Chapter 9, 10 and 11

Momentum, Collision and Rotation

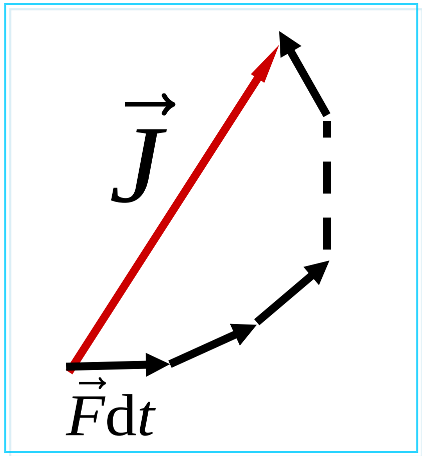
§ 1 Impulse and Momentum

- **Definition of impulse of a force** $\vec{F}$

$$\vec{J} = \int_{t_i}^{t_f} \vec{F} dt$$

SI unit N·s

The impulse of a force is a vector. It depends on the strength of the force and on its duration.



Impulse and Momentum

- **Another form of Newton's second law in terms of momentum**

$$\vec{F} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

- **Definition of momentum or linear momentum of an object**

$$\vec{p} = m\vec{v}$$

SI unit kg•m/s

- **The form** $\vec{F} = m\vec{a}$ **is the special case for Newton's second law when the mass of the object remains constant.**

Impulse-momentum theorem (P206)

- **The impulse-momentum theorem for a particle**

$$\vec{J}=\int_{t_i}^{t_f}\vec{F}dt=\int_{t_i}^{t_f}\frac{d\vec{p}}{dt}dt=\int_{\vec{p}_i}^{\vec{p}_f}d\vec{p}=\vec{p}_f-\vec{p}_i=\Delta\vec{p}$$

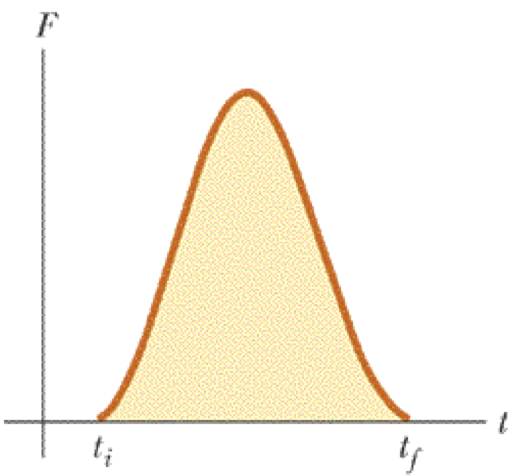
$$\boxed{\vec{J}=\Delta\vec{p}}$$

The impulse of the net force acting on a particle during a given time interval is equal to the change in momentum of the particle during that interval. (Valid only in inertial frame of reference)

Time - averaged impulsive force (P205 § 9-3)

Impulsive force

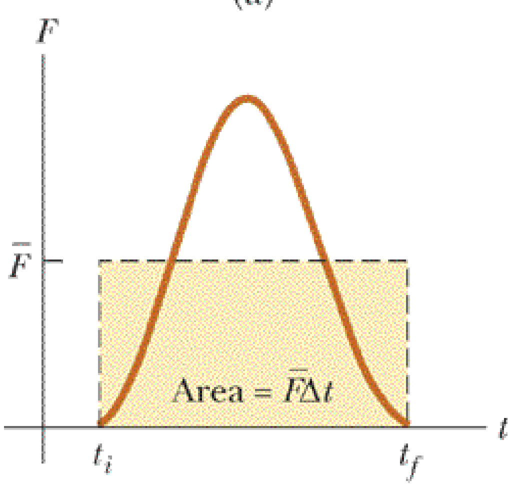
- When a time-varying net force $\vec{F}(t)$ is difficult to measure, we can use a **time-averaged net force** as the substitute provided that it would give the same impulse to the particle in same time interval.



(a)

$$\overline{\vec{F}} = \frac{\int_{t_i}^{t_f} \vec{F} dt}{\Delta t} = \frac{\Delta \vec{p}}{\Delta t}$$

$$\vec{J} = \Delta \vec{p} = \overline{\vec{F}} \Delta t$$

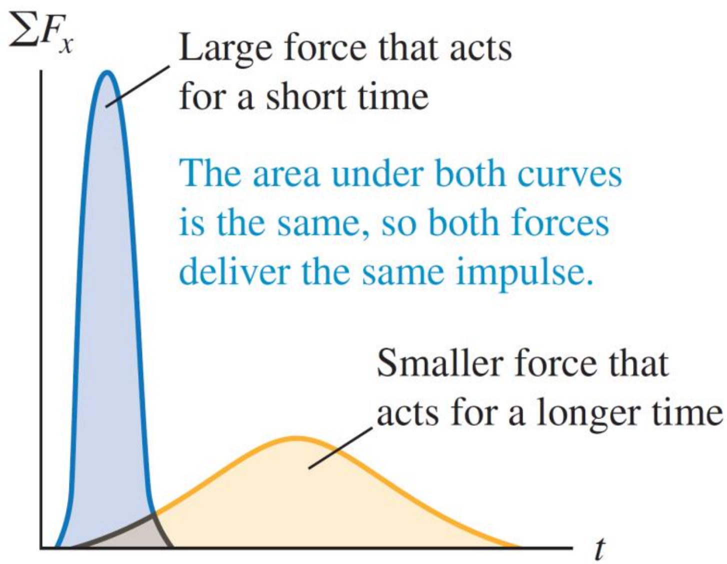


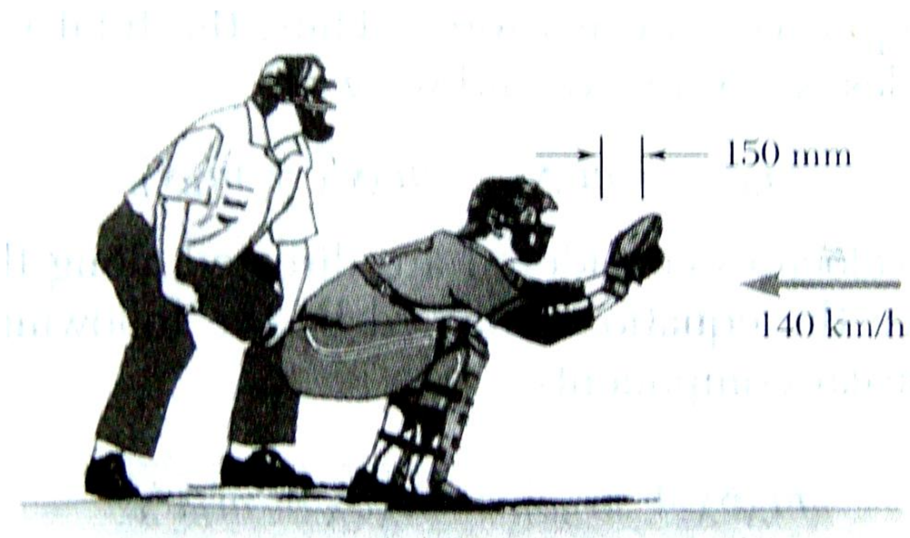
(b)

- When a particle experiences a impact in a very short time, the non-impulse forces such as **gravitational** force and **friction** force are **negligible** compared to **impulsive** force.

Time - averaged impulsive force

- For a given amount of momentum change, we can delay the time interval to decrease the impulsive force.
- A baseball player catching a ball can soften the impact by pulling his hand back.

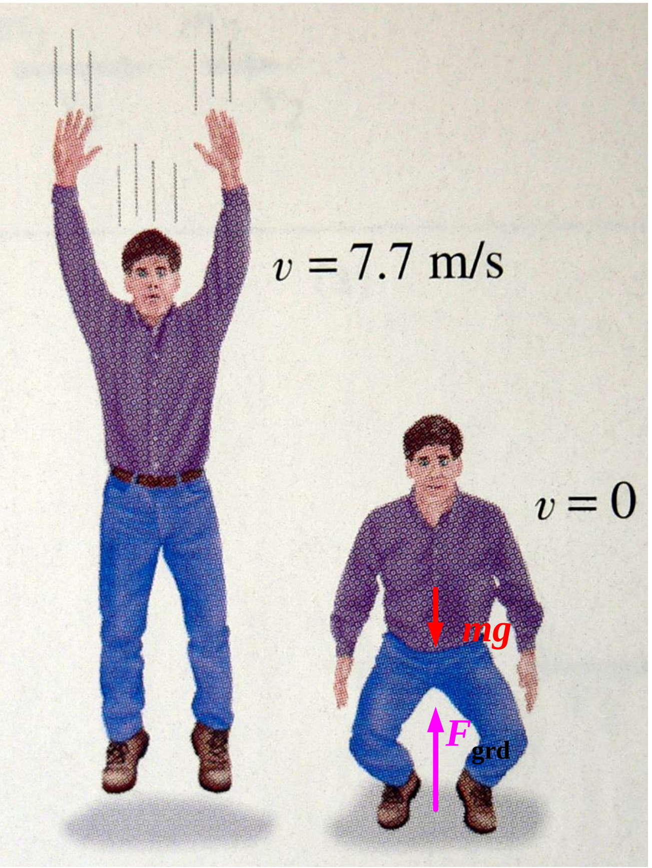




Example
(P207 Ex. 9-6)

Bend your knees when landing.

(a) Calculate the impulse experienced when a 70 kg person lands on firm ground after jumping from a height of 3.0 m. Then estimate the average force exerted on the person's feet by the ground, if the landing is (b) stiff-legged (body moves 1.0 cm during impact), and (c) with bent legs (about 50 cm).



Solution

(a) $v = \sqrt{2gh} = 7.7\text{m/s}$

$$J = p_f - p_i = 0 - (70\text{kg})(7.7\text{m/s}) = -540\text{N}\cdot\text{s}$$

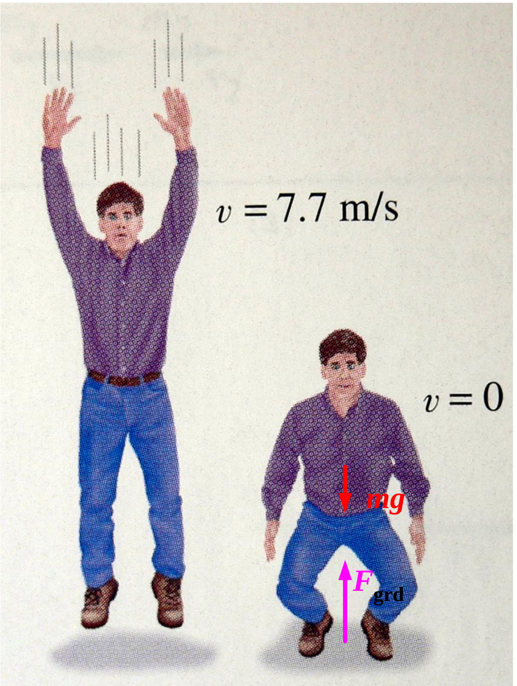
(b) $d = 1.0\ \text{cm} = 1.0\times10^{-2}\ \text{m}$

$$\bar{v} = (7.7+0)/2 = 3.8\text{m/s},\quad \Delta t = d/\bar{v} = 2.6\times10^{-3}\text{s}$$

$$F_{\text{grd}} + mg = \frac{J}{\Delta t} = \frac{-540}{2.6\times10^{-3}} = -2.1\times10^{5}\,\text{N}$$

$mg=(70\text{kg})(9.8\text{m/s}^2)=690\text{N}$

$$\left|F_{\text{grd}}\right| = 2.1\times10^{5}\,\text{N} + 690\,\text{N} \approx 2.1\times10^{5}\,\text{N} \gg mg$$



The person's legs would likely break in such a stiff landing.

