1.2 *Basic Time Signals*

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Outline

- Elementary Signals
  - Exponential Signals (指数信号)
  - Sinusoid Signals (正弦信号)
  - The Unit-Step Function (单位阶跃函数)
  - The Unit-Impulse Function (单位冲激函数)

Exponential Signals（指数信号）

$$x(t) = Be^{at}$$

B and a are real parameters

- $a < 0$: **decaying exponential**
- $a > 0$: **growing exponential**

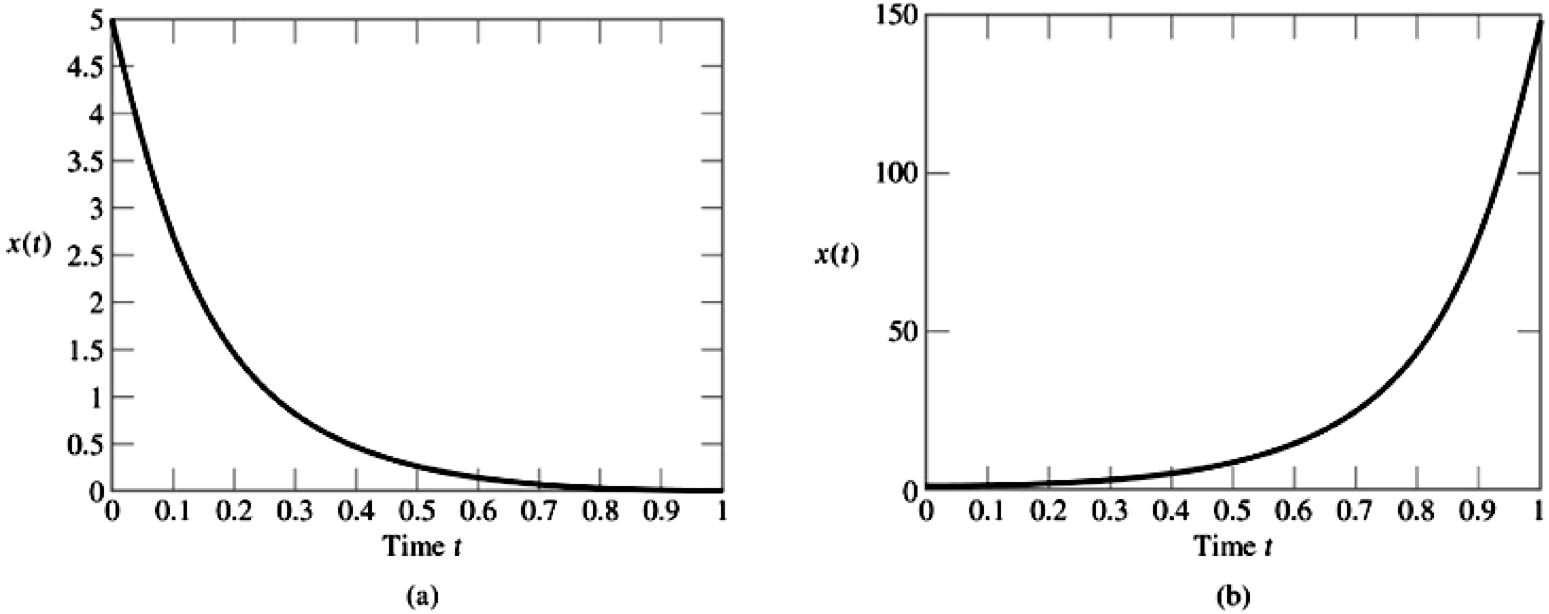


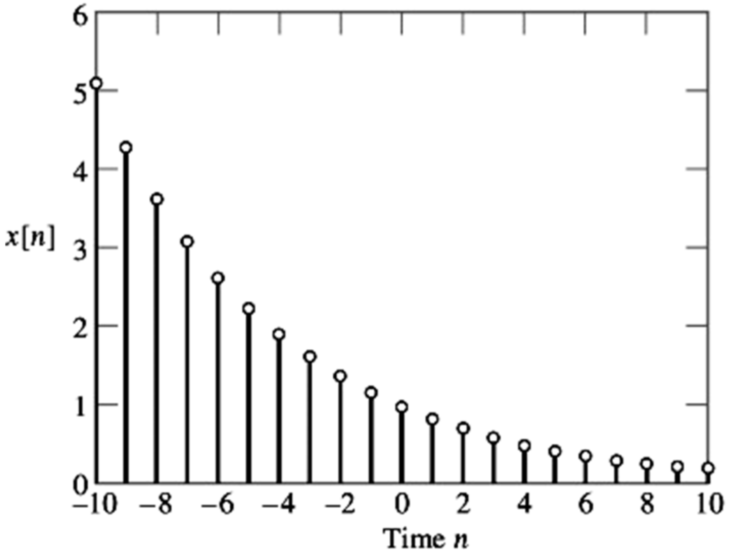
Figure 1.28 (a) Decaying exponential form of continuous-time signal. (b) Growing exponential form of continuous-time signal.

