1.3 *Systems classification and Properties*

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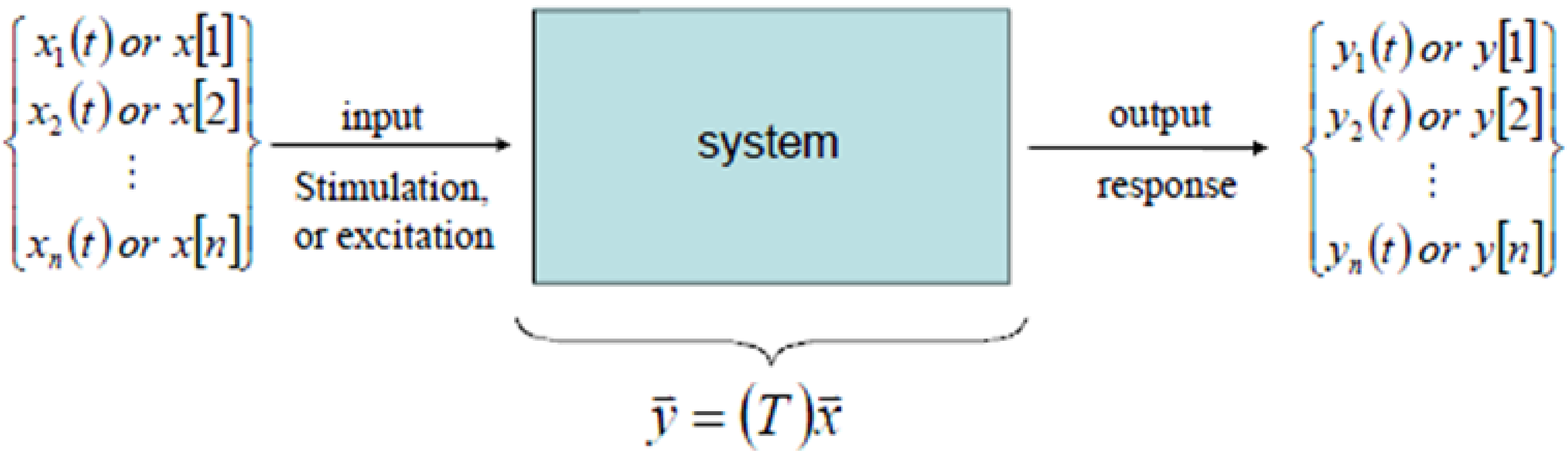
Outline

- **Systems Classification and Properties**
  - Continuous-time and Discrete-time Systems
  - Systems with and without memory
  - Causal and Non-causal Systems
  - Linear and Nonlinear Systems
  - Time-variant and Time-invariant Systems
  - Stable Systems
  - Feedback Systems
  - Invertibility

System Representation

- A system is a **mathematical model** of a physical process that relates the input (or excitation) signal ***x*** to the output (or response) signal ***y***.
- The system can be viewed as an **interconnection of operations** that transforms an input signal x into an output signal y with properties different from those of x.



Continuous-time and Discrete-time Systems

- ***continuous-time system***: the input x and output y are continuous-time signals.

$$y(t) = H\{x(t)\}$$

where the overall operator H denote the action of a system.

