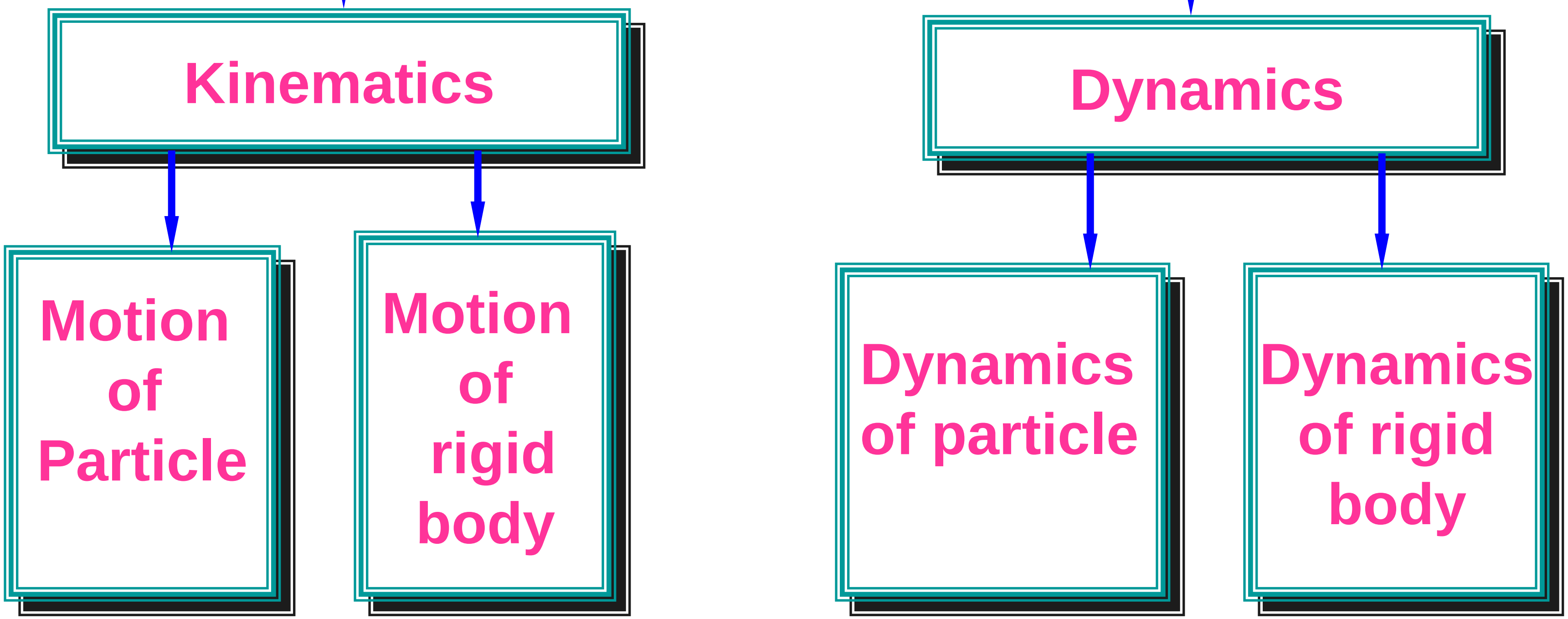
Part 1 Classic Mechanic



- Kinematics: the part of mechanics that deals with the description of motion.
- Dynamics: the relation of motion to its causes.

Chapter 2 Kinematics in One, Two and Three Dimensions

§ 1 Frame of Reference, Coordinate system

p. 17

- Frame of Reference
  - To describe the position of an object, the other object referred to (Reference Frame 参照系）should be chosen. It is arbitrary.
  - Observers in different reference frames may measure different velocities or accelerations.

Observer in the truck: the ball moves in a vertical path;

Observer on the Earth: the ball is in projection motion.

§ 1 Frame of Reference, Coordinate system

Cartesian coordinate system

(rectangular coordinate system)

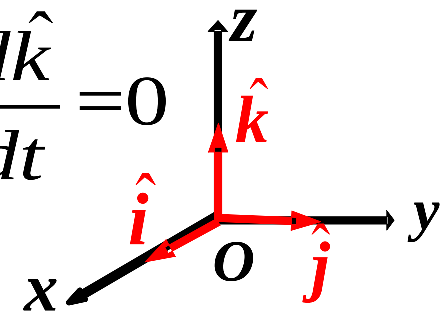
x-y-z constitute a set of orthogonal bases of unit vectors （正交基矢量） $\hat{i},\ \hat{j},\ \hat{k}$

A point is described by (x, y, z)

$$\hat{i}\cdot\hat{j}=\hat{i}\cdot\hat{k}=\hat{j}\cdot\hat{k}=0$$

$$\hat{i}\times\hat{j}=\hat{k},\ \ \hat{j}\times\hat{k}=\hat{i},\ \ \hat{k}\times\hat{i}=\hat{j}$$

$$\frac{d\hat{i}}{dt}=\frac{d\hat{j}}{dt}=\frac{d\hat{k}}{dt}=0$$



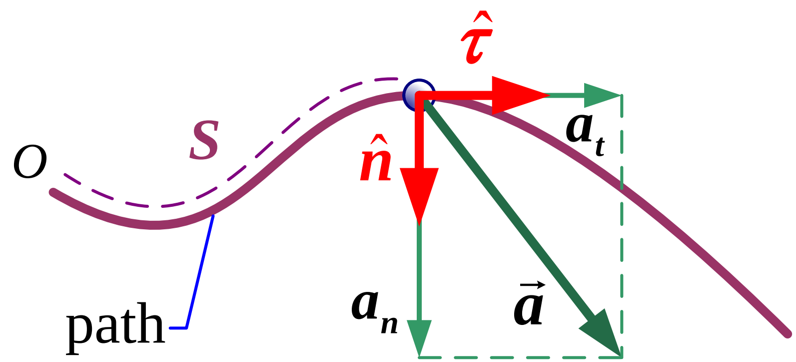
Natural coordinate:

Orthogonal bases: $\hat{n},\ \ \hat{\tau}$

tangential and normal

$$\hat{n}\cdot\hat{\tau}=0$$

usually $\quad \dfrac{d\hat{n}}{dt}\neq 0,\qquad \dfrac{d\hat{\tau}}{dt}\neq 0$

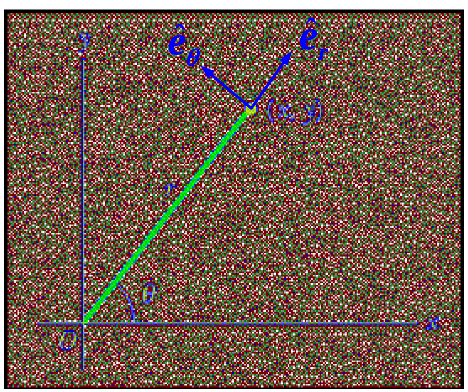


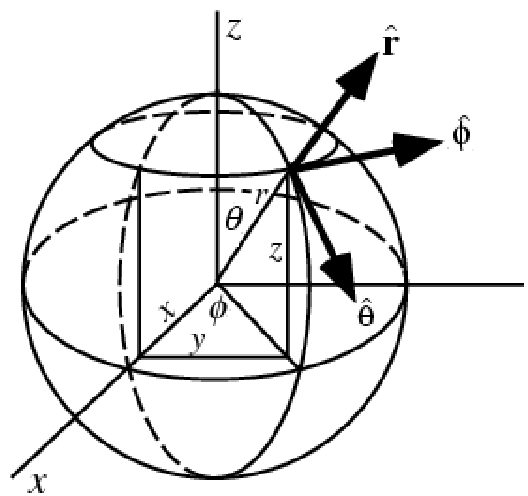
§ 1 Frame of Reference, Coordinate system

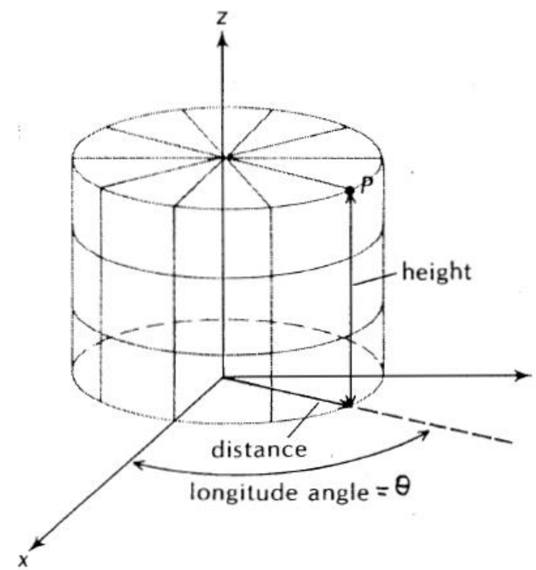
Other Coordinate System (不要求)

- Plane polar coordinate
- Spherical coordinate: (r, ϕ, θ)

Cylindrical coordinate: (r, θ, z)







