Chapter 10-11 Rotation and Rigid Bodies

§ 1 Kinematics of Rigid Bodies

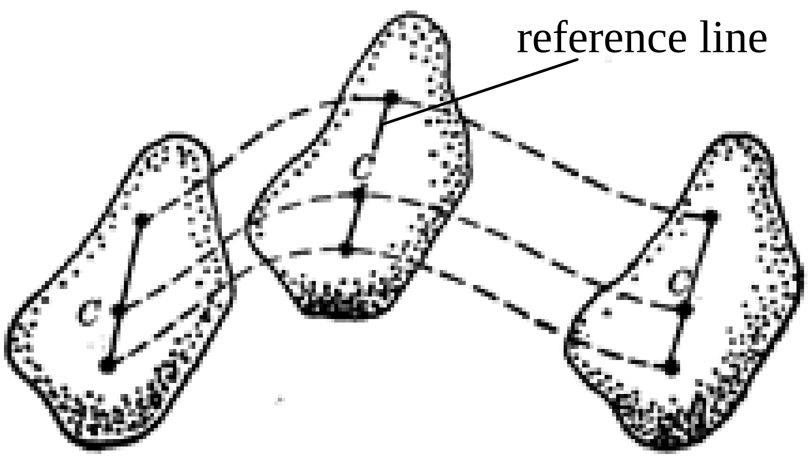
P234, p239-240

- Rigid Body
  - Definition: is the body has a perfectly definite and unchanging shape and size. The distance between any two arbitrary points in the body is a constant.

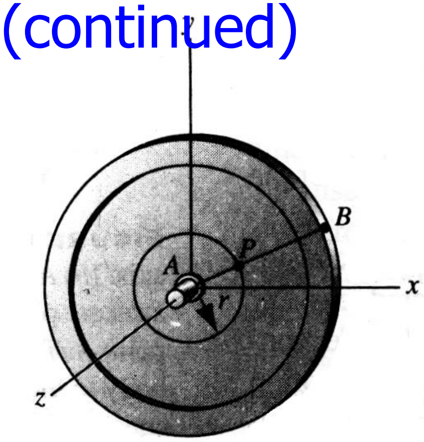
$$\left|\vec{r}_i - \vec{r}_j\right| = d_{ij} = \text{constant}$$

    - Idealized model: the external forces that act on the real-world bodies can deform them —— stretching, twisting, and squeezing.
    - If these deformations are so little that can be ignored, such bodies can be treated as rigid bodies.
  - Why introduce the rigid body model?
    - Any body can be viewed as a system of *N* numbers of particles.
    - Generally need 3N motional equations to describe its motion.
    - The rigid body model simplifies the description of body's motion.

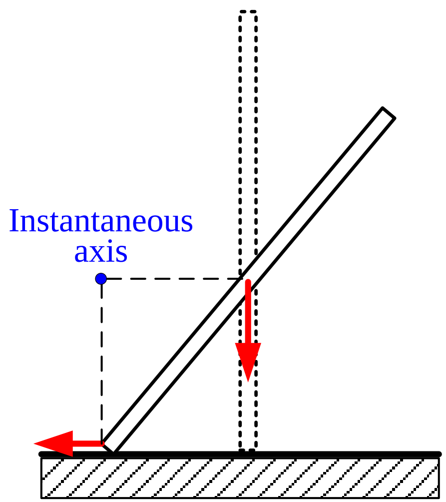
- Translational and Rotational Motion of Rigid Bodies
  - Translational motion of a rigid body
    - The trajectories of all the points of a rigid body are the same, or the line between any two points of a rigid body keeps its orientation unchanged all the time.

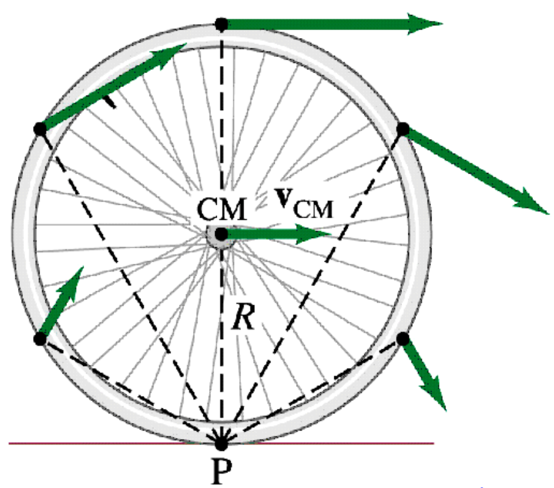


- Translational and Rotational Motion of Rigid Bodies (continued)
  - Rotational motion of a rigid body
    - Rotation about a fixed axis: every point of the body moves in a circular path. The centers of these circles must lie on a common straight line called the axis of rotation.

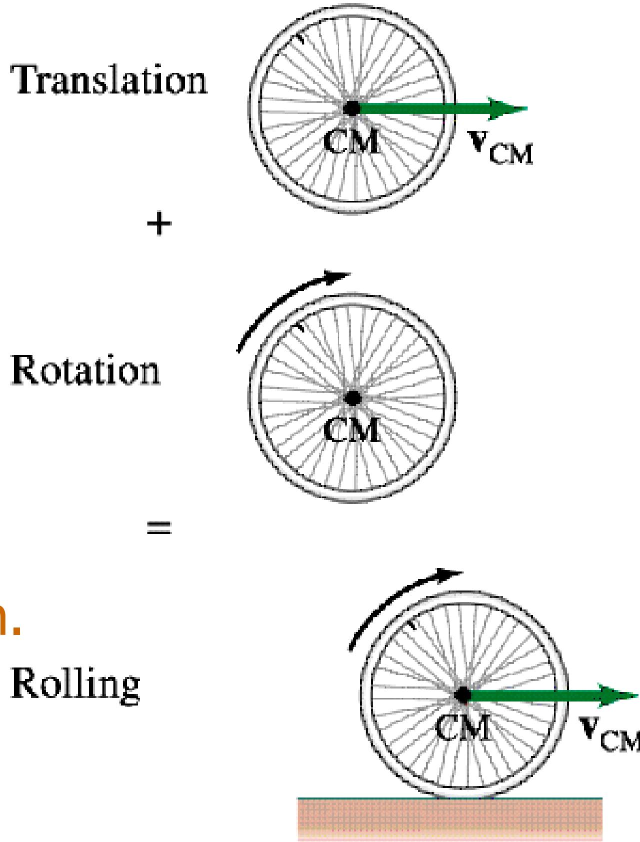


    - Rotation about a non-fixed axis: the position or the orientation of the rotational axis varied with time. An instantaneous rotational axis must exist that the instantaneous velocity of any point in the body is perpendicular to the axis.



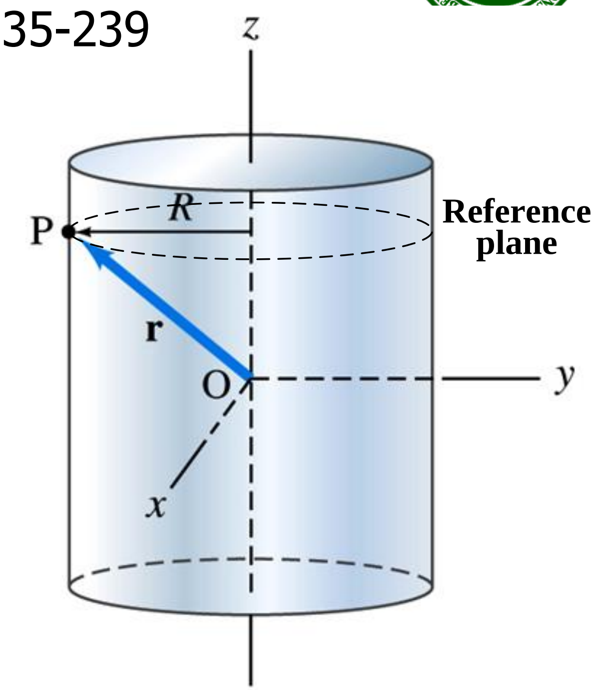


- Translational and Rotational Motion of Rigid Bodies (continued)
  - The general motion of a rigid body will include both rotational and translational components.
  - Why does the rigid body model simplifies the description of body's motion?
    - Generally, N particle system needs 3N motional equation to describe its motion.
    - But for a rigid body, we only need 6 coordinates:
      - Three to locate the center of mass.
      - Two angles to orient the axis of rotation.
      - One angle to describe rotation about the axis.



§ 2 Angular Quantities for rigid bodies

P235-239

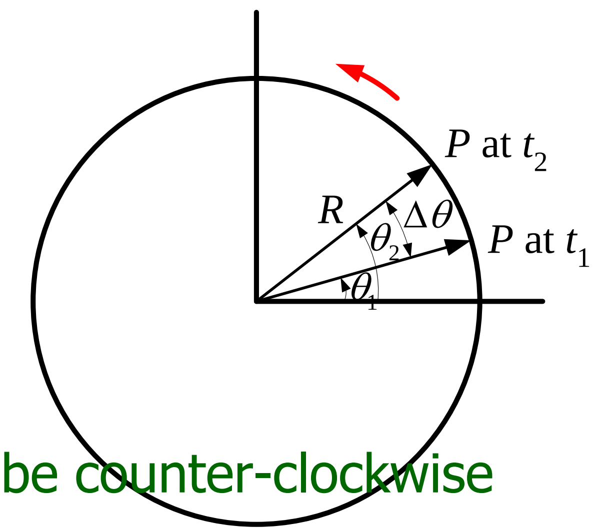


- Angular velocity
  - Rotational radius ***R***
    - The perpendicular distance of point P in the reference plane from the axis of rotation.
  - Angular position and angular displacement
    - Angular position: θ_1, θ_2
    - Angular displacement: $\Delta\theta = \theta_2 - \theta_1$.
  - Angular velocity
    - Average angular velocity:

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

    - Instantaneous angular velocity:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$