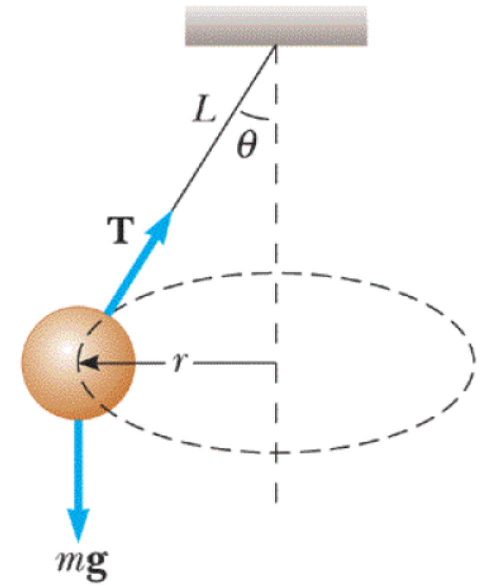
(c) $d = 0.50$ m, $\Delta t = 0.13$ s,

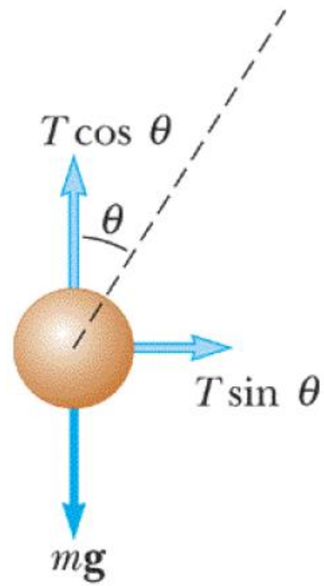
$$F_{\text{grd}} + mg = \frac{540}{0.13} = -4.2\times10^{3}\,\text{N},\qquad F_{\text{grd}} = -4.9\times10^{3}\,\text{N}$$

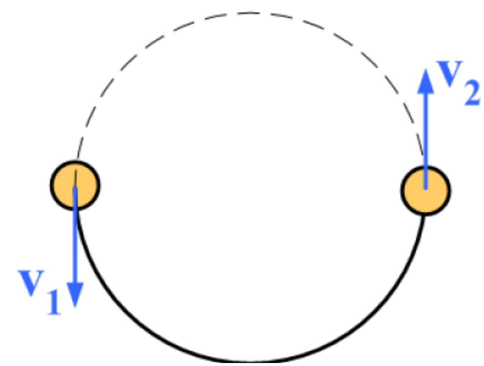
Example

Conical Pendulum

A small object of mass m is suspended from a string. The object revolves in a horizontal circle of radium r with constant speed v. Determine the **impulse** exerted (1) by **gravity**, (2) by string **tension** on the object, during the time in which the object has passed **half** of the circle.







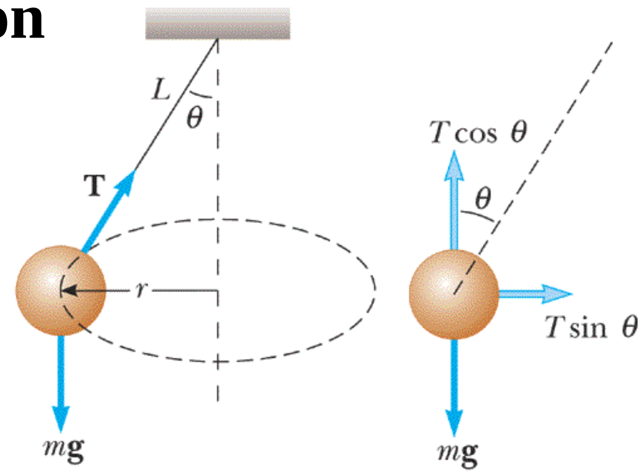
Example

Solution: **(1) The impulse exerted by gravity on the object**

$$\vec{J}_{mg}=\int_{t_1}^{t_2} m\vec{g}dt=m\vec{g}\int_{t_1}^{t_2}dt=m\vec{g}\left(\frac{1}{2}\frac{2\pi r}{v}\right)=\frac{\pi r}{v}m\vec{g}$$

(2) The impulse exerted by string tension on the object

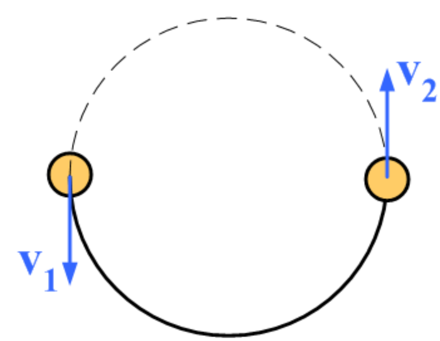
$$\vec{J}_T=\vec{J}_{net}-\vec{J}_{mg}$$



From impulse-momentum theorem

$$\vec{J}_{net}=\Delta\vec{p}=m\vec{v}_2-m\vec{v}_1=2m\vec{v}$$

$$\vec{J}_T=2m\vec{v}-\frac{\pi r}{v}m\vec{g}$$



$$J_T=\sqrt{(2mv)^2+\left(\frac{\pi rmg}{v}\right)^2}=m\sqrt{4v^2+\frac{\pi^2r^2g^2}{v^2}}$$

§ 2 Impulse-momentum theorem for a system of particles

Consider a system of N interacting particles

For i-th particle:

the net external force $\vec{F}_i$

the internal force exerted by j-th particle $\vec{f}_{ij}$

$$(\vec{F}_i + \sum_{j \neq i} \vec{f}_{ij})dt = d\vec{p}_i$$

For the system of particles: $\displaystyle \sum_i (\vec{F}_i + \sum_{j \neq i} \vec{f}_{ij})dt = \sum_i d\vec{p}_i$

According to Newton's third law, the internal forces cancel in pairs. $\displaystyle \sum_i \sum_{j \neq i} \vec{f}_{ij} = 0$

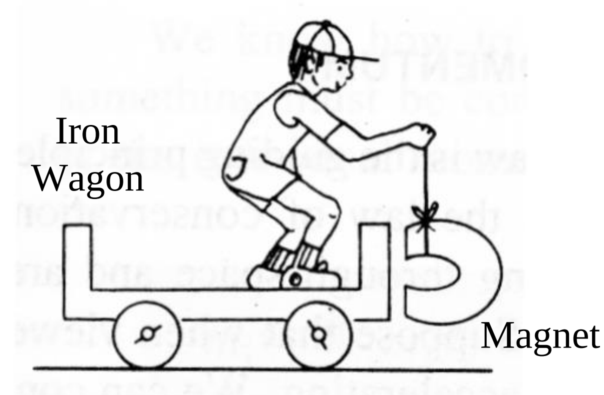
The total external force acting on the system: $\displaystyle \sum_i \vec{F}_i$

The total momentum of the system: $\displaystyle \sum_i d\vec{p}_i = d\left(\sum_i \vec{p}_i\right) = d\vec{p}_{\text{tot}}$

Impulse-momentum theorem for a system of particles

Conclusion: $$\left(\sum_i \vec{F}_i\right) dt = d\vec{p}_{\text{tot}}$$

Can you get the wagon to move by hanging a huge magnet in front of you?



The derivative form:

$$\boxed{\sum_i \vec{F}_{i-\text{ext}} = \frac{d\vec{p}_{\text{tot}}}{dt}}$$

The integral form:

$$\boxed{\int_{t_1}^{t_2} \sum_i \vec{F}_{i-\text{ext}}\, dt = \vec{p}_{\text{tot}2} - \vec{p}_{\text{tot}1}}$$

- **The total external force applied to a system of particles equals to the change in total momentum of the system.**
- **The internal forces can exchange the momenta between particles within system, but can not influence the total momentum of the system.**

Conservation of momentum

When $\sum_i \vec{F}_{i-\text{ext}} = 0, \quad \frac{d\vec{p}_{\text{tot}}}{dt} = 0$ **or** $\vec{p}_{\text{tot}} = \sum_i \vec{p}_i = \text{constant}$

- **When the vector sum of external forces on a system is zero, the total momentum of the system is constant.**

- **Notice the difference between conservation of momentum and conservation of mechanical energy.**

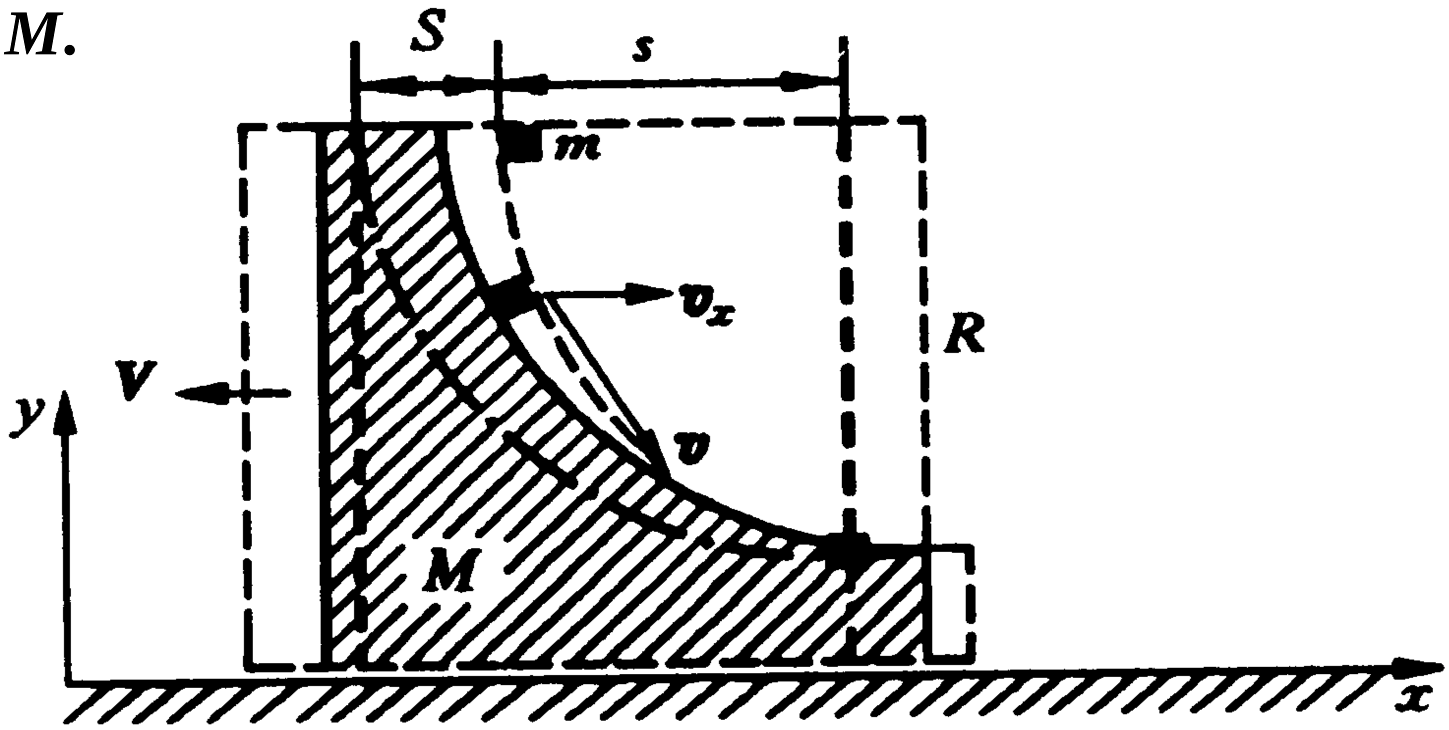
  For an isolated system, the mechanical energy is conserved only when the internal forces are conservative. But conservation of momentum is valid even when the internal forces are not conservative.

- **Conservation of momentum in component form**

When $\sum_i F_{i-\text{ext-}x} = 0$ **then** $p_{\text{tot-}x} = \sum_i p_{i-x} = \text{constant}$

Example

A small cube of mass *m* slides down a circular path of radius *R* cut into a large block of mass *M*. *M* rests on a frictionless table. *M* and *m* are initially at rest. *m* starts from the top of the path. Find the distance traveled by *M* when the cube *m* leaves the block *M*.



