Lecture 16

Chapter 7 Quadratic Form and Applications

7.1 Quadratic Form and its Matrix Representation

Quadratic polynomials in 1 variable

$$p(x) = ax^2 + bx + c$$

Quadratic polynomials in 2 variables

$$p(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + b_1x_1 + b_2x_2 + c$$

If we introduce

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

symmetric

then

$$p(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + c.$$

Quadratic polynomials in 3 variables

$$p(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2$$
$$+2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$
$$+b_1x_1 + b_2x_2 + b_3x_3 + c$$

If we introduce

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

symmetric

then
$$p(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + c.$$

Quadratic polynomials in n variables

$$p(x_1, \dots, x_n) = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2$$
$$+2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots + 2a_{n-1,n}x_{n-1}x_n$$
$$+b_1x_1 + b_2x_2 + \cdots + b_nx_n + c$$

If we denote **symmetric**

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

then quadratic polynomial of n variables can be written as

$$p(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + c.$$

Notice that A is a **symmetric** matrix.

Definition 1. A **quadratic form [二次型]** or **purely quadratic real function [纯二次实值函数]** is a homogeneous polynomial of degree two in n variables, which is denoted by

$$q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x},$$

where A is a real $n \times n$ **symmetric** matrix and $\boldsymbol{x} \in \mathbf{R}^n$.

Example. (the number of variables $n = 1,2,3$)

Unary [一元二次型]: $q(x) = ax^2$;

Binary [二元二次型]: $q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$;

Ternary [三元二次型]: $q(x_1, x_2, x_3) = ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2ex_2x_3 + 2fx_1x_3$.

Exercises.

(1) Find the matrix associated with the following quadratic form

$$q(x,y,z) = x^2 - y^2 + 3z^2 + 2xy + 4yz - 3zx.$$

(2) Find the quadratic form associated with the following symmetric matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 3 & 5 \\ -2 & 5 & 10 \end{pmatrix}.$$

Definition 2. Let $q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$ be a quadratic form and the coefficient matrix A be a real $n \times n$ symmetric matrix. We say

(1) $q(x)$ and A are **definite [定的]** if $q(\boldsymbol{x}) \geq 0$ or $q(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in \mathbf{R}^n$; otherwise, they are **indefinite [不定的];**

(2) $q(x)$ and A are **positive definite [正定的]** if $q(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$;

(3) $q(x)$ and A are **positive semidefinite [半正定的]** if $q(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbf{R}^n$;

(4) $q(x)$ and A are **negative definite [负定的]** if $q(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \neq \mathbf{0}$;

(5) $q(x)$ and A are **positive semidefinite [半负定的]** if $q(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in \mathbf{R}^n$.

Example 1. Let $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. Show that $q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$ is positive definite.

Proof. (I. Factorization**)** Notice that

$$q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} = (x_1, x_2) \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 3x_1^2 - 2x_1 x_2 + 3x_2^2$$

$$= (x_1 + x_2)^2 + 2(x_1 - x_2)^2,$$

so that, $q(\boldsymbol{x}) > 0$ for any $\boldsymbol{x} \neq \boldsymbol{0}$.

Therefore, $q(\boldsymbol{x})$ is positive definite.

Example 1. Let $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. Show that $q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$ is positive definite.

Proof. (II. Diagonalization**)** Consider the characteristic polynomial of matrix A

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4).$$

Therefore, A has two eigenvalues $\lambda_1 = 2, \lambda_2 = 4$.

For $\lambda_1 = 2$, we have to solve $(A - 2I)x = 0$

$$A - 2I = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

and $$N(A - 2I) = \mathrm{Span}\{(1,1)^T\}.$$

We can take $\boldsymbol{x_1} = (1,1)^T$ as the eigenvector belonging to the eigenvalue $\lambda_1 = 2$.

Example 1. Let $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. Show that $q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$ is positive definite.

