Introduction to Electronic Systems

Zheng Feng

Part 3

Sinusoidal Steady-State Analysis

(AC+Dynamic/Resistive Element+Topology)

Part 3: Sinusoidal Steady-State Analysis

8. **Sinusoidal Steady-State Analysis**
9. Sinusoidal Steady-State Power Calculations
10. Frequency Selective Circuits *

Chapter 8 (1)

- Complex Number Review
- Sinusoidal Source
- Sinusoidal Response
- The Phasor
- VCR of Passive Elements in Frequency Domain

Chapter 8 (2)

- KCL and KVL in Frequency Domain
- Mesh-Current Method
- Node-Voltage Method
- Source Transformation
- Superposition Theorem
- Thévenin and Norton Equivalents

8-1 Complex Number Review

- **Mathematic Representations of Complex Number**
- **Basic Operations of Complex Number**

Mathematic Representations

- Algebraic Representation
- Geometric Representation
- Trigonometric Form
- Exponential Form
- Polar Form

Algebraic Representation

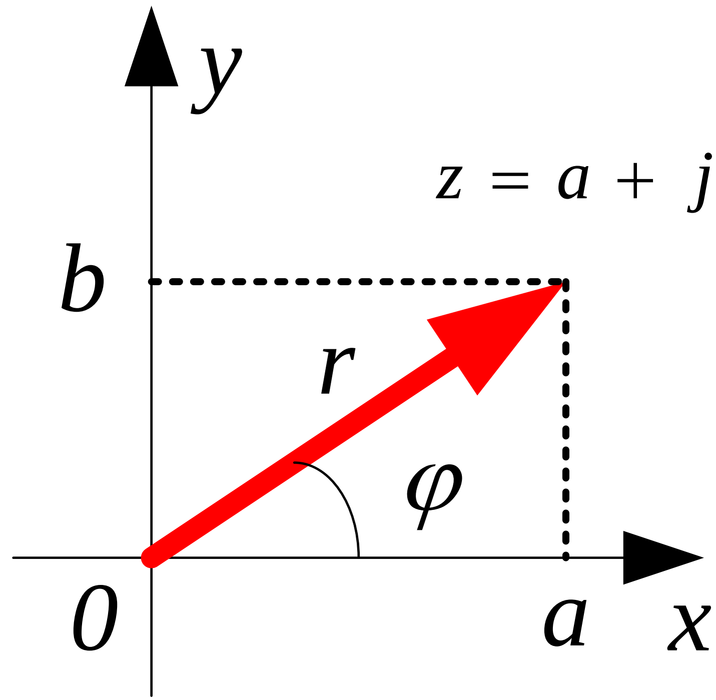
$$z = a + jb \qquad z^* = a - jb$$

$a = \mathrm{Re}[z]$ **: Real part** $b = \mathrm{Im}[z]$ **: Imaginary part**

$r = |z| = \sqrt{a^2 + b^2}$ **: Modulus** $\varphi = \arctan \dfrac{b}{a}$ **: Phase**

$j = \sqrt{-1}$ **: Imaginary unit**

Geometric Representation

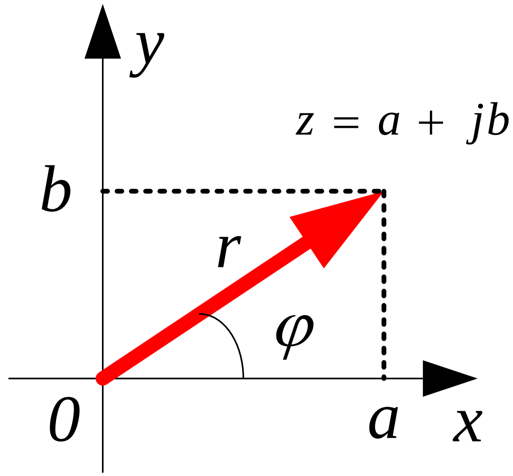


Representation of complex number on the complex plane

Trigonometric Representation

$$z = a + jb$$

$$\begin{cases} a = r\cos\varphi \\ b = r\sin\varphi \end{cases}$$



$$z = a + jb$$
$$= r\cos\varphi + jr\sin\varphi$$
$$= r\left(\cos\varphi + j\sin\varphi\right)$$

$$\begin{cases} r = |z| = \sqrt{a^2 + b^2} \\ \varphi = \arctan\dfrac{b}{a} \end{cases}$$

Exponential Form

- **Euler's formula** $\begin{cases} e^{j\varphi} = \cos\varphi + j\sin\varphi \\ e^{-j\varphi} = \cos\varphi - j\sin\varphi \end{cases}$

$$z = r\left(\cos\varphi + j\sin\varphi\right)$$

$$= re^{j\varphi}$$

Exponential Form

Polar Form

$$z = r\left(\cos\varphi + j\sin\varphi\right)$$

$$= re^{j\varphi}$$

$$= r\angle\varphi$$

Polar Form

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Basic Operations of Complex Number

- Addition and Subtraction
- Multiplication
- Division

Addition and Subtraction Operation

If $z_1 = a_1 + jb_1 \qquad z_2 = a_2 + jb_2$

Then
$$z_1 \pm z_2 = (a_1 + jb_1) \pm (a_2 + jb_2)$$
$$= (a_1 \pm a_2) + j(b_1 \pm b_2)$$

$$z_1 = z_2 \Longleftrightarrow \begin{cases} a_1 = a_2 \\ b_1 = b_2 \end{cases}$$

Multiplication Operation

If $z_1 = a_1 + jb_1 \qquad z_2 = a_2 + jb_2$

$$z_1 \cdot z_2 = (a_1 + jb_1) \cdot (a_2 + jb_2) = a_1 a_2 + jb_1 a_2 + ja_1 b_2 + j^2 b_1 b_2$$
$$= (a_1 a_2 - b_1 b_2) + j(b_1 a_2 + a_1 b_2)$$

$$z_1 \cdot z_2 = r_1(\cos\varphi_1 + j\sin\varphi_1) \cdot r_2(\cos\varphi_2 + j\sin\varphi_2)$$
$$= r_1 r_2 [\cos(\varphi_1 + \varphi_2) + j\sin(\varphi_1 + \varphi_2)]$$

Specially,

$$j^2 = -1, \quad j^3 = -j, \quad j^4 = 1, \quad j^5 = j, \quad \cdots$$

If $z_1 = r_1 e^{j\varphi_1}$ $\qquad z_2 = r_2 e^{j\varphi_2}$

Then $z_1 \cdot z_2 = r_1 e^{j\varphi_1} \cdot r_2 e^{j\varphi_2} = r_1 r_2 e^{j(\varphi_1+\varphi_2)}$

If $z_1 = r_1 \underline{/\varphi_1}$ $\qquad z_2 = r_2 \underline{/\varphi_2}$

Then $z_1 \cdot z_2 = r_1 \underline{/\varphi_1} \cdot r_2 \underline{/\varphi_2} = r_1 r_2 \underline{/\varphi_1 + \varphi_2}$

$$z \cdot z^* = (a + jb) \cdot (a - jb) = a^2 + b^2 = r^2$$

Division of Complex Number

If $z_1 = a_1 + jb_1$ $z_2 = a_2 + jb_2$

Then $$\frac{z_1}{z_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{(a_1 + jb_1)\cdot(a_2 - jb_2)}{(a_2 + jb_2)\cdot(a_2 - jb_2)}$$

$$= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + j\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}$$

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If $z_1 = r_1 e^{j\varphi_1}$ $z_2 = r_2 e^{j\varphi_2}$

Then $$\frac{z_1}{z_2} = \frac{r_1 e^{j\varphi_1}}{r_2 e^{j\varphi_2}} = \frac{r_1}{r_2} e^{j(\varphi_1 - \varphi_2)}$$

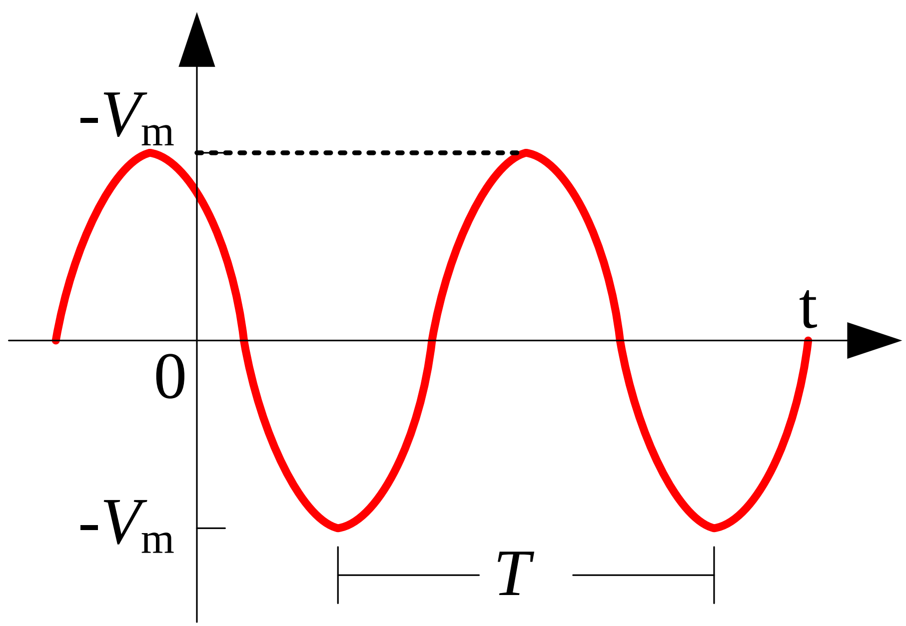
If $z_1 = r_1 \angle \varphi_1$ $z_2 = r_2 \angle \varphi_2$

Then $$\frac{z_1}{z_2} = \frac{r_1 \angle \varphi_1}{r_2 \angle \varphi_2} = \frac{r_1}{r_2} \angle \varphi_1 - \varphi_2$$

8-2 Sinusoidal Source

- **Sinusoidal Voltage/Current Source:**

Voltage/Current varies sinusoidally with time.



Alternative Current (AC)

Parameters of Sinusoidal Function

$$v(t) = V_m \cos(\omega t + \phi)$$

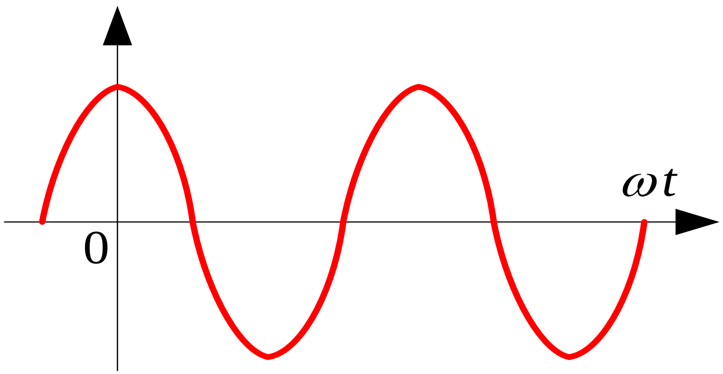
ϕ : Initial phase angle determines the value at t = 0.

ω : Angular frequency determines the rate varying with time.

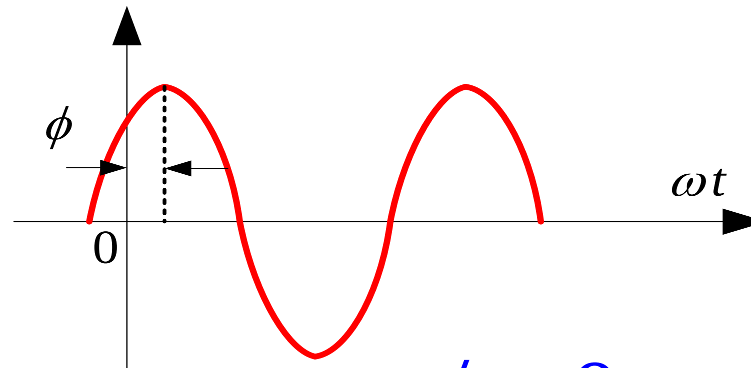
V_m : Amplitude determines the maximum value.

Phase Angle

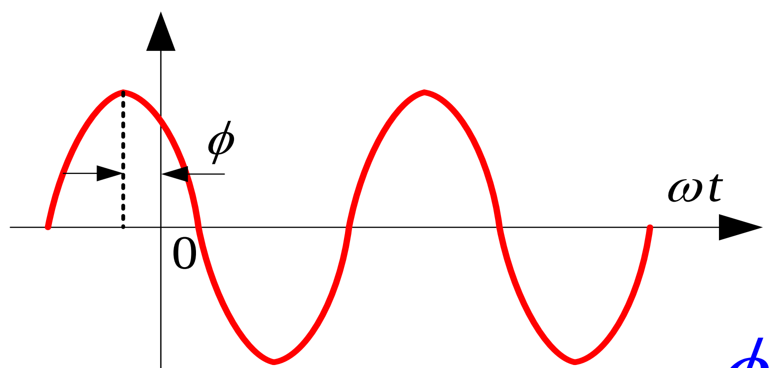
ϕ : Initial phase angle determines the value at t=0.



$\phi = 0$



$\phi < 0$



$\phi > 0$

Angular Frequency

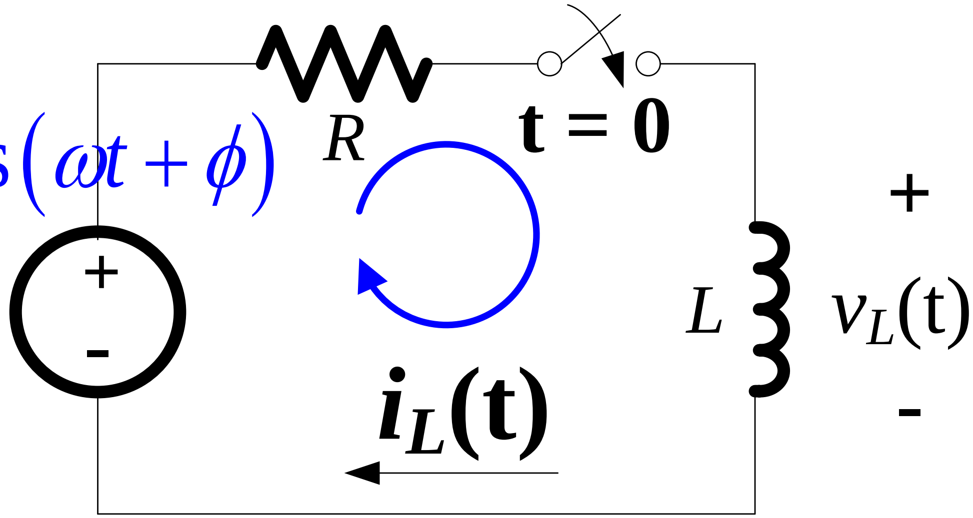
$$\omega = 2\pi f = \frac{2\pi}{T}$$: measured in radians/second

$$f = \frac{1}{T}$$: measured in hertz (Hz)

T : measured in seconds

8-3 Sinusoidal Response

$$v(t) = V_m \cos(\omega t + \phi)$$



$$i_L(0^-) = 0$$

- The initial current of the inductor is zero.
- The switch is closed at t = 0;
- Find the inductive current.

