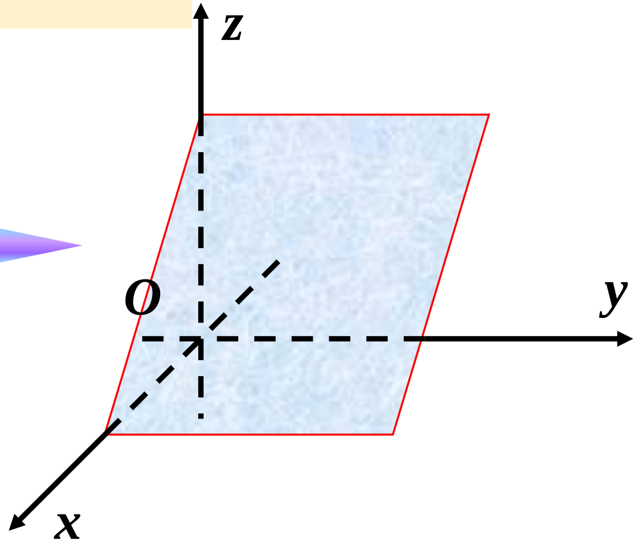
(3) Inconsistent.

$m \times 3$ Systems

Thinking:

1. What is the graph of a linear equation in 3 variables?
2. How many kinds of solutions of a 2×3 system?
3. How many kinds of solutions of a 3×3 system?
4. How many kinds of solutions of a $m \times 3$ system?

$$\boldsymbol{ax + by + cz = d}$$



Equivalent Systems

Definition 4. Two linear systems are said to be **equivalent [等价]** if they have the same solution set.

Theorem 1. (Properties of Equivalence) Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be three linear systems, then

(1) If $\boldsymbol{A}$ is equivalent to $\boldsymbol{B}$ and $\boldsymbol{B}$ is equivalent to $\boldsymbol{C}$, then $\boldsymbol{A}$ is equivalent to $\boldsymbol{C}$;

(2) If $\boldsymbol{A}$ is equivalent to $\boldsymbol{C}$ and $\boldsymbol{B}$ is equivalent to $\boldsymbol{C}$, then $\boldsymbol{A}$ is equivalent to $\boldsymbol{B}$.

Equivalent Systems

Example 3. Consider the following two systems

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 \qquad\quad &= \ \ 3 \\ 2x_3 &= \ \ 4 \end{aligned}$$

and

$$\begin{aligned} 3x_1 + 2x_2 - \ \ x_3 &= -2 \\ -3x_1 - \ \ x_2 + \ \ x_3 &= \ \ 5 \\ 3x_1 + 2x_2 + \ \ x_3 &= \ \ 2 \end{aligned}$$

These two systems have the same solution set $\{(-2,3,2)\}$. Thus they are equivalent systems.

Equivalent Systems

Theorem 2. (Operations to Obtain Equivalent Linear Systems) There are three basic operations that can be used to obtain an equivalent system from a given system:

(I) interchanging two equations;

(II) multiplying an equation by a nonzero real number;

(III) adding a constant multiple of one equation to another.

Operations **(I), (II), (III)** are generally used to derive an equivalent linear system, which is easy to be solved, from a given system.

$n \times n$ Systems

We now restrict ourselves to $n \times n$ linear systems.

Definition 5. (Strict Triangular System) An $n \times n$ linear system is said to be in **strict triangular form [严格三角形式]** if and only if in the k-th equation the coefficients of the previous $k - 1$ variables are all zero and the coefficient of the k-th variable x_k is nonzero ($k = 1,2, \dots, n$).

$n \times n$ Systems

Example 4. The system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1, \\ x_2 - x_3 &= 2, \\ 2x_3 &= 4, \end{aligned}$$

is in triangular form. It is easy to solve this system. Actually, from the last equation, we have $x_3 = 2$. Using this value in the second equation, we obtain

$$x_2 - 2 = 2$$

so $x_2 = 4$. Using $x_2 = 4$ and $x_3 = 2$ in the first equation, we end up with

$$3x_1 + 2 \cdot 4 + 2 = 1,$$

Then $x_1 = -3$. Thus the solution to the system is $(-3,4,2)$.

$n \times n$ Systems

Remark: The last example shows that if a system is in a triangular form, it is easy to be solved. The progress of solving system of triangular form is called **back substitution [回代法]**.

In general, if a system is not triangular, we are suggested to use operations **(I)-(III)** to try to change the system equivalently into strict triangular form, so that we can find the solution by back substitution.

Operations

(I) Interchange two equations.

(II) Multiply an equation by a nonzero scalar.

(III) Add a constant multiple of one equation to another.