- ***discrete-time system***: the input x and output y are discrete-time signals

$$y[n] = H\{x[n]\}$$

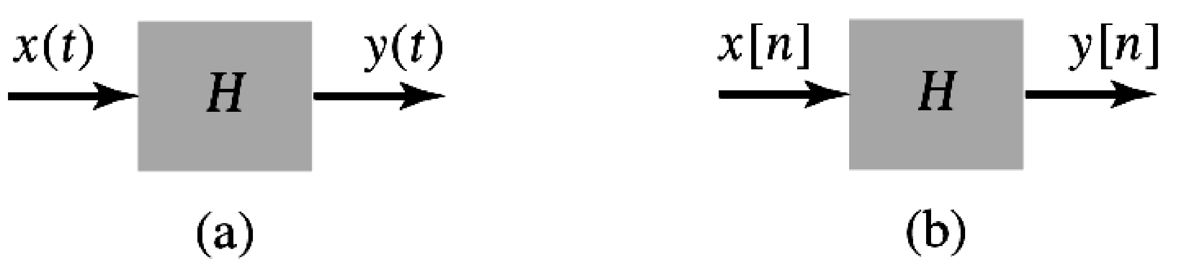


Figure 1.49 Block diagram representation of operator *H* for (a) continuous time and (b) discrete time.

Continuous-time and Discrete-time Systems

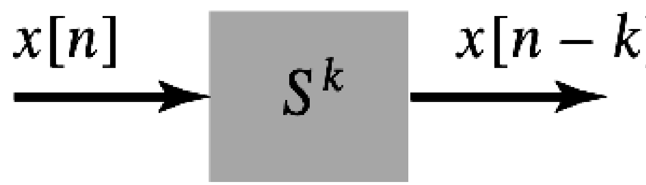
Example 1.12 Moving-average system. Consider a discrete-time system whose output signal $y[n]$ is the average of the three most recent values of the input signal $x[n]$, that is

$$y[n]=\frac{1}{3}(x[n]+x[n-1]+x[n-2])$$

Formulate the operator *H* for this system; hence, develop a block diagram representation for it.

<Sol.>

- Discrete-time-shift operator S^k: Shifts the input $\boldsymbol{x[n]}$ by $\boldsymbol{k}$ time units to produce an output equal to $\boldsymbol{x[n-k]}$.

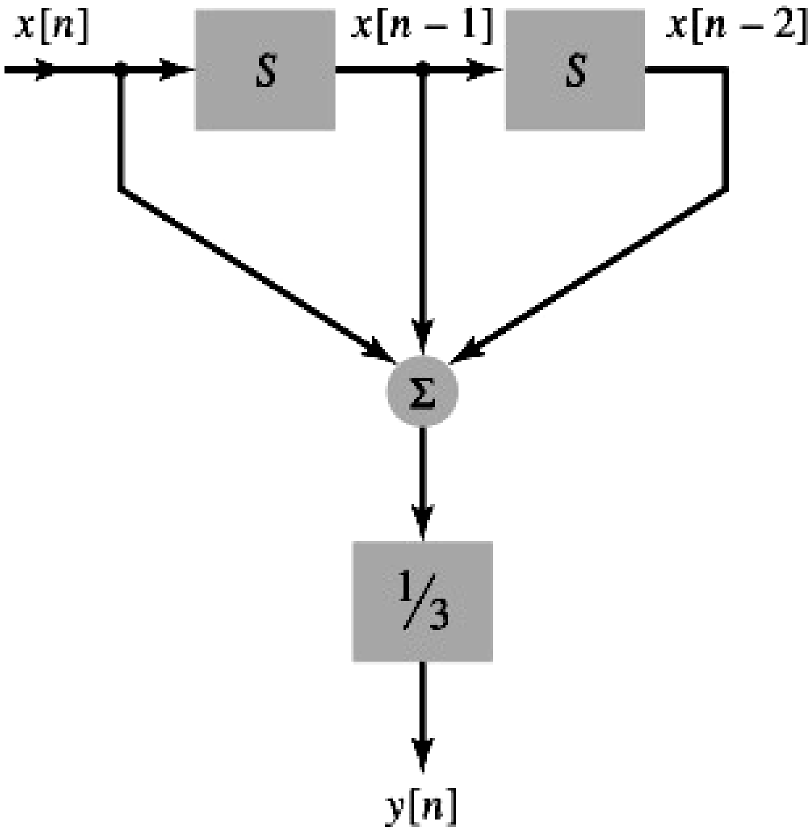


- Overall operator *H* for the moving-average system:

