Chapter 12 Oscillations

§ 1 The Causes of Oscillation

- The systems tends to return to equilibrium when slightly displaced
  - Existence of a point of stable equilibrium

    For a block-spring system

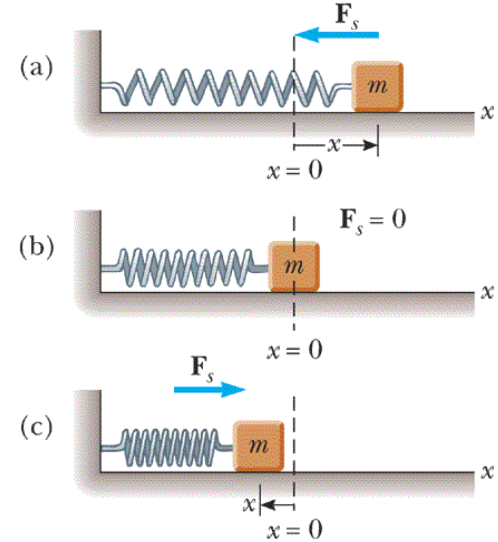
$$U(x) = \frac{1}{2}kx^2$$

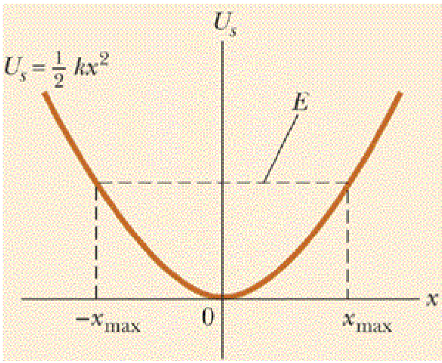
  - Existence of a restoring force

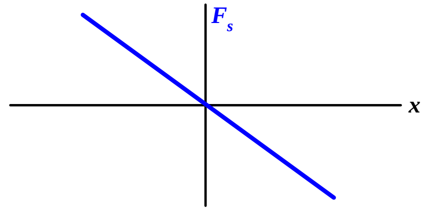
    No matter what the direction of the displacement, the force always acts in a direction to restore the system to its equilibrium position.

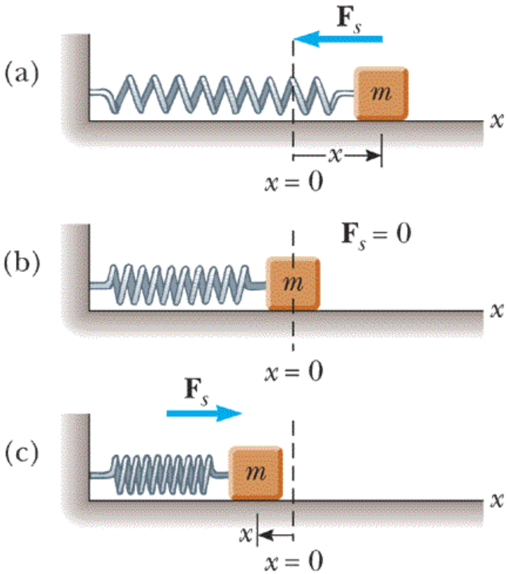
    For a block-spring system

$$F_s = -\frac{dU}{dx} = -kx$$









- The block-spring system
  - Newton's second law for block-spring system

$$-kx = m\frac{d^2x}{dt^2}$$

  - Dynamics' equation

Denote the ratio ***k*/*m*** with symbol ω^2

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \qquad \textbf{(1)}$$

Dynamics' equation for SHM

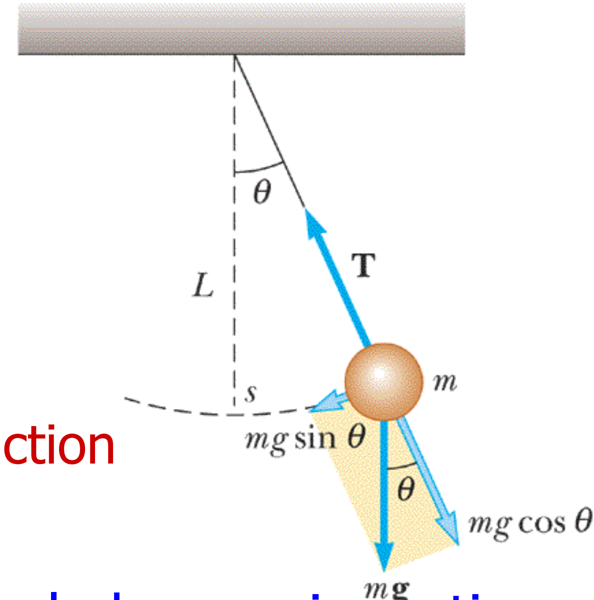
Take a tentative solution to Eq.(1)

$$\boxed{x = A\cos(\omega t + \phi)}$$

Kinematics' equation for SHM

A and ϕ arise from the integral constants

Simple Harmonic Motion



- The simple harmonic motion
  - The motion action is governed by Eq. (1)

$$\boxed{\frac{d^2x}{dt^2}+\omega^2x=0}$$

  - Can be described in terms of sine and cosine function

$$\boxed{x=A\cos(\omega t+\phi)}$$

- The simple pendulum – Another example of simple harmonic motion
  - Newton's second law for the simple pendulum

$$-mg\sin\theta=m\frac{d^2s}{dt^2}\qquad s=L\theta\qquad \Rightarrow\qquad \frac{d^2\theta}{dt^2}=-\frac{g}{L}\sin\theta$$

Let $\omega^2=\dfrac{g}{L}$, and for small angles $\sin\theta\approx\theta$

  - We get also a equation of motion of SHM

$$\boxed{\frac{d^2\theta}{dt^2}+\omega^2\theta=0}\qquad \boxed{\theta=\theta_m\cos(\omega t+\phi)}$$

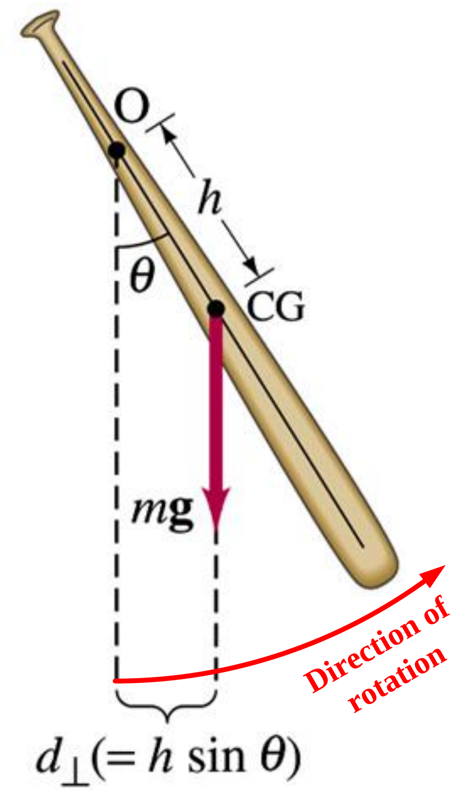
- The Physical Pendulum (复摆)
  - Newton's second law for rigid body:

$$\tau_{\text{net-axis}} = I\alpha$$

$$\boxed{-mgh\sin\theta = I\frac{d^2\theta}{dt^2}}$$

  - It follows that:

$$\frac{d^2\theta}{dt^2} + \frac{mgh}{I}\sin\theta = 0 \qquad \sin\theta \approx \theta$$



$$\frac{d^2\theta}{dt^2} + \left(\frac{mgh}{I}\right)\theta = 0 \Rightarrow \boxed{\theta = \theta_{\max}\cos(\omega t + \phi)} \qquad \omega = \sqrt{\frac{mgh}{I}}$$

§ 3 The Characteristic Quantities for SHM

P300-304

- Angular Frequency, Frequency, and Period
  - The period, T, is the time for oscillator to go though one circle of motion
  - The frequency, f, is the number of circles in a unit of time. (SI unit: Hz)

$$f = \frac{1}{T}$$

  - The angular frequency, ω, is 2π times the frequency. (SI unit: rad/s)

$$\omega = 2\pi f = \frac{2\pi}{T}$$

  - T , f, ω relate to the essential nature of an oscillator, which often called natural (intrinsic) period, natural frequency, and natural angular frequency.
    - For a block-spring oscillator: $\boxed{\omega = \sqrt{\frac{k}{m}}}$
    - For a simple pendulum: $\boxed{\omega = \sqrt{\frac{g}{L}}}$

All determined by the essential natures of two different oscillators

- The amplitude ***A***
  - Maximum magnitude of displacement from equilibrium

$$A = |x_{\max}|$$

- The phase $(\omega t + \phi)$, phase constant (or phase angle) ϕ
  - The phase $(\omega t + \phi)$ can reflect entirely the motion state of an oscillator

Phase —— $\omega t + \phi$ $\Longleftrightarrow$ $\begin{Bmatrix} x \\ v \end{Bmatrix}$ —— **State of motion**

$$x = A\cos(\omega t + \phi), \qquad v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

  - When t=0, ϕ reflect the initial motion state of the oscillator
  - A and ϕ are determined by initial conditions (How the motion starts?)