Exponential Signals

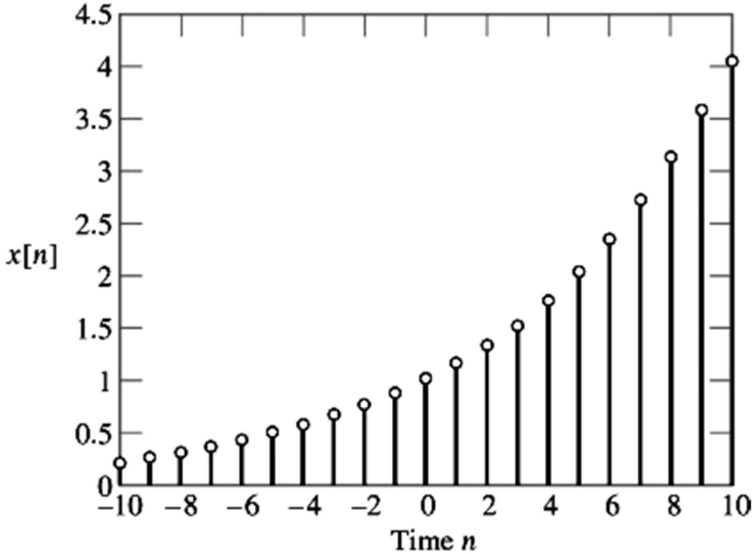
- **Discrete-time Case**

$$x[n] = Br^n \qquad \text{where} \quad r = e^a$$

  - $0 < r < 1$: decaying exponential
  - $r > 1$: growing exponential
  - $r < 0$: alternating signs



(a)



(b)

Sinusoidal Signals (正弦信号)

- **Continuous-time Case**

$$x(t) = A\cos(\omega t + \phi)$$

  - A: amplitude
  - ω: angular frequency in rad/s
  - ϕ: phase angle in radians
  - period: $T = \dfrac{2\pi}{\omega}$

$$\begin{aligned} x(t+T) &= A\cos(\omega(t+T)+\phi) \\ &= A\cos(\omega t + \omega T + \phi) \\ &= A\cos(\omega t + 2\pi + \phi) \\ &= A\cos(\omega t + \phi) \\ &= x(t) \end{aligned}$$

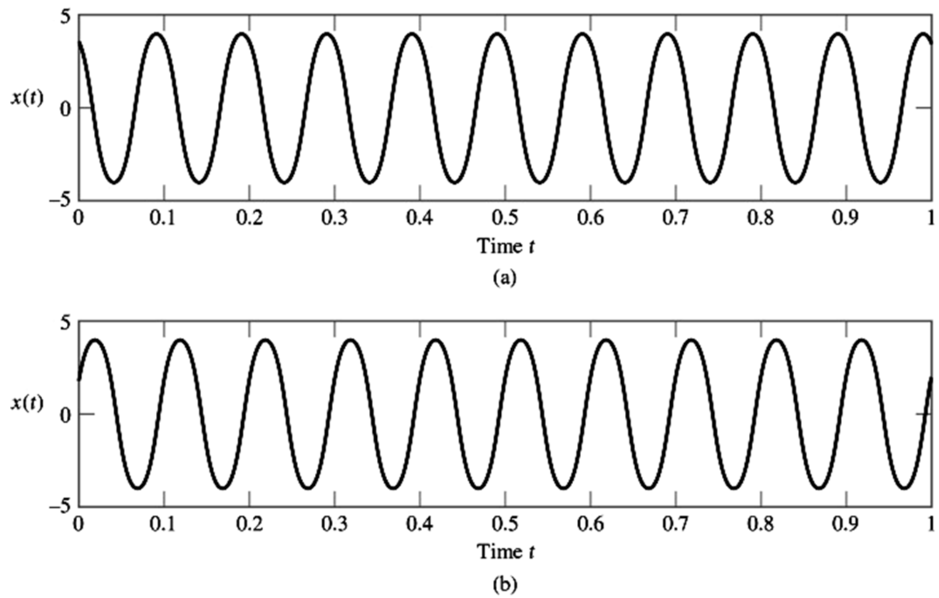


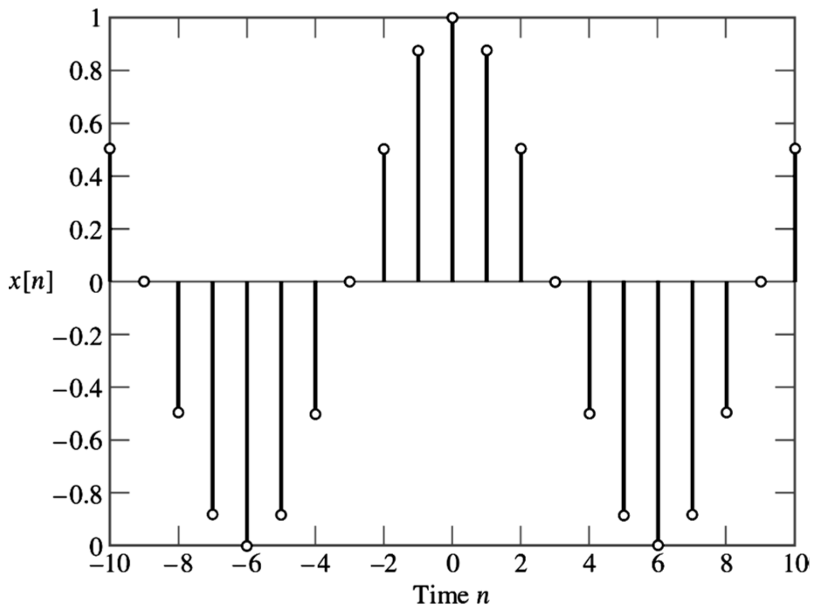
Figure 1.31
(a) Sinusoidal signal $A\cos(\omega t + \Phi)$ with phase $\Phi = +\pi/6$ radians.
(b) Sinusoidal signal $A\sin(\omega t + \Phi)$ with phase $\Phi = +\pi/6$ radians.

