Ch 3.5 Frequency Representation of LTI systems

Prof. Shaoshi Yang

E-mail: shaoshi.yang@bupt.edu.cn

Web: https://teacher.bupt.edu.cn/yangshaoshi

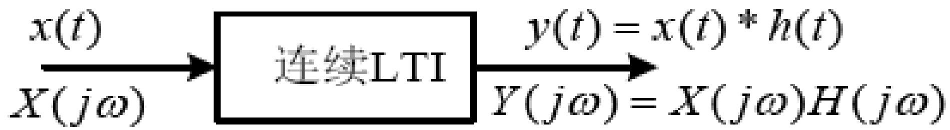
School of Information & Communication Engineering

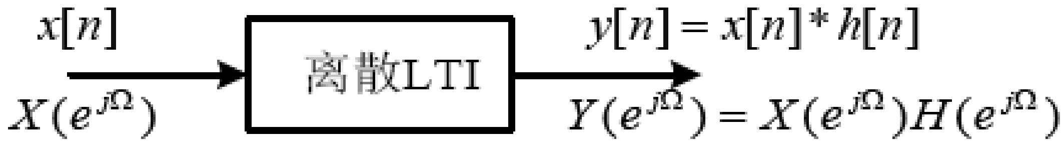
BUPT

Outline

- Frequency Representations of LTI system
  - Frequency response of LTI systems
  - Representations and solutions of LTI systems in frequency domain
  - Filtering

Frequency Response of LTI System





- **For CT system:** $H(j\omega)=\int_{-\infty}^{\infty}h(t)e^{-j\omega t}dt=\dfrac{Y(j\omega)}{X(j\omega)}$
- **For DT system:** $H(e^{j\Omega})=\sum_{n=-\infty}^{\infty}h[n]e^{-j\Omega n}=\dfrac{Y(e^{j\Omega})}{X(e^{j\Omega})}$

- The convolution property implies that the frequency response of a system may be expressed as the ratio of the FT or DTFT of the output to the input.

Frequency Response of LTI System

Example 3.34 ***Identifying a System, Given Its Input and Output***

The output of an LTI system in response to an input $x(t) = e^{-2t}u(t)$ is $y(t) = e^{-t}u(t)$. Find the frequency response and the impulse response of this system.

<Sol.>

$$X(j\omega) = \frac{1}{j\omega + 2}$$

$$Y(j\omega) = \frac{1}{j\omega + 1}$$

$$\Rightarrow \quad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 2}{j\omega + 1} = 1 + \frac{1}{j\omega + 1}$$

$$\Rightarrow \quad h(t) = \delta(t) + e^{-t}u(t)$$

- **System equation in terms of differential equation**

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

$$\Rightarrow \sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Example **The LTI system is**

$$y''(t)+3y'(t)+2y(t)=x'(t)+4x(t) \qquad x(t)=\mathrm{e}^{-3t}u(t)$$

Find (1) impulse response $h(t)$ of the system; (2) the output $y_{zs}(t)$ in response to the input $x(t)$.

<Sol.> $(\mathrm{j}\omega)^2 Y(j\omega)+3\mathrm{j}\omega Y(\mathrm{j}\omega)+2Y(\mathrm{j}\omega)=\mathrm{j}\omega X(\mathrm{j}\omega)+4X(\mathrm{j}\omega)$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{4+\mathrm{j}\omega}{(\mathrm{j}\omega)^2+3(\mathrm{j}\omega)+2}=\frac{-2}{\mathrm{j}\omega+2}+\frac{3}{\mathrm{j}\omega+1}$$

$$\Rightarrow \quad h(t)=-2\mathrm{e}^{-2t}u(t)+3\mathrm{e}^{-t}u(t)$$

$$Y_{zs}(j\omega)=X(j\omega)H(j\omega)=\frac{1}{\mathrm{j}\omega+3}\bullet\frac{\mathrm{j}\omega+4}{(\mathrm{j}\omega+2)(\mathrm{j}\omega+1)}$$

$$=\frac{1/2}{j\omega+3}+\frac{-2}{j\omega+2}+\frac{3/2}{j\omega+1}$$

$$\Rightarrow \quad y_{zs}(t)=\frac{1}{2}e^{-3t}u(t)-2e^{-2t}u(t)+\frac{3}{2}e^{-t}u(t)$$

Example **The LTI system is**

$$y[n] = x[n] + x[n\text{-}1] + x[n\text{-}2]$$

Find (1) impulse response $h[n]$ of the system; (2) frequency response $H(e^{j\Omega})$ of the system.