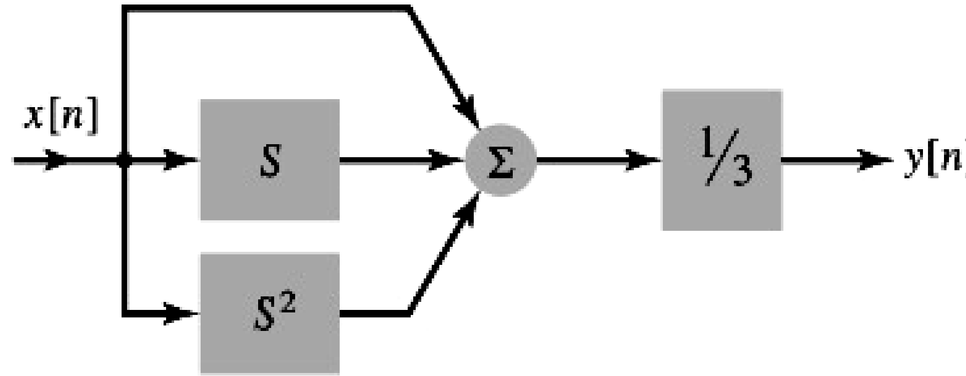
$$H=\frac{1}{3}(1+S+S^2)$$

Continuous-time and Discrete-time Systems

$$H = \frac{1}{3}(1 + S + S^2)$$



(a)

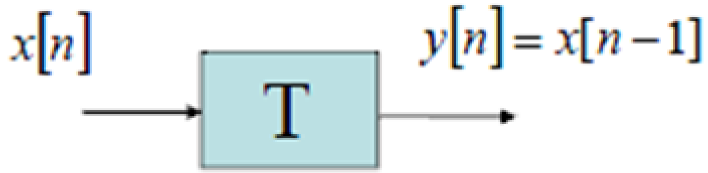


(b)

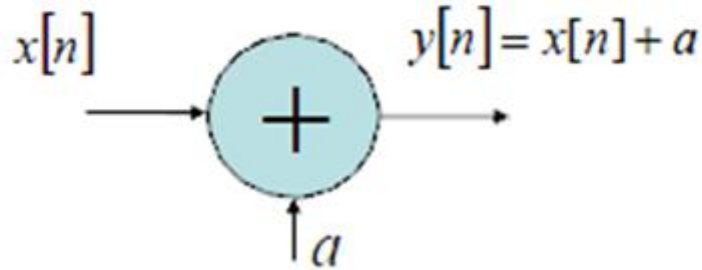
Figure 1.51 Two different (but equivalent) implementations of the moving-average system: (a) cascade form of implementation and (b) parallel form of implementation.

Continuous-time and Discrete-time Systems

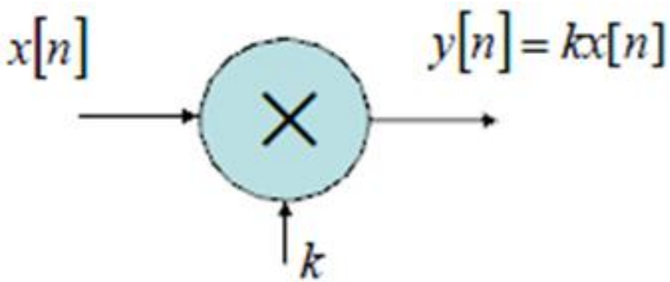
- **Representation of discrete-time operations**



