Chapter 21 Electric Potential

§ 1 Electric Potential Energy

- **The similarity of electrostatic and gravitational force**

$$\overrightarrow{F_e} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{r} \qquad \text{electrostatic}$$

$$\overrightarrow{F_g} = -G\frac{Mm}{r^2}\hat{r} \qquad \text{gravitational}$$

➧ **Both forces depend on the inverse square of the separation distance between the two objects.**

Electrostatic vs. gravitational

$$\overrightarrow{F_g} = -G\frac{Mm}{r^2}\hat{r} \qquad \text{gravitational}$$

- **The work done by the gravitational force on the object m depends only on the starting and finishing points and does not depend on the path taken between the points —— gravitational force is a conservative force.**

$$\Delta U = U_f - U_i = -W_{if} = -\int_i^f \overrightarrow{F_c} \cdot d\vec{s}$$

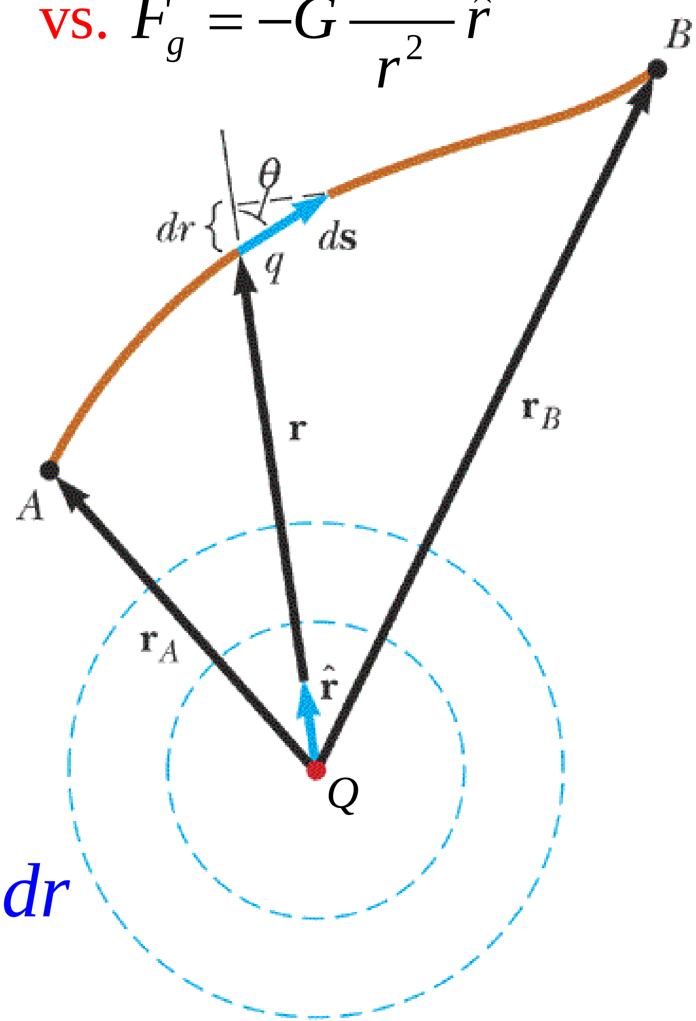
the gravitational potential energy difference

$$\Delta U = \left(-G\frac{Mm}{r_f}\right) - \left(-G\frac{Mm}{r_i}\right)$$

- **The electric potential energy**
  - **Because of the similarity of the electrostatic and gravitational force laws, the electrostatic force is also conservative, and there is a potential energy associated with the configuration of a system (the relative locations of the charges).**

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{r}$$

vs. $\vec{F}_g = -G\frac{Mm}{r^2}\hat{r}$



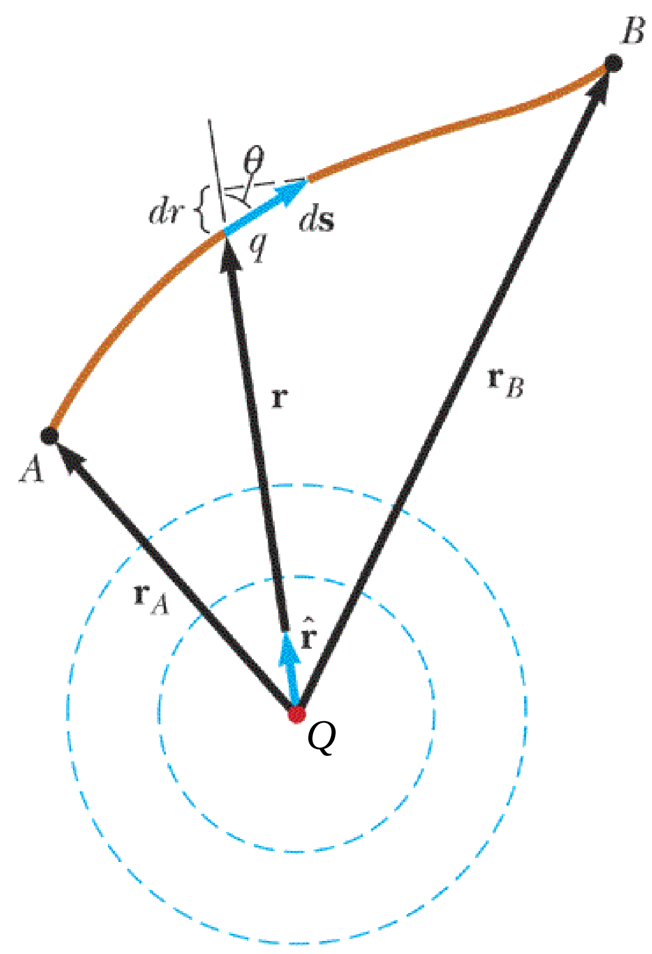
$$\vec{F}_e \cdot d\vec{s} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{r}\cdot d\vec{s}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2} ds\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2} dr$$

The electric potential energy difference between A and B

$$U_B - U_A = -\int_A^B \overrightarrow{F_e} \cdot d\vec{s} = -\int_{r_A}^{r_B} \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}dr$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\frac{Qq}{r_B}\right) - \left(\frac{1}{4\pi\varepsilon_0}\frac{Qq}{r_A}\right)$$



If we set $U_A(\infty) = 0$ as our reference potential energy, the potential energy at any point in space is

$$\boxed{U(r) = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r}}$$

Electric potential

- **Consider a test charge q_0 in the field of charge Q. The potential energy U associates with both the test charge q_0 and the source charge Q, which means that the U characterizes the interaction of two charges with one another.**

$$\Delta U_{BA} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q_0 Q}{r_B}\right) - \left(\frac{1}{4\pi\varepsilon_0}\frac{q_0 Q}{r_A}\right) = -\int_A^B \overrightarrow{F_e}\cdot d\vec{s} = -q_0\int_A^B \vec{E}\cdot d\vec{s}$$

- **The potential energy per unit test charge, which is symbolized as $\Delta U/q_0$, is independent of the test charge q_0, and is characteristic only of the field of due to source charge Q which we are investigating —— we define the electric potential difference ΔV to be the electric potential energy difference per unit test charge.**

$$\boxed{\Delta V_{BA} = \frac{\Delta U_{BA}}{q_0} = -\frac{1}{q_0}\int_A^B \overrightarrow{F_e}\cdot d\vec{s} = -\int_A^B \vec{E}\cdot d\vec{s}}$$

Electric potential

$$\boxed{\Delta V_{BA} = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}}$$

- **If we set** $V_A(\infty) = 0$ **as our reference potential**

$$V_B = -\int_{\infty}^{B} \vec{E} \cdot d\vec{s} = \int_B^{\infty} \vec{E} \cdot d\vec{s}$$

$$\boxed{V_P = -\int_{"0"}^{P} \vec{E} \cdot d\vec{s} = \int_P^{"0"} \vec{E} \cdot d\vec{s}}$$