When t=0 $x=x_0$, $v=v_0$

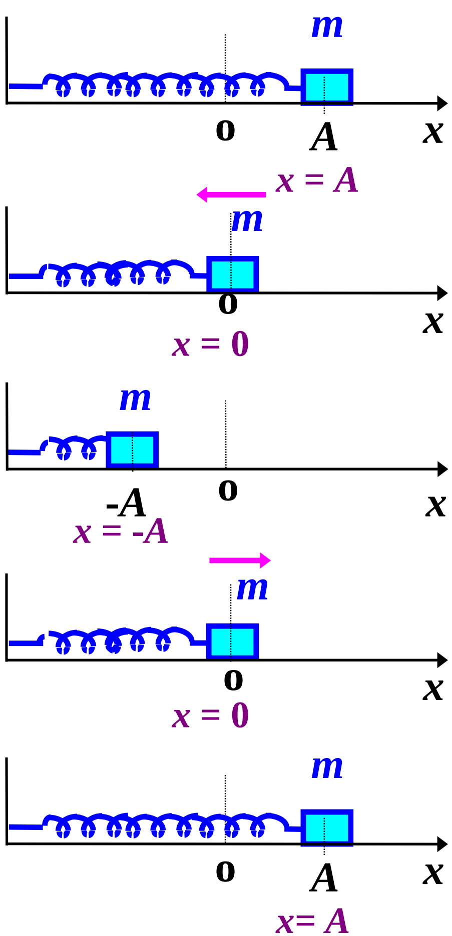
$$x_0 = A\cos\phi$$
$$v_0 = -\omega A\sin\phi$$

$$\Rightarrow \boxed{A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \quad \phi = \arctan\left(-\frac{v_0}{\omega x_0}\right)}$$

The relationship between motion state and phase

$$x(t) = A\cos(\omega t + \phi), \quad v = -\omega A\sin(\omega t + \phi)$$

Motion state ⟷ **Phase**



0

π/2

π

3π/2或-π/2

2π

Phase difference

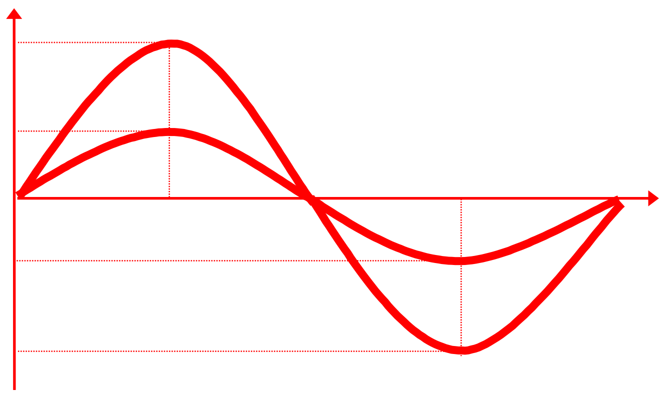
- Phase difference play a an important role for oscillator
  - Two oscillators with phases: $\theta_1=\omega t+\phi_1$, $\theta_2=\omega t+\phi_2$

$$\Delta\theta=\theta_2-\theta_1>0$$

Ahead in phase

$$\Delta\theta=\theta_2-\theta_1<0$$

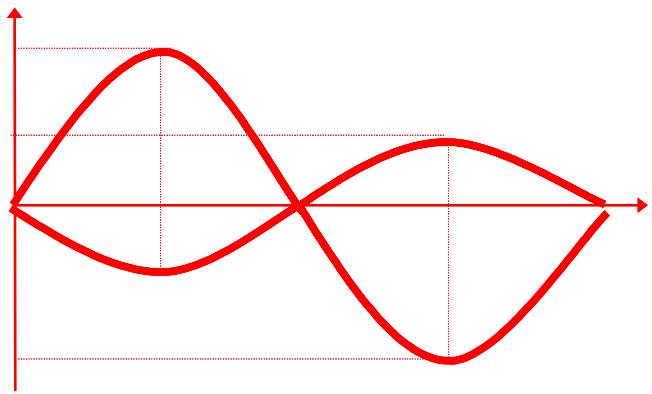
Lag in phase



$$\Delta\theta=\theta_2-\theta_1=2k\pi$$

$$k=0,\pm1,\pm2\cdots$$

In phase



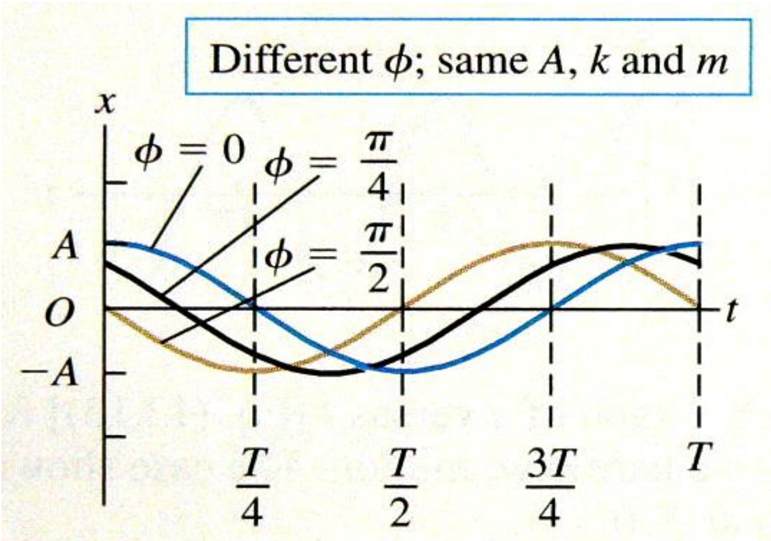
$$\Delta\theta=\theta_2-\theta_1=(2k+1)\pi$$