Solution:

- **By KVL for the loop:** $L\dfrac{di_L}{dt}+Ri_L=V_m\cos(\omega t+\phi)$

$$i_L(t)=\frac{-V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\phi-\theta)e^{-(R/L)t}$$

$$+\frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t+\phi-\theta)$$

in which, $\theta=\arctan(\omega L/R)$

Sinusoidal Response

$$i_L(t) = \frac{-V_m}{\sqrt{R^2+\omega^2 L^2}}\cos(\phi-\theta)e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2+\omega^2 L^2}}\cos(\omega t+\phi-\theta)$$

Transient Response | **Steady State Response**

- **The transient response becomes infinitesimal as time elapses;**
- **The steady state response exists as long as the source continues to supply the voltage.**

Sinusoidal Response

- The transient response becomes infinitesimal as time elapses;
- The steady state response exists as long as the source continues to supply the voltage;
- Because the transient response vanishes as time elapses, the steady state response must also satisfy the differential equation.

Steady State Response

- In a linear circuit driven by sinusoidal sources, the steady state response is also a sinusoidal function;
- The frequency of the steady state response is identical to the frequency of the sinusoidal source;
- The amplitude and phase angle differ from those of the sinusoidal source.

Steady State Response

- We focus on the steady state response in this chapter;
- Is there any method for calculating the steady state response without solving the differential equation?
- If yes, what is it?

8-4 Phasors

- **Euler's formula:** $e^{j\varphi} = \cos\varphi + j\sin\varphi$

$$\cos\varphi = \mathrm{Re}\left[e^{j\varphi}\right]$$

$$V_m \cos(\omega t + \phi) = \mathrm{Re}\left[V_m e^{j(\omega t + \phi)}\right]$$

$$= \mathrm{Re}\left[V_m e^{j\phi} e^{j\omega t}\right]$$

Phasor

$$V_m \cos(\omega t + \phi) = \mathrm{Re}\left[V_m e^{j\phi} e^{j\omega t}\right]$$

$$V_m \cos(\omega t + \phi) \Longrightarrow \hat{V} = V_m e^{j\phi}$$

Sinusoidal Voltage | **Phasor**

$$\hat{V} = V_m e^{j\phi} = V_m \angle\phi = V_m(\cos\phi + j\sin\phi)$$

(Inverse) Phasor Transform

- **Phasor Transform**

$$V_m \cos(\omega t + \phi) \Rightarrow \hat{V} = V_m e^{j\phi}$$

$$P\left[V_m \cos(\omega t + \phi)\right] = V_m e^{j\phi} = \hat{V}$$

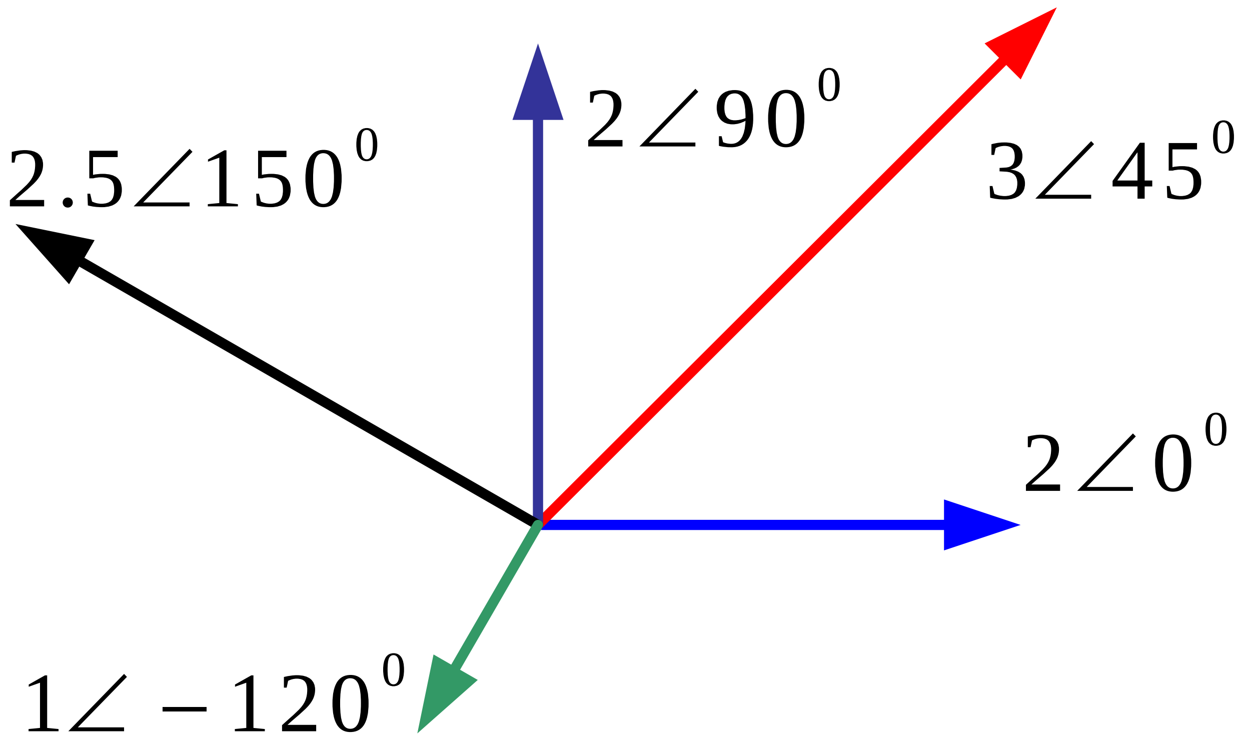
- **Inverse Phasor Transform**

$$V_m \cos(\omega t + \phi) \Leftarrow \hat{V} = V_m e^{j\phi}$$

$$P^{-1}\left[\hat{V} = V_m e^{j\phi}\right] = V_m \cos(\omega t + \phi)$$

Phasor Diagram

$$\hat{V} = V_m e^{j\phi} = V_m \angle\phi = V_m \cos\phi + jV_m \sin\phi$$



Phasor Transform

- A phasor is a complex number;
- A phasor represents a sinusoidal function, and shows up as the coefficient of $e^{j\omega t}$;
- A phasor has two elements: Amplitude and Phase angle;
- Phasor transform transfers the sinusoidal function from time domain to phasor domain, which is also called frequency domain.

Example

$$v(t) = 5\cos\left(100\pi t + 30^0\right)$$

$$\hat{V} = P\left[5\cos\left(100\pi t + 30^0\right)\right] = P\left[\mathrm{Re}\left(5e^{j30^0}e^{j100\pi t}\right)\right]$$

$$= 5e^{j30^0} = 5\angle 30^0$$

$$= 5\left(\cos 30^0 + j\sin 30^0\right) = \frac{5\sqrt{3}}{2} + j\frac{5}{2}$$

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Example

$$v(t) = 5\cos\left(500\pi t + 30^0\right)$$

$$\hat{V} = P\left[5\cos\left(500\pi t + 30^0\right)\right] = P\left[\mathrm{Re}\left(5e^{j30^0} e^{j500\pi t}\right)\right]$$

$$= 5e^{j30^0} = 5\angle 30^0$$

$$= 5\left(\cos 30^0 + j\sin 30^0\right) = \frac{5\sqrt{3}}{2} + j\frac{5}{2}$$

Example

$$i(t) = -7\sin\left(100\pi t + 30^0\right)$$

$$\hat{I} = P\left[-7\sin\left(100\pi t + 30^0\right)\right] = P\left[7\cos\left(100\pi t + 30^0 + 90^0\right)\right]$$

$$= P\left[7\cos\left(100\pi t + 120^0\right)\right] = P\left[\mathrm{Re}\left(7e^{j120^0} e^{j100\pi t}\right)\right]$$

$$= 7e^{j120^0} = 7\angle 120^0$$

$$= 7\left(\cos 120^0 + j\sin 120^0\right) = -\frac{7}{2} + j\frac{7\sqrt{3}}{2}$$

Inverse Phasor Transform

- Phasor is NOT equal to sinusoidal function.
- A sinusoidal voltage/current can NOT be completely determined by a phasor;
- Sinusoidal voltage/current is determined by both a phasor and corresponding radian angular frequency.

$$\hat{V} = V_m e^{j\phi} \Longleftrightarrow v(t) = \mathrm{Re}\left[V_m e^{j\phi} e^{j\omega t}\right] = V_m \cos(\omega t + \phi)$$

Example

$$f = 50\text{Hz}, \quad \hat{V} = 50\angle -30^0\,\text{V}$$

$$\omega = 2\pi f = 2\pi \times 50\text{Hz} = 100\pi\left(\text{rad/s}\right)$$

$$v(t) = P^{-1}\left[50\angle -30^0\right] = \text{Re}\left[50e^{j\phi}e^{j\omega t}\right]$$

$$= \text{Re}\left[50e^{j\left(-30^0\right)}e^{j100\pi t}\right] = 50\cos\left(100\pi t - 30^0\right)\text{V}$$

Example

$$f = 100\text{Hz}, \quad \hat{V} = 50\angle -30^0\,\text{V}$$

$$\omega = 2\pi f = 2\pi \times 100\text{Hz} = 200\pi\left(\text{rad/s}\right)$$

$$v(t) = P^{-1}\left[50\angle -30^0\right] = \text{Re}\left[50e^{j\left(-30^0\right)}e^{j200\pi t}\right]$$

$$= 50\cos\left(200\pi t - 30^0\right)\text{V}$$

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Basic Operation of Phasors

- Addition and Subtraction Operation
- Differential Operation
- Integral Operation

Addition and Subtraction

If $v_n(t) = V_{mn}\cos(\omega t + \phi_n)$

Then $P\left[\sum_{n=1}^{N} v_n(t)\right] = \sum_{n=1}^{N} P\left[v_n(t)\right]$

Proof:

$$\sum_{n=1}^{N} v_n(t) = \sum_{n=1}^{N} V_{mn}\cos\left(\omega t + \phi_n\right)$$

$$= \sum_{n=1}^{N} \mathrm{Re}\left[V_{mn} e^{j\phi_n} e^{j\omega t}\right] = \mathrm{Re}\left\{\sum_{n=1}^{N}\left[V_{mn} e^{j\phi_n} e^{j\omega t}\right]\right\}$$

$$= \mathrm{Re}\left\{\left[\sum_{n=1}^{N} V_{mn} e^{j\phi_n}\right] e^{j\omega t}\right\} = \mathrm{Re}\left\{\sum_{n=1}^{N} P\left[v_n(t)\right] e^{j\omega t}\right\}$$

Differential Operation

If $$v(t) = V_m \cos(\omega t + \phi)$$

Then $$P\left[\frac{dv(t)}{dt}\right] = j\omega P[v(t)]$$

Proof:

$$\frac{dv(t)}{dt}=\frac{d\left[V_m\cos(\omega t+\phi)\right]}{dt}=\frac{d\left\{\mathrm{Re}\left[V_m e^{j\phi}e^{j\omega t}\right]\right\}}{dt}$$

$$=\mathrm{Re}\left[\frac{d\left(V_m e^{j\phi}e^{j\omega t}\right)}{dt}\right]=\mathrm{Re}\left[V_m e^{j\phi}\frac{d\left(e^{j\omega t}\right)}{dt}\right]$$

$$=\mathrm{Re}\left[V_m e^{j\phi}\left(j\omega e^{j\omega t}\right)\right]=\mathrm{Re}\left[j\omega V_m e^{j\phi}e^{j\omega t}\right]$$

$$=\mathrm{Re}\left[j\omega\hat{V}e^{j\omega t}\right]$$

Integral Operation

If $$v(t) = V_m \cos(\omega t + \phi)$$

Then $$P\left[\int v(t)\,dt\right] = \frac{1}{j\omega} P\left[v(t)\right]$$

Proof:

$$\int v(t)\,dt = \int V_m \cos(\omega t + \phi)\,dt = \int \mathrm{Re}\left[V_m e^{j\phi} e^{j\omega t}\right] dt$$

$$= \mathrm{Re}\left[\int V_m e^{j\phi} e^{j\omega t}\,dt\right] = \mathrm{Re}\left[V_m e^{j\phi} \int e^{j\omega t}\,dt\right]$$

$$= \mathrm{Re}\left[V_m e^{j\phi} \cdot \frac{1}{j\omega} e^{j\omega t}\right] = \mathrm{Re}\left[\frac{1}{j\omega} V_m e^{j\phi} e^{j\omega t}\right]$$

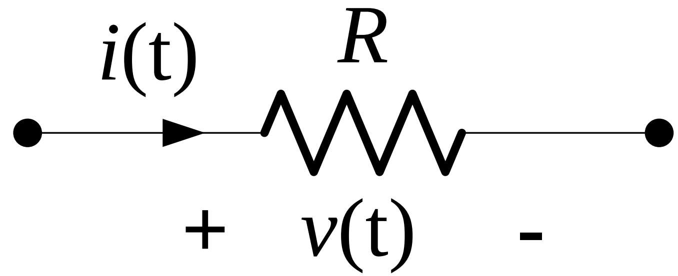
$$= \mathrm{Re}\left[\frac{1}{j\omega} \hat{V} e^{j\omega t}\right]$$

8-5 VCR of Passive Elements

- VCR of Resistor in Phasors
- VCR of Capacitor in Phasors
- VCR of Inductor in Phasors