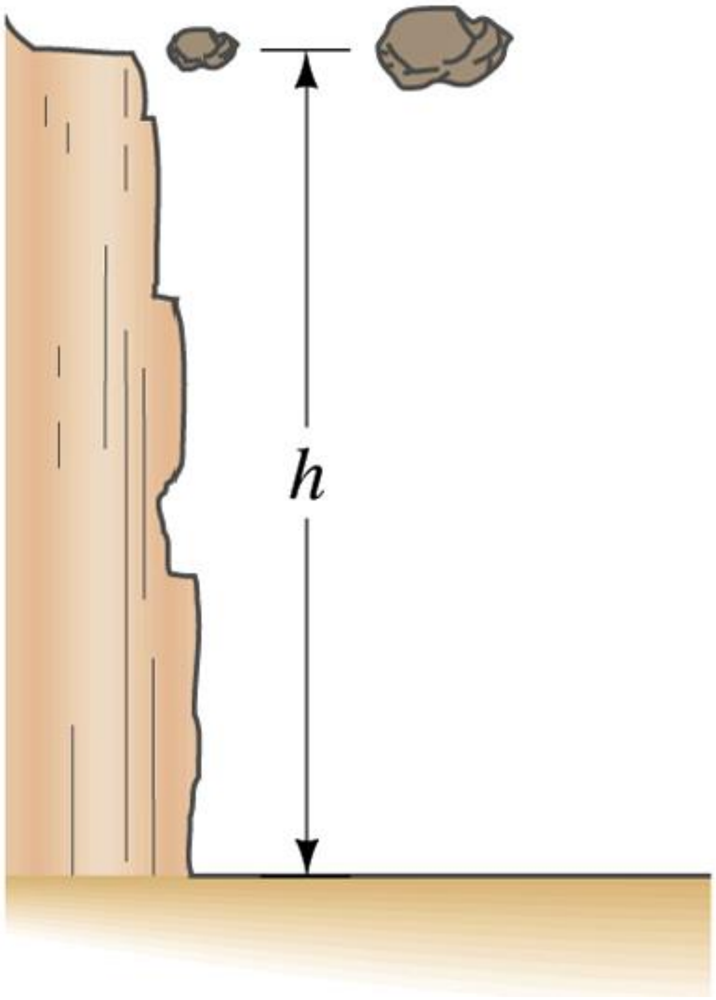
- **SI unit:** 1V=1 J/C

$$V_P = \int_P^{"0"} \vec{E} \cdot d\vec{s}$$

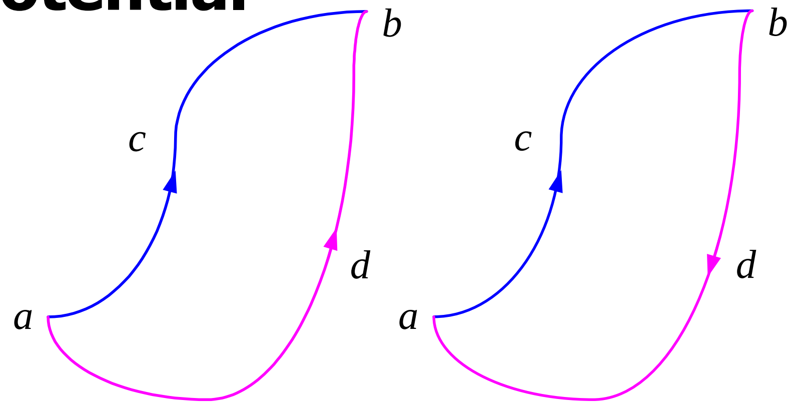
$$= -\int_{"0"}^{P} \vec{E} \cdot d\vec{s}$$

$$mgh = \int_P^{"0"} (m\vec{g}) \cdot d\vec{s}$$

$$= -\int_{"0"}^{P} (m\vec{g}) \cdot d\vec{s}$$



- **The circulation law of electric potential**



$$\int_{acb} \vec{E}\cdot d\vec{s} = \int_{adb} \vec{E}\cdot d\vec{s}$$

$$\int_{acb} \vec{E}\cdot d\vec{s} - \int_{adb} \vec{E}\cdot d\vec{s} = \int_{acb} \vec{E}\cdot d\vec{s} + \int_{bda} \vec{E}\cdot d\vec{s} = 0$$

$$\boxed{\oint_L \vec{E}\cdot d\vec{s} = 0}$$

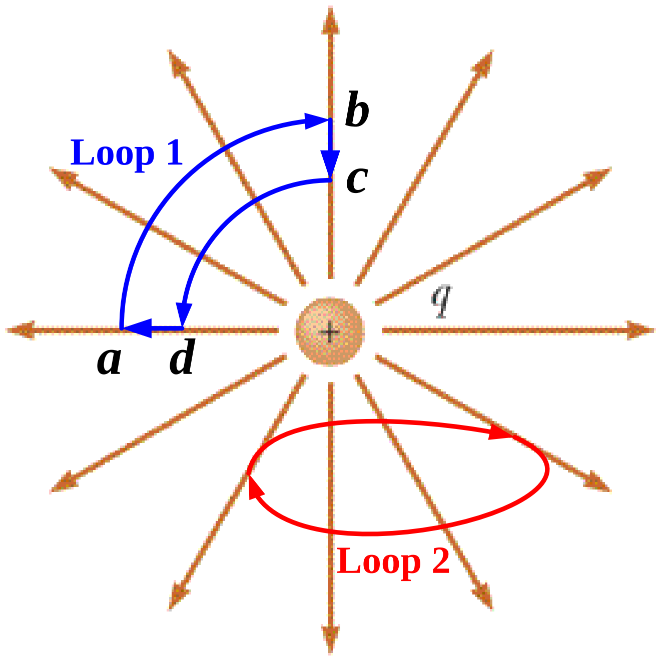
The circulation law of electric potential

This law means that the electrostatic field is a conservative field !

The Circulation law of electric potential

$$\oint_L \vec{E}\cdot d\vec{s} = 0$$

Example: For an electric field of a point charge *q*.



$$\underset{\text{Loop 1}}{\oint} \vec{E}\cdot d\vec{s} = \int_a^b \vec{E}\cdot d\vec{s} + \int_b^c \vec{E}\cdot d\vec{s} + \int_c^d \vec{E}\cdot d\vec{s} + \int_d^a \vec{E}\cdot d\vec{s} = 0$$

$$\underset{\text{Loop 2}}{\oint} \vec{E}\cdot d\vec{s} = 0$$

- **Summary of the laws for electrostatic field**
  - **Gauss' Law:**

    The electrostatic charge is the source of the electrostatic field.

$$\oiint_S \vec{E} \cdot d\vec{A} = \frac{q_{inside}}{\varepsilon_0}$$

  - **Circulation Law:**

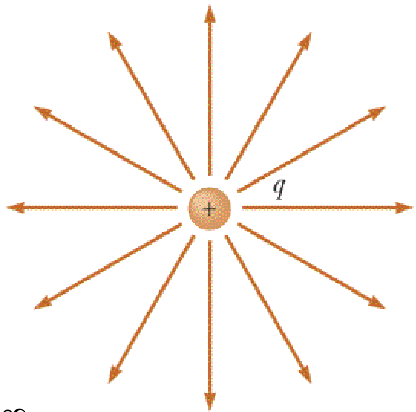
    The electrostatic field is a conservative field. Therefore we can introduce a scalar field (electric potential) correlated to the electrostatic field.

$$\oint_L \vec{E} \cdot d\vec{s} = 0$$

§ 2 Calculating the Electric Potential

- **If the electric field is known**

$$V_P = \int_P^{\infty} \vec{E} \cdot d\vec{s}$$



For a point charge q

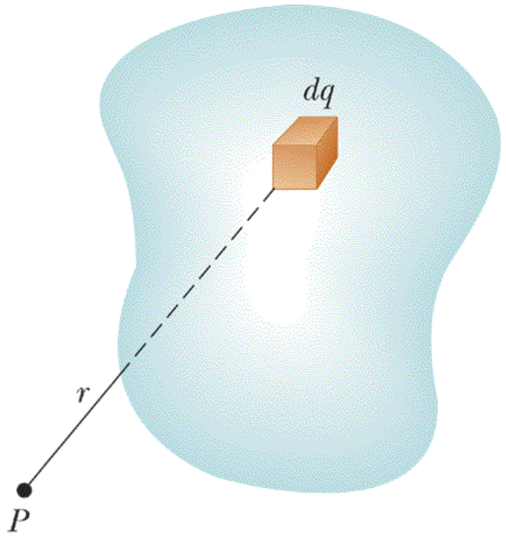
$$V_P = \int_P^{\infty} \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} \cdot d\vec{s} = \frac{q}{4\pi\varepsilon_0} \int_P^{\infty} \frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_0} \left(\frac{-1}{r} \right) \Bigg|_{r_P}^{\infty} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r_P}$$

- **If the charge distribution is known**
  - **The electric potential due to individual charge particles**

