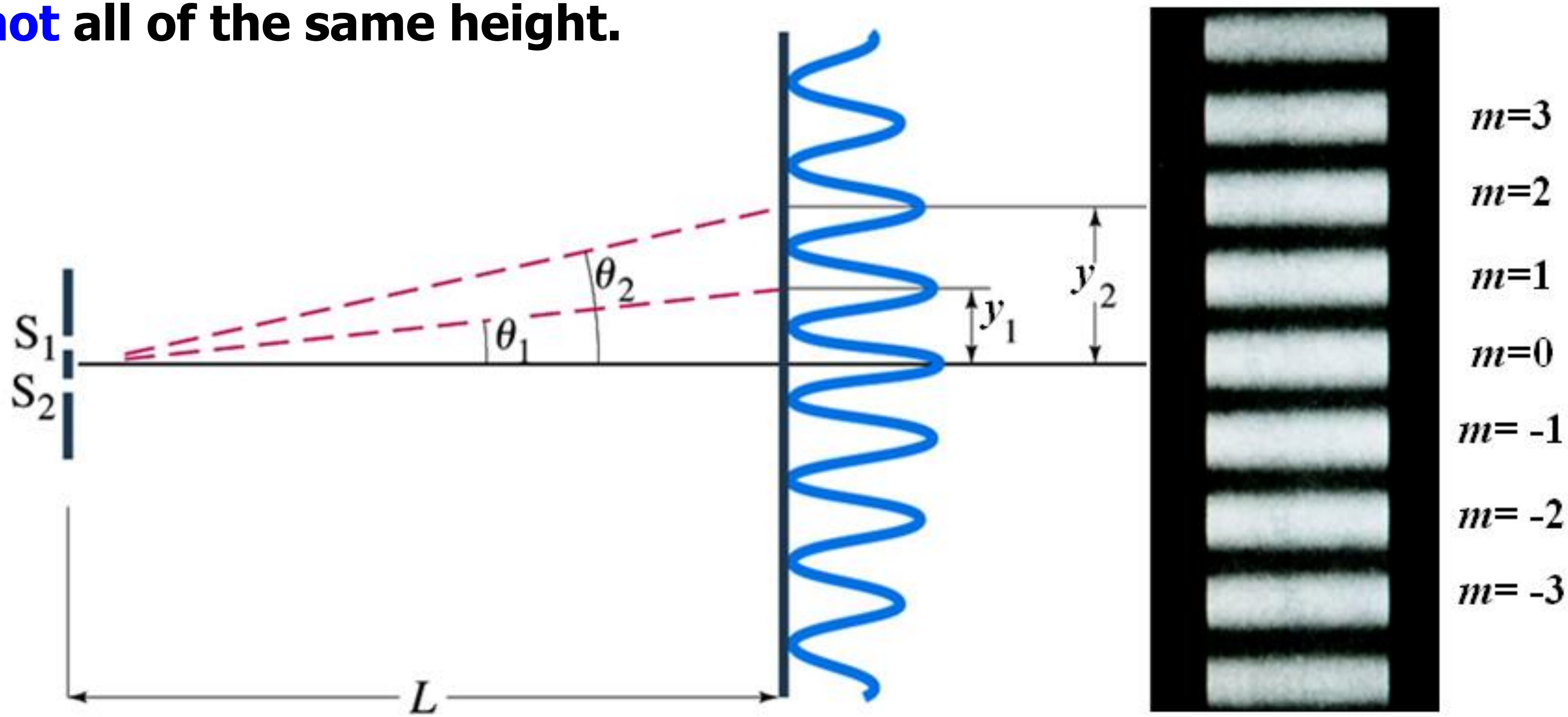
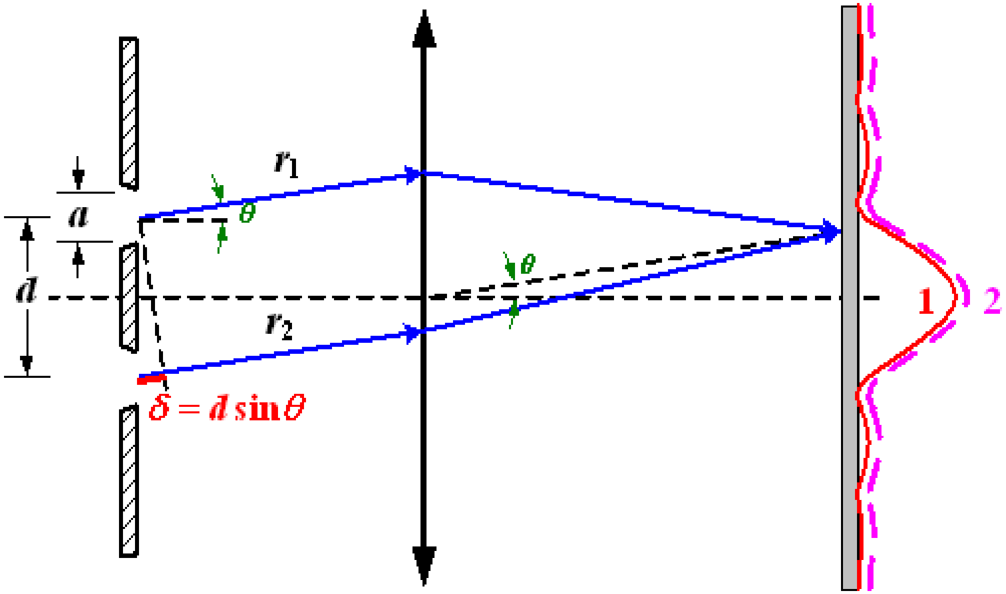
§ 4 Diffraction in the Double-Slit Experiment

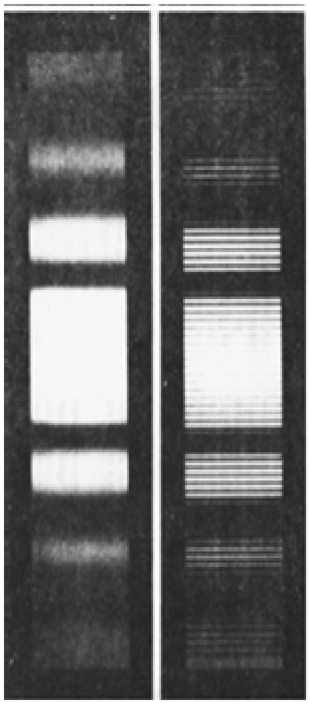
- In our previous analysis of Young's double-slit experiment, we assumed that the slits were infinitesimally narrow, so that the central diffraction maximum was spread out over the whole screen.
- For the case of real slits, the diffraction reduces the intensity of the bright interference fringes to the side of center so they are not all of the same height.

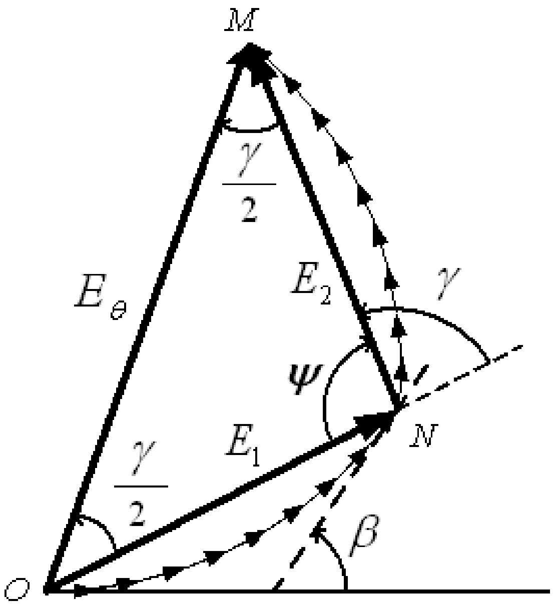


Intensity of double-slit diffraction

- The diffraction pattern is the interference fringes **modulated** by the single-slit diffraction.





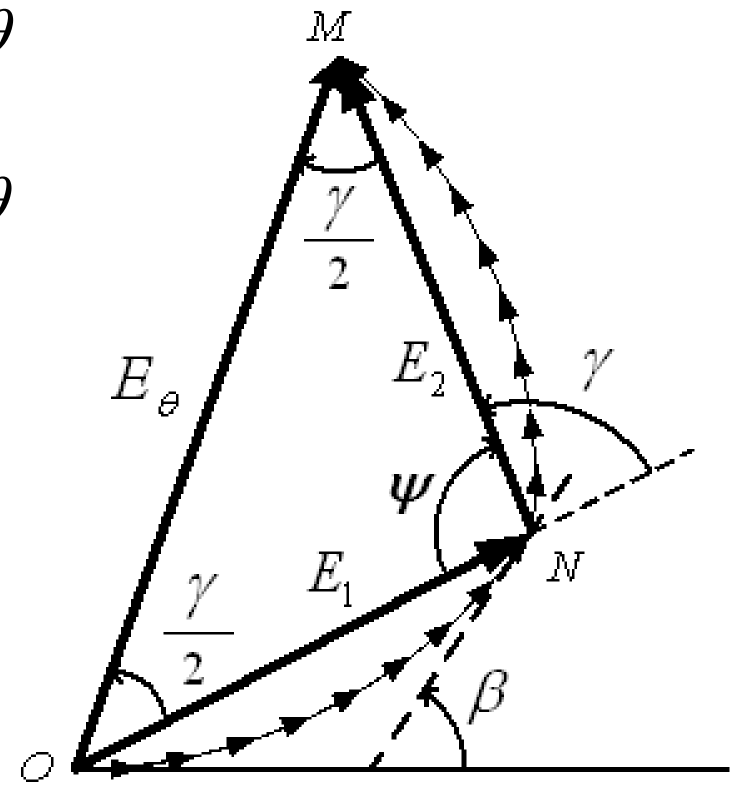


Intensity of double-slit diffraction

$$E_1 = E_2 = E_0 \frac{\sin \beta/2}{\beta/2}, \qquad \beta = \frac{2\pi}{\lambda} a \sin\theta$$

$$E_\theta = 2E_1 \cos\frac{\gamma}{2}, \qquad \gamma = \frac{2\pi}{\lambda} d \sin\theta$$

$$E_\theta = 2E_0 \left(\frac{\sin \beta/2}{\beta/2}\right) \cos\frac{\gamma}{2}$$



$$I_\theta = (4I_0)\left(\frac{\sin \beta/2}{\beta/2}\right)^2 \left(\cos\frac{\gamma}{2}\right)^2$$

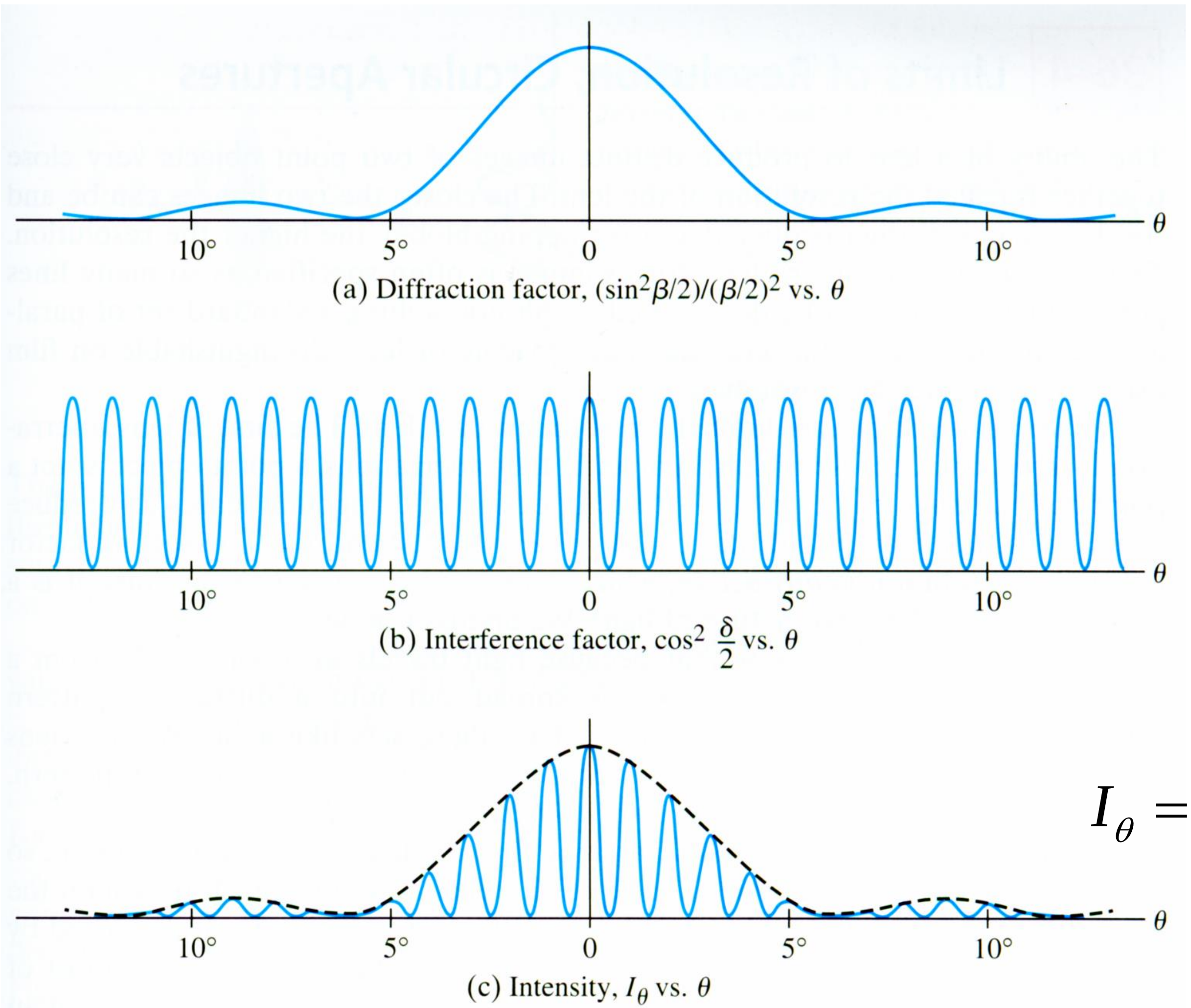
diffraction factor **interference factor**