    - Choose the positive sense of the rotation to be counter-clockwise

Angular velocity

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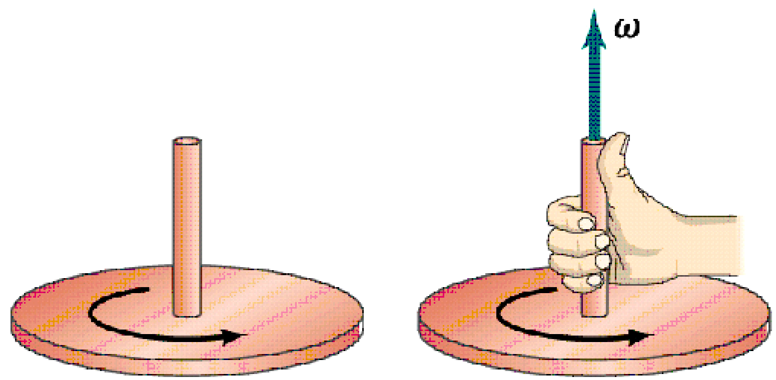
- Angular velocity as a vector
  - The direction of angular velocity vector —— right-hand rule
    - The right-hand rule: when the fingers of right hand curl in direction of rotation, the thumb position is the direction of $\vec{\omega}$

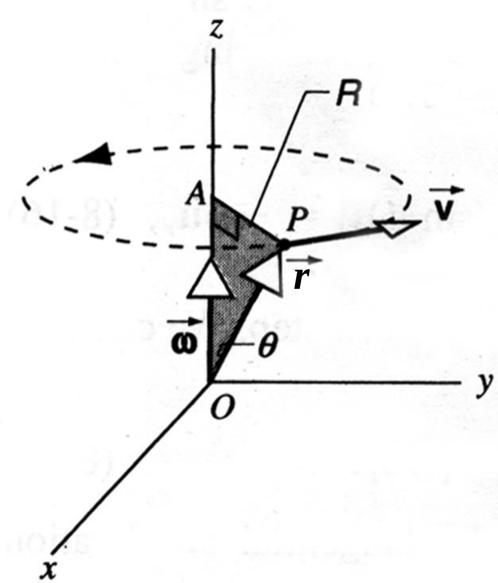


  - Relationship between linear and angular velocities (only for rotation about a fixed axis)
    - Magnitude:

$$v = \frac{ds}{dt} = \frac{d(R\theta)}{dt} = R\frac{d\theta}{dt} = R\omega$$

    - Considering the direction:

$$\vec{v} = \vec{\omega} \times \vec{R} = \vec{\omega} \times \vec{r}$$



- Angular acceleration
  - Average angular acceleration:

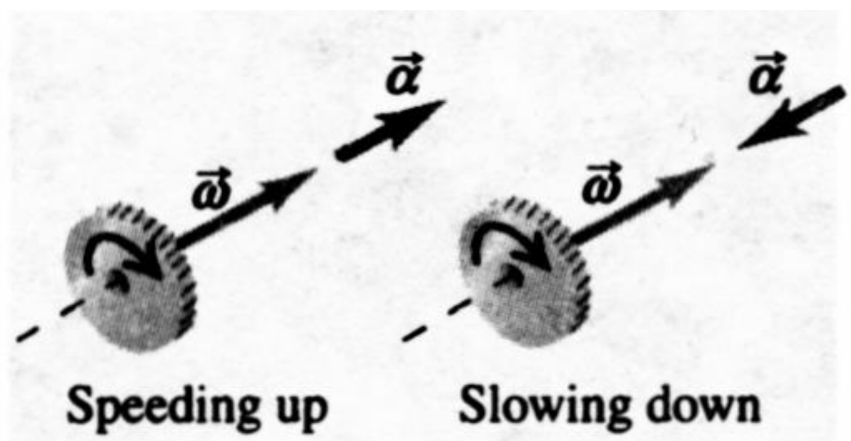
$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

  - Instantaneous angular acceleration

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt}$$

  - Angular acceleration as a vector

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

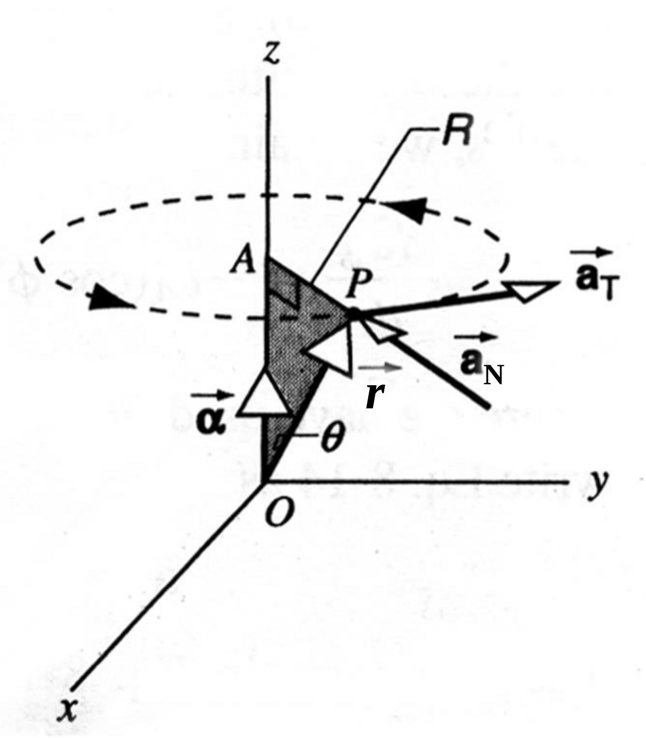


- Relationship between linear and angular accelerations (only for rotation about a fixed axis)

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\vec{\omega}\times\vec{R}\right) = \frac{d\vec{\omega}}{dt}\times\vec{R} + \vec{\omega}\times\frac{d\vec{R}}{dt}$$

$$= \vec{\alpha}\times\vec{R} + \vec{\omega}\times\vec{v}$$

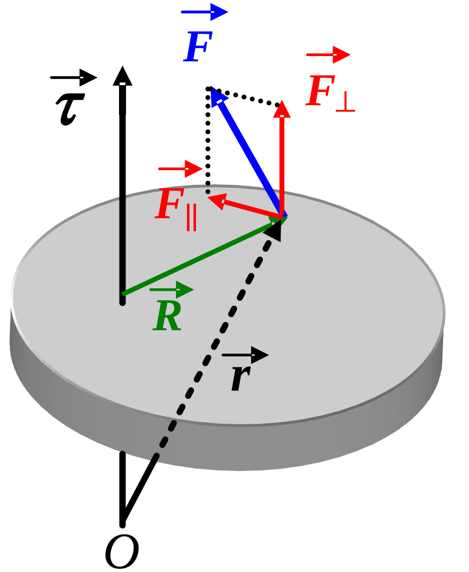
- Tangential acceleration:
  - Magnitude: $a_t = R\alpha$
- Normal acceleration:
  - Magnitude: $a_n = \omega v = \omega^2 R$



§ 3 The Rotational Form of Newton's Second Law

P280-281, p241-248

- The torque about a fixed axis —— torque component along the axis of rotation
  - The force $\vec{F}$ can be resolved into the parallel component $\vec{F}_{\parallel}$ lying in the reference plane, and the perpendicular component $\vec{F}_{\perp}$.
    - The perpendicular component $\vec{F}_{\perp}$ does not contribute to the torque about the rotation axis, since it can not tend to change the body's rotation about that axis. (or there must be a opposite torque exerted on the axis to balance it)

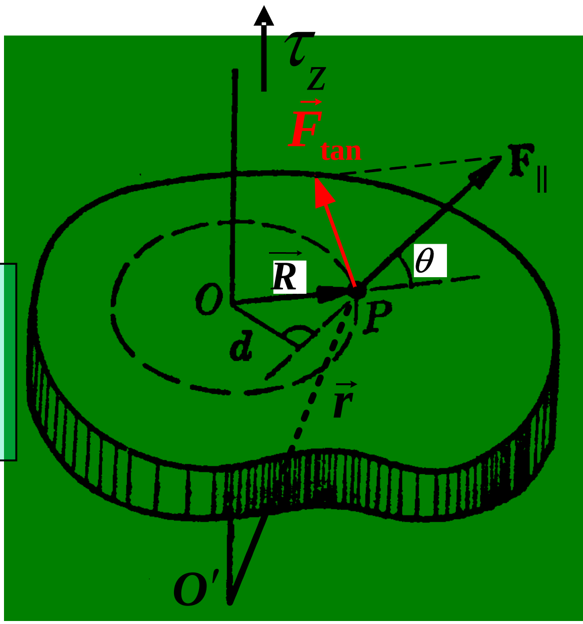


- So the torque about the fixed rotation axis:

$$\vec{\tau} = \vec{r} \times \vec{F}_{\parallel} = (\overrightarrow{O'O} + \vec{R}) \times \vec{F}_{\parallel} = \overrightarrow{O'O} \times \vec{F}_{\parallel} + \vec{R} \times \vec{F}_{\parallel}$$

$$= \vec{R} \times \vec{F}$$

Perpendicular to the rotation axis ***O′O***, and will be balanced by another torque acting on the axis.



$$\tau_z = \tau_{axis} = RF_{\parallel} \sin\theta = F_{\parallel} d = F_{\tan} R$$

- The torque about the axis *O′O* is actually the projection of the torque about the point *O′* on the axis *O′O*.

