6 *The Laplace Transform*

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Outline

- The Laplace Transform
  - Introduction
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Introduction

不知道在说什么 也没必要吧

- The Laplace Transform is a more general continuous-time signal and system representation based on complex exponential signals.
  - There are some functions of interest, such as the ramp function which do not have a Fourier transform.
  - We wish to determine a system's response from a specific time, and also include any initial conditions in the system's response.
- Main usage: transient and stability analysis of causal LTI system.
  - Unilateral (one sided) Laplace Transform: solving differential equations with initial conditions.
  - Bilateral (two sided) Laplace Transform: analysis on the system characteristics such as stability, causality, and frequency response.

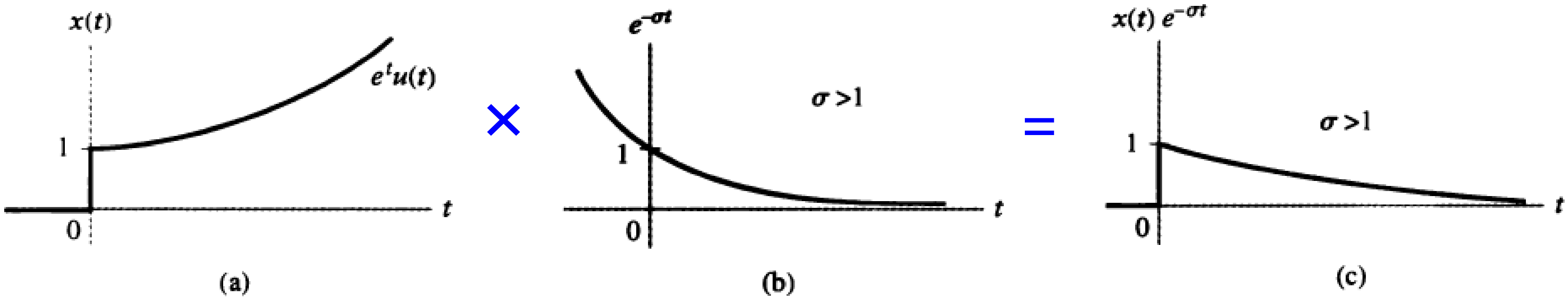
From Fourier Transform to Laplace Transform

- Eg. The Fourier transform of $x(t)=e^{at}u(t),\ a>0$ non-exists.

$$F\left[x(t)e^{-\sigma t}\right]=\int_{-\infty}^{\infty}x(t)e^{-\sigma t}e^{-j\omega t}dt=\int_{0}^{\infty}e^{at}e^{-(\sigma+j\omega)t}dt$$

$$\xrightarrow{s=\sigma+j\omega}=\int_{0}^{\infty}e^{-(s-a)t}dt=\frac{-1}{s-a}e^{-(s-a)t}\Big|_{0}^{\infty}=\frac{1}{s-a}\quad \text{if } \sigma>a.$$

For $a=\mathbf{1}$:



From Fourier Transform to Laplace Transform

To be generalized

$$F\left[x(t)e^{-\sigma t}\right]=\int_{-\infty}^{\infty}x(t)e^{-\sigma t}e^{-j\omega t}dt=\int_{-\infty}^{\infty}x(t)e^{-(\sigma+j\omega)t}dt$$

$$\xrightarrow{s=\sigma+j\omega}=\int_{-\infty}^{\infty}x(t)e^{-st}dt \triangleq X(s)$$

- The Laplace transform

$$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt \quad \text{or} \quad X(s)=L\left[x(t)\right]$$

- The inverse Laplace transform

$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds \quad \text{or} \quad x(t)=L^{-1}\left[X(s)\right]$$

$$x(t)\xleftrightarrow{L}X(s)$$

Complex Exponentials

$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds$$

represents ***x(t)*** as a weighted superposition of complex exponentials ***e^{st}***.

$$e^{st}=e^{(\sigma+j\omega)t}=e^{\sigma t}\cos\omega t+je^{\sigma t}\sin\omega t$$

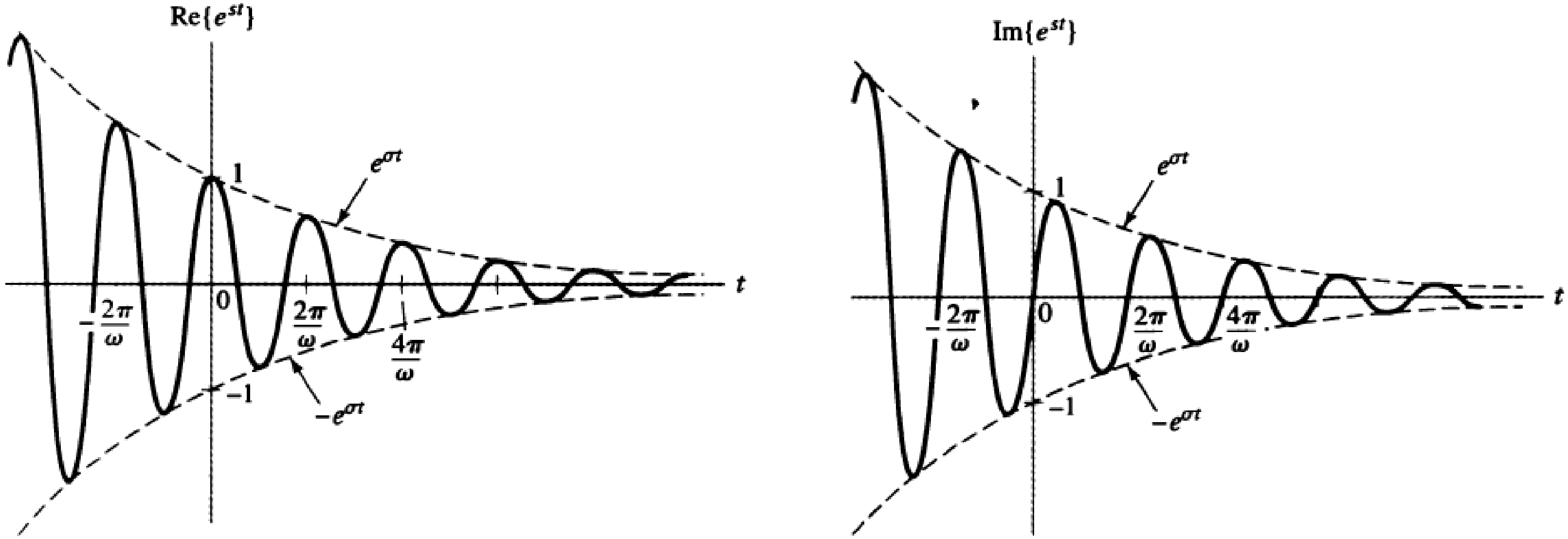


Figure 6.1 Real and imaginary parts of the complex exponential e^{st}, $s=\sigma+j\omega$.

