§ 1 Kinematics of Rigid Bodies

- **What Rigid Body**

The body that has a perfectly definite and unchanging shape and size.

$$\left|\vec{r}_i - \vec{r}_j\right| = d_{ij} = \text{constant}$$

The distance between any two arbitrary points in the body is a constant.

- **Idealized model: the external forces that act on the real-world bodies can deform them —— stretching, twisting, and squeezing.**
- **If these deformations are so little that can be ignored, such bodies can be treated as rigid bodies.**

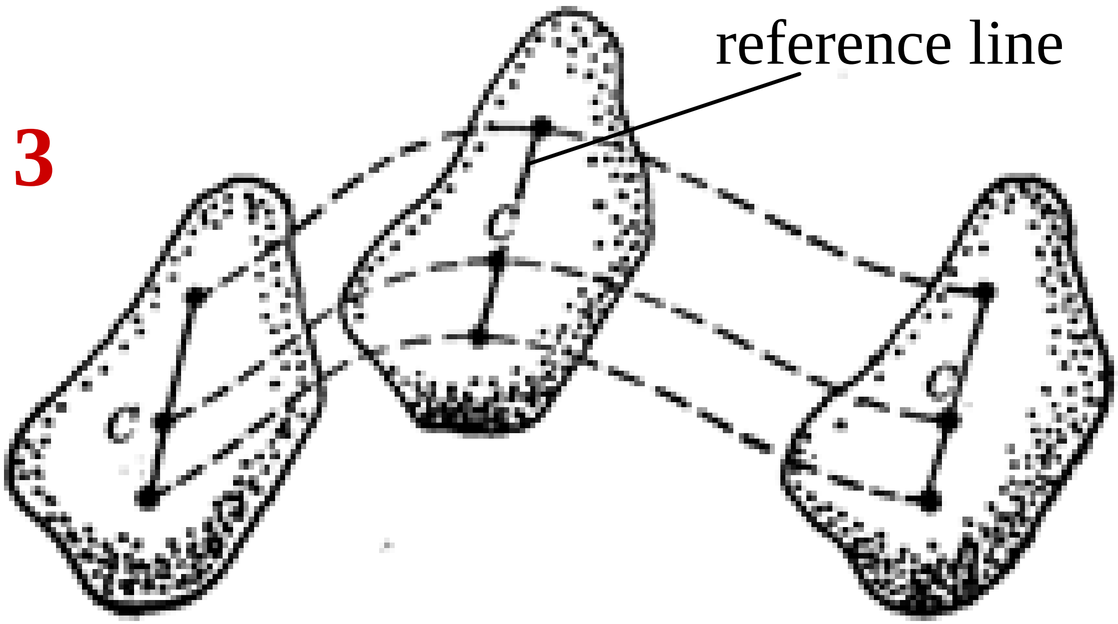
Why Rigid Body

- **Any body can be viewed as a system of N numbers of particles.**
- **Generally need $3N$ motional equations to describe its motion.**
- **The rigid body model simplifies the description of body's motion.**

$$\left|\vec{r}_i - \vec{r}_j\right| = d_{ij} = \text{constant}$$

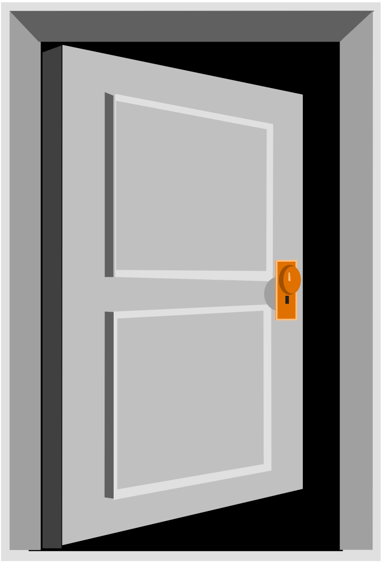
- **Translational motion of a rigid body**
  - **The trajectories of all the points of a rigid body are the same, or the line between any two points of a rigid body keeps its orientation unchanged all the time.**

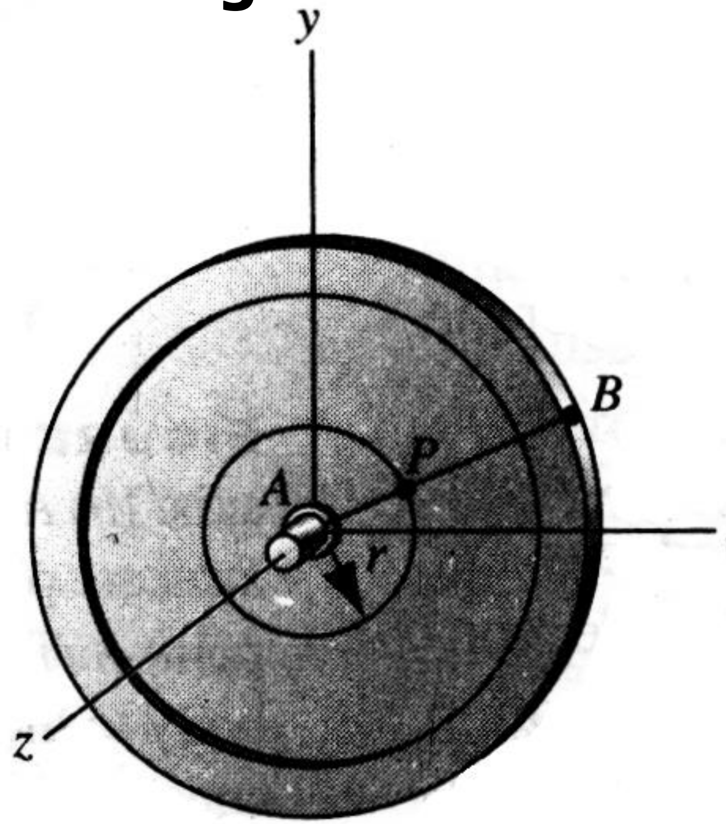
$i = 3$



- **Rotational motion of a rigid body**
  - **Rotation about a fixed axis: every point of the body moves in a circular path. The centers of these circles must lie on a common straight line called the axis of rotation.**

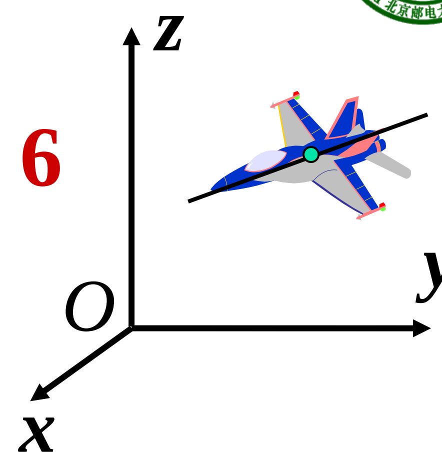
$$i = 1$$





- The **general** motion of a rigid body will include both rotational and translational components.

$i = 6$

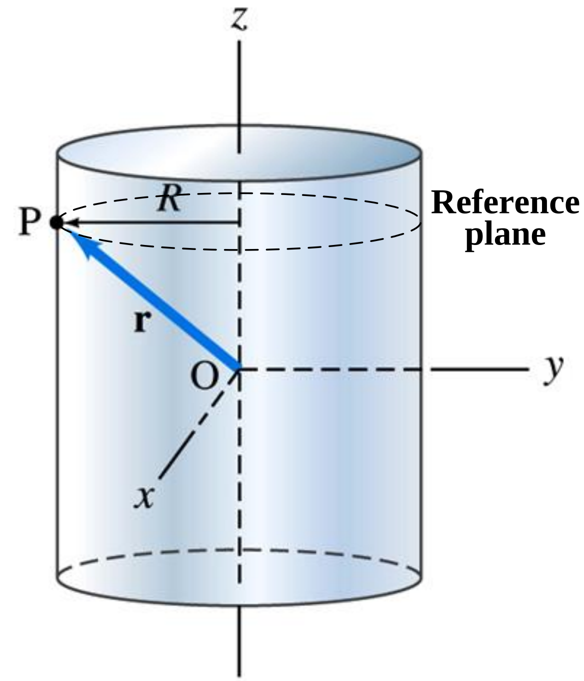


- ✓ **Three** to locate the center of mass.
- ✓ **Two** angles to orient the axis of rotation.
- ✓ **One** angle to describe rotation about the axis.

- The rigid body model **simplifies** the description of body's motion.
  - For a rigid body, we only need **6** coordinates.

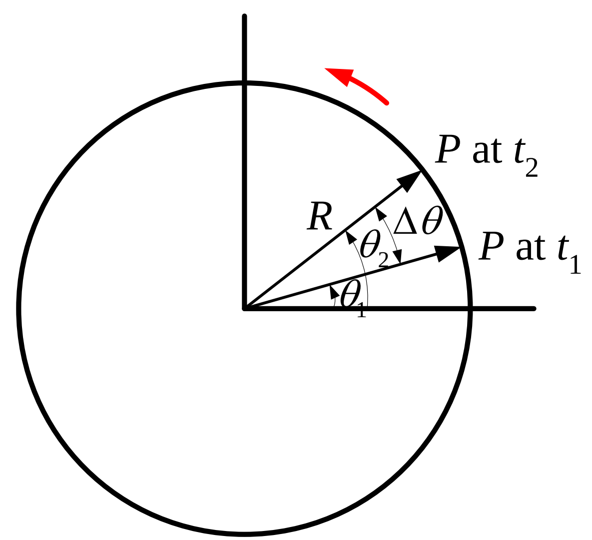
§ 2 Angular Quantities for rigid bodies

- **Rotational radius *R***
  - **The perpendicular distance of point *P* in the reference plane from the axis of rotation.**



- **Angular position and angular displacement**
  - **Angular position: θ_1, θ_2**
  - **Angular displacement:**

    $\Delta\theta = \theta_2 - \theta_1$.

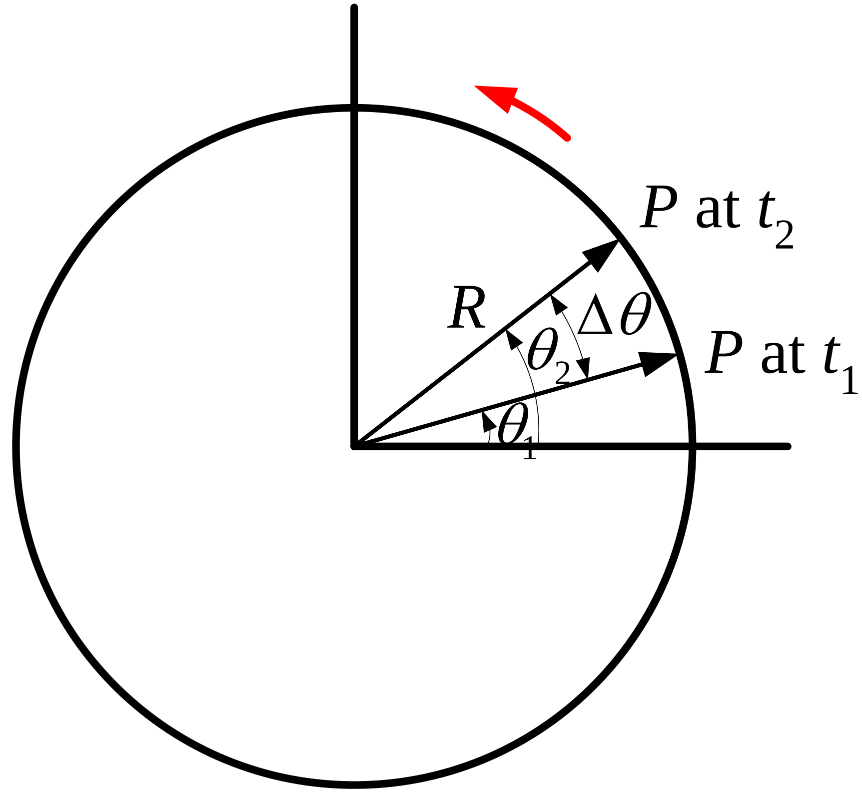


- **Average angular velocity:**

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

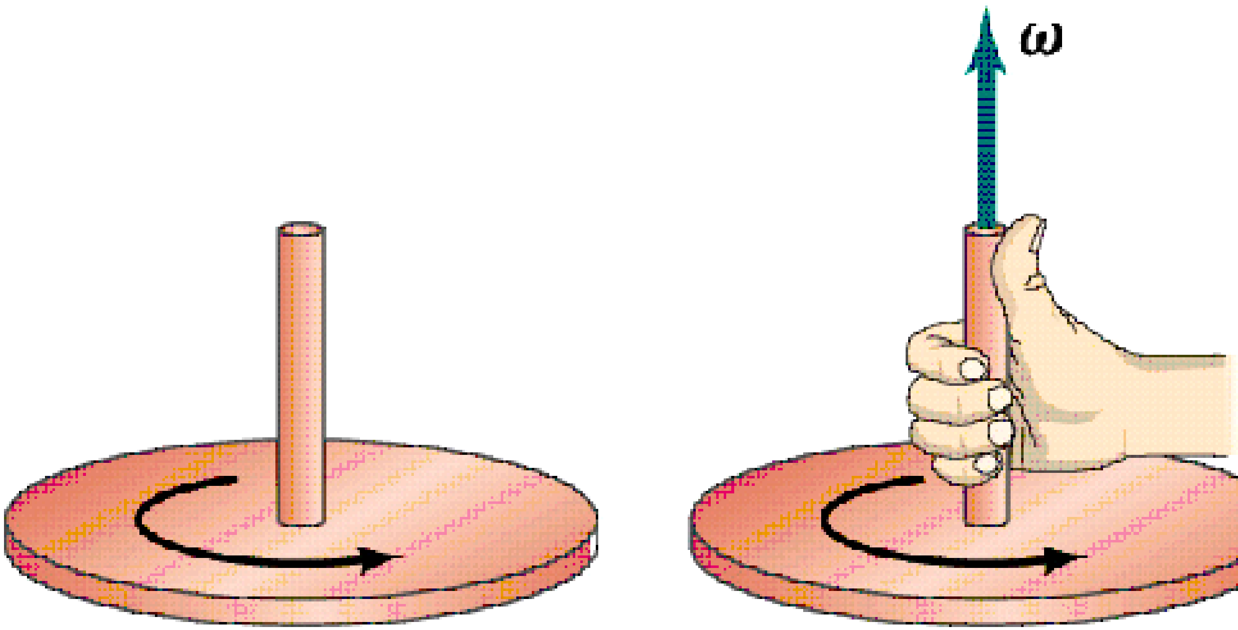
- **Instantaneous angular velocity:**

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$



- **Choose the positive sense of the rotation to be counter-clockwise.**

- **Angular velocity as a vector**
  - **The direction of angular velocity vector —— right-hand rule**
    - **The right-hand rule: when the fingers of right hand curl in direction of rotation, the thumb position is the direction of $\vec{\omega}$**



