SOLUTIONS

Module:	Engineering Mathematics		
Module Code	BBC4111	Paper	A
Time allowed	2hrs	Filename	**Solutions_2122_ BC4111_A**
Rubric	ANSWER ALL QUESTIONS		
Examiners	Dr Xia Shi	Dr Huixia Mo	

Question1. [30 marks]

Fill in all the following blanks. Only the final results are required to be written down.

a) The modulus of the complex number $z = \frac{(3+i)(2-i)}{(2+i)(3-i)(1+i)}$ is ( ).

Solution. $\frac{1}{\sqrt{2}}$

b) The function $f(z) = \begin{cases} 0, & z = 0 \\ \frac{(\bar{z})^2}{z}, & z \neq 0 \end{cases}$ is ( ) (continuous or discontinuous) at $z = 0$.

Solution. continuous

c) The period of the function $f(z) = e^{\frac{z}{5}}$ is ( ).

Solution. $10\pi i$

d) $\operatorname*{Res}_{z=0} \cot z$ =( ).

Solution. 1

e) The Laurent series of $f(z) = \frac{1}{z^2-1}$ in the annular domain $0 < |z-1| < 2$ is ( ).

Solution. $\sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^{n-1}}{(2)^{n+1}}$

f) The standard form of the linear second order PDE $y^2 u_{xx} - x^2 u_{yy} = 0, (xy \neq 0)$ is ( ).

Solution. $u_{\xi\eta} = \frac{\eta}{2(\xi^2-\eta^2)} u_\xi - \frac{\xi}{2(\xi^2-\eta^2)} u_\eta$

g) Given that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, we can get $J_{\frac{3}{2}}(x) =$ ( ) by the recurrence formula.

Solution. $\sqrt{\frac{2}{\pi x}}\left(-\cos x + \frac{1}{x}\sin x\right)$

h) Suppose that $\mathcal{F}[f(x)] = F(\lambda)$, where $\mathcal{F}[f(x)]$ is the Fourier integral transformation of $f(x)$, then for any constant $c \in R$, $\mathcal{F}[f(x-c)] =$ ( ).

Solution. $e^{i\lambda c} F(\lambda)$

i) The improper integral $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx =$ ( ).

Solution. $\frac{\pi}{2}$

j) The Laplace integral transformation of $f(t) = e^t$ is ( ).

Solution. $\frac{1}{p-1}$

Question 2. [10 marks]

Please determine whether the following statements are true. Put "T" if the statement is true or "F" if it's wrong.

a) If $f(z) = u(x,y) + iv(x,y)$ is an analytic function, then $-u(x,y)$ is the harmonic conjugate of $v(x, y)$.

Solution. T

b) $\overline{e^z} = e^{\bar{z}}$.

Solution. T

c) $z = 0$ is a pole of order 3 of the function $f(z) = \frac{e^z - 1}{z^3}$.

Solution. F

d) The Strurm-Liourville eigenvalue problem must have a finite number of real eigenvalues and eigenfunctions.

Solution. F

e) Let $J_n(x)$ be the first kind of Bessel function of order n and $Y_n(x)$ be the second kind of Bessel function of order n, then $J_n(x)$ and $Y_n(x)$ are all have finite values at $x = 0$.

Solution. F

Question 3. [10 marks]

Please choose the correct answers for the following questions. Only one is correct.

(1) The coefficients of the Laurent series of $f(z)$ in the annular domain $0 < |z - b| < 2$ is ( ).

A. $C_k = \frac{f^{(k)}(b)}{k!}$ B. $C_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-b)^{k+1}} dz$

C. $C_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-b} dz$ D. $\frac{k!}{2\pi i} \oint_C \frac{f(z)}{(z-b)^{k+1}} dz$

Solution. B

(2) For the eigenvalue problem $\begin{cases} (1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y(x) = 0, |x| < 1 \\ |y(x)|\big|_{x=\pm 1} < +\infty \end{cases}$, the eigenvalues are ( ).

A. $\lambda = n(n+1), n = 0, 1, 2, \cdots\cdots$ **B.** $\lambda = n, n = 0, 1, 2, \cdots\cdots$

C. $\lambda = n(n+1), n = 1, 2, \cdots\cdots$ **D.** $\lambda = n, n = 1, 2, \cdots\cdots$

Solution. A

(3) The type and the characteristic curves of the equation $y^2 u_{xx} - x^2 u_{yy} = 0, (xy \neq 0)$ are ( ).