$$I_0 \propto (E_0)^2, \quad a \to 0 \Rightarrow \beta \to 0 \Rightarrow \frac{\sin(\beta/2)}{\beta/2} \to 1, \quad I = (4I_0)(\cos\gamma/2)^2$$

$$d \to 0 \Rightarrow \gamma \to 0 \Rightarrow \cos\frac{\gamma}{2} \to 1, \quad I = (4I_0)\left(\frac{\sin \beta/2}{\beta/2}\right)^2$$

Intensity of double-slit diffraction

- **The diffraction pattern is the interference fringes modulated by the single-slit diffraction.**

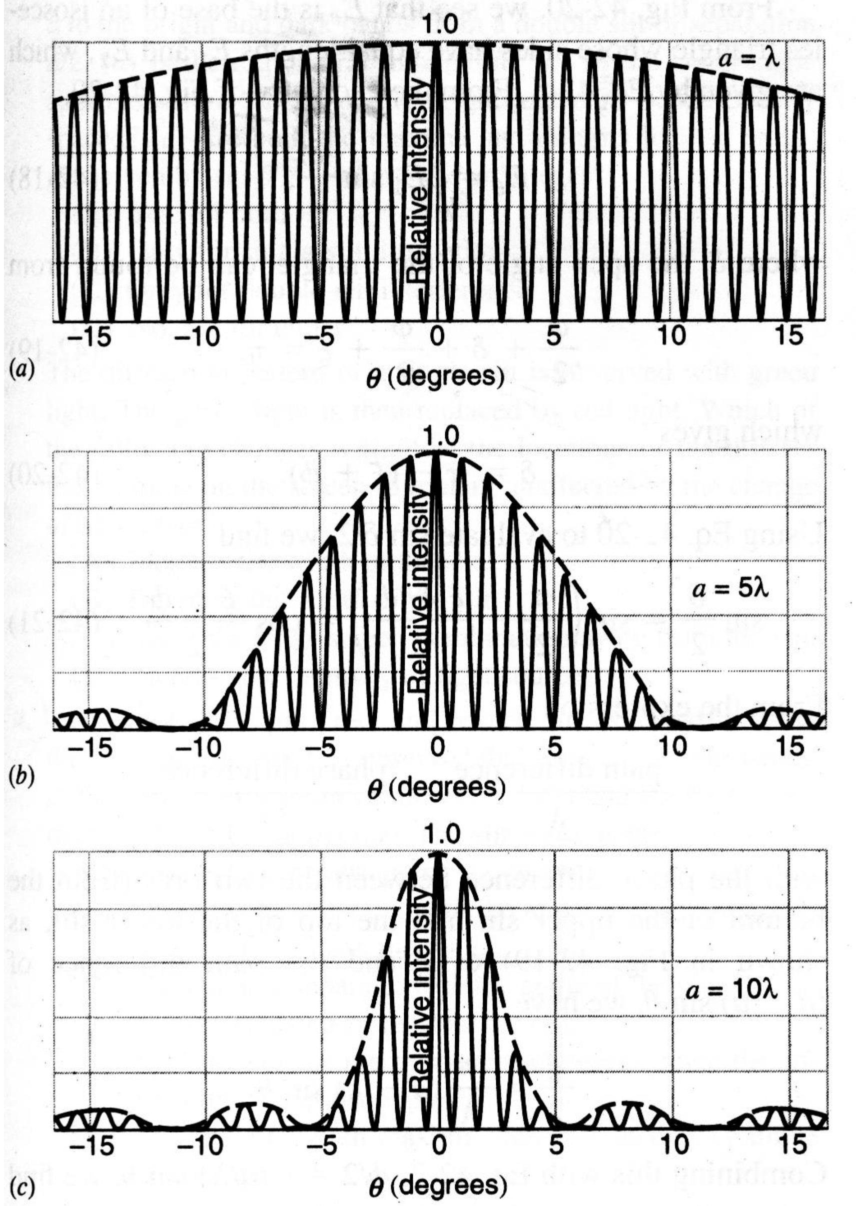


$$\beta = \frac{2\pi}{\lambda} a \sin\theta$$

$$\gamma = \frac{2\pi}{\lambda} d \sin\theta$$

$$I_\theta = (4I_0)\left(\frac{\sin \beta/2}{\beta/2}\right)^2 \left(\cos\frac{\gamma}{2}\right)^2$$

Intensity of double-slit diffraction



$$\boxed{I_\theta = 4I_0\left(\frac{\sin\beta/2}{\beta/2}\right)^2\left(\cos\frac{\gamma}{2}\right)^2}$$

$$\beta = \frac{2\pi}{\lambda}a\sin\theta,\ \ \gamma = \frac{2\pi}{\lambda}d\sin\theta$$

Diffraction minimum:

$$a\sin\theta = \pm n\lambda, \quad n = 1, 2, 3, \ldots$$

Interference maximum:

$$d\sin\theta = \pm m\lambda, \quad m = 0, 1, 2, \ldots$$

Example

A diffraction pattern is produced by a light of wavelength **580 nm** from a distant source incident on two identical parallel slits separated by a distance of **0.530 mm**. (a) What is the angular positions of the first and second order maxima? (b) Let the slits have width **0.320** mm. In terms of the intensity at the center of the central maximum, what is the intensity at each of the angular positions in part (a)?

Solution: (a) λ=580 nm, $\boldsymbol{d}$=0.530 mm.

For the first maximum:

$$\sin\theta_1 = \lambda / d = 580\times10^{-9}\,\mathrm{m} / 0.530\times10^{-3}\,\mathrm{m} = 1.09\times10^{-3}$$

$$\theta_1 \approx \sin\theta_1 = 1.09\times10^{-3}\ \mathrm{rad}$$

for the second maximum

$$\theta_2 \approx \sin\theta_2 = 2\lambda / d = 2\times580\times10^{-9}\,\mathrm{m} / 0.530\times10^{-3}\,\mathrm{m} = 2.19\times10^{-3}\ \mathrm{rad}$$

Example (Cont'd)

$$I_\theta = (4I_0)\left(\frac{\sin\beta/2}{\beta/2}\right)^2\left(\cos\frac{\gamma}{2}\right)^2, \qquad \beta = \frac{2\pi}{\lambda}a\sin\theta, \quad \gamma = \frac{2\pi}{\lambda}d\sin\theta$$

(b) Considering the effect of single-slit diffraction:

$$I = (4I_0)\left(\frac{\sin\beta/2}{\beta/2}\right)^2, \qquad \frac{\beta}{2} = \frac{\pi}{\lambda}a\sin\theta \approx \frac{\pi}{\lambda}a\theta = 1.73\times10^3\theta$$

$$I_1 = (4I_0)\left[\frac{\sin(1.73\times10^3\times1.09\times10^{-3})}{1.73\times10^3\times1.09\times10^{-3}}\right]^2 = 0.252(4I_0)$$

$$I_2 = (4I_0)\left[\frac{\sin(1.73\times10^3\times2.19\times10^{-3})}{1.73\times10^3\times2.19\times10^{-3}}\right]^2 = 0.0253(4I_0)$$

Discussion: the first diffraction minimum

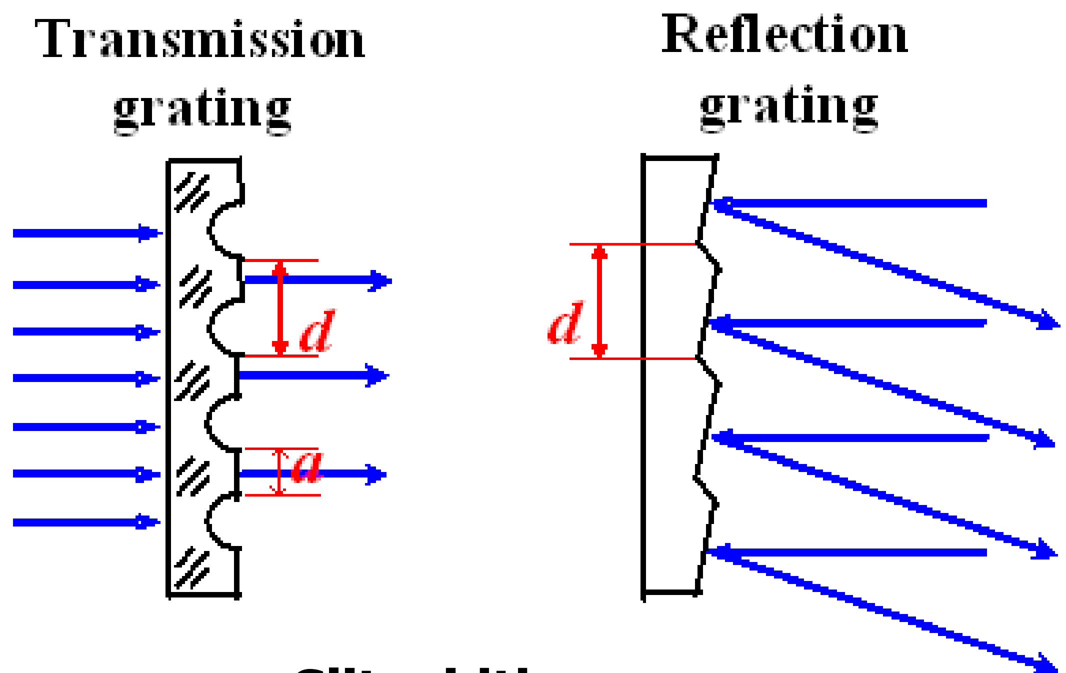
$$\theta_1' \approx \frac{\lambda}{a} = 1.81\times10^{-3} = 1.66\frac{\lambda}{d} \approx 1.66\theta_1, \qquad \frac{d}{a} = \frac{1.66}{1}$$

Prob.5

P727, Prob.14, 16, 18

§ 5 Diffraction Grating

- An array of a large number of parallel slits, all with the same width a and spaced equal distance d between centers, is called a diffraction grating.
- The spacing d between centers of adjacent slits is called the grating spacing.
- For grating, what we have been calling slits are often called rulings (划线) or lines.