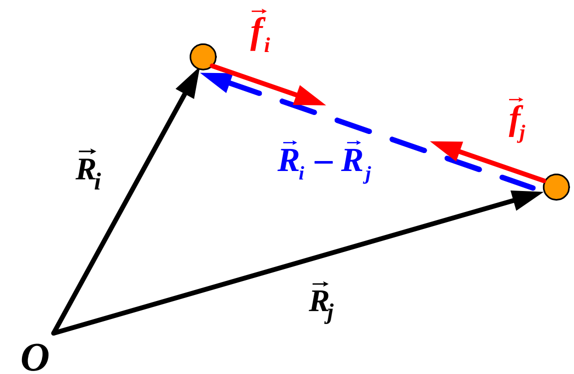
The Rotational Form of Newton's Second Law

- The Rotational Form of Newton's Second Law (转动定律)
  - Imagine the body as being made up of a large number of particles.
    - For i-th particle Δm_i

      external force: $\vec{F}_i$

      internal force: $\vec{f}_i$

$$\vec{F}_i + \vec{f}_i = \Delta m_i \vec{a}_i$$



  - It is followed that:

$$\sum_i \vec{R}_i \times \vec{F}_i + \sum_i \vec{R}_i \times \vec{f}_i = \sum_i \vec{R}_i \times \Delta m_i \vec{a}_i$$

    - The torques of each pair of internal forces are vanished

$$\vec{R}_i \times \vec{f}_i + \vec{R}_j \times \vec{f}_j = (\vec{R}_i - \vec{R}_j) \times \vec{f}_i = 0 \quad \Rightarrow \quad \sum_i \vec{R}_i \times \vec{f}_i = 0$$

    - The external torque:

zero

$$\vec{R}_i \times \vec{F}_i = \vec{R}_i \times \vec{F}_{it} + \vec{R}_i \times \vec{F}_{in} = R_i F_{it} \hat{k}$$

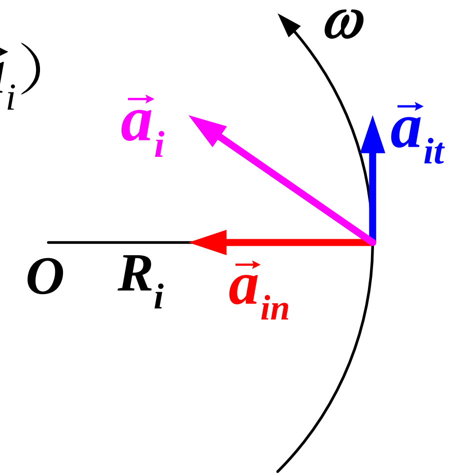
    - The net torque about rotation axis that acts on the body:

$$\vec{\tau}_{\text{net}} = \sum_i R_i F_{it} \hat{k}$$

The Rotational Form of Newton's Second Law (cont'd)

- The Rotational Form of Newton's Second Law (cont'd)
  - It is followed that:

$$\vec{\tau}_{\text{net}} = \sum_i R_i F_{it} \hat{k} = \sum_i \vec{R}_i \times \Delta m_i \vec{a}_i = \sum_i \Delta m_i (\vec{R}_i \times \vec{a}_i)$$



  - The right side of the equation:

$$\begin{aligned}\vec{R}_i \times \vec{a}_i &= \vec{R}_i \times \vec{a}_{it} + \underbrace{\vec{R}_i \times \vec{a}_{in}}_{\text{zero}} \\ &= R_i a_{it} \hat{k} \\ &= R_i^2 \alpha \hat{k}\end{aligned}$$

  - The rotational form of Newton's II Law:

$$\vec{\tau}_{\text{net}} = \sum_i R_i F_{it} \hat{k} = \left(\sum_i \Delta m_i R_i^2 \right) \alpha \hat{k} = I \alpha \hat{k}$$

$$\boxed{\sum \tau_{\text{net-axis}} = I\alpha}$$

  - The quantity $\boxed{I = \sum_i \Delta m_i R_i^2}$ is defined as the moment of inertia of the body.

转动惯量

The Rotational Form of Newton's Second Law (cont'd)

- Some Comments for the Rotational Form of Newton's II Law.

$$\boxed{\sum \tau_{\text{net-axis}} = I\alpha} \quad \text{analog of} \quad \sum F_{z-\text{ext}} = ma_z$$

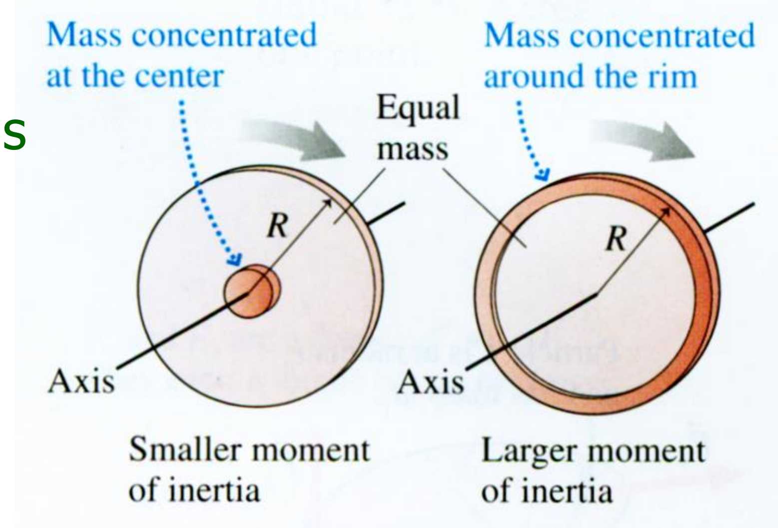
  - It relates the net external torque about a particular fixed axis to the angular acceleration about that axis. The moment of inertia ***I*** must be calculated about that same axis.
  - The moment of inertia reflects the tendency of a rigid body to resist angular acceleration, just like the mass reflecting the tendency of a object to resist linear acceleration.
  - Generally, this equation is valid for the rotation of a rigid body about a fixed axis in an inertial reference frame.
  - It is also valid for the rotation about an axis fixed in the center of mass of the body, although the CM is not an inertial reference frame.

$$\boxed{\sum \tau_{\text{ext-CM}} = I_{\text{CM}}\alpha}$$

§ 4 The Moment of Inertia

P248-251

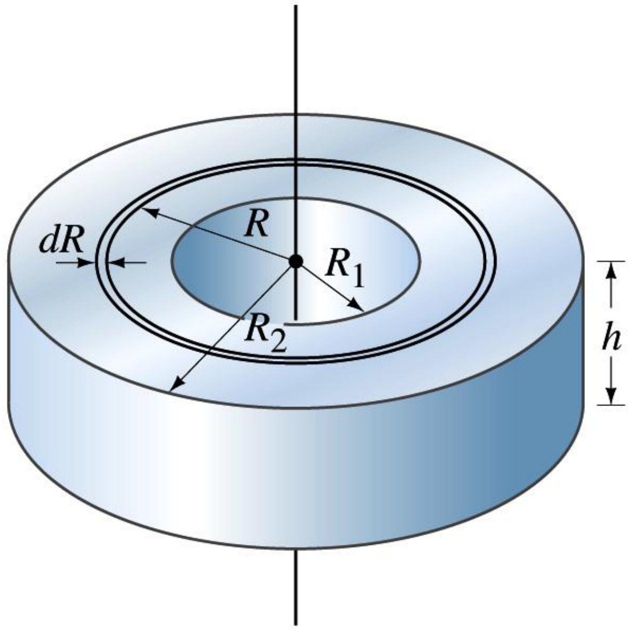
- The Moment of Inertia (rotational inertia) of a Rigid Body
  - The definition: $\boxed{I = \sum_i \Delta m_i R_i^2}$
    - For continuous distribution bodies: $\boxed{I = \int R^2 dm}$ where $dm = \begin{cases} \rho dV \\ \sigma dS \\ \lambda dl \end{cases}$
  - Better understanding of the moment of inertia:
    - The moment of inertia is the equivalent of mass.
      It play the same role in $\alpha = \tau_{\text{net}} / I$ as mass in $\vec{a} = \vec{F}_{\text{net}} / m$.
      The larger the moment of inertia, the more effort it takes and the slower her angular acceleration.
    - An object's moment of inertia depends not only on the object's mass *but on how the mass is distributed around the axis*.



Example for the moment of inertia

Example: The moment of inertia of a uniform hollow cylinder of inner radius ***R_1***, outer radius ***R_2***, and mass ***M***, if the rotation axis is though the center along the axis of symmetry.

Solution: Divided the cylinder into thin concentric cylindrical rings or hoops of thickness ***dR***,



$$dm = \rho dV = \frac{M}{\pi(R_2^2 - R_1^2)h} 2\pi R h dR = \frac{2M}{R_2^2 - R_1^2} R dR$$

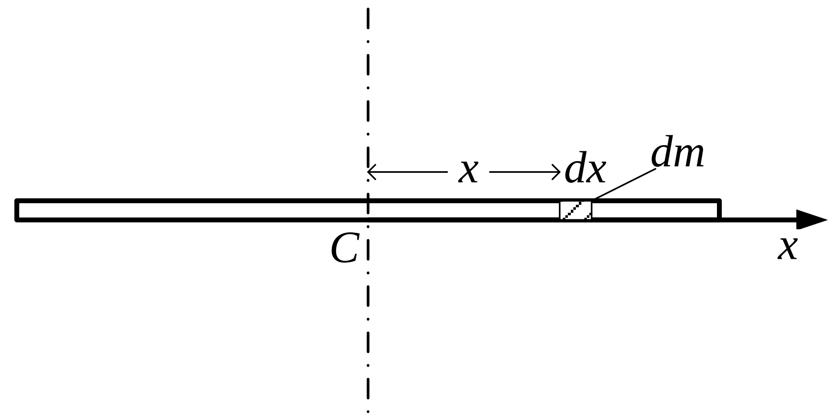
$$I = \int R^2 dm = \frac{2M}{R_2^2 - R_1^2} \int_{R_1}^{R_2} R^3 dR$$

$$= \frac{2M}{R_2^2 - R_1^2} \frac{R_2^4 - R_1^4}{4}$$

$$= \frac{1}{2} M (R_1^2 + R_2^2)$$

Example for the moment of inertia

Example: Uniform thin rod with mass ***M*** and length ***l***. Calculate the moment of inertia about the axis located (1) at the CM, (2) at an arbitrary distance ***h*** from the CM.

