Lecture 15

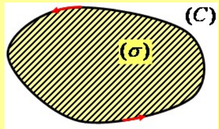
Gauss' Formula and Stokes' Formula

Green's Theorem

Theorem Suppose there is a closed bounded domain $(\sigma)\subset \mathbf{R}^2$ bounded by a piecewise smooth simple curve (C), and functions $P(x,y),Q(x,y)\in C^{(1)}((\sigma))$. Then the following relation holds:

$$\iint_{(\sigma)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma=\oint_{(+C)}\left[P(x,y)dx+Q(x,y)dy\right],$$

where $(+C)$ indicates that the integration is in the positive direction of (C).



Gauss' Formula

Theorem **Gauss' Theorem**

Suppose that a region (V) in space is bounded by a piecewise smooth closed simple surface (S), and $P(x,y,z),Q(x,y,z),R(x,y,z)\in C^{(1)}((V))$.

Then $$\iiint_{(V)}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV=\oint\!\!\oint_{(+S)}Pdydz+Qdzdx+Rdxdy,$$

where $(+S)$ denote the normal vector of (S) pointing to the outside of (V).

The Gauss' Theorem gives the relationship between a triple integral over a region (V) in space and a surface integral of the second type over the boundary surface (S) of the region (V).

Gauss' Formula

Proof. We only prove that

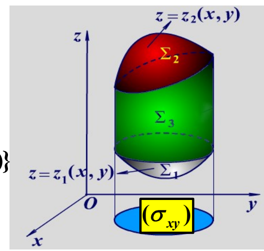
$$\iiint_{(V)}\frac{\partial R}{\partial z}dV=\oint\!\!\oint_{(+S)}Rdxdy$$

(1) Suppose that the region is

$$V=\{(x,y,z)\,|\,z_1(x,y)\le z\le z_2(x,y),(x,y)\in(\sigma_{xy})\}$$

$$\iiint_{(V)}\frac{\partial R}{\partial z}dV=\iint_{(\sigma_{xy})}d\sigma\int_{z_1(x,y)}^{z_2(x,y)}\frac{\partial R}{\partial z}dz$$

$$=\iint_{(\sigma_{xy})}\{R(x,y,z_2(x,y))-R(x,y,z_1(x,y))\}d\sigma$$



(2) For the surface integral $\oint\!\!\oint_{(+S)}Rdxdy=\iint_{\Sigma_2}Rdxdy+\iint_{\Sigma_3}Rdxdy+\iint_{\Sigma_1}Rdxdy$

$$\oint\!\!\oint_{(+S)}Rdxdy=\iint_{(\sigma_{xy})}R(x,y,z_2(x,y))d\sigma+0+\iint_{(\sigma_{xy})}-R(x,y,z_1(x,y))d\sigma$$

Gauss' Formula

Note:

$$\oint\!\!\oint_{(+S)}\vec{\mathbf{F}}\cdot\vec{\mathbf{dS}}=\oint\!\!\oint_{(+S)}\vec{\mathbf{F}}\cdot\vec{\mathbf{e}}_n dS=\oint\!\!\oint_{(S)}(P,Q,R)\cdot\vec{\mathbf{e}}_n dS$$

$$=\oint\!\!\oint_{(S)}(P\cos\alpha+Q\cos\beta+R\cos\gamma)dS$$

$$=\oint\!\!\oint_{(S)}Pdy\wedge dz+Qdz\wedge dx+Rdx\wedge dy$$

$$=\iiint_{(V)}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV=\iiint_{(V)}\nabla\cdot\vec{\mathbf{F}}dV,\qquad \nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right).$$

where the normal vector of (S) points the outside of (V). α,β,γ are the angles between the outward normal vector at (x,y,z) on the surface and the three axes.

Gauss' Theorem

Example 1 Evaluate the surface integral $I=\iint_{(S)}x^3dydz+y^3dzdx+z^3dxdy$ where (S) is: (1) the outside of the sphere $x^2+y^2+z^2=R^2$.

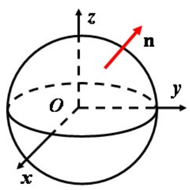
(2) the upper side of the upper hemisphere $z=\sqrt{R^2-x^2-y^2}$.

Solution (I) Suppose that the region bounded by (S) is (V).

According to the Gauss' theorem

$$I=\iiint_{(V)}3(x^2+y^2+z^2)dV$$

$$=3\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^{R}r^4\sin\varphi\,dr=\frac{12}{5}\pi R^5$$



Gauss' Theorem

Example 1 Evaluate the surface integral $I=\iint\limits_{(S)} x^3dydz+y^3dzdx+z^3dxdy$

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Solution (II) The surface (S) is not closed. We adjoin the circular surface $(S_1): z=0, x^2+y^2\le R^2$ with downward normal.