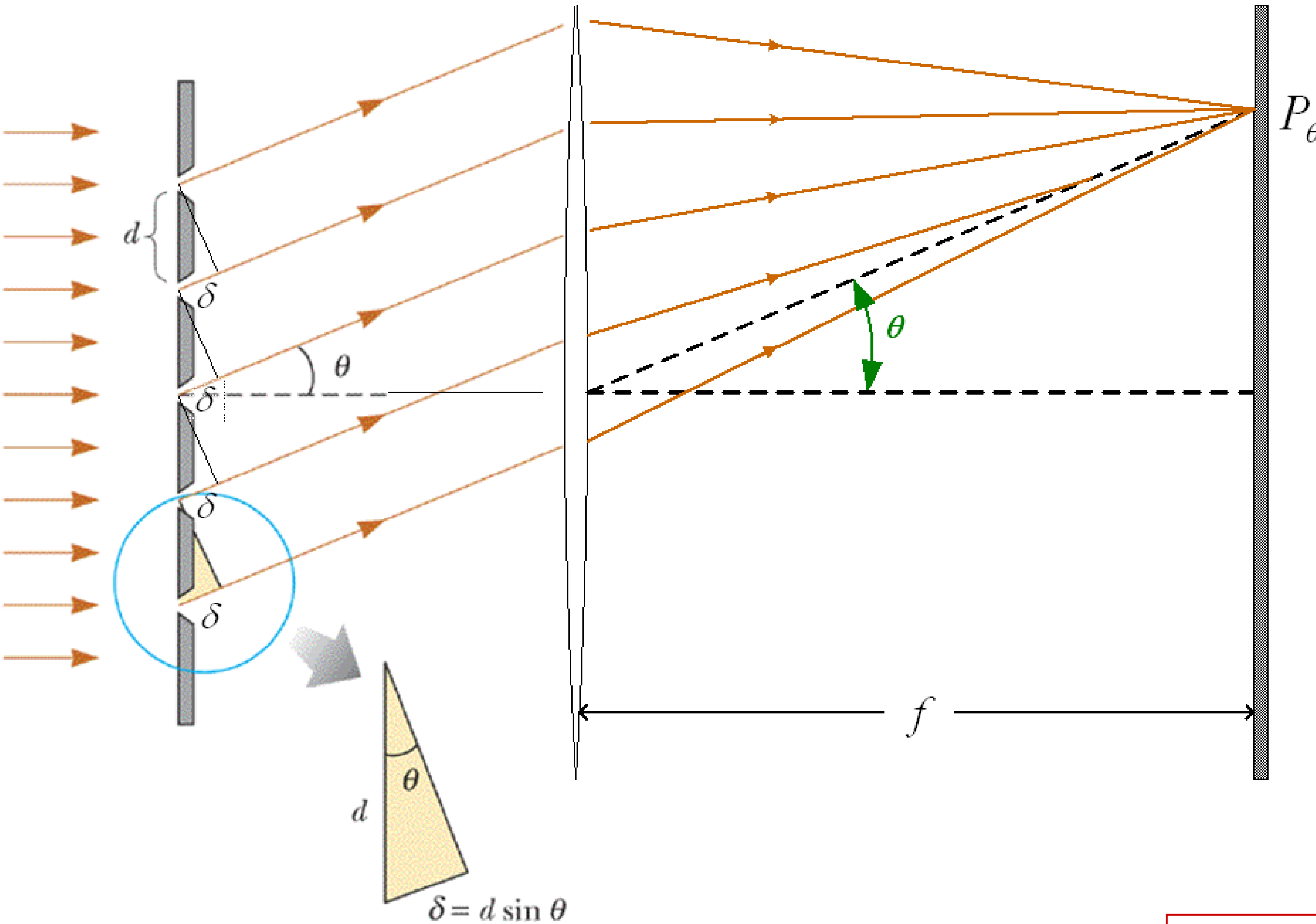


Slit width: a

Grating spacing: d

Number of slit: N

Principal maxima

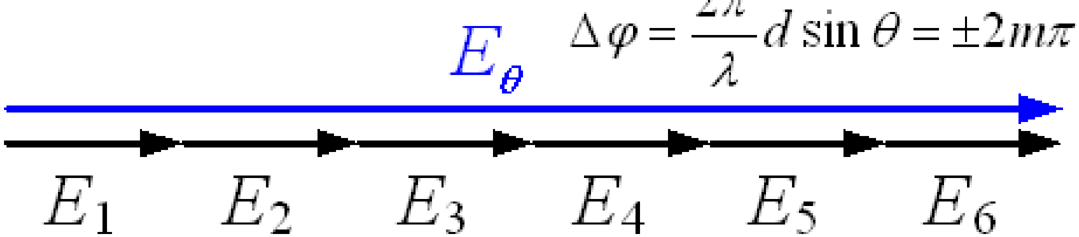


- **Constructive interference occurs at the angle when the optical path length difference between rays from any pair of adjacent slits is equal to an integer number of wavelengths:**

$$d\sin\theta_{\text{principal max}} = \pm m\lambda$$

$$(m = 0, 1, 2, \cdots)$$

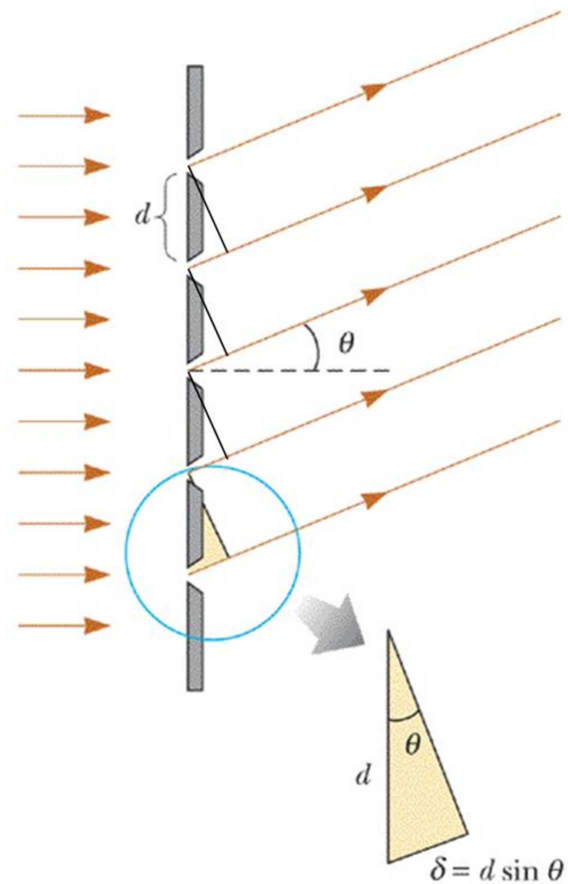
$$\Delta\varphi = \frac{2\pi}{\lambda} d\sin\theta = \pm 2m\pi$$



Principal maxima

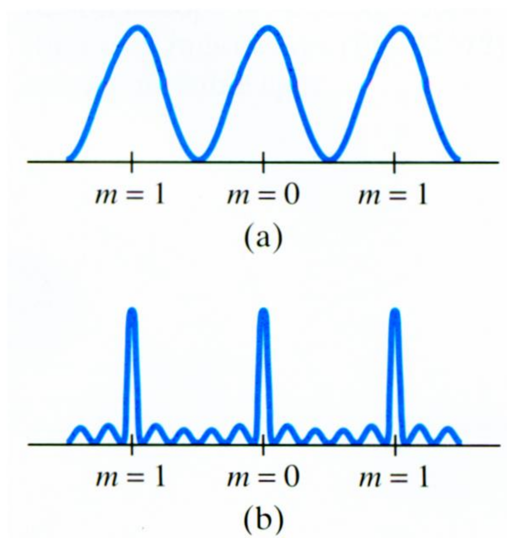
$$\boxed{d\sin\theta_{\text{principal max}} = \pm m\lambda, \quad (m = 0, 1, 2, \cdots)}$$

- That is the **same direction** as for the double-slit situation.
- The other two parameters, a and N, of grating do not affect the locations of **principal maxima**.
- The important difference between a double-slit and multiple-slit diffraction pattern is that the principal maxima are **much sharper** and narrower for a grating.

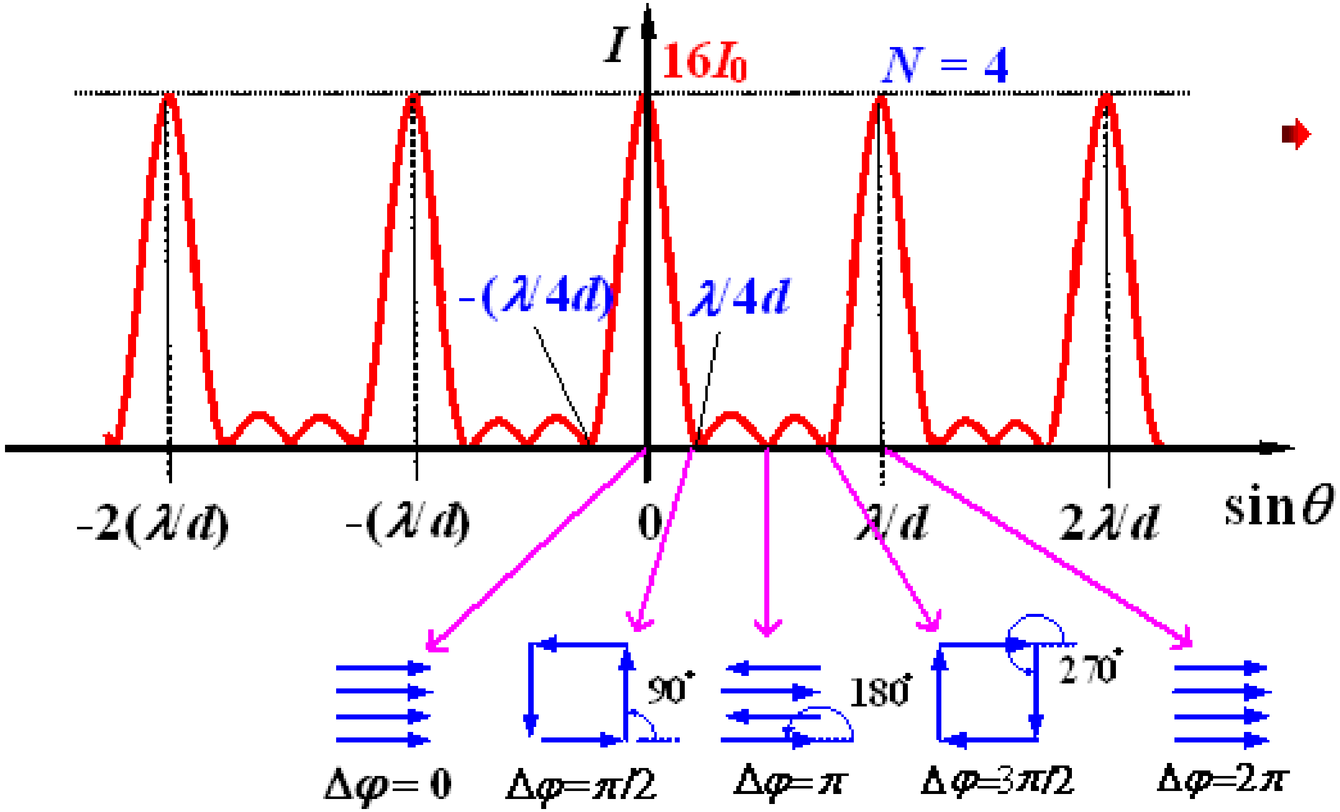


two-slit

six-slit



Minima and secondary maxima



- **For the principal maxima, the rays from adjacent slits are in phase.**

$$\Delta\varphi = \pm 2m\pi$$

- **The minima occur when phase difference between the rays from adjacent slits is integer multiple of $2\pi/N$.**