Sinusoidal Signals

- **Discrete-Time Case** $x[n] = A\cos(\Omega n + \phi)$
  - Periodic condition: $x[n+N] = A\cos(\Omega n + \Omega N + \phi)$

$$\Omega N = 2\pi m \quad \Rightarrow \quad \Omega = \frac{2\pi m}{N} \text{ radians/cycle, integer } m,\ N$$

Ex. A discrete-time sinusoidal signal: $A = 1$, $\phi = 0$, and $N = 12$.



$$x[n] = \cos(\Omega n)$$

$$= \cos\left(\frac{2\pi}{12} n\right) = \cos\left(\frac{n\pi}{6}\right)$$

Figure 1.33 Discrete-time sinusoidal signal.

Sinusoidal Signals

Example 1.7 Discrete-Time Sinusoidal Signal

A pair of sinusoidal signals with a common angular frequency is defined by

$$x_1[n] = \sin[5\pi n] \quad \textbf{and} \quad x_2[n] = \sqrt{3}\cos[5\pi n]$$

(a) Both $x_1[n]$ and $x_2[n]$ are periodic. Find their common fundamental period.

(b) Express the composite sinusoidal signal

$$y[n] = x_1[n] + x_2[n]$$

In the form $y[n] = A\cos(\Omega n + \phi)$, and evaluate the amplitude A and phase ϕ.

<Sol.>

(a) Angular frequency of both $x_1[n]$ and $x_2[n]$:

$$\Omega = 5\pi \quad \text{radians/cycle} \Rightarrow N = \frac{2\pi m}{\Omega} = \frac{2\pi m}{5\pi} = \frac{2m}{5}$$

This can be only for $m = 5, 10, 15, \ldots$, which results in $N = 2, 4, 6, \ldots$

Sinusoidal Signals

(b) $y[n] = x_1[n] + x_2[n] = \sin[5\pi n] + \sqrt{3}\cos[5\pi n]$

Trigonometric identity:

$$A\cos(\Omega n + \phi) = A\cos(\Omega n)\cos(\phi) - A\sin(\Omega n)\sin(\phi)$$

$$A\sin(\phi) = -1 \quad \text{and} \quad A\cos(\phi) = \sqrt{3}$$

$$\Rightarrow \quad \tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{-1}{\sqrt{3}} \quad \Rightarrow \quad \phi = -\pi / 6$$

$$A\sin(\phi) = -1 \quad \Rightarrow \quad A = \frac{-1}{\sin\left(-\pi/6\right)} = 2$$

Accordingly, $$y[n] = 2\cos\left(5\pi n - \frac{\pi}{6}\right)$$

Sinusoidal Signals

Prob 1.17 **Determine whether each $x[n]$ is periodic, and if it is, find its fundemental period.**

$$a)\ x[n]=5\sin[2n] \Rightarrow N=\frac{2\pi m}{\Omega}=\frac{2\pi m}{2}=m\pi$$ **nonperiodic**

$$b)\ x[n]=5\cos[0.2\pi n] \Rightarrow N=\frac{2\pi m}{\Omega}=\frac{2\pi m}{0.2\pi}=10m$$ **periodic**

Fundamental period=10 for m=1

$$c)\ x[n]=5\cos[6\pi n] \Rightarrow N=\frac{2\pi m}{\Omega}=\frac{2\pi m}{6\pi}=\frac{m}{3}$$ **periodic**

Fundamental period=1 for m=3

$$d)\ x[n]=5\cos[6\pi n/35] \Rightarrow N=\frac{2\pi m}{\Omega}=\frac{35m}{3}$$ **periodic**

Fundamental period=35 for m=3

Sinusoidal Signals

Prob 1.18 **Find the smallest angular frequencies for which discrete-time sinusoidal signals with the following periods would be periodic:**

$$a)\ N = 8 \Rightarrow \Omega = \frac{2\pi m}{N} = \frac{2\pi m}{8} = \frac{\pi}{4} m = \frac{\pi}{4} \ \text{ when } m = 1.$$

$$b)\ N = 32 \Rightarrow \Omega = \frac{2\pi m}{32} = \frac{\pi}{16} m = \frac{\pi}{16} \ \text{ when } m = 1.$$

$$c)\ N = 64 \Rightarrow \Omega = \frac{2\pi m}{64} = \frac{\pi}{32} m = \frac{\pi}{32} \ \text{ when } m = 1.$$

$$d)\ N = 128 \Rightarrow \Omega = \frac{2\pi m}{128} = \frac{\pi}{64} m = \frac{\pi}{64} \ \text{ when } m = 1.$$

Relation between sinusoidal and complex exponential signals

- **Euler's identity**

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\Rightarrow \quad x(t) = A\cos(\omega t + \phi) \quad = \mathrm{Re}\{Be^{j\omega t}\} \quad \text{where } B = Ae^{j\phi}.$$

- Complex exponential signal: $Be^{j\omega t}$

$$\begin{aligned} Be^{j\omega t} &= Ae^{j\phi}e^{j\omega t} \\ &= Ae^{j(\phi+\omega t)} \\ &= A\cos(\omega t+\phi) + jA\sin(\omega t+\phi) \end{aligned}$$

- Continuous-time signal in terms of sine function:

$$\begin{aligned} x(t) = A\sin(\omega t+\phi) &= \mathrm{Im}\{Be^{j\omega t}\} \\ &= A\cos(\omega t+\phi-\pi/2) \end{aligned}$$

