Chapter 9 Momentum and Collision

P200-207

§ 1 Impulse and Momentum

- From Newton's second law
  - Another form of Newton's second law in terms of momentum

$$\sum \vec{F} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

  - Definition of Momentum or Linear Momentum of an object

$$\vec{p} = m\vec{v}$$

SI unit kg•m/s

  - The form $\sum \vec{F} = m\vec{a}$ is the special case for Newton's second law when the mass of the object remains constant.

Impulse-momentum theorem

- Impulse-momentum theorem

$$\int_{t_i}^{t_f} \sum \vec{F} dt = \int_{t_i}^{t_f} \frac{d\vec{p}}{dt} dt = \int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \vec{p}_f - \vec{p}_i = \Delta \vec{p}$$

- Definition of Impulse of a Force $\vec{F}$

$$\vec{J} = \int_{t_i}^{t_f} \vec{F} dt$$

SI unit N•s

- The impulse-momentum theorem for a particle

$$\boxed{\sum \vec{J} = \vec{p}_f - \vec{p}_i = \Delta \vec{p}}$$

The impulse of the net force acting on a particle during a given time interval is equal to the change in momentum of the particle during that interval.

Impulse-momentum theorem

About Impulse-momentum theorem

- *F* should be the sum of all external forces.
- Valid only in inertial frame of reference.
- Both $\vec{J}$ and $\vec{P}$ are vectors. They have same units and dimensions
- In reality, one often use its component form:

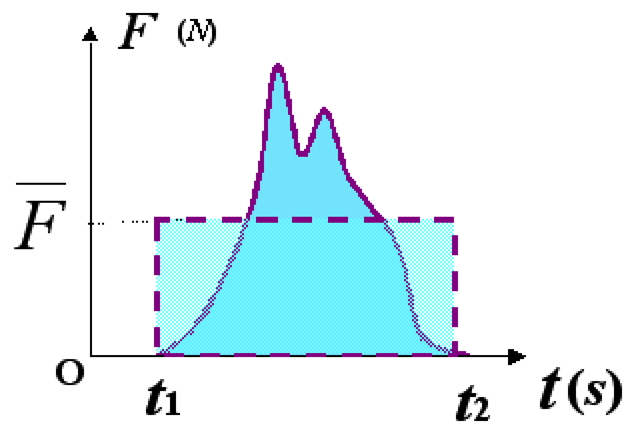
$$\begin{cases} J_x = \int_{t_i}^{t_f} F_x dt = mv_{fx} - mv_{ix} \\ J_y = \int_{t_i}^{t_f} F_y dt = mv_{fy} - mv_{iy} \\ J_z = \int_{t_i}^{t_f} F_z dt = mv_{fz} - mv_{iz} \end{cases}$$

Impulse-momentum theorem

About Impulse of Forces

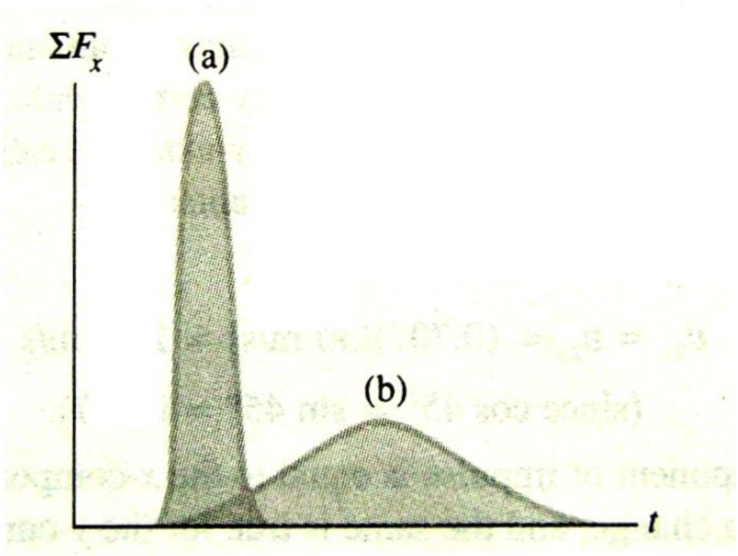
- The impulse of a force is a vector. It depends on the strength of the force and on its duration. When the force is constant, the direction of $\vec{J}$ is as same as the force; if $\vec{F}$ is variable, the direction of $\vec{J}$ is determined by the integral of $\int_{t_1}^{t_2} \vec{F} dt$
- The magnitude of the impulse is equal to the area under the $F(t)$ curve
- When a time-varying net force $\sum \vec{F}(t)$ is difficult to measure, we can use a time-averaged net force as the substitute provided that it would give the same impulse to the particle in same time interval.

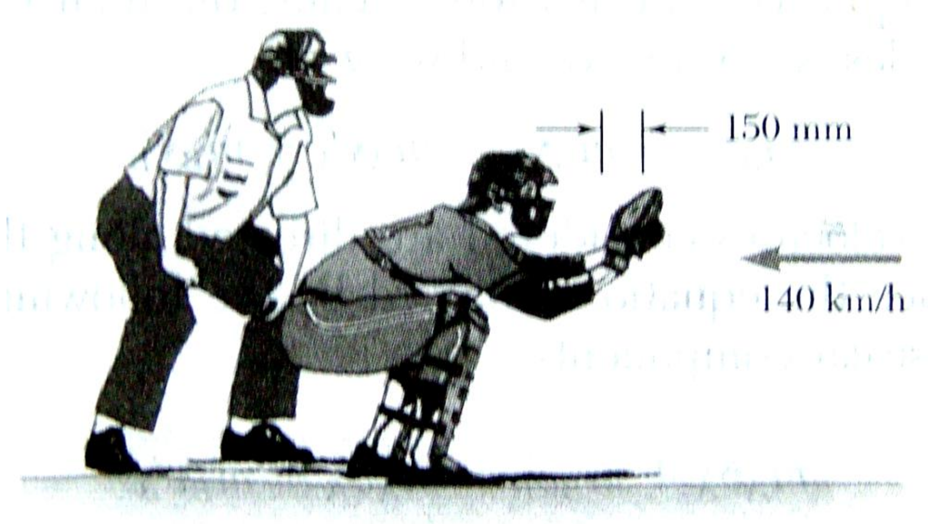
$$\bar{\vec{F}} = \frac{1}{\Delta t}\int_{t_i}^{t_f} \vec{F} dt$$



Time - averaged net force

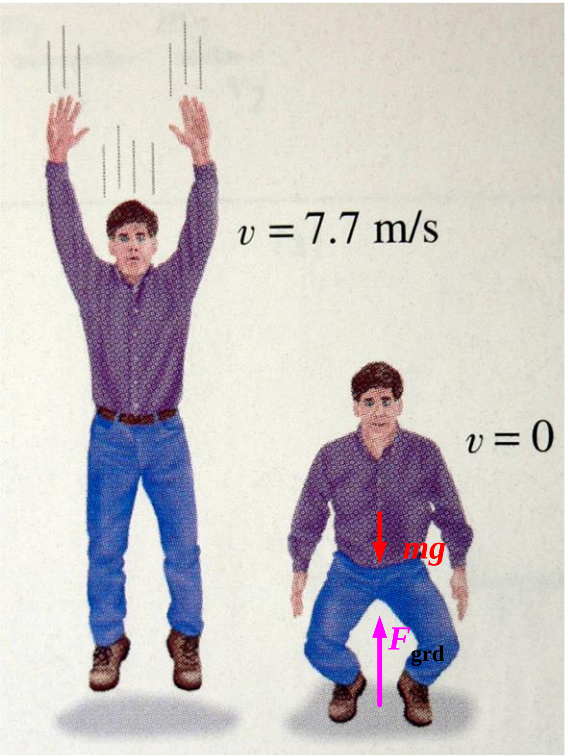
- For a given amount of momentum change, we can delay the time interval to decrease the impulsive force.
- A baseball player catching a ball can soften the impact by pulling his hand back.





Example

Example: Bend your knees when landing. (a) Calculate the impulse experienced when a 70kg person lands on firm ground after jumping from a height of 3.0m. Then estimate the average force exerted on the person's feet by the ground, if the landing is (b) stiff-legged (body moves 1.0cm during impact), and (c) with bent legs (about 50cm).



Solution：

(a) $$v = \sqrt{2gh} = 7.7\text{m/s}$$

$$J = p_f - p_i = 0 - (70\text{kg})(7.7\text{m/s}) = -540\text{N}\cdot\text{s}$$

(b) d=1.0cm=1.0×10^{-2}m $\quad \overline{v} = (7.7+0) = 3.8\text{m/s}$

$$\Delta t = d/\overline{v} = 2.6\times10^{-3}\text{s}$$

$$F_{\text{grd}} - mg = \frac{J}{\Delta t} = \frac{540}{2.6\times10^{-3}} = 2.1\times10^{5}\,\text{N}$$

mg=(70kg)(9.8m/s^2)=690N

$$F_{grd} = 2.1\times10^{5}\,\text{N} + 690\text{N} \approx 2.1\times10^{5}\,\text{N} >> mg$$

The person's legs would likely break in such a stiff landing.

(c) d=0.50m, Δt=0.13s $\quad F_{\text{grd}} - mg = \frac{540}{0.13} = 4.2\times10^{3}\,\text{N}$

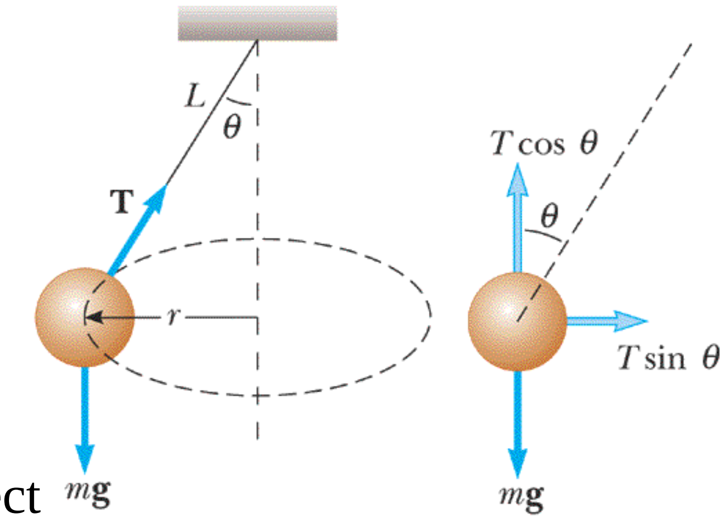
$$F_{\text{grd}} = 4.9\times10^{3}\,\text{N}$$

Example

Example: Conical Pendulum. A small object of mass ***m*** is suspended from a string of length ***L***. The object revolves in a horizontal circle of radius ***r*** with constant speed ***v***. Determine the impulse exerted (1) by gravity, (2) by string tension on the object, during the time in which the object has passed half of the circle.