§ 2 Position and Displacement

P17, p52

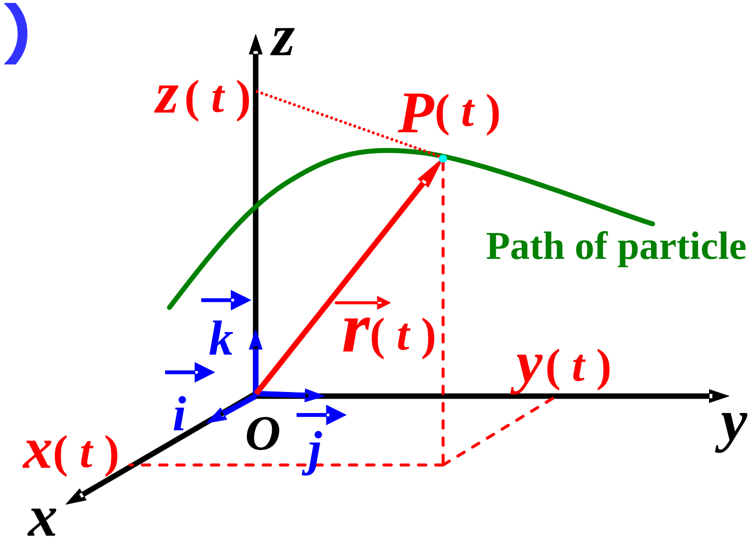
Position Vector(位矢）—The location of a particle relative to the origin of a coordinate system. $\vec{r}$

For a Cartesian system: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Motion Function (运动方程)

$$\vec{r} = \vec{r}(t)$$

$$= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$



Trajectory Equation

——轨道方程

$$f(x, y, z) = 0$$

§ 2 Position and Displacement

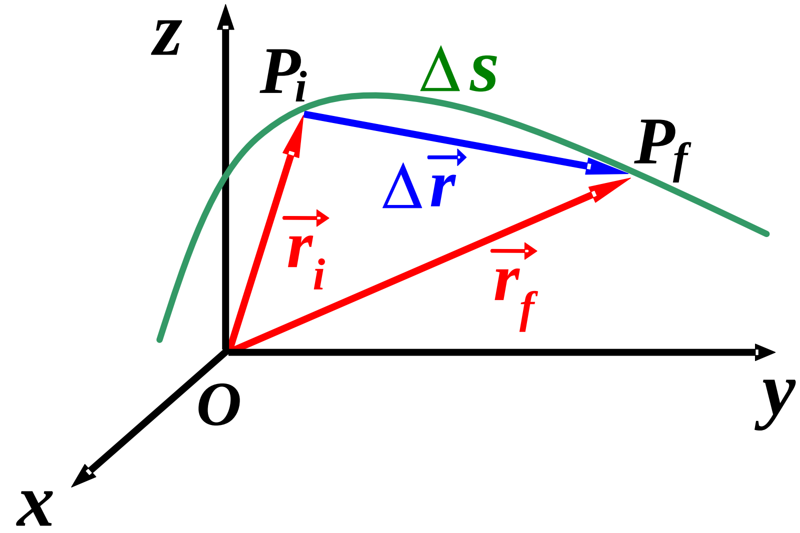
Displacement (位移) ——change in position

$$\Delta\vec{r} = \vec{r}_f(t+\Delta t) - \vec{r}_i(t) = \vec{r}_f - \vec{r}_i$$

$$= (x_f - x_i)\hat{i} + (y_f - y_i)\hat{j} + (z_f - z_i)\hat{k}$$

$$= \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$$

The displacement vector extends from the head of the initial position vector to the head of the later position vector.



§ 2 Position and Displacement

Notes about Displacement:

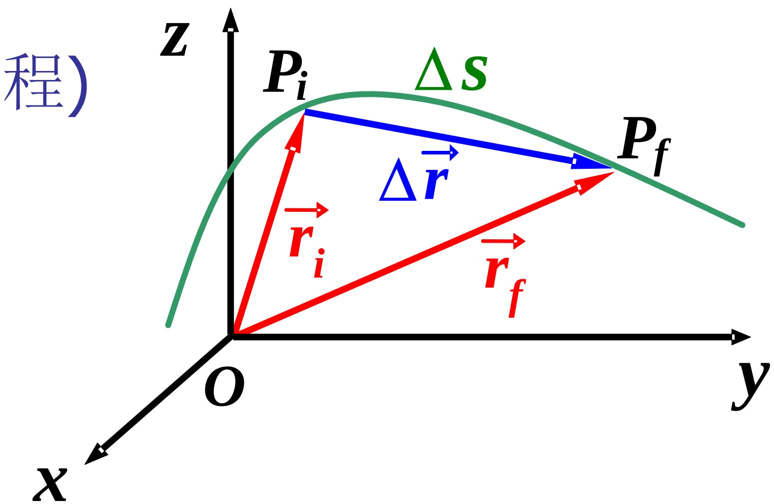
(i). Vector —— The magnitude of vector should be the length of this vector, i.e.

$$|\Delta\vec{r}| = \sqrt{(x_f - x_i)^2 + (y_f - y_i)^2 + (z_f - z_i)^2}$$

(ii). The displacement is independent on the choice of origin

(iii). Different from the distance(路程)

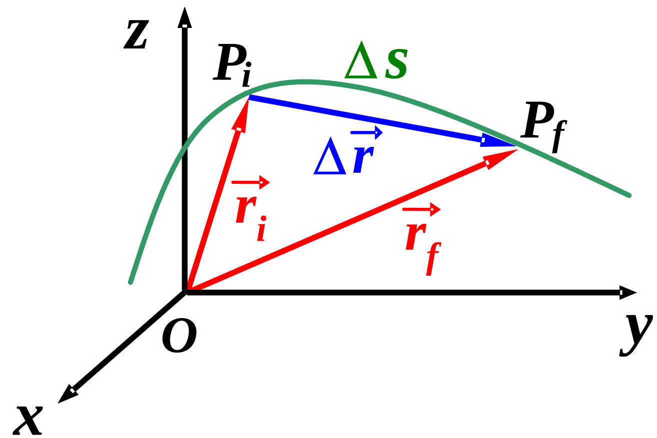
Distance ΔS is the total lengths of the path curve， scalar，



§ 2 Position and Displacement

Generally: $\Delta s \neq |\Delta \vec{r}|$

but for infinitesimal: $\mathbf{d}s = |\mathbf{d}\vec{r}|$

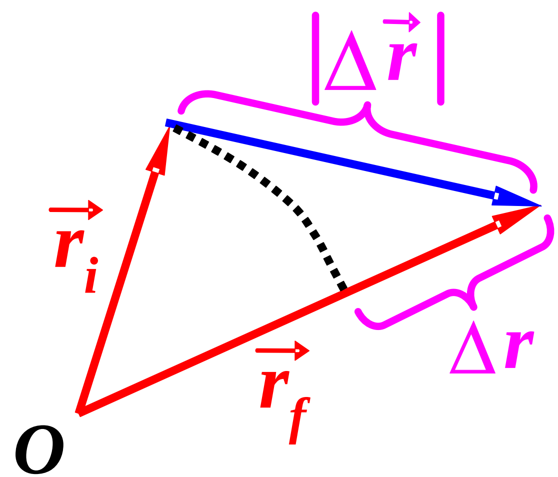


Caution: compare two quantities

$$|\Delta\vec{r}| \quad \Delta r$$

$$|\Delta\vec{r}| = |\vec{r}_f - \vec{r}_i| \neq \Delta r = r_f - r_i = |\vec{r}_f| - |\vec{r}_i|$$

$$dr = d\,|\vec{r}| \neq |d\vec{r}|$$

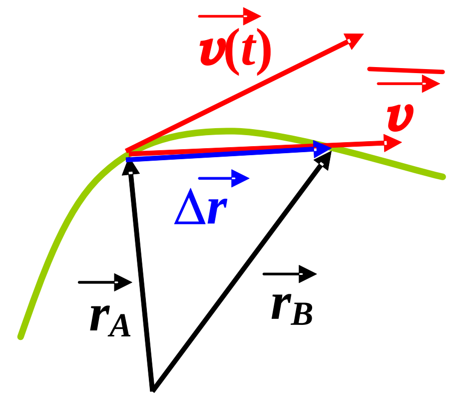


§ 3 Velocity and speed

P18-21, p52

- Velocity and speed
  - Average velocity:

$$\bar{\vec{v}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \bar{\vec{v}} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k}$$



  - Instantaneous velocity:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

In Cartesian coordinate

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$\begin{cases} \text{Direction is along tangent line} \\ \text{Magnitude is } v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} \end{cases}$$

Average and Instantaneous Speed (速率):

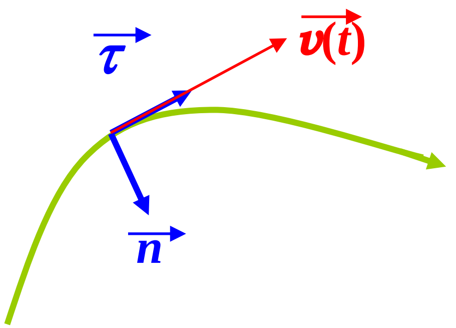
Average Speed: $$\bar{v} = \frac{\Delta S}{\Delta t}$$

Instantaneous Speed: $$\boxed{v = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \frac{dS}{dt} = \frac{|d\vec{r}|}{dt} = |\vec{v}|}$$

The magnitude of instantaneous velocity equals to instantaneous speed (瞬时速度的大小等于瞬时速率)

