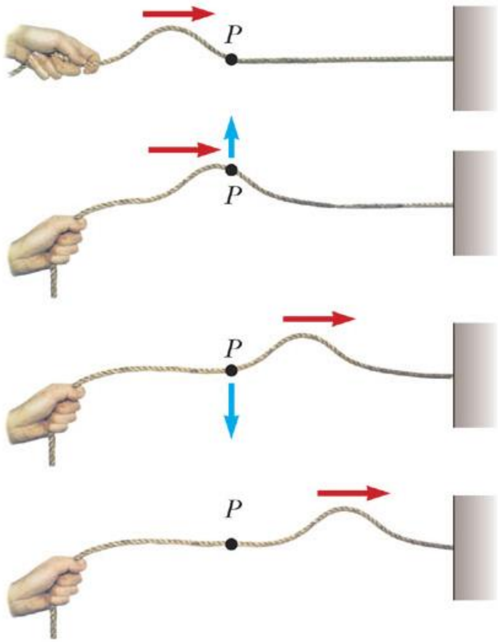
Chapter 13-14 Mechanical waves

§ 1 Conceptual ideas of Waves



- Requirements for mechanical waves

  A mechanical wave is the propagation of a disturbances in a medium.

  - Source of disturbance (origin of wave).
  - Medium through which the wave can propagate.

- The essence of wave motion
  - Wave transports the disturbance (also state of motion and energy) through space without accompanying the transfer of matter.
  - The particles of the medium do not experience any net displacement as the wave passes, the particles simply move back and forth through small distance about their equilibrium positions.

§ 2 Categories of Waves

- Mechanical waves —— require an elastic medium
  - Sound wave, water wave, earthquakes.
- Electromagnetic waves —— do not require any medium
  - Lightwave, radio wave, microwave.
- Matter waves —— any matter has wave-like and particle-like behaviors

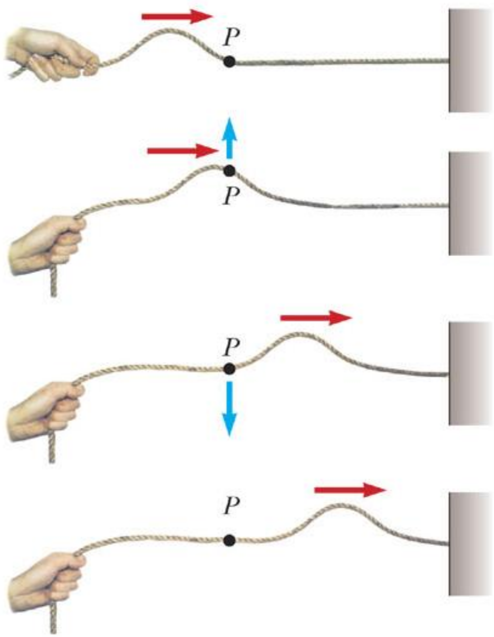
All types of waves use similar mathematical descriptions. We can therefore learn a great deal about waves in general by making a careful study of one type of wave —— For example, mechanical wave.

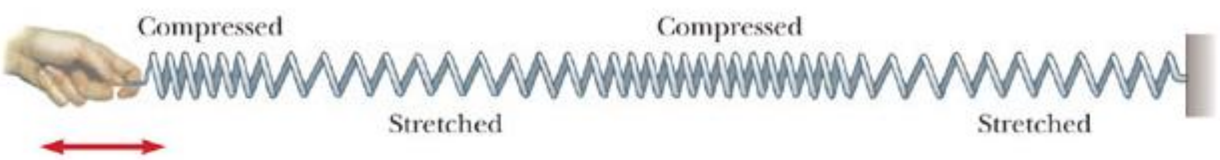
Transverse and longitudinal waves

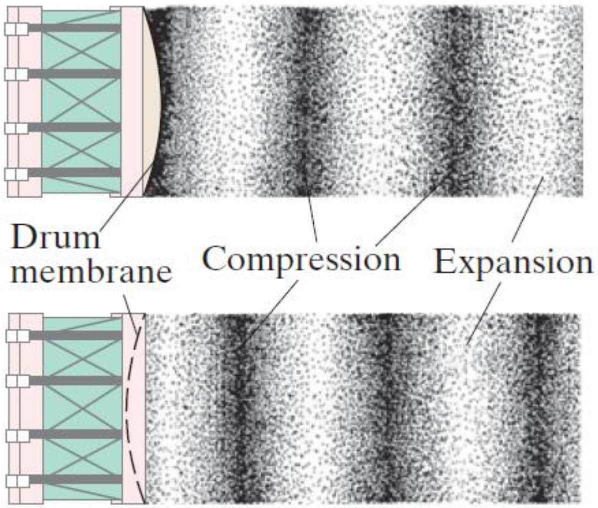
- Transverse and longitudinal waves
  - Transverse wave: the motion of the particles of the medium is perpendicular to direction of propagation.

    Ex. string wave, electromagnetic wave.
  - Longitudinal wave: the motion of the particles is back and forth parallel to the direction of propagation.

    Ex. sound wave, spring compress and stretch wave.







§ 3 Wave Function for Traveling Wave

- Wave function for the wave traveling to the right

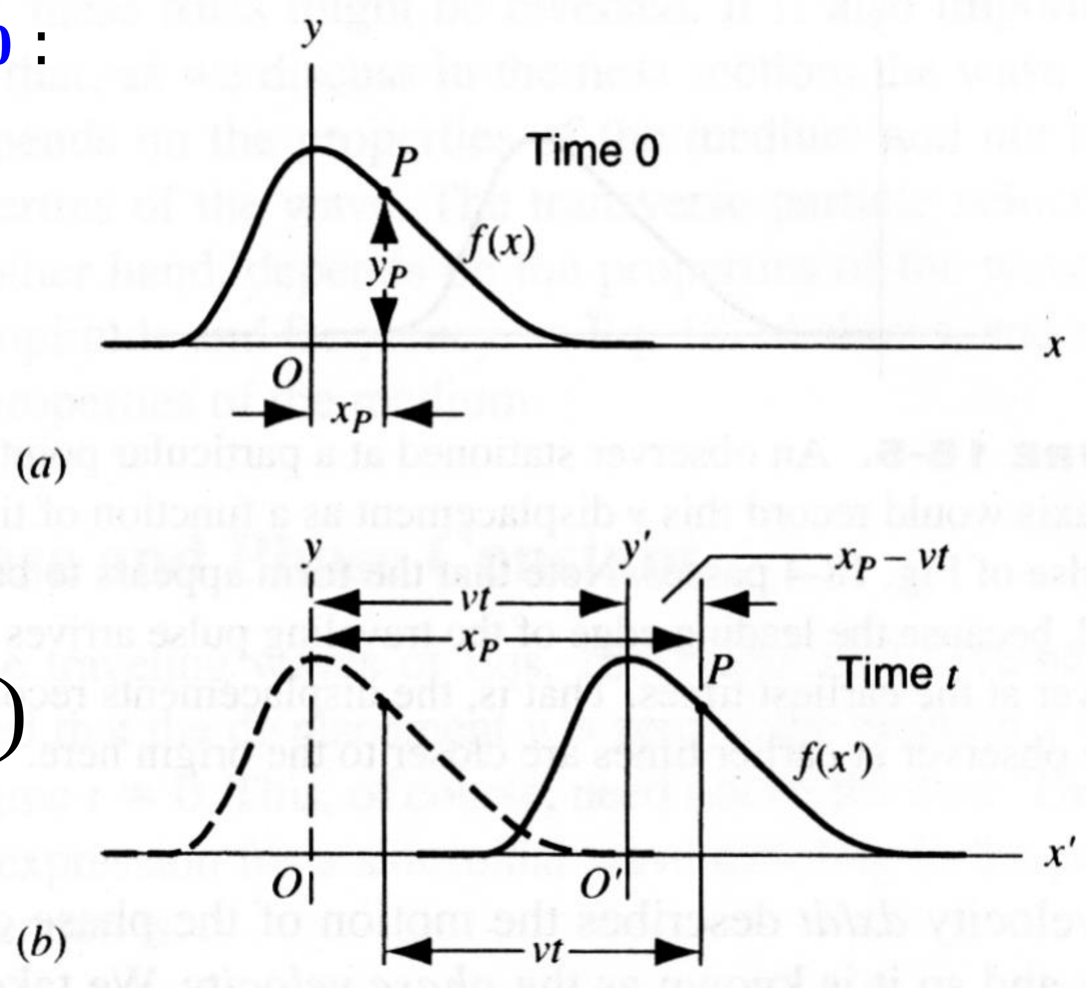
$$y(x,t) = f(x - vt)$$

  - Wave shape at time $t=0$:

$$y(x,0) = f(x)$$

  - The element of the string at x at time t has the same y position as the element located at $(x-vt)$ had at time $t=0$.

$$y(x_p,t) = y(x_p - vt, 0)$$
$$= f(x_p - vt)$$



Example

A pulse moving to the right along the *x* axis is represented by the wave function

$$y(x,t)=\frac{2}{\left(x-3.0t\right)^2+1}$$

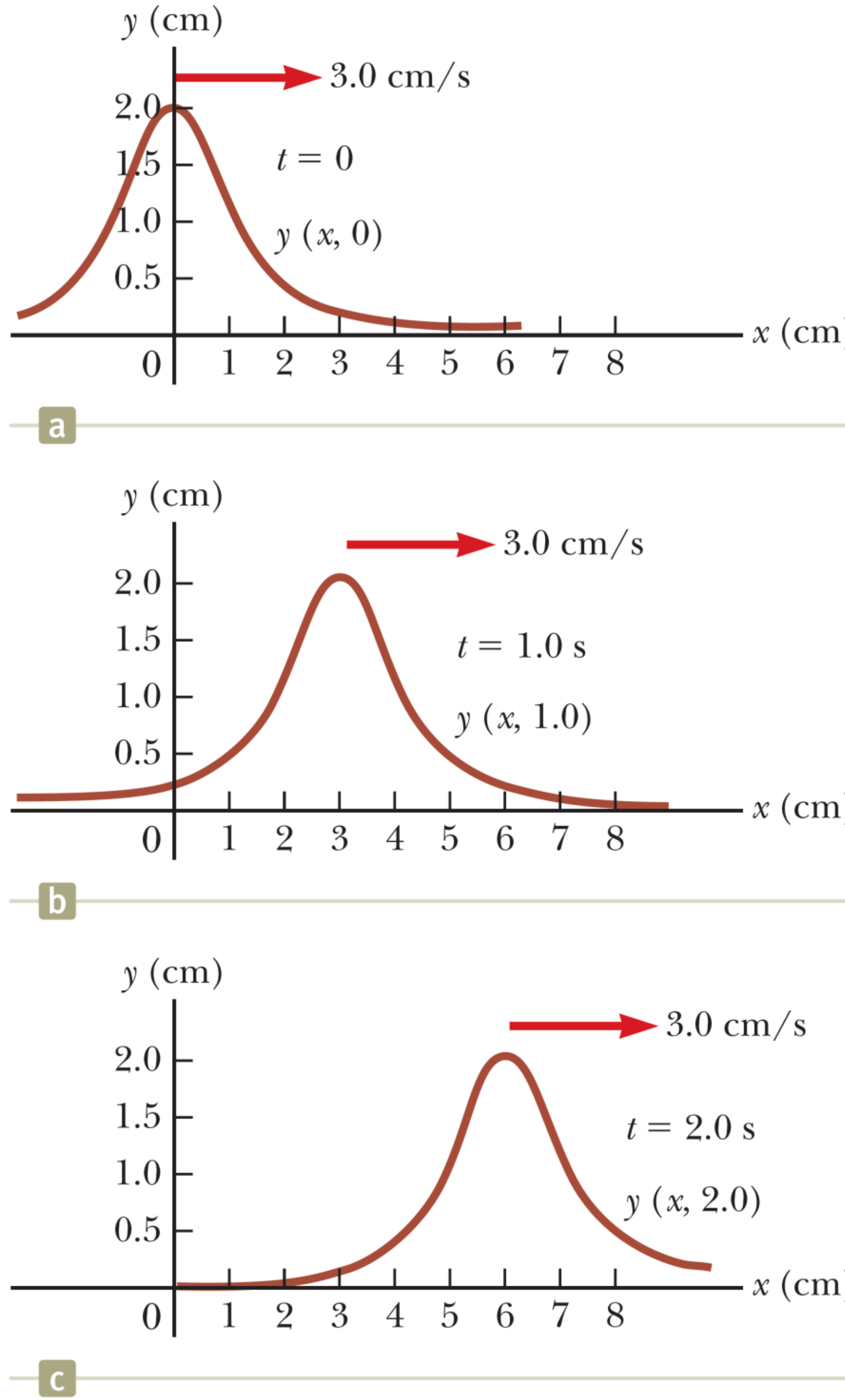
where *x* and *y* are measured in centimeters and *t* is measured in seconds. Find expressions for the wave function at *t*=0 s, *t*=1.0 s, and *t*=2.0 s.

Solution: $$y(x,t)=\frac{2}{(x-3.0t)^2+1}$$

$$y(x,0)=\frac{2}{x^2+1}$$

$$y(x,1.0)=\frac{2}{(x-3.0)^2+1}$$

$$y(x,2.0)=\frac{2}{(x-6.0)^2+1}$$

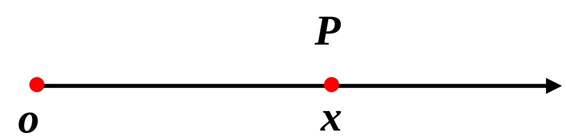


Wave Function for Traveling Wave

- With the view of time:
  - The motion of origin at $x = 0$: $y(0,t) = g(t)$
  - The motion of point x at time t is the same as the motion of point $x=0$ at the earlier time $t-x/v$.

$$y(x_p,t) = y\left(0, t - \frac{x_p}{v}\right) = g\left(t - \frac{x_p}{v}\right)$$

$$y(x,t) = g\left(t - \frac{x}{v}\right)$$



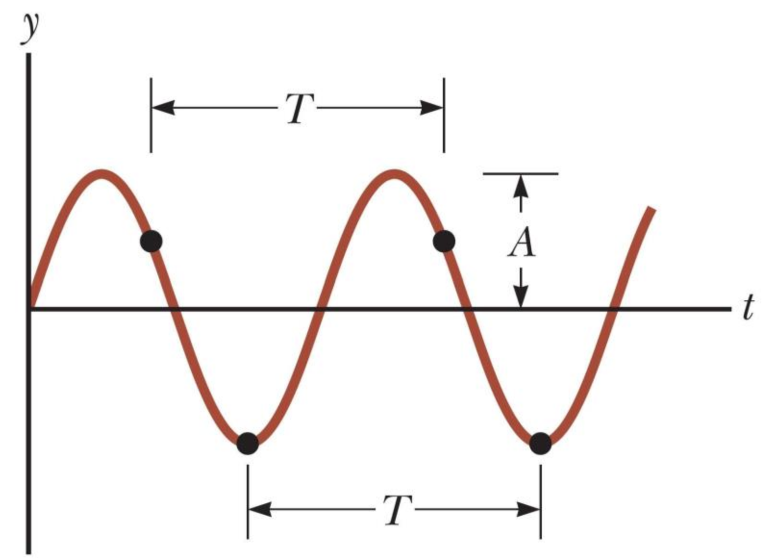
- Wave function for the wave travels to the **left**

$$y(x,t) = f(x + vt) = g\left(t + \frac{x}{v}\right)$$

§ 4 Harmonic Waves

$$y(0,t) = g(t), \qquad y(x,t) = g\left(t - \frac{x}{v}\right)$$

- Suppose the origin of wave at point $x = 0$ is disturbed with sine or cosine function.



