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Section 9.3

Partial Derivatives and Total Differentials of Multi-variable Functions

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Partial Derivatives

We had seen that the derivative of a function of one variable, $f'(x_0)$, represents the rate of change of the function f at the point x_0; it reflects the speed of change of the function $f(x)$ when x varies from the point x_0.

When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a **'partial' derivative.**

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Partial Derivatives

From now on, we will see how partial derivatives arise and how to calculate partial derivatives by applying the rules for differentiating functions of a single variable.

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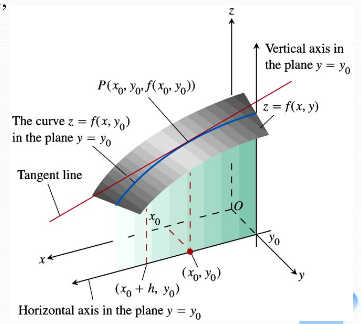
Partial Derivatives of a Function of Two Variables

If (x_0, y_0) is a point in the domain of a function $f(x,y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x,y)$ in the curve $z = f(x, y_0)$. The horizontal coordinate in this plane is x; the vertical coordinate is z.

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$.



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Partial Derivatives of a Function of Two Variables

Definition Partial Derivative with Respect to x

Suppose that the function $z = f(x,y)$ is defined in a neighbourhood $U(x_0, y_0)$ of (x_0, y_0). We fix the independent variable y at y_0, i.e, $y = y_0$. When the independent variable x has an increment Δx, and $(x_0 + \Delta x, y_0) \in U(x_0, y_0)$ the corresponding function $f(x)$ has an increment

$$f(x_0 + \Delta x, y_0) - f(x_0, y_0).$$

If the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

exists, then this limit is called the **partial derivative of the function $z = f(x,y)$ with respect to x at the point (x_0, y_0).**

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Partial Derivatives of a Function of Two Variables

Definition (continued) Partial Derivative with Respect to x

The partial derivative of function $z = f(x,y)$ respect to x at (x_0, y_0) is denoted by

$$f_x(x_0, y_0), \quad z_x(x_0, y_0), \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} \text{ or } \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$$

i.e.

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

The stylized " ∂ " (similar to the lowercase Greek letter "δ" used in the limit definition) is just another kind of "d". It is convenient to have this distinguishable way of extending the Leibniz differential notation into a multivariable context.

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Partial Derivatives of a Function of Two Variables

Similarly, we can define the partial derivative of the function $z=f(x,y)$ at the point (x_0,y_0) with respect to y as follows

$$f_y(x_0,y_0)=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}.$$

It may also be denote by

$$z_y(x_0,y_0),\quad \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}\quad \text{or}\quad \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}.$$

Partial Derivatives of a Function of Two Variables

If the partial derivative of the function $z=f(x,y)$ with respect to x at every point $(x,y)\in D\subseteq \mathbf{R}^2$ exists, then the partial derivative

$$f_x(x,y)$$

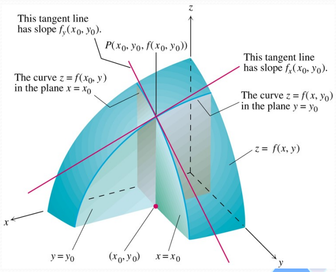
is also a function of two variable (x,y), called the partial derived function with respect to x, denoted by $f_x, z_x, \dfrac{\partial z}{\partial x}$ or $\dfrac{\partial f}{\partial x}$.

$$e.g.\quad P=R\frac{T}{V},\quad \frac{\partial P}{\partial T}\bullet\frac{\partial T}{\partial V}\bullet\frac{\partial V}{\partial P}=1?$$

Partial Derivatives of a Function of Two Variables

The slope of the curve $z=f(x,y_0)$ at the point $P(x_0,y_0,f(x_0,y_0))$ in the plane $y=y_0$ is the value of the partial derivative of f with respect to x at (x_0,y_0).

