Chapter 7-8 Work and Energy

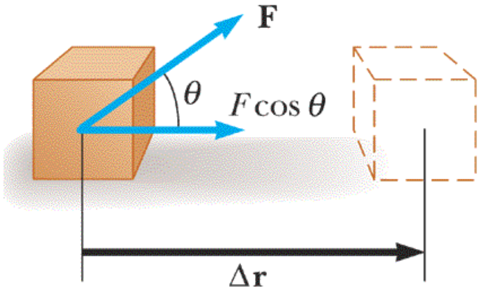
P147-155

§1 Work and Power

1. Work: 研究力和运动的空间过程关系

Work Done by a Constant Force along a Straight-line Displacement

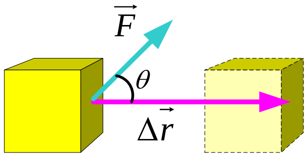
$$W = F\,|\Delta\vec{r}\,|\cos\theta = \vec{F}\cdot\Delta\vec{r}$$



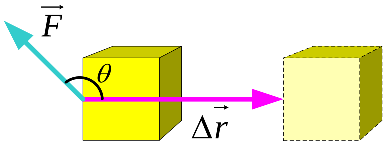
Work is a scalar quantity, no direction.

The SI unit of work: Newton•meter or Joule

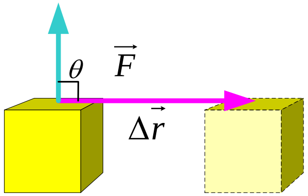
W is positive when θ <90°



W is negative when θ >90°



W is zero when θ =90°



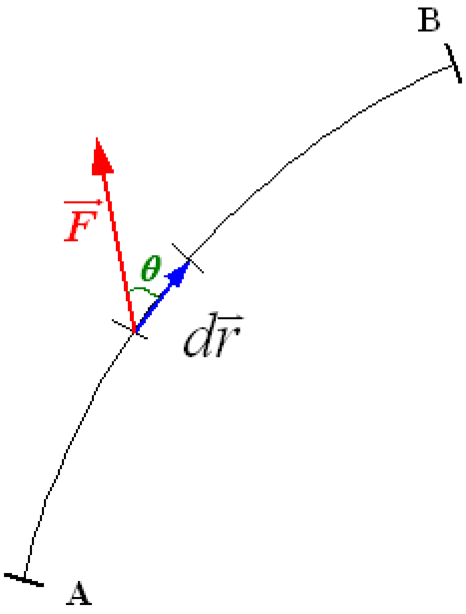
§ 1 Work and Power

Work Done by a Varying Force along a Curve Path

- Divide the path into a large number of small displacement $d\vec{r}$

  In each segment of displacement the work can be considered as done by a constant force along a straight-line displacement.

$$\mathrm{dW} = \vec{F} \cdot \mathrm{d}\vec{r}$$



the total work done by the force

$$\boxed{W = \int_A^B \overrightarrow{F} \cdot d\vec{r}}$$

Line Integral or path integral

§ 1 Work and Power

- Work is a scalar quantity, no direction
- Work is a process quantity. Generally, depends on the path followed by the particle. Different path corresponds to different work done by the same force.
- Calculation of work relates to the reference frame.
- Work done by multiple forces.

  Total work done is the scalar addition of the work done by each force.

$$W_{net} = \int_A^B \vec{F}_{net} \cdot d\vec{l} = \int_A^B \left(\sum_i \vec{F}_i \right) \cdot d\vec{l} = \sum_i \int_A^B \vec{F}_i \cdot d\vec{l} = \sum_i W_i$$

- In Cartesian Coordinate

$$W = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B (F_x dx + F_y dy + F_z dz)$$

$$= \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz$$

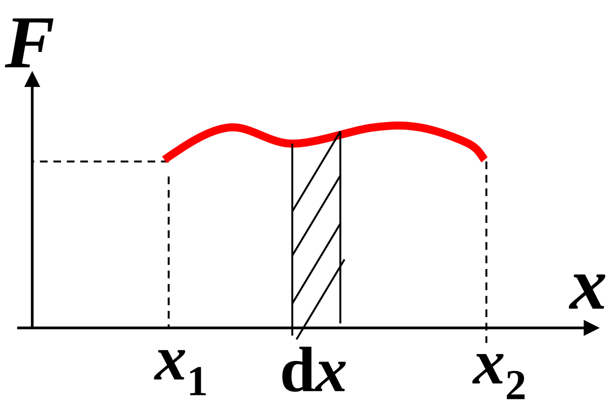
§ 1 Work and Power

In Natural Coordinate

$$W=\int_A^B(\vec{F}_n+\vec{F}_\tau)\cdot ds\hat{\tau}=\int_A^B F_\tau ds$$

Geometrical Representation of Work:

The work done by a force equals the area under the F versus x curve —— 变力曲线与位移轴在极限x_1, x_2 之间所包围的面积.

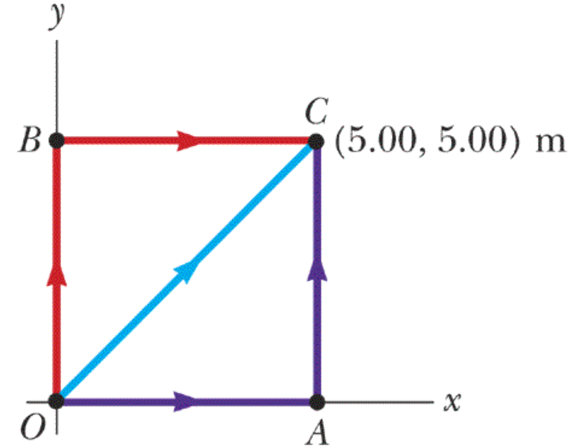


Example: **A force acting on a particle moving in the *xy* plane is given by**

$$\vec{F} = 2y\hat{i} + x^2\hat{j} \quad \text{(SI)}$$

The particle moves from the origin to a final position *C* (5.00m, 5.00m). Calculate the work done by $\vec{F}$ along (1) *OAC*, (b) *OBC*, (3) *OC*.

Solution:



(1) Along path *OAC*:

$$\int_{OAC} \vec{F}\cdot d\vec{l} = \int_{OA} \vec{F}\cdot d\vec{l} + \int_{AC} \vec{F}\cdot d\vec{l}$$

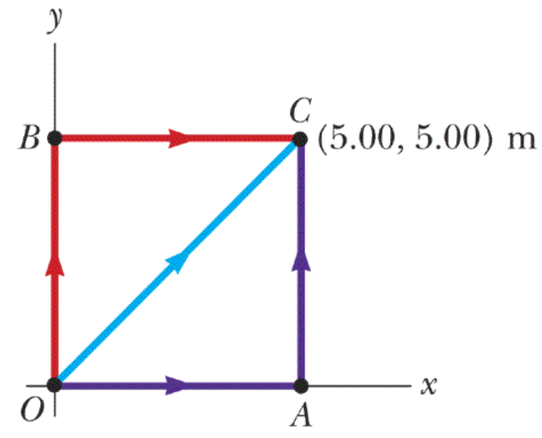
$$= \int_0^5 (2y\hat{i} + x^2\hat{j})\cdot dx\,\hat{i} + \int_0^5 (2y\hat{i} + x^2\hat{j})\cdot dy\,\hat{j}$$

$$= \underbrace{\int_0^5 2y(=0)dx}_{\text{zero}} + \int_0^5 \left(x(=5)\right)^2 dy = \int_0^5 25dy = 125\ \text{J}$$

Example (continued)

$$\vec{F} = 2y\hat{i} + x^2\hat{j}$$

(2) Along path *OBC*:



$$\int_{OBC} \vec{F} \cdot d\vec{l} = \int_{OB} \vec{F} \cdot d\vec{l} + \int_{BC} \vec{F} \cdot d\vec{l}$$

$$= \int_0^5 (2y\hat{i} + x^2\hat{j}) \cdot dy\,\hat{j} + \int_0^5 (2y\hat{i} + x^2\hat{j}) \cdot dx\,\hat{i}$$

$$= \int_0^5 \underbrace{\left(x(=0)\right)^2 dy}_{\text{zero}} + \int_0^5 2y(=5)dx = \int_0^5 2\times 5dx = 50 \text{ J}$$

(3) Along path *OC*: $y = x, \quad d\vec{l} = dx\hat{i} + dy\,\hat{j}$

$$\int_{OC} \vec{F} \cdot d\vec{l} = \int_{OC} (2y\hat{i} + x^2\hat{j}) \cdot (dx\hat{i} + dy\,\hat{j}) = \int_{OC} (2ydx + x^2dy)$$

$$= \int_0^5 2xdx + \int_0^5 y^2dy = 25 + \frac{125}{3} = 66.7 \text{ J}$$

Power (P186)

- The power: The rate at which work is done
  - Average power:

$$\bar{P} = \frac{\Delta W}{\Delta t}$$

  - Instantaneous power:

$$P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

  - SI unit: watt.