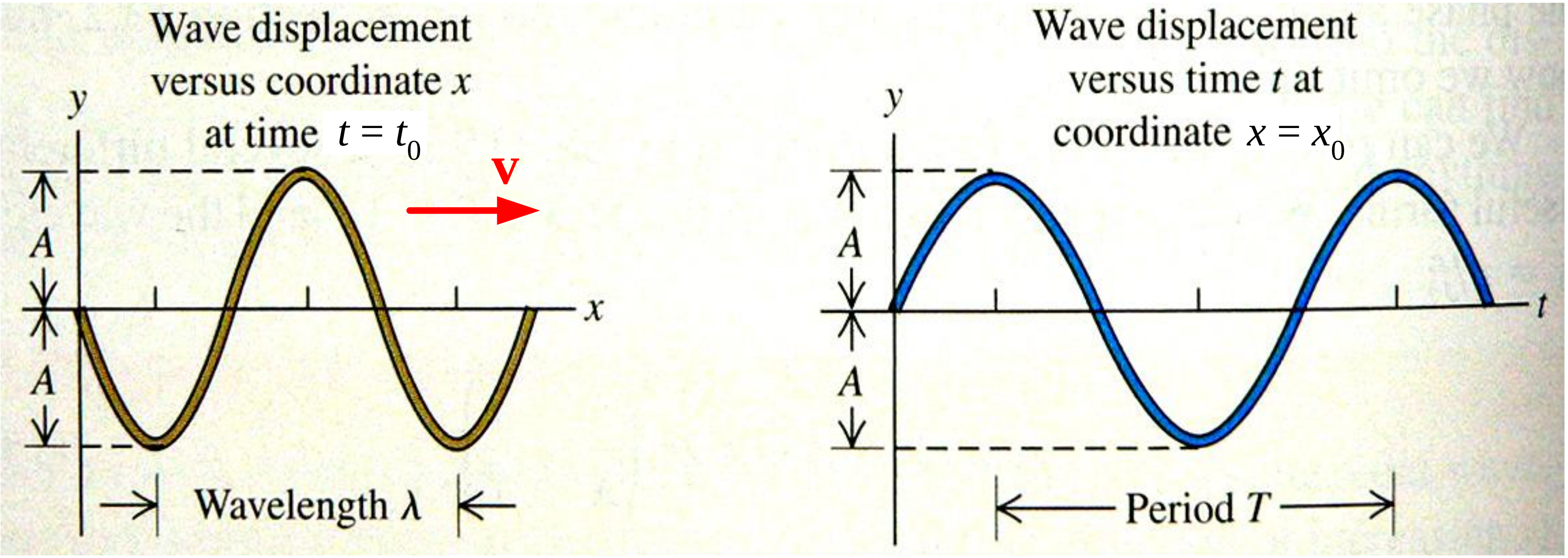
$$y(0,t) = A\cos(\omega t + \varphi)$$

- Wave function moving in $+x$-direction

$$y(x,t) = A\cos\left[\omega\left(t - \frac{x}{v}\right) + \varphi\right]$$

Wave function

$$y(x,t) = A\cos\left[\omega\left(t-\frac{x}{v}\right)+\varphi\right]$$



Wave function

$$y(x,t) = A\cos\left[\omega\left(t - \frac{x}{v}\right) + \varphi\right]$$

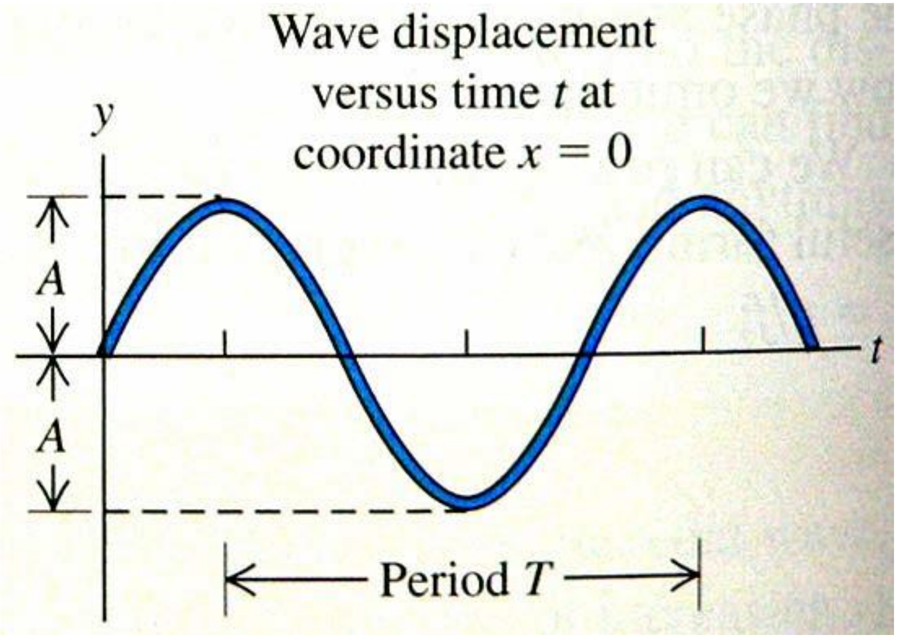
- The meaning of wave function

  The wave function $y(x,t)$ represents the y coordinate of any point P located at position x at any time t.

  - If t is fixed ($t=t_0$), the wave function $y=y(x,t_0)$ defines a curve representing the actual geometric shape of wave at that time, called the waveform (the photo of all group of particles in the medium at the same time).
  - If x is fixed ($x=x_0$), the wave function $y=y(x_0,t)$ is actually the kinematics' equation for particles located at $x=x_0$.

Harmonic Wave

- Basic quantities for harmonic wave.
  - Period: T (Time periodicity)
    - Frequency: f
    - Angular frequency: ω



$$f = \frac{1}{T}, \quad \omega = 2\pi f = \frac{2\pi}{T}$$

$$y(x,t) = A\cos\left[\omega\left(t - \frac{x}{v}\right) + \varphi\right]$$

$$= A\cos\left[2\pi f\left(t - \frac{x}{v}\right) + \varphi\right] = A\cos\left[\frac{2\pi}{T}\left(t - \frac{x}{v}\right) + \varphi\right]$$

Harmonic Wave

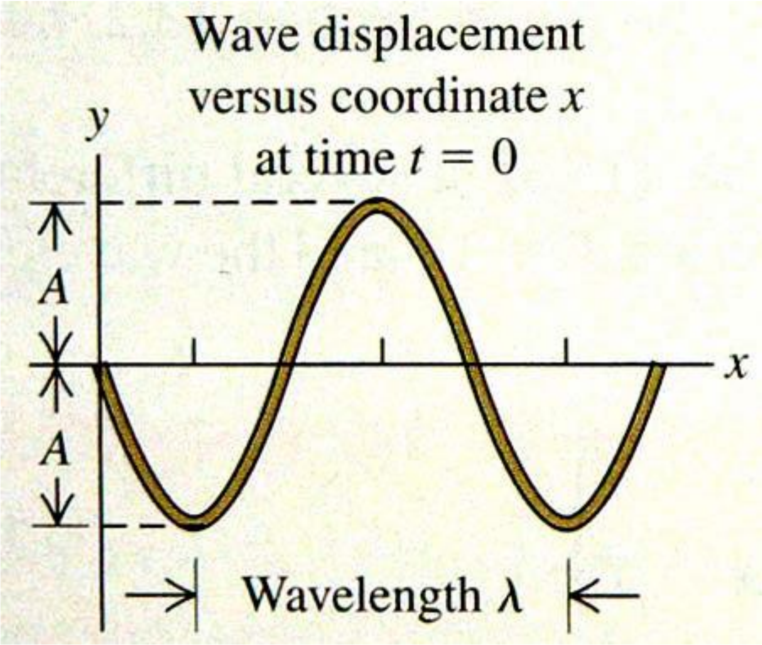
- Basic quantities for harmonic wave.
  - Wavelength: λ (Space periodicity)

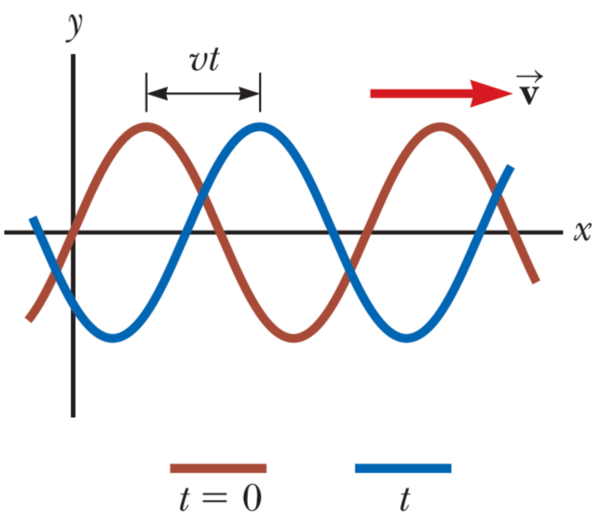
$$\lambda = vT, \qquad v = \frac{\lambda}{T}$$

  - Angular wave number: k

$$k = \frac{2\pi}{\lambda}, \qquad v = \frac{\omega}{k}$$

$$\boxed{\begin{aligned} y(x,t) &= A\cos\left[\omega\left(t - \frac{x}{v}\right) + \varphi\right] \\ &= A\cos\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + \varphi\right] \\ &= A\cos(\omega t - kx + \varphi) \end{aligned}}$$





The speed of wave and the speed of oscillation particle

- The speed of particle: change rate of its displacement with time t.

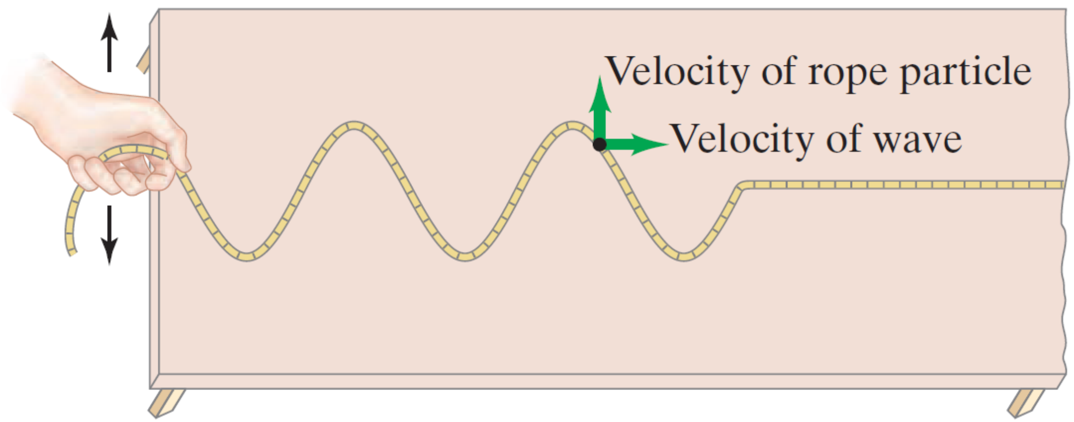
$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t}$$

- The speed of wave: the speed of the phase propagation (phase velocity) or the transfer speed of the disturbance status.

Keeping phase constant:

$$\omega t - kx = \text{constant}$$

$$v = \frac{dx}{dt} = \frac{\omega}{k}$$



How to get a wave function

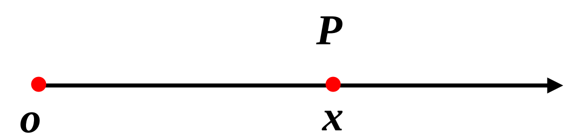
- Wave function obtained by the view of phase retardation
  - Suppose the origin of wave at point $x=0$ is disturbed with:

$$y(0,t) = A\cos(\omega t + \phi)$$

  - The phase at point x is retarded with amount of $-2\pi \dfrac{x}{\lambda}$
  - The wave function:

$$y(x,t) = A\cos\left(\omega t - 2\pi \frac{x}{\lambda} + \phi\right)$$

$$= A\cos(\omega t - kx + \phi)$$



- Wave function obtained with reference point not at $x=0$
  - Suppose an oscillation is disturbed at point $x=x_0$ with

$$y(x_0,t) = A\cos(\omega t + \phi)$$

  - The phase at point x is retarded with amount of $-2\pi \dfrac{x - x_0}{\lambda}$
  - The wave function:

$$y(x,t) = A\cos\left(\omega t - 2\pi \frac{x - x_0}{\lambda} + \phi\right)$$

Example

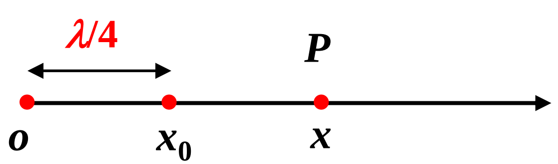
A harmonic wave with wavelength λ travels in $+x$ direction. The particle at $x_0=\lambda/4$ oscillates with the function:

$$y(x_0,t)=A\cos\omega t$$

Write the wave function describing the wave.

Solution I: **By phase comparison with the reference point x_0. The phase at point x is retarded with respect to x_0.**

$$-\frac{2\pi}{\lambda}(x-x_0)$$



$$y(x,t)=A\cos\left[\omega t-\frac{2\pi}{\lambda}(x-x_0)\right]$$

$$=A\cos\left(\omega t-\frac{2\pi}{\lambda}x+\frac{\pi}{2}\right)$$

Example

A harmonic wave with wavelength λ travels in $+x$-direction. The particle at $x_0=\lambda/4$ oscillates with the function:

$$y(x_0,t) = A\cos\omega t$$

Write the wave function describing the wave.

Solution II: by comparison with the standard wave function

Suppose the wave function has the form:

$$y(x,t) = A\cos(\omega t - kx + \phi)$$

At $x=x_0=\lambda/4$,

$$y(\frac{\lambda}{4},t) = A\cos(\omega t - \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4} + \phi)$$

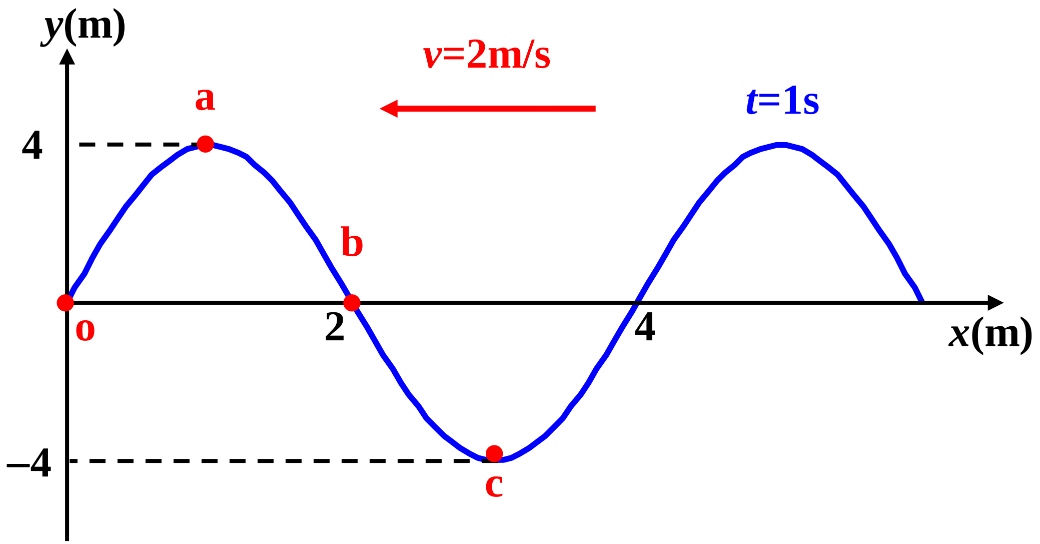
$$= A\cos(\omega t - \frac{\pi}{2} + \phi)$$

Compare it with $y(x_0,t) = A\cos\omega t$, **We have** $-\frac{\pi}{2} + \phi = 0 \Rightarrow \phi = \frac{\pi}{2}$

$$y(x,t) = A\cos\left(\omega t - \frac{2\pi}{\lambda}x + \frac{\pi}{2}\right)$$

Example

A harmonic wave travels in $-x$-direction. The waveform at time $t = 1$ s is shown in the figure.



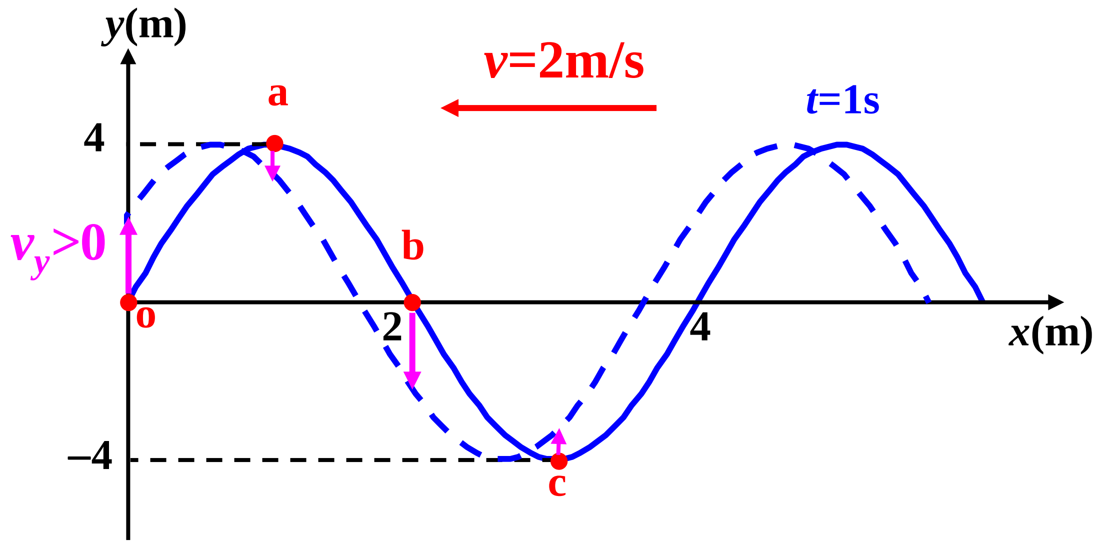
(1) Draw the direction of motion of particle marked with o, a, b, c.

(2) Write the wave function.

(3) Draw the waveform graph at time $t = 2$ s.