Solution: From impulse-momentum theorem

$$\vec{J}_{net} = \int_{t_1}^{t_2} \vec{F}_{net} dt = m\vec{v}_2 - m\vec{v}_1 = 2m\vec{v}_2$$



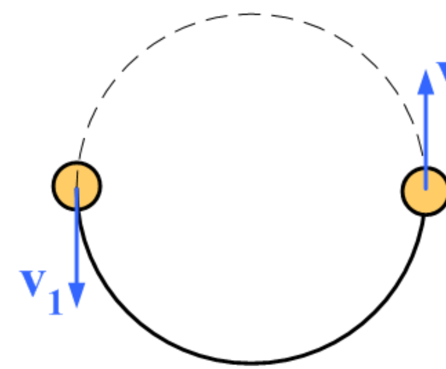
The impulse exerted by gravity on the object

$$\vec{J}_{mg} = \int_{t_1}^{t_2} m\vec{g}dt = m\vec{g}\frac{T}{2} = m\vec{g}\frac{1}{2}\frac{2\pi r}{v} = \frac{\pi r}{v}m\vec{g}$$

The impulse exerted by string intension on the object

$$\vec{J}_T = \vec{J}_{net} - \vec{J}_{mg}$$

$$J_T = \sqrt{(2mv)^2 + \left(\frac{\pi rmg}{v}\right)^2} = m\sqrt{4v^2 + \frac{\pi^2 r^2 g^2}{v^2}}$$



§ 2 Impulse-momentum theorem for a system of particles

P202-208

- Impulse-momentum theorem for a system of particles

Consider a system of N interacting particles with masses m_1, m_2, … , m_N.

For i-th particle:

the net external force $\vec{F}_i$

the internal force exerted by j-th particle $\vec{f}_{ij}$

$$(\vec{F}_i + \sum_{i \neq j} \vec{f}_{ij})dt = d\vec{p}_i$$

For the system of particles: $\sum_i (\vec{F}_i + \sum_{j \neq i} \vec{f}_{ij})dt = \sum_i d\vec{p}_i$

According to Newton's third law, the internal forces cancel in pairs.

$$\sum_i \sum_{j \neq i} \vec{f}_{ij} = 0$$

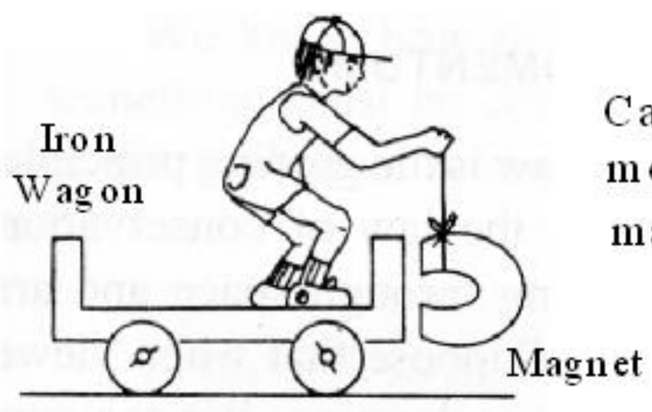
The total external force acting on the system: $\sum_i \vec{F}_i$

The total momentum of the system: $\vec{p}_{\text{tot}} = \sum_i \vec{p}_i$

Impulse-momentum theorem for a system of particles

- Impulse-momentum theorem for a system of particles (continued)
  - The derivative form: $$\boxed{\sum_i \vec{F}_{i-\text{ext}} = \frac{d\vec{p}_{\text{tot}}}{dt}}$$
  - The integral form: $$\boxed{\int_{t_1}^{t_2} \sum_i \vec{F}_{i-\text{ext}}\, dt = \vec{p}_{\text{tot}2} - \vec{p}_{\text{tot}1}}$$

The total external force applied to a system of particles equals to the change in total momentum of the system. (The internal forces can exchange the momenta between particles within system, but can not influence the total momentum of the system.)



Can you get the wagon to move by hanging a huge magnet in front of you ?

- Conservation of Momentum
  - When the vector sum of external forces on a system is zero, the total momentum of the system is constant.

When $\sum_i \vec{F}_{i-\text{ext}} = 0$ $\quad \dfrac{d\vec{p}_{\text{tot}}}{dt} = 0$ or $\vec{p}_{\text{tot}} = \sum_i \vec{p}_i = \text{constant}$

  - Notice the difference between conservation of momentum and conservation of mechanical energy

For an isolated system, the mechanical energy is conserved only when the internal forces are conservative. But conservation of momentum is valid even when the internal forces are not conservative.

  - Conservation of momentum in component form.

Since momentum is vector quantity, we can treat its component independently. When one of the component of sum of external forces is zero, the component of total momentum is conserved in this direction.

When $\sum_i F_{i-\text{ext-}x} = 0$ then $p_{\text{tot-}x} = \sum_i p_{i-x} = \text{constant}$

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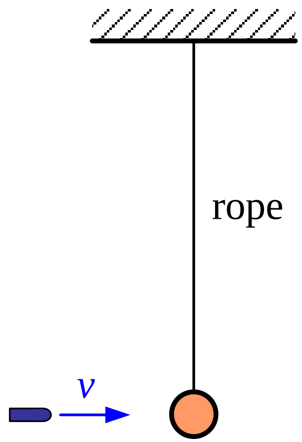
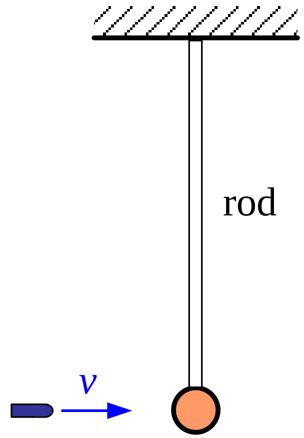
Which case of collision satisfies the conservation of momentum? Or conservation of component momentum?



A



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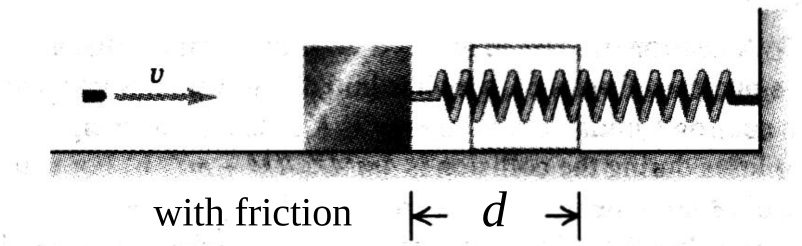

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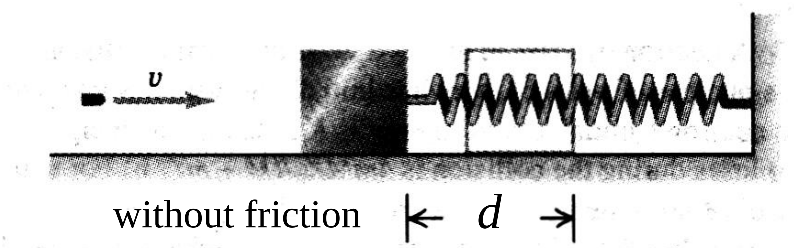
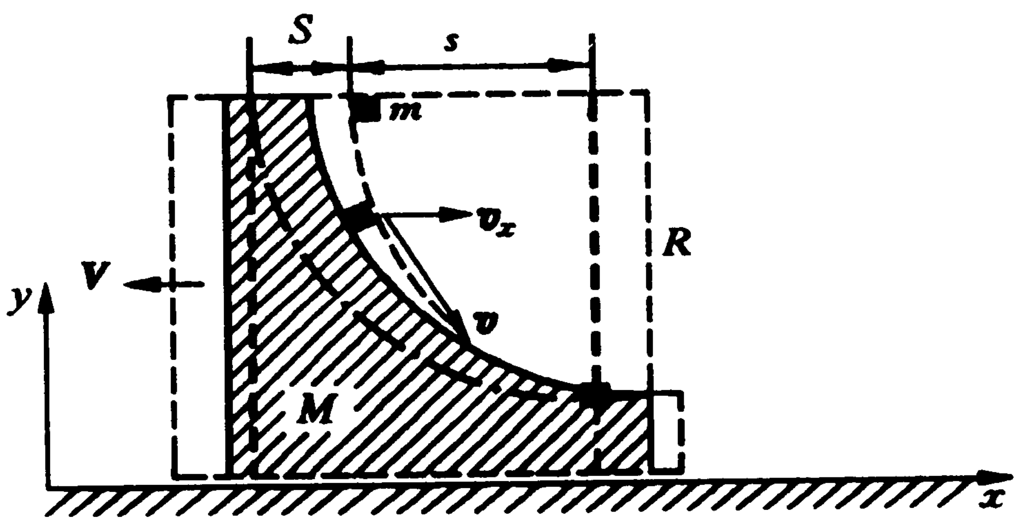


D

提交

Example:

A small cube of mass ***m*** slides down a circular path of radius R cut into a large block of mass ***M***. ***M*** rests on a table. ***M*** and ***m*** are initially at rest. ***m*** starts from the top of the path. Find the distance traveled by ***M*** when the cube ***m*** leaves the block ***M***. All surfaces are frictionless.