$$V = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{r_i}$$

  - **The electric potential due to continuous charge distributions**

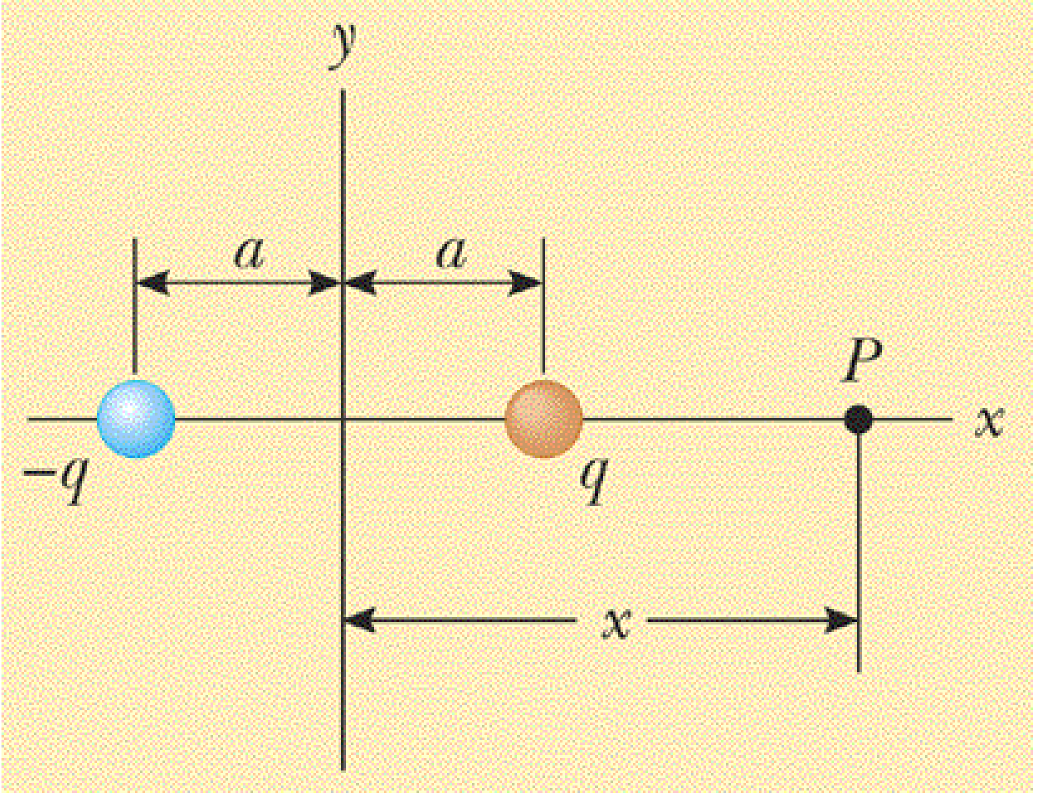
$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r}$$



Example —— The Electric Dipole

The electric potential of a dipole

The dipole is along the x axis and is centered at the origin. Calculating the electric potential at any point P along the x axis.



Example —— The Electric Dipole

The electric potential of a dipole

The dipole is along the x axis and is centered at the origin. Calculating the electric potential at any point P along the x axis.

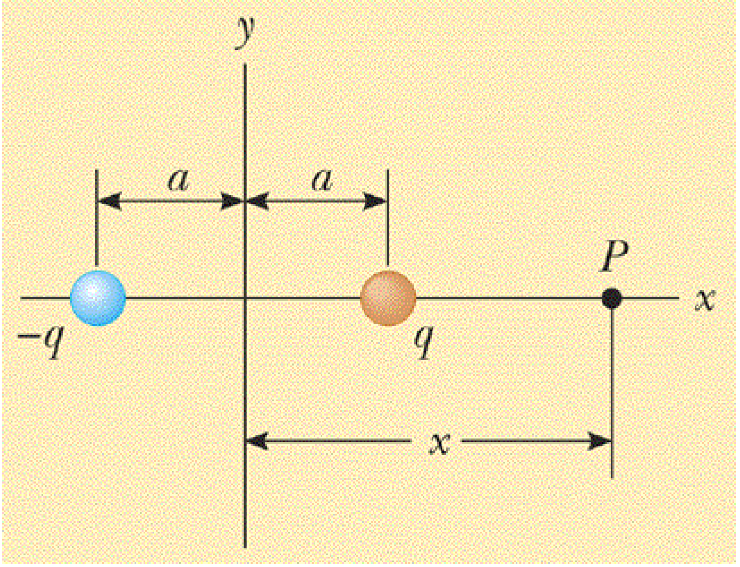
Solution:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{x-a} + \frac{-q}{x+a}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2aq}{x^2-a^2}$$

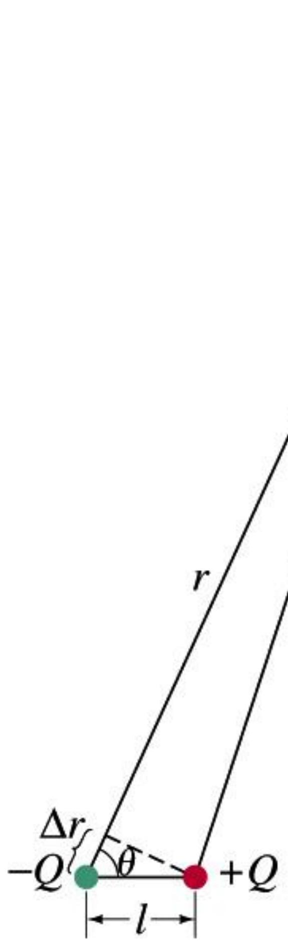
$$= \frac{1}{4\pi\varepsilon_0}\frac{p}{x^2-a^2}$$

$$\approx \frac{1}{4\pi\varepsilon_0}\frac{p}{x^2}, \qquad (x >> a)$$



Example —— The Electric Dipole

The electric potential of a dipole



$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{-Q}{r+\Delta r}\right)$$

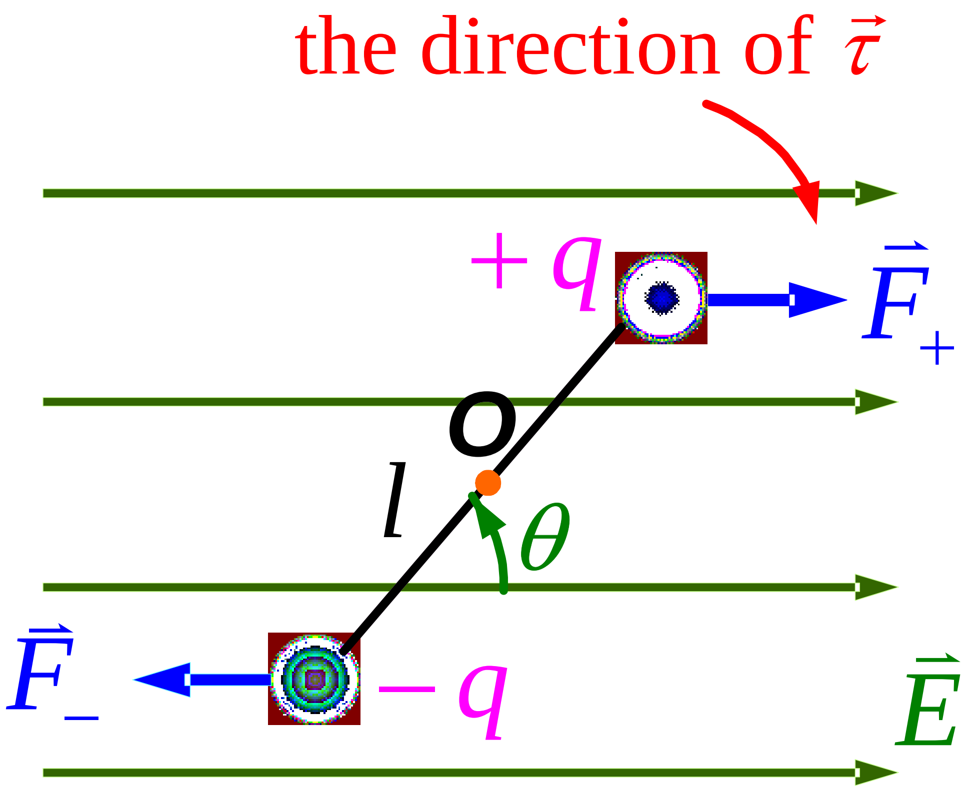
$$= \frac{Q}{4\pi\varepsilon_0}\frac{\Delta r}{r(r+\Delta r)}$$

$$\Delta r \approx l\cos\theta, \quad r \gg \Delta r$$

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{l\cos\theta}{r^2}$$

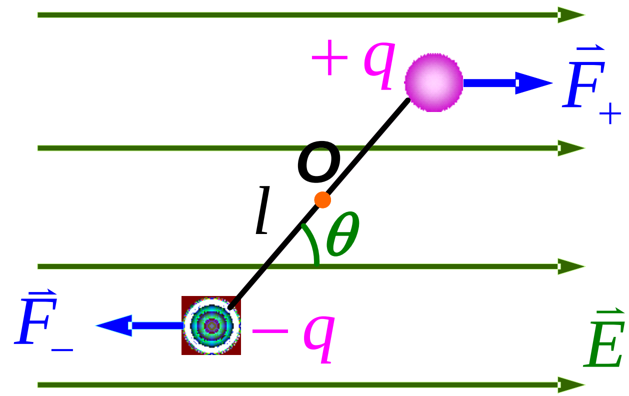
$$= \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

Example: Find the potential energy of an electric dipole in a uniform external field.