Example Cont'd

Solution: (1)



(2) From the waveform graph, we get A=4m; λ=4m, $k=2\pi/\lambda=\pi/2$ m^{-1};

$T=\lambda/v$=2s, $\omega=2\pi/T=\pi$ s^{-1}

$$y(x,t)=A\cos(\omega t+kx+\phi)=4\cos(\pi t+\frac{\pi}{2}x+\phi)\ \mathrm{m}$$

At x=0, t=1, $y(0,1)=4\cos(\pi\times1+\frac{\pi}{2}\times0+\phi)$

$$=4\cos(\pi+\phi)=0,\qquad \pi+\phi=\frac{\pi}{2}\ \text{or}\ \frac{3\pi}{2}$$

$$v_y=\left.\frac{\partial y}{\partial t}\right|_{x=0,t=1}=-A\pi\sin(\pi+\phi)>0,$$

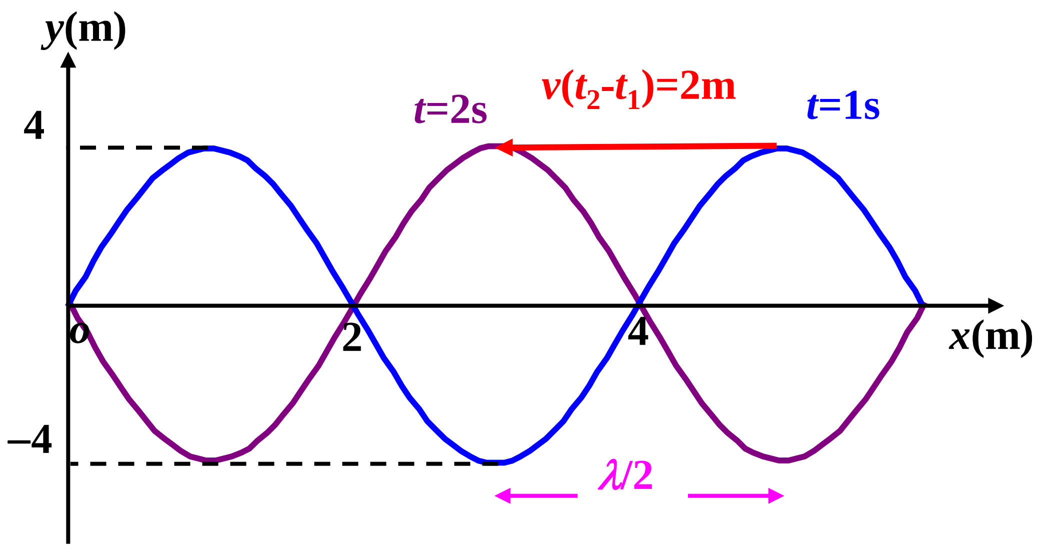
$$\pi+\phi=\frac{3\pi}{2},\quad \phi=\frac{\pi}{2},$$

$$\boxed{y(x,t)=4\cos(\pi t+\frac{\pi}{2}x+\frac{\pi}{2})\quad \mathrm{m}}$$

Example Cont'd

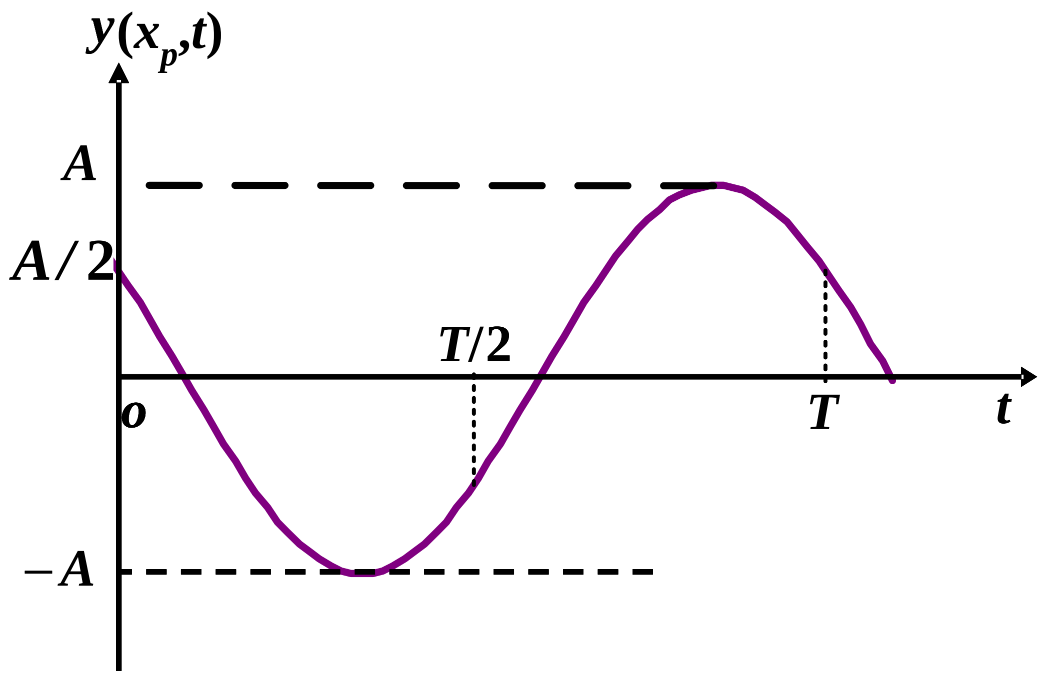
Solution: (3)

$$y(x,t) = 4\cos(\pi t + \frac{\pi}{2}x + \frac{\pi}{2}) \quad \text{(SI)}$$



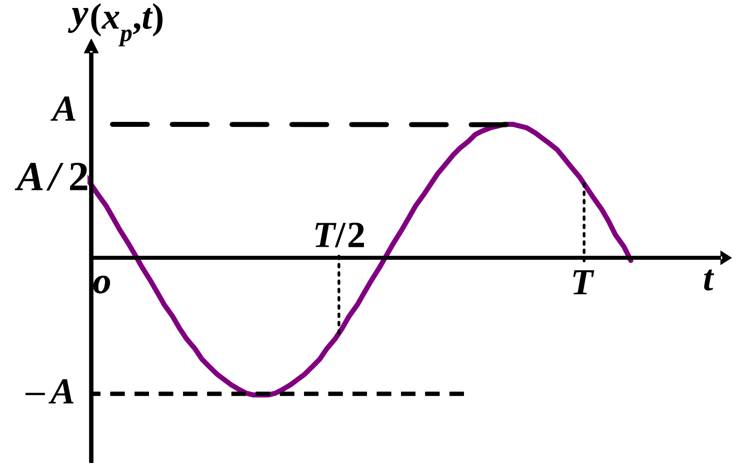
Example

A harmonic wave with the amplitude of A, the wavelength λ, and the period of T travels in $+x$-direction. The y-t graph of the particle P at $x_p = \lambda/2$ is shown in the figure. Find (1) the **wave function**; (2) the **waveform** at time $t = T/2$.



Example

A harmonic wave with the amplitude of A, the wavelength λ, and the period of T travels in $+x$-direction. The ***y*-*t*** graph of the particle ***P*** at $x_p = \lambda/2$ is shown in the figure. Find (1) the wave function; (2) the waveform at time $t = T/2$.



Solution: (1) The initial phase for the oscillation at point *P* is $\pi/3$

$$y(x_P,t) = A\cos\left(\frac{2\pi}{T}t + \frac{\pi}{3}\right)$$

Phase retardation

$$y(x,t) = A\cos\left[\frac{2\pi}{T}t - \frac{2\pi}{\lambda}\left(x - \frac{\lambda}{2}\right) + \frac{\pi}{3}\right]$$

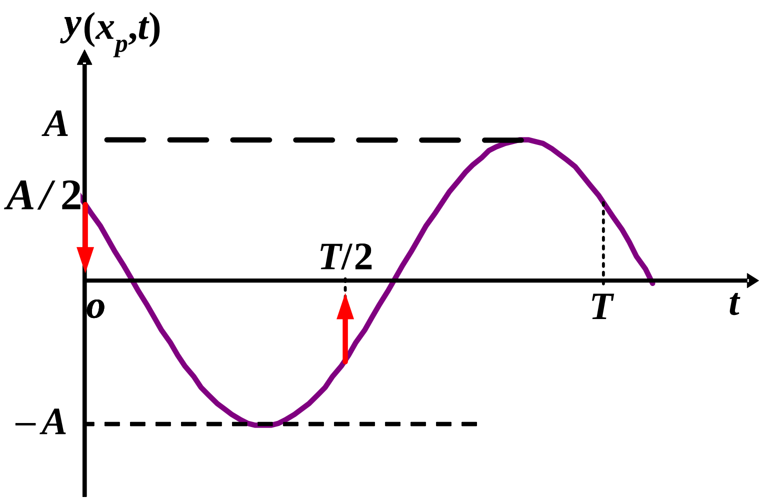
$$= A\cos\left[\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x + \frac{4\pi}{3}\right]$$

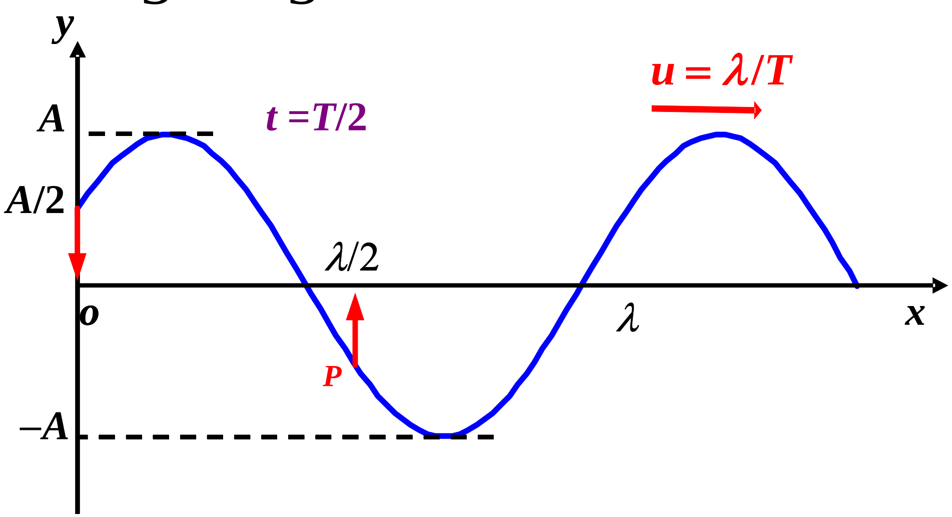
Example Cont'd

$$y(x,t) = A\cos\left[\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x + \frac{4\pi}{3}\right]$$

(2) At $t=T/2$, the phase angle at $x=0$:

$$\frac{2\pi}{T}\cdot\frac{T}{2} + \frac{4\pi}{3} = 2\pi + \frac{\pi}{3} \leftrightarrow \frac{\pi}{3}$$





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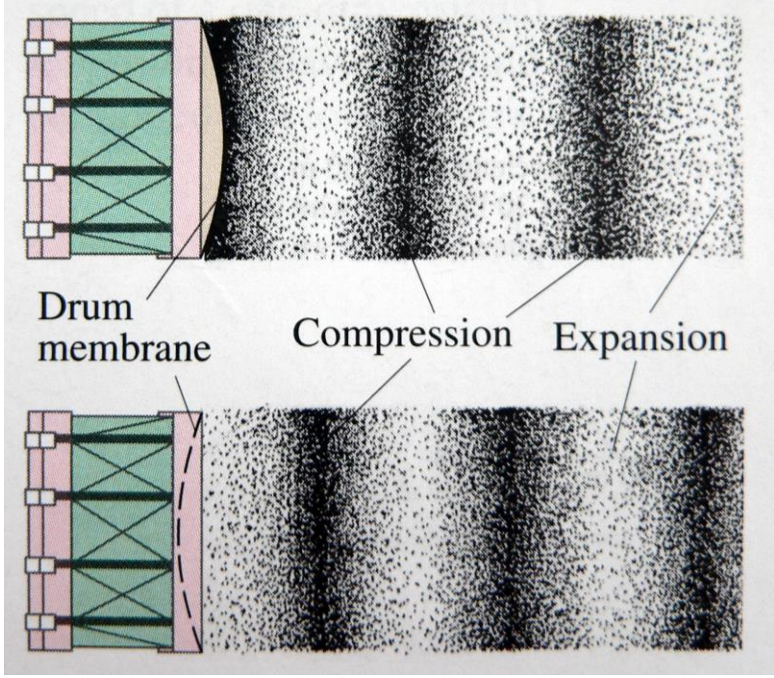
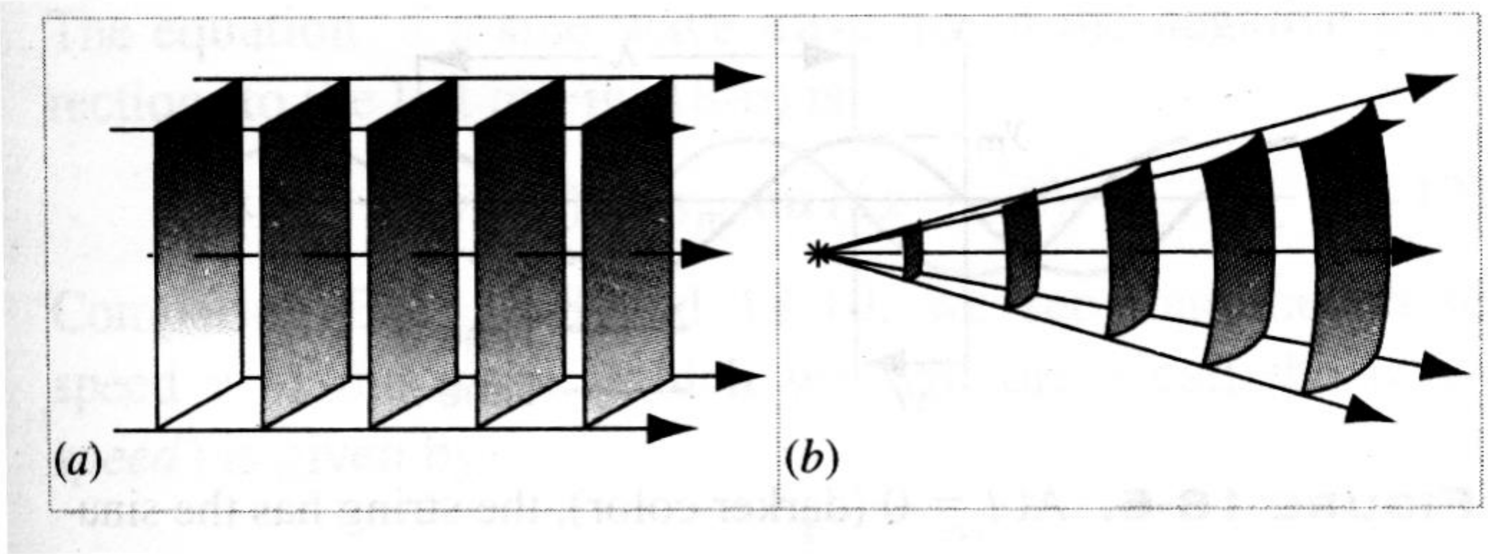
§ 5 Plane Wave and Spherical Wave

- **Wavefronts and Rays**
  - **Wavefront: the surface composed of all the points having the same state of motion (with equal phase)**
  - **Ray: A line normal to the wavefronts, indicating the direction of motion of the wave.**





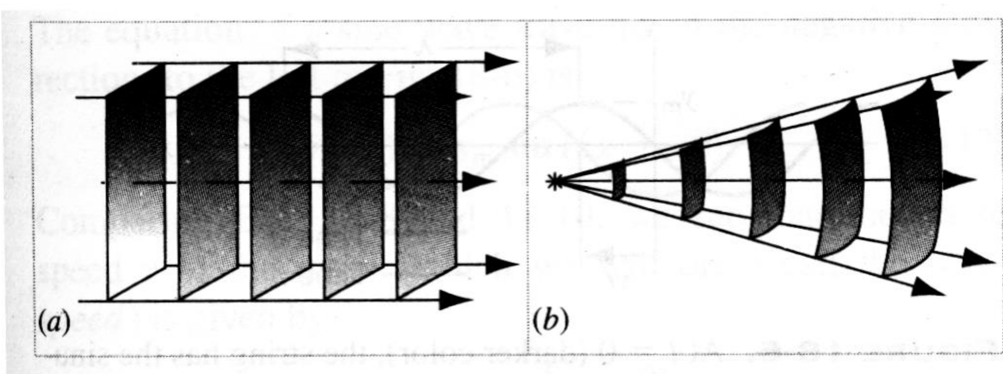
Plane Wave and Spherical Wave

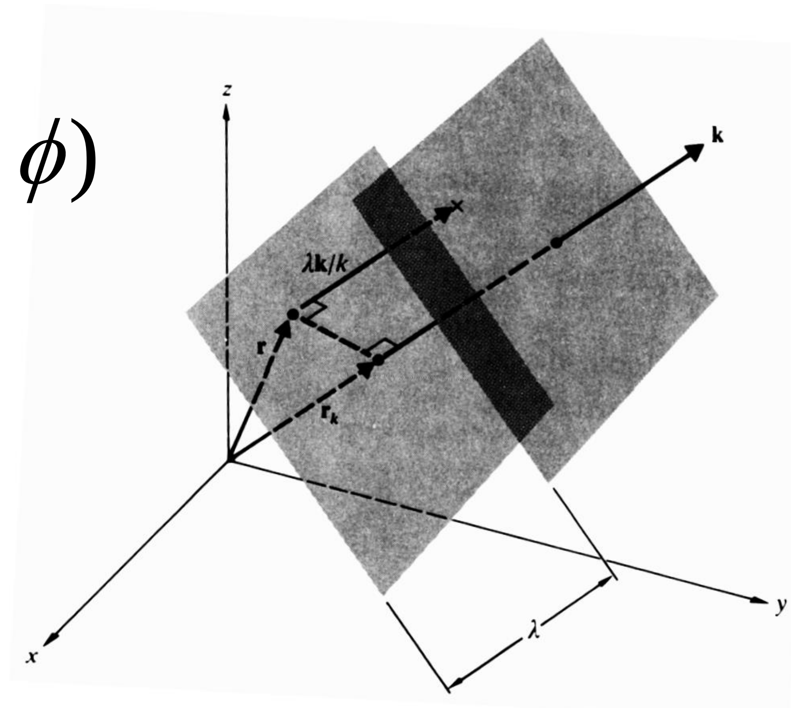
- A **plane** wave traveling in the direction of $\vec{k}$:

$$\psi(\vec{r},t) = A\cos(\omega t - \vec{k}\cdot\vec{r} + \phi)$$

- A **spherical** wave traveling in radial direction:

$$\psi(\vec{r},t) = \frac{A}{r}\cos(\omega t - kr + \phi)$$





§ 6 The Linear Wave Equation

The wave function: $$y(x,t) = A\cos(\omega t - kx)$$

$$\frac{\partial y}{\partial t} = -\omega A\sin(\omega t - kx), \qquad \frac{\partial y}{\partial x} = kA\sin(\omega t - kx)$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\cos(\omega t - kx), \qquad \frac{\partial^2 y}{\partial x^2} = -k^2 A\cos(\omega t - kx)$$

$$\frac{\partial^2 y}{\partial x^2} = -\frac{k^2}{\omega^2}\omega^2 A\cos(\omega t - kx) = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}, \qquad v = \frac{\omega}{k}$$

Linear wave equation: $$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

The general solutions of linear wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$y(x,t) = f(x \pm vt)$ **are the general solutions of linear wave Equation**

$$z = x \pm vt$$

$$\frac{\partial y}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}, \quad \frac{\partial y}{\partial t} = \frac{df}{dz}\frac{\partial z}{\partial t} = \pm v\frac{df}{dz}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d}{dz}\left(\frac{df}{dz}\right)\frac{\partial z}{\partial x} = \frac{d^2 f}{dz^2}, \quad \frac{\partial^2 y}{\partial t^2} = \frac{d}{dz}\left(\pm v\frac{df}{dz}\right)\frac{\partial z}{\partial t} = v^2\frac{d^2 f}{dz^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}, \qquad \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}$$