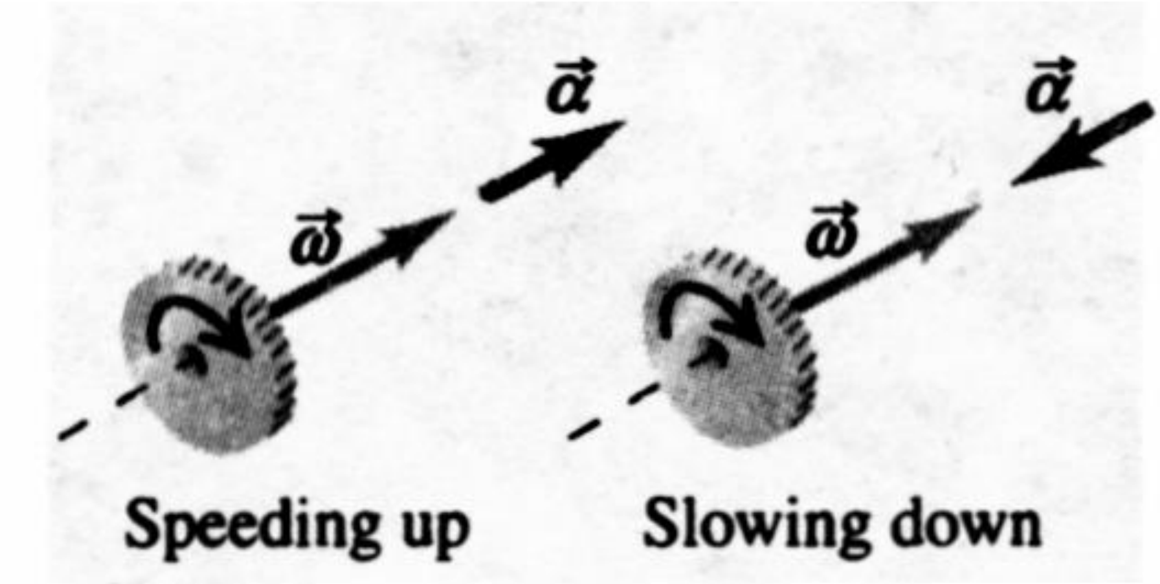
Angular acceleration

- **Average angular acceleration:** $$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

- **Instantaneous angular acceleration:** $$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt}$$

- **Angular acceleration as a vector**

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$



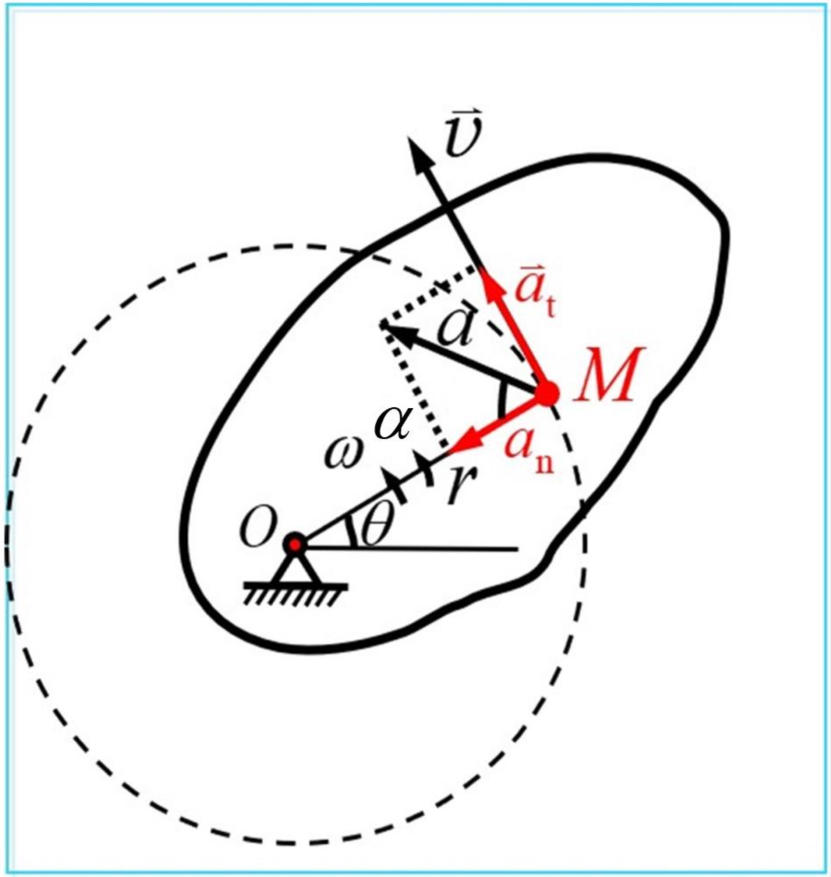
Linear quantities versus angular quantities

$$s = r\theta$$

$$v = r\omega$$

$$a_t = r\alpha$$

$$a_n = \omega^2 r = \omega v$$

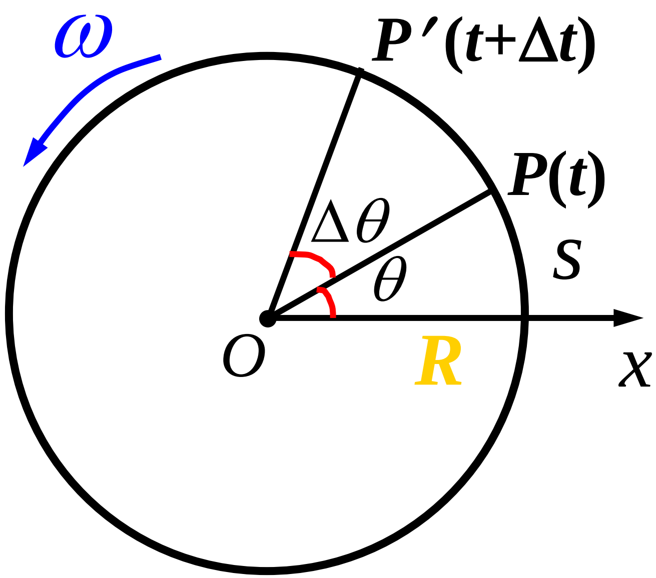


Uniformly accelerated rotational motion

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\left(\theta - \theta_0\right)$$



vs.

$$v = v_0 + a t$$

$$S = S_0 + v_0 t + \frac{1}{2} a t^2$$

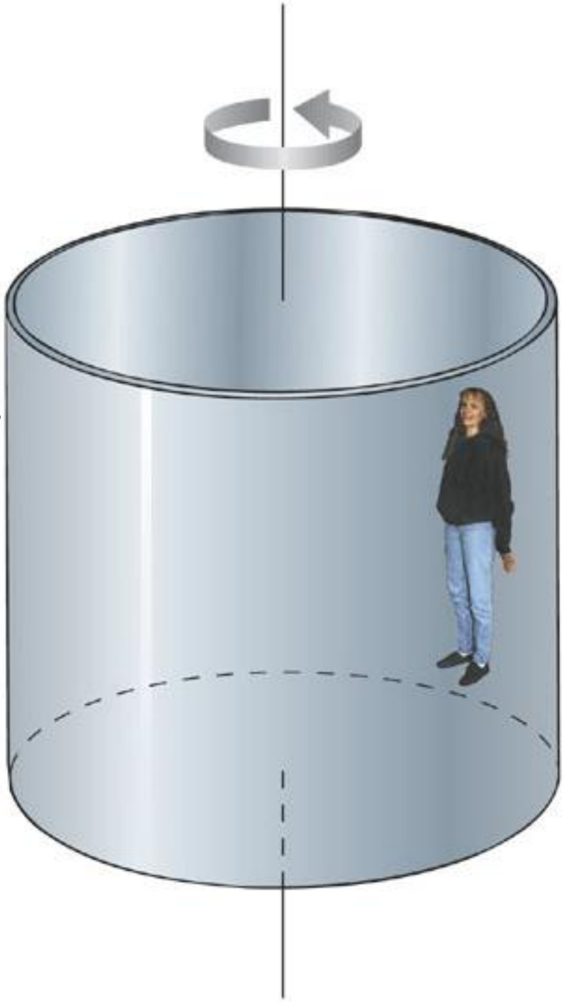
$$v^2 = v_0^2 + 2a\left(S - S_0\right)$$

Example

While you are operating a Rotor, you spot a passenger in acute distress and decrease the angular speed of the cylinder from 3.40 rad/s to 2.00 rad/s in 20.0 rev, at constant angular acceleration.

(a) What is the constant angular acceleration during this decrease in angular speed?

(b) How much time did the speed decrease take?

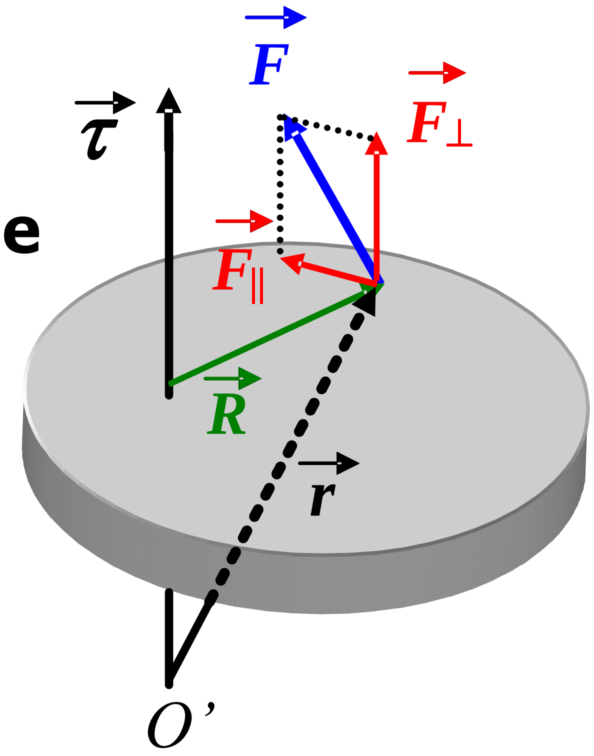


Solution:

(a) $\omega^2 = \omega_0^2 + 2\alpha\left(\theta - \theta_0\right)$

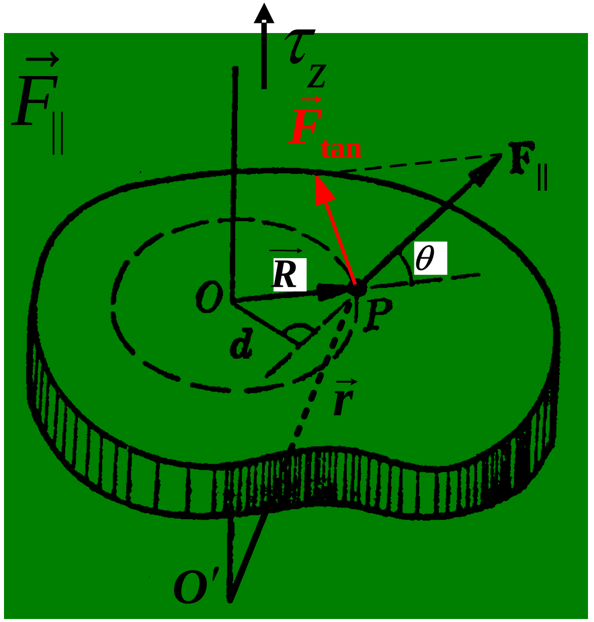
(b) $\omega = \omega_0 + \alpha t$

- **The torque about a fixed axis —— torque component along the axis of rotation**
  - **The force $\vec{F}$ can be resolved into the parallel component $\vec{F}_{\parallel}$ lying in the reference plane, and the perpendicular component $\vec{F}_{\perp}$.**
  - **The perpendicular component $\vec{F}_{\perp}$ does not contribute to the torque about the rotation axis, since it can not tend to change the body's rotation about that axis. (or there must be an opposite torque exerted on the axis to balance it)**



- **So the torque about the fixed rotation axis:**

$$\vec{\tau} = \vec{r} \times \vec{F}_{\parallel} = (\overrightarrow{O'O} + \vec{R}) \times \vec{F}_{\parallel} = \overrightarrow{O'O} \times \vec{F}_{\parallel} + \vec{R} \times \vec{F}_{\parallel}$$



Perpendicular to the rotation axis $O'O$, and will be balanced by another torque acting on the axis.

$$\tau_z = \tau_{axis} = RF_{\parallel} \sin\theta = F_{\parallel} d = RF_{\tan}$$

