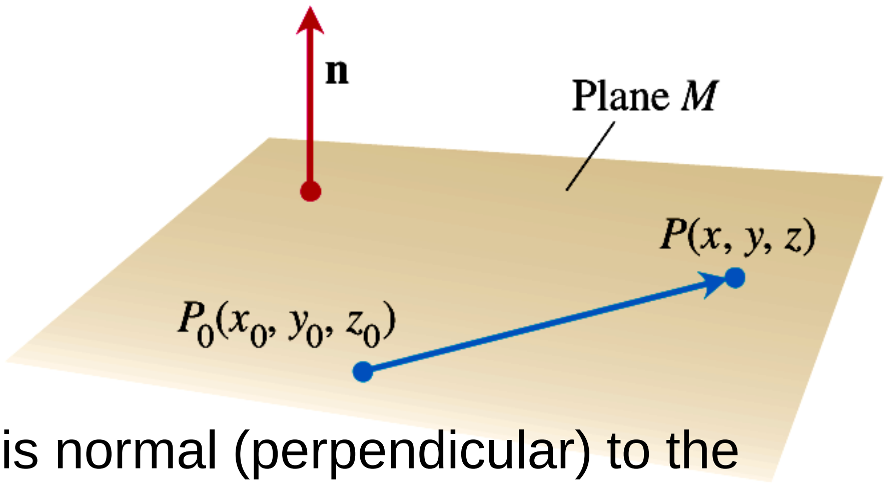
Lecture 11

Planes and Lines in Space

Equations for Planes in Space



A plane in space is determined by knowing a point on the plane and its "tilt" or orientation.

Suppose that plane $\boldsymbol{M}$ passes through a point $\boldsymbol{P_0(x_0, y_0, z_0)}$ and is normal (perpendicular) to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}.$ Then $\boldsymbol{M}$ is the sets of all points $\boldsymbol{P(x, y, z)}$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$. Thus, the dot product $\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$

This equation is equivalent to

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot \left[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}\right] = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Equations for Planes in Space

Equation for a plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

General form of equation:

$$Ax + By + Cz = D, \quad \text{where} \quad D = Ax_0 + By_0 + Cz_0$$

Two planes are parallel if and only if their normals are parallel, or

$\mathbf{n}_1 = k\mathbf{n}_2$ for some scalar k.

Equations for Planes in Space

Equation for a plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

General form of equation:

$Ax + By + Cz = D,$ where $D = Ax_0 + By_0 + Cz_0$

The normal vector of a plane is not unique: if $\boldsymbol{n}$ is a normal vector, then $k\boldsymbol{n}$ is also a normal vector ($k \neq 0$).

Finding an Equation for a Plane

Example. Find an equation for the plane through $P_0(-3,0,7)$

perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$

Solution The component equation is

$$5(x-(-3)) + 2(y-0) + (-1)(z-7) = 0.$$

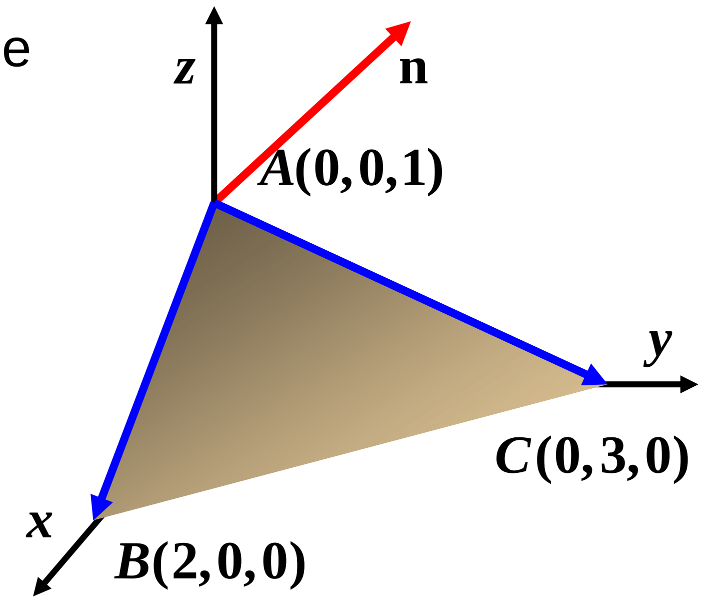
Simplifying, we obtain

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22.$$

Finding an Equation for a Plane Through Three Points

Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$.



Solution (I) We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&0&-1\\0&3&-1\end{vmatrix}=3\mathbf{i}+2\mathbf{j}+6\mathbf{k}=\mathbf{n}.$$

Finding an Equation for a Plane Through Three Points

Solution (I) (continued)

It is easy to see that $\mathbf{n}$ is normal to the plane.

We substitute the components of this vector

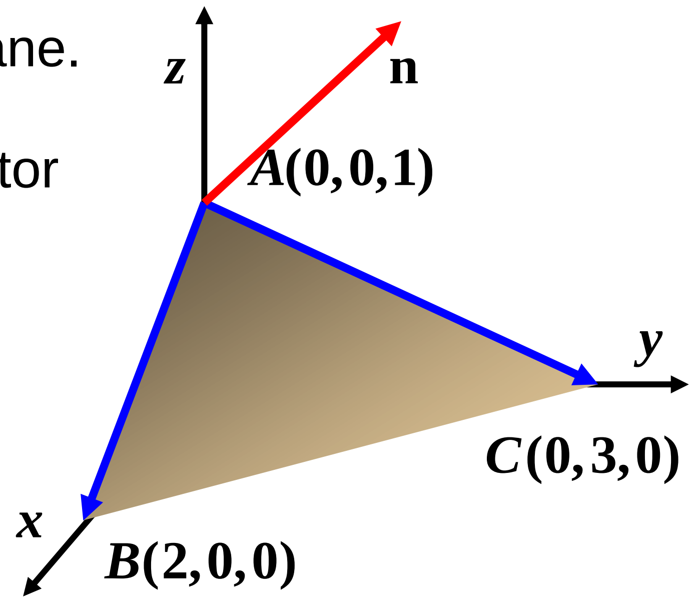
and the coordinate of $A(0,0,1)$ into

the component form of the equation

to obtain

$$3(x-0)+2(y-0)+6(z-1)=0$$

$$3x+2y+6z=6.$$



$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0.$$

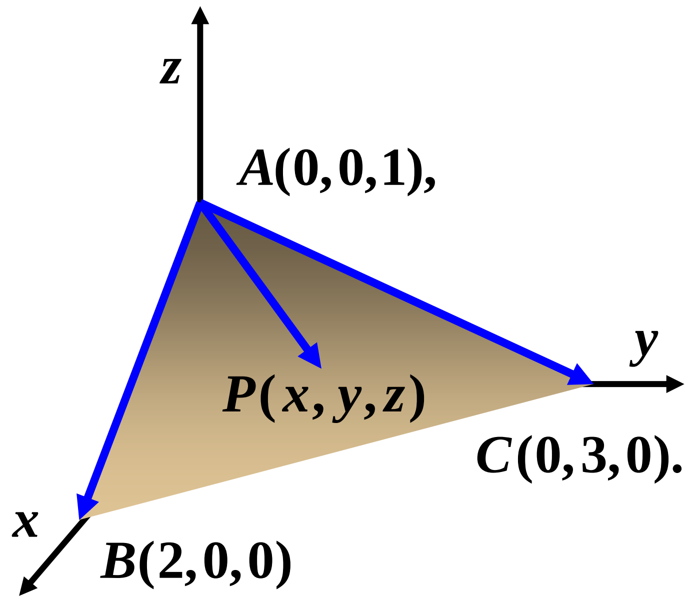
Finding an Equation for a Plane Through Three Points