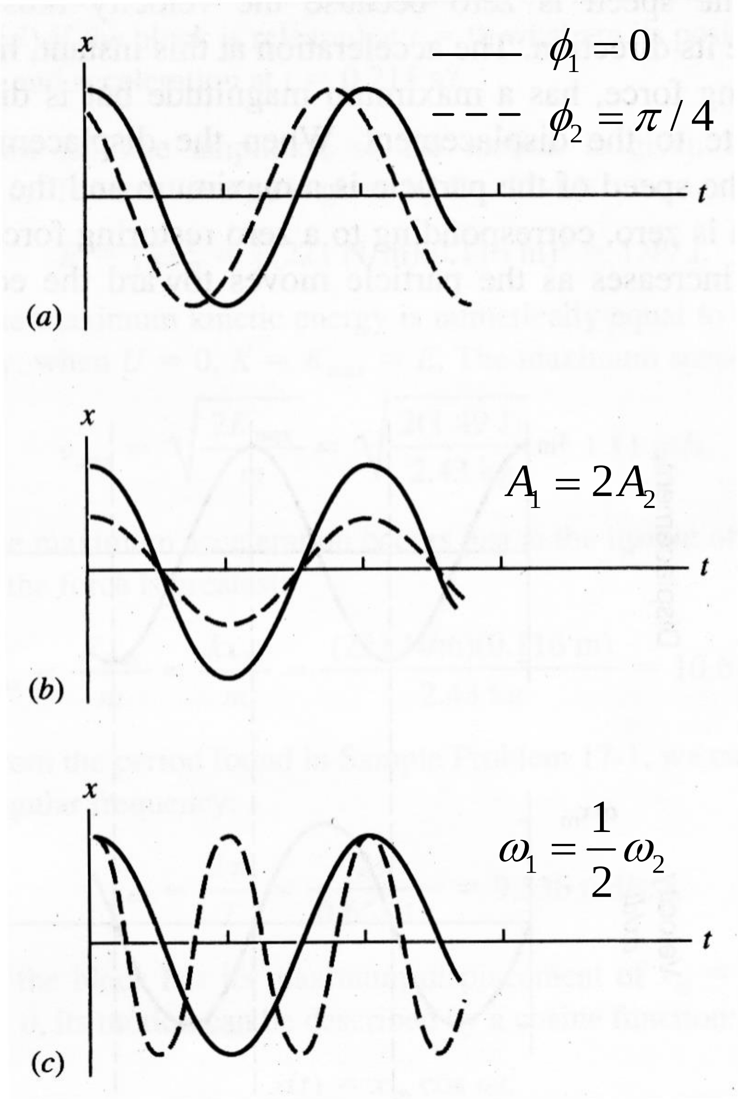
$$k=0,\pm1,\pm2\cdots$$

Out of phase

The Roles Characteristic Quantities

Several simple harmonic motion with different characteristic quantities





Different ϕ

Different A

Different ω

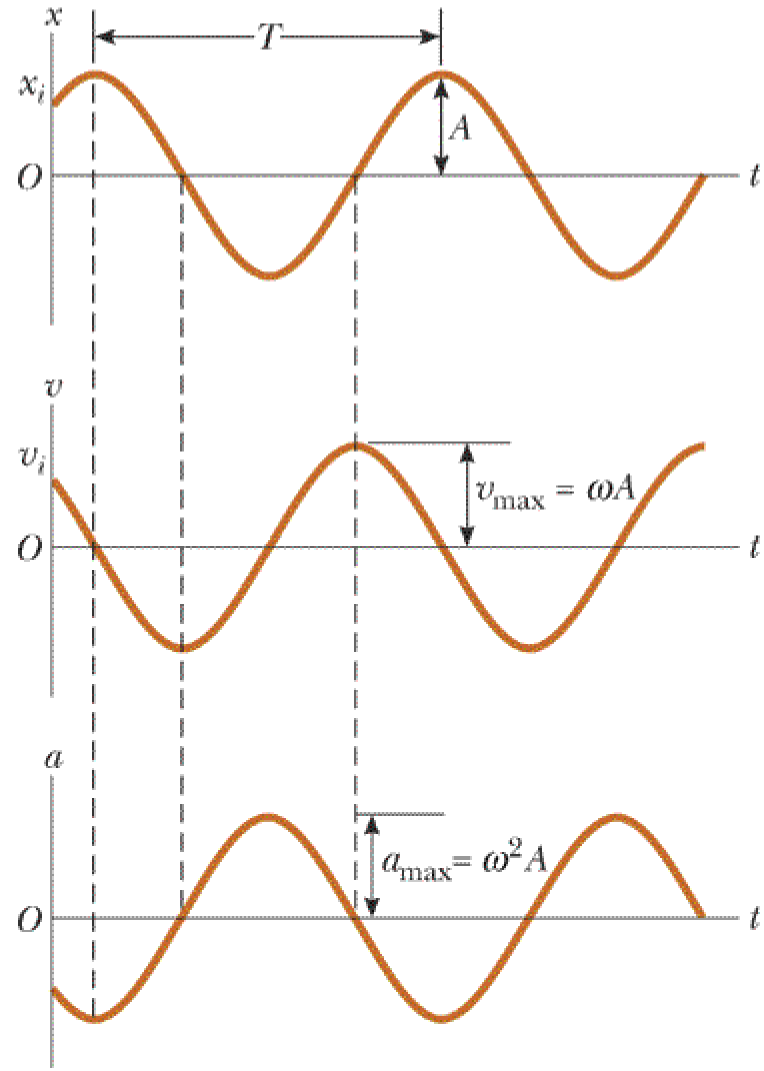
The relations among the position, velocity, and acceleration

- The relations among the position, velocity, and acceleration
  - The velocity is π/2 ahead in phase of the position.
  - The acceleration is π out of phase with the position.

$$x = A\cos(\omega t + \phi)$$

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) = \omega A\cos(\omega t + \phi + \frac{\pi}{2})$$

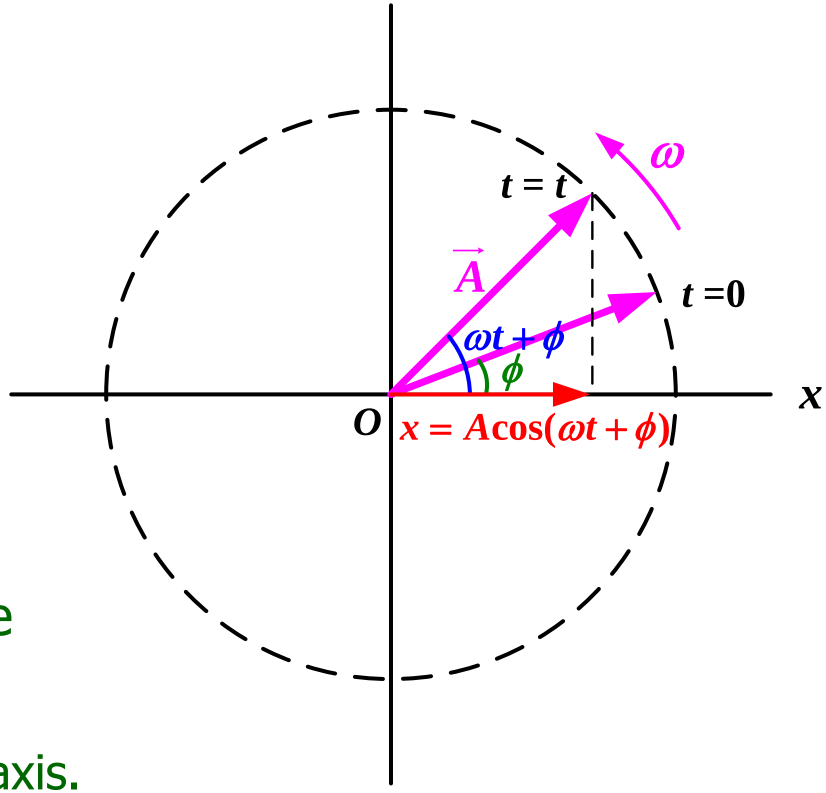
$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi) = \omega^2 A(\omega t + \phi + \pi)$$



§ 4 The Circle of Reference

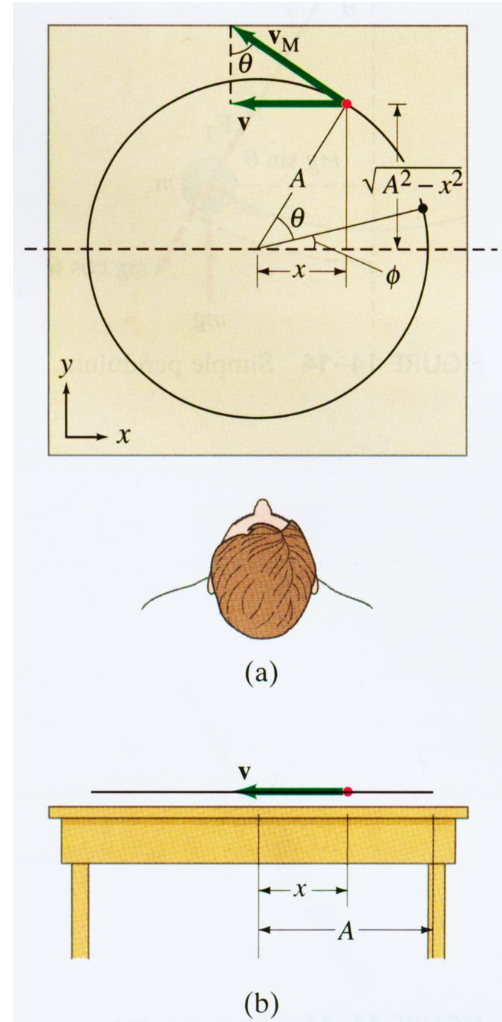
P306-307

- The corresponding relation between SHM and uniform circular motion —— Circle of Reference (参考圆) or Phasor (旋转矢量)
  - Simple Harmonic Motion is the projection of uniform circular motion of phasor $\vec{A}$ onto x axis.
  - The circle in which the phasor moves so that the projection of phasor's top matches the motion of the oscillating body is called the circle of reference.
  - The phasor $\vec{A}$ rotates with constant angular speed ω, and makes an angle $\omega t + \phi$ with the x axis. When t=0, the phasor $\vec{A}$ makes an angle ϕ with the x axis.