Convergence (收斂)

- necessary condition for convergence: absolutely integrability of $x(t)e^{-\sigma t}$.

$$\int_{-\infty}^{\infty}\left|x(t)e^{-\sigma t}\right|dt<\infty \quad \Longrightarrow \quad \lim_{t\to\infty}x(t)e^{-\sigma t}=0$$

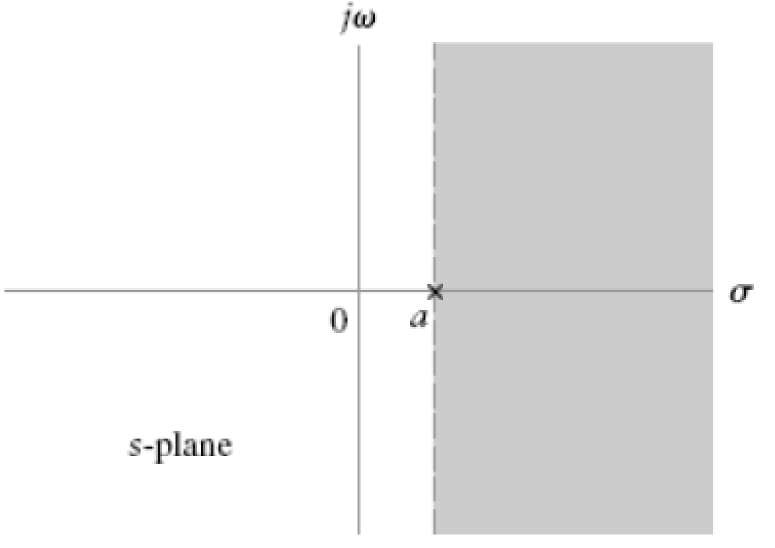
- **Region of convergence(ROC):** the region of σ which the Laplace transform converges.

Ex.6.1 $x(t) = e^{at}u(t)$

$$X(s)=\int_{0}^{\infty}e^{-(s-a)t}dt=\frac{-1}{s-a}e^{-(s-a)t}\Big|_{0}^{\infty}$$

$$=\frac{1}{s-a},$$

$$\mathrm{Re}(s)=\sigma>a.$$



Region of convergence

Example 6.2 An anticausal signal is zero for t>0. Determine the Laplace transform and ROC for the anticausal signal

$$y(t) = -e^{at}u(-t)$$

<Sol.>

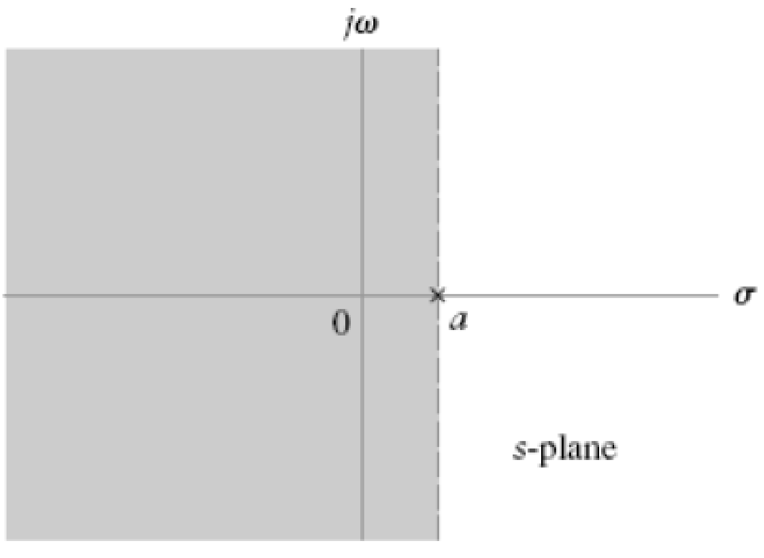
$$Y(s) = \int_{-\infty}^{\infty} -e^{at}u(-t)e^{-st}dt$$

$$= -\int_{-\infty}^{0} e^{-(s-a)t}dt$$

$$= \frac{1}{s-a}e^{-(s-a)t}\Big|_{-\infty}^{0}$$

$$= \frac{1}{s-a},$$

$$\mathrm{Re}(s) = \sigma < a.$$

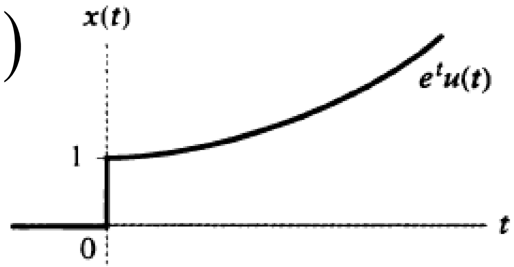


Region of convergence

- Laplace transforms of left-and right-sided exponentials have the same form; with left-and right-sided ROCs, respectively.

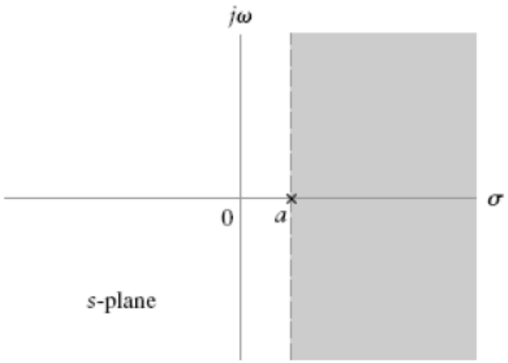
Time function | **Laplace transform**

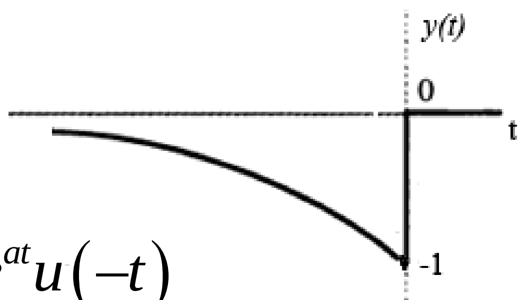
$$x(t)=e^{at}u(t)$$



(a)

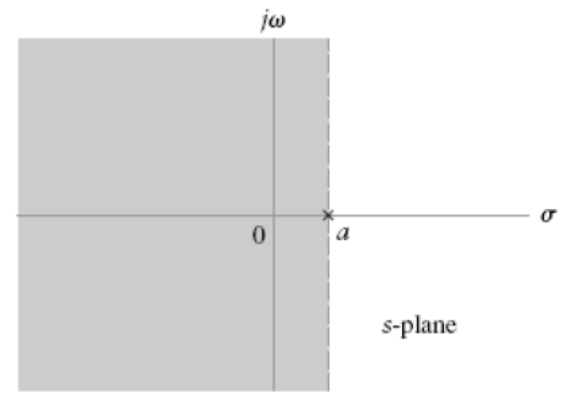
$$\frac{1}{s-a}$$





$$y(t)=-e^{at}u(-t)$$

$$\frac{1}{s-a}$$



Relations between Laplace and Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

- If ROC includes the imaginary axis($\sigma=0$), both Laplace transform and Fourier transform for $x(t)$ exist.