The wave equation for a string wave

- **Deriving the wave equation for a string wave**

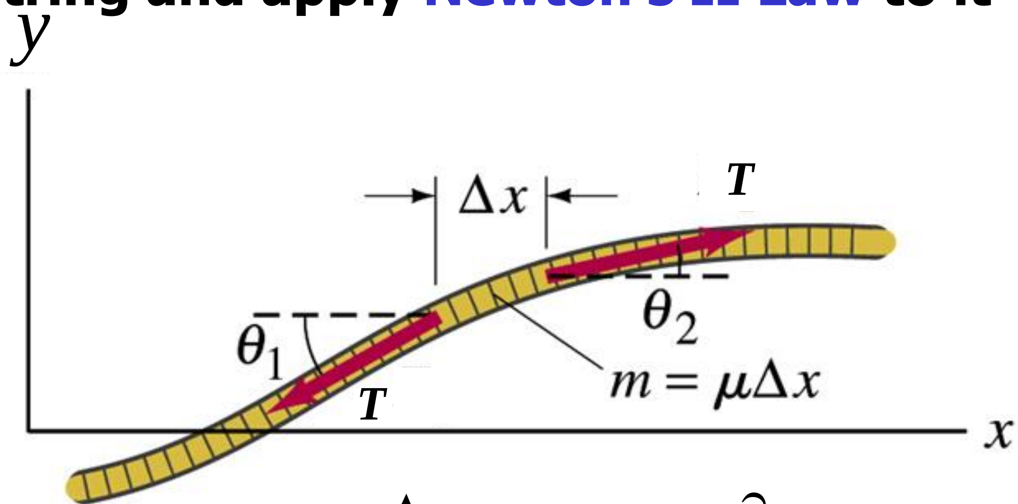
  Take a tiny segment of the string and apply Newton's II Law to it

  - **Horizontal:**

$$T\cos\theta_2 - T\cos\theta_1 = \Delta m a_x$$

  - **Vertical:**

$$T\sin\theta_2 - T\sin\theta_1 = \Delta m a_y$$



$$\theta_1 \ll 1, \quad \theta_2 = \theta_1 + \Delta\theta \ll 1$$

$$\cos\theta \approx 1, \quad \sin\theta \approx \theta$$

$$\theta \approx \tan\theta = \frac{\Delta y}{\Delta x} \xrightarrow{\text{取极限}} \frac{\partial y}{\partial x}$$

$$\begin{cases} 0 = (\Delta m) a_x \\ T(\Delta\theta) = (\Delta m) a_y = \mu(\Delta x) a_y \end{cases}$$

$$\frac{\Delta\theta}{\Delta x} \xrightarrow{\text{取极限}} \frac{\partial\theta}{\partial x} = \frac{\partial^2 y}{\partial x^2}$$

$$T\frac{\Delta\theta}{\Delta x} = \mu\frac{\partial^2 y}{\partial t^2}, \qquad \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}, \qquad \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}, \qquad \boxed{v = \sqrt{\frac{T}{\mu}}}$$

* § 7 The speeds of some kinds of waves

- **The speed of longitudinal wave in a fluid:**

$$v = \sqrt{\frac{B}{\rho}}$$

B : the bulk modulus; ρ : density of medium

- **The speed of longitudinal wave in a solid rod:**

$$v = \sqrt{\frac{Y}{\rho}}$$

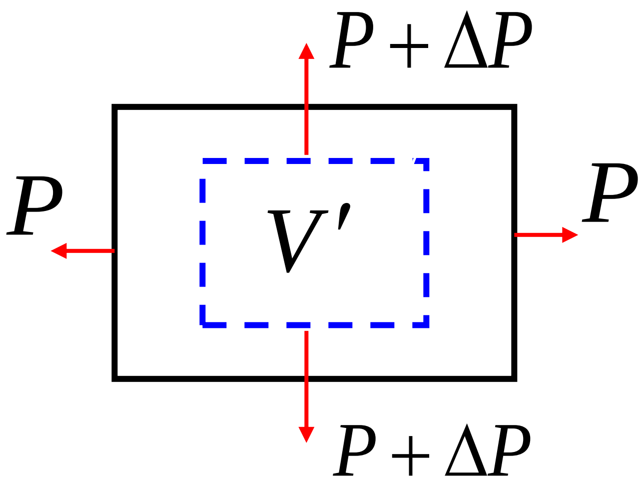
Y : Young's modulus; ρ : density of medium

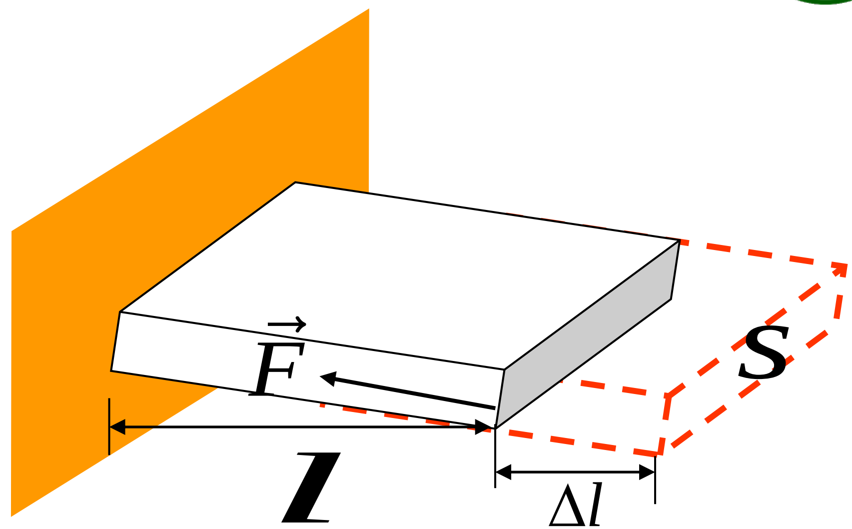
- **The speed of sound in an ideal gas:**

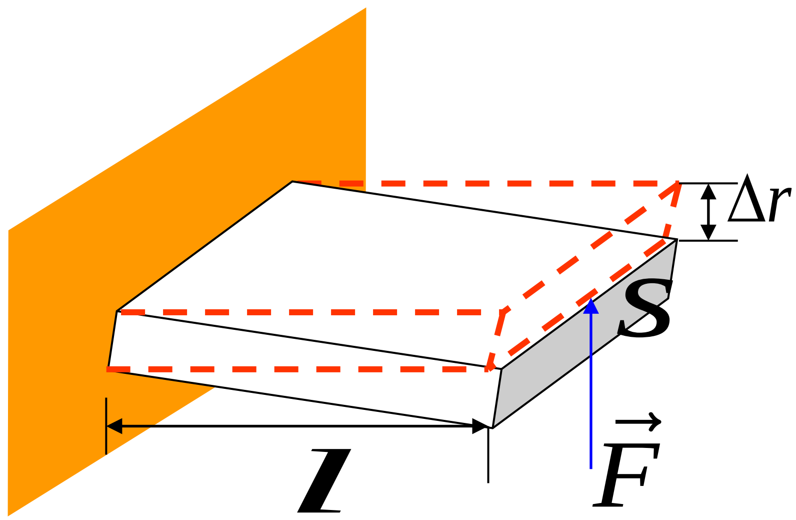
$$v = \sqrt{\frac{\gamma RT}{M}}$$

$\gamma = C_p/C_{v:}$ dimensionless ratio of heat capacity; R: the gas constant (8.315J/(mol·K)); M: molar mass

The speeds of some kinds of waves







Example

A uniform rope of mass *m* and length *L* is suspended vertically. Analysis shows that for short transverse pulses, the waves above a short distance from the free end of the rope can be represented to a good approximation by the linear wave equation. Find the time interval in that a transverse pulse travels from the bottom to the top of the rope.

Solution:

When the pulse is at position *x* above the lower end of the rope, the wave speed of the pulse is

$$v=\sqrt{\frac{T}{\mu}}=\sqrt{\frac{(\mu x)g}{\mu}}=\sqrt{gx},$$

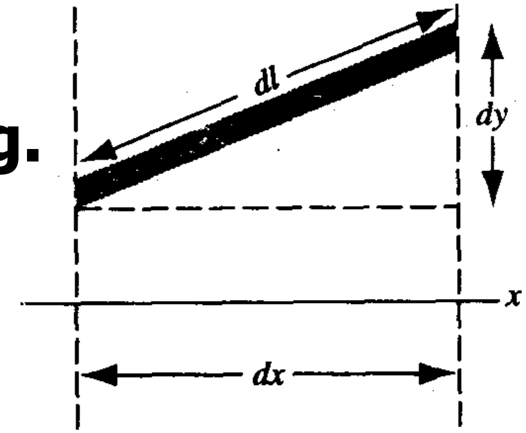
$$v=\frac{dx}{dt}=\sqrt{gx}, \qquad \int_0^t dt=\int_0^L \frac{dx}{\sqrt{gx}}, \qquad t=\frac{1}{g}\left.\frac{\sqrt{gx}}{\frac{1}{2}}\right|_0^L=2\sqrt{\frac{L}{g}}$$

§ 8 Energy Transfer in Wave Motion

- **Total energy density of a wave.**

  Consider a wave traveling along a string.

  A segment of string $\mathrm{d}x$



  - **The kinetic energy:**

$$dK = \frac{1}{2}(dm)v_y^2 = \frac{1}{2}(\mu\, dx)\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}(\mu\, dx)\omega^2 A^2 \sin^2(\omega t - kx)$$

  - **The potential energy:**

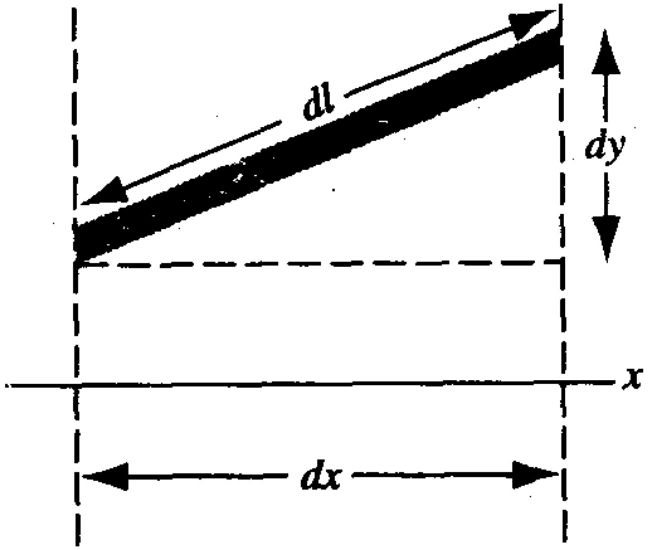
$$x \ll 1,\ \ (1+x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x$$

$$dU = T(dl - dx) = T\left[\sqrt{(dx)^2 + (dy)^2} - dx\right] = Tdx\left[\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1\right] \approx \frac{1}{2}Tdx\left(\frac{\partial y}{\partial x}\right)^2$$

$$T = \mu v^2 = \mu\left(\frac{\omega}{k}\right)^2, \qquad dU = \frac{1}{2}(\mu dx)\omega^2 A^2 \sin^2(\omega t - kx) = dK$$

Energy density

$$dU = dK = \frac{1}{2}(\mu dx)\omega^2 A^2 \sin^2(\omega t - kx)$$



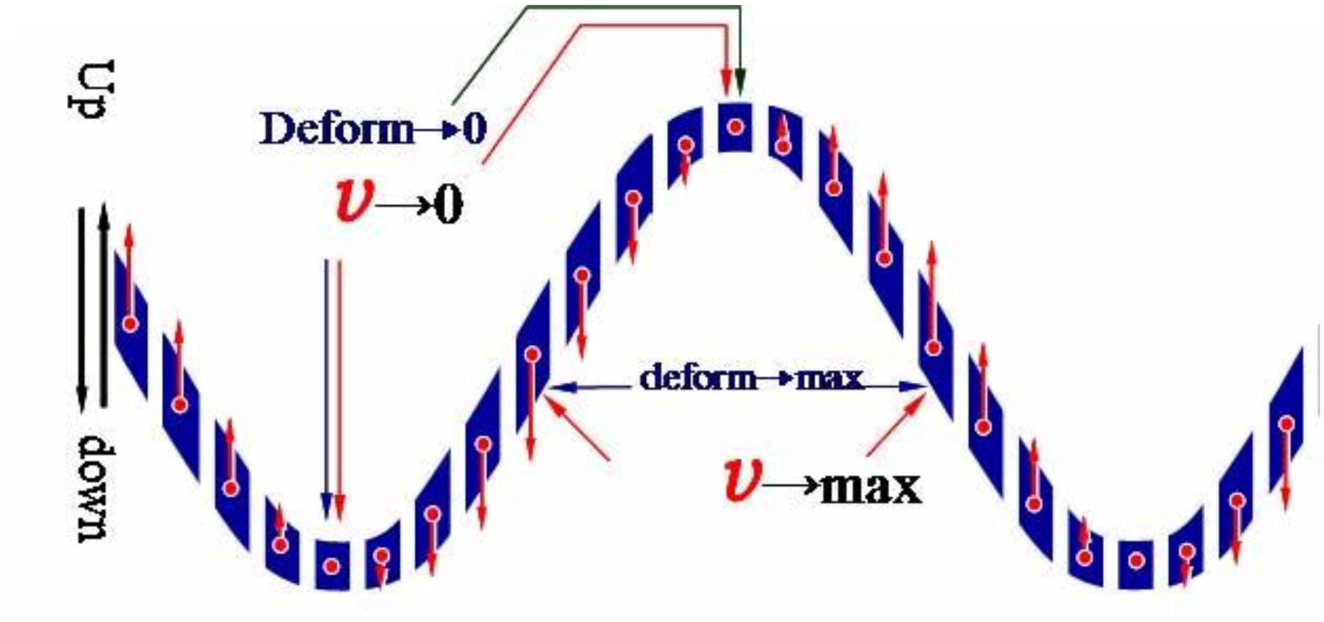
- **Energy density:**

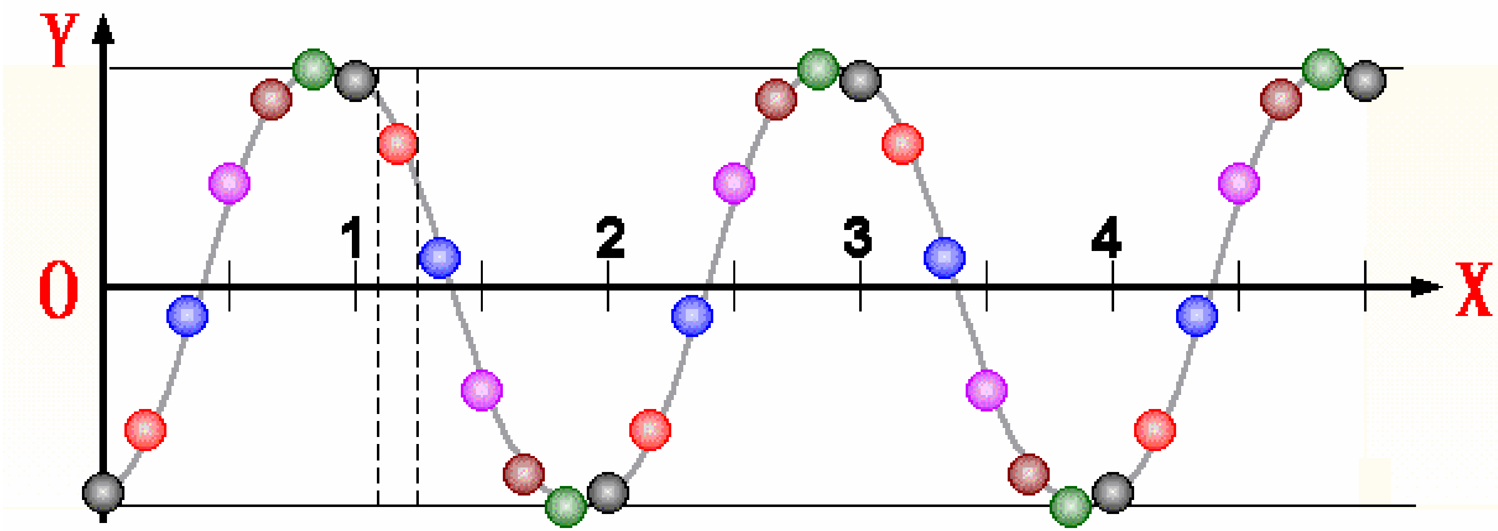
$$\boxed{w = \frac{dE}{dx} = \frac{dK + dU}{dx} = \mu\omega^2 A^2 \sin^2(\omega t - kx)}$$

Energy density for volume mass distribution:

$$\boxed{w = \frac{dE}{dV} = \rho\omega^2 A^2 \sin^2(\omega t - kx)}$$

The description of the energy characteristics in wave motion



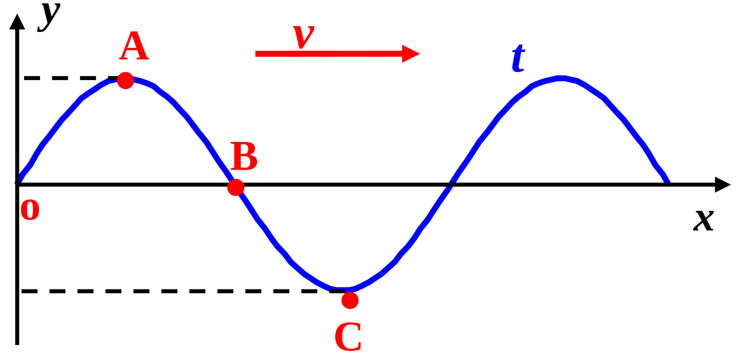


Energy characteristics in wave motion

$$dU = dK = \frac{1}{2}(\mu dx)\omega^2 A^2 \sin^2(\omega t - kx)$$

- **For a particle in medium, the kinetic energy and potential energy are in phase —— They reach their maximum simultaneously.**