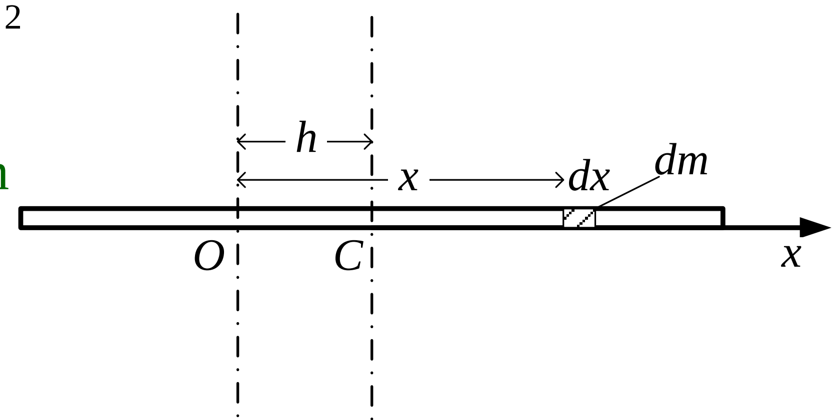


Solution: (1) The axis locates at the CM

Take a small element of mass:

$$dm = \lambda dx = \frac{M}{l} dx \qquad dI = x^2 dm = \lambda x^2 dx$$

$$I = \int dI = \int_{-l/2}^{l/2} \lambda x^2 dx = \frac{1}{3}\lambda x^3 \Big|_{-l/2}^{l/2} = \frac{1}{12} Ml^2$$



(2) The axis locates at arbitrary distance ***h*** from the CM.

$$I = \int_{-(l/2-h)}^{l/2+h} \lambda x^2 dx = \frac{1}{3}\lambda x^3 \Big|_{-l/2+h}^{l/2+h}$$

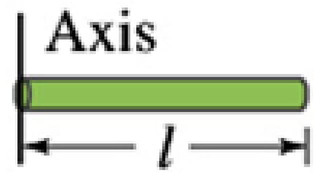
$$= \frac{1}{3}\frac{M}{l}\left[\left(\frac{l}{2}+h\right)^3 - \left(-\frac{l}{2}+h\right)^3\right] = \frac{1}{12} Ml^2 + Mh^2$$

The parallel-axis theorem

- The Parallel-axis Theorem

$$\boxed{I = I_{\mathrm{CM}} + Mh^2}$$

  - Long uniform rod of length ***l***, axis though one end:

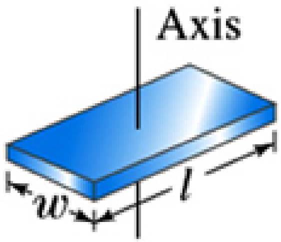


$$I_{\mathrm{end}} = I_{\mathrm{CM}} + M\left(\frac{l}{2}\right)^2 = \frac{1}{12}Ml^2 + \frac{1}{4}Ml^2 = \frac{1}{3}Ml^2$$

- The Perpendicular-axis Theorem
  - Only valid for the plane figures

$$\boxed{I_z = I_x + I_y}$$

  - Rectangular thin plate, of length ***l*** and width ***w***.



$$I_z = \frac{1}{12}M(l^2 + w^2)$$

  - Circular thin plate?

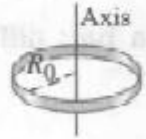
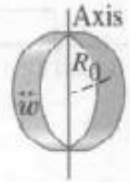
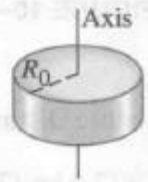
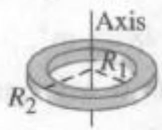
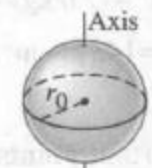
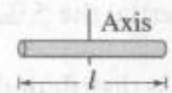
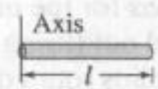
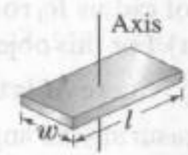
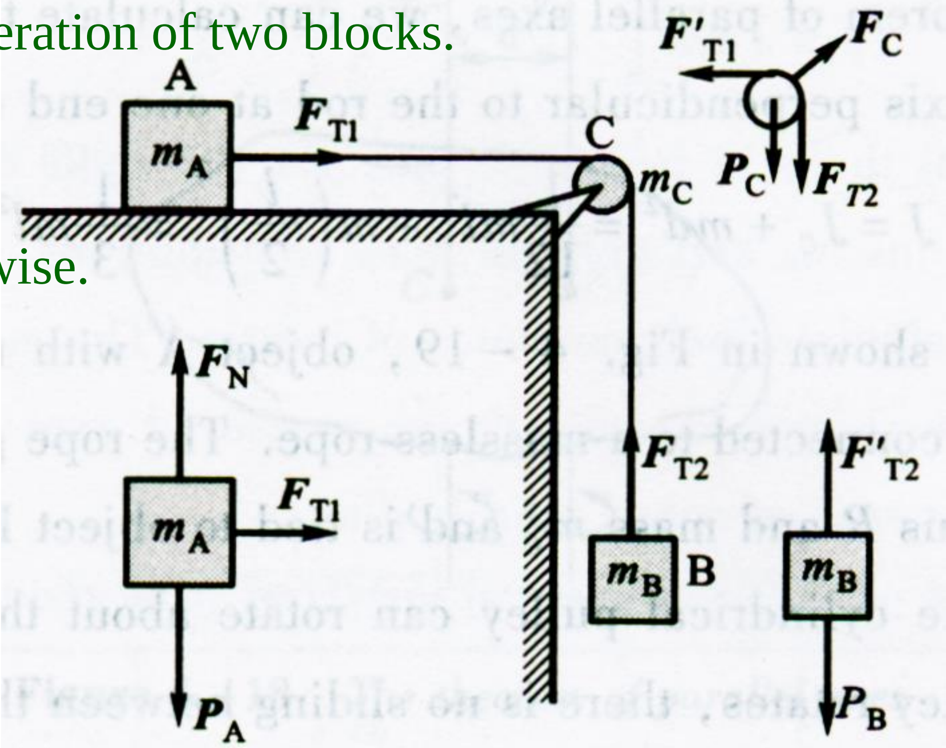
	Object	Location of axis		Moment of inertia
(a)	Thin hoop of radius R_0	Through center		MR_0^2
(b)	Thin hoop of radius R_0 and width w	Through central diameter		$\frac{1}{2}MR_0^2+\frac{1}{12}Mw^2$
(c)	Solid cylinder of radius R_0	Through center		$\frac{1}{2}MR_0^2$
(d)	Hollow cylinder of inner radius R_1 and outer radius R_2	Through center		$\frac{1}{2}M(R_1^2+R_2^2)$
(e)	Uniform sphere of radius r_0	Through center		$\frac{2}{5}Mr_0^2$
(f)	Long uniform rod of length l	Through center		$\frac{1}{12}Ml^2$
(g)	Long uniform rod of length l	Through end		$\frac{1}{3}Ml^2$
(h)	Rectangular thin plate, of length l and width w	Through center		$\frac{1}{12}M(l^2+w^2)$

FIGURE 10–21 Moments of inertia for various objects of uniform composition.

Example: Two blocks and a pulley: Two blocks of masses m_A and m_B are connected by a light cord running over a pulley. The pulley are considered as a uniform cylindrical disk of mass m_C and radius R. There is no sliding between the pulley and the cord. Find the acceleration of two blocks.



Solution: (1) Draw free-body diagrams.

(2) Newton's II law for every object:

The positive direction of rotation is clockwise.

$$F_{T1} = m_A a$$

$$R(F_{T2} - F_{T1}) = \left(\frac{1}{2} m_c R^2\right)\alpha$$

$$m_B g - F_{T2} = m_B a$$

4 unknowns. The restriction condition: no sliding between the pulley and the cord.

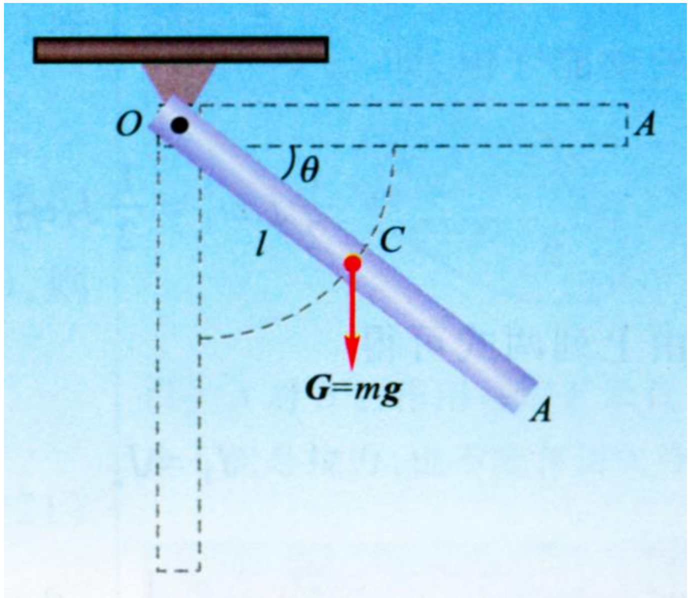
$$a = R\alpha$$

$$\boxed{a = \frac{m_B g}{m_A + m_B + \frac{1}{2} m_C}}$$

Example

Example: A uniform rod of mass ***m*** and length ***l*** can pivot freely (no friction on the pivot) about a hinge to the ceiling. The rod is held horizontally and released. Determine: (1) The angular acceleration and angular velocity of the rod as the function of θ. (2) The force on the hinge exerted by the rod.