$$\Delta\varphi = \pm m' \frac{2\pi}{N}, \quad m' \neq mN$$

$$m' = 1, 2, \cdots, (N-1);$$

$$(N+1), \cdots, (2N-1);$$

$$\cdots\cdots$$

Minima and secondary maxima

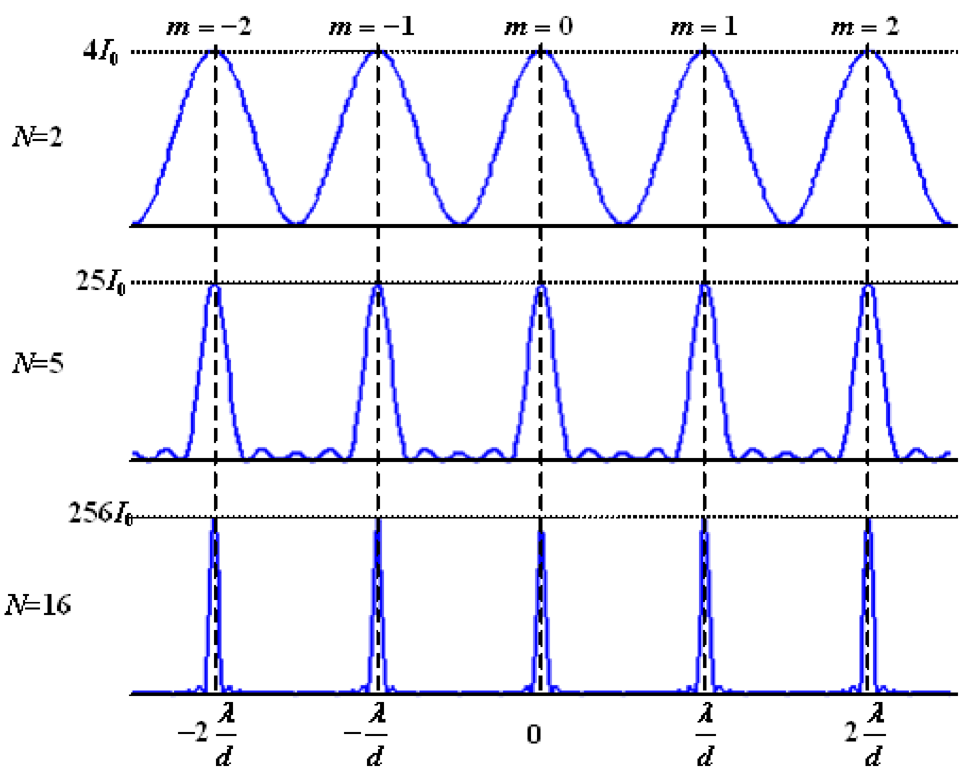
$$\Delta\varphi = \pm 2m\pi$$

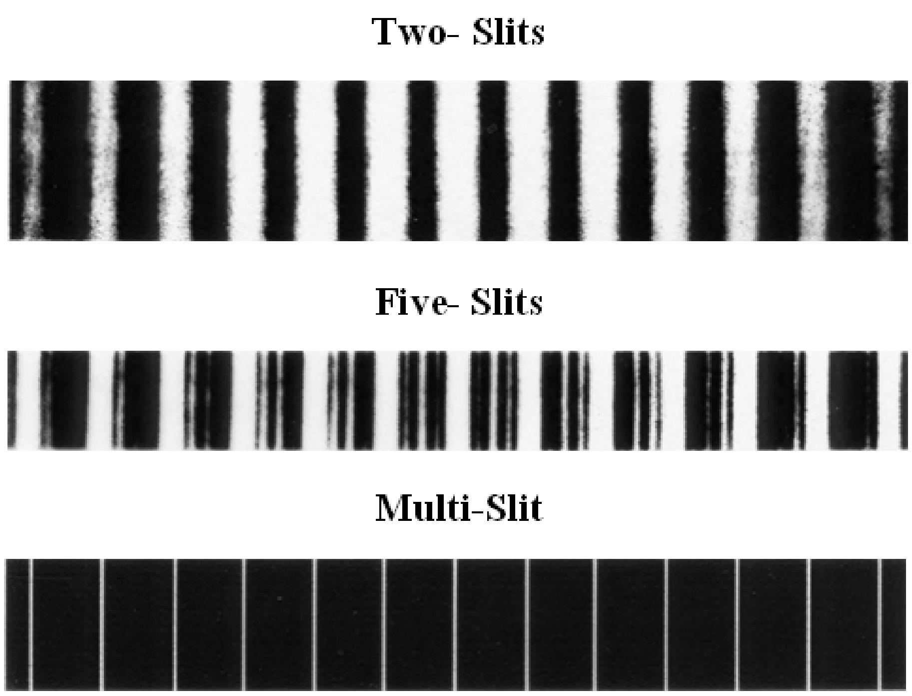
[principal maxima]

$$\Delta\varphi = \pm m'\frac{2\pi}{N}, \quad m' \neq mN$$

[minima]

- The number of the minima between two adjacent maxima is N-1.
- The number of the secondary maxima between two adjacent maxima is N-2.

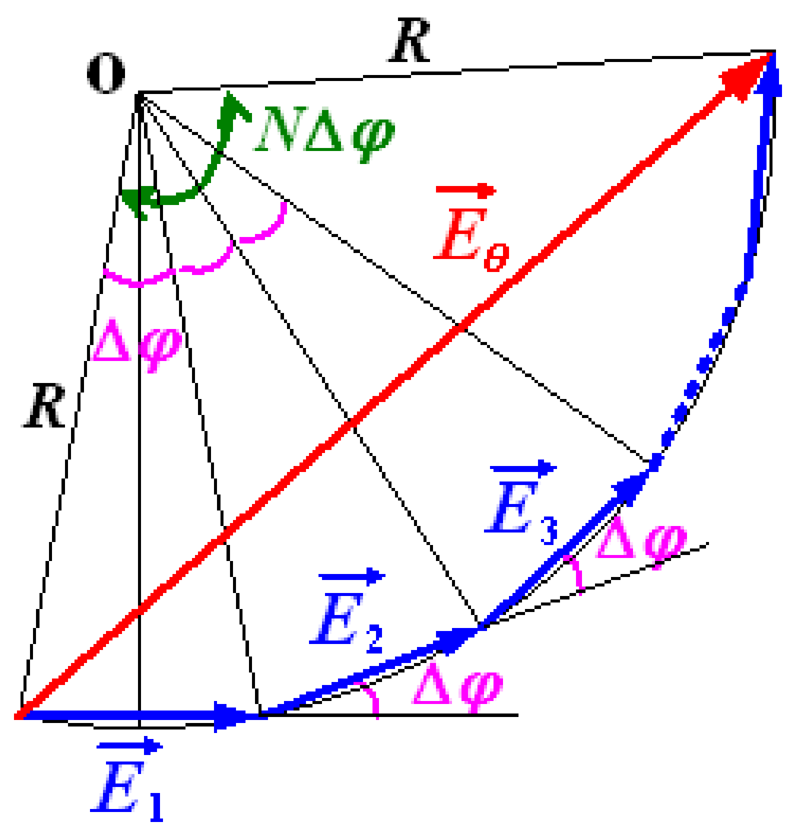




The intensity of diffraction pattern for a grating

$$E_1 = E_0 \frac{\sin\beta/2}{\beta/2}$$

$$\left.\begin{aligned} E_\theta &= 2R\sin\frac{N\Delta\varphi}{2} \\ E_1 &= 2R\sin\frac{\Delta\varphi}{2} \end{aligned}\right\} \quad E_\theta = E_1 \frac{\sin\dfrac{N\Delta\varphi}{2}}{\sin\dfrac{\Delta\varphi}{2}}$$



let $\gamma = \Delta\varphi = \frac{2\pi}{\lambda} d\sin\theta, \quad \beta = \frac{2\pi}{\lambda} a\sin\theta$

$$E_\theta = E_0\left(\frac{\sin\beta/2}{\beta/2}\right)\left(\frac{\sin N\gamma/2}{\sin\gamma/2}\right), \quad I_\theta = I_0\left(\frac{\sin\beta/2}{\beta/2}\right)^2\left(\frac{\sin N\gamma/2}{\sin\gamma/2}\right)^2$$

$$I_0 \propto (E_0)^2$$

diffraction factor

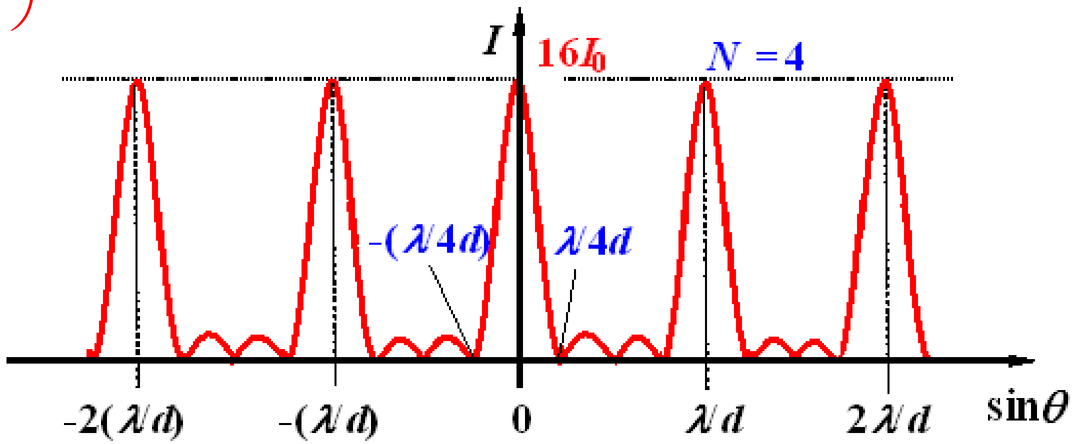
interference factor

The locations of principal maxima

$$I = I_0\left(\frac{\sin\beta/2}{\beta/2}\right)^2\left(\frac{\sin N\gamma/2}{\sin\gamma/2}\right)^2, \qquad \beta = \frac{2\pi}{\lambda}a\sin\theta,\quad \gamma = \frac{2\pi}{\lambda}d\sin\theta$$

$$a\to 0, \beta\to 0$$

$$I = I_0\left(\frac{\sin N\gamma/2}{\sin\gamma/2}\right)^2$$



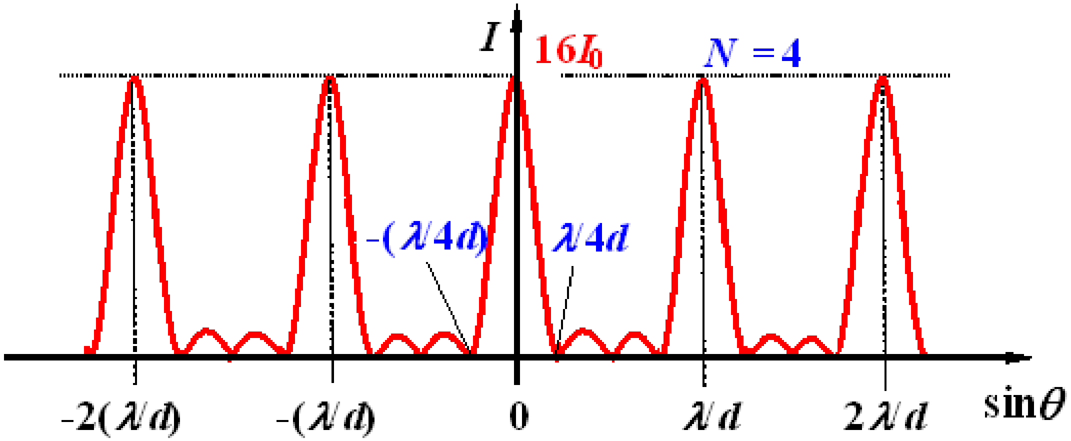
$$\gamma = \pm 2m\pi \Leftrightarrow d\sin\theta = \pm m\lambda,\quad m = 0,1,2,\cdots$$ **[maxima]**

$$\left.\frac{\sin\frac{N\gamma}{2}}{\sin\frac{\gamma}{2}}\right|_{\gamma=\pm 2m\pi} = \lim_{\gamma\to\pm 2m\pi}\frac{\left(\sin\frac{N\gamma}{2}\right)'}{\left(\sin\frac{\gamma}{2}\right)'} = \lim_{\gamma\to\pm 2m\pi}\frac{\frac{N}{2}\cos\frac{N\gamma}{2}}{\frac{1}{2}\cos\frac{\gamma}{2}} = N,$$

$$I = N^2 I_0$$

The locations of principal maxima

- **The highest order which can be seen in the range from +90° to –90°**



$$d\sin\theta = \pm m\lambda, \quad m = 0, 1, 2, \ldots$$

$$|\sin\theta| = \left| m\frac{\lambda}{d} \right| < 1, \qquad |m_{\max}| < \frac{d}{\lambda}$$