§ 2 Work – Kinetic Energy Theorem

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$$W_{net} = \int_A^B \sum_i \vec{F}_i \cdot d\vec{r} = \int_A^B \sum_i F_{it} ds = \int_A^B m\frac{dv}{dt} ds = \int_{v_A}^{v_B} mvdv = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

- Kinetic energy: $K = \frac{1}{2}mv^2$

Process quantity

The change of state quantity

- Work – kinetic energy theorem: $\boxed{W_{net} = K_f - K_i}$
- The work done by the net force on a particle equals the change in kinetic energy.

(1) Work is the measurements for the change in the kinetic energy "*K*" of a body.

$If\ W > 0, K_f > K_i$ ——Body gains the kinetic energy.

$W < 0, K_f < K_i$ —Body loses "*K*" & does work outside.

- (2) Both work and "*K*" are scalars. They have same units and dimensions; "*K*" only depends on the speed of initial and final states, but work depends on the real process(动能是状态量,功是过程量).

- (3) valid in the inertial frame of reference

- For a system with more particles, work-kinetic energy theorem should be: The sum work done on a system by all external and internal forces equal to the change in kinetic energy of the system (质点系动能定理系统外力和内力做功总和等于系统动能的增量).

$$\boxed{\sum W_{i-\text{external}} + \sum W_{i-\text{internal}} = \sum K_f - \sum K_i}$$

Generally, the works done by internal forces between particles cannot be canceled (the displacements of particles are different).

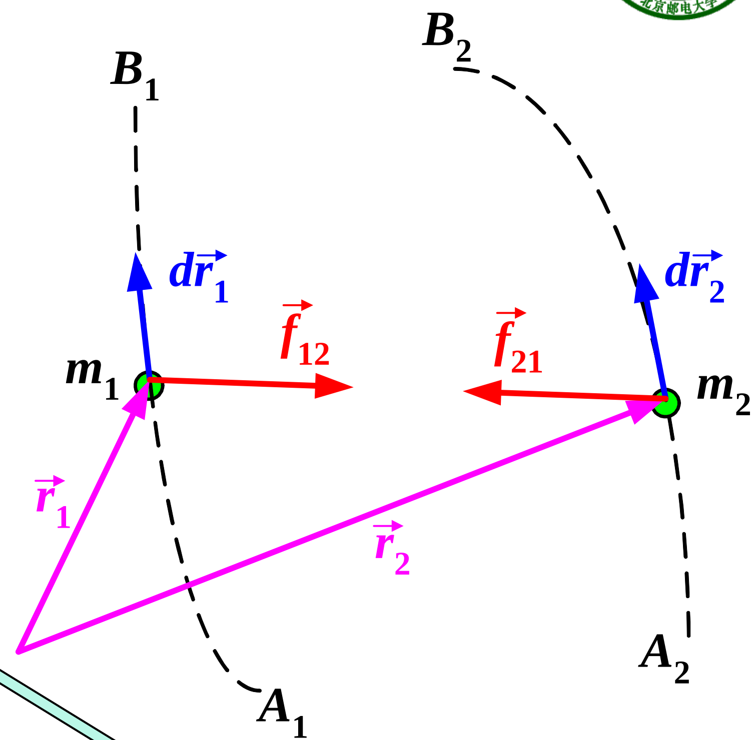
The work done by a pair of internal forces

- The work done by a pair of internal forces

$$\vec{f}_{12} = -\vec{f}_{21}$$

For a infinitesimal process

$$dW = \vec{f}_{12} \cdot d\vec{r}_1 + \vec{f}_{21} \cdot d\vec{r}_2$$
$$= \vec{f}_{21} \cdot (d\vec{r}_2 - d\vec{r}_1) = \vec{f}_{21} \cdot d\underbrace{(\vec{r}_2 - \vec{r}_1)}$$
$$= \vec{f}_{21} \cdot d\vec{r}_{21}$$



The displacement of 2 relative to 1

- The calculation of net work done by a pair of internal forces on two particles is equivalent to —— in the reference frame of particle 1, the calculation of work done by one force acting on particle 2.

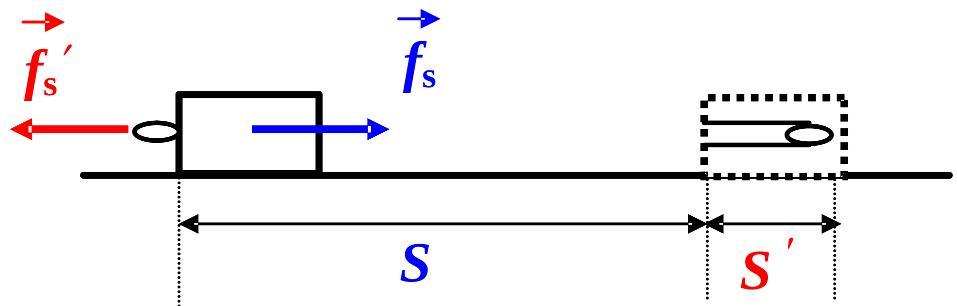
Example

Example: **A bullet coming from left is shot into a wooden block and passes through a length of *S '* in the block. The system of bullet-block comes to a halt after sliding a distance of *S*. Calculate the net work done by a pair of friction forces f_s and f_s' between the bullet and the block.**

Solution: $\vec{f}_s = -\vec{f}_s', \qquad |\vec{f}_s| = |\vec{f}_s'| = f_s$

For the block:

$$W_s = f_s S$$



For the bullet:

$$W_{s'} = -f_s(S + S')$$

The net work:

$$W_s^{\text{net}} = W_s + W_{s'} = f_s S - f_s(S + S') = -f_s S'$$

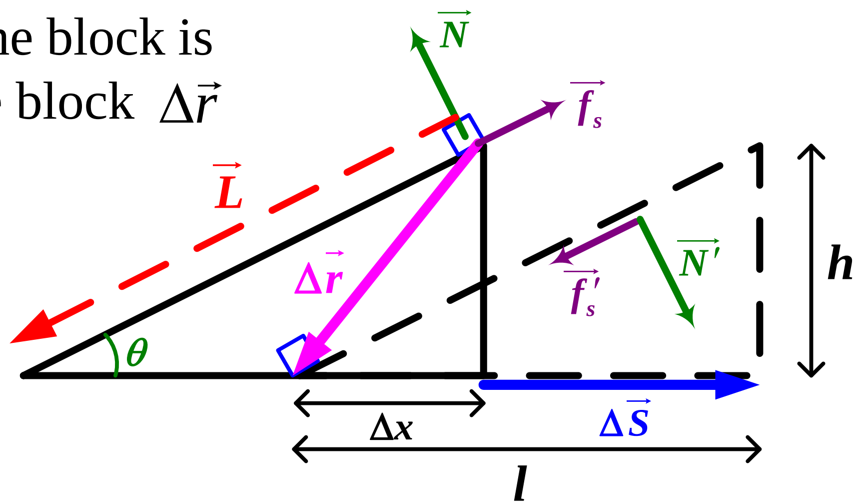
Example

Example: Calculate the net works done respectively by a pair of normal forces ***N*** and ***N'*** , f_s and f'_s between the block and the wedge.