Solution

No horizontal external force acts on the system consisting of the cube and the block. The total momentum of the system is conserved in horizontal direction.

$$0 = mv_x + M(-V) \quad \Rightarrow \quad mv_x = MV$$

Integrations on both side: $m\int_0^t v_x dt = M\int_0^t V dt, \quad ms = MS$ ①

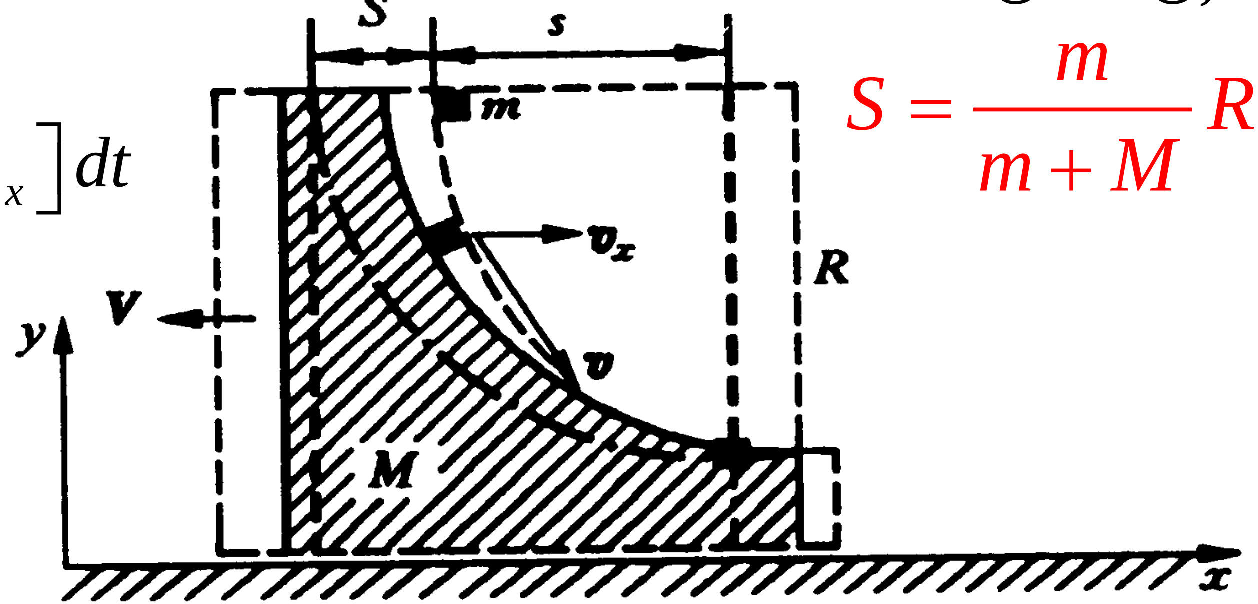
In the reference frame of M: the horizontal displacement of m is

$$R = \int_0^t (v_{mM})_x dt$$

$$= \int_0^t \left[(v_{m地})_x - (v_{M地})_x \right] dt$$

$$= \int_0^t \left[v_x - (-V) \right] dt$$

$$= s + S \qquad ②$$

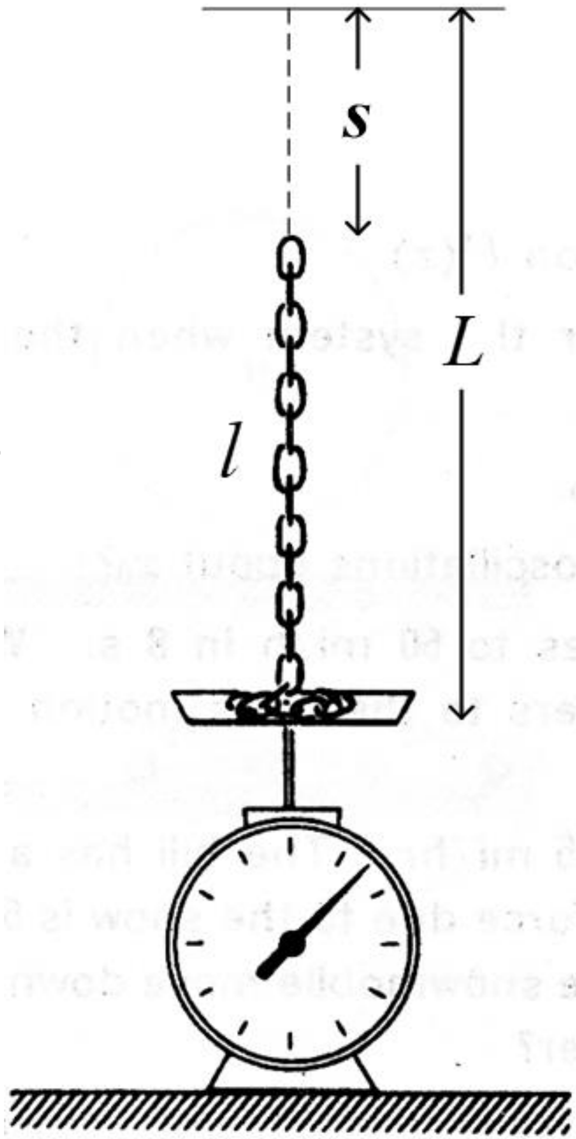


From ① and ②,

$$S = \frac{m}{m+M} R$$

Example

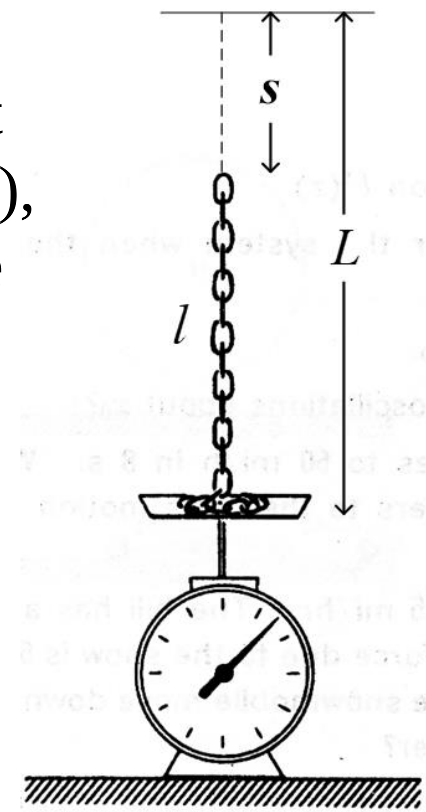
A chain of mass M length L is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale. What is the **reading** of the scale when a length of chain, s, has fallen? (Neglect the size of individual links.)



Example

Solution (I) : Using impulse-momentum theorem for a particle

Assuming a length of chain ***s*** has been already in the scale. Take a infinitesimal process during ***dt***, a segment chain of length of ***vdt*** impacts with the scale ($v=\sqrt{2gs}$), and comes to a halt. The impulse that the surface of the scale acting on this segment is:



$$Fdt = 0 - \left(\frac{M}{L}vdt\right)(-v),$$

$$F = \frac{M}{L}v^2 = \frac{M}{L}(2gs) = 2Mg\frac{s}{L}$$

$$F' = -F = -2Mg\frac{s}{L}$$

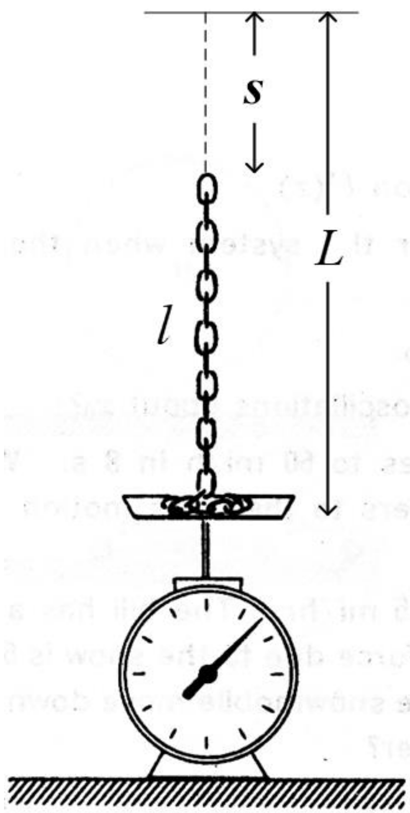
The reading of the scale $=$ the weight that has already in the scale + $|F'|$

$$= Mg\frac{s}{L} + 2Mg\frac{s}{L} = 3Mg\frac{s}{L}$$

Example

Solution (II) : Using impulse-momentum theorem for a system

$$F_{net} - Mg = \frac{d(-p)}{dt}$$



$$\frac{dp}{dt} = \frac{d}{dt}\left(\frac{L-s}{L}Mv\right) = \frac{d}{dt}\left[\frac{M}{L}(L-s)\sqrt{2gs}\right]$$

$$= \frac{d}{ds}\left[\frac{M}{L}(L-s)\sqrt{2gs}\right]\frac{ds}{dt}$$

$$= \frac{M}{L}\left[L\frac{g}{\sqrt{2gs}} - \sqrt{2gs} - s\frac{g}{\sqrt{2gs}}\right]\sqrt{2gs}$$

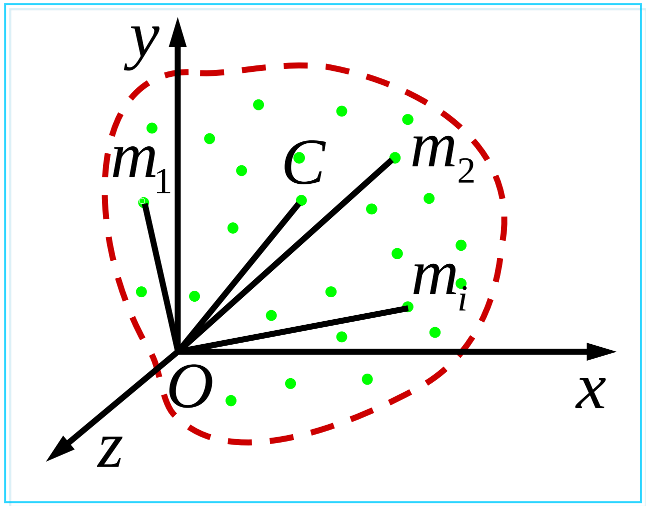
$$= Mg - 3\frac{s}{L}Mg, \qquad F_{net} = 3\frac{s}{L}Mg$$

§ 3 Center of Mass

Describe the motion of a **system** of particles
- by every motion for individual particles
- by overall motion in terms of center of mass

- Center of mass
  - For the system of discretely distributed particles

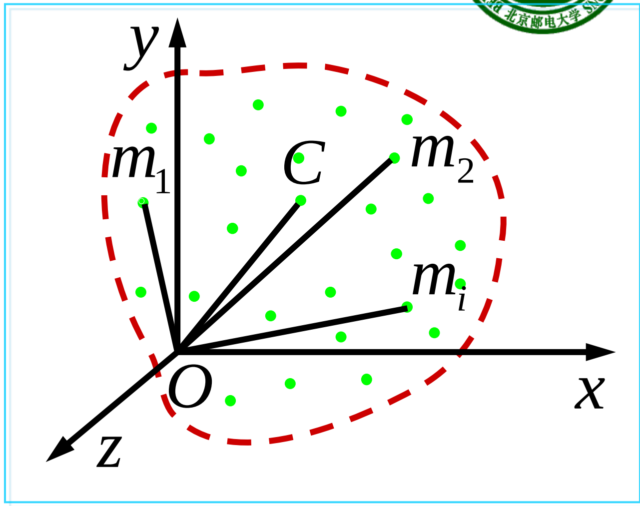
$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M}$$



$$x_{CM} = \frac{\sum_i m_i x_i}{M}, \quad y_{CM} = \frac{\sum_i m_i y_i}{M}, \quad z_{CM} = \frac{\sum_i m_i z_i}{M}$$

The Newton's Second Law for the motion of CM

(P219 § 9-9)



- **Motion of the center of mass**

Starting from $$M\,\vec{r}_{\mathrm{CM}} = \sum_i m_i \vec{r}_i$$

by derivative $$M\,\frac{d\vec{r}_{\mathrm{CM}}}{dt} = \sum_i m_i \frac{d\vec{r}_i}{dt}$$

$$M\,\vec{v}_{\mathrm{CM}} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i = \vec{p}_{\mathrm{tot}}$$

The total momentum of the system of particles is equal to its total mass times the velocity of center of mass, just as though all the mass were concentrated at center of mass.

The Newton's Second Law for the motion of CM

$$M\vec{v}_{\mathrm{CM}} = \vec{p}_{\mathrm{tot}}, \qquad \sum_i \overrightarrow{F}_{i-\mathrm{ext}} = \frac{d\vec{p}_{\mathrm{tot}}}{dt} = M\frac{d(\vec{v}_{\mathrm{CM}})}{dt} = M\vec{a}_{\mathrm{CM}}$$

- **The Newton's Second Law for the motion of center of mass**

$$\boxed{\sum_i \overrightarrow{F}_{i-\mathrm{ext}} = M\vec{a}_{\mathrm{CM}}}$$

The overall translational motion of a system of particles can be analyzed using Newton's Law as if all the mass were concentrated at the center of mass and total external force were applied at that point.