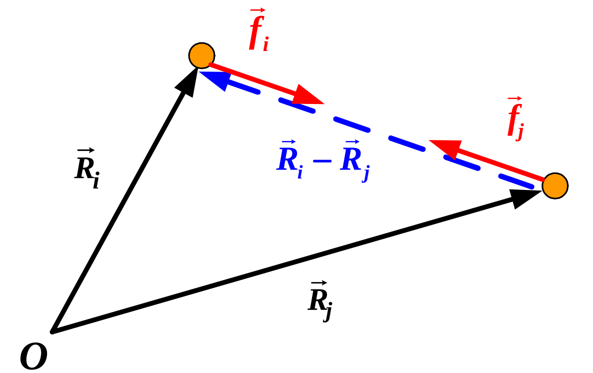
- **The torque about the axis *O'O* is actually the projection of the torque about the point *O'* on the axis *O'O*.**

The Rotational Form of Newton's Second Law (转动定律)

- **Imagine the body as being made up of a large number of particles.**
  - **For i-th particle Δm_i** **external force: $\vec{F}_i$** **internal force: $\vec{f}_i$**

$$\vec{F}_i + \vec{f}_i = \Delta m_i \vec{a}_i$$

$$\sum_i \vec{R}_i \times \vec{F}_i + \sum_i \vec{R}_i \times \vec{f}_i = \sum_i \vec{R}_i \times \left(\Delta m_i \vec{a}_i\right)$$



The torques of each pair of internal forces are vanished.

$$\vec{R}_i \times \vec{f}_{ij} + \vec{R}_j \times \vec{f}_{ji} = (\vec{R}_i - \vec{R}_j) \times \vec{f}_{ij} = 0 \quad \Rightarrow \quad \sum_i \vec{R}_i \times \vec{f}_i = 0$$

The external torque:

zero

$$\vec{R}_i \times \vec{F}_i = \vec{R}_i \times \vec{F}_{it} + \vec{R}_i \times \vec{F}_{in} = R_i F_{it} \hat{k}$$

The net torque about rotation axis that acts on the body:

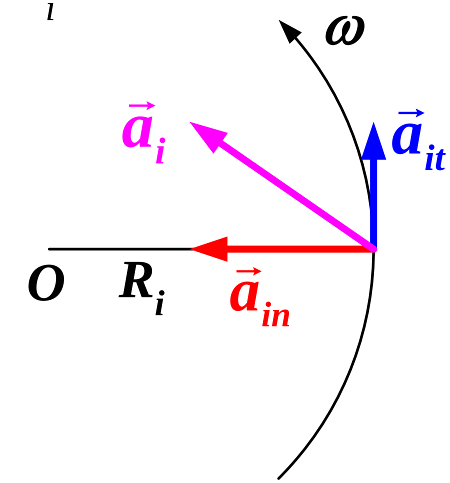
$$\vec{\tau}_{\text{net}} = \sum_i R_i F_{it} \hat{k}$$

The Rotational Form of Newton's Second Law

- **It is followed that:** $\vec{\tau}_{\text{net}} = \sum_i R_i F_{it}\hat{k} = \sum_i \vec{R}_i \times \left(\Delta m_i \vec{a}_i\right) = \sum_i \Delta m_i (\vec{R}_i \times \vec{a}_i)$

- **The right side of the equation:**

$$\vec{R}_i \times \vec{a}_i = \vec{R}_i \times \vec{a}_{it} + \underbrace{\vec{R}_i \times \vec{a}_{in}}_{\text{zero}} = R_i a_{it}\,\hat{k} = R_i^2 \alpha\,\hat{k}$$



$$\vec{\tau}_{\text{net}} = \sum_i R_i F_{it}\hat{k} = \left(\sum_i \Delta m_i R_i^2\right)\alpha\hat{k}$$

- **The moment of inertia of the body (转动惯量)** $I = \sum_i \Delta m_i R_i^2$

$$\sum \tau_{\text{net-axis}} = I\alpha$$

- **The rotational form of Newton's II Law**

Some Comments

$$\boxed{\sum \tau_{\text{net-axis}} = I\alpha} \qquad \sum F_{z-\text{ext}} = ma_z$$

- **It relates the net external torque about a particular fixed axis to the angular acceleration about that axis. The moment of inertia I must be calculated about that same axis.**
- **The moment of inertia reflects the tendency of a rigid body to resist angular acceleration, just like the mass reflecting the tendency of a object to resist linear acceleration.**
- **Generally, this equation is valid for the rotation of a rigid body about a fixed axis in an inertial reference frame.**
- **It is also valid for the rotation about an axis fixed in the center of mass of the body, although the CM is not an inertial reference frame.**

$$\sum \tau_{\text{ext-CM}} = I_{\text{CM}}\alpha$$

§ 4 The Moment of Inertia

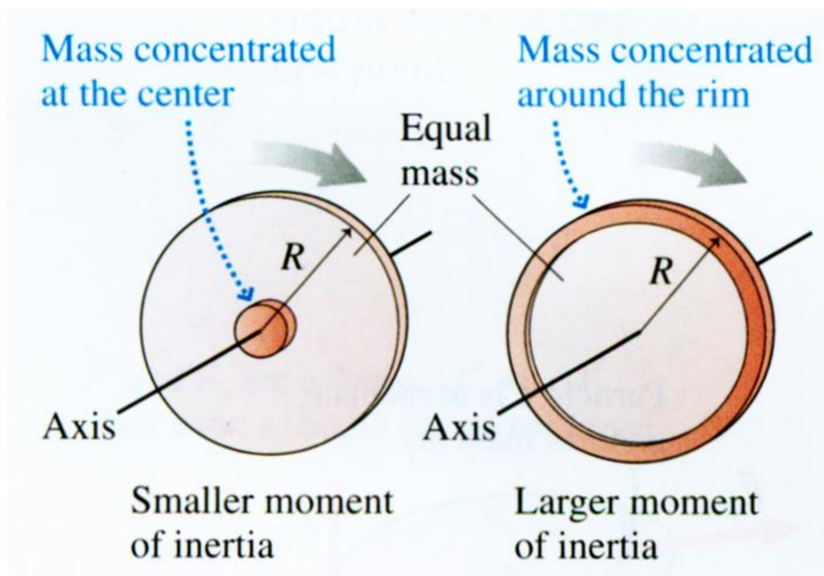
- **The definition:** $$\boxed{I = \sum_i \Delta m_i R_i^2}$$

It plays the same role in $\alpha = \tau_{\text{net}} / I$ **as mass in** $\vec{a} = \vec{F}_{\text{net}} / m$ **.**
The larger the moment of inertia, the more effort it takes and the slower its angular acceleration.

For continuous distribution bodies: $$\boxed{I = \int R^2 dm}$$

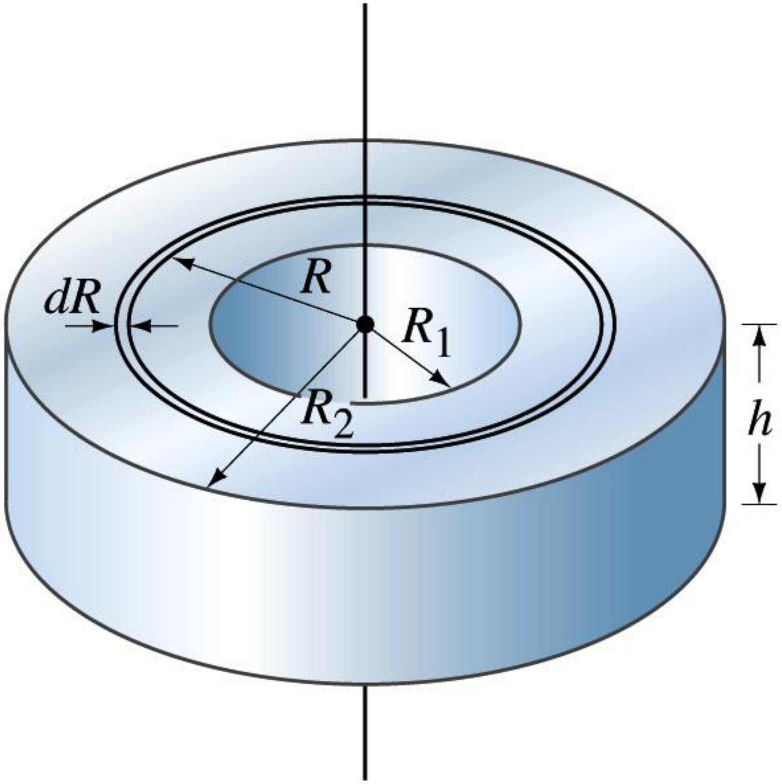
$$dm = \begin{cases} \rho dV \\ \sigma dS \\ \lambda dl \end{cases}$$

An object's moment of inertia depends not only on the object's mass but on how the mass is distributed around the axis.



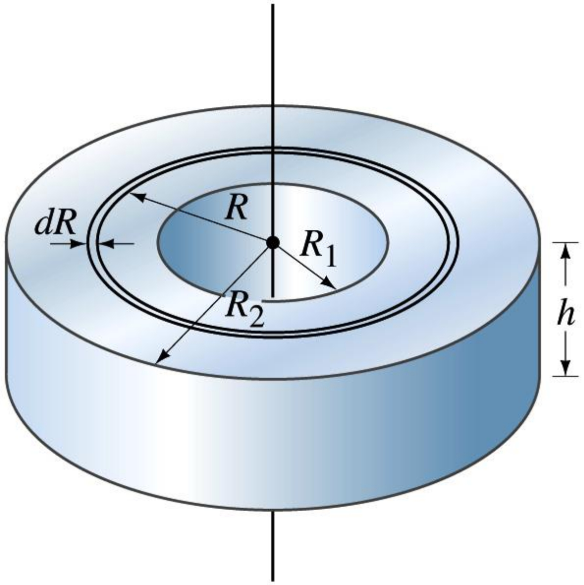
Example (P249 Ex.10-10)

The moment of inertia of a uniform hollow cylinder of inner radius R_1, outer radius R_2, and mass M, if the rotation axis is through the center along the axis of symmetry.



Example

Solution: Divided the cylinder into thin concentric cylindrical rings or hoops of thickness *dR*



$$dI = R^2 dm$$

$$dm = \rho dV$$

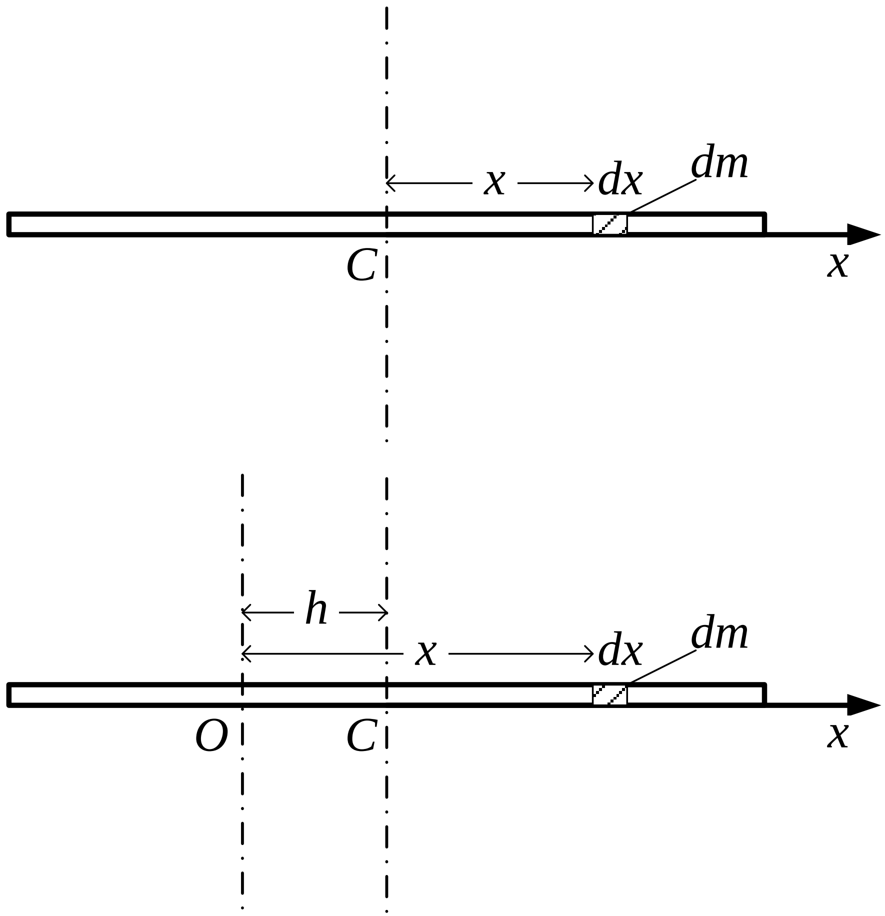
$$= \frac{M}{\pi(R_2^2 - R_1^2)h}(2\pi R)hdR$$

$$= \frac{2M}{R_2^2 - R_1^2} RdR$$

$$I = \int R^2 dm = \frac{2M}{R_2^2 - R_1^2}\int_{R_1}^{R_2} R^3 dR = \frac{1}{2}M(R_1^2 + R_2^2)$$

Example

Uniform thin rod with mass ***M*** and length ***l***. Calculate the moment of inertia about the axis located (1) at the CM, (2) at an arbitrary distance ***h*** from the CM.



