Chapter 29 Maxwell's Equations

§ 1 Displacement Current and The Extended Ampére's Law （p661-664)

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}}$$

electric current → magnetic field

Another way → magnetic field ?

$$\oint_L \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

Changing magnetic field → Electric field

Magnetic field ← Changing electric field ?

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}} + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} \ ?$$

The Contradiction of Ampére's Law

- Question: Does Ampére's law need to be modified?
- The contradiction in applying Ampére's law to a charging capacitor
  - Apply Ampére's law to a circular loop that surrounding the wire. Consider two surface bounded by the same Ampérian loop.

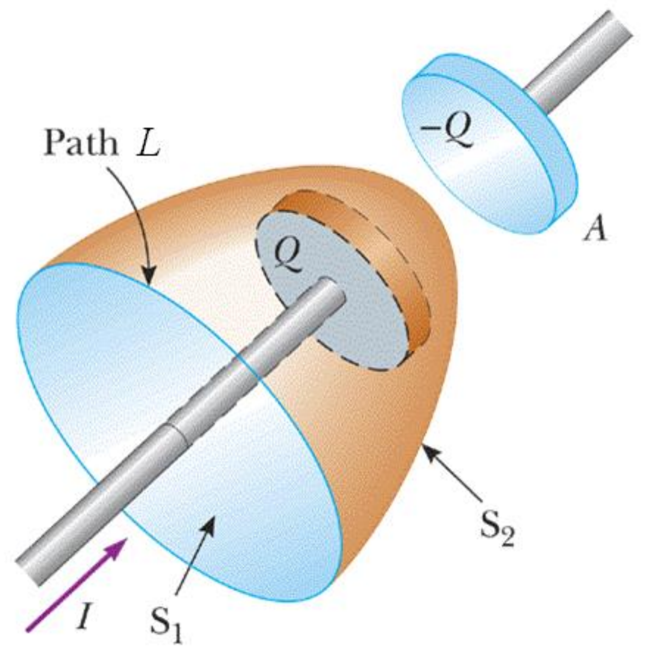
    Surface S_1: the circular area in which the *conduction current I* penetrates.

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}} = \mu_0 \iint_{S_1} \vec{J}\cdot d\vec{A} = \mu_0 I$$

    Surface S_2: the paraboloid passing between the capacitor's plates.

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}} = \mu_0 \iint_{S_2} \vec{J}\cdot d\vec{A} = 0$$

  - Ampére's law in this form is valid only if the conduction current is continuous in space.

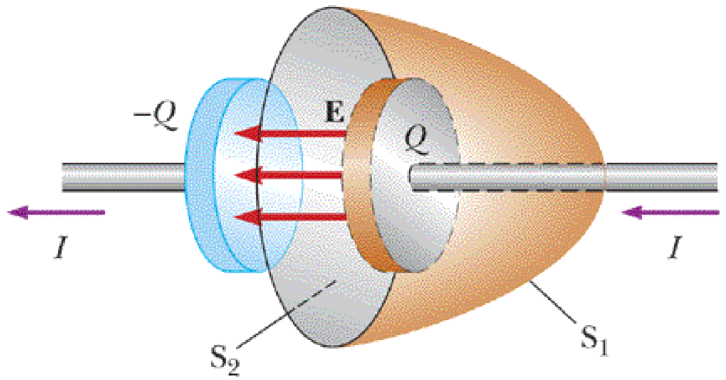


- How to save Ampére's law from the contradiction?
  - The contradiction comes from the discontinuity of the conduction current.

    The *conduction current I* is interrupted in the region between capacitor's two plates, there is also a changing electric field $\vec{E}$ or a changing electric flux Φ_E in this region.

$$I=\frac{dQ}{dt}=\frac{d(\sigma A)}{dt}=\frac{d\sigma}{dt}A=\frac{d}{dt}\left(\varepsilon_0 E\right)A$$

$$=\varepsilon_0\frac{d}{dt}\left(EA\right)=\varepsilon_0\frac{d\Phi_E}{dt}$$



  - To keep the continuity of the current,

    Maxwell made a postulation that there exists a fictitious current in the region between the plates, called the *displacement current I_d*.