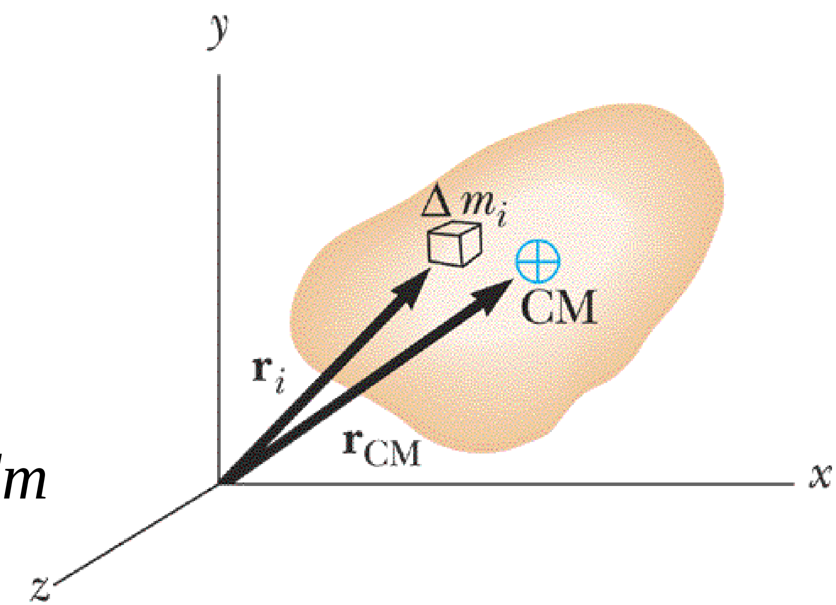
- **If net external force is zero, the center of mass moves with constant velocity**

$$\sum_i \overrightarrow{F}_{i-\mathrm{ext}} = 0 \quad \Rightarrow \quad \vec{p}_{\mathrm{tot}} = M\vec{v}_{\mathrm{CM}} = \text{constant}$$

Center of Mass

- For the extended object with uniformly distribution of mass

$$\boxed{\vec{r}_{\mathrm{CM}} = \frac{1}{M}\int \vec{r}\, dm}$$



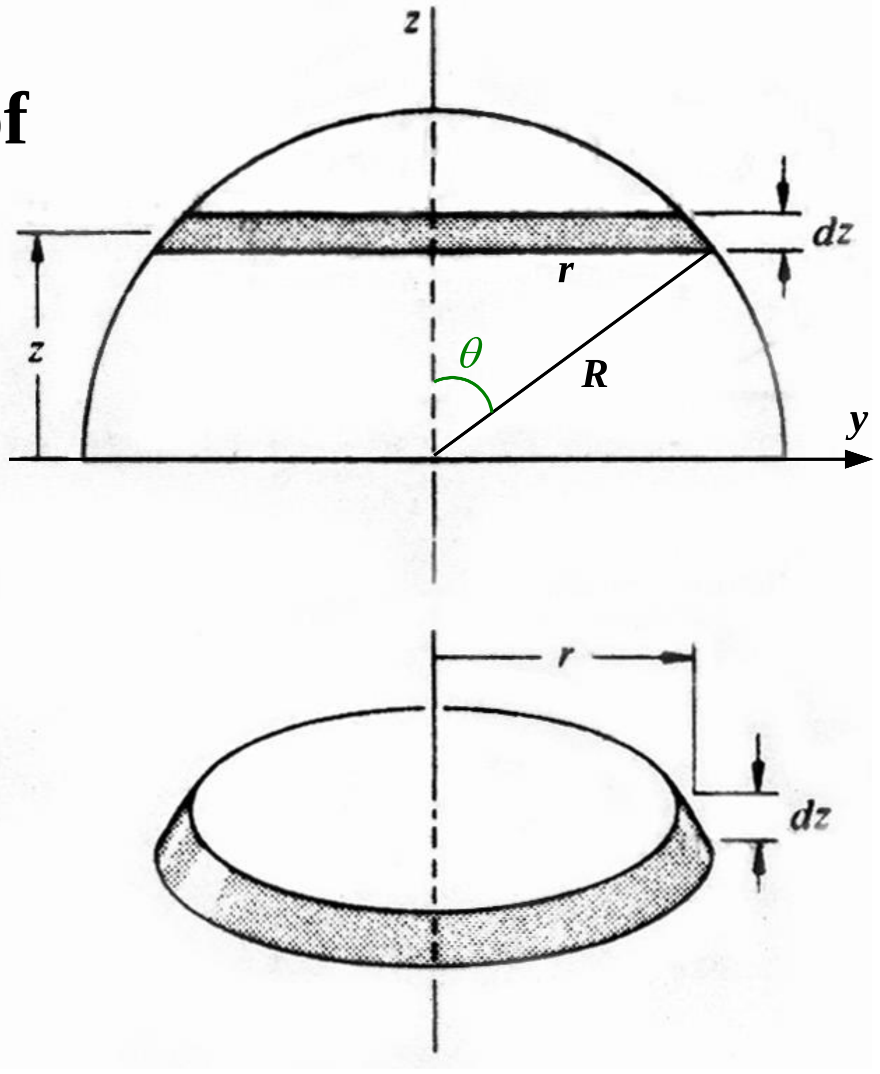
$$x_{\mathrm{CM}} = \frac{\displaystyle\lim_{\substack{N\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{N} x_i \Delta m_i}{\displaystyle\lim_{\substack{N\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{N} \Delta m_i} = \frac{\int x\, dm}{\int dm} = \frac{1}{M}\int x\, dm$$

$$y_{\mathrm{CM}} = \frac{1}{M}\int y\, dm$$

$$z_{CM} = \frac{1}{M}\int z\, dm$$

Example

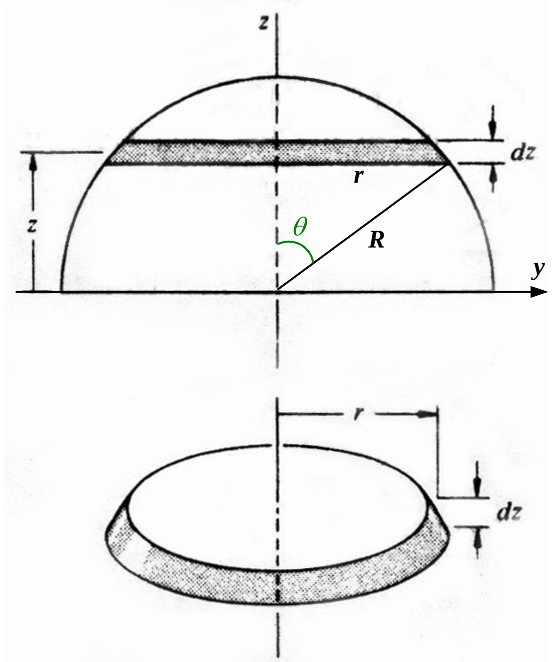
Find the center of mass of a uniform solid hemisphere of radius R and mass M.



Example

Solution: From symmetry it is apparent that the center of mass lies on the z axis. $\boldsymbol{x_{CM}=0}$, $\boldsymbol{y_{CM}=0}$.

$$z_{\mathrm{CM}} = \frac{1}{M}\int z\,dm = \frac{1}{M}\int z\rho\,dV$$



The three-dimensional integral can be treated as an one-dimensional integral. Subdivide the hemisphere into a pile of thin disk.

$$\begin{cases} dV = \pi r^2 dz \\ \rho = M\Big/\left(\dfrac{2}{3}\pi R^3\right) \end{cases}$$

Find $\boldsymbol{r}$, $\boldsymbol{z}$ in terms of $\boldsymbol{\theta}$.

$$r = R\sin\theta,$$

$$z = R\cos\theta,$$

$$dz = -R\sin\theta\, d\theta$$

$$z_{\mathrm{CM}} = \frac{3}{2R^3}\int_{\frac{\pi}{2}}^{0}(R\cos\theta)(R^2\sin^2\theta)(-R\sin\theta)d\theta$$

$$= \frac{3}{2}R\int_0^{\frac{\pi}{2}}\cos\theta\sin^3\theta\, d\theta = \frac{3}{2}R\int_0^{\frac{\pi}{2}}\sin^3\theta\, d(\sin\theta)$$

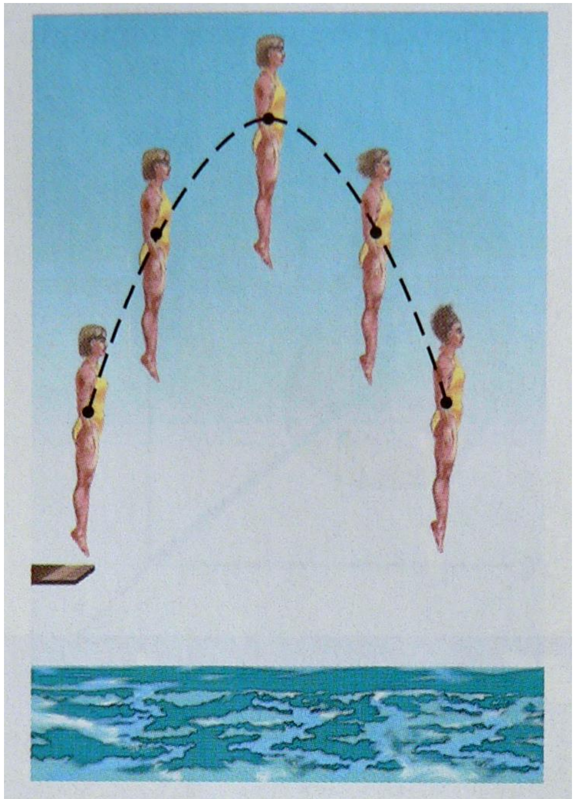
$$= \frac{3}{2}R\times\frac{1}{4} = \frac{3}{8}R$$

Applications of center of mass

- **For a rigid body**

  We can describe a rigid body as a combination of translational motion of the center of mass and rotational motion about an axis through the center of mass.

Applications of center of mass



(a)

(b)

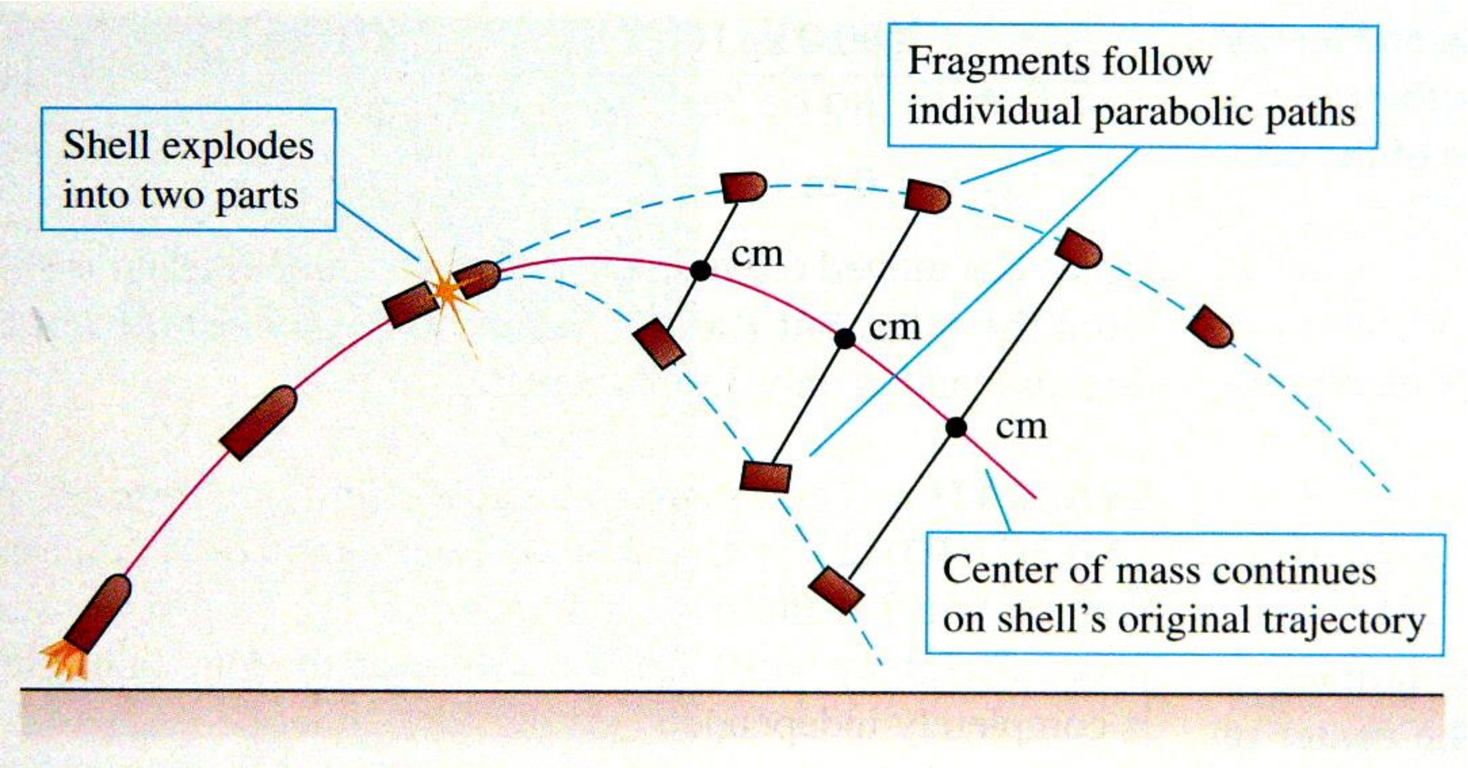
Fig. (a) The motion of the diver is pure translation.

Fig. (b) The motion of the diver is translation plus rotation.