At point A, C,



$$\frac{\partial y}{\partial t} = 0, \quad \frac{\partial y}{\partial x} = 0, \quad dK = dU = 0$$

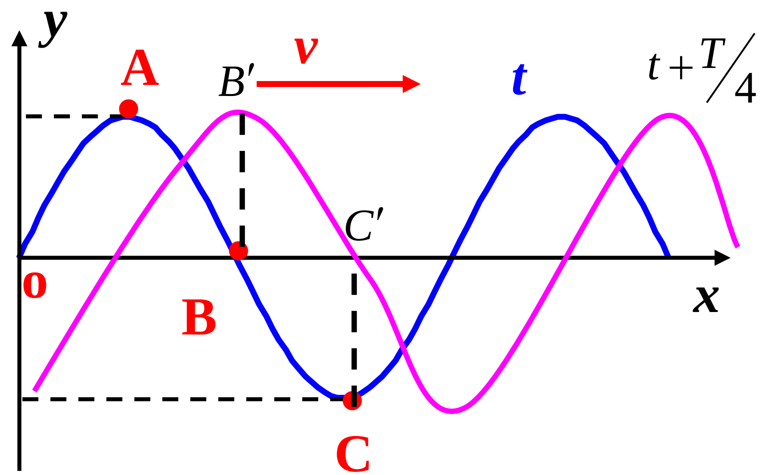
At point B, dK **and** dU **reach their maximum.**

- **The energy in the volume** dV **is not conserved. Sometime the energy is net input, sometime is net output— energy is transported.**

The energy features for wave and SHM

$$w = \frac{dE}{dV} = \rho\omega^2 A^2 \sin^2(\omega t - kx)$$

	t	$t+T/4$
B	$w\rightarrow$max	$w\rightarrow$0
C	$w\rightarrow$0	$w\rightarrow$max



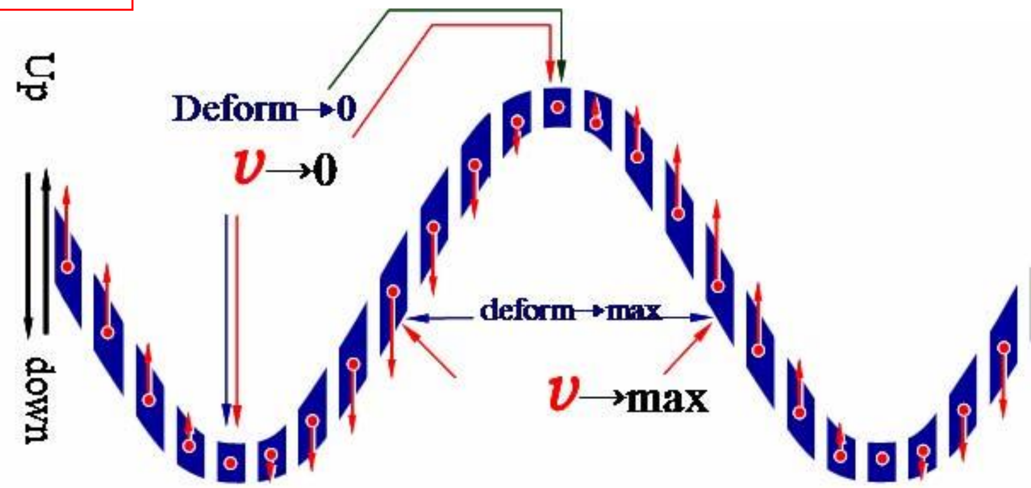
Wave	SHM
For segment, E doesn't conserve	$\Delta E=0$
Transfer energy	Doesn't transfer energy

The energy in volume dV is **not conserved**. Sometime the energy is net input, sometime is net output — energy is **transported**.

Energy characteristics in wave motion

$$\boxed{w = \frac{dE}{dV} = \rho\omega^2 A^2 \sin^2(\omega t - kx)}$$

- **In volume $\mathrm{d}V$, the average value of energy in one period is constant.**



$$\overline{w} = \frac{1}{T}\int_{-T/2}^{T/2} w\,dt$$

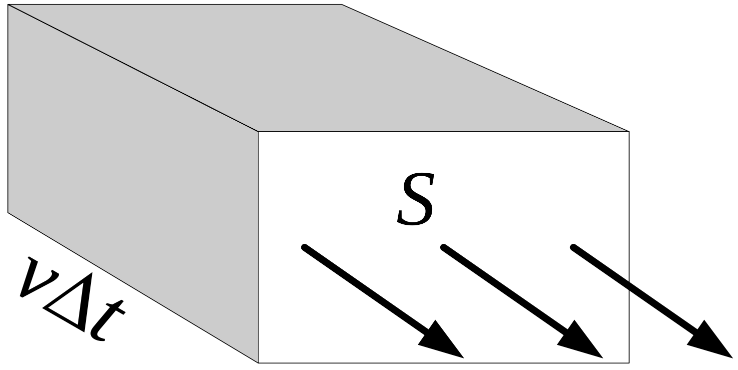
$$= \frac{1}{T}\int_{-T/2}^{T/2} \rho\omega^2 A^2 \sin^2(\omega t - kx)\,dt = \frac{1}{2}\rho\omega^2 A^2 \propto \begin{cases} \omega^2 \\ A^2 \end{cases}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} \sin^2(\omega t)dt = \frac{1}{T}\int_{-T/2}^{T/2} \frac{1-\cos(2\omega t)}{2}dt = \frac{1}{2}$$

- **Energy current: the net energy that flows through a cross-sectional area per unit time interval.**

In time interval Δt, the energy that can flow through the surface S is the energy in the cuboid volume $Sv\Delta t$.

$$P=\frac{wV}{\Delta t}=\frac{wSv\Delta t}{\Delta t}=wvS$$



- **Energy current density:**

$$J=\frac{P}{S}=wv$$

- **Intensity of wave: time average of energy current density**

$$\boxed{I=\overline{J}=\frac{\overline{P}}{S}=\overline{w}v=\frac{1}{2}\rho\omega^{2}A^{2}v}$$

Example

A taut string for which $\mu = 5.00 \times 10^{-2}$ kg/m is under a tension of 80.0 N. How much power must be supplied to the string to generate sinusoidal waves at a frequency of 60.0 Hz and an amplitude of 6.00 cm?

Solution:

$$P = IS = \left(\frac{1}{2}\rho\omega^2 A^2 v\right)S = \frac{1}{2}(\rho S)\omega^2 A^2 v$$

$$= \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2}\mu(2\pi f)^2 A^2\left(\sqrt{\frac{T}{\mu}}\right) = 2\pi^2 f^2 A^2\sqrt{\mu T}$$

$$= 2\pi^2(60.0\ \text{Hz})^2(0.0600\ \text{m})^2\sqrt{(0.0500\ \text{kg/m})(80.0\ \text{N})} = 512\ \text{W}$$

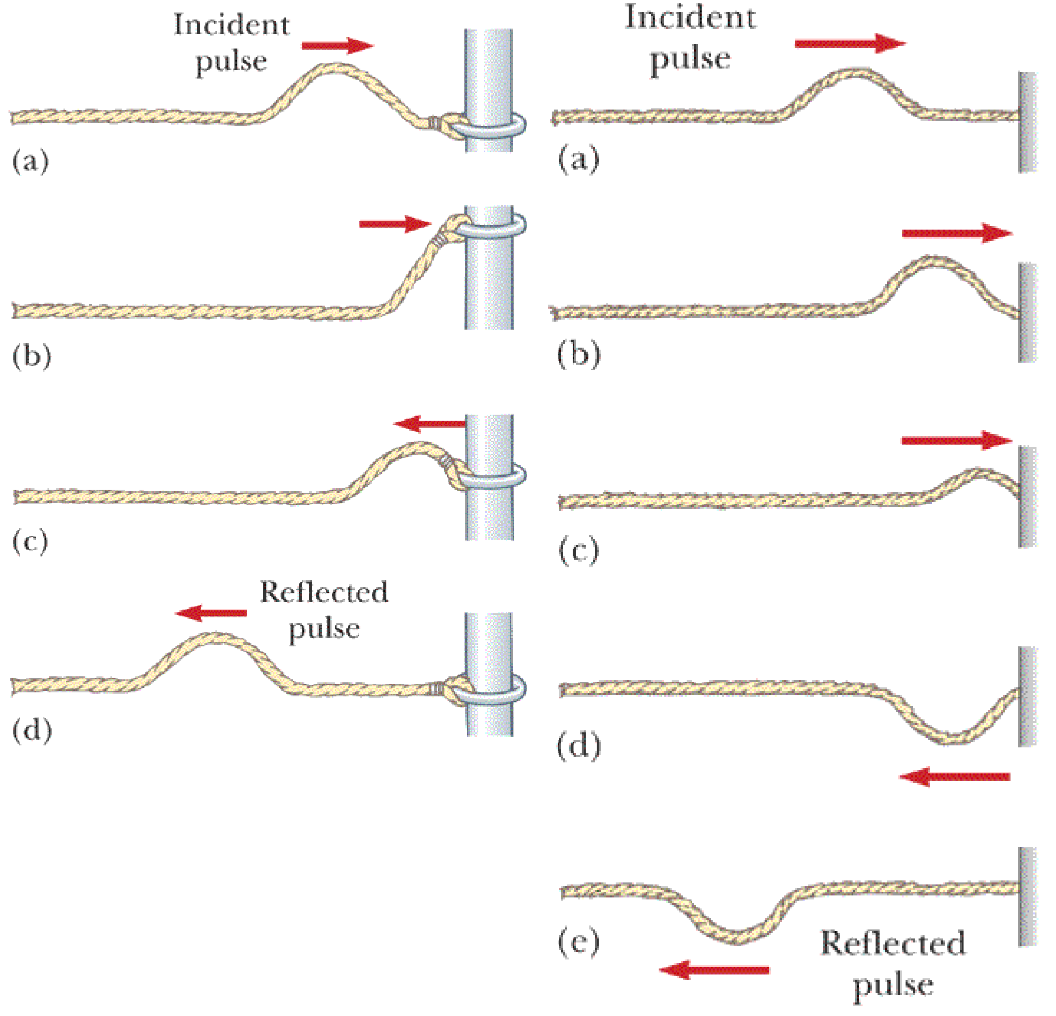
Problem

Ch13 (P348): 13, 14, 17

§ 9 Reflection and Transmission of Waves

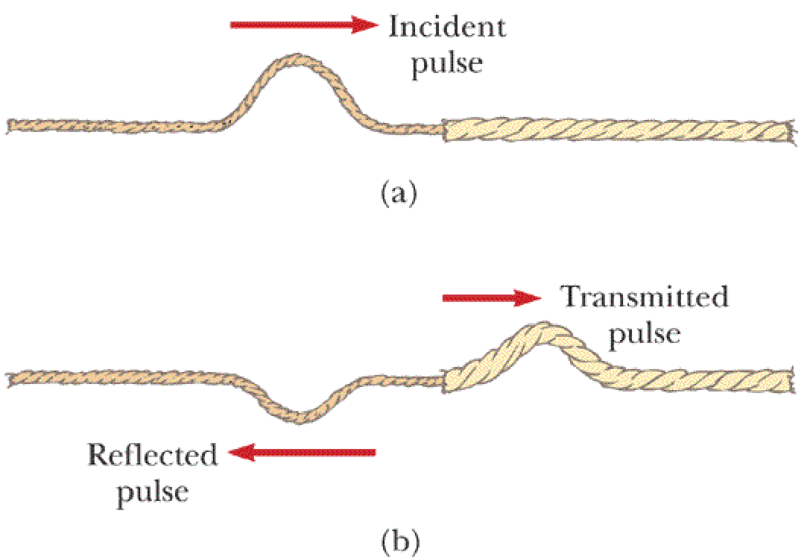
- Reflection through the **free** boundary
  - The reflected wave is **not** inverted.

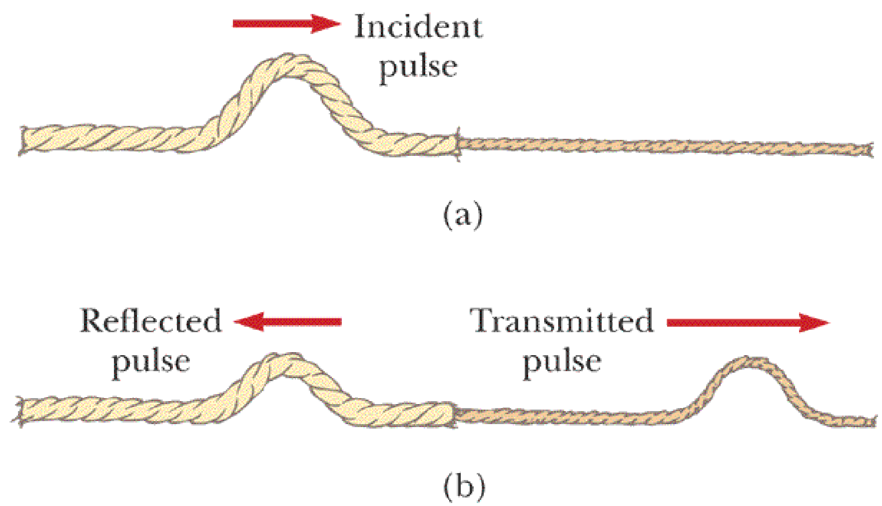
- Reflection through the **fixed** boundary
  - The reflection at a rigid end causes to invert on reflection.
  - For a sinusoidal wave, the inversion of a wave causes to a π phase shift.



Reflection and Transmission of Waves

- **Boundary of light string attached to a heavier (more dense) string**
  - **The inversion in the reflected wave is similar to the behavior of a wave meeting a fixed boundary, but partially reflected.**
  - **The transmitted wave has the same shape of the incident wave.**
- **Boundary of heavy string attached to a lighter (less dense) string**
  - **The incident wave is partially reflected and partially transmitted. The reflected wave is not inverted.**

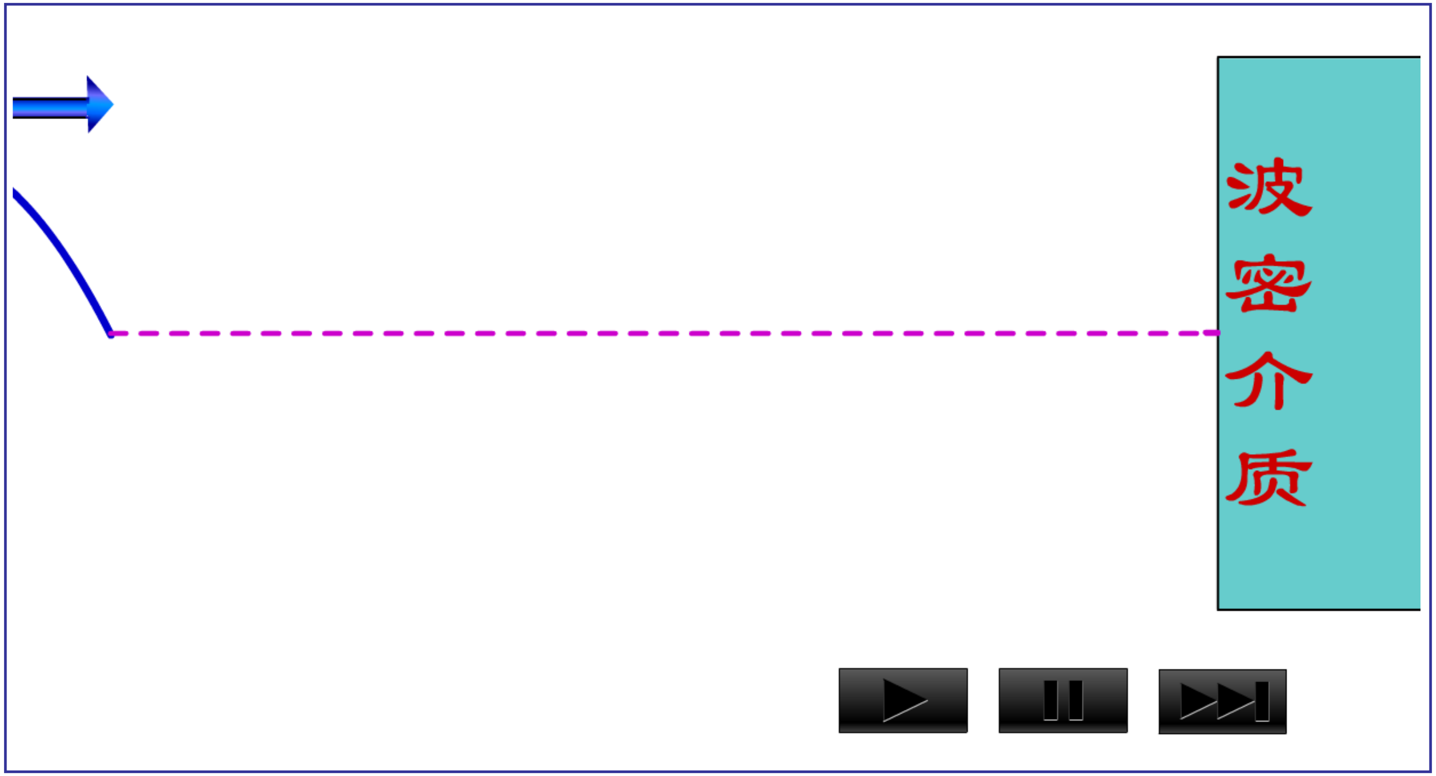




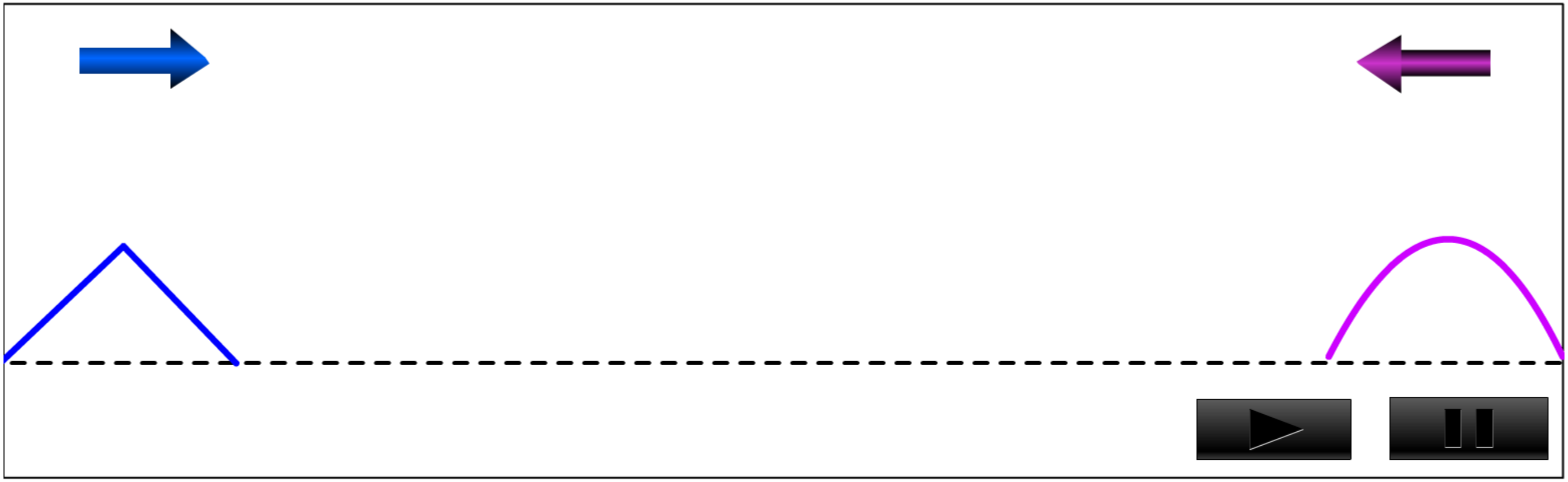
Reflection and Transmission of Waves

For mechanical wave, the larger is ρv, more dense is the medium.