<Sol.>

$$\left(1 + e^{-j\Omega} + e^{-j2\Omega}\right) X\left(e^{j\Omega}\right) = Y\left(e^{j\Omega}\right)$$

$$\Rightarrow \quad H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = 1 + e^{-j\Omega} + e^{-j2\Omega}$$

$$\Rightarrow \quad h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

Filtering (滤波)

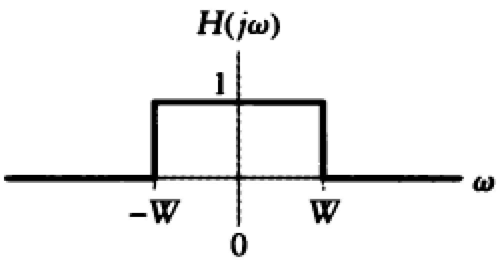
$$y(t)=h(t)*x(t)\xleftrightarrow{FT}Y(j\omega)=X(j\omega)H(j\omega)$$

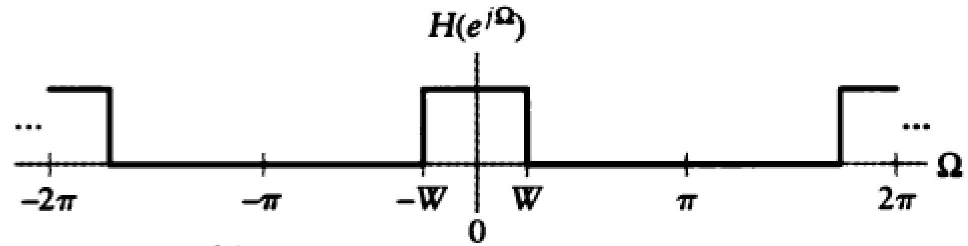
$$y[n]=x[n]*h[n]\xleftrightarrow{DTFT}Y(e^{j\Omega})=X(e^{j\Omega})H(e^{j\Omega})$$

- **Filtering ↔ Multiplication in frequency domain**
  - **The term “filtering” implies that some frequency components of the input are eliminated while others are passed by the system unchanged.**
- **System Types of filtering:**
  - **Low-pass filter (LPF)**
  - **High-pass filter (HPF)**
  - **Band-pass filter (BPF)**

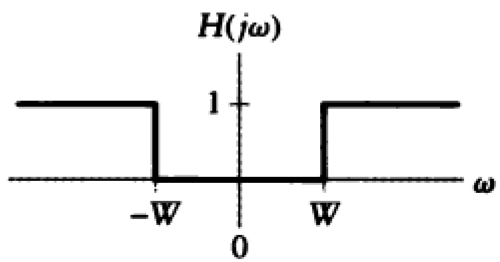
Filtering

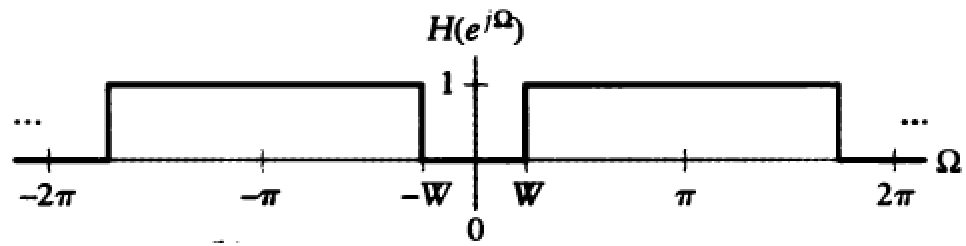
- Low-pass filter



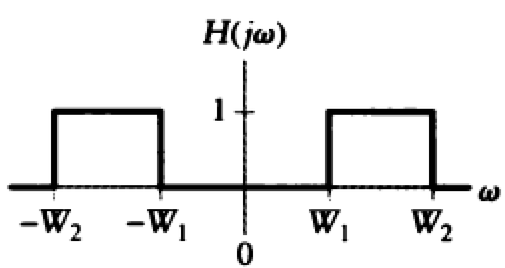


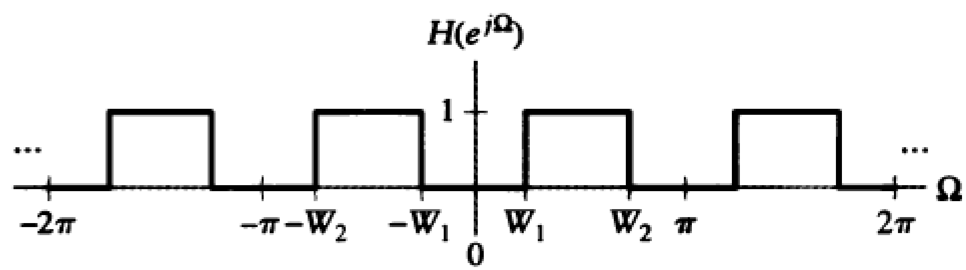
- High-pass filter





- Band-pass filter





Filtering

- **Passband (通带)** of a filter: the band of frequencies that are passed by the system.
- **Stopband (阻带)** of a filter: the range of frequencies that are attenuated by the system.
- Realistic filter has gradual **transition band (过渡带)**, and nonzero gain of stop band.
- Magnitude response of filter: $20\log|H(j\omega)|$ or $20\log|H(e^{j\Omega})|$ **[dB]**
  - ♣ **Unity gain = 0dB**
- The edge of the passband is usually defined by the frequencies for which the response is **–3dB**, corresponding to a magnitude response of ($1/\sqrt{2}$).