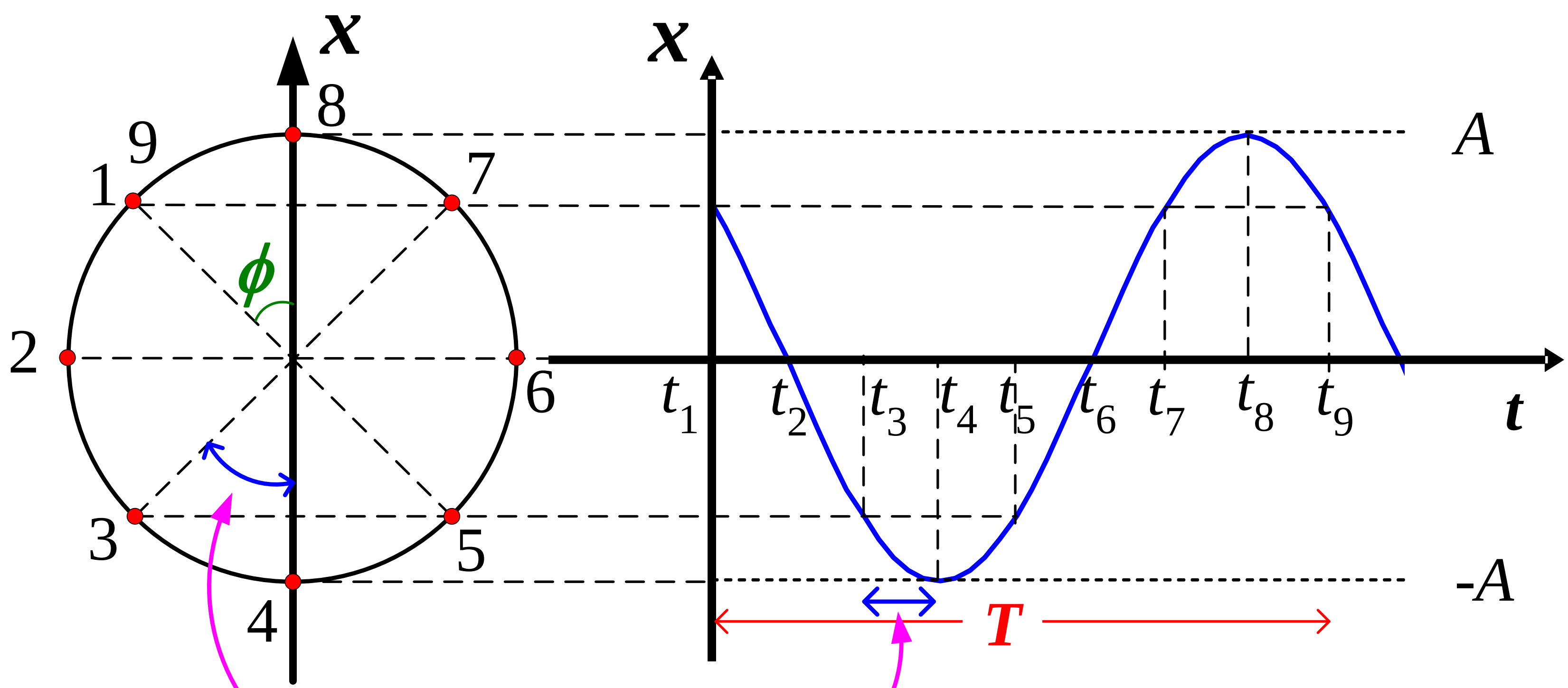
Corresponding Relation Between SHM and UCM

	For Simple Harmonic Motion	For Uniform Circular Motion
A	Amplitude	Radius
x	Displacement	Projection
ω	Angular Frequency	Angular Velocity
$\theta = \omega t + \phi$	Phase	Angle between Phasor and x axis



The simple harmonic motion is the side view of circular motion.

Draw x-t Diagram Using Circle of Reference



$$\frac{|\Delta\phi|}{2\pi}T = \Delta t$$

Example

Example: An object of mass 4 kg is attached to a spring of $\boldsymbol{k}$ =100N.m-1. The object is given an initial velocity of $\boldsymbol{v_0} = -5\text{m.s}^{-1}$ and an initial displacement of $\boldsymbol{x_0}$=1. Find the kinematics equation.

Solution: $$x = A\cos(\omega t + \phi)$$

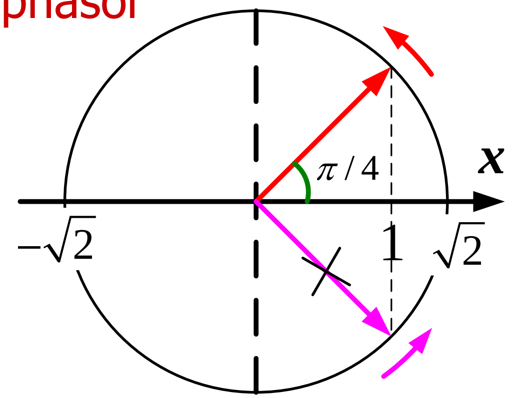
$$\because \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{4}} = 5 \text{ rad/s}, \qquad \therefore A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} = \sqrt{2} = 1.4m$$

$$\because tg\phi = -\frac{v_0}{\omega x_0} = 1$$ ϕ locates in I or III quadrant $\phi = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

with $v_0 = -\omega A \sin\phi < 0$ $\qquad \therefore \phi = \frac{\pi}{4}$

$$\therefore \boldsymbol{x = 1.4\cos(5t + \frac{\pi}{4})}$$

Using the phasor



Example

Example: A particle undergoes SHM with A=4cm， f = 0.5Hz. The displacement x = –2cm when t = 1s, and is moving in the positive x-axis. Write the kinematics equation.

Solution: changed initial conditions: $x = x'_0$, $v = v'_0$, when $t = t'_0$.

$$A = 4\text{cm},\ f = 0.5\text{Hz} \quad T = 1/f = 2\ \text{s}$$

$$x = 0.04\cos(\pi t + \phi)\ (\text{SI}) \quad \phi = ?$$

When t=1s

$$-0.02 = 0.04\cos(\pi + \phi) = -0.04\cos\phi$$

$$\cos\phi = 1/2 \Rightarrow \phi = \pm\pi/3$$

locates in I or IV quadrant

$$v = -0.04\sin(\pi + \phi) = 0.04\sin\phi > 0$$

ϕ locates in I quadrant. $\phi = \dfrac{\pi}{3}$

Too complicated !

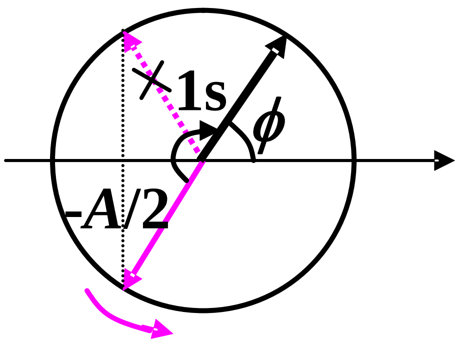
Using the phasor:

One revolution corresponds to one period T=2s, and half a revolution corresponds to Δt = 1 s