A. hyperbolic, $\frac{1}{2}y^2 + \frac{1}{2}x^2 = C$ **B.** elliptic, $\frac{1}{2}y^2 + \frac{1}{2}x^2 = C$

C. hyperbolic, $\frac{1}{2}y^2 \pm \frac{1}{2}x^2 = C$ **D.** elliptic, $\frac{1}{2}y^2 \pm \frac{1}{2}x^2 = C$

Solution. C

(4) Suppose that $\mathcal{F}[f(x)] = F(\lambda)$, where $\mathcal{F}[f(x)]$ is the Fourier integral transformation of $f(x)$, then $\mathcal{F}[f'(x) - 3f(x)]$ is ( ).

A. $i\lambda F(\lambda) - 3F(\lambda)$ B. $-i\lambda F(\lambda) + 3F(\lambda)$

C. $i\lambda F(\lambda) + 3F(\lambda)$ D. $-i\lambda F(\lambda) - 3F(\lambda)$

Solution. D

(5) Suppose that $f(t) = e^{-2t}\cos 3t$, then its Laplace integral transformation $\mathcal{L}[f(t)]$ is ( ).

(Given that $\mathcal{L}[\cos 3t] = \frac{s}{s^2+9}$)

A. $\frac{3}{(s+2)^2+9}$ B. $\frac{s+2}{(s+2)^2+9}$

C. $\frac{3s}{(s+2)^2+9}$ D. $\frac{3(s+2)}{(s+2)^2+9}$

Solution. B

Question 4. [12 marks]
Evaluate the following contour integral

a) $\oint_C (|z| - e^z \sin z)dz$, where C is the positively oriented circle $|z| = a\ (a > 0)$.

b) $\oint_{|z|=3} \frac{z^5}{(z^2+1)(z^4+2)} dz$ with positive orientation.

Solution.

a) We know $\oint_C (|z| - e^z \sin z)dz = \oint_C (|z|)dz - \oint_C (e^z \sin z)dz.$

Since $e^z \sin z$ is analytic in the whole complex plane, then it is obvious $\oint_C (e^z \sin z)dz = 0.$

[2 marks]

For $\oint_C (|z|)dz$, let $z = ae^{i\theta}$, $0 \le \theta \le 2\pi$, then we can easily derive

$\oint_C (|z|)dz = \int_0^{2\pi} a \cdot aie^{i\theta} d\theta = 0.$ **[4 marks]**

Accordingly, $\oint_C (|z| - e^z \sin z)dz = 0.$

b) Let $f(z) = \frac{z^5}{(z^2+1)(z^4+2)}$, then

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z(1+z^2)(1+2z^4)}$$

It is obvious that $\operatorname*{Res}_{z=0} \frac{1}{z(1+z^2)(1+2z^4)} = 1$, and $\oint_{|z|=3} \frac{z^5}{(z^2+1)(z^4+2)} dz = 2\pi i.$ **[6 marks]**

Question 5. [8 marks]

Find out all points at which the function $f(z) = x^3 - y^3 + 2x^2y^2 i$ is **differentiable** and **analytic** (give the explanation), and then find its derivatives.

Solution.

For $\boldsymbol{f(z)}$, we know $\boldsymbol{u(x,y) = x^3 - y^3, v(x,y) = 2x^2y^2}$.

Then the Cauchy-Riemann equations look like

$$\boldsymbol{3x^2 = 4x^2y, \qquad 4xy^2 = 3y^2.}$$

The Cauchy-Riemann equations hold only when $\boldsymbol{x = y = 0}$ and $\boldsymbol{x = y = \frac{3}{4}}$. **[6 marks]**

Hence, the function are differentiable at $\boldsymbol{x = y = 0}$ and $\boldsymbol{x = y = \frac{3}{4}}$, and it is analytic nowhere.

Moreover, $\boldsymbol{f'(0) = u_x + v_x = 0}$

$$\boldsymbol{f'\left(\frac{3}{4} + \frac{3}{4}i\right) = u_x + v_x = \frac{27}{16}(1+i)}$$ **[2 marks]**

Question 6. [8 marks]

Solve the following problem by D'Alembert's formula:

$$\begin{cases} \boldsymbol{u_{tt} - a^2u_{xx} = 0,} & \boldsymbol{t \geq 0, -\infty < x < \infty,} \\ \boldsymbol{u(x,0) = \cos x,\ u_t(x,0) = e^{-1},} & \boldsymbol{-\infty < x < \infty.} \end{cases}$$

Solution.