Example

Solution: No horizontal external force acts on the system consisting of the cube and the block. The total momentum of the system is conserved in horizontal direction.

$$0 = mv_x + M(-V) \quad \Rightarrow \quad mv_x = MV$$

Integrations on both side: $m\int_0^t v_x dt = M\int_0^t V dt \qquad ms = MS$ ①

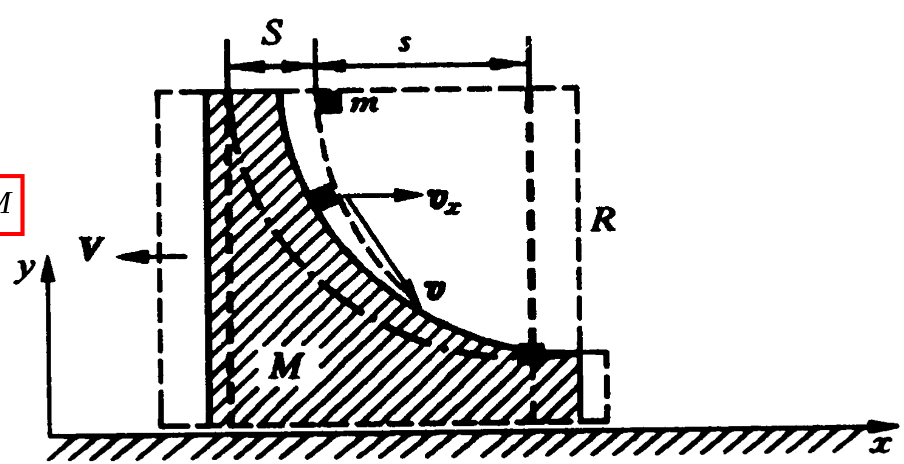
In the reference frame of M: The horizontal displacement of m is

$$R = \int_0^t v'_x dt = \int_0^t (v_x + V)dt = s + S \qquad ②$$

From ① and ②

$$\boxed{S = \frac{m}{m+M}R}$$

$\boxed{F = (3M + 2m)mg/M}$



Expansion:

- The speed of cube when it leaves the block M?

$$\boxed{v = \sqrt{2MgR/(M+m)}}$$

- The normal force acting on the cube when it reaches the bottom of the block ?

$$\boxed{F = (3M+2m)mg/M}$$

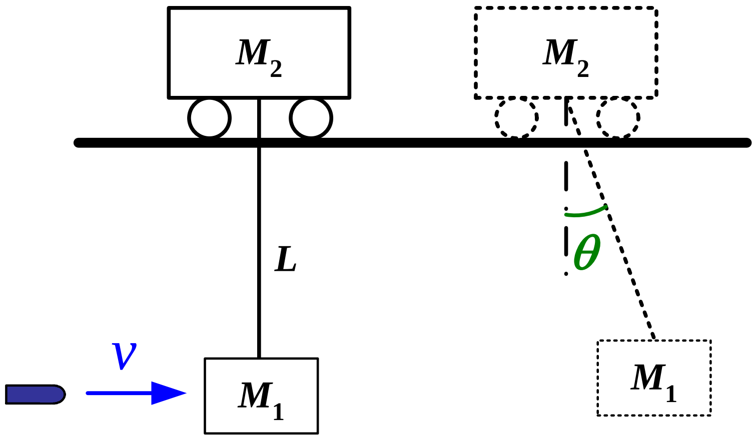
- If the surface between block M and cube m is friction, what about the result of the question 1 ?

Example:

A wooden block of mass M_1 is suspended from a cord of length L attached to a cart of mass M_2 which can roll freely on a frictionless horizontal track. A bullet of mass m is fired into the block from left. After the impact of the bullet, the block swings up with the maximum angle of θ. What is the initial speed v of the bullet?

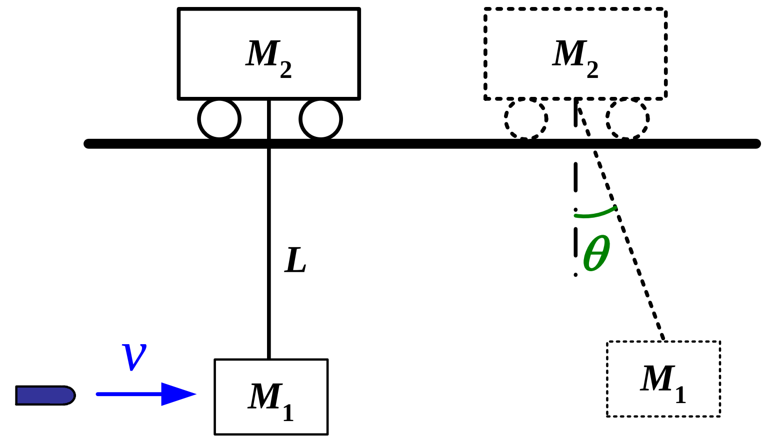
Solution: (1) Stage 1: For the system consisting of m and M_1, the momentum is conserved in horizontal during a small interval time of impact.

$$mv = (M_1 + m)v_1$$



Example (continued)

(1) Stage 1: $mv = (M_1 + m)v_1$



(2) Stage 2: The block plus bullet swing up with initial speed v_1, and drive the cart sliding forward in the track. At the instant when the block-bullet swing at maximum angle, (M_1+m), M_2 have the same horizontal speed of v_2, and the mechanical energy of the system of (M_1+m) and M_2 is conserved.

$$\frac{1}{2}(M_1 + m)v_1^2 = \frac{1}{2}(M_1 + M_2 + m)v_2^2 + (M_1 + m)gL(1 - \cos\theta)$$

(3) During the whole Stage1+Stage2: The momentum of system consisting of M_1, m, M_2 is conserved in horizontal.

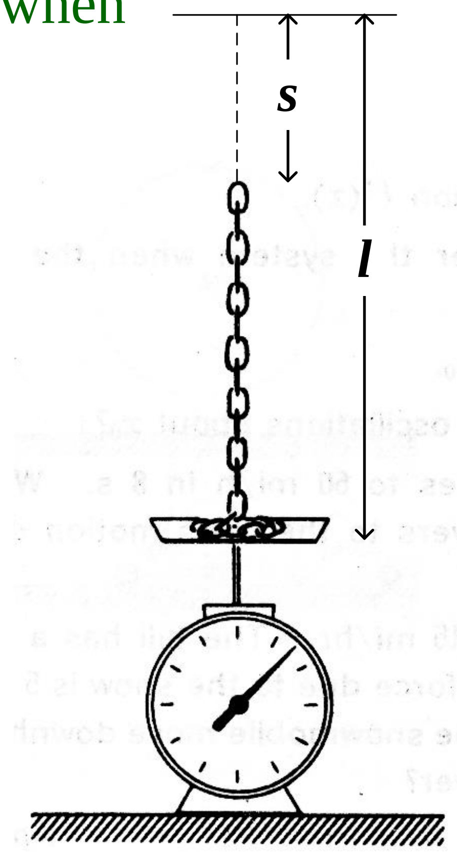
$$mv = (M_1 + M_2 + m)v_2$$

Final answer:

$$\boxed{v = \frac{M_1 + m}{m}\sqrt{\frac{M_1 + M_2 + m}{M_1}2gL(1 - \cos\theta)}}$$

Example:

A chain of mass ***M*** length ***l*** is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale. What is the reading of the scale when a length of chain, ***s***, has fallen? (Neglect the size of individual links.)



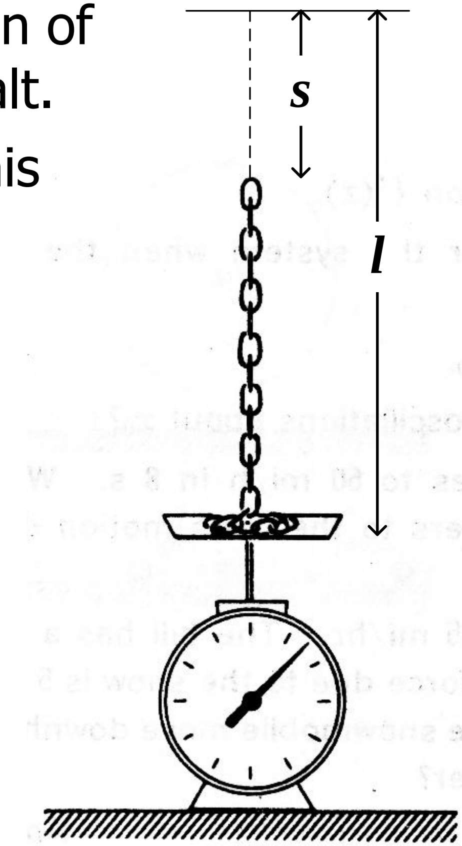
Example

Solution: (1) Using impulse-momentum theorem:

Assuming a length of chain s has been already in the scale.

Take a infinitesimal process during dt, a segment chain of length of ds impacts with the scale, and comes to a halt.

The impulse that the surface of the scale acting on this segment is:



$$Fdt = 0-(-dm\,v) = v\frac{M}{l}ds$$

$$F = \frac{M}{l}v\frac{dy}{dt} = \frac{M}{l}v^2 = \frac{M}{l}(2gs) = 2Mg\frac{s}{l}$$

The reading of the scale

= the weight that has already in the scale+F

$$= Mg\frac{s}{l} + 2Mg\frac{s}{l} = 3Mg\frac{s}{l}$$

§ 3 Center of Mass

P214-218

Describe the motion of a system of particles
- by every motion for individual particles
- by overall motion in terms of center of mass

- Center of mass
  - For the system of discretely distributed particles

$$\vec{r}_{\mathrm{CM}} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M}$$

represented by components

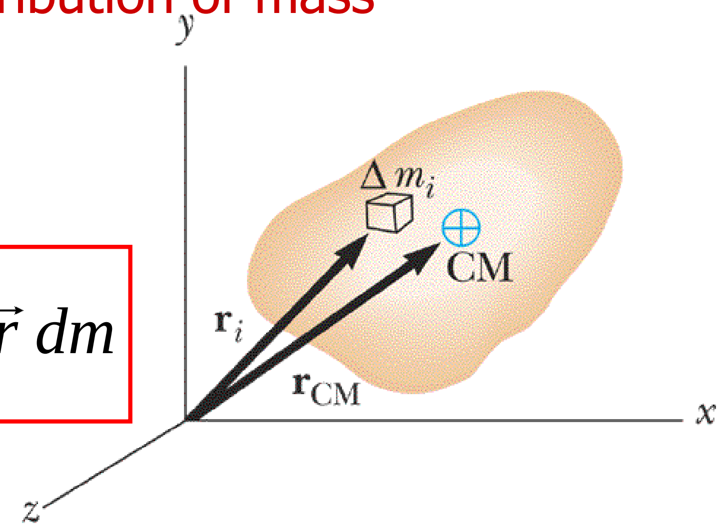
$$x_{\mathrm{CM}} = \frac{\sum_i m_i x_i}{M} \qquad y_{\mathrm{CM}} = \frac{\sum_i m_i y_i}{M} \qquad z_{\mathrm{CM}} = \frac{\sum_i m_i z_i}{M}$$