Proof. (II. Diagonalization, continue**)** For $\lambda_2 = 4$, we have

$$A - 4I = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

and

$$N(A - 4I) = \text{Span}\{(-1,1)^T\}.$$

We can take $\boldsymbol{x_2} = (-1,1)^T$ as the eigenvector belonging to the eigenvalue $\lambda_2 = 4$.

Let

$$\boldsymbol{v_1} = \left(\frac{1}{\|\boldsymbol{x_1}\|}\right)\boldsymbol{x_1} = \frac{1}{\sqrt{2}}(1,1)^T,$$

$$\boldsymbol{v_2} = \left(\frac{1}{\|\boldsymbol{x_2}\|}\right)\boldsymbol{x_2} = \frac{1}{\sqrt{2}}(-1,1)^T$$

Then $\boldsymbol{v_1}, \boldsymbol{v_2}$ are eigenvectors of A and they form an **orthonormal** set.

Example 1. Let $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. Show that $q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$ is positive definite.

Proof. (II. Diagonalization, continue**)** The **orthogonal** matrix

$$S = (\boldsymbol{v_1}, \boldsymbol{v_2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

diagonalizes the matrix A:

$$\Lambda = S^{-1} A S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= S^T A S \quad = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

Example 1. Let $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. Show that $q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$ is positive definite.

Proof. (II. Diagonalization, continue**)**

$$\Lambda = S^T A S = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

If we take $\boldsymbol{x} = S\boldsymbol{y}$, or equivalently $\boldsymbol{y} = S^T\boldsymbol{x}$, the quadratic form $q(\boldsymbol{x})$ can be changed into

$$q(\boldsymbol{x}) = (S\boldsymbol{y})^T A (S\boldsymbol{y}) = \boldsymbol{y}^T(S^T A S)\boldsymbol{y} = \boldsymbol{y}^T \Lambda \boldsymbol{y} = \bar{q}(\boldsymbol{y}),$$

and $\bar{q}(\boldsymbol{y}) = 2y_1^2 + 4y_2^2$. It is clear that $\bar{q}(\boldsymbol{y}) > 0$ for all $\boldsymbol{y} \neq 0$. Therefore, $q(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$, q is a positive definite quadratic form.

$\bar{q}(\boldsymbol{y}) = 2y_1^2 + 4y_2^2$**: Standard form [标准型]**

Example 2. Let $A_2 = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$. Then the quadratic form $q(\boldsymbol{x}) = \boldsymbol{x}^T A_2 \boldsymbol{x}$ is negative definite.

Example 3. Let $A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$. Then the quadratic form $q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$ is indefinite.

7.2 Diagonalization of Real Symmetric Matrices

Theorem 1. Let A be a real $n \times n$ symmetric matrix and λ is one of its eigenvalues, then λ must be a real number.

Proof. $$A\boldsymbol{x} = \lambda\boldsymbol{x}, \qquad \boldsymbol{x} \neq \boldsymbol{0}.$$

Then $$A\overline{\boldsymbol{x}} = \bar{A}\overline{\boldsymbol{x}} = \overline{A\boldsymbol{x}} = \overline{\lambda\boldsymbol{x}} = \bar{\lambda}\overline{\boldsymbol{x}},$$

which means that $\bar{\lambda}$ is also an eigenvalue of A with eigenvector $\overline{\boldsymbol{x}}$.

$$(A\overline{\boldsymbol{x}})^T\boldsymbol{x} = \overline{\boldsymbol{x}}^{\boldsymbol{T}}A\boldsymbol{x} = \lambda\,\overline{\boldsymbol{x}}^{\boldsymbol{T}}\boldsymbol{x},$$

$$(A\overline{\boldsymbol{x}})^T\boldsymbol{x} = \left(\bar{\lambda}\overline{\boldsymbol{x}}\right)^T\boldsymbol{x} = \bar{\lambda}\overline{\boldsymbol{x}}^{\boldsymbol{T}}\boldsymbol{x},$$

and therefore $$\left(\lambda - \bar{\lambda}\right)\overline{\boldsymbol{x}}^{\boldsymbol{T}}\boldsymbol{x} = 0.$$

Since $\boldsymbol{x} \neq \boldsymbol{0}$, then $\overline{\boldsymbol{x}}^{\boldsymbol{T}}\boldsymbol{x} \neq 0$, which implies $\lambda = \bar{\lambda}$, λ is a real number.

