Lecture 3

Chapter 2 Matrix Algebra

2.1 Notations and Operations

2.2 Inverse and Transpose of Matrices

Overview

Matrices are one of the most important tools in mathematics.

Till now, we know that a linear system can be represented by using its corresponding augmented matrix. The process of solving linear system can be simplified by using notations of matrices. In this chapter, we shall introduce some new notations for matrices and define some operations of matrices using these notations.

One thing should be pointed out is that matrices can be obtained from many kinds of questions, not only be obtained from linear equation systems.

2.1 Notations and Operations

Matrix Notations

A matrix is a rectangular array of numbers, symbols, expressions or even other matrices. Each entry of a matrix is called **scalar [标量].** For the most part we will be working with matrices whose entries are real numbers.

We use capital letters A, B, C and so on to denote matrices, and a_{ij} denotes the entry of the matrix A that is in the ith row and jth column. We will refer to this entry as the (i, j) entry of A. Thus, is A is an $m \times n$ matrix, then

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

We will sometimes shorten the notation to $A = \left(a_{ij}\right)_{m\times n} = \left(a_{ij}\right)$.

Vectors

We will refer to an n-tuple of real numbers as a **vector [向量].** If an n-tuple is represented in terms of a $1 \times n$ matrix, then we will refer to it as a **row vector [行向量].** Alternatively, if the n-tuple is represented by an $n \times 1$ matrix, we will refer to it as a **column vector [列向量].**

a row vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$

a column vector $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

In working with matrix equations, it is generally more convenient to represent the solutions in terms of column vectors (i.e. $n \times 1$ matrices). The set of all $n \times 1$ matrices of real numbers is called **Euclidean n-space [n维欧几里得空间]** and is usually denoted by $\mathbf{R}^n$.

We will always refer to column vectors or elements of $\mathbf{R}^n$ simply as **vectors**.

Vectors

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

If A is an $m \times n$ matrix, the ith row vector of A will be denoted by $\boldsymbol{a}(i,:)$

$$\boldsymbol{a}(i,:) = (a_{i1}, a_{i2}, \dots a_{in}),$$

and the jth column vector will be denoted by $\boldsymbol{a}(:,j) = \boldsymbol{a}_j$

$$\boldsymbol{a}_j = \boldsymbol{a}(:,j) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

Vectors

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

$$\boldsymbol{a}(i,:) = (a_{i1}, a_{i2}, \ldots, a_{in}), \qquad i = 1,2,\ldots,m.$$

$$\boldsymbol{a}_j = \boldsymbol{a}(:,j) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}, \qquad j = 1,2,\ldots,n.$$

The matrix A can be represented in terms of either its column vectors or its row vectors

$$A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n) \qquad \text{or} \quad A = \begin{pmatrix} \boldsymbol{a}(1,:) \\ \boldsymbol{a}(2,:) \\ \vdots \\ \boldsymbol{a}(m,:) \end{pmatrix}.$$

Matrix Operations

- Addition/Subtraction
- Scalar multiplication
- Matrix multiplication*

Equality

Definition 1. (Equality) Two $m \times n$ matrices A and B are said to be **equal** **[相等]** if and only if $a_{ij} = b_{ij}$ for each i and j.

Remark. Two matrices are equal if and only if

(1) they must have the same number of rows and the same number of columns;

(2) if two entries of each matrix is at the same position of each matrix, they must be equal.

Example. Matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and matrix $B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ are not equal, since they are in different shapes.

Scalar Multiplication

Definition 2. (Scalar Multiplication) If A is an $m \times n$ matrix and α is a scalar, then αA is the matrix whose (i, j) entry is αa_{ij}.

Example. If

$$A = \begin{pmatrix} 4 & 8 & 2 \\ 6 & 8 & 10 \end{pmatrix},$$

then

$$3A = \begin{pmatrix} 3 \times 4 & 3 \times 8 & 3 \times 2 \\ 3 \times 6 & 3 \times 8 & 3 \times 10 \end{pmatrix} = \begin{pmatrix} 12 & 24 & 6 \\ 18 & 24 & 30 \end{pmatrix}$$

$$\frac{1}{2}A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 4 & 5 \end{pmatrix}$$

Matrix Addition

Definition 3. (Addition) If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrices, then the **sum [和]** of them, denoted by $A + B$, is the $m \times n$ matrix whose (i, j) entry is $a_{ij} + b_{ij}$ for each i and j.