Solution: (1) The axis locates at the CM

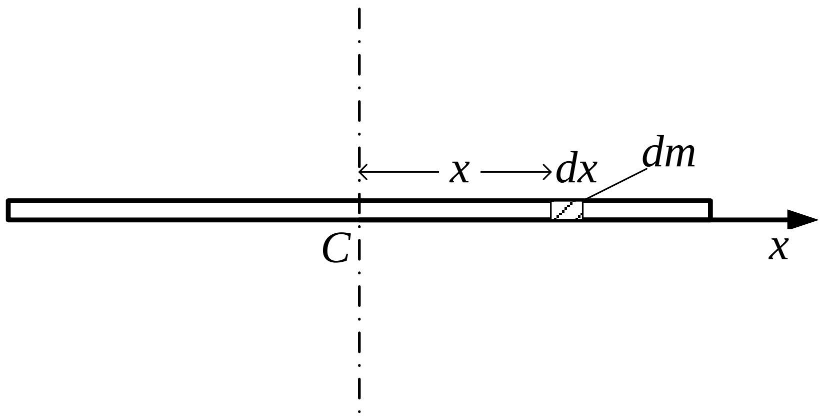
Take a small element of mass:

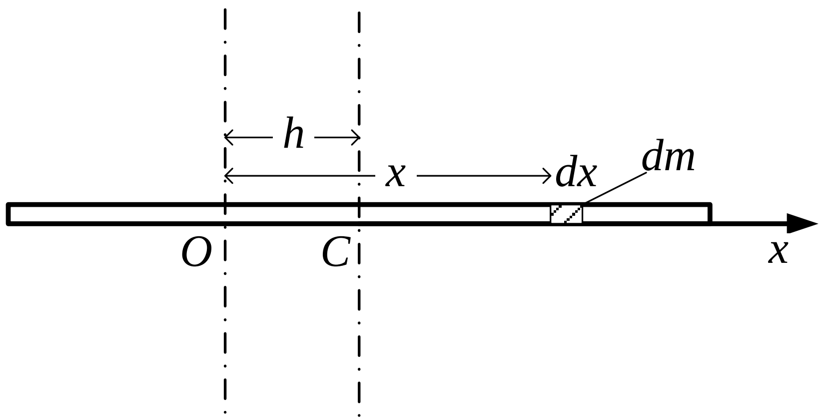
$$dm = \lambda dx = \frac{M}{l} dx$$

$$dI = x^2 dm = \lambda x^2 dx$$

$$I = \int dI = \int_{-l/2}^{l/2} \lambda x^2 dx$$

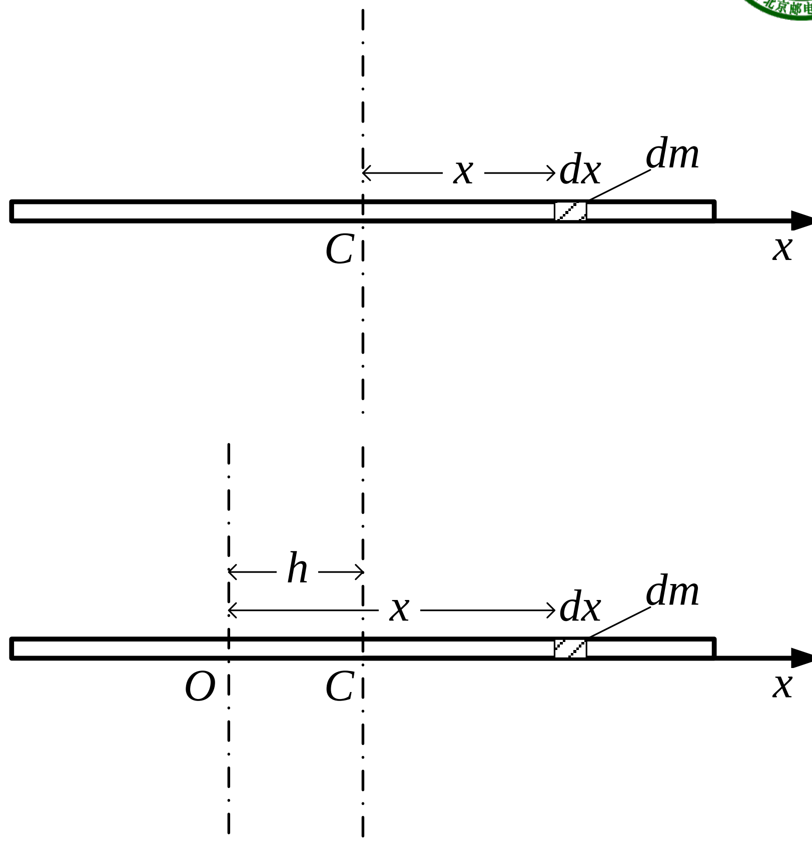
$$= \frac{1}{3} \lambda x^3 \Big|_{-l/2}^{l/2} = \frac{1}{12} M l^2$$





Example

(2) The axis locates at arbitrary distance *h* from the CM.



$$I=\int_{-(l/2-h)}^{l/2+h}\lambda x^2 dx$$

$$=\frac{1}{3}\lambda x^3\Big|_{-l/2+h}^{l/2+h}$$

$$=\frac{1}{12}Ml^2+Mh^2$$

Or $I=\int_{-l/2}^{l/2}(x+h)^2(\lambda \mathrm{d}x)=\frac{1}{12}Ml^2+Mh^2 \quad \left(\int_{-l/2}^{l/2}x\mathrm{d}x=0\right)$

The parallel-axis theorem

The Parallel-axis and Perpendicular-axis Theorems

(P249,250)

- ## The Parallel-axis Theorem

$$\boxed{I = I_{\mathrm{CM}} + Mh^2}$$

Long uniform rod of length *l*, axis through one end:

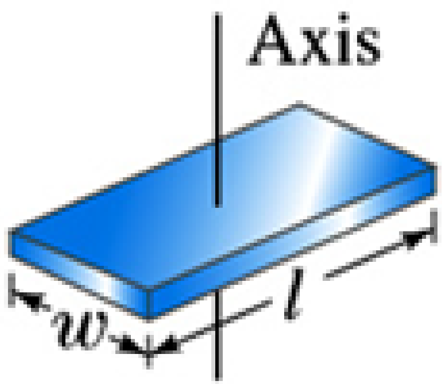
$$I_{\mathrm{end}} = I_{\mathrm{CM}} + M\left(\frac{l}{2}\right)^2 = \frac{1}{12}Ml^2 + \frac{1}{4}Ml^2 = \frac{1}{3}Ml^2$$

- **The Perpendicular-axis Theorem**
  - **The sum of the moment of inertia of a plane body about any two perpendicular axes in the plane of the body is equal to the moment of inertia about an axis through their point of intersection perpendicular to the plane of the object.**

$$\boxed{I_z = I_x + I_y}$$

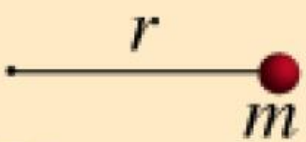
  - **Rectangular thin plate, of length l and width w.**

$$I_z = \frac{1}{12} M (l^2 + w^2)$$



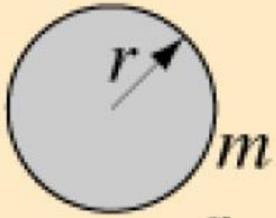
  - **Circular thin plate?**

The moment of inertia



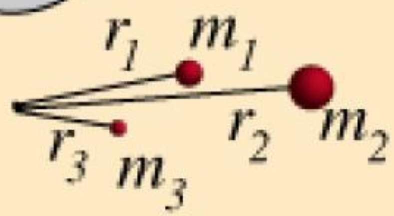
$$I = mr^2$$

For a point mass the moment of inertia is just the mass times the radius from the axis squared. For a collection of point masses (below) the moment of inertia is just the sum for the masses.



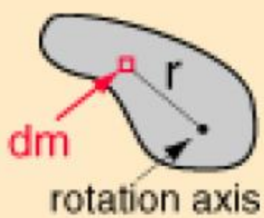
$$I = kmr^2$$

For an object with an axis of symmetry, the moment of inertia is some fraction of that which it would have if all the mass were at the radius r.



$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots$$

Sum of the point mass moments of inertia.



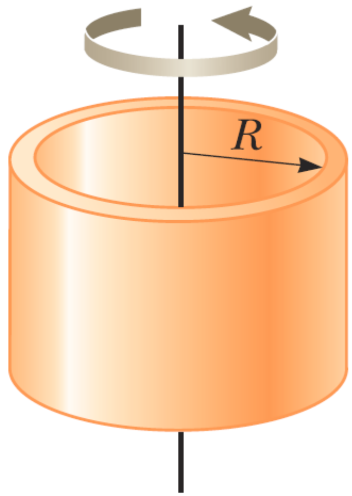
$$I = \int_0^M r^2 dm$$

Continuous mass distributions require an infinite sum of all the point mass moments which make up the whole. This is accomplished by an integration over all the mass.

The moment of inertia

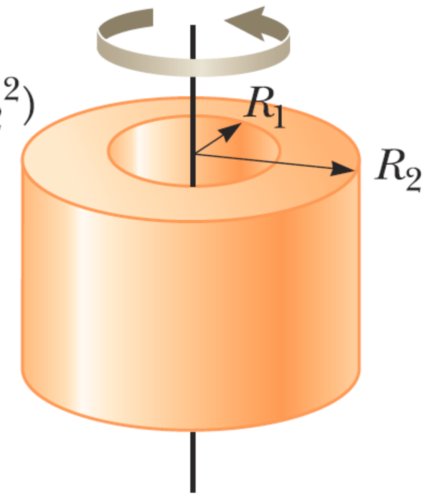
Hoop or thin cylindrical shell

$I_{CM} = MR^2$



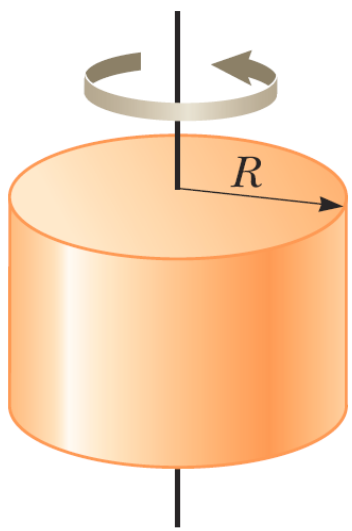
Hollow cylinder

$I_{CM} = \frac{1}{2}M(R_1{}^2 + R_2{}^2)$



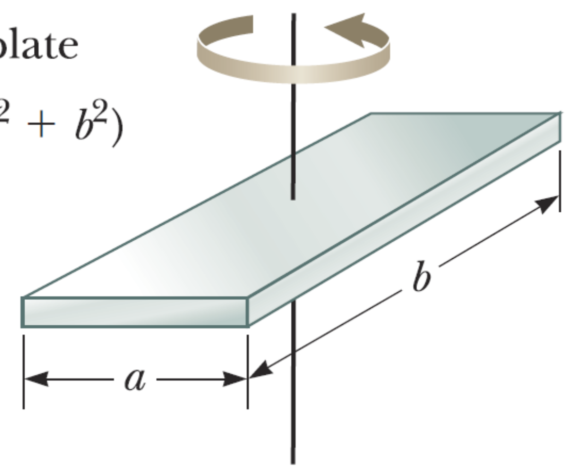
Solid cylinder or disk

$I_{CM} = \frac{1}{2}MR^2$



Rectangular plate

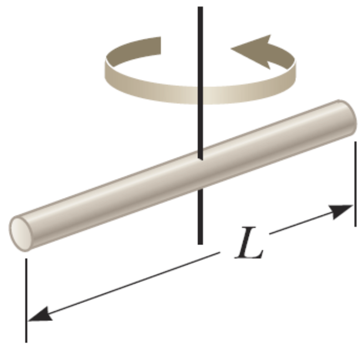
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



The moment of inertia

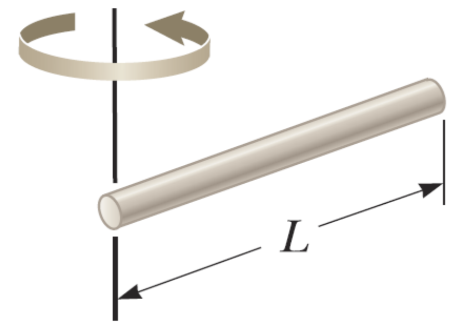
Long, thin rod with rotation axis through center

$$I_{CM} = \frac{1}{12} ML^2$$



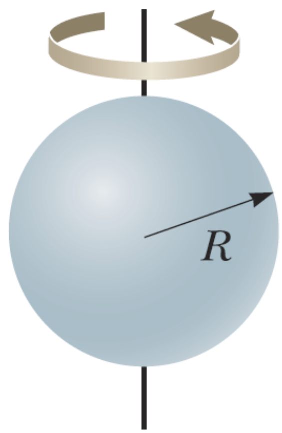
Long, thin rod with rotation axis through end

$$I = \frac{1}{3} ML^2$$



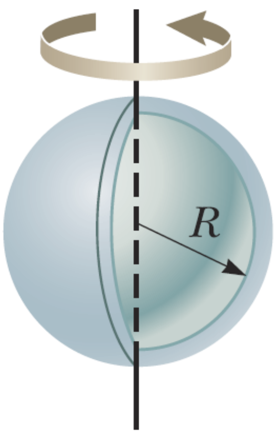
Solid sphere

$$I_{CM} = \frac{2}{5} MR^2$$



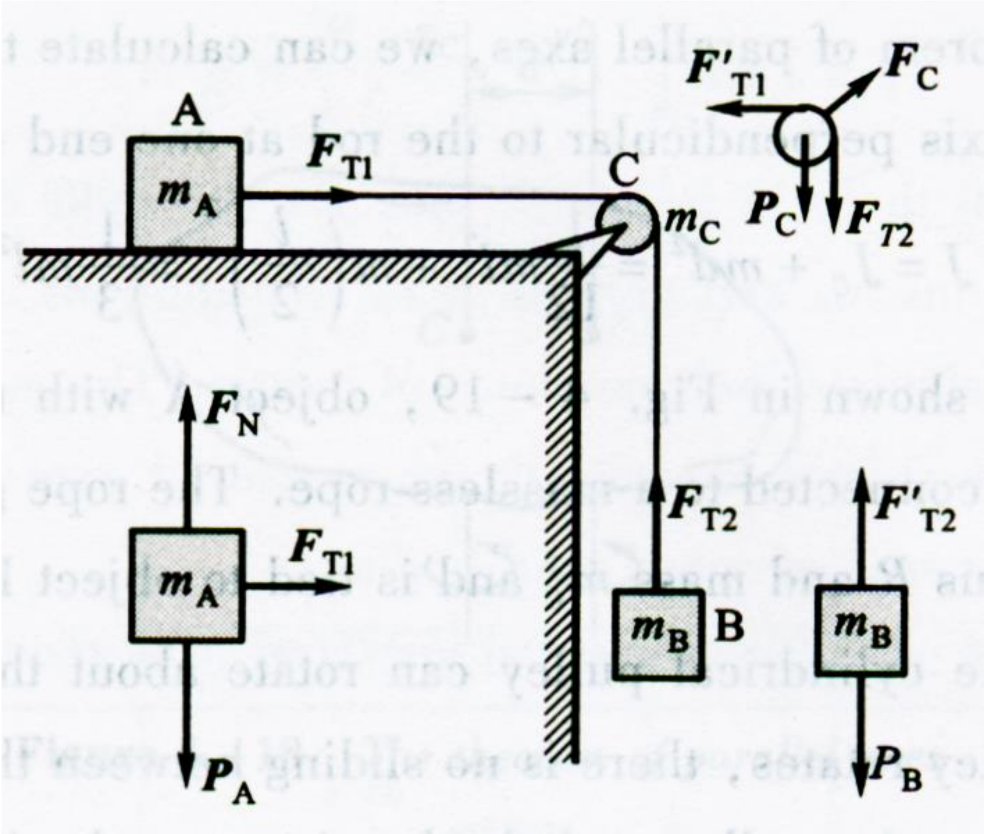
Thin spherical shell

$$I_{CM} = \frac{2}{3} MR^2$$



Two blocks and a pulley:

Two blocks of masses m_A and m_B are connected by a light cord running over a pulley. The pulley are considered as a uniform cylindrical disk of mass m_C and radius R. There is no sliding between the pulley and the cord. Find the acceleration of two blocks.