Speed: $\boxed{v = \frac{dS}{dt}}$ and $v \neq \frac{dr}{dt}$

In natural coordinate $$\vec{v} = \frac{ds}{dt}\hat{\tau}$$

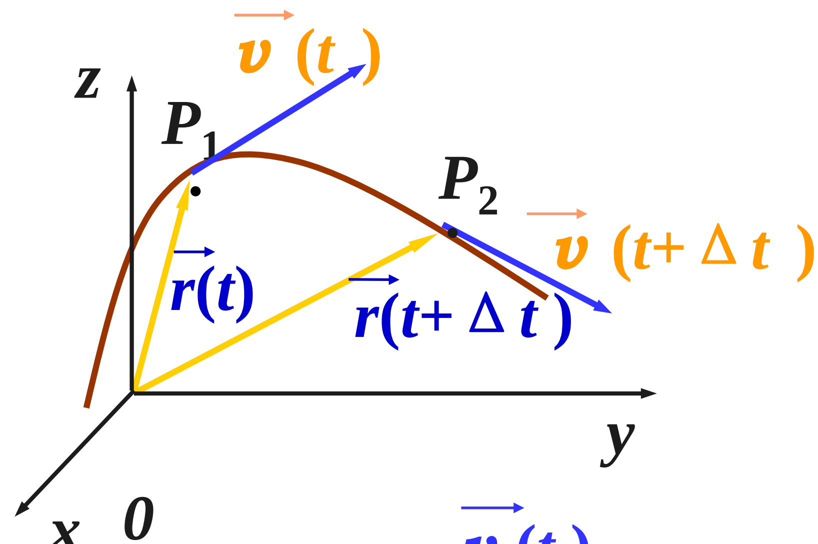


§ 4 Acceleration

P22-25, p53

- Acceleration
  - Average acceleration:

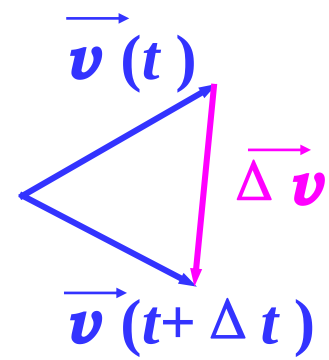
$$\bar{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t}$$



  - Instantaneous acceleration:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

direction: limiting direction of $\Delta \vec{v}$



  - Acceleration in Cartesian coordinate:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$= \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

Example

Example: A particle moves with the motional function as:

$$\vec{r} = 2t\,\hat{i} + (2 - t^2)\,\hat{j} \qquad \text{(SI)}$$

Find: (1) its trajectory function;

(2) its velocities at t=1s and 2s respectively;

(3) its accelerations at t=1s and 2s respectively;

(4) the path distance it travels during this time interval.

Example

Example: A particle moves with the motional function as:

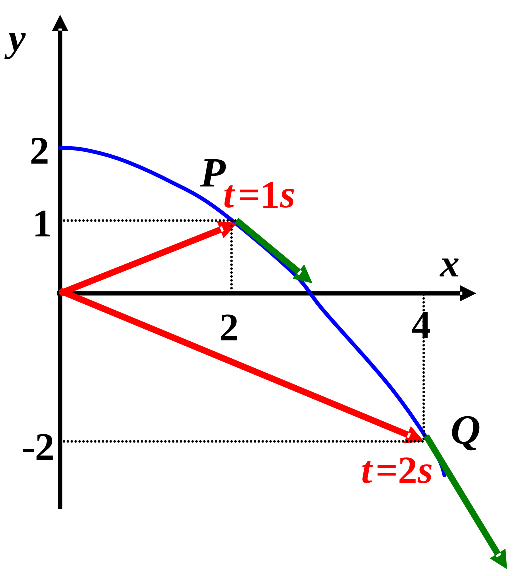
$$\vec{r} = 2t\,\hat{i} + (2 - t^2)\,\hat{j} \quad \text{(SI)}$$

Solution: (1) $x = 2t$, $y = 2 - t^2$ by canceling t ⇨ $y = 2 - \frac{1}{4}x^2$ parabola

(2) $\vec{v} = \dot{\vec{r}} = 2\hat{i} - 2t\,\hat{j}$

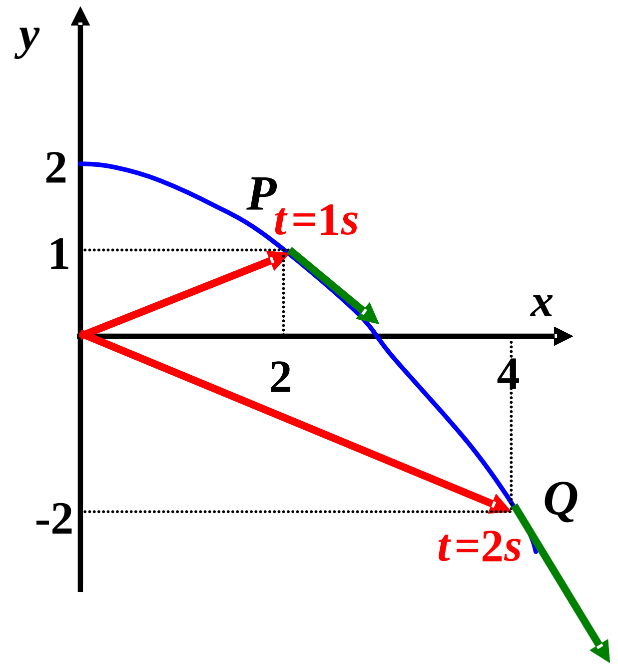
t=1s, $\vec{r}_1 = 2\hat{i} + 1\hat{j}$ $\quad \vec{v}_1 = 2\hat{i} - 2\,\hat{j}$

t=2s, $\vec{r}_2 = 4\hat{i} - 2\,\hat{j}$ $\quad \vec{v}_2 = 2\hat{i} - 4\,\hat{j}$



(3) $\vec{a} = \dot{\vec{v}} = -2\hat{j}$ Constant acceleration

(4) The path distance $PQ = \int_P^Q dS$



$$dS = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{1 + \frac{1}{4}(2t)^2}\ 2dt = 2\sqrt{1+t^2}\,dt$$

$$PQ = \int_1^2 2\sqrt{1+t^2}\,dt = 3.62\text{m}$$

$$\vec{r}_1 = 2\hat{i} + 1\hat{j}$$

$$|\Delta\vec{r}| = \sqrt{2^2 + 3^2} = 3.61\ \text{m} < \Delta S$$

$$\vec{r}_2 = 4\hat{i} - 2\hat{j}$$

§ 5 Two categories of problems in Kinematics

P34-35, p56-61

- Two categories of problems in Kinematics
  - The position of particle is known quantity, Find its velocity and acceleration——By way of derivatives
  - The acceleration of particle is known quantity, Find its velocity and position——By way of integrals.

Known quantities **Unknown quantities**

$$\vec{r}(t) \xrightarrow{\text{derivative}} \vec{v}(t), \quad \vec{a}(t)$$

$$\vec{a}(t) + \text{initial conditions } (\vec{v}(0), \vec{r}(0)) \xrightarrow{\text{integral}} \vec{v}(t), \quad \vec{r}(t)$$

Example

Example: For uniformly accelerated rectilinear motion, find the relationships between velocity and time, position and time, velocity and position.

Starting with $$\frac{dv}{dt} = a = \text{constant}$$ **or** $$dv = a\,dt$$

By integration $$\int_{v_0}^{v} dv = a\int_0^t dt$$ $$\boxed{v - v_0 = at}$$

Starting with $$\frac{dx}{dt} = v_0 + at \qquad \int_{x_0}^{x} dx = \int_0^t (v_0 + at)dt$$ $$\boxed{x - x_0 = v_0 t + \frac{1}{2}at^2}$$

Introducing x as intermediate variable

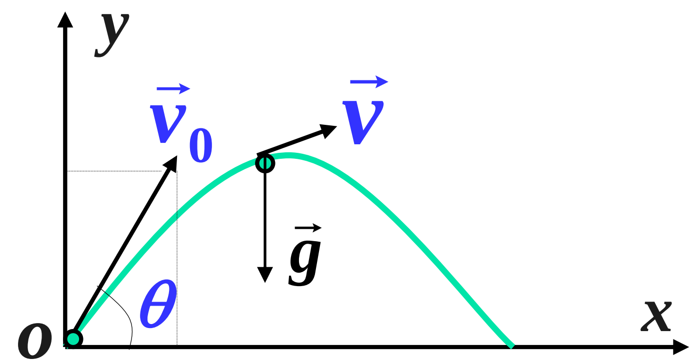
$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = a \qquad \int_{v_0}^{v} v\,dv = a\int_{x_0}^{x} dx$$

$$\boxed{v^2 - v_0^2 = 2a(x - x_0)}$$

Example

Projectile Motion (P54): initial velocity $\vec{v}_0$, initial position $\vec{r}_0 = 0$

$$acceleration \qquad \vec{a} = \vec{g}$$



$$\because \quad \vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \quad \text{and} \quad \vec{a} = -g\vec{j}$$

$$\therefore \quad \int_{\vec{v}_0}^{\vec{v}} \mathrm{d}\vec{v} = \int_{t_0}^{t} \vec{a}\,\mathrm{d}t = \int_{t_0}^{t} (-g\vec{j})\,\mathrm{d}t$$

