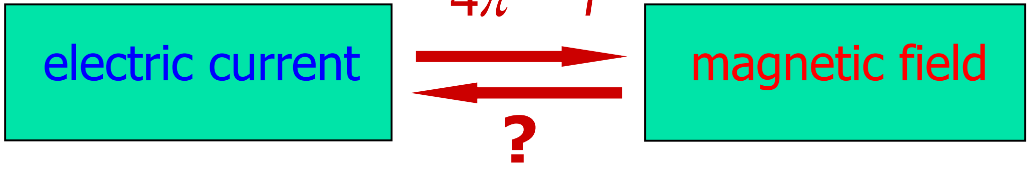
Chapter 27-28 Faraday's Law and Inductance

§ 1 Faraday's Law of Induction and Lenz's Law

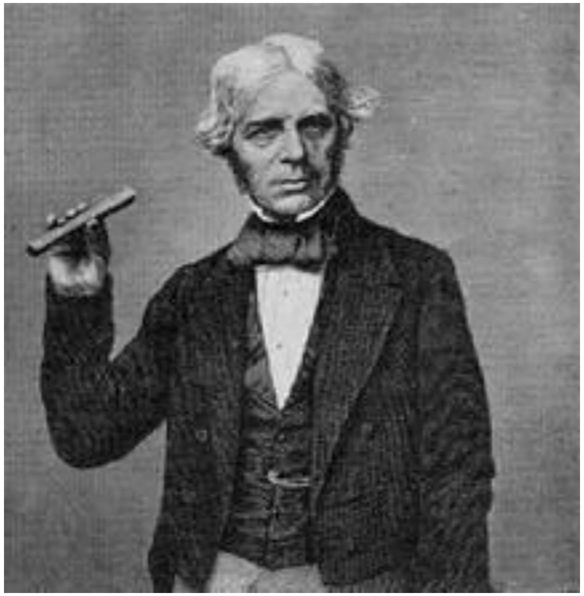
(p629-633)

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{s}\times\hat{r}}{r^2}$$

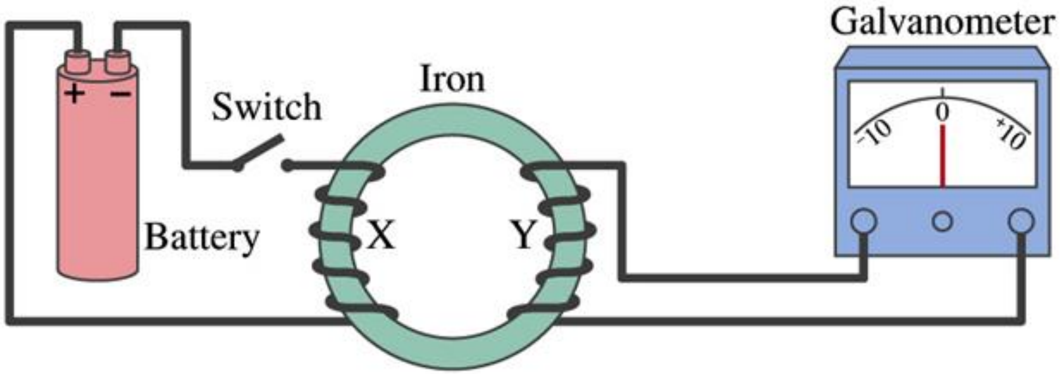

electric current

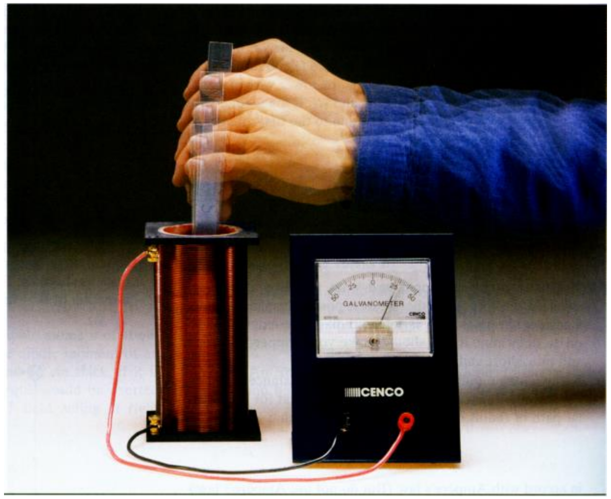
magnetic field

?



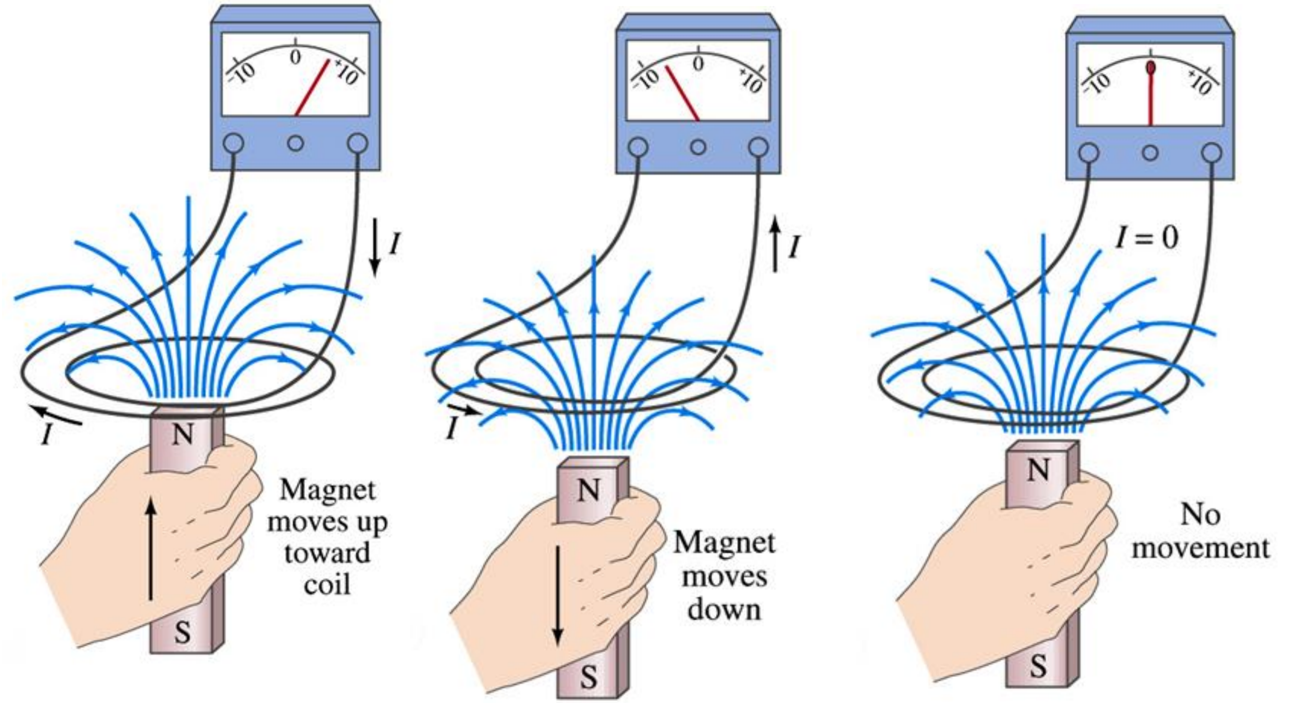
- Question: Can an electric current be produced by a magnetic field?
  - M. Faraday (1791-1867) answered this question in 1831.

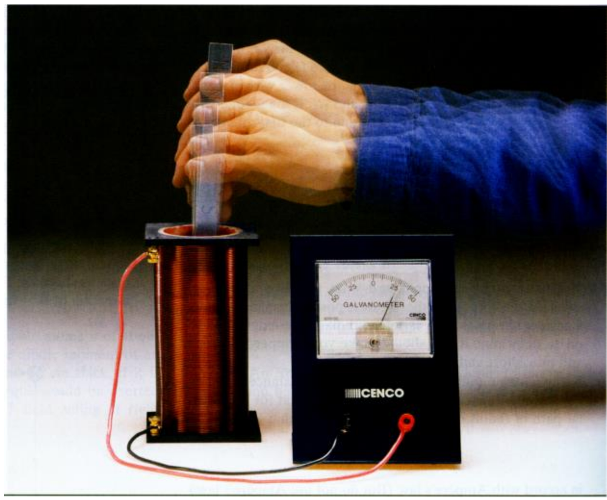




Evaluate The Experiment of Induction

- From the experiment:
  - Steady magnetic field can not produce any current.
  - A time-varying magnetic field can induce an electric current.
  - The galvanometer shows a lager induced current when the relative motion of the magnet is faster。
  - It is the rate of change in the number of the magnetic field lines passing trough the loop that determine the induced emf in the loop.

