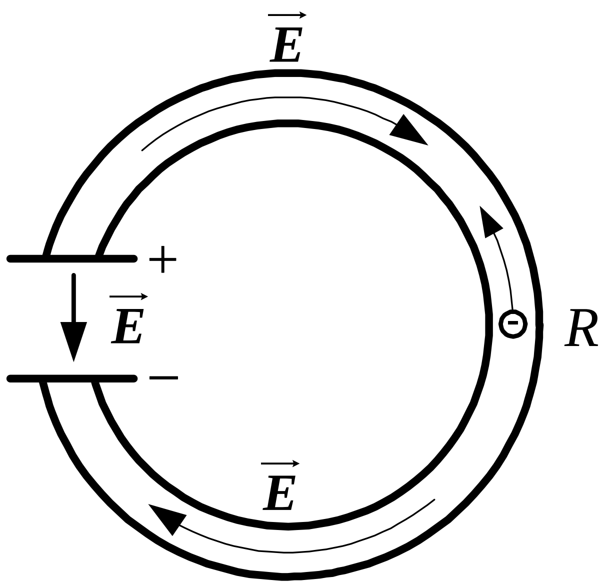
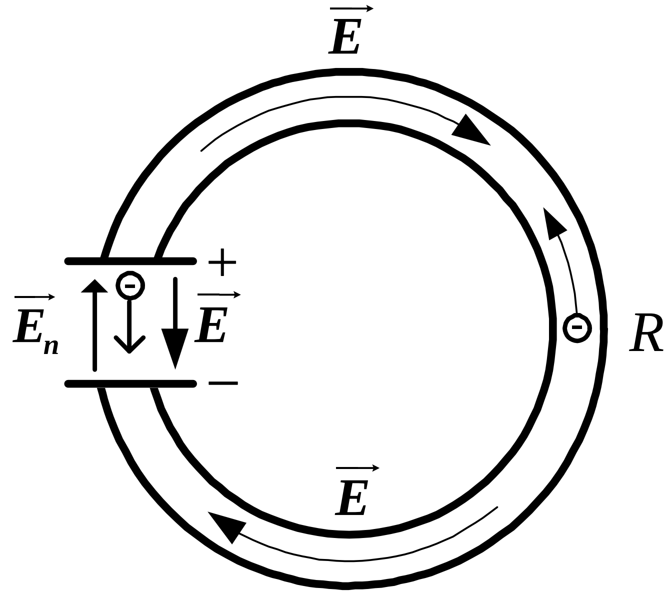
The Electromotive Force (*emf*) (P566 § 24-1)

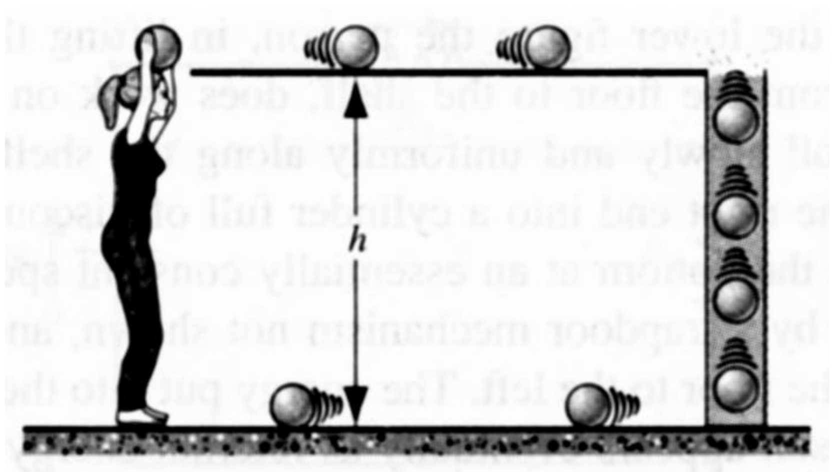
- **How to maintain a steady current?**



Nonsteady current loop

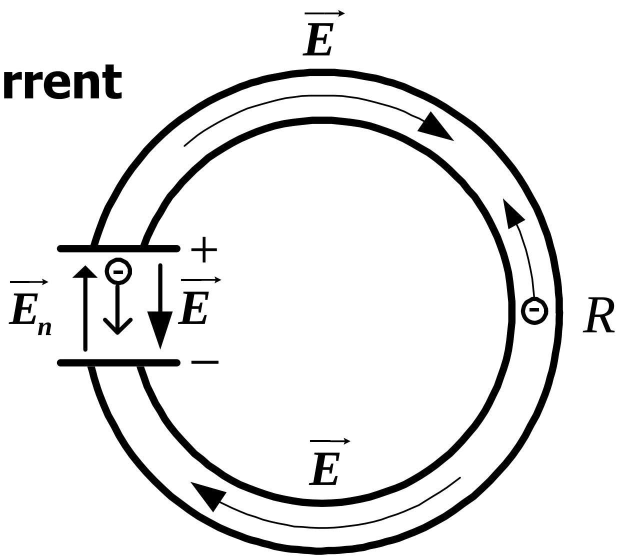


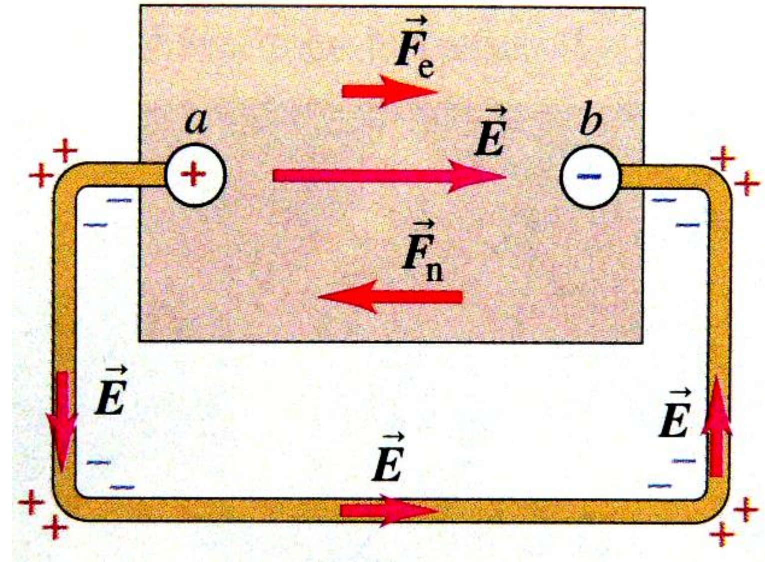
Steady current loop



Electromotive force —— *emf*

Steady current loop





$$W_n = \int_{-}^{+} \vec{F}_n \cdot d\vec{s} = \int_{-}^{+} q\vec{E}_n \cdot d\vec{s}$$

$$\mathcal{E} = \frac{W_n}{q} = \int_{(-)}^{(+)} \vec{E}_n \cdot d\vec{s}, \qquad \mathcal{E} = \oint \vec{E}_n \cdot d\vec{s}$$

Ideal source: $q\mathcal{E} = qV_{ab}$ ⇨ $V_{ab} = \mathcal{E}$

Chapter 27, 28 Faraday's Law and Inductance

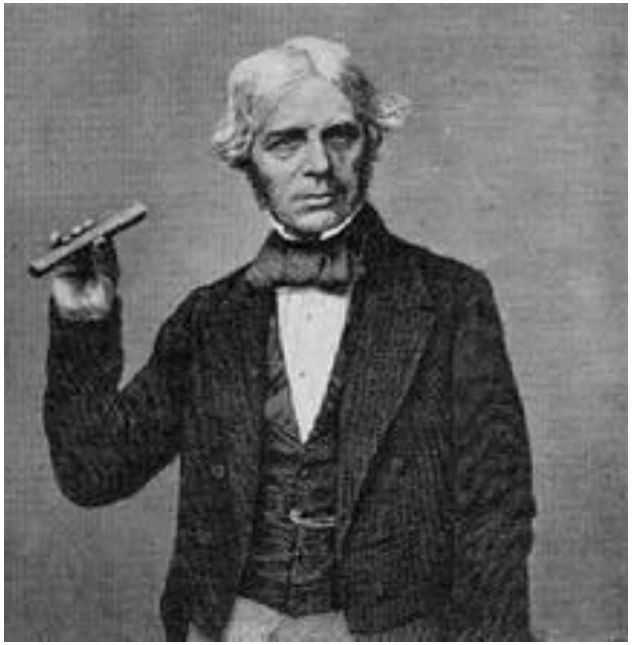
§ 1 Faraday's Law of Induction and Lenz's Law

$$\oint_L \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{encl}}$$

electric current ⇄ **magnetic field**

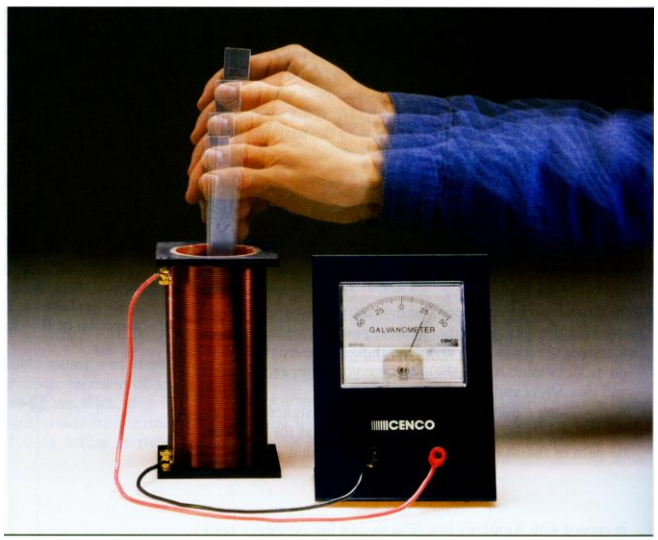
?

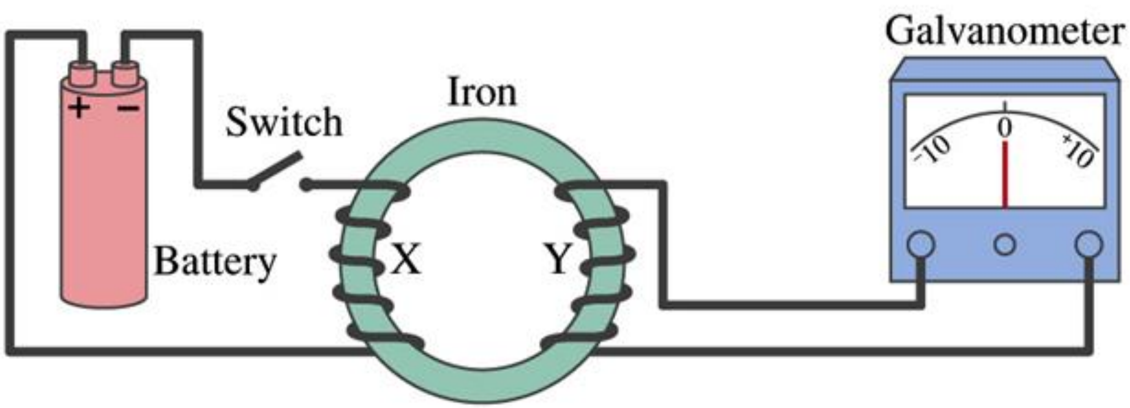
- **Question: Can an electric current be produced by a magnetic field?**
  - **M. Faraday (1791-1867) answered this question in 1831.**