Applications of center of mass

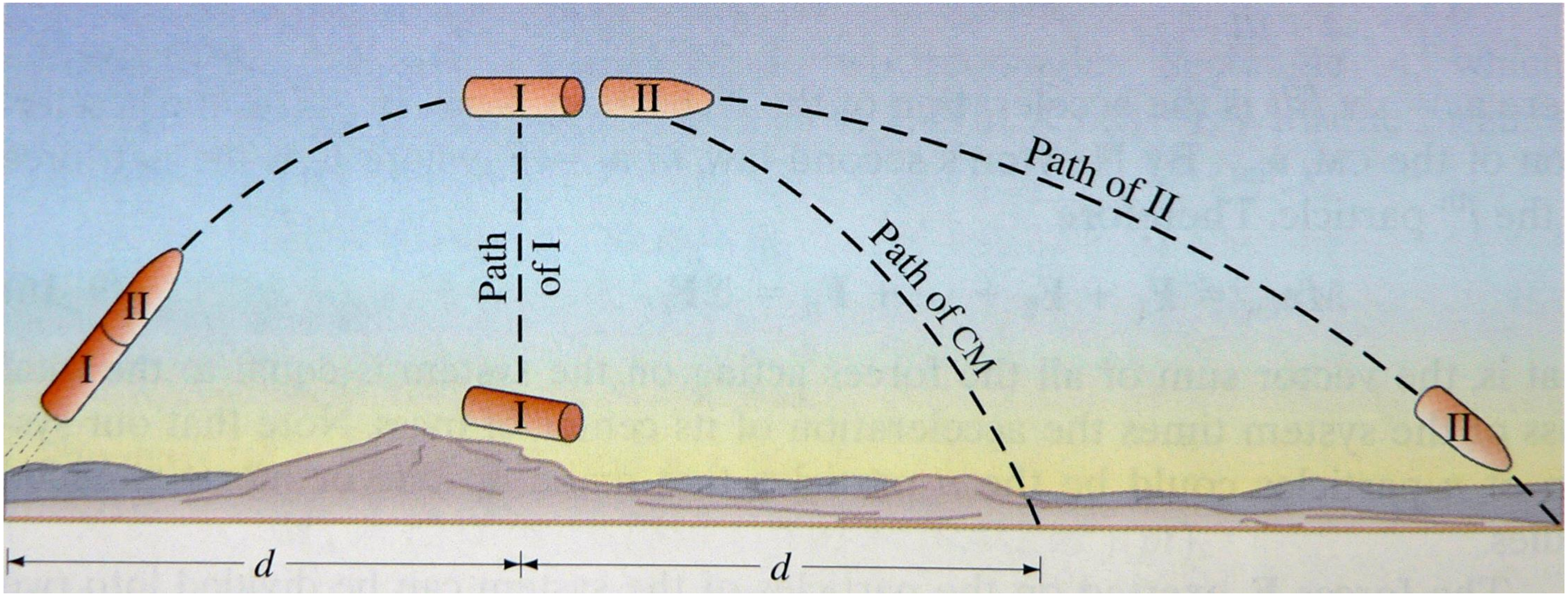
- **For a system of discrete particles**

  A cannon shell in a parabolic trajectory explodes in flight, splitting into two fragments. The fragments follow new paths, but center of mass continues on the original parabolic trajectory.

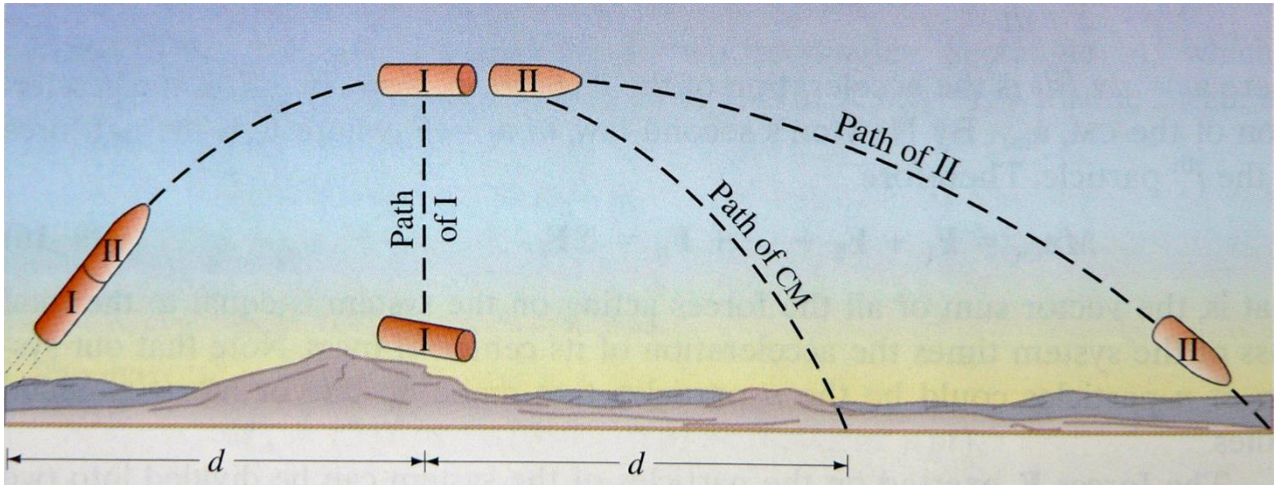


Example (P220 Ex. 9-16)

A rocket is fired into the air. At the moment it reaches its highest point, a horizontal distance ***d*** from its starting point, an explosion separates it into two parts of **equal mass**. Part I is stopped in midair by explosion and falls **vertically** to Earth. Where does part II land?



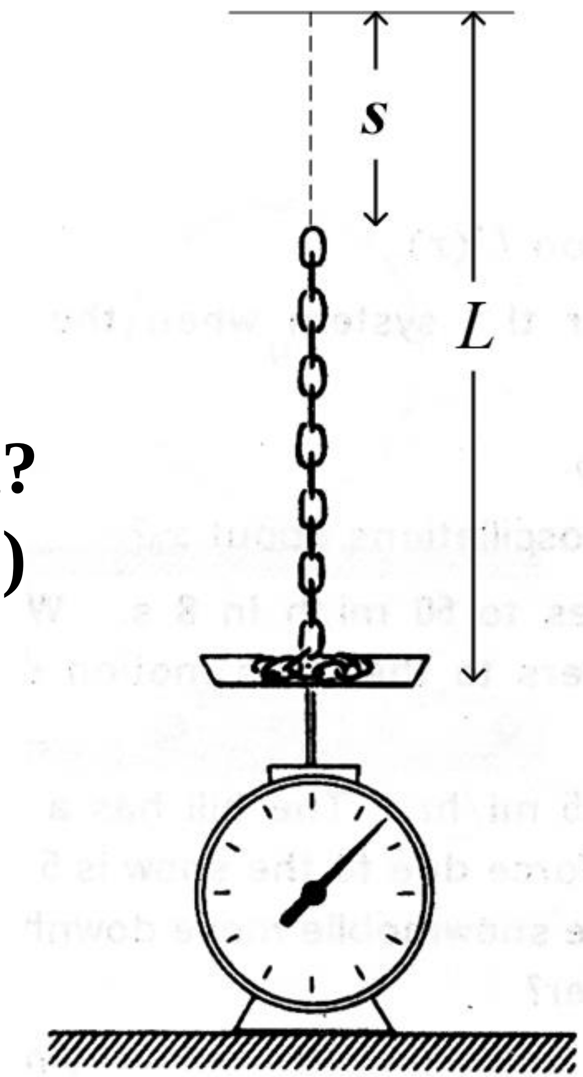
Example



Solution: After the rocket is fired, the path of the center of mass of the system continues to follow the parabolic trajectory of a projectile acted on only by a constant gravitational force. The center of mass will thus arrive at a point 2*d* from the starting point. Since the masses of I and II are equal, the center of mass must be midway between them. Therefore, II lands a distance 3*d* from the starting point.

Example

A chain of mass *M* length *L* is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale. What is the reading of the scale when a length of chain, *s*, has fallen? (Neglect the size of individual links.)



Example

Solution (III): **Using the Center of Mass**

$$N - Mg = M\frac{d^2 y_{\mathrm{CM}}}{dt^2}$$

$$y_{\mathrm{CM}} = \frac{M_1 y_1 + M_2 y_2}{M}$$

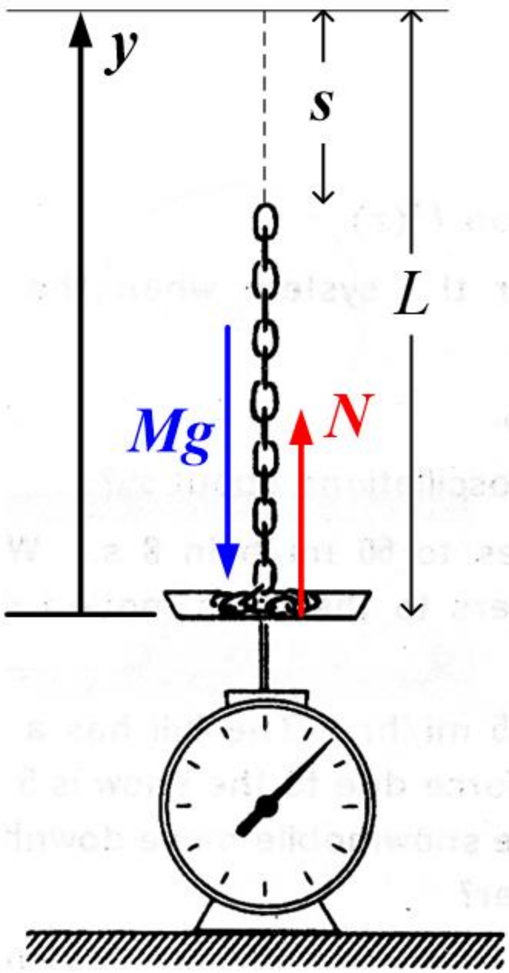
Two part:

$$M_1 = \lambda(L-s), \qquad y_1 = (L-s)/2$$

$$M_2 = \lambda s, \qquad y_2 = 0$$

$$\lambda = M/L$$

$$y_{\mathrm{CM}} = \frac{\lambda(L-s)\frac{L-s}{2}}{\lambda L} = \frac{(L-s)^2}{2L}$$



Example (continued)

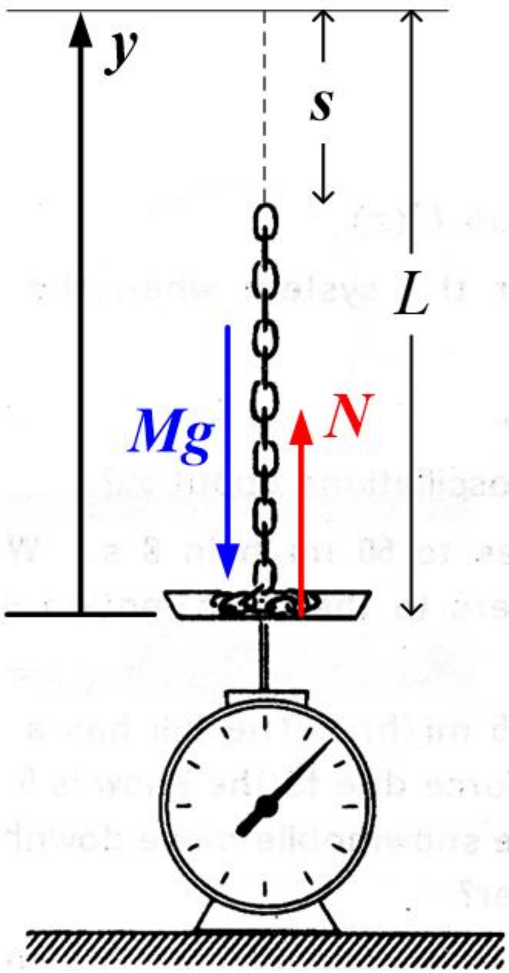
$$y_{\mathrm{CM}} = \frac{(L-s)^2}{2L}$$

For the part in the air: $s = \frac{1}{2}gt^2$

$$\frac{d\,y_{\mathrm{CM}}}{dt} = \frac{d}{dt}\left[\frac{(L-s)^2}{2L}\right] = -\frac{(L-s)}{L}\frac{ds}{dt}$$

$$= -\frac{\left(L-\frac{1}{2}gt^2\right)}{L}gt = -\frac{gt}{L}\left(L-\frac{1}{2}gt^2\right)$$

$$\frac{d^2 y_{\mathrm{CM}}}{dt^2} = \frac{g\left(\frac{3}{2}gt^2 - L\right)}{L} = \frac{g(3s-L)}{L}$$