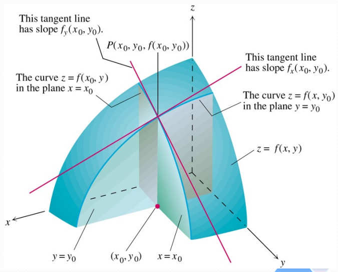
The tangent line to the curve at P is the line in the plane $y=y_0$ that passes through P with this slope.



Partial Derivatives of a Function of Two Variables

The partial derivative $\partial f/\partial x$ at (x_0,y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0.

Similarly, the derivative $\partial f/\partial y$ at (x_0,y_0) gives the rate of change of f with respect to y when x is held fixed at the value x_0.



Partial Derivatives of a Function of Two Variables

Example Finding Partial Derivatives at a Point

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4,-5)$ if

$$f(x,y)=x^2+3xy+y-1.$$

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2+3xy+y-1)=2x+3\cdot 1\cdot y+0-0=2x+3y.$$

The value of $\partial f/\partial x$ at $(4,-5)$ is $2(4)+3(-5)=-7$. Similarly, to find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2+3xy+y-1)=0+3\cdot x\cdot 1+1-0=3x+1.$$

The value of $\partial f/\partial y$ at $(4,-5)$ is $3(4)+1=13$.

Partial Derivatives of a Function of Two Variables

Example Finding Partial Derivatives as a Function

Find $\partial f/\partial y$ if $f(x,y)=y\sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial y}(y\sin xy)=y\frac{\partial}{\partial y}\sin xy+(\sin xy)\frac{\partial}{\partial y}(y)$$

$$=(y\cos xy)\frac{\partial}{\partial y}(xy)+(\sin xy)=xy\cos xy+\sin xy.$$

DIY Find the $\partial f/\partial x$, where $f(x,y)=y\sin xy$.

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Partial Derivatives of a Function of Two Variables

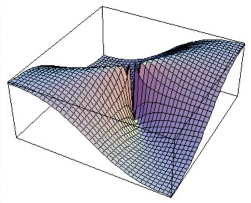
Example Discuss the existence of partial derivatives of the function

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$ at the origin.

Solution Since

$$\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=0,$$

$$\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=0.$$



The partial derivatives of f at the origin both exist and $f_x(0,0)=0, f_y(0,0)=0.$

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Partial Derivatives of a Function of Two Variables

Definition If both partial derivatives of the function $f(x,y)$ at the point (x_0,y_0) exist, then we say that the function $f(x,y)$ is partial derivable at the point (x_0,y_0).

Note We know that if a single variable function $f(x)$ is derivable at $x=x_0$ implies that the function f is continuous at $x=x_0$, but this may not be true with functions of two variables. As an example, discuss the continuity of function $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$ at point $(0,0)$.

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Total Differentials

Definition (Total differential) Suppose that a function $z=f(x,y)$ is defined in a neighbourhood $U((x_0,y_0))$ of the point (x_0,y_0). If for $(x_0+\Delta x,y_0+\Delta y)\in U((x_0,y_0))$, the increment of the function f at (x_0,y_0)

$$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$$

can be expressed in the form

$$\Delta z=a_1\Delta x+a_2\Delta y+o(\rho),$$

where a_1,a_2 are constants independent of Δx and $\Delta y, \rho=\sqrt{\Delta x^2+\Delta y^2}$ and $o(\rho)$ is an infinitesimal of higher order with respect to ρ as $\rho\to 0$, then the function f is said to be differentiable at the point (x_0,y_0), and $a_1\Delta x+a_2\Delta y$ is called the total differential of the function f at the point (x_0,y_0), denoted by $dz|_{(x_0,y_0)}$ or $df(x_0,y_0)$.

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P81 / Definition 6.2.2

Total Differentials

Definition (continued) (Total differential) Thus

$$dz|_{(x_0,y_0)}=a_1\Delta x+a_2\Delta y.$$

We define the differentials of the independent variables to be equal to their increments, that is $\Delta x=dx, \Delta y=dy$; then the total differential of the function f at the point (x_0,y_0) can be written as

$$dz|_{(x_0,y_0)}=a_1dx+a_2dy.$$

Obviously, when r is sufficiently small, the total differential is the linear and main part of the increment of the function f at the point (x_0,y_0).