The Experiment of Induction

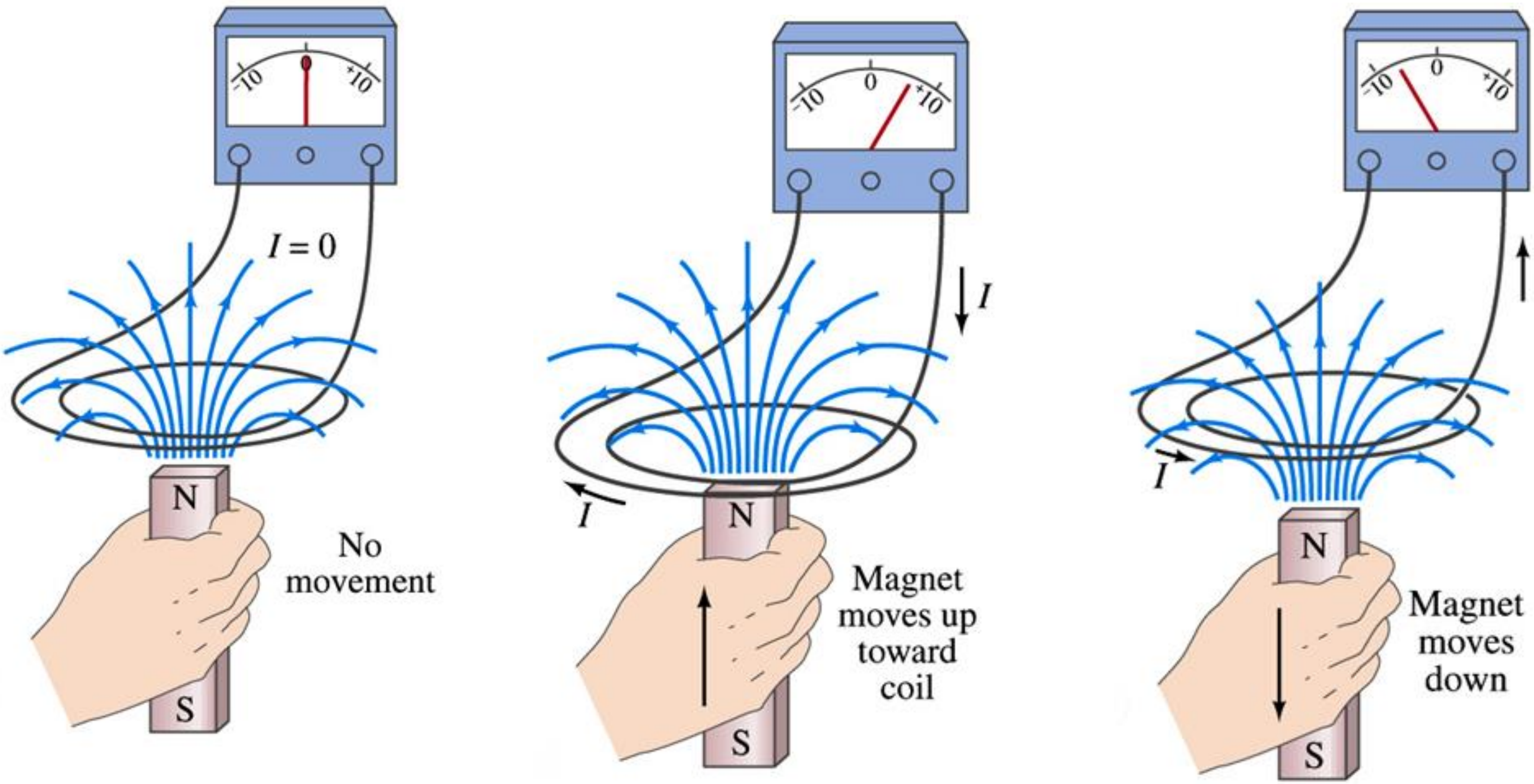
- From the experiment:
  - **Steady** magnetic field **can not** produce any current.
  - A **time-varying** magnetic field **can** induce an electric current.
  - The galvanometer shows a **larger** induced current when the relative motion of the magnet is **faster**.





The Experiment of Induction

- It is the **rate of change** in the number of the **magnetic field lines** **passing through the loop** that determine the induced **E**lectro**m**otive **f**orce (*emf*) in the loop.



- **Faraday's law:**
  - **The *emf* induced in a circuit is equal to the time rate of change of magnetic flux through the circuit.** ⇒ $$|\mathcal{E}| = \left| \frac{d\Phi_B}{dt} \right|$$
  - **If the circuit is a coil consists of N turns:** ⇒ $$|\mathcal{E}| = N \left| \frac{d\Phi_B}{dt} \right|$$
  - **How about the direction of the induced *emf* ?**

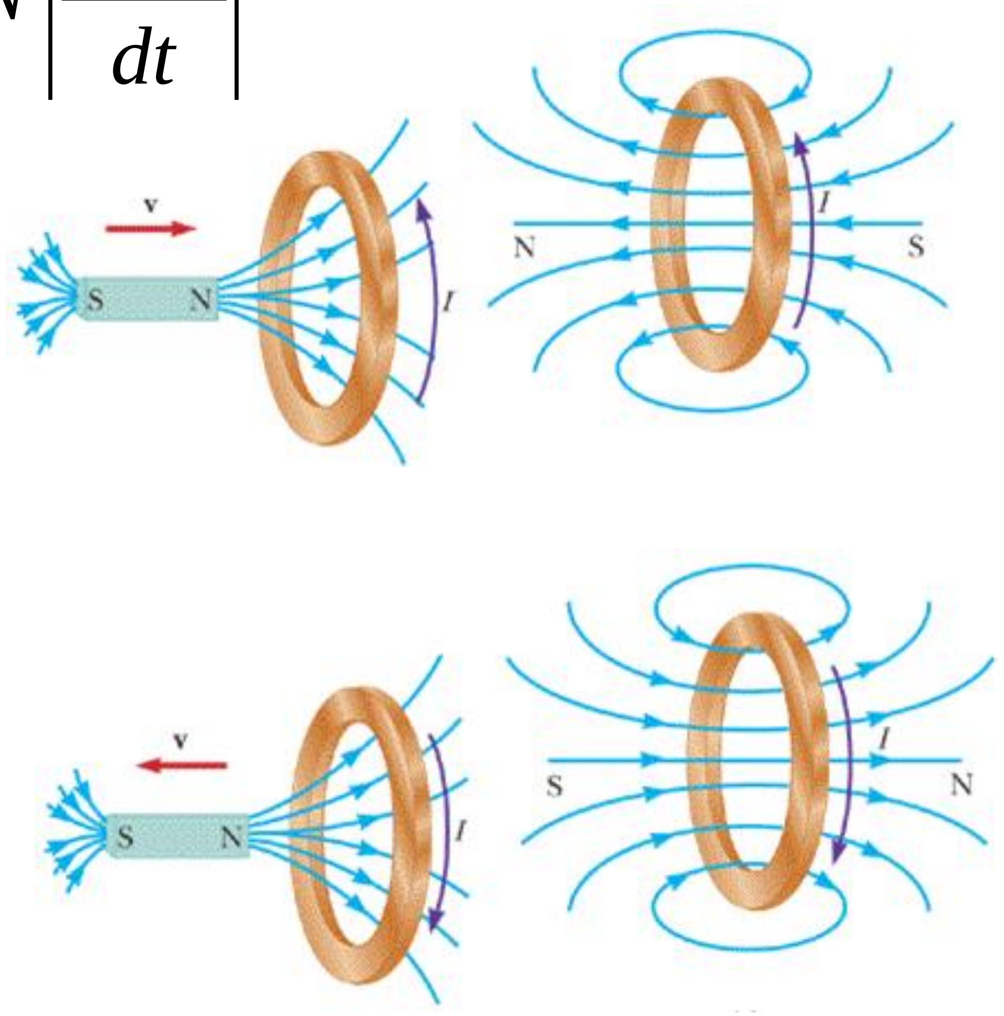
—— Lenz's law

$$|\mathcal{E}| = N\left|\frac{d\Phi_B}{dt}\right|$$

- **Lenz's law**
  - **The polarity of the induced emf in a loop is such that it produces a current whose magnetic field opposes the change in magnetic flux through the loop.**

    Another statement:

  - **The induced current is in a direction such that the induced magnetic field attempts to maintain the original flux through the loop.**



- **Complete Faraday's law:**

$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint\limits_{\substack{\text{surrounding}\\ \text{surface}}} \vec{B}\cdot d\vec{A}}$$

- **A coil consists of *N* turns:**

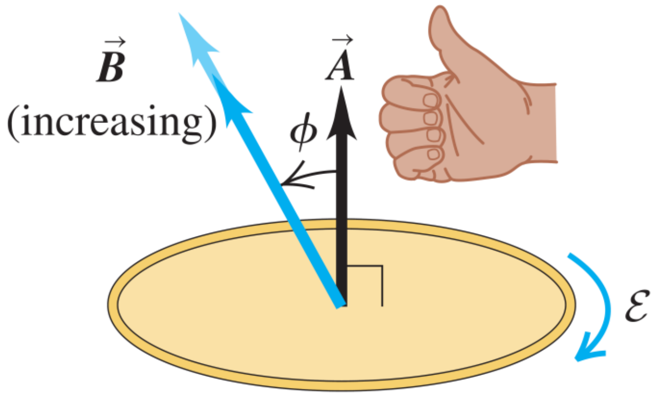
$$\boxed{\mathcal{E} = -N\frac{d\Phi_B}{dt}}$$

How to Determine the Sign of Induced emf

- **Using the right-hand rule to determine the sign of Φ_B and the sign of emf $\mathcal{E}$.**

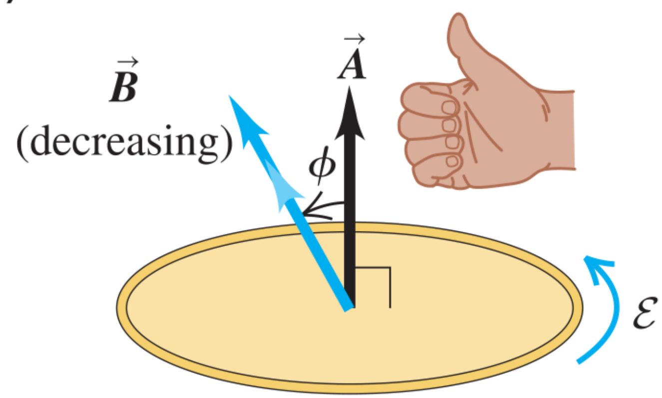
$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_{\substack{\text{surrounding}\\ \text{surface}}} \vec{B}\cdot d\vec{A}$$

(a)



- Flux is positive ($\Phi_B > 0$) ...
- ... and becoming more positive ($d\Phi_B/dt > 0$).
- Induced emf is negative ($\mathcal{E} < 0$).

(b)

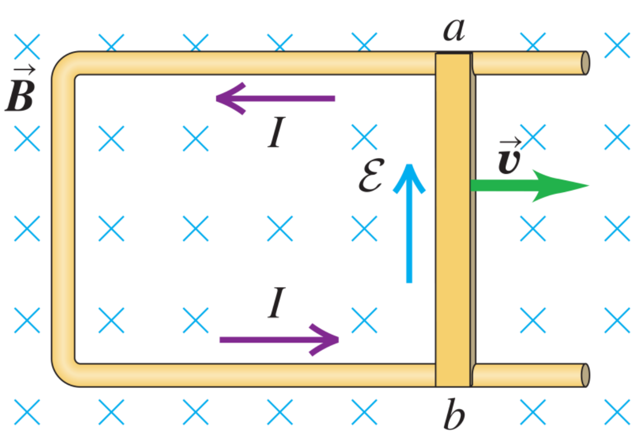


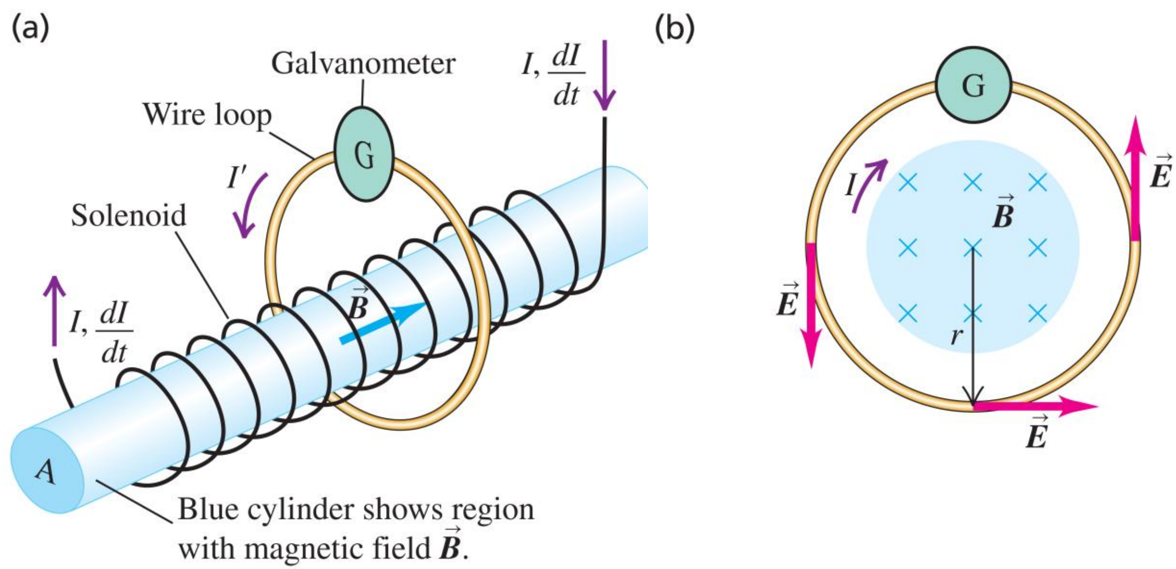
- Flux is positive ($\Phi_B > 0$) ...
- ... and becoming less positive ($d\Phi_B/dt < 0$).
- Induced emf is positive ($\mathcal{E} > 0$).

What makes the magnetic flux change?

