SOLUTIONS

Module:	ENGINEERING MATHEMATICS		
Module Code	BBC4111	PAPER	A
Time allowed	2HRS	FILENAME	SOLUTIONS_2223_ BC4111_A
Rubric	ANSWER ALL EIGHT QUESTIONS		
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Question1. [24 marks total, 2 marks for each blank]

Fill in all the following blanks. Only the final results are required to be written down.

a). The exponential form of $\dfrac{(\cos 5\varphi + i\sin 5\varphi)^2}{(\cos 3\varphi - i\sin 3\varphi)^3}$ is ($e^{i19\varphi}$).

b). Suppose that $Arg(z+2)=\dfrac{\pi}{3}$ and $Arg(z-2)=\dfrac{5\pi}{6}$. Then $z=$($-1+i\sqrt{3}$).

c). $\lim\limits_{z\to i}\dfrac{z-i}{z(1+z^2)}=$ ($-\dfrac{1}{2}$).

d). If $\cos(2+z)=3$, then $z=$($-2+2k\pi - i\ln(3\pm 2\sqrt{2})$).

e). Let C denote the semicircle $|z|=1$ from 1 to -1. Then $\oint_C \left(z^2+z\bar{z}\right)dz=$ ($-\dfrac{8}{3}$).

f). $\displaystyle\int_{-\infty}^{+\infty}\frac{1}{\left(1+x^2\right)^2}dx=$ ($\pi/2$).

g). The PDE $\dfrac{\partial^2 u}{\partial x^2}+5\dfrac{\partial^2 u}{\partial x\partial t}+6xu+5t=0$ is of (hyperbolic) type.

h). The general solution of the equation $(1-x^2)y''(x)-2xy'(x)+12y(x)=0$ is

($c_1P_3(x)+c_2Q_3(x)$)

i). The characteristic curves of $\dfrac{\partial^2 u}{\partial x^2}-3\dfrac{\partial^2 u}{\partial x\partial y}+2\dfrac{\partial^2 u}{\partial y^2}=\sin(x^2+y^2)$ are

($x+y=C_1,\ 2x+y=C_2$).

j). The eigenvalues of the problem

$$\begin{cases} X''(x)+\lambda X(x)=0,\ 0<x<l \\ X'(0)=0,\quad X(l)=0 \end{cases}$$

are ($\dfrac{\left[(n-\frac{1}{2})\pi\right]^2}{l^2}, n=1,2,\cdots$), and the corresponding eigenfunctions are

($X_n(x)=\cos\dfrac{(n-\frac{1}{2})\pi x}{l}, n=1,2,\cdots$).

k). $\displaystyle\int_0^x x^4J_1(x)dx=$($x^4J_2(x)-2x^3J_3(x)$), where $J_1(x)$ is the first kind of 1st order Bessel function.

Question2. [6 marks total, 2 marks for each one]

a). The convergence domain of the power series $\sum_{n=1}^{+\infty}\frac{(z-i)^n}{n^3}$ is (B).

A. $|z-i|<\frac{1}{n^3}$ B. $|z-i|<1$ C. $|z|<1$ D. $|z-i|<\frac{1}{n}$

b). Which equation is not correct? (D)

A. $\int_{|z|=2}\frac{3z-1}{z(z-1)}dz=6\pi i$ B. $\oint_{|z-i|=0.5}\frac{e^z dz}{z^2+1}=\pi(\cos 1+i\sin 1)$

C. $\int\limits_{|z|=1}\frac{\cos z dz}{z^3}=-\pi i$ D. $\int_0^i (z-1)e^{-z}dz=-\sin 1+i\cos 1$

c). Which one is correct? (B)

A. $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly dependent.

B. The first kind of n order Bessel function $J_n(x)$ and the Bessel function of second kind $Y_n(x)$ are linearly independent.

C. $\lim\limits_{x\to\infty}J_n(x)=0$ when n is positive integer.

D. $J_n(0)=0$ for positive integer, and $J_\upsilon(0)=\infty$ when ν is nonnegative.

Question3. [18 marks total, 6 marks for each part]

a). Find out all points at which the function $f(z)$ is differentiable and analytic (please give the explanation), when $f(z)=\frac{\sin z\cdot \mathrm{Log}(1+z)}{z}$.

b). If the real part of entire function $f(z)$ is $u(x,y)=e^x(x\cos y-y\sin y)$, and $f(0)=0$, then find the imaginary part of $f(z)$ and calculate the value of $f'(1)$.

c). Give the Laurent series expansions for the function $f(z)=\frac{1}{z(z+1)}$ in the following annular domain $1<|z-1|<2$.