Example. If $A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

then $A + B = \begin{pmatrix} 3+2 & 2+2 & 1+2 \\ 4+1 & 5+2 & 6+3 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 3 \\ 5 & 7 & 9 \end{pmatrix}$.

Matrix Addition

Remark. We define $A - B$ to be $A + (-1)B$.

Example. If $A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$

then
$$\begin{aligned} A - B &= \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} + (-1)\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} -4 & -5 \\ -2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}. \end{aligned}$$

Matrix Addition

Let O denote the matrix, with the same size as A, whose entries are all zero. Then

$$A + O = O + A = A.$$

We refer to O as the **zero matrix [零矩阵].** It acts as an **additive identity [加法不动元]** on the set of all $m \times n$ matrices. (sameness, equality)

Furthermore, each $m \times n$ matrix A has an **additive inverse [加法逆元].** Indeed,

$$A + (-1)A = O = (-1)A + A.$$

It is customary to denote the additive inverse by $-A$. Thus

$$-A = (-1)A.$$

Matrix Multiplication and Linear Systems

It is clear, if we have a system of one linear equation in one unknown, it can be written in the form

$$ax = b.$$

We generally think of a, x and b as being scalars; however, they could also be treated as 1×1 matrices. Then the last equation can be represented as

$$A\boldsymbol{x} = \boldsymbol{b},$$

where A is a 1×1 matrix and $\boldsymbol{x}$ is an unknown vector in $\mathbf{R}^1$ and $\boldsymbol{b}$ is in $\mathbf{R}^1$.

Question: How about the case of one equation in several unknowns?

Matrix Multiplication and Linear Systems

Case 1: One equation in several unknowns

Consider the equation

$$3x_1 + 2x_2 + 5x_3 = 4.$$

$$A\boldsymbol{x} = 4.$$

We set

$$A = \begin{pmatrix} 3 & 2 & 5 \end{pmatrix} \text{ and } \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and define the product $A\boldsymbol{x}$ by

$$A\boldsymbol{x} = \begin{pmatrix} 3 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3x_1 + 2x_2 + 5x_3$$

Matrix Multiplication and Linear Systems

Case 2: m equations in n unknowns

Consider now an $m \times n$ linear system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$
A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
$$

Coefficient matrix

Matrix Multiplication and Linear Systems

If we denote $\quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

and define the product

$$A\boldsymbol{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

then the linear system can be written as a matrix equation $\quad A\boldsymbol{x} = \boldsymbol{b}.$

Matrix Multiplication and Linear Systems

Example. Write the following system of equations as a matrix equation

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 5 \\ x_1 - 2x_2 + 5x_3 &= -2 \\ 2x_1 + x_2 - 3x_3 &= 1 \end{aligned}$$

Solution.

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -2 & 5 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

Finish.

Matrix Multiplication and Linear Systems

Remark. Another important way to represent a linear system

$$A\boldsymbol{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{pmatrix} + \begin{pmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{pmatrix} + \cdots + \begin{pmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Thus we have

$$A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n.$$

Matrix Multiplication and Linear Systems

Definition. (Linear Combination) If $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are vectors in $\mathbf{R}^m$ and $c_1, c_2, \ldots, c_n$ are scalars, then a sum of the form

$$c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \cdots + c_n\boldsymbol{a}_n$$

is said to be a **linear combination [线性组合]** of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$.

Remark. It follows that the left hand side of a linear system of n unknowns, $A\boldsymbol{x}$, is a linear combination of the column vectors of the coefficient matrix A. Therefore, if A is an $m \times n$ matrix and $\boldsymbol{x}$ is a vector in $\mathbf{R}^n$, then

$$A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots x_n\boldsymbol{a}_n.$$

Matrix Multiplication and Linear Systems

Theorem. (Consistency theorem for linear systems) A linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ can be written as a linear combination of the column vectors of A.