VCR of Resistor in Phasors

$$i(t) = I_m \cos(\omega t + \phi)$$

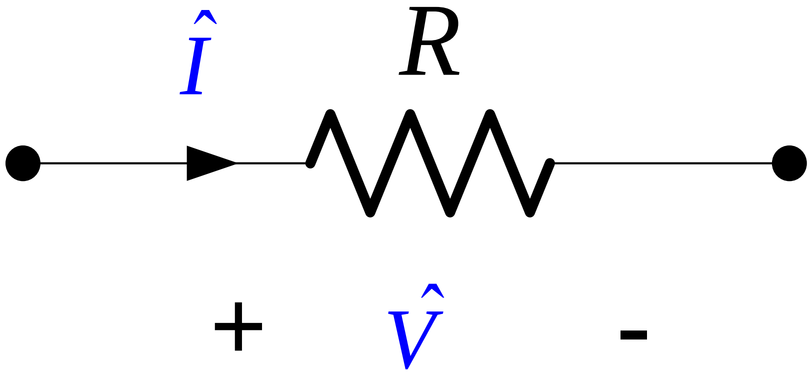


$$v(t) = RI_m \cos(\omega t + \phi)$$

$$= R \cdot \mathrm{Re}\left[I_m e^{j\phi} e^{j\omega t} \right] = \mathrm{Re}\left[R \cdot \hat{I} e^{j\omega t} \right]$$

$$\boxed{\hat{V} = R \cdot \hat{I}}$$

VCR of Resistor in Phasors



$$\boxed{\hat{V} = R \cdot \hat{I}}$$

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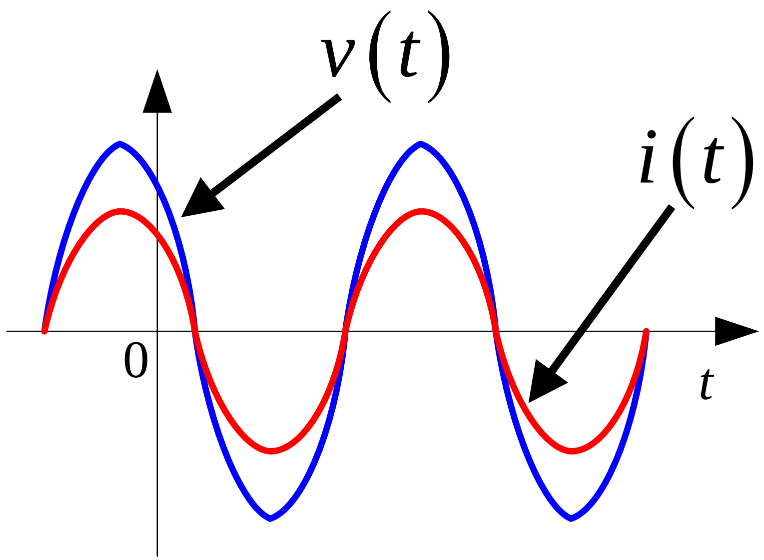
VCR of Resistors in Phasor

$$v(t) = \mathrm{Re}\left[R \cdot \hat{I} e^{j\omega t} \right]$$

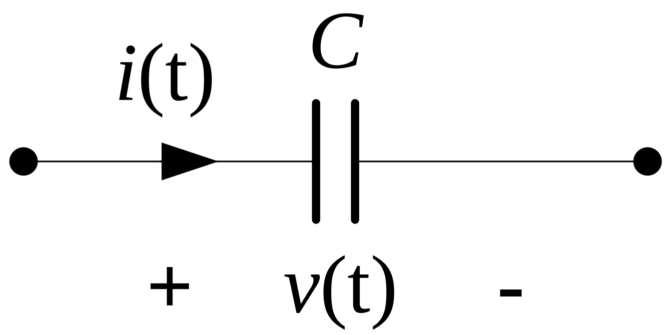
$$= \mathrm{Re}\left[R \cdot I_m e^{j\phi} e^{j\omega t} \right] = \mathrm{Re}\left[V_m e^{j\phi} e^{j\omega t} \right]$$



- **For resistor, *v*(t) and *i*(t) are in phase.**

VCR of Capacitors in Phasor

$$v(t) = V_m \cos(\omega t + \phi)$$



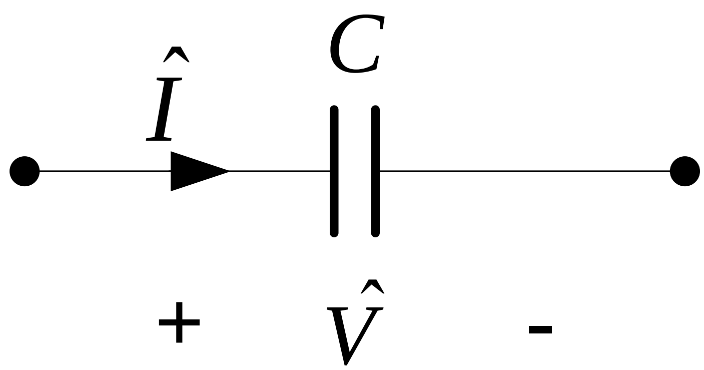
$$i(t) = C\frac{dv}{dt} = \omega C V_m \cos\left(\omega t + \phi + 90^0\right)$$

$$= \mathrm{Re}\left[\omega C V_m e^{j\left(\phi+90^0\right)} e^{j\omega t}\right] = \mathrm{Re}\left[\omega C V_m e^{j\phi} e^{j90^0} e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[j\omega C V_m e^{j\phi} e^{j\omega t}\right] = \mathrm{Re}\left[j\omega C \hat{V} e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[\hat{I} e^{j\omega t}\right]$$

VCR of Capacitors in Phasor

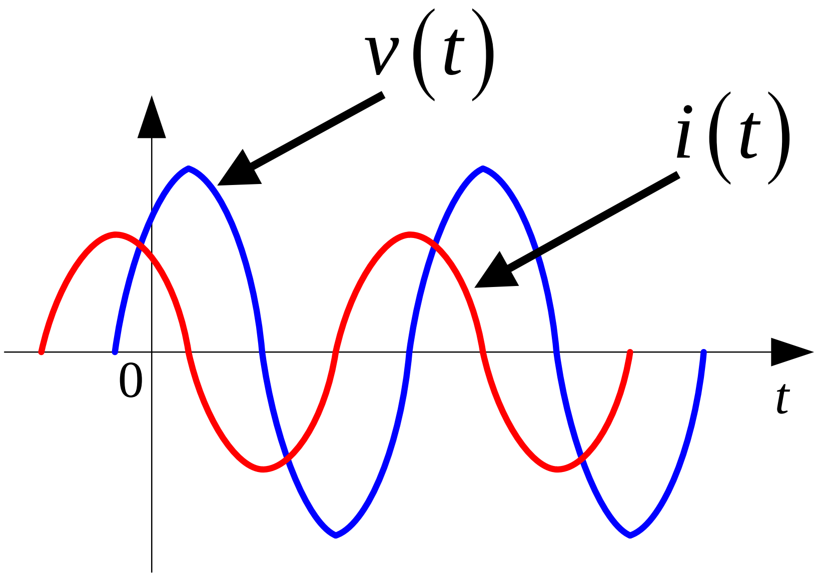


$$\boxed{\hat{V} = \frac{1}{j\omega C}\hat{I}}$$

VCR of Capacitors in Phasor

$$i(t) = \mathrm{Re}\left[\hat{I}e^{j\omega t}\right] = \mathrm{Re}\left[j\omega C\hat{V}e^{j\omega t}\right] = \mathrm{Re}\left[\omega C V_m e^{j\left(\phi+90^0\right)} e^{j\omega t}\right]$$

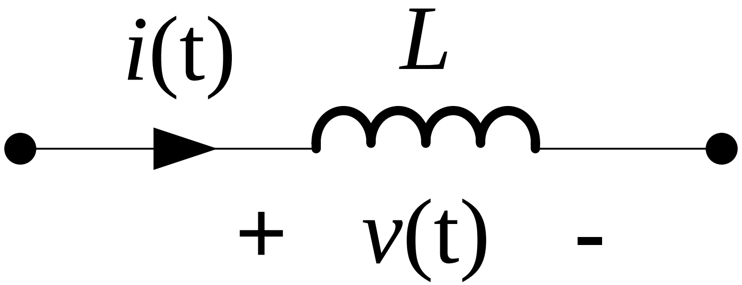
$$v(t) = V_m \cos\left(\omega t + \phi\right)$$



- **For capacitor, *v*(t) lags *i*(t) by 90⁰.**

VCR of Inductors in Phasor

$$i(t) = I_m \cos(\omega t + \phi)$$



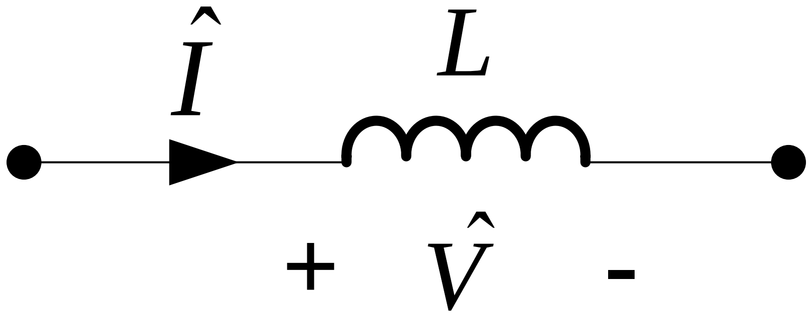
$$v(t) = L\frac{di}{dt} = \omega L I_m \cos\left(\omega t + \phi + 90^0\right)$$

$$= \mathrm{Re}\left[\omega L I_m e^{j\left(\phi + 90^0\right)} e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[\omega L I_m e^{j\phi} e^{j90^0} e^{j\omega t}\right] = \mathrm{Re}\left[j\omega L I_m e^{j\phi} e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[j\omega L \hat{I} e^{j\omega t}\right] = \mathrm{Re}\left[\hat{V} e^{j\omega t}\right]$$

VCR of Inductors in Phasor

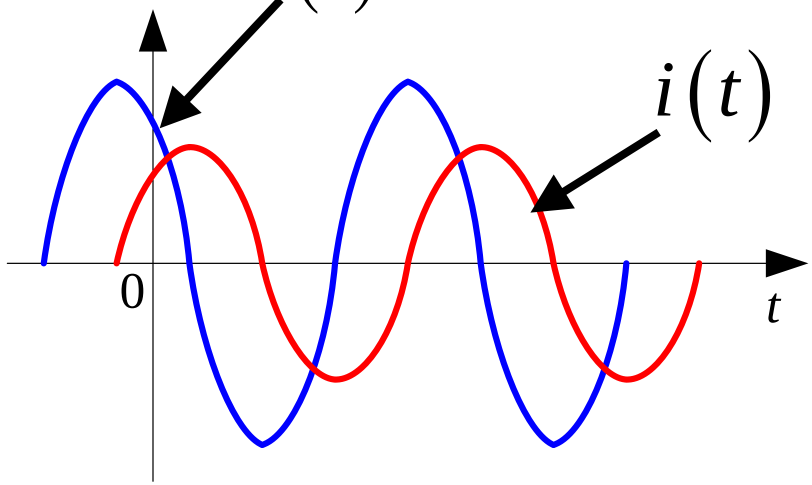


$$\boxed{\hat{V} = j\omega L\hat{I}}$$

VCR of Inductors in Phasor

$$v(t)=\mathrm{Re}\left[\hat{V}e^{j\omega t}\right]=\mathrm{Re}\left[\,j\omega L I_m e^{j\phi}e^{j\omega t}\right]=\mathrm{Re}\left[\omega L I_m e^{j\left(\phi+90^0\right)}e^{j\omega t}\right]$$

$$i(t)=I_m\cos(\omega t+\phi)$$

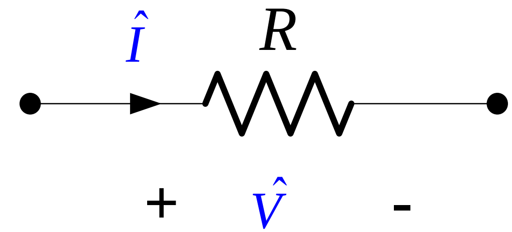


- **For inductor, *v*(t) leads *i*(t) by 90⁰.**

Summary

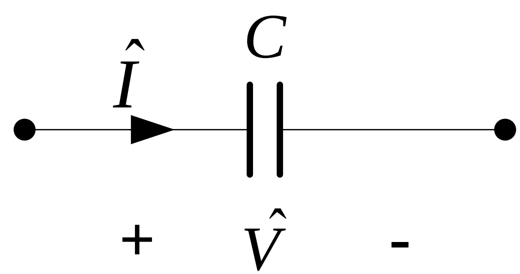
Resistor:

$$\boxed{\hat{V} = R \cdot \hat{I}}$$



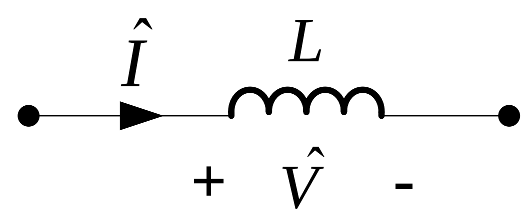
Capacitor:

$$\boxed{\hat{V} = \frac{1}{j\omega C}\hat{I}}$$



Inductor:

$$\boxed{\hat{V} = j\omega L\hat{I}}$$



Impedance and Admittance

- **Impedance:**

defined as the ratio of a circuit element's voltage phasor to its current phasor.

$$Z = \frac{\hat{V}}{\hat{I}}$$ **:measured in Ω**

- **Admittance:**

defined as the reciprocal of impedance.

$$Y = \frac{1}{Z} = \frac{\hat{I}}{\hat{V}}$$ **:measured in S**

	VCR	Impedance	Admittance
R:	$\hat{V} = R \cdot \hat{I}$	$Z = R$	$Y = 1/R = G$
C:	$\hat{V} = \frac{1}{j\omega C}\hat{I}$	$Z = \frac{1}{j\omega C}$	$Y = j\omega C$
L:	$\hat{V} = j\omega L\hat{I}$	$Z = j\omega L$	$Y = 1/j\omega L$

Ohm's Law in Frequency Domain

$$\hat{V} = Z\hat{I}$$

$$\hat{I} = Y\hat{V}$$

$$\boxed{Z = R}$$

$$\boxed{Y = 1/R = G}$$

$$\boxed{Z = \frac{1}{j\omega C}}$$

$$\boxed{Y = j\omega C}$$

$$\boxed{Z = j\omega L}$$

$$\boxed{Y = 1/j\omega L}$$

Reactance and Susceptance

- **Reactance:**

defined as the imaginary part of the impedance.

$$Z = R + jX$$

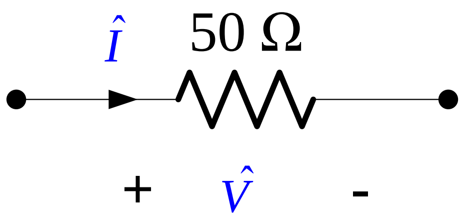
- **Susceptance:**

defined as the imaginary of the admittance.