1. What conditions will differentiable function satisfy?

2. If a function is differentiable, what are the values of a_1 and a_2?

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Total Differentials

Theorem (Necessary Conditions for Differentiability)

Suppose that a function $z=f(x,y)$ is differentiable at a point (x_0,y_0). Then

(1) f must be continuous at the point (x_0,y_0);

(2) both partial derivatives of the function f at the point (x_0,y_0) exist and $a_1=f_x(x_0,y_0), a_2=f_y(x_0,y_0)$, where a_1 and a_2 are expressed by $dz|_{(x_0,y_0)}=a_1dx+a_2dy$, that is, the total differential of the function f at the point (x_0,y_0) is

$$dz|_{(x_0,y_0)}=f_x(x_0,y_0)dx+f_y(x_0,y_0)dy.$$

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Total Differentials

Proof of (1) f must be continuous at the point

If the function f is differentiable at the point (x_0,y_0), then the expression $\Delta z=a_1\Delta x+a_2\Delta y+o(\rho)$ holds. Let $\rho\to 0$, (i.e. $\Delta x\to 0, \Delta y\to 0$). We have

$$\lim_{\rho\to 0}\Delta z=0,$$

or

$$\lim_{\Delta x\to 0,\Delta y\to 0}f(x_0+\Delta x,y_0+\Delta y)=f(x_0,y_0).$$

Thus, $f(x,y)$ is continuous at the point (x_0,y_0).

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Total Differentials

Proof of (2) The value of a_1 and a_2 are just the partial derivatives of f

Since $z = f(x,y)$ is differentiable at the point (x_0, y_0), then

$$\Delta z = a_1\Delta x + a_2\Delta y + o(\rho).$$

Let $\Delta y = 0$, we have

$$f(x_0+\Delta x, y_0) - f(x_0, y_0) = a_1\Delta x + o(\Delta x),$$

so that

$$\frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x} = a_1 + \frac{o(\Delta x)}{\Delta x}.$$

Notice that a_1 is independent of Δx. Letting $\Delta x \to 0$, we have

$$f_x(x_0, y_0) = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x} = a_1 + \lim_{\Delta x\to 0}\frac{o(\Delta x)}{\Delta x} = a_1.$$

Similarly, we have $f_y(x_0, y_0) = a_2$.

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Total Differentials

Definition (Differentiable Function)

If the function $z = f(x,y)$ is differentiable at every point in the region $\Omega \subseteq \mathbf{R}^2$, then f is said to be a differentiable function in Ω. If Ω is the domain of the function f, then f is called a differentiable function.

In this case, the total differential of the function f at the point (x, y) can be denoted by df or dz, and

$$dz = f_x dx + f_y dy.$$

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Total Differentials

Example Discuss the differentiability of the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0, \end{cases}$$

at the point $(0,0)$.

Solution By the definition of the partial differential of the function, it is easy to see that $f_x(0,0) = f_y(0,0) = 0$, then

$$\Delta z - [f_x(0,0)\Delta x + f_y(0,0)\Delta y] = \frac{\Delta x \cdot \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}.$$

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Total Differentials

Solution (continued)

Thus

$$\frac{\Delta z - [f_x(0,0)\Delta x + f_y(0,0)\Delta y]}{\rho} = \frac{\dfrac{\Delta x \cdot \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}}{\rho} = \frac{\Delta x \cdot \Delta y}{(\Delta x)^2 + (\Delta y)^2}.$$

It is easy to see that the limit

$$\lim_{\rho\to 0}\frac{\Delta x \cdot \Delta y}{(\Delta x)^2 + (\Delta y)^2}$$

does not exist. This means that the given function is not differentiable at the point (0,0).

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Total Differentials

Theorem (Sufficient Condition for Differentiability)

If the partial derivatives of a function $z = f(x,y)$, $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ both exist in the neighbourhood of point (x_0, y_0) and are continuous at the point (x_0, y_0) then the function f is differentiable at the point (x_0, y_0).