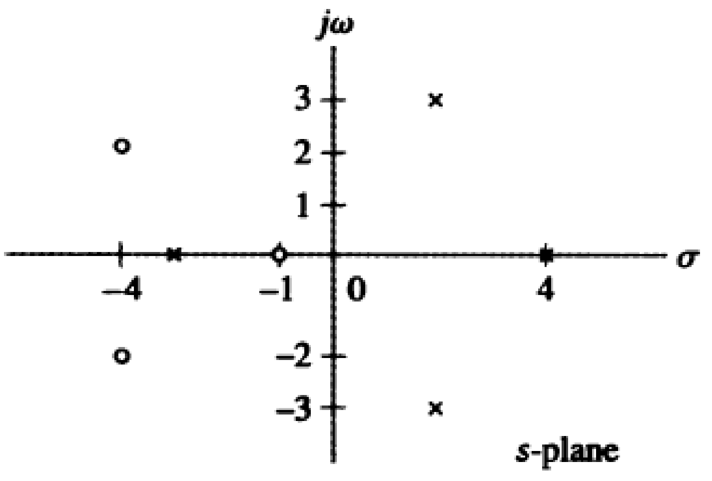
$$\boxed{X(j\omega) = X(s)\big|_{\sigma=0}}$$

- If ROC does not include the imaginary axis, Laplace transform exists while Fourier transform is nonexistent.

Poles and Zeros

$$X(s)=\frac{b_M s^M + b_{M-1}s^{M-1}+\cdots+b_0}{s^N + a_{N-1}s^{N-1}+\cdots a_1 s + a_0} = \frac{b_M\prod_{k=1}^{M}(s-c_k)}{\prod_{k=1}^{N}(s-d_k)}$$

- Zeros of $X(s)$: the roots of the numerator polynomial c_k. "o"
- Poles of $X(s)$: the roots of the denominator polynomial d_k. " ×"



Zeros:

$s=-1, \quad s=-4\pm 2j$

Poles:

$s=-3, \quad s=2\pm 3j, \quad s=4$

Laplace Transform for Elementary Signals

$$e^{\lambda t}u(t) \xleftrightarrow{L} \frac{1}{s-\lambda} \quad \mathrm{Re}\{s\} > \lambda.$$

$$\mathrm{e}^{-\lambda t}u(t) \xleftrightarrow{L} \frac{1}{s+\lambda} \quad \mathrm{Re}(s) > -\lambda$$

$$\mathrm{e}^{-\mathrm{j}\omega_0 t} \; u(t) \xleftrightarrow{L} \frac{1}{s+\mathrm{j}\omega_0} \quad \mathrm{Re}(s) > 0$$

$$\mathrm{e}^{\mathrm{j}\omega_0 t} \; u(t) \xleftrightarrow{L} \frac{1}{s-\mathrm{j}\omega_0} \quad \mathrm{Re}(s) > 0$$

$$\cos\omega_0 t \; u(t) = \frac{\mathrm{e}^{\mathrm{j}\omega_0 t} + \mathrm{e}^{-\mathrm{j}\omega_0 t}}{2} u(t) \xleftrightarrow{L} \frac{1}{2}\left(\frac{1}{s-\mathrm{j}\omega_0} + \frac{1}{s+\mathrm{j}\omega_0}\right) = \frac{s}{s^2 + \omega_0{}^2} \quad \mathrm{Re}(s) > 0$$

$$\sin\omega_0 t \; u(t) = \frac{\mathrm{e}^{\mathrm{j}\omega_0 t} - \mathrm{e}^{-\mathrm{j}\omega_0 t}}{2\mathrm{j}} u(t) \xleftrightarrow{L} \frac{1}{2\mathrm{j}}\left(\frac{1}{s-\mathrm{j}\omega_0} - \frac{1}{s+\mathrm{j}\omega_0}\right) = \frac{\omega_0}{s^2 + \omega_0{}^2} \quad \mathrm{Re}(s) > 0$$

Laplace Transform for Elementary Signals

$$\delta(t) \xleftrightarrow{L} 1 \qquad \text{Re(s)} > -\infty$$

$$\delta^{(n)}(t) \xleftrightarrow{L} s^n \qquad \text{Re(s)} > -\infty$$

$$u(t) \xleftrightarrow{L} \frac{1}{s} \qquad \text{Re}(s) > 0$$

$$tu(t) \xleftrightarrow{L} \frac{1}{s^2} \qquad \text{Re(s)} > 0$$

$$t^n u(t) \xleftrightarrow{L} \frac{n!}{s^{n+1}} \qquad \text{Re(s)} > 0$$

$$te^{-\lambda t}u(t) \xleftrightarrow{L} \frac{1}{(s+\lambda)^2} \qquad \text{Re(s)} > -\lambda$$

$$t^n e^{-\lambda t}u(t) \xleftrightarrow{L} \frac{n!}{(s+\lambda)^{n+1}} \qquad \text{Re}(s) > -\lambda$$

Laplace Transform for Elementary Signals

$$e^{-\sigma_0 t}\cos\omega_0 t\, u(t) \xleftrightarrow{L} \frac{s+\sigma_0}{(s+\sigma_0)^2+\omega_0^2} \quad \mathrm{Re(s)} > -\sigma_0$$

$$e^{-\sigma_0 t}\sin\omega_0 t u(t) \xleftrightarrow{L} \frac{\omega_0}{(\mathrm{s}+\sigma_0)^2+\omega_0^2} \quad \mathrm{Re(s)} > -\sigma_0$$

$$t\cos\omega_0 t u(t) \xleftrightarrow{L} \frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2} \quad \mathrm{Re(s)} > 0$$

$$t\sin\omega_0 t u(t) \xleftrightarrow{L} \frac{2\omega_0 s}{(s^2+\omega_0^2)^2} \quad \mathrm{Re(s)} > 0$$

Unilateral Laplace Transform (单边拉氏变换)

$$X(s)=\int_{0^-}^{\infty} x(t)e^{-st}dt$$

$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$x(t)\xleftrightarrow{L_u} X(s)$$

- Lower limit of 0^- implies to include discontinuities and impulses that occur at $t = 0$ in the integral.
- The unilateral and bilateral Laplace transforms are equivalent for signals that are zero for $t < 0$.