- **What makes the magnetic flux change?**
  - **Is the loop or coil changing orientation or part of the loop moving? —— Motional *emf*.**
  - **Is the magnetic field changing? —— Induced electric field as the non-electrostatic field.**

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_{\text{surrounding surface}} \vec{B}\cdot d\vec{A}$$





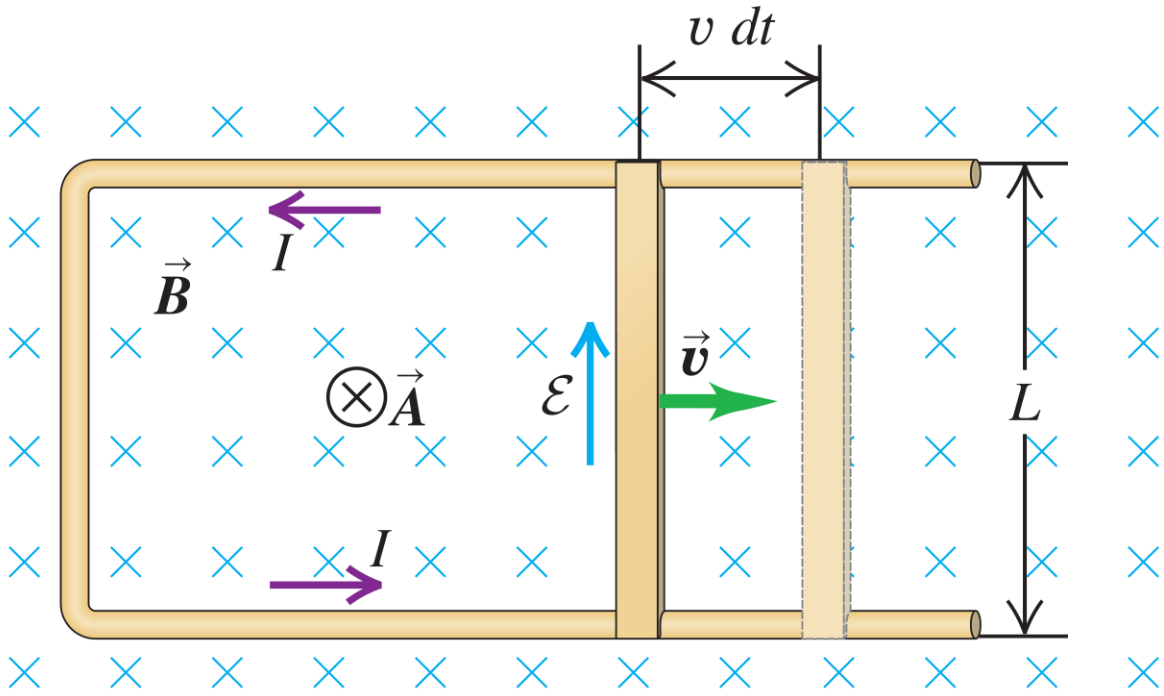
Motional *emf*

Induced *emf*

§ 2 Motional *emf*

- **Staring with the slide-wire generator**

A U-shaped conductor in a uniform magnetic field $\vec{B}$ perpendicular to the plane, directed into page. A metal rod with length L across the two arms of the conductor, forming a circuit. The metal rod slides to the right with a constant velocity $\vec{V}$. Find the motional *emf*.



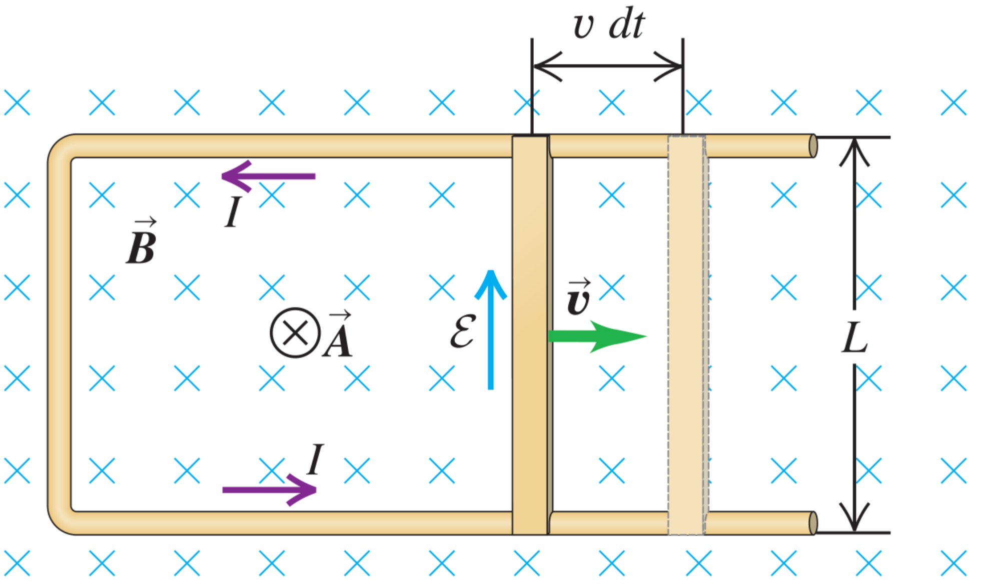
Choose the direction of area $\vec{A}$ as directing into the page.

- **The magnetic flux through the circuit:**

$$\Phi_B = \vec{B}\cdot\vec{A} = B(Lvt)$$

- **The induced emf:**

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -BLv$$



> **The negative sign means that direction of *emf* is counterclockwise.**

The Origin of the Motional *emf*

- **The magnetic force exerting on the moving charge in rod acts as the non-electric force that produces the *emf*.**

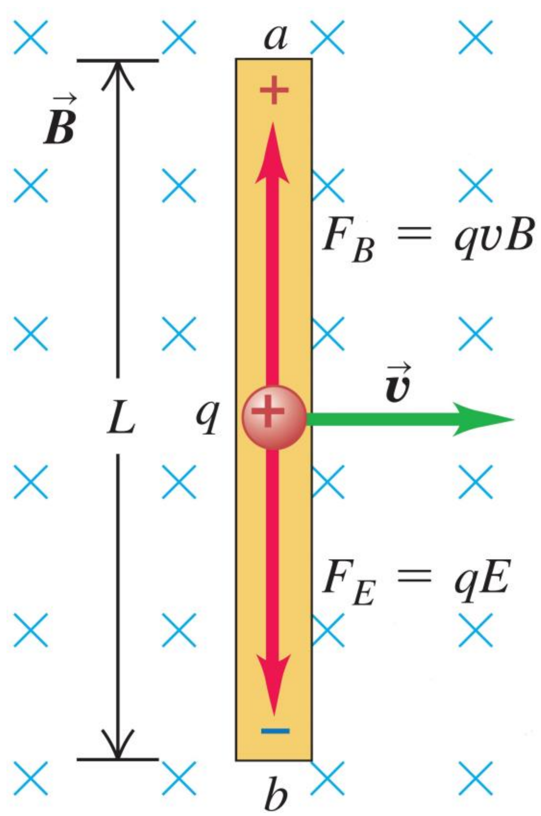
- **The magnetic force:** $\vec{F} = q\vec{v} \times \vec{B}$

- **The *emf* along the rod:**

$$\mathcal{E} = \int_a^b \vec{E}_n \cdot d\vec{s} = \int_a^b \frac{\vec{F}_n}{q} \cdot d\vec{s} = \int_a^b (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

$$= -\int_0^L vBds = -vBL$$

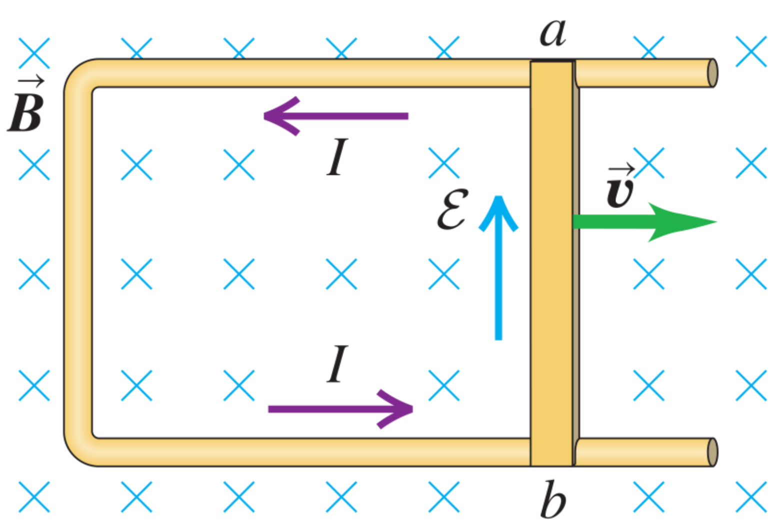
- **The *emf* is induced in a conductor moving through a magnetic field, called motional *emf*.**



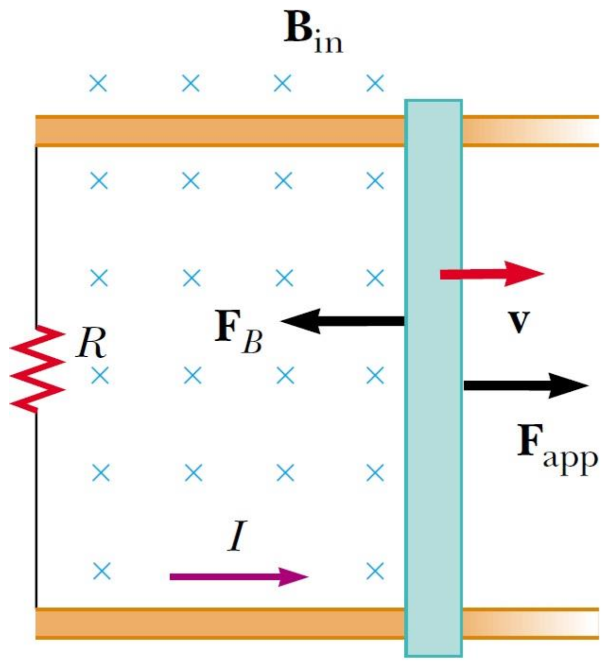
Isolated moving rod

The Origin of the Motional *emf*

- **With Faraday's law, we cannot know which part of the circuit is the source of the emf. Here we know that the moving rod is the source of *emf* ; within it, positive charge moves from lower to higher potential, and in the remainder of the circuit, charge moves from higher to lower potential.**



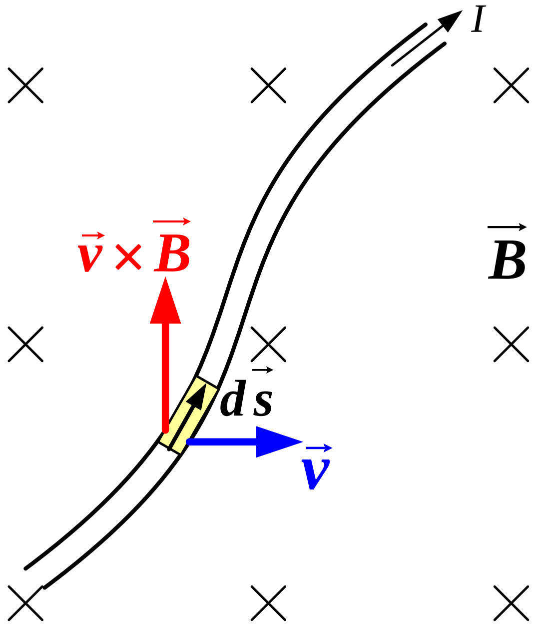
Rod connected to stationary conductor



- **Definition of motional *emf* :**
  - **For moving current-carrying wire of any shape in a magnetic field**

$$d\mathcal{E} = (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

$$\boxed{\mathcal{E} = \int_L (\vec{v} \times \vec{B}) \cdot d\vec{s}}$$



  - **For any closed conducting loop:**

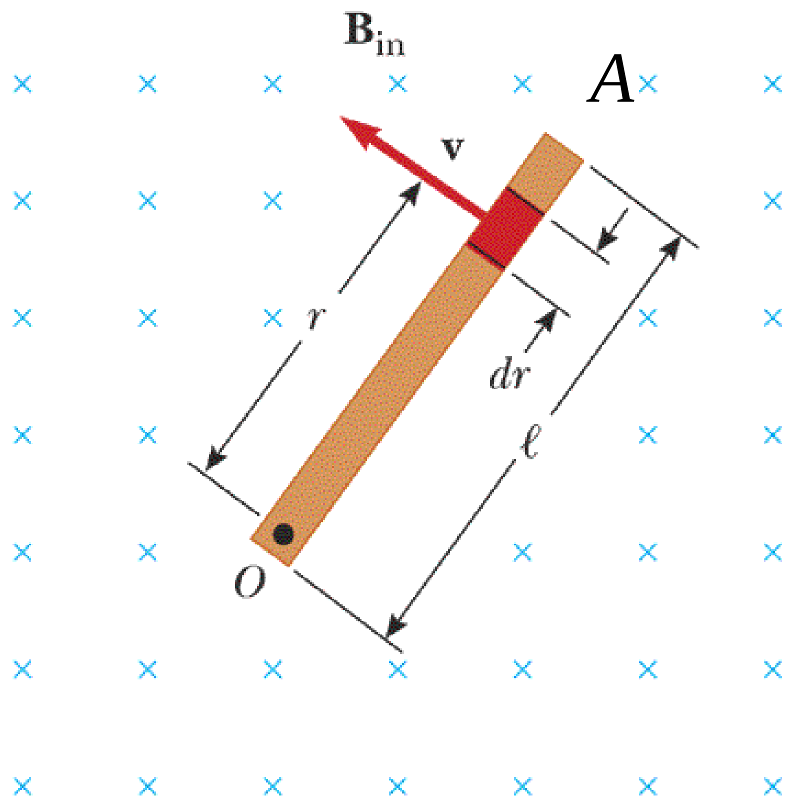
$$\mathcal{E} = \oint_L (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

  - **The direction of motional *emf* : determined by the projection direction of** $\vec{v} \times \vec{B}$

Example

Motional *emf* induced in a rotating bar

A conducting bar of length l rotates with an angular speed ω about a pivot at one end. B is uniform and perpendicular to the plane of rotation. Find the *emf* induced between the ends of the bar.

