Ch.7 *Representing Signals by Using Discrete-Time Complex Exponentials: The z-Transform*

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Outline

- z-Transform
  - Introduction
  - The z-Transform
  - Properties of the z-Transform
  - Inversion of the z-Transform
  - The Transfer Function
  - Causality and Stability

Introduction

- The z-Transform is a more general discrete-time signal and system representation based on complex exponential signals.
  - To study a much broader class of discrete-time LTI systems and signals, e.g. the impulse response for unstable LTI systems.
- Main usage
  - study of system characteristics and the derivation of computational structures for implementing discrete-time system on computers.
  - to solve difference equations subject to initial conditions.

From DTFT to z-Transform

$$F\left\{x[n]r^{-n}\right\} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\Omega}\right)^{-n}$$

$$\xrightarrow{z = re^{j\Omega}} \quad = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \triangleq X(z)$$

- The z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad \text{or} \quad X(z) = Z\{x[n]\}$$

- The inverse z-Transform

$$x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1} dz \quad \text{or} \quad x[n] = Z^{-1}\{X(z)\}$$

Integration around a circle of radius **$|z|=r$** in a counter-clockwise direction

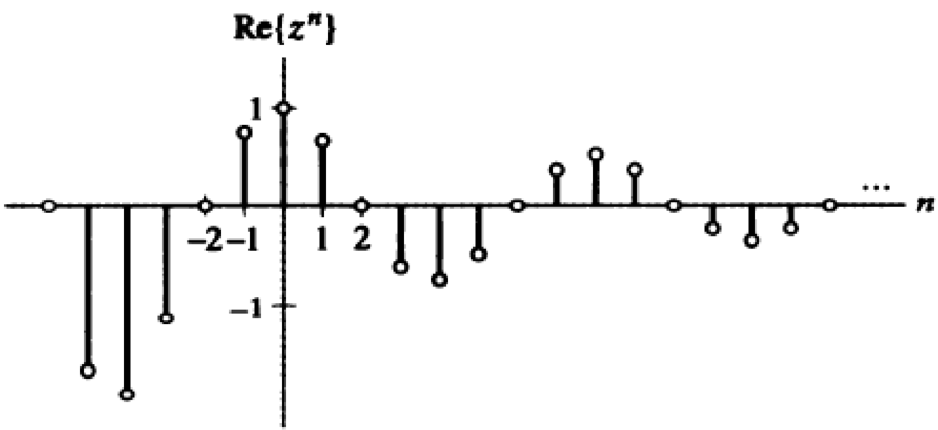
$$x[n] \overset{z}{\longleftrightarrow} X(z)$$

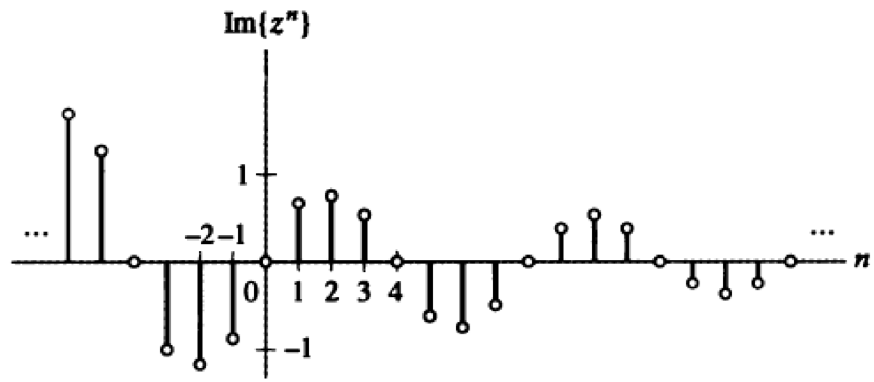
From DTFT to z-Transform

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

- The inverse z-Transform represents $\boldsymbol{x[n]}$ as a weighted superposition of complex exponentials $\boldsymbol{z^n}$.

