Solutions to Assignment of Dec. 13.

Exercise 5.3.

1. (a) $A=(1,0)$, (b) $A=(1,1)$. (e) $A=\begin{pmatrix}1&0\\0&2\end{pmatrix}$

(f) $A=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$. (h) $A=\begin{pmatrix}1&0\\0&-1\\1&1\end{pmatrix}$.

2. $E=\left\{v_1=\begin{pmatrix}1\\-1\end{pmatrix}, v_2=\begin{pmatrix}1\\1\end{pmatrix}\right\}$ is a basis of $\mathbb{R}^2$: since $\begin{vmatrix}1&1\\-1&1\end{vmatrix}=2\neq 0$.

consider the linear transformation $L(x)=2x_1e_1+(x_1+x_2)e_2$

we have
$$L(v_1)=\begin{pmatrix}2\\0\end{pmatrix}=v_1+v_2=(v_1,v_2)\begin{pmatrix}1\\1\end{pmatrix} \qquad [L(v_1)]_E=\begin{pmatrix}1\\1\end{pmatrix}$$
$$\Rightarrow$$
$$L(v_2)=\begin{pmatrix}2\\2\end{pmatrix}= 2v_2=(v_1,v_2)\begin{pmatrix}0\\2\end{pmatrix} \qquad [L(v_2)]_E=\begin{pmatrix}0\\2\end{pmatrix}$$

the matrix representing L relative to basis E is

$$A=\left([L(v_1)]_E, [L(v_2)]_E\right)=\begin{pmatrix}1&0\\1&2\end{pmatrix}.$$

Exercise 5.4.

6. Consider the linear transformation $L:\mathbb{R}^2\to\mathbb{R}^2$

$$L(x)=(x_1\cos\alpha+x_2\sin\alpha)e_1+(x_2\cos\alpha-x_1\sin\alpha)e_2$$

(a). $E=\{e_1,e_2\}$ standard basis of $\mathbb{R}^2$.

$$L(e_1)=\begin{pmatrix}\cos\alpha\\-\sin\alpha\end{pmatrix},\quad L(e_2)=\begin{pmatrix}\sin\alpha\\\cos\alpha\end{pmatrix}.$$

$\Rightarrow$ the matrix representing L relative to basis E is

$$A=(L(e_1),L(e_2))=\begin{pmatrix}\cos\alpha&\sin\alpha\\-\sin\alpha&\cos\alpha\end{pmatrix}.$$

(b) $F = \left\{ v_1 = \begin{pmatrix} \cos\beta \\ -\sin\beta \end{pmatrix}, v_2 = \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^2$: $\begin{vmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{vmatrix} = 1 \neq 0$

we have $L(v_1) = \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha \\ -\sin\beta\cos\alpha - \cos\beta\sin\alpha \end{pmatrix} = \cos\alpha \begin{pmatrix} \cos\beta \\ -\sin\beta \end{pmatrix} - \sin\alpha \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix}$

$= (v_1, v_2) \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}$. $\Rightarrow [L(v_1)]_F = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}$

$L(v_2) = \begin{pmatrix} \sin\beta\cos\alpha + \cos\beta\sin\alpha \\ \cos\beta\cos\alpha - \sin\beta\sin\alpha \end{pmatrix} = \sin\alpha \begin{pmatrix} \cos\beta \\ -\sin\beta \end{pmatrix} + \cos\alpha \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix}$

$= (v_1, v_2) \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$ $\Rightarrow [L(v_2)]_F = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$.

$\Rightarrow$ matrix representing L relative to basis F is

$$B = \left([L(v_1)]_F, [L(v_2)]_F \right) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

<u>Remark</u>. We can also compute B using the transition matrix S from basis F to standard basis E.

$S = (v_1, v_2) = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$, $S^{-1} = S^T = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$

then $B = S^{-1}AS = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

(c). The invertible matrix S can be taken as the transition matrix from F to E.; or simply taken $S = I$. (since $A = B$ in this example.).

8. $B \sim A \Rightarrow \exists\, S$ invertible, $B = S^{-1}AS$.

A is invertible $\Rightarrow |A| \neq 0$. $A^{-1} = \frac{1}{|A|}\operatorname{adj} A$., $\operatorname{adj} A = |A| \cdot A^{-1}$.

$|B| = |S^{-1}AS| = |S^{-1}| \cdot |A| \cdot |S| = |A| \neq 0$. so B is also invertible.

$$\begin{aligned}\operatorname{adj} B = |B| \cdot B^{-1} &= |A| \cdot (S^{-1}AS)^{-1} \\ &= |A| \cdot S^{-1}A^{-1}S \\ &= S^{-1}(|A|A^{-1})S = S^{-1}(\operatorname{adj} A)S\end{aligned}$$

$\Rightarrow \operatorname{adj} B \sim \operatorname{adj} A$.

<u>Exercise 6.1.</u>

6. (a). Indeed, we have checked the positivity of $\langle \cdot, \cdot \rangle$ in class.

- $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx = \int_{-\pi}^{\pi} g(x) f(x)\, dx = \langle g, f \rangle$.

- $$\begin{aligned}\langle \alpha f + \beta g, h \rangle &= \int_{-\pi}^{\pi} (\alpha f + \beta g)(x)\, h(x)\, dx \\ &= \alpha \int_{-\pi}^{\pi} f(x) h(x)\, dx + \beta \int_{-\pi}^{\pi} g(x) h(x) dx \\ &= \alpha \langle f, h \rangle + \beta \langle g, h \rangle.\end{aligned}$$

$\Rightarrow \langle \cdot, \cdot \rangle$ is an inner product on $C[-\pi, \pi]$.

(b). $$\begin{aligned}\langle v_m, u_n \rangle &= \int_{-\pi}^{\pi} \cos(mx) \sin(nx)\, dx \\ &= \int_{-\pi}^{\pi} \frac{1}{2}\left(\sin(n+m)x - \sin(n-m)x\right) dx. \\ &= 0. \quad \forall n, m \in \mathbb{N},\end{aligned}$$

<u>Remark</u>. One can show that in $C[a,b]$,

- $v_m = \cos(mx) \perp v_n = \cos(nx)$, $\forall\, m \neq n$.

- $u_m = \sin(mx) \perp u_n = \sin(nx)$, $\forall\, m \neq n,\ m, n \geq 1$.