Solution: (1) Newton's II law for the rotation of rod.



$$\frac{l}{2}mg\cos\theta = I\alpha = \left(\frac{1}{3}ml^2\right)\alpha$$

$$\boxed{\alpha = \frac{3}{2}\frac{g}{l}\cos\theta}$$

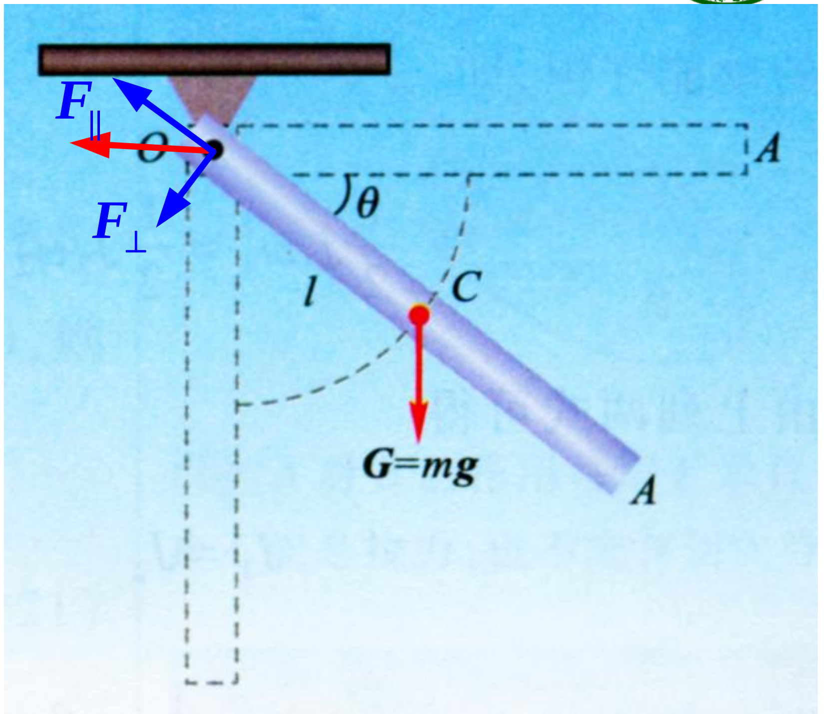
$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta} = \frac{3}{2}\frac{g}{l}\cos\theta$$

$$\int_0^\omega \omega\, d\omega = \frac{3}{2}\frac{g}{l}\int_0^\theta \cos\theta\, d\theta \Rightarrow \boxed{\omega = \sqrt{\frac{3g}{l}\sin\theta}}$$

Example cont'd

$$\boxed{\alpha = \frac{3}{2}\frac{g}{l}\cos\theta}$$

$$\boxed{\omega = \sqrt{\frac{3g}{l}\sin\theta}}$$



Solution: (2) Newton's II law for the CM of the rod.

Normal:

$$F_{\parallel} - mg\sin\theta = ma_{\text{n-CM}} = m\frac{l}{2}\omega^2$$

Tangential:

$$F_{\perp} + mg\cos\theta = ma_{\text{t-CM}}$$

$$= m\frac{dv_{\text{CM}}}{dt} = m\frac{l}{2}\frac{d\omega}{dt} = m\frac{l}{2}\alpha$$

$$\boxed{F_{\parallel} = \frac{5}{2}mg\sin\theta}$$

$$\boxed{F_{\perp} = -\frac{1}{4}mg\cos\theta}$$

§ 5 Angular Momentum for a Rigid Body

P251-254

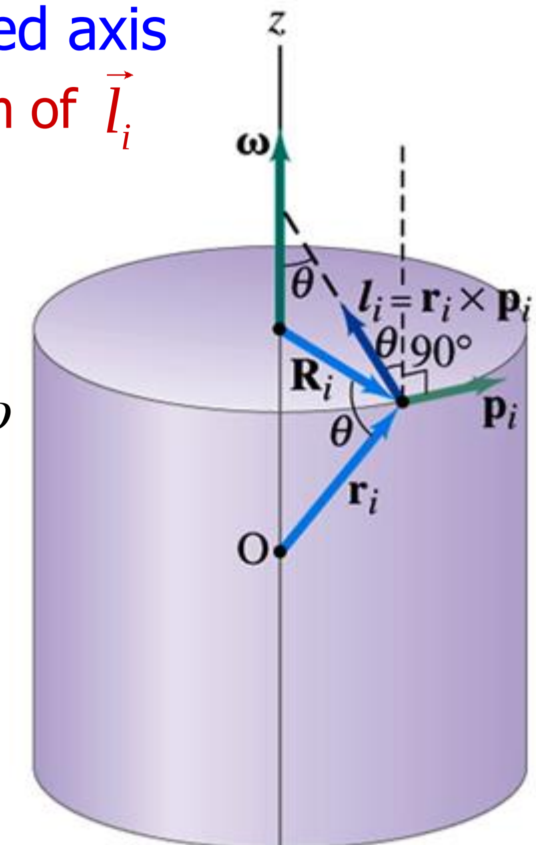
- Angular Momentum for a Rigid Body about a fixed axis
  - The total angular momentum $\vec{L}$ is the vector sum of $\vec{l}_i$ for each particle of the rigid body.
    - For i-th particle: $\vec{l}_i = \vec{r}_i \times \vec{p}_i$

      its component along the axis:

$$l_{i\omega} = r_i p_i \cos\theta_i = m_i v_i (r_i \cos\theta_i) = m_i(\omega R_i) R_i = (m_i R_i^2)\omega$$

Sum over all the particles:

$$\boxed{L_\omega = \sum_i l_{iz} = \left(\sum_i m_i R_i^2\right)\omega = I\omega}$$



  - If the rigid body rotates about a symmetry axis though the CM (or if the body is thin and flat, and rotates about a perpendicular axis)

$$\boxed{\vec{L} = I\vec{\omega}}$$

Angular Momentum for a Rigid Body

- Rotational Form of Newton's II Law
  - Starting from the Torque-angular momentum theorem.

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \quad \Rightarrow \quad \boxed{\sum \tau_{\text{ext-axis}} = \frac{dL_\omega}{dt} = \frac{d}{dt}(I\omega) = I\alpha}$$

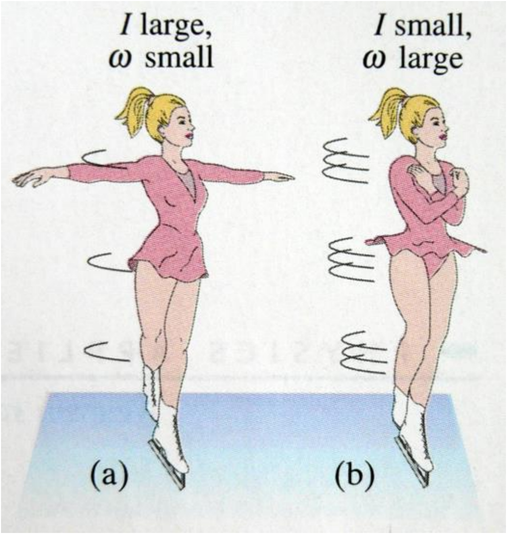
    - The Rotational Form of Newton's II Law can be considered as a special case of Torque-angular momentum theorem for a rigid body rotation about a fixed axis.

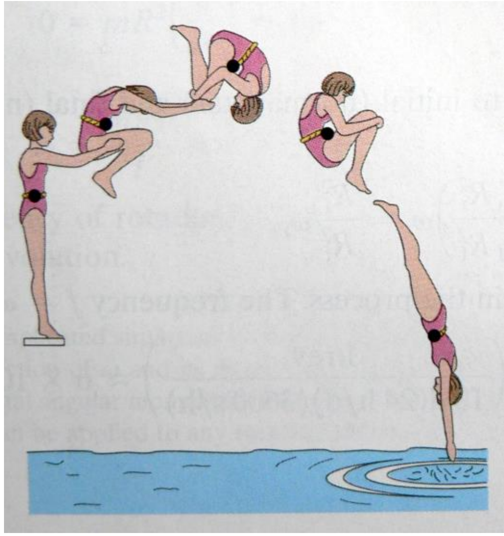
- The Conservation of Angular Momentum for Rigid Body
  - The total angular momentum of rotating body remains constant if the net external torque acting on it is zero.

If $\sum \tau_{\text{ext-axis}} = 0$

$$\Downarrow$$

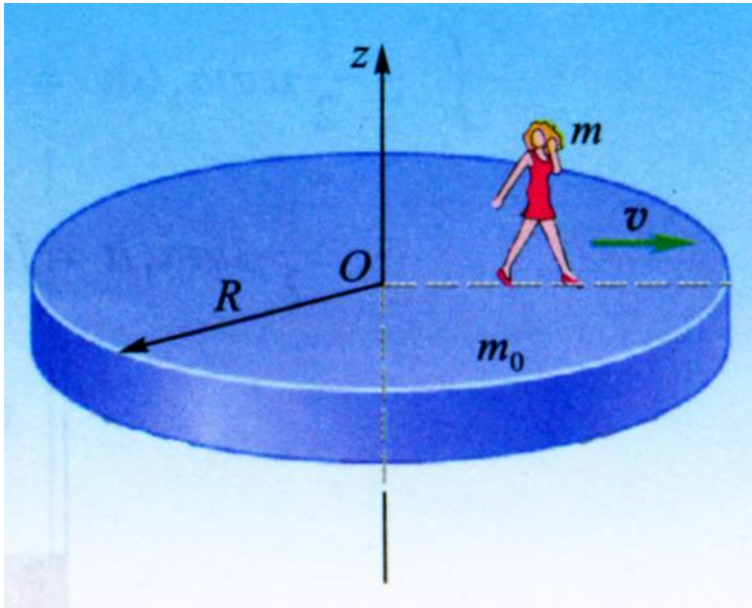
$$\boxed{I\omega = I_0\omega_0}$$





Example

Example: A circular platform of mass m_0 and radius R rotates friction-free about an axis through its center. A woman standing on the platform a distance $R/2$ from the center. At beginning, the system of platform and woman rotates at the angular velocity ω_0 about the axis. The woman starts to walk to the edge of the platform. Determine the final angular velocity ω of the system when the woman arrives at the edge.