Solution:

The potential energy of a dipole is the sum of the potential energies of positive and negative charges in the field.



$$U = U_+ + U_- = qV(P_+) + (-q)V(P_-)$$

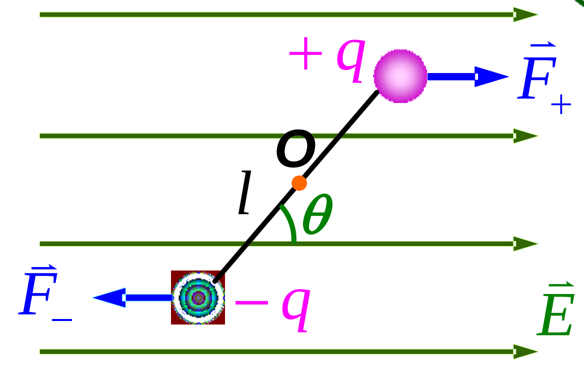
$$= q[V(P_+) - V(P_-)]$$

$$= q\int_{P_+}^{P_-} \vec{E}\cdot d\vec{s} = q(-El\cos\theta) = -pE\cos\theta = -\vec{p}\cdot\vec{E}$$

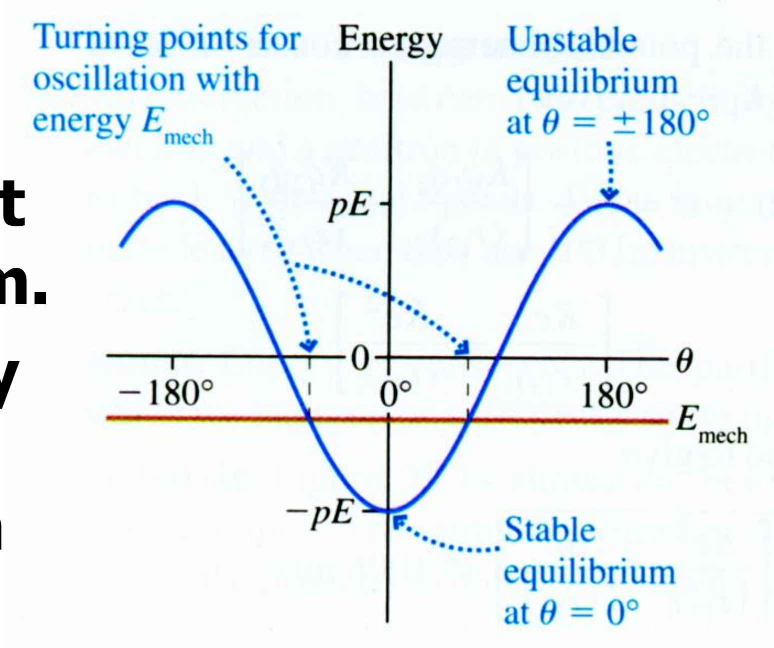
The potential energy of the system of an object in the Earth's gravitational field: $U_g = mgy$

The Potential Energy of a Dipole in an External Field

$$\boxed{U = -\vec{p} \cdot \vec{E}}$$

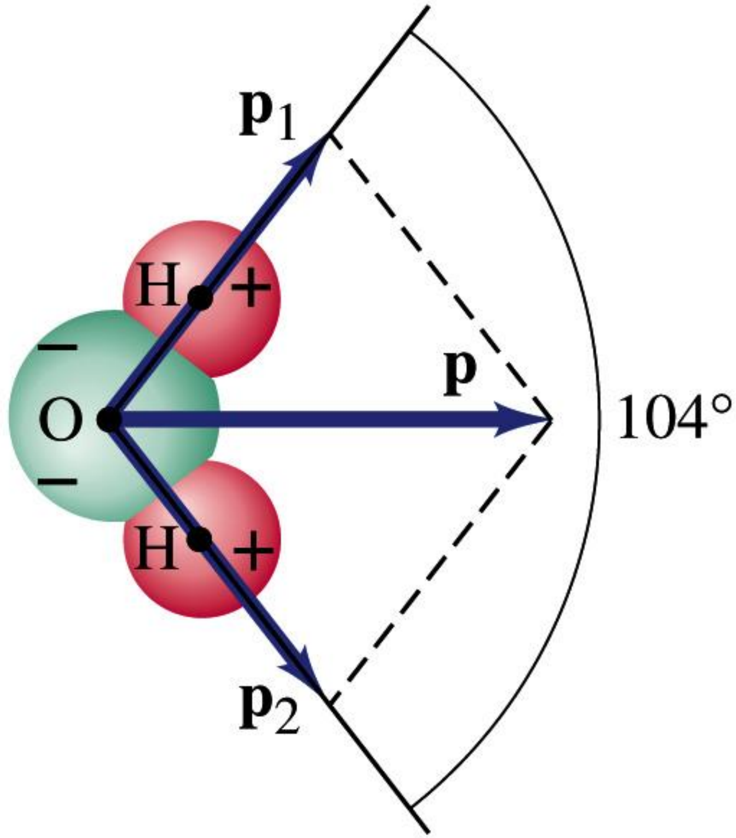


- The potential energy is minimum at $\theta = 0°$. This is the a point of **stable** equilibrium.
- The potential energy is maximum at $\theta = \pm 180°$, which is at the point of **unstable** equilibrium.
- A dipole with mechanical energy E_{mech} will **oscillates** back and forth between turning points on either side of $\theta = 0°$.



Prob. 43 (P521)

The dipole moment of a water molecule

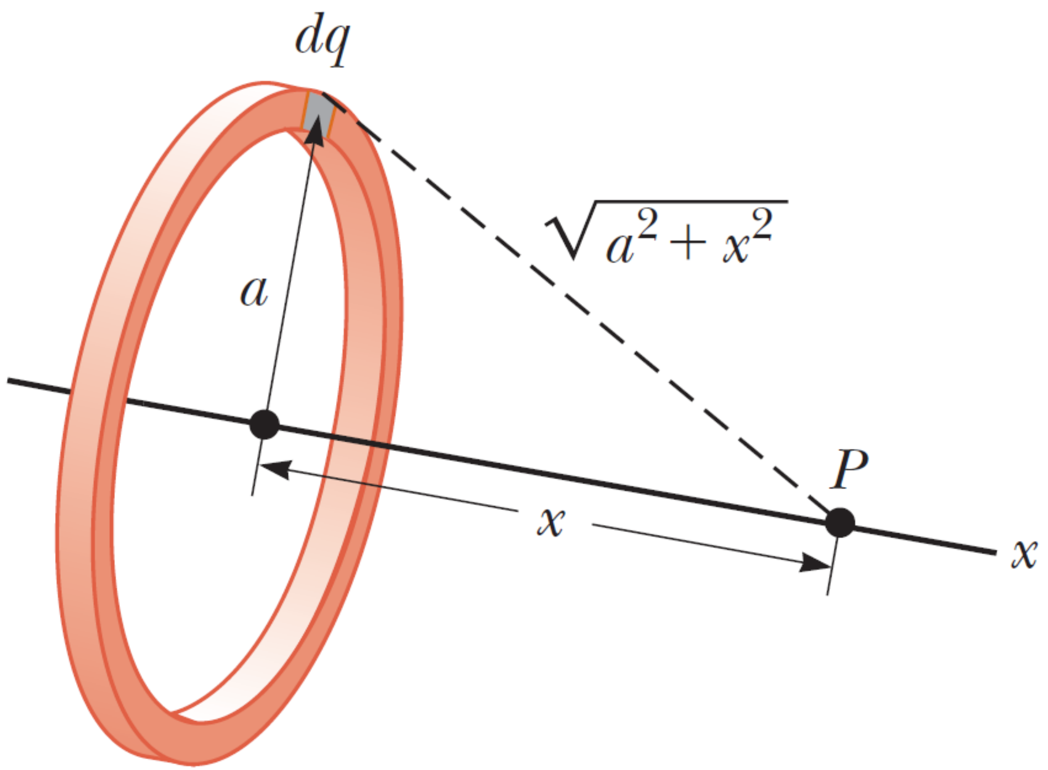


The water molecule has a permanent polarization resulting from its nonlinear geometry. We can model the water molecule and other polar molecules as dipoles.

Example (P510 Ex. 21-8)

The electric potential due to a uniformly charged ring

Find the electric potential at a point P located on the axis of a uniformly charged ring of radius a and total charge Q.



The electric potential due to a uniformly charged ring

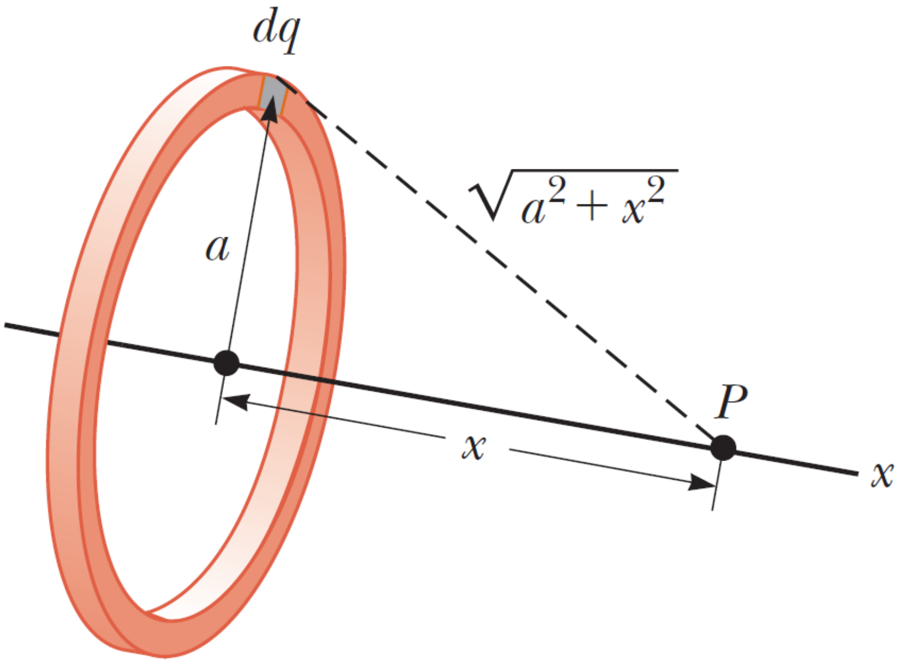
Find the electric potential at a point P located on the axis of a uniformly charged ring of radius a and total charge Q.