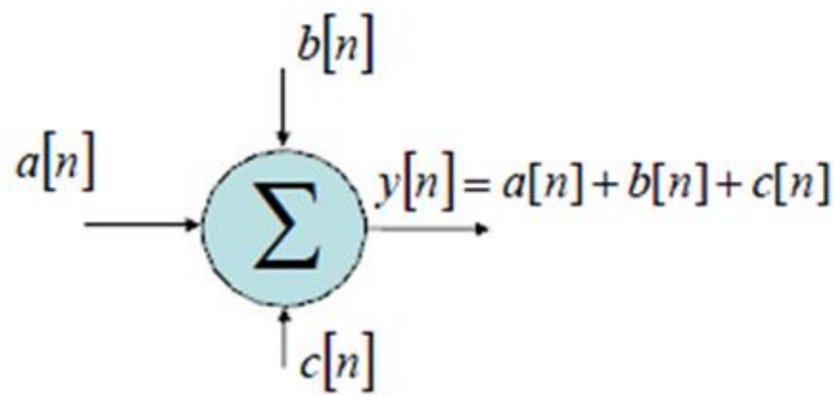
The delay block



The addition of a constant



The scaling of a constant



The summation of sequences

Systems with and without memory

- **A system is said to be memoryless if the output at any time depends on only the input at that same time. Otherwise, the system is said to have memory.**
  - **Resistor:** $i(t) = \frac{1}{R} v(t)$ ⇒ ***Memoryless !***
  - **Inductor:** $i(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau) d\tau$ ⇒ ***Memory !***
  - **Moving-average system:**

    $$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

    ⇒ ***Memory* ! (2 time units)**
  - **A system described by the input-output relation:**

    $y[n] = x^2[n]$ ⇒ ***Memoryless !***

Causal and Non-causal Systems

- A system is said to be ***causal*** if its present value of the output signal depends only on the present or past values of the input signal.
- A system is said to be ***noncausal*** if its output signal depends on one or more future values of the input signal.
  - **Moving-average system:**

$$y[n]=\frac{1}{3}(x[n]+x[n-1]+x[n-2])$$ ➡ ***Causal*** **!**

$$y[n]=\frac{1}{3}(x[n+1]+x[n]+x[n-1])$$ ➡ ***Noncausal*** **!**

*** Causality is required for a system to be capable of operating in *real time*.**

Linear and Nonlinear Systems

- If a system is ***linear***, it has to satisfy the following two conditions:
  - **Superposition (叠加性):**

$$y_1(t) = H\{x_1(t)\} \quad y_2(t) = H\{x_2(t)\} \Longrightarrow y(t) = H\{x_1(t) + x_2(t)\} = y_1(t) + y_2(t)$$

  - **Homogeneity(倍增性):**

$$y(t) = H\{x(t)\} \Longrightarrow H\{\alpha x(t)\} = \alpha y(t)$$

α: constant factor

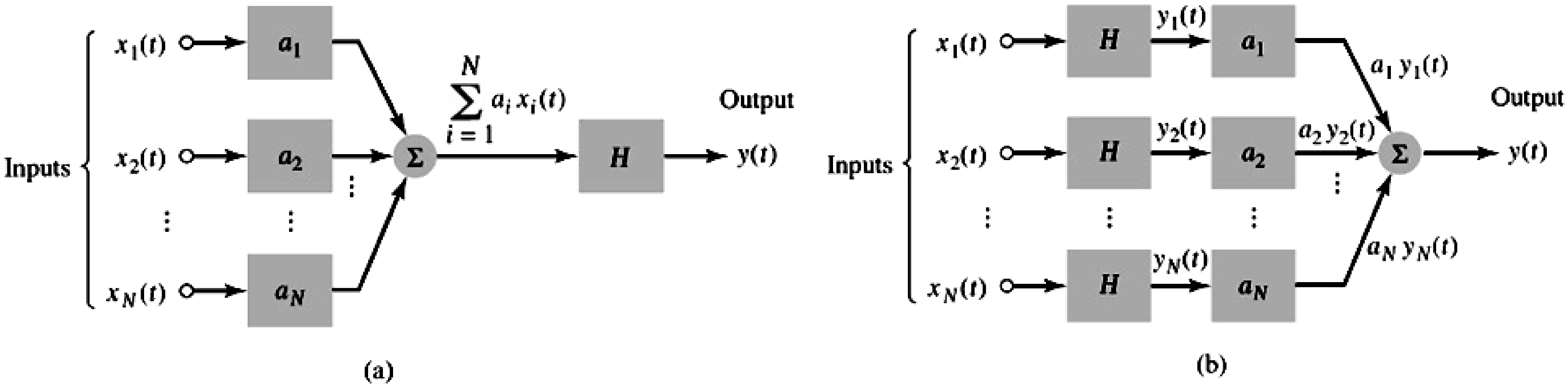
- Any system not satisfying these conditions is classified ***nonlinear system***.
- Conditions 1. and 2. may be combined into the single condition