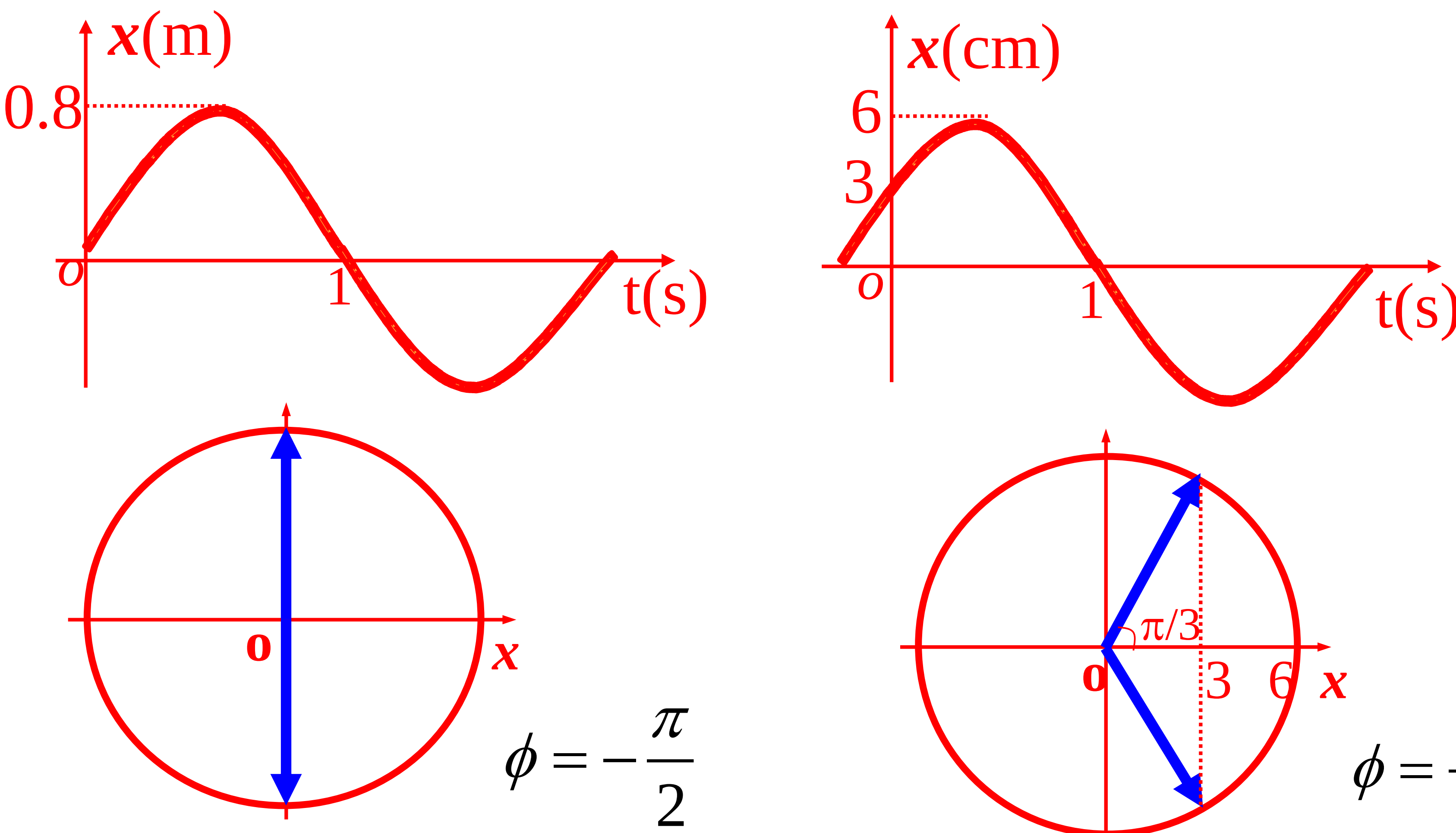
$$v > 0$$

$$\omega\Delta t + \phi = \pi + \phi = 4\pi/3$$

$$\phi = \frac{\pi}{3}$$



Example: Find the initial phase of the two oscillations



$$\phi = -\frac{\pi}{2}$$

$$\phi = -\frac{\pi}{3}$$

Example

Example: SHM: From given x-t graph, find ϕ, θ_a, θ_b, and the angular frequency ω.

Solution:

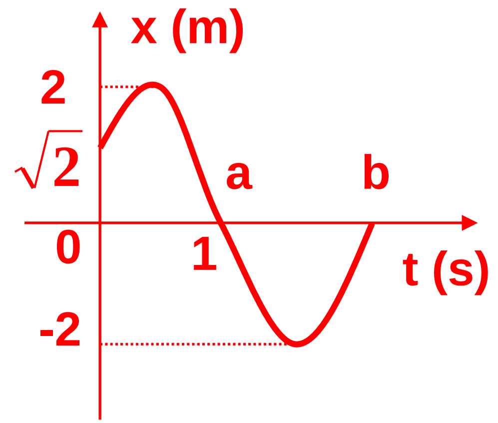
From circle of reference

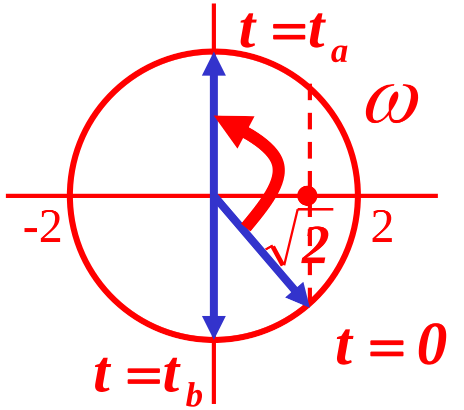
$$\therefore \phi = -\frac{\pi}{4}, \qquad \theta_a = \frac{\pi}{2} \qquad \theta_b = \frac{3\pi}{2}$$

for $\theta = \omega t + \phi$

$$\Delta\theta = \omega\,\Delta t$$

$$\therefore \omega = \frac{\Delta\theta}{\Delta t} = \frac{\theta_a - \phi}{\Delta t} = \frac{\pi/2 - (-\pi/4)}{1} = \frac{3\pi}{4}\ \text{rad/s}$$

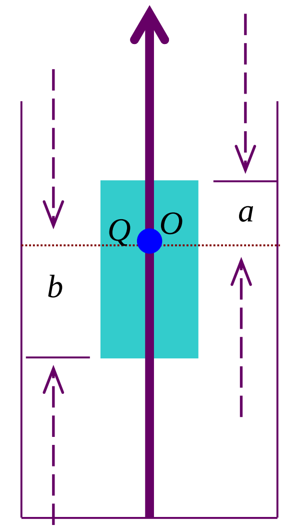




Example

Example: A wooden block floats in water. We press it until its upper surface just under water, and release. Will the motion of the wooden block be SHM?

Solution: Take the point O at the surface of water be the origin of x-axis. When the block is in equilibrium, the point Q of block coincides with origin point O.

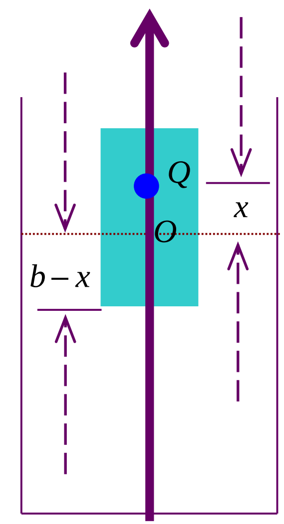


When block is in equilibrium.

$$Sl\rho_{block}g = Sb\rho_{water}g$$

where S is the area of block's cross section, and $\boldsymbol{l=a+b}$

The net force:
$$\sum F = S(b-x)\rho_{water}g - Sl\rho_{block}g$$
$$= -S\rho_{water}gx$$



$$-S\rho_{water}gx = (Sl\rho_{block})\frac{d^2x}{dt^2} \quad \Rightarrow \quad \frac{d^2x}{dt^2} + \frac{g}{b}x = 0$$

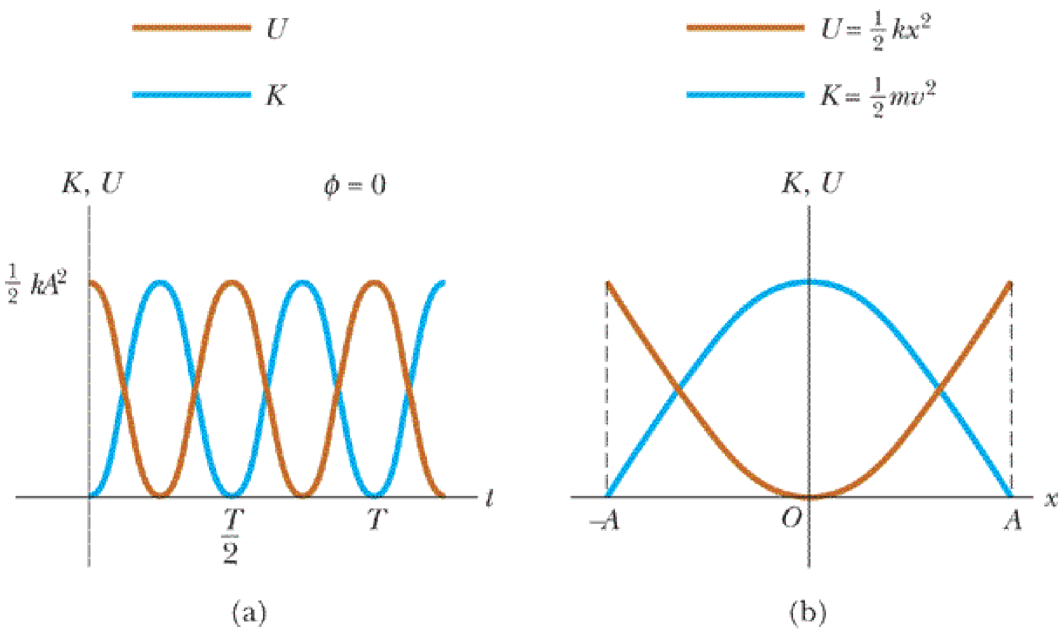
$$x = a\cos\left(\sqrt{\frac{g}{b}}t + \pi\right) = a\cos\left(\sqrt{\frac{g}{l-a}}t + \pi\right)$$