The locations of minima

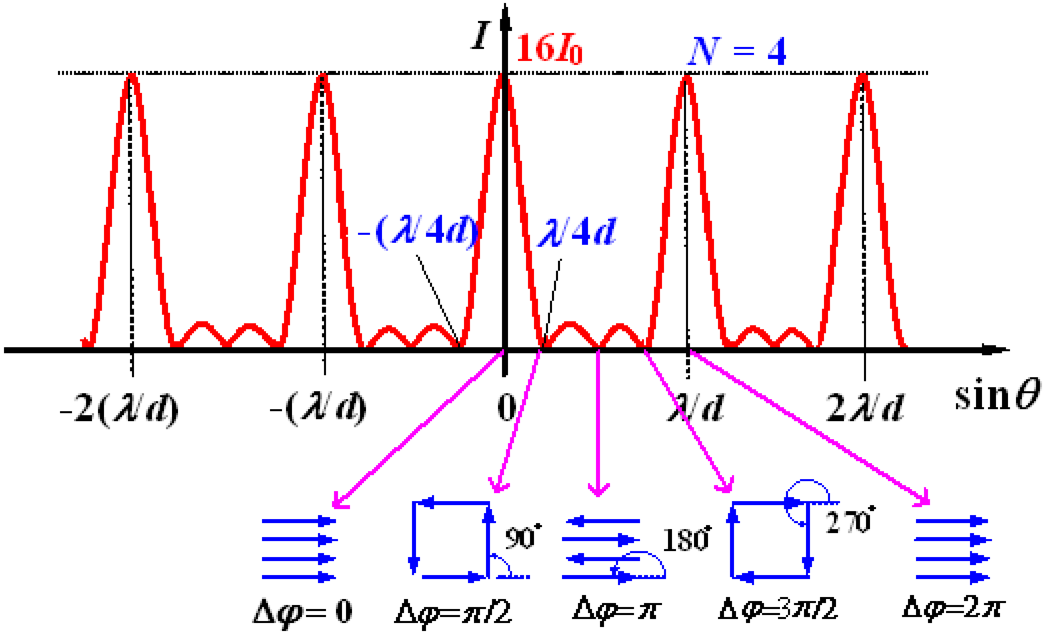
$$I = I_0\left(\frac{\sin N\gamma/2}{\sin\gamma/2}\right)^2$$

$$\gamma = \frac{2\pi}{\lambda} d\sin\theta$$

$$\frac{N\gamma}{2} = \pm n'\pi, \text{ but } \frac{\gamma}{2} \neq \pm n\pi$$

$$\gamma = \pm n'\frac{2\pi}{N}, \text{ but } n' \neq nN$$

$$d\sin\theta = \pm\frac{n'}{N}\lambda, \quad n' \neq 0, N, 2N, \cdots$$



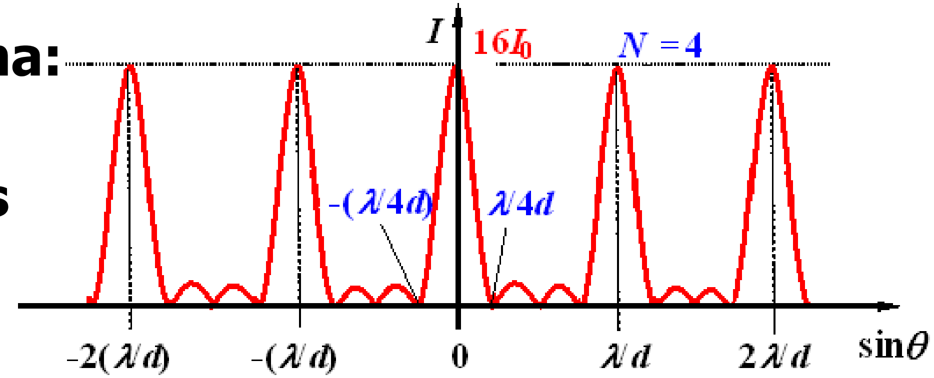
$$\Leftrightarrow d\sin\theta = \pm\left(m + \frac{m'}{N}\right)\lambda, \quad m = 0, 1, 2, \cdots \quad m' = 1, 2, \cdots, N-1$$

[minima]

There are N-1 minima between two adjacent principal maxima, and N-2 secondary maxima.

Width of the principal maxima

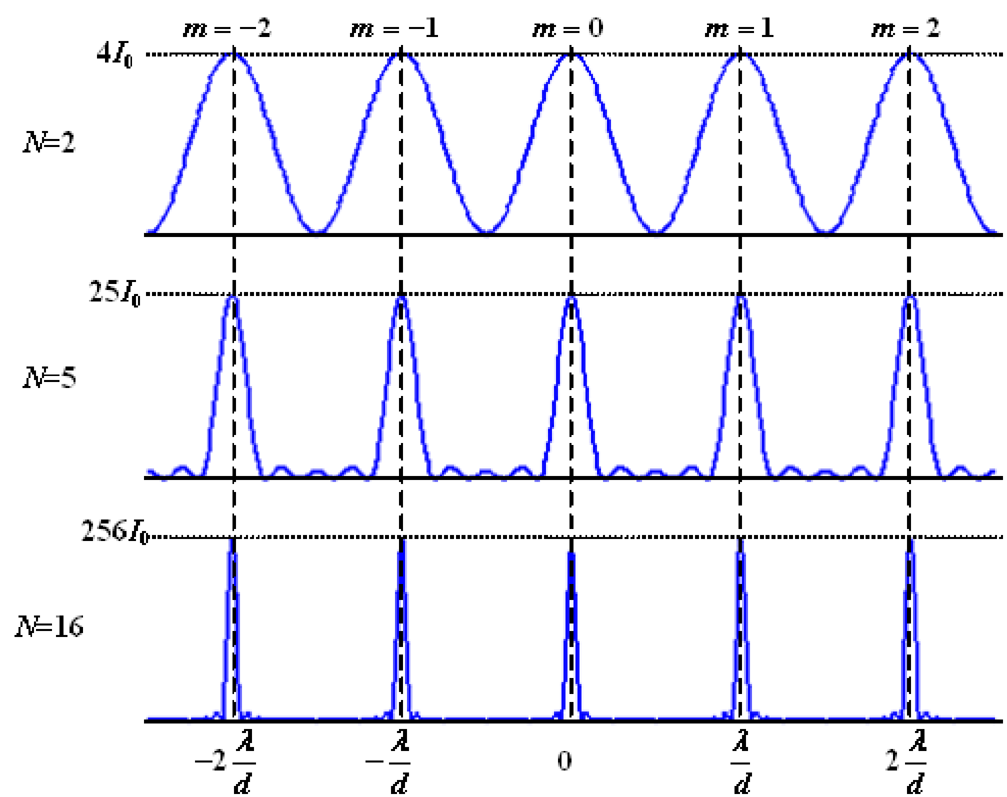
- **Half-width of the principal maxima:**
  The angular separation between the principal maximum and its nearest minimum.



$$\begin{cases} d\sin\theta_m = m\lambda \\ d\sin(\theta_m+\Delta\theta_m) = \left(m+\dfrac{1}{N}\right)\lambda \end{cases}$$

$$d\left(\cos\theta_m\right)\Delta\theta_m = \frac{\lambda}{N}$$

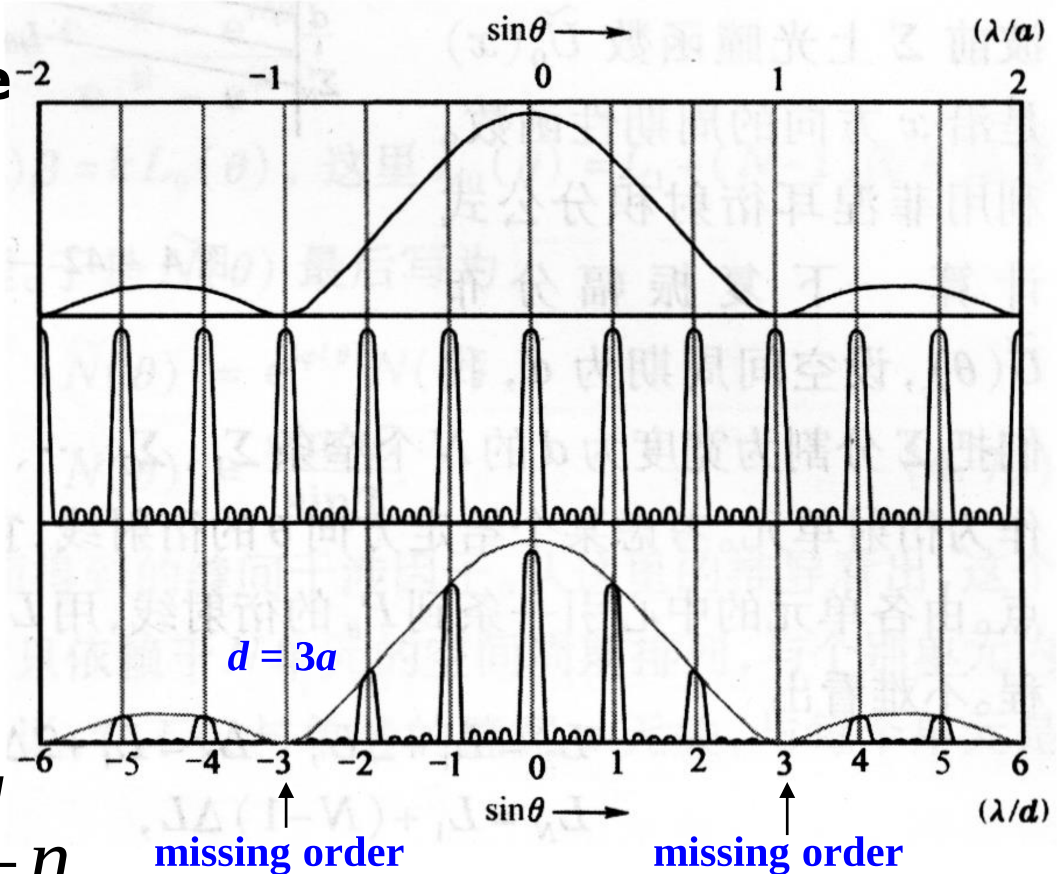
$$\boxed{\Delta\theta_m = \frac{\lambda}{Nd\cos\theta_m}}$$



- **The greater the number of slits N, the sharper the principal maximum.**

The effect of diffraction factor

- The multiple-slit interference pattern is modulated by the diffraction factor. So the heights of the principal maxima are not same.
- The diffraction factor may induce the missing order.



$$d\sin\theta = m\lambda$$

$$a\sin\theta = n\lambda$$

If $\frac{d}{a} = \frac{m}{n} \Rightarrow m = \frac{d}{a}n$

The m-th principal maxima will miss because an interference maximum and a diffraction minimum corresponds to the same θ-value.

Summary of grating

	Principal maxima				Minima		Secondary maxima	
	Location	**Intensity**	**Half-width $\Delta\theta$**	**Missing order**	**Location**	**Number**	Location	Number
d	$\sin\theta = m\dfrac{\lambda}{d}$	×	$\Delta\theta = \dfrac{\lambda}{Nd\cos\theta}$	$\dfrac{d}{a} = \dfrac{m}{n}$	$\sin\theta = \dfrac{m'}{N}\dfrac{\lambda}{d}$ $m' \neq 0, \pm N, \pm 2N, \cdots$	×	√	×
a	×	**Modulation**	×	$\dfrac{d}{a} = \dfrac{m}{n}$	×	×	×	×
N	×	$N^2 I_0$	$\Delta\theta = \dfrac{\lambda}{Nd\cos\theta}$	×	$\sin\theta = \dfrac{m'}{N}\dfrac{\lambda}{d}$ $m' \neq 0, \pm N, \pm 2N, \cdots$	N–1	√	N–2

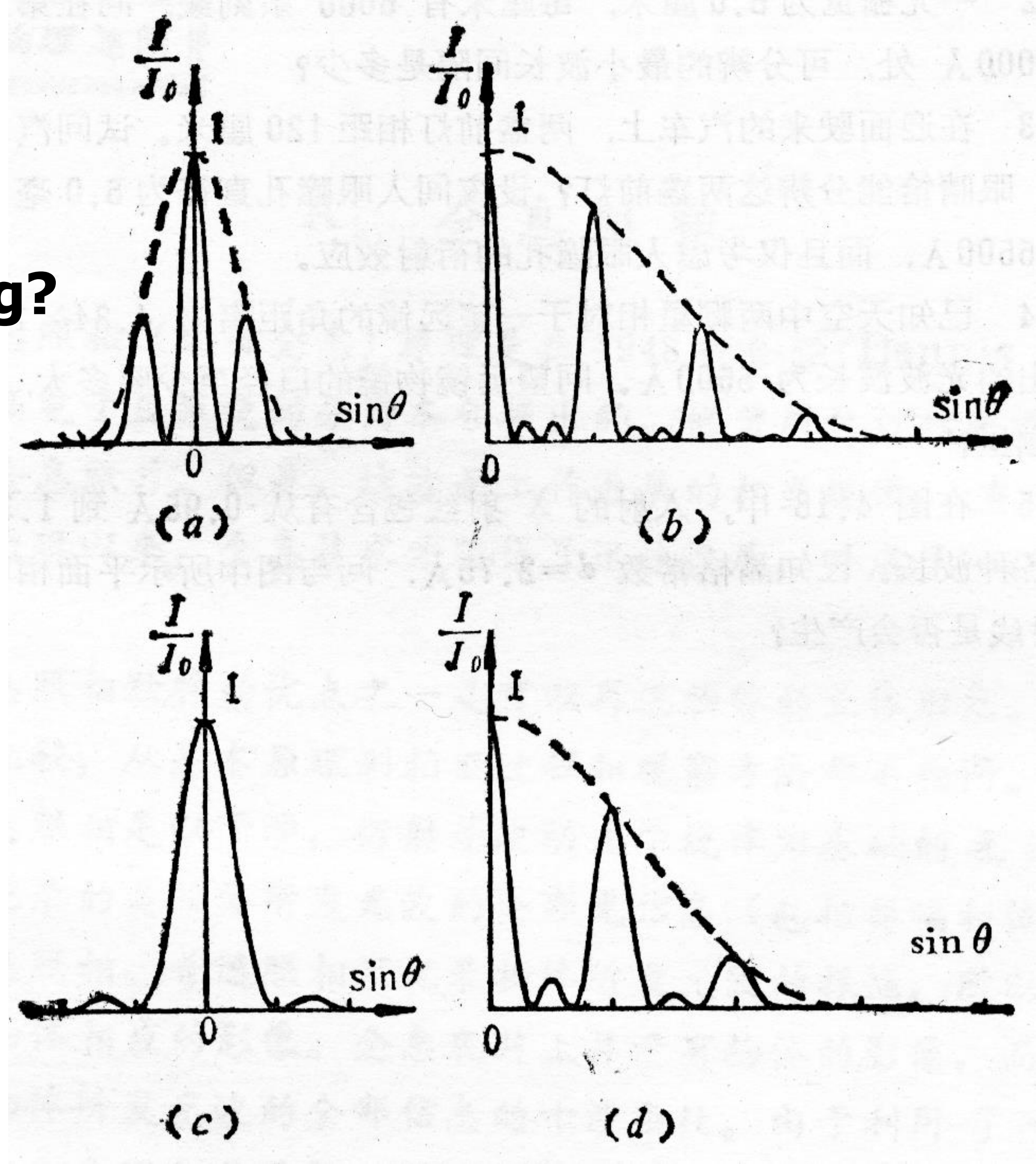
Example

For each diffraction pattern,

(1) how many slits are there for the grating?