Relation between sinusoidal and complex exponential signals

- **Discrete-Time Case**

$$A\cos(\Omega n+\phi)=\mathrm{Re}\{Be^{j\Omega n}\}$$

$$A\sin(\Omega n+\phi)=\mathrm{Im}\{Be^{j\Omega n}\}$$

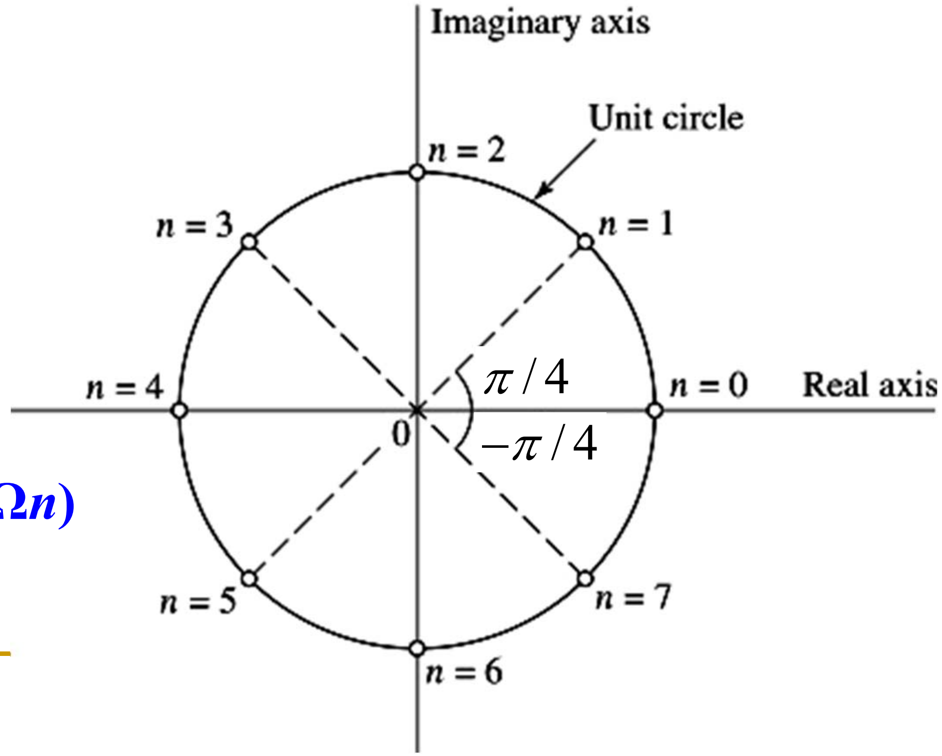
where $B = Ae^{j\phi}$.

  - Ex. Two-dimensional representation of the complex exponential $e^{j\Omega n}$ for $\Omega = \pi/4$ and $n = 0, 1, 2, \ldots, 7$.

$$e^{j\Omega n} = \cos\Omega n + j\sin\Omega n$$
$$= \cos\frac{\pi n}{4} + j\sin\frac{\pi n}{4}$$

Projection on real axis: **$\cos(\Omega n)$**

Projection on imaginary axis: **$\sin(\Omega n)$**



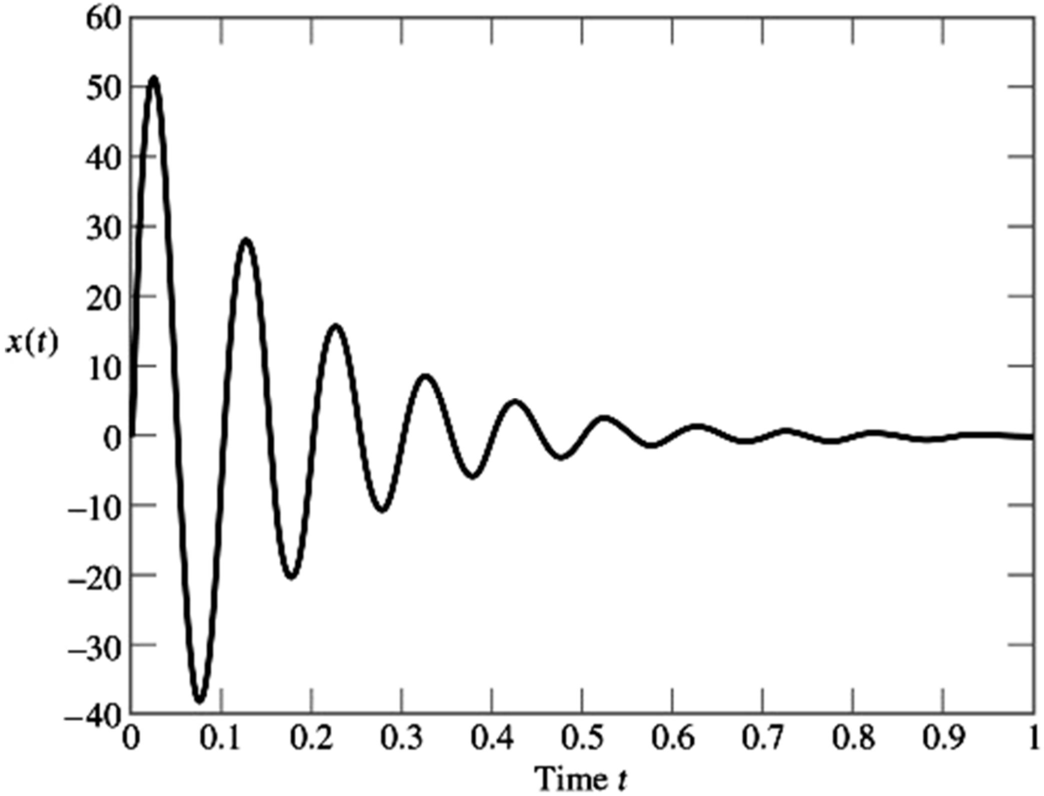
Exponential Damped Sinusoidal Signals (指数阻尼正弦信号)

$$x(t) = Ae^{-\alpha t}\sin(\omega t + \phi), \quad \alpha > 0$$

$A = 60$,

$\alpha = 6$, and $\phi = 0$

Figure 1.35 Exponentially damped sinusoidal signal $Ae^{-at}\sin(\omega t)$, with $A = 60$ and $a = 6$.