For optical wave, the larger is the index of refraction n, more dense is the medium.



§ 10 The Principle of Superposition

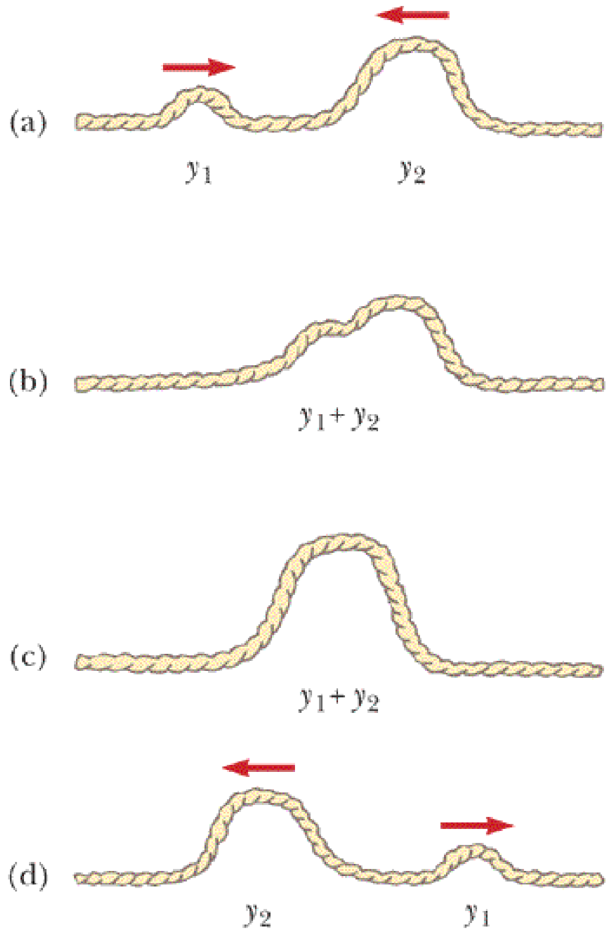


The Principle of Superposition

- If two or more traveling waves are moving through a medium and combine at a given point, the resultant displacement of the medium at that point is the **sum** of the displacements of the individual waves.

  In the region they overlap:

$$y(x,t) = y_1(x,t) + y_2(x,t)$$



- Two traveling waves can pass through each other **without** being destroyed or even altered.

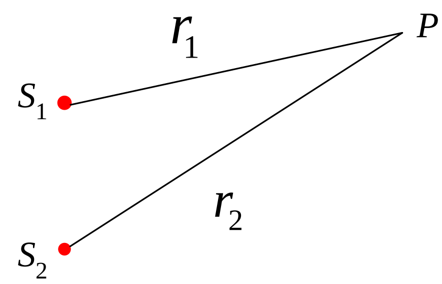
§ 11 Interference of waves

- The overlapping of waves is called interference
  - If two wave overlap in a region:

$$y_1 = A_1\cos(\omega t - kr_1 + \phi_1),$$

$$y_2 = A_2\cos(\omega t - kr_2 + \phi_2),$$

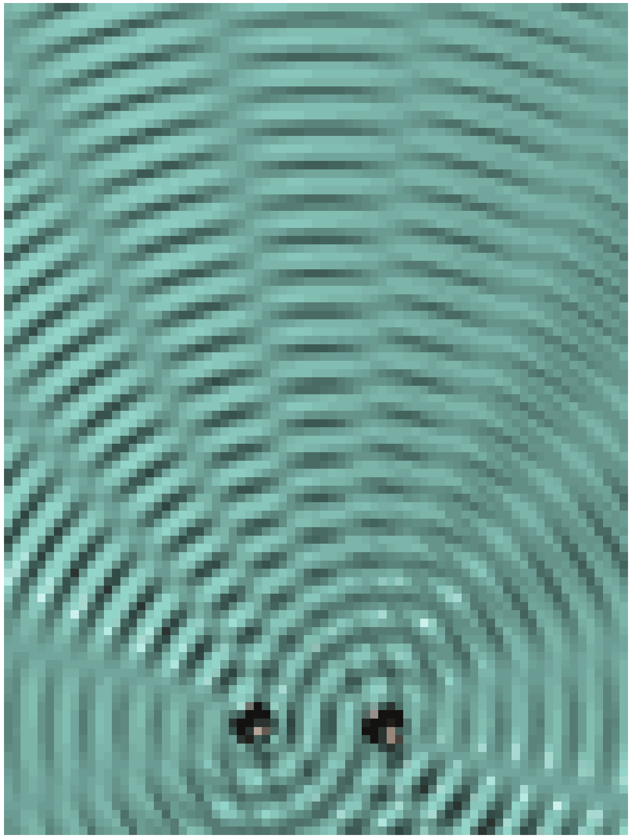
$$y = y_1 + y_2$$



$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\left[k(r_2 - r_1) - (\phi_2 - \phi_1)\right]$$

$$I = I_1 + I_2 + \underbrace{2\sqrt{I_1 I_2}\cos\Delta\varphi}_{\text{Coherent term}}$$

Phase difference: $\Delta\varphi = k(r_2 - r_1) - (\phi_2 - \phi_1)$



- For some point:

$$\Delta\varphi = \pm 2m\pi, \quad m = 0, 1, 2, \ldots$$

Two waves are in phase

$I = I_{max}$ —— constructive interference.

- For some point:

$$\Delta\varphi = \pm(2m+1)\pi, \quad m = 0, 1, 2, \ldots$$

Two waves are out of phase

$I = I_{min}$ —— destructive interference.

The interference pattern

Interference produces a redistribution of energy.

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2} \xrightarrow{I_1 = I_2} 4I_1$$

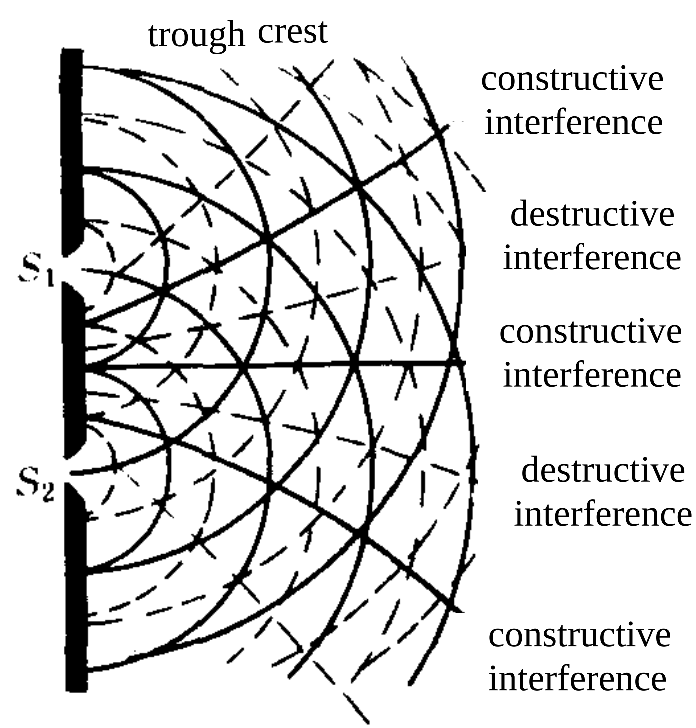
$$I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2} \xrightarrow{I_1 = I_2} 0$$

$$\Delta\varphi = k(r_2 - r_1) - (\phi_2 - \phi_1)$$

The trajectories of all points for both constructive or destructive are governed by

$$r_2 - r_1 = \text{constant}$$

The surfaces of hyperboloid.



The conditions for coherent interference

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\Delta\varphi$$

- The conditions for coherent interference

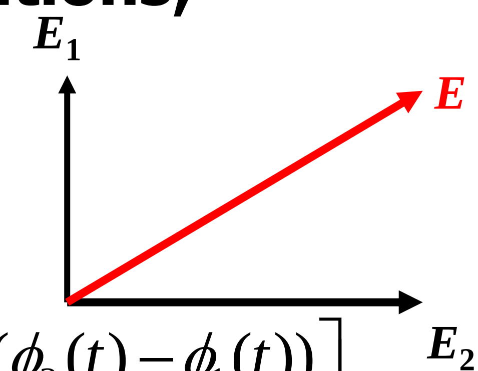
  Coherent interference: when $2\sqrt{I_1 I_2}\cos\Delta\varphi \neq 0$

  Incoherent interference: when $2\sqrt{I_1 I_2}\cos\Delta\varphi = 0$

  - **Have the same components of vibrations;**

$$E_1^2 + E_2^2 = E^2$$

$$I_1 + I_2 = I$$ without an interference term



  - **Have the same frequencies;**

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left[\left(\omega_2 - \omega_1\right)t - k(r_2 - r_1) + (\phi_2(t) - \phi_1(t))\right]$$

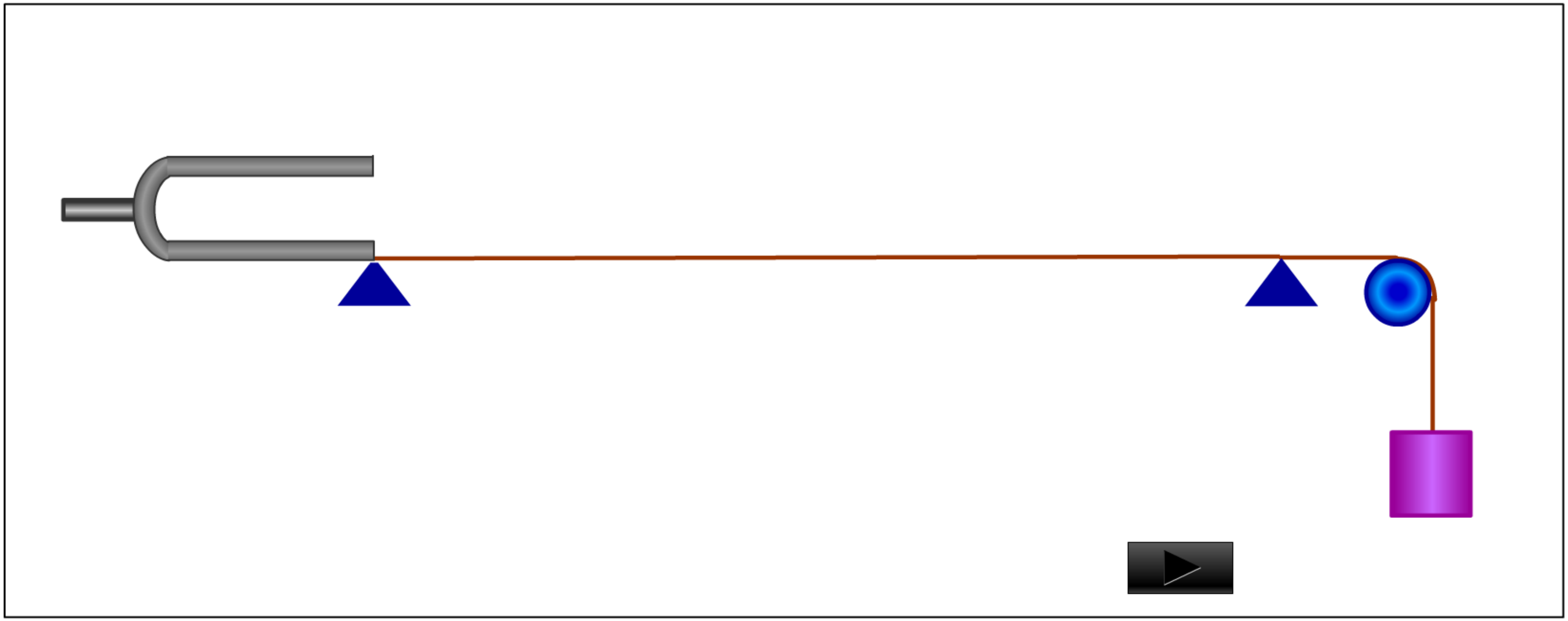
$\omega_1 \neq \omega_2$ $\quad \omega \sim 10^6\,\text{Hz},\quad \Delta\omega \sim 10^5 - 10^6\,\text{Hz},$ $\quad \overline{I} = \frac{1}{\tau}\int_0^\tau I\,dt = I_1 + I_2$

  - **Phase difference is unchanged (stable).**

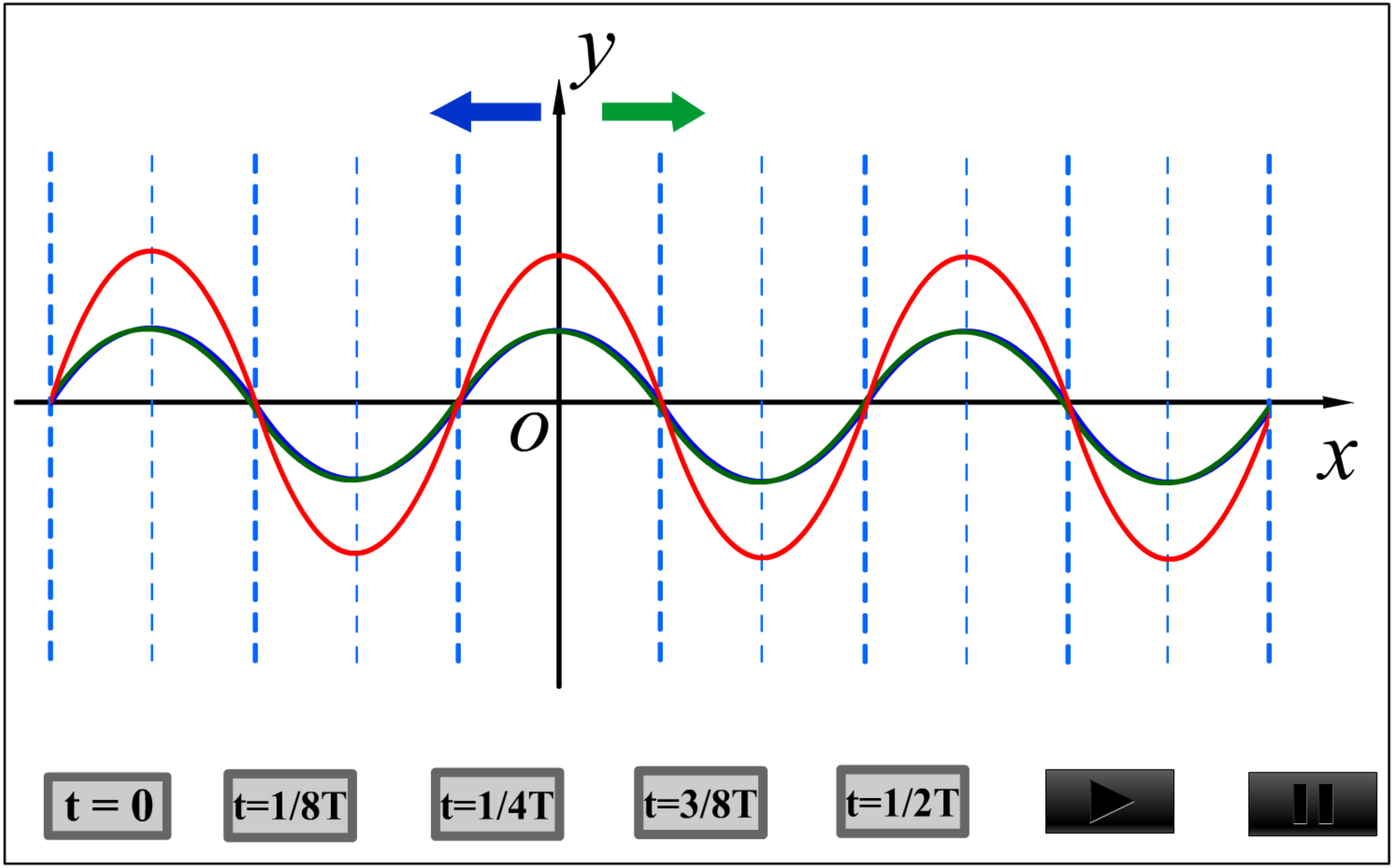
$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left[\underbrace{\Delta\phi(t)}_{\text{Varies with time}} - k(r_2 - r_1)\right], \quad \overline{I} = \frac{1}{\tau}\int_0^\tau I\,dt = I_1 + I_2$$

§ 12 Standing waves

What ?