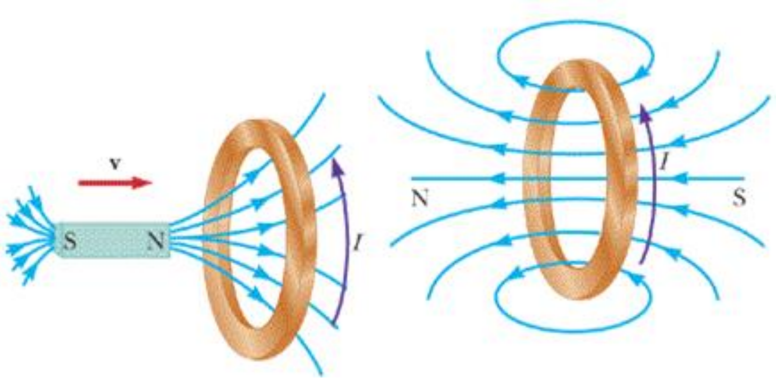


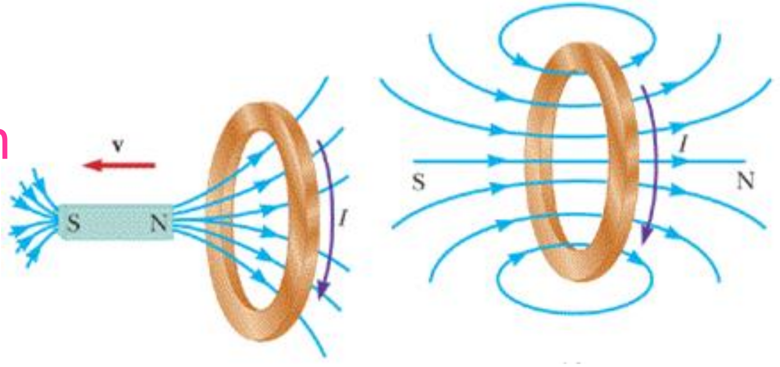


Faraday's Law and Lenz's Law

- Faraday's law:
  - The emf induced in a circuit is equal to the time rate of change of magnetic flux through the circuit. ⇨ $|\varepsilon| = \left|\frac{d\Phi_B}{dt}\right|$
  - If the circuit is a coil consists of N turns: ⇨ $|\varepsilon| = N\left|\frac{d\Phi_B}{dt}\right|$
  - How about the direction of the induced emf? —— determined by Lenz's law
- Lenz's law
  - The polarity of the induced emf in a loop is such that it produces a current whose magnetic field opposes the change in magnetic flux through the loop.

  Another statement:
  - The induced current is in a direction such that the induced magnetic field attempts to maintain the original flux through the loop.





How to Determine the Sign of Induced emf

- Complete Faraday’s law:

$$\boxed{\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_{\text{surrounding surface}} \vec{B}\cdot d\vec{A}}$$

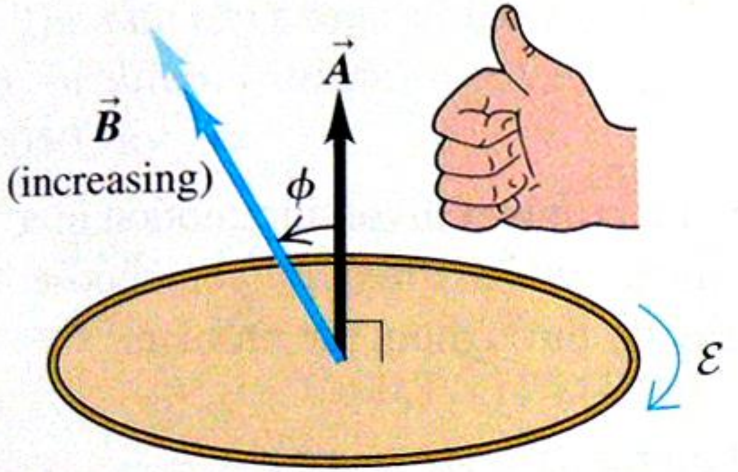
  - A coil consists of N turns:

$$\boxed{\varepsilon = -N\frac{d\Phi_B}{dt}}$$

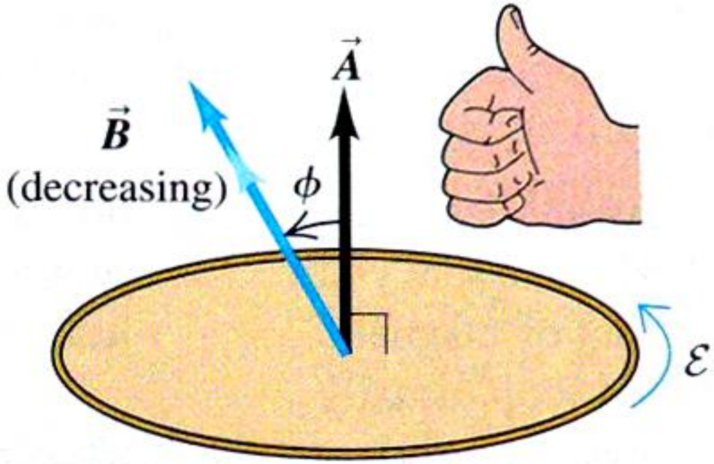
- The relationship between the direction of emf $\mathcal{E}$ and the sign of Φ_B
  - Using the right-hand rule to determine the sign of Φ_B and the sign of emf $\mathcal{E}$.

Φ_B takes the sign "+"

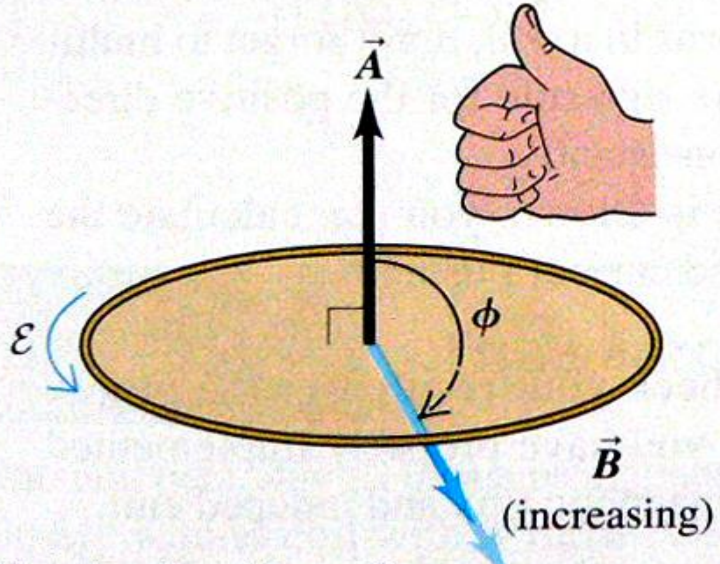
The direction of the sign "+" that $\mathcal{E}$ takes



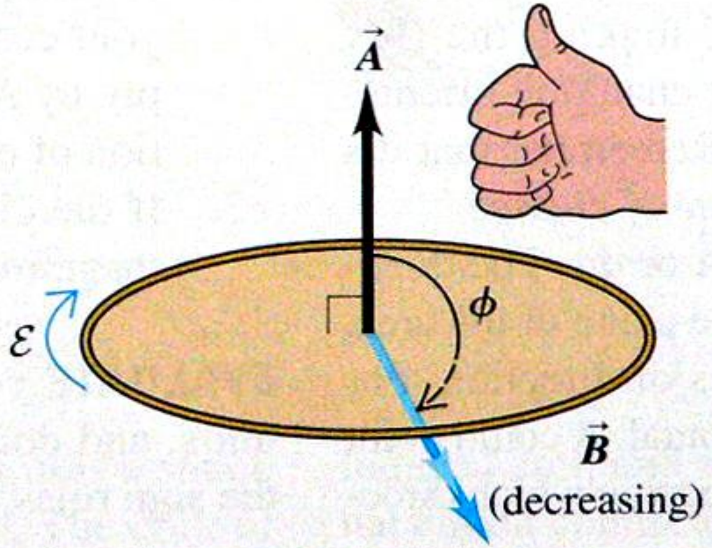
Positive flux ($\Phi_B > 0$)
Flux becoming more positive $\left(\frac{d\Phi_B}{dt} > 0\right)$
Induced emf is negative ($\mathcal{E} < 0$)



Positive flux ($\Phi_B > 0$)
Flux becoming less positive $\left(\frac{d\Phi_B}{dt} < 0\right)$
Induced emf is positive ($\mathcal{E} > 0$)



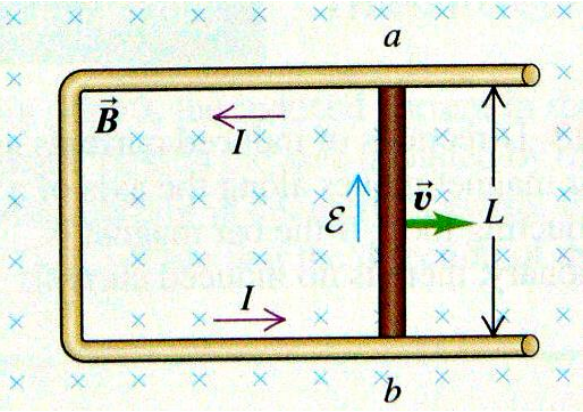
Negative flux ($\Phi_B < 0$)
Flux becoming more negative $\left(\frac{d\Phi_B}{dt} < 0\right)$
Induced emf is positive ($\mathcal{E} > 0$)



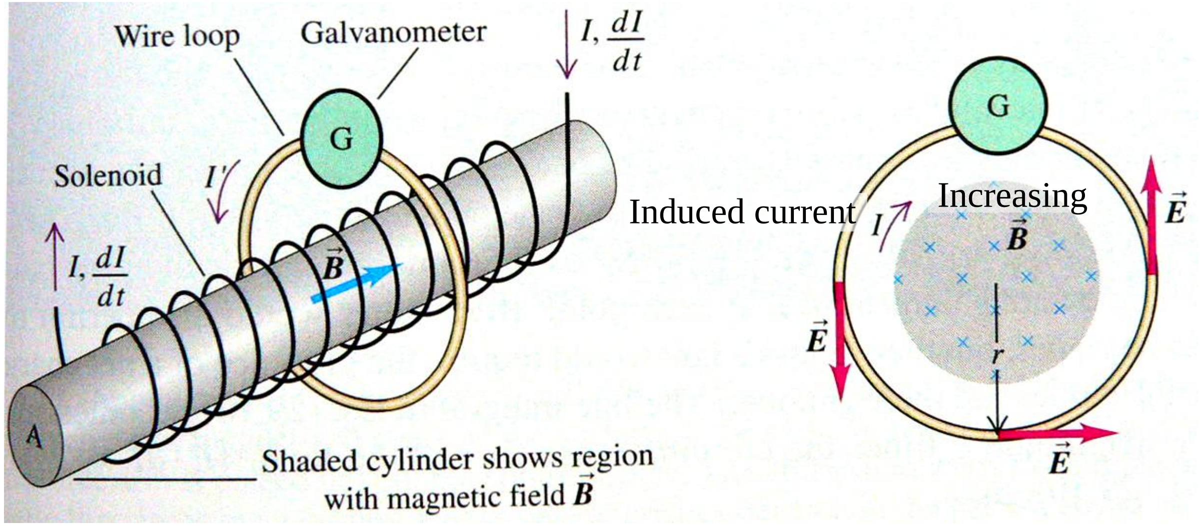
Negative flux ($\Phi_B < 0$)
Flux becoming less negative $\left(\frac{d\Phi_B}{dt} > 0\right)$
Induced emf is negative ($\mathcal{E} < 0$)

What makes the magnetic flux change?