Exponential Damped Sinusoidal Signals

Prob 1.20 evaluate the real and imaginary components of complex-valued exponential signal $x(t)$ for the following cases:

(a) $\alpha = \alpha_1$, real; (b) $\alpha = j\omega_1$, imaginary; (c) $\alpha = \alpha_1 + j\omega_1$, complex.

$$x(t) = Ae^{\alpha t + j\omega t}$$

$$a)\; x(t) = Ae^{\alpha_1 t + j\omega t} = Ae^{\alpha_1 t}\left(\cos\omega t + j\sin\omega t\right)$$

$$\Rightarrow \operatorname{Re}\{x(t)\} = Ae^{\alpha_1 t}\cos\omega t,\quad \operatorname{Im}\{x(t)\} = Ae^{\alpha_1 t}\sin\omega t$$

$$b)\; x(t) = Ae^{j\omega_1 t + j\omega t} = Ae^{j(\omega_1+\omega)t}$$

$$\Rightarrow \operatorname{Re}\{x(t)\} = A\cos(\omega_1+\omega)t,\quad \operatorname{Im}\{x(t)\} = A\sin(\omega_1+\omega)t$$

$$c)\; x(t) = Ae^{\alpha_1 t + j\omega_1 t + j\omega t} = Ae^{\alpha_1 t}e^{j(\omega_1+\omega)t}$$

$$\Rightarrow \operatorname{Re}\{x(t)\} = Ae^{\alpha_1 t}\cos(\omega_1+\omega)t,\quad \operatorname{Im}\{x(t)\} = Ae^{\alpha_1 t}\sin(\omega_1+\omega)t$$

Unit-Step Function (单位阶跃函数)

- **Discrete-Time Case**

$$u[n]=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$$

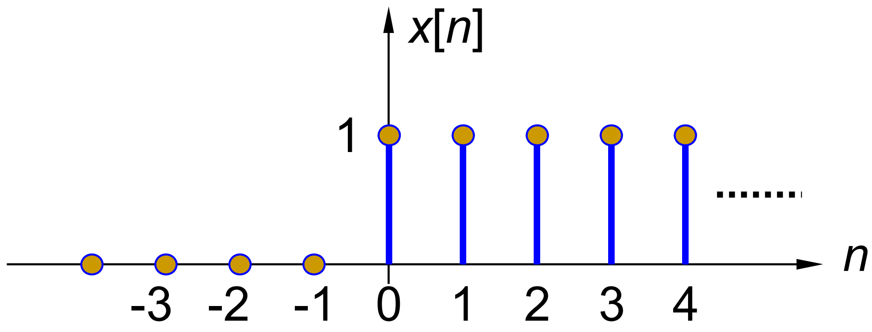


Figure 1.37 Discrete-time version of step function of unit amplitude.

- **Continuous-Time Case**

$$u(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$$

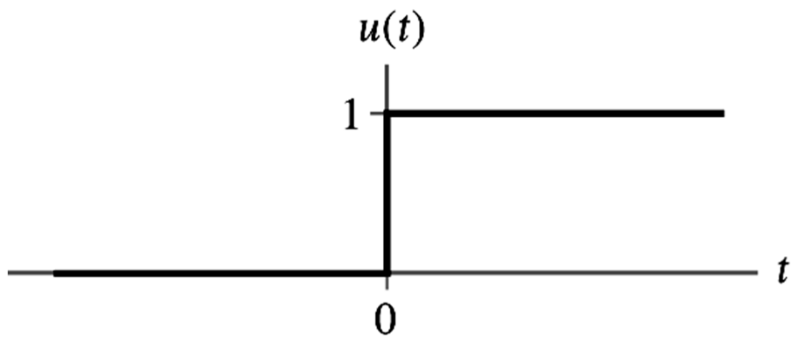


Figure 1.38 Continuous-time version of the unit-step function of unit amplitude.