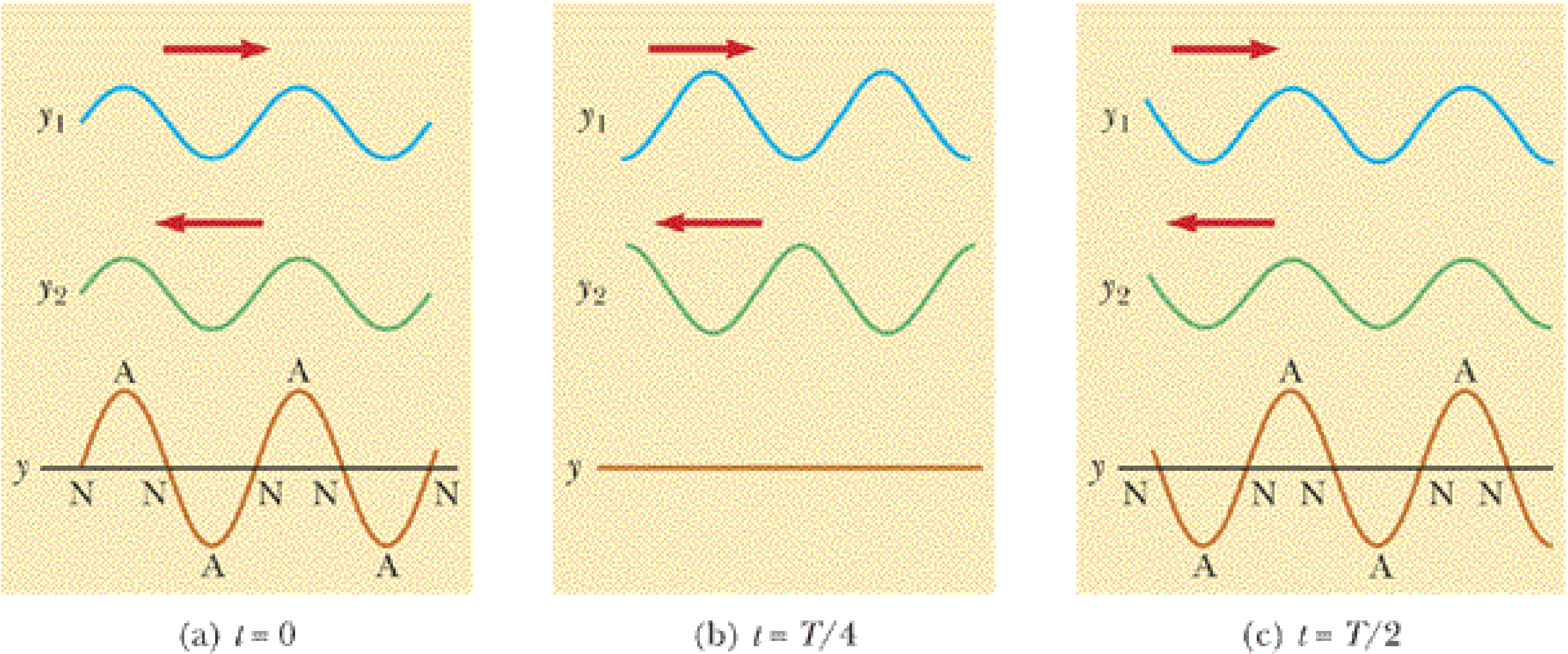
How is a standing wave produced?



- **Consider two waves that are identical except for traveling in opposite direction**

$$+x:\ y_1 = A\cos(\omega t - kx), \quad -x:\ y_2 = A\cos(\omega t + kx)$$

Resultant wave: $$\boxed{y = y_1 + y_2 = 2A\cos(kx)\cos(\omega t)}$$

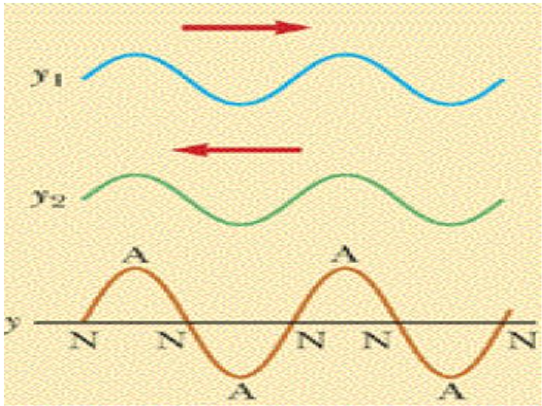


$$y = y_1 + y_2 = 2A\cos(kx)\cos(\omega t)$$

- **Features of standing wave**
  - x **and** t **appear separately, not in the combination** $(x \pm vt)$ **required for a traveling wave. The equation looks like more a simple harmonic motion than a wave motion.**
  - **In traveling wave each particle of the string vibrates with the same amplitude. In standing wave, however, the amplitude is not the same for different particles but varies with the location** x **of the particle.**

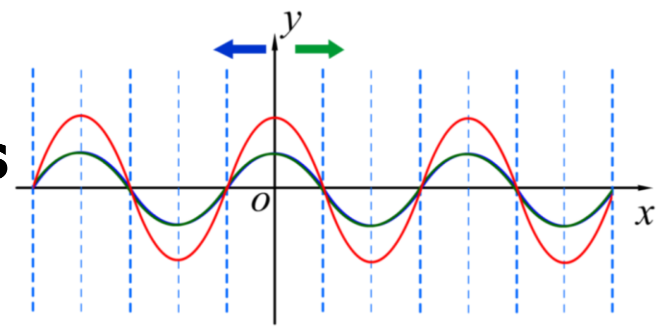
The amplitude for the particle located at x **is**

$$|2A\cos kx|$$



The nodes and the antinodes in a standing wave

$$|2A\cos kx|$$



- **Nodes**: the amplitude $|2A\cos kx|$ has a minimum value of zero at positions where

$$kx = \frac{2\pi}{\lambda}x = \pm(2m+1)\frac{\pi}{2}, \quad x = \pm(m+\frac{1}{2})\frac{\lambda}{2}, \quad m = 0,1,2,\ldots$$

The adjacent nodes are spaced one-half wavelength apart.

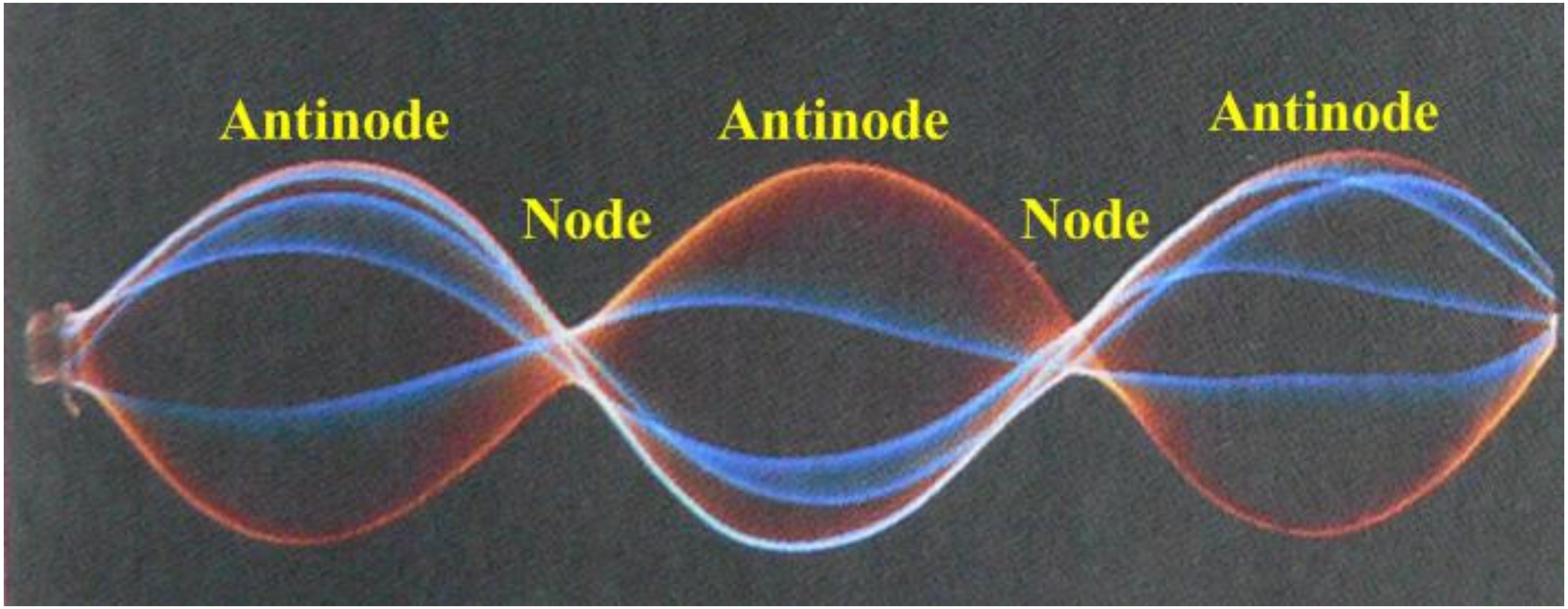
- **Antinodes**: the amplitude has a maximum value of 2A at positions where

$$kx = \frac{2\pi}{\lambda}x = \pm m\pi, \quad x = \pm m\frac{\lambda}{2}, \quad m = 0,1,2,\ldots$$

The adjacent antinodes are also spaced one-half wavelength apart.

The nodes and the antinodes in a standing wave

- **All the particles within two nodes are in phase, The particles at two sides of a node are π out of phase.**



Three Loop

The distance between adjacent antinodes is equal to $\lambda/2$.
The distance between adjacent nodes is equal to $\lambda/2$.
The distance between a node and an adjacent antinode is $\lambda/4$.

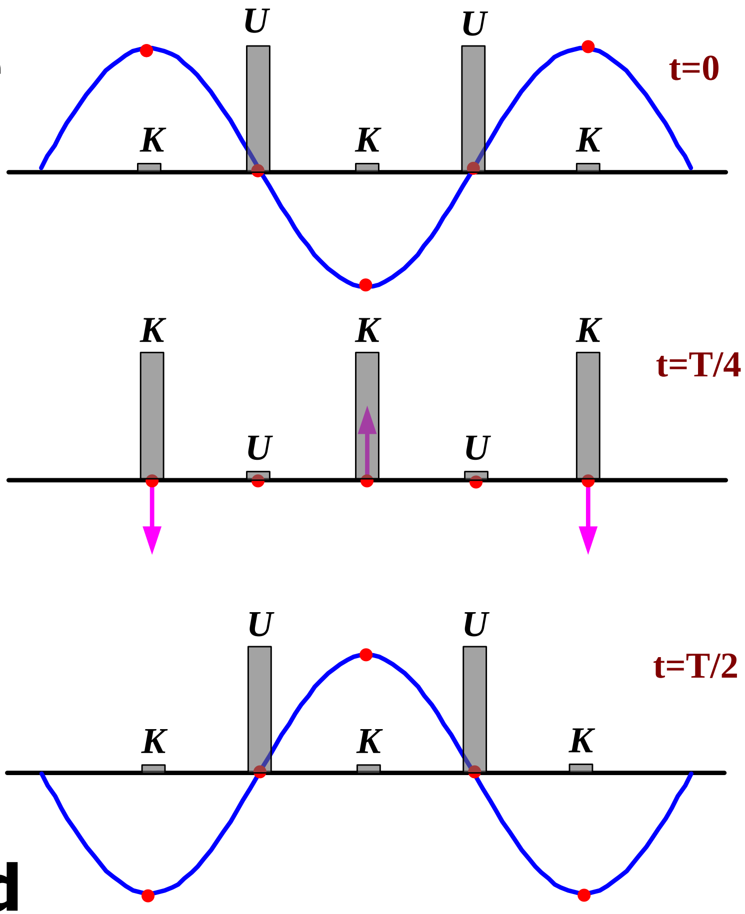
Energy feature of a standing wave

- $t = 0, T/2$

  The kinetic energies for all the particles are zero. The potential energies of particles at nodes reach maximum.

- $t = T/4$

  The potential energies for all the particles are zero. The kinetic energies of particles at antinodes reach maximum.



- The energy can only exchange between node and antinode, and cannot be transported along the string to the right or to the left.

§ 13 Standing waves in Strings

- **Normal modes** (简正模) **for standing waves in strings**
  - **Normal modes: the possible natural patterns of vibration.**

$$L = n\frac{\lambda}{2}, \qquad n = 1, 2, 3, \ldots$$

for string fixed at both ends

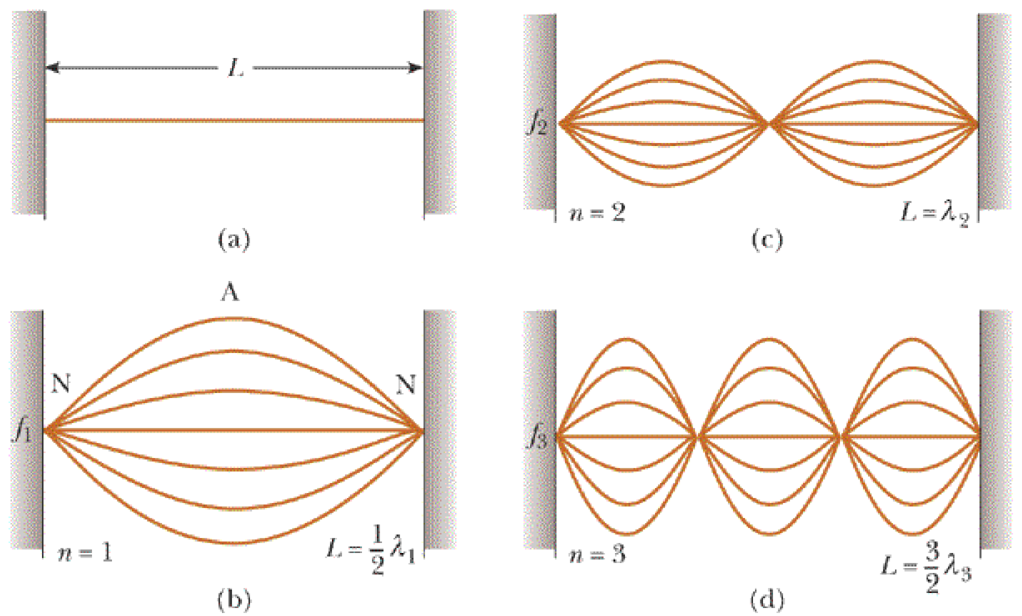
- **Fundamental frequency** (基频) $n=1$ **and harmonic series** (谐频系列)

$$\lambda_n = \frac{2L}{n},$$

$$f_n = \frac{v}{\lambda_n} = \frac{n}{2L}v = \frac{n}{2L}\sqrt{\frac{F_T}{\mu}},$$

$$n = 1, 2, 3, \ldots$$

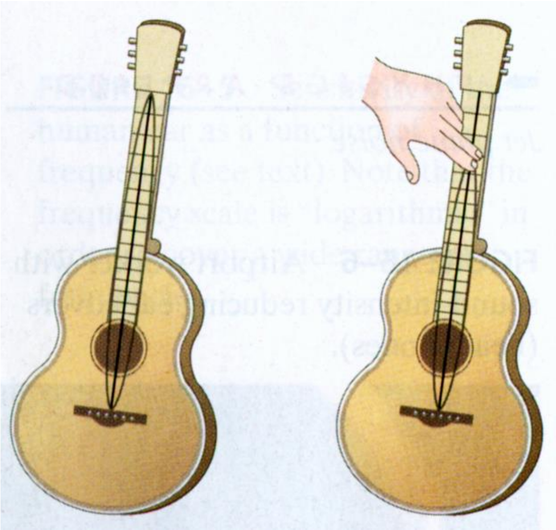
$$f_n = nf_1, \quad n = 1, 2, 3, \ldots$$



Musical Instruments

- How a string musical instrument works?
  - The fundamental frequency of a vibrating string:

$$f_1 = \frac{1}{2L}\sqrt{\frac{F_T}{\mu}}$$



  - What factors you can adjust?

Line density μ :

The lower-pitched strings are "fat";

Tension F_T :

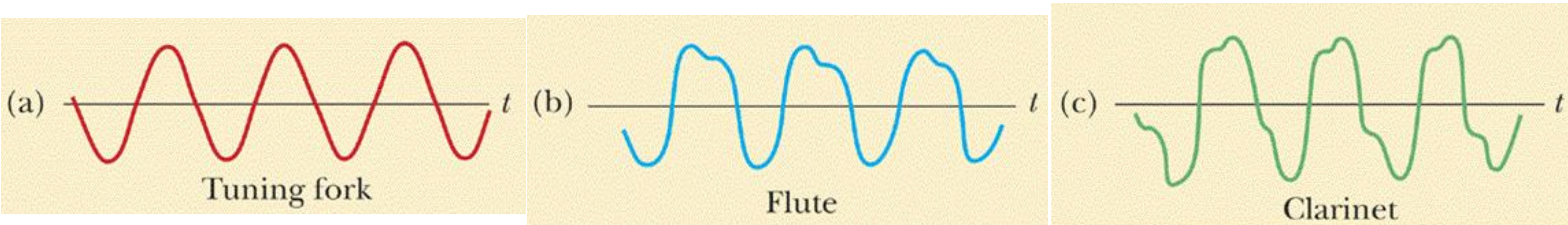
Adjust the tension to bring each string to the exact desired frequency;

The length of the string L:

Move your finger tips to alter the effective length of the string.