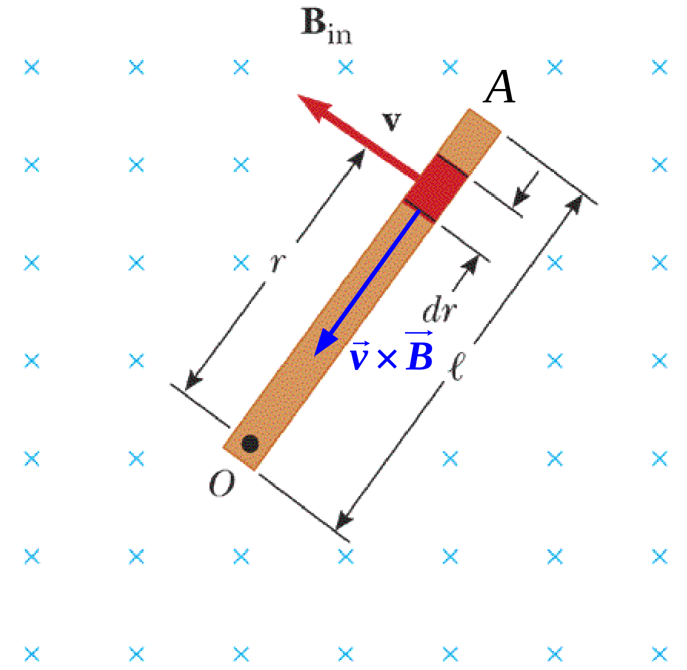


Motional emf induced in a rotating bar

Solution: Choose the direction of integration to be from end O to end A.

$$\mathcal{E} = \int_O^A (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

$$= \int_0^l (-Bv)dr = -\int_0^l B(\omega r)dr = -\frac{1}{2}B\omega l^2$$



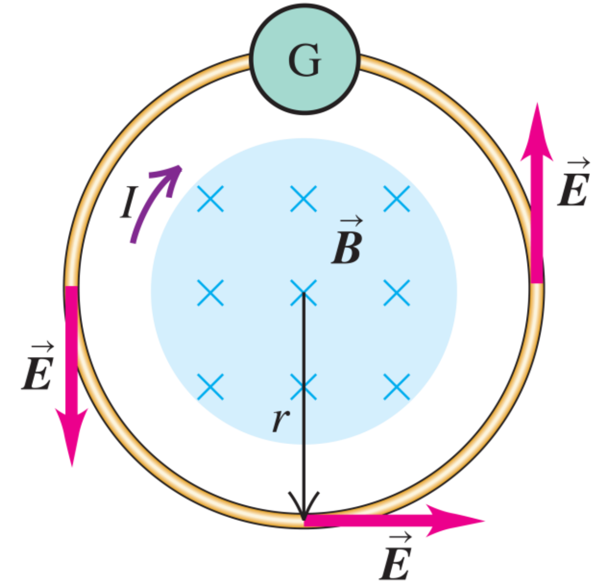
The negative sign means that the real direction of *emf* is opposite to the direction of integration, and potential at end A is lower than end O.

Ch27 Prob. 11, 26, 27 (P640)

§ 3 Induced Electric Field

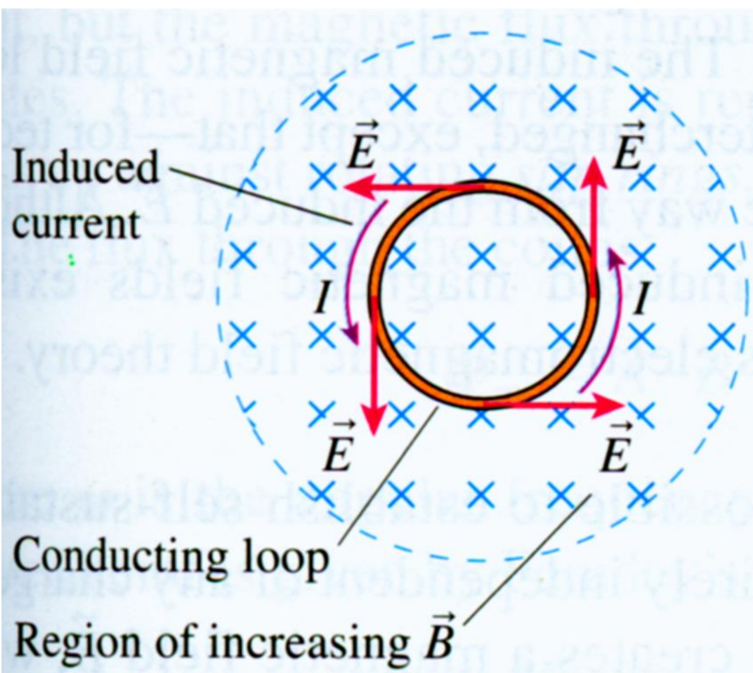
- **What is the basis of induced *emf* when there is a changing flux through a stationary conducting loop?**
  - **Now we can understand that magnetic force is the reason of the induced *emf* in a moving conductor.**
  - **By Faraday's law, we only know the result that an induced *emf* also occurs when there is a changing flux through a stationary conducting loop.**
  - **But up to now, we don't know what *force* makes the charges moving around the loop. It can't be a magnetic force because the conductor is not moving in the magnetic field.**

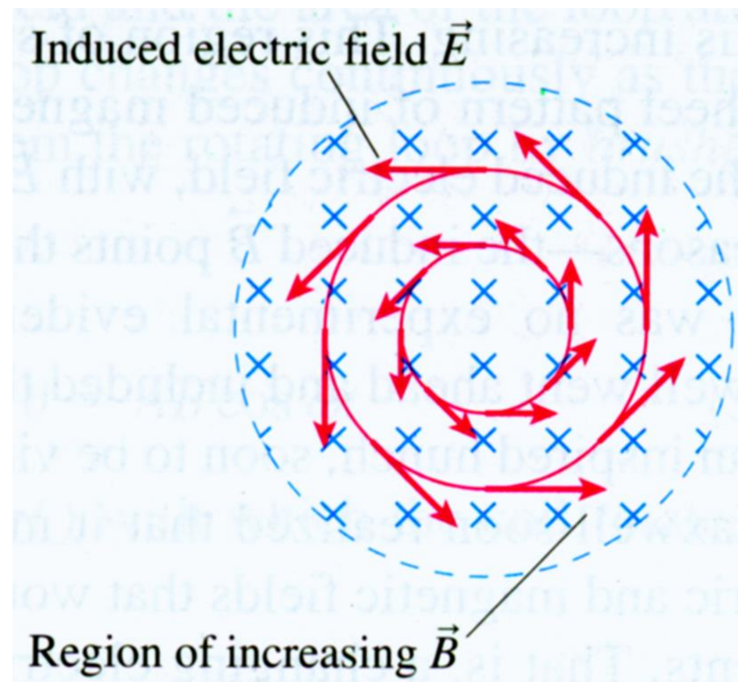
$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$



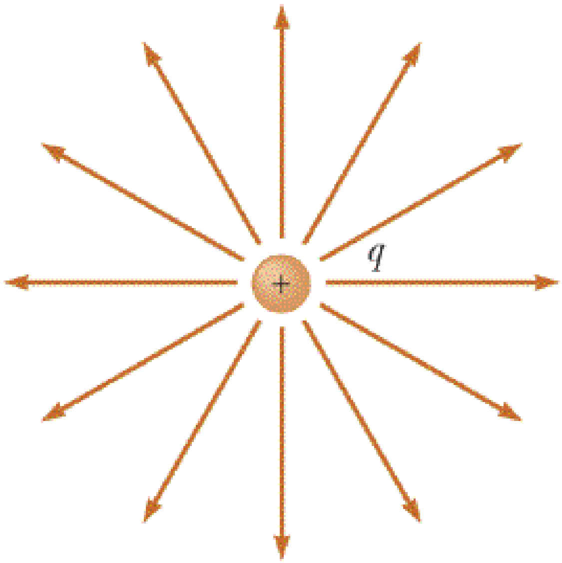
The Induced Electric Field as the Source of Induced *emf*

- **Maxwell's suggestion: induced electric field**
  - **There must be an induced electric field (non-electrostatic field) created in the conductor as a result of changing magnetic flux.**
  - **This kind of electric field is induced even when no conductor is present.**

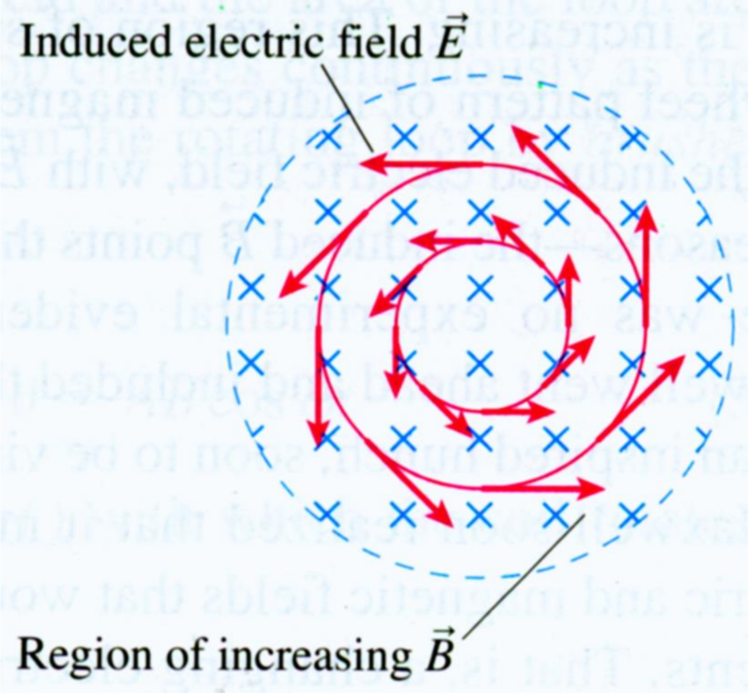




- **Confused points**
  - **We were accustomed to thinking about electric field as being caused by electric charges. Now we know that a changing magnetic field can also act as a source of electric field.**



Electrostatic field



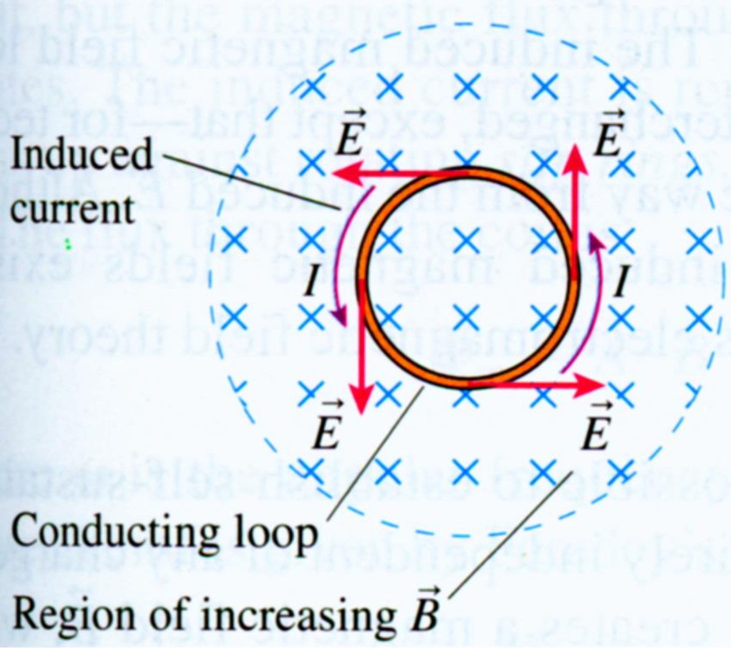
Induced electric field

The Confused Points for Induced *emf*

- **By the definition of *emf*, $\mathcal{E}$ is equal to the work done by a non-electrostatic field, induced electric field $\vec{E}_i$, per unit charge.**

$$\mathcal{E} = \oint_L \vec{E}_i \cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_{\text{the surface surround the loop}} \vec{B}\cdot d\vec{A}$$

$$= -\iint_{\text{the surface surround the loop}} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$