- What makes the magnetic flux change?
  - Is the loop or coil moving or changing orientation? —— Motional emf.
  - Is the magnetic field changing? —— Induced electric field as the non-electric field.



Motional emf



Induced emf

Example

Example: A plane loop of area ***A*** is placed in a region where a uniform magnetic field is an angle θ to the normal to the plane. The magnitude of the magnetic field varies with time according to the expression $B = B_{max} e^{-\alpha t}$. Find the induced emf in the loop as a function of time.

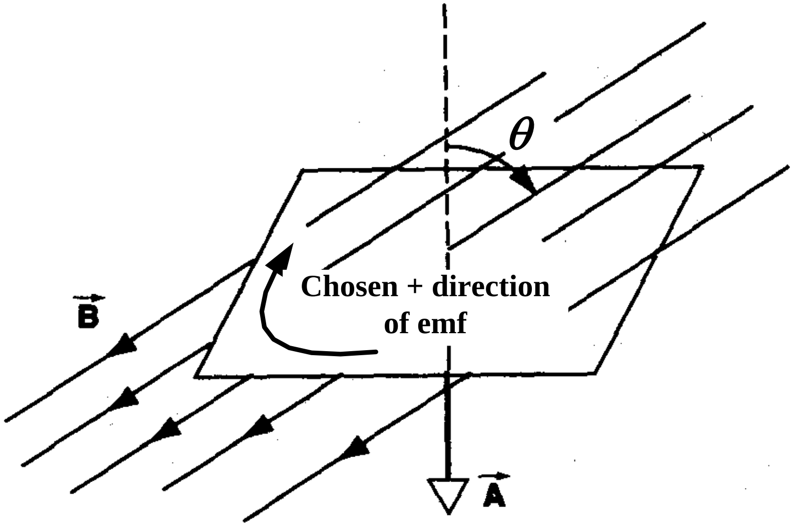
Solution: Choose the direction of area vector point to downward.

$$\Phi_B = \vec{B} \cdot \vec{A} = BA\cos\theta$$

$$= AB_{\max} e^{-\alpha t} \cos\theta$$

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\left(-\alpha AB_{\max} e^{-\alpha t} \cos\theta\right)$$

$$= \alpha AB_{\max} \cos\theta e^{-\alpha t}$$

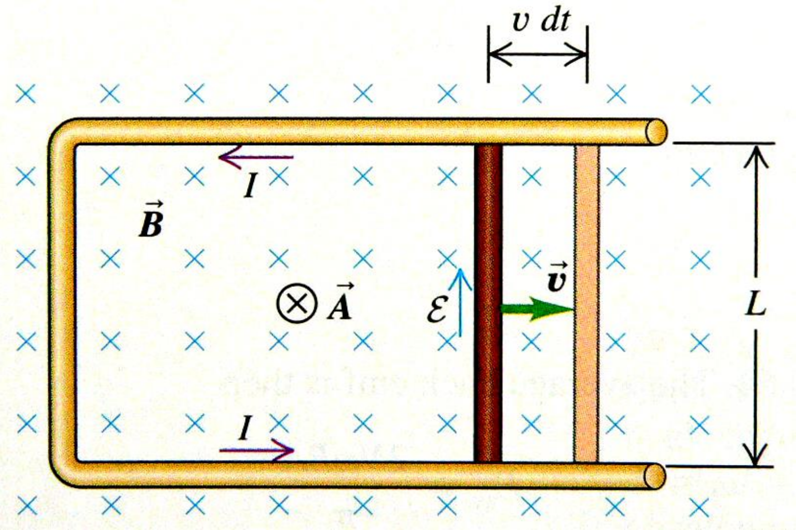


§ 2 Motional emf (p634-635)

- Staring with the slide-wire generator

A U-shaped conductor in a uniform magnetic field $\vec{B}$ perpendicular to the plane, directed into page. A metal rode with length L across the two arms of the conductor, forming a circuit. The metal rode slides to the right with a constant velocity $\vec{v}$. Find the induced emf.

Choose the direction of area $\vec{A}$ as directing into the page.



- The magnetic flux through the circuit:

$$\Phi_B = BLvt$$

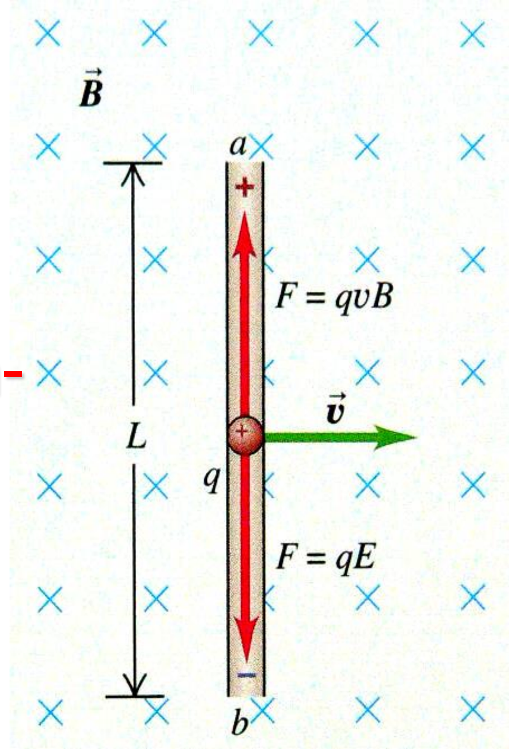
- The induced emf:

$$\varepsilon = -\frac{d\Phi_B}{dt} = -BLv$$

- The negative sign means that direction of emf is counterclockwise.

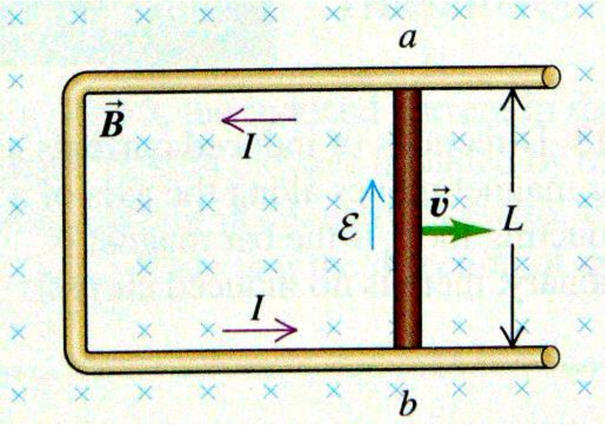
The Origin of the Motional emf

- Additional insight into the origin of the induced emf:
  - The magnetic force exerting on the moving charge in rode acts as the **non-electric force** that produces the emf.
  - The magnetic force: $\overrightarrow{F} = q\vec{v} \times \overrightarrow{B}$
  - The emf along the rode:



$$\varepsilon = \int_a^b \overrightarrow{E}_n \cdot d\vec{s} = \int_a^b \frac{\overrightarrow{F}}{q} \cdot d\vec{s} = \int_a^b (\vec{v} \times \overrightarrow{B}) \cdot d\vec{s}$$

$$= -\int_0^L vBds = -BLv$$



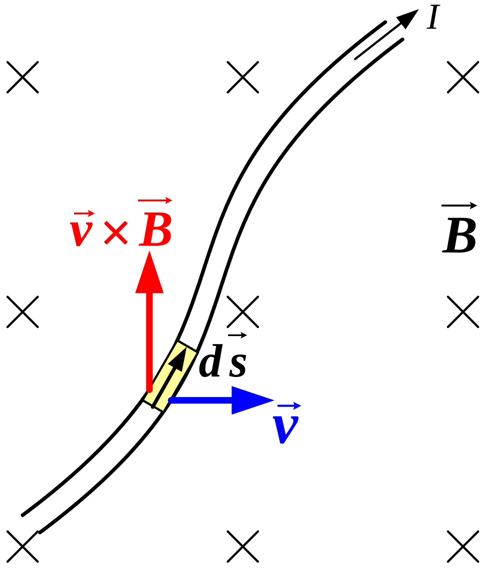
- The emf is induced in a conductor moving through a magnetic field, called motional emf.
- With Faraday’s law, we cannot know which part of the circuit is the source of the emf. Here we know that the moving rode is the source of emf; within it, positive charge moves from lower to higher potential, and in the remainder of the circuit, charge moves from higher to lower potential.