Remark. The eigenvectors of a real symmetric matrix can be taken as real vectors.

Theorem 2. Let A be a real $n \times n$ symmetric matrix, and λ_1, λ_2 be two distinct eigenvalues of A. If $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are eigenvectors belonging to λ_1 and λ_2, respectively, then $\boldsymbol{x_1} \perp \boldsymbol{x_2}$.

Proof. $$A\boldsymbol{x_1} = \lambda_1 \boldsymbol{x_1}, \qquad A\boldsymbol{x_2} = \lambda_2 \boldsymbol{x_2}.$$

Notice that

$$\lambda_1 \boldsymbol{x_1^T} \boldsymbol{x_2} = (A\boldsymbol{x_1})^T \boldsymbol{x}_2 = \boldsymbol{x_1^T} A^T \boldsymbol{x_2} = \boldsymbol{x_1^T} A \boldsymbol{x_2} = \lambda_2 \boldsymbol{x_1^T} \boldsymbol{x_2},$$

implying $$(\lambda_1 - \lambda_2)\boldsymbol{x_1^T}\boldsymbol{x_2} = 0.$$

Since $\lambda_1 \neq \lambda_2$, we deduce that $\boldsymbol{x_1} \perp \boldsymbol{x_2}$.

Theorem 3. Let A be a real $n \times n$ symmetric matrix. There exists an **orthogonal** matrix Q such that

$$\Lambda = Q^T A Q$$

is a diagonal matrix.

Remark. Indeed, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A. The matrix Q is an orthogonal matrix which takes the corresponding orthonormal eigenvectors as column vectors.

Example 1. Let $A = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find the orthogonal matrix Q and the diagonal matrix Λ, such that $\Lambda = Q^T A Q$.

Solution. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} \frac{3}{2} - \lambda & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = -(\lambda - 1)^2(\lambda - 2).$$

Therefore the eigenvalues of A are $\lambda_1 = \lambda_2 = 1, \lambda_3 = 2$.

Example 1. Let $A = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find the orthogonal matrix Q and the diagonal matrix Λ, such that $\Lambda = Q^T A Q$.

Solution. (continue) For $\lambda_1 = \lambda_2 = 1$, we have

$$A - I = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that $N(A - I) = \mathrm{Span}\{(1,1,0)^T, (0,0,1)^T\}$.

By using **Gram-Schmidt process**, we can find two orthonormal vectors as basis of $N(A - I)$:

$$\boldsymbol{q_1} = \frac{1}{\sqrt{2}}(1,1,0)^T, \qquad \boldsymbol{q_2} = (0,0,1)^T.$$

Example 1. Let $A = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find the orthogonal matrix Q and the diagonal matrix Λ, such that $\Lambda = Q^T A Q$.

Solution. (continue) For $\lambda_3 = 2$, we have

$$A - 2I = \begin{pmatrix} -\dfrac{1}{2} & -\dfrac{1}{2} & 0 \\ -\dfrac{1}{2} & -\dfrac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so that $N(A - 2I) = \mathrm{Span}\{(1, -1, 0)^T\}$. We take

$$\boldsymbol{q_3} = \frac{1}{\sqrt{2}}(1, -1, 0)^T.$$

Example 1. Let $A = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find the orthogonal matrix Q and the diagonal matrix Λ, such that $\Lambda = Q^T A Q$.

Solution. (continue) We take the orthogonal matrix

$$Q = (\boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

and the diagonal matrix $\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(Check directly that $\Lambda = Q^T A Q$.)