Solution (II) Suppose that $P(x,y,z)$

is any point in the plane, then

$$\overrightarrow{AB}=(2,0,-1) \qquad \overrightarrow{AC}=(0,3,-1)$$

$$\overrightarrow{AP}=(x,y,z-1)$$



Since these three vector are coplanar

if and only if the point P lies in the plane,

so we have

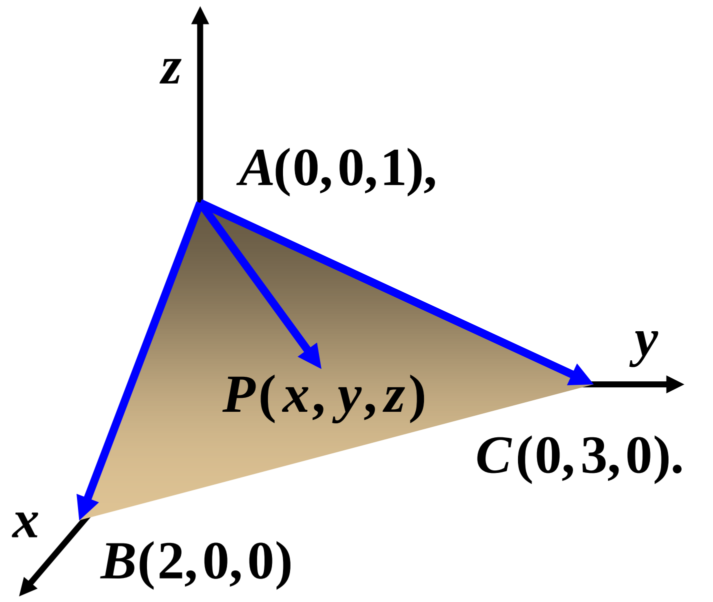
$$\left[\overrightarrow{AP},\overrightarrow{AB},\overrightarrow{AC}\right]=0,$$

Finding an Equation for a Plane Through Three Points

Solution (II) (continued)

that is

$$\begin{vmatrix} x & y & z-1 \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 0.$$



Expanded the determinant on the left

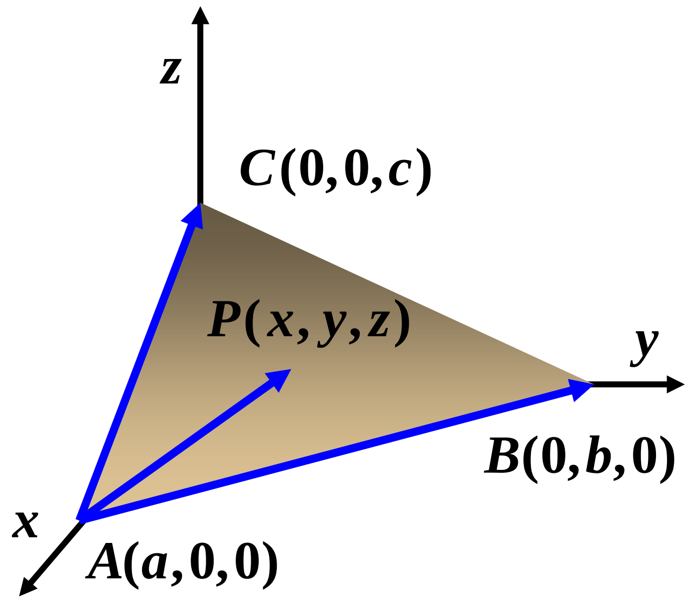
side of above equation, we have

$$3x + 2y + 6z = 6.$$

Intercept Form of the Equation for a Plane

In general, if the intercepts of the plane with the x-axis, y-axis and z-axis are $OA = a,\ OB = b$ and $OC = c,$ respectively. Then, just as the last example, we can obtain the equation for the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$



Equations for Planes in Space

General Equation for a plane

The equation can be rewritten in the form

$$\boldsymbol{Ax + By + Cz = D,} \quad \text{where} \quad \boldsymbol{D = Ax_0 + By_0 + Cz_0}$$

Therefore, the equation of any plane is a linear equation in three variables. Conversely, any linear equation in three variables represents a plane with normal vector $\mathbf{n} = (\boldsymbol{A, B, C})$. if $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are not all 0.

In fact, if $\boldsymbol{C \neq 0,}$ the equation can be written as

$$A(x-0) + B(y-0) + C\left(z - \frac{D}{C}\right) = 0.$$

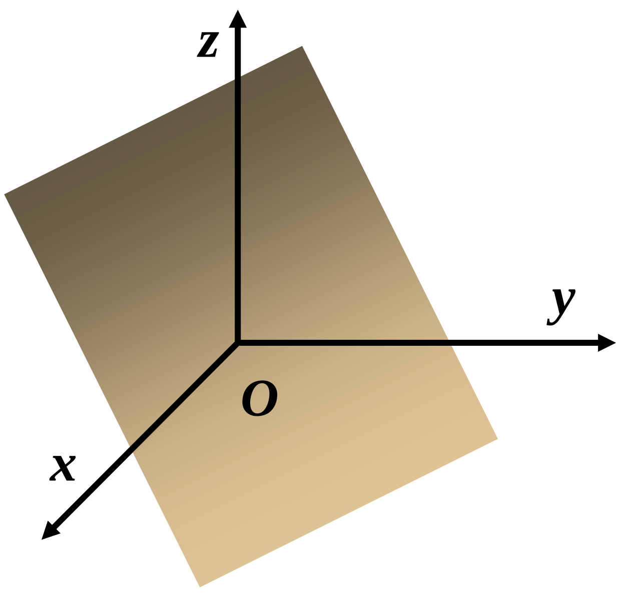
Some Planes with Special Locations

(1) If a given plane passes through the origin $O(0,0,0)$, then

$x = y = z = 0$ satisfy the general equation for the plane, so that $D = 0$.

Therefore, the equation of the plane through the origin is

$$Ax + By + Cz = 0.$$

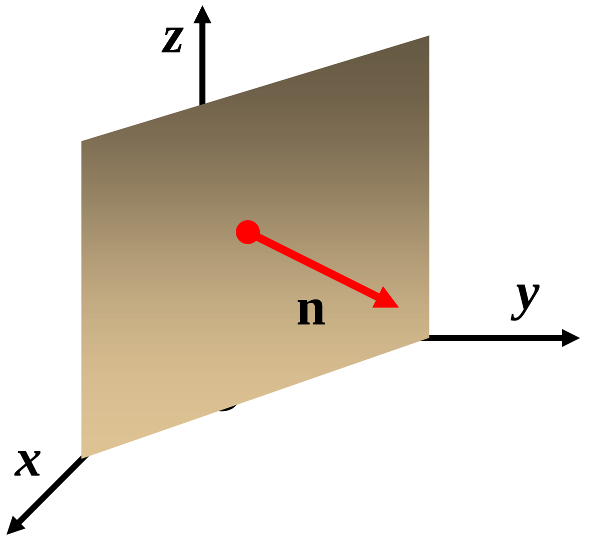


Some Planes with Special Locations

(2) If a given plane is parallel to the z-axis, then the normal vector $\mathbf{n}=(A,B,C)$ is orthogonal to $\mathbf{k}=(0,0,1)$, and $\mathbf{n}\cdot\mathbf{k}=C=0.$

Therefore, the equation of this plane is

$$Ax+By+D=0.$$



Similarly, the equations of planes which are parallel to the x-axis or y-axis are

$$By+Cz+D=0,\quad Ax+Cz+D=0,$$

respectively.

Some Planes with Special Locations

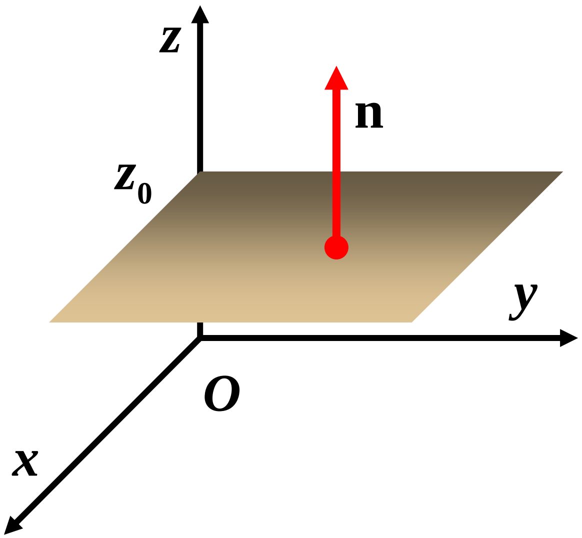
(3) If a given plane is orthogonal to the z-axis, then $\mathbf{n} // \mathbf{k}$ and so

$A = B = 0.$ Therefore, the equation of this plane is

$$Cz + D = 0$$

or

$$z = -\frac{D}{C} = z_0.$$



Similarly, the equations of planes which are orthogonal to the x-axis or y-axis are

$$Ax + D = 0, \quad By + D = 0,$$

respectively.