According to the Gauss' theorem

$$\iint\limits_{(S)} x^3dydz+y^3dzdx+z^3dxdy+\iint\limits_{(S_1)} x^3dydz+y^3dzdx+z^3dxdy$$

$$=\iiint\limits_{(V_1)} 3(x^2+y^2+z^2)dV=\frac{6}{5}\pi R^5$$

$$\iint\limits_{(S_1)} x^3dydz+y^3dzdx+z^3dxdy=0+0-\iint\limits_{(\sigma_{xy})} 0^3dxdy=0$$

Therefore $I=\frac{6}{5}\pi R^5-0=\frac{6}{5}\pi R^5$

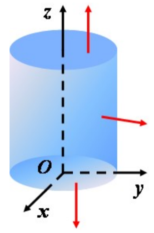
Gauss' Theorem

Example 2 Evaluate $I=\oiint\limits_{\Sigma}(x-y)dxdy+(y-z)xdydz$, where Σ is the outside of the surface of the cylinder $x^2+y^2=1$ and $z=0,\ z=3.$

Solution Since $P=(y-z)x, Q=0, R=x-y,$ we have

$$\frac{\partial P}{\partial x}=y-z,\quad \frac{\partial Q}{\partial y}=0,\quad \frac{\partial R}{\partial z}=0.$$

Therefore, by the Gauss' Theorem, we obtain



$$I=\iiint\limits_{(V)}(y-z)dxdydz$$

$$=\int_0^{2\pi}d\theta\int_0^1 d\rho\int_0^3(\rho\sin\theta-z)\rho dz=-\frac{9\pi}{2}.$$

Gauss' Theorem

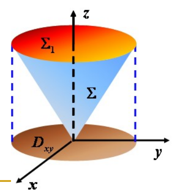
Example 3. Evaluate $I=\iint\limits_{\Sigma}(x^2\cos\alpha+y^2\cos\beta+z^2\cos\gamma)dS$, where Σ is the surface of the cone $x^2+y^2=z^2$ between $z=0$ and $z=h(h>0)$, and α,β,γ are the angles between the normal vector oriented downward at (x,y,z) on the surface and the three axes.

Solution By adding an additional plane $\Sigma_1: z=h,(x,y)\in D_{xy}: x^2+y^2\le h^2$ with upperward norm, we can use the Gauss' Theorem.

$$\iint\limits_{\Sigma+\Sigma_1}(x^2\cos\alpha+y^2\cos\beta+z^2\cos\gamma)dS$$

$$=\iint\limits_{\Sigma+\Sigma_1} x^2dy\wedge dz+y^2dz\wedge dx+z^2dx\wedge dy$$

$$=2\iiint\limits_{(V)}(x+y+z)dV,$$



Gauss' Theorem

Solution (continued) By the symmetry, we know that $\iiint\limits_{(V)}(x+y)dV=0,$

then $\iint\limits_{\Sigma+\Sigma_1}(x^2\cos\alpha+y^2\cos\beta+z^2\cos\gamma)dS=2\iiint\limits_{(V)}(x+y+z)dV$

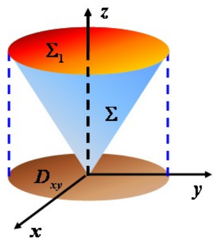
$$=2\iiint\limits_{(V)} zdV=2\int_0^{2\pi}d\theta\int_0^h\rho d\rho\int_\rho^h zdz=\frac{1}{2}\pi h^4.$$

Moreover, since

$$\iint\limits_{\Sigma_1}(x^2\cos\alpha+y^2\cos\beta+z^2\cos\gamma)dS$$

$$=\iint\limits_{\Sigma_1} z^2dx\wedge dy=\iint\limits_{D_{xy}} h^2d\sigma=\pi h^4,$$

we have $I=\frac{1}{2}\pi h^4-\pi h^4=-\frac{1}{2}\pi h^4.$



Flux and Flux Density

Suppose that $\vec{\mathbf{F}}$ be a vector field on $(G)\subseteq\mathbf{R}^3$ and (S) be a directed surface in (G). Then the flux of the fluid passing through the surface (S) to the given side is

$$\Phi=\iint\limits_{(S)}\vec{\mathbf{F}}\cdot\vec{\mathbf{dS}}=\iint\limits_{(S)}Pdydz+Qdzdx+Rdxdy$$

Flux density is defined as $\lim\limits_{(\Delta V)\to M}\frac{\Delta\Phi}{\Delta V}=\lim\limits_{(\Delta V)\to M}\frac{1}{\Delta V}\oiint\limits_{(\Delta S)}\vec{\mathbf{F}}\cdot\vec{\mathbf{dS}}$