Proof Since

$$\begin{aligned}\Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) \\ &= [f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)] \\ &\quad + [f(x_0, y_0+\Delta y) - f(x_0, y_0)],\end{aligned}$$

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Total Differentials

$$\begin{aligned}\Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) \\ &= [f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)] \\ &\quad + [f(x_0, y_0+\Delta y) - f(x_0, y_0)]\end{aligned}$$

Proof (continued)

By the Lagrange theorem we have

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y) = \frac{\partial f(x_0+\theta_1\Delta x, y_0+\Delta y)}{\partial x}\Delta x,$$

and because the partial derivatives is continuous at point (x_0, y_0), then

$$\frac{\partial f(x_0+\theta_1\Delta x, y_0+\Delta y)}{\partial x}\Delta x = f_x(x_0, y_0)\Delta x + \varepsilon_1\Delta x,$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$.

Similarly,

$$f(x_0, y_0+\Delta y) - f(x_0, y_0) = \frac{\partial f(x_0, y_0+\theta_2\Delta y)}{\partial y}\Delta y = f_y(x_0, y_0)\Delta y + \varepsilon_2\Delta y,$$

where $\varepsilon_2 \to 0$ as $\Delta y \to 0$.

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Total Differentials

Proof (continued)

$f(x_0+\Delta x, y_0+\Delta y)-f(x_0, y_0+\Delta y)=f_x(x_0,y_0)\Delta x+\varepsilon_1\Delta x,$
$f(x_0, y_0+\Delta y)-f(x_0,y_0)=f_y(x_0,y_0)\Delta y+\varepsilon_2\Delta y.$

Thus,

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y.$$

Since

$$\left|\frac{\varepsilon_1\Delta x+\varepsilon_2\Delta y}{\rho}\right| \le |\varepsilon_1|\left|\frac{\Delta x}{\rho}\right| + |\varepsilon_2|\left|\frac{\Delta y}{\rho}\right| \le |\varepsilon_1|+|\varepsilon_2| \to 0 \quad \text{as} \quad \Delta x\to 0, \Delta y \to 0,$$

Hence

$$\lim_{\rho\to 0}\frac{\Delta z - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y}{\rho}=0.$$

Therefore, f is differentiable at the point (x_0, y_0).

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Total Differentials

Example Find the total differentials of function $z=e^{xy}$ at the point **(2,1).**

$dz|_{(x_0,y_0)} = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy$

Solution Since

$$\frac{\partial z}{\partial x} = ye^{xy}, \qquad \frac{\partial z}{\partial y} = xe^{xy},$$

then

$$\left.\frac{\partial z}{\partial x}\right|_{(2,1)} = e^2, \qquad \left.\frac{\partial z}{\partial y}\right|_{(2,1)} = 2e^2,$$

Therefore, the total differentials of the given function at point $(2,1)$ is

$$dz|_{(2,1)} = e^2dx + 2e^2dy.$$

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Total Differentials

NOTE The definition of total differential, the condition for differentiability and the computational formula for the total differential mentioned above can all be extended to functions of n variables.

$$u = f(x_1, x_2, \cdots, x_n)$$
$$\Downarrow$$
$$du = f_{x_1}dx_1 + f_{x_2}dx_2 + \cdots + f_{x_n}dx_n$$

$\mathbf{x}^0 = (x_1, x_2, \cdots, x_n)$ $\quad f(x_1, x_2, \cdots, x_n) = f(\mathbf{x}^0)$ $\quad df(\mathbf{x}^0) = f_{x_1}dx_1 + f_{x_2}dx_2 + \cdots + f_{x_n}dx_n$

DIY Find the du, where $u = zy\sin(xy) + xy\cos(xz).$

BUPT
The Applications of the Total Differential to Approximate Computation and Estimation of Errors

If a function f of n variables is differentiable at point $\mathbf{x}^0$, then we have

$$\Delta f = f(\mathbf{x}^0+\Delta\mathbf{x}) - f(\mathbf{x}^0) = df(\mathbf{x}^0) + o(\rho),$$

so if $\rho = \|\Delta\mathbf{x}\| \ll 1,$ we have

$$f(\mathbf{x}^0+\Delta\mathbf{x}) - f(\mathbf{x}^0) \approx df(\mathbf{x}^0) = \sum_{i=1}^{n} f_{x_i}(\mathbf{x}^0)\Delta x_i,$$

or

$$f(\mathbf{x}) \approx f(\mathbf{x}^0) + \sum_{i=1}^{n} f_{x_i}(\mathbf{x}^0)\Delta x_i.$$

The right side of the expression is a linear function of the n variables.