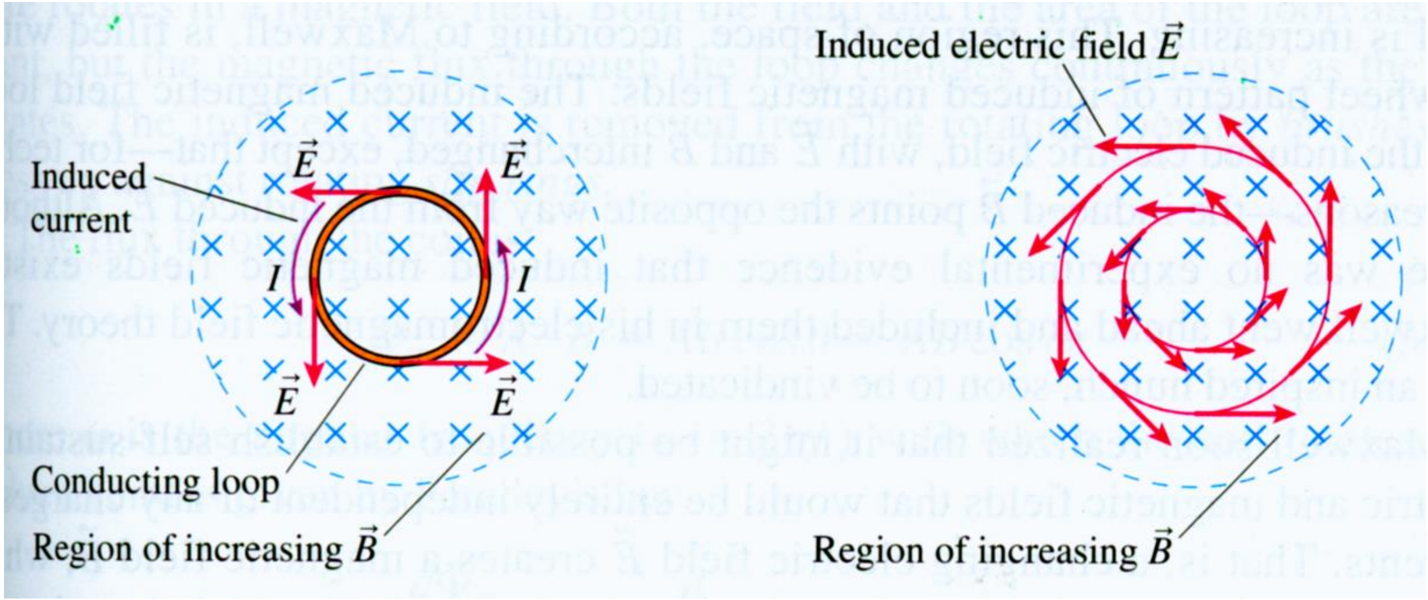
- **The line integral around a closed path is not zero. So the induced electric field is not conservative.**

General Form of Faraday's Law

- The relationship between the induced electric field and the changing magnetic field

$$\boxed{\oint_L \vec{E_i} \cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}}$$

Valid not only in conductors, but in any region of space.



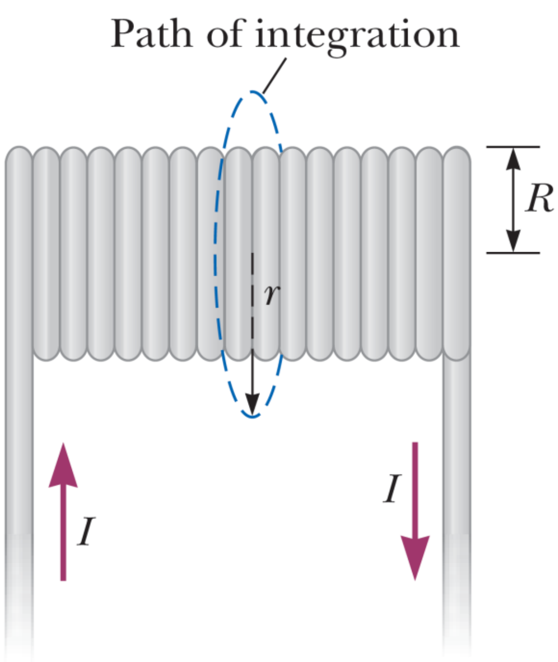
Electrostatic field vs. Induced electric field

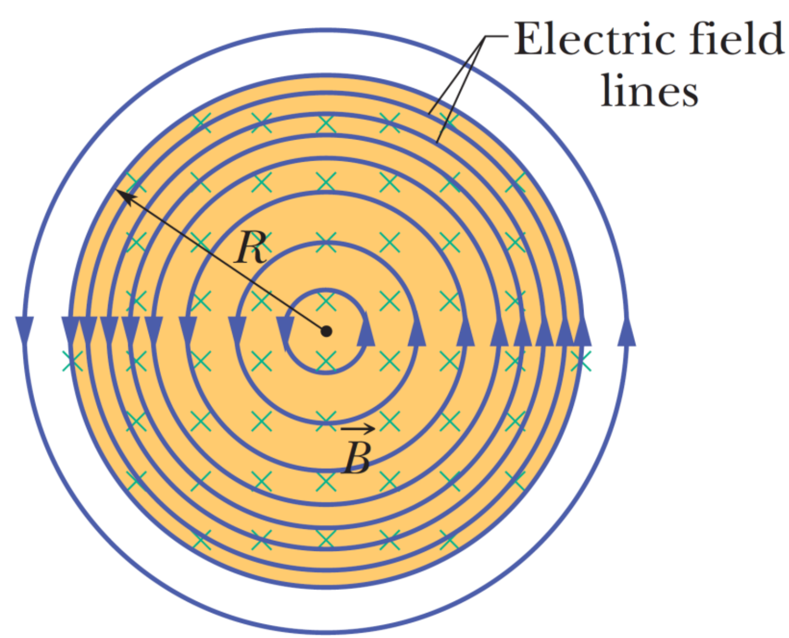
	Electrostatic field $\vec{E}_s$	Induced electric field $\vec{E}_i$
The source of the field	The charges	The changing magnetic field
Line integral around a closed path	$\oint_L \vec{E}_s \cdot d\vec{s} = 0$ Conservative	$\oint_L \vec{E}_i \cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$ Non-conservative
Gauss's law	$\oiint_S \vec{E}_s \cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0}$ Field lines begin and end on charge	$\oiint_S \vec{E}_i \cdot d\vec{A} = 0$ Field lines form closed loops

Example

Electric field induced by a changing magnetic field in a solenoid

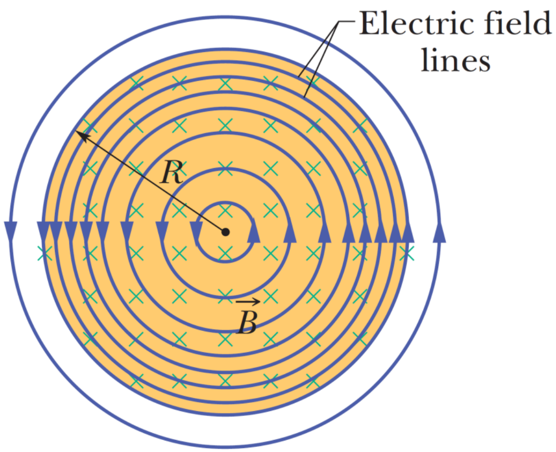
A long solenoid of radius R has n turns of wire per unit length and carries a time-varying current that varies sinusoidally as $I = I_{max}\cos\omega t$. (1) Determine the magnitude of the induced electric field outside the solenoid, a distance $r > R$ from its long central axis. (2) Find the induced electric filed magnitude inside the solenoid, a distance $r < R$ from its axis.





Electric field induced by a changing magnetic field in a solenoid

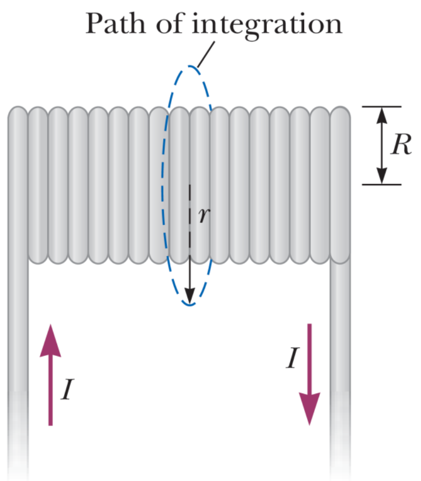
Solution: Choose a path for the line integral to be a circle of radius r centered on the solenoid. By symmetry, the $\vec{E}$ is tangent to the circle and has constant magnitude on it.



$$\left|\oint_L \vec{E}\cdot d\vec{s}\right| = \left|\oint_L E ds\right| = \left|E\oint_L ds\right| = E(2\pi r)$$

$$= \left|-\frac{d\Phi_B}{dt}\right| = \left|\frac{d}{dt}\left(B\pi R^2\right)\right| = \pi R^2\left|\frac{dB}{dt}\right|$$

$$E = \frac{R^2}{2r}\left|\frac{dB}{dt}\right| \qquad \textbf{(for } r > R\textbf{)}$$



$$= \frac{R^2}{2r}\left|\frac{d}{dt}(\mu_0 n I_{\max}\cos\omega t)\right| = \frac{\mu_0 n I_{\max}\omega R^2}{2r}\left|\sin\omega t\right|$$

For an interior point ($r < R$)

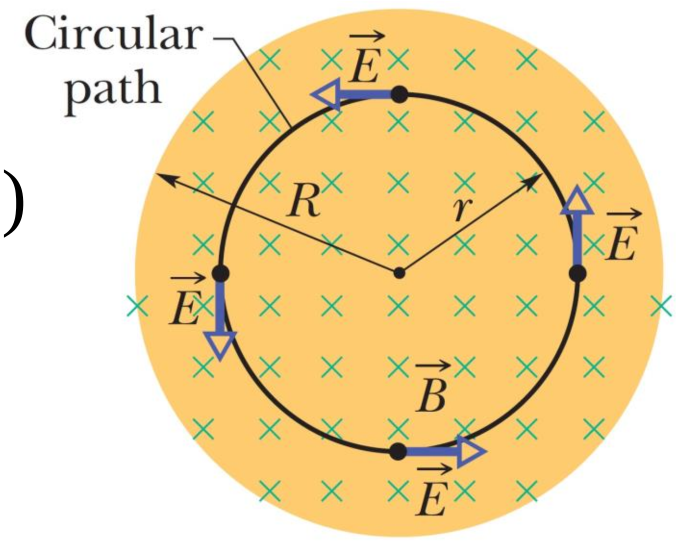
$$\left|\oint_L \vec{E}\cdot d\vec{s}\right| = \left|\oint_L E ds\right| = \left|E\oint_L ds\right| = E(2\pi r)$$

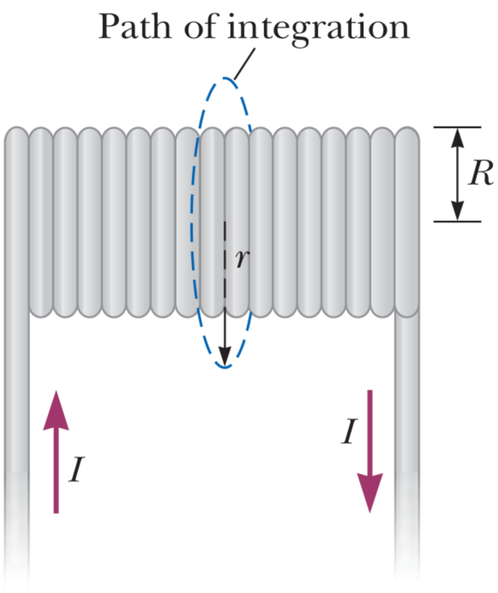
$$= \left|-\frac{d\Phi_B}{dt}\right| = \left|\frac{d}{dt}\left(B\pi r^2\right)\right| = \pi r^2\left|\frac{dB}{dt}\right|$$

$$E = \frac{r}{2}\left|\frac{dB}{dt}\right| \qquad \textbf{(for } r < R \textbf{)}$$

$$= \frac{r}{2}\left|\frac{d}{dt}\left(\mu_0 n I_{\max}\cos\omega t\right)\right|$$

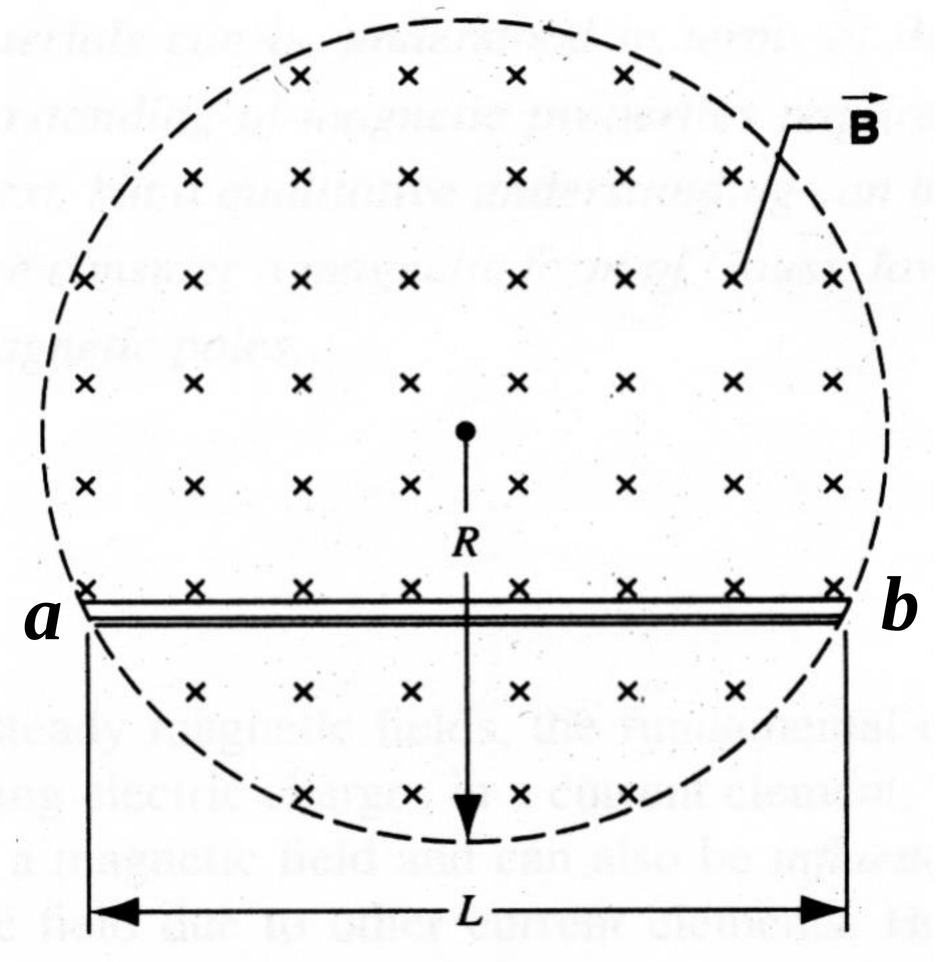
$$= \frac{\mu_0 n I_{\max}\omega}{2} r\left|\sin\omega t\right|$$