Relative Positions of two Planes

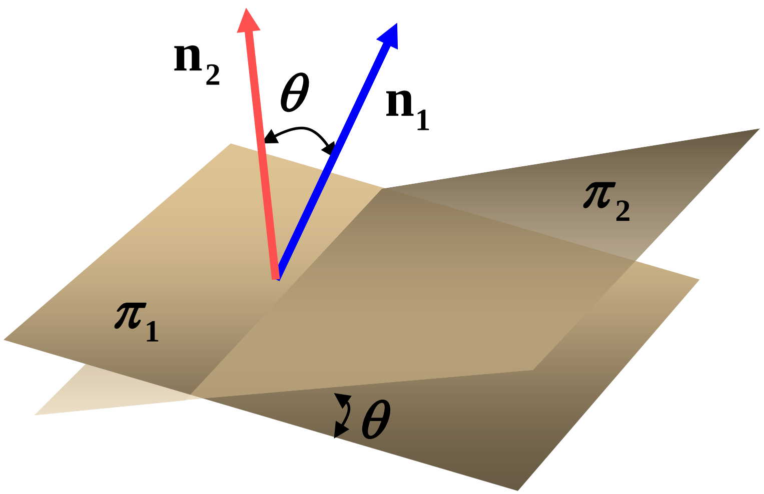
(1) Parallel but not coincident;

(2) Coincident;

(3) Intersecting (the intersection is a line).

Angle Between Planes

The angle between two intersecting planes is defined to be the (acute) angle determined by the normal vectors as shown in the figure.



Let

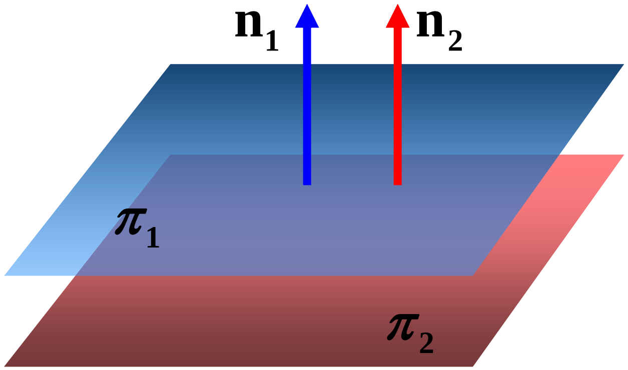
$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0, \qquad \pi_2 : A_2x + B_2y + C_2z + D_2 = 0.$$

There normal vectors can be chosen as $\mathbf{n}_1 = (A_1, B_1, C_1)$ and $\mathbf{n}_2 = (A_2, B_2, C_2),$ respectively. Then

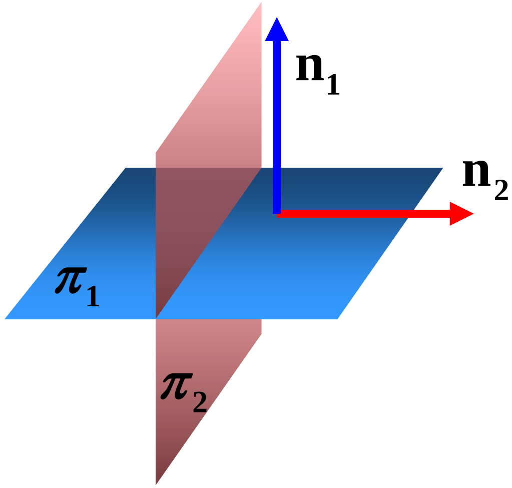
$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

Angle Between Planes

If two planes are parallel or orthogonal, their normal vector are also parallel or orthogonal.



$$\pi_1 // \pi_2 \iff \mathbf{n}_1 // \mathbf{n}_2 \iff \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$



$$\pi_1 \perp \pi_2 \iff \mathbf{n}_1 \perp \mathbf{n}_2$$

$$\iff A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

Relative Positions between two Planes

Example Discuss the relative position between the following planes:

$$\textbf{(1)}\ \pi_1: -x + 2y - z + 1 = 0, \qquad \pi_2: y + 3z - 1 = 0$$

Solution (1) Since

$$\cos\theta = \frac{|-1\times 0 + 2\times 1 - 1\times 3|}{\sqrt{(-1)^2 + 2^2 + (-1)^2}\cdot\sqrt{1^2 + 3^2}} = \frac{1}{\sqrt{60}}.$$

then these two planes intersect and the angle between them is

$$\theta = \arccos\frac{1}{\sqrt{60}}.$$

Relative Positions between two Planes

Example Discuss the relative position between the following planes:

$$(2)\ \ \pi_1 : 2x - y + z - 1 = 0, \qquad \pi_2 : -4x + 2y - 2z - 1 = 0$$

Solution (2) Since $\mathbf{a}_1 = \{2, -1, 1\}$, $\mathbf{a}_2 = \{-4, 2, -2\}$ and

$$\frac{2}{-4} = \frac{-1}{2} = \frac{1}{-2},$$

then these two planes are parallel. Meanwhile, $M(1,1,0) \in \pi_1$ but $M(1,1,0) \notin \pi_2$, these two planes are not coincident.

Relative Positions between two Planes

Example Discuss the relative position between the following planes:

(3) $\pi_1: 2x - y - z + 1 = 0, \qquad \pi_2: -4x + 2y + 2z - 2 = 0$

Solution (3) Since $\dfrac{2}{-4} = \dfrac{-1}{2} = \dfrac{-1}{2},$ these two planes are parallel.

Moreover, $M(1,1,0) \in \pi_1$ and $M(1,1,0) \in \pi_2,$ these two planes are coincident.

Relative Positions between two Planes

Example Find an equation for the plane π that passes through the point $(1,-2,0)$ and is parallel to the plane $\frac{1}{2}x+3y-4z+6=0.$

Solution Let the normal vector to the plane π be $\mathbf{n}$; then

$\mathbf{n}//\left(\frac{1}{2},3,-4\right)$ and so $\mathbf{n}=(1,6,-8)$ can be taken as the normal vector of π. Thus the equation of the plane π is

$$(x-1)+6(y+2)-8(z-0)=0$$

or

$$x+6y-8z+11=0.$$

Relative Positions between two Planes

Example Find an equation for the plane π that passes through the two points $P_1(-1,0,2)$, $P_2(1,1,1)$, and is perpendicular to the plane $x+y+z+1=0$.

Solution (I) Let $Ax+By+Cz+D=0$ be the equation for the plane π.

Since the two point P_1 and P_2 lie in the plane, we have

$$A-2C-D=0, \quad \text{and} \quad A+B+C+D=0.$$

Because π is perpendicular to the plane $x+y+z+1=0$, we have

$$A+B+C=0.$$

Then we have $A=2C, B=-3C, D=0.$ Therefore, the equation for π is

$$2x-3y+z=0.$$

Relative Positions between two Planes

Example Find an equation for the plane π that passes through the two points $P_1(-1,0,2)$, $P_2(1,1,1)$, and is perpendicular to the plane $x+y+z+1=0$.

Solution (II) Let the normal vector to the plane π be $\mathbf{n}$. Then $\mathbf{n} \perp \overrightarrow{P_1P_2}$, where $\overrightarrow{P_1P_2} = (2,1,-1)$. Also, $\mathbf{n} \perp (1,1,1)$ by the given conditions, then

$$\mathbf{n} = \overrightarrow{P_1P_2} \times (1,1,1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = (2,-3,1).$$

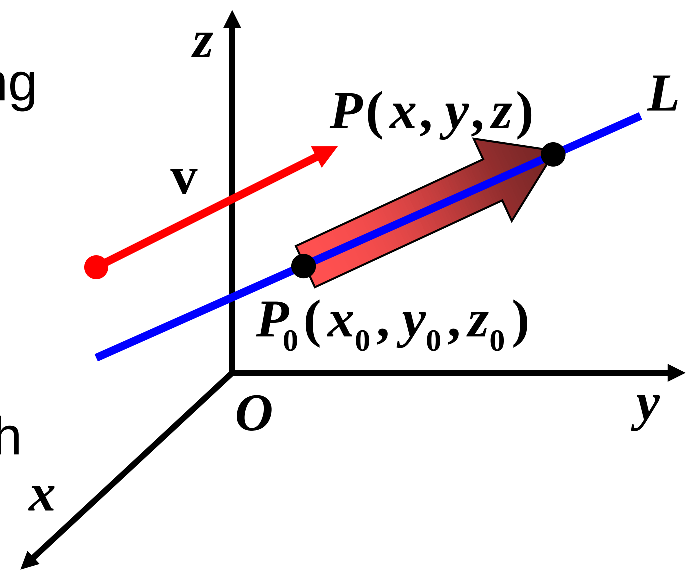
Thus, the equation of the plane π is

$$2x - 3y + z = 0.$$

Equations for Lines in Space

In the plane, a line is determined by a point and a number giving the slope of the line. Analogously, in space a line is determined by a point and a vector giving the direction of the line.