$$z^n = \left(re^{j\Omega}\right)^n = r^n \cos\Omega n + jr^n \sin\Omega n$$





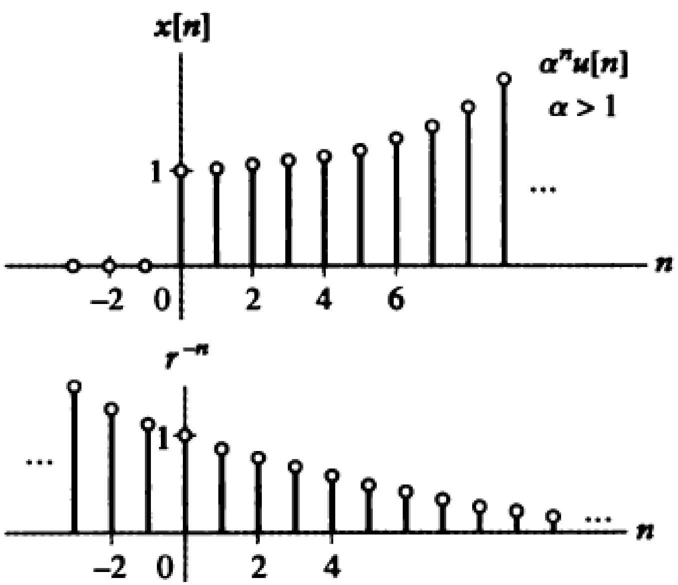
Convergence

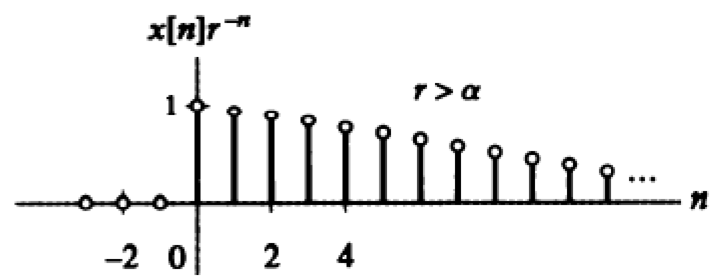
- necessary condition for convergence: absolutely summability of $x[n]\boldsymbol{z^{-n}}$.

$$\sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right| < \infty \quad \Longrightarrow \quad \sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty$$

- **Region of convergence(ROC):** the range of $\boldsymbol{r}$ which the z-Transform converges.

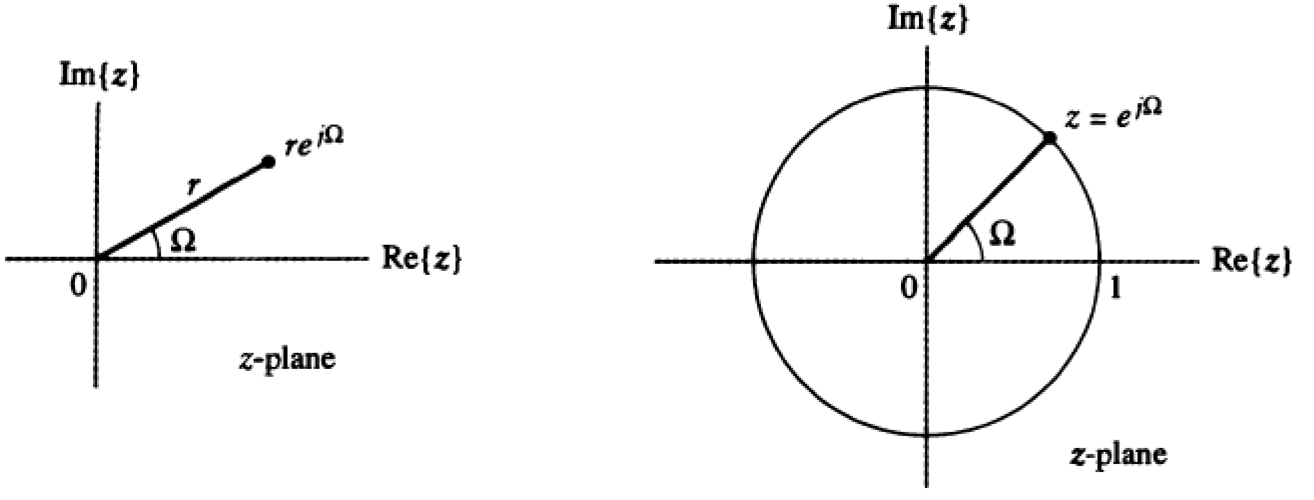
Ex. $x[n]=\alpha^{n} u[n] \quad \Longrightarrow \quad |z| = r > \alpha.$