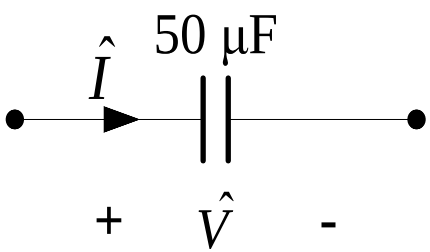
$$Y = G + jB$$

Example

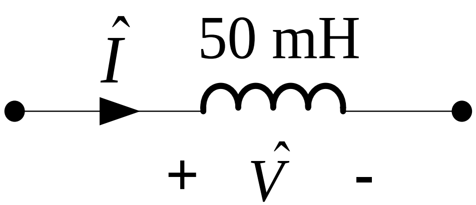
$$\omega = 1000 rad/s$$



$$Z = R = 50\Omega$$

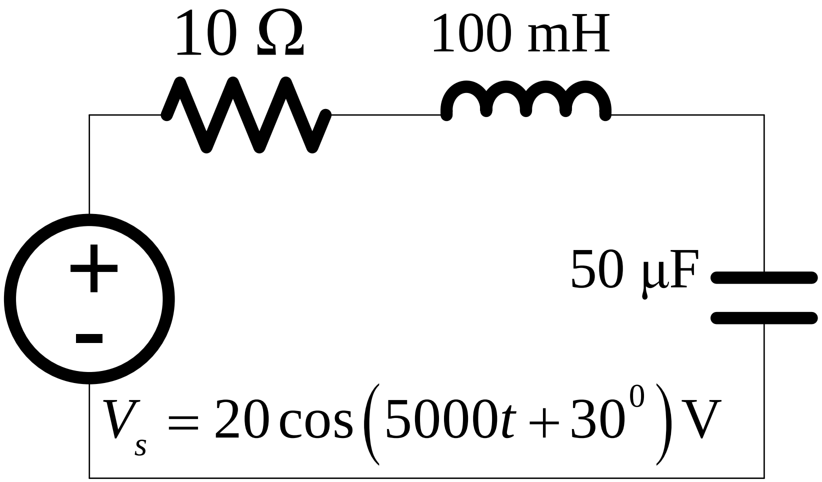


$$Z = \frac{1}{j\omega C} = -j\frac{1}{1000 \times 50 \times 10^{-6}} = -j20\Omega$$



$$Z = j\omega L = j1000 \times 50 \times 10^{-3} = j50\Omega$$

Example



$V_s = 20\cos\left(5000t + 30^0\right)\text{V}$

$$\omega = 5000 rad / s$$

$$\hat{V}_s = P\left[20\cos\left(5000t + 30^0\right)\right]$$

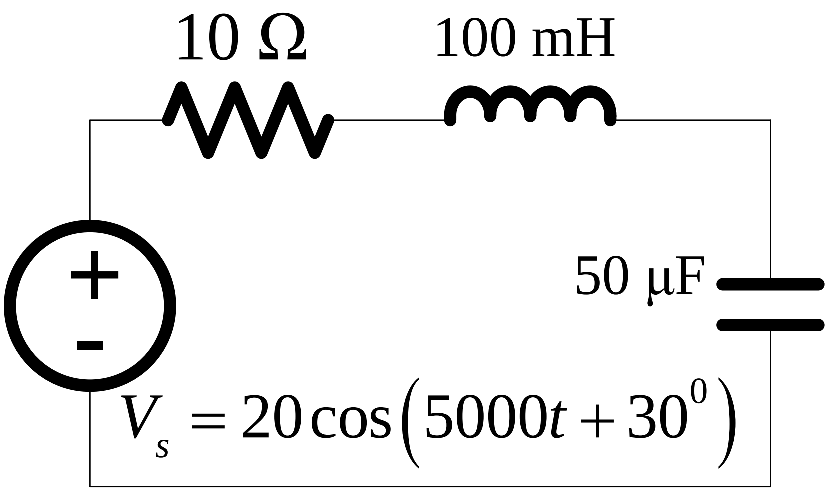
$$= 20\angle 30^0\ \text{V}$$

$$Z_R = 10\Omega$$

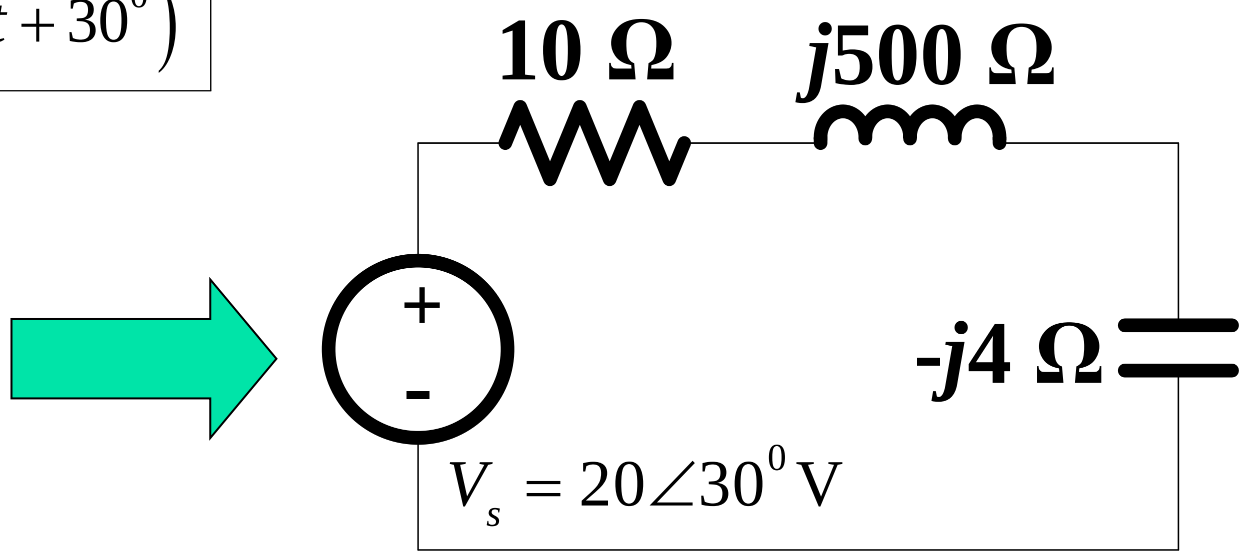
$$Z_L = j\omega L = j \times 5000 \times 100 \times 10^{-3} = j500\Omega$$

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{5000 \times 50 \times 10^{-6}} = -j4\Omega$$

Circuits in Frequency Domain



$$\omega = 5000 rad/s$$



8-6 KCL and KVL in Phasors

- KCL in frequency domain
- KVL in frequency domain

KCL in Frequency Domain

- **The algebraic sum of all the phasor currents entering any node in a circuit is ZERO.**

$$\sum_{n=1}^{N} \hat{I}_n = 0$$

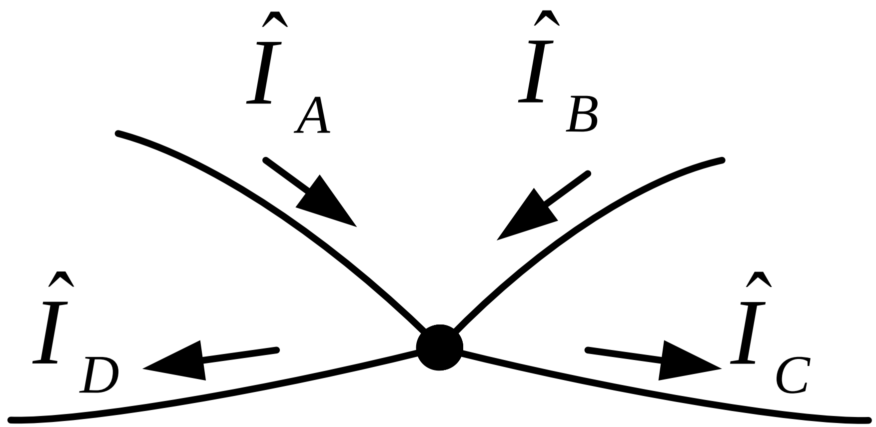
Proof:

$$\sum_{n=1}^{N} i_n(t) = \sum_{n=1}^{N} \mathrm{Re}\left[I_{mn} e^{j\theta_n} e^{j\omega t} \right] = \mathrm{Re}\left[e^{j\omega t} \cdot \sum_{n=1}^{N} I_{mn} e^{j\theta_n} \right]$$

$$= \mathrm{Re}\left[e^{j\omega t} \cdot \sum_{n=1}^{N} \hat{I}_n \right] = 0$$

$$\Rightarrow \sum_{n=1}^{N} \hat{I}_n = 0$$

KCL in Frequency Domain



$$\hat{I}_A + \hat{I}_B = \hat{I}_C + \hat{I}_D$$

$$\hat{I}_A + \hat{I}_B + \left(-\hat{I}_C\right) + \left(-\hat{I}_D\right) = 0$$

$$\left(-\hat{I}_A\right) + \left(-\hat{I}_B\right) + \hat{I}_C + \hat{I}_D = 0$$

KVL in Frequency Domain

- **The algebraic sum of all the phasor voltages around any loop in a circuit is ZERO.**

$$\sum_{n=1}^{N} \hat{V}_n = 0$$

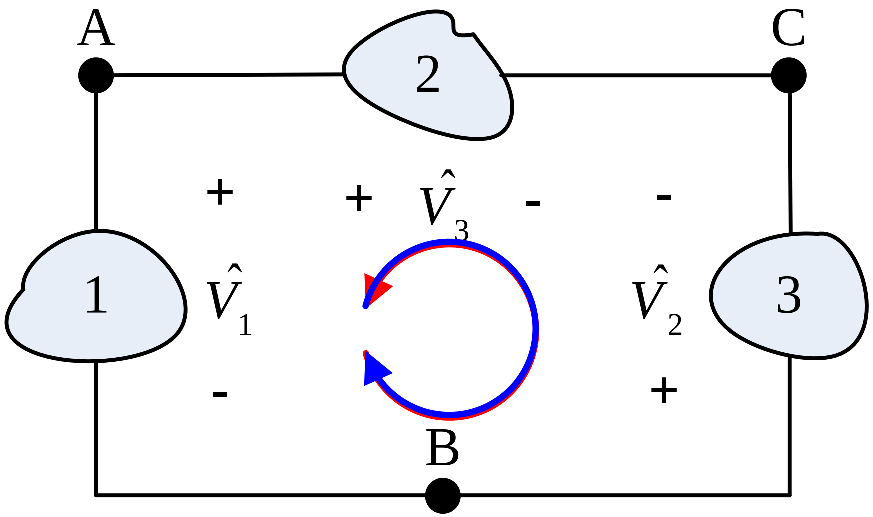
Proof:

$$\sum_{n=1}^{N} v_n(t) = \sum_{n=1}^{N} \mathrm{Re}\left[V_{mn} e^{j\theta_n} e^{j\omega t}\right] = \mathrm{Re}\left[\sum_{n=1}^{N} V_{mn} e^{j\theta_n} e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[e^{j\omega t} \cdot \sum_{n=1}^{N} \hat{V}_n\right] = 0$$

$$\Rightarrow \sum_{n=1}^{N} \hat{V}_n = 0$$

KVL in Frequency Domain

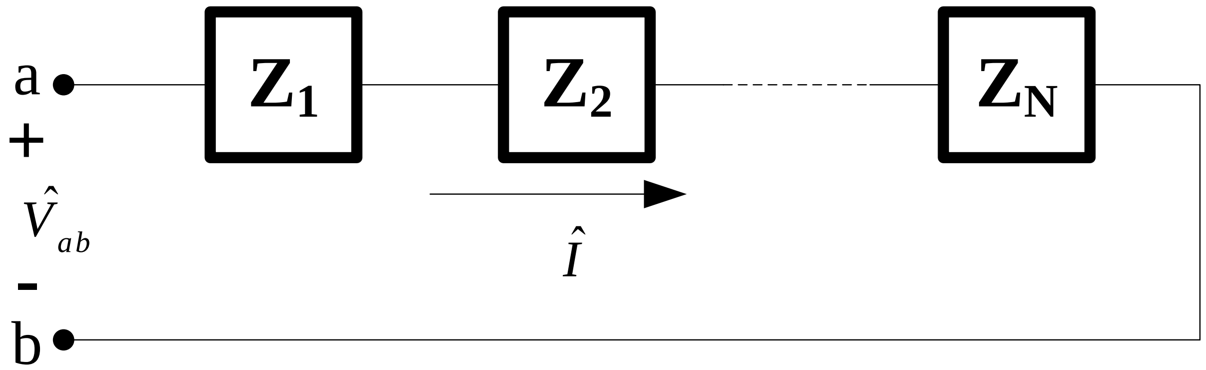


$$\hat{V}_1 = \hat{V}_3 - \hat{V}_2$$

$$\hat{V}_1 + \hat{V}_2 - \hat{V}_3 = 0$$

$$-\hat{V}_1 + \hat{V}_3 - \hat{V}_2 = 0$$

Impedances in Series and Parallel



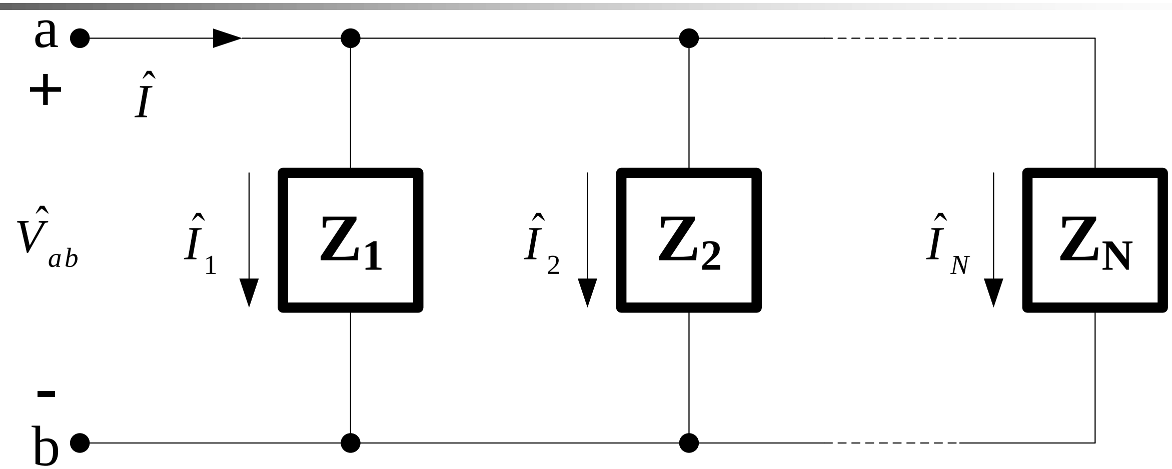
By KVL in phasor:

$$\hat{V}_{ab} = \hat{V}_1 + \hat{V}_2 + \cdots + \hat{V}_N = \sum_{n=1}^{N} \hat{V}_n$$

$$= \sum_{n=1}^{N} \left(Z_n \hat{I} \right) = \hat{I} \sum_{n=1}^{N} Z_n$$

$$\Rightarrow \quad Z_{ab} = \frac{\hat{V}_{ab}}{\hat{I}_{ab}} = \sum_{n=1}^{N} Z_n$$

Impedances in Series and Parallel

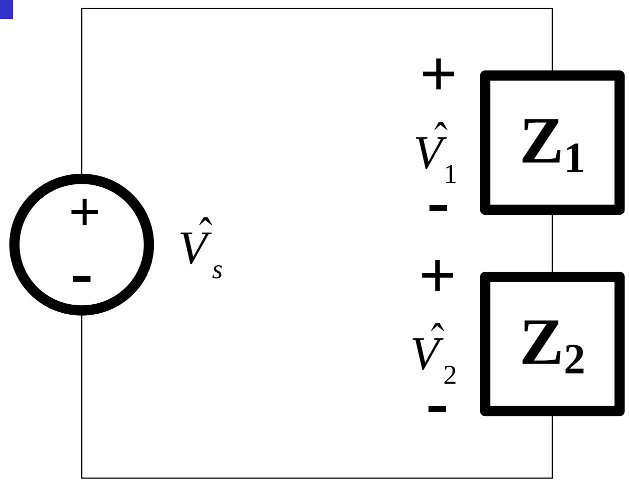


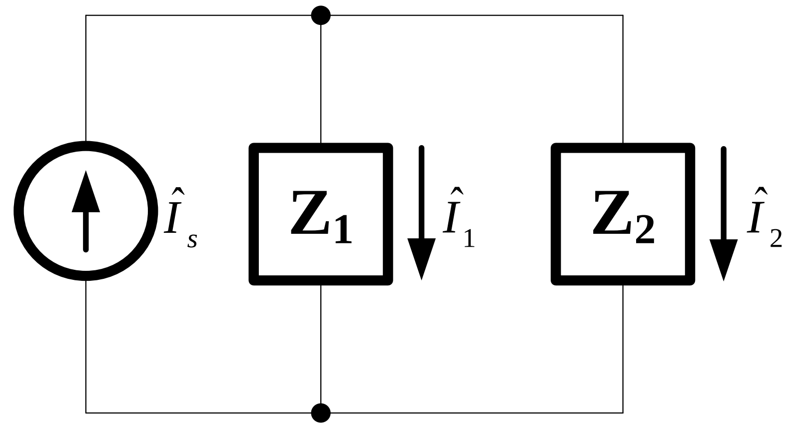
By KCL in phasor:

$$\hat{I} = \hat{I}_1 + \hat{I}_2 + \cdots + \hat{I}_N = \sum_{n=1}^{N} \hat{I}_n$$

$$= \sum_{n=1}^{N} \left(\frac{\hat{V}_{ab}}{Z_n} \right) = \hat{V}_{ab} \sum_{n=1}^{N} \left(\frac{1}{Z_n} \right) \Rightarrow \frac{1}{Z_{ab}} = \frac{\hat{I}}{\hat{V}_{ab}} = \sum_{n=1}^{N} \frac{1}{Z_n}$$