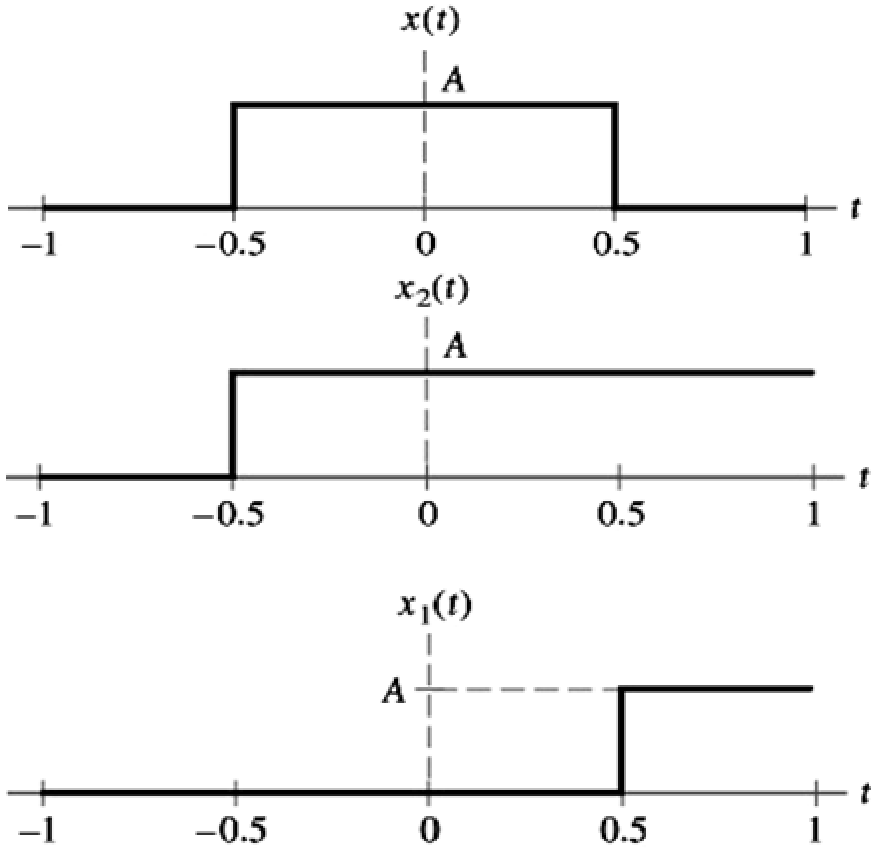
Unit-Step Function

Example 1.8 **Consider the rectangular pulse *x*(*t*) shown in Fig. 1.39 (a). This pulse has an amplitude *A* and duration of 1 second. Express *x*(*t*) as a weighted sum of two step functions.**

<Sol.>

$$x(t) = \begin{cases} A, & 0 \le |t| < 0.5 \\ 0, & |t| > 0.5 \end{cases}$$

$$= Au\left(t + \frac{1}{2}\right) - Au\left(t - \frac{1}{2}\right)$$



Unit-Impulse Function (单位冲激函数)

- **Discrete-time version of unit impulse**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

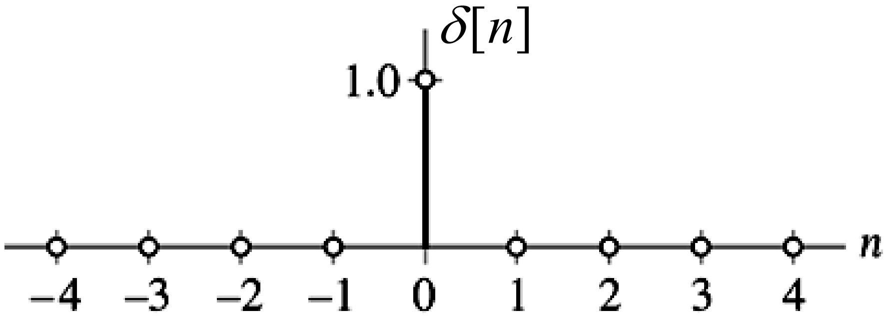


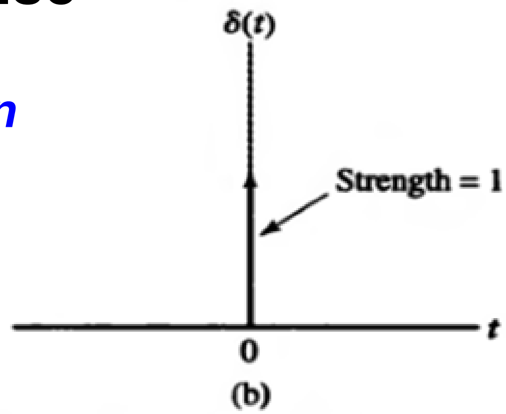
Figure 1.41 Discrete-time form of impulse.

- **Continuous-time version of unit impulse**

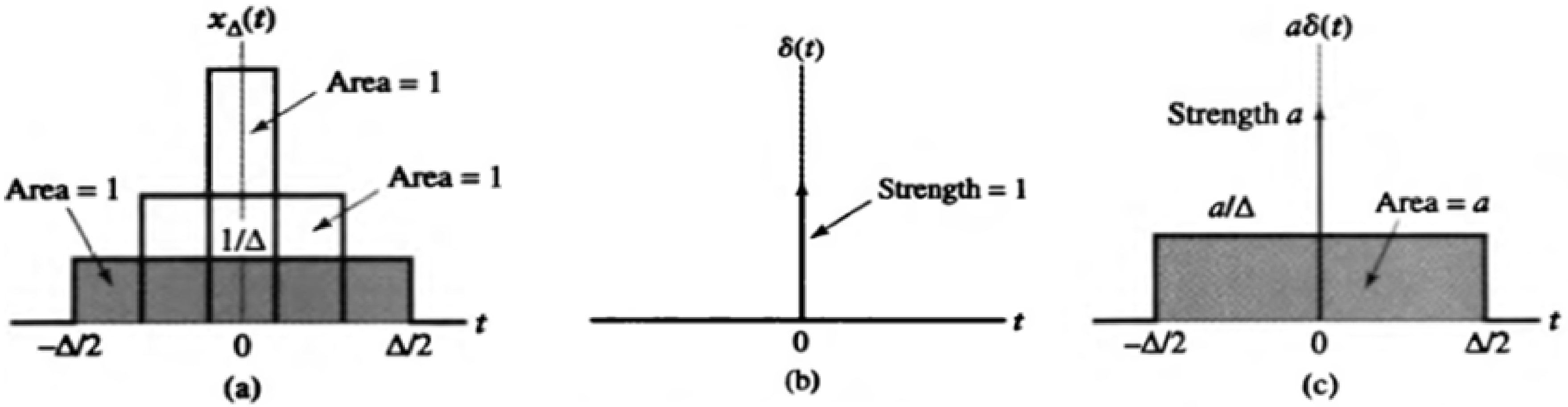
$$\delta(t) = 0 \quad \text{for} \quad t \neq 0$$