§ 5 Energy in Simple Harmonic Motion

P304-306

- The total mechanical energy for an isolated simple harmonic oscillator
  - Kinetic energy: $$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$
  - Potential energy: $$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$
  - Total mechanical energy: $$E = K + U = \frac{1}{2}kA^2 = \text{costant}$$

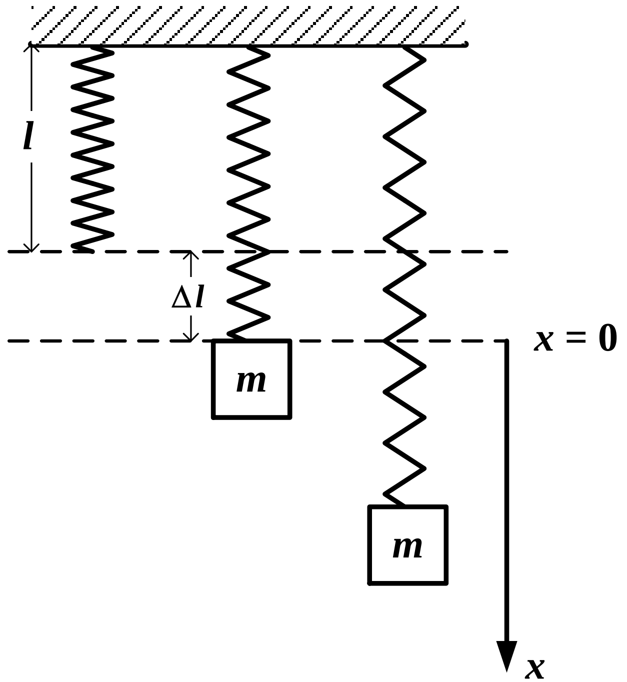
$$\omega^2 = \frac{k}{m}$$



Example

Example —— Vertical SHM: Suppose we hang a spring with force constant k and suspend from it a body with mass m. Oscillation will now be vertical. Will it still be SHM?

Solution I: by Newton' second law



When the body hangs at rest, in equilibrium

$$k\Delta l = mg$$

Take x=0 to be the equilibrium position, and

take the positive x-direction to be downward.

$$F_{net} = -k(x+\Delta l) + mg = -kx - k\Delta l + mg$$

$$= -kx = m\frac{d^2x}{dt^2}$$

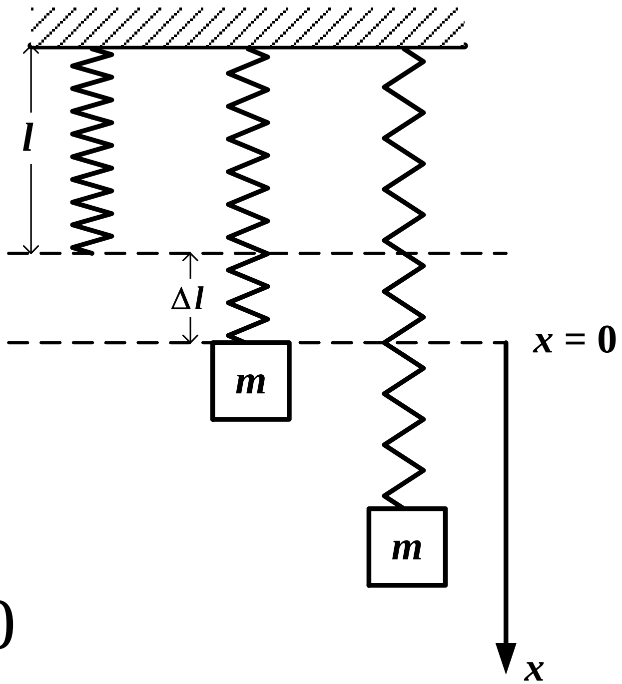
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{d^2x}{dt^2} + \omega^2 x = 0$$

The body's motion is still SHM with the angular frequency:

$$\omega = \sqrt{\frac{k}{m}}$$

Solution II: by energy analysis

When the body is at the position x, the total mechanical energy is



$$\frac{1}{2}mv^2 + \frac{1}{2}k(x+\Delta l)^2 - mgx = \text{costant}$$

by derivative on both sides

$$mv\frac{dv}{dt} + k(x+\Delta l)\frac{dx}{dt} - mg\frac{dx}{dt} = 0$$

$$\frac{dv}{dt} = \frac{d^2x}{dt^2}, \qquad \frac{dx}{dt} = v \qquad m\frac{d^2x}{dt^2} + kx + (k\Delta l - mg) = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{d^2x}{dt^2} + \omega^2 x = 0$$

§ 8 Superposition of SHM

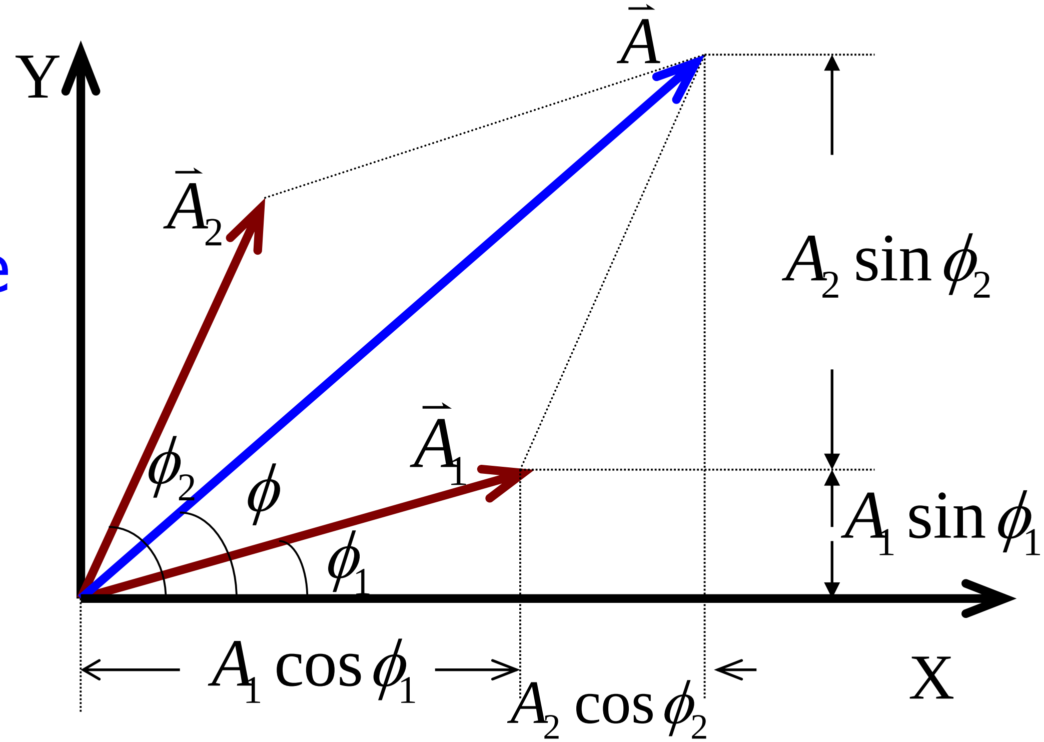
- An object experiences two SHMs simultaneously.
  - Two SHMs
$$x_1 = A_1 \cos(\omega t + \phi_1)$$
$$x_2 = A_2 \cos(\omega t + \phi_2)$$
  - Resultant motion which is superposed by the two SHMs is also a SHM
$$x = x_1 + x_2 = A \cos(\omega t + \phi)$$

Resultant Amplitude ?

Resultant Phase angle ?