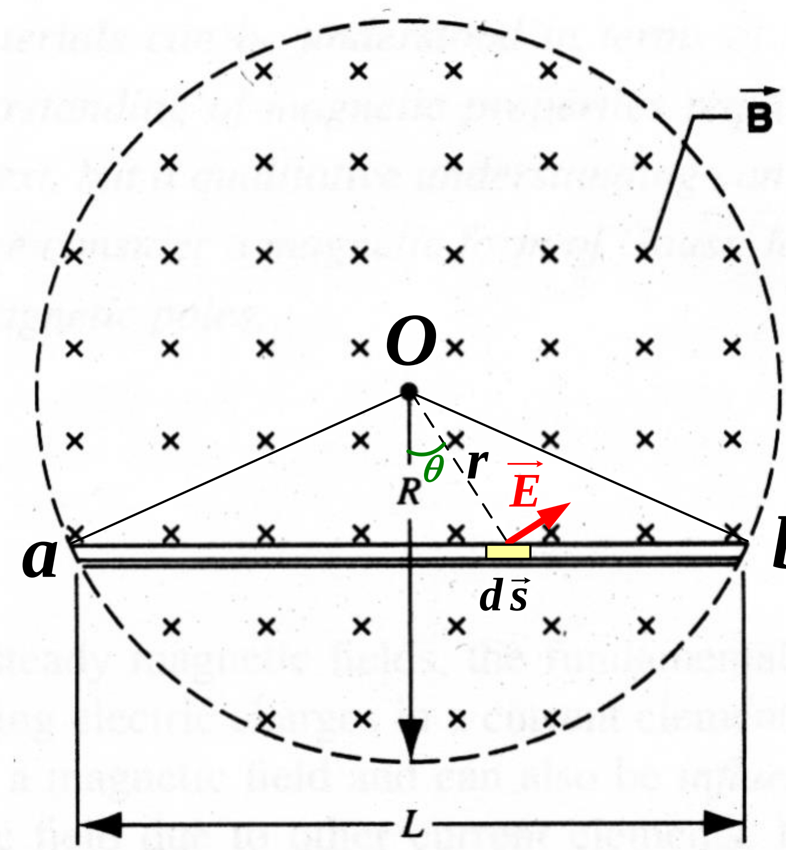
Example

A uniform magnetic field B fill with cylindrical volume of radius R. A metal rod ab of length L is placed as shown in the figure. If B is changing at the constant rate $(dB/dt) > 0$, find the *emf* acting between the end a and b of the rod.



Solution I: By line integration of induced electric field.

For $(dB/dt) > 0$ **,**
we have
know that : $E = \begin{cases} \dfrac{R^2}{2r}\dfrac{dB}{dt} & \text{for } r > R \\ \dfrac{r}{2}\dfrac{dB}{dt} & \text{for } r < R \end{cases}$



$$\mathcal{E}_{ab} = \int_a^b \vec{E}\cdot d\vec{s} = \int_{-L/2}^{L/2} E\cos\theta ds$$

$$= \int_{-L/2}^{L/2}\left(\frac{r}{2}\frac{dB}{dt}\right)\cos\theta ds$$

$$= \frac{1}{2}\frac{dB}{dt}\int_{-L/2}^{L/2} r\cos\theta ds$$

$$r\cos\theta = \sqrt{R^2 - \frac{L^2}{4}},\qquad \mathcal{E}_{ab} = \frac{1}{2}\sqrt{R^2 - \frac{L^2}{4}}\frac{dB}{dt}\int_{-L/2}^{L/2} ds = \frac{L}{2}\sqrt{R^2 - \frac{L^2}{4}}\frac{dB}{dt}$$

Solution II: Using Faraday's law

Choose the loop *abO*.

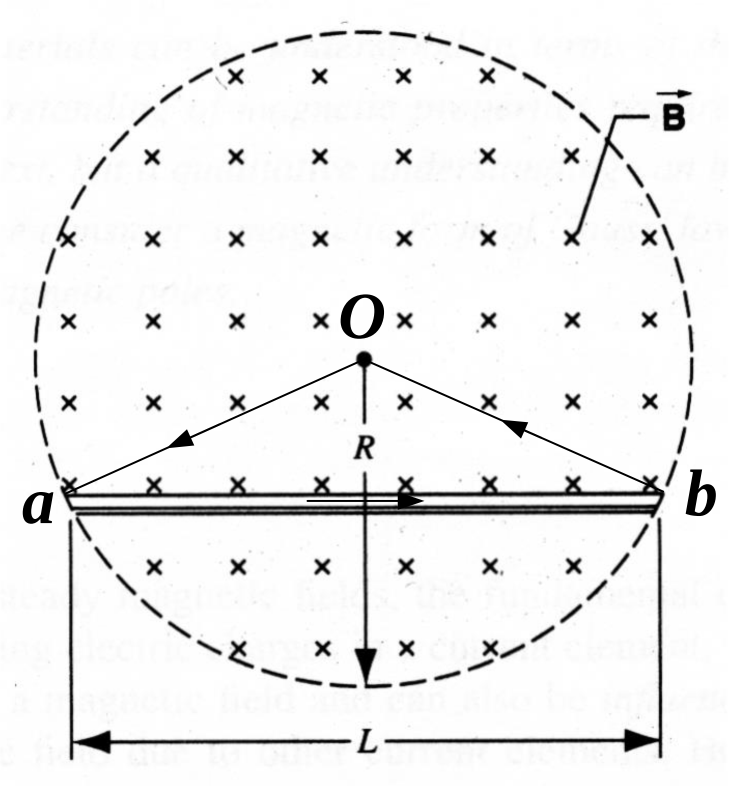
$$\Phi_B = \vec{B}\cdot\vec{A}_{abO} = -BA_{abO}$$

$$\mathcal{E}_{OabO} = \mathcal{E}_{Oa} + \mathcal{E}_{ab} + \mathcal{E}_{bO} = -\frac{d\Phi_B}{dt}$$

$$= A_{abO}\frac{dB}{dt}$$

$$\mathcal{E}_{Oa} = \int_O^a \vec{E}_n \cdot d\vec{s} = 0, \qquad \mathcal{E}_{bO} = 0$$

$$\mathcal{E}_{ab} = A_{abO}\frac{dB}{dt} = \frac{L}{2}\sqrt{R^2 - \frac{L^2}{4}}\frac{dB}{dt}$$



When $\dfrac{dB}{dt} > 0,\ \varepsilon_{ab} > 0$

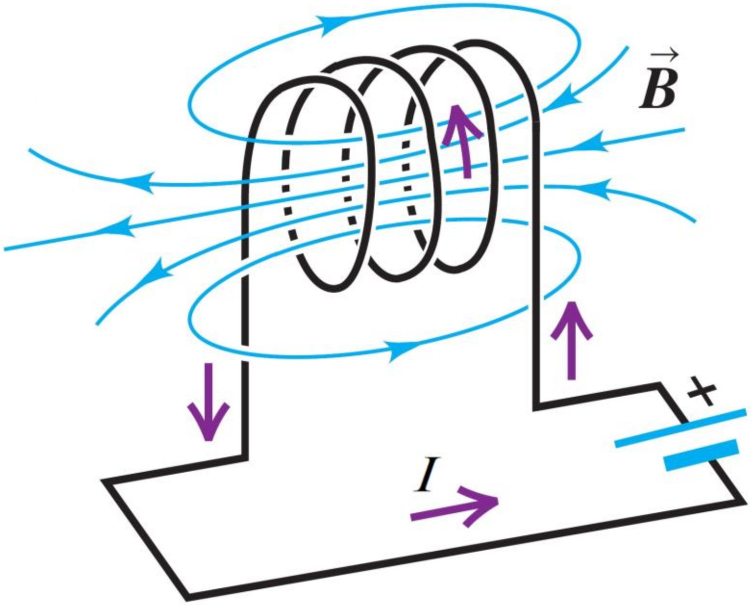
The potential at end *b* is higher than end *a*.

§ 4 Self-Inductance

- **Inductor and self-induced *emf* :**
  - **For a circuit including a solenoid**

Current I in wire of solenoid → Flux Φ_B through the solenoid

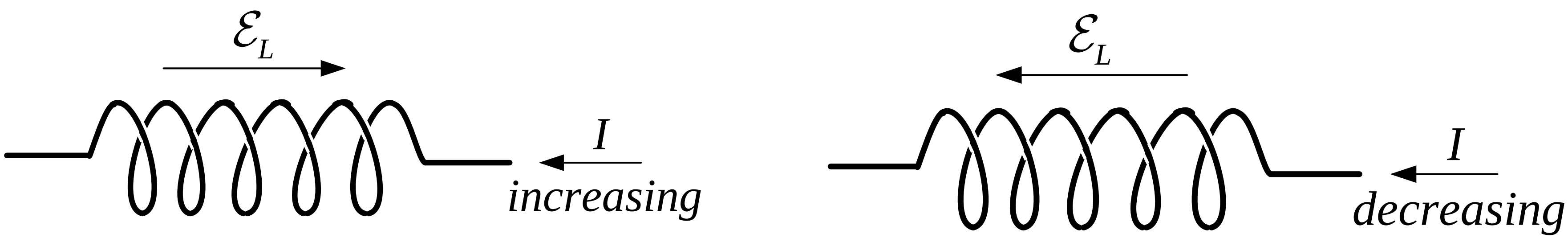
I changes → Φ_B changes → Induced $\mathcal{E}_L$ → (I changes)



- **An inductor is a circuit element such as solenoid that stores energy in the magnetic field surrounding its current-carrying wires, just as a capacitor store energy in the electric field between its charged plates.**

Inductor and self-induced emf

- The emf set up by changing self-current is called **self-induced *emf*** $\mathcal{E}_L$
- By Lenz's law a self-induced emf always opposes the change in the current that caused the emf, and then tends to make it more difficult for variation in current to occur.



- Definition of Self-induced *emf* :

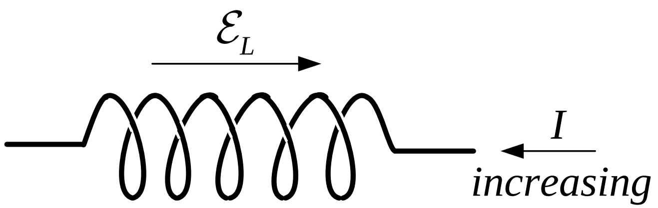
$$\boxed{\mathcal{E}_L = -L\frac{dI}{dt}}$$

- $L > 0$
- The negative sign reflects Lenz's law.