Solution: In the whole process that the woman walk to the edge of platform, the external torque is zero. Using the conservation of angular momentum of the system:

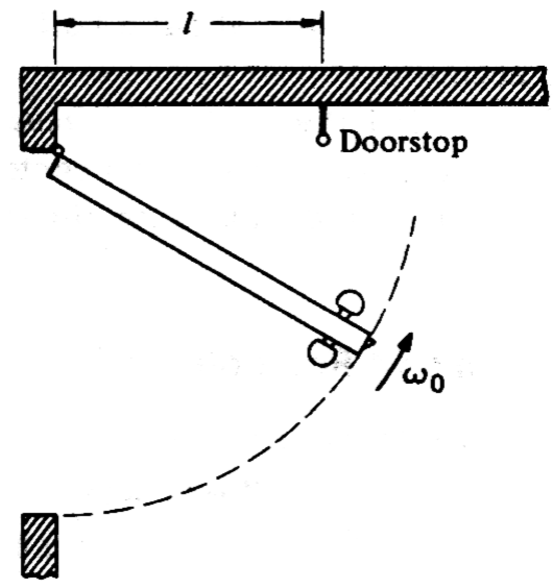
Initial state:
$$L_0 = \frac{1}{2}m_0R^2\omega_0 + m\left(\frac{R}{2}\right)^2\omega_0$$

Final state:
$$L = \frac{1}{2}m_0R^2\omega + mR^2\omega$$

$$L_0 = L \;\Rightarrow\; \boxed{\omega = \frac{2m_0 + m}{2m_0 + 4m}\omega_0}$$

Example

Example: The banging of a door against its stop can tear loose the hinges (合页). By the proper choice of ***l***, the impact forces on the hinge can be made to vanish. Determine the ***l***.



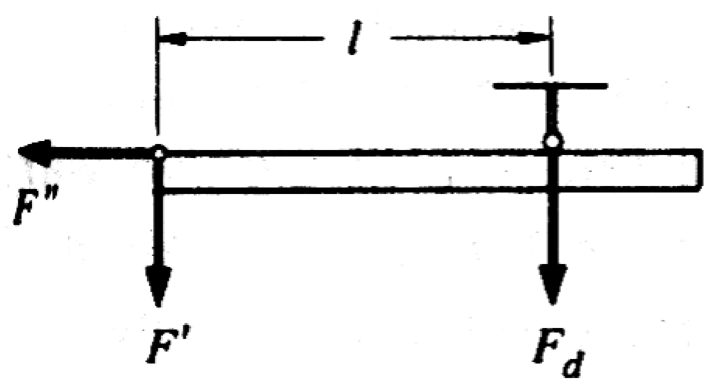
Solution: The forces on the door during impact are $\boldsymbol{F_d}$, due to the stop, and $\boldsymbol{F'}$ and $\boldsymbol{F''}$ due to the hinge. $\boldsymbol{F''}$ is the small radial force which provides the centripetal acceleration of swinging door. $\boldsymbol{F'}$ and $\boldsymbol{F_d}$ are the large impact forces which bring the door to rest when it bangs against the stop. To minimize the stress on the hinges, we must make $\boldsymbol{F'}$ as small as possible.

During the impact: using Torque-Angular Momentum Theorem

$$L_{\text{final}} - L_{\text{initial}} = \int_{t_i}^{t_f} \tau dt$$

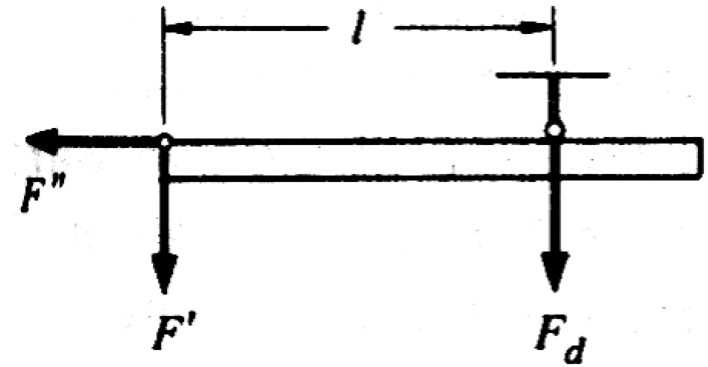
$$L_{\text{initial}} = I\omega_0, \quad L_{\text{final}} = 0, \quad \tau = -lF_d$$

$$\boxed{I\omega_0 = l\int_{t_i}^{t_f} F_d dt} \qquad ①$$



Example cont'd

$$\boxed{I\omega_0 = l\int_{t_i}^{t_f} F_d dt} \quad ①$$

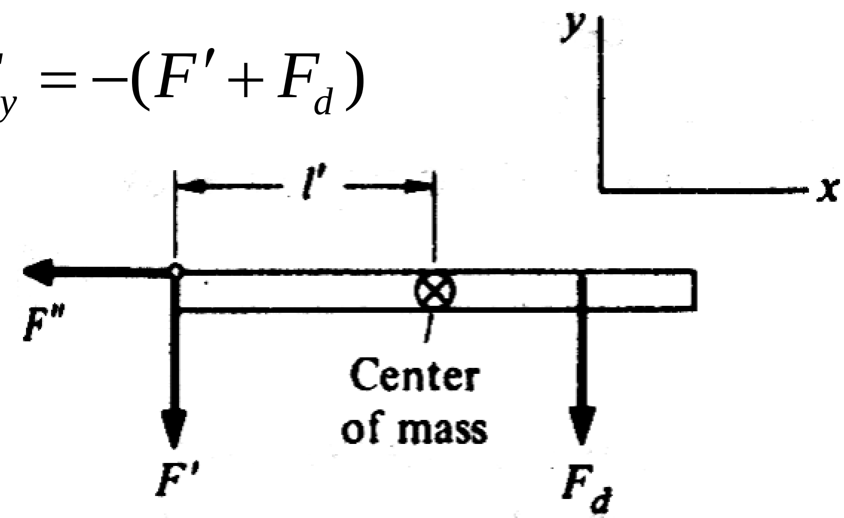


For the CM, using Impulse-Momentum Theorem in y- direction

$$p_{\text{y-final}} - p_{\text{y-initial}} = \int_{t_i}^{t_f} F_y dt$$

$$p_{\text{y-initial}} = MV_y = Ml'\omega_0, \quad p_{\text{y-final}} = 0, \quad F_y = -(F' + F_d)$$

$$\boxed{Ml'\omega_0 = \int_{t_i}^{t_f} (F' + F_d) dt} \quad ②$$



Combine ① and ②: $\int_{t_i}^{t_f} F' dt = \left(Ml' - \frac{I}{l} \right) \omega_0$

$$F' = 0 \Rightarrow l = \frac{I}{Ml'}$$

If the door is uniform, and of width *a*.

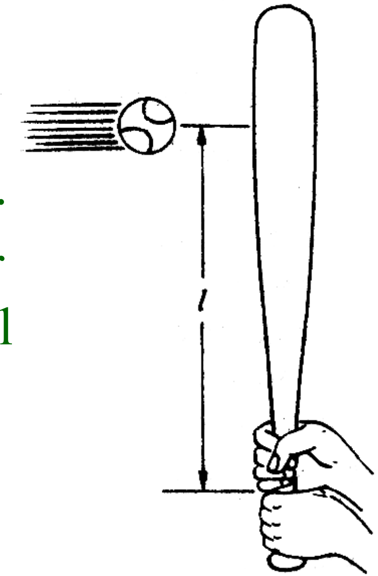
$$\boxed{I = \frac{1}{3}Ma^2 \text{ and } l' = \frac{a}{2}, \quad \Rightarrow l = \frac{2}{3}a}$$

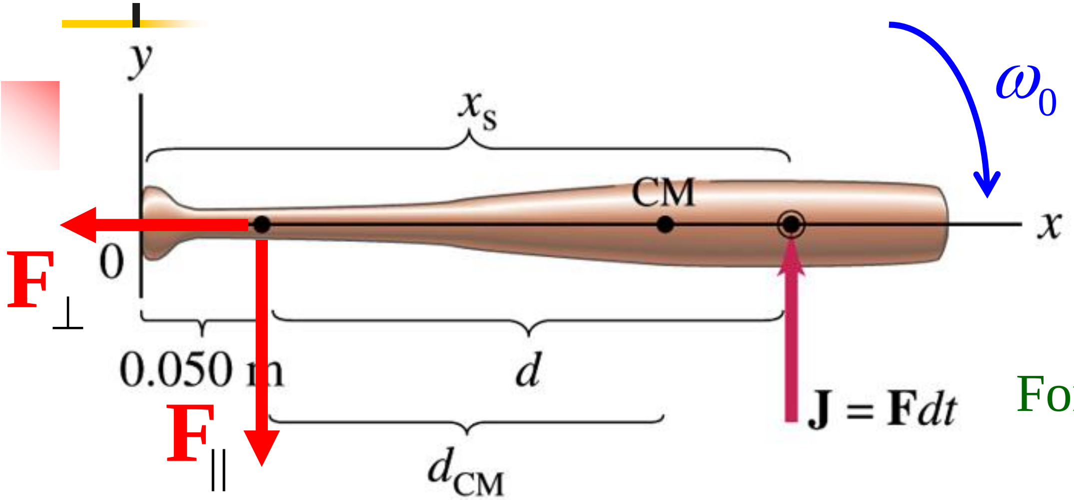
Example cont'd

The distance ***l*** specified by: $l = \dfrac{I}{Ml'}$

is called "the center of percussion" (打击中心) or "sweet spot".

In batting a baseball it is important to hit the ball at the bat's center of percussion to avoid a reaction on batter's hands and a painful sting.