Solution: The normal force $\vec{N}$ acting on the block is not perpendicular to the displacement of the block $\Delta\vec{r}$

Therefore: $W_{N'} \neq 0$

The normal force $\vec{N}'$ acting on the wedge is not perpendicular to the displacement of the wedge $\Delta\vec{S}$. Therefore:



$$W_N \neq 0$$

$$\Delta\vec{r} = -\Delta x\,\hat{i} - h\,\hat{j} \qquad \vec{N} = -N\sin\theta\,\hat{i} + N\cos\theta\,\hat{j}$$

$$\Delta\vec{S} = (l - \Delta x)\,\hat{i} \qquad \vec{N}' = N\sin\theta\,\hat{i} - N\cos\theta\,\hat{j}$$

$$W_N = \vec{N}\cdot\Delta\vec{r} = \Delta x N\sin\theta - hN\cos\theta$$

$$W_{N'} = \vec{N}'\cdot\Delta\vec{S} = lN\sin\theta - \Delta x N\sin\theta$$

$$\tan\theta = \frac{h}{l}, \quad h\cos\theta = l\sin\theta$$

$$\boxed{W_N^{\text{net}} = W_N + W_{N'} = 0 = \vec{N}\cdot\vec{L}}$$

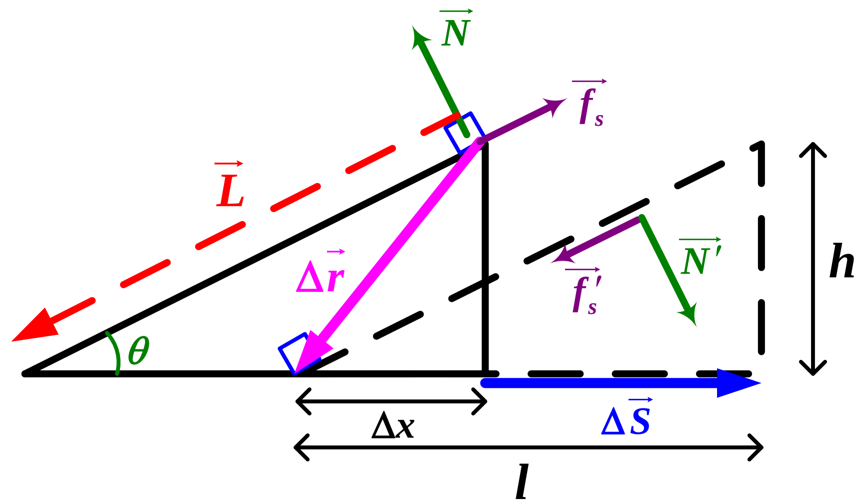
Example (continued)

$$\Delta\vec{r} = -\Delta x\,\hat{i} - h\,\hat{j}$$

$$\Delta\vec{S} = (l - \Delta x)\,\hat{i}$$

$$\vec{f}_s = f_s \cos\theta\,\hat{i} + f_s \sin\theta\,\hat{j}$$

$$\vec{f}\,' = -f_s \cos\theta\,\hat{i} - f_s \sin\theta\,\hat{j}$$



$$W_{f_s} = \vec{f}_s \cdot \Delta\vec{r} = -\Delta x\, f_s \cos\theta - h\, f_s \sin\theta$$

$$W_{f_s'} = \vec{f}_s' \cdot \Delta\vec{S} = -l\, f_s \cos\theta + \Delta x\, f_s \cos\theta$$

$$W_{f_s}^{\text{net}} = W_{f_s} + W_{f_s'} = -(l\cos\theta + h\sin\theta)\, f_s$$

$$= -(L\cos^2\theta + L\sin^2\theta)\, f_s = -f_s L = \vec{f}_s \cdot \vec{L}$$

$$\vec{L} = -l\,\hat{i} - h\,\hat{j}$$

also $\boxed{W_{f_s}^{\text{net}} = \vec{f}_s \cdot \vec{L} = -(l\cos\theta + h\sin\theta)\, f_s}$

Example: **A small object of mass m is suspended from a string of length of L. The object is pulled sideways by a force F that is always horizontal, until the string finally makes an angle ϕ_m. The displacement is accomplished at a very small constant speed. Find the work done by all the forces that act on the object.**

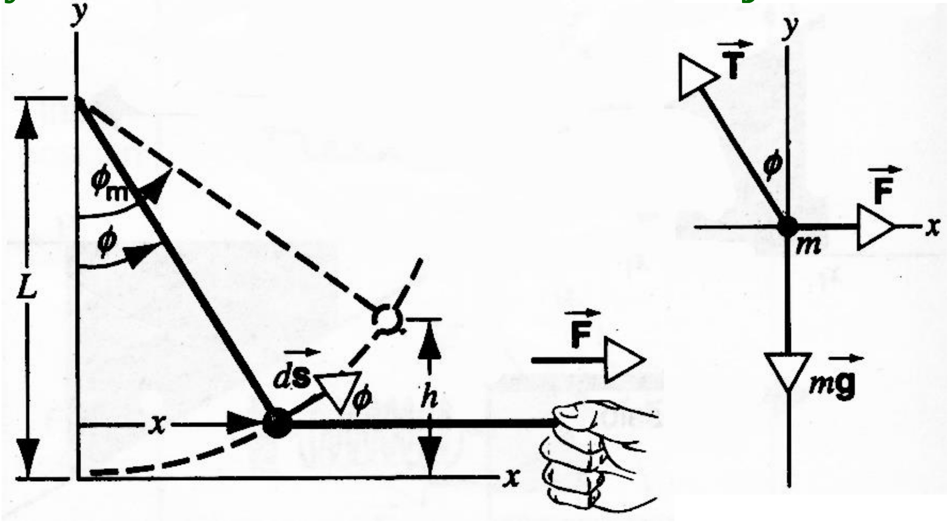
Solution:

x component: $F - T\sin\phi = 0$

y component: $T\cos\phi - mg = 0$

$$F = mg\tan\phi$$

$$x = L\sin\phi \qquad dx = L\cos\phi\, d\phi$$



$$W_F = \int_i^f \vec{F}\cdot d\vec{s} = \int_i^f F ds\cos\phi = \int_i^f F dx$$

$$= \int_0^{\phi_m} mg\tan\phi L\cos\phi\, d\phi = mgL\int_0^{\phi_m}\sin\phi\, d\phi = mgL(1-\cos\phi_m) = mgh$$

$$W_g = \int_i^f -mg\hat{j}\cdot(dx\hat{i} + dy\hat{j}) = \int_0^h -mgdy = -mgh$$

$\vec{T}$ is perpendicular to the displacement $d\vec{s}$ at every point of the motion.

$$W_{net} = W_F + W_g + W_T = mgh - mgh + 0 = 0$$

§ 4 Conservative Forces and Potential Energy

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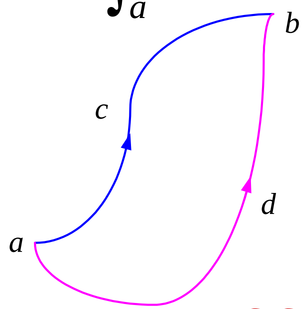
$$W = \int_a^b \vec{F} \cdot d\vec{r}$$



Work done by a force is a line integral or path integral.

Generally, depends on the path followed by the particle.

Different path corresponds to different work done by the same force.

A category of forces which have the special property, that the work done by such a force is independent of the path —— are conservative forces.

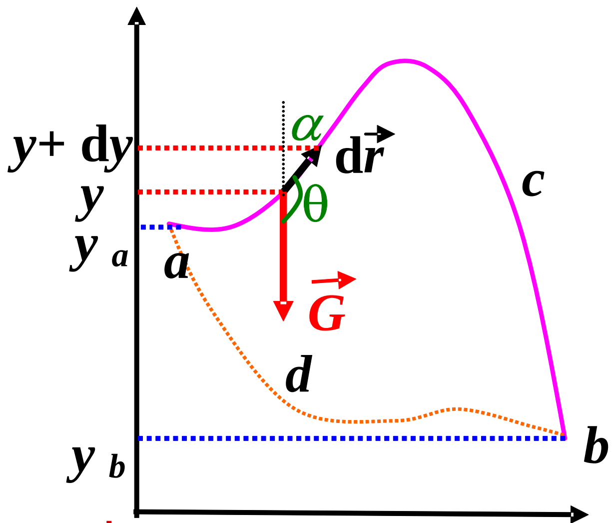
1. Work Done by weight. $\vec{G} = m\vec{g}$

$$dW = \vec{G} \cdot d\vec{r} = G\cos\theta\, ds$$

$$= G\cos(180^\circ - \alpha)ds$$

$$= -mgds\cos\alpha = -mg\,dy$$

$$W = \int_a^b dW = \int_{y_a}^{y_b} -mg\,dy = -(mgy_b - mgy_a)$$



- Only depends on the initial and final positions, and does not depend on the path taken by the particle.