Solution:

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{x^2+a^2}}$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+a^2}}\int dq$$

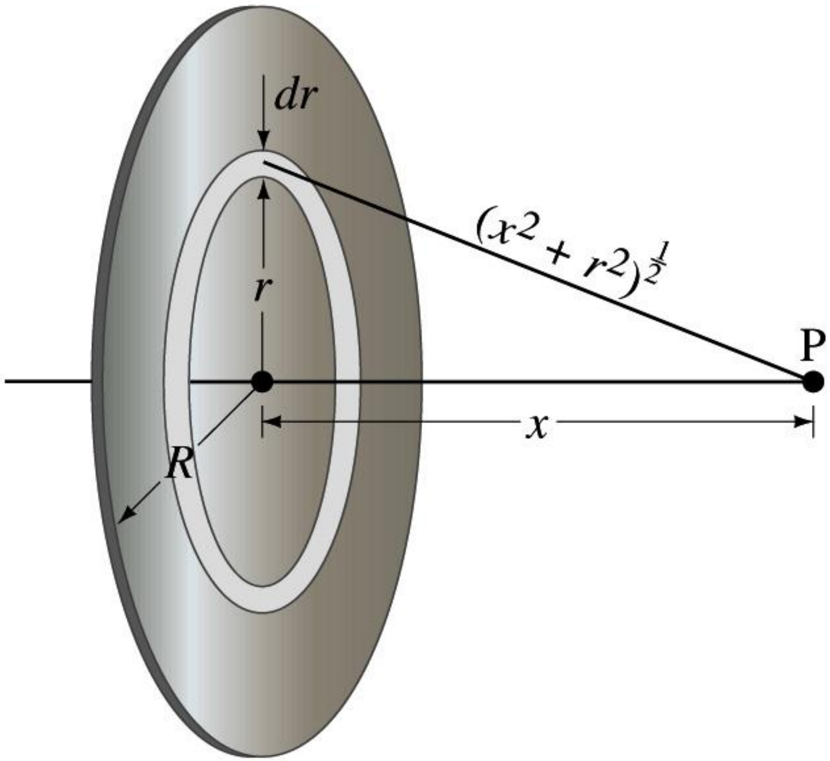
$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{x^2+a^2}}$$



Example (P510 Ex. 21-9)

The electric potential due to a uniformly charged disk

A thin flat disk, of radius R, carries a uniformly distributed charge Q. Determine the potential at a point P on the axis of the disk, a distance x from its center.



Example (P510 Ex. 21-9)

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{x^2+r^2}}$$

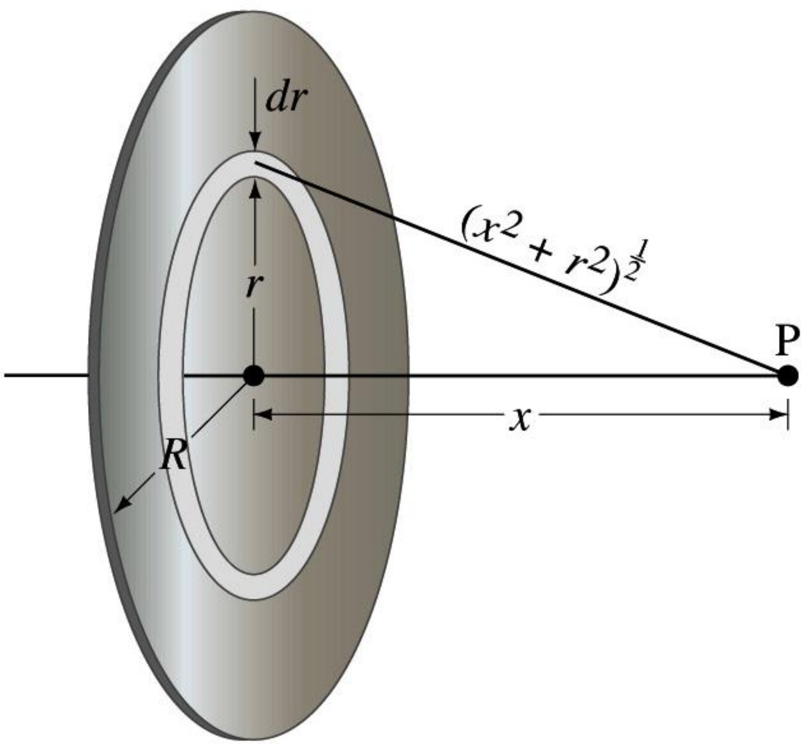
$$dA = (2\pi r)(dr)$$

$$dq = \frac{Q}{\pi R^2}dA = \frac{2Q}{R^2}rdr$$

$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\sqrt{x^2+r^2}}$$

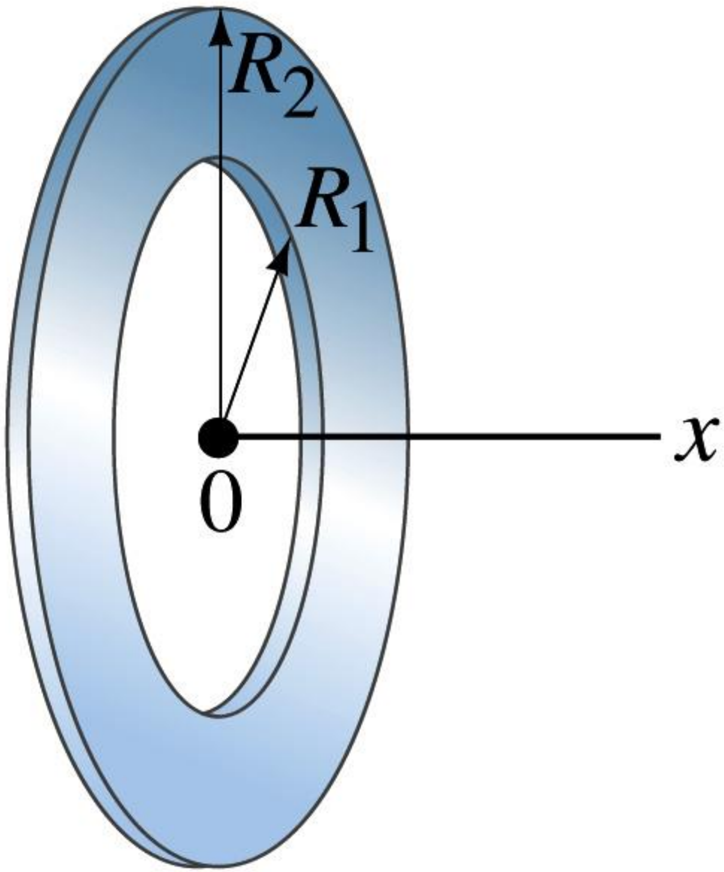
$$= \frac{1}{4\pi\varepsilon_0}\frac{2Q}{R^2}\int_0^R\frac{rdr}{\sqrt{x^2+r^2}}$$

$$= \frac{1}{2\pi\varepsilon_0}\frac{Q}{R^2}\left(\sqrt{x^2+R^2}-x\right)$$



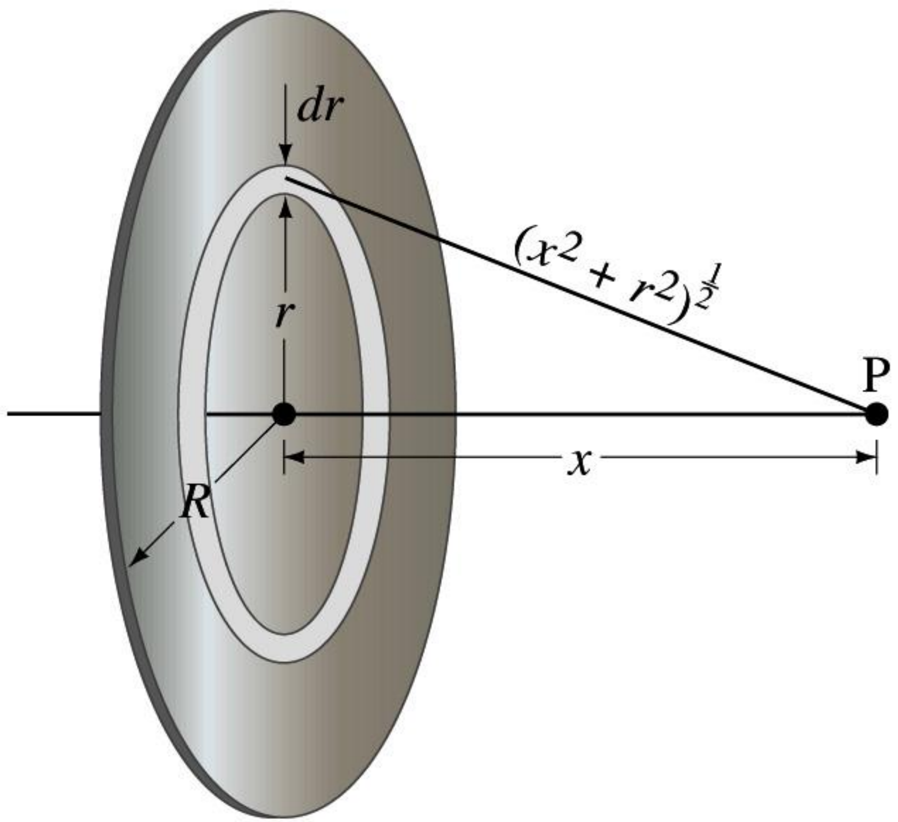
Prob. 31 (Ch21 P520)

A flat ring of inner radius R_1 and outer radius R_2 carries a uniform surface charge density σ. Determine the electric potential at points along the x axis.