Energy spectrum of filter output: $|Y(j\omega)|^2 = |H(j\omega)|^2 |X(j\omega)|^2$

The –3dB point corresponds to frequencies at which the filter passes only half of the input power.

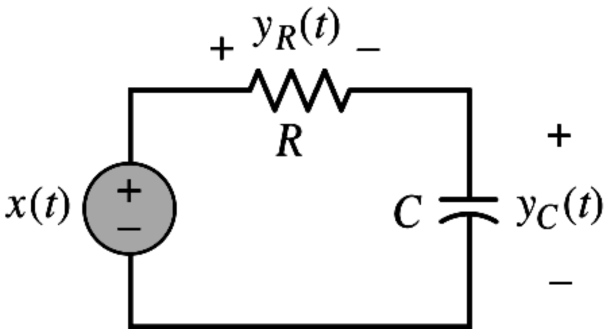
–3dB point ⟹ **Cutoff frequency (截止频率)**

Filtering

Example 3.33 ***RC Circuit: Filtering***

For the *RC* circuit depicted in Fig. 3.54, the impulse response for the case where $y_C(t)$ is the output is given by

$$h_C(t) = \frac{1}{RC} e^{-t/RC} u(t)$$



Since $y_R(t) = x(t) - y_C(t)$, the impulse response for the case where $y_R(t)$ is the output is given by

$$h_R(t) = \delta(t) - \frac{1}{RC} e^{-t/RC} u(t)$$

Plot the magnitude responses of both systems on a linear scale and in dB, and characterize the filtering properties of the systems.

<Sol.>

- **Frequency response corresponding to $h_C(t)$:** $H_C(j\omega) = \dfrac{1}{j\omega RC + 1}$

- **Frequency response corresponding to $h_R(t)$:** $H_R(j\omega) = \dfrac{j\omega RC}{j\omega RC + 1}$

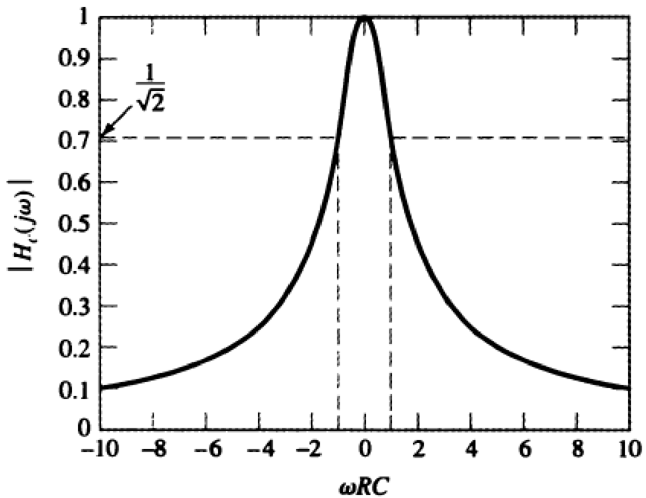
Filtering

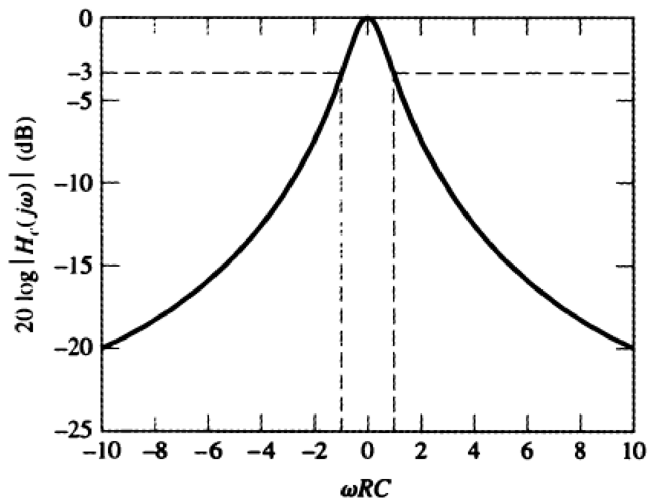
$$H_C(j\omega) = \frac{1}{j\omega RC + 1}$$

Low-pass filter

Cutoff frequency:

$\omega_c = 1/(RC)$





$$H_R(j\omega) = \frac{j\omega RC}{j\omega RC + 1}$$

High-pass filter

Cutoff frequency:

$\omega_c = 1/(RC)$