$$\vec{v} - \vec{v}_0 = -gt\,\vec{j}$$

$$\because \vec{r} = \int_0^{\vec{r}} \mathrm{d}\vec{r} = \int_0^{t} \vec{v}\,\mathrm{d}t$$

$$\therefore \vec{r} = \vec{v}_0 t + \frac{1}{2}\vec{g}t^2$$

初速度方向的匀速直线运动 +
竖直方向的自由落体运动

In Cartesian coordinate

$$v_{0x} = v_0 \cos\theta \quad ; \quad v_{0y} = v_0 \sin\theta$$

$$a_x = 0 \quad ; \quad a_y = -g$$

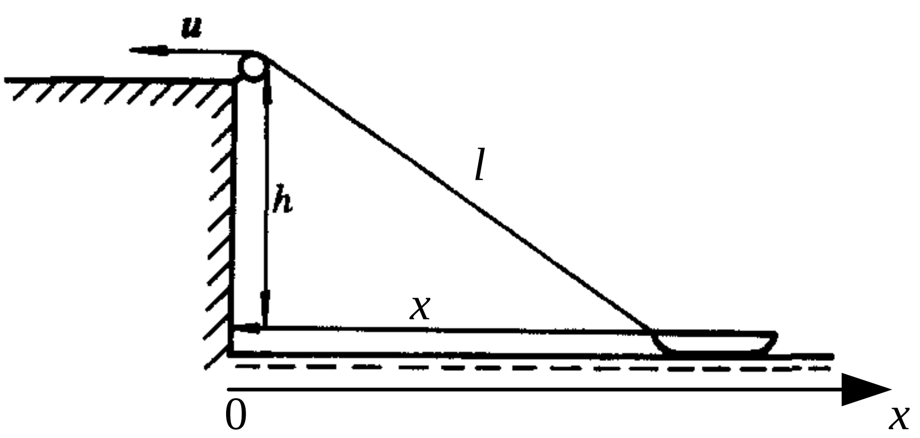
$$\vec{v} = (v_0 \cos\theta)\vec{i} + (v_0 \sin\theta - gt)\vec{j} = v_x \vec{i} + v_y \vec{j}$$

$$\therefore \vec{r} = (v_o t \cos\theta)\vec{i} + (v_o t \sin\theta - \frac{1}{2} g t^2)\vec{j}$$

Trajectory: $$y = x\tan\theta - \frac{1}{2}\frac{g x^2}{v_o^2 \cos^2\theta}$$

Example

Example: A person on a cliff pulls a boat floating in water with a constant velocity ***u*** through a rope over a pulley fixed on the edge of the cliff. The height of cliff above water is ***h***, and the horizontal distance between the cliff and the boat is ***x***. Find the velocity and acceleration of the boat in water.

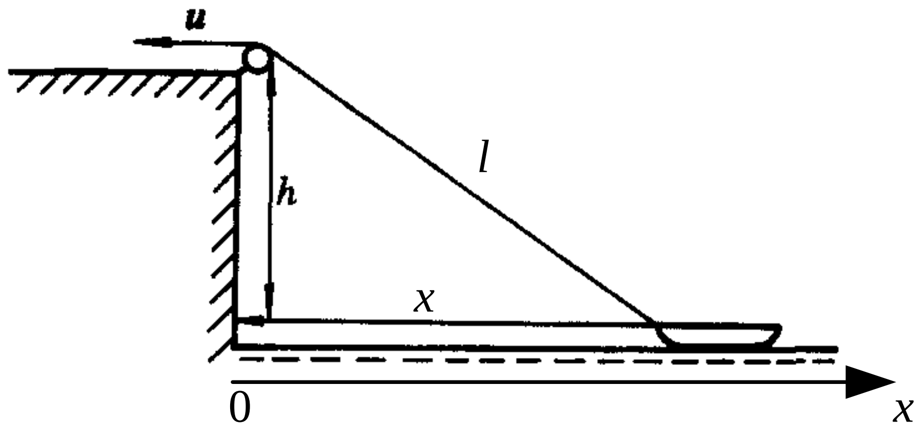


Example

Solution : Take right side to be positive.

Staring from the relation: $l^2 = h^2 + x^2$ $\qquad 2l\dfrac{dl}{dt} = 2x\dfrac{dx}{dt}$

Notice: $\dfrac{dl}{dt} = -u \qquad \dfrac{dx}{dt} = v$



$$\boxed{v = -\frac{l}{x}u = -\frac{\sqrt{h^2+x^2}}{x}u = -\frac{u}{\cos\theta}}$$

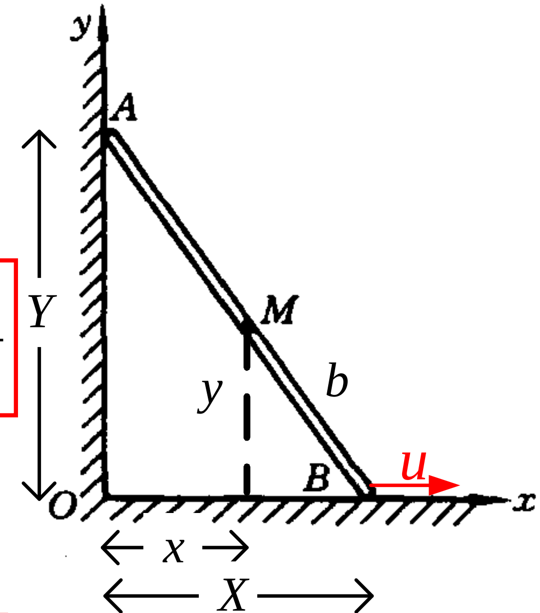
$$\boxed{a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = -\frac{h^2}{x^3}u^2}$$

Note

$$\boxed{\begin{aligned}\frac{dv}{dx} &= -\left[\frac{1}{x}\frac{d\left(\sqrt{h^2+x^2}\right)}{dx} + \sqrt{h^2+x^2}\,\frac{d\left(\frac{1}{x}\right)}{dx}\right]u \\ &= -u\left[\frac{1}{x}\frac{2x}{2\sqrt{h^2+x^2}} + \sqrt{h^2+x^2}\left(-\frac{1}{x^2}\right)\right] = \frac{h^2}{x^2\sqrt{h^2+x^2}}u\end{aligned}}$$

Example

Example: A ladder of length ***l*** leans against a vertical wall. The bottom end of the ladder slides to the right with the constant speed of ***u***. Find the velocities and accelerations of points A and M (|MB|=*b*) when |OB|=*X*.



Solution:

(1) Point A. From relation: $X^2 + Y^2 = l^2 \quad 2X\dfrac{dX}{dt} + 2Y\dfrac{dY}{dt} = 0$

$$\boxed{v_{Ay} = -\frac{X}{Y}u = -\frac{X}{\sqrt{l^2 - X^2}}u} \quad \boxed{a_{Ay} = \frac{dv_{Ay}}{dt} = \frac{dv_{Ay}}{dX}\frac{dX}{dt} = -\frac{l^2u^2}{(l^2 + X^2)^{3/2}}}$$

from relation: $\dfrac{X - x}{b} = \dfrac{X}{l} \qquad X - x = \dfrac{b}{l}X$

$$\boxed{v_{Mx} = \frac{l-b}{l}u} \quad \boxed{a_{Mx} = \frac{dv_{Mx}}{dt} = \frac{l-b}{l}\frac{du}{dt} = 0}$$

$$\frac{Y}{l} = \frac{y}{b} \quad \boxed{v_{My} = \frac{b}{l}v_{Ay} = -\frac{bX}{l\sqrt{l^2 + X^2}}u} \quad \boxed{a_{My} = \frac{dv_{My}}{dt} = \frac{b}{l}\frac{dv_{Ay}}{dt} = -\frac{blu^2}{(l^2 + X^2)^{3/2}}}$$

Example

Example: A particle moves in xy-plane. Its motional equations are:

$$x(t) = R\cos\omega t \qquad y(t) = R\sin\omega t$$

where ***R*** and ***ω*** are constant.

(1) Show that the particle moves in a circle of radius ***R***.

(2) Show that the magnitude of the particle's velocity is constant and equals ***ωR***.

(3) Show that the particle's acceleration is always opposite to its position vector and has the magnitude of ***ω²R***.

Solution:

(1) Its path equation: $x^2 + y^2 = R^2$, So it moves in a circle of radius ***R***.