Definition of Motional emf

- Definition of motional emf:
  - For moving current-carrying wire of any shape in a magnetic field

$$d\varepsilon = (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

$$\boxed{\varepsilon = \int_L (\vec{v} \times \vec{B}) \cdot d\vec{s}}$$



  - For any closed conducting loop:

$$\varepsilon = \oint_L (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

  - The direction of motional emf: determined by the projection direction of $\vec{v} \times \vec{B}$

Example

Motional emf induced in a rotating bar

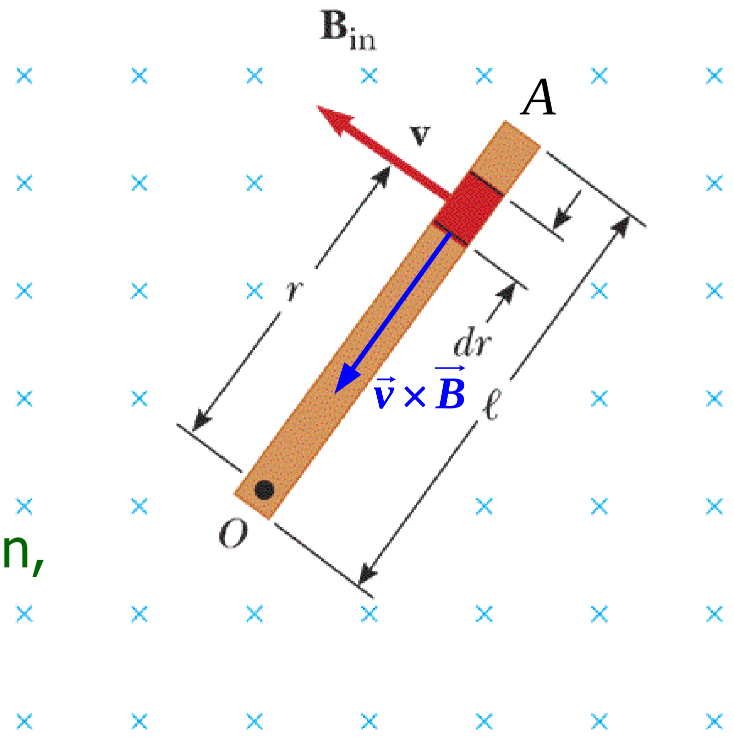
Example: A conducting bar of length ***l*** rotates with a angular speed ***ω*** about a pivot at one end. ***B*** is perpendicular to the plane of rotation. Find the emf induced between the ends of the bar.

Solution: Choose the direction of integration to be from end O to end A.

$$\varepsilon = \int_O^A (\vec{v} \times \vec{B}) \cdot d\vec{s} = \int_0^l (-Bv)dr$$

$$= -\int_0^l B\omega r dr = -\frac{1}{2}B\omega l^2$$

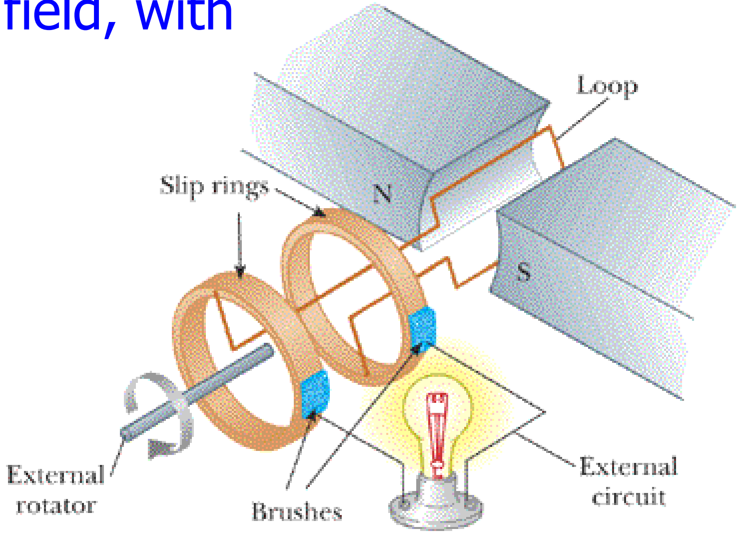
The negative sign means that the real direction of emf is opposite to the direction of integration, and potential at end A is lower than end O.



Example

The Alternating-current generator

Example: A ***N*** turns rectangular loop of area ***A*** is made to rotate in an external uniform magnetic field, with a angular velocity ***ω*** about the axis.



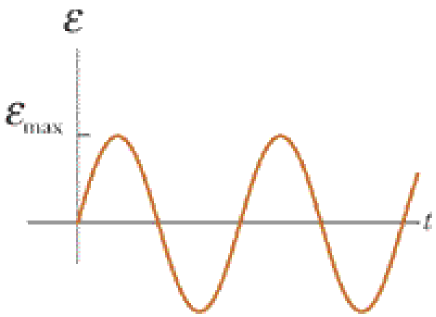
Solution: Assume at time t=0, the direction of area $\vec{A}$ is in alignment with $\vec{B}$.

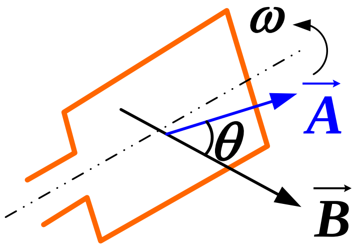
The flux through the loop

$$\Phi_B = \vec{B}\cdot\vec{A} = BA\cos\theta = BA\cos\omega t$$

By Faraday's law,

$$\varepsilon = -N\frac{d\Phi_B}{dt} = \omega NAB\sin\omega t = \varepsilon_{\max}\sin\omega t$$





Example

Example: A rod with length ***l***, mass ***m***, and resistance ***R*** slides without friction down parallel conducting rails of negligible resistance. The rails are connected together at the bottom, forming a conducting loop with the rod as the top member. The plane of the rails makes an angle ***θ*** with the horizontal, and a uniform vertical magnetic field ***B*** exists throughout the region. (1) What is the terminal speed of the rod? (2) What is the induced current in the rod when the terminal speed has been reached?

Solution: (1) Newton's law for the rod

$$m\frac{dv}{dt} = mg\sin\theta - F_B\cos\theta$$

The motional emf:

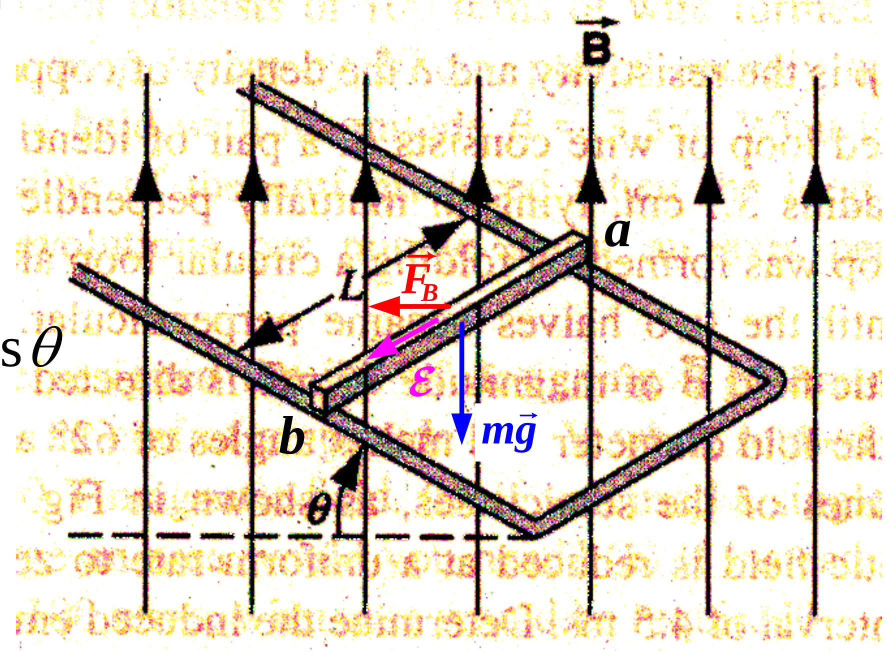
$$\varepsilon = |(\vec{v}\times\vec{B})\cdot\vec{L}| = vBL\sin(90^\circ+\theta) = vBL\cos\theta$$

The current in the loop:

$$I = \frac{\varepsilon}{R} = \frac{vBL\cos\theta}{R}$$

The magnetic force acts on the rod

$$F_B = I\,|\vec{L}\times\vec{B}| = ILB = \frac{vB^2L^2\cos\theta}{R}$$



Example Cont'd

Newton's law for the rod becomes: $$m\frac{dv}{dt}=mg\sin\theta-\frac{vB^2L^2\cos^2\theta}{R}$$

When the rod reaches its terminal speed: $$\frac{dv}{dt}=0$$

The terminal speed: $$v=\frac{mgR}{B^2L^2}\frac{\sin\theta}{\cos^2\theta}$$

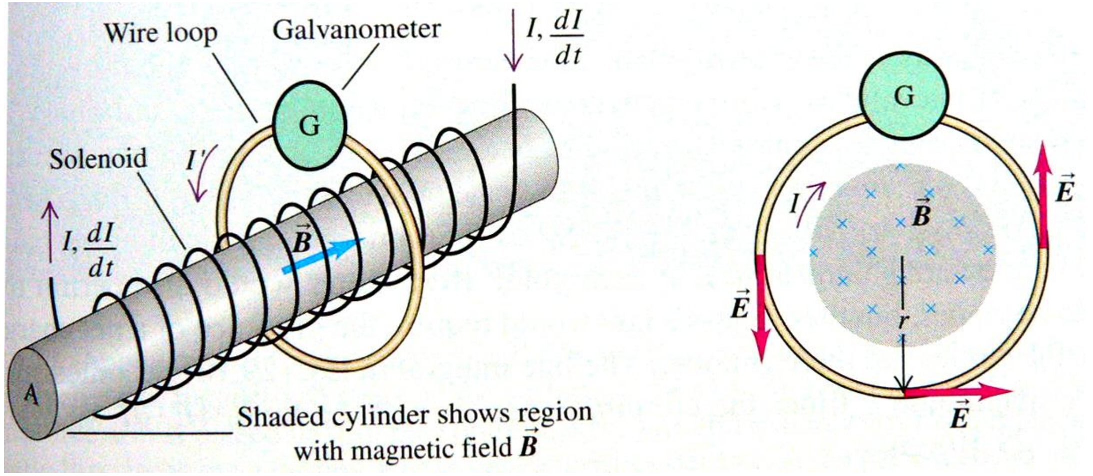
(2) When the rod reaches the terminal speed, the induced current is:

$$I=\frac{vBL\cos\theta}{R}=\frac{mg}{BL}\tan\theta$$

§ 3 Induced Electric Field (p635-637)

- What is the basis of induced emf when there is a changing flux trough a stationary conducting loop?
  - Now we can understand that magnetic force is the reason of the induced emf in a moving conductor.
  - By Faraday’s law, we only know the result that an induced emf also occurs when there is a changing flux through a stationary conducting loop.