If $\mathbf{x}\in\mathbf{R}^2$, this is a plane; if $\mathbf{x}\in\mathbf{R}^n, n\ge 3,$ this is called a hyperplane.

This expression shows that when $\rho \ll 1,$ the increment of function f can be approximated by its total differential.

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The Applications of the Total Differential to Approximate Computation and Estimation of Errors

1) Approximate computation of functional values

Example Find the approximate value of $\sqrt{(1.97)^3+(1.01)^3}.$

Solution Let $f(x,y)=\sqrt{x^3+y^3}, (x_0,y_0)=(2,1), \Delta x = -0.03, \Delta y = 0.01.$

Then

$$f_x(2,1) = \left.\frac{3x^2}{2\sqrt{x^3+y^3}}\right|_{(2,1)} = 3, \qquad f_y(2,1) = \left.\frac{3y^2}{2\sqrt{x^3+y^3}}\right|_{(2,1)} = \frac{1}{2}.$$

Thus,

$$\sqrt{(1.97)^3+(1.01)^3} = f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + df(x_0,y_0)$$
$$= f(2,1) + f_x(2,1)\Delta x + f_y(2,1)\Delta y = 2.945.$$

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The Applications of the Total Differential to Approximate Computation and Estimation of Errors

2) Estimation of errors

If z is determined by a function $z = f(x,y)$ through measurement of the quantities x and y. Let the measured value of x and y be x_0 and y_0, and the maximum absolute errors of the measurement are δ_x and δ_y. Thus, $|\Delta x| < \delta_x, |\Delta y| < \delta_y$. Then the value $z_0 = f(x_0, y_0)$, which is obtained by computation from the formula $z = f(x,y)$ using the approximate value x_0 and y_0, is also an approximate value for the quantity z.

The question is what the error is, when we replace the actual value z with the approximate value z_0.

$$|\Delta z| = |f(x,y) - f(x_0,y_0)| \le ?$$

The Applications of the Total Differential to Approximate Computation and Estimation of Errors

Since $|\Delta x|$ and $|\Delta y|$ are very small, the approximate equality $\Delta z \approx dz$ can be used.

$$|\Delta z| \approx |dz| = |f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y|$$
$$\le |f_x(x_0,y_0)||\Delta x| + |f_y(x_0,y_0)||\Delta y|$$
$$< |f_x(x_0,y_0)|\delta_x + |f_y(x_0,y_0)|\delta_y,$$

so that the absolute error of the value z_0 may be taken as

$$\delta_z = |f_x(x_0,y_0)|\delta_x + |f_y(x_0,y_0)|\delta_y,$$

and the relative error of z_0 may be taken as

$$\frac{\delta_z}{|z_0|} = \left|\frac{f_x(x_0,y_0)}{f(x_0,y_0)}\right|\delta_x + \left|\frac{f_y(x_0,y_0)}{f(x_0,y_0)}\right|\delta_y.$$

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The Applications of the Total Differential to Approximate Computation and Estimation of Errors

Example Let $z = xy$. Find the absolute error and relative error of the approximate value z_0 produced by the computation of z from the values of the measurement x_0 and y_0.

Solution Since $z_x = y, z_y = x$, substituting these into the expression of the absolute error and the relative error, we obtain

$$\delta_z = |f_x(x_0,y_0)|\delta_x + |f_y(x_0,y_0)|\delta_y,$$

the absolute error: $\delta_z = |y_0|\delta_x + |x_0|\delta_y,$

$$\frac{\delta_z}{|z_0|} = \left|\frac{f_x(x_0,y_0)}{f(x_0,y_0)}\right|\delta_x + \left|\frac{f_y(x_0,y_0)}{f(x_0,y_0)}\right|\delta_y$$

the relative error: $\dfrac{\delta_z}{|z_0|} = \underline{\dfrac{\delta_x}{|x_0|}} + \underline{\dfrac{\delta_y}{|y_0|}}.$