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$.



Equations for Lines in Space

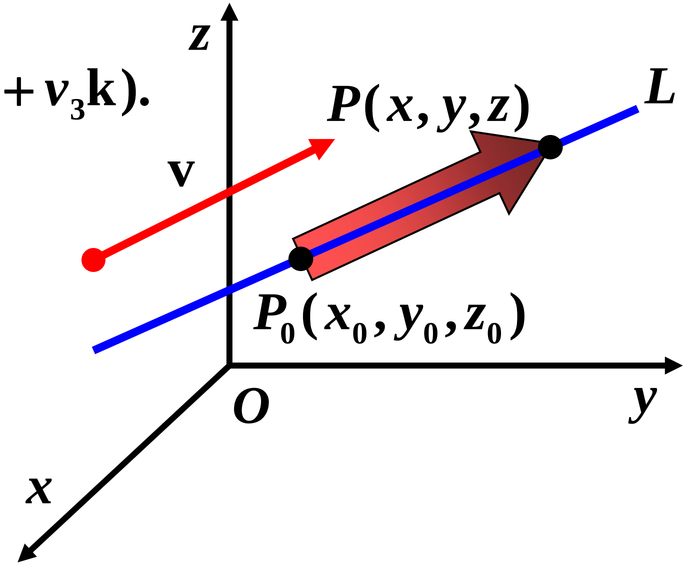
Thus $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar parameter t. The value of t depends on the location of the point P along the line, and the domain of t is $(-\infty, +\infty)$.

The expanded form of the equation $\overrightarrow{P_0P} = t\mathbf{v}$ is

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}).$$

and this last equation can be rewritten as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$= x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}). \quad (1)$$



Equations for Lines in Space

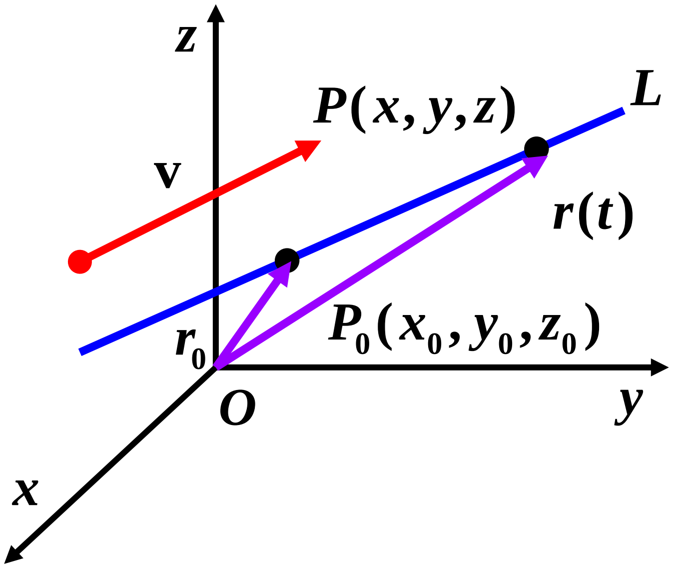
If $\mathbf{r}(t)$ is the position vector of a point $P(x, y, z)$ on the line and $\mathbf{r}_0$ is the position vector of point $P_0(x_0, y_0, z_0)$, then we have the following vector form for the equation of a line in space.

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$, parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < +\infty,$$

$\mathbf{r}(t)$ and $\mathbf{r}_0$ are the position vector of point $P(x, y, z)$ and $P_0(x_0, y_0, z_0)$ on the line, respectively.



Equations for Lines in Space

Equating the corresponding components of the two sides of Equation (1) Gives three scalar equations involving the parameter t :

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

These equations give us the standard parametrization of the line for the parameter interval $-\infty < t < +\infty$.

Parametric Equation for a Line

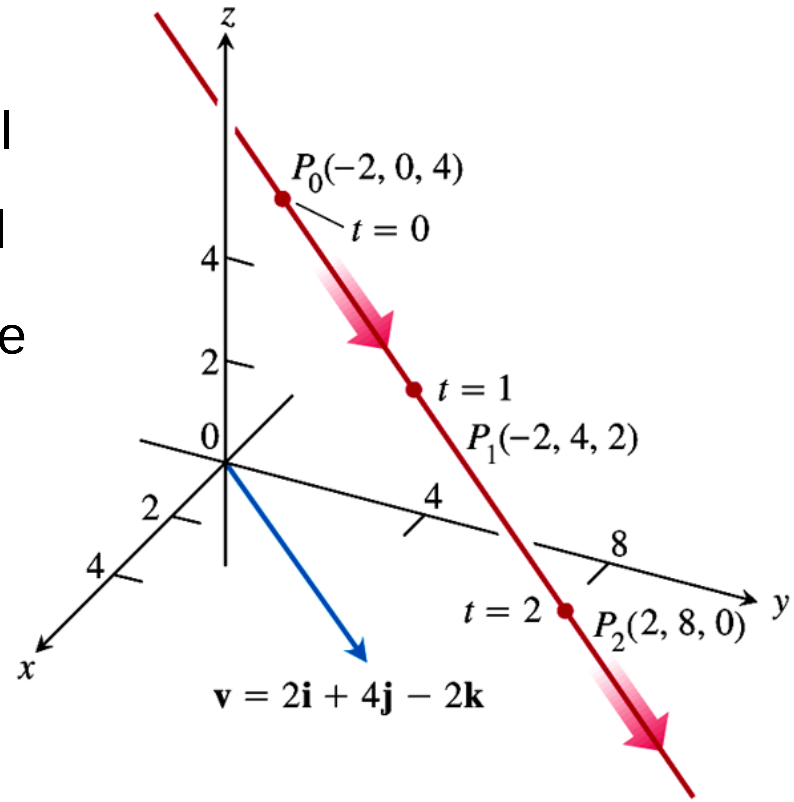
The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < +\infty. \qquad (2)$$

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}). \qquad \textbf{(1)}$$

Parametrizing a Line Through a Point Parallel to a Vector

Find parametric equations for the line through $(-2,0,4)$ parallel to $\mathbf{v}=2\mathbf{i}+4\mathbf{j}-2\mathbf{k}.$



Solution With $P_0(x_0,y_0,z_0)$ equal to $(-2,0,4)$ and $v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$ equal to $2\mathbf{i}+4\mathbf{j}-2\mathbf{k}$, equations (2) become

$$x=-2+2t,$$
$$y=4t,$$
$$z=4-2t.$$

$$x=x_0+tv_1,\quad y=y_0+tv_2,\quad z=z_0+tv_3,\quad -\infty<t<+\infty. \qquad \textbf{(2)}$$

Parametrizing a Line Through Two Points

Find parametric equations for the line through $P(-3,2,-3)$ and $Q(1,-1,4)$.

Solution The vector

$$\overrightarrow{PQ} = (1-(-3))\mathbf{i} + (-1-2)\mathbf{j} + (4-(-3))\mathbf{k} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$$

is parallel to the line, and equation (2) with $(x_0, y_0, z_0) = (-3,2,-3)$ give

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

We could have chosen $Q(1,-1,4)$ as the "base point" and written

$$x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of t .

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < +\infty. \qquad \textbf{(2)}$$

Equations for Lines in Space

Obviously, a point $P(x,y,z)$ lies on the line L if and only if the coordinates x,y,z of P satisfy the equations

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

If we eliminate the parameter t in the equations, we obtain the equivalent forms

$$\frac{x - x_0}{\underline{v_1}} = \frac{y - y_0}{\underline{v_2}} = \frac{z - z_0}{\underline{v_3}}.$$

the direction numbers of L

called the symmetric form equations of L.