(2) Which pattern corresponds to largest slit width?

(3) what is d/a for each pattern?



Example

A light of wavelength 600nm falls normally on a grating. The second and third order maxima appear at the angular positions of $\sin\theta=0.20$ and $\sin\theta=0.30$. It is found that the fourth order maximum is missing from the diffraction pattern.

(a) Determine the slit separation of the grating,

(b) and the smallest width of each slit.

(c) List all the order of maxima that will appear on the viewing screen.

Solution: (a) For the second order maximum:

$$\sin\theta = 0.20 = 2\frac{\lambda}{d}, \qquad d = 2\times 600\text{nm} / 0.20 = 6000\ \text{nm} = 6.00\ \mu\text{m}$$

(b)

$$\frac{d}{a} = 4, \qquad a = 1.50\ \mu\text{m}$$

(c)

$$|m_{\max}| < \frac{d}{\lambda} = \frac{6000\text{nm}}{600\text{nm}} = 10$$

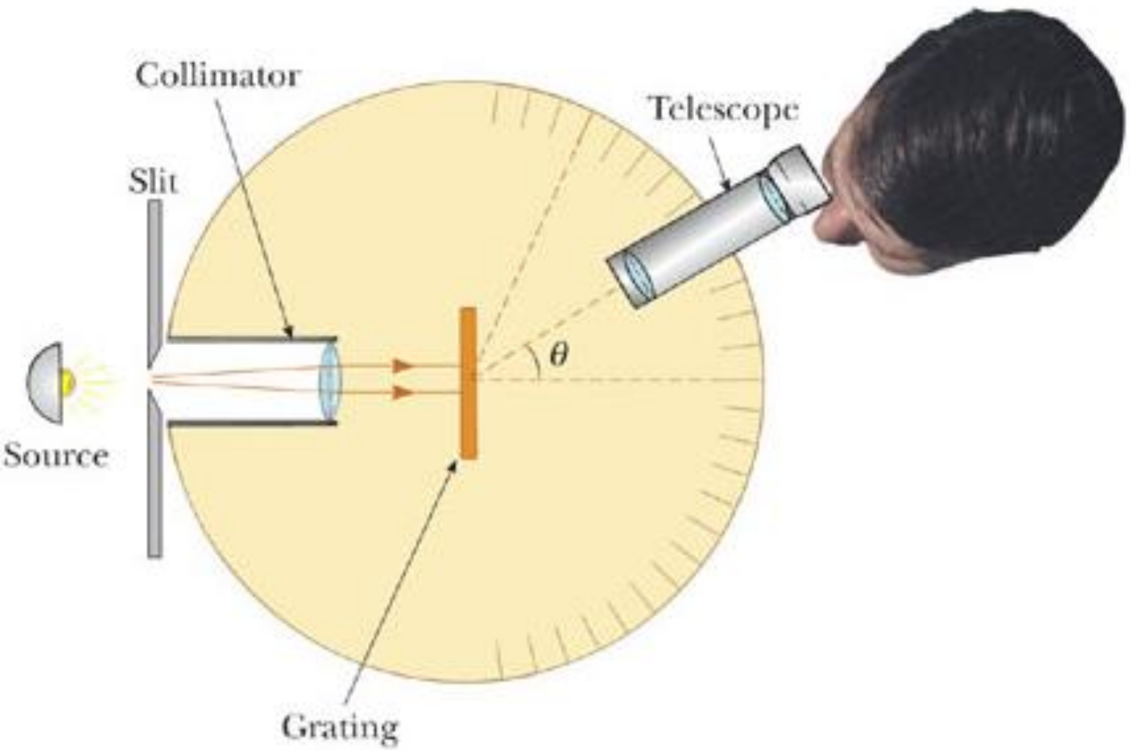
On the viewing screen, the order of $0, \pm 1, \pm 2, \pm 3; \pm 5, \pm 6, \pm 7; \pm 9$ maxima will appear.

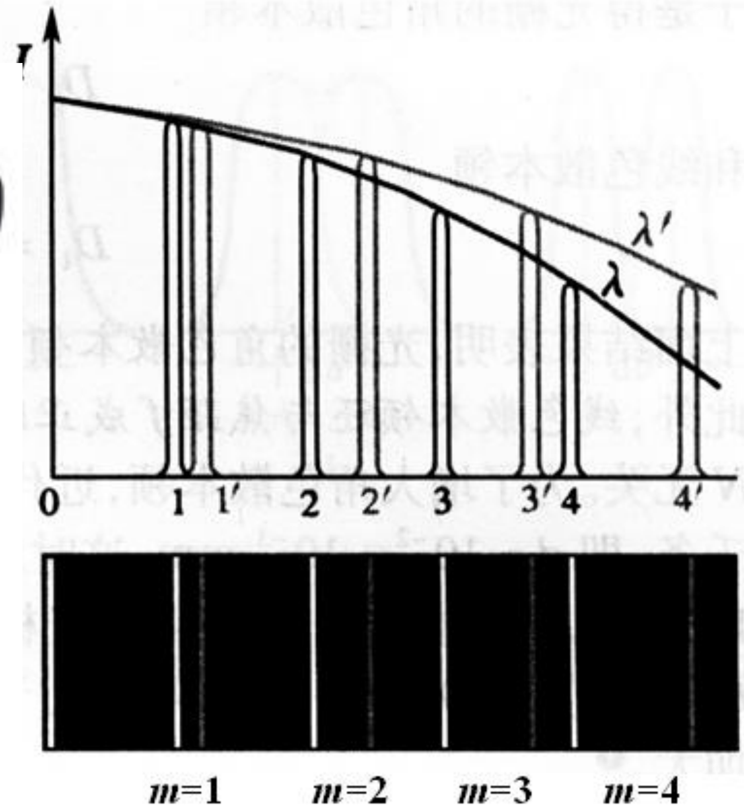
Prob.6

P728, Prob.29, 32, 33

§ 6 Grating Spectrometers

- The principle of grating spectrometer

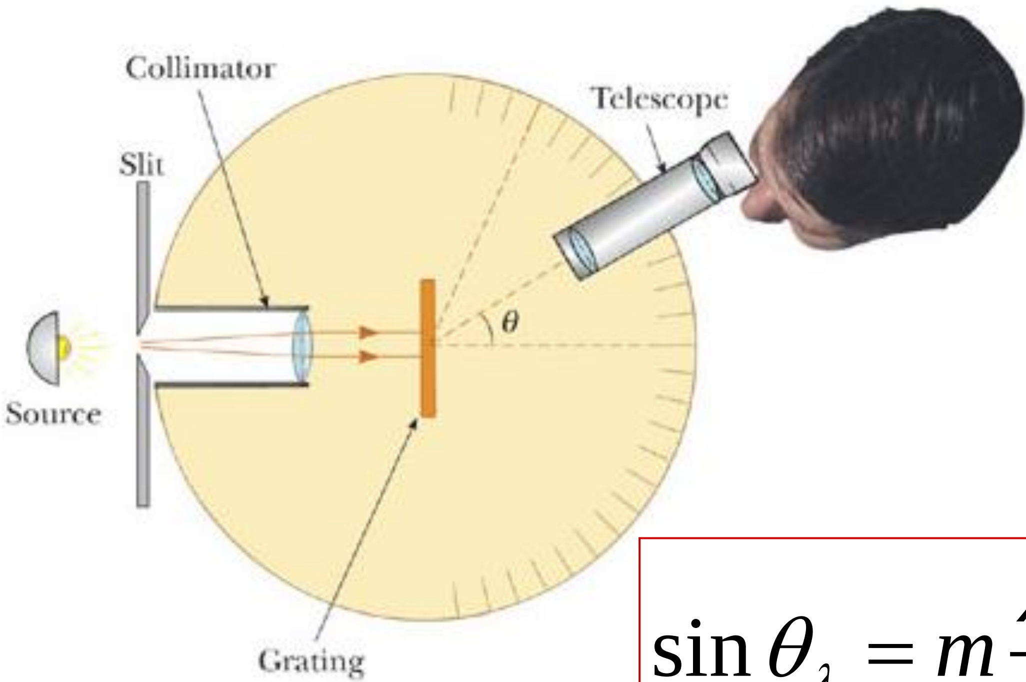




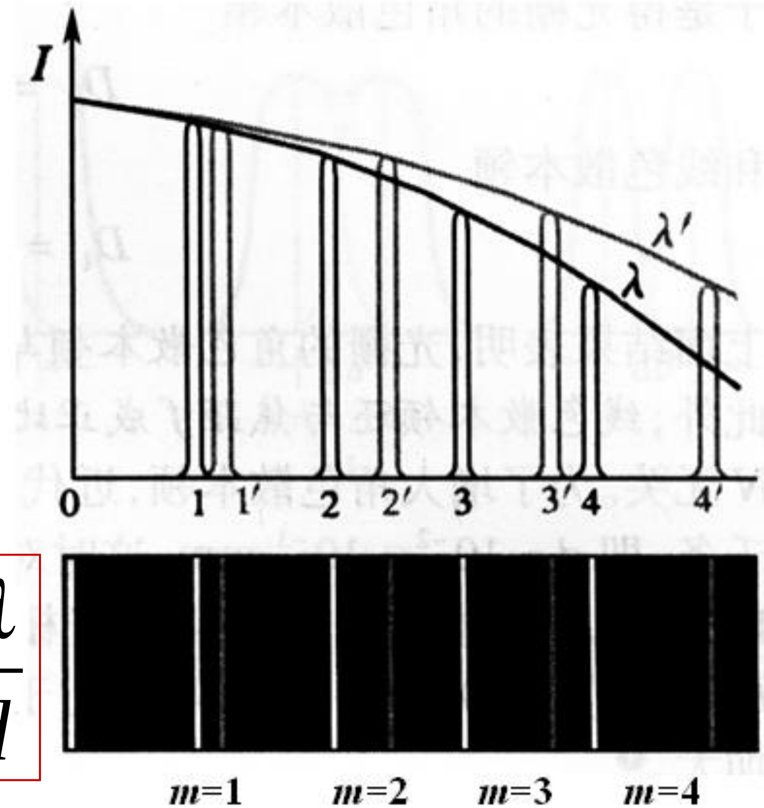
- **The angular position of the diffraction maxima (or diffraction lines) of a grating is proportional to the light wavelength.**