Divergence

Definition (Divergence)

Consider a continuous vector field $\vec{\mathbf{F}}$ defined on $(V)\subseteq\mathbf{R}^3$ and construct an arbitrary closed surface $(\Delta S)\subseteq(V)$ in the neighborhood of M which contains the point M with normal vector of (ΔS) pointing outwards. The region bounded by (ΔS) is denoted by (ΔV) with volume ΔV. If the limit of the ratio $\frac{\Delta\Phi}{\Delta V}=\frac{1}{\Delta V}\oiint\limits_{(\Delta S)}\vec{\mathbf{F}}\cdot\vec{dS}$ exists when (ΔV) shrinks to the points M in any way, then this limiting value is said to be the **divergence** of $\vec{\mathbf{F}}$ at the point M, denoted by $div\vec{\mathbf{F}}(M)=\lim\limits_{(\Delta V)\to M}\frac{1}{\Delta V}\oiint\limits_{(\Delta S)}\vec{\mathbf{F}}\cdot\vec{dS}.$

Computation of Divergence

Construct a rectangular coordinate system and let

$$div\vec{F}(M)=\lim_{(\Delta V)\to M}\frac{\Delta\Phi}{\Delta V}$$

$$\vec{F}=(P(x,y,z),Q(x,y,z),R(x,y,z))$$

where $P,Q,R\in C^{(1)}$. Then according to the definition of the flux, Gauss' formula and the mean value theorem for the integral, we have

$$\Delta\Phi=\oiint_{(\Delta S)}\vec{F}\cdot\vec{dS}=\oiint_{(\Delta S)}Pdydz+Qdzdx+Rdxdy$$

$$=\iiint_{(\Delta V)}(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z})dV=\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Bigg|_{M^*}\Delta V$$

Therefore, $$div\ \vec{F}(M)=\lim_{(\Delta V)\to M}\frac{1}{\Delta V}\oiint_{(\Delta S)}\vec{F}\cdot\vec{dS}=\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Bigg|_M$$

Computation of Divergence

The divergence of a vector field

$$\vec{F}(x,y,z)=P(x,y,z)\mathbf{i}+Q(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$$

is the scalar function

$$\boxed{div\ \vec{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\nabla\cdot\vec{F}.}$$

Flux of a vector field $\vec{F}$ across a closed oriented surface (S)

$$\oiint_{(+S)}\vec{F}\cdot\vec{dS}=\oiint_{(+S)}\vec{F}\cdot\vec{e}_n dS=\iiint_{(V)}\nabla\cdot\vec{F}dV=\iiint_{(V)}div\vec{F}dV.\quad \vec{F}=(P,Q,R)$$

Example 4 Find the divergence of $\vec{F}(x,y,z)=2xz\mathbf{i}-xy\mathbf{j}-z\mathbf{k}$.

Solution The divergence of $\vec{F}$ is

$$\nabla\cdot\vec{F}=\frac{\partial}{\partial x}(2xz)+\frac{\partial}{\partial y}(-xy)+\frac{\partial}{\partial z}(-z)\ =2z-x-1.$$

Stokes' Theorem

Theorem Stokes' Theorem

Suppose $P,Q,R\in C^{(1)}(G)$ and (C) is a piecewise smooth directed simply closed curve, and (S) is a piecewise smooth oriented surface in G whose boundary is (C), and the direction of (C) and the normal vector of the surface (S) accord with the right-hand rule. Then

$$\oint_{(C)}Pdx+Qdy+Rdz=\iint_{(S)}(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})dydz+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})dzdx+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy$$

$$=\iint_{(S)}\begin{vmatrix}dydz & dzdx & dxdy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

Stokes' Theorem

Theorem Stokes' Theorem

The circulation of a vector field $\vec{F}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ around the boundary (C) of an oriented surface (S) in the direction counterclockwise with respect to the surface's unit normal vector $\vec{e}_n$ equals the integral of $\nabla\times\vec{F}\cdot\vec{e}_n$ over surface (S).

Counterclockwise circulation $\boxed{\oint_{(C)}\vec{F}\cdot\vec{ds}}=\boxed{\iint_{(S)}\boxed{\nabla\times\vec{F}\cdot\vec{e}_n}dS}$ Curl Integral

circulation density

Note If two different oriented surface$(S_1)$and(S_2) have the same boundary (C), their curl integral are same.