If $v_1 = 0,$ this implies $x - x_0 = 0$ or $x = x_0.$ In this case, we write

$$x = x_0, \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}.$$

Lines of Intersection

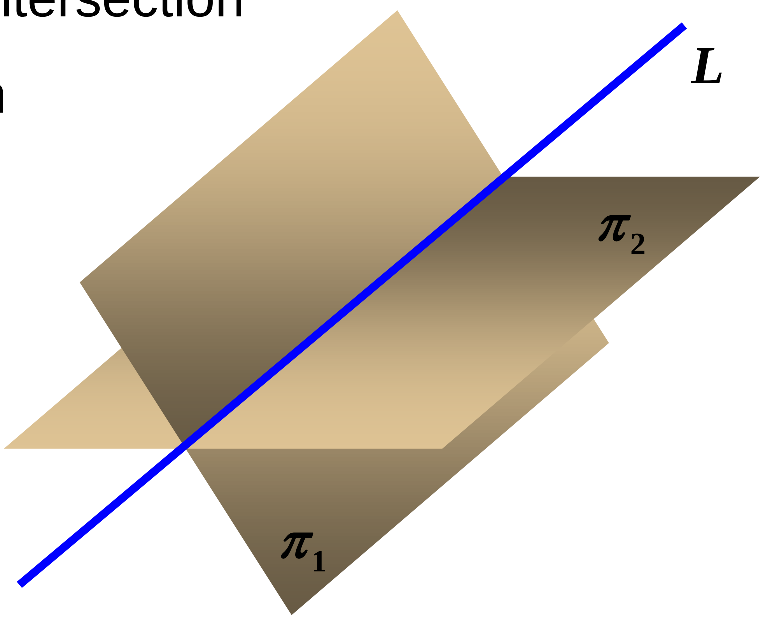
Two planes that are not parallel intersect in a line.

Suppose that the equations of two planes are

$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0 \quad \text{and} \quad \pi_2 : A_2x + B_2y + C_2z + D_2 = 0.$$

Then the equation for the line of intersection can be represented by the system of equations

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}.$$



This is called the general form of the equations of the line.

Finding a Vector Parallel to the Line of Intersection of Two Planes

Find a vector parallel to the line of intersection of the planes

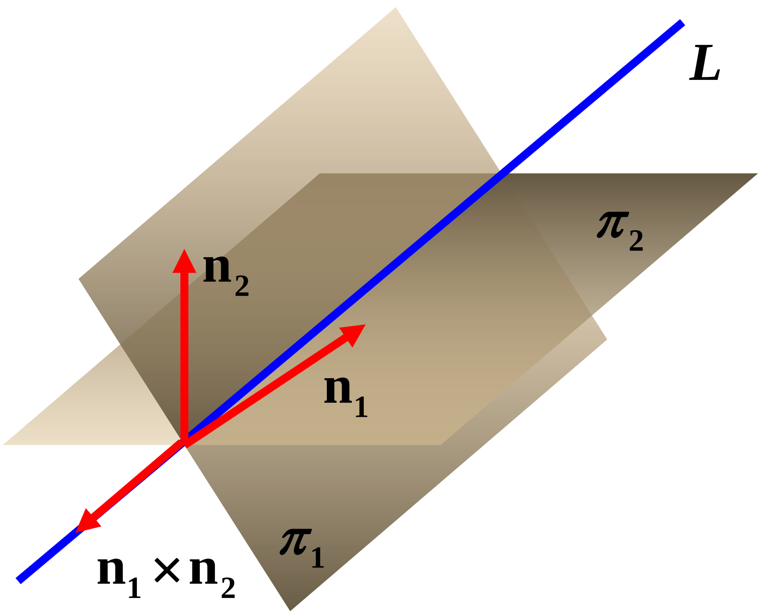
$3x - 6y - 2z = 15$ and $2x + y - 2z = 5.$

Solution As in the right figure,

the required vector is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$



Parametrizing the Line of Intersection of Two Planes

Find parametric equations for the line in which the planes

$3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Solution We find a vector parallel to the line and a point on the line and use equation (2). The last example identifies $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ as a vector parallel to the line. To find a point on the line, we can take any point common to the two planes. substituting $z = 0$ in the plane equations and solving for x and y simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = 3 + 14t, \quad y = -1 + 2t, \quad z = 15t.$$

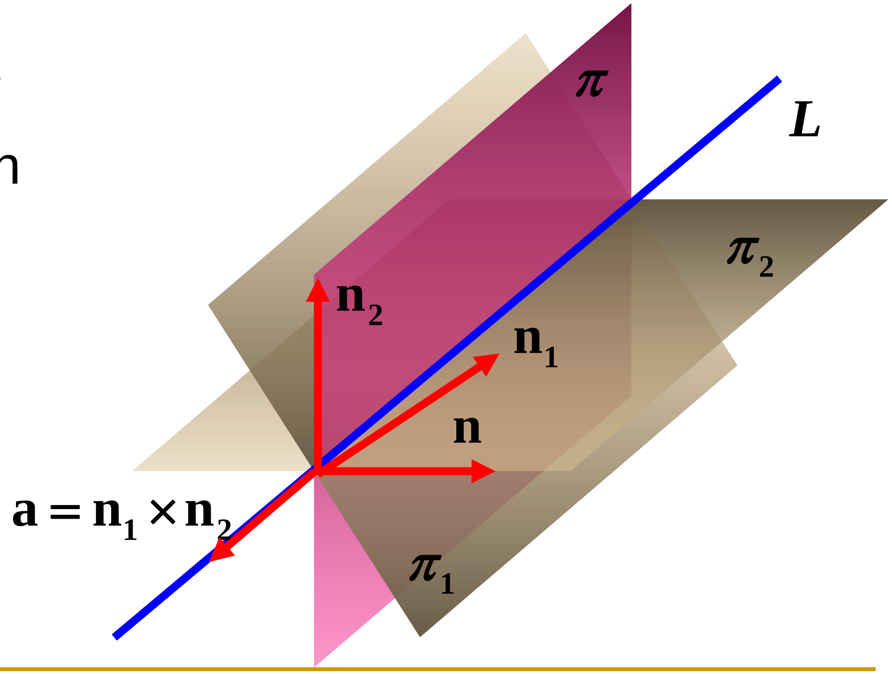
$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < +\infty. \qquad (2)$$

Finding the Equation for a Plane

Find the equation of a plane π that passes through the line L of intersection of the two planes $\pi_1: 2x+5y-3z+4=0$ and $\pi_2: -x-3y+z-1=0$ and is perpendicular to the plane π_2.

Solution (I) It is easy to see that $P_0(-7,2,0)$ is on L and the direction vector of L is $\mathbf{a}=(4,-1,1)$. By the assumptions, we know $\mathbf{n}\perp\mathbf{a}$.

Notice that $\mathbf{n}\perp\mathbf{n}_2$, then $\mathbf{n}$ can be chosen as



Finding the Equation for a Plane

Solution (I) (continued)

$$\mathbf{n} = \mathbf{a} \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ -1 & -3 & 1 \end{vmatrix} = (2, -5, -13).$$

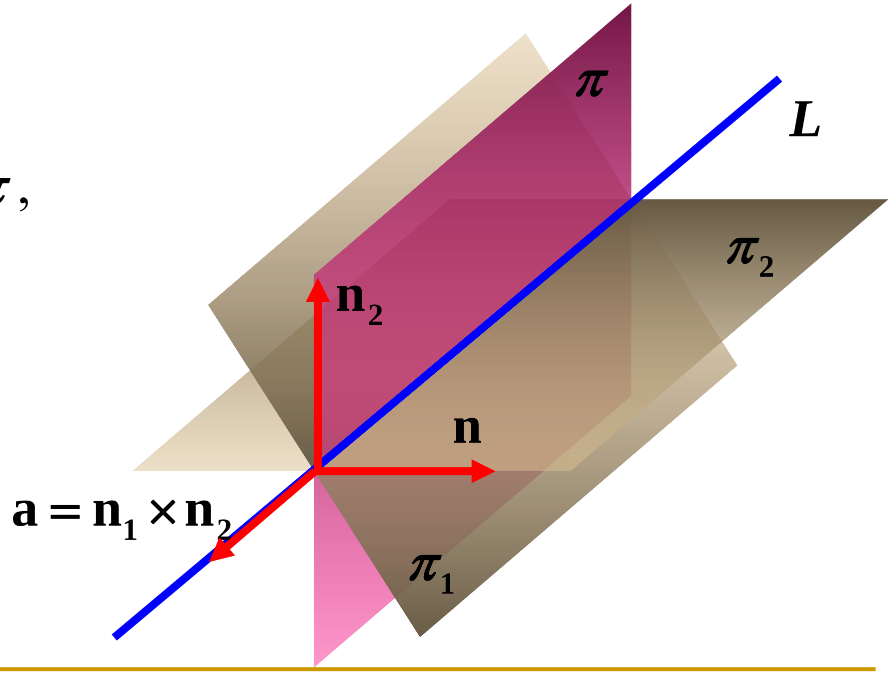
Since $P_0(-7, 2, 0)$ lies in the plane π,