Ex.

$$e^{at}u(t)\xleftrightarrow{L_u}\frac{1}{s-a} \quad \text{and} \quad e^{at}u(t)\xleftrightarrow{L}\frac{1}{s-a} \text{ with ROC } \mathrm{Re}\{s\}>a.$$

Properties of Unilateral Laplace Transform

$$x(t) \xleftrightarrow{L_u} X(s) \qquad y(t) \xleftrightarrow{L_u} Y(s)$$

- **Linearity**

$$ax(t)+by(t) \xleftrightarrow{L_u} aX(s)+bY(s)$$

- **Scaling**

$$x(at) \xleftrightarrow{L_u} \frac{1}{a}X\left(\frac{s}{a}\right) \text{ for } a>0.$$

- **Time shift**

$$x(t-\tau) \xleftrightarrow{L_u} e^{-s\tau}X(s)$$

$$\text{for all } \tau \text{ such that } x(t-\tau)u(t)=x(t-\tau)u(t-\tau).$$

Properties of Unilateral Laplace Transform

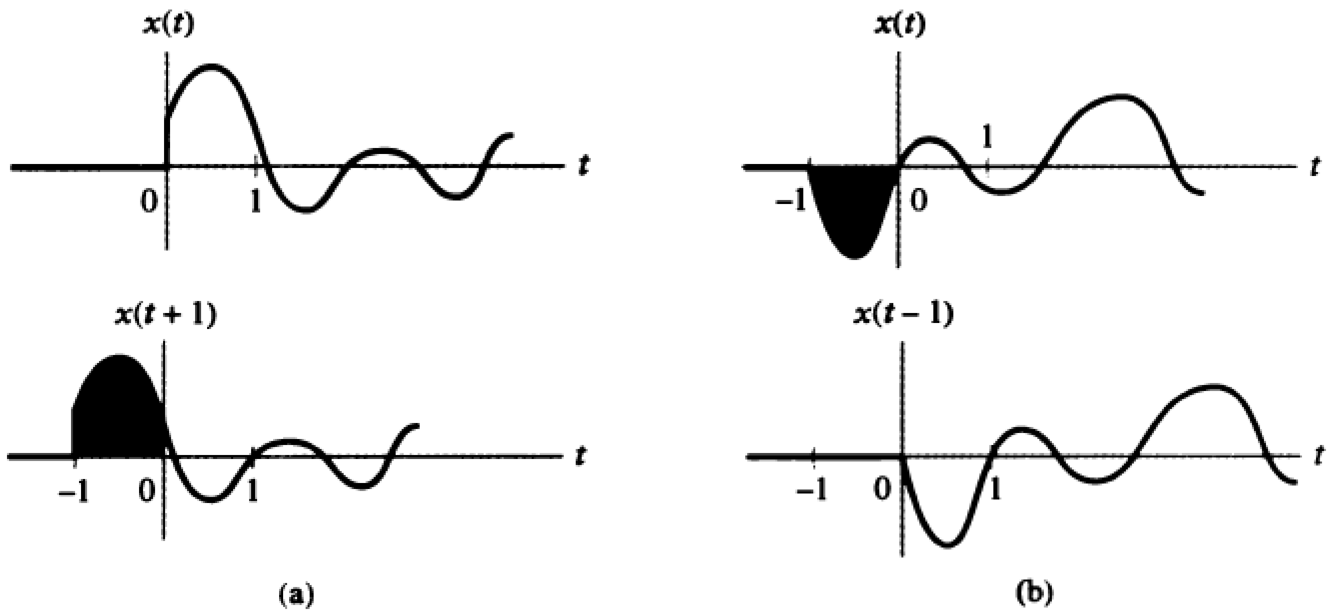


Figure 6.6 Time shifts for which unilateral Laplace transform time-shift property does not apply.

Ex. $u(t) \xleftrightarrow{L_u} 1/s$

$$u(t+3) \xleftrightarrow{L_u} \int_{0^-}^{\infty} u(t+3)e^{-st}dt = \int_{0^-}^{\infty} u(t)e^{-st}dt = 1/s$$

$$u(t-3) \xleftrightarrow{L_u} \int_{0^-}^{\infty} u(t-3)e^{-st}dt = \int_{3^-}^{\infty} e^{-st}dt = e^{-3s}/s$$

Properties of Unilateral Laplace Transform

- **s-Domain Shift**

$$e^{s_0 t} x(t) \xleftrightarrow{L_u} X(s - s_0)$$

Ex. $$\cos \omega_0 t u(t) \xleftrightarrow{L_u} \frac{s}{s^2 + \omega_0^2}$$

$$\Rightarrow \quad e^{-\lambda t} \cos \omega_0 t u(t) \xleftrightarrow{L_u} \frac{s+\lambda}{(s+\lambda)^2 + \omega_0^2}$$

- **Convolution**

$$x(t) * y(t) \xleftrightarrow{L_u} X(s) Y(s)$$

only when $x(t) = 0$ and $y(t) = 0$ for $t < 0$.

Properties of Unilateral Laplace Transform

- **Differentiation in the s-Domain**

$$-tx(t) \xleftrightarrow{L_u} \frac{d}{ds}X(s)$$

Ex. $u(t) \xleftrightarrow{L_u} \dfrac{1}{s}$

$$tu(t) \xleftrightarrow{L_u} -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$t^2u(t) \xleftrightarrow{L_u} -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

$$t^n u(t) \xleftrightarrow{L_u} \frac{n!}{s^{n+1}} \qquad t^n e^{-\lambda t}u(t) \xleftrightarrow{L_u} \frac{n!}{(s+\lambda)^{n+1}}$$

Properties of Unilateral Laplace Transform

Example 6.3. Find the unilateral Laplace transform of

$$x(t) = \left(-e^{3t}u(t)\right) * \left(tu(t)\right).$$

<Sol.>

$$u(t) \xleftrightarrow{L_u} \frac{1}{s}$$

$$\Rightarrow \quad x_1(t) = -e^{3t}u(t) \xleftrightarrow{L_u} X_1(s) = -\frac{1}{s-3}$$

$$x_2(t) = tu(t) \xleftrightarrow{L_u} X_2(s) = \frac{1}{s^2}$$

$$x(t) = x_1(t) * x_2(t) \xleftrightarrow{L_u} X(s) = -\frac{1}{s-3} \cdot \frac{1}{s^2} = -\frac{1}{s^2(s-3)}$$

Properties of Unilateral Laplace Transform

- **Differentiation in the time domain**

$$\frac{d}{dt}x(t) \xleftrightarrow{L_u} sX(s) - x(0^-)$$

<p.f.>

$$L\left[\frac{d}{dt}x(t)\right] = \int_{0^-}^{\infty}\left(\frac{d}{dt}x(t)\right)e^{-st}dt$$

$$= x(t)e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty}x(t)\left(-se^{-st}\right)dt$$

$$= -x(0^-) + s\int_{0^-}^{\infty}x(t)e^{-st}dt \;\; = sX(s) - x(0^-)$$

$$\frac{d^2x(t)}{dt} \xleftrightarrow{L_u} s^2X(s) - sx(0^-) - x'(0^-)$$

$$\frac{d^nx(t)}{dt} \xleftrightarrow{L_u} s^nX(s) - s^{n-1}x(0^-) - s^{n-2}x'(0^-) - \cdots - x^{n-1}(0^-)$$

Properties of Unilateral Laplace Transform

- **Integration property**

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{L_u} \frac{X(s)}{s} + \frac{x^{(-1)}(0^-)}{s} \quad \text{where } x^{(-1)}(0^-) = \int_{-\infty}^{0^-} x(\tau)\,d\tau.$$