  - For the extended object with uniformly distribution of mass

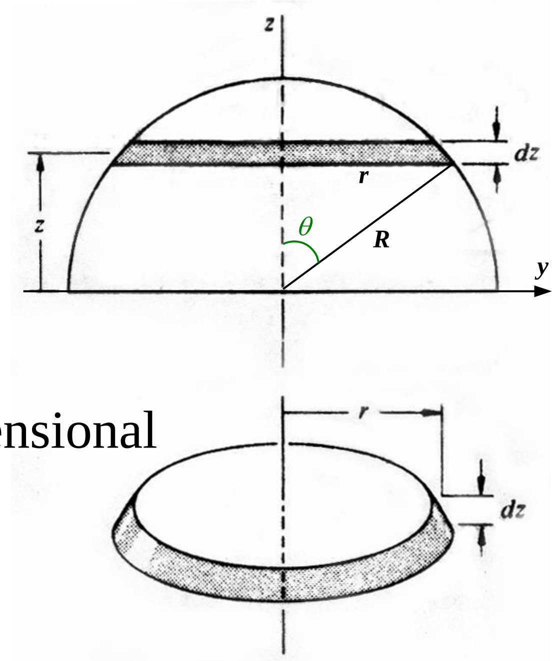
$$x_{\mathrm{CM}} = \frac{\sum_i \lim_{\Delta m_i \to 0} x_i \Delta m_i}{\sum_i \lim_{\Delta m_i \to 0} \Delta m_i} = \frac{\int x\, dm}{\int dm} = \frac{1}{M}\int x\, dm,$$

$$y_{\mathrm{CM}} = \frac{1}{M}\int y\, dm, \quad z_{CM} = \frac{1}{M}\int z\, dm$$

$$\boxed{\vec{r}_{\mathrm{CM}} = \frac{1}{M}\int \vec{r}\, dm}$$



Example: Find the center of mass of a uniform solid hemisphere of radius ***R*** and mass ***M***.



Solution: From symmetry it is apparent that the center of mass lies on the z axis. $\boldsymbol{x_{CM}=0}$, $\boldsymbol{y_{CM}=0}$.

$$z_{\text{CM}} = \frac{1}{M}\int z\,dm = \frac{1}{M}\int z\rho\,dV$$

The three-dimensional integral can be treated as an one-dimensional integral. Subdivide the hemisphere into a pile of thin disk.

$$\begin{cases} dV = \pi r^2 dz \\ \rho = M \Big/ \left(\dfrac{2}{3}\pi R^3\right) \end{cases}$$

Find ***r***, ***z*** in terms of ***θ***.

$$r = R\sin\theta$$

$$z = R\cos\theta$$

$$dz = -R\sin\theta\,d\theta$$

$$z_{\text{CM}} = \frac{3}{2R^3}\int_{\frac{\pi}{2}}^{0} R\cos\theta R^2\sin^2\theta(-R\sin\theta)d\theta$$

$$= \frac{3}{2}R\int_0^{\frac{\pi}{2}}\cos\theta\sin^3\theta\,d\theta = \frac{3}{2}R\int_0^{\frac{\pi}{2}}\sin^3\theta\,d(\sin\theta)$$

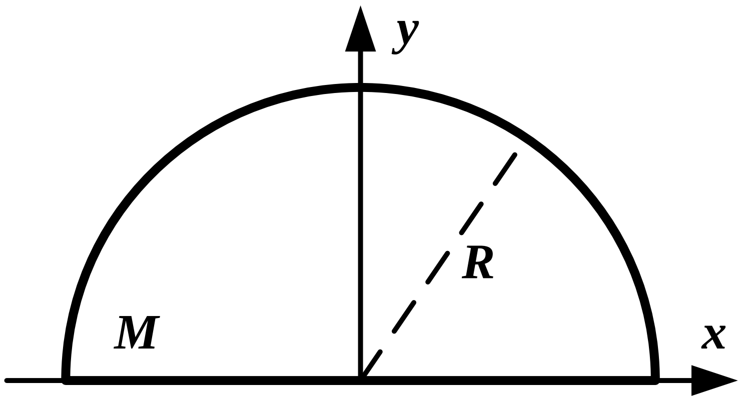
$$= \frac{3}{2}R\times\frac{1}{4} = \frac{3}{8}R$$

Supplement problem: Find the center of mass of a uniform semicircular plate of radius ***R*** and mass ***M***.

$$y_{\text{CM}} = \frac{1}{M}\int y\sigma dA = \frac{1}{M}\int y\sigma 2r dy$$

$$= \frac{2}{\pi R^2}\int_0^{\frac{\pi}{2}} R\sin\theta 2R\cos\theta R\cos\theta d\theta$$

$$= \frac{4}{3\pi}R$$



The Newton's Second Law for the motion of CM

P219-220

- Motion of the center of mass

  Starting from $M\,\vec{r}_{\text{CM}} = \sum_i m_i \vec{r}_i$ by derivative $M\dfrac{d\vec{r}_{\text{CM}}}{dt} = \sum_i m_i \dfrac{d\vec{r}_i}{dt}$

$$M\,\vec{v}_{\text{CM}} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i = \vec{p}_{\text{tot}}$$

  The total momentum of the system of particles is equal to its total mass times the velocity of center of mass, just as though all the mass were concentrated at center of mass.

- The Newton's Second Law for the motion of center of mass

$$\boxed{\sum_i \vec{F}_{i-\text{ext}} = \frac{d\,\vec{p}_{\text{tot}}}{dt} = M\frac{d(\vec{v}_{\text{CM}})}{dt} = M\,\vec{a}_{\text{CM}}}$$

  The overall translational motion of a system of particles can be analyzed using Newton's Law as if all the mass were concentrated at the center of mass and total external force were applied at that point.

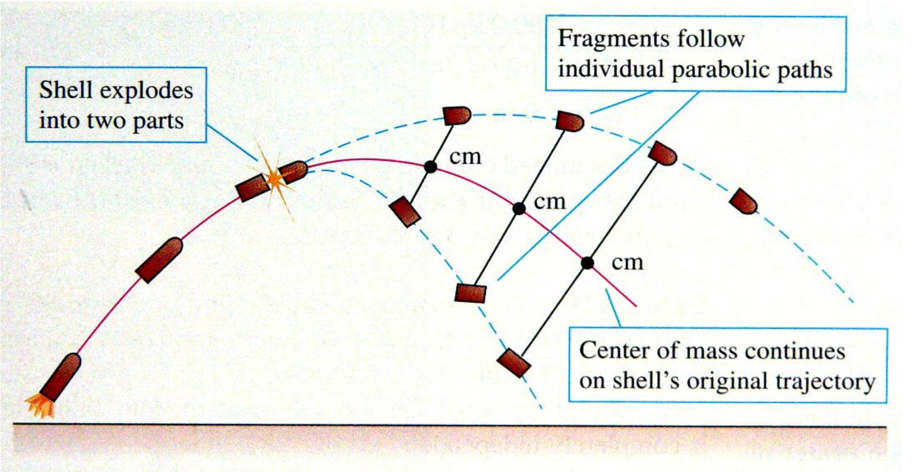
  - If net external force is zero, the center of mass moves with constant velocity

$$\sum_i \vec{F}_{i-\text{ext}} = 0 \quad \Rightarrow \quad \vec{p}_{\text{tot}} = M\,\vec{v}_{\text{CM}} = \text{constant}$$

Applications of center of mass

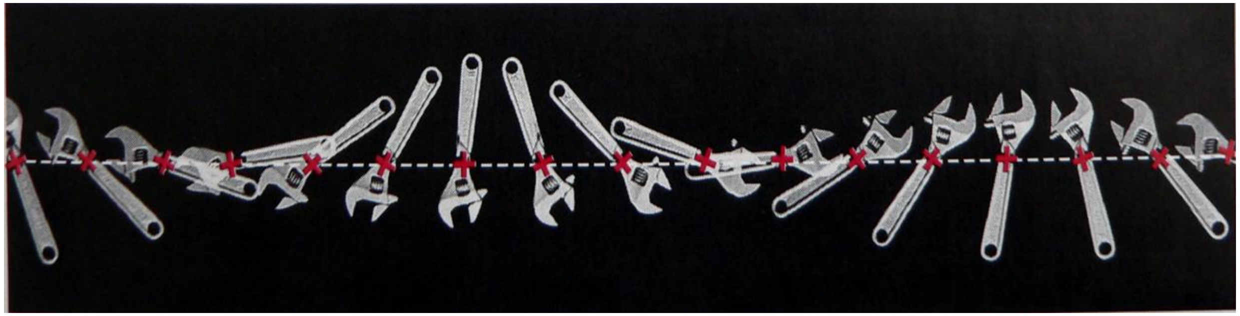
- Applications of center of mass
  - For a system of discrete particles

    A cannon shell in a parabolic trajectory explodes in flight, splitting into two fragments. The fragments follow new paths, but center of mass continues on the original parabolic trajectory.



  - For a rigid body

    We can describe a rigid body as a combination of translational motion of the center of mass and rotational motion about an axis through the center of mass.