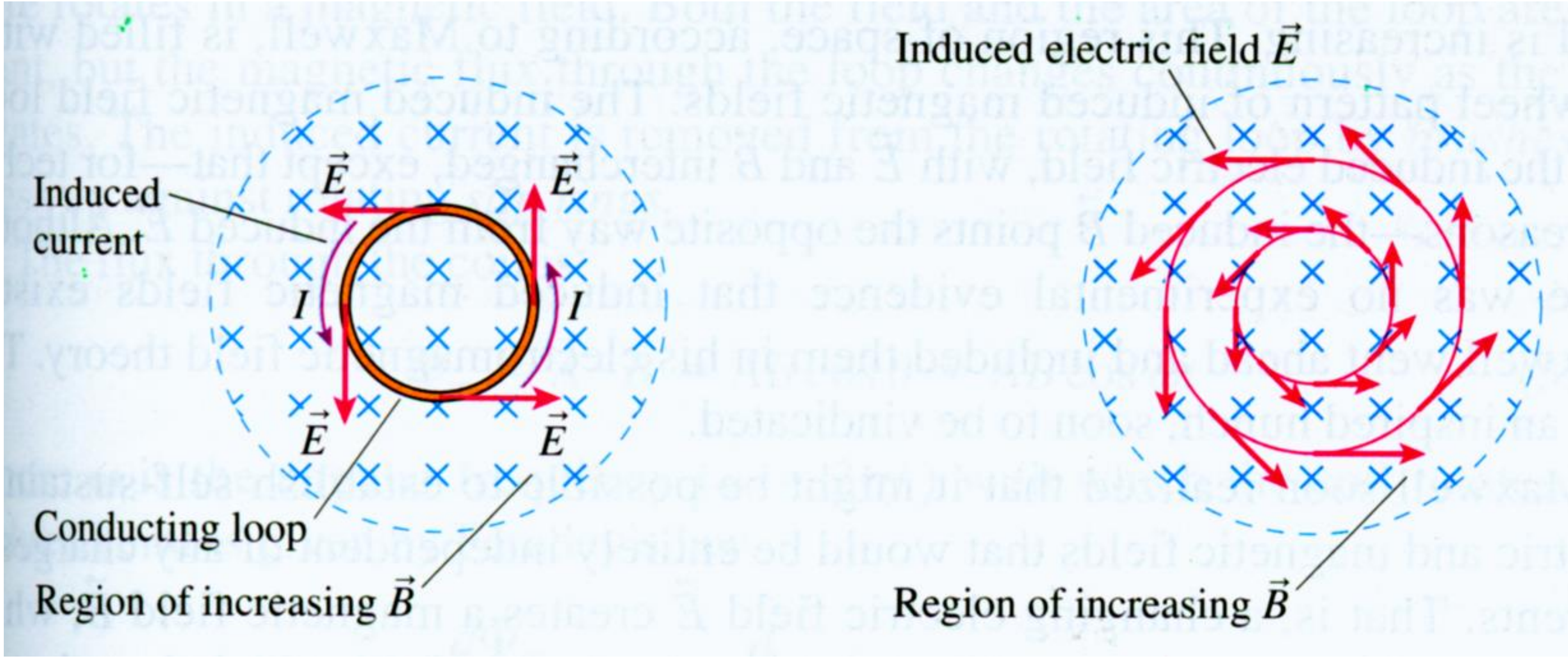
$$\varepsilon = -\frac{d\Phi_B}{dt}$$



  - But up to now, we don’t know what *force* makes the charges moves around the loop. It can’t be a magnetic force because the conductor is not moving in the magnetic field. In fact it is not even in the magnetic field.

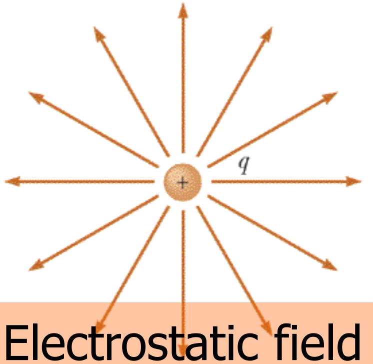
The Induced Electric Field as the Source of Induced emf

- Maxwell's suggestion: induced electric field
  - There must be an induced electric field (non-electrostatic field) created in the conductor as a result of changing magnetic flux.
  - This kind of electric field is induced even when no conductor is present.

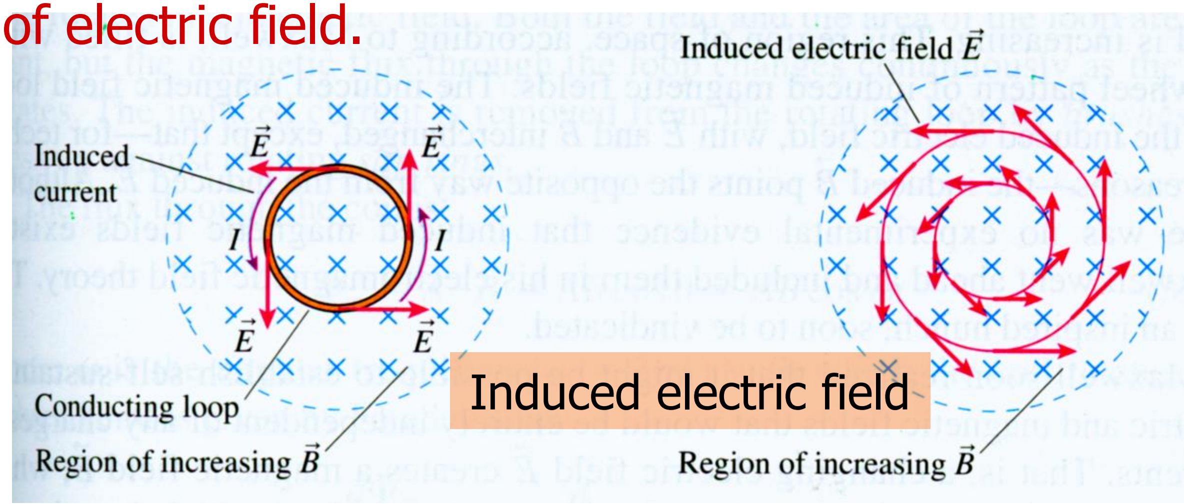


The Easy Confused Points for Induced emf

- Easy confused points:
  - We accustomed to thinking about electric field as being caused by electric charges. Now we know that a changing magnetic field can also act as a source of electric field.



Electrostatic field



Induced electric field

  - By the definition of emf, ε is equal to the work done by a non-electrostatic field, induced electric field $\vec{E}_i$, per unit charge.

$$\varepsilon = \oint_L \vec{E}_i \cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_{\substack{\text{the surface}\\ \text{suround the loop}}} \vec{B}\cdot d\vec{A} = \iint_{\substack{\text{the surface}\\ \text{suround the loop}}} -\frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

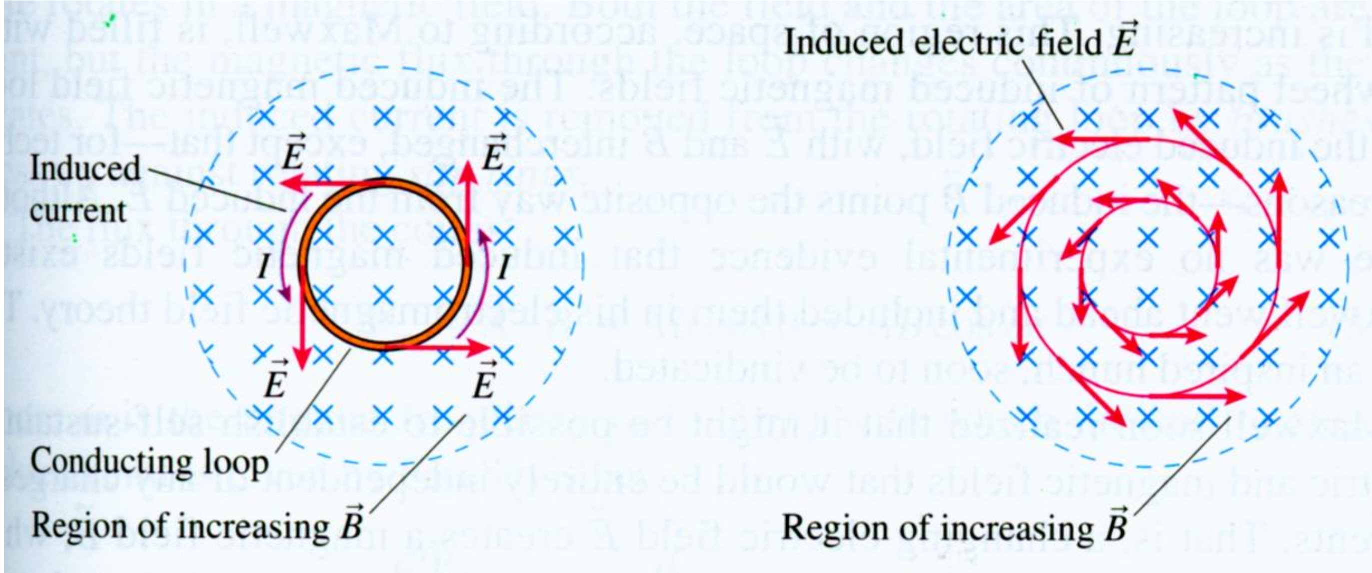
  - The line integral around a closed path is not zero. So the induced electric field is not conservative.