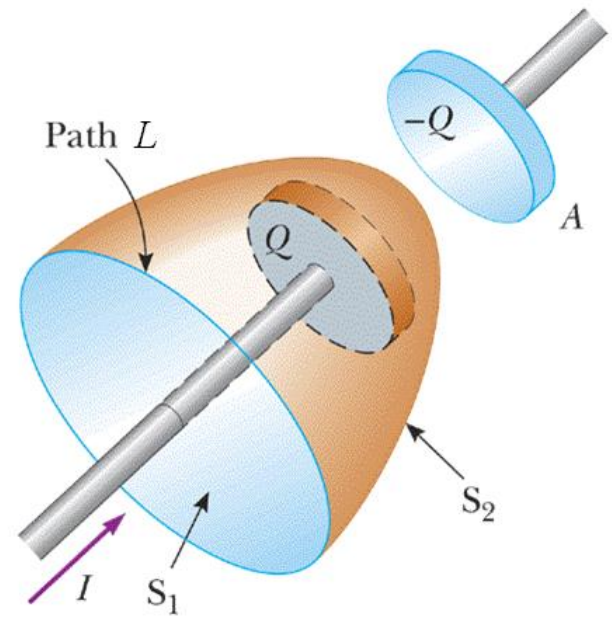
$$I_d=\varepsilon_0\frac{d\Phi_E}{dt}=\varepsilon_0\frac{d}{dt}\iint_S\vec{E}\cdot d\vec{A}=\varepsilon_0\iint_S\frac{\partial\vec{E}}{\partial t}\cdot d\vec{A}$$

  - Displacement current density: $\vec{J}_d=\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$ $\quad I_d=\iint_S\vec{J}_d\cdot d\vec{A}$

Extended Ampére's law

- Extended Ampére's law or Ampére-Maxwell law:
  - The postulation of displacement current solved the discontinuity of the conduction current.

$$\boxed{\oint_L \vec{B}\cdot d\vec{l} = \mu_0 (I + I_d)_{\text{encl}} = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}}$$



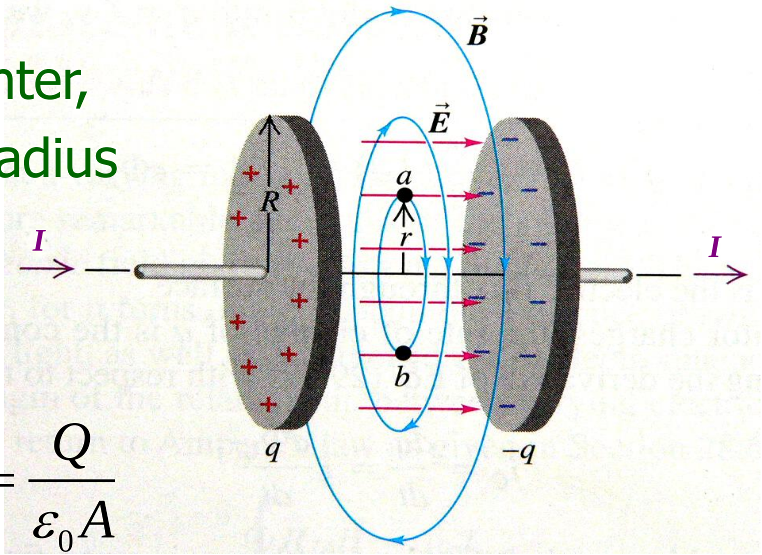
- The displacement current is also a source of magnetic field

$$\left.\begin{array}{l}\boxed{\text{Conduction current } I} \\ \\ \boxed{\text{Displacement current } I_d \text{ or changing electric field}}\end{array}\right\} \Rightarrow \boxed{\text{Magnetic field}}$$

  - Magnetic field are produced both by conduction current and by changing electric field.

Example

Example: Calculate the magnetic field in the region between the two capacitor's plates while the capacitor is charging with a increasing current ***I***. The radius of plate is ***R***.



Solution: For a point a distance ***r*** from the center, we apply Ampére's law to a circular path of radius ***r*** passing through the point.

$$\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

The electric field between the plates: $E = \dfrac{\sigma}{\varepsilon_0} = \dfrac{Q}{\varepsilon_0 A}$

$$\Phi_E = \begin{cases} E\pi r^2 = \dfrac{1}{\varepsilon_0}\dfrac{r^2}{R^2}Q & \text{for } r < R \\ EA = \dfrac{Q}{\varepsilon_0} & \text{for } r > R \end{cases}$$

For r<R: $\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0 \dfrac{r^2}{R^2}\dfrac{dQ}{dt} = \mu_0 \dfrac{r^2}{R^2} I$ $\qquad B = \dfrac{\mu_0}{2\pi}\dfrac{r}{R^2} I$

For r>R: $\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0 \dfrac{dQ}{dt} = \mu_0 I$ $\qquad B = \dfrac{\mu_0 I}{2\pi r}$

§ 2 Maxwell's Equations (p664)

$$\oint\!\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0}$$

Gauss's law for electricity

$$\oint\!\!\!\oint_S \vec{B}\cdot d\vec{A} = 0$$

Gauss's law for magnetism

$$\oint_L \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

Faraday's law of induction

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}} + \varepsilon_0\mu_0\frac{d\Phi_E}{dt} = \mu_0 I_{\text{encl}} + \varepsilon_0\mu_0\iint_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$

Ampére-Maxwell law

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

Lorentz force

Maxwell's equations and Lorentz force give the fundamental relations of electromagnetism! They are fundamental in the sense that Newton's three laws are for mechanics.