Ex. $$tu(t) = \int_{-\infty}^{t} u(\tau)\,d\tau \xleftrightarrow{L_u} \frac{L\left[u(t)\right]}{s} + \frac{1}{s}\int_{-\infty}^{0^-} u(\tau)\,d\tau = \frac{1}{s^2}$$

- **Initial-value theorem**

$$\lim_{s\to\infty} sX(s) = x(0^+)$$

- **Final-value theorem**

$$\lim_{s\to 0} sX(s) = x(\infty)$$

Inversion of Unilateral Laplace Transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

- Direct inversion by contour integration
- Inversion by partial-fraction expansion (部分分式展开法)

$$X(s) = \frac{b_M s^M + b_{M-1}s^{M-1} + \cdots + b_0}{s^N + a_{N-1}s^{N-1} + \cdots a_1 s + a_0} \qquad (M \geq N)$$

$$= c_0 + c_1 s + c_2 s^2 + \cdots + c_{M-N}s^{M-N} + \frac{D(s)}{A(s)} \quad \Rightarrow$$

$$x(t) = c_0\delta(t) + c_1\delta'(t) + c_2\delta''(t) + \cdots + c_{M-N}\delta^{(M-N)}(t) + L^{-1}\left[\frac{D(s)}{A(s)}\right]$$

Inversion of Unilateral Laplace Transform

$$\frac{D(s)}{A(s)} = \frac{b_P s^P + b_{P-1} s^{P-1} + \cdots + b_0}{\prod_{k=1}^{N} (s - d_k)} \qquad (P < N)$$

- Case1: all the poles are distinct: $s = d_1, d_2, \cdots, d_N$

$$\frac{D(s)}{A(s)} = \frac{A_1}{s-d_1} + \frac{A_2}{s-d_2} + \cdots + \frac{A_N}{s-d_N} \quad \text{where} \quad A_k = (s-d_k)\frac{D(s)}{A(s)}\bigg|_{s=d_k}$$

$$A_k e^{d_k t} u(t) \xleftrightarrow{L_u} \frac{A_k}{s-d_k}$$

$$\Rightarrow \quad L^{-1}\left[\frac{D(s)}{A(s)}\right] = \left(A_1 e^{d_1 t} + A_2 e^{d_2 t} + \cdots + A_N e^{d_N t}\right) u(t)$$

Inversion of Unilateral Laplace Transform

Ex. **Find the inverse Laplace transform of** $X(s)=\dfrac{s+2}{s^3+4s^2+3s}$.

<Sol.> $$X(s)=\frac{s+2}{s^3+4s^2+3s}=\frac{s+2}{s(s+1)(s+3)}=\frac{A_1}{s}+\frac{A_2}{s+1}+\frac{A_3}{s+3}$$

$$A_1=(s-0)X(s)\big|_{s=0}=\left.\frac{s+2}{(s+1)(s+3)}\right|_{s=0}=\frac{2}{3}$$

$$A_2=(s+1)X(s)\big|_{s=-1}=\left.\frac{s+2}{s(s+3)}\right|_{s=-1}=-\frac{1}{2}$$

$$A_3=(s+3)X(s)\big|_{s=-3}=\left.\frac{s+2}{s(s+1)}\right|_{s=-3}=-\frac{1}{6}$$

$\Rightarrow$ $$x(t)=\left(2/3-e^{-t}/2-e^{-3t}/6\right)u(t)$$

Inversion of Unilateral Laplace Transform

- Case 2: a pole is repeated N times $\quad s = d_1 = d_2 = \cdots = d_N = d$

$$\frac{D(s)}{A(s)} = \frac{D(s)}{(s-d)^N} = \frac{A_1}{s-d} + \frac{A_2}{(s-d)^2} + \cdots + \frac{A_N}{(s-d)^N}$$

$$\text{where} \quad A_k = \frac{1}{(N-k)!} \cdot \frac{d^{N-k}}{ds^{N-k}} \left[(s-d)^N \frac{D(s)}{A(s)} \right] \Bigg|_{s=d}$$

$$\frac{A_k t^{n-1}}{(n-1)!} e^{dt} u(t) \xleftrightarrow{L_u} \frac{A_k}{(s-d)^n}$$

$$\Rightarrow \quad L^{-1}\left[\frac{D(s)}{A(s)} \right] = \left(A_1 + A_2 t + \cdots + \frac{A_N t^{N-1}}{(N-1)!} \right) e^{dt} u(t)$$

Inversion of Unilateral Laplace Transform

Ex. **Find the inverse Laplace transform of** $X(s)=\dfrac{s-2}{s(s+1)^3}$.

<Sol.>

$$X(s)=\frac{s-2}{s(s+1)^3}=\frac{A_1}{s}+\frac{A_2}{s+1}+\frac{A_3}{(s+1)^2}+\frac{A_4}{(s+1)^3}$$

$$A_1=sX(s)\big|_{s=0}=\left.\frac{s-2}{(s+1)^3}\right|_{s=0}=-2$$

$$A_4=\frac{1}{0!}(s+1)^3X(s)\big|_{s=-1}=\left.\frac{s-2}{s}\right|_{s=-1}=3$$

$$A_3=\frac{1}{1!}\cdot\frac{d}{ds}(s+1)^3X(s)\big|_{s=-1}=\left.\left(\frac{s-2}{s}\right)'\right|_{s=-1}=2$$

$$A_2=\frac{1}{2!}\cdot\frac{d^2}{ds^2}(s+1)^3X(s)\big|_{s=-1}=\left.\frac{1}{2}\left(\frac{s-2}{s}\right)''\right|_{s=-1}=2$$

$$\Rightarrow\quad x(t)=\left(-2+2e^{-t}+2te^{-t}+3t^2e^{-t}/2\right)u(t)$$

Inversion of Unilateral Laplace Transform

Ex 6.8 Inverting an improper rational Laplace transform. **Find the inverse Laplace transform of**

$$X(s)=\frac{2s^3-9s^2+4s+10}{s^2-3s-4}.$$

<Sol.>

$$X(s)=\frac{2s^3-9s^2+4s+10}{s^2-3s-4}$$

$$=2s-3+\frac{3s-2}{s^2-3s-4}$$

$$=2s-3+\frac{1}{s+1}+\frac{2}{s-4}$$

$$\begin{array}{r|l} & \mathbf{2s-3} \\ \hline \mathbf{s^2-3s-4} & \mathbf{2s^3-9s^2+4s+10} \\ & \mathbf{2s^3-6s^2-8s} \\ \cline{2-2} & \mathbf{-3s^2+12s+10} \\ & \mathbf{-3s^2+9s+12} \\ \cline{2-2} & \mathbf{3s-2} \end{array}$$

$$\Rightarrow \quad x(t)=2\delta^{(1)}(t)-3\delta(t)+e^{-t}u(t)+2e^{4t}u(t)$$

Inversion of Unilateral Laplace Transform

- Case 3: a pair of complex-conjugate poles $s = \sigma \pm j\omega$

$$\frac{D(s)}{A(s)} = \frac{A_1}{s-(\sigma + j\omega)} + \frac{A_2}{s-(\sigma - j\omega)}$$

In order for this sum to represent a real-valued signal, A_1 and A_2 must be complex conjugates of each other.

$$\frac{D(s)}{A(s)} = \frac{B_1 s + B_2}{(s-\sigma-j\omega)(s-\sigma+j\omega)} = \frac{C_1(s-\sigma)}{(s-\sigma)^2+\omega^2} + \frac{C_2\omega}{(s-\sigma)^2+\omega^2}$$

where $C_1 = B_1, \quad C_2 = \dfrac{B_1\sigma + B_2}{\omega}.$

$$\Rightarrow \quad L^{-1}\left[\frac{D(s)}{A(s)}\right] = C_1 e^{\sigma t}\cos(\omega t)u(t) + C_2 e^{\sigma t}\sin(\omega t)u(t)$$

Inversion of Unilateral Laplace Transform

Ex. **Find the inverse Laplace transform of** $X(s)=\dfrac{4s^2+6}{s^3+s^2-2}$.