Solution

(1) Draw free-body diagrams.

(2)Newton's II law for every object:

The positive direction of rotation is clockwise.

$$F_{T1} = m_A a$$

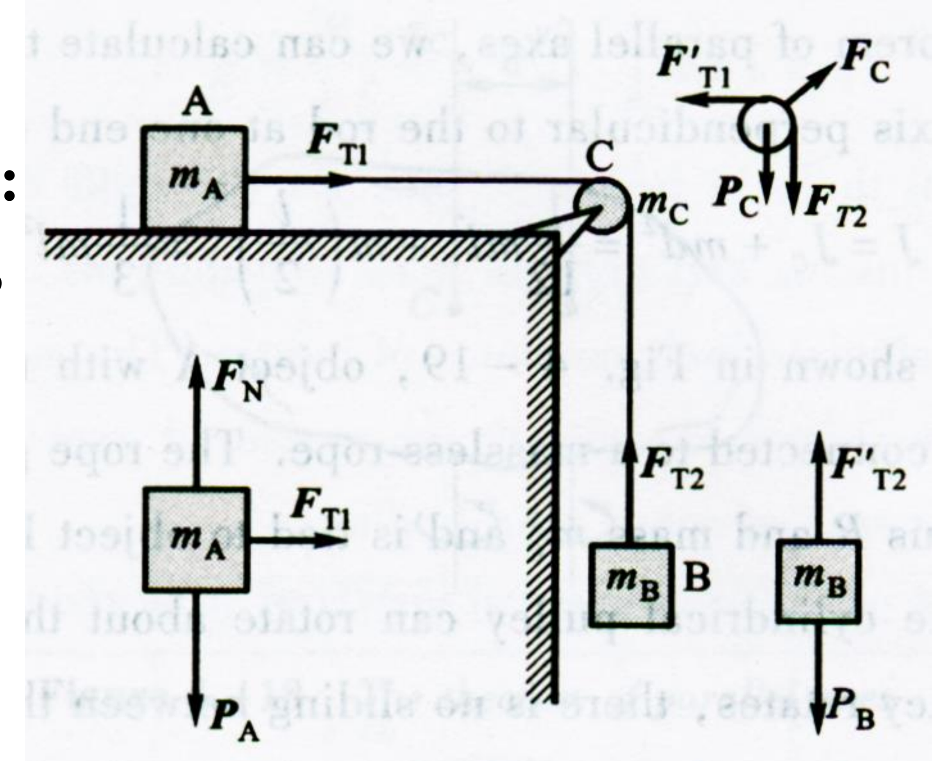
$$m_B g - F_{T2} = m_B a$$

$$RF_{T2} - RF_{T1} = \left(\frac{1}{2} m_c R^2\right)\alpha$$

4 unknowns.

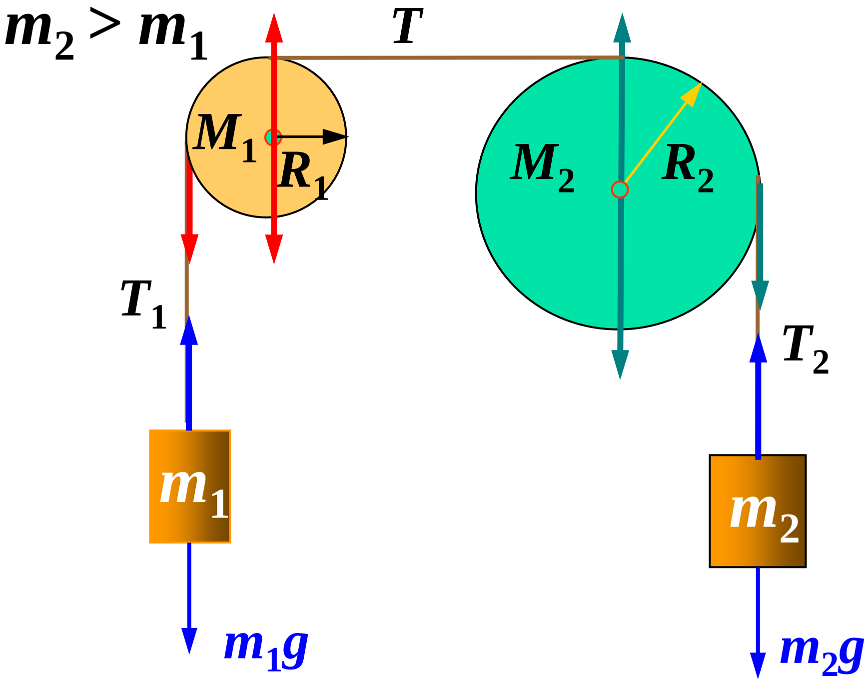
The restriction condition: no sliding between the pulley and the cord.

$$a = R\alpha$$



$$\boxed{a = \frac{m_B g}{m_A + m_B + \frac{1}{2} m_C}}$$

Example



$$T_1 - m_1 g = m_1 a$$

$$m_2 g - T_2 = m_2 a$$

$$TR_1 - T_1 R_1 = I_1 \alpha_1$$

$$T_2 R_2 - TR_2 = I_2 \alpha_2$$

$$I_1 = \frac{1}{2} M_1 R_1^2$$

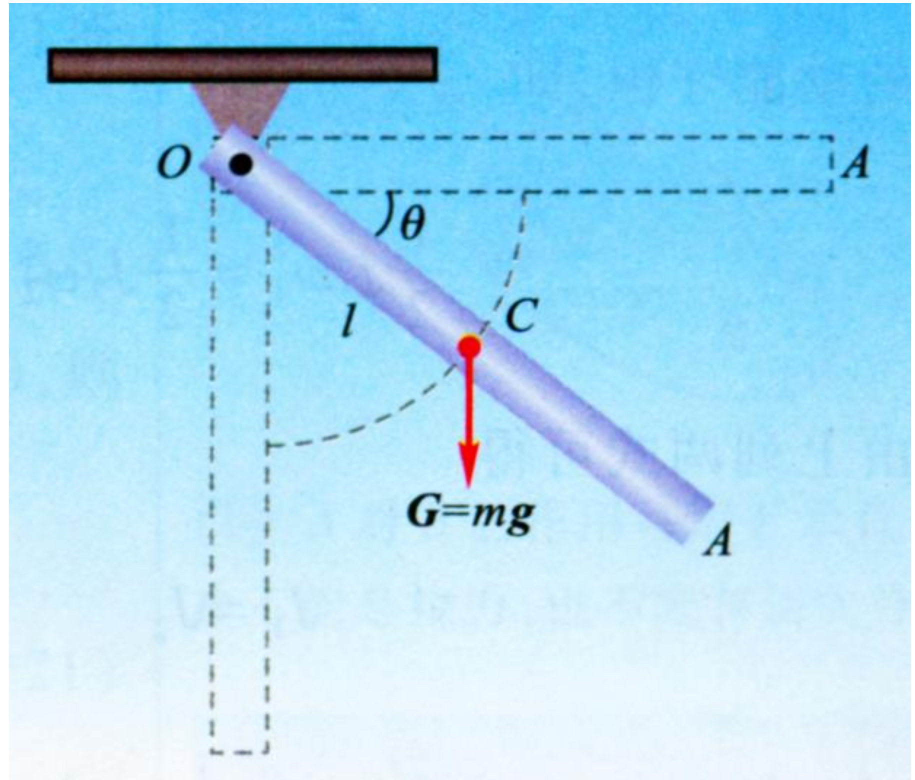
$$I_2 = \frac{1}{2} M_2 R_2^2$$

$$a = R_1 \alpha_1 = R_2 \alpha_2$$

Example

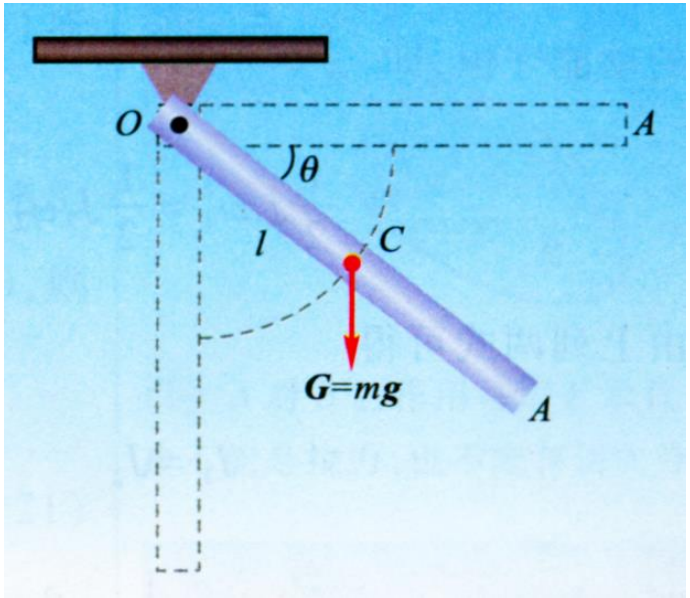
(vs. P248 Ex. 10-9)

A uniform rod of mass m and length l can pivot freely (no friction on the pivot) about a hinge to the ceiling. The rod is held horizontally and released. Determine: (1) The angular acceleration and angular velocity of the rod as the function of θ. (2) The force on the hinge exerted by the rod.



Solution: (1) Newton's II law for the rotation of rod.

$$\frac{l}{2}(mg)\cos\theta = I\alpha, \qquad I = \frac{1}{3}ml^2$$



$$\boxed{\alpha = \frac{3}{2}\frac{g}{l}\cos\theta}$$

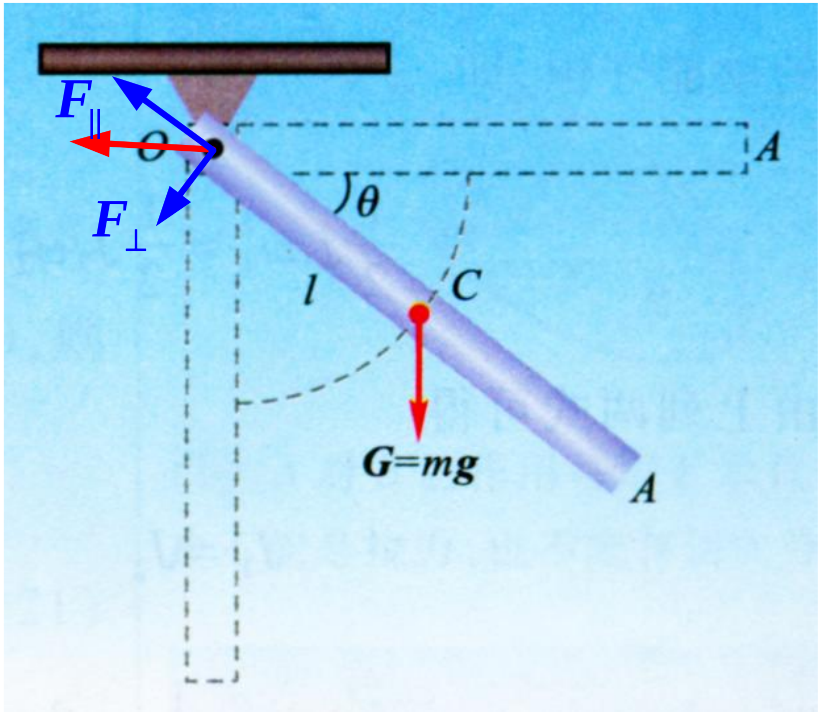
$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta} = \frac{3}{2}\frac{g}{l}\cos\theta$$

$$\int_0^{\omega}\omega\, d\omega = \frac{3}{2}\frac{g}{l}\int_0^{\theta}\cos\theta\, d\theta$$

$$\boxed{\omega = \sqrt{\frac{3g}{l}\sin\theta}}$$

Solution

$$\alpha = \frac{3}{2}\frac{g}{l}\cos\theta, \qquad \omega = \sqrt{\frac{3g}{l}\sin\theta}$$



Solution: (2) Newton's II law for the CM of the rod.