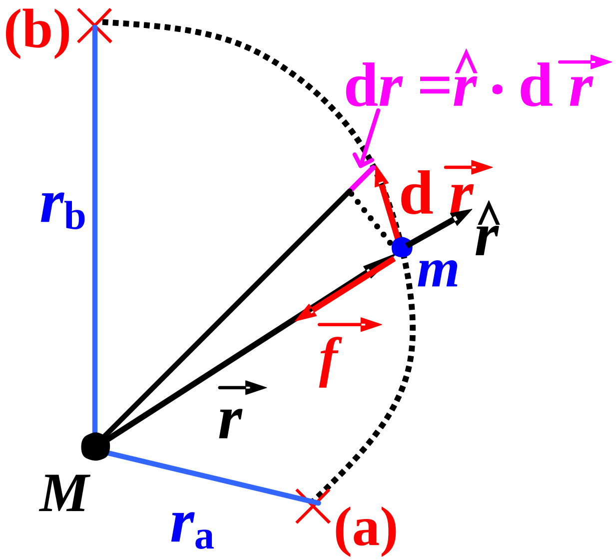
2. Work Done by the universal gravitational force.

$$\vec{f} = -G\frac{Mm}{r^2}\hat{r}$$

$$W = \int_a^b \vec{f}\cdot d\vec{r} = -\int_{r_a}^{r_b} G\frac{Mm}{r^2}\hat{r}\cdot d\vec{r}$$

$$= -\int_{r_a}^{r_b} G\frac{Mm}{r^2}|d\vec{r}|\cos\theta = -\int_{r_a}^{r_b} G\frac{Mm}{r^2}dr$$

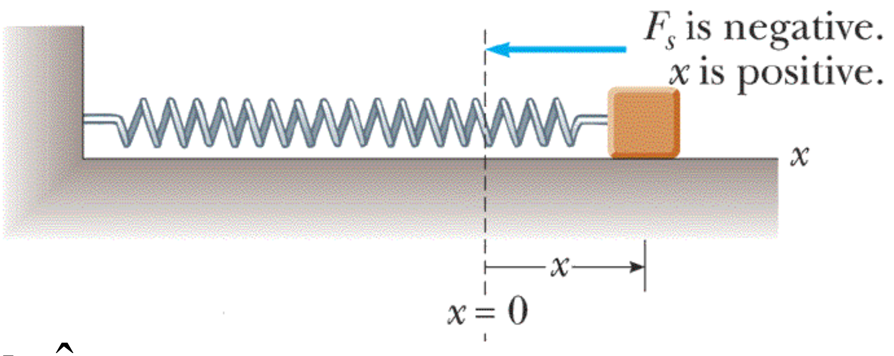
$$= -\left[\left(-G\frac{Mm}{r_b}\right) - \left(-G\frac{Mm}{r_a}\right)\right]$$



- Only depends on the initial and final positions, and does not depend on the path taken by the particle.

3. Work Done by the spring force.

$$\vec{F}_s = -kx\,\hat{i}$$



$$W = \int_{x_a}^{x_b} \vec{F}_s \cdot d\vec{r} = \int_{x_a}^{x_b} (-kx\,\hat{i}) \cdot dx\,\hat{i}$$

$$= -\int_{x_a}^{x_b} kx\,dx = -\left(\frac{1}{2}kx_b^2 - \frac{1}{2}kx_a^2\right)$$

- Only depends on the initial and final positions, and does not depend on the path taken by the particle.

The conservative force

- Conclusion: The conservative force has properties that
  - The work done by a conservative force dose not depend on the path followed by the particle, and depends only on the initial and final positions.

$$\boxed{W = \int_a^b \vec{F} \cdot d\vec{r} = -\left[U(\vec{r}_b) - U(\vec{r}_a)\right]}$$

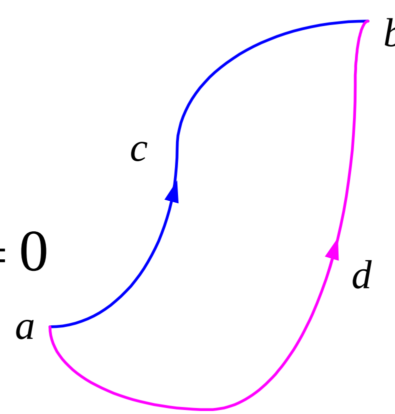
Equivalent statement:

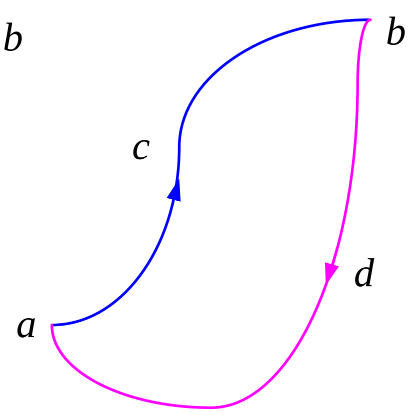
- The total work done by a conservative force is zero, as the particle moves around a close path and returns to its starting point (round trip).

$$\int_{acb} \vec{F} \cdot d\vec{r} = \int_{adb} \vec{F} \cdot d\vec{r}$$

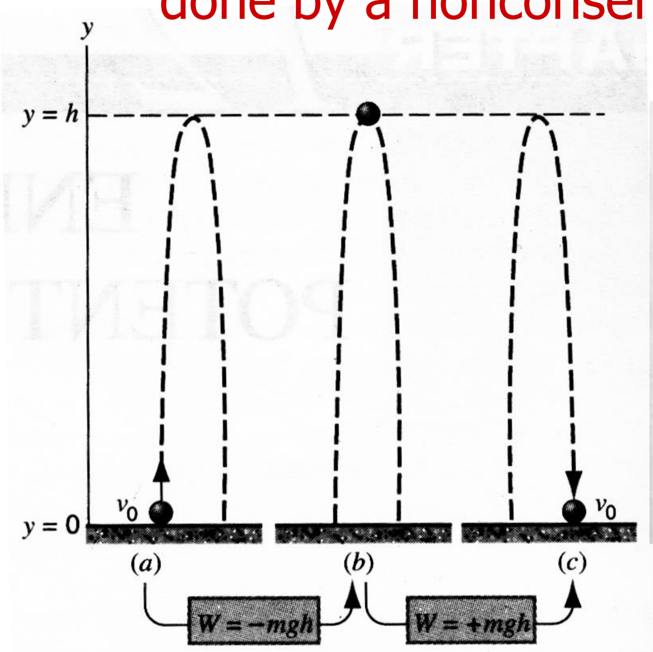
$$\int_{acb} \vec{F} \cdot d\vec{r} - \int_{adb} \vec{F} \cdot d\vec{r} = \int_{acb} \vec{F} \cdot d\vec{r} + \int_{bda} \vec{F} \cdot d\vec{r} = 0$$

$$\oint \vec{F} \cdot d\vec{r} = 0$$



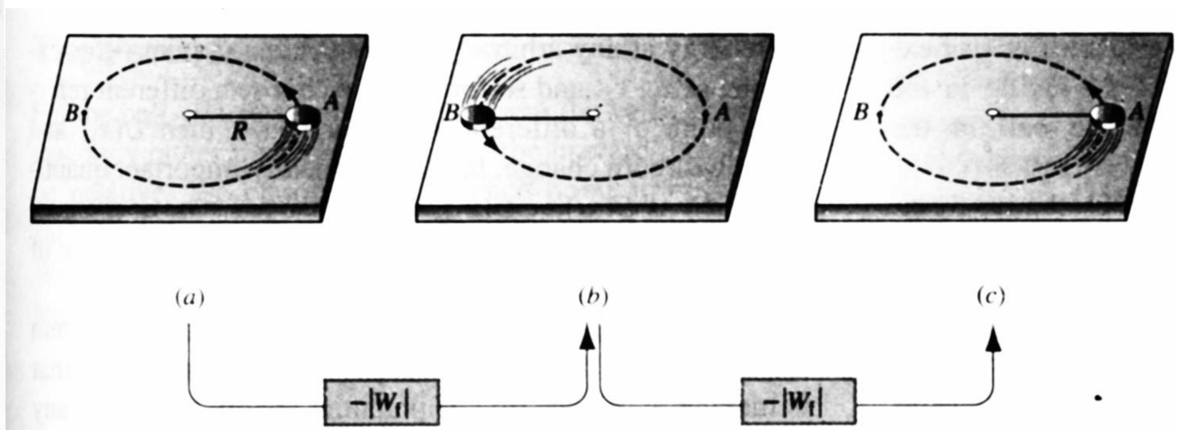


- The conservative force and non-conservative force
  - The total work done by a conservative force is zero as the particle moves along a round trip.
  - But when the particle moves along a round trip, the total work done by a nonconservative force is not zero.