General Form of Faraday's Law

- The relationship between the induced electric field and the changing magnetic field

$$\boxed{\oint_L \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}}$$

  - The form $\varepsilon = -d\Phi_B / dt$ is always true. But the equation above is valid only if the path around which we integrate is stationary.



The Features of Induced Electric Field

The Comparison between the electrostatic field and induced electric field

	Electrostatic field $\vec{E}_s$	Induced electric field $\vec{E}_i$
The source of the field	The charges	The changing magnetic field
Line integral around a closed path	$\oint_L \vec{E}_s \cdot d\vec{s} = 0$ Conservative	$\oint_L \vec{E}_i \cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$ Non-conservative
Gauss's law	$\oiint_S \vec{E}_s \cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0}$ Field lines begin and end on charge	$\oiint_S \vec{E}_i \cdot d\vec{A} = 0$ Field lines form closed loops

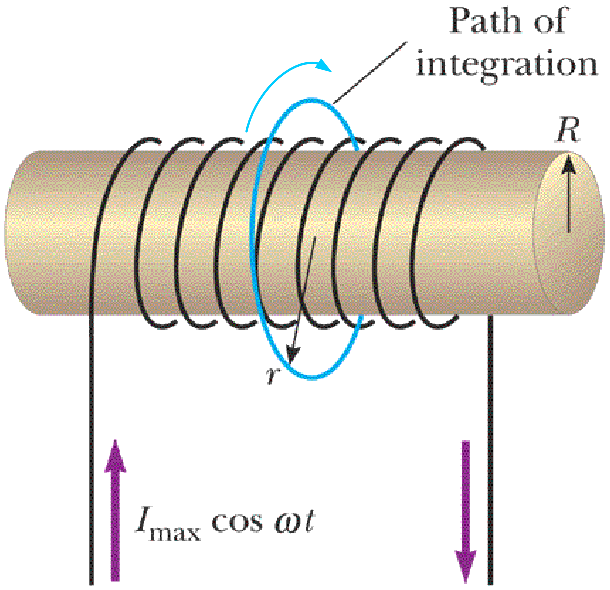
Example

Electric field induced by a changing magnetic field in a solenoid

Example: A long solenoid of radius ***R*** has ***n*** turns of wire per unit length and carries a time-varying current that varies sinusoidally as $I = I_{max}\cos\omega t$. (1) Determine the magnitude of the induced electric field outside the solenoid, a distance $r > R$ from its long central axis. (2) Find the induced electric filed inside the solenoid, a distance $r < R$ from its axis.

Solution: Choose a path for the line integral to be a circle of radius ***r*** centered on the solenoid.

By symmetry, the $\vec{E}$ is tangent to the circle and has constant magnitude on it.



$$\oint_L \vec{E}\cdot d\vec{s} = E\oint_L ds = E(2\pi r)$$

$$= -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left(B\pi R^2\right) = -\pi R^2\frac{dB}{dt}$$

$$E = -\frac{R^2}{2r}\frac{dB}{dt} = -\frac{R^2}{2r}\frac{d}{dt}(\mu_0 n I_{\max}\cos\omega t) = \frac{\mu_0 n I_{\max}\omega R^2}{2r}\sin\omega t \qquad \text{for } r > R$$

Example Cont'd

For an interior point (***r*<*R***)

$$E\left(2\pi r\right)=-\frac{d\Phi_B}{dt}=-\frac{d}{dt}\left(B\pi r^2\right)=-\pi r^2\frac{dB}{dt}$$

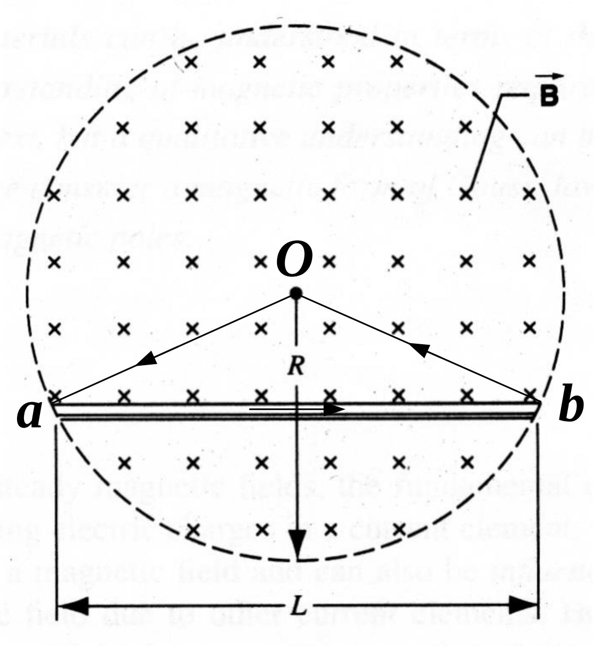
$$E=-\frac{r}{2}\frac{dB}{dt}=-\frac{r}{2}\frac{d}{dt}\left(\mu_0 n I_{\max}\cos\omega t\right)=\frac{\mu_0 n I_{\max}\omega}{2}r\sin\omega t\quad \text{for } r<R$$

Example

Example: A uniform magnetic field ***B*** fill with cylinrical volume of radius ***R***. A metal rod ***ab*** of length ***L*** is placed as shown in the figure. If ***B*** is changing at the rate **d*B*/dt**, find the emf acting between the end ***a*** and ***b*** of the rod.

Solution I: Using Faraday’s law

Choose the loop ***abO***.



$$\Phi_B = -BA_{abO} = -\frac{1}{2}BL\sqrt{R^2-\frac{L^2}{4}}$$

$$\varepsilon = \oint_{Oab}\vec{E}\cdot d\vec{s} = \int_O^a + \int_a^b + \int_b^O \vec{E}\cdot d\vec{s}$$

$$= \int_a^b \vec{E}\cdot d\vec{s} = \varepsilon_{ab} = -\frac{d\Phi_B}{dt} = A_{abO}\frac{dB}{dt}$$

$$\varepsilon_{ab} = \frac{L}{2}\sqrt{R^2-\frac{L^2}{4}}\frac{dB}{dt}$$

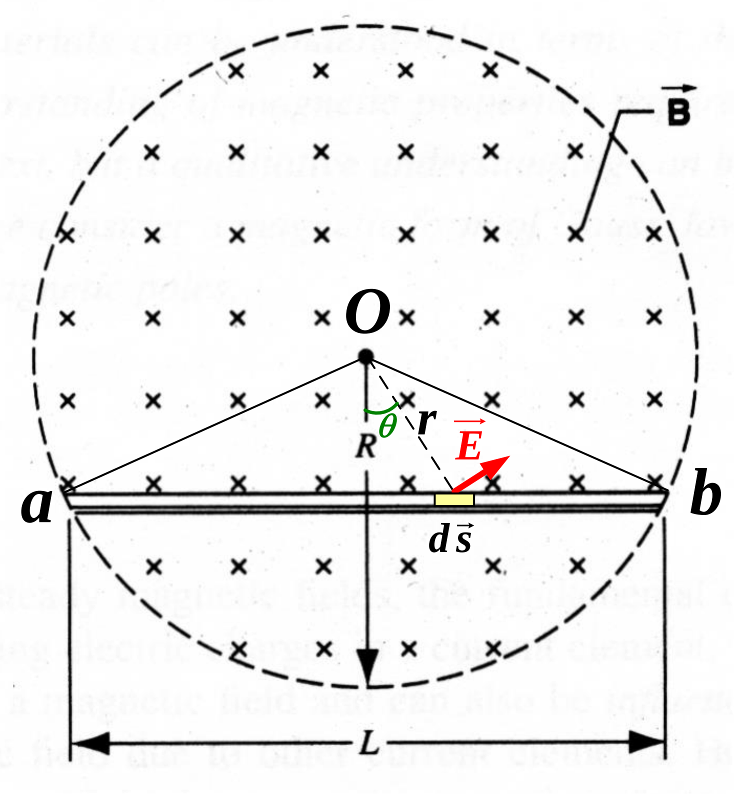
The potential at end ***b*** is higher than end ***a***.

Example Cont'd

Solution II: By line integration of induced electric field.