Theorem 4. Let A be a real $n \times n$ symmetric matrix. It is positive definite if and only if all its eigenvalues are positive.

Proof. Let λ_i, $i = 1, \dots, n$ be eigenvalues of A (the multiplicity are counted), $\boldsymbol{q_i}$ be eigenvectors belonging to eigenvalue λ_i such that $\{\boldsymbol{q_1}, \dots, \boldsymbol{q_n}\}$ is an orthonormal basis of $\mathbf{R}^n$ (consequence of **Theorem 3**).

We have for any $\boldsymbol{x} \in \mathbf{R}^n$,

$$\boldsymbol{x} = \alpha_1 \boldsymbol{q_1} + \cdots + \alpha_n \boldsymbol{q_n}.$$

Therefore, the quadratic form

$$\begin{aligned} q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} &= (\alpha_1 \boldsymbol{q_1} + \cdots + \alpha_n \boldsymbol{q_n})^T A (\alpha_1 \boldsymbol{q_1} + \cdots + \alpha_n \boldsymbol{q_n}) \\ &= \sum_{i,j=1}^{n} (\alpha_i \boldsymbol{q_i})^T A (\alpha_j \boldsymbol{q_j}) = \sum_{i,j=1}^{n} \alpha_i \alpha_j \lambda_j \boldsymbol{q_i^T} \boldsymbol{q_j} = \sum_{i=1}^{n} \lambda_i \alpha_i^2 . \end{aligned}$$

As a result, q is positive definite if and only if $\lambda_i > 0$ for all $i = 1,2, \dots, n$.

Remark.

- A is **positive definite** $\Leftrightarrow$ all its eigenvalues are **positive**.
- A is **negative definite** $\Leftrightarrow$ all its eigenvalues are **negative**.
- A is **indefinite** $\Leftrightarrow$ some of its eigenvalues are **positive** and some are **negative**.

Exercise. Determine the following symmetric matrices are positive definite, negative definite or indefinite.

$$\begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} -1 & 3 \\ 3 & -10 \end{pmatrix}, \quad \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}.$$

7.3 Conic Sections and Quadric Surfaces

Conic Sections

On the Cartesian plane, graphs of quadratic equations in two variables are curves called **conic sections [圆锥曲线].**

$$ax_1^2 + 2bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0,$$

where a, b, c, e, d, e, f are all real numbers.

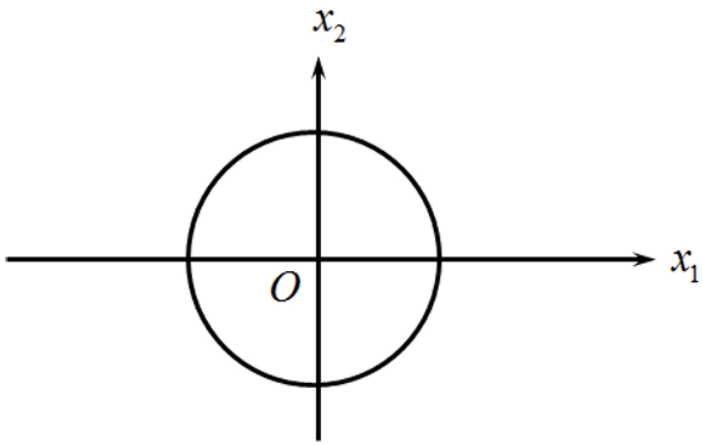
Standard forms of conic sections:

(1)Circle [圆]: $x_1^2 + x_2^2 = r^2$;