For the force of gravity:

$$W_{\text{round trip}} = 0$$



For the friction force:

$$W_{f\ \text{round trip}} = -2\pi R\mu_k mg$$

Why introduce potential energy?

$$\Delta U = U(\vec{r}_b) - U(\vec{r}_a) = -W = -\int_a^b \vec{F} \cdot d\vec{r}$$

- The work done by a conservative force can be represented in terms of the change in potential energy.
- Notice:
  - The potential energy U is the energy associated with the configuration of a system. Here "*configuration*" means how the parts of a system are located or arranged with respect to one another (the compression or stretching of the spring in the block-spring system, or height of the ball in the ball-Earth system.)
  - The potential energy belongs to the system. We should properly speak of "the elastic potential energy of the block-spring system" or "the gravitational potential energy of the ball-Earth system", not "the elastic potential energy of the spring" or "the gravitational energy of the ball".

How to get the absolute value of potential energy?

$$U(\vec{r}_b) - U(\vec{r}_a) = -\int_a^b \vec{F} \cdot d\vec{r}$$ the definition of potential energy only gives the change in potential energy, or the relative value of potential energy. We can choose a position $\vec{r}_0 = \vec{r}_a$ as the reference point, define $U(\vec{r}_0) = 0$ at the reference point. The choice of reference point is arbitrarily.

New definition of potential energy: $\boxed{U(\vec{r}) = U(\vec{r}) - 0 = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r}}$

- For gravitational potential energy near the Earth's surface, it is accustomed to choose the reference point $y_0=0$ as surface of the Earth.

$$U(y) = mgy$$

- For gravitational potential energy associate with two particles, it is accustomed to take $U(r_0 = \infty) = 0$.

$$U(r) = -G\frac{Mm}{r}$$

- For elastic potential energy, it is accustomed to choose the reference position to be that in which the spring is in its relaxed state.

$$U(x) = \frac{1}{2}kx^2$$

§5 Work-Energy Theorem and Conservation of Mechanical Energy

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- Starting with work – kinetic energy theorem for the system of particles

$$\sum W_{i-\text{external}} + \sum W_{i-\text{internal}} = K_f - K_i$$

  - The internal forces can be divided into conservative and non-conservative.

$$\sum W_{i-\text{external}} + \sum W_{i-\text{internal-conserv}} + \sum W_{i-\text{internal-nonconserv}} = K_f - K_i$$

  - The work done by conservative forces can be described by the change in potential energy $\sum W_{i-\text{internal-conserv}} = -(U_f - U_i)$

$$\sum W_{i-\text{external}} + \sum W_{i-\text{internal-nonconserv}} = (K_f + U_f) - (K_i + U_i)$$

  - Define $E_{\text{mech}} = K + U$ to be total mechanical energy of the system.
- Work – energy theorem:

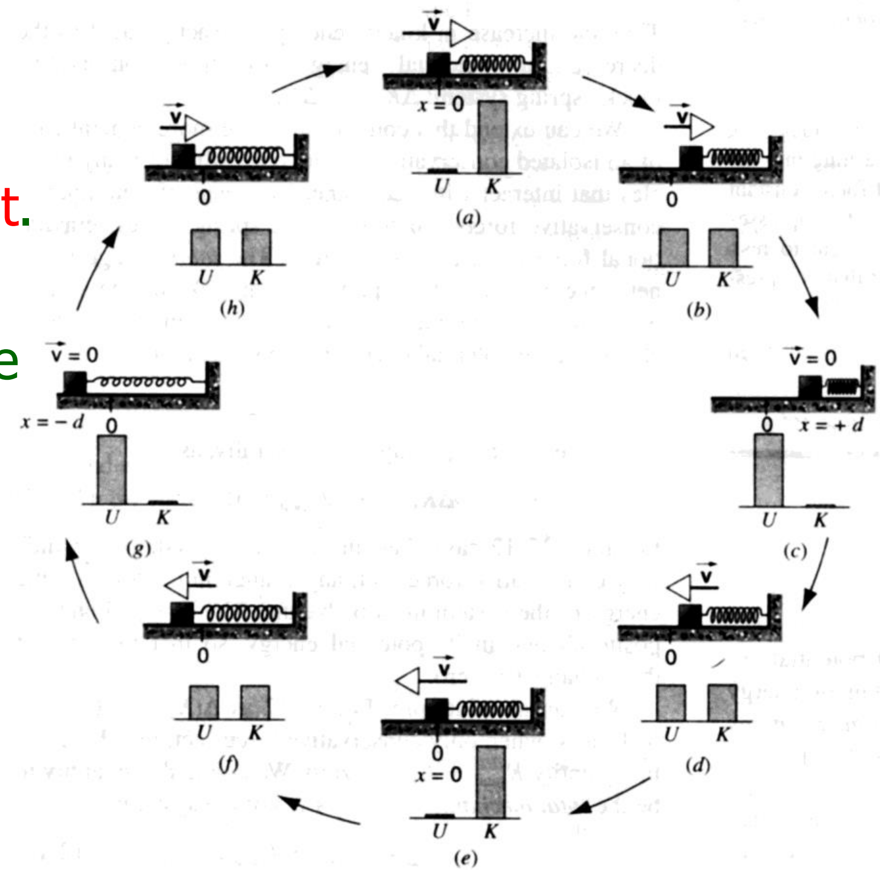
$$\sum W_{i-\text{external}} + \sum W_{i-\text{internal-nonconserv}} = \Delta E_{\text{mech}} = E_{\text{mech}\,f} - E_{\text{mech}\,i}$$

  - The work done by all the external forces and internal forces other than internal conservative forces acting in a system of particles equals the change in total mechanical energy of the system.

Conservation of Mechanical Energy

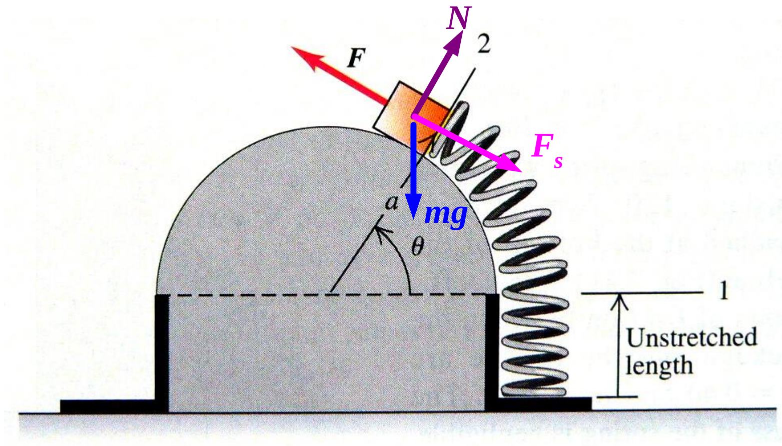
- Conservation of Mechanical Energy
  - For a system, if $\sum W_{i-\text{external}} + \sum W_{i-\text{internal-nonconserv}} = 0$
    then $\Delta E_{\text{mech}} = 0$ or $K_f + U_f = K_i + U_i = \text{constant}$
  - In a system in which only internal conservative forces act, the total mechanical energy remains constant.
  - When $\Delta E_{\text{mech}}=0$, it is the internal conservative forces acting within the system that change kinetic into potential or potential into kinetic energy.

$$U \underset{W_{\text{conservative}}<0}{\overset{W_{\text{conservative}}>0}{\rightleftarrows}} K$$