Applications of center of mass

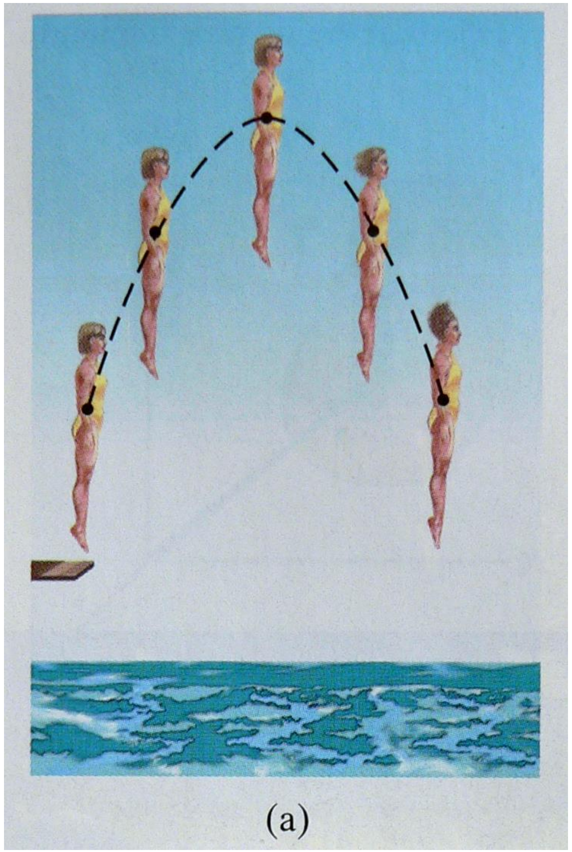
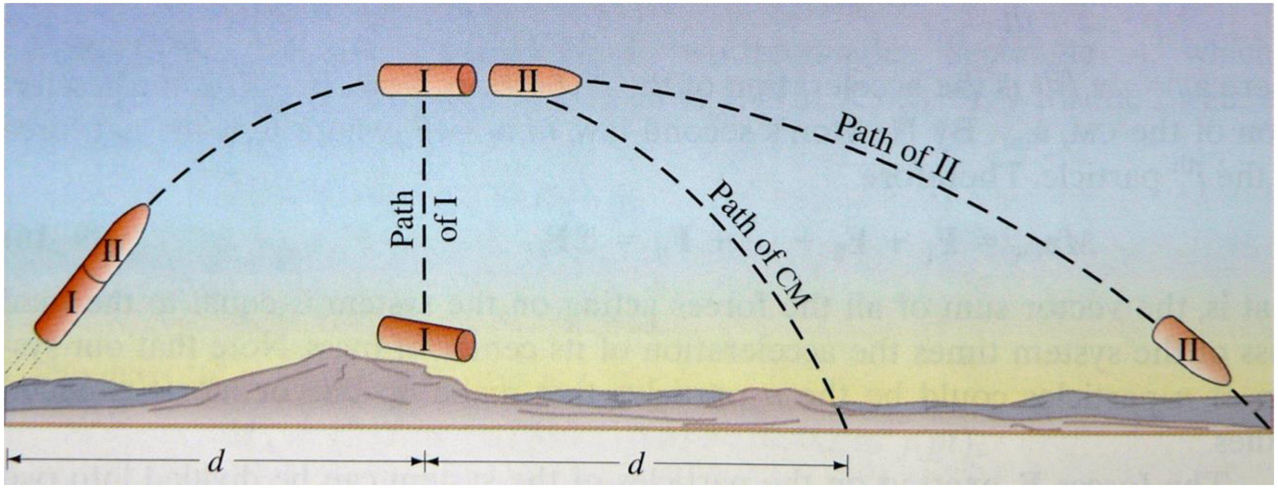


Fig. (a) The motion of the diver is pure translation.

Fig. (b) The motion of the diver is translation plus rotation.

Example

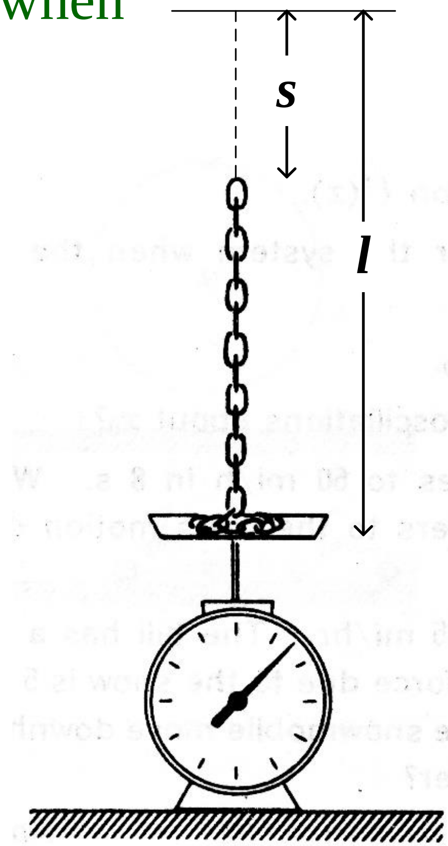
Example: A rocket is fired into the air. At the moment it reaches its highest point, a horizontal distance ***d*** from its starting point, an explosion separates it into two parts of equal mass. Part I is stopped in midair by explosion and falls vertically to Earth. Where does par II land?



Solution: After the rocket is fired, the path of the center of mass of the system continues to follow the parabolic trajectory of a projectile acted on only by a constant gravitational force. The center of mass will thus arrive at a point **2*d*** from the starting point. Since the masses of I and II are equal, the center of mass must be midway between them. Therefore, II lands a distance **3*d*** from the starting point.

Example:

A chain of mass ***M*** length ***l*** is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale. What is the reading of the scale when a length of chain, ***s***, has fallen? (Neglect the size of individual links.)

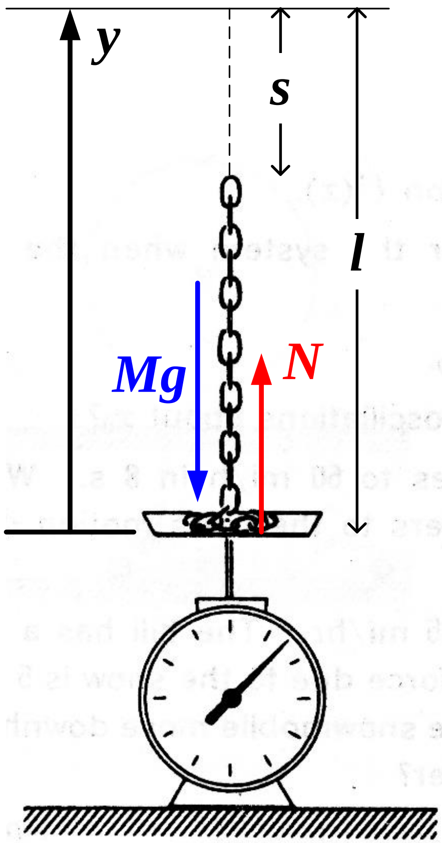


Example

Solution: (2) Using the Center of Mass:

Two part,: $M_1 = \lambda(l-s)$ $\quad y_1 = (l-s)/2$

$M_2 = \lambda s$ $\quad y_2 = 0$ $\quad \lambda = M/l$



$$N - Mg = M\frac{d^2 y_{\mathrm{CM}}}{dt^2} = Mg\left(\frac{3s}{l} - 1\right)$$

For the part in the air: $s = \frac{1}{2}gt^2$

$$\frac{d\,y_{\mathrm{CM}}}{dt} = \frac{d}{dt}\left[\frac{(l-s)^2}{2l}\right] = -\frac{(l-s)}{l}\frac{ds}{dt} = -\frac{gt}{l}\left(l - \frac{1}{2}gt^2\right)$$

$$\frac{d^2 y_{\mathrm{CM}}}{dt^2} = \frac{g(3s-l)}{l}$$

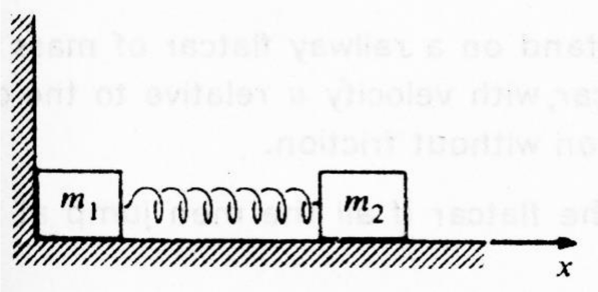
Newton's II law for CM:

$$y_{\mathrm{CM}} = \frac{M_1 y_1 + M_2 y_2}{M} = \frac{\lambda(l-s)(l-s)/2}{\lambda l} = \frac{(l-s)^2}{2l}$$

$$\boxed{N = 3Mg\frac{s}{l}}$$

Example

Example: A system is composed of two blocks of mass m_1 and m_2 connected by a massless spring with spring constant k . The blocks slide on a frictionless plane. The unstretched length of the spring is l . Initially is m_2 is held so that the spring is compressed to $l / 2$ and m_1 is forced against a wall, as shown in the figure. m_2 is released at $t = 0$. Find the motion of the center of mass of the system as function of time.

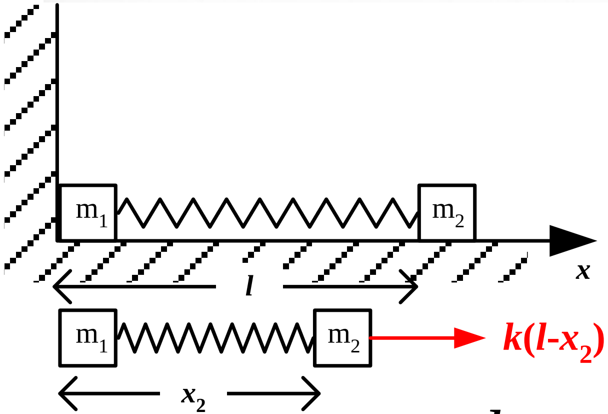


Solution I: Treat two blocks separately.