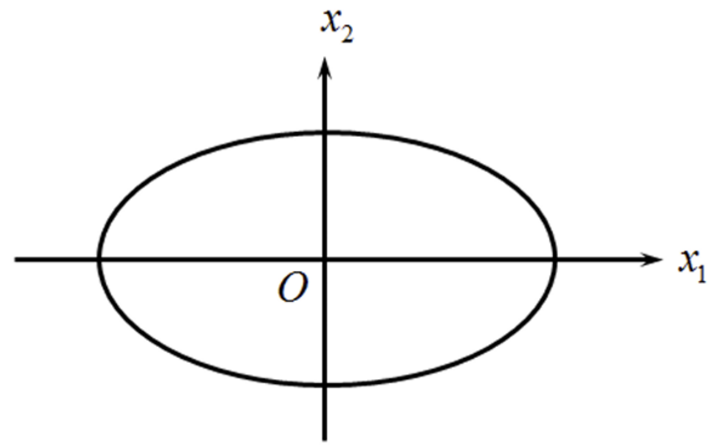
(2)Ellipse [椭圆]: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$;

(3)Hyperbola [双曲线]: $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$ or $-\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$;

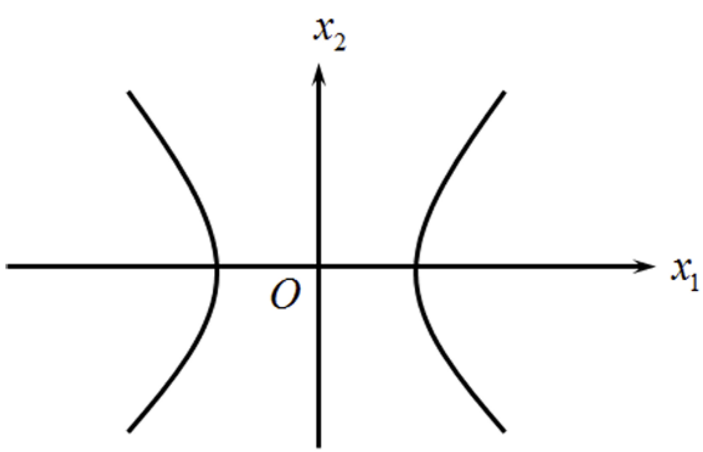
(4)Parabola [抛物线]: $x_2^2 = ax_1$ or $x_1^2 = ax_2$.



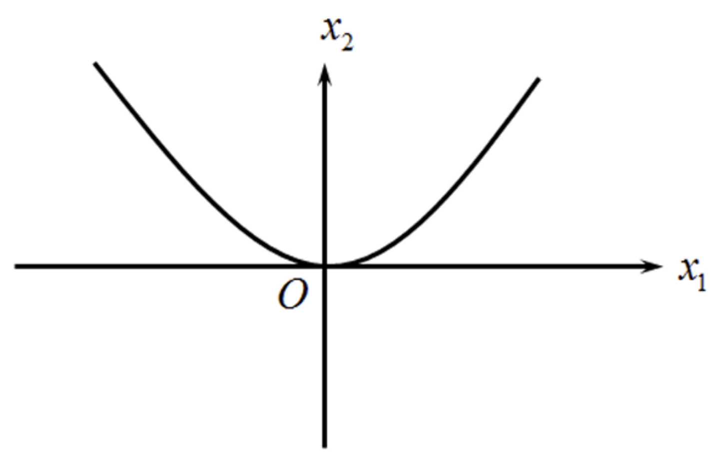
(a) circle



(b) ellipse



(c) hyperbola



(d) parabola

Quadric Surfaces

Consider quadratic equation in 3 variables. Graphs of these equations are called **quadric surfaces [二次曲面].**

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + gx + hy + iz + \alpha = 0,$$

or

$$\boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + \alpha = 0,$$

where

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad A = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} g \\ h \\ i \end{pmatrix}$$

Standard forms of quadric surfaces in $Oxyz$ coordinate system.