Example:

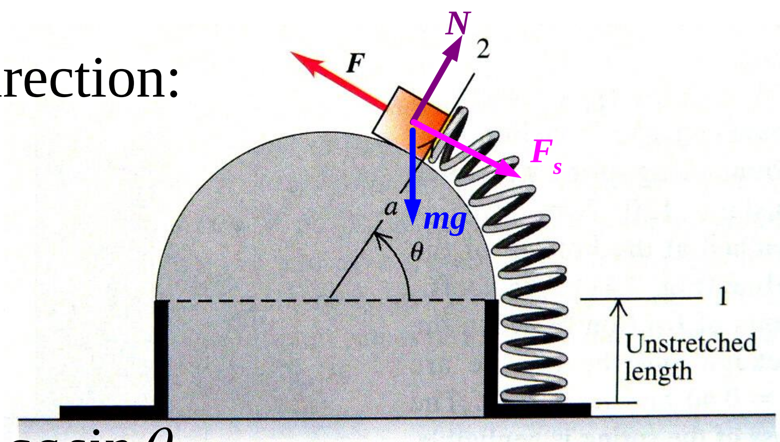
Variable force *F* is maintained tangent to a frictionless semicircular surface. By a slowly varying force *F*, a block with mass of *m* is moved, and spring to which it is attached is stretched from position 1 to position 2. The spring has negligible mass and force constant *k*. The end of the spring moves in an arc of radius *a*. Calculate the work done by the force *F*.



Example

Solution I: by integration directly.

The block is in equilibrium in tangential direction:



$$F = ks + mg\cos\theta$$

$$W_F = \int_1^2 \vec{F}\cdot d\vec{s} = \int_0^s (ks + mg\cos\theta)ds$$

$$= \int_0^{a\theta} ks\,ds + \int_0^{\theta} mga\cos\theta d\theta = \frac{1}{2}ka^2\theta^2 + mga\sin\theta$$

Solution II: by using work-energy theorem.

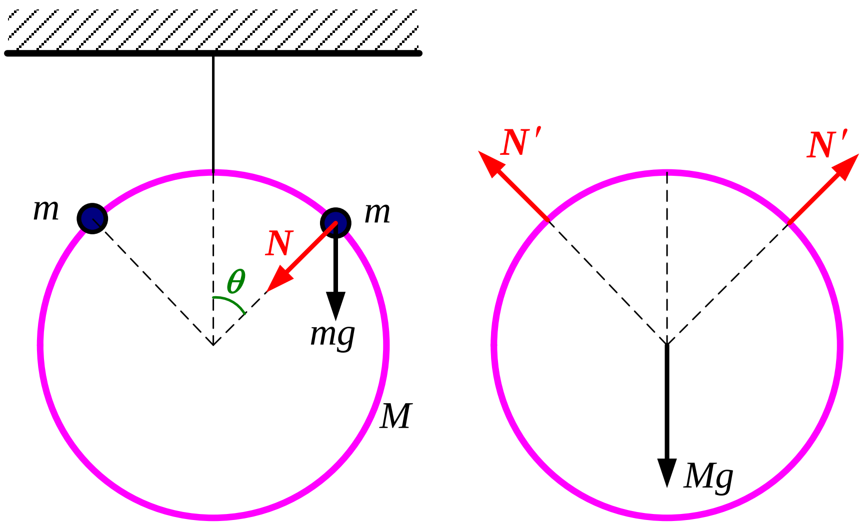
External force: ***F***; Internal forces: ***N*** (non-conservative, does no work), ***mg*** and F_s (conservative)

Choose the reference point at position 1 both for gravitational and elastic energy of block-spring-Earth system.

$$W_F = \Delta E = \Delta U = mga\sin\theta + \frac{1}{2}ks^2 = mga\sin\theta + \frac{1}{2}ka^2\theta^2$$

Example:

A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction. The beads are released simultaneously from the top of the ring and slide down opposite sides. Show that the ring will start to rise if $m>3M/2$, and find the angle at which this occurs.



Example

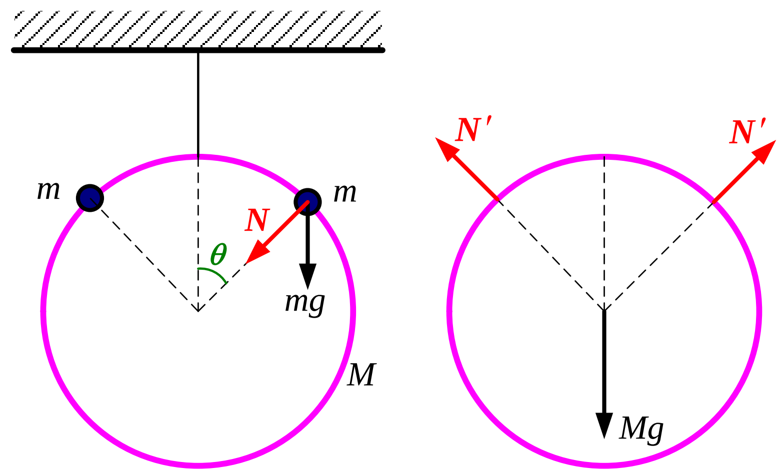
Solution: for beads

Normal component: $$mg\cos\theta + N = m\frac{v^2}{R} \quad (1)$$

Conservation of mechanical energy: $$mgR(1-\cos\theta) = \frac{1}{2}mv^2 \quad (2)$$

For ring, the condition under which it will rise:

$$2N'\cos\theta = 2N\cos\theta \ge Mg \quad (3)$$



Unknowns: v, N, θ

$$6m\cos^2\theta - 4m\cos\theta + M \le 0$$

$$\frac{1}{3}-\frac{1}{3}\sqrt{1-\frac{3}{2}\frac{M}{m}} \le \cos\theta \le \frac{1}{3}+\frac{1}{3}\sqrt{1-\frac{3}{2}\frac{M}{m}}$$

(1)After the ring starts to rise, the beads is no longer in circular motion. So the value of θ is chosen to be minimum.

$$\cos\theta = \frac{1}{3}+\frac{1}{3}\sqrt{1-\frac{3}{2}\frac{M}{m}}$$

（2）In order for **cos***θ* be real, it must be

$$1-\frac{3}{2}\frac{M}{m} > 0 \quad \text{namely:} \quad m > \frac{3}{2}M$$

EXAMPLE: (P184)

Find the escape velocity v of a satellite (the minimum initial speed needed to prevent it from returning to the Earth 第二宇宙速度) of mass m, is projected into the air from the Earth (M_E, R_E).

Solution: System: Earth + satellite $\rightarrow E_{mec} = C$

$$E = \frac{1}{2}mv_1^2 - \frac{GM_E m}{R_E} = \frac{1}{2}mv^2 - \frac{GM_E m}{R_E + h}$$

$$v_1^2 = v^2 - 2\frac{GM_E}{R_E + h} + 2\frac{GM_E}{R_E} \qquad (1)$$

$$m\frac{v^2}{R_E + h} = \frac{GM_E m}{(R_E + h)^2} \quad ; \qquad v^2 = \frac{GM_E}{R_E + h} \qquad (2)$$

$$v_1 = \sqrt{2\frac{GM_E}{R_E} - \frac{GM_E}{R_E + h}}$$

Near the Earth, $g = G\dfrac{M_E}{R_E^2}$ $\quad v_1 = \sqrt{gR_E\left(2 - \dfrac{R_E}{R_E + h}\right)}$

The greater the velocity, the higher the satellite can reach.

When $R_E >> h$, $\quad v_1 = \sqrt{gR_E} = 7.9\times 10^3 m/s$ 第一宇宙速度

Minimum initial speed of rotating around Earth.