(2) $v_x = \dfrac{dx}{dt} = -\omega R\sin\omega t, \quad v_y = \dfrac{dy}{dt} = \omega R\cos\omega t \qquad v = \sqrt{v_x^2 + v_y^2} = \omega R$

(3) $\vec{a} = \dfrac{dv_x}{dt}\hat{i} + \dfrac{dv_y}{dt}\hat{j} = -\omega^2 R\cos\omega t\,\hat{i} - \omega^2 R\sin\omega t\,\hat{j}$

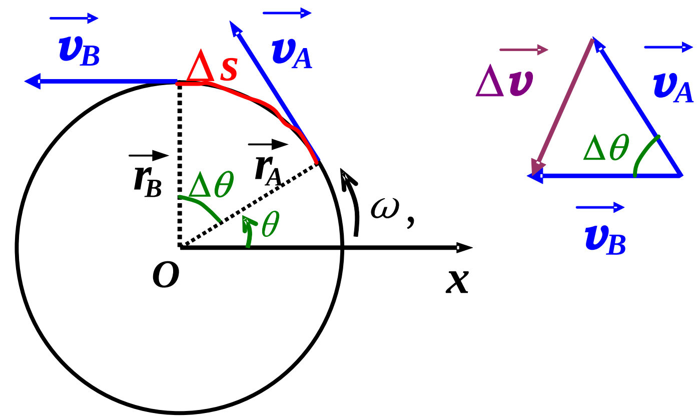
$$= -\omega^2(R\cos\omega t\,\hat{i} + R\sin\omega t\,\hat{j}) = -\omega^2\vec{r}$$ is opposite to the position vector

$$a = \sqrt{a_x^2 + a_y^2} = \omega^2 R$$

§ 6 Circular Motion

P62-64, p119-120

(1) Uniform Circular Motion——Centripetal acceleration



- Characteristics
  - Moves in a circle with constant speed:

    $$|\vec{v}| = v = \text{constant}$$

  - Change in direction, has an acceleration:
- Centripetal acceleration

  (meaning "seeking center")

  $$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$$

$$\Delta\vec{v} = \vec{v}_B - \vec{v}_A \begin{cases} \textbf{Magnitude } |\Delta\vec{v}| = 2v\sin\dfrac{\Delta\theta}{2} \xrightarrow{\Delta t \to 0} 2v\dfrac{\Delta\theta}{2} = v\Delta\theta = \dfrac{v}{r}\Delta s \\ \textbf{Limiting direction: perpendicular to } \vec{v}_A\textbf{, point toward the center} \end{cases}$$

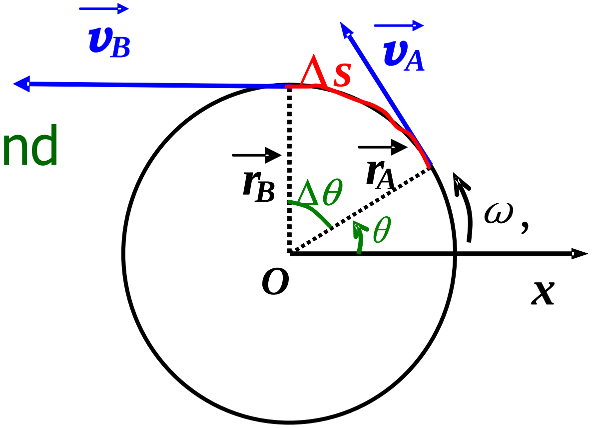
$$a = \lim_{\Delta t \to 0} \frac{v}{r}\frac{\Delta s}{\Delta t} = \frac{v}{r}\frac{ds}{dt} = \frac{v^2}{r}$$

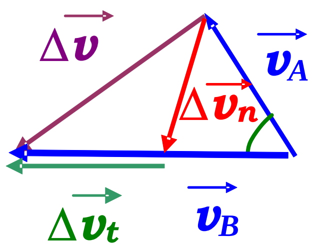
In natural coordinate: $\boxed{\vec{a} = \dfrac{v^2}{r}\hat{n}}$

(2) Non-Uniform Circular Motion—tangential and normal acceleration

- Characteristics
  - Changes both in magnitude and direction

$$\Delta\vec{v} = \Delta\vec{v}_t + \Delta\vec{v}_n$$





$\Delta\vec{v}_n$ represents the change in direction— same as in uniform circular motion.

$$\vec{a}_n = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}_n}{\Delta t} = \frac{v^2}{r}\hat{n}$$

Normal acceleration—due to the change in direction of the velocity vector.

$\Delta\vec{v}_t$ represents the change in magnitude. $|\Delta\vec{v}_t| = |\vec{v}_B| - |\vec{v}_A| = \Delta|\vec{v}|$

$$\vec{a}_t = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}_t}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta|\vec{v}|}{\Delta t}\hat{\tau} = \frac{dv}{dt}\hat{\tau}$$

Tangential acceleration—arises from the change in magnitude of the velocity vector (change rate of speed).

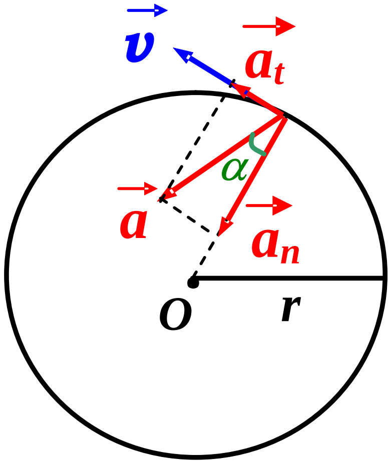
Circular Motion

Total acceleration vector:

$$\vec{a} = \vec{a}_n + \vec{a}_t = \frac{v^2}{r}\hat{n} + \frac{dv}{dt}\hat{\tau}$$

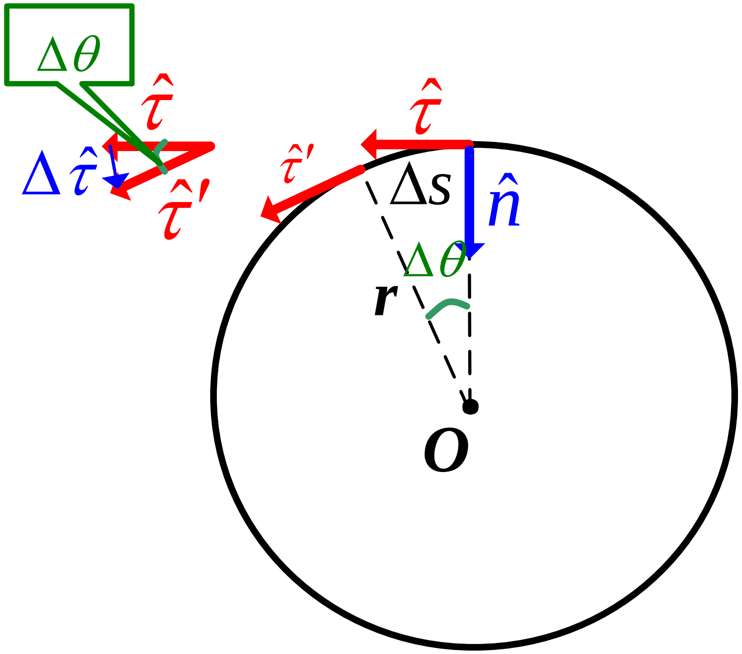
$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

$$\alpha = \arctan\frac{a_t}{a_n}\begin{cases} >0, \text{ also } a_t>0, \text{ if the speed increases.} \\ <0, \text{ also } a_t<0, \text{ if the speed decreases.} \end{cases}$$



Circular Motion

Another explanation of acceleration vector:



$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\hat{\tau}) = \frac{dv}{dt}\hat{\tau} + v\frac{d\hat{\tau}}{dt}$$

$$\frac{d\hat{\tau}}{dt} = \frac{d\hat{\tau}}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt}$$

$$\lim_{\Delta\theta\to 0}\frac{\Delta\hat{\tau}}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{\Delta\theta\cdot 1}{\Delta\theta}\hat{n} = \hat{n} \qquad \frac{d\theta}{ds} = \lim_{\Delta\theta\to 0}\frac{\Delta\theta}{r\cdot\Delta\theta} = \frac{1}{r} \qquad \frac{ds}{dt} = v$$

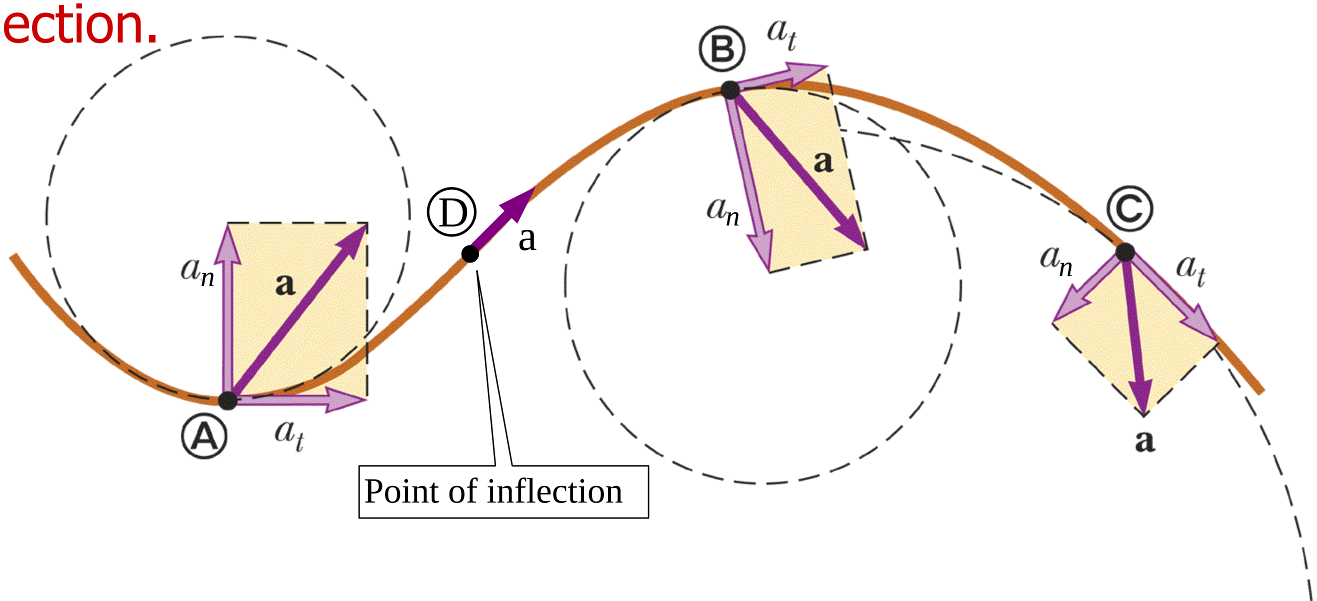
$$\boxed{\vec{a} = \frac{dv}{dt}\hat{\tau} + \frac{v^2}{r}\hat{n}}$$

(3) Motion Along a Arbitrary Curved Path in Plane

- Tangential acceleration and normal acceleration

$$\vec{a} = \vec{a}_t + \vec{a}_n = \frac{dv}{dt}\hat{\tau} + \frac{v^2}{\rho}\hat{n}$$

  - Tangential acceleration— same as circular motion.
  - Normal acceleration— same as circular motion except that ρ is the radius of curvature of the path at the point.— always directs toward the center of the curvature. — be zero when particle passes through a point of inflection.