The Physical Meaning Embodied in Maxwell's Equations

$$\oint\!\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0}$$ —— Charged particles create an electric field (electrostatic).

$$\oint_L \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$ —— An electric field (non-electrostatic) can also be created by a changing magnetic field.

$$\oint\!\!\!\oint_S \vec{B}\cdot d\vec{A} = 0$$ —— There are no magnetic monopoles.

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}} + \varepsilon_0\mu_0\frac{d\Phi_E}{dt} = \mu_0 I_{\text{encl}} + \varepsilon_0\mu_0\iint_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$

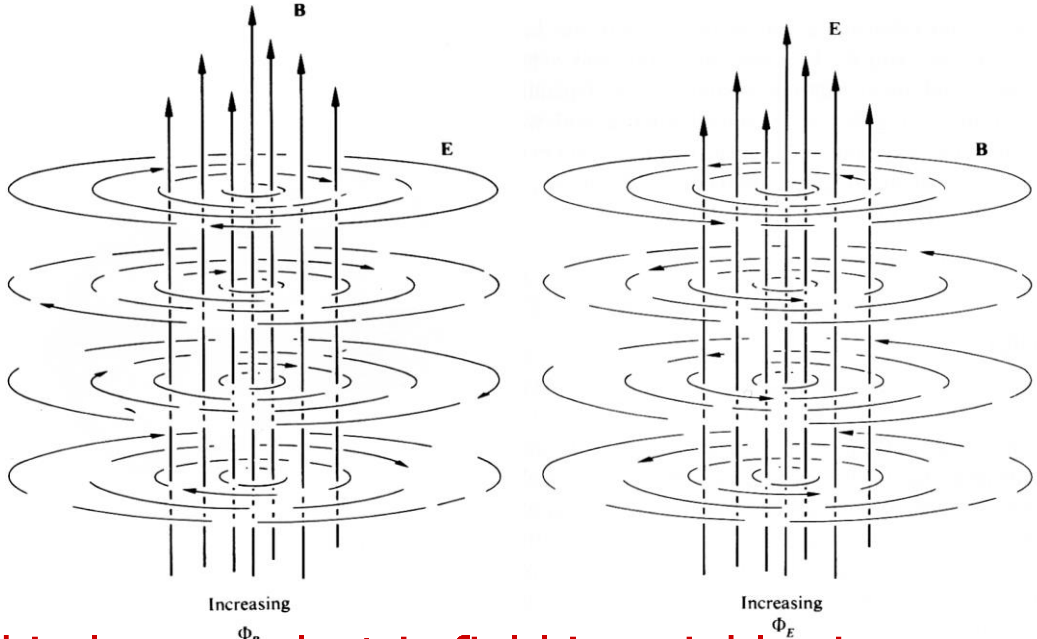
—— A magnetic field can either be created by currents or a changing electric field.

§ 3 Electromagnetic Waves (p665-666)

- The relationship between electric and magnetic field in empty space.

$$\oint_L \vec{E}\cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

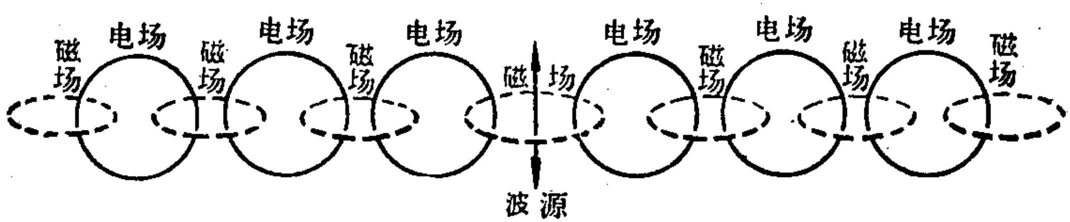
$$\oint_L \vec{B}\cdot d\vec{s} = \varepsilon_0\mu_0\iint_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$