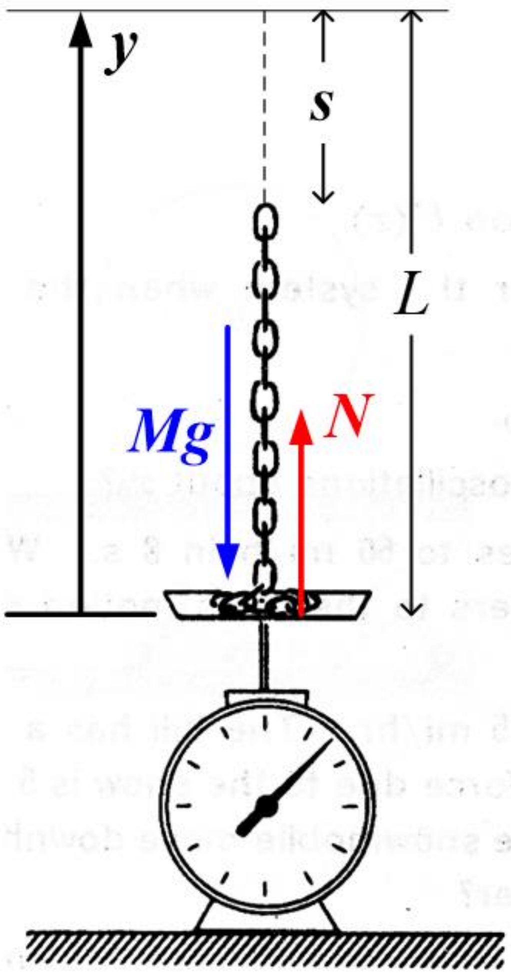
Example (continued)

$$\frac{d^2 y_{\mathrm{CM}}}{dt^2} = \frac{g(3s - L)}{L}$$

Newton's II law for CM:

$$N - Mg = M\frac{d^2 y_{\mathrm{CM}}}{dt^2} = Mg\left(\frac{3s}{L} - 1\right)$$

$$N = 3Mg\frac{s}{L}$$



Problem

Ch9 (P225)

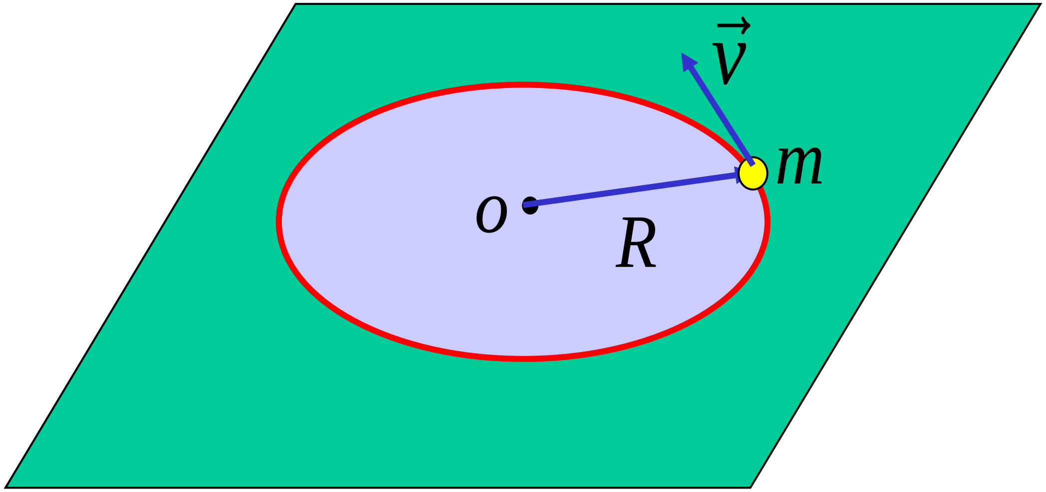
Prob. 2, 5, 67, 70

§ 4 Angular Momentum

(P277 § 11-3)

Why?

Example:



$$\left(m, v, R\right)$$

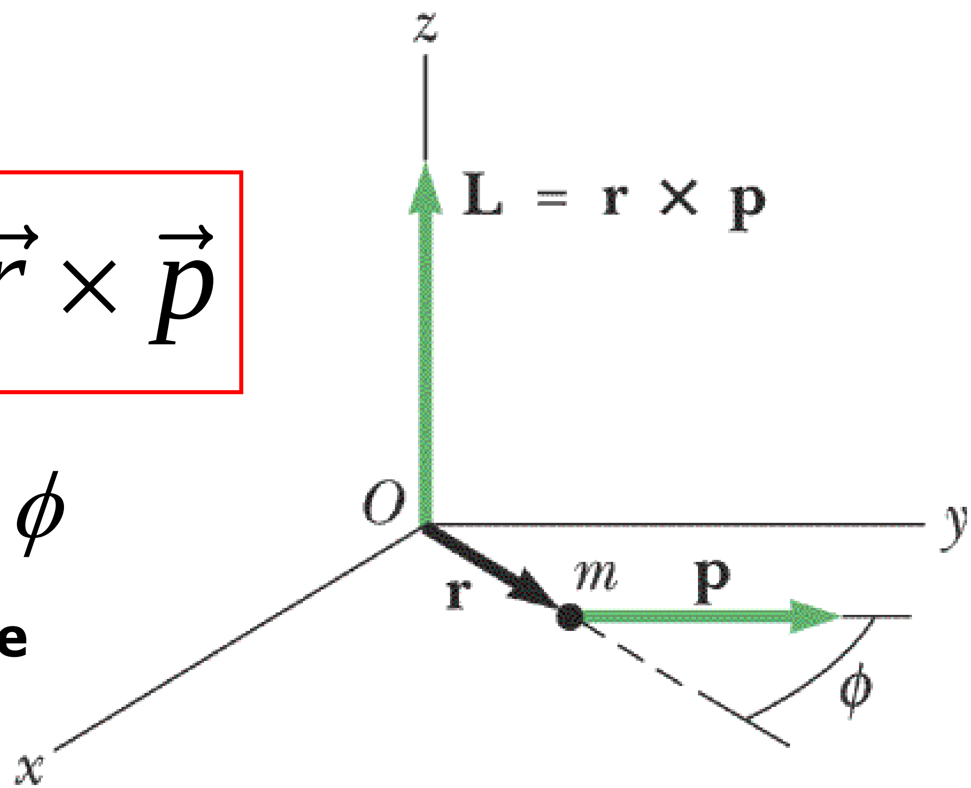
What?

- **Definition**

$$\boxed{\vec{L} = \vec{r} \times (m\vec{v}) = \vec{r} \times \vec{p}}$$

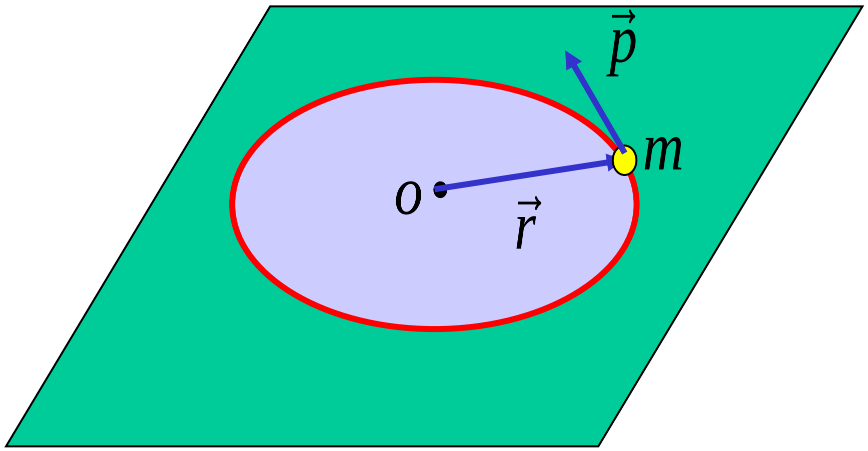
Magnitude: $L = mvr\sin\phi$

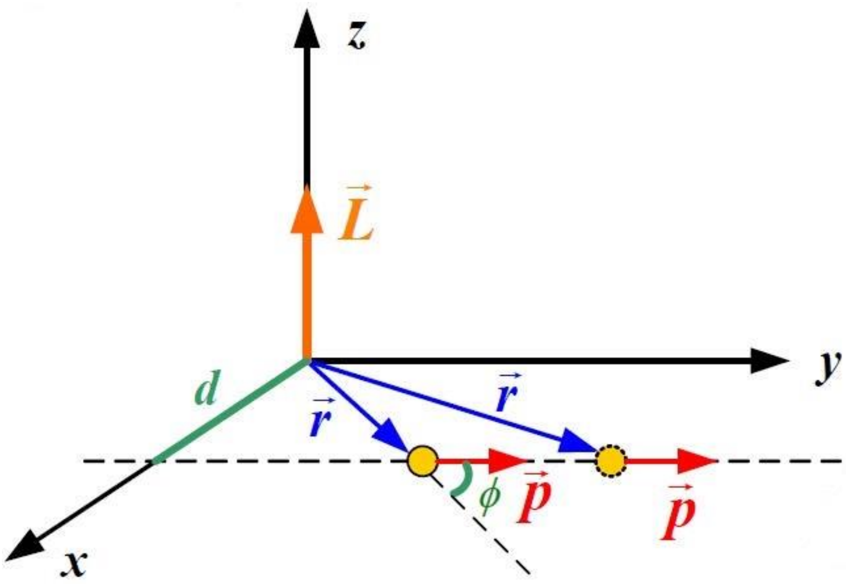
Direction: the right-hand rule



- **Depends on the choice of origin *O*.**
- **SI unit:** kg•m^2/s

Example





The particle that moves in the circular orbit.

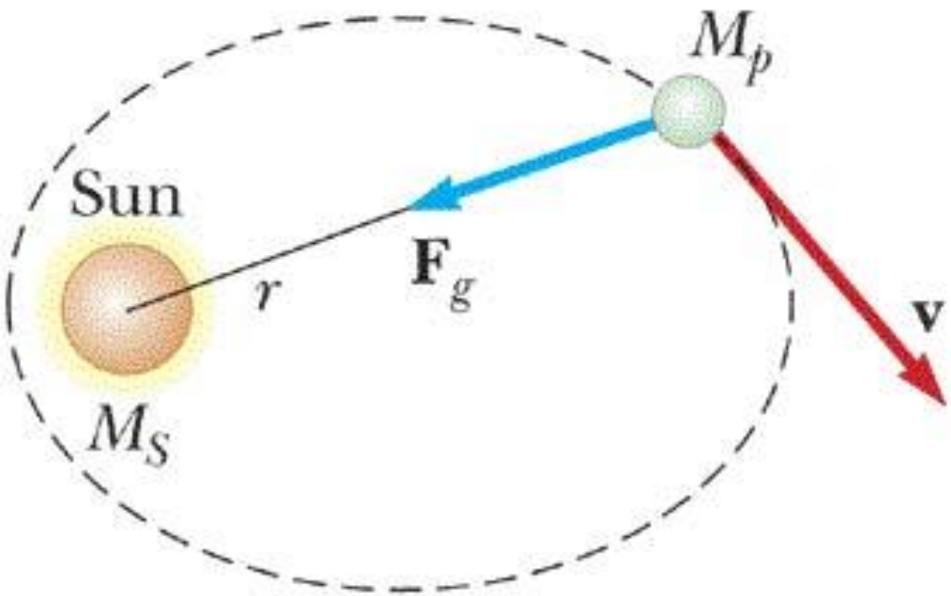
$$L = |\vec{r} \times \vec{p}| = mvr$$

The particle that moves in a straight line at constant velocity

$$L = |\vec{r} \times \vec{p}|$$

$$= mvr \sin\phi = mvd$$

Example



$$L = |\vec{r} \times \vec{p}\,|$$
$$= mvr\sin\phi$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F} = \vec{r} \times \vec{F}$$

Torque

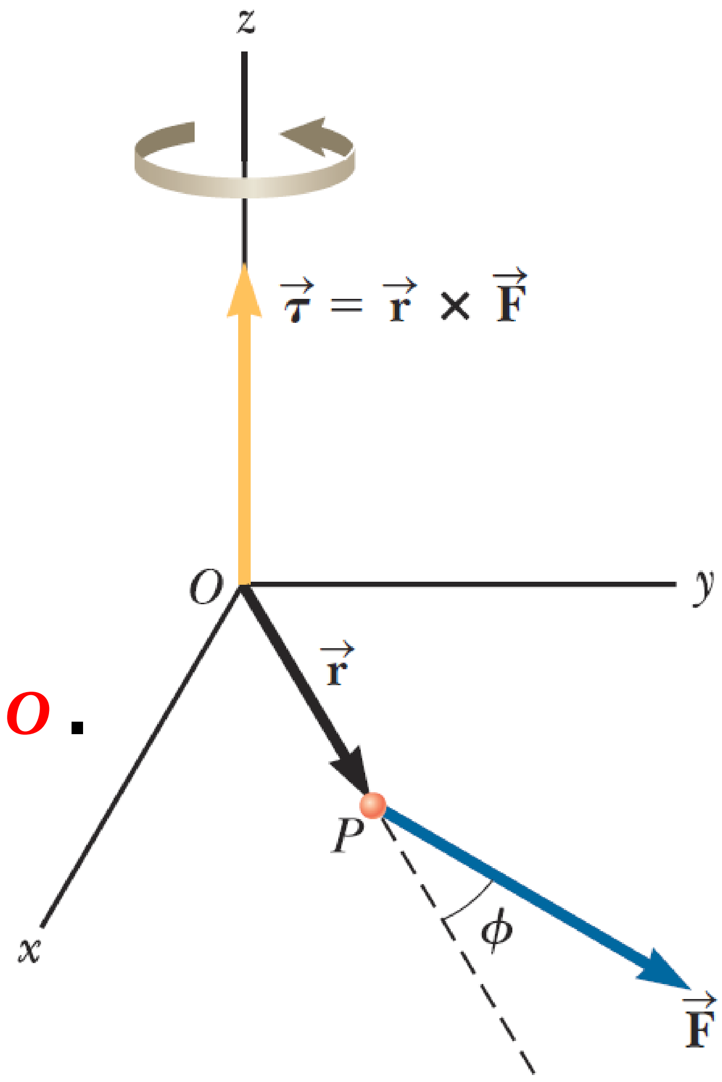
(P276 § 11-2)