(the relative error w.r.t. x and y)

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Higher-Order Partial Derivatives

Suppose that $u = f(\mathbf{x}), \mathbf{x} \in \mathbf{R}^n$. If the partial derivative of $\dfrac{\partial f}{\partial x_i}$ w.r.t. x_j at $\mathbf{x}^0$ exists, then this partial derivative is called the second order partial derivative (or second partial derivative) of the function f at the point $\mathbf{x}^0$ w.r.t. x_i first and then x_j, denoted by $\dfrac{\partial^2 f(\mathbf{x}^0)}{\partial x_j \partial x_i} = \dfrac{\partial}{\partial x_j}\left(\dfrac{\partial f}{\partial x_i}\right)_{\mathbf{x}=\mathbf{x}^0}$, or $f_{x_ix_j}(\mathbf{x}^0)$ or $f_{ij}(\mathbf{x}^0), (1 \le i \le n, 1 \le j \le n)$.

For example, a function of two variables $z = f(x,y)$ has the four second order partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy},$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

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Higher-Order Partial Derivatives

Similarly, the nth order partial derivatives of a function f may be defined from the partial derivatives of n–1-st order of the function f. For example

$$f_{x_ix_jx_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial^2 f}{\partial x_j \partial x_i}\right).$$

The second order or higher order partial derivatives are called by the joint name higher-order partial derivative, and the partial derivatives of the function are sometimes called first-order partial derivatives of the function or partial derivatives for short.

The operations of calculating higher-order partial derivatives are actually the operations of calculating derivatives of functions of one variables.

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Higher-Order Partial Derivatives

Example Find all second derivatives of the function $z = x^y (x > 0)$.

Solution The first order derivatives are

$$\frac{\partial z}{\partial x} = yx^{y-1}, \frac{\partial z}{\partial y} = x^y \ln x. \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

Another derivation with respect to x and y respectively gives the second partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = y(y-1)x^{y-2}, \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = x^{(y-1)} + yx^{(y-1)}\ln x,$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = yx^{(y-1)}\ln x + x^{(y-1)}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = x^y(\ln x)^2.$$

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Higher-Order Partial Derivatives

Example Let

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \ne 0, \\ 0, & x^2+y^2 = 0. \end{cases}$$

Prove that $f_{xy}(0,0) \ne f_{yx}(0,0)$.

Proof It is easy to find the first order partial derivatives

$$f_x(x,y) = \begin{cases} y\left[\dfrac{x^2-y^2}{x^2+y^2} + \dfrac{4x^2y^2}{(x^2+y^2)^2}\right], & x^2+y^2 \ne 0, \\ 0, & x^2+y^2 = 0; \end{cases}$$

$$f_y(x,y) = \begin{cases} x\left[\dfrac{x^2-y^2}{x^2+y^2} - \dfrac{4x^2y^2}{(x^2+y^2)^2}\right], & x^2+y^2 \ne 0, \\ 0, & x^2+y^2 = 0. \end{cases}$$

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Higher-Order Partial Derivatives

Proof (continued)

Thus

$$f_x(0,y) = -y, \quad f_y(x,0) = x.$$

Again, from the definition of the partial derivative we have

$$f_{xy}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-\Delta y}{\Delta y} = -1,$$

$$f_{yx}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x,0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

Thus

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

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Higher-Order Partial Derivatives

In general, the order of derivation may affect the result, but it is possible to prove that if f_{xy} and f_{yx} are both continuous at a point P then $f_{xy} = f_{yx}$ at a point P.

This result can be extended to higher partial derivatives. If all the m^{th} order mixed partial derivatives are continuous, then the order of derivatives are continuous, then the order of derivation does not affect the result. For instance, if all third order mixed partial derivatives of the function $f(x,y,z)$ are continuous, then we have

$$f_{xxy} = f_{xyx} = f_{yxx};$$

$$f_{xyz} = f_{yzx} = f_{zxy} = f_{yxz} = f_{xzy} = f_{zyx}.$$