We have know that:

$$E = \begin{cases} \frac{r}{2}\frac{dB}{dt} & \text{for } r < R \\ \frac{R^2}{2r}\frac{dB}{dt} & \text{for } r > R \end{cases}$$



$$\varepsilon_{ab} = \int_a^b \vec{E} \cdot d\vec{s} = \int_{-L/2}^{L/2} E\cos\theta ds$$

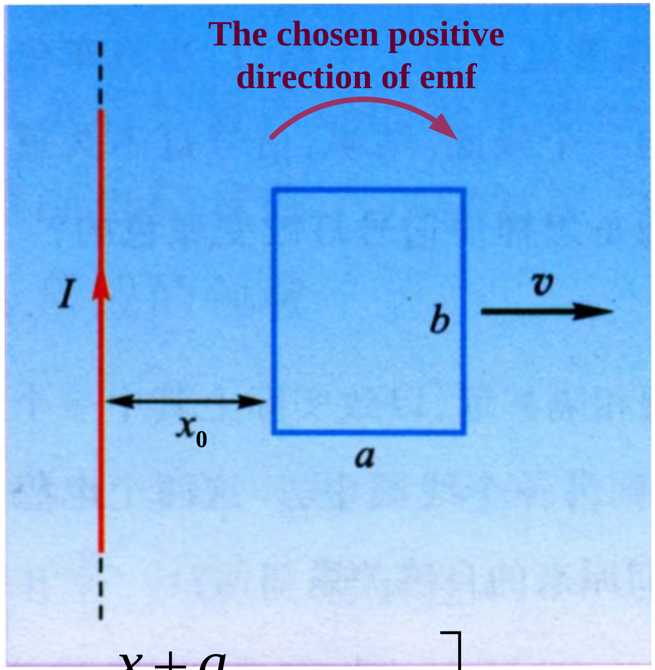
$$= 2\int_0^{L/2} \frac{r}{2}\frac{dB}{dt}\cos\theta ds = \frac{dB}{dt}\int_0^{L/2} r\cos\theta ds$$

$$r = \frac{1}{\cos\theta}\sqrt{R^2 - \frac{L^2}{4}}$$

$$\varepsilon_{ab} = \sqrt{R^2 - \frac{L^2}{4}}\frac{dB}{dt}\int_0^{L/2} r\cos\theta ds = \frac{L}{2}\sqrt{R^2 - \frac{L^2}{4}}\frac{dB}{dt}$$

Example

Example: A long, straight wire carries a time-varying current $I = I_0 \sin\omega t$. A rectangular wire loop of sides a and b is placed in the same plane as the straight current is, and a distance x_0 from the straight current. The wire loop starts to move to the right at the speed of v at $t = 0$. Determine the induced emf in the wire loop at time t.


The chosen positive direction of emf

Solution I: Using Faraday’s law

Choose the loop direction as shown in the Fig.

$$x = x_0 + vt$$

$$\Phi_B = \int_x^{x+a} \frac{\mu_0 I}{2\pi x} b dx = \frac{b\mu_0 I_0}{2\pi} \ln\frac{x+a}{x} \sin\omega t$$

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{b\mu_0 I_0}{2\pi}\left[\frac{x}{x+a}\frac{x-(x+a)}{x^2}\frac{dx}{dt}\sin\omega t + \ln\frac{x+a}{x}\omega\cos\omega t\right]$$

$$= \frac{b\mu_0 I_0}{2\pi}\left[\frac{av}{x(x+a)}\sin\omega t - \ln\frac{x+a}{x}\omega\cos\omega t\right]$$

Example Cont'd

$$\varepsilon = \frac{b\mu_0 I_0}{2\pi}\left[\frac{av}{(x_0+vt)(x_0+a+vt)}\sin\omega t - \ln\frac{x_0+a+vt}{x_0+vt}\,\omega\cos\omega t\right]$$

Solution II: By calculation of motional emf and induced electric field.

$$\varepsilon = \varepsilon_m + \varepsilon_i$$

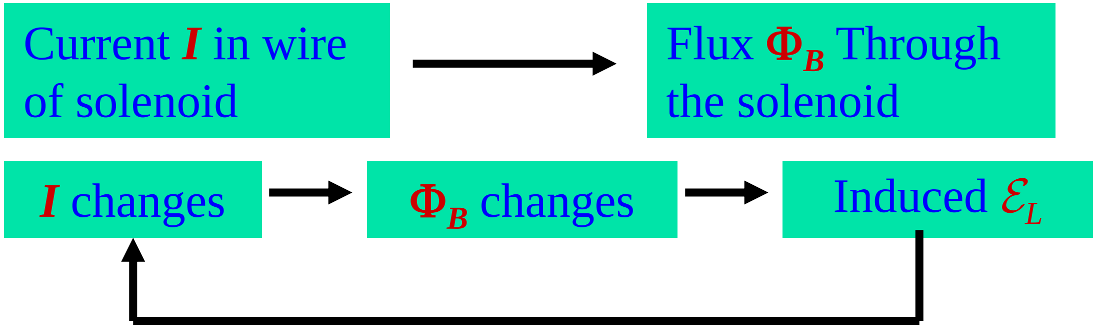
$$\varepsilon_m = vbB_x - vbB_{x+a} = \frac{vb\mu_0 I}{2\pi}\left(\frac{1}{x}-\frac{1}{x+a}\right) = \frac{vb\mu_0 I}{2\pi}\,\frac{a}{x(x+a)}$$

$$= \underline{\frac{b\mu_0 I_0}{2\pi}\,\frac{av}{(x_0+vt)(x_0+a+vt)}\sin\omega t}$$

$$\varepsilon_i = -\frac{d\Phi_B}{dt}\bigg|_{x=\text{const}} = -\frac{b\mu_0 I_0}{2\pi}\ln\frac{x+a}{x}\,\omega\cos\omega t = \underline{-\frac{b\mu_0 I_0}{2\pi}\ln\frac{x_0+a+vt}{x_0+vt}\,\omega\cos\omega t}$$

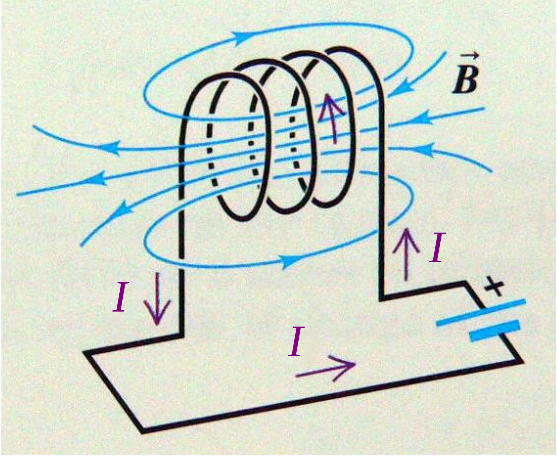
§ 4 **Self-Inductance** (p645-647)

- Inductor and self-induced emf:
  - An inductor is a circuit element such as solenoid that stores energy in the magnetic field surrounding its current-carrying wires, just as a capacitor store energy in the electric field between its charged plates.
  - For a circuit including a solenoid

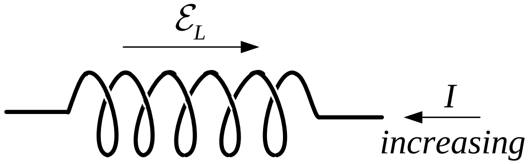


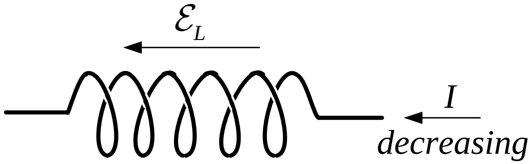
Current ***I*** in wire of solenoid → Flux Φ_B Through the solenoid

I changes → Φ_B changes → Induced $\mathcal{E}_L$



  - The emf set up by changing self-current is called self-induced emf $\mathcal{E}_L$
  - By Lenz's law a self-induced emf always opposes the change in the current that caused the emf, and so tends to make it more difficult for variation in current to occur.





Definition of the Self-inductance

- Self-induced emf:

$$\boxed{\varepsilon_L = -L\frac{dI}{dt}}$$

  - The negative sign reflects Lenz’s law.
- The self-inductance
  - The proportionality constant L is called the self-inductance.
  - From Faraday’s law

$$\varepsilon_L = -\frac{d(N\Phi_B)}{dt} \quad \Rightarrow \quad L\frac{dI}{dt} = \frac{d(N\Phi_B)}{dt}$$

  - Integrating with respect to the time, and assuming that Φ_B=0 when I=0

$$\boxed{L = \frac{N\Phi_B}{I}}$$

SI unit: H (henry)

  - Note that, since Φ_B is proportional to the current, the self-inductance is independent of I. (Like the capacitance) The self-inductance depends only on the geometry of the device.

Example

Inductance of a solenoid

Example: Find the inductance of a uniformly wound solenoid having ***N*** turns and length ***l***. Assume that ***l*** is long compared with the radius and the core of the solenoid is.

Solution: For an ideal solenoid, the interior magnetic field is uniform.

$$B = \mu_0 nI = \mu_0 \frac{N}{l} I$$

The magnetic flux through each turn is

$$\Phi_B = BA = \mu_0 \frac{NA}{l} I$$

$$L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 A}{l} = \mu_0 \frac{N^2}{l^2}(Al) = \mu_0 n^2 V$$

Example

Inductance of a coaxial cable

Example: A long coaxial cable consists of two concentric cylindrical conductors of radii a and b and length ***l***. The conductors carry current ***I*** in opposite directions. Find the self-inductance of this cable.

Solution: Firstly, we find the magnetic flux through cross section between the two conductors.