- **Definition**

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

Magnitude: $$\tau = rF\sin\phi$$

Direction: the right-hand rule



- **Depends on the choice of origin *O* .**
- **SI unit: Newton • m.**

Torque-Angular Momentum Theorem

- **For one particle**

$$\vec{L} = \vec{r} \times \vec{p},$$

$$\vec{\tau} = \vec{r} \times \vec{F},$$

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}$$

The torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

- **Valid only if the origins of $\vec{L}$ and $\vec{\tau}$ are the same.**
- **Valid in inertial frame.**

$$\boxed{\vec{\tau} = 0 \quad \Rightarrow \quad \vec{L} = \text{const.}}$$

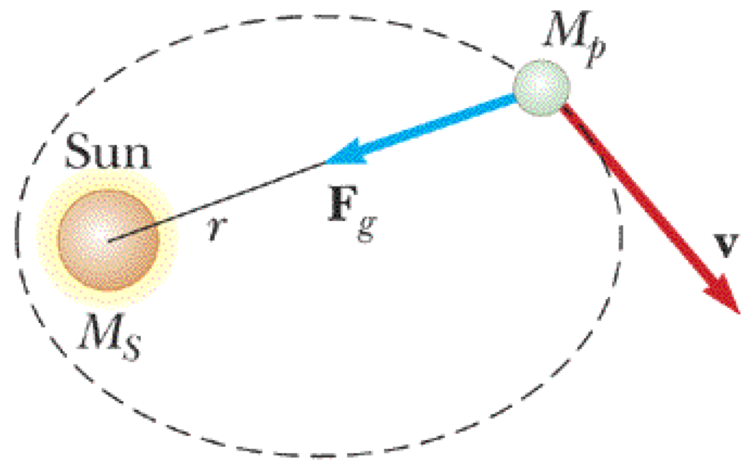
$$\left.\begin{array}{l} \text{(i) } \vec{F} = 0 \\ \text{(ii) } \vec{r} = 0 \\ \text{(ii) } \vec{r} \,/\!/\, \vec{F} \end{array}\right\} \Rightarrow \vec{\tau} = 0$$

Example: Kepler's Second Law
(P284 Ex.11-6)

"The radius vector drawn from the Sun to any planet sweeps out equal areas in equal time intervals."

$$dA = \frac{1}{2}\left|\vec{r} \times d\vec{r}\right| = \frac{1}{2}\left|\vec{r} \times \vec{v}\,dt\right|$$

$$\frac{dA}{dt} = \frac{1}{2}\left|\vec{r} \times \vec{v}\right| = \frac{1}{2}\left|\frac{L}{M_p}\right|$$



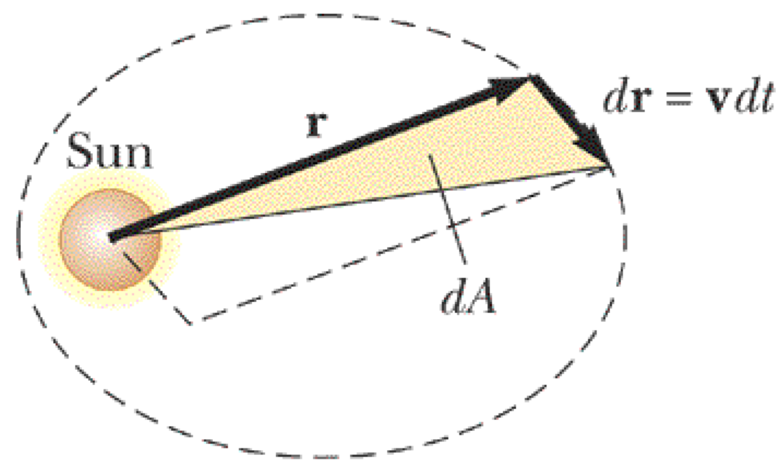
(a)

For a planet of mass M_p moving about the Sun

$$\vec{\tau} = \vec{r} \times \vec{F}_g = 0$$

$$\Rightarrow \quad \vec{L} = \text{constant}$$

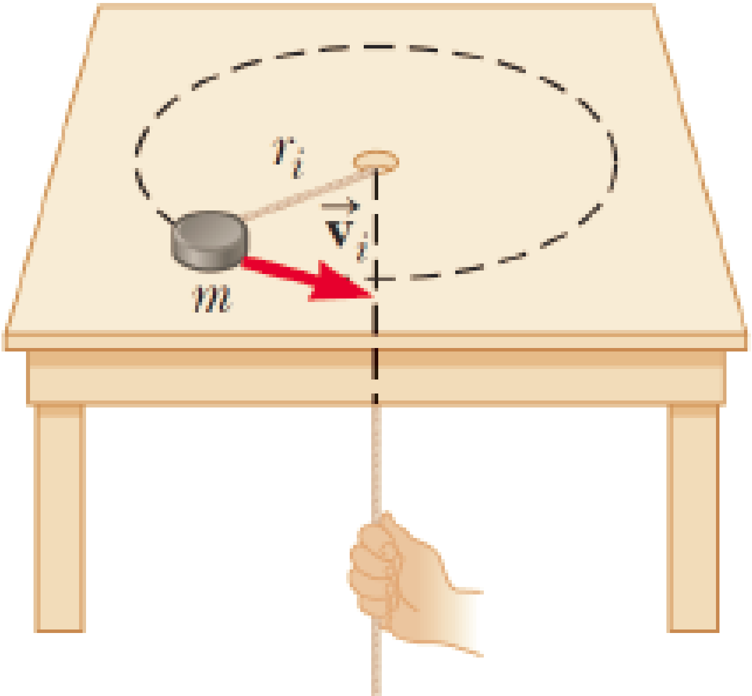
$$\Rightarrow \quad \frac{dA}{dt} = \text{constant}$$



(b)

Example

A puck of mass $m = 50.0$ g is attached to a taut cord passing through a small hole in a frictionless, horizontal surface as shown. The puck is initially orbiting with speed $v_i = 1.50$ m/s in a circle of radius $r_i = 0.300$ m. The cord is then slowly pulled from below, decreasing the radius of the circle to $r_f = 0.100$ m.



(a) What is the puck's speed at the smaller radius?

(b) Find the tension in the cord at the smaller radius.

(c) How much work is done by the hand in pulling the cord so that the radius of the puck's motion changes from r_i to r_f?

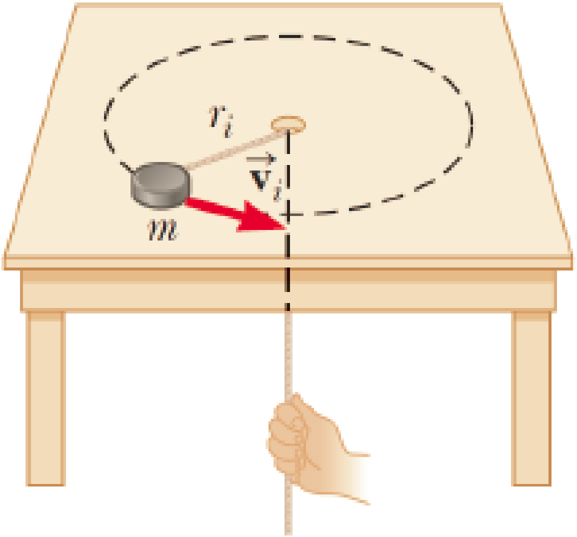
Solution

(a) Although an **external force** acts on the puck, no **external torques** act. Therefore, angular momentum conservation leads to

$$r_f\left(mv_f\right)=r_i\left(mv_i\right)$$

and

$$v_f=\frac{v_i r_i}{r_f}=4.50\text{ m/s}$$



(b) From Newton's second law, the tension is

$$T=m\frac{v_f^2}{r_f}=10.1\text{ N}$$

(c) The work-kinetic energy theorem identifies the work as

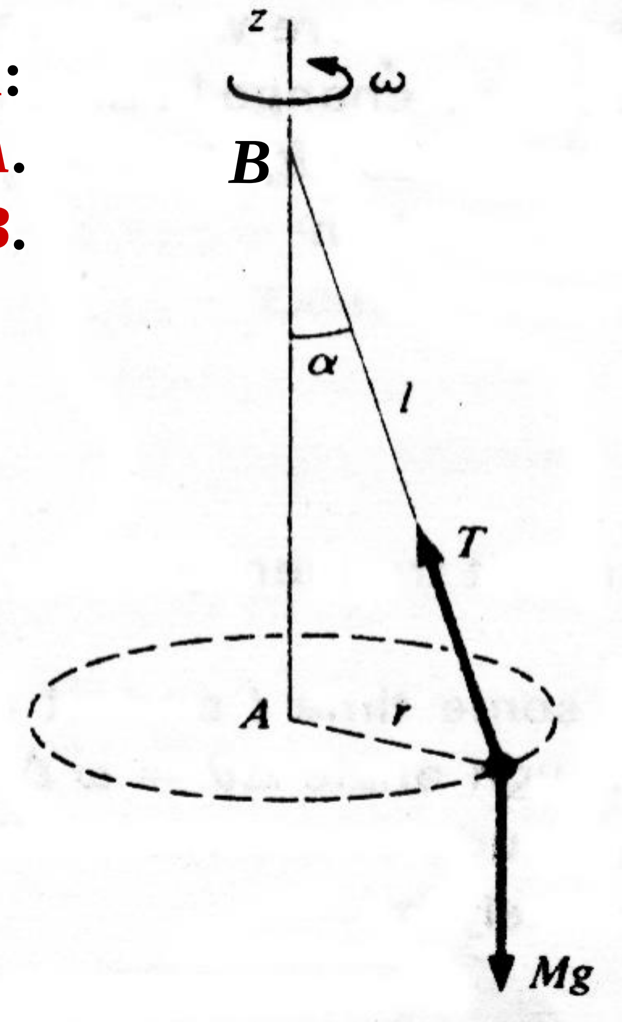
$$W=\Delta K=\frac{1}{2}mv_f^2-\frac{1}{2}mv_i^2=0.450\text{ J}$$

Example

Angular momentum of the conical pendulum:

(1) The angular momentum about origin *A*.

(2) The angular momentum about origin *B*.