Normal: $$F_{\parallel} - mg\sin\theta = m\left(a_{\mathrm{CM}}\right)_n$$

$$= m\frac{l}{2}\omega^2$$

Tangential:

$$F_{\perp} + mg\cos\theta = m\left(a_{\mathrm{CM}}\right)_\tau$$

$$= m\frac{l}{2}\alpha$$

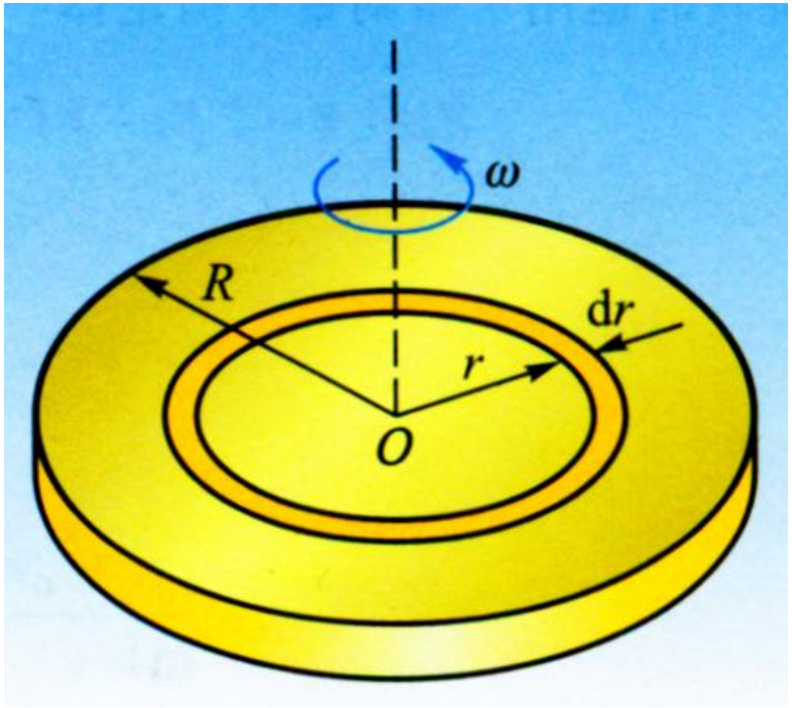
$$\boxed{F_{\parallel} = \frac{5}{2}mg\sin\theta}$$

$$\boxed{F_{\perp} = -\frac{1}{4}mg\cos\theta}$$

Check the results: $\theta = 0;\ \theta = \pi/2.$

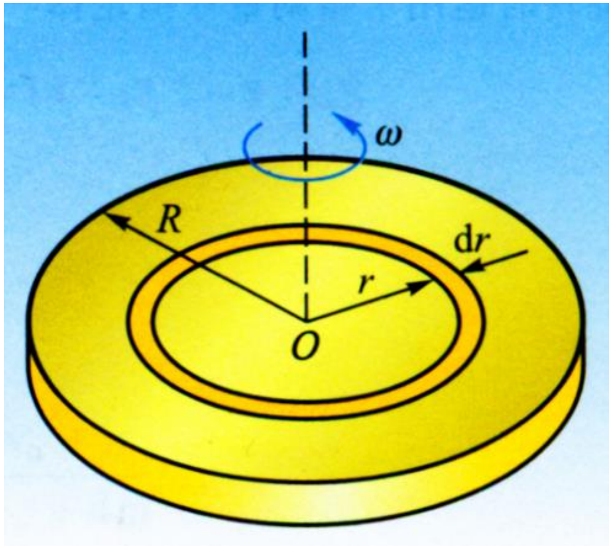
Example

A circular platform of mass M and radius R rotates initially at an angular velocity ω_0 about its central axis. Then the platform is placed on a rough horizontal surface. The coefficient of friction between the platform and the surface is μ . Determine (1) the torque acting on the platform by the friction force; (2) the time before the platform comes to a halt.



Solution

(1) **The friction force is distributed in the whole area of the platform. Divide the whole platform into many circular rings with a radius of r and width dr:**



$$dm = \sigma dS = \sigma(2\pi r)\,dr, \qquad \sigma = \frac{M}{\pi R^2}$$

$$dF_f = \mu(dm)g, \quad d\tau_f = -rdF_f = -\mu rgdm$$

$$\tau_f = -\int_m \mu rgdm = -\int_0^R \mu gr(\sigma 2\pi rdr) = -\frac{2}{3}\pi\mu gR^3\sigma = -\frac{2}{3}\mu MgR$$

(2) **The Newton's II law for rotation:** $\tau_f = I\alpha$

$$-\frac{2}{3}\mu MgR = \left(\frac{1}{2}MR^2\right)\frac{d\omega}{dt}, \qquad t = \int_0^t dt = -\frac{3R}{4\mu g}\int_{\omega_0}^0 d\omega = \frac{3R}{4\mu g}\omega_0$$

or $$t = \frac{\omega_0}{\alpha} = \frac{3R}{4\mu g}\omega_0$$

Problem

§3 The Rotational Form of Newton's Second Law

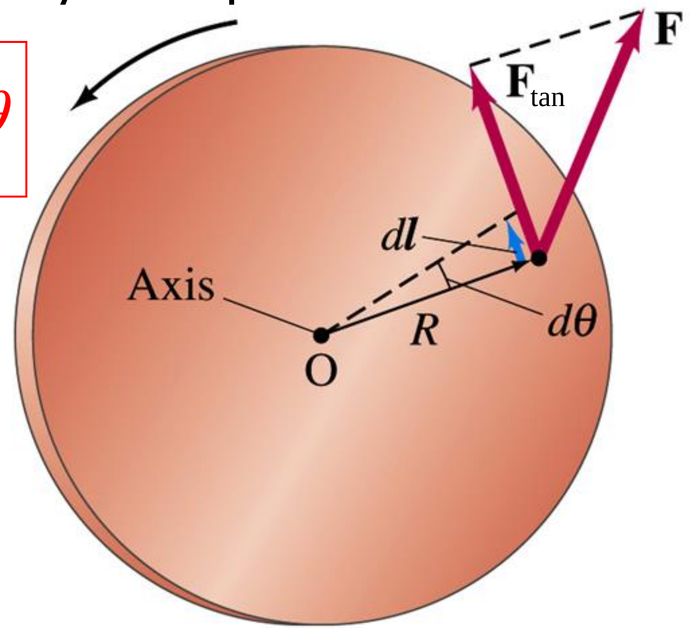
Ch10 (P266)

Prob. 17, 40, 47

§ 5 Work-Energy Theorem for a Rigid Body

- Work done by a torque
  - For a fixed axis rotation of a rigid body, the work done by a force can appear in the form of torque —— work done by a torque.

$$\boxed{W = \int_1^2 \vec{F}\cdot d\vec{l} = \int_1^2 F_{\tan} dl = \int_1^2 F_{\tan} R d\theta = \int_{\theta_1}^{\theta_2} \tau d\theta}$$



- The Power of a torque

$$\boxed{P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega}$$

- Rotational Kinetic Energy
  - For a fixed axis rotation of a rigid body, the kinetic energy can appear in another form:

$$\boxed{K = \sum_i \left(\frac{1}{2} m_i v_i^2\right) = \sum_i \left(\frac{1}{2} m_i R_i^2 \omega^2\right) = \frac{1}{2}\left(\sum_i m_i R_i^2\right)\omega^2 = \frac{1}{2} I\omega^2}$$

Work-Energy Theorem for a Rigid Body

- Work-kinetic energy theorem for a body rotating about a fixed axis
  - Starting from the rotational form of Newton's II law

$$\tau_{\text{net}} = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta}$$

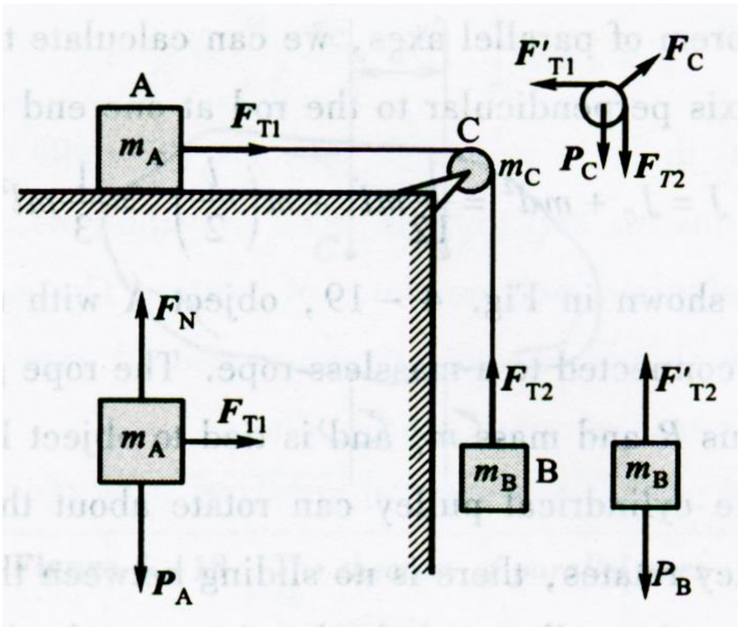
$$\boxed{W_{\text{net}} = \int_{\theta_1}^{\theta_2} \tau_{\text{net}}\, d\theta = \int_{\omega_1}^{\omega_2} I\omega\, d\omega = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2}$$

- **The work done in rotating a body through an angle $\theta_2 - \theta_1$ is equal to the change in rotational kinetic energy of the body.**

- **In addition to the work–kinetic energy theorem, other energy principles can also be applied to rotational situations.**

Example

Two blocks of masses m_A and m_B are connected by a light cord running over a pulley. The pulley are considered as a uniform cylindrical disk of mass m_C and radius R. There is no sliding between the pulley and the cord. Find the acceleration of two blocks.



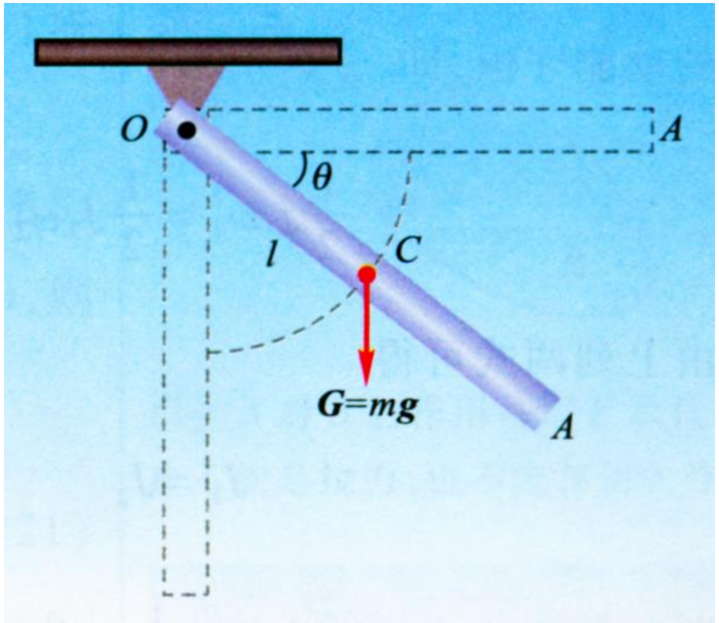
Solution (II): **Conservation of mechanical energy**

$$0=\frac{1}{2}m_A\upsilon_A^2+\frac{1}{2}m_B\upsilon_B^2+\frac{1}{2}I_C\omega_C^2-m_Bgh,\quad I_C=\frac{1}{2}m_CR^2,\quad \upsilon_A=\upsilon_B=\omega_CR$$

$$\omega_C=\frac{1}{R}\sqrt{\frac{2m_Bgh}{m_A+m_B+\frac{1}{2}m_C}},\qquad a=\frac{\mathrm{d}\upsilon}{\mathrm{d}t}=\frac{\mathrm{d}(\omega_CR)}{\mathrm{d}t}=\frac{m_Bg}{m_A+m_B+\frac{1}{2}m_C}$$

Example

A uniform rod of mass *m* and length *l* can pivot freely (no friction on the pivot) about a hinge to the ceiling. The rod is held horizontally and released. Determine the angular velocity and angular acceleration of the rod as the function of θ.