<Sol.> $$X(s)=\frac{4s^2+6}{\left(s^3-1\right)+\left(s^2-1\right)}=\frac{A}{s-1}+\frac{B_1 s+B_2}{(s+1)^2+1}$$

$$A=(s-1)X(s)\Big|_{s=1}=\left.\frac{4s^2+6}{(s+1)^2+1}\right|_{s=1}=2$$

$$4s^2+6=2\left((s+1)^2+1\right)+\left(B_1 s+B_2\right)(s-1) \Rightarrow B_1=2,\quad B_2=-2$$

$$\begin{cases} C_1=2, \\ C_2=\dfrac{B_1\sigma+B_2}{\omega}=-4 \end{cases} \Rightarrow X(s)=\frac{2}{s-1}+2\frac{s+1}{(s+1)^2+1}-4\frac{1}{(s+1)^2+1}$$

$$\Rightarrow x(t)=\left(2e^t+2e^{-t}\cos t-4e^{-t}\sin t\right)u(t)$$

Inversion of Unilateral Laplace Transform

Ex. **Find the inverse Laplace transform of** $X(s)=\dfrac{1}{3s^2(s^2+4)}$.

<Sol.> $$X(s)=\frac{A_1}{s}+\frac{A_2}{s^2}+\frac{B_1 s+B_2}{s^2+4}$$

$$A_2=s^2X(s)\big|_{s=0}=\left.\frac{1}{3(s^2+4)}\right|_{s=0}=\frac{1}{12};\qquad A_1=\frac{1}{1!}\cdot\frac{d}{ds}s^2X(s)\big|_{s=0}=\left.\left(\frac{1}{3(s^2+4)}\right)'\right|_{s=0}=0$$

$$1/3=(s^2+4)/12+(B_1 s+B_2)s^2 \Rightarrow B_1=0,\quad B_2=-\frac{1}{12}$$

$$C_1=0,\ C_2=\frac{B_1\sigma+B_2}{\omega}=-\frac{1}{24} \Rightarrow X(s)=\frac{1}{12}\cdot\frac{1}{s^2}-\frac{1}{24}\cdot\frac{2}{s^2+4}$$

$$\Rightarrow x(t)=\frac{1}{12}\left(t-\frac{1}{2}\sin 2t\right)u(t)$$

Inversion of Unilateral Laplace Transform

Ex. **Find the inverse Laplace transform of** $X(s)=\dfrac{1-e^{-2s}}{s(s^2+4)}$.

<Sol.> $X(s)=\dfrac{1}{s(s^2+4)}+\dfrac{-e^{-2s}}{s(s^2+4)}$

$$X_1(s)=\frac{1}{s(s^2+4)}=\frac{A_1}{s}+\frac{B_1s+B_2}{s^2+4},\qquad A_1=sX(s)\big|_{s=0}=\frac{1}{(s^2+4)}\bigg|_{s=0}=\frac{1}{4};$$

$$1=(s^2+4)/4+(B_1s+B_2)s \;\Longrightarrow\; B_1=-1/4,\quad B_2=0$$

$$C_1=-1/4,\; C_2=(B_1\sigma+B_2)/\omega=0$$

$$X_1(s)=\frac{1}{4}\cdot\frac{1}{s}-\frac{1}{4}\cdot\frac{s}{s^2+4} \;\Longrightarrow\; x_1(t)=\frac{1}{4}(1-\cos 2t)u(t)$$

$$x(t)=x_1(t)-x_1(t-2)=\frac{1}{4}(1-\cos 2t)u(t)-\frac{1}{4}\left[1-\cos 2(t-2)\right]u(t-2)$$

Solving Differential Equations with Initial Conditions

$$\frac{d^2y(t)}{dt^2}+a_1\frac{dy(t)}{dt}+a_2y(t)=b_0\frac{d^2x(t)}{dt^2}+b_1\frac{dx(t)}{dt}+b_2x(t)$$

Determine $y(t)$ with specified $x(t)$ and initial conditions $y(0^-), y'(0^-)$.

$$\left[s^2Y(s)-sy(0^-)-y'(0^-)\right]+a_1\left[sY(s)-y(0^-)\right]+a_2Y(s)$$

$$=b_0s^2X(s)+b_1sX(s)+b_2X(s)$$

$$Y(s)=\frac{b_0s^2+b_1s+b_2}{s^2+a_1s+a_2}X(s)+\frac{sy(0^-)+y'(0^-)+a_1y(0^-)}{s^2+a_1s+a_2}$$

$$=Y^{(f)}(s)+Y^{(n)}(s) \quad \Longrightarrow \quad y(t)=y^{(f)}(t)+y^{(n)}(t)$$

- Forced response $Y^{(f)}(s)$: response to the input
- Natural response $Y^{(n)}(s)$: response to the initial conditions

Solving Differential Equations with Initial Conditions

Ex. Use the unilateral Laplace transform to determine the output of a system

$$y''(t)+5y'(t)+6y(t)=2x'(t)+8x(t)$$

in response to input $x(t)=e^{-t}u(t)$, and initial conditions $y(0^-)=3,\ y'(0^-)=2.$

<Sol.>

$$\left[s^2Y(s)-sy(0^-)-y'(0^-)\right]+5\left[sY(s)-y(0^-)\right]+6Y(s)=2sX(s)+8X(s)$$

$$Y(s)=\frac{2s+8}{s^2+5s+6}X(s)+\frac{sy(0^-)+y'(0^-)+5y(0^-)}{s^2+5s+6}$$

$$Y^{(f)}(s)=\frac{2s+8}{s^2+5s+6}\cdot\frac{1}{s+1}=\frac{3}{s+1}-\frac{4}{s+2}+\frac{1}{s+3}$$

$$\Rightarrow\quad y^{(f)}(t)=\left(3e^{-t}-4e^{-2t}+e^{-3t}\right)u(t)$$

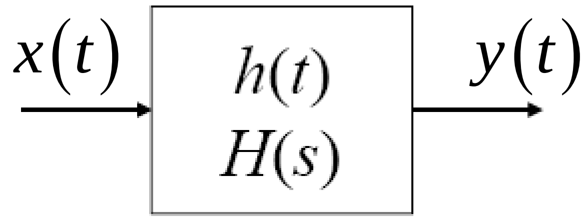
$$Y^{(n)}(s)=\frac{3s+17}{s^2+5s+6}=\frac{11}{s+2}-\frac{8}{s+3}\quad\Rightarrow\quad y^{(n)}(t)=\left(11e^{-2t}-8e^{-3t}\right)u(t)$$

$$\Rightarrow\quad y(t)=y^{(f)}t+y^{(n)}(t)=\left(3e^{-t}+7e^{-2t}-7e^{-3t}\right)u(t)$$