Relations between the z-Transform and DTFT

- The z-Plane



- If $\boldsymbol{x[n]}$ is absolutely summable, the DTFT is obtained from the z-transform by setting $\boldsymbol{r = 1}$, i.e.

$$X\left(e^{j\Omega}\right) = X\left(z\right)\Big|_{z=e^{j\Omega}}$$

- If ROC does not include the unit circle, z-Transform exists while DTFT is nonexistent.

Poles and Zeros

$$X(z)=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+\cdots+a_Nz^{-N}}=\frac{\tilde{b}\prod_{k=1}^{M}\left(1-c_kz^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_kz^{-1}\right)},\quad \tilde{b}=\frac{b_0}{a_0}.$$

- Zeros of $X(z)$: the roots of the numerator polynomial c_k. "o"
- Poles of $X(z)$: the roots of the denominator polynomial d_k. " ×"

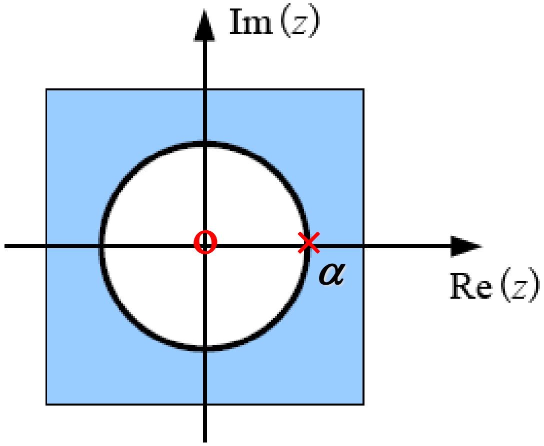
z-Transform of Signals

Example 7.2 Find the z-Transform of $x_1[n]=\alpha^n u[n]$ and $x_2[n]=-\alpha^n u[-n-1]$.

<Sol.> $X_1(z)=\sum_{n=-\infty}^{\infty}\alpha^n u[n]z^{-n} = \sum_{n=0}^{\infty}\alpha^n z^{-n}$

$$=\sum_{n=0}^{\infty}\left(\alpha z^{-1}\right)^n = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$$

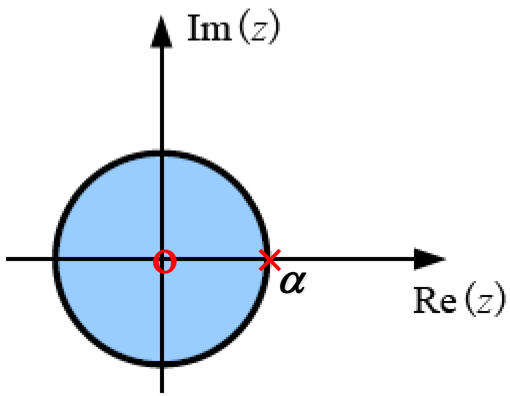
ROC: $|z|>|\alpha|$



$$X_2(z)=\sum_{n=-\infty}^{\infty}-\alpha^n u[-n-1]z^{-n}=\sum_{n=-\infty}^{-1}-\alpha^n z^{-n}$$

$$=-\sum_{n=1}^{\infty}\left(\alpha z^{-1}\right)^{-n}=1-\sum_{n=0}^{\infty}\left(\alpha^{-1}z\right)^n$$