$$\boxed{\sin\theta_\lambda = m\frac{\lambda}{d}}$$

The principle of grating spectrometer

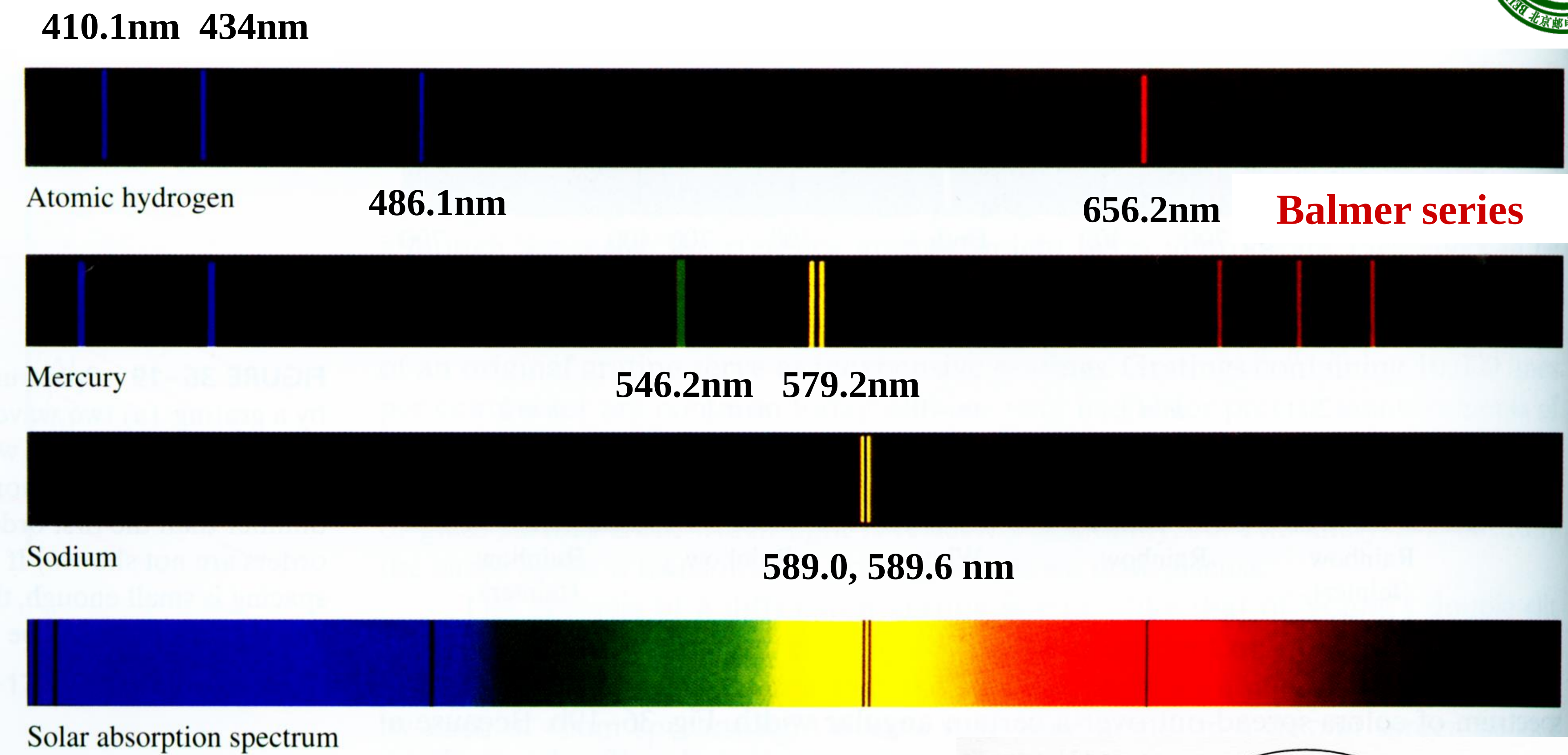


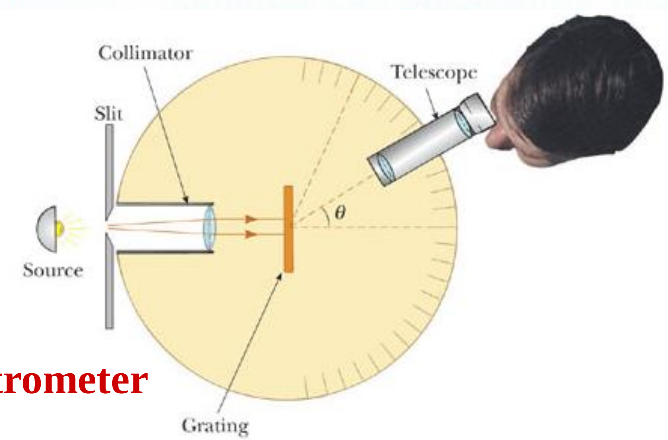
$$\sin\theta_\lambda = m\frac{\lambda}{d}$$



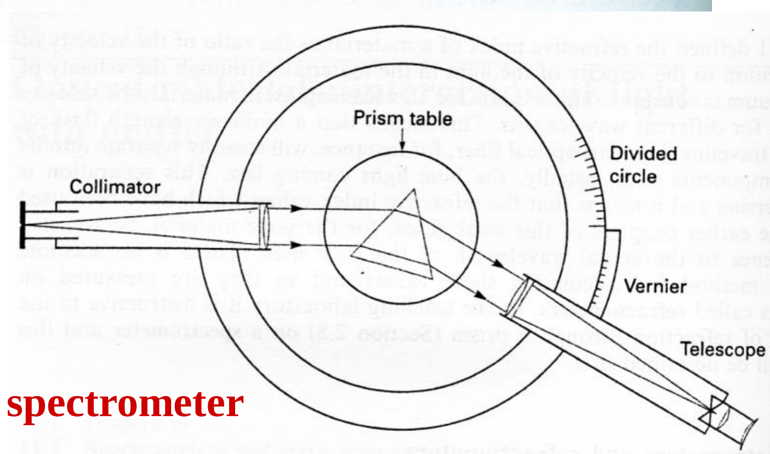
- **Gratings are therefore used as spectrometers to determine the wavelength.**
- **For a grating with large number of slits, the maxima are very narrow and bright, which allows their position to be measured with great precision.**

Grating Spectrometers





Grating spectrometer



Prism spectrometer

The resolving power for a diffraction grating

- **Dispersion:**

  the angular separation $\Delta\theta$ per unit wavelength interval $\Delta\lambda$

$$D \equiv \frac{\Delta\theta}{\Delta\lambda}$$

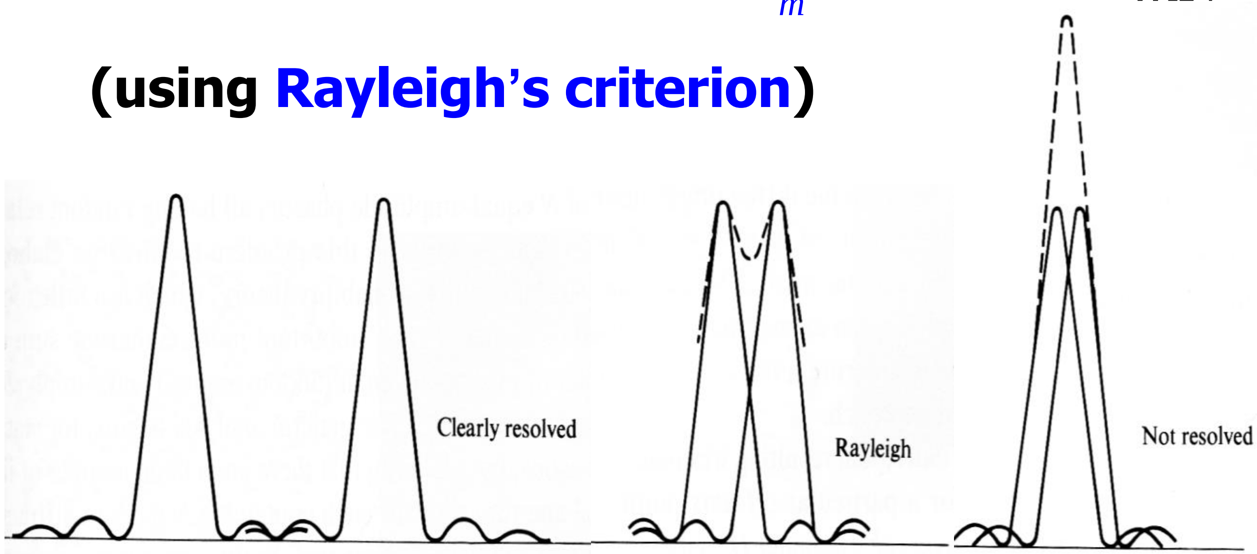
$$d\sin\theta_m = m\lambda \Rightarrow d\cos\theta_m(\Delta\theta) = m(\Delta\lambda),\ D = \frac{\Delta\theta}{\Delta\lambda} = \frac{m}{d\cos\theta_m}$$

- **Resolving power**

$$R \equiv \frac{\lambda}{\Delta\lambda_{\min}},\qquad \Delta\lambda_{\min} = \frac{\Delta\theta_{\min}}{D} = \Delta\theta_{\mathrm{m}}\cdot\frac{1}{D} = \frac{\lambda}{Nd\cos\theta_m}\cdot\frac{d\cos\theta_m}{m} = \frac{\lambda}{mN}$$

(using Rayleigh's criterion)

$$R = mN$$



Light from a laboratory sodium lamp has two strong yellow doublet (光谱的双重线), λ_1= 589.0 nm, λ_2= 589.6 nm. (1) What is the total number of lines a grating must have in order just to separate this doublet in the third order?(2) If we can only make 500-line/cm diffraction grating, what is the width of this grating we must make?

Solution:

$$R \equiv \frac{\lambda}{\Delta\lambda_{\min}} = mN, \quad \Delta\lambda_{\min} = 0.6\text{ nm}, \quad \bar{\lambda} = 589.3\text{ nm}, \quad m = 3$$

$$N \geq \frac{1}{m}\frac{\bar{\lambda}}{\Delta\lambda_{\min}} = \frac{1}{3}\frac{589.3\text{ nm}}{0.6\text{ nm}} = 327.4$$

$$N \geq 328, \qquad \text{Width} = \frac{N}{500\text{ line/cm}} = 6.56\text{ mm}$$