For m_1, $x_1=0$

For m_2: $k(l-x_2)=m_2\dfrac{d^2x_2}{dt^2}$

$$\frac{d^2x_2}{dt^2}+\frac{k}{m_2}(x_2-l)=0 \qquad x_2-l=A\cos\left(\sqrt{\frac{k}{m_2}}t+\phi\right)$$



Initial condition: $t=0$, $x2=l/2$, $v2=0$

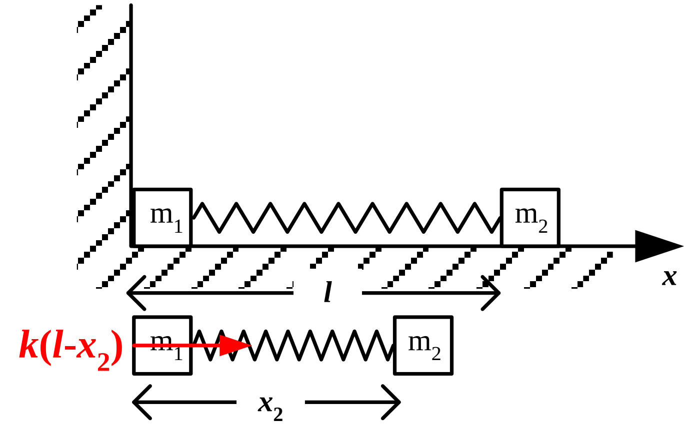
$$\frac{l}{2}-l=A\cos\phi$$

$$v_2=0=-A\sqrt{\frac{k}{m_2}}\sin\left(\sqrt{\frac{k}{m_2}}t+\phi\right)\Bigg|_{t=0}=-A\sqrt{\frac{k}{m_2}}\sin\phi$$

$$\Rightarrow \quad \phi=\pi,\quad A=\frac{l}{2}$$

Example (continued)

$$x_2 = l + \frac{l}{2}\cos\left(\sqrt{\frac{k}{m_2}}t + \pi\right) = l - \frac{l}{2}\cos\left(\sqrt{\frac{k}{m_2}}t\right)$$



Find the turning point: As soon as $\mathbf{x_2=l}$, there will be no external force acting on the two-block system. After that instant, the center of mass will moves at a constant velocity.

When: $$\sqrt{\frac{k}{m_2}}t = \frac{\pi}{2},\ t = \frac{\pi}{2}\sqrt{\frac{m_2}{k}},\ x_2 = l,\ v_2 = \frac{l}{2}\sqrt{\frac{k}{m_2}}\sin\left(\frac{\pi}{2}\right) = \frac{l}{2}\sqrt{\frac{k}{m_2}}$$

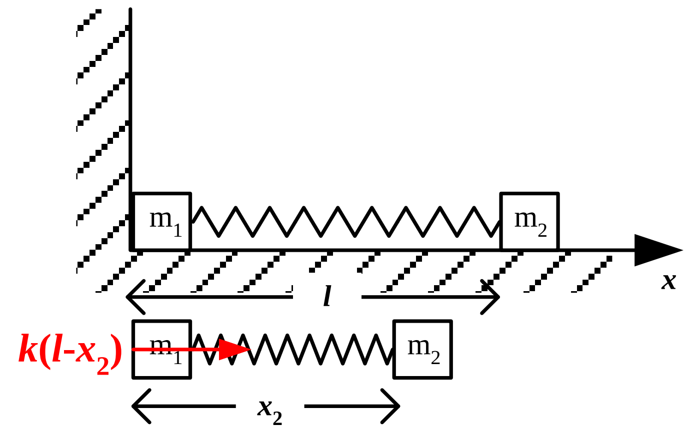
$$v_{\mathrm{CM}} = \frac{m_2}{m_1 + m_2}v_2 = \frac{m_2}{m_1 + m_2}\frac{l}{2}\sqrt{\frac{k}{m_2}}$$

$$\boxed{x_{\mathrm{CM}} = \begin{cases} \dfrac{m_2}{m_1 + m_2}x_2 = \dfrac{m_2}{m_1 + m_2}\left[l - \dfrac{l}{2}\cos\left(\sqrt{\dfrac{k}{m_2}}t\right)\right], & t \le \dfrac{\pi}{2}\sqrt{\dfrac{m_2}{k}} \\ \dfrac{m_2}{m_1 + m_2}l + \dfrac{m_2}{m_1 + m_2}\dfrac{l}{2}\sqrt{\dfrac{k}{m_2}}\left(t - \dfrac{\pi}{2}\sqrt{\dfrac{m_2}{k}}\right), & t > \dfrac{\pi}{2}\sqrt{\dfrac{m_2}{k}} \end{cases}}$$

Example (continued)

Solution II: Treat the two blocks as a whole.

External force: exerted by wall on m_2.



$$k(l - x_2) = (m_1 + m_2)\frac{d^2 x_{CM}}{dt^2} \qquad x_{CM} = \frac{m_2}{m_1 + m_2} x_2$$

$$k\left(l - \frac{m_1 + m_2}{m_2} x_{CM}\right) = (m_1 + m_2)\frac{d^2 x_{CM}}{dt^2}$$

$$\frac{d^2 x_{CM}}{dt^2} + \frac{k}{m_2}\left(x_{CM} - \frac{m_2}{m_1 + m_2} l\right) = 0$$

$$x_{CM} - \frac{m_2}{m_1 + m_2} l = A\cos\left(\sqrt{\frac{k}{m_2}} t + \phi\right)$$

Example (continued)

$$x_{\mathrm{CM}}-\frac{m_2}{m_1+m_2}l=A\cos\left(\sqrt{\frac{k}{m_2}}t+\phi\right)$$

Initial condition: ***t*=0**, $x_{\mathrm{CM}}=\frac{m_2}{m_1+m_2}\frac{l}{2},\quad v_{\mathrm{CM}}=0$

$$\frac{m_2}{m_1+m_2}\frac{l}{2}-\frac{m_2}{m_1+m_2}l=A\cos\phi$$

$$v_{\mathrm{CM}}=0=-A\sqrt{\frac{k}{m_2}}\sin\phi$$

$$\Rightarrow\quad \phi=\pi,\quad A=\frac{m_2}{m_1+m_2}\frac{l}{2}$$

$$\boxed{x_{\mathrm{CM}}=\begin{cases}\dfrac{m_2}{m_1+m_2}\left[l+\dfrac{l}{2}\cos\left(\sqrt{\dfrac{k}{m_2}}t+\pi\right)\right]=\dfrac{m_2}{m_1+m_2}\left[l-\dfrac{l}{2}\cos\left(\sqrt{\dfrac{k}{m_2}}t\right)\right], & t\le\dfrac{\pi}{2}\sqrt{\dfrac{m_2}{k}}\\[2ex] \dfrac{m_2}{m_1+m_2}l+\dfrac{m_2}{m_1+m_2}\dfrac{l}{2}\sqrt{\dfrac{k}{m_2}}\left(t-\dfrac{\pi}{2}\sqrt{\dfrac{m_2}{k}}\right), & t>\dfrac{\pi}{2}\sqrt{\dfrac{m_2}{k}}\end{cases}}$$

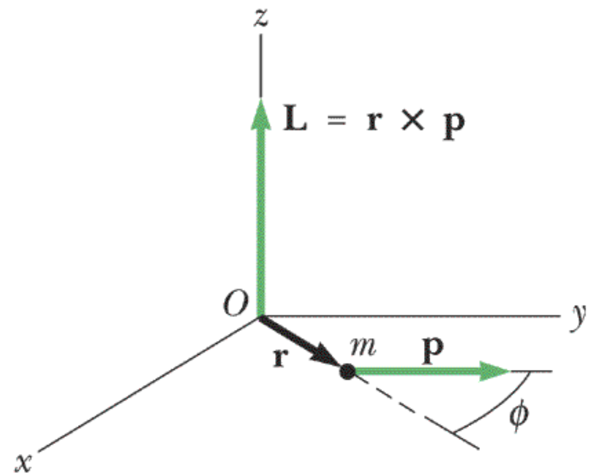
§ 5 Angular Momentum

P277-278

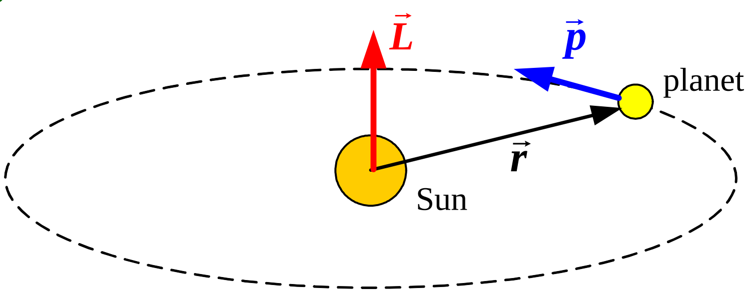
- Angular Momentum
  - Definition

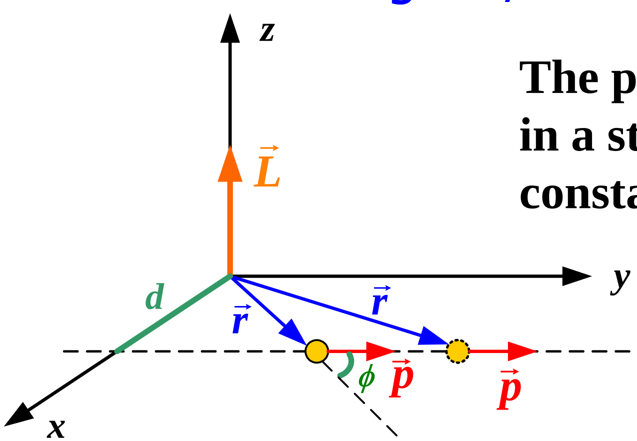
$$\boxed{\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}}$$

  - Magnitude: $L = mvr\sin\phi$
  - Direction: perpendicular to the plane formed by $\vec{r}$ and $\vec{p}$, and sense of $\vec{L}$ is governed by the right-hand rule.