Superposition of SHMs using phasor diagram

Using Circle of Reference



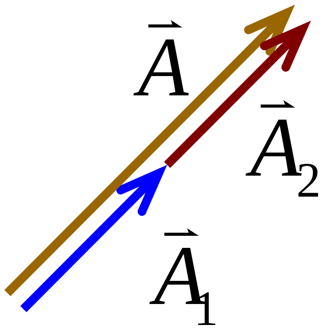
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1)}$$

$$\varphi = \arctan\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}$$

Superposition of SHMs under different phase differences

- The phase difference $\Delta\phi=\phi_2-\phi_1$.
  - When $\Delta\phi=\phi_2-\phi_1=2k\pi$, $k=0, \pm1, \pm2, \ldots$

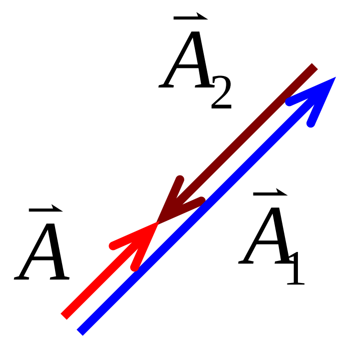
    The two SHMs are in phase, the resultant amplitude take its maximum.



$$A = A_1 + A_2$$

  - When $\Delta\phi=\phi_2-\phi_1=(2k+1)\pi$, $k=0, \pm1, \pm2, \ldots$

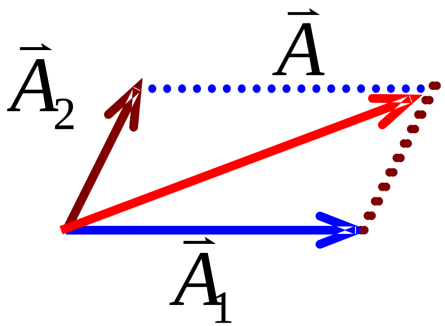
    The two SHMs are out of phase, the resultant amplitude take its minimum.



$$A = | A_1 - A_2 |$$

  - Generally, $\Delta\phi=\phi_2-\phi_1\neq k\pi$

$$| A_1 - A_2 | < A < A_1 + A_2$$



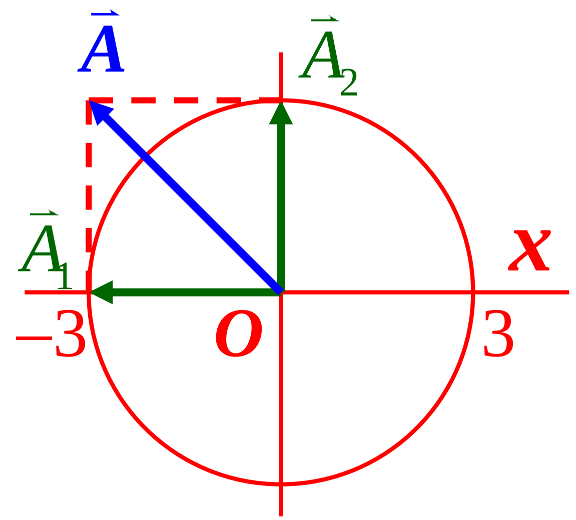
Example: $x_1=3\cos(2\pi t+\pi)$cm, $x_2=3\cos(2\pi t+\pi/2)$cm, find the superposition displacement of x_1 and x_2.

Solution:

Draw a circle of reference,

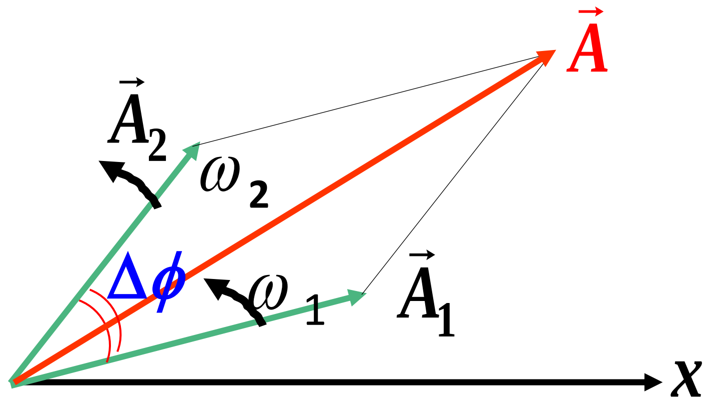
$$x = x_1 + x_2 = A\cos(\omega t + \phi)$$

$$= 3\sqrt{2}\cos(2\pi t + \frac{3\pi}{4}) \quad \text{cm}$$



2.Superposition of Two SHM in Same Direction With Different frequencies

a. Vibration equation



$$x_1 = A_0 \cos \omega_1 t$$

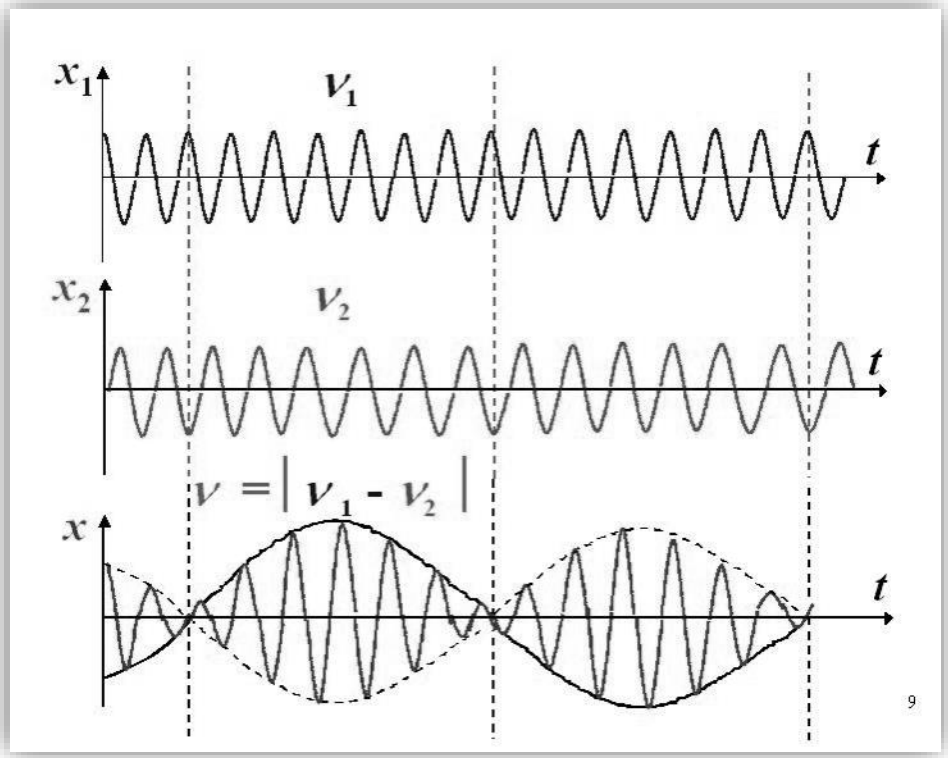
$$x_2 = A_0 \cos \omega_2 t$$

$$x = x_1 + x_2 = 2A_0 \cos \frac{\omega_1 - \omega_2}{2} t \cos \frac{\omega_1 + \omega_2}{2} t$$

$$\omega_1 \approx \omega_2 \Rightarrow \Delta\omega = \omega_1 - \omega_2 << \omega_1 + \omega_2 = \bar{\omega}$$