Prob. 35 (Ch21 P521)

Suppose the flat circular disk has a nonuniform surface charge density $\sigma = ar^2$, where r is measured from the center of the disk. Find the potential at points along the x axis, relative to $V=0$ at $x=\infty$.



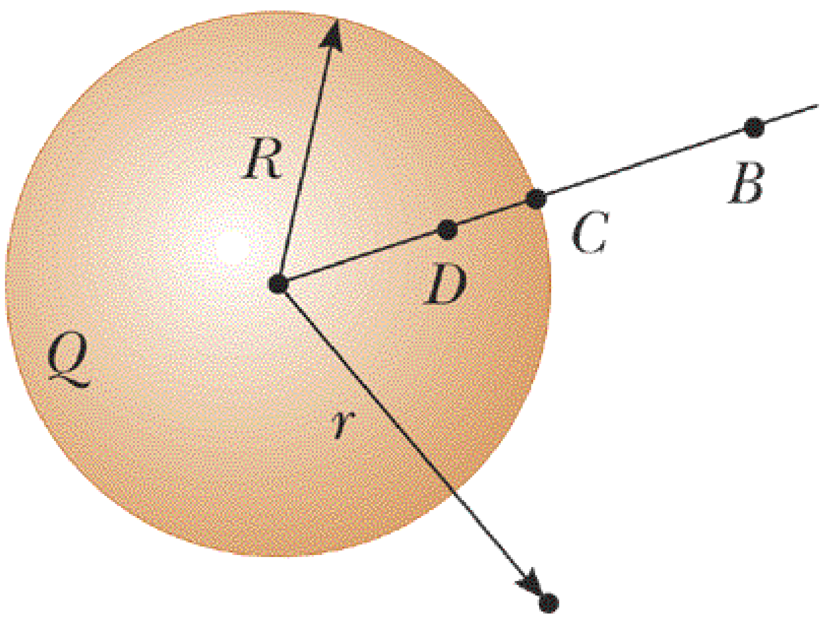
Example

The electric potential of a uniformly charged sphere

An insulating solid sphere of radius R has a total charge Q, which is distributed uniformly throughout the volume of the sphere.

(1) Find the electric potential at a point for $r > R$.

(2) Find the electric potential at a point for $r < R$.

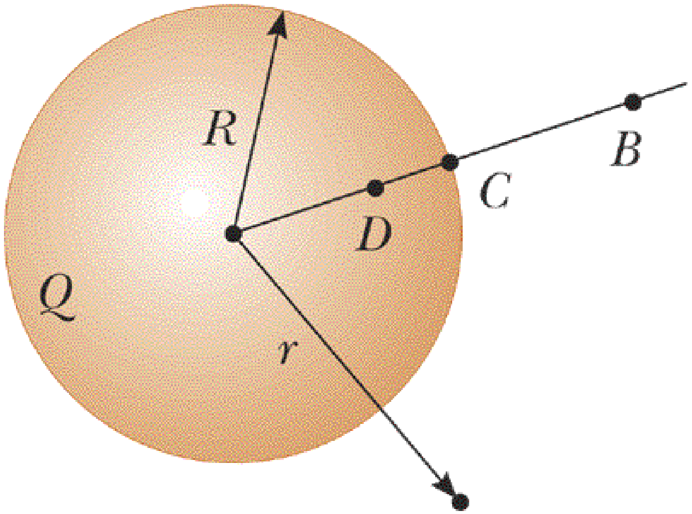


Example

The electric potential of a uniformly charged sphere

Solution 1: $V = \dfrac{1}{4\pi\varepsilon_0}\int \dfrac{dq}{r}$

Solution 2: $V_P = \int_P^{\infty} \vec{E} \cdot d\vec{s}$

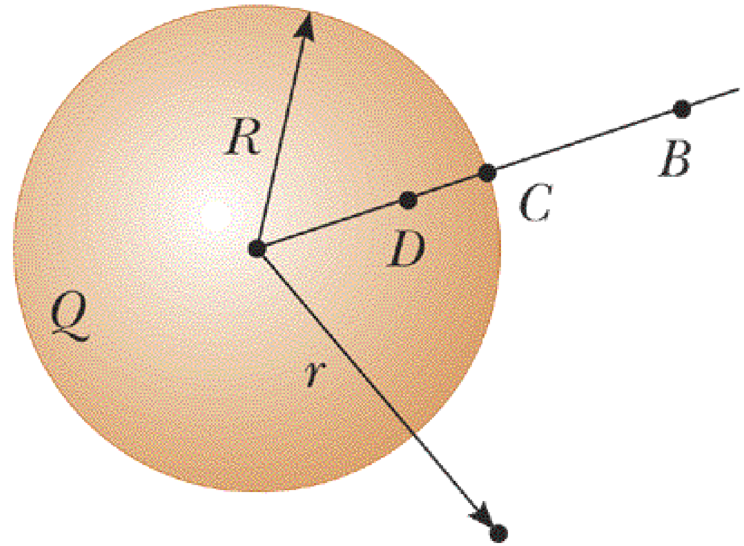


$$E = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2} & \text{for } r > R \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{r}{R^3}Q & \text{for } r < R \end{cases}$$

Example – cont'd

For *r* > *R*, $V_B = \int_r^\infty \vec{E}\cdot d\vec{s}$

$$= \frac{Q}{4\pi\varepsilon_0}\int_r^\infty \frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$



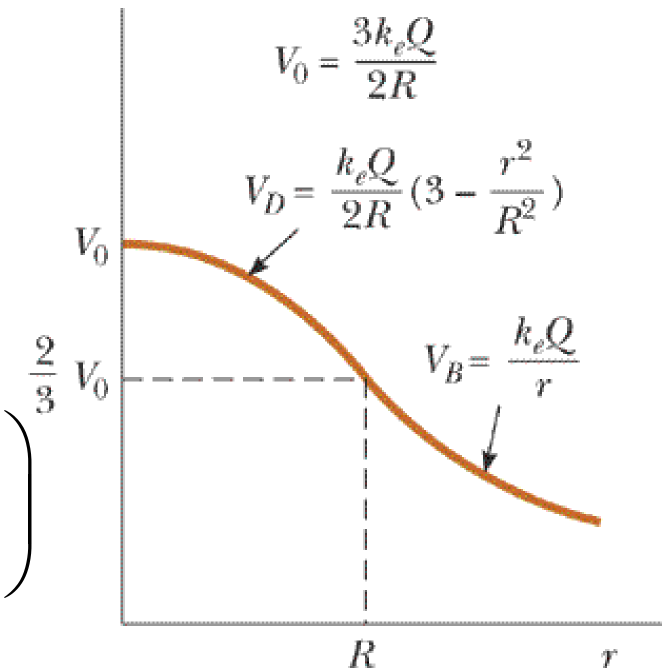
For *r* < *R*

$$V_D = \int_r^R \vec{E}\cdot d\vec{s} + \int_R^\infty \vec{E}\cdot d\vec{s}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^3}\int_r^R r\,dr + \frac{Q}{4\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{2R^3}\left(R^2 - r^2\right) + \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{2R}\left(3 - \frac{r^2}{R^2}\right) = \frac{Q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$