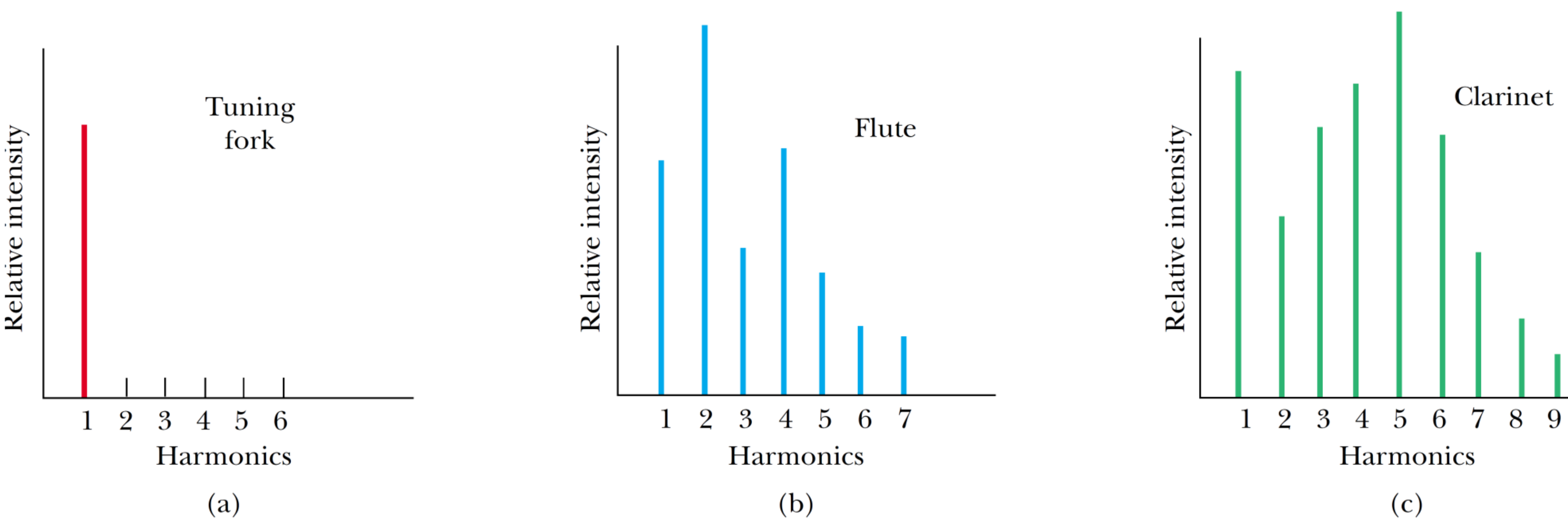
The Waveforms and Harmonics for some musical instruments

In time domain



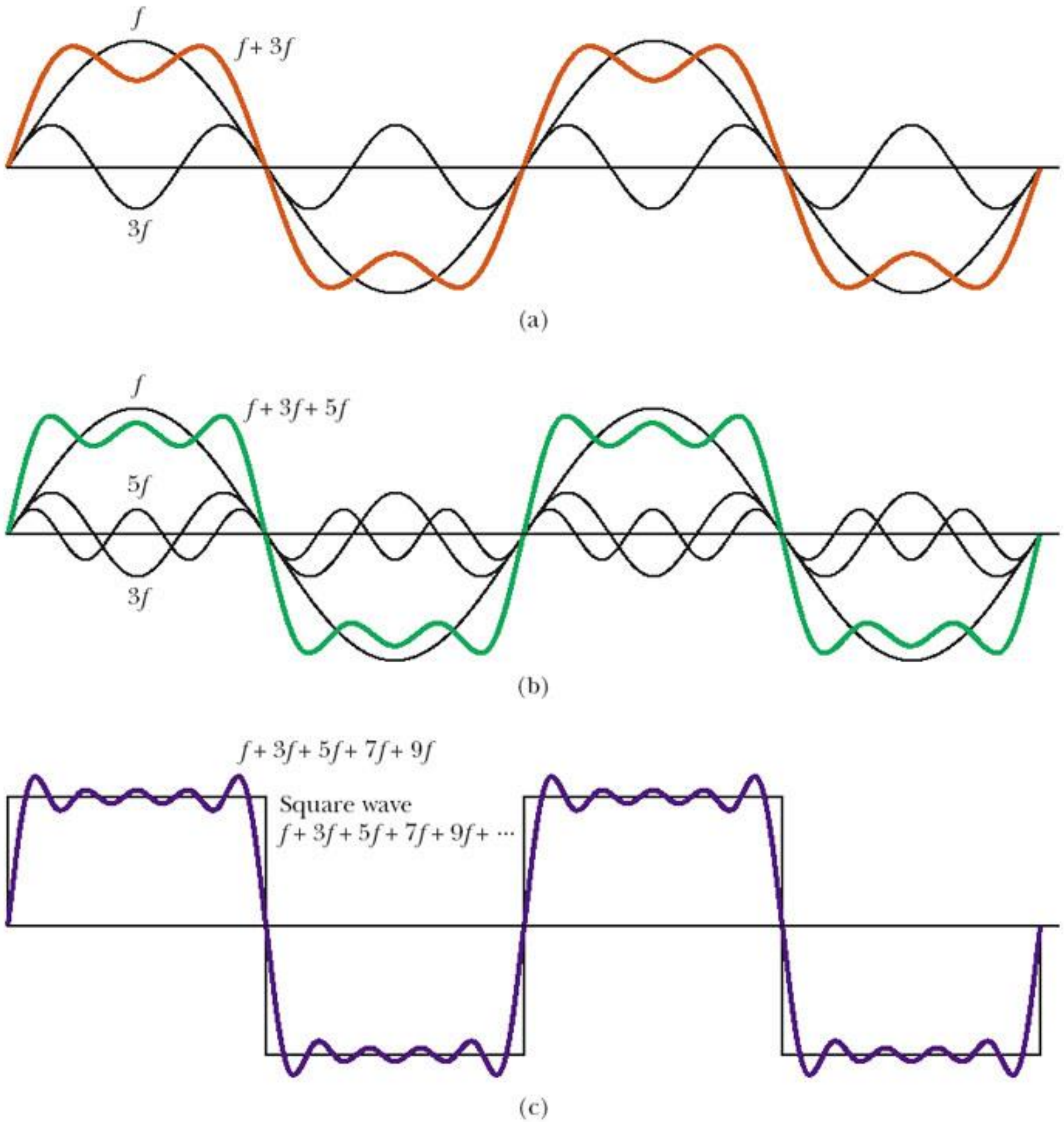
音叉　　长笛　　单簧管

In frequency domain



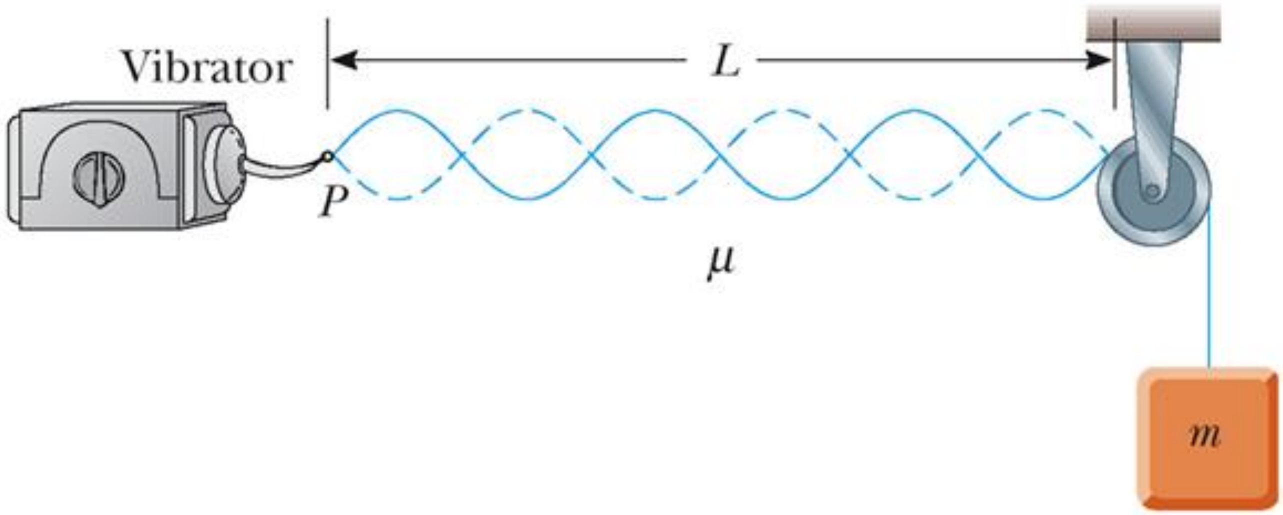
How to construct a square wave

$$y = A\sin\omega t + \frac{A}{3}\sin 3\omega t + \frac{A}{5}\sin 5\omega t + \cdots$$



Example

An object is hung from a string (with mass density μ) that pass over a light pulley. The string is connected to a vibrator of frequency f, and the length of the string between point P and the pulley is L. (1) What should the mass of the object be in order to stimulate a clear standing wave in the string? (2) What is the largest mass for which standing waves could be observed?



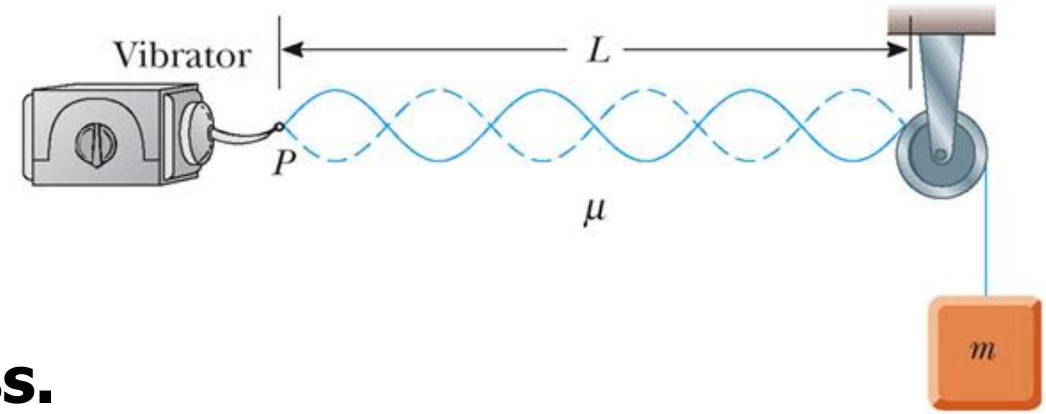
Example

Solution: In order to generate a clear standing wave,

$$L = n\frac{\lambda}{2}$$

$$\lambda = \frac{v}{f}, \quad v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{mg}{\mu}} \Rightarrow L = \frac{n}{2f}\sqrt{\frac{mg}{\mu}}$$

$$m = \frac{4\mu f^2 L^2}{n^2 g}$$

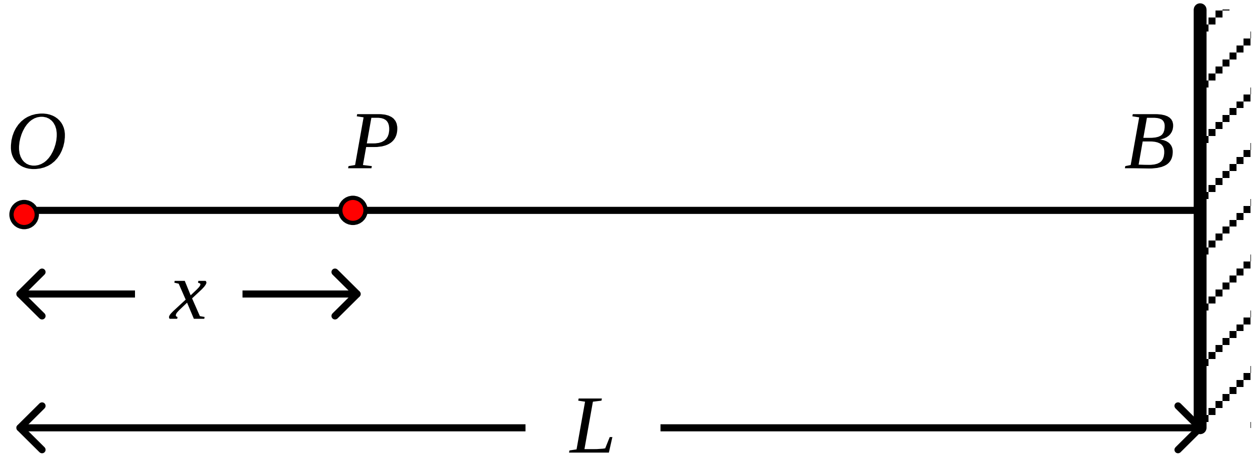


When $n=1$, we get the maximum value of mass.

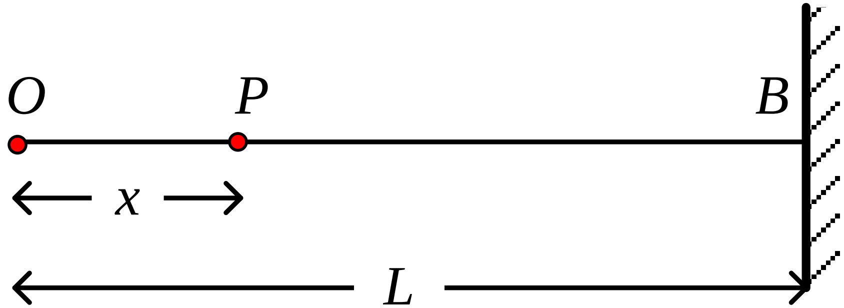
$$m_{\max} = \frac{4\mu f^2 L^2}{g}$$

Example

A string of length L and mass m is pulled under the tension F_T. One end is connected to a vibrator of angular frequency ω, amplitude A, and initial phase angle ϕ, while the other end is fixed to a wall. (1) Write the **incident** wave function; (2) Write the **reflected** wave function by wall (assume the amplitude is same as the incident wave); (3) Write the **resultant** wave function due to the superposition.



Example



Solution: Take point O as origin, and positive x-direction to the right. The wave velocity on the string:

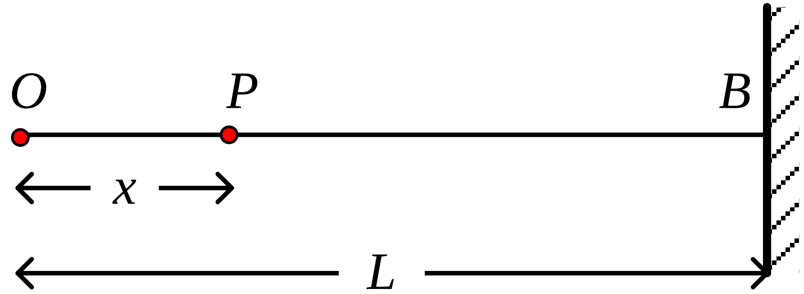
$$v = \sqrt{F_T / \mu} = \sqrt{F_T L / m}$$

(1) The wave function for incident wave:

$$y_{in} = A\cos\left(\omega t + \phi - kx\right), \qquad k = \frac{\omega}{v}$$

$$= A\cos\left[\omega t - \frac{\omega}{\sqrt{F_T L / m}} x + \phi\right]$$

Example



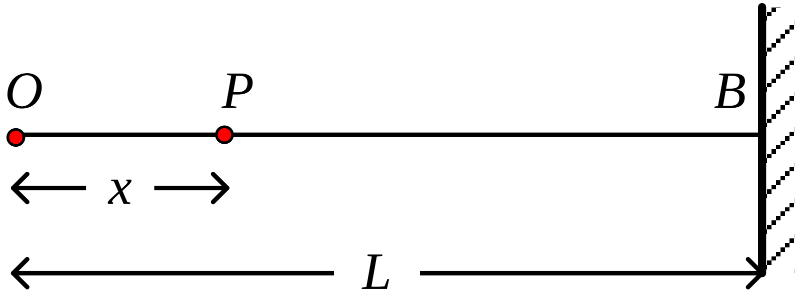
(2) The phase retardation at arbitrary point P due to reflected wave with respect to point O is:

$$-k(2L-x)\pm\pi=-\frac{\omega}{\sqrt{F_T L/m}}(2L-x)\pm\pi$$

The reflected wave:

$$y_{reflect}=A\cos\left[\omega t+\phi-\frac{\omega}{\sqrt{F_T L/m}}(2L-x)-\pi\right]$$

$$=A\cos\left[\omega t+\frac{\omega}{\sqrt{F_T L/m}}x-\frac{2\omega L}{\sqrt{F_T L/m}}+\phi-\pi\right]$$



(3) The resultant wave function:

$$y = y_{in} + y_{reflect}$$

$$= 2A\cos\left[\frac{\omega}{\sqrt{F_T L / m}}(L - x) + \frac{\pi}{2}\right]\cos\left[\omega t - \left(\frac{\omega}{\sqrt{F_T L / m}}L - \phi + \frac{\pi}{2}\right)\right]$$

How many nodes exist in the string? Their locations?

Problem

Ch13 (P349): 27, 30; 43

Review

Simple Harmonic Oscillation		Harmonic Wave
Kinematics,	$y = A\cos(\omega t + \phi)$,	$y(x,t) = A\cos(\omega t - kx + \phi)$
Dynamics,	$\dfrac{d^2 y}{dt^2} + \omega^2 y = 0$,	$\dfrac{\partial^2 y}{\partial t^2} = v^2 \dfrac{\partial^2 y}{\partial x^2}$
Energy,	$E_{\text{mech}} = \text{constant}$,	$I = \dfrac{\bar{P}}{S} = \bar{w}v = \dfrac{1}{2}\rho\omega^2 A^2 v$
Superposition,	$y = A_1\cos(\omega t + \phi_1) + A_2\cos(\omega t + \phi_2) = A\cos(\omega t + \phi)$,	$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\Delta\phi$
	$A^2 = A_1^2 + A_2^2 + 2A_1 A_2\cos\Delta\phi$,	$y = y_1 + y_2 = 2A\cos(kx)\cos(\omega t)$
		$f_n = \dfrac{v}{\lambda_n} = \dfrac{n}{2L}v = \dfrac{n}{2L}\sqrt{\dfrac{F_T}{\mu}}$, $n = 1, 2, 3, \ldots$