Solution.

a). For $f(z)=\frac{\sin z\cdot \mathrm{Log}(1+z)}{z}$, we know that $\sin z$ and z are all differentiable and analytic throughout the entire complex plane. Since **Logz** is differentiable and analytic throughout $r>0,-\pi<\theta<\pi$, we know **Log(1+z)** is differentiable and analytic except the points $z=x+iy$, $x\le -1, y=0.$ Thus, $f(z)$ is differentiable and analytic except 0 and $z=x+iy$ where $x\le -1, y=0.$ **[6 marks]**

b). $f(z)=u(x,y)+iv(x,y)$ satisfies the C-R equation, then

$$v_y = u_x = e^x(x\cos y - y\sin y + \cos y),$$
$$-v_x = u_y = -e^x(x\sin y + \sin y + y\cos y).$$

Then using the equations $v_x = e^x(x\sin y + \sin y + y\cos y)$, we have

$$v(x, y) = e^x(x\sin y + y\cos y) + \varphi(y).$$ **[4 marks]**

By using $v_y = e^x(x\cos y - y\sin y + \cos y)$, $\varphi(y) = C$. The corresponding analytic function is

$$f(z) = u(x, y) + iv(x, y) = e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y + C).$$

Since $f(0) = 0$, $C = 0$.

And $f(z) = e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y) = ze^z$ and $f'(1) = 2e$.

[2 marks]

3) In the annular domain $1 < |z-1| < 2$,

$$f(z) = \frac{1}{z} - \frac{1}{z+1} = \frac{1}{z-1+1} - \frac{1}{z-1+2} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(z-1)^{n+1}} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{2^{n+1}}.$$ **[6 marks]**

Question 4. [10 marks total, 5 marks for each part]

Determine all the isolated singular points of the following two functions and identify their types, explaining each type. Hence, select one isolated point and calculate its residue.

a). $f(z) = \dfrac{1}{z\sin\left(\dfrac{1}{z}\right)}$; **b).** $f(z) = \dfrac{\sin z - z}{\cos z - 1}$.

Solution.

a). $f(z)$ has singular points: 0, $\dfrac{1}{k\pi}$, $k = \pm 1, \pm 2, \cdots$.

Obviously, $z = 0$ is not isolated.

Since $z = \dfrac{1}{k\pi}(k \neq 0)$ is a zero of order 1 of $\sin\left(\dfrac{1}{z}\right)$, $z = \dfrac{1}{k\pi}$ is a simple pole of $f(z)$.

For $z = \dfrac{1}{k\pi}(k \neq 0)$,

$$\operatorname{Res}\left[f(z), \frac{1}{k\pi}\right] = (-1)^{k+1}\frac{1}{k\pi},$$

$k = \pm 1, \pm 2, \cdots$. **[5 marks]**

b). $f(z)$ has singular points: $2k\pi, k = 0, \pm 1, \pm 2, \cdots$.

Since $z=0$ is a zero of order 2 of $\cos z-1$, and a zero of order 3 of $\sin z-z$, we know $z=0$ is a removable singular point. Thus $\mathrm{Res}\left[f(z),0\right]=0.$

Since $z=2k\pi(k\neq 0)$ are zeros of order 2 of $\cos z-1$, and $\sin z-z$ is analytic and nonzero at these points, we know $z=2k\pi(k\neq 0)$ are poles of order 2.

[5marks]

Question 5. [10 marks]

Solve the following problem of small oscillation of semi-infinite unloaded string with rigidly free end $x=0.$

$$\begin{cases} u_{tt}-4u_{xx}=0, & 0<x<+\infty,\ t>0, \\ u_x(0,t)=0, & t>0, \\ u(x,0)=x^2, u_t(x,0)=x, & 0<x<+\infty. \end{cases}$$

Solution. Consider the Cauchy problem of the one dimensional infinite string oscillation equation:

$$\begin{cases} u_{tt}=a^2u_{xx}, -\infty<x<+\infty, t>0 \\ u|_{t=0}=\varphi(x), u_t|_{t=0}=\psi(x) \end{cases}.$$

The D'Alembert formula tells us that $u(x,t)=\dfrac{\varphi(x-at)^2+\varphi(x+at)^2}{2}+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(\xi)d\xi$. **[5 marks]**

For the problem of small oscillation of semi-infinite unloaded string with rigidly free end $x=0$, we need prolong the functions $\varphi(x)$ and $\psi(x)$ as even functions, that is,

$\varphi(x)=x^2, \psi(x)=|x|, \quad a=2$.

Then $u(x)=\dfrac{(x-2t)^2+(x+2t)^2}{2}+\dfrac{1}{2\times 2}\displaystyle\int_{x-2t}^{x+2t}|\xi|\,d\xi=\begin{cases} x^2+xt+4t^2, & x\geq 2t \\ \dfrac{5(x^2+4t^2)}{4}, & x<2t \end{cases}$. **[10 marks]**

Question 6. [10 marks]

Determine the type of the PDE $u_{xx}-6u_{xy}+9u_{yy}=xy^2$ and transform it into its standard form.

Solution. The discriminant is $\Delta=(-6)^2-4\times 9=0$, so this PDE is of parabolic type.

The characteristic equations is $dy^2+6dxdy+9dy^2=0$. **[5 marks]**

It has a real solution $\dfrac{dy}{dx}=-3$. That is, $y=-3x+C$

So we may let $\begin{cases} \xi=y+3x \\ \eta=y \end{cases}$. **[8 marks]**

The Jacobian of the transformation is $J=\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}=\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}=3\neq 0$.