What does the particle motion?

(1) $a_n = 0, \quad a_t = 0$

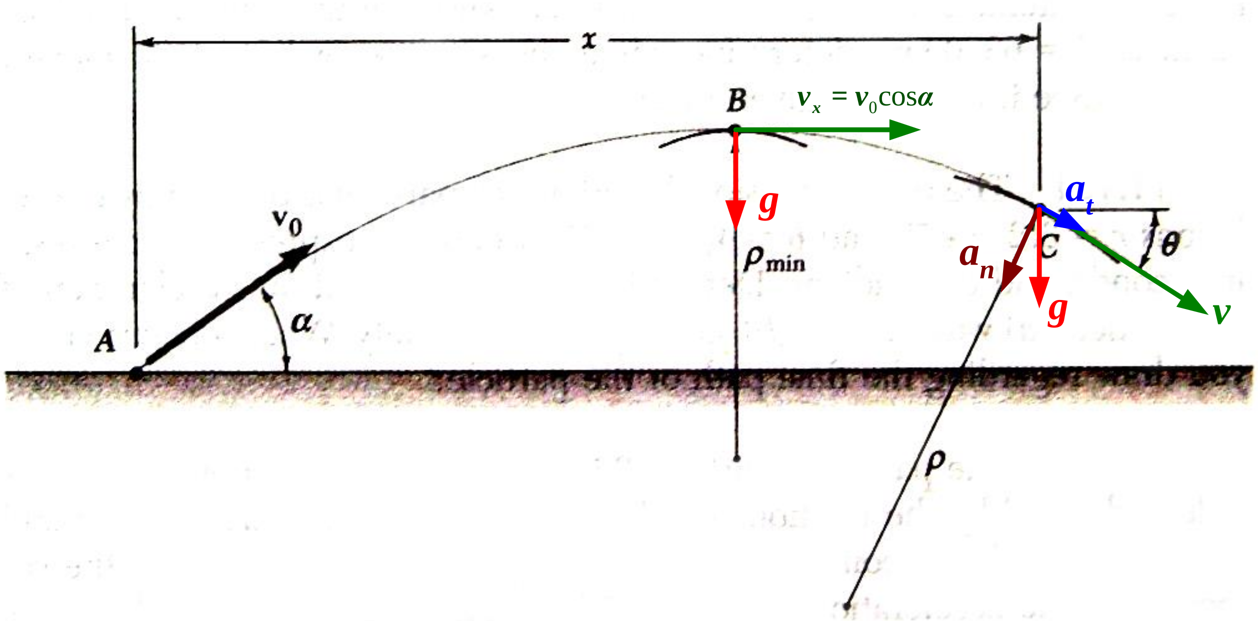
(2) $a_n = 0, \quad a_t \neq 0$

(3) $a_n \neq 0, \quad a_t = 0$

(4) $a_n \neq 0, \quad a_t \neq 0$

Example

Example: A projectile is fired from point A with an initial velocity $\mathbf{v_0}$ which forms an angle α with the horizontal. Find the radii of curvature of the trajectory of the projectile at point B and C.



Example

Solution: At point B, $\boldsymbol{g}$ is the normal acceleration. Therefore:

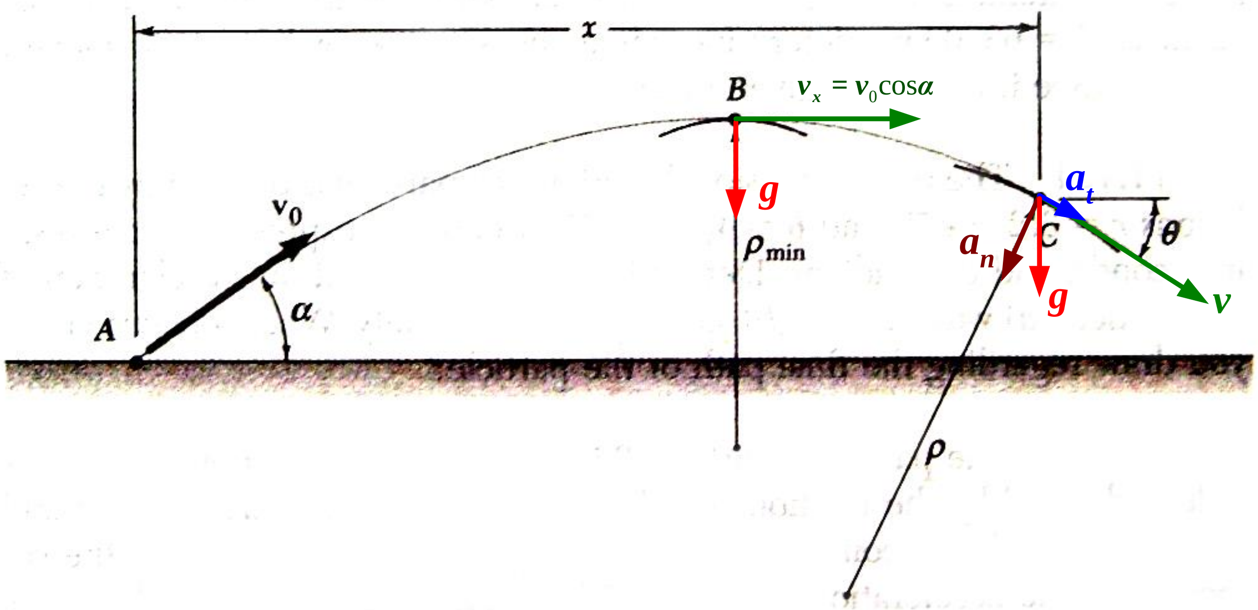
$$\rho_B = \frac{v_x^2}{g} = \frac{v_0^2 \cos^2 \alpha}{g}$$

At point C, $a_t = g\sin\theta$, $a_n = g\cos\theta$

$$v = \frac{v_x}{\cos\theta} = \frac{v_0 \cos\alpha}{\cos\theta}$$

$$\boxed{\rho_C = \frac{v^2}{a_n} = \frac{v_x^2}{g\cos^3\theta} = \frac{v_0^2 \cos^2\alpha}{g\cos^3\theta}}$$

We can see that at point B, $\boldsymbol{\rho}$ reaches its minimum.



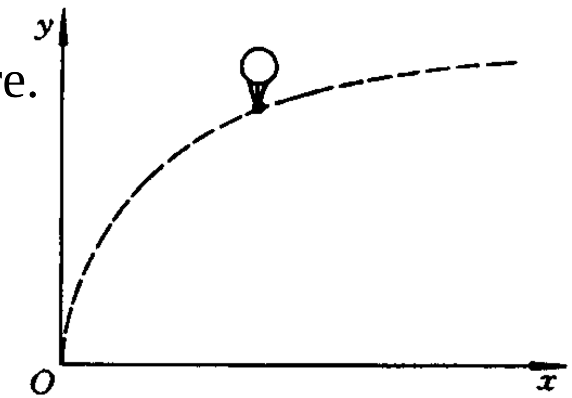
Example

Example: A balloon moves up from ground with an initial vertical velocity of $\boldsymbol{v_0}$. For the reason of wind, in the air the balloon is blew to the right with horizontal velocity $\boldsymbol{v_x = by}$ ($\boldsymbol{b}$ is a positive constant, $\boldsymbol{y}$ is the height of the balloon). Choose the right side to be positive for x axis.

(1) Find the motional equation of the balloon.

(2) Find the path (trajectory) equation of balloon.

(3) Determine the tangential acceleration and the radius of the curvature of the trajectory with respect to height y.



Solution: Establish a coordinate system shown in the Figure. Let the balloon locates at origin point O when t=0.