Linear and Nonlinear Systems

- Superposition property

$$x(t)=\sum_{i=1}^{N}a_i x_i(t), \quad y_i(t)=H\{x_i(t)\}, \quad i=1,\ 2,\ \ldots,\ N.$$

$$\Rightarrow \quad y(t)=H\{x(t)\}=H\left\{\sum_{i=1}^{N}a_i x_i(t)\right\}=\sum_{i=1}^{N}a_i H\{x_i(t)\}=\sum_{i=1}^{N}a_i y_i(t)$$



- If these two configurations produce the same output ***y(t)***, the operator ***H*** is linear.

Linear and Nonlinear Systems

Example 1.19 Linear Discrete-Time system. Consider a discrete-time system described by the input-output relation

$$y[n] = nx[n]$$

Show that this system is linear.

<p.f.>

1. Input: $x[n] = \sum_{i=1}^{N} a_i x_i[n]$

2. Resulting output signal:

$$y[n] = n\sum_{i=1}^{N} a_i x_i[n] = \sum_{i=1}^{N} a_i n x_i[n] = \sum_{i=1}^{N} a_i y_i[n] \qquad \textbf{where} \quad y_i[n] = nx_i[n]$$

⇒ ***Linear system*!**

Linear and Nonlinear Systems

Example 1.20 Nonlinear Continuous-Time System

Consider a continuous-time system described by the input-output relation

$$y(t) = x(t)x(t-1)$$

Show that this system is nonlinear.

<p.f.>

1. Input: $x(t) = \sum_{i=1}^{N} a_i x_i(t)$

2. Output:

$$y(t) = \sum_{i=1}^{N} a_i x_i(t) \sum_{j=1}^{N} a_j x_j(t-1) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j x_i(t) x_j(t-1)$$

since

$$\sum_{i=1}^{N} a_i y_i(t) = \sum_{i=1}^{N} a_i x_i(t) x_i(t-1) \neq y(t) \Rightarrow$$ ***Nonlinear system*!**

Linear and Nonlinear Systems

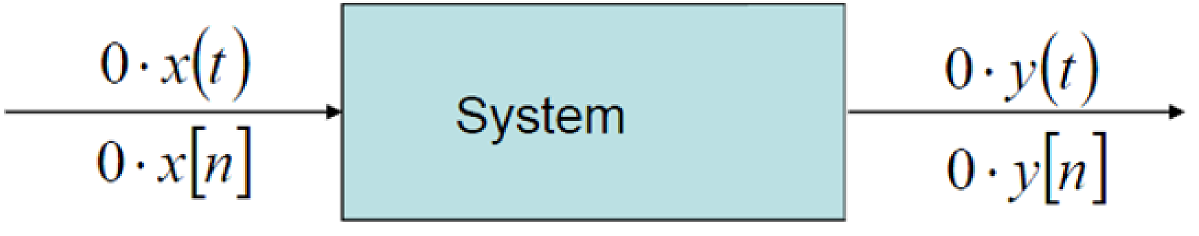
- More examples

$v(t) = Ri(t)$ **Linear systems**

$y = x^2(t);$ **Nonlinear systems**

- NB: a consequence of the *scaling* property of linear-systems is

zero in = zero out



Linear-system

Time-variant and Time-invariant Systems

- A system is time-invariant if a time-shift (advance or delay) at the input causes an identical shift at the output.
- For a continuous-time system, time-invariance exists if:

$$H\{x(t)\} = y(t) \Longrightarrow H\{x(t \pm \tau)\} = y(t \pm \tau)$$

- For a discrete-time system, the system is time-invariant if

$$H\{x[n]\} = y[n] \Longrightarrow H\{x[n \pm k]\} = y[n \pm k], \quad k \in Z$$

- A system not satisfying above equations is time-varying.