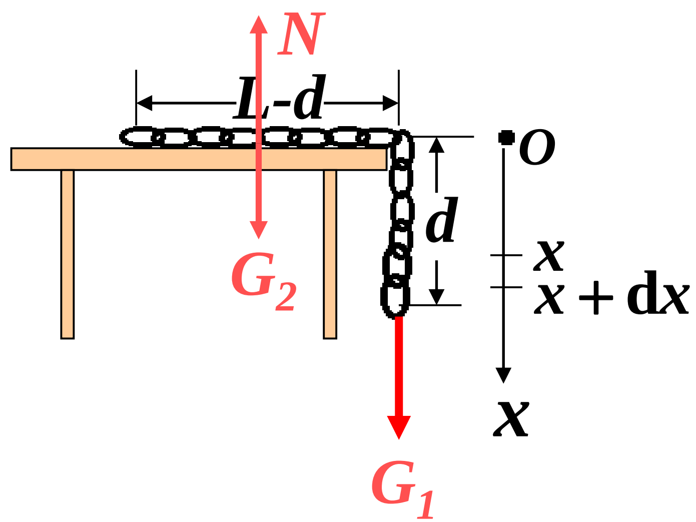
If the speed is high enough, it will continue out into space never to return to Earth， then $h\to\infty$, with merely zero speed,

$$\frac{1}{2}mv_{\mathrm{esc}}^2 - G\frac{M_E m}{R_E} = 0 + 0$$

$$\therefore\ v_{\mathrm{esc}} = \sqrt{2GM_E/R_E} = \sqrt{2gR_E} = \sqrt{2}v_1 = 11.2\times 10^3 \mathrm{m/s}$$

EXAMPLE

A chain has length L and mass M, and was put on a frictionless table. At $t = 0$, the chain is stationary and length d is hanged over the edge. At the moment that the whole chain leave the table, what is the (a) W_g and (b) the velocity of the chain ?



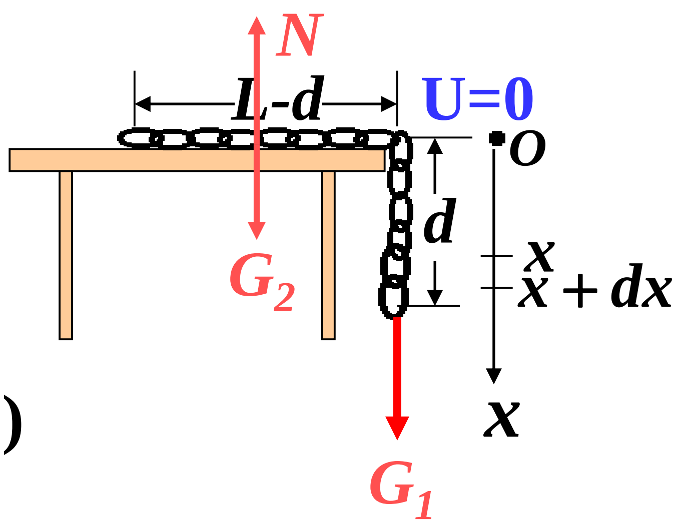
Solution: (a)

$$\because\ dW = G_1 dx = \frac{M}{L} g x dx \qquad \therefore\ W_G = \int_d^L \frac{M}{L} g x dx = \frac{M}{2L} g(L^2 - d^2)$$

Earth, table and chain as a system, $E_{mec} = C$

$$E_1 = -\frac{M}{L} d g \frac{d}{2} \quad \text{and} \quad E_2 = \frac{1}{2} M v^2 - M g \frac{L}{2}$$

$$E_1 = E_2 \quad \text{yields,} \quad v = \sqrt{g(L^2 - d^2)/L}$$



Applying the Kinetic-energy theorem, only W_g,

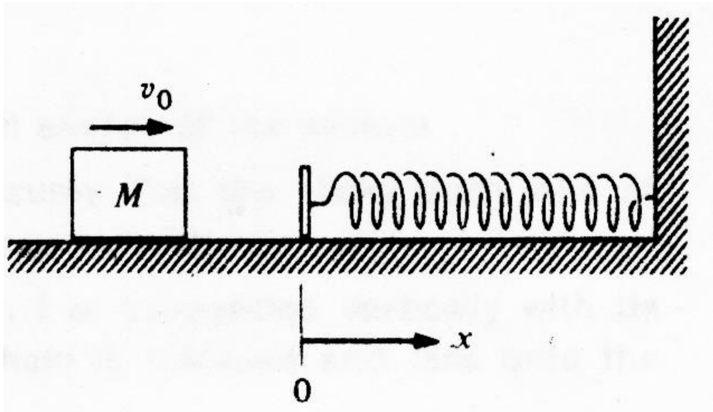
$$\therefore \quad \Delta K = W_{net}$$

$$W_{net} = W_G = \Delta K = \frac{1}{2} M v^2 = \frac{Mg}{2L}(L^2 - d^2)$$

If the friction coefficient of table's surface is μ, how about the v ?

Example:

A block of mass ***M*** slide along a horizontal table with speed v_0. At $x=0$ it hits a spring with spring constant ***k*** and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu = bx$, where ***b*** is a constant. Find the loss in mechanical energy when the block has first come momentarily to rest.



Example

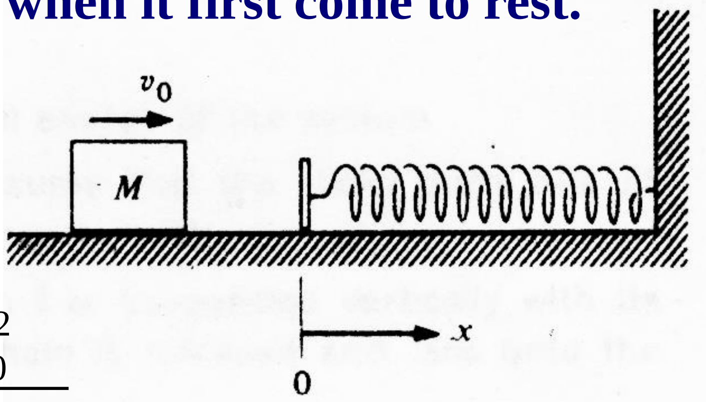
Solution: **Take block-spring-Earth as a system.**

Internal conservative forces: spring force, gravitational force

Internal non-conservative forces: normal force (does no work), friction force.

Using work-energy theorem: $W_{f_s} = E_f - E_i = -E_{loss}$

Suppose the block's position is x_f at the moment when it first come to rest.



$$W_{f_s}(x = 0 \to x_f) = \int_0^{x_f} -bxMg\,dx = -\frac{1}{2}bMgx_f^2$$

$$-\frac{1}{2}bMgx_f^2 = \frac{1}{2}kx_f^2 - \frac{1}{2}Mv_0^2 \qquad x_f = \frac{Mv_0^2}{k - bMg}$$

$$E_{loss} = E_i - E_f = -W_{f_s} = \frac{1}{2}bMgx_f^2 = \frac{bgM^2v_0^2}{2(k - bMg)}$$

The conservative force and potential energy

For an infinitesimal process,

$$-dU = -\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right) = \vec{F}\cdot d\vec{r}$$

$$= F_x dx + F_y dy + F_z dz$$

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$$

$$\vec{F} = -\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right)$$

$$= -\nabla U$$

∇U means the gradient of the potential-energy function. The gradient of a scalar function is a vector function. ∇ is a gradient operator.

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The conservative force and potential energy

- For force of gravity.

$$U(y) = mgy \qquad F_y = -\frac{\partial U}{\partial y} = -mg$$

- For universal gravitational force.

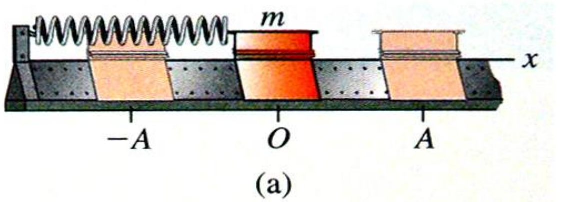
$$U(r) = -\frac{GMm}{r} \qquad F_r = -\frac{\partial U}{\partial r} = -\frac{GMm}{r^2}$$

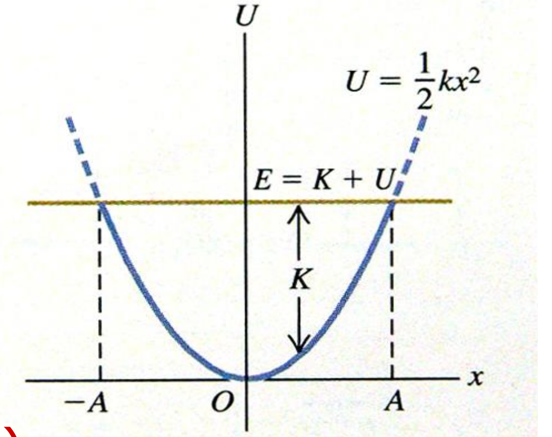
- For spring force.