(1) $$v_y = \frac{dy}{dt} = v_0 \qquad y = v_0 t$$

$$v_x = \frac{dx}{dt} = by \qquad \frac{dx}{dt} = bv_0 t \qquad \int_0^x dx = \int_0^t bv_0 t\, dt$$

$$x = \frac{1}{2} bv_0 t^2$$

Motional equation: $\boxed{\vec{r} = \frac{1}{2} bv_0 t^2 \hat{i} + v_0 t\ \hat{j}}$

Example

(2) By canceling time t, we get the path equation of balloon.

$$\boxed{x = \frac{b}{2v_0} y^2}$$

(3) The speed of balloon:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{b^2 v_0^2 t^2 + v_0^2} = \sqrt{b^2 y^2 + v_0^2}$$

$$\boxed{a_t = \frac{dv}{dt} = \frac{b^2 v_0 t}{\sqrt{b^2 t^2 + 1}} = \frac{b^2 v_0 y}{\sqrt{b^2 y^2 + v_0^2}}}$$

$$a^2 = \left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2 = (bv_0)^2 + 0^2 = b^2 v_0^2 \qquad a_n = \sqrt{a^2 - a_t^2} = \frac{bv_0^2}{\sqrt{b^2 y^2 + v_0^2}}$$

$$\boxed{\rho = \frac{v^2}{a_n} = \frac{(b^2 y^2 + v_0^2)^{3/2}}{bv_0^2}}$$

Example

Example: **A particle moves along a circle of radius of *R*. The path it follows is $s = v_0 t - \frac{1}{2} b t^2$. v_0 and *b* are positive constant ($v_0^2 > Rb$).**

(1) When will $|a_t| = a_n$?

(2) When will the magnitude of acceleration equals *b*?

(3) How many revolutions that the particle have completed when magnitude of acceleration reaches to *b*?

Example

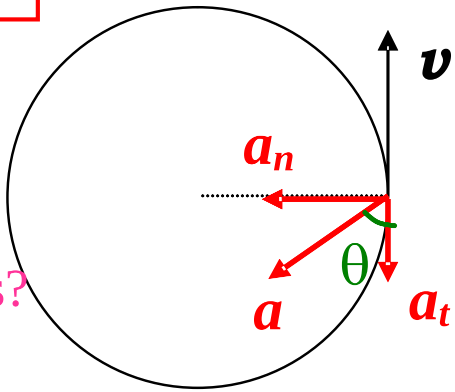
Solution:

(1) $v = \frac{ds}{dt} = v_0 - bt$ $\quad a_t = \frac{dv}{dt} = -b$ $\quad a_n = \frac{v^2}{R} = \frac{(v_0 - bt)^2}{R}$

$a_t = a_n$ $\quad b = \frac{(v_0 - bt)^2}{R}$ $\quad \boxed{t = \frac{v_0}{b} \pm \sqrt{\frac{R}{b}}}$

(2) $a = \sqrt{b^2 + \frac{(v_0 - bt)^4}{R^2}} = b$ $\quad \boxed{t = \frac{v_0}{b}}$

Why two values?



(3) When

$t = \frac{v_0}{b}$ $\quad s = v_0 \frac{v_0}{b} - \frac{1}{2} b \frac{v_0^2}{b^2} = \frac{v_0^2}{2b}$

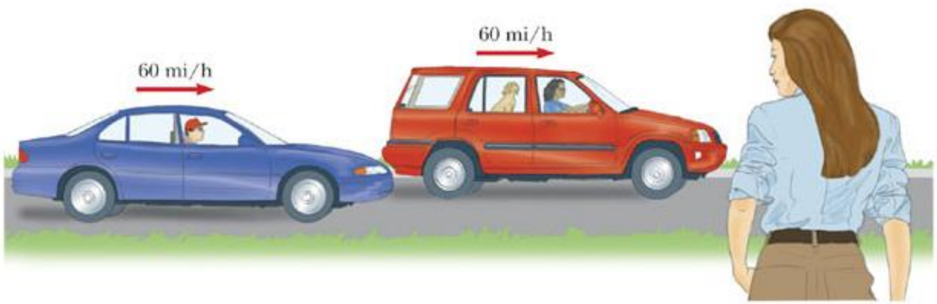
$$\boxed{N = \frac{s}{2\pi R} = \frac{v_0^2}{4\pi bR}}$$

§ Relative Velocity p.64-67

- The descriptions of the motion are different in different frames of reference.

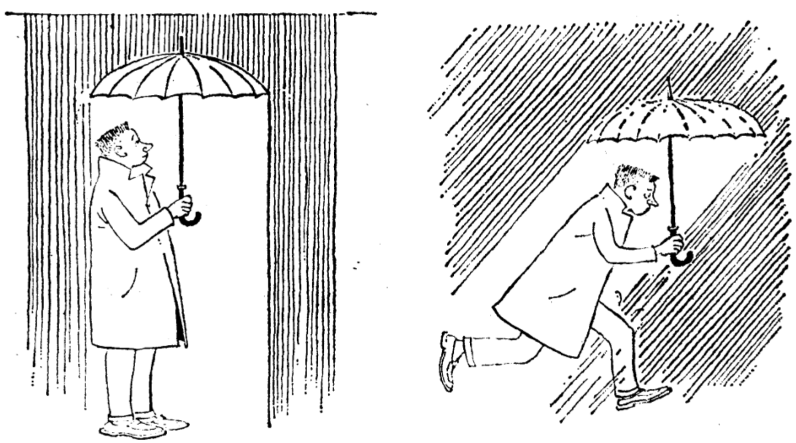
Example 1.

- The lady observer measures a speed for red car of 60mi/h.
- The observer in blue car measures a speed for red car of zero.



Example 2.

- The man in rest feels that the rain falling vertically.
- The man in motion feels that the rain inclines towards him.

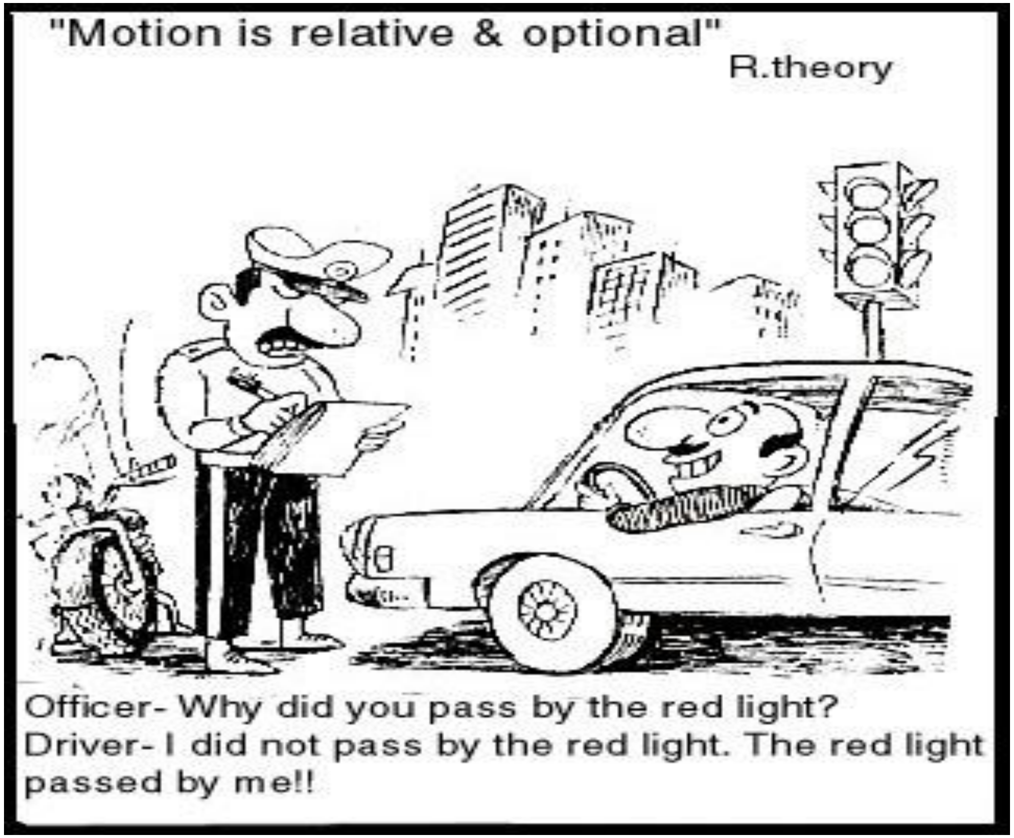


Motion is relative

- The descriptions of the motion are different in different frames of reference.

Example 3.

- Officer – Why did you pass by the red light?
- Driver – I did not pass by the red light. The red light passed by me.