the equation of π is

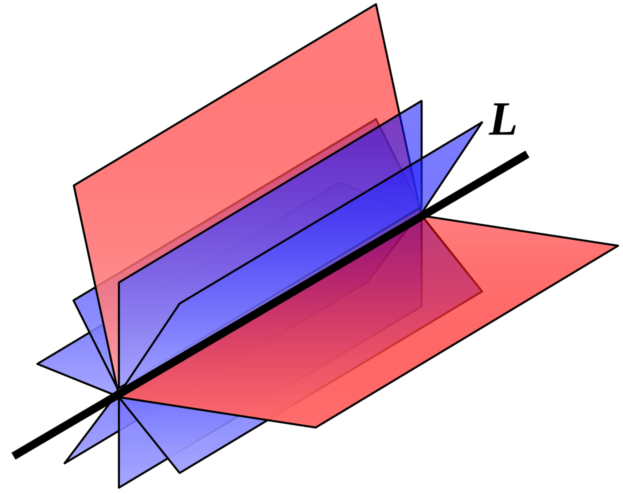
$$2(x+7) - 5(y-2) - 13z = 0,$$

or

$$2x - 5y - 13z + 24 = 0.$$



The Method of Pencil of Planes



Let the equation of L be

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$$

Then the equation

$$A_1x + B_1y + C_1z + D_1 + t(A_2x + B_2y + C_2z + D_2) = 0,$$

or

The equation of the pencil of planes through line L

$$(A_1 + tA_2)x + (B_1 + tB_2)y + (C_1 + tC_2)z + (D_1 + tD_2) = 0,$$

where the parameter t is an arbitrary real constant, represents all planes through line L except the plane $A_2x + B_2y + C_2z + D_2 = 0.$

Finding the Equation for a Plane

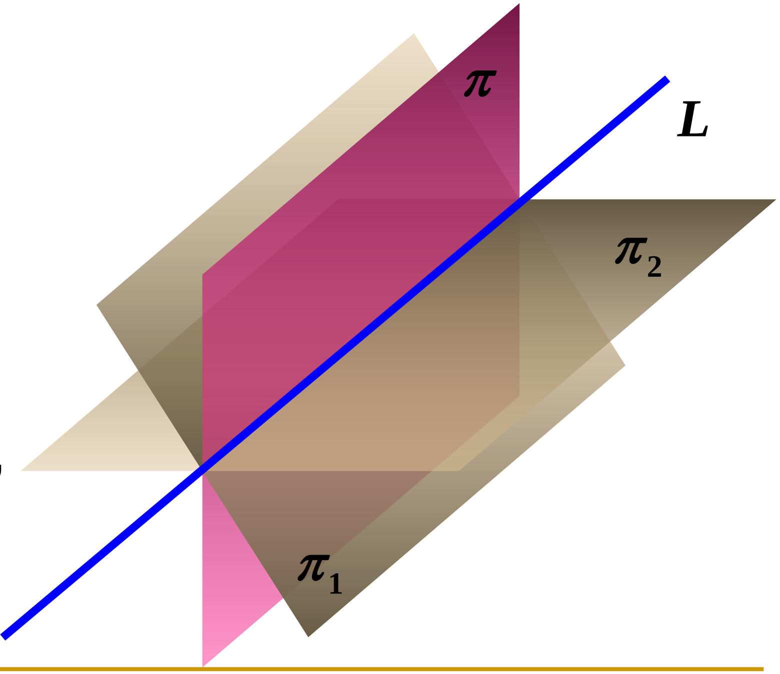
Find the equation of a plane π that passes through the line L of intersection of the two planes $\pi_1: 2x+5y-3z+4=0$ and $\pi_2: -x-3y+z-1=0$ and is perpendicular to the plane π_2.

Solution (II)

The equation of the pencil of planes through L is

$$(2-t)x+(5-3t)y+(-3+t)z+(4-t)=0.$$

Since π is perpendicular to the plane π_2, we have



Finding the Equation for a Plane

$$(2-t)x+(5-3t)y+(-3+t)z+(4-t)=0.$$

Solution (II) (continued)

$$(2-t)(-1)+(5-3t)(-3)+(-3+t)(1)=0.$$

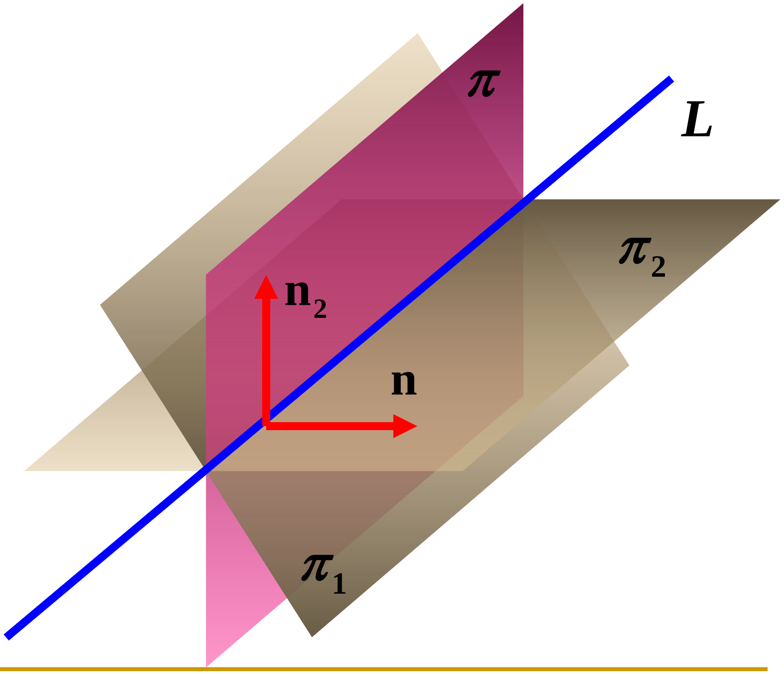
Solving this equation, we obtain that

$t=\dfrac{20}{11},$ and therefore the equation of

π is

$$2x-5y-13z+24=0.$$

Compare this result with the previous result, it is easy to see that this method serves just as the previous method.



$\pi_2 : -x-3y+z-1=0$

Relative Positions Between Two Lines

(1) Parallel and distinct;

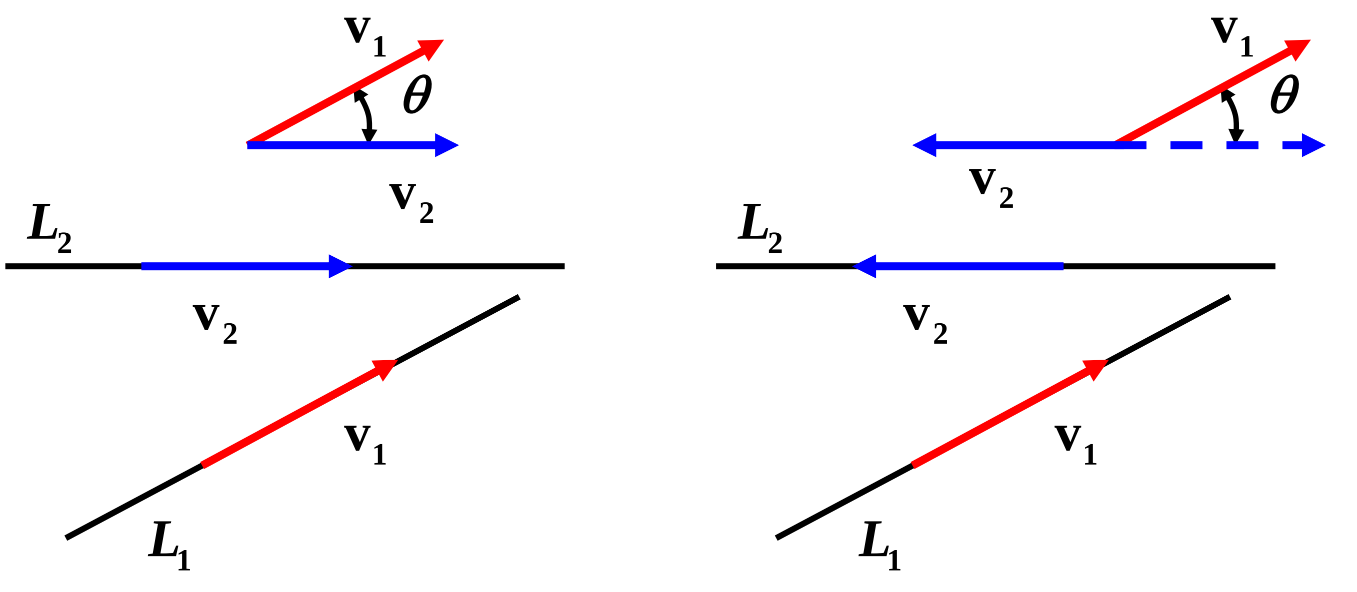
(2) Parallel and coincident;

(3) Intersecting (the intersection is a point);

(4) Skewing.