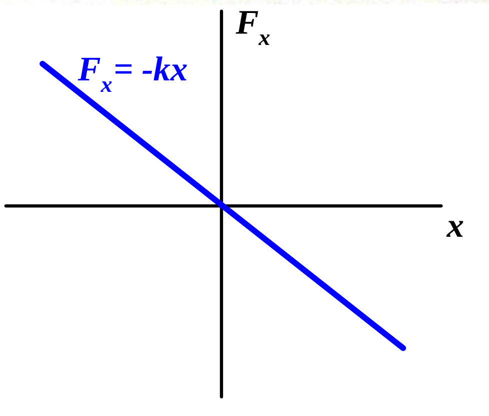
$$U(x) = \frac{1}{2}kx^2 \qquad F_x = -\frac{\partial U}{\partial x} = -kx$$

The force is equal to the negative of the slope of U(x)

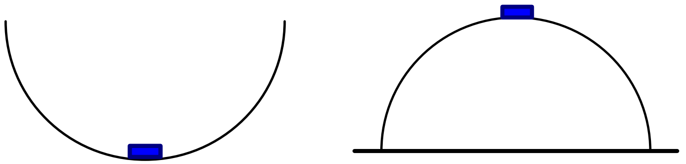
- Because of conservation of mechanical energy, E as a function of x is a straight horizontal line $E = K + U$
- The glider can only move in the range between $x = \pm A$, since the kinetic energy in this range is positive.
- At x=0, the slope of $U(x)$ and the force are zero, so it is an equilibrium position.





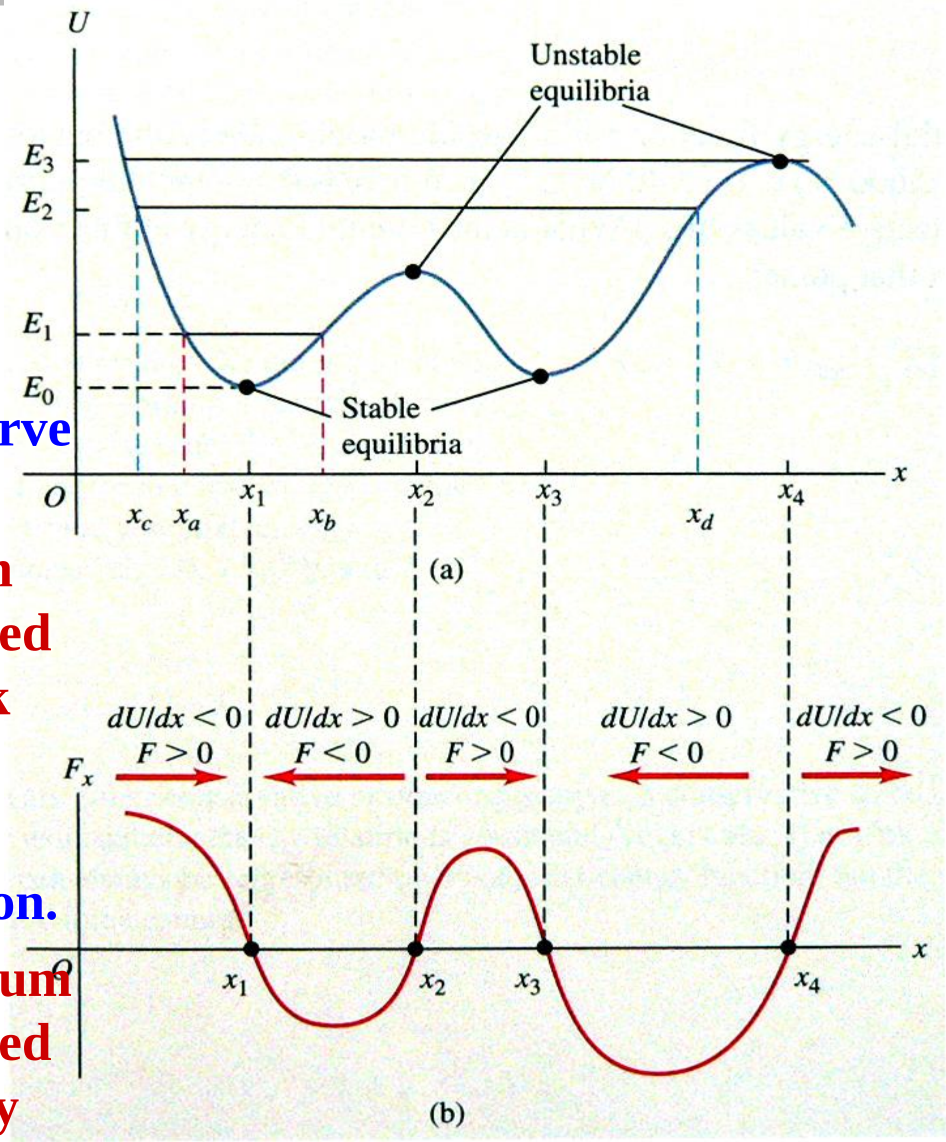


Stable and unstable equilibrium

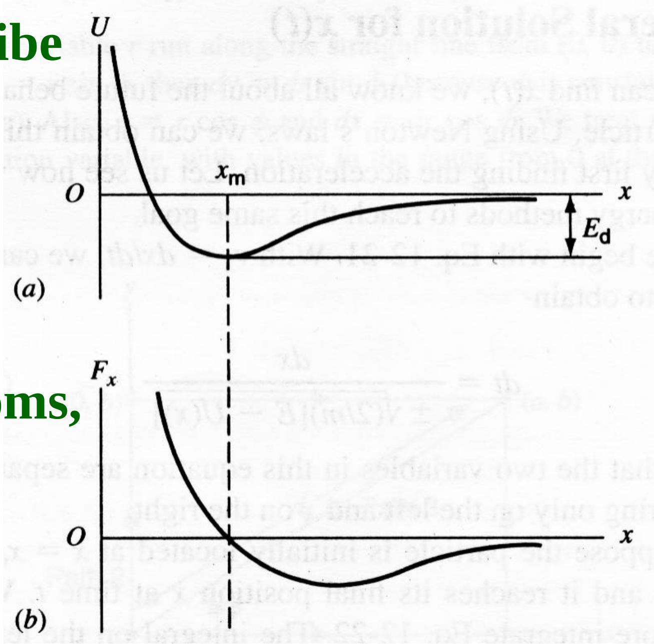


The particle is in stable equilibrium (left) and in unstable equilibrium (right).

- **Any minimum in a potential-energy curve is a stable equilibrium position.**
  - Points x_1 and x_3 are stable equilibrium points. When the particle is displaced to either side, the force pushes back toward the equilibrium point.
- **Any maximum in a potential-energy curve is an unstable equilibrium position.**
  - Points x_2 and x_4 are unstable equilibrium points. When the particle is displaced to either side, the force pushes away from the equilibrium point.



Example: **A commonly used potential function to describe the interaction between the two atoms in a diatomic molecule is the Lennard-Jones 6-12 potential**



$$U(x)=\varepsilon\left[\left(\frac{x_0}{x}\right)^{12}-2\left(\frac{x_0}{x}\right)^{6}\right]$$

Find (a) the equilibrium separation between the atoms,

(b) the force between the atoms,

(c) the minimum energy necessary to break the molecule apart.

Solution: (a) Equilibrium occurs at the position where U(x) is minimum which is found from

$$\left(\frac{dU(x)}{dx}\right)_{x=x_m}=0 \qquad \varepsilon\left(-12\frac{x_0^{12}}{x_m^{13}}+12\frac{x_0^{6}}{x_m^{7}}\right)=0 \qquad x_m=x_0$$

(b)

$$F(x)=-\frac{dU(x)}{dr}=12\varepsilon\left(\frac{x_0^{12}}{x^{13}}-\frac{x_0^{6}}{x^{7}}\right)$$

(c) The minimum energy needed to break up the molecule into separate atoms is called *dissociation energy*, E_d.

$$U(x_0)+E_d=0, \qquad E_d=-U(x_0)=\varepsilon$$