~ ***Dirac delta function***

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$



Unit-Impulse Function



- $x_\Delta(t)$: even function of t with duration Δ and unit area

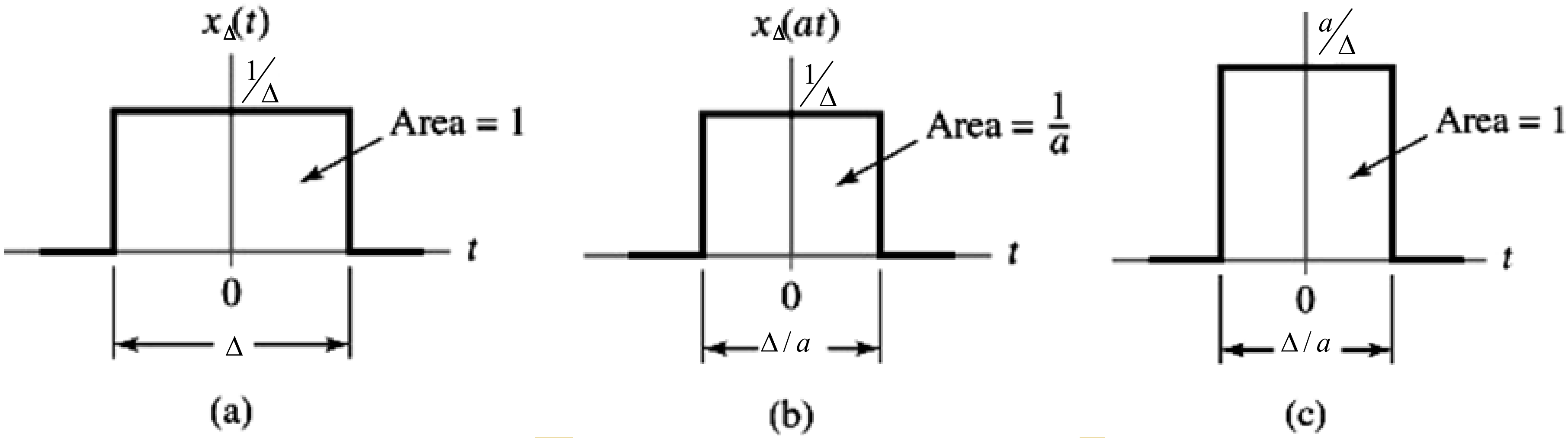
$$\delta(t) = \lim_{\Delta \to 0} x_\Delta(t) = \lim_{\Delta \to 0} \frac{1}{\Delta}\left[u\left(t+\frac{\Delta}{2}\right) - u\left(t-\frac{\Delta}{2}\right)\right]$$

- strength of the impulse: area under the pulse
- relations between impulse and unit step function

$$\delta(t) = \frac{d}{dt}u(t) \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

Unit-Impulse Function

- **Properties of impulse function**
  - Even function: $\delta(-t) = \delta(t)$
  - Sampling property: $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
  - Shifting property: $\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$
  - Time-scaling property: $\delta(at) = \frac{1}{a}\delta(t), \quad a > 0$



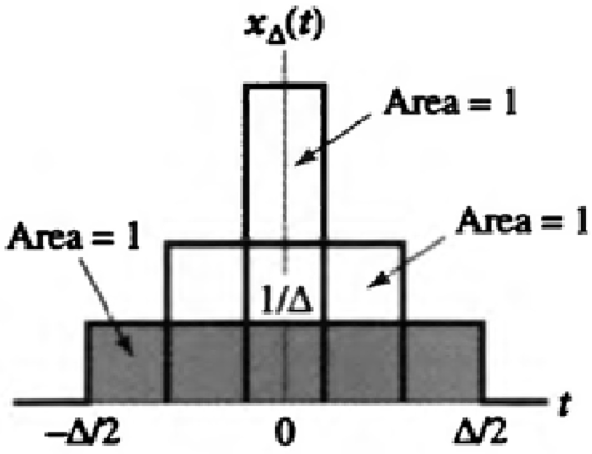
$$\delta(at) = \lim_{\Delta \to 0} x_\Delta(at) = \frac{1}{a}\delta(t) \quad \text{while} \quad \lim_{\Delta \to 0} a x_\Delta(at) = \delta(t).$$