Relative Positions Between Two Lines

The included angle between two lines is defined as the acute angle between their direction vectors. Thus, two lines are parallel (or orthogonal) iff their direction vectors are parallel (or orthogonal).



Relative Positions Between Two Lines

Let

$$L_1: \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad \text{and} \quad L_2: \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

be two given lines. Then their direction vectors can be chosen as

$\mathbf{a}_1 = (l_1, m_1, n_1)$ and $\mathbf{a}_2 = (l_2, m_2, n_2)$, respectively. By the formula of

include angle between two vectors, we have

$$\cos\theta = \frac{|\mathbf{a}_1 \cdot \mathbf{a}_2|}{\|\mathbf{a}_1\|\|\mathbf{a}_2\|} = \frac{|l_1 l_2 + m_1 m_2 + n_1 n_2|}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}.$$

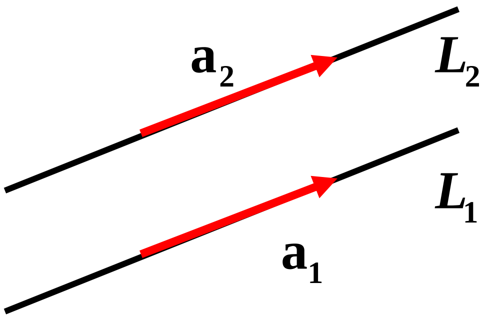
By this formula, we can easily find the angle between two lines.

Relative Positions Between Two Lines

Then, by the necessary and sufficient condition for two vectors to be parallel and orthogonal, we obtain:

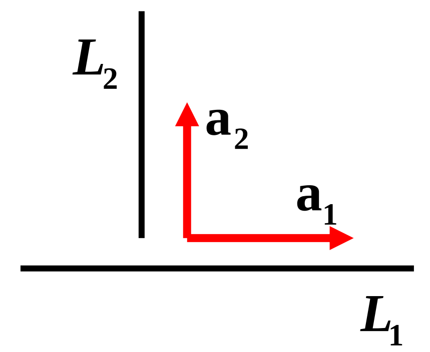
$$L_1 \,//\, L_2 \iff \mathbf{a}_1 \,//\, \mathbf{a}_2$$

$$\iff l_1 : m_1 : n_1 = l_2 : m_2 : n_2$$



$$L_1 \perp L_2 \iff \mathbf{a}_1 \perp \mathbf{a}_2$$

$$\iff l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$



Relative Positions Between Two Lines

Discuss the position relationships between the two lines

$$L_1: \frac{x-2}{-1} = \frac{y+2}{2} = \frac{z+1}{3} \quad \text{and} \quad L_2: \frac{x-5}{2} = \frac{y-2}{1} = \frac{z-4}{1}$$

If they intersect, find the point of intersection. If they are coplanar, find the plane π in which they lie.

Solution The direction vectors of L_1 and L_2 are

$$\mathbf{a}_1 = (-1, 2, 3) \quad \text{and} \quad \mathbf{a}_2 = (2, 1, 1)$$

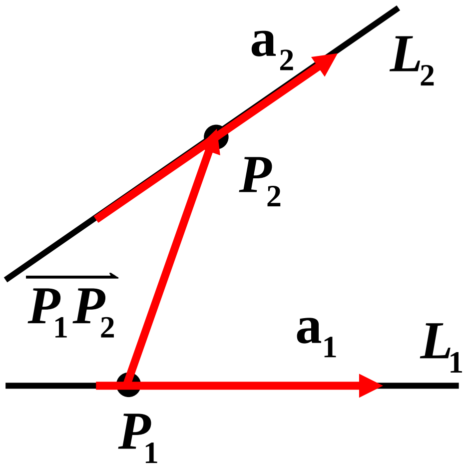
respectively. Then the cosine of the angle between these two line is

$$\cos\theta = \frac{|l_1 l_2 + m_1 m_2 + n_1 n_2|}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}} = \frac{|(-1)(2) + (2)(1) + (3)(1)|}{\sqrt{(-1)^2 + 2^2 + 3^2}\sqrt{2^2 + 1^2 + 1^2}} = \frac{3}{2\sqrt{21}}.$$

Then these two lines are neither parallel nor orthogonal.

Relative Positions Between Two Lines

Solution (continued) Are they coplanar?



It is easy to see that point $P_1(2,-2,-1)$ is on the line L_1 and $P_2(5,2,4)$ is on L_2.

Then, L_1 and L_2 are coplanar, if and only if $\mathbf{a}_1, \mathbf{a}_2$ and $\overrightarrow{P_1P_2}$ are coplanar. Since

$$\left[\overrightarrow{P_1P_2}, \mathbf{a}_1, \mathbf{a}_2\right] = \begin{vmatrix} 3 & 4 & 5 \\ -1 & 2 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 0,$$

Do they intersect?

Thus L_1 and L_2 are coplanar. Therefore, these two line intersect.

$$L_1: \frac{x-2}{-1} = \frac{y+2}{2} = \frac{z+1}{3} \qquad L_2: \frac{x-5}{2} = \frac{y-2}{1} = \frac{z-4}{1}$$

Relative Positions Between Two Lines

Solution (continued) Where is the point of intersection?

Parametrizing L_1 and L_2, we have

$$L_1 : \begin{cases} x = 2 - t \\ y = -2 + 2t \\ z = -1 + 3t \end{cases} \quad \text{and} \quad L_2 : \begin{cases} x = 5 + 2s \\ y = 2 + s \\ z = 4 + s \end{cases}.$$

Solving the equations

$$2 - t = 5 + 2s, \quad -2 + 2t = 2 + s, \quad -1 + 3t = 4 + s,$$

we find $t = 1, s = -2.$ Therefore, the point of intersection of these two lines is $(1, 0, 2).$

$$L_1 : \frac{x-2}{-1} = \frac{y+2}{2} = \frac{z+1}{3} \qquad L_2 : \frac{x-5}{2} = \frac{y-2}{1} = \frac{z-4}{1}$$

Relative Positions Between Two Lines

Solution (continued) What is the equation of the plane?

Since L_1 and L_2 are coplanar, we will find the equation of the plane.

The normal vector of the plane is

$$\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ 2 & 1 & 1 \end{vmatrix} = (-1, 7, -5).$$

Then the equation of the plane is

$$(-1)(x-2) + (7)(y+2) + (-5)(z+1) = 0,$$

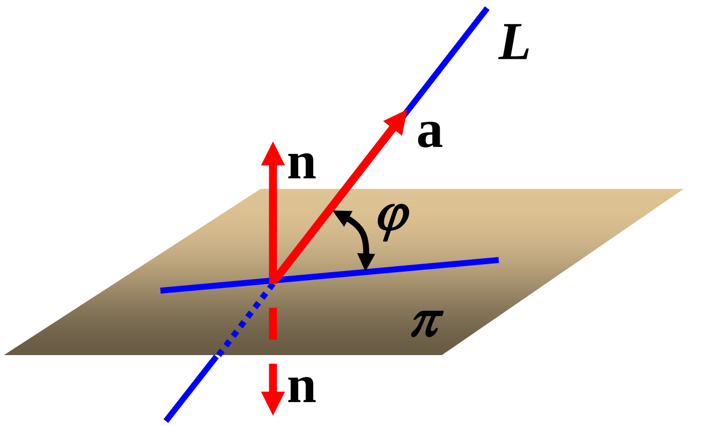
or

$$x - 7y + 5z - 11 = 0.$$

$$L_1 : \frac{x-2}{-1} = \frac{y+2}{2} = \frac{z+1}{3} \qquad L_2 : \frac{x-5}{2} = \frac{y-2}{1} = \frac{z-4}{1}$$

Angle between a Line and a Plane

The included angle between the line L and the plane π is defined as the acute angle φ between L and its projection vector in the plane π.