$n \times n$ Systems

Example 5. Solve the system

$$\begin{aligned} x_1 &+ 2x_2 &+ x_3 &= 3 \\ 3x_1 &- x_2 &- 3x_3 &= -1 \\ 2x_1 &+ 3x_2 &+ x_3 &= 4 \end{aligned}$$

Solution:

$$2x_1 + 4x_2 + 2x_3 = 6$$

$$3x_1 + 6x_2 + 3x_3 = 9$$

$3\times$ $2\times$

$$x_1 + 2x_2 + x_3 = 3$$

$$3x_1 - 7x_2 - 6x_3 = -10$$

$\frac{1}{7}\times$

$$2x_1 + 3x_2 + x_3 = 4$$

$$-\frac{1}{7}x_3 = -\frac{4}{7}$$

$$-x_2 - \frac{6}{7}x_3 = -\frac{10}{7}$$

Matrix Notation

Remark. The last example also shows that the coefficient of a system is very important while using back substitution. To make it simple, we associate the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

with a 3×3 array of numbers whose entries are the coefficients of the x_i's.

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{pmatrix}$$

This array is named as the **coefficient matrix [系数矩阵]** of the system and this matrix has 3 rows and 3 columns is said to be 3×3.

Matrix Notation

Remark.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

If we attach to the coefficient matrix an additional column whose entries are the numbers on the right-hand side of the system, we obtain the new matrix

$$\left(\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right)$$

We will refer to this new matrix as the **augmented matrix [增广矩阵]**.

Matrix Notation

More generally, for an $m \times n$ linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

we associate it with the **coefficient matrix [系数矩阵]** A and the **augmented matrix [增广矩阵]** A'

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, A' = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

A has m rows and n columns, which is an $m \times n$ matrix. A' has m rows and $(n+1)$ columns, which is an $m \times (n+1)$ matrix.

Matrix Notation

If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mr} \end{pmatrix}$, then we can obtain a new $m \times (n + r)$ matrix which is denoted by

$$(A \mid B) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} & b_{12} & \cdots & b_{1r} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{m1} & b_{m2} & \cdots & b_{mr} \end{array}\right).$$

This matrix is also called **augmented matrix**.

Solving Linear Systems

Operations **(I), (II), (III)** of a linear system used to obtain equivalent systems can be corresponding to three row operations applied to the augmented matrix.

Operations used to obtain equivalent systems

(I) Interchange two equations.

(II) Multiply an equation by a nonzero scalar.

(III) Add a constant multiple of one equation to another.

Definition 1. (Elementary Row Operations) For a given matrix, there are three types of **elementary row operations [初等行变换]:**

(I) Interchange two rows $(r_i \leftrightarrow r_j)$

(II) Multiply a row by a nonzero scalar $(a \times r_i)$

(III) Add a constant multiple of one row to another $(r_i + a \times r_j)$

Solving Linear Systems

Remark: In general, the solving process of an equation systems can be expressed as three steps:

$$\begin{cases} 2x_1 + 3x_2 - 3x_3 = 9 \\ x_1 + 2x_2 + x_3 = 4 \\ 3x_1 + 7x_2 + 4x_3 = 19 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 2 & 3 & -3 & 9 \\ 1 & 2 & 1 & 4 \\ 3 & 7 & 4 & 19 \end{array}\right)$$

$\Updownarrow$ **Equivalent Systems** $\qquad\Downarrow$ **Elementary Row Operations**

$$\begin{cases} \mathbf{x_1} + \mathbf{2x_2} + \mathbf{x_3} = \mathbf{4} \\ \quad - \mathbf{x_2} - \mathbf{5x_3} = \mathbf{1} \\ \quad\quad - \mathbf{4x_3} = \mathbf{8} \end{cases} \Leftarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & -5 & 1 \\ 0 & 0 & -4 & 8 \end{array}\right)$$

Solving Linear Systems

Example 1. Solve the system

$$\begin{aligned} x_1 + 2x_2 + \quad x_3 &= \quad 3, \\ 3x_1 - \quad x_2 - 3x_3 &= -1, \\ 2x_1 + 3x_2 + \quad x_3 &= \quad 4. \end{aligned}$$

Solution. The augmented matrix of this system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right).$$

We can use the elementary row operations to solve this system. We refer to the first line as the **pivotal row [主行].** The first nonzero entry of the pivotal row is called the **pivot [主元].**

Solving Linear Systems

Example 1. Solve the system

$$\begin{aligned} x_1 + 2x_2 + \quad x_3 &= \quad 3, \\ 3x_1 - \quad x_2 - 3x_3 &= -1, \\ 2x_1 + 3x_2 + \quad x_3 &= \quad 4. \end{aligned}$$

Solution. (continue)

$$\begin{array}{r} \textbf{pivot } \boldsymbol{a_{11}} = \mathbf{1} \\ \textbf{Entries to be eliminated} \\ \boldsymbol{a_{21}} = \mathbf{3} \textbf{ and } \boldsymbol{a_{31}} = \mathbf{2} \end{array} \quad \left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right) \quad \begin{array}{l} \leftarrow \textbf{pivotal row} \\ \\ \\ \end{array}$$

By using row operation III, 3 times the first row is subtracted from the second row and two times the first row is subtracted from the third row. Then we have