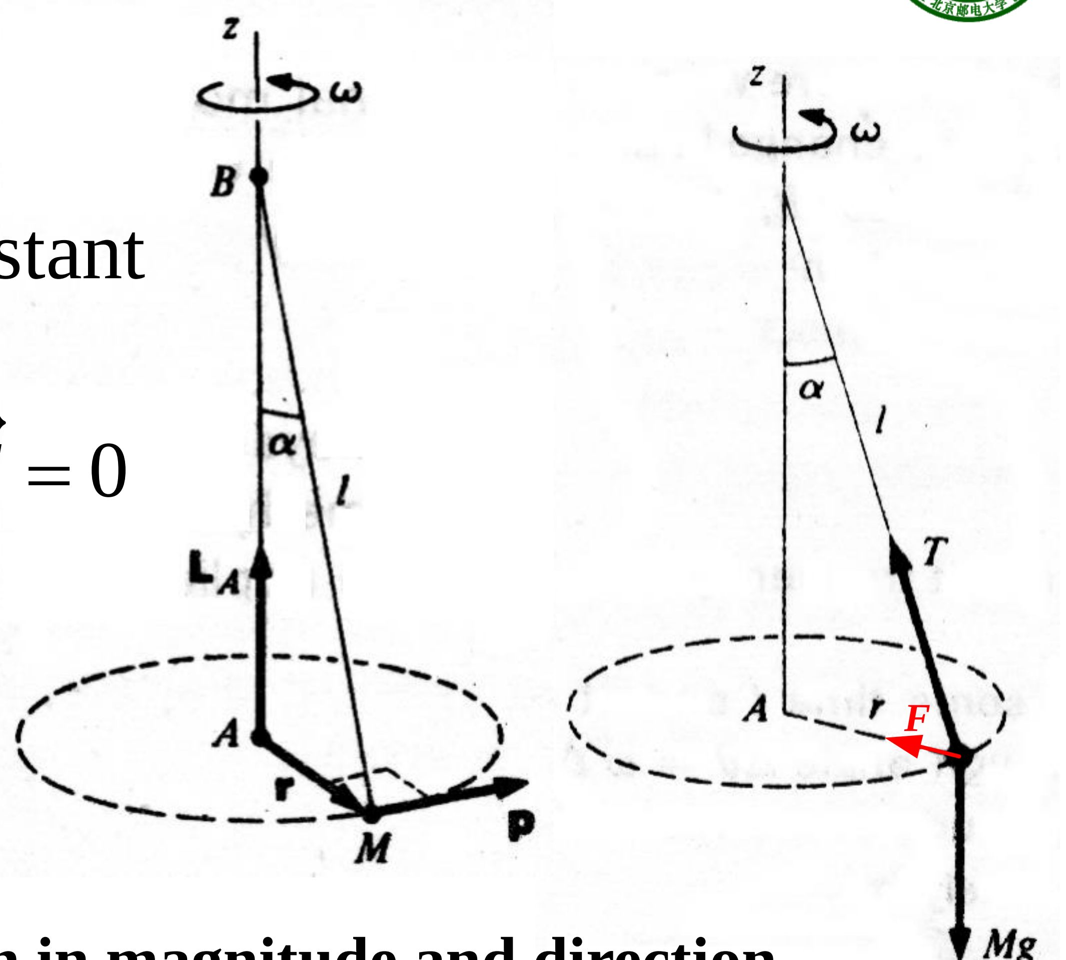
Solution

(1) For origin *A*

$$\vec{L}_A = \vec{r} \times \vec{p} = rMv\hat{k} = \text{constant}$$

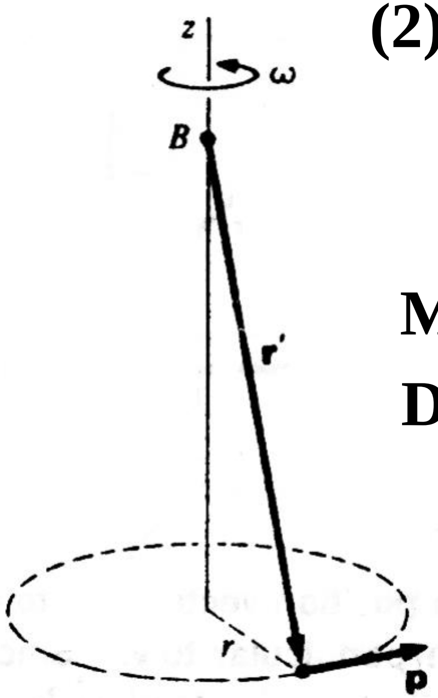
$$\vec{\tau}_A = \vec{r} \times (\vec{T} + M\vec{g}) = \vec{r} \times \vec{F} = 0$$

$$\vec{\tau}_A = \frac{d\vec{L}_A}{dt} = 0$$



$\vec{L}_A$ **remains constant, both in magnitude and direction.**

Solution



(2) For origin *B*

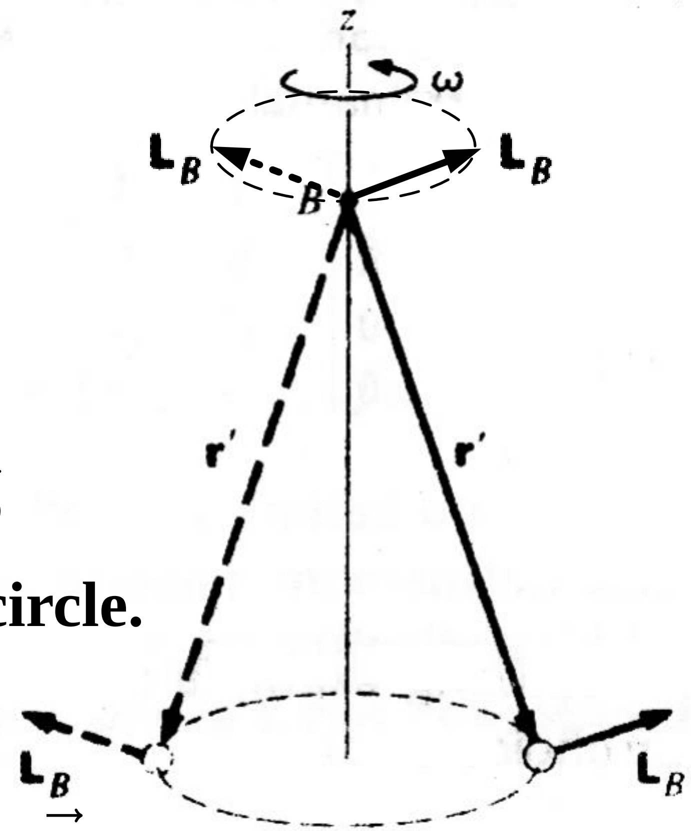
$$|\vec{L}_B| = |\vec{r}' \times \vec{p}| = r'Mv$$

Magnitude: constant.

Direction:

perpendicular to $\vec{r}'$ and $\vec{p}$

Its tip draws a horizontal circle.



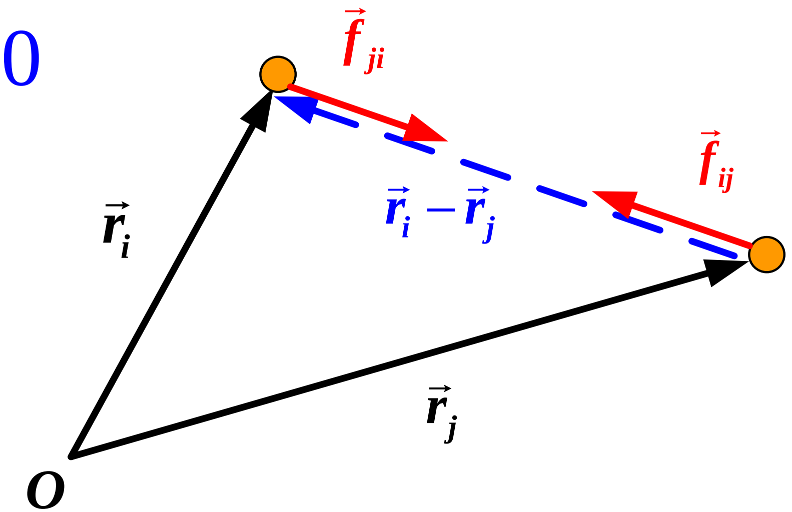
$$\vec{\tau}_B = \vec{r}' \times (\vec{T} + M\vec{g}) = \vec{r}' \times \vec{F} \neq 0, \qquad \vec{\tau}_B = \frac{d\vec{L}_B}{dt} \neq 0$$

$$\left(|\vec{L}_B|\right)_z = |\vec{L}_B| \sin\alpha = r'\sin\alpha Mv = rMv = |\vec{L}_A|, \qquad \left(|\vec{L}_B|\right)_z = |\vec{L}_A|$$

Torque-Angular Momentum Theorem for a system of particles

- The torques of each pair of internal forces are vanished.

$$\vec{r}_i \times \vec{f}_{ji} + \vec{r}_j \times \vec{f}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{f}_{ji} = 0$$



$$\sum_i \left(\vec{\tau}_{\text{in}} + \vec{\tau}_{\text{ext}}\right)_i = \sum_i \frac{d\vec{L}_i}{dt}$$

$$= \frac{d}{dt}\sum_i \vec{L}_i = \frac{d\vec{L}_{\text{tot}}}{dt}$$

$$\boxed{\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}_{\text{tot}}}{dt}}$$

The net external torque acting on the system is equal to the time rate of change of the total angular momentum of the system.

Valid in inertial frame and the reference frame of the center of mass.
Valid only if all the origins of $\vec{L}$ and $\vec{\tau}$ in the system are the same.

Conservation of Angular Momentum

(P284 § 11-7)

For a system of particles

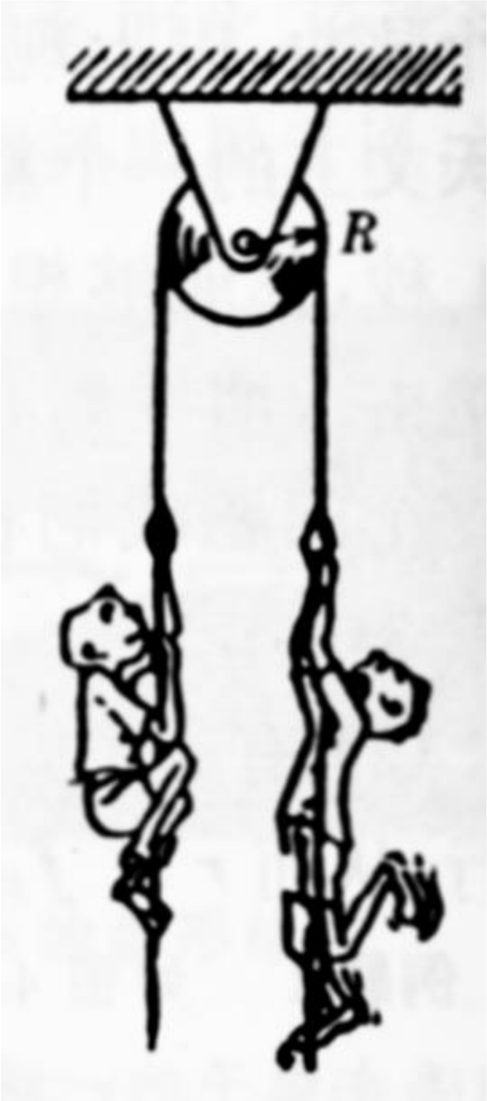
$$\sum \vec{\tau}_{\text{ext}} = 0 \;\Rightarrow\; \frac{d\vec{L}_{\text{tot}}}{dt} = 0 \quad \textbf{or} \quad \vec{L}_{\text{tot}} = \text{constant}$$

The total angular momentum of a system remains constant if the net external torque acting on the system is zero.

$$\sum \tau_{z\text{-ext}} = 0 \;\Rightarrow\; \frac{dL_{z\text{-tot}}}{dt} = 0 \quad \textbf{or} \quad L_{z\text{-tot}} = \text{constant}$$

Example

Two boys, with same mass of *m* , suspend to the two side of a pulley with a light rope. The boy on the left makes an effort to climb up, but the other boy keeps at rest without any action. Which boy is the first to approach pulley? Neglecting the mass of the pulley and the friction on the axis of the pulley.

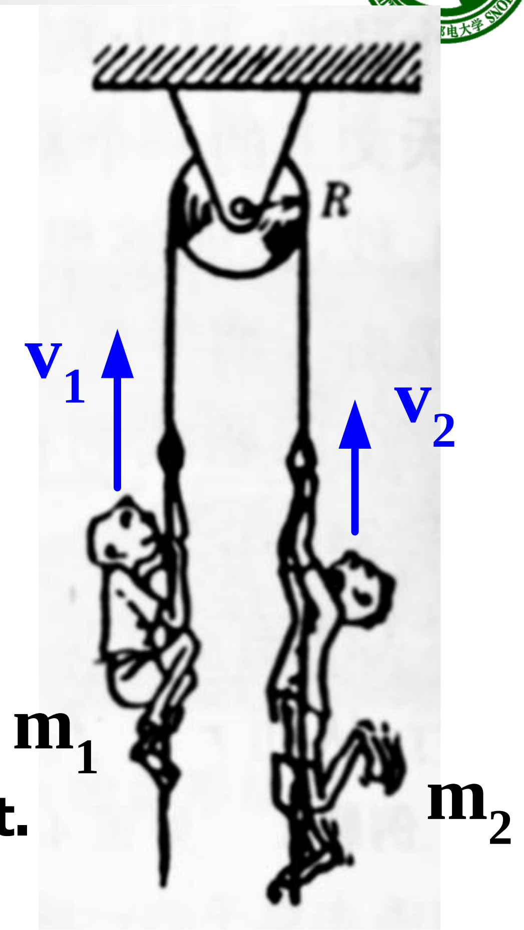


Solution

For the two-boy system, the **net external torque**.

$$\sum \tau_{\text{ext}} = Rm_1 g - Rm_2 g = Rmg - Rmg = 0$$

(Take the direction of torque consistent with anti-clockwise.)



The **angular momentum** of two-boy system is **conserved**.

$$L_f = mR(v_2 - v_1) = L_i = 0$$

$$v_2 - v_1 = 0$$

Two boy approach the pulley at same time, whoever makes an effort.

But if $m_1 > m_2$, $\sum \tau_{\text{ext}} > 0, \quad \frac{dL}{dt} > 0, \quad L_i = 0, \ L_f > 0$

$$v_1 < v_2$$

Problem

Ch11 (P292)

Prob. 7, 10, 18