Solution (II):

Conservation of mechanical energy

$$0=\frac{1}{2}\left(\frac{1}{3}ml^2\right)\omega^2-mg\frac{l}{2}\sin\theta,\qquad \omega=\sqrt{\frac{3g}{l}\sin\theta}$$

$$\alpha=\frac{d\omega}{dt}=\frac{d}{d\theta}\left(\sqrt{\frac{3g}{l}\sin\theta}\right)\frac{d\theta}{dt}=\sqrt{\frac{3g}{l}}\frac{\cos\theta}{2\sqrt{\sin\theta}}\sqrt{\frac{3g}{l}\sin\theta}=\frac{3g}{2l}\cos\theta$$

补充讨论

系统总动能：

$$K_{sys}=\frac{1}{2}\left(\frac{1}{3}ml^2\right)\omega^2=\frac{1}{6}ml^2\omega^2$$

质心动能：

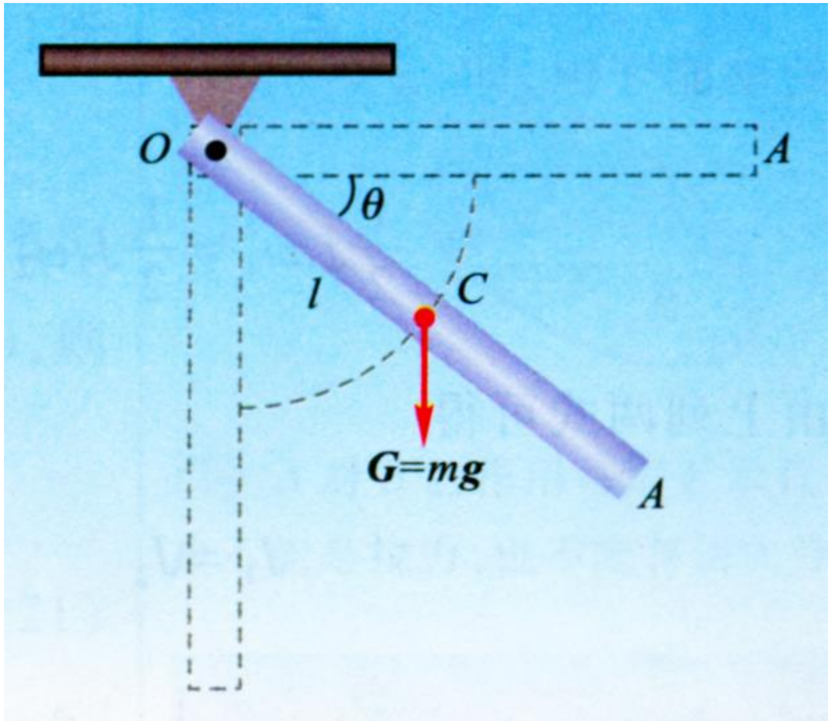
$$K_C=\frac{1}{2}mv_C^2=\frac{1}{2}m\left(\frac{l}{2}\omega\right)^2=\frac{1}{8}ml^2\omega^2$$

（平动动能）

绕质心转动动能：

$$K_{相对C}=\frac{1}{2}\left(\frac{1}{12}ml^2\right)\omega^2=\frac{1}{24}ml^2\omega^2$$

刚体转动角速度的绝对性　（转动动能）

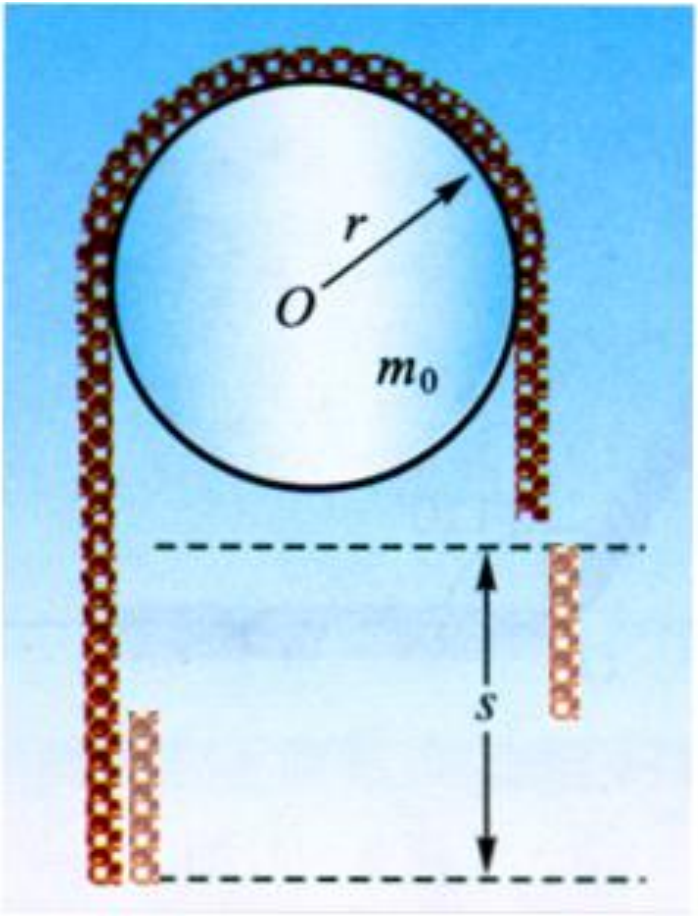


$$K_{sys}=K_C+K_{相对C}$$

柯尼希定理

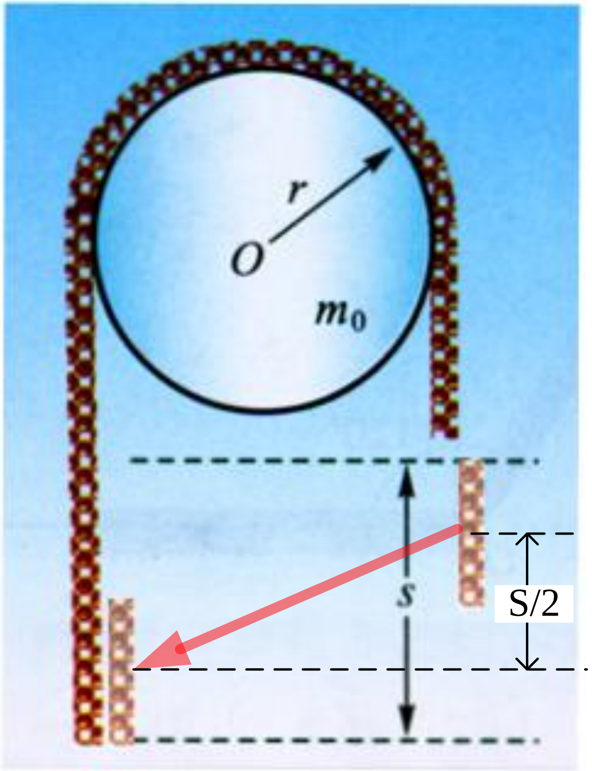
Example

A heavy steel chain of mass m and length l passes over a pulley of mass m_0 and radius r. The pulley is fixed with a frictionless pivot O. There is no slide between the chain and pulley. At beginning, the chain passes over the pulley with the lengths of both side equal. And then with a small perturbation, the chain slides to the left. Find the velocity and acceleration of the chain when the height difference of two end is s.



Solution

Take the chain, the pulley and the Earth as a system, the mechanical energy of the system is conserved.



$$0=\frac{1}{2}mv^2+\frac{1}{2}\left(\frac{1}{2}m_0r^2\right)\omega^2-\left(\frac{m}{l}\frac{s}{2}\right)g\frac{s}{2}$$

$$v=\omega r,\qquad v=\sqrt{\frac{mgs^2}{2\left(m+\frac{1}{2}m_0\right)l}}$$

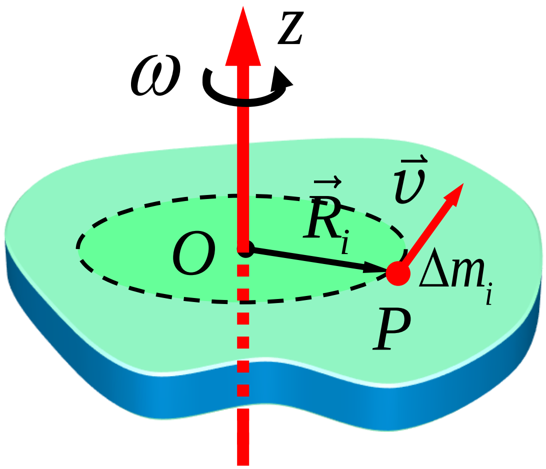
The acceleration: $$a=\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=2v\frac{dv}{ds}$$

$$=2\sqrt{\frac{mgs^2}{2\left(m+\frac{1}{2}m_0\right)l}}\cdot\sqrt{\frac{mg}{2\left(m+\frac{1}{2}m_0\right)l}}=\frac{mgs}{\left(m+\frac{1}{2}m_0\right)l}$$

§ 6 Angular Momentum for a Rigid Body (P281)

- **The total angular momentum L is the vector sum of l_i for each particle of the rigid body.**

$$l_{iz} = R_i\left(\Delta m_i \upsilon_i\right) = R_i\left(\Delta m_i\right)\left(R_i\omega\right) = \left(\Delta m_i R_i^2\right)\omega$$



(about a fixed axis)

Sum over all the particles:

$$\boxed{L_z = \sum_i l_{iz} = \left(\sum_i \Delta m_i R_i^2\right)\omega = I\omega}$$

Angular Momentum for a Rigid Body

- Rotational Form of Newton's II Law
  - Starting from the Torque-angular momentum theorem.

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \quad \Rightarrow \quad \boxed{\sum \tau_{\text{ext-axis}} = \frac{dL_z}{dt}} = \frac{d}{dt}(I\omega) = I\alpha$$

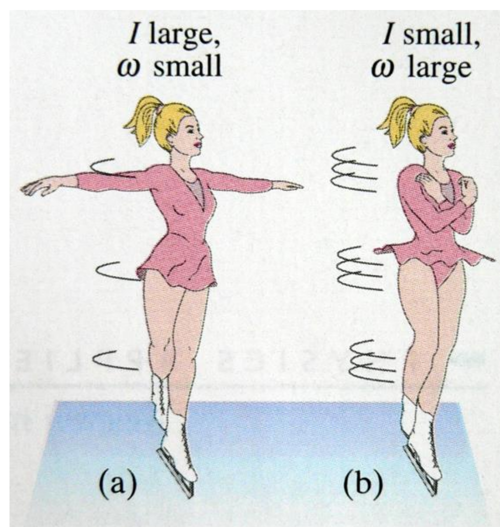
    - The Rotational Form of Newton's II Law can be considered as a special case of Torque-angular momentum theorem for a rigid body rotation about a fixed axis.

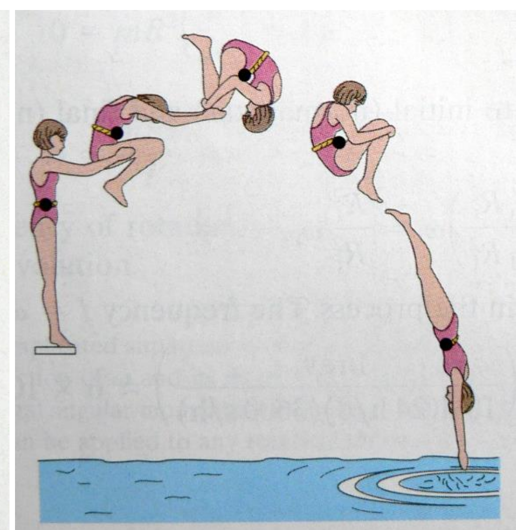
- The Conservation of Angular Momentum for Rigid Body
  - The total angular momentum of rotating body remains constant if the net external torque acting on it is zero.

If $\sum \tau_{\text{ext-axis}} = 0$

$$\Downarrow$$

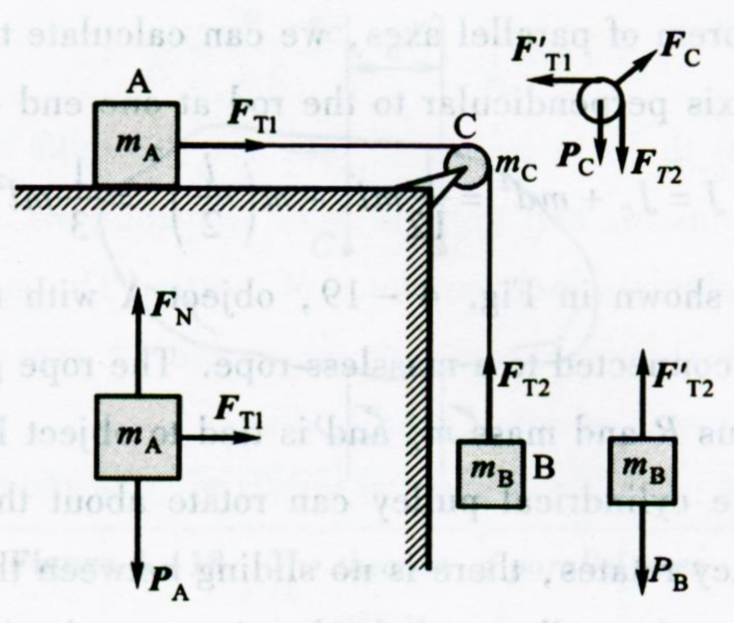
$$\boxed{I\omega = I_0\omega_0}$$





Example

Two blocks of masses m_A and m_B are connected by a light cord running over a pulley. The pulley are considered as a uniform cylindrical disk of mass m_C and radius R. There is no sliding between the pulley and the cord. Find the acceleration of two blocks.



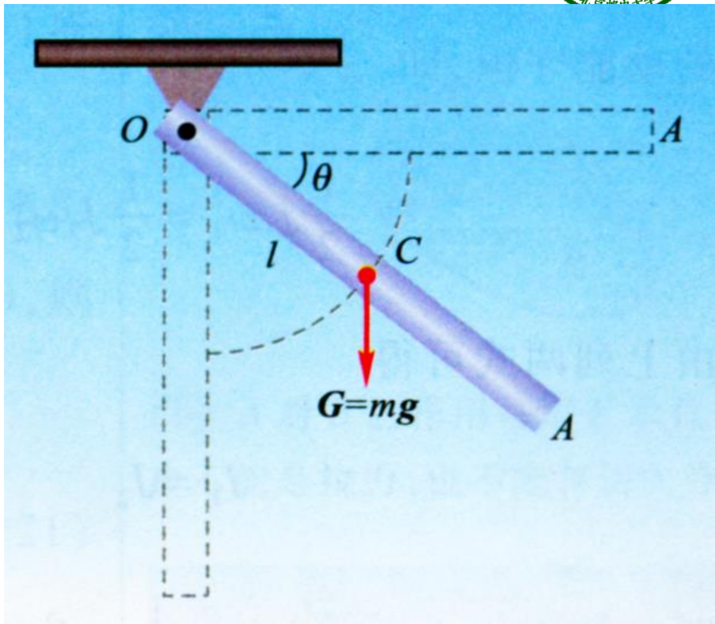
Solution (III): Torque-angular momentum theorem

$$\sum \tau_{\text{ext}} = \frac{dL_{\text{tot}}}{dt}, \quad \tau_{\text{ext}} = R(m_B g), \quad L_{\text{tot}} = R(m_A \upsilon_A) + R(m_B \upsilon_B) + I_C \omega_C$$

$$\upsilon = \upsilon_A = \upsilon_B = \omega_C R, \quad I_C = \frac{1}{2} m_C R^2$$

$$m_B g R = \frac{d}{dt}\left[\left(m_A + m_B + \frac{1}{2} m_C\right)\upsilon R\right], \quad a = \frac{d\upsilon}{dt} = \frac{m_B g}{m_A + m_B + \frac{1}{2} m_C}$$

A uniform rod of mass m and length l can pivot freely (no friction on the pivot) about a hinge to the ceiling. The rod is held horizontally and released. Determine the angular acceleration of the rod as the function of θ.



