Lecture 07

Chapter 3 Vector Spaces

3.2 Subspaces

3.3 Linear Independence

3.2 Subspaces

Construct new vector spaces from a given vector space V.

Example 1. Let $\boldsymbol{S} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_2 = 2x_1 \right\} \subset \mathbf{R}^2$.

(1) If $\begin{pmatrix} c \\ 2c \end{pmatrix} \in \boldsymbol{S}$ and α is any scalar, then $\alpha \begin{pmatrix} c \\ 2c \end{pmatrix} = \begin{pmatrix} \alpha c \\ 2\alpha c \end{pmatrix} \in \boldsymbol{S}$.

(2) If $\begin{pmatrix} a \\ 2a \end{pmatrix}, \begin{pmatrix} b \\ 2b \end{pmatrix} \in \boldsymbol{S}$, their sum $\begin{pmatrix} a+b \\ 2a+2b \end{pmatrix} = \begin{pmatrix} a+b \\ 2(a+b) \end{pmatrix} \in \boldsymbol{S}$.

By the definition of vector space, the mathematical system consisting of the set $\boldsymbol{S}$ together with the operations induced from $\mathbf{R}^2$, is itself a vector space.

Definitions

Definition 1. If $\boldsymbol{S}$ is a nonempty subset of a vector space $\boldsymbol{V}$ and $\boldsymbol{S}$ satisfies the following conditions:

(1) $\alpha\boldsymbol{x} \in \boldsymbol{S}$ whenever $\boldsymbol{x} \in \boldsymbol{S}$ and for any scalar α;

(2) $\boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{S}$ whenever $\boldsymbol{x} \in \boldsymbol{S}$ and $\boldsymbol{y} \in \boldsymbol{S}$;

then $\boldsymbol{S}$ is said to be a **subspace [子空间]** of $\boldsymbol{V}$.

Remark **(closure properties [封闭性])**

(1): $\boldsymbol{S}$ is closed under **scalar multiplication**;

(2): $\boldsymbol{S}$ is closed under **addition**.

If we do arithmetic using operations from $\boldsymbol{V}$ and elements of $\boldsymbol{S}$, we will always end up with elements of $\boldsymbol{S}$. A subspace of $\boldsymbol{V}$, is a subset $\boldsymbol{S}$ that is closed under the operations of $\boldsymbol{V}$.

Remark. In a vector space $\boldsymbol{V}$, it is easy to verify that $\{\mathbf{0}\}$ and $\boldsymbol{V}$ are subspaces of $\boldsymbol{V}$.

- Subspaces $\{\mathbf{0}\}$ and $\boldsymbol{V}$ are called **trivial subspaces [平凡子空间]** of vector space $\boldsymbol{V}$.
- All other subspaces are referred to as **proper subspaces [真子空间].**
- The subspace $\{\mathbf{0}\}$ is also called **zero subspace [零子空间].**

Example 2. Let $\boldsymbol{S} = \left\{ (x_1, x_2, x_3)^T \middle| x_1 = x_2 \right\} \subset \mathbf{R}^3$.

The subset $\boldsymbol{S}$ is nonempty since $\boldsymbol{x} = (1,1,0)^T \in \boldsymbol{S}$.

(1) If $\boldsymbol{x} = (a, a, b)^T \in \boldsymbol{S}$, then $\alpha\boldsymbol{x} = (\alpha a, \alpha a, \alpha b)^T \in \boldsymbol{S}$.

(2) If $(a, a, b)^T$ and $(c, c, d)^T \in \boldsymbol{S}$, then

$$(a, a, b)^T + (c, c, d)^T = (a + c, a + c, b + d)^T \in \boldsymbol{S}.$$

Since $\boldsymbol{S}$ is nonempty and satisfies the two closure conditions, it follows that $\boldsymbol{S}$ is a subspace of $\mathbf{R}^3$.

Example 3. Let $W = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \middle| x \text{ is a real number} \right\} \subset \mathbf{R}^2$.

The subset W is not a subspace of $\mathbf{R}^2$. In this case, both conditions fail:

(1) W is not closed under scalar multiplication

since $\alpha \begin{pmatrix} x \\ 1 \end{pmatrix} \notin \boldsymbol{S}$ unless $\alpha = 1$.