- time-invariance can be tested by correlating the shifted output with the output produced by a shifted input.

Time-variant and Time-invariant Systems

- Continuous-time system $y_1(t) = H\{x_1(t)\}$
  - Input signal $x_1(t)$ is shifted in time by t_0 seconds:

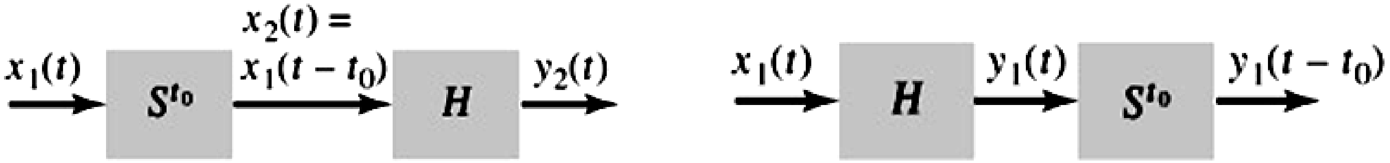
    $$x_2(t) = x_1(t - t_0) = S^{t_0}\{x_1(t)\}$$ **~ S^{t_0} = operator of a time shift equal to t_0**

    $$y_2(t) = H\{x_1(t - t_0)\} = H\{S^{t_0}\{x_1(t)\}\} = HS^{t_0}\{x_1(t)\}$$

  - Output signal $y_1(t)$ is shifted in time by t_0 seconds:

    $$y_1(t - t_0) = S^{t_0}\{y_1(t)\} = S^{t_0}\{H\{x_1(t)\}\} = S^{t_0}H\{x_1(t)\}$$

  - Condition for time-invariant system: $HS^{t_0} = S^{t_0}H$



These two situations are equivalent, provided that *H* is time invariant.

Time-variant and Time-invariant Systems

Example 1.17 Inductor. **The inductor shown in figure is described by the input-output relation:**

$$y_1(t) = \frac{1}{L}\int_{-\infty}^{t} x_1(\tau)d\tau$$

where *L* is the inductance. Show that the inductor is time invariant.

<Sol.>

- $x_1(t) \longrightarrow x_1(t - t_0) \longrightarrow$ **Response** $y_2(t)$ **of the inductor to** $x_1(t - t_0)$

$$y_2(t) = \frac{1}{L}\int_{-\infty}^{t} x_1(\tau - t_0)d\tau \xrightarrow{\tau' = \tau - t_0} y_2(t) = \frac{1}{L}\int_{-\infty}^{t-t_0} x_1(\tau')d\tau'$$

- $x_1(t) \longrightarrow y_1(t) \longrightarrow y_1(t - t_0)$

$$y_1(t - t_0) = \frac{1}{L}\int_{-\infty}^{t-t_0} x_1(\tau)d\tau = y_2(t)$$

$\Rightarrow$ **Inductor is time invariant.**

Time-variant and Time-invariant Systems

Example 1.18 Thermistor. **Let $R(t)$ denote the resistance of the thermistor, expressed as a function of time. We may express the input-output relation of the device as**

$$y_1(t) = x_1(t)/R(t)$$

Show that the thermistor so described is time variant.