$$x = A(t) \cos \bar{\omega} t$$

b. Features of figure $x = A(t)\cos\bar{\omega}t$



$$A(t) = 2A_0\cos\left(\frac{\omega_2 - \omega_1}{2}\right)t$$

Amplitude modulation factor

$\dfrac{\omega_2 - \omega_1}{2}$: frequency modulation

$\dfrac{\omega_2 + \omega_1}{2}$: carrier frequency

3.Superposition of two SHM in different directions

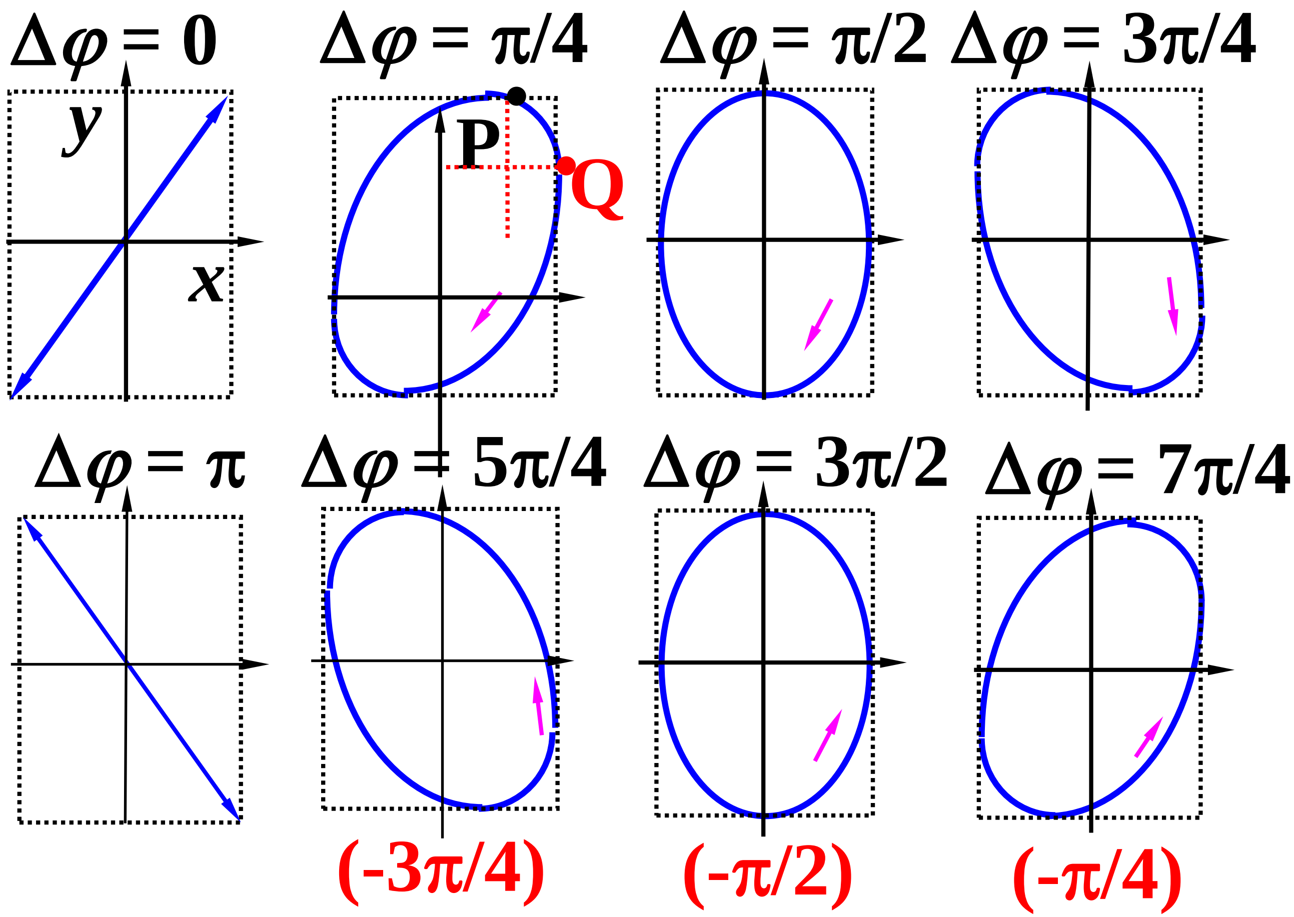
a. Two SHMs

$$x = A_1 \cos(\omega t + \phi_1)$$
$$y = A_2 \cos(\omega t + \phi_2)$$

$$\Rightarrow \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - 2\frac{x}{A_1}\frac{y}{A_2}\cos(\varphi_2 - \varphi_1) = \sin^2(\varphi_2 - \varphi_1)$$

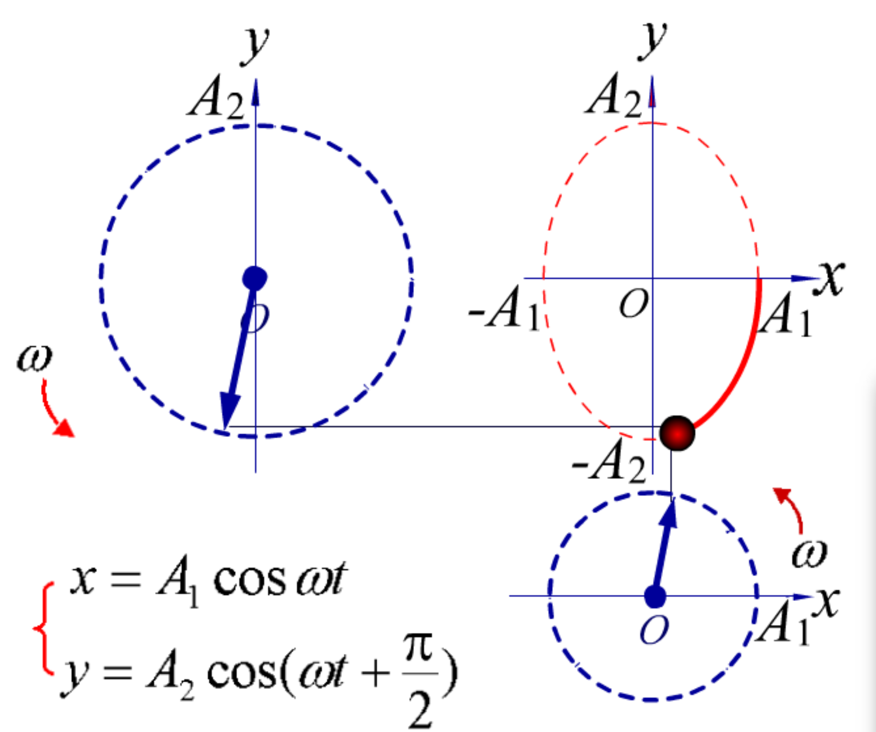
$$\Delta\varphi = k\pi \Rightarrow \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} \pm 2\frac{x}{A_1}\frac{y}{A_2} = 0 \Rightarrow \frac{x}{A_1} \pm \frac{y}{A_2} = 0$$

$$\Delta\varphi = (2k+1)\frac{\pi}{2} \Rightarrow \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

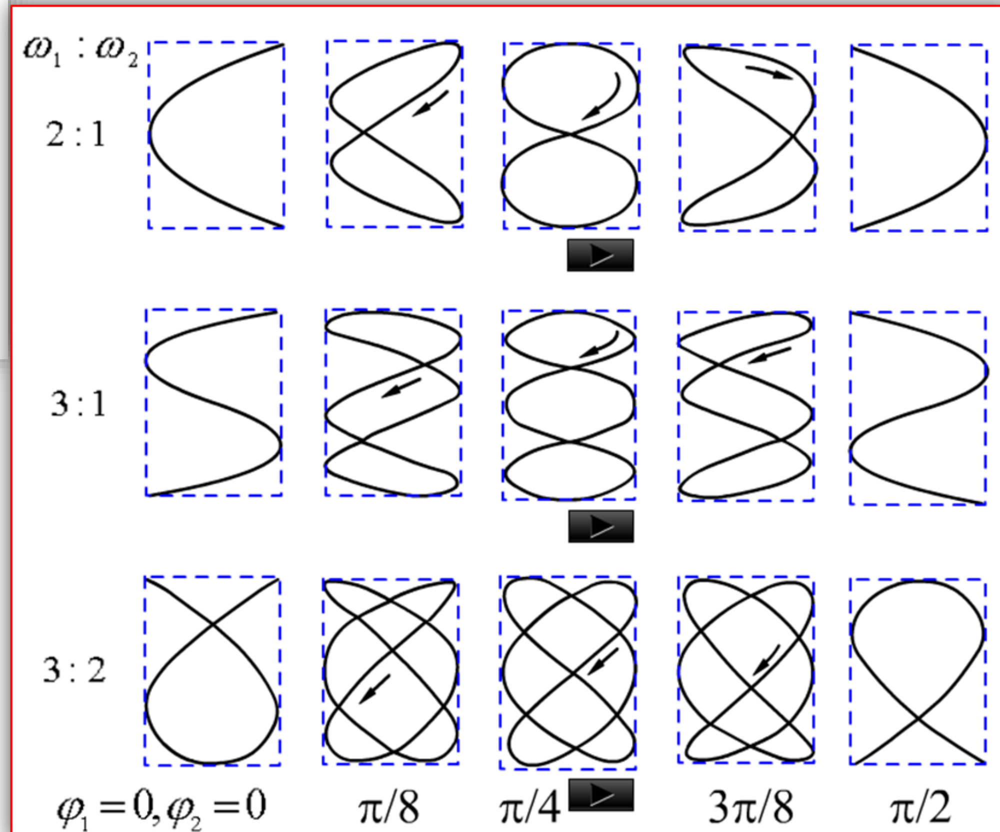


$\Delta\varphi = 0$
y
x
$\Delta\varphi = \pi/4$
P
Q
$\Delta\varphi = \pi/2$
$\Delta\varphi = 3\pi/4$
$\Delta\varphi = \pi$
$\Delta\varphi = 5\pi/4$
$\Delta\varphi = 3\pi/2$
$\Delta\varphi = 7\pi/4$
$(-3\pi/4)$
$(-\pi/2)$
$(-\pi/4)$

Two vertical simple harmonic oscillator



$$\begin{cases} x = A_1 \cos \omega t \\ y = A_2 \cos(\omega t + \dfrac{\pi}{2}) \end{cases}$$



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