(2) W is not closed under addition,

since $\begin{pmatrix} x \\ 1 \end{pmatrix} + \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} x + y \\ 2 \end{pmatrix} \notin \boldsymbol{S}$.

Example 4. Let $\boldsymbol{S} = \{A \in \mathbf{R}^{2\times 2} | a_{12} = -a_{21}\} \subset \mathbf{R}^{2\times 2}$.

The set $\boldsymbol{S}$ form a subspace of $\mathbf{R}^{2\times 2}$, since

(1) If $A \in \boldsymbol{S}$, then A must be of the form

$$A = \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \quad \text{and hence} \quad \alpha A = \begin{pmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{pmatrix}.$$

Since the (2,1) entry of αA is the negative of the (1,2) entry, $\alpha A \in \boldsymbol{S}$.

(2) If $A, B \in \boldsymbol{S}$, then they must be of the form

$$A = \begin{pmatrix} a & b \\ -b & c \end{pmatrix}, \qquad B = \begin{pmatrix} d & e \\ -e & f \end{pmatrix}.$$

It follows that $$A + B = \begin{pmatrix} a+d & b+e \\ -(b+e) & c+f \end{pmatrix} \in \boldsymbol{S}.$$

Example 5. Let $C^1[a, b]$ be the set of all functions f that have a continuous derivative on $[a, b]$. Then $C^1[a, b]$ is a proper subspace of $C[a, b]$.

Example 6. Let S be the set of all polynomials of degree less than n with the property that $p(0) = 0$. Then S is a proper subspace of P_n.

Nullspace of a Matrix

Definition 2. The **nullspace [零空间]** or **kernel [核]** of an $m \times n$ matrix A, denoted by $N(A)$, is defined as the set of all solutions to the homogeneous system $A\boldsymbol{x} = \mathbf{0}$.

$$N(A) = \{\boldsymbol{x} \in \mathbf{R}^n | A\boldsymbol{x} = \mathbf{0}\}.$$

- Since $\mathbf{0} \in N(A)$, $N(A)$ is nonempty.
- If $\boldsymbol{x} \in N(A)$ and α is a scalar, then $A(\alpha\boldsymbol{x}) = \alpha A\boldsymbol{x} = \alpha\mathbf{0} = \mathbf{0}$, and hence $\alpha\boldsymbol{x} \in N(A)$.
- If $\boldsymbol{x}, \boldsymbol{y} \in N(A)$, then $A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Therefore $\boldsymbol{x} + \boldsymbol{y} \in N(A)$.

The nullspace $N(A)$ is a subspace of $\mathbf{R}^n$.

Example 7. Determine $N(A)$ if

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}.$$

Solution. Using Gauss-Jordan reduction to solve $A\boldsymbol{x} = \mathbf{0}$, we obtain

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right).$$

The reduced row echelon form involves two free variables, x_3 and x_4.

$$x_1 = x_3 - x_4,$$

$$x_2 = -2x_3 + x_4.$$

Example 7. Determine $N(A)$ if

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}.$$

Solution. (continue) Thus if we set $x_3 = \alpha$ and $x_4 = \beta$, then

$$\boldsymbol{x} = \begin{pmatrix} \alpha - \beta \\ -2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

is a solution to $A\boldsymbol{x} = \boldsymbol{0}$. The vector space $N(A)$ consists of vectors of the form

$$\alpha(1,-2,1,0)^T + \beta(-1,1,0,1)^T$$

where α and β are arbitrary scalars. Finish.

The Span of vectors

Definition 3. Let $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ be vectors in a vector space $\boldsymbol{V}$. A sum of the form

$$\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n},$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars, is called a **linear combination [线性组合]** of $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$.