Definition of the Self-inductance

- **The self-inductance**
  - **The proportionality constant L is called the self-inductance.**

$$\mathcal{E}_L = -L\frac{dI}{dt}$$



  - **From Faraday's law**

$$\mathcal{E}_L = -\frac{d(N\Phi_B)}{dt} \quad \Rightarrow \quad L\frac{dI}{dt} = \frac{d(N\Phi_B)}{dt}$$

  - **Integrating with respect to the time, and assuming that $\Phi_B=0$ when $I=0$.**

$$\boxed{L = \frac{N\Phi_B}{I}}$$

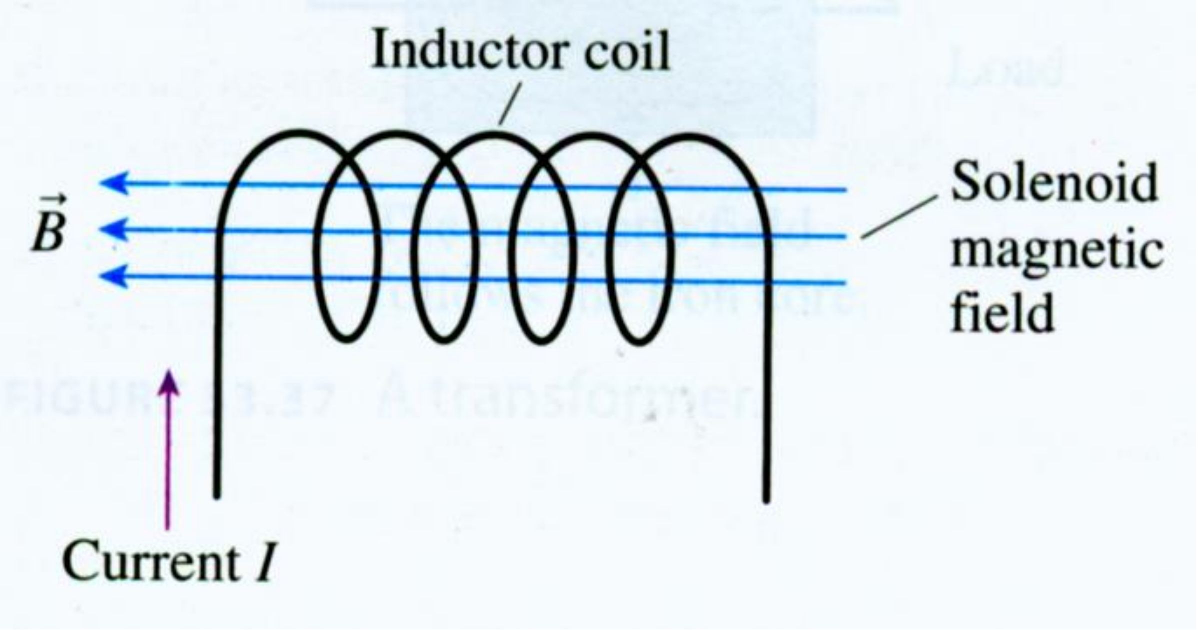
SI unit: H (Henry)

  - **Note that, since Φ_B is proportional to the current, the self-inductance is independent of I. Just as the capacitance, the self-inductance depends only on the geometry of the device.**

Example

Inductance of a solenoid

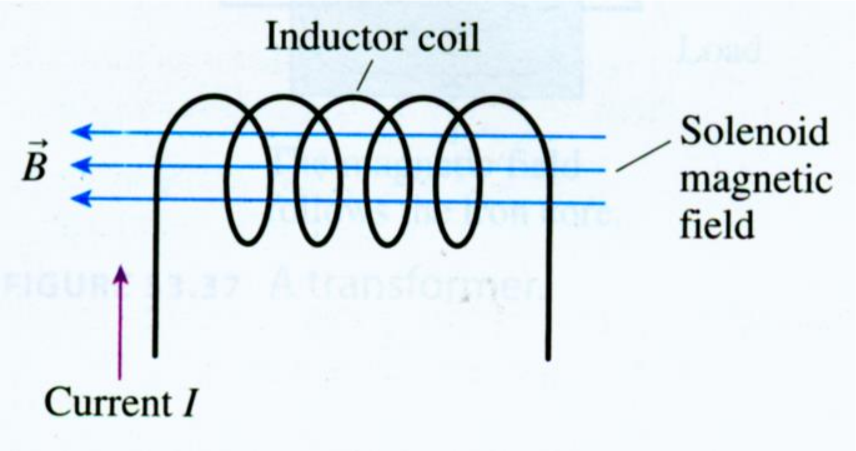
Find the inductance of a uniformly round solenoid having N turns and length l. Assume that l is long compared with the radius and the core of the solenoid.



Solution:

For an ideal solenoid, the interior magnetic field is uniform.

$$B = \mu_0 n I = \mu_0 \frac{N}{l} I$$



The magnetic flux through each turn is

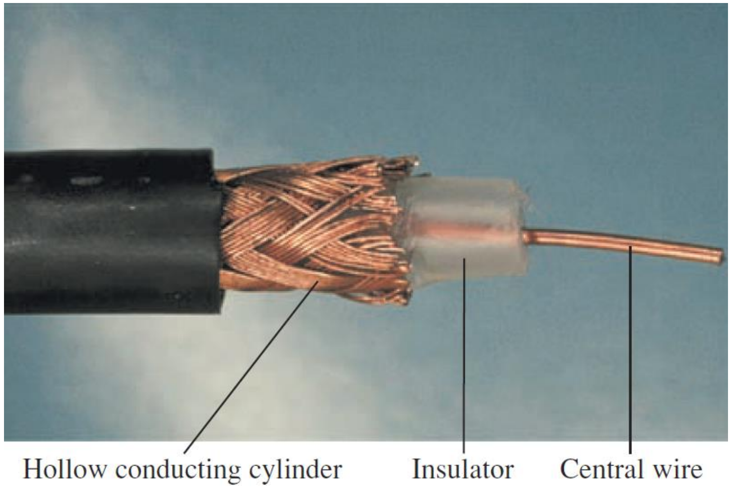
$$\Phi_B = BA = \mu_0 \frac{NA}{l} I$$

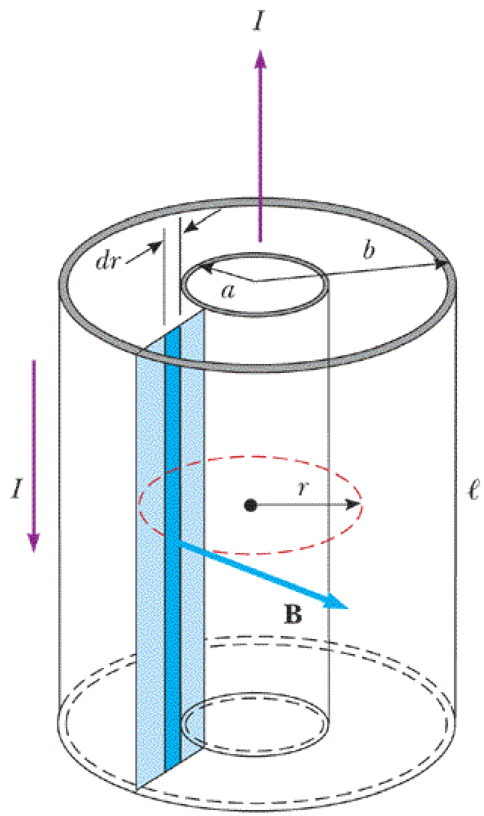
The inductance is

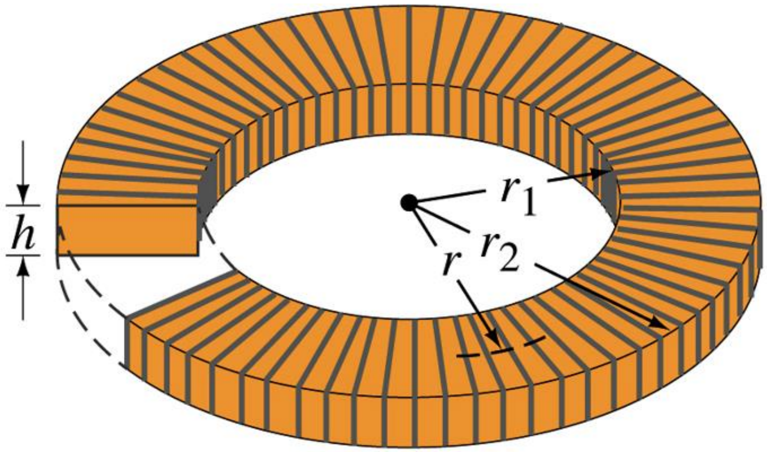
$$L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 A}{l} = \mu_0 \frac{N^2}{l^2}(Al) = \mu_0 n^2 V$$

Example - A coaxial cable

A long coaxial cable consists of two concentric cylindrical conductors of radii *a* and *b* and length *l*. The conductors carry current *I* in opposite directions. Find the **self-inductance** of this cable.







Inductance of a coaxial cable

Solution:

The magnetic field between the conductors:

$$B = \frac{\mu_0 I}{2\pi r}$$

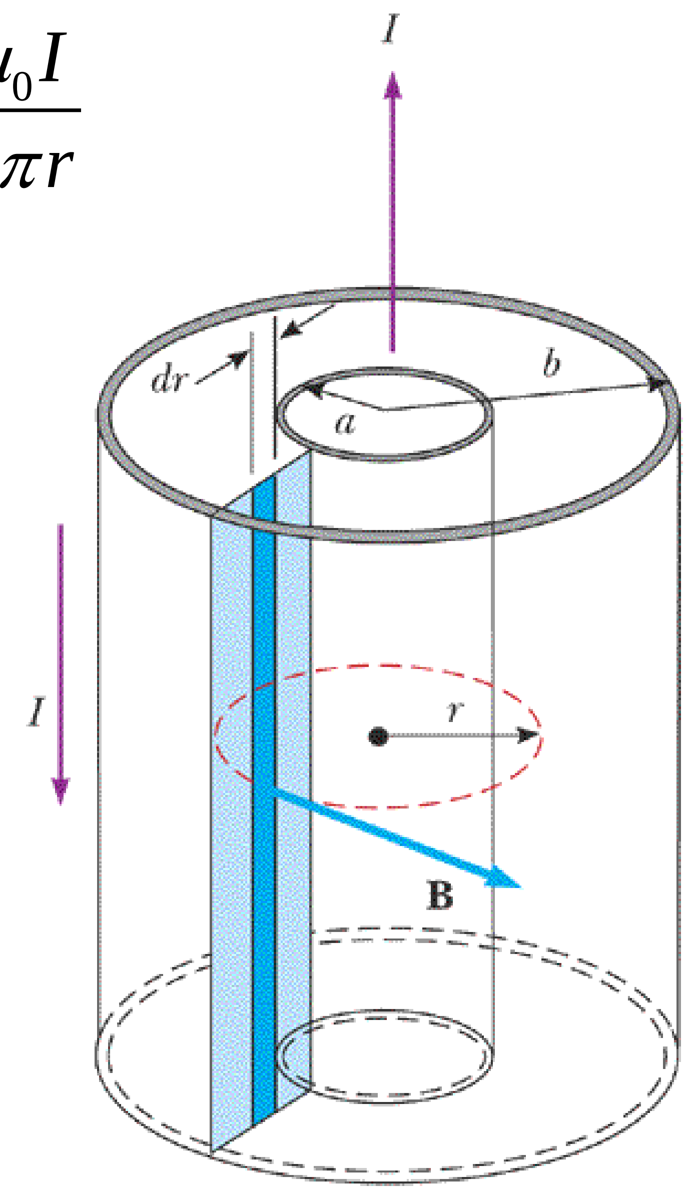
To find the magnetic flux through cross-section between the two conductors, we divide the rectangular cross section into strips of width d*r*.

$$\Phi_B = \iint \vec{B} \cdot d\vec{A} = \int_a^b \left(\frac{\mu_0 I}{2\pi r}\right)(l dr)$$

$$= \frac{\mu_0 I l}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

The inductance is

$$L = \frac{\Phi_B}{I} = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)$$

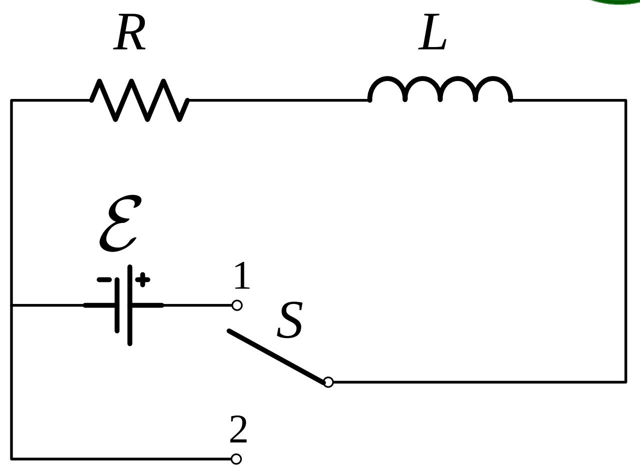


Problems

Ch28 Prob. 16, 48 (P656)

* § 5 *RL* Circuit

- ***RL* circuit:**
  - **The switch jumps to 1 from 2. From Kirchhoff's loop rule**