Example 1. The linear system

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 1 \end{aligned}$$

is inconsistent since the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ cannot be written as a linear combination of the column vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

Matrix Multiplication and Linear Systems

Example 2. Revisit of the 2×2 system

$$\begin{aligned} 2x - \ \ y &= 0, \\ -x + 2y &= 3. \end{aligned}$$

$$x\begin{pmatrix} 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

The system has solution $(1,2)$, therefore

$$1 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Solving a linear system $A\boldsymbol{x} = \boldsymbol{b}$ is equivalent to finding the group of coefficients $x_1, \ldots, x_n$ such that

$$\boldsymbol{b} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n.$$

Matrix Multiplication

Definition 4. (Matrix Multiplication) If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times r$ matrix, then the **matrix multiplication [矩阵乘积]** of A and B, denoted by AB, is the $m \times r$ matrix whose entries are defined by

$$c_{ij} = \boldsymbol{a}(i,:)\boldsymbol{b}_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Matrix Multiplication

Example 1. If $A = \begin{pmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{pmatrix}$

then $AB = \begin{pmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{pmatrix}$

$$= \begin{pmatrix} 3 \cdot (-2) - 2 \cdot 4 & 3 \cdot 1 - 2 \cdot 1 & 3 \cdot 3 - 2 \cdot 6 \\ 2 \cdot (-2) + 4 \cdot 4 & 2 \cdot 1 + 4 \cdot 1 & 2 \cdot 3 + 4 \cdot 6 \\ 1 \cdot (-2) - 3 \cdot 4 & 1 \cdot 1 - 3 \cdot 1 & 1 \cdot 3 - 3 \cdot 6 \end{pmatrix}$$

$$= \begin{pmatrix} -14 & 1 & -3 \\ 12 & 6 & 30 \\ -14 & -2 & -15 \end{pmatrix}.$$

Matrix Multiplication

Example 2. If $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \end{pmatrix}$

then $AB = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{pmatrix}$

Algebraic Rules of Matrix Operations

Just as in ordinary algebra, if an expression involves both multiplication and addition and there are no parentheses to indicate the order of the operations, multiplications are carried out before additions. For example, if

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix},$$

then

$$\begin{aligned} A + BC = A + (BC) &= \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} + \left(\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 7 & 7 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 11 \\ 0 & 6 \end{pmatrix}. \end{aligned}$$

$$3A + B = (3A) + B = \begin{pmatrix} 9 & 12 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 15 \\ 5 & 7 \end{pmatrix}.$$

Algebraic Rules of Matrix Operations

Theorem 1. (Algebraic Rules of Matrix Operations) Let A, B, C be matrices and α, β be scalars. The dimensions of A, B, C are carefully chosen so that the following operations are well defined.

Then the following statements are valid:

(1) Commutative Law [交换律]: $A + B = B + A$;

(2) Associative Law [结合律]:

$$(A + B) + C = A + (B + C); \quad (\alpha\beta)A = \alpha(\beta A);$$

$$(AB)C = A(BC); \quad \alpha(AB) = (\alpha A)B = A(\alpha B);$$

(3) Distributive Law [分配律]:

$$A(B + C) = AB + AC; \quad (A + B)C = AC + BC;$$

$$(\alpha + \beta)A = \alpha A + \beta A; \quad \alpha(A + B) = \alpha A + \alpha B.$$

Algebraic Rules of Matrix Operations

Proof of $(AB)C = A(BC)$

Notation: If k is a positive integer and A is a square matrix ($m = n$), then $A^k = \underbrace{A \cdot A \cdot \ldots A}_{k \text{ times}}$.

Question: commutative law for matrix multiplication? $AB = BA$?

Algebraic Rules of Matrix Operations

Example. Consider

$$A = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix}, \qquad B = \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Evaluate AB, AC and BA.

Solution. We have $AB = AC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, BA = \begin{pmatrix} 2ab & 2ab \\ -2ab & -2ab \end{pmatrix}.$

Remark. In general, $AB \neq BA$.