Voltage/Current Division

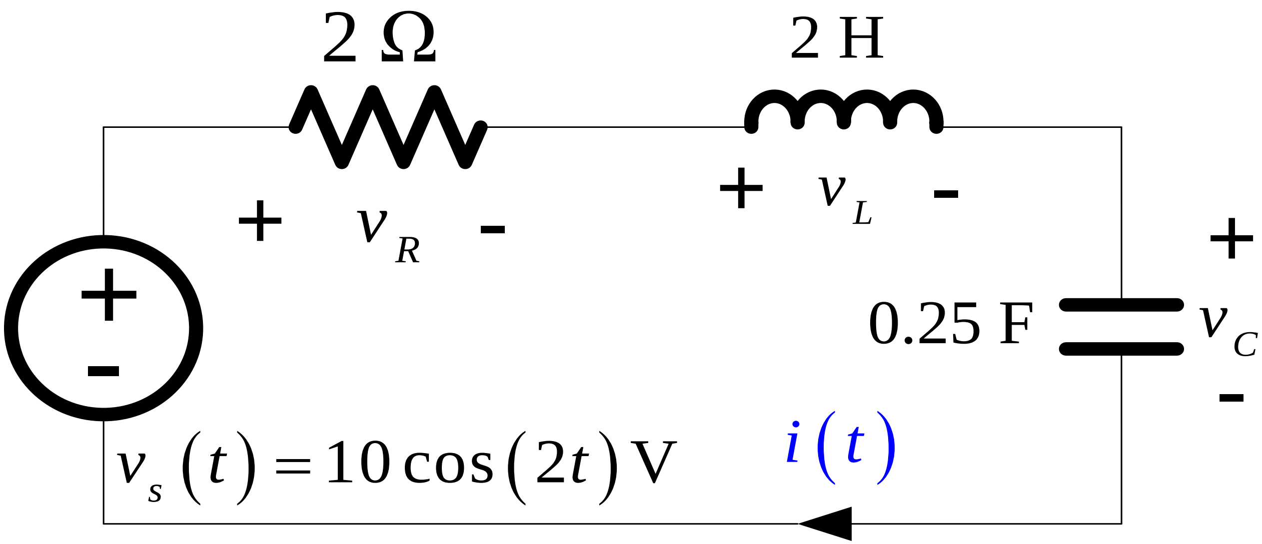




$$\begin{cases} \hat{V}_1 = \dfrac{Z_1}{Z_1 + Z_2}\hat{V}_s \\ \hat{V}_2 = \dfrac{Z_2}{Z_1 + Z_2}\hat{V}_s \end{cases}$$

$$\begin{cases} \hat{I}_1 = \dfrac{Z_2}{Z_1 + Z_2}\hat{I}_s \\ \hat{I}_2 = \dfrac{Z_1}{Z_1 + Z_2}\hat{I}_s \end{cases}$$

Example

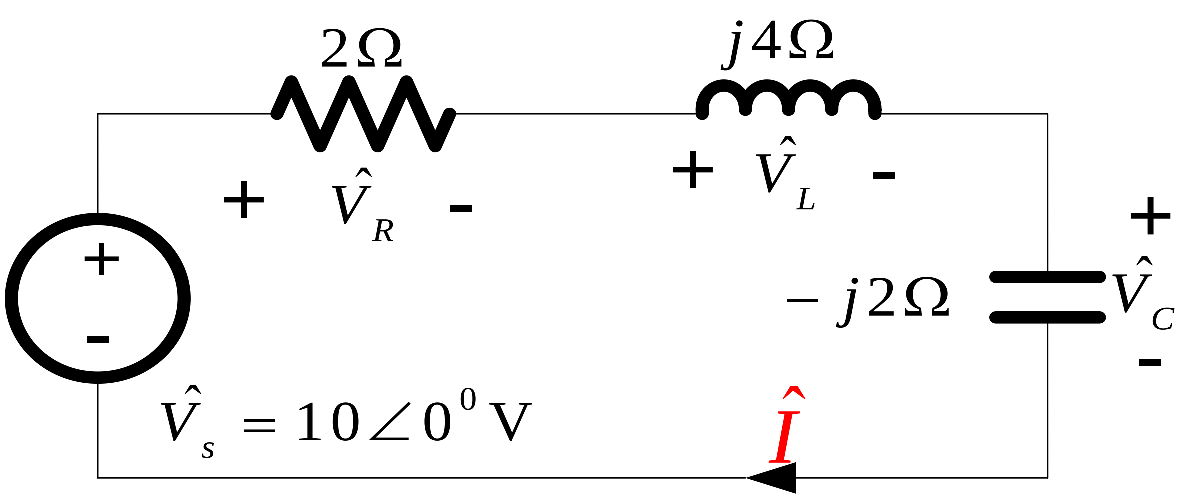


Find the steady-state current *i*(t) and voltages across all passive elements.

Solution:

$$\omega = 2rad/s \qquad \hat{V}_s = 10\angle 0^0\,\mathrm{V}, \quad Z_R = R = 2\Omega$$

$$Z_L = j\omega L = j4\Omega, \quad Z_C = \frac{1}{j\omega C} = -j2\Omega$$



Apply KVL to the loop: $\hat{V}_s = \left(Z_R + Z_L + Z_C\right)\hat{I}$

$$\hat{I} = \frac{\hat{V}}{Z_R + Z_L + Z_C} = \frac{10\angle 0^0}{2 + j4 - j2} = \frac{10\angle 0^0}{2 + j2}$$

$$= \frac{10\angle 0^0}{2\sqrt{2}\angle 45^0} = \frac{5}{\sqrt{2}}\angle -45^0\,\mathrm{A}$$

$$i(t) = \mathrm{Re}\left[\hat{I}e^{j\omega t}\right] = \mathrm{Re}\left[\frac{5}{\sqrt{2}}e^{j\left(-45^0\right)}e^{j\omega t}\right] = \frac{5}{\sqrt{2}}\cos\left(2t - 45^0\right)\mathrm{A}$$

$$\begin{cases} \hat{V}_R = Z_R \hat{I} = \dfrac{10}{\sqrt{2}} \angle -45^0 \mathrm{V} \\ \hat{V}_L = Z_L \hat{I} = j4 \times \dfrac{5}{\sqrt{2}} \angle -45^0 = \dfrac{20}{\sqrt{2}} \angle 45^0 \mathrm{V} \\ \hat{V}_C = Z_C \hat{I} = -j2 \times \dfrac{5}{\sqrt{2}} \angle -45^0 = \dfrac{10}{\sqrt{2}} \angle -135^0 \mathrm{V} \end{cases}$$

$$\begin{cases} v_R = \dfrac{10}{\sqrt{2}} \cos\left(2t - 45^0\right) \mathrm{V} \\ v_L = \dfrac{20}{\sqrt{2}} \cos\left(2t + 45^0\right) \mathrm{V} \\ v_C = \dfrac{10}{\sqrt{2}} \cos\left(2t - 135^0\right) \mathrm{V} \end{cases}$$

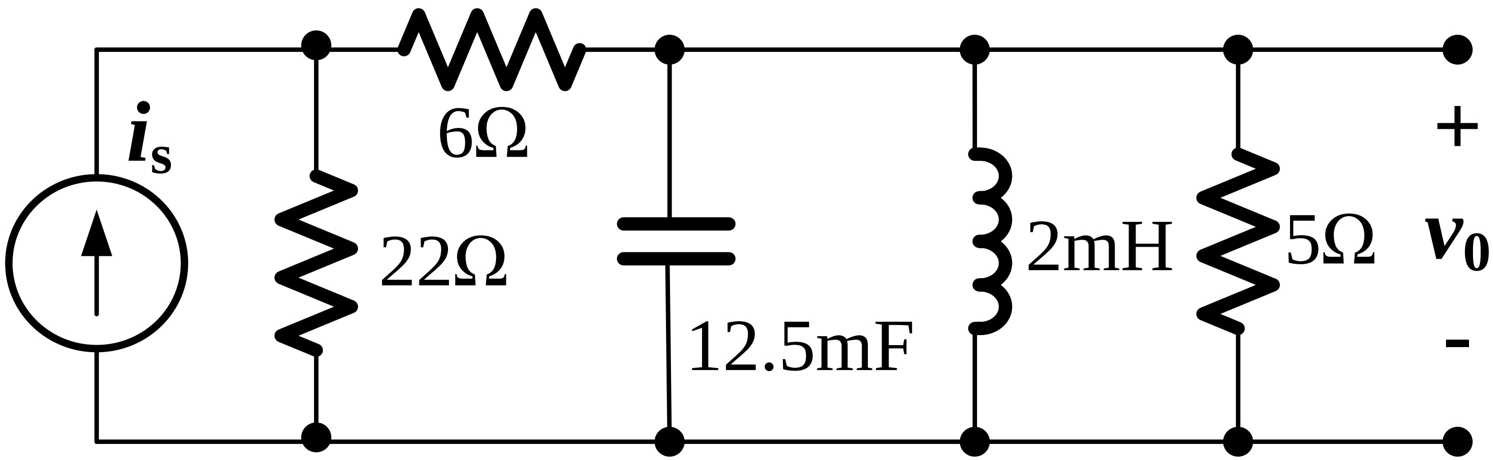
$$\omega = 2rad/s$$

Steps of Sinusoidal Steady State Analysis:

1. Represent all voltage/current source by phasors;
2. Find the phasor model of the given circuit by phasor transform;

3. **Find the sinusoidal steady-state response in frequency domain by using KCL/KVL, or mesh-current/node-voltage method, or circuit theorems;**
4. **Find the corresponding response in time domain by inverse phasor transform if necessary.**

Example



The circuit shown above is operating in the sinusoidal steady state. Find the voltage v_0 if i_s=3cos(200t)mA.

$$v_0 = 10\cos(200t)\,\mathrm{mV}$$

Solution

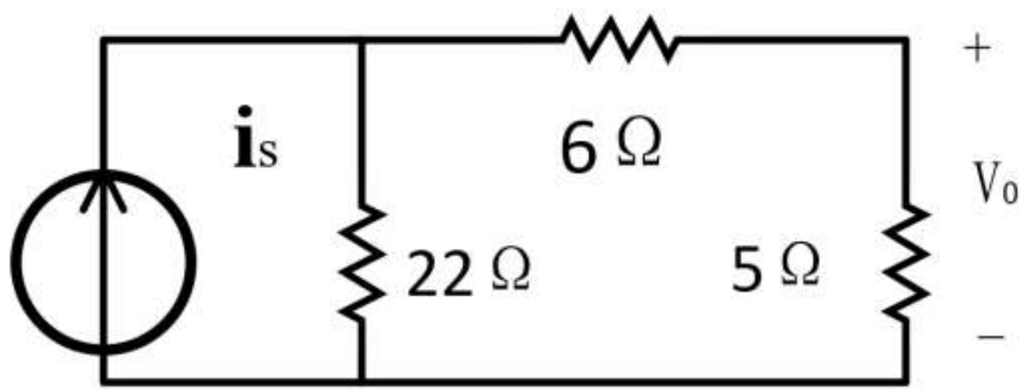
$w = 200 rad/s$， R=5Ω

$$Z_c = \frac{1}{jwc} = \frac{1}{j\times 200\times 12.5\times 10^{-3}} = -0.4j\Omega$$

$$Z_L = jwl = j \times 200 \times 2 \times 10^{-3} = 0.4j\Omega$$

$\frac{1}{Z} = \frac{1}{Z_c} + \frac{1}{Z_L} + \frac{1}{Z_R}$,则Z=$Z_R$,那么并联电阻Z=5Ω

电路可以等效为右图的电路：



Solution

电路为22Ω电阻与11Ω电阻并联电路

则流过v_0的电流为

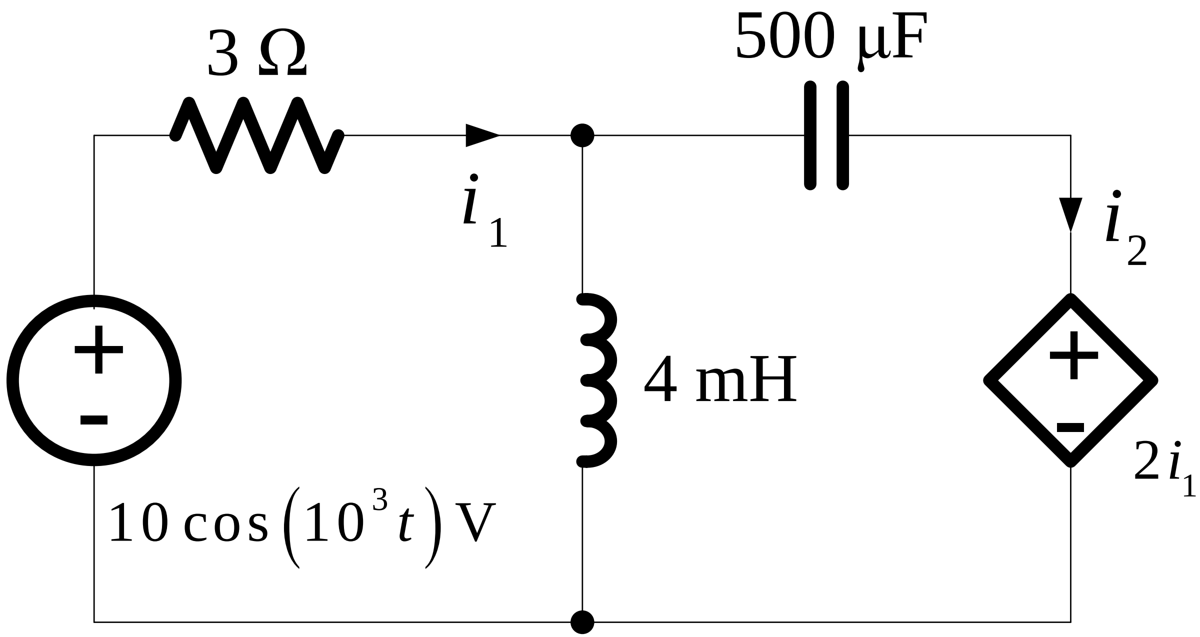
$$i_{v_0}=\frac{22}{22+6+5}i_s=\frac{2}{3}i_s=2cos(200t)mA$$

$$v_0=Ri_{v_0}=10cos(200t)mV$$

8-7 Mesh-Current and Node-Voltage Method

- Mesh-Current Method in Frequency Domain
- Node-Voltage Method in Frequency Domain

Example



Find the current of i_1(t) and i_2(t) for the circuit shown above by mesh-current method.