Solution (III):

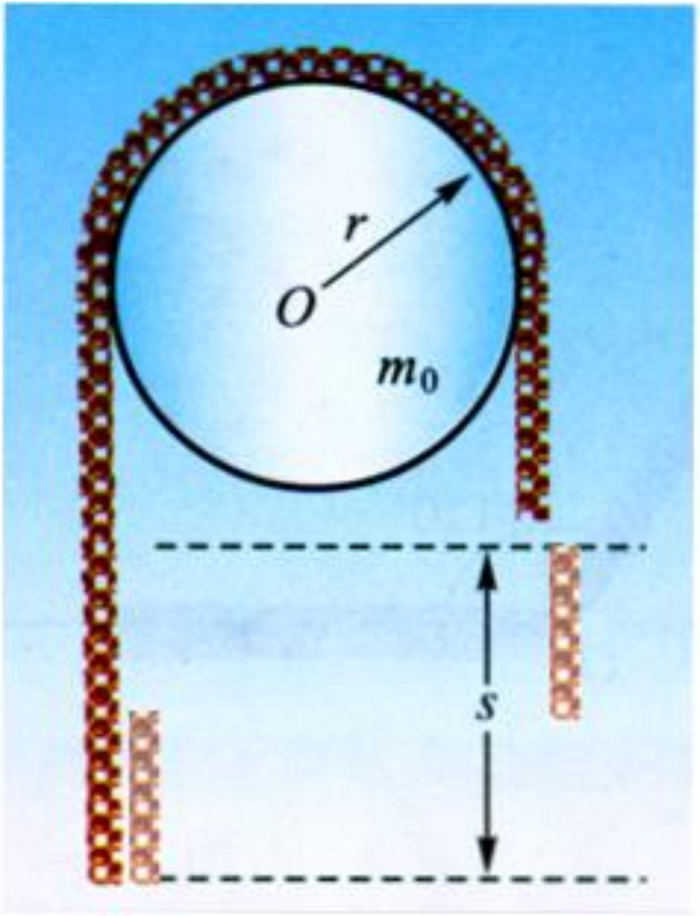
Torque-angular momentum theorem

$$\sum \tau_{\text{ext}} = \frac{dL_{\text{tot}}}{dt}, \qquad \tau_{\text{ext}} = \frac{l}{2}(mg)\cos\theta, \qquad L_{\text{tot}} = \left(\frac{1}{3}ml^2\right)\omega$$

$$\frac{l}{2}(mg)\cos\theta = \left(\frac{1}{3}ml^2\right)\frac{d\omega}{dt}, \qquad \alpha = \frac{d\omega}{dt} = \frac{3g}{2l}\cos\theta$$

Example

A heavy steel chain of mass m and length l passes over a pulley of mass m_0 and radius r. The pulley is fixed with a frictionless pivot O. There is no slide between the chain and pulley. At beginning, the chain passes over the pulley with the lengths of both side equal. And then with a small perturbation, the chain slides to the left. Find the **acceleration** of the chain when the height difference of two end is s. (Using the **Torque-angular momentum theorem**.)



Example

Solution: Torque-angular momentum theorem

$$\tau = \frac{dL}{dt},$$

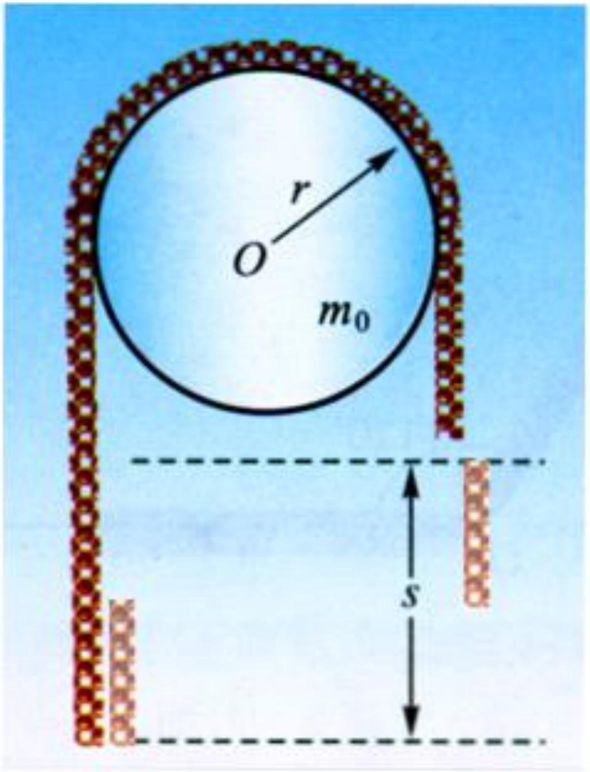
$$\tau = r\left(\frac{s}{l}mg\right)$$

$$L = r(mv) + \left(\frac{1}{2}m_0 r^2\right)\omega$$

$$v = \omega r$$

$$r\left(\frac{s}{l}mg\right) = \frac{d}{dt}\left[\left(mr + \frac{1}{2}m_0 r\right)v\right]$$

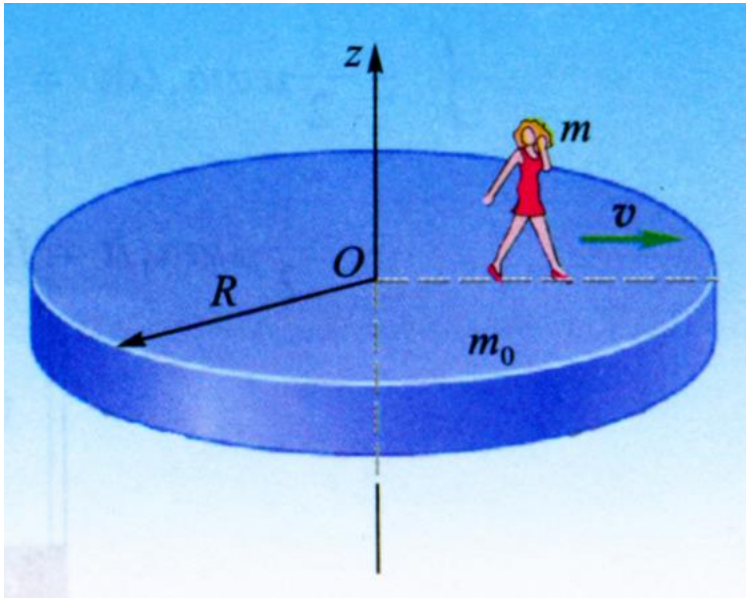
$$= \left(mr + \frac{1}{2}m_0 r\right)\frac{dv}{dt}$$



The acceleration: $$a = \frac{dv}{dt} = \frac{r\frac{s}{l}mg}{mr + \frac{1}{2}m_0 r} = \frac{mgs}{\left(m + \frac{1}{2}m_0\right)l}$$

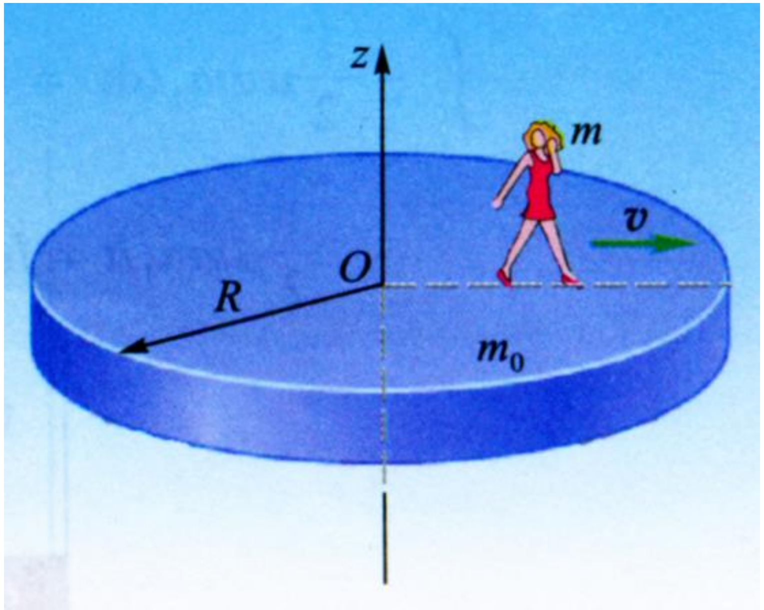
Example

A circular platform of mass m_0 and radius R rotates friction-free about an axis through its center. A woman of mass m standing on the platform a distance $R/2$ from the center. At beginning, the system of platform and woman rotates at the angular velocity ω_0 about the axis. The woman starts to walk to the edge of the platform. Determine the final angular velocity ω of the system when the woman arrives at the edge.



Solution

In the whole process that the woman walk to the edge of platform, the external torque is zero. Using the conservation of angular momentum of the system:



Initial state:

$$L_i = \left(\frac{1}{2}m_0R^2\right)\omega_0 + m\left(\frac{R}{2}\right)^2\omega_0$$

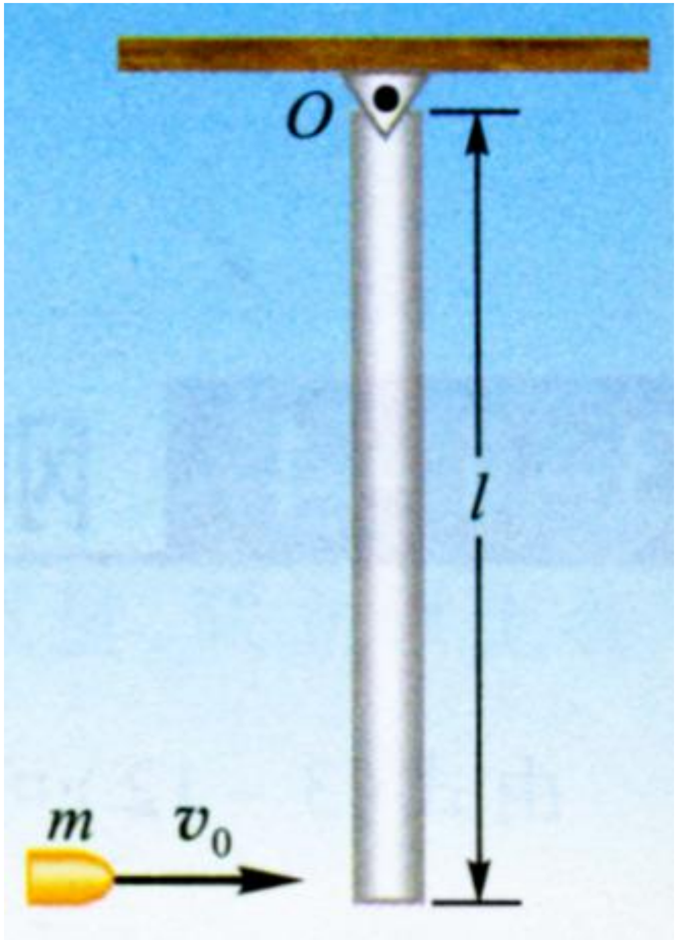
Final state:

$$L_f = \left(\frac{1}{2}m_0R^2\right)\omega + mR^2\omega$$

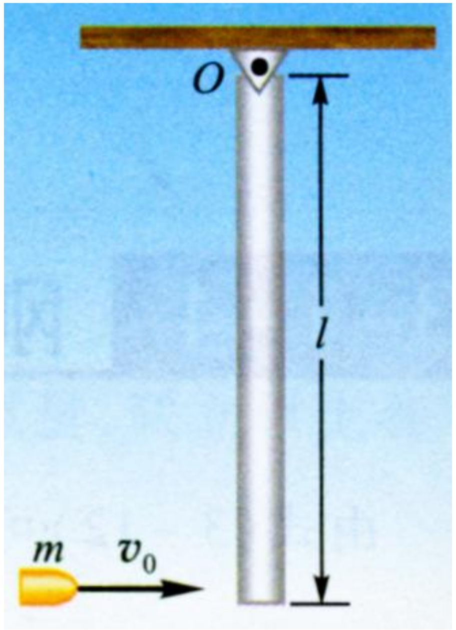
$$L_i = L_f \Rightarrow \boxed{\omega = \frac{2m_0 + m}{2m_0 + 4m}\omega_0}$$

Example

A rod of mass M and length l can rotate about pivot O freely, a bullet of mass m and speed v_0 is shot into the lower end of the rod and embedded in the rod. What is the angle θ when the rod swings to its highest position?



Solution



(i) Take the bullet and the rod as a system.

The external forces: the constraint force exerted by the pivot; gravity. They go through the origin O. So the external torque about O is zero, and the angular momentum of the system should be conserved in the process of shouting.

$$l\left(mv_0\right)=\left(\frac{1}{3}Ml^2+ml^2\right)\omega,\qquad \omega=\frac{3mv_0}{(M+3m)l}$$

(ii) Take the bullet, the rod and the Earth as a system. In the process of the system swinging up, the mechanical energy is conserved.

$$\frac{1}{2}\left(\frac{1}{3}Ml^2+ml^2\right)\omega^2=mgl(1-\cos\theta)+Mg\frac{l}{2}(1-\cos\theta)$$

$$\cos\theta=1-\frac{3m^2}{(M+3m)(M+2m)}\frac{v_0^2}{gl}$$

Problem

§ 5 Work-Energy Theorem for a Rigid Body

Ch10 (P270): 65

§ 6 Angular Momentum for a Rigid Body

Ch10 (P270): 60, 68