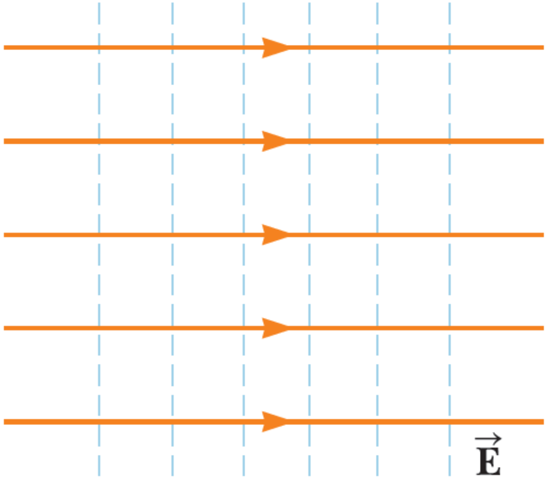
Problems

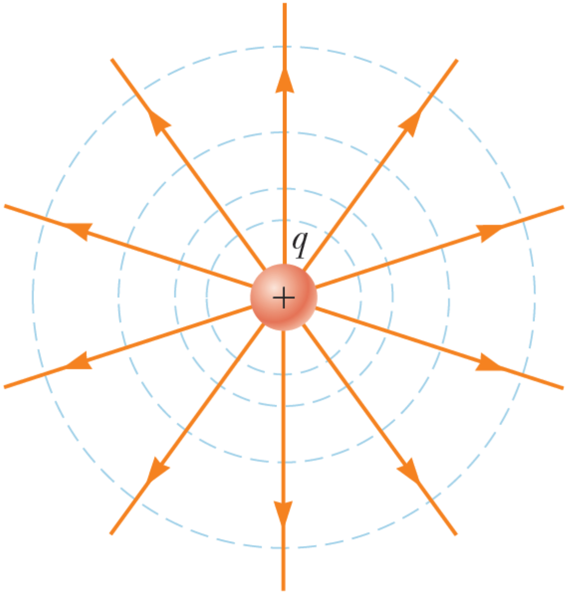
Ch21 Prob. 18, 34, 35 (P520)

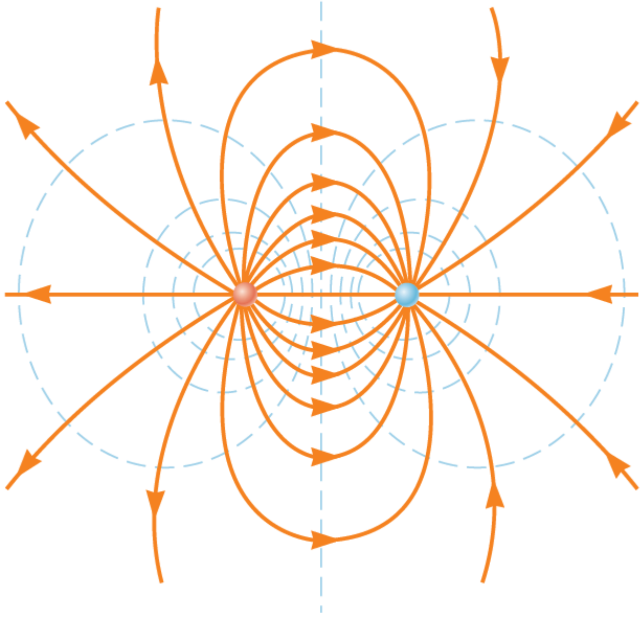
§ 3 Equipotential Surfaces

(P511, § 21-5)

- **The equipotential surface**
  - **An equipotential surface is a three-dimensional surface on which the electric potential V is the same at every point.**

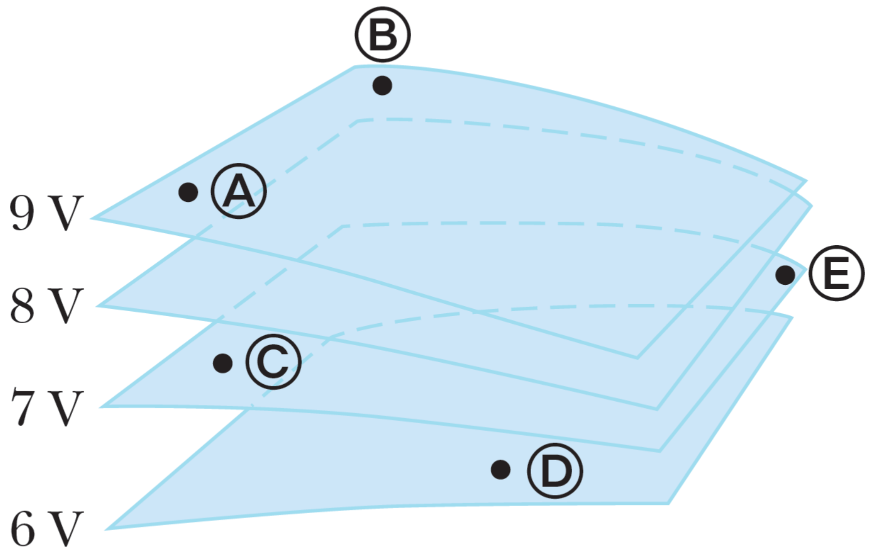






- **The properties of the equipotential surface**
  - **If a test charge moves over an equipotential surface, the electric field can do no work on such a charge.**

$$W_{ab} = -q_0 (V_b - V_a) = 0$$



- **Field lines and equipotential surface are always mutually perpendicular.**

  A test charge q_0 moves a distance $d\vec{l}$ on an equipotential surface

$$dW = q_0 \vec{E} \cdot d\vec{l} = q_0 E \cos\theta dl = 0 \quad \Rightarrow \quad \vec{E} \perp d\vec{l}$$

- **In regions where the magnitude of $\vec{E}$ is large, the equipotential surface are close together.**

§ 4 Potential Gradient

(P513 § 21-7)

$$-dV = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right)$$

$$= \vec{E}\cdot d\vec{s} = E_x dx + E_y dy + E_z dz$$

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)V$$

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

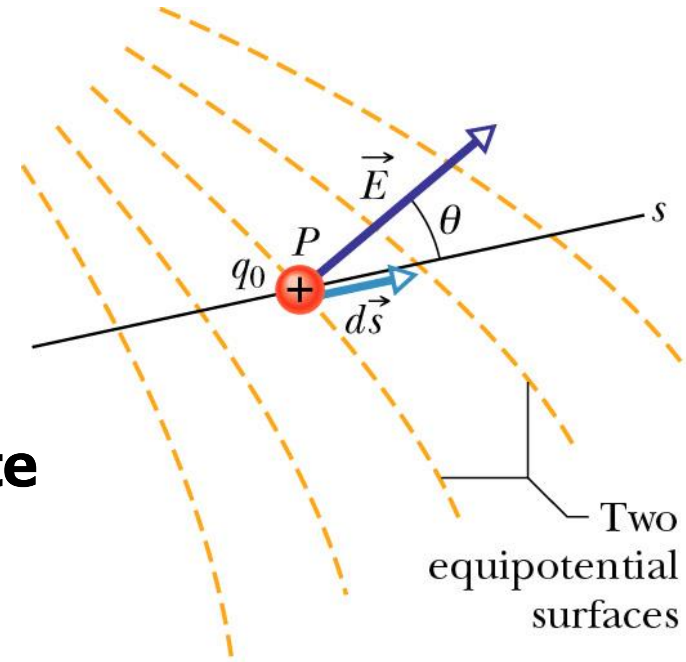
$\vec{E}$ **is the negative of the gradient of** V**.**

The meaning of the gradient

- **Make a displacement $d\vec{s}$ from one equipotential surface to the adjacent surface**

$$-dV = \vec{E} \cdot d\vec{s} = E\cos\theta ds$$

$$E\cos\theta = -\frac{dV}{ds}, \qquad \boxed{E_s = -\frac{\partial V}{\partial s}}$$



- **The component of $\vec{E}$ in any direction is the negative of the rate of change of the electric potential with distance in that direction.**

- **Take the s axis to be, in turn, x, y, and z axis, we get the x, y, z components of $\vec{E}$ at any point are**

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

The Methods of Calculating the Electric Field

- **By Coulomb's law:**

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{r}}$$

  - **The most general method.**

- **By Gauss' law:**

$$\boxed{\oiint_S \vec{E}\cdot d\vec{A} = \frac{q_{inside}}{\varepsilon_0}}$$

  - **If charge distribution possesses a high degree of symmetry**

- **By gradient of V:**

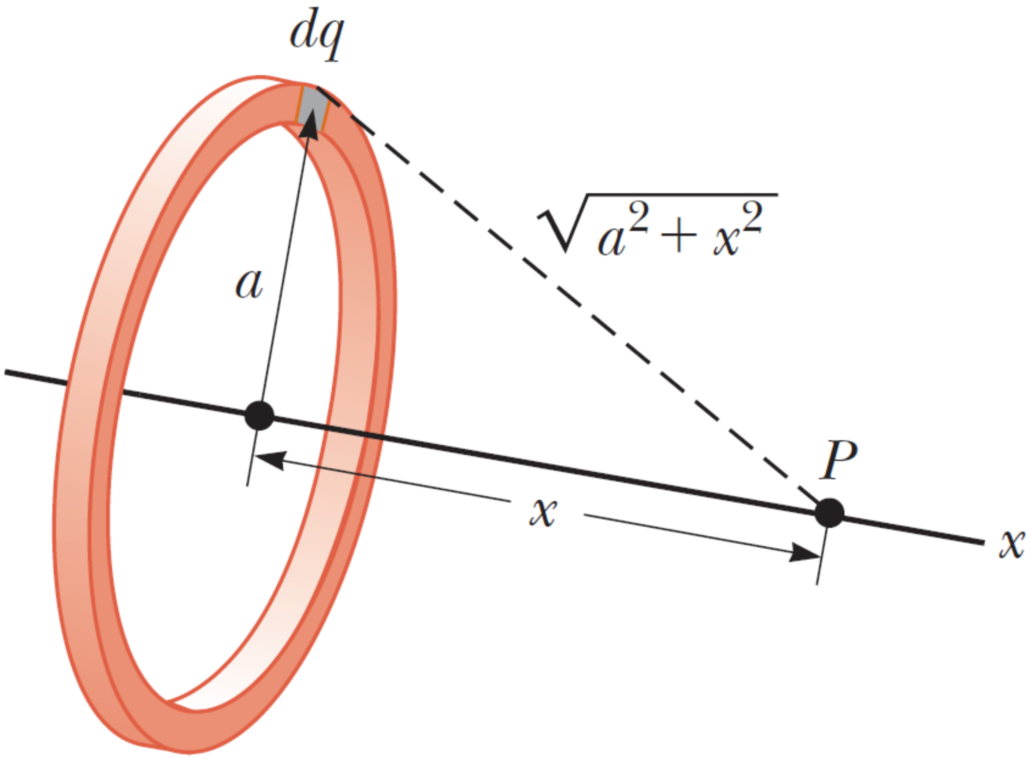
$$\boxed{\vec{E} = -\vec{\nabla} V}$$

  - **If the potential is easy to obtain.**

Example

A uniformly charged ring (P514 Ex. 21-11)