Solution:

$$\omega = 10^3 rad / s$$

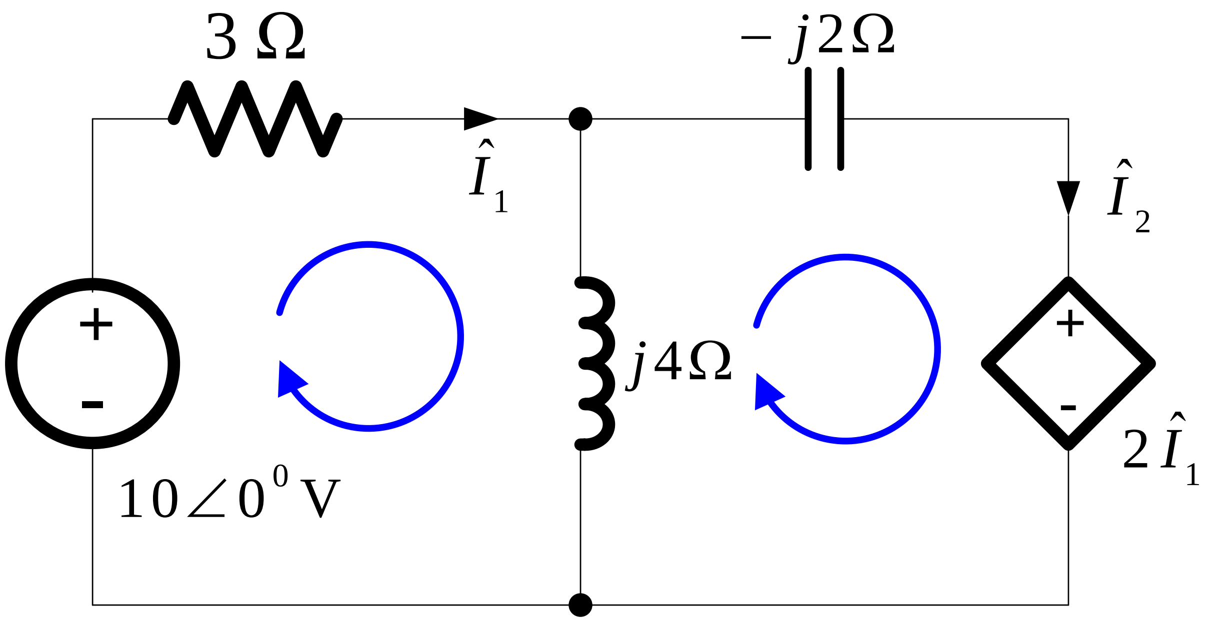
$$\hat{V}_s = 10\angle 0^0$$

$$Z_R = R = 2\Omega$$

$$Z_L = j\omega L = j10^3 \times 4 \times 10^{-3} = j4\Omega$$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j10^3 \times 500 \times 10^{-6}} = -j2\Omega$$

The frequency domain equivalent circuit represented by phasors is:



By mesh current method:

$$\begin{cases} 3\hat{I}_1 + j4\left(\hat{I}_1 - \hat{I}_2\right) = 10\angle 0^0 \\ j4\left(\hat{I}_2 - \hat{I}_1\right) - j2\hat{I}_2 = -2\hat{I}_1 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{I}_1 = \dfrac{10}{7 - j4} = 1.24\angle 29.7^0\,\text{A} \\ \hat{I}_2 = \dfrac{20 + j30}{13} = 2.77\angle 56.3^0\,\text{A} \end{cases}$$

$$\omega = 10^3 rad/s$$

The required currents of i_1(t) and i_2(t) are:

$$\begin{cases} i_1 = 1.24\cos\left(10^3 t + 29.7^0\right)\mathrm{A} \\ i_2 = 2.77\cos\left(10^3 t + 56.3^0\right)\mathrm{A} \end{cases}$$

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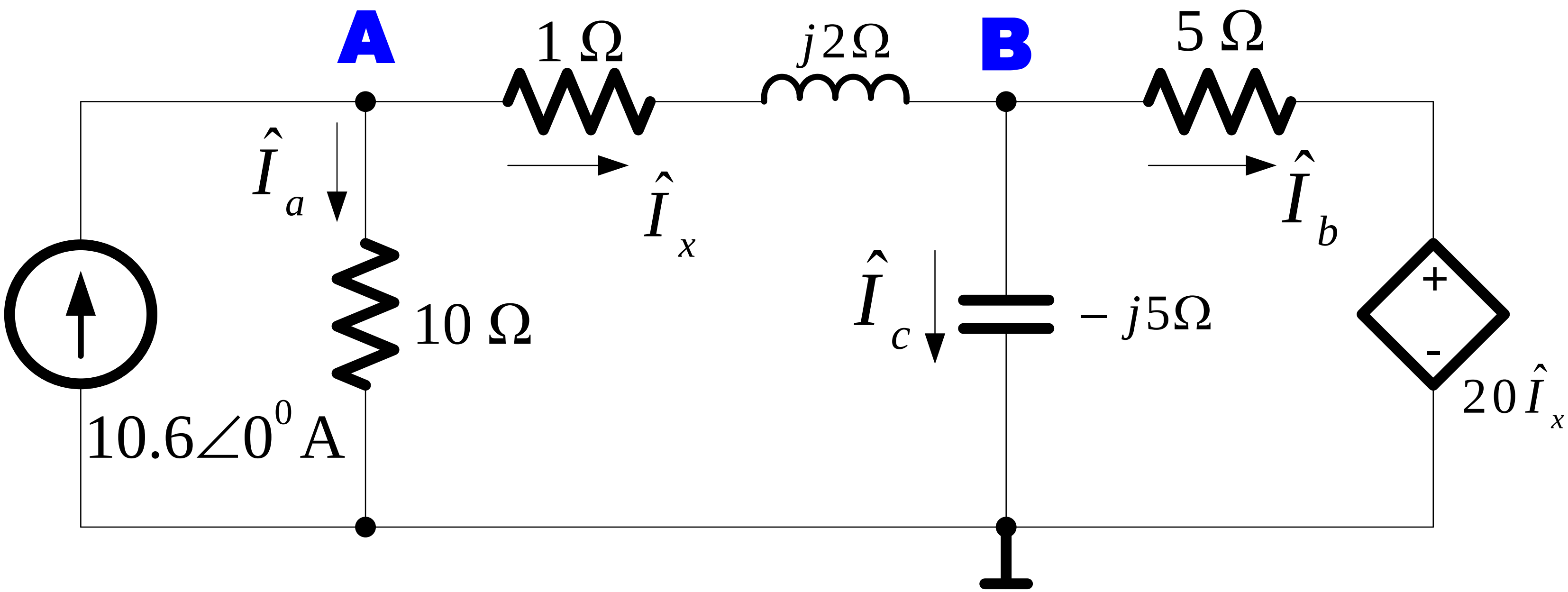
Example



Find the current of $\hat{I}_a$, $\hat{I}_b$, and $\hat{I}_c$ for the circuit shown above by node-voltage method.

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Solution:

Apply KCL for node A and node B:

$$\begin{cases} \dfrac{\hat{V}_1}{10} + \dfrac{\hat{V}_1 - \hat{V}_2}{1 + j2} - 10.6 = 0 \\ \dfrac{\hat{V}_2 - \hat{V}_1}{1 + j2} + \dfrac{\hat{V}_2}{-j5} + \dfrac{\hat{V}_2 - 20\hat{I}_x}{5} = 0 \\ \hat{I}_x = \dfrac{\hat{V}_1 - \hat{V}_2}{1 + j2} \end{cases} \Rightarrow \begin{cases} \hat{V}_1 = 68.4 - j16.8\text{V} \\ \hat{V}_2 = 68 - j26\text{V} \end{cases}$$

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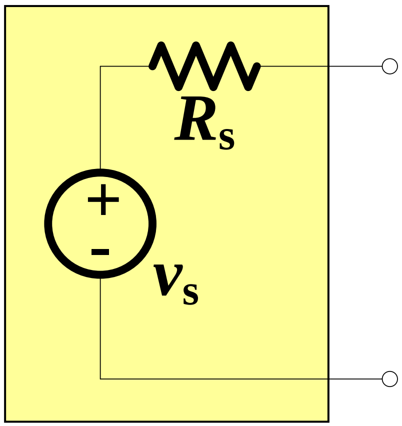
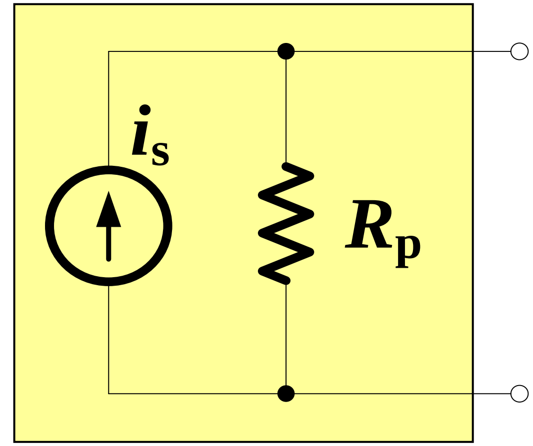
Hence the required currents are:

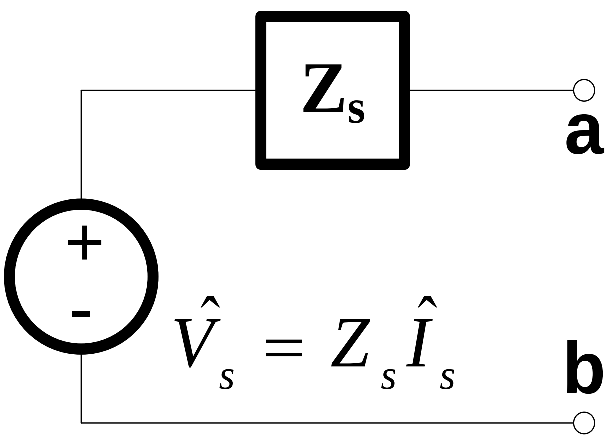
$$\begin{cases} \hat{I}_a = \dfrac{\hat{V}_1}{10} = 6.84 - j1.68\text{A} \\ \hat{I}_x = \dfrac{\hat{V}_1 - \hat{V}_2}{1 + j2} = 3.76 + j1.68\text{A} \\ \hat{I}_b = \dfrac{\hat{V}_2 - 20\hat{I}_x}{5} = -1.44 - j11.92\text{A} \\ \hat{I}_c = \dfrac{\hat{V}_2}{-j5} = 5.2 + j13.6\text{A} \end{cases}$$

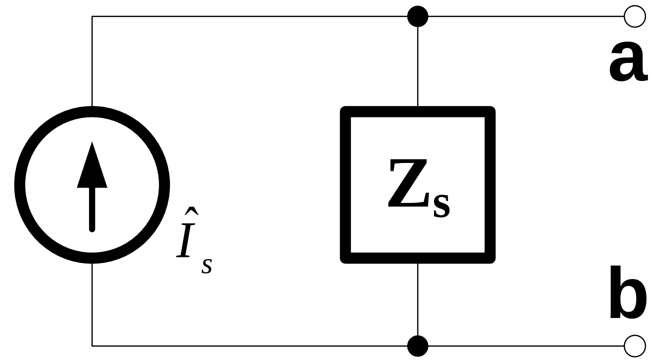
8-8 Source Transformation



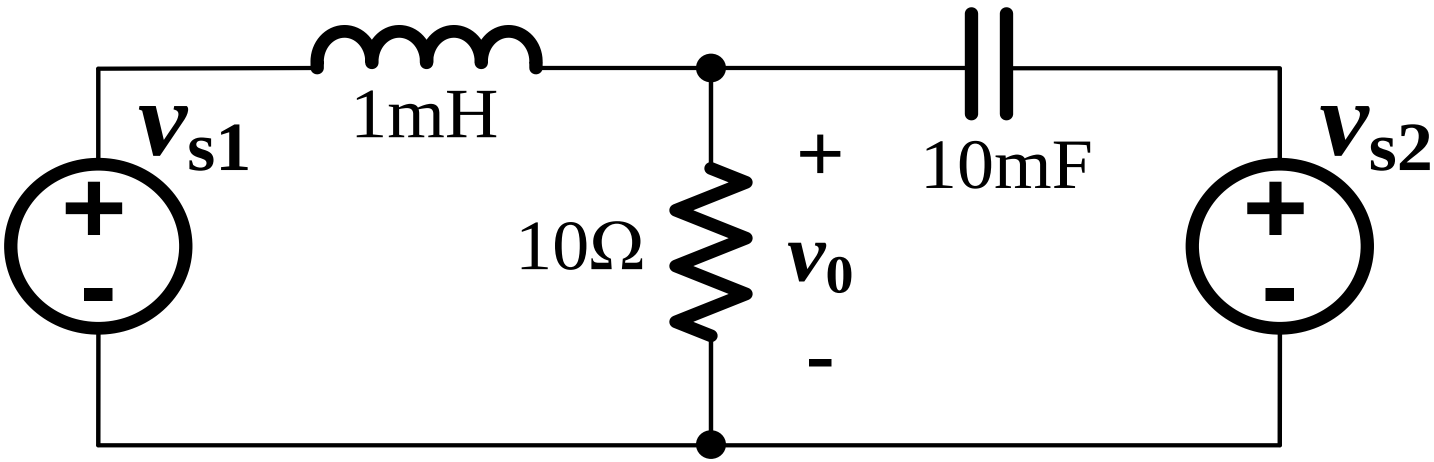
$$\begin{cases} R_s = R_p \\ v_s = i_s R_s \end{cases}$$







Example



Use source transformation to find the steady state expression v_0 if v_{s1}=20cos(200t)V and v_{s2}=50cos(200t)V.