<Sol.>

- **response $y_2(t)$ of the thermistor to $x_1(t - t_0)$**

$$y_2(t) = \frac{x_1(t-t_0)}{R(t)}$$

- **Let $y_1(t - t_0)$ = the original output of the thermistor, shifted by t_0 seconds:**

$$y_1(t-t_0) = \frac{x_1(t-t_0)}{R(t-t_0)} \neq y_2(t) \quad \text{for} \;\; t_0 \neq 0$$

➡ ***Time variant*!**

Stability of a system

- A system is **bounded-input/bounded-output(BIBO) stable** if for any bounded input $x(t)$ defined by

$$|x(t)| \leq M_x < \infty \qquad \text{for all } t$$

the corresponding output $y(t)$ is also bounded defined by

$$|y(t)| \leq M_y < \infty \qquad \text{for all } t$$

- Note that there are many other definitions of stability.
- It is important that a system of interest remain stable under all possible operating conditions.

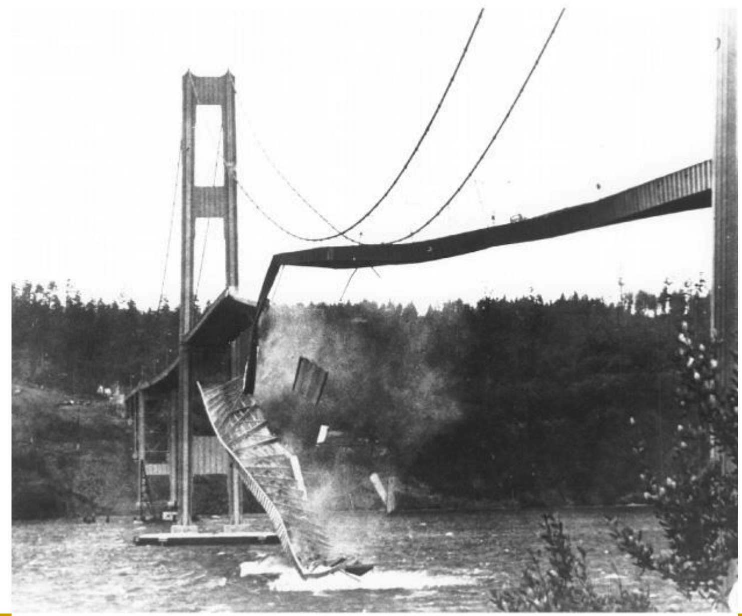
Stability of a system

One famous example of an unstable system:

Figure 1.52a (p. 56)
Dramatic photographs showing the collapse of the Tacoma Narrows suspension bridge on November 7, 1940. (a) Photograph showing the twisting motion of the bridge's center span just before failure.
(b) A few minutes after the first piece of concrete fell, this second photograph shows a 600-ft section of the bridge breaking out of the suspension span and turning upside down as it crashed in Puget Sound, Washington. Note the car in the top right-hand corner of the photograph.

(Courtesy of the Smithsonian Institution.)

(a)



(b)

Stability of a system

Example 1.13 Moving-average system (continued)

Show that the moving-average system described in Ex. 1.12 is BIBO stable.

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

<p.f.>

Assume that: $\left|x[n]\right| \le M_x < \infty \quad \text{for all} \quad n$

$$\Rightarrow \quad \left|y[n]\right| = \frac{1}{3}\left|x[n] + x[n-1] + x[n-2]\right|$$

$$\le \frac{1}{3}\left(\left|x[n]\right| + \left|x[n-1]\right| + \left|x[n-2]\right|\right)$$

$$\le \frac{1}{3}\left(M_x + M_x + M_x\right) = M_x < \infty$$

$\Rightarrow$ **The moving-average system is stable.**

Stability of a system

Example 1.14 Consider a discrete-time system whose input-output relation is defined by

$$y[n] = r^n x[n]$$

where $r > 1$. Show that this system is unstable.