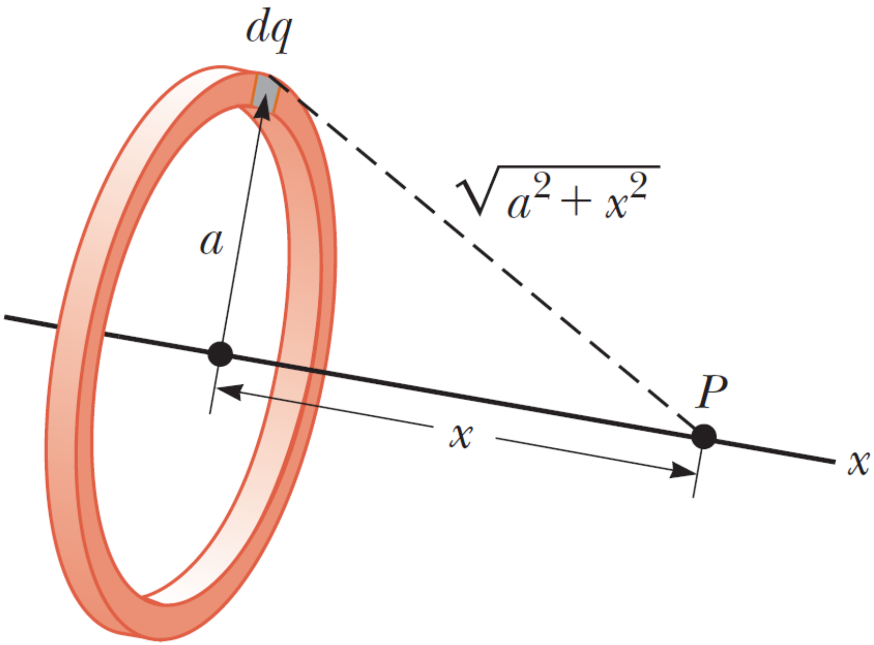
Find the electric field at a point P located on the axis of a uniformly charged ring of radius a and total charge Q.



Example

Solution: Based on the electric potential,

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{x^2+a^2}}$$



$$E_x = -\frac{\partial V}{\partial x}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\frac{d}{dx}\left(x^2+a^2\right)^{-1/2}$$

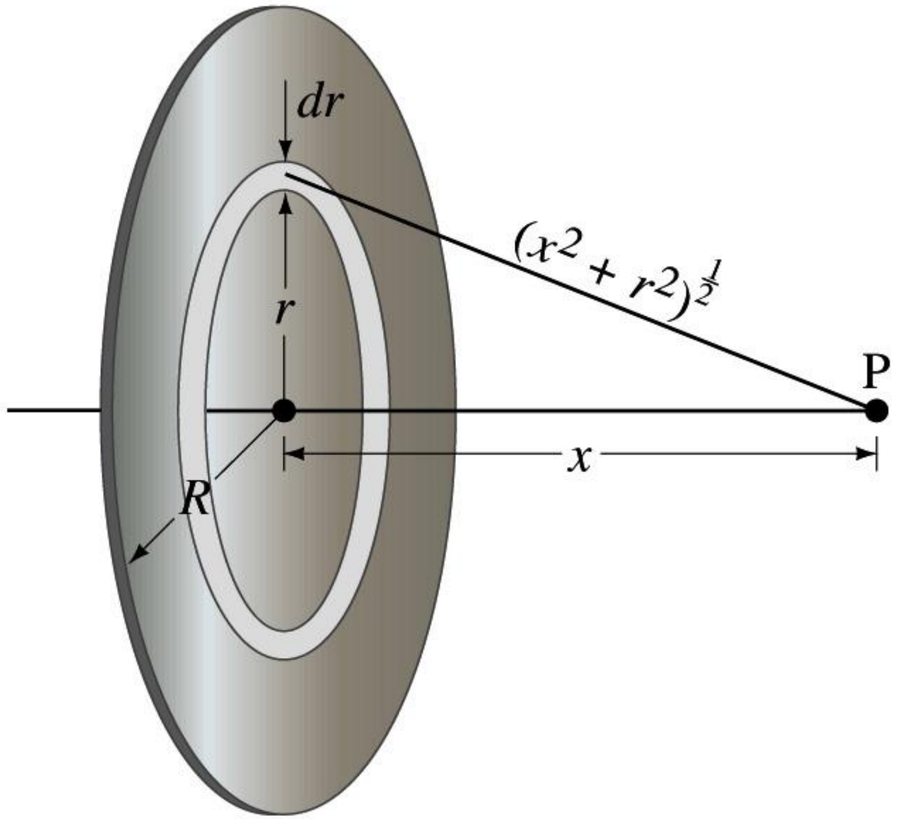
$$= -\frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{2}\right)\left(x^2+a^2\right)^{-3/2}(2x)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{xQ}{\left(x^2+a^2\right)^{3/2}}$$

Example

A uniformly charged disk (P514 Ex. 21-11)

Find the electric field at a point *P* located on the axis of a uniformly charged disk of radius *R* and total charge *Q*.



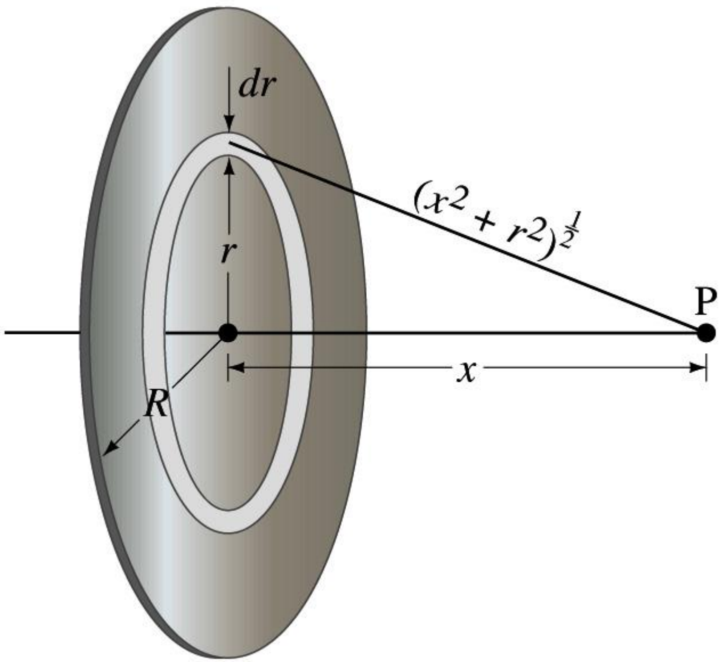
Solution:

$$V=\frac{1}{2\pi\varepsilon_0}\frac{Q}{R^2}\left(\sqrt{x^2+R^2}-x\right)$$

$$E_x=-\frac{\partial V}{\partial x}$$

$$=\frac{Q}{2\pi\varepsilon_0 R^2}\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$$

$$E_y=E_z=0$$



$$x \ll R,\qquad E_x\approx\frac{Q}{2\pi\varepsilon_0 R^2}=\frac{\sigma}{2\varepsilon_0}$$

Prob. 46 (Ch21 P522)

The electric potential in a region of space varies as

$$V = \frac{ay}{b^2 + y^2}$$

Determine $\vec{E}$

Solution:

The components of the electric field is

$$E_x = -\frac{\partial V}{\partial x} = 0$$

$$E_y = -\frac{\partial V}{\partial y} = -\frac{a(b^2 + y^2) - (ay)(2y)}{(b^2 + y^2)^2} = \frac{a(y^2 - b^2)}{(b^2 + y^2)^2}$$

$$E_z = -\frac{\partial V}{\partial z} = 0$$

The electric field is $\vec{E} = \dfrac{a(y^2 - b^2)}{(b^2 + y^2)^2}\vec{j}$

Problems

Ch21 Prob. 38, 47 (P521)