With this variable transformation, the PDE is transformed into $\boldsymbol{u_{\eta\eta} = \frac{1}{9}xy^2 = \frac{1}{27}\xi\eta^2 - \frac{1}{27}\eta^3}$.

[10 marks]

Question 7. [12 marks]

Solve the following problem by means of separation of variables:

$$\begin{cases} \boldsymbol{u_{tt} - 4u_{xx} = 0, \quad 0 < x < 1, \ t > 0,} \\ \boldsymbol{u_x(0,t) = 0, \ u_x(1,t) = 0, \quad t \ge 0,} \\ \boldsymbol{u(x,0) = \sin \pi x, \ u_t(x,0) = 0, \quad 0 \le x \le 1.} \end{cases}$$

Solution.

Let $u(x,t) = X(x)T(t)$ and substitute it into the equation, we have

$X(x)T''(t) = 4X''(x)T(t)$.

Dividing it by $a^2X(x)T(t)$, we have $\dfrac{T''(t)}{4T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$.

Then $\begin{cases} X''(x) + \lambda X(x) = 0, \\ T''(t) + 4\lambda T(t) = 0 \end{cases}$ **[5 marks]**

And the boundary conditions become $X'(0) = X'(1) = 0$.

Solving the eigenvalue problem $\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(1) = 0 \end{cases}$, we obtain the eigenvalues

$\lambda_n = n^2\pi^2 (n = 0,1,2,\cdots)$ and the eigenfunctions $X(x) = \cos n\pi x$.

Solving the other problem $T''(t) + 4\lambda T(t) = 0$ for $\lambda_n = n^2\pi^2 (n = 0,1,2,\cdots)$, we obtain

$$T_0(t) = A_0 t + B_0; \quad T_n(t) = A\cos 2n\pi t + B\sin 2n\pi t, \quad n = 1,2,\cdots.$$

So, $u(x,t) = A_0 t + B_0 + \sum_{n=1}^{\infty}(A_n \cos 2n\pi t + B_n \sin 2n\pi t)\cos n\pi x$. **[10 marks]**

According to the initial condition, we have $\begin{cases} \sin \pi x = B_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x \\ 0 = A_0 + \sum_{n=1}^{\infty} B_n 2n\pi \cos n\pi x \end{cases}$,

Then $A_0 = \int_0^1 \sin \pi x dx = \dfrac{2}{\pi}$, $A_n = 2\int_0^1 \sin \pi x \cos n\pi x dx = \begin{cases} 0 & n \text{ is odd,} \\ \dfrac{-4}{(n^2-1)\pi} & n \text{ is even,} \end{cases}$ and $B_n = 0$.

Hence the solution is $u(x,t) = \dfrac{2}{\pi} - \sum_{n=1}^{\infty} \dfrac{4}{(4n^2-1)\pi} \cos 4n\pi t \cos 2n\pi x$. **[12 marks]**

Question 8. [10 marks]

Solve the vibration problem of a half infinite string:

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \\ u(0,t) = f(t), \quad \lim\limits_{x\to+\infty} u_x(x,t) = 0. \end{cases}$$

Solution. Let $V(x,p) = \mathcal{L}[u(x,t)]$, $L(p) = \mathcal{L}[f(t)]$.

Then by equation $u_{tt} = c^2 u_{xx}$, we have

$$\mathcal{L}[u_{tt}] - c^2\mathcal{L}[u_{xx}] = 0. \qquad (1)$$

Moreover, by $u(x,0) = 0, \quad u_t(x,0) = 0,$

$$\mathcal{L}\left[u_{tt}\right] = p^2 L[u] - pu(x,0) - u_t(x,0) = p^2 V. \qquad (2)$$

Thus, combining (1) and (2), we obtain

$$\frac{d^2V}{dx^2} - \frac{p^2}{c^2}V = 0.$$

Solving this equation, then

$$V(x,p) = C_1(p)e^{-\frac{p}{c}x} + C_2(p)e^{\frac{p}{c}x}. \qquad (3)$$

[4 marks]

From (3)， it is obvious that

$$V(0,p) = L(p)\,,\ \lim_{x\to+\infty}\frac{dV}{dx}(x,p) = 0. \qquad (4)$$

Thus, substituting (4) into (3), we have

$$C_1(p) = L(p), \quad C_2(p) = 0.$$

Then

$$V(x,p) = L(p)e^{-\frac{p}{c}x}.$$

[6 marks]

Finally, by the Laplace inverse transformation we have

$$u(x,t) = \mathcal{L}^{-1}\left[V(x,p)\right] = \mathcal{L}^{-1}\left[L(p)e^{-\frac{p}{c}x}\right]$$

[8 marks]

$$= \mathcal{L}^{-1}\left[\int_0^{+\infty} f(t)e^{-pt}e^{-\frac{p}{c}x}dt\right]$$

$$= \mathcal{L}^{-1}\left[\mathcal{L}\left[f\left(t-\frac{x}{c}\right)\right]\right] = f\left(t-\frac{x}{c}\right).$$

[10 marks]