For the CM, using Impulse-Momentum Theorem in y- direction

$$p_{\text{y-final}} - p_{\text{y-initial}} = \int_{t_i}^{t_f} F_y dt$$

$$p_{\text{y-initial}} = -MV_c = -Md_{\text{CM}}\omega_0,$$

$$p_{\text{y-final}} = 0, \quad F_y = -F_\parallel + F$$

$$\boxed{Md_{\text{CM}}\omega_0 = \int_{t_i}^{t_f} (-F_\parallel + F)dt} \qquad ②$$

During the impact: using Torque-Angular Momentum Theorem

$$L_{\text{final}} - L_{\text{initial}} = \int_{t_i}^{t_f} \tau dt$$

$$L_{\text{initial}} = -I\omega_0, \quad L_{\text{final}} = 0, \quad \tau = dF$$

$$\boxed{I\omega_0 = d\int_{t_i}^{t_f} Fdt} \qquad ①$$

Combine ① and ②:

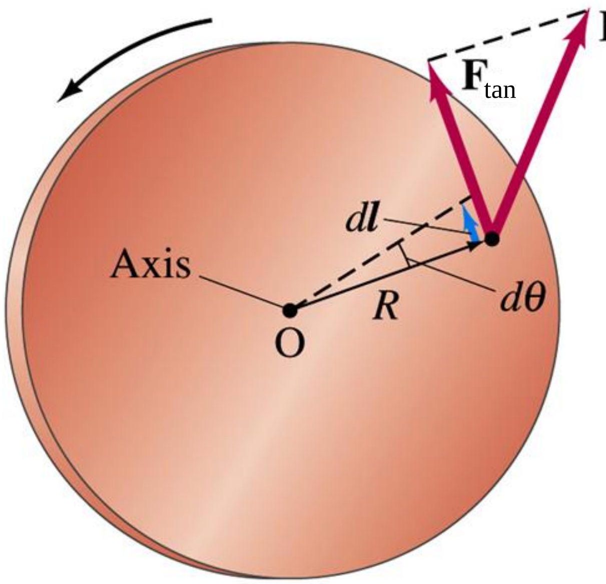
$$\int_{t_i}^{t_f} F_\parallel dt = \left(Md_{\text{CM}} - \frac{I}{d}\right)\omega_0 \qquad F_\parallel = 0 \quad \Rightarrow \quad d = \frac{I}{Md_{\text{CM}}}$$

§ 6 Work-Energy Theorem for a Rigid Body

P254-256

- Work done by a torque
  - For a fixed axis rotation of a rigid body, the work done by a force can appear in the form of torque —— work done by a torque.

$$W=\int_1^2 \vec{F}\cdot d\vec{l}=\int_1^2 F_{\tan} R d\theta=\int_{\theta_1}^{\theta_2}\tau d\theta$$



- The Power of a torque

$$P=\frac{dW}{dt}=\tau\frac{d\theta}{dt}=\tau\omega$$

- Rotational Kinetic Energy
  - For a fixed axis rotation of a rigid body, the kinetic energy can appear in another form:

$$K=\sum_i\left(\frac{1}{2}m_i v_i^2\right)=\sum_i\left(\frac{1}{2}m_i R_i^2\omega^2\right)=\frac{1}{2}\sum_i\left(m_i R_i^2\right)\omega^2=\frac{1}{2}I\omega^2$$

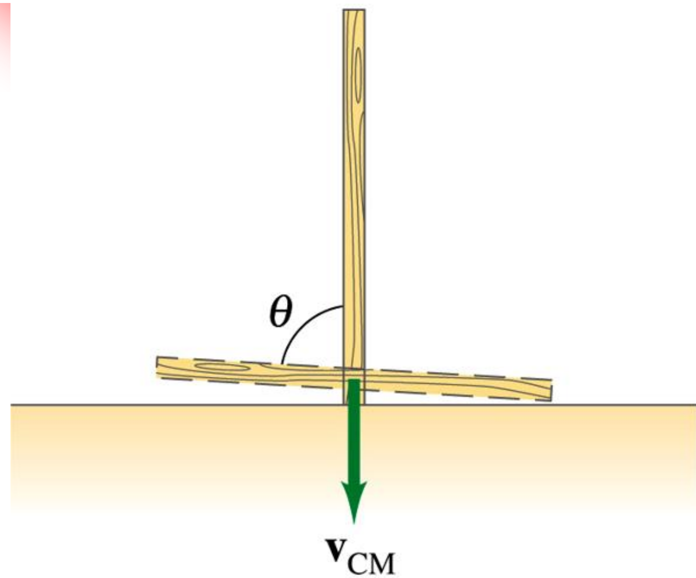
Work-Energy Theorem for a Rigid Body

- Work-kinetic energy theorem for a body rotating about a fixed axis
  - Starting from the rotational form of Newton's II law.

$$\tau_{\text{net}} = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta}$$

$$\boxed{W_{\text{net}} = \int_{\theta_1}^{\theta_2} \tau_{\text{net}} d\theta = \int_{\omega_1}^{\omega_2} I\omega d\omega = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2}$$

  - The work done in rotating a body through an angle $\theta_2 - \theta_1$ is equal to the change in rotational kinetic energy of the body.



Conservation of mechanical energy

$$\frac{1}{2}mgl = \frac{1}{2}mgl\cos\theta + \frac{1}{2}I_c\omega^2 + \frac{1}{2}mv_{CM}^2$$

$$v_{CM} = -\frac{dy}{dt} = -\frac{d}{dt}\left(\frac{l}{2}\cos\theta\right)$$

$$= \frac{l}{2}\sin\theta\frac{d\theta}{dt} = \frac{l}{2}\sin\theta\omega$$

$$\frac{1}{2}mgl(1-\cos\theta) = \frac{1}{2}\left(\frac{1}{12}ml^2\right)\omega^2 + \frac{1}{2}m\left(\frac{l}{2}\sin\theta\omega\right)^2$$

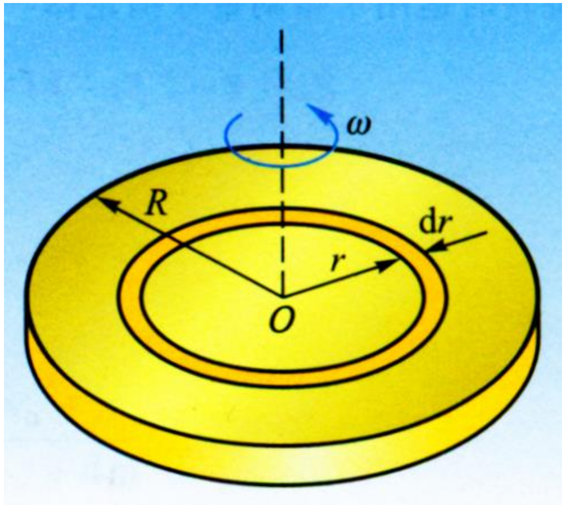
Kinetic energy = kinetic energy of CM

+rotation kinetic energy about the CM

Example

Example: A circular platform of mass ***m*** and radius ***R*** rotates initially at an angular velocity ω_0 about its central axis. Then the platform is placed on a rough horizontal surface. Determine (1) the torque acting on the platform by the friction force; (2) the time before the platform comes to a halt. The coefficient of friction between the platform and the surface is μ.

Solution: (1) The friction force is distributed in the whole area of the platform. Divide the whole platform into many circular rings with a radius of ***r*** and width ***dr***:



$$dm = \sigma dS = \sigma \cdot 2\pi r\,dr \qquad dF_f = \mu g dm$$

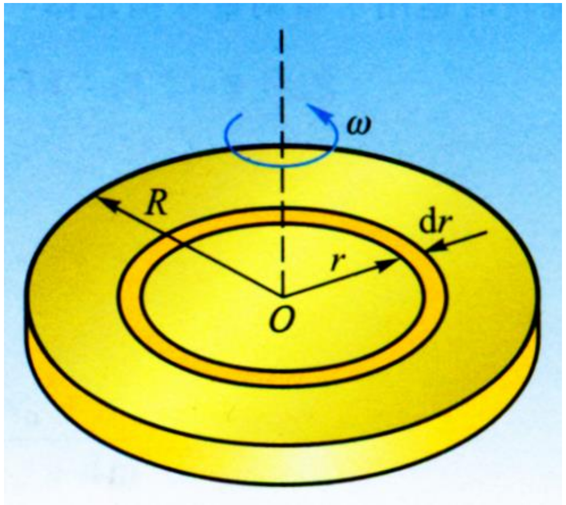
$$d\tau_f = -r dF_f = -\mu r g dm$$

$$\tau_f = -\int_m \mu r g dm = -\int_0^R \mu g r \sigma 2\pi r dr$$

$$= -\frac{2}{3}\pi\mu g R^3\sigma = -\frac{2}{3}\mu m g R$$

Example cont'd

$$\boxed{\tau_f = -\frac{2}{3}\mu mgR}$$



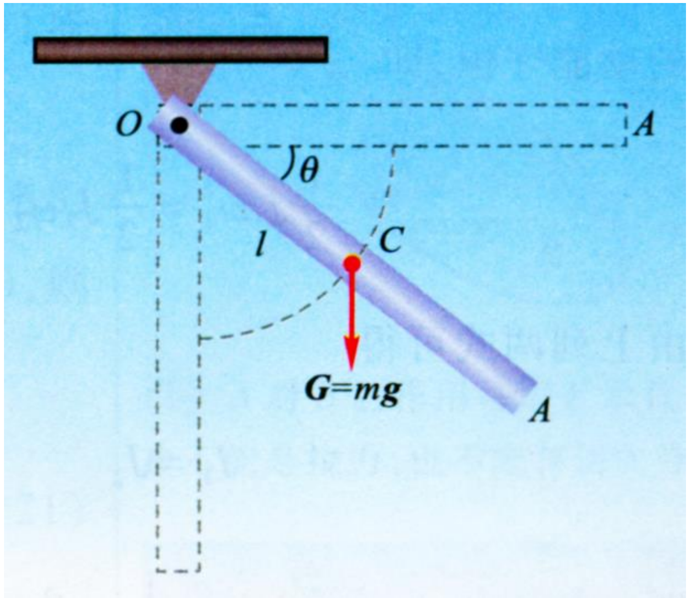
(2) The Newton's II law for rotation: $\tau_f = I\alpha$

$$-\frac{2}{3}\mu mgR = \frac{1}{2}mR^2\frac{d\omega}{dt}$$

$$\boxed{t = \int_0^t dt = -\frac{3R}{4\mu g}\int_{\omega_0}^{0} d\omega = \frac{3R}{4\mu g}\omega_0}$$

Example

Example: A uniform rod of mass ***m*** and length ***l*** can pivot freely (no friction on the pivot) about a hinge to the ceiling. The rod is held horizontally and released. Determine the angular acceleration and angular velocity of the rod as the function of ***θ***.