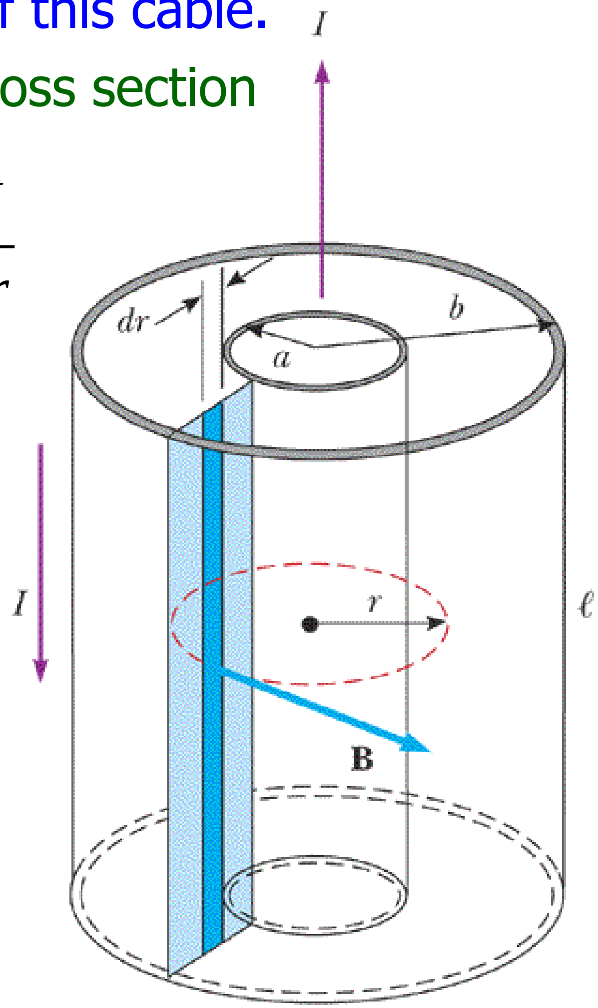
The magnetic field between the conductors: $B = \dfrac{\mu_0 I}{2\pi r}$

Divide the rectangular cross section into strips of width dr.

$$\Phi_B = \iint \vec{B} \cdot d\vec{A} = \int_a^b \left(\frac{\mu_0 I}{2\pi r} \right) (l dr)$$

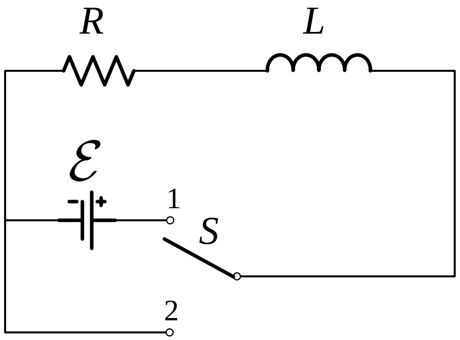
$$= \frac{\mu_0 I l}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I l}{2\pi} \ln\left(\frac{b}{a} \right)$$

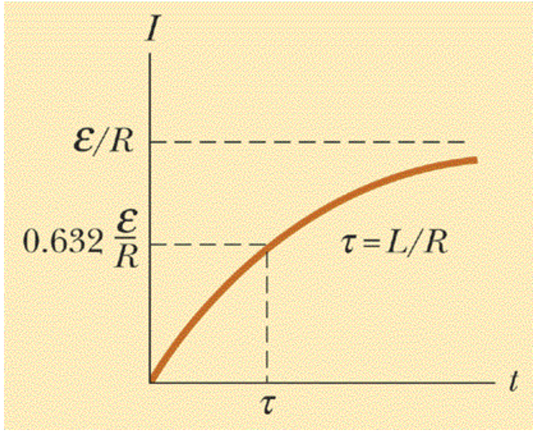
$$L = \frac{\Phi_B}{I} = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a} \right)$$



§ 5 RL Circuit (648-649)

- RL circuit:
  - The switch jumps to 1 from 2. From Kirchhoff's loop rule





$$\varepsilon + \varepsilon_L - IR = 0$$

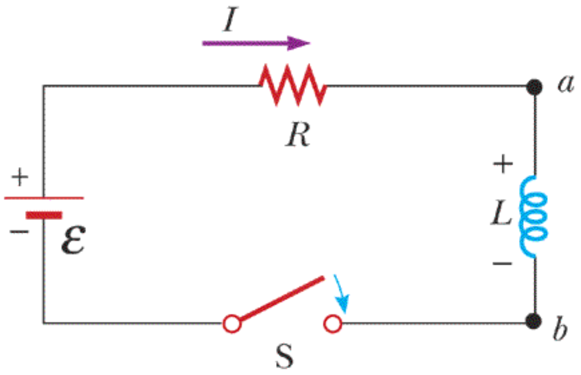
$$\varepsilon - L\frac{dI}{dt} - IR = 0 \qquad \frac{dI}{dt} = \frac{R}{L}\left(\frac{\varepsilon}{R} - I\right)$$

$$\int_0^I \frac{dI}{I - \frac{\varepsilon}{R}} = -\int_0^t \frac{R}{L}dt \qquad I = \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right) = \frac{\varepsilon}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

  - Time constant of the RL circuit: $\tau = \frac{L}{R}$

§ 6 Energy Stored in A Magnetic Field (647-648)

- Starting with a RL circuit:
  - The switch jumps to 1 from 2. $\varepsilon = IR + L\dfrac{dI}{dt}$



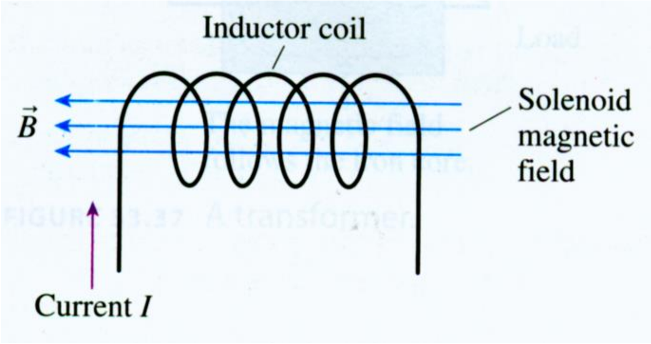
$$\int_0^t \varepsilon I dt = \int_0^t I^2 R dt + \int_0^t LI\frac{dI}{dt}dt$$

  - The term on left side:

    The energy is supplied by the source.
  - The first term on right side:

    The energy is dissipated in the resistor.
  - The second term on right side:

    The energy that is delivered to the inductor and is stored in the magnetic field through the coil.



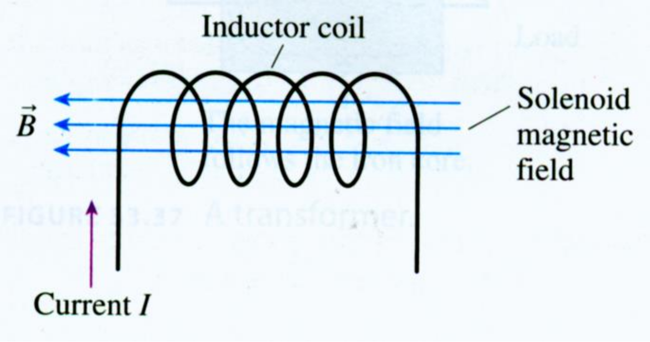
- Energy stored in the inductor

$$U_B = \int_0^t LI\frac{dI}{dt}dt = \int_0^I LIdI = \frac{1}{2}LI^2$$

  - Which one is the storehouse of the energy, the inductor or the magnetic field?

The Energy Density in Magnetic Field

- Energy stored in magnetic field.
  - Take a solenoid as an example.



$$B = \mu_0 nI \quad L = \mu_0 n^2 V$$

$$U_B = \frac{1}{2}LI^2 = \frac{1}{2}\left(\mu_0 n^2 V\right)\left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0}V \propto \begin{cases} B^2 \\ V \end{cases}$$

  - Energy is indeed stored in the space where the magnetic field exists.
- Energy density

$$\boxed{u_B = \frac{U_B}{V} = \frac{B^2}{2\mu_0}}$$

  - For a non-uniformed magnetic field

$$U_B = \iiint du_B = \iiint_V \left(\frac{B^2}{2\mu_0}\right) dV$$

Energy in Electric and Magnetic Field

	Electric field	Magnetic field
Energy stored in the device	A capacitor stores energy $U=\frac{1}{2}C\left(\Delta V\right)^2$	An inductor stores energy $U=\frac{1}{2}LI^2$
Energy density in the field	$u_E=\frac{1}{2}\varepsilon_0 E^2$	$u_B=\frac{1}{2\mu_0}B^2$

Example

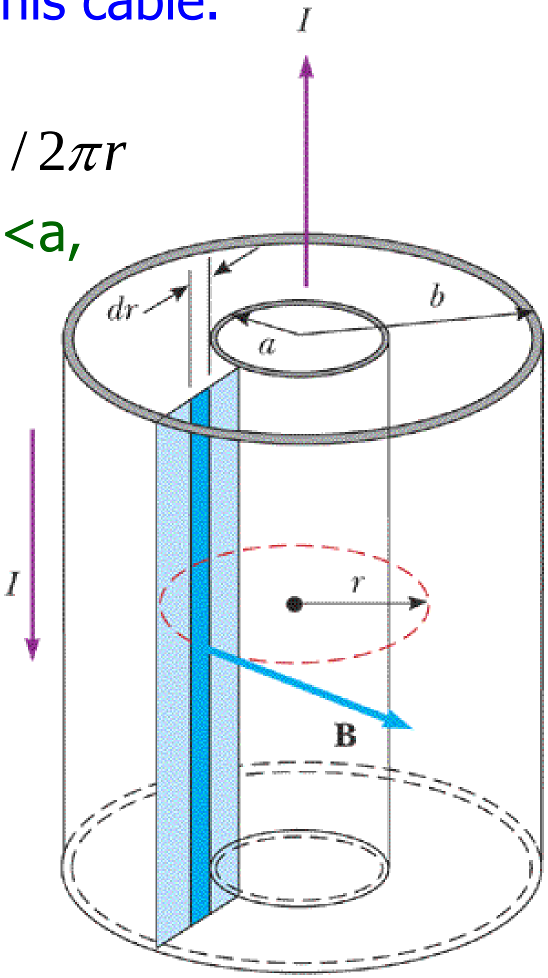
The energy stored in a coaxial cable

Example: A long coaxial cable consists of two concentric cylindrical conductors of radii a and b and length ***l***. The conductors carry current ***I*** in opposite directions. Find the energy stored in this cable.

Solution:

The magnetic field between the conductors is $B = \mu_0 I / 2\pi r$

The magnetic field is zero inside the inner conductor r<a, and outside the outer conductor r>b.



$$U_B = \iiint \left(\frac{B^2}{2\mu_0} \right) dV = \int_a^b \left[\frac{1}{2\mu_0} \left(\frac{\mu_0 I}{2\pi r} \right)^2 \right] (2\pi r l dr)$$

$$= \frac{\mu_0 I^2 l}{4\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a} \right)$$

$$U_B = \frac{1}{2} L I^2 = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a} \right) \qquad L = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a} \right)$$