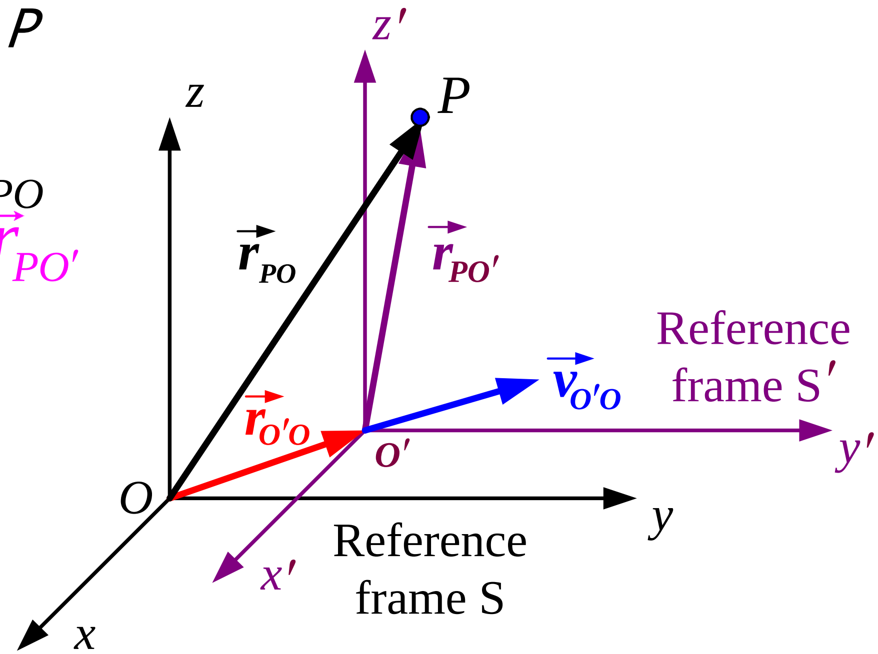


The Relative Motion

The Relative Motion respect to two the Frames in Translation

- The relationship between positions of *P* in two reference frames:
  - The position of *P* relative to *S* is $\vec{r}_{PO}$
  - The position of *P* relative to *S'* is $\vec{r}_{PO'}$

$$\vec{r}_{PO} = \vec{r}_{PO'} + \vec{r}_{O'O}$$



- The relationship between velocities of the particle in the two frames:

$$\frac{d}{dt}(\vec{r}_{PO}) = \frac{d}{dt}(\vec{r}_{PO'}) + \frac{d}{dt}(\vec{r}_{OO'})$$

$$\vec{v}_{PO} = \vec{v}_{PO'} + \vec{v}_{O'O}$$

absolute velocity

relative velocity

attached velocity

The Relative Motion -- subscript rule

- Conventional subscript rule for the equation relating velocities in different reference frame:
  - On the right-hand side: inner subscripts are the same,
  - Whereas the outer subscripts on the right are the same as the two subscripts for the "absolute vector"

$$\vec{v}_{PO} = \vec{v}_{PO'} + \vec{v}_{O'O}$$

The first

The last

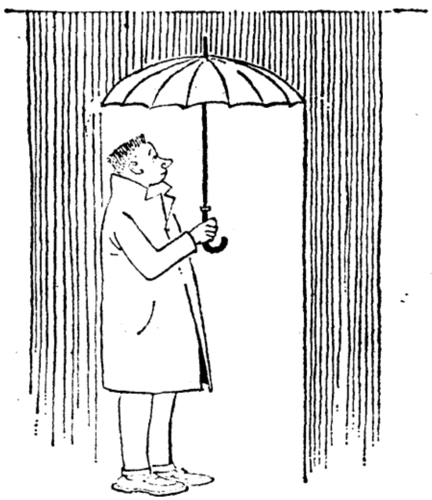
  - Also valid for Position Vectors and Acceleration Vectors

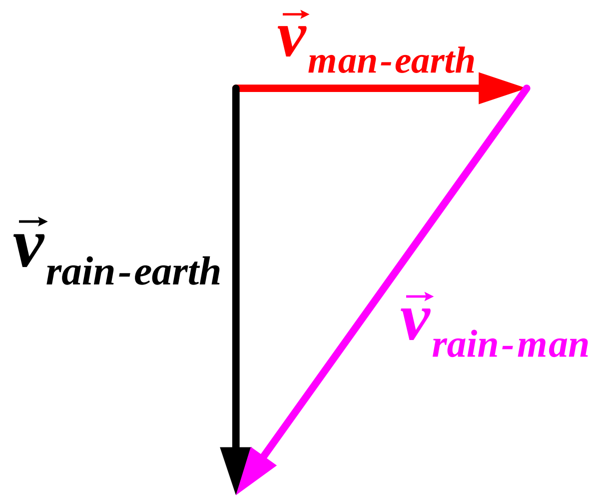
$$\vec{r}_{PO} = \vec{r}_{PO'} + \vec{r}_{O'O}$$

The man in the rain

Example: The man in the rain.

$$\vec{v}_{rain-earth} = \vec{v}_{rain-man} + \vec{v}_{man-earth}$$





Example

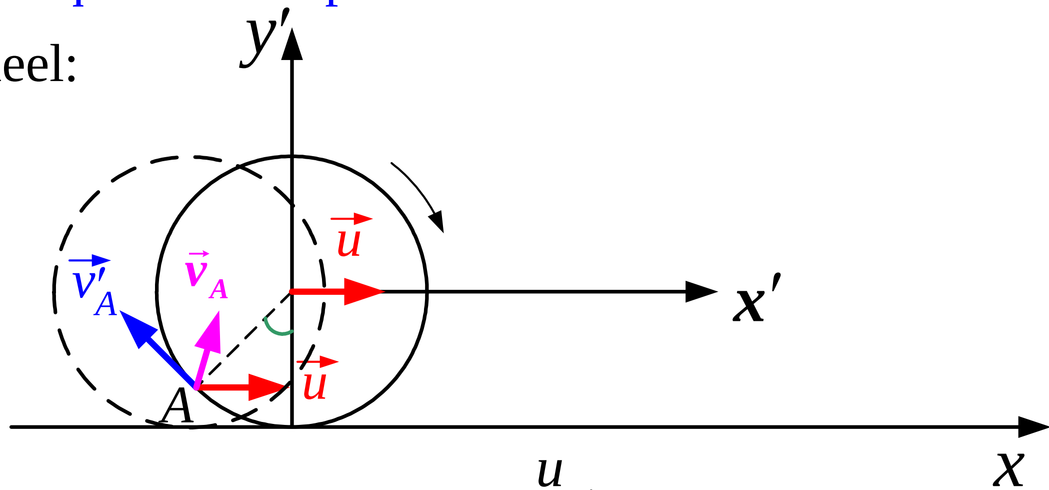
Example: A wheel of radius ***R*** rolls on the ground without slipping. Its center moves with a constant ***u***. Find the motional equation of a point A on the rim of the wheel.

Solution: In the reference frame of wheel:

$$v'_{Ax} = -\omega R \cos \omega t$$

$$v'_{Ay} = \omega R \sin \omega t$$

In the reference frame of ground



$$v_{Ax} = v'_{Ax} + u = -\omega R \cos \omega t + u$$

$$v_{Ay} = v'_{Ay} = \omega R \sin \omega t$$

$$v_{Ax} = -u \cos \frac{u}{R} t + u$$

When t=0 $\quad v_A(0) = 0 \quad$ We get $\quad u = \omega R$

$$v_{Ay} = u \sin \frac{u}{R} t$$

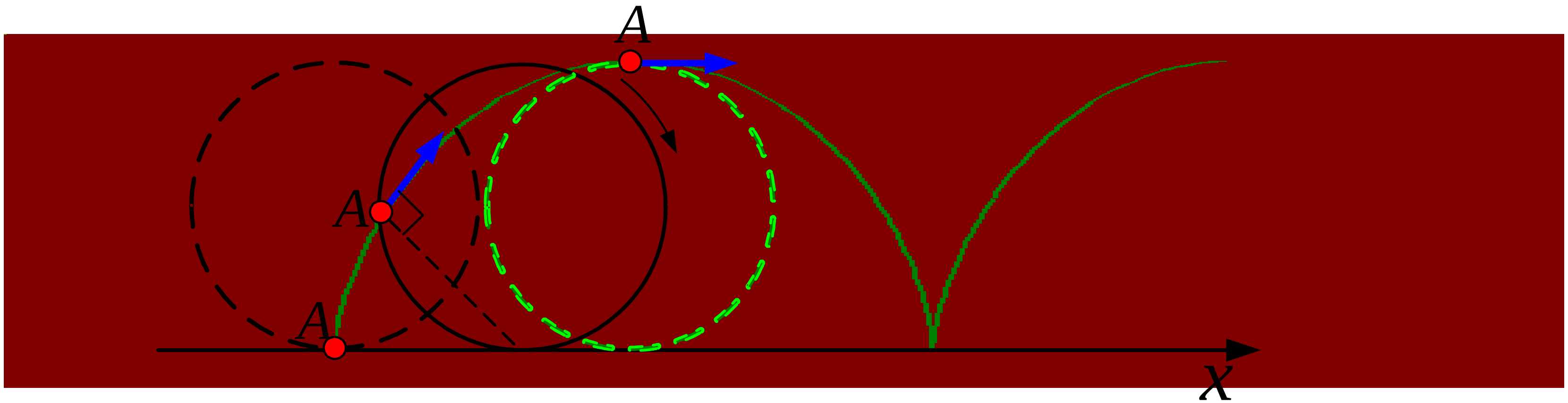
$$x = \int_0^R v_{ax} dt + x_0 = \int_0^t \left(-u \cos \frac{u}{R} t + u \right) dt = ut - R \sin \frac{u}{R} t$$

$$y = \int_0^R v_{ay} dt + y_0 = \int_0^t \left(u \sin \frac{u}{R} t \right) dt = R \left(1 - \cos \frac{u}{R} t \right)$$

The rim moves in the path of a cycloid.

Example

The path of the cycloid:



The program of the cycloid using MatLab:

```
R=1;T=0:0.05:3*pi;
x=T-R*sin(T/R);
y=R*(1-cos(T/R));
plot(x,y)
axis off
```