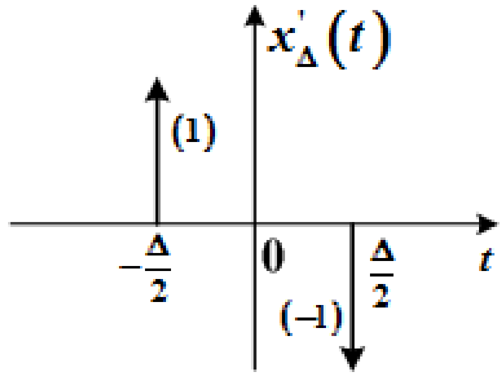
Unit-Impulse Function

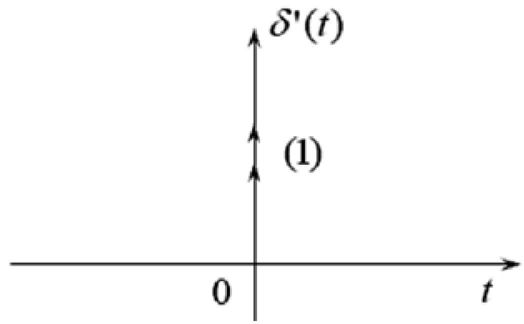
- **derivatives of the impulse**

$$\delta'(t)=\frac{d\delta(t)}{dt}=\lim_{\Delta\to 0}\frac{d}{dt}x_\Delta(t)=\lim_{\Delta\to 0}\frac{1}{\Delta}\cdot\frac{d}{dt}\left[u\left(t+\frac{\Delta}{2}\right)-u\left(t-\frac{\Delta}{2}\right)\right]$$

$$=\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\delta(t+\Delta/2)-\delta(t-\Delta/2)\right]$$







  - doublet: the first derivative of $\delta(t)$ is the limiting form of the first derivative of the same rectangular pulse

$$\delta^{(1)}(t)=\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\delta(t+\Delta/2)-\delta(t-\Delta/2)\right]$$

Unit-Impulse Function

- Fundamental property of the doublet

$$\int_{-\infty}^{\infty} \delta^{(1)}(t)dt = 0$$

$$\int_{-\infty}^{\infty} f(t)\delta^{(1)}(t-t_0)dt = \lim_{\Delta\to 0}\frac{1}{\Delta}\int_{-\infty}^{\infty} f(t)\left[\delta(t-t_0+\Delta/2)-\delta(t-t_0-\Delta/2)\right]dt$$

$$= \lim_{\Delta\to 0}\frac{1}{\Delta}\left[f(t_0-\Delta/2)-f(t_0+\Delta/2)\right] = -\frac{d}{dt}f(t)\Big|_{t=t_0}$$

- Second derivative of impulse

$$\delta^{(2)}(t) = \frac{d^2}{dt^2}\delta(t) = \frac{d}{dt}\delta^{(1)}(t) = \lim_{\Delta\to 0}\frac{\delta^{(1)}(t+\Delta/2)-\delta^{(1)}(t-\Delta/2)}{\Delta}$$

$$\int_{-\infty}^{\infty} f(t)\delta^{(2)}(t-t_0)dt = \frac{d^2}{dt^2}f(t)|_{t=t_0}$$

$$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t-t_0)dt = \left(-1\right)^n\frac{d^n}{dt^n}f(t)|_{t=t_0}$$

Ramp Function (斜坡函数)

- **Continuous-time Case**

$$r(t)=\begin{cases} t, & t\geq 0 \\ 0, & t<0 \end{cases}$$

$$= tu(t)$$

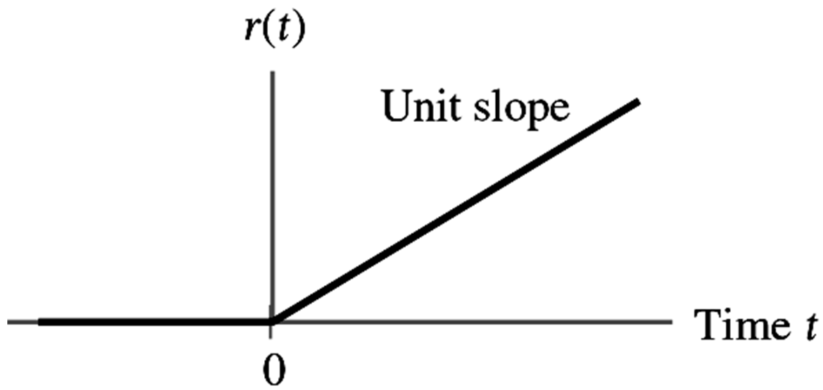


Figure 1.46 Ramp function of unit slope.

- **Discrete-time Case**

$$r[n]=\begin{cases} n, & n\geq 0 \\ 0, & n<0 \end{cases}$$

$$= nu[n]$$

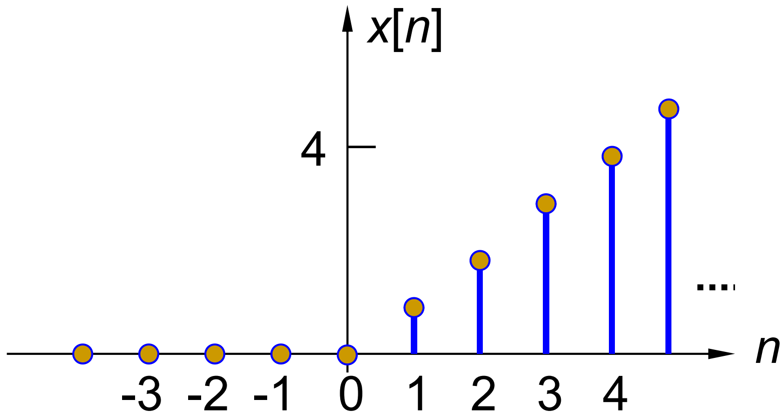


Figure 1.47 Discrete-time version of the ramp function.

Summary

- Elementary Signals
  - Exponential Signals
  - Sinusoid Signals
  - The Unit-Step Function
  - The Unit-Impulse Function

- Reference in textbook: 1.6
- Homework: 1.58, 1.60