$$=1-\frac{1}{1-\alpha^{-1}z} = \frac{1}{1-\alpha z^{-1}}$$



ROC: $|z|<|\alpha|$

z-Transform of Signals

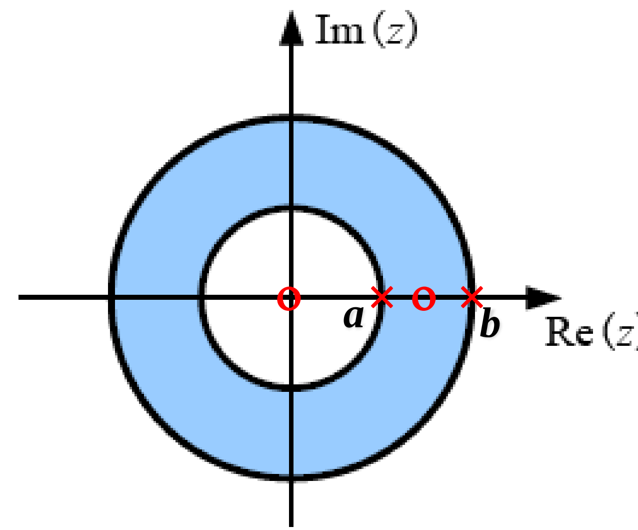
Example 7.4 Find the z-Transform of $x[n] = a^n u[n] - b^n u[-n-1]$.

<Sol.> $X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} -b^n u[-n-1] z^{-n}$

$$= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}}$$

$$= \frac{2\left[1 - \frac{a+b}{2} z^{-1}\right]}{\left(1-az^{-1}\right)\left(1-bz^{-1}\right)}$$

ROC: $|a| < |z| < |b|$



The z-Transform only exists when **$|b| > |a|$**.

z-Transform for Elementary Signals

$$\delta[n] \xleftrightarrow{z} 1, \quad \text{All } z$$

$$u[n] \xleftrightarrow{z} \frac{1}{1-z^{-1}}, \quad |z|>1; \qquad nu[n] \xleftrightarrow{z} \frac{z^{-1}}{\left(1-z^{-1}\right)^{2}}, \quad |z|>1$$

$$(n+1)u[n] \xleftrightarrow{z} \frac{1}{\left(1-z^{-1}\right)^{2}}, \quad |z|>1$$

$$a^{n}u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}}, \quad |z|>|a|; \qquad na^{n}u[n] \xleftrightarrow{z} \frac{az^{-1}}{\left(1-az^{-1}\right)^{2}}, \quad |z|>|a|$$

$$(n+1)a^{n}u[n] \xleftrightarrow{z} \frac{1}{\left(1-az^{-1}\right)^{2}}, \quad |z|>|a|$$

Properties of z-Transform

$x[n] \xleftrightarrow{z} X(z)$ with ROC R_x $\quad y[n] \xleftrightarrow{z} Y(z)$ with ROC R_y

- **Linearity**

$$ax[n]+by[n] \xleftrightarrow{z} aX(z)+bY(z) \quad \text{with ROC at least } R_x \cap R_y.$$

- **Time shift**

$$x[n-n_0] \xleftrightarrow{z} z^{-n_0}X(z) \text{ with ROC } R_x, \text{except possibly } z=0 \text{ or } |z|=\infty.$$

Ex. Find the z-Transform of $x[n]=u[n]-u[n-5]$.

$$X(z)=\frac{1}{1-z^{-1}}-\frac{z^{-5}}{1-z^{-1}}=\frac{1-z^{-5}}{1-z^{-1}},\quad |z|>1$$

Ex. If $X(z)=\dfrac{1}{z-a},\quad |z|>a,\quad$ determine $x[n]$.

$$X(z)=z^{-1}\frac{1}{1-az^{-1}} \quad \Longrightarrow \quad x[n]=a^{n-1}u[n-1]$$

Properties of z-Transform

- **Convolution**

$$x[n]*y[n] \overset{z}{\longleftrightarrow} X(z)Y(z) \quad \text{with ROC at least } R_x \cap R_y.$$

- **Multiplication by an exponential sequence**

$$a^n x[n] \overset{z}{\longleftrightarrow} X\left(\frac{z}{a}\right) \quad \text{with ROC } |a| R_x.$$