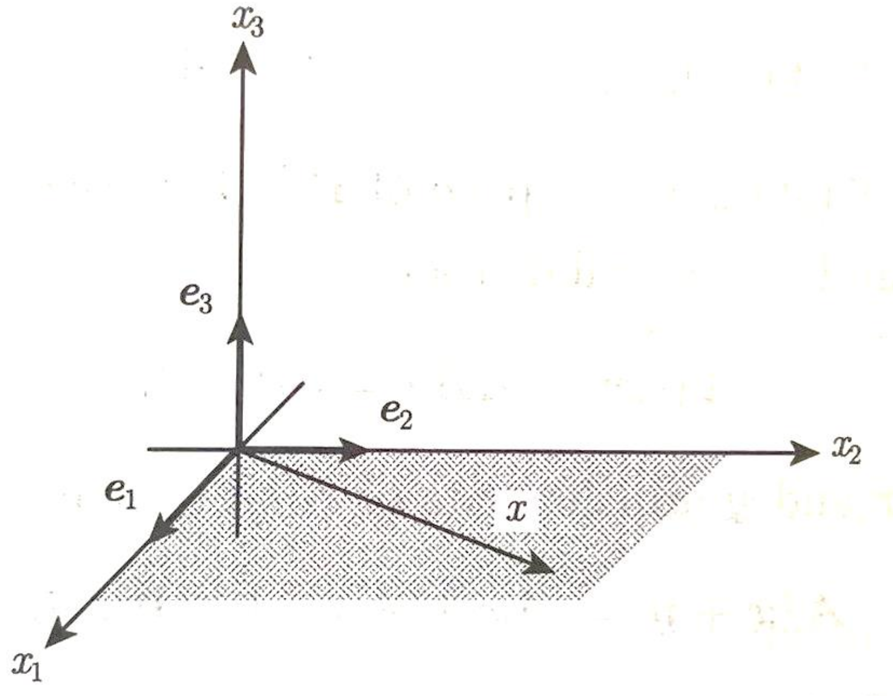
The set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ is called the **span [张成的集合]** of $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$. The span of $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ will be denoted by $\mathrm{Span}\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$.

In **Example 7**, the nullspace of A is the span of the vectors

$$(1, -2, 1, 0)^T, (-1, 1, 0, 1)^T.$$

Example 8. In $\mathbf{R}^3$, the span of $\mathbf{e_1}$ and $\mathbf{e_2}$ is the set of all vectors of the form $\alpha\mathbf{e_1} + \beta\mathbf{e_2} = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}$.

Verify that Span$\{\mathbf{e_1}, \mathbf{e_2}\}$ is a subspace of $\mathbf{R}^3$.



Theorem 1. If $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ are elements of a vector space $\boldsymbol{V}$, then

$$\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$$

is a subspace of $\boldsymbol{V}$.

Proof. We know that $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ is a subset of $\boldsymbol{V}$.

- Let $\mathbf{v} = \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n} \in$ $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ and β be a scalar .

$$\beta\mathbf{v} = (\beta\alpha_1)\mathbf{v_1} + (\beta\alpha_2)\mathbf{v_2} + \cdots + (\beta\alpha_n)\mathbf{v_n}$$
$$\in \text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}.$$

- Let $\mathbf{v} = \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n}$ and $\mathbf{w} = \beta_1\mathbf{v_1} + \beta_2\mathbf{v_2} + \cdots \beta_n\mathbf{v_n}$.

$$\mathbf{v} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{v_1} + (\alpha_2 + \beta_2)\mathbf{v_2} + \cdots + (\alpha_n + \beta_n)\mathbf{v_n}$$
$$\in \text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}.$$

Therefore, $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ is a subspace of $\boldsymbol{V}$.

Question: Given a vector space $\boldsymbol{V}$, can we find vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ such that they span $\boldsymbol{V}$?

Definition 4. The set $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ is a **spanning set [生成集]** for the vector space $\boldsymbol{V}$ **if and only if** every vector in $\boldsymbol{V}$ can be written as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$.

Example 9. For any vector $\boldsymbol{x} \in \mathbf{R^2}$,

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Therefore, $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a spanning set for the vector space $\mathbf{R^2}$.

Example 10. Which of the following are spanning sets for $\mathbf{R}^3$?

A. $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, (1,2,3)^T\}$;

B. $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$;

C. $\{(1,0,1)^T, (0,1,0)^T\}$;

D. $\{(1,2,4)^T, (2,1,3)^T, (4,-1,1)^T\}$.

To determine whether a set of vectors spans $\mathbf{R}^3$, we must determine whether an arbitrary vector $(a, b, c)^T$ in $\mathbf{R}^3$ can be written as a linear combination of the vectors in the set.