Solution: Using the law of conservation of mechanical energy.

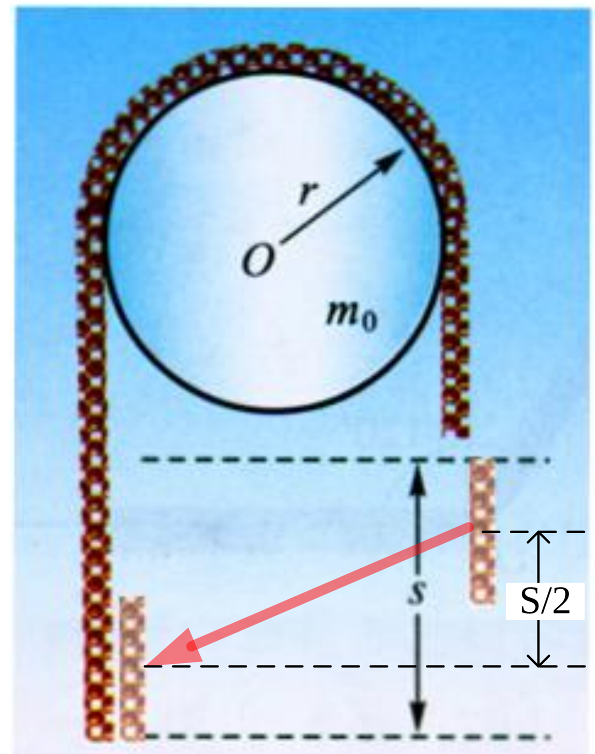
$$0 = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\omega^2 + \left(-mg\frac{l}{2}\sin\theta\right)$$

$$\boxed{\omega = \sqrt{\frac{3g}{l}\sin\theta}}$$

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d}{d\theta}\left(\sqrt{\frac{3g}{l}\sin\theta}\right)\frac{d\theta}{dt} = \sqrt{\frac{3g}{l}}\frac{\cos\theta}{2\sqrt{\sin\theta}}\sqrt{\frac{3g}{l}\sin\theta} = \frac{3g}{2l}\cos\theta}$$

Example

Example: A heavy steel chain of mass ***m*** and length ***l*** passes over a pulley of mass m_0 and radius ***r***. The pulley is fixed with a frictionless pivot O. There is no slide between the chain and pulley. At beginning, the chain passes over the pulley with the lengths of both side equal. And then with a small perturbation, the chain slides to the left. Find the velocity and acceleration of the chain when the height difference of two end is ***s***.



Solution: Take the chain, the pulley and the Earth as a system, the mechanical energy of the system is conserved.

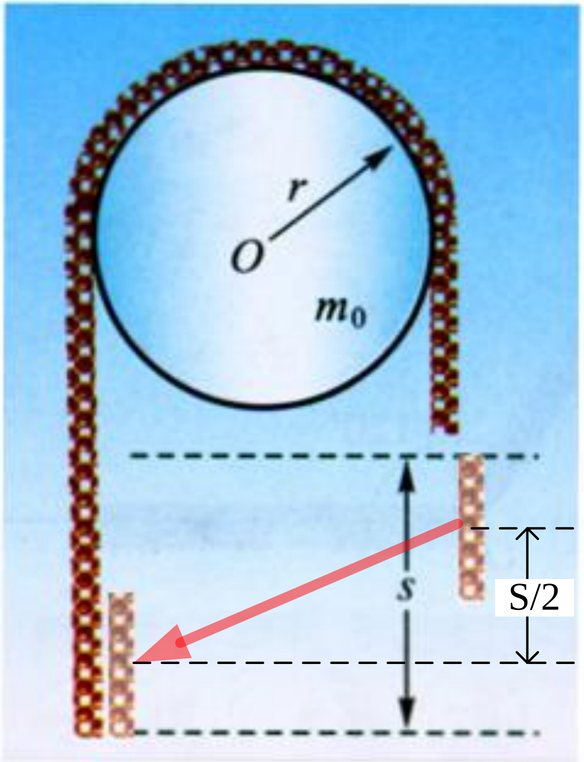
$$-m\frac{s/2}{l}g\frac{s}{2}+\frac{1}{2}mv^2+\frac{1}{2}\left(\frac{1}{2}m_0r^2\right)\omega^2=0$$

$$v=\omega r$$

$$\boxed{v=\sqrt{\frac{mgs^2}{2\left(m+\frac{1}{2}m_0\right)l}}}$$

Example cont'd

$$\boxed{v=\sqrt{\frac{mgs^2}{2\left(m+\frac{1}{2}m_0\right)l}}}$$



The acceleration:

$$a=\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=2v\frac{dv}{ds}$$

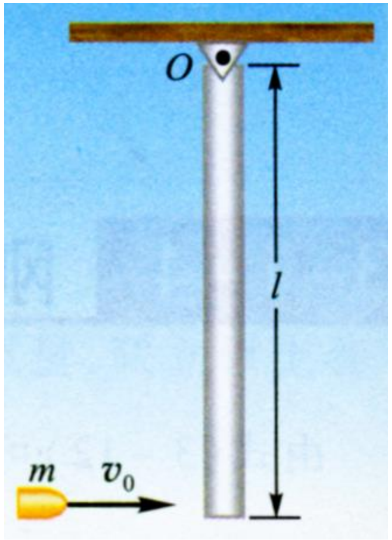
$$=2\sqrt{\frac{mgs^2}{2\left(m+\frac{1}{2}m_0\right)l}}\cdot\sqrt{\frac{mg}{2\left(m+\frac{1}{2}m_0\right)l}}=\frac{mgs}{\left(m+\frac{1}{2}m_0\right)l}$$

Example

Example: A rod of mass m' and length l can rotate about pivot O freely, a bullet of mass m and speed v_0 is shot into the lower end of the rod and embeded in the rod. What is the angle θ when the rod swings to its highest position?

Solution: Take the bullet and the rod as a system.

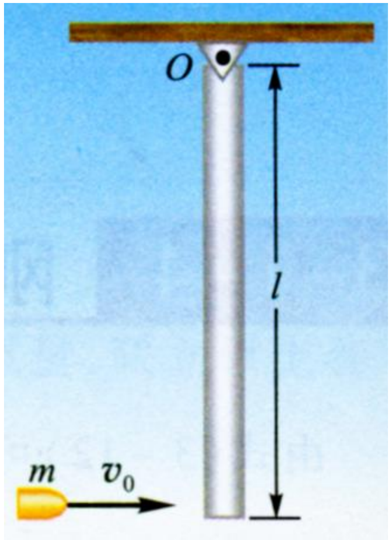
The external forces: the constraint force exerted by the pivot; gravity. They go through the origin O. So the external torque about O is zero, and the angular momentum of the system should be conserved in the process of shouting.



$$lmv_0 = \left(\frac{1}{3}m'l^2 + ml^2\right)\omega$$

$$\boxed{\omega = \frac{3mv_0}{(m'+3m)l}}$$

$$\boxed{\omega = \frac{mv_0}{\left(\frac{1}{3}m' + m\right)l}}$$

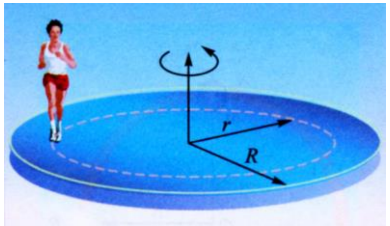


Take the bullet, the rod and the Earth as a system.

In the process of the system swinging up, the mechanical energy is conserved.

$$\frac{1}{2}\left(\frac{1}{3}m'l^2 + ml^2\right)\omega^2 = mgl(1-\cos\theta) + m'g\frac{l}{2}(1-\cos\theta)$$

$$\boxed{\cos\theta = 1 - \frac{3m^2}{(m'+3m)(m'+2m)}\frac{v_0^2}{gl}}$$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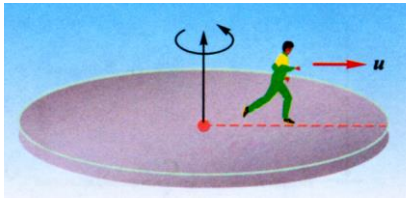
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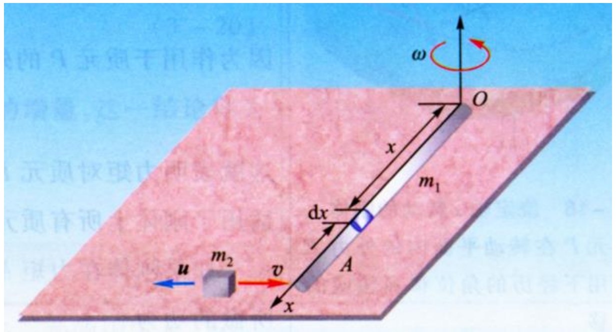
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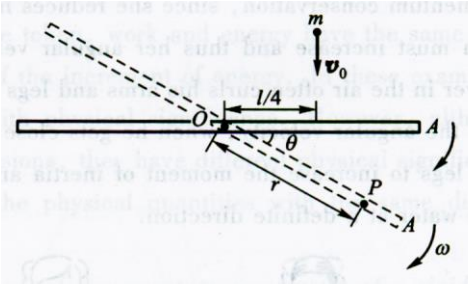
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