- **Differential in the z-Domain**

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}X(z) \quad \text{with ROC } R_x.$$

Ex. $u[n] \overset{z}{\longleftrightarrow} \dfrac{1}{1-z^{-1}}, \quad |z|>1$

$$\Rightarrow \quad nu[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right) = -z\frac{-z^{-2}}{\left(1-z^{-1}\right)^2} = \frac{z^{-1}}{\left(1-z^{-1}\right)^2}$$

Inversion of z-Transform

$$x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1} dz$$

- Direct inversion by residue computation
- Power series expansion
- Inversion by partial-fraction expansion (部分分式法)

$$X(z) = \frac{b_M z^{-M} + \cdots + b_1 z^{-1} + b_0}{a_N z^{-N} + \cdots a_1 z^{-1} + a_0} \qquad (M \geq N)$$

$$= c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{D(z)}{A(z)} \Rightarrow$$

$$x[n] = c_0 \delta[n] + c_1 \delta[n-1] + \cdots + c_{M-N} \delta[n-(M-N)] + Z^{-1}\left[\frac{D(z)}{A(z)}\right]$$

Inversion by Partial-fraction Expansions

$$\frac{D(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_P z^{-P}}{a_0 \prod_{k=1}^{N} \left(1 - d_k z^{-1}\right)} = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \qquad (P < N)$$

where $A_k = \left(1 - d_k z^{-1}\right) \left.\dfrac{D(z)}{A(z)}\right|_{z=d_k}$

$$A_k (d_k)^n u[n] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}} \quad \text{with ROC } |z| > d_k.$$

or $$-A_k (d_k)^n u[-n-1] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}} \quad \text{with ROC } |z| < d_k.$$

$$A_k (n+1) d_k^n u[n] \xleftrightarrow{z} \frac{A_k}{\left(1 - d_k z^{-1}\right)^2} \quad \text{with ROC } |z| > d_k.$$

or $$-A_k (n+1) d_k^n u[-n-1] \xleftrightarrow{z} \frac{A_k}{\left(1 - d_k z^{-1}\right)^2} \quad \text{with ROC } |z| < d_k.$$

Inversion by Partial-fraction Expansions

Ex. Find the inverse z-Transform of $X(z) = \dfrac{1}{\left(1-2z^{-1}\right)\left(1-3z^{-1}\right)}$.

<Sol.> $X(z) = \dfrac{A_1}{1-2z^{-1}} + \dfrac{A_2}{1-3z^{-1}} = \dfrac{-2}{1-2z^{-1}} + \dfrac{3}{1-3z^{-1}}$

$$A_1 = \left(1-2z^{-1}\right)X(z)\Big|_{z=2} = -2$$

$$A_2 = \left(1-3z^{-1}\right)X(z)\Big|_{z=3} = 3$$

- $|z| > 3$: $x[n] = \left(-2\cdot 2^n + 3\cdot 3^n\right)u[n] = \left(-2^{n+1} + 3^{n+1}\right)u[n]$
- $2 < |z| < 3$: $x[n] = -2^{n+1}u[n] - 3^{n+1}u[-n-1]$
- $|z| < 2$: $x[n] = 2^{n+1}u[-n-1] - 3^{n+1}u[-n-1]$

Inversion by Partial-fraction Expansions

Ex. **Find the inverse z-Transform of** $X(z)=\dfrac{1}{\left(1-2z^{-1}\right)^2\left(1-4z^{-1}\right)},\ |z|>4.$

<Sol.> $$X(z)=\frac{A_1}{1-2z^{-1}}+\frac{A_2}{\left(1-2z^{-1}\right)^2}+\frac{A_3}{1-4z^{-1}}$$