The Transfer Function (系统/传递函数)

- Transfer function: for an LTI system with impulse response $h(t)$

$$H(s)=\int_{-\infty}^{\infty} h(t)e^{-st}\,dt$$



$$y(t)=h(t)*x(t)$$

$$Y(s)=H(s)X(s) \Longrightarrow H(s)=\frac{Y(s)}{X(s)}$$

- Furthermore, for an input $x(t)=e^{st}$ to the LTI system

$$y(t)=h(t)*x(t)=\int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau=e^{st}H(s)$$

  - Eigenfunction of the system: e^{st}
  - Eigenvalue: $H(s)$

Transfer Function and Differential Equation

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

If initial conditions equal zero, and $x(t) = e^{st}$

$$\left(\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} \left\{ e^{st} \right\} \right) H(s) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} \left\{ e^{st} \right\}$$

$$\frac{d^k}{dt^k} \left\{ e^{st} \right\} = s^k e^{st} \quad \Longrightarrow \quad H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Transfer function $\Longleftrightarrow$ Differential equation system description

Transfer Function and Differential Equation

Ex. **Find the transfer function of the LTI system described by the differential equation**

$$y''(t)+7y'(t)+10y(t)=2x'(t)+x(t)$$

<Sol.>

$$\left(s^2+7s+10\right)Y(s)=(2s+1)X(s) \Longrightarrow H(s)=\frac{Y(s)}{X(s)}=\frac{2s+1}{s^2+7s+10}$$

Ex. **Find a differential-equation description of the systems described by the following transfer function**

(a) $$H(s)=\frac{s^2-2}{s^3-3s+1} \Longrightarrow y'''(t)-3y'(t)+y(t)=x''(t)-2x(t)$$

(b) $$H(s)=\frac{2(s+1)(s-1)}{s(s+2)(s+1)}=\frac{2s^2-2}{s^3+3s^2+2s}$$

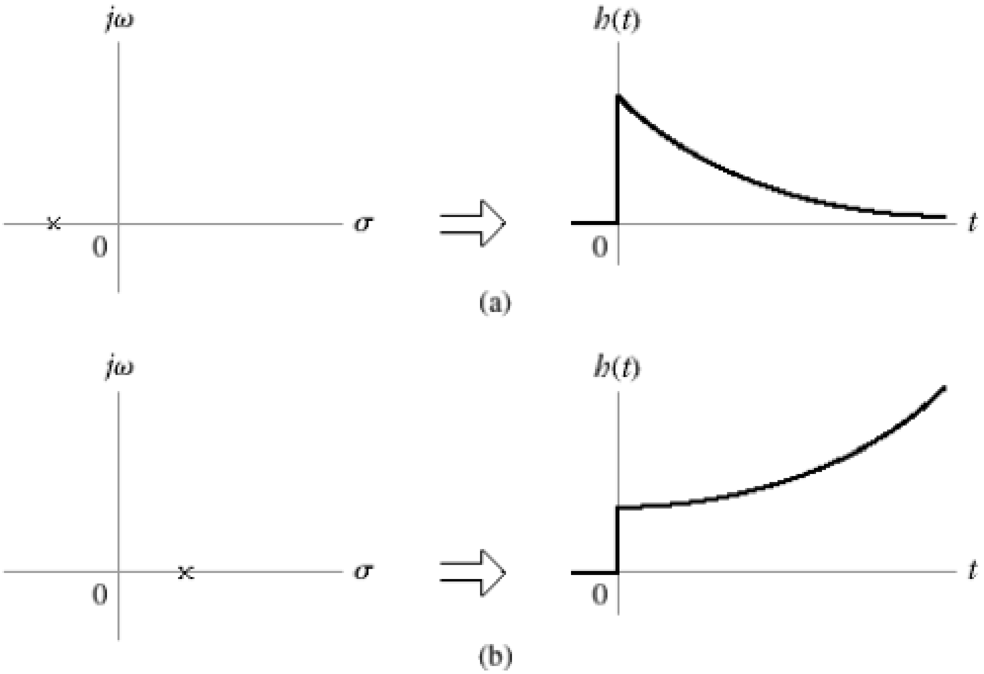
$$\Longrightarrow y'''(t)+3y''(t)+2y'(t)=2x''(t)-2x(t)$$

Causality and Stability (因果性与稳定性)

- For a causal system: $h(t) = 0 \text{ for } t < 0.$
  - A pole in the left half of the s-plane corresponds to an exponentially decaying impulse response.
  - A pole in the right half of the s-plane corresponds to an exponentially increasing impulse response --> unstable.

$$x(t) = e^{at}u(t) \leftrightarrow \frac{1}{s-a}$$

$$\mathrm{Re}(s) = \sigma > a.$$



Causality and Stability (因果性与稳定性)

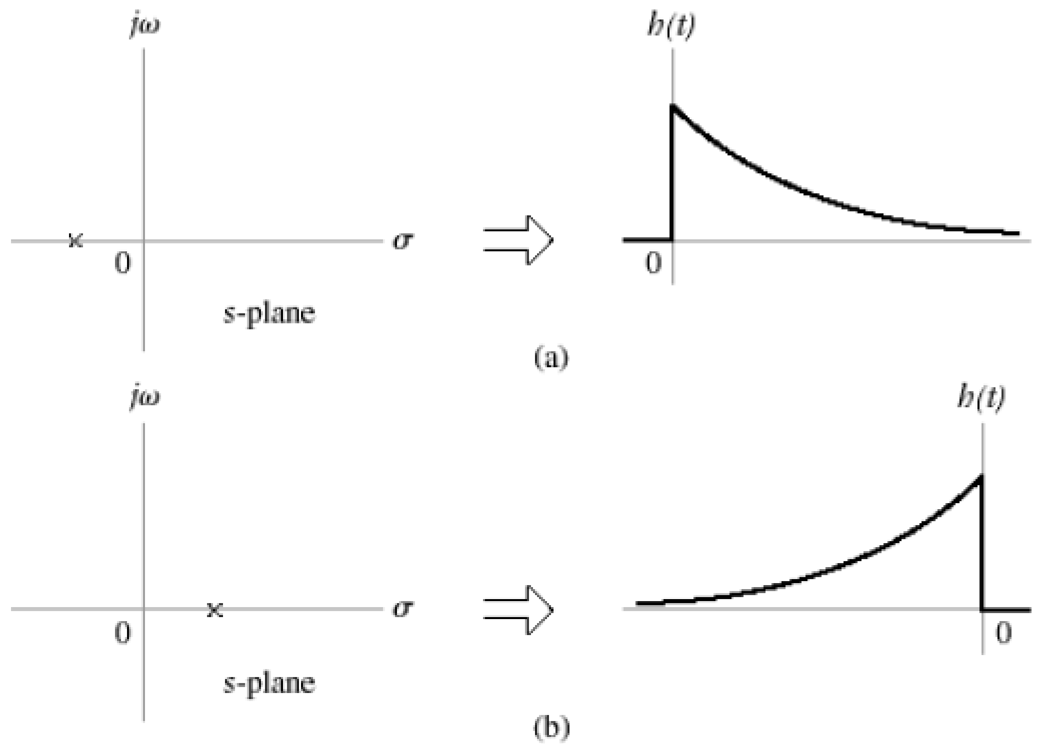
- For a stable system: $\int_{-\infty}^{\infty} |h(\tau)| d\tau = S < \infty.$
  - A pole in the left half of the *s-plane corresponds to a right-sided impulse response.*
  - A pole in the right half of the s-plane corresponds to an left-sided impulse response → noncausal.