According to the D'Alembert's formula, we have

$$u(x,t) = \frac{1}{2}[\varphi(x-at) + \varphi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi. \qquad \text{[4 marks]}$$

$$= \frac{1}{2}[\cos(x-at) + \cos(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} e^{-1}d\xi$$

$$= -\cos x \cos at + e^{-1}t$$ **[4 marks]**

Question 7. [12 marks]

Solve the following problem by means of separation of variables:

$$\begin{cases} \boldsymbol{u_t - a^2u_{xx} = 0,} & \boldsymbol{t > 0, 0 < x < \pi,} \\ \boldsymbol{u(0,t) = u(\pi,t) = 0,} & \boldsymbol{t \geq 0} \\ \boldsymbol{u(x,0) = \sin x + 7\sin 5x,} & \boldsymbol{0 \leq x \leq \pi.} \end{cases}$$

Solution.

Let $u(x,t) = X(x)T(t)$ and substitute it into the equation, we have

$$T'(t)X(x) = a^2X''(x)T(t).$$

Dividing it by $a^2X(x)T(t)$, we have $\frac{T'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda$, that is,

$$X''(x) + \lambda X(x) = 0 \text{ and } T'(t) + a^2\lambda T(t) = 0.$$ **[4 marks]**

And the boundary conditions become $X(0) = X(\pi) = 0$.

Then solving the eigenvalue problem $\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(\pi) = 0 \end{cases}$, we obtain the eigenvalues $\lambda_n = n^2$

and the eigenfunctions $X_n(x) = \sin nx,\ n = 1,2,\cdots$. **[2 marks]**

Solving the other problem $T'(t) + a^2\lambda T(t) = 0$ for $\lambda_n = n^2$, we obtain

$$T_n(t) = c_ne^{-a^2n^2t}, \quad n = 1,2,\cdots.$$

So,

$$u_n(x,t) = c_n e^{-a^2 n^2 t} \sin n\,x, \quad n = 1,2,\cdots$$

Assume that

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-a^2 n^2 t} \sin n\,x.$$

According to the initial condition, we have

$u(x,0) = \sum_{n=1}^{\infty} c_n \sin n\,x = \sin x + 7 \sin 5x\,, \quad \Rightarrow a_1 = 1, a_5 = 7, a_n = 0, n = 2,3,4,6,7\cdots,$

Hence the solution is $u(x,t) = e^{-a^2 t} \sin x + 7e^{-25a^2 t} \sin 5x$.

[6 marks]

Question 8. [10 marks]

Evaluate the following integral:

a) $\boldsymbol{I = \int_0^x x^4 J_1(x)\,dx}$ with $\boldsymbol{J_1(x)}$ being the first order of the first kind Bessel function.

b) $\boldsymbol{I = \int_{-1}^{1} x^3\, P_2(x) dx}$ with $\boldsymbol{P_2(x)}$ being the second order Legendre polynomial.

Solution.

a) According to the formula $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$, we have

$$I = \int_0^x x^4 J_1(x)\,dx = \int_0^x x^2 x^2 J_1(x)\,dx = \int_0^x x^2 d[x^2 J_2(x)]$$

$$= x^4 J_2(x) - 2\int_0^x x^3 J_2(x) dx$$

$$= x^4 J_2(x) - 2x^3 J_3(x)$$ **[5 marks]**

b) We know that

$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1).$

Let $x^3 = C_1 P_1(x) + C_3 P_3(x) = C_1 x + \frac{C_3}{2}(5x^3 - 3x)$, then we can easily derive

$$C_1 = \frac{3}{5}, \quad C_3 = \frac{2}{5}$$

That is, $x^3 = \frac{3}{5} P_1(x) + \frac{2}{5} P_3(x)$.

$$I = \int_{-1}^{1} x^3\, P_2(x) dx = \int_{-1}^{1} \left[\frac{3}{5} P_1(x) + \frac{2}{5} P_3(x)\right] P_2(x) dx$$

$$= \int_{-1}^{1} \frac{3}{5} P_1(x)\, P_2(x) dx + \int_{-1}^{1} \frac{2}{5} P_3(x)\, P_2(x) dx$$

$$= 0$$ **[5 marks]**