  - Depends on the choice of origin O.
  - SI unit: kg•m^2/s





The particle that moves in a straight line at constant velocity

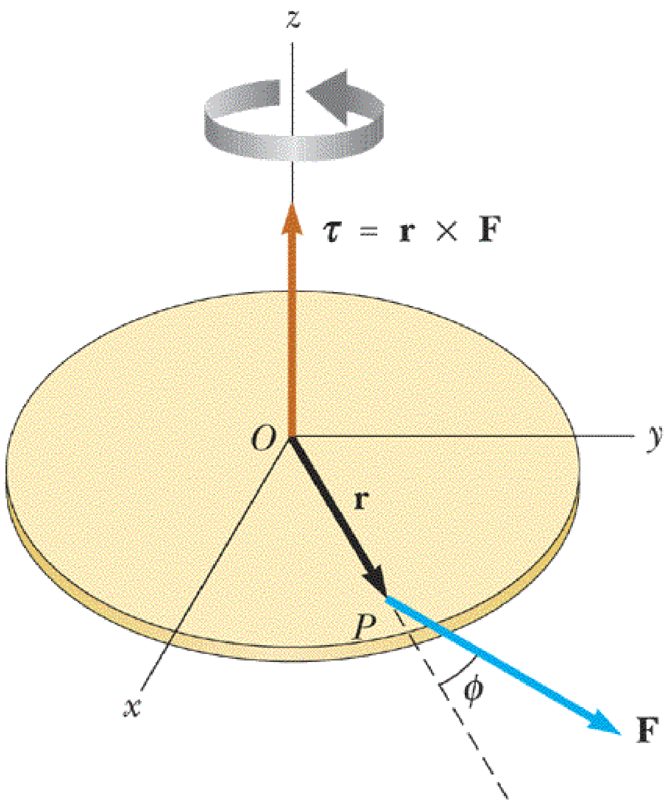
$$L = |\vec{r} \times \vec{p}|$$
$$= mvr\sin\phi = mvd$$

The planet that moves in the circular orbit around the sun.

$$L = |\vec{r} \times \vec{p}| = mvr$$

Torque

- Torque
  - Definition $\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$
    - Magnitude: $\tau = rF\sin\phi$
    - Direction: perpendicular to the plane formed by $\vec{r}$ and $\vec{F}$, and sense of $\vec{\tau}$ is governed by the right-hand rule.
  - Depends on the choice of origin O.
  - SI unit: Newton • m.



- Torque-Angular Momentum Theorem
  - For one particle

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\,\vec{p}}{dt}$$

$$\frac{d\,\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\,\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\,\vec{p}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \frac{d\,\vec{p}}{dt} = \vec{r} \times \frac{d\,\vec{p}}{dt}$$

$$\boxed{\vec{\tau} = \frac{d\,\vec{L}}{dt}}$$

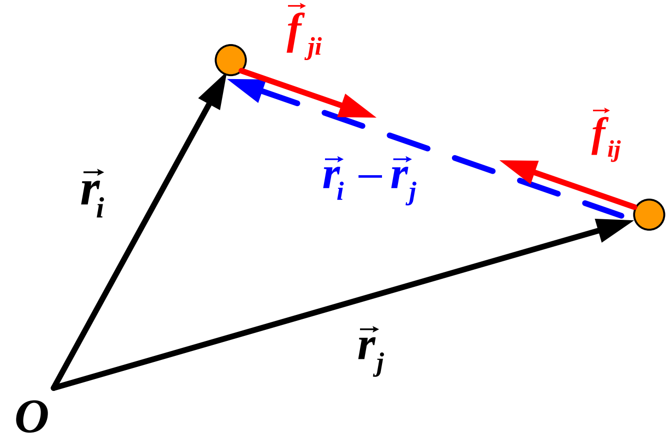
The torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

- Valid only if the origins of $\vec{L}$ and $\vec{\tau}$ are the same.
- Valid in inertial frame.

Torque-Angular Momentum Theorem for a system of particles

- Torque-Angular Momentum Theorem for a system of particles
  - The torques of each pair of internal forces are vanished.

$$\vec{r}_i \times \vec{f}_{ji} + \vec{r}_j \times \vec{f}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{f}_{ji} = 0$$



$$\boxed{\sum \vec{\tau}_{\text{ext}} = \sum_i \frac{d\vec{L}_i}{dt} = \frac{d}{dt}\sum_i \vec{L}_i = \frac{d\vec{L}_{\text{tot}}}{dt}}$$

The net external torque acting on the system is equal to the time rate of change of the total angular momentum of the system.

- Valid only if all the origins of $\vec{L}$ and $\vec{\tau}$ in the system are the same.
- Valid in inertial frame and the reference frame of the center of mass.

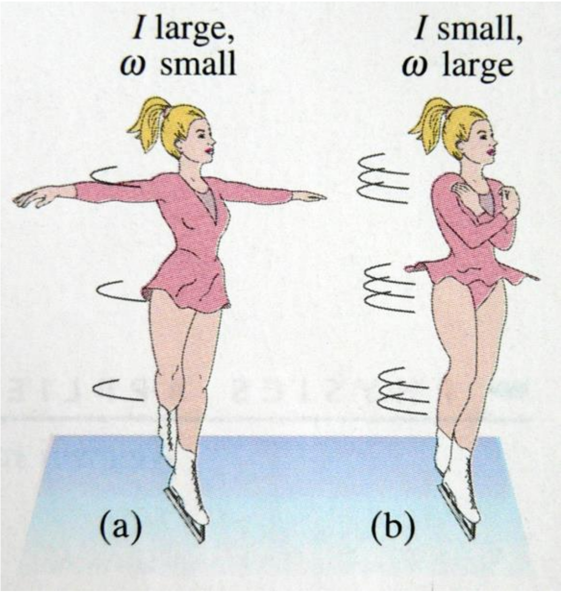
$$\boxed{\sum \vec{\tau}_{\text{ext-CM}} = \frac{d\vec{L}_{\text{tot-CM}}}{dt}}$$

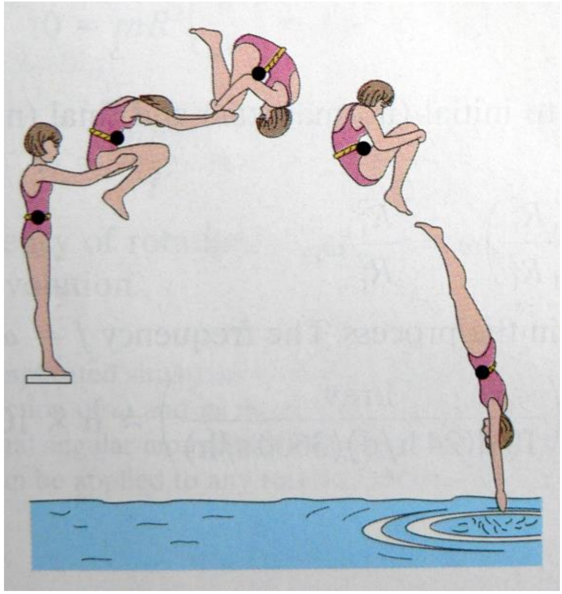
- Conservation of Angular Momentum

  For a system of particles

$$\sum \vec{\tau}_{\text{ext}} = 0 \quad \Longrightarrow \quad \frac{d\vec{L}_{\text{tot}}}{dt} = 0 \quad \text{or} \quad \vec{L}_{\text{tot}} = \text{constant}$$

The total angular momentum of a system remains constant if the net external torque acting on the system is zero.



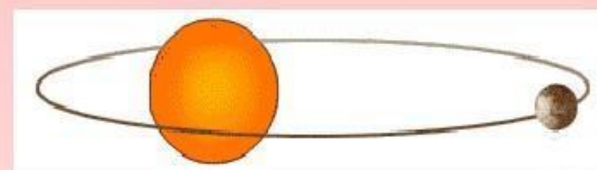


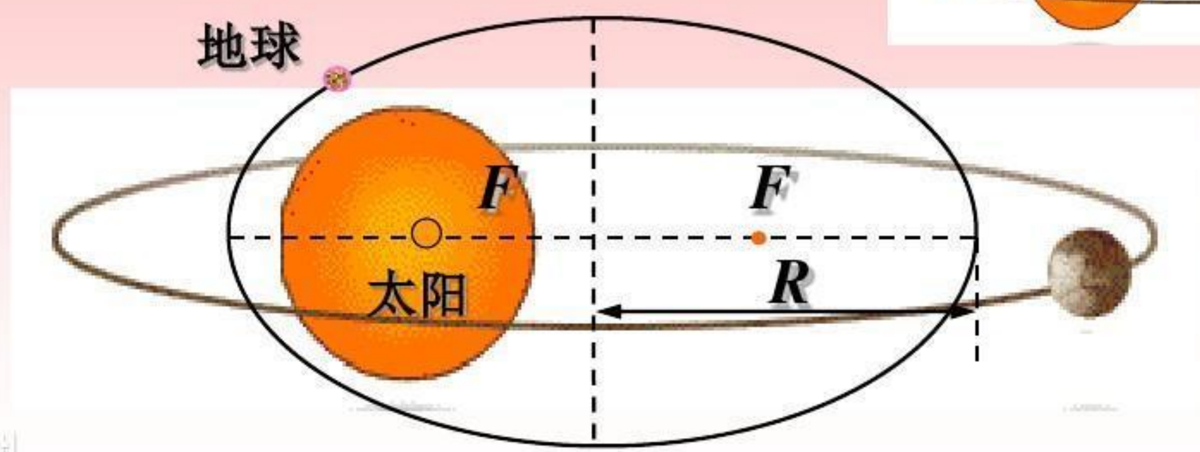
Two examples to indicate the conservation of angular momentum

Kepler's Laws

1. ***The Law of Orbits*****: All planets move in elliptical (*椭圆*) orbits, with the Sun at one focus.**

开普勒第一定律(轨道定律)所有的行星围绕太阳运动的轨道都是椭圆，太阳处在所有椭圆的一个焦点上。





- The Application of Conservation of Angular Momentum ——
  Kepler's Second Law

  The radius vector drawn from the Sun to any planet sweeps out equal areas in equal time intervals.