$$v_0 = 0$$

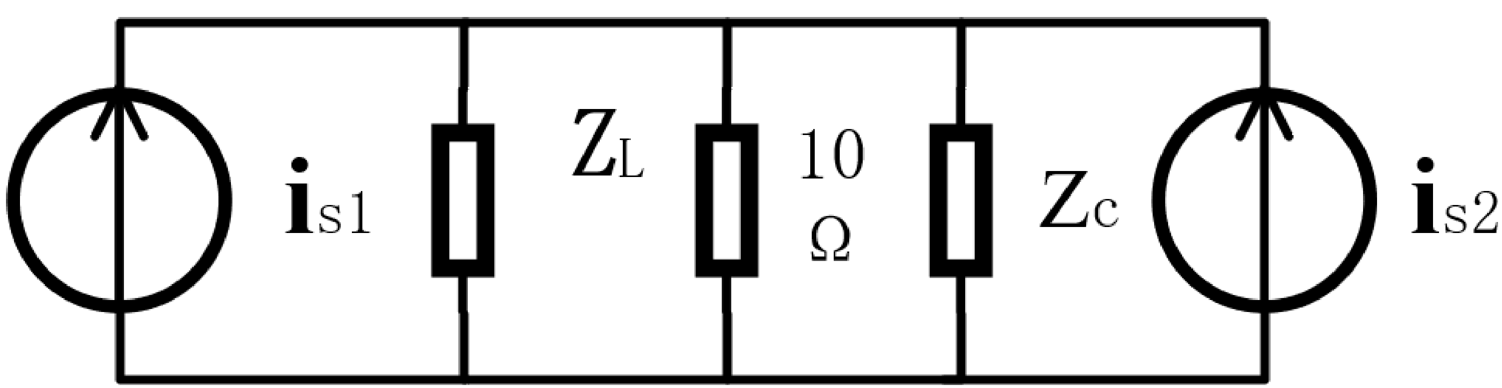
Solution

$$w = 200rad/s$$

$$Z_L = jwl = j \times 200 \times 1 \times 10^{-3} = 0.2j\Omega$$

$$Z_c = \frac{1}{jwc} = \frac{1}{j\times200\times10\times10^{-3}} = -0.5j\Omega$$

经过电源变换后的电路图如下：



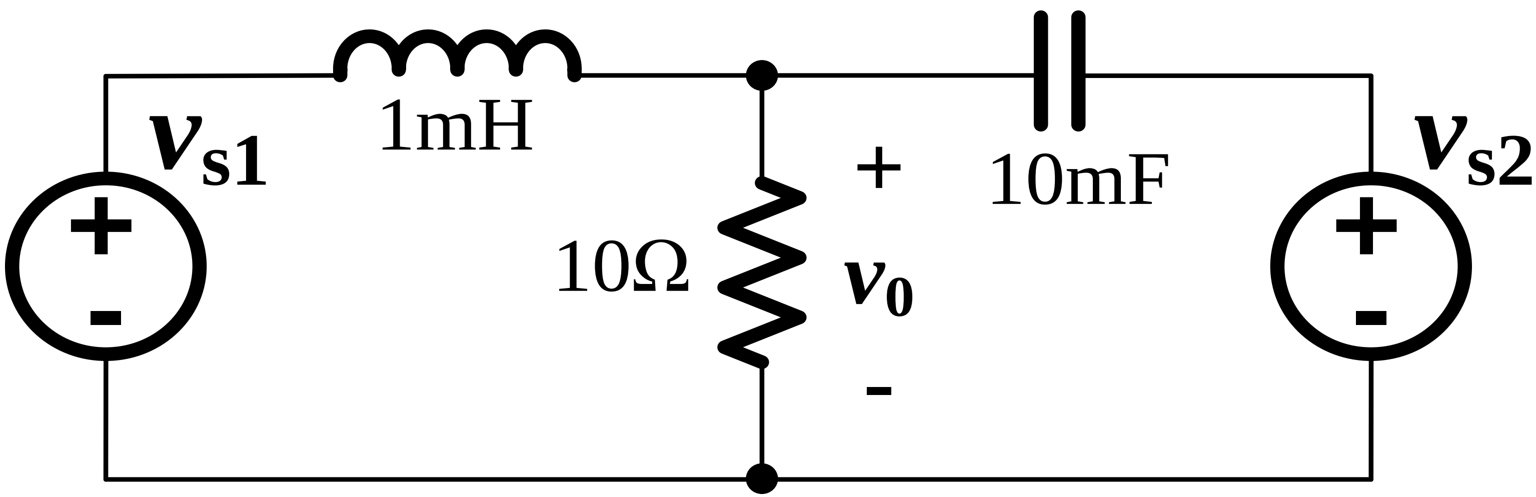
其中$i_{s1}=\frac{v_{s1}}{Z_L}=\frac{20cos(200t)}{0.2j}=-j100cos(200t)A$

$$i_{s2}=\frac{v_{s2}}{Z_c}=\frac{50cos(200t)}{-0.5j}=j100cos(200t)A$$

电流源合并则经过v_0的电流为i =0A

所以$v_0=iR=0V$

Superposition Theorem



You can try it by Superposition Theorem.

Solution

叠加定理是各个电源单独作用时对电路的影响，**当电源单独作用时其他电压源看作短路，其他电流源看作断路。**

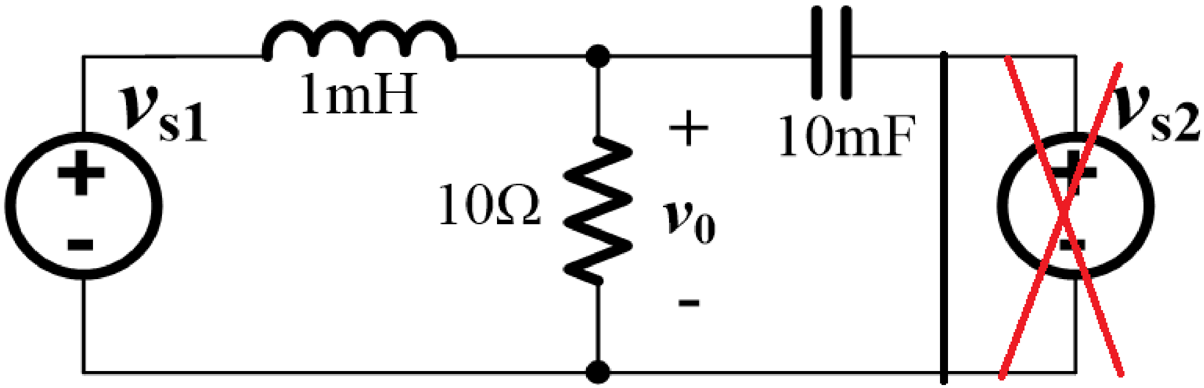
电容与电感的电阻为：

$$w = 200rad/s$$

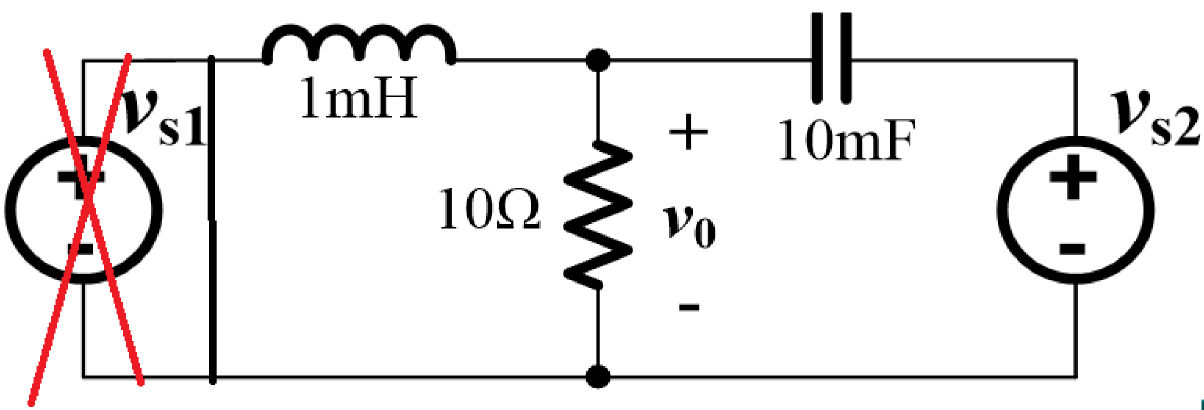
$$Z_L = jwl = j \times 200 \times 1 \times 10^{-3} = 0.2j\Omega$$

$$Z_c = \frac{1}{jwc} = \frac{1}{j\times200\times10\times10^{-3}} = -0.5j\Omega$$

当v_{s1}单独作用时：



当v_{s2}单独作用时：



当v_{s1}单独作用时v_0等于电阻与电容的并联电压

电阻与电容的并联电阻为：$Z_1=\frac{Z_R Z_c}{Z_R+Z_c}$，

当v_{s1}作用时$v_0=\frac{Z_1}{Z_1+Z_L}v_{s1}=\frac{250cos(200t)}{3j-0.1}$V

当v_{s2}单独作用时v_0等于电阻与电感的并联电压

电阻与电感的并联电阻为：$Z_2=\frac{Z_R Z_L}{Z_R+Z_L}$，

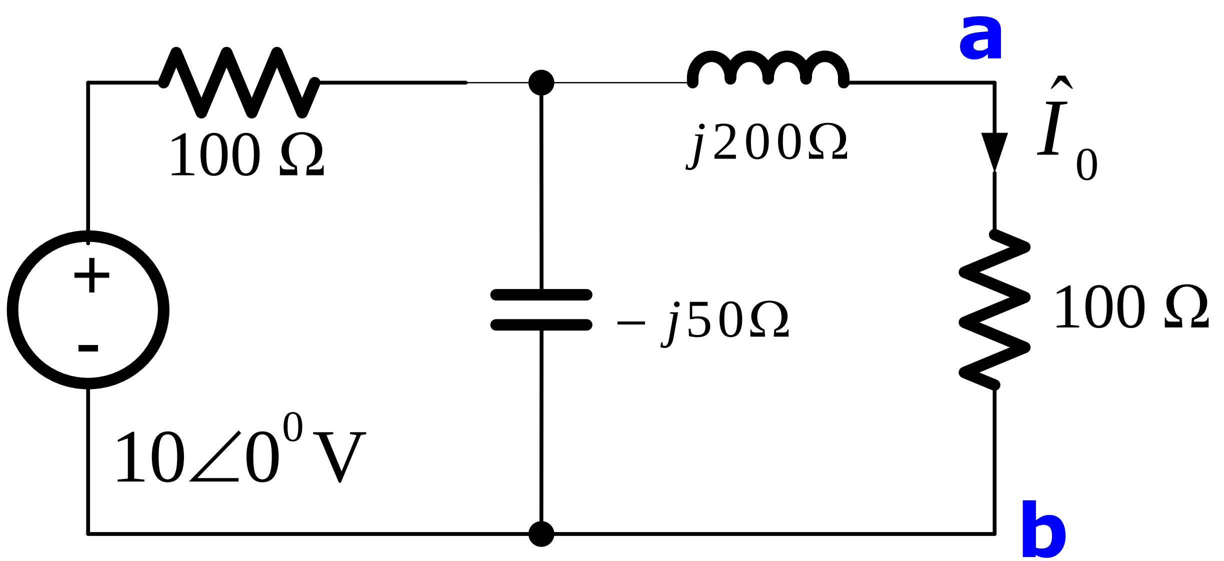
当v_{s2}作用时$v_0=\frac{Z_2}{Z_2+Z_c}v_{s1}=\frac{250cos(200t)}{0.1-3j}$V

所以v_0在v_{s1}和v_{s2}同时作用时为0V

8-9 Thévenin Equivalents and Norton Equivalents

- Thévenin Equivalents in Frequency Domain
- Norton Equivalents in Frequency Domain

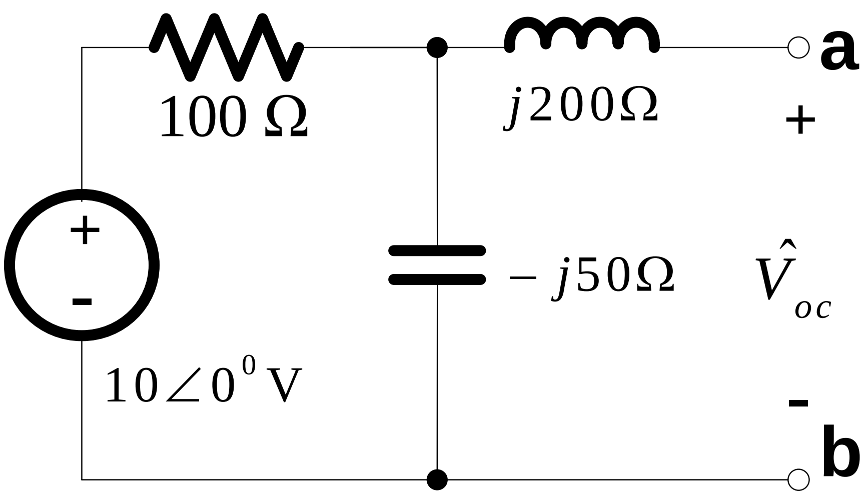
Example



Find the current of $\hat{I}_0$ for the circuit shown above by Thévenin equivalent.

Solution:

1. **Find the open circuit voltage between terminal a and b:**



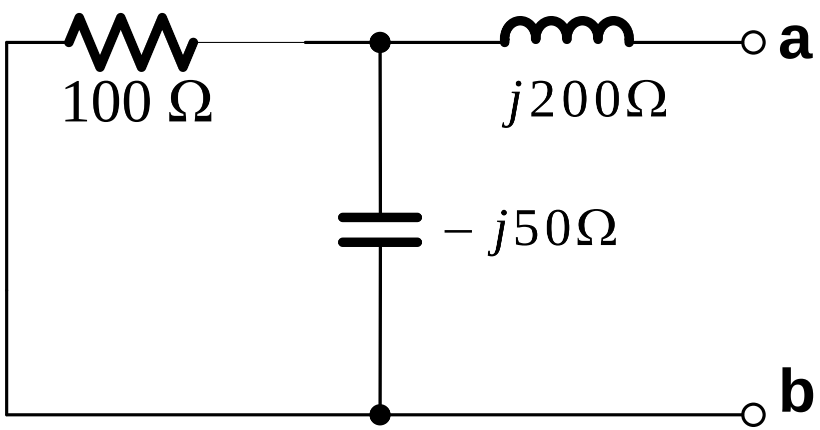
$$\hat{V}_{oc} = \frac{-j50}{-j50+100} \times 10\angle 0^0$$

$$= \frac{-j}{-j+2} \times 10\angle 0^0$$

$$= \frac{\angle -90^0}{\sqrt{5}\angle -26.6^0} \times 10\angle 0^0$$

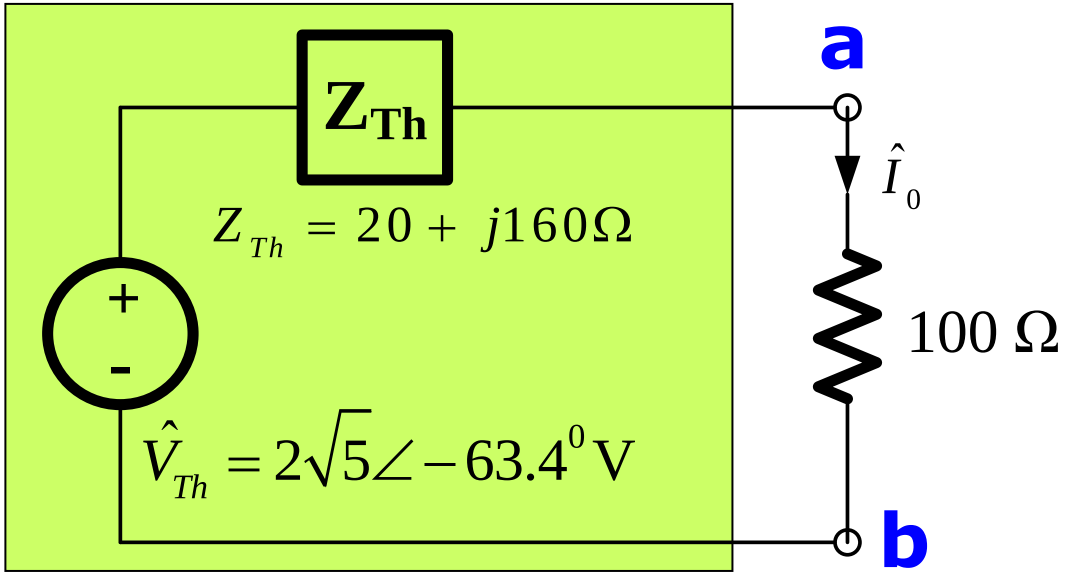
$$= 2\sqrt{5}\angle -63.4^0\,\mathrm{V}$$