$$x(t) = e^{at}u(t) \leftrightarrow \frac{1}{s-a}$$

$$\mathrm{Re}(s) = \sigma > a.$$

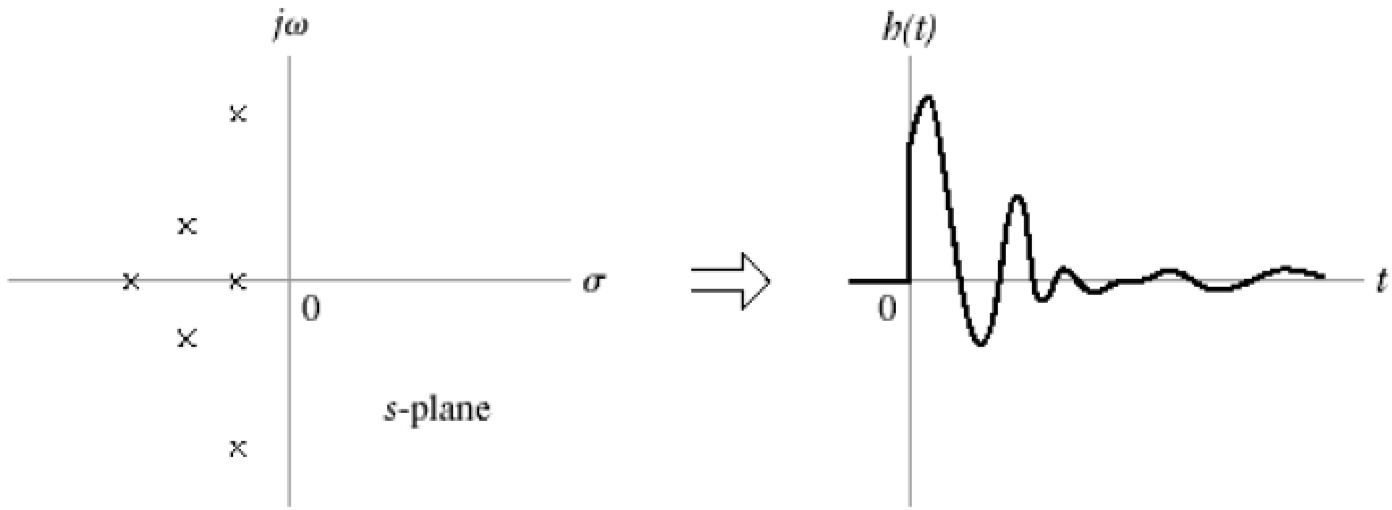
$$y(t) = -e^{at}u(-t) \leftrightarrow \frac{1}{s-a}$$

$$\mathrm{Re}(s) = \sigma < a$$



Causality and Stability (因果性与稳定性)

- A system that is both stable and causal must have a transfer function with all of its poles in the left half of the *s-plane*.



Causality and Stability (因果性与稳定性)

Ex. A system has the transfer function $H(s)=\dfrac{2}{s+3}+\dfrac{1}{s-2}$

Find the impulse response: a) the system is stable; b) the system is causal. Can this system be both stable and causal?

<Sol.> The system has two poles: $s=-3, \quad s=2.$

It cannot be both stable and causal!

a) the system is stable

$$y(t)=2e^{-3t}u(t)-e^{2t}u(-t)$$

b) the system is causal

$$y(t)=2e^{-3t}u(t)+e^{2t}u(t)$$

Inverse Systems

$$h^{inv}(t) * h(t) = \delta(t)$$

$$\Rightarrow \quad H^{inv}(s) H(s) = 1$$

$$\text{or} \quad H^{inv}(s) = \frac{1}{H(s)} = \frac{\prod_{k=1}^{N}(s-d_k)}{b_M \prod_{k=1}^{M}(s-c_k)}$$

- The zeros of the inverse system are the poles of $H(s)$, and the poles of the inverse system are the zeros of $H(s)$.
- Minimum phase system: have a transfer function with all of its poles and zeros in the left half of the *s-plane.*
  - Unique relationship between the magnitude and phase response.

Inverse Systems

Ex. Consider an LTI system described by

a) differential equation: $y'(t)+3y(t)=x''(t)+x'(t)-2x(t)$

b) Impulse response: $h(t)=\delta(t)+e^{-3t}u(t)+2e^{-t}u(t)$

Find the transfer function of the inverse system. Does a stable and causal inverse system exist?

<Sol.>

a) $H(s)=\dfrac{s^2+s-2}{s+3}$ $\Rightarrow$ $H^{inv}(s)=\dfrac{s+3}{s^2+s-2}=\dfrac{s+3}{(s-1)(s+2)}$

The system has two poles: $s=1,\quad s=-2.$ **cannot be both stable and causal!**

b) $H(s)=1+\dfrac{1}{s+3}+\dfrac{2}{s+1}=\dfrac{s^2+7s+10}{(s+3)(s+1)}$

$\Rightarrow$ $H^{inv}(s)=\dfrac{(s+3)(s+1)}{s^2+7s+10}=\dfrac{s^2+4s+3}{(s+2)(s+5)}$

The system has two poles: $s=-2,\quad s=-5.$ **both stable and causal!**

Summary

- The Laplace Transform
  - Introduction
  - Definition
  - The Unilateral Laplace Transform
  - Property of The Unilateral Laplace Transform
  - Inversion of The Unilateral Laplace Transform
  - Solving Differential Equations with Initial Conditions
  - The Transfer Function
  - Causality and Stability
- Reference in textbook:6.1~6.6
- Homework: 6.29, 6.36, 6.37(a,c,f,h), 6.38(a,c)