Example 10. Which of the following are spanning sets for $\mathbf{R}^3$?

A. $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, (1,2,3)^T\}$;

Solution. It is easily seen that $(a, b, c)^T$ can be written as

$$(a, b, c)^T = a\mathbf{e_1} + b\mathbf{e_2} + c\mathbf{e_3} + 0(1,2,3)^T$$

so that A is a spanning set for $\mathbf{R}^3$.

Example 10. Which of the following are spanning sets for $\mathbf{R}^3$?

B. $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$;

Solution. (continue) In part (B), we must determine whether it is possible to find constants $\alpha_1, \alpha_2, \alpha_3$ such that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

This leads to the system of equations

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= a \\ \alpha_1 + \alpha_2 \qquad &= b \\ \alpha_1 \qquad\qquad &= c \end{aligned}$$

Example 10. Which of the following are spanning sets for $\mathbf{R^3}$?

B. $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$;

Solution. (continue)

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= a \\ \alpha_1 + \alpha_2 \quad\quad &= b \\ \alpha_1 \quad\quad\quad\quad &= c \end{aligned}$$

Since the coefficient matrix of the system is nonsingular, the system has a unique solution. In fact, we find that $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} c \\ b-c \\ a-b \end{pmatrix}$. Thus

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (b-c)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (a-b)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So the three vectors in B span $\mathbf{R^3}$.

Example 10. Which of the following are spanning sets for $\mathbf{R}^3$?

C. $\{(1,0,1)^T, (0,1,0)^T\}$;

Solution. (continue)

The span of the vectors in C is the set of vectors of the form

$$\begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix}, \qquad \alpha, \beta \in \mathbf{R},$$

which is clearly not the entire $\mathbf{R}^3$, since the vector $(1,0,0)^T$ is not in the span.

Part D is left as **an exercise**.

Remark. The last example shows that

- the spanning set of a vector space $\boldsymbol{V}$ may **not** be unique, since the vector sets in A, B are both spanning set of $\mathbf{R}^{3}$;
- the number of vectors of a spanning set of $\boldsymbol{V}$ may also be **not** unique, because in A, B the number of vectors is four and three respectively.

Example 11. Show that the vector set

$$S = \left\{ E_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, E_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

is a spanning set of the vector space $\mathbf{R}^{2\times 2}$.

Proof. See the textbook.

Example 12. Show that the vectors $1-x^2, x+2, x^2$ span P_3.

Proof. Any vector p in P_3 can be written as

$$p(x) = ax^2 + bx + c.$$

It is enough to show that p can be represented as a linear combination of the given vectors, i.e. to find constants $\alpha_1, \alpha_2, \alpha_3$ such that

$$ax^2 + bx + c = \alpha_1\left(1 - x^2\right) + \alpha_2(x + 2) + \alpha_3 x^2.$$

This leads to the following system

$$\begin{aligned} \alpha_3 - \alpha_1 &= a, \\ \alpha_2 &= b, \\ 2\alpha_2 - \alpha_1 &= c. \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} \alpha_1 &= c - 2b \\ \alpha_2 &= b, \\ \alpha_3 &= a + c - 2b \end{aligned}$$

Therefore, the three vectors span P_3.

Linear System revisited

Let S be the solution set to the linear system $A\boldsymbol{x} = \boldsymbol{b}$, where A is $m \times n$, $\boldsymbol{b} \in \mathbf{R}^m$.

- If $\boldsymbol{b} = \mathbf{0}$, then $S = N(A)$ is a subspace of $\mathbf{R}^n$.
- If $\boldsymbol{b} \neq \mathbf{0}$, then S is not a subspace of $\mathbf{R}^n$.

Theorem. Assume that the linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent and $\boldsymbol{x}_0$ is a particular solution. Then a vector $\boldsymbol{y}$ is in the solution set S **if and only** if

$$\boldsymbol{y} = \boldsymbol{x}_0 + \boldsymbol{z},$$

where $\boldsymbol{z} \in N(A)$.

Linear System revisited

Solution set of $A\boldsymbol{x} = \boldsymbol{b}$