2. **Find the Thévenin equivalent impedance between terminal a and b:**



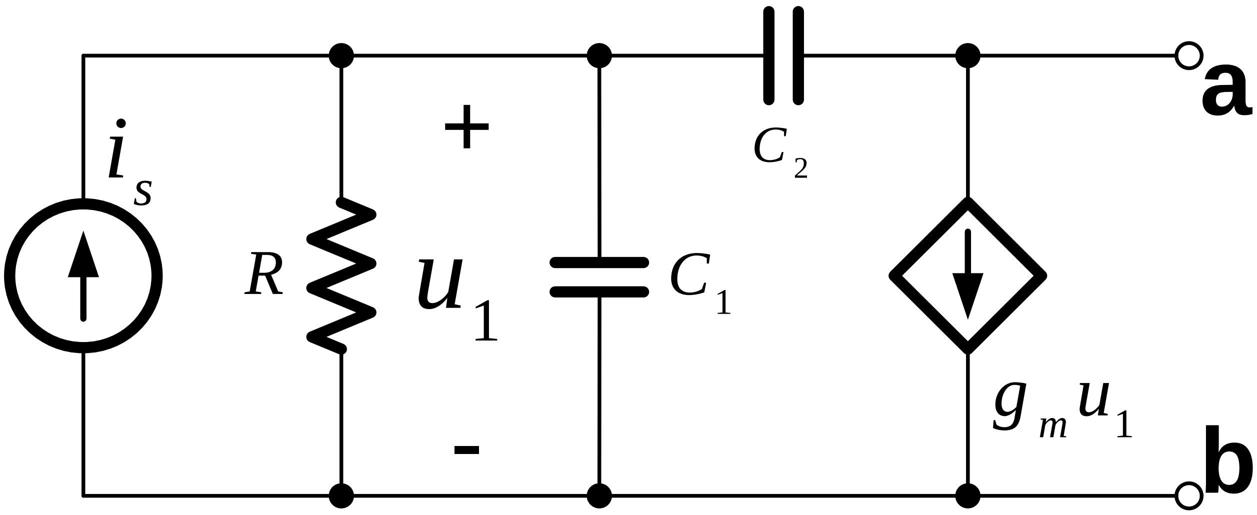
$$Z_{Th} = j200 + \frac{100(-j50)}{100 - j50} = (20 + j160)\Omega$$

Hence, the Thévenin equivalent between terminal a and b is:



$$\hat{I}_0 = \frac{\hat{V}_{Th}}{Z_{Th} + 100\Omega} = \frac{2\sqrt{5}\angle -63.4^0}{20 + j160 + 100} = 0.0224\angle -116.53^0\,\mathrm{A}$$

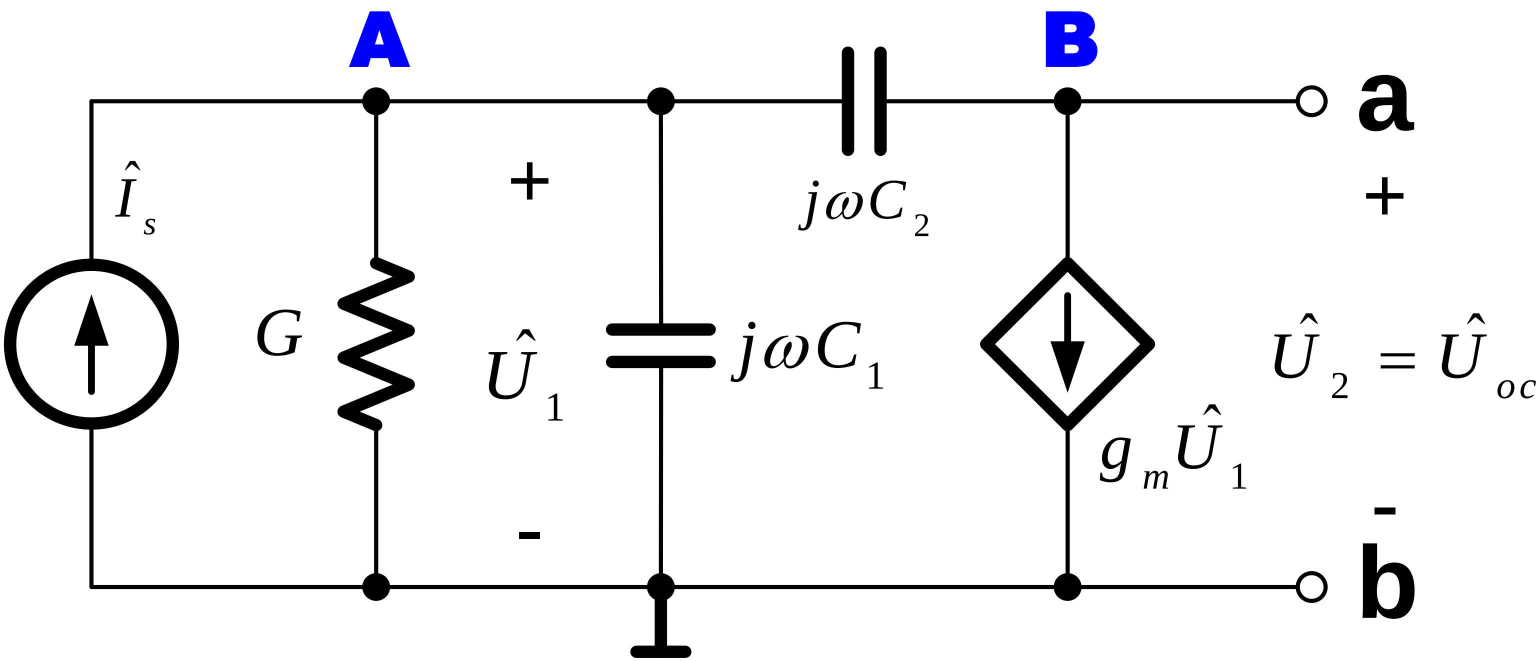
Example



Find the Thévenin equivalent for the two terminals of a and b in the circuit shown above.

Solution:

The frequency domain equivalent circuit represented by phasors is:



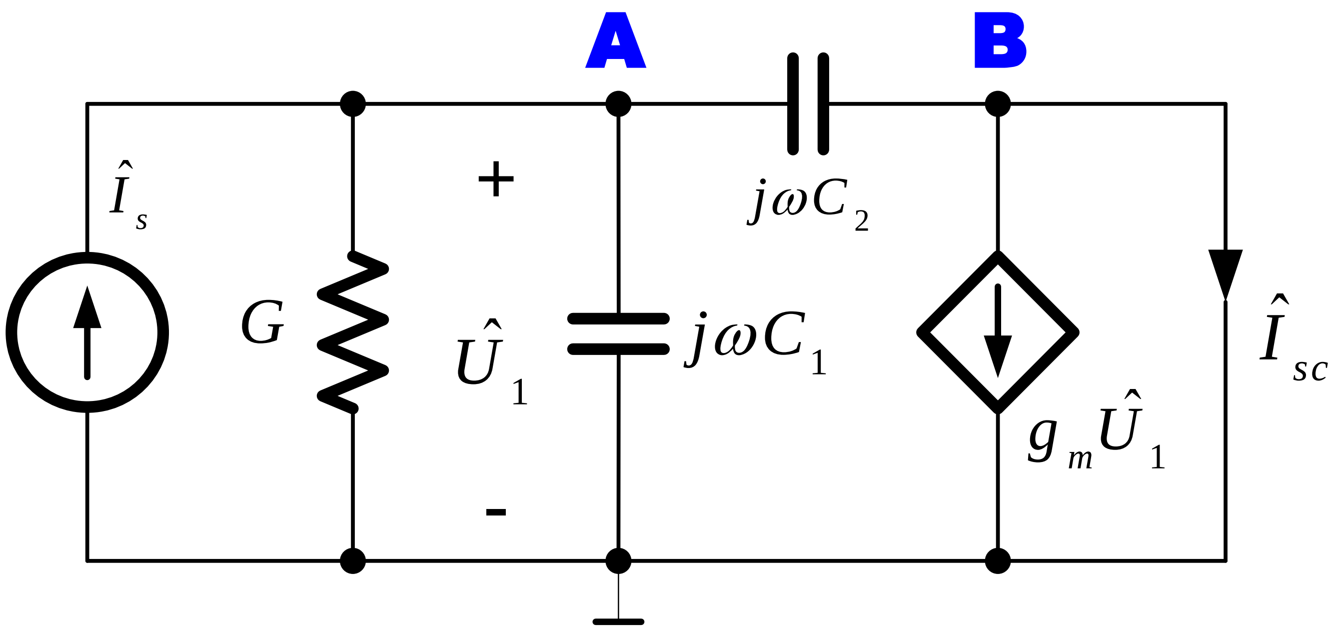
1. **Find the open circuit voltage between the terminal a and b:**

Apply node voltage method:

$$\begin{cases} G\hat{U}_1 + j\omega C_1\hat{U}_1 + j\omega C_2\left(\hat{U}_1 - \hat{U}_2\right) = \hat{I}_s \\ j\omega C_2\left(\hat{U}_2 - \hat{U}_1\right) + g_m\hat{U}_2 = 0 \end{cases}$$

Then, $$\hat{U}_{oc} = \hat{U}_2 = \frac{\left(j\omega C_2 - g_m\right)\hat{I}_s}{-\omega^2 C_1 C_2 + j\omega C_2\left(g_m + G\right)}$$

2. **Find the short circuit current between the terminal a and b:**



Apply node-voltage method for the circuit:

$$\begin{cases} \hat{I}_{sc} = j\omega C_2 \hat{U}_1 - g_m \hat{U}_1 \\ \hat{I}_s = G\hat{U}_1 + j\omega C_1 \hat{U}_1 + j\omega C_2 \hat{U}_1 \end{cases}$$

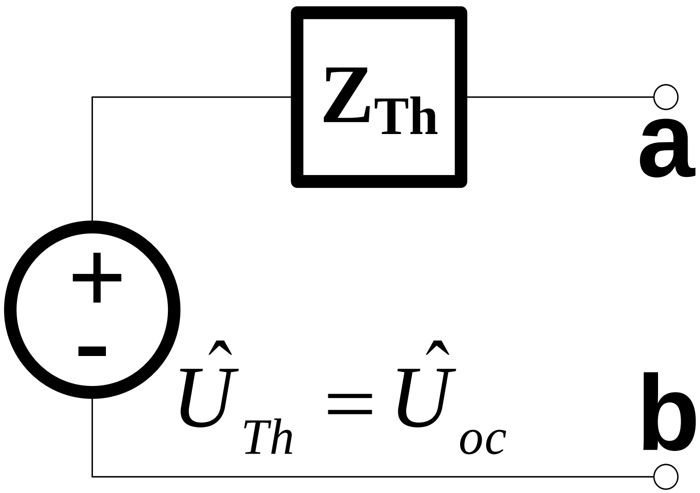
$$\Rightarrow \quad \hat{I}_{sc} = \frac{\left(j\omega C_2 - g_m\right)\hat{I}_s}{G + j\omega\left(C_1 + C_2\right)}$$

3. Find the Thévenin Equivalent Impedance:

$$\hat{U}_{Th} = \hat{U}_{oc} = \frac{\left(j\omega C_2 - g_m\right)\hat{I}_s}{-\omega^2 C_1 C_2 + j\omega C_2\left(g_m + G\right)}$$

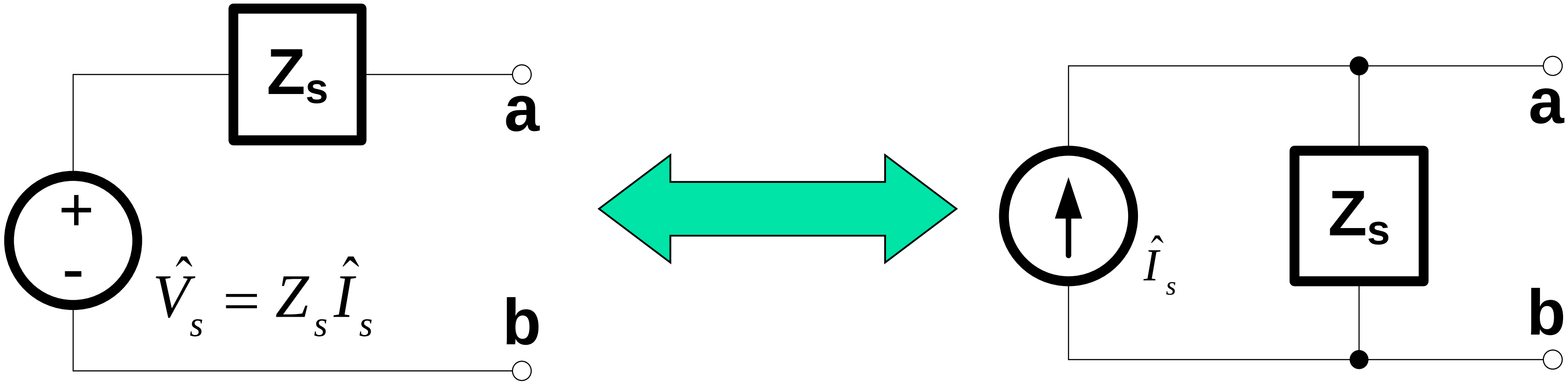
$$Z_{Th} = \frac{\hat{U}_{oc}}{\hat{I}_{sc}} = \frac{G + j\omega\left(C_1 + C_2\right)}{-\omega^2 C_1 C_2 + j\omega C_2\left(g_m + G\right)}$$

Hence, the Thévenin equivalent for the two terminals of a and b is:



$$
\begin{cases}
\hat{U}_{Th} = \dfrac{\left(j\omega C_2 - g_m\right)\hat{I}_s}{-\omega^2 C_1 C_2 + j\omega C_2\left(g_m + G\right)} \\[2ex]
Z_{Th} = \dfrac{G + j\omega\left(C_1 + C_2\right)}{-\omega^2 C_1 C_2 + j\omega C_2\left(g_m + G\right)}
\end{cases}
$$

Norton Equivalents



$$\hat{V}_s = Z_s \hat{I}_s$$

$\hat{I}_s$

Norton equivalent can be derived from Thévenin equivalent by source transformation; and vice versa.

Summary of Chapter 8

- Conception of phasor;
- Phasor transform and inverse phasor transform
- Circuit in frequency domain by phasors
- KCL and KVL in frequency domain
- Node-voltage and mesh-current method, Source transformation, Superposition, Thévenin and Norton Equivalents in frequency domain