$$A_3=\left(1-4z^{-1}\right)X(z)\Big|_{z=4}=4$$

$$A_2=\frac{1}{0!}\left(1-2z^{-1}\right)^2X(z)\Big|_{z=2}=-1$$

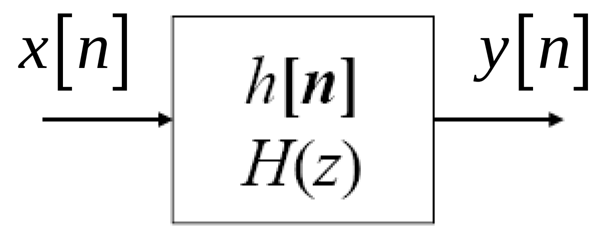
$$A_1=\frac{1}{1!}\cdot\frac{d}{dz}\left[\left(1-2z^{-1}\right)^2X(z)\Big|\right]_{z=2}=-2$$

$$\Rightarrow\quad x[n]=\left\{-2\cdot2^n-(n+1)2^n+4\cdot4^n\right\}u[n]$$

The Transfer Function (系统函数)

- Transfer function: for an LTI system with impulse response $h[n]$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$



$$y[n] = h[n] * x[n]$$

$$Y(z) = H(z)X(z) \Longrightarrow H(z) = \frac{Y(z)}{X(z)}$$

- Furthermore, for an input $x[n] = z^n$ to the LTI system

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} = H(z) z^n$$

  - Eigenfunction of the system: z^n
  - Eigenvalue: $H(z)$

Transfer Function and Difference Equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

If Initial conditions equal zero, and $x[n] = z^n$

$$z^n \sum_{k=0}^{N} a_k z^{-k} H(z) = z^n \sum_{k=0}^{M} b_k z^{-k}$$

$$\Rightarrow \quad H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Transfer function $\Leftrightarrow$ Difference equation system description

Transfer Function and Differential Equation

Ex. **Find the transfer function of the LTI system described by the difference equation**

$$y[n]-0.7y[n-1]+0.1y[n-2]=x[n-1],\quad n\geq 0$$

<Sol.>

$$\left(1-0.7z^{-1}+0.1z^{-2}\right)Y(z)=z^{-1}X(z) \Rightarrow H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}}{1-0.7z^{-1}+0.1z^{-2}}$$

Ex. **Find the impulse response of the LTI system described by the following difference equation**

$$y[n]+5y[n-1]+6y[n-2]=x[n-1],\quad n\geq 0$$

<Sol.> $\left(1+5z^{-1}+6z^{-2}\right)Y(z)=z^{-1}X(z)$

$$\Rightarrow H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}}{1+5z^{-1}+6z^{-2}}=\frac{1}{1+2z^{-1}}-\frac{1}{1+3z^{-1}}$$

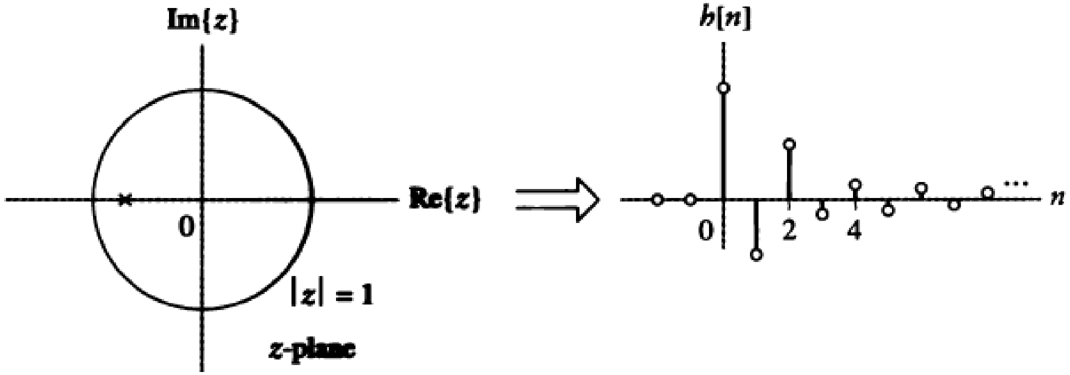
$$\Rightarrow h[n]=\left\{(-2)^n-(-3)^n\right\}u[n]$$