(1) Ellipsoid [椭球面]: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$

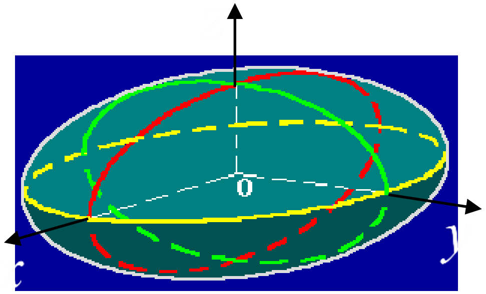
(2) Elliptic paraboloid [椭圆抛物面]: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c};$

(3) Elliptic cone [椭圆锥面]: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2};$

(4) Hyperboloid of one sheet [单叶双曲面]: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1;$

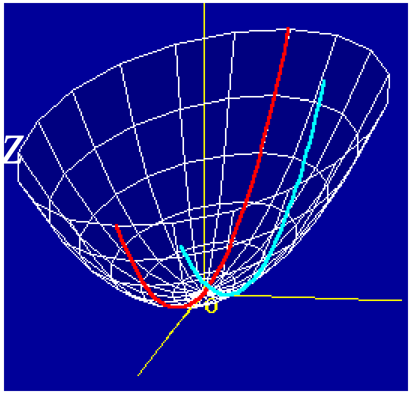
(5) Hyperboloid of two sheets [双叶双曲面]: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1;$

(6) Hyperbolic paraboloid [双曲抛物面]: $\frac{y^2}{b^2}-\frac{x^2}{a^2}=\frac{z}{c}, c>0.$



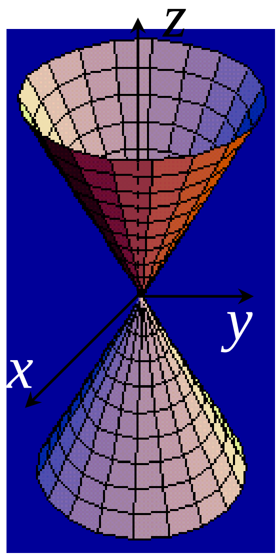
Ellipsoid [椭球面]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



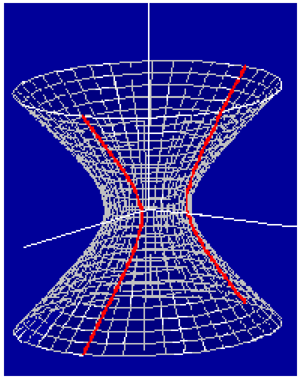
Elliptic paraboloid [椭圆抛物面]:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$



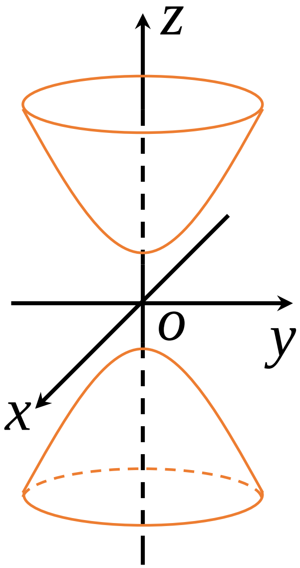
Elliptic cone [椭圆锥面]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$



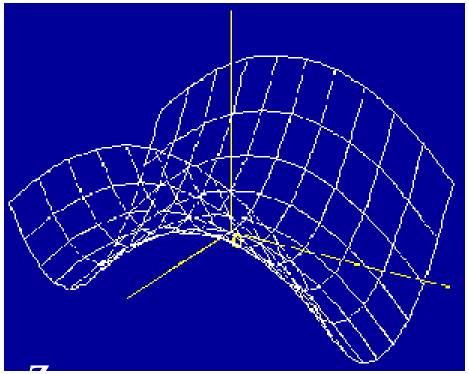
Hyperboloid of one sheet
[单叶双曲面]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



Hyperboloid of two sheets
[双叶双曲面]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$



Hyperbolic paraboloid
[双曲抛物面]:

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=\frac{z}{c}, \qquad c>0$$

Review

- Quadratic Form and its matrix representation
- Diagonalization of Real Symmetric Matrices
- Conic Sections and Quadric Surfaces

Preview

The END