For a planet of mass M_p moving about the Sun

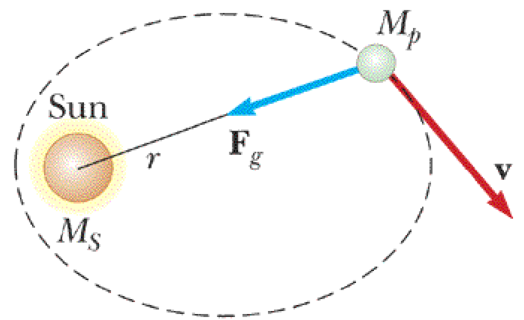
$$\vec{\tau} = \vec{r} \times \vec{F}_g = 0$$

$$\Downarrow$$

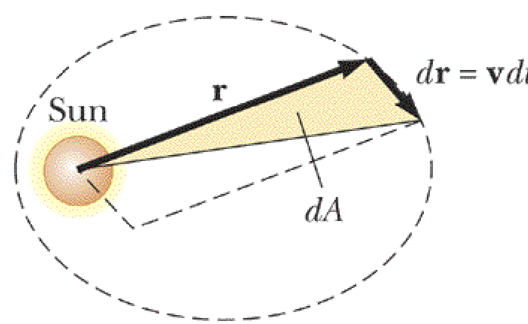
$$\vec{L} = \vec{r} \times \vec{p} = M_p \vec{r} \times \vec{v} = \text{constant}$$

$$dA = \frac{1}{2}\left|\vec{r} \times d\vec{r}\right| = \frac{1}{2}\left|\vec{r} \times \vec{v}\,dt\right| = \frac{L}{2M_p}dt$$

$$\boxed{\frac{dA}{dt} = \frac{L}{2M_p} = \text{constant}}$$



(a)



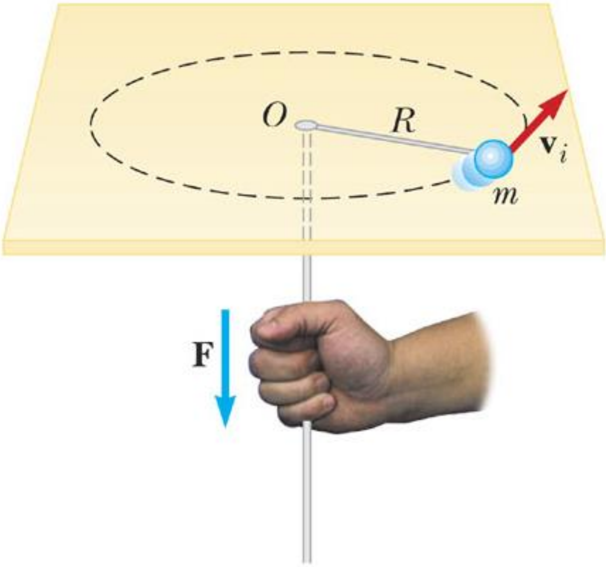
(b)

Example: A ball of mass ***m*** on a horizontal, frictionless table is connected to a string that passes through a small hole in the table. The ball is set into circular motion of radius ***R***, at which time its speed is v_i. If the string is pulled from the bottom so that the radius of the circular path is decreased to r, what is final speed v_f of the ball?

Solution: The net torque for the ball system is zero. Therefore the angular momentum of ball remains constant.

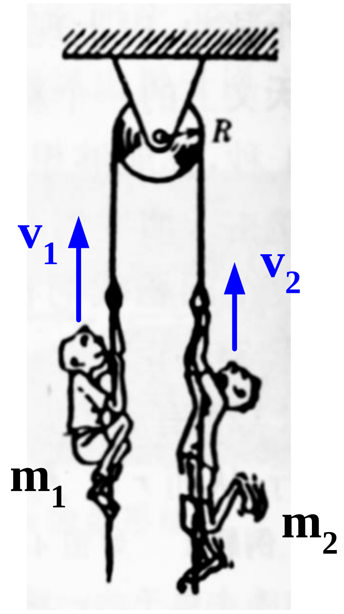
$$Rmv_i = rmv_f$$

$$v_f = \frac{v_i R}{r} > v_i$$



Example

Example: Two boys, with same mass of ***m*** , suspend to the two side of a pulley with a light rope. The boy on the left makes an effort to climb up, but the other boy keeps at rest without any action. Which boy is the first to approach pulley? Neglecting the mass of the pulley and the friction on the axis of the pulley.



Solution: For the two-boy system, the net external torque.

Take the direction of torque consistent with anti-clockwise.

$$\sum \tau_{\text{ext}} = Rm_1 g - Rm_2 g = 0$$

The angular momentum of two-boy system is conserved.

$$L_f = mR(v_2 - v_1) = L_i = 0$$

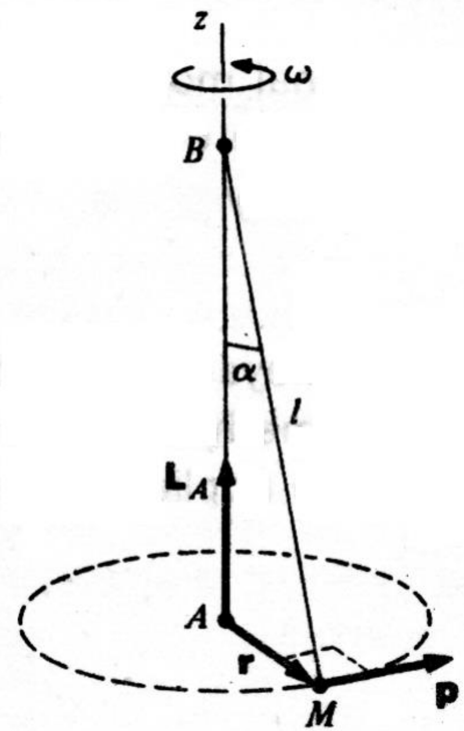
Two boy approach the pulley at same time, whoever makes an effort.

But if $m_1 > m_2$, $\sum \tau_{\text{ext}} > 0$ $\quad \dfrac{dL}{dt} > 0,\ L_i = 0,\ L_f > 0,\ v_2 > v_1$

Example: Angular momentum of the conical pendulum:

(1) The angular momentum about origin A.

(2) The angular momentum about origin B.



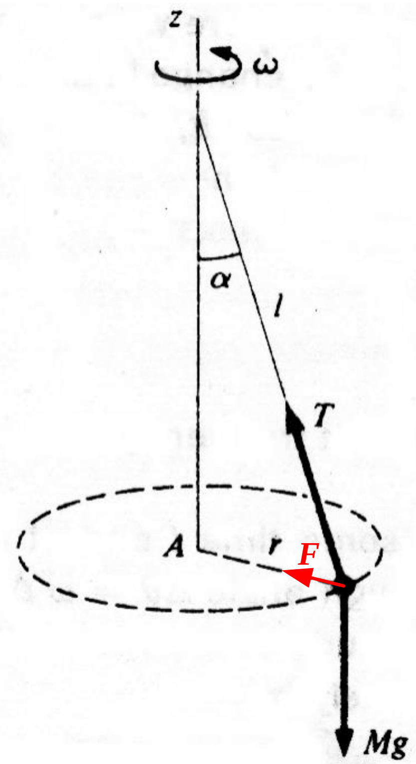
Solution: (1) For origin A

$$\vec{L}_A = \vec{r} \times \vec{p} = rMv\hat{k} = r^2 M \omega \hat{k} = \text{constant}$$

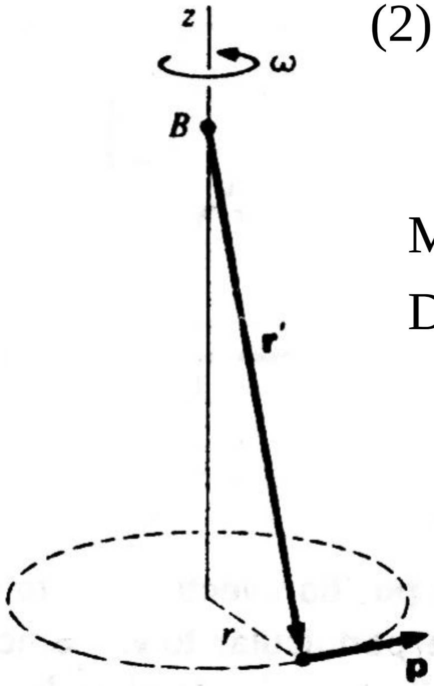
$$\vec{\tau}_A = \vec{r} \times (\vec{T} + M\vec{g}) = 0$$

$$\vec{\tau}_A = \frac{d\vec{L}_A}{dt} = 0$$

$\vec{L}_A$ remains constant, both in magnitude and direction.



Example cont'd



(2) For origin B

$$|\vec{L}_B| = |\vec{r}' \times \vec{p}| = l\,p = Mlr\omega$$

Magnitude: constant.

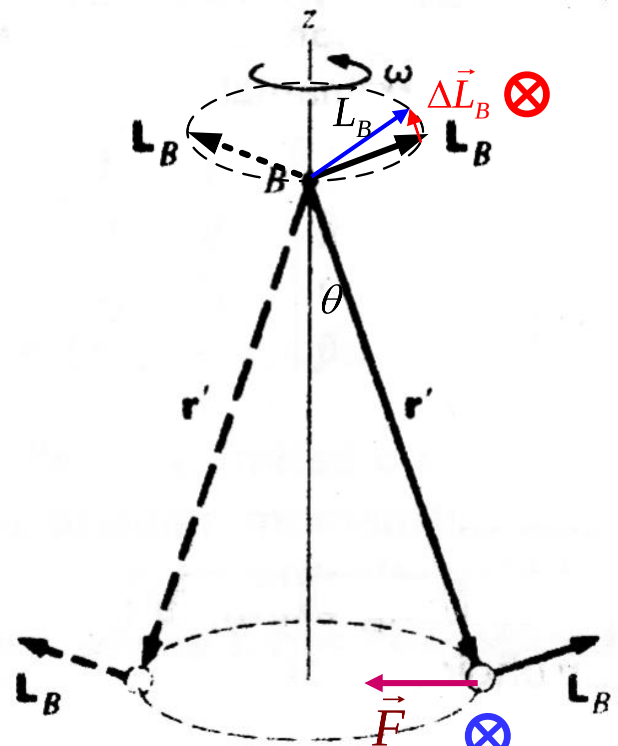
Direction: perpendicular to $\vec{r}'$ and $\vec{p}$

Its tip draws a horizontal circle.

Calculate the torque τ_B about B.

Evaluate the validity of $\vec{\tau}_B = \dfrac{d\vec{L}_B}{dt}$?

$$|\vec{\tau}_B| = r'\sin\theta \cdot F = rF$$



$$\vec{\tau}_B = \vec{r}' \times \vec{F}$$

Circular motion: $F = Mr\omega^2$ $\quad |\vec{\tau}_B| = r'\sin\theta \cdot F = Mr^2\omega^2$

$$\frac{|\Delta L_B|}{\Delta t} = \frac{L_B\cos\theta \cdot \Delta\varphi}{\Delta t} \rightarrow L_B\cos\theta \cdot \omega = Mr\omega^2\left(l\cos\theta\right) = Mr^2\omega^2$$