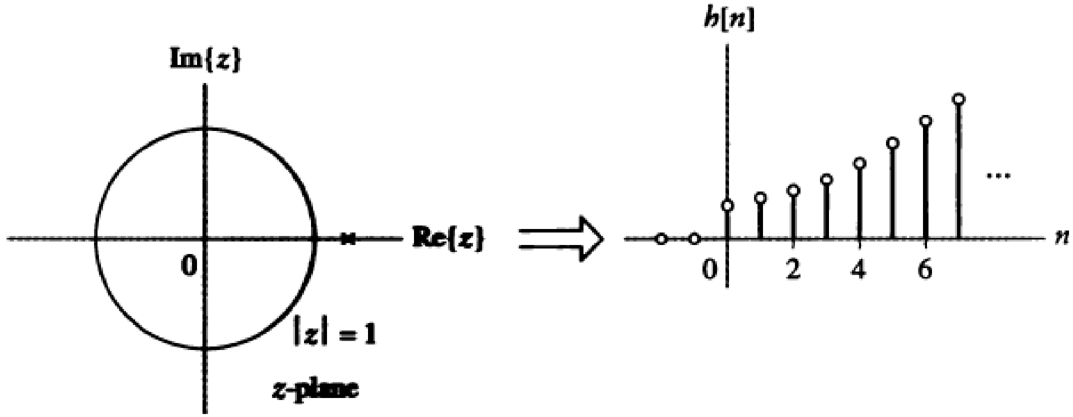
Causality and Stability (因果性与稳定性)

- For a causal system: $h[n]=0$ for $n<0$.
  - A pole inside the unit circle in the z-plane ($|\alpha|<1$) corresponds to an exponentially decaying impulse response.
  - A pole inside the unit circle in the z-plane ($|\alpha|>1$) corresponds to an exponentially increasing impulse response --> unstable.

$$\alpha^n u[n] \xleftrightarrow{Z} \frac{1}{1-\alpha z^{-1}}$$

ROC: $|z|>|\alpha|$



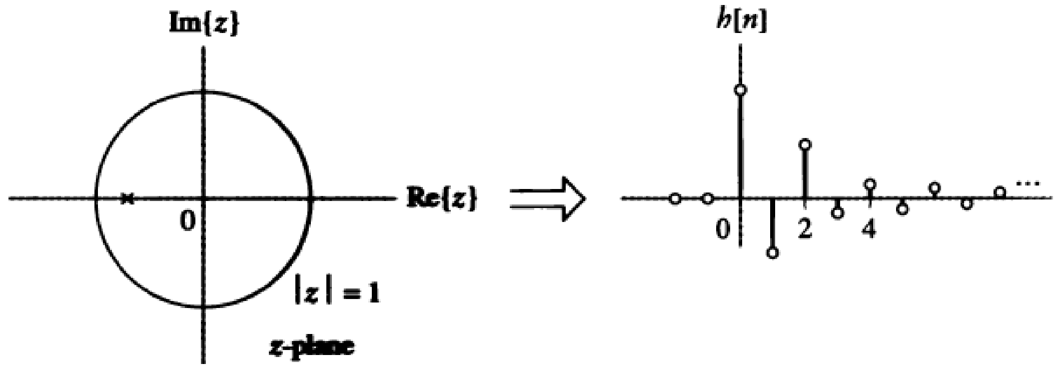


Causality and Stability (因果性与稳定性)

- For a stable system: $\sum_{k=-\infty}^{\infty} |h[k]| < \infty.$
  - A pole inside the unit circle in the z-plane *corresponds to a right-sided impulse response.*
  - A pole outside the unit circle in the z-plane corresponds to a left-sided impulse response → noncausal.