<p.f.>

Assume that: $|x[n]| \le M_x < \infty \quad \text{for all} \quad n$

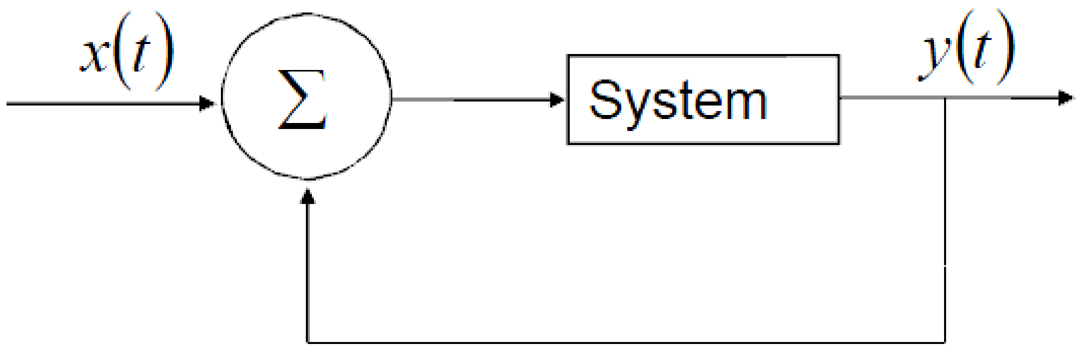
$$|y[n]| = |r^n x[n]| = |r^n| . |x[n]|$$

With $r > 1$, the multiplying factor r^n diverges for increasing n.

The system is unstable.

Feedback Systems

- A special class of systems of great importance consists of systems having ***feedback*.**
- In a ***feedback system,*** the output signal is fed back and added to the input to the system.



System feedback

Invertibility Systems

- A system is said to be **invertible** if the input of the system can be recovered from the output.
  - **For a continuous-time system H:**

$$H^{inv}\{y(t)\} = H^{inv}\{H\{x(t)\}\}$$
$$= H^{inv}H\{x(t)\}$$

Condition for invertible system:

$$\boxed{H^{inv}H = I}$$

I **: identity operator**

H^{inv} **: inverse operator of H**

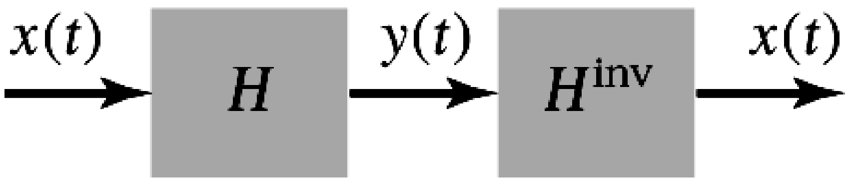


Figure 1.54 The notion of system invertibility.

  - **A system is not invertible unless distinct inputs applied to the system produce distinct outputs (one-to-one mapping between input and output).**

Invertibility Systems

Example 1.15 Inverse of System. **Consider the following time-shift system**

$$y(t) = x(t - t_0) = S^{t_0}\{x(t)\}$$

where the operator S^{t_0} represents a time shift of t_0 seconds. Find the inverse of this system.

<Sol.> **Inverse operator of S^{t_0}: time shift of $-t_0$ ~ S^{-t_0}**

$$S^{-t_0}\{y(t)\} = S^{-t_0}\{S^{t_0}\{x(t)\}\} = S^{-t_0}S^{t_0}\{x(t)\}$$

$$\Rightarrow \quad S^{-t_0}S^{t_0} = I$$

Example 1.16 Non-Invertible System. **Show that a square-law system is not invertible.**

$$y(t) = x^2(t)$$

<p.f.> **Since the distinct inputs $x(t)$ and $-x(t)$ produce the same output $y(t)$. Accordingly, the square-law system is not invertible.**

Summary

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  - Systems with and without memory
  - Causal and Non-causal Systems
  - Linear and Nonlinear Systems
  - Time-variant and Time-invariant Systems
  - Linear Time-invariant Systems
  - Stable Systems
  - Feedback Systems
  - Invertibility

- Reference in textbook: 1.7, 1.8
- Homework: 1.64(a, c, f, g), 1.76