If $AB = O$, we cannot claim that $A = O$ or $B = O$.

If $AB = AC$, we cannot claim that $A = O$ or $B = C$.

2.2 Inverse and Transpose of Matrices

Identity Matrix

Just like the number 1, if there is a special matrix I satisfies

$$IA = A = AI,$$

where A is any $n \times n$ matrix, we refer to it as the **identity matrix**.

Actually, I has to be an $n \times n$ matrix with 1's on the main diagonal and 0's elsewhere.

Identity Matrix

Definition 1. (Identity Matrix) The $n \times n$ **identity matrix [单位矩阵]** is the matrix $I = (\delta_{ij})$, where δ_{ij} is the Dirac Delta function

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

such as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The identity matrix can also be written as

$$I = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n).$$

Matrix Inverse

Definition 2. (Inverse Matrix) An $n \times n$ matrix A is said to be **nonsingular [非奇异]** or **invertible [可逆]** if there exists a matrix B such that

$$AB = BA = I$$

The matrix B is said to be a **multiplicative inverse [乘法逆]** or **inverse matrix [逆矩阵]** of A, which is denoted by A^{-1}.

A is said to be **singular [奇异]** if it does not have a multiplicative inverse.

Matrix Inverse

Remark. If B and C are both multiplicative inverse of A, then

$$B = BI = B(AC) = (BA)C = IC = C.$$

The multiplicative inverse of A is **unique** if it exists.

Remark. Only square matrices can have inverse matrices.

Matrix Inverse

Example 1. The matrices $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix}$ are inverses of each other.

Example 2. The matrices $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$ are inverses of each other.

Example 3. (Find inverse matrix) Let $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, find A^{-1} and B^{-1}.

Matrix Inverse

Theorem 1. If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof.

$$\left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}B = I,$$
$$(AB)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} = AA^{-1} = I.$$

The end.

Matrix Inverse

Remark. It follows that if $A_1, A_2, \ldots, A_k$ are nonsingular $n \times n$ matrices, then

$$(A_1 A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1} A_1^{-1}.$$

Remark. If $AB = AC$ and A is nonsingular, then $B = C$.

If $BA = CA$ and A is nonsingular, then $B = C$.

Transpose of a Matrix

Definition 3. (Transpose of a Matrix) The **transpose [转置]** of an $m \times n$ matrix A, is the $n \times m$ matrix B, defined by

$$b_{ji} = a_{ij}$$

for j=1,2,…, n, i=1,2,…, m. The transpose of A is denoted by A^T or A'.

Example. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \end{pmatrix}$. Then

$$A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}.$$

Transpose of a Matrix

Theorem 2. (Algebraic rules of transposes)

1. $\left(A^T\right)^T = A$;
2. $(\alpha A)^T = \alpha A^T$;
3. $(A+B)^T = A^T + B^T$;
4. $(AB)^T = B^T A^T$.

Proof of 4.

Remark. In general, $(A_1 A_2 \dots A_k)^T = A_k^T \dots A_2^T A_1^T$.

Transpose of a Matrix

Definition 4. (Symmetric matrix) Matrix A is called **symmetric matrix [对称矩阵]** if $A^T = A$.

Example. The nth order identity matrix I_n is a symmetric matrix.

Triangular and diagonal Matrices

Definition 5. The **main diagonal [主对角线]** of a matrix A is the collection of entries a_{ij}, where $i = j$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Lower triangular matrix [下三角]

$$L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix},$$

Upper triangular matrix [上三角]

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

Triangular and diagonal Matrices

Definition 6. A matrix is called **diagonal matrix [对角矩阵]** if it is both upper and lower triangular matrix.

$$D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$$

Example: The nth order identity matrix I_n is a diagonal matrix.

Review

- Matrix Notations
- Matrix Operations (Addition, Scalar Multiplication, **Matrix Multiplication**, **Inversion**, Transpose)

Preview

- Partitioned Matrices
- Elementary Matrices

Exercises

P37: 1, 3(d)(e), 7(b)(d), 10;
P43: 1(a)(d)(e), 3, 5, 6.