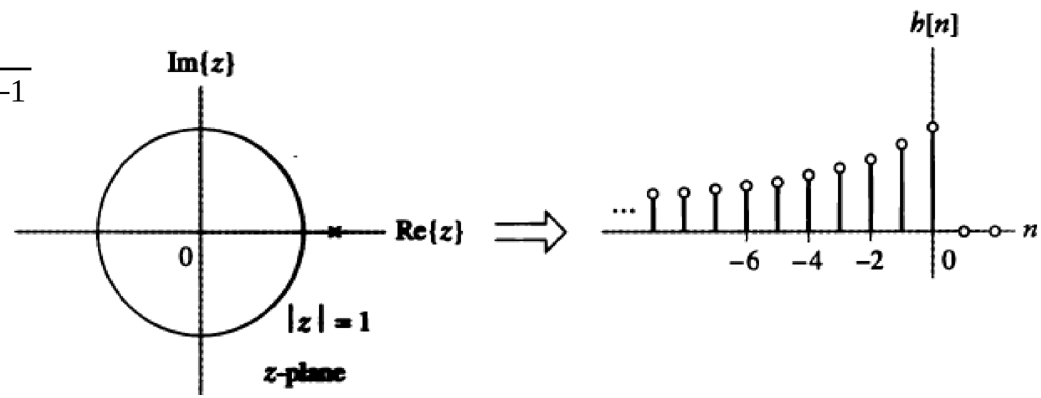
$$\alpha^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}$$

ROC: $|z| > |\alpha|$



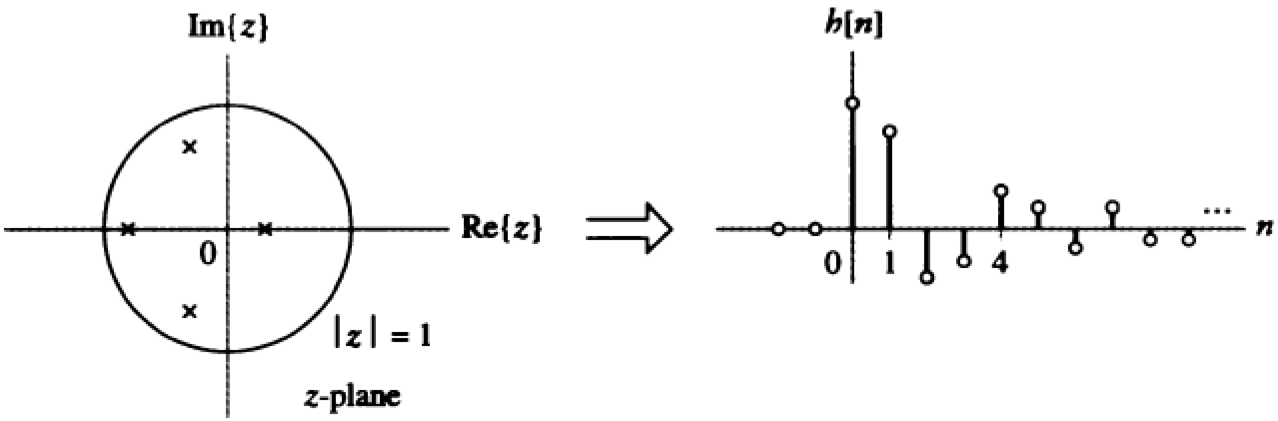
$$-\alpha^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}$$

ROC: $|z| < |\alpha|$



Causality and Stability (因果性与稳定性)

- A system that is both stable and causal must have a transfer function with all of its poles inside the unit circle.



Causality and Stability (因果性与稳定性)

Ex. A causal system has the transfer function $H(z)=\dfrac{2+z^{-1}}{1-0.7z^{-1}+0.1z^{-2}}$

Find the impulse response. Is the system stable?

<Sol.> $H(z)=\dfrac{2+z^{-1}}{1-0.7z^{-1}+0.1z^{-2}}=\dfrac{-14/3}{1-0.2z^{-1}}+\dfrac{20/3}{1-0.5z^{-1}}$

$$\Rightarrow\quad h[n]=\left\{-\frac{14}{3}(0.2)^n+\frac{20}{3}(0.5)^n\right\}u[n]$$

The system has two poles: $z=0.2,\quad z=0.5.$

The system is stable.

Summary

- z-Transform
  - Introduction
  - The z-Transform
  - Properties of the z-Transform
  - Inversion of the z-Transform
  - The Transfer Function
  - Causality and Stability

- Reference in textbook:7.1~7.7
- Homework: 7.17(e,g), 7.22, 7.31(a,b)