Solving Linear Systems

Example 1. Solve the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3, \\ 3x_1 - x_2 - 3x_3 &= -1, \\ 2x_1 + 3x_2 + x_3 &= 4. \end{aligned}$$

Solution. (continue)

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & \boxed{-7} & -6 & -10 \\ 0 & -1 & -1 & -2 \end{array}\right) \leftarrow \textbf{pivotal row}$$

Again, by row operation III, $\frac{1}{7}$ times the second row is subtracted from the third row, Then we have

Solving Linear Systems

Example 1. Solve the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3, \\ 3x_1 - x_2 - 3x_3 &= -1, \\ 2x_1 + 3x_2 + x_3 &= 4. \end{aligned}$$

Solution. (continue)

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{array}\right)$$

It is clear that the solution is $x_3 = 4, x_2 = -2$ and $x_1 = 3$. This is the end.

Solving Linear Systems

Example 1. The above process of elimination can be written in terms of augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right) \xrightarrow[r_3+(-2)r_1]{r_2+(-3)r_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{r_3+(-\frac{1}{7})r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & 10 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{array}\right).$$

Strict Triangular Form

Solving Linear Systems

Example 2. Solve the system

$$\begin{aligned} -\ x_2 - \ x_3 + \ x_4 &= 0, \\ x_1 + \ x_2 + \ x_3 + \ x_4 &= 6, \\ 2x_1 + 4x_2 + \ x_3 - 2x_4 &= -1, \\ 3x_1 + \ x_2 - 2x_3 + 2x_4 &= 3. \end{aligned}$$

Solution. The augmented matrix for this system is

0 can not eliminate any entry.

$$\left(\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right).$$

So we will use operation I to interchange the first two rows and the pivot element will be 1

Solving Linear Systems

Example 2. Solve the system

$$\begin{aligned} - x_2 - x_3 + x_4 &= 0, \\ x_1 + x_2 + x_3 + x_4 &= 6, \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -1, \\ 3x_1 + x_2 - 2x_3 + 2x_4 &= 3. \end{aligned}$$

Solution. (continue) Then the new augmented matrix for this system is

$$\textbf{(pivot } \boldsymbol{a_{11}} = \textbf{1)} \quad \left(\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right). \quad \begin{array}{l} \leftarrow \textbf{pivot row} \\ \\ \\ \\ \end{array}$$

By elementary row operations III three times, we have

Solving Linear Systems

Example 2. Solve the system

$$\begin{aligned} - \quad x_2 - \quad x_3 + \quad x_4 &= \ \ 0, \\ x_1 + \quad x_2 + \quad x_3 + \quad x_4 &= \ \ 6, \\ 2x_1 + 4x_2 + \quad x_3 - 2x_4 &= -1, \\ 3x_1 + \quad x_2 - 2x_3 + 2x_4 &= \ \ 3. \end{aligned}$$

Solution. (continue)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right).$$

Then by back substitution, the solution is $(2, -1, 3, 2)$.

Solving Linear Systems

Remarks. In general, if an $n \times n$ linear system can be reduced to triangular form, then it will have a unique solution that can be obtained by performing back substitution on the triangular system. The reduction process can be thought as an algorithm with $n - 1$ steps.

Solving Linear Systems

Remarks. However, this procedure will break down if, at any step, all possible choices for a pivot element are equal to 0. When this happens, we can reduce the system to certain special echelon or staircase-shaped forms.

This form will also be used for $m \times n$ systems, where $m \neq n$.

Matrix

Definition 2. (Row and Column Vector) An $m \times n$ matrix is a rectangular arrangement of numbers with m **rows [行]** and n **columns [列]** and is denoted by

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad or \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where m and n are positive integers, a_{ij} $(i = 1,2, \dots, m, j = 1,2, \dots, n)$ is called the ith row and jth column entry. A matrix with one row (a $1 \times n$ matrix) is called a **row vector [行向量],** and a matrix with one column (an $m \times 1$ matrix) is called a **column vector [列向量].** An $n \times n$ matrix is called a **square matrix [方阵].**

Matrix

Example. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \end{pmatrix}$ is a 2×3 matrix, $\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 4 \end{pmatrix}$ is a 3×2 matrix,

$\begin{pmatrix} 1 & 2 \end{pmatrix}$ is a 1×2 matrix or a row vector,

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a 2×1 matrix or a column vector,

and (-1) is a 1×1 square matrix.

Elimination Process

The elimination process can be illustrated as follows

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$

where "$*$" represents any possible entries.

- Elementary row operations
- Back substitution

Review

- **Definitions:** Linear systems, Solution, Equivalent system, Triangular form
- **Terms:** elementary row operations, back substitution, coefficient matrix, augmented matrix, consistent, inconsistent

Preview

- Reduced Row Echelon Form
- Consistency of Linear Systems

Exercises

P9: 2(e), 4, 6;
P14: 3(e)(f), 4(a)(c), 5.