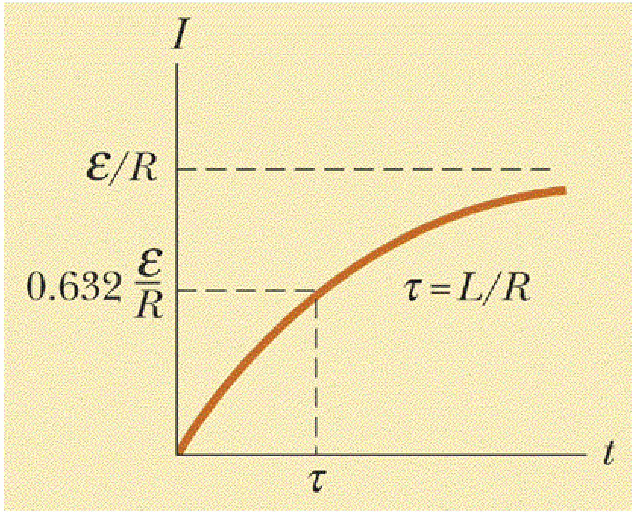


$$\mathcal{E} + \mathcal{E}_L - IR = 0$$

$$\mathcal{E} - L\frac{dI}{dt} - IR = 0, \qquad \frac{dI}{dt} = \frac{R}{L}\left(\frac{\mathcal{E}}{R} - I\right)$$

$$\int_0^I \frac{dI}{I - \frac{\mathcal{E}}{R}} = -\int_0^t \frac{R}{L}dt$$

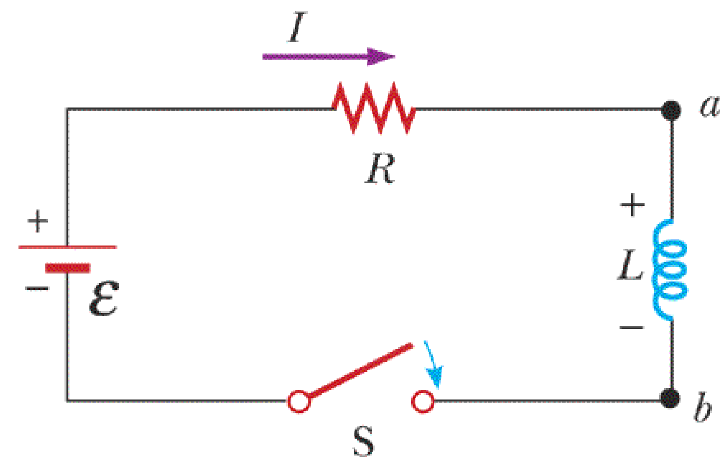
$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$



- **Time constant of the *RL* circuit:** $\tau = \dfrac{L}{R}$

§ 6 Energy Stored in a Magnetic Field

- **Starting with a *RL* circuit:**
  - **The switch jumps to 1 from 2.**



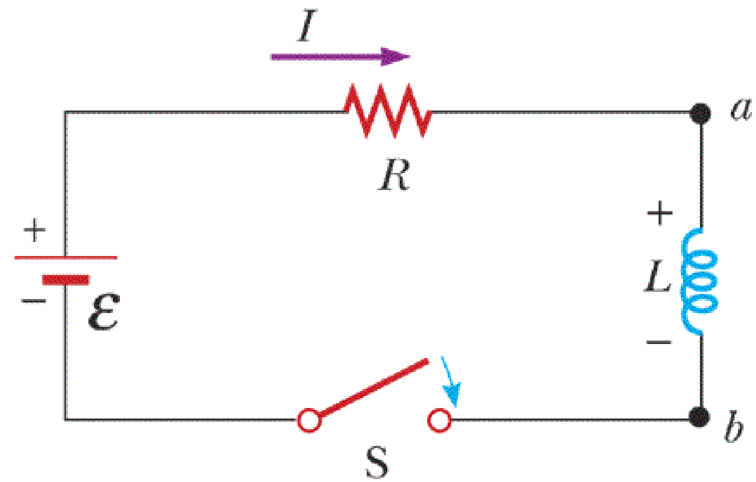
$$\mathcal{E} = IR + L\frac{dI}{dt}$$

$$\int_0^t \mathcal{E} I dt = \int_0^t I^2 R dt + \int_0^t LI\frac{dI}{dt}dt$$

  - **The term on left side:**
    The energy is supplied by the source.
  - **The first term on right side:**
    The energy is dissipated in the resistor.
  - **The second term on right side:**
    The energy that is delivered to the inductor and is stored in the magnetic field through the coil.

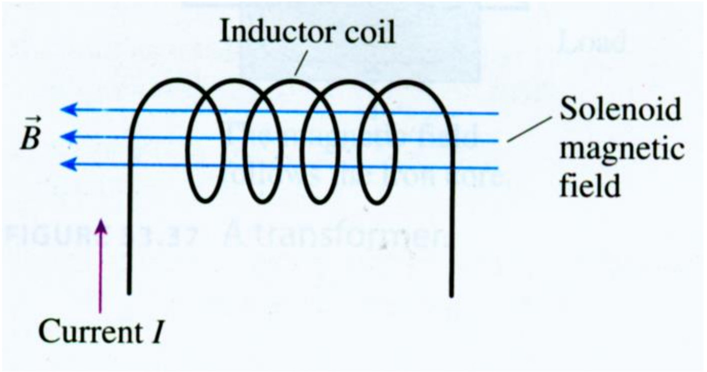
Energy stored in an inductor

$$\int_0^t \mathcal{E} I dt = \int_0^t I^2 R dt + \int_0^t LI \frac{dI}{dt} dt$$

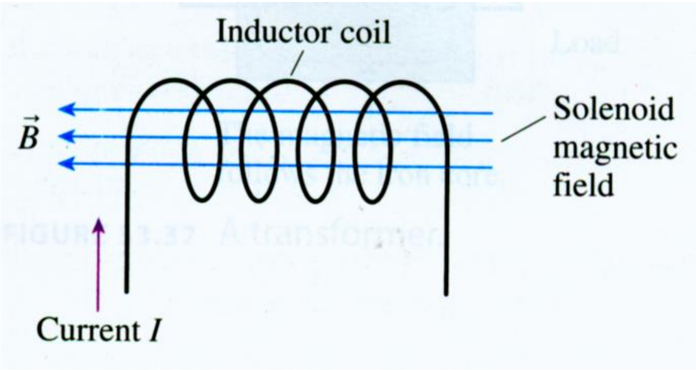


- **Energy stored in the inductor**

$$U_B = \int_0^t LI \frac{dI}{dt} dt = \int_0^I LIdI = \frac{1}{2} LI^2$$



- **Which one is the storehouse of the energy, the inductor or the magnetic field?**



- **Energy stored in magnetic field.**
  - **Take a solenoid as an example.**

$$L = \mu_0 n^2 V, \quad B = \mu_0 n I$$

$$U_B = \frac{1}{2} L I^2 = \frac{1}{2}\left(\mu_0 n^2 V\right)\left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0} V \propto \begin{cases} B^2 \\ V \end{cases}$$

  - **Energy is indeed stored in the space where the magnetic field exists.**

- **Energy density** $$u_B = \frac{U_B}{V} = \frac{B^2}{2\mu_0}$$
  - **For a non-uniformed magnetic field**

$$U_B = \iiint u_B dV = \iiint_V \left(\frac{B^2}{2\mu_0}\right) dV$$

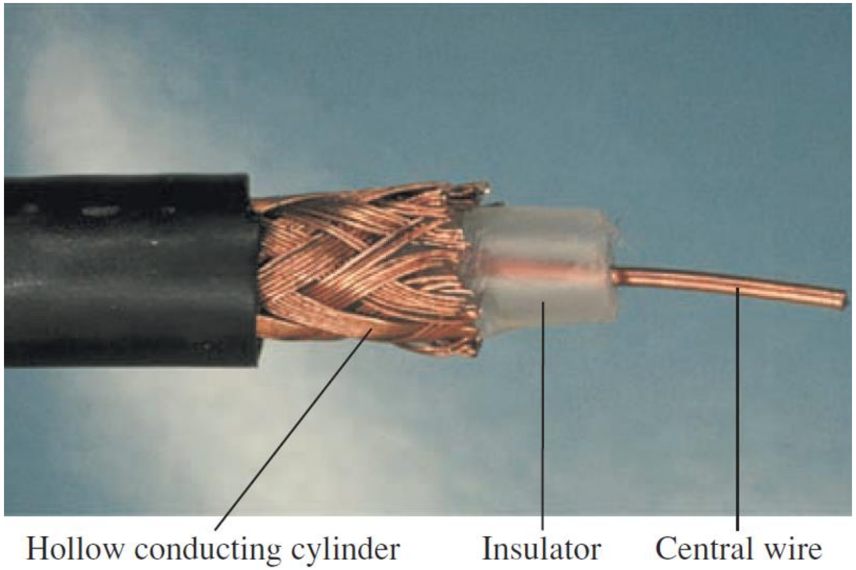
Energy in Electric and Magnetic Field

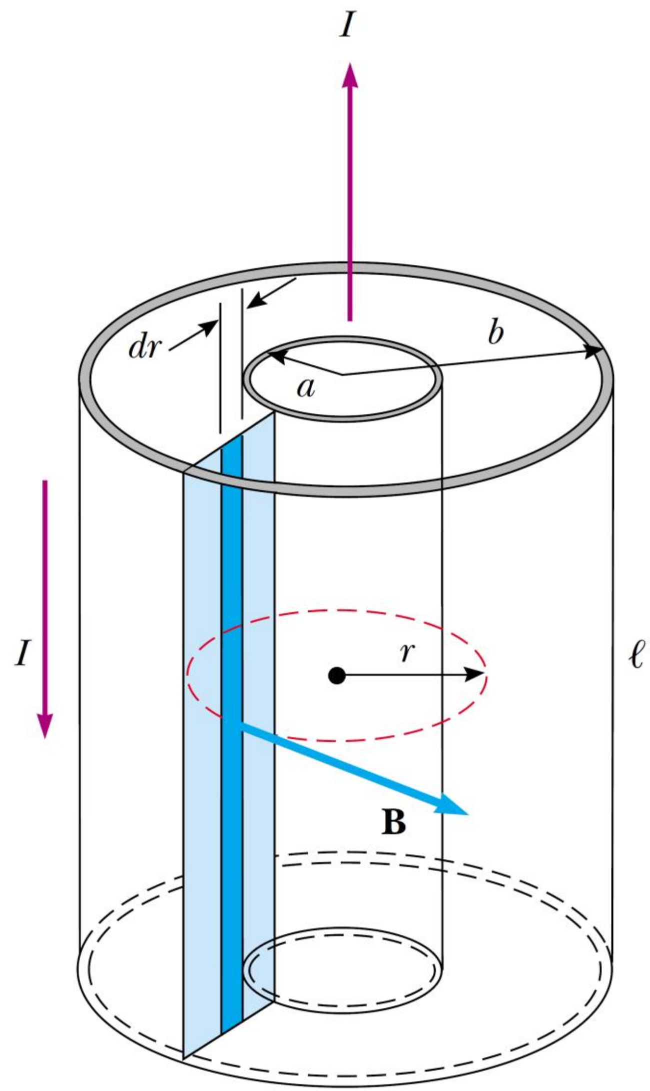
	Electric field	Magnetic field
Energy stored in the device	A capacitor stores energy $U_E = \frac{1}{2}C(\Delta V)^2$	An inductor stores energy $U_B = \frac{1}{2}LI^2$
Energy density in the field	$u_E = \frac{1}{2}\varepsilon_0 E^2$	$u_B = \frac{1}{2\mu_0}B^2$

Example

The energy stored in a coaxial cable

A long coaxial cable consists of two concentric cylindrical conductors of radii a and b and length l. The conductors carry current I in opposite directions. Find the energy stored in this cable.





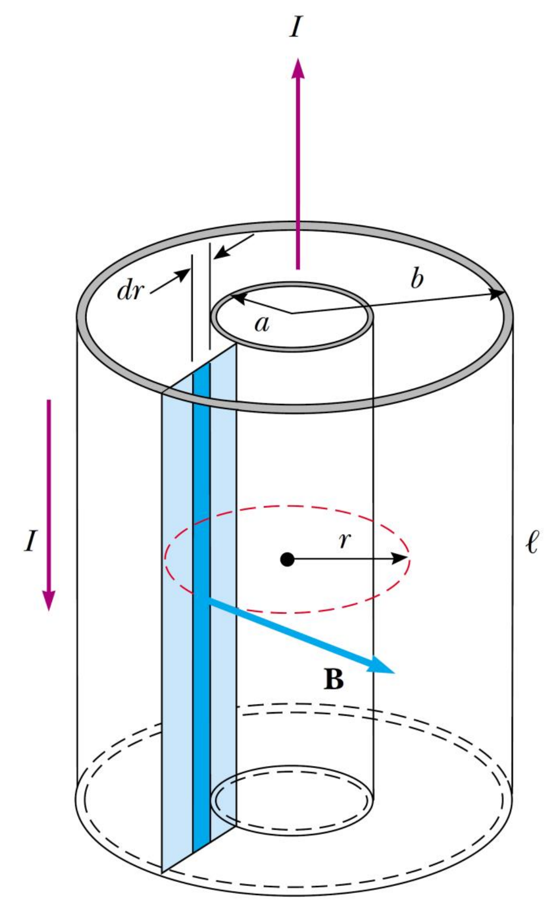
The energy stored in a coaxial cable

Solution I:

The magnetic field between the conductors is $B = \dfrac{\mu_0 I}{2\pi r}$

The magnetic field is zero inside the inner conductor $r < a$**,**

and outside the outer conductor $r > b$**.**



$$U_B = \iiint \left(\frac{B^2}{2\mu_0} \right) dV = \int_a^b \left[\frac{1}{2\mu_0} \left(\frac{\mu_0 I}{2\pi r} \right)^2 \right] (2\pi r l dr)$$

$$= \frac{\mu_0 I^2 l}{4\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a} \right)$$

Solution II:

$$U_B = \frac{1}{2} L I^2 = \frac{1}{2} \left[\frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a} \right) \right] I^2 = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a} \right)$$

Problems

Ch28 Prob. 22 (P657)