Prob. 7

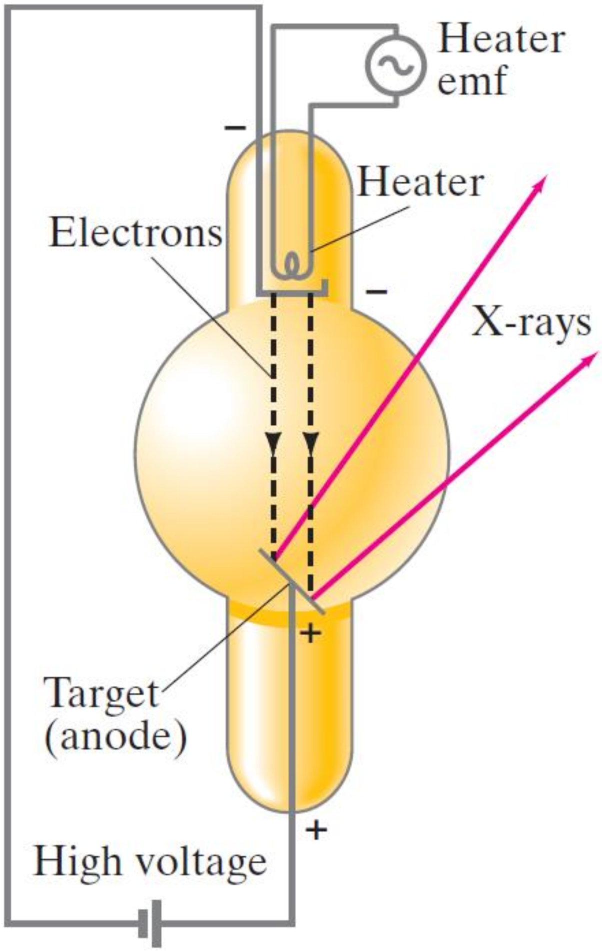
P728, Prob.35, 36, 37

§ 7 X-Ray Diffraction

X-ray

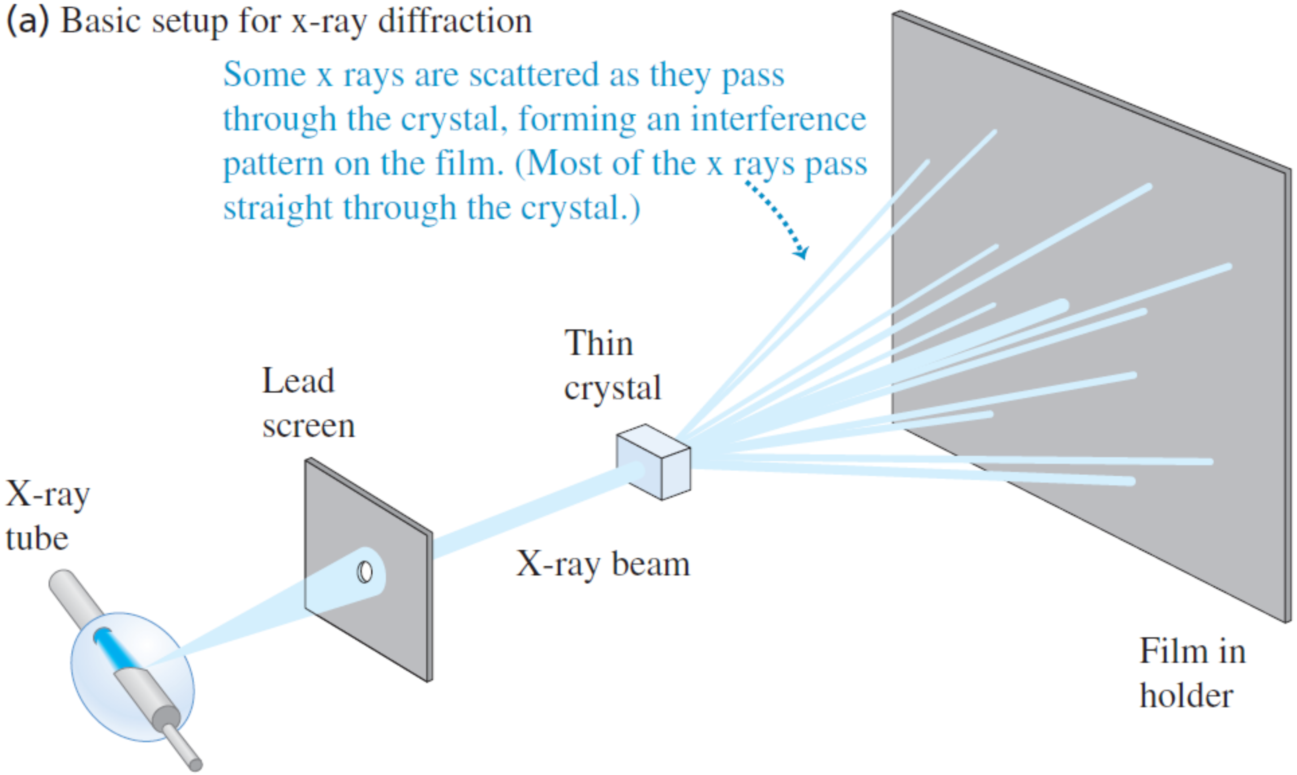
In 1895, W. C. Roentgen (1845 - 1923)

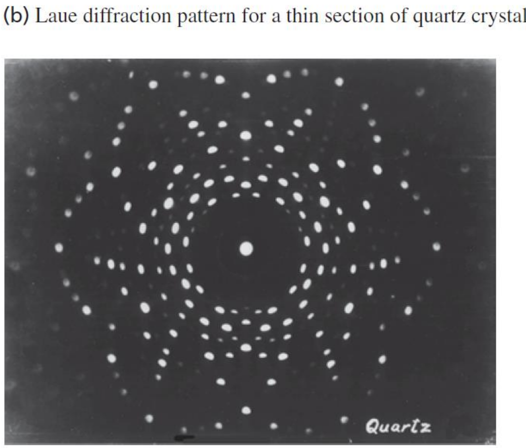
- **X-rays are now recognized as electromagnetic radiation with wavelengths in the range of about 10^{-2} nm to 10 nm, the range readily produced in an X-ray tube.**



X-ray tube

X-Ray Diffraction

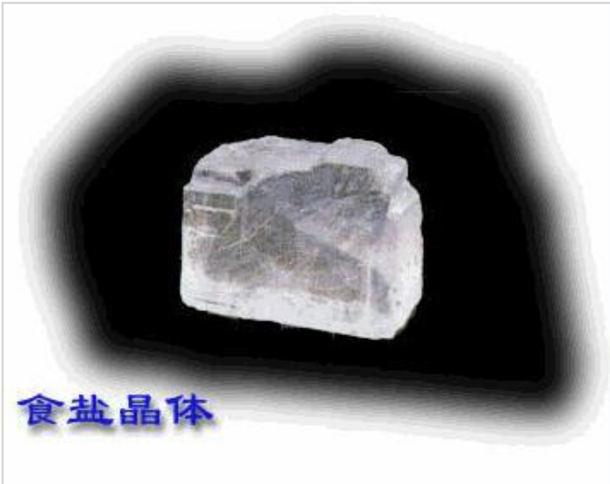


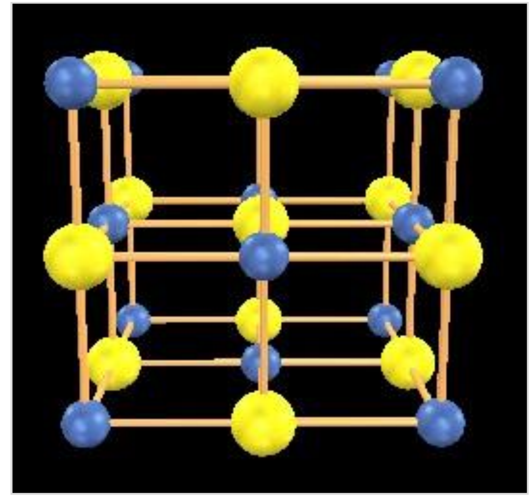


X-ray diffraction pattern

- **Around 1912, Max von Laue (1879 - 1960) suggested that if the atoms in a crystal were arranged in a regular array, such a crystal might serve as a diffraction grating for very short wavelengths on the order of the spacing between atoms, estimated to be about 10^{-1} nm.**

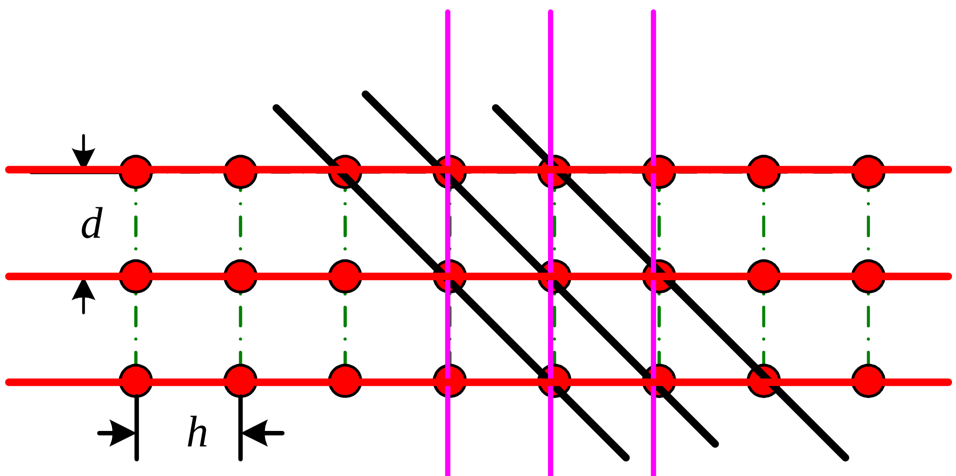
X-Ray Diffraction



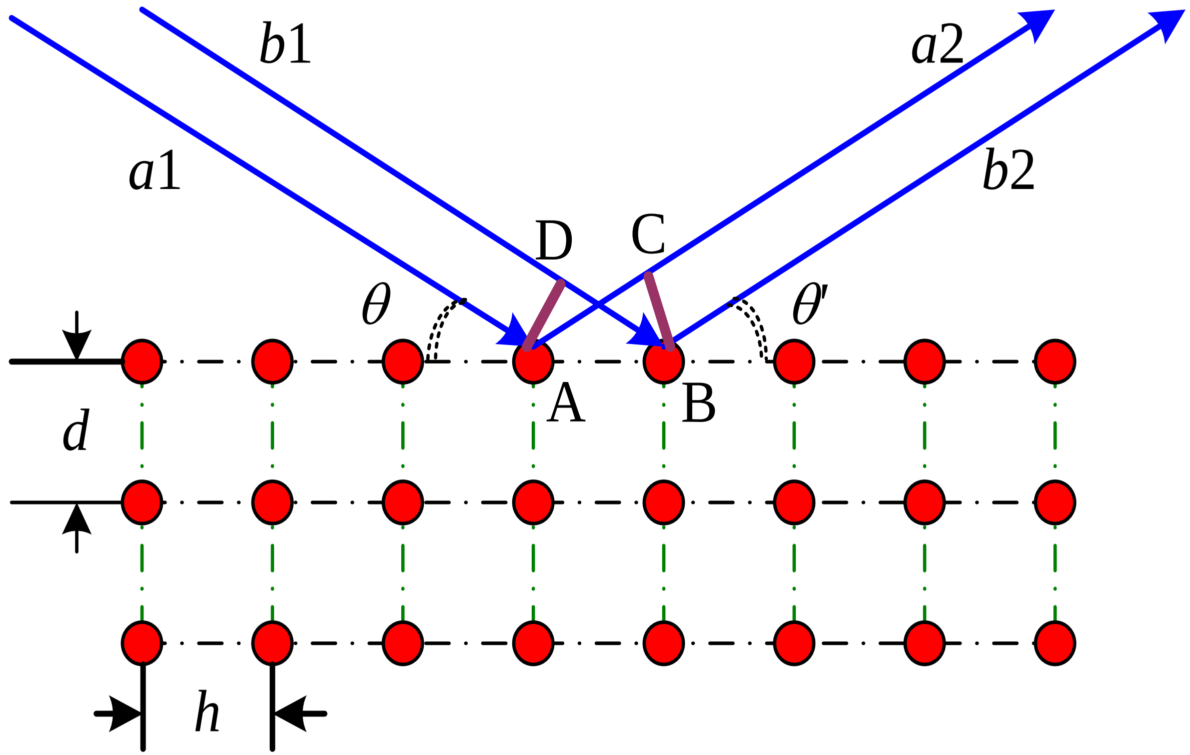


Lattice model of salt crystal

- A crystal is a three-dimensional object, and X-rays can be diffracted from different planes at different angles within the crystal.



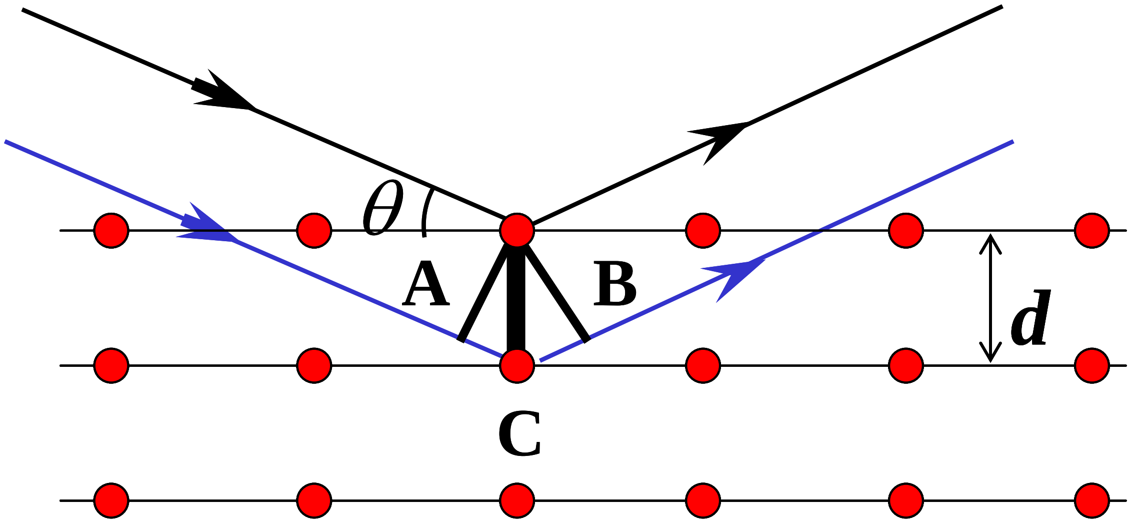
Bragg equation



Difference of optical path $$\delta = AC - BD = h\cos\theta' - h\cos\theta$$

Zero-order constructive interference $$\delta = 0 \quad \Rightarrow \quad \theta' = \theta$$

Bragg equation



d : Interplanar spacing (Lattice constant)

Difference of optical path $\delta = \overline{AC} + \overline{CB} = 2d\sin\theta$

Constructive interference $2d\sin\theta = m\lambda, \quad m = 1, 2, 3, \cdots$

- **X-ray crystallography**

If the X-ray wavelength λ is known and the angle θ is measured, the distance d between atoms can be obtained.

Example

First-order Bragg diffraction is observed at 26.8° relative to the crystal surface, with spacing between atoms of 0.24 nm. (a) At what angle will second order be observed? (b) What is the wavelength of the X-rays?

Solution:

(a) $2d\sin\theta_1 = 1\lambda,\qquad \sin\theta_2 = 2\sin\theta_1 = 2\sin 26.8^\circ$

$2d\sin\theta_2 = 2\lambda,\qquad \theta_2 = 64.4^\circ$

(b) $\lambda = 2d\sin\theta_1 = 2(0.24\ \mathrm{nm})\sin 26.8^\circ = 0.22\ \mathrm{nm}$

Prob. 8

P728, Prob. 40, 41