Green's Theorem versus Stokes' Theorem

If (C) is a curve in the xy – plane, oriented counterclockwise, and (σ) is the region in the xy – plane bounded by (C), then $dS=d\sigma=dxdy$ and

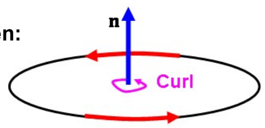
$$(\nabla\times\vec{F})\cdot\vec{e}_n=(\nabla\times\vec{F})\cdot\vec{k}=\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right).$$

Under these conditions, Stokes' equation becomes

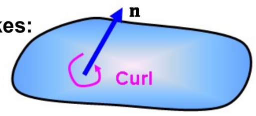
$$\oint_{(C)}Pdx+Qdy=\oint_{(C)}\vec{F}\cdot\vec{ds}=\iint_{(S)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dS=\iint_{(\sigma)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy.$$

Green's Theorem

Green:



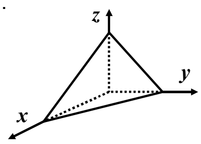
Stokes:



Find Line Integral by Stokes' Theorem

Example 5 Find $\oint_\Gamma zdx+xdy+ydz$ where Γ cuts from the plane $x+y+z=1$ by the three coordinates plane and its positive direction has right – hand relation with the normal up vector.

Solution By Stokes' Theorem, we have



$$\oint_\Gamma zdx+xdy+ydz\ =\iint_\Sigma\begin{vmatrix}dydz & dzdx & dxdy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z & x & y\end{vmatrix}$$

$$=\iint_\Sigma dydz+dzdx+dxdy$$

Method I By the symmetry, we have $\iint_\Sigma dydz+dzdx+dxdy=3\iint_{(\sigma_{xy})}d\sigma=\frac{3}{2}.$

Find Line Integral by Stokes' Theorem

Solution(Cont.) Method II $\vec{e}_n=\frac{1}{\sqrt{3}}(1,1,1), z=1-x-y, z_x=z_y=-1,$

$$\iint_{\Sigma} dydz+dzdx+dxdy=\iint_{\Sigma}\sqrt{3}dS=3\iint_{(\sigma_{xy})} d\sigma=\frac{3}{2}$$

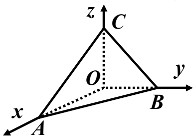
Method III $\iint_{\Sigma} dydz+dzdx+dxdy=\iint_{(\sigma_{yz})} d\sigma+\iint_{(\sigma_{zx})} d\sigma+\iint_{(\sigma_{xy})} d\sigma=\frac{3}{2}$

Method IV

$$\overrightarrow{AB}: y=1-x, x:1\to 0; \overrightarrow{BC}: z=1-y, y:1\to 0; \overrightarrow{CA}: x=1-z, z:1\to 0;$$

$$\oint_{\Gamma} zdx+xdy+ydz=(\int_{AB}+\int_{BC}+\int_{CA})zdx+xdy+ydz$$

$$=\int_1^0 x(-1)dx+\int_1^0 y(-1)dy+\int_1^0 z(-1)dz=\frac{3}{2}.$$



Curl (Rotation)

If the velocity field is $\vec{\mathbf{F}}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$

Define $\mathbf{curl}\,\vec{\mathbf{F}}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}.$

If we let $\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z},$

the $\mathbf{curl}\,\vec{\mathbf{F}}$ is $\nabla\times\vec{\mathbf{F}}$:

$$\nabla\times\vec{\mathbf{F}}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}=\mathbf{curl}\,\vec{\mathbf{F}}.$$

$$\oint_{(C)}\vec{\mathbf{F}}\cdot\vec{ds}=\iint_{(S)}\nabla\times\vec{\mathbf{F}}\cdot\vec{e}_n dS=\iint_{(S)}\nabla\times\vec{\mathbf{F}}\cdot\vec{\mathbf{dS}}=\iint_{(S)}\mathbf{curl}\,\vec{\mathbf{F}}\cdot\vec{\mathbf{dS}}$$

Curl (Rotation)

Example 6 Find the curl of $\vec{\mathbf{F}}=(x^2-y)\mathbf{i}+4z\mathbf{j}+x^2\mathbf{k}.$

Solution

$$\mathbf{curl}\,\vec{\mathbf{F}}=\nabla\times\vec{\mathbf{F}}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

$$=\left(\frac{\partial}{\partial y}(x^2)-\frac{\partial}{\partial z}(4z)\right)\mathbf{i}+\left(\frac{\partial}{\partial z}(x^2-y)-\frac{\partial}{\partial x}(x^2)\right)\mathbf{j}$$

$$+\left(\frac{\partial}{\partial x}(4z)-\frac{\partial}{\partial y}(x^2-y)\right)\mathbf{k}$$

$$=-4\mathbf{i}-2x\mathbf{j}+\mathbf{k}.$$