Let $\pi : Ax + By + Cz + D = 0$ be a plane and

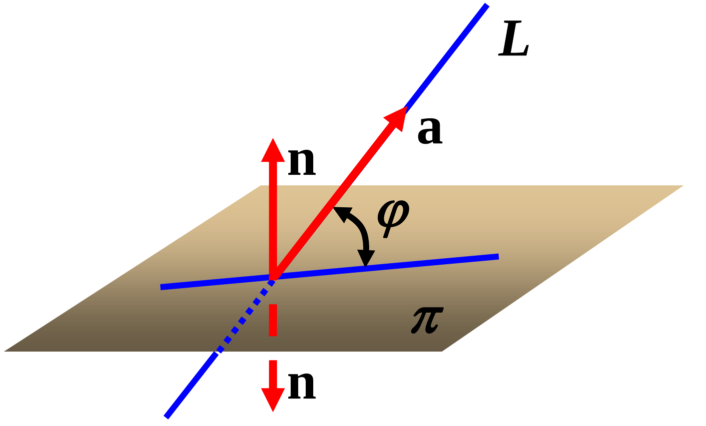
$L : \dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$ be a line. Then the include angle between the direction vector $\mathbf{a} = (l, m, n)$ of L and the normal vector $\mathbf{n}$ of π is $\dfrac{\pi}{2} - \varphi$ or $\dfrac{\pi}{2} + \varphi$.

Position Relationships Between a Line and a Plane

Therefore

$$\sin\varphi = \left|\cos\left(\frac{\pi}{2} \pm \varphi\right)\right| = |\cos(\mathbf{a},\mathbf{n})|$$

$$= \frac{|Al + Bm + Cn|}{\sqrt{A^2 + B^2 + C^2}\sqrt{l^2 + m^2 + n^2}}.$$



Then, it is easy to see that

$$L // \pi \Longleftrightarrow \mathbf{a} \perp \mathbf{n} \Longleftrightarrow Al + Bm + Cn = 0,$$

$$L \perp \pi \Longleftrightarrow \mathbf{a} // \mathbf{n} \Longleftrightarrow l : m : n = A : B : C.$$

$\pi : Ax + By + Cz + D = 0 \quad L : \dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$

Position Relationships Between a Line and a Plane

Find the point of intersection of the line L and the plane π, and the include angle between L and π, where

$$L: \frac{x-1}{1} = \frac{y+2}{-2} = \frac{z}{2}, \quad \pi: x+4y-z+1=0.$$

Solution The parametric equations of L are

$$x = 1+t, \quad y = -2-2t, \quad z = 2t, \quad -\infty < t < +\infty.$$

Suppose that $P(x, y, z)$ is a point of intersection of L and π. Then the coordinates of P must satisfy both the equations of L and of π. Substituting the expression for x, y and z from the parametric equations into the equation of π, we obtain

Position Relationships Between a Line and a Plane

Solution (continued) $x = 1 + t, \quad y = -2 - 2t, \quad z = 2t, \quad -\infty < t < +\infty.$

$$(1 + t) + 4(-2 - 2t) - 2t + 1 = 0.$$

Solving the equation, we obtain $t = -\frac{2}{3}$. Substituting this value into the parametric equations of L, we find $x = \frac{1}{3}, \quad y = -\frac{2}{3}, \quad z = -\frac{4}{3}.$

Therefore, the point of intersection is $\left(\frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}\right).$

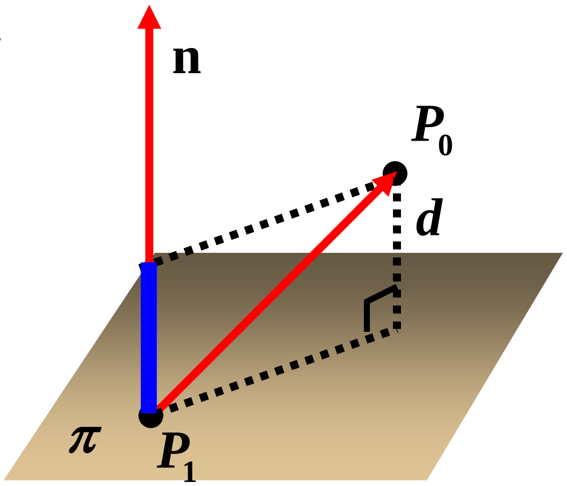
Since the direction vector of L is $\mathbf{a} = (1, -2, 2)$ and the normal vector to π is $\mathbf{n} = (1, 4, -1),$ the included angle between L and π is

$$\varphi = \arcsin \frac{|1 \times 1 + 4 \times (-2) + (-1) \times 2|}{\sqrt{1^2 + 4^2 + (-1)^2}\sqrt{1^2 + (-2)^2 + 2^2}} = \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

$L: \frac{x-1}{1} = \frac{y+2}{-2} = \frac{z}{2}, \quad \pi: x + 4y - z + 1 = 0.$

The Distance from a Point to a Plane

Let $\pi: Ax + By + Cz + D = 0$ be a given plane, and $P_0(x_0, y_0, z_0)$ be a given point which does not lie in the plane π. We choose arbitrarily a point $P_1(x_1, y_1, z_1)$ in the plane π and draw a vector $\overrightarrow{P_1P_0}$. Then the distance from a point to the plane π is equal to the absolute value of the projection of the vector $\overrightarrow{P_1P_0}$ onto the normal vector $\mathbf{n}$ of π.



Thus, by the formula of projection, we get

$$d = |\overrightarrow{P_1P_0} \cdot \mathbf{n}^\circ|.$$

The Distance from a Point to a Plane

Since

$$\overrightarrow{P_1P_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1) \quad \text{and} \quad \mathbf{n}^\circ = \frac{1}{\sqrt{A^2 + B^2 + C^2}}(A, B, C).$$

Therefore

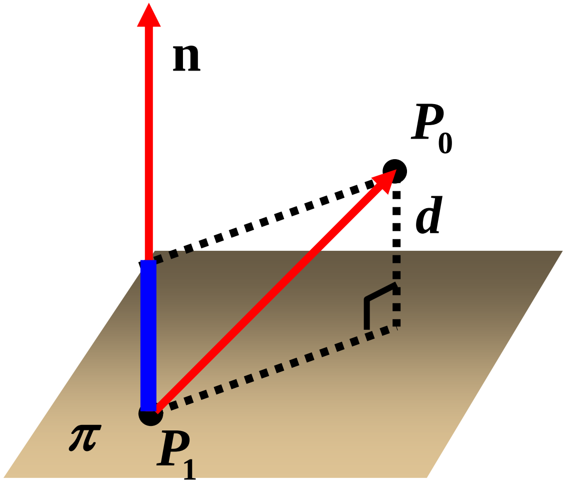
$$d = \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{\sqrt{A^2 + B^2 + C^2}}.$$

Notice that P_1 lies in the plane π, so that

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

Thus the formula of the distance is

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$



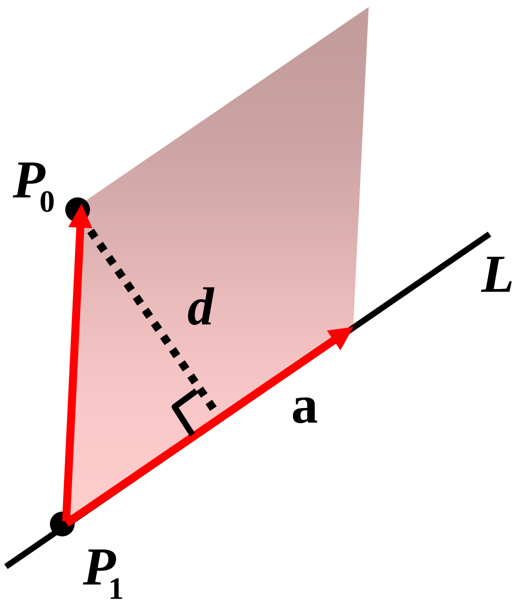
$d = |\overrightarrow{P_1P_0} \cdot \mathbf{n}^\circ|$

The Distance from a Point to a Line

If $P_0(x_0, y_0, z_0)$ does not lie on the line L and $P_1(x_1, y_1, z_1)$ is an arbitrary point on L, then the distance from P_0 to the line L is

$$d = \frac{\| \overrightarrow{P_1P_0} \times \mathbf{a} \|}{\| \mathbf{a} \|},$$

where $\mathbf{a}$ is the direction vector of L.



Review

- Equations for planes in space
- The position relationships between two planes
- Equations for lines in space
- The position relationships between two lines
- The distance from a point to a plane (line)