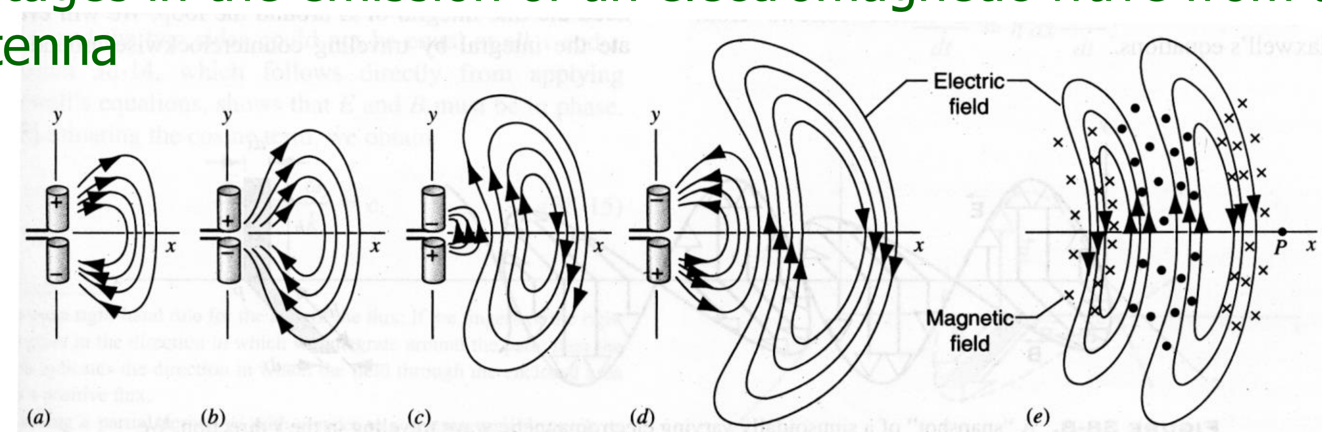
- A time varying magnetic field induces a electric field in neighboring regions;
- A time varying electric field induces a magnetic field in neighboring regions.

These relationships predicts the existence of electromagnetic waves consisting of time-varying electric and magnetic fields that travel from one region of space to another, even if no charge or current are present in space.

- The mechanism for maintaining the propagation of the electromagnetic wave.
  - Unlike mechanical waves, which need a medium such as water or air to transit a wave, electromagnetic waves require no medium. The changing electric and magnetic fields create each other to maintain the propagation of the waves.
  - A exhibition map (not real) for propagation of electromagnetic waves



  - The real stages in the emission of an electromagnetic wave from a dipole antenna

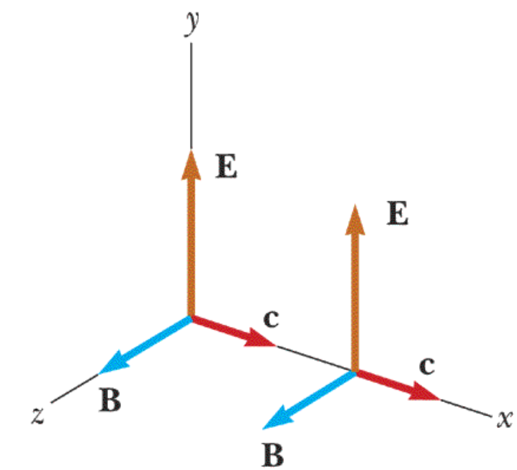


- The important features of electromagnetic waves.
  - The wave equation:

    From Maxwell's equations, we can obtain the wave equation for a wave which propagates in x-direction

$$\frac{\partial^2 E}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 B}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2}$$

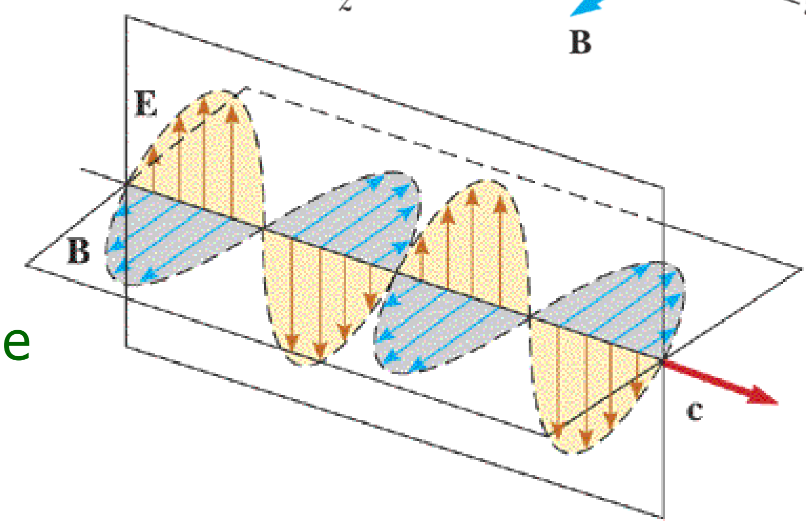


  - The wave speed:

    Generally, the wave equation

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi(x,t)}{\partial t^2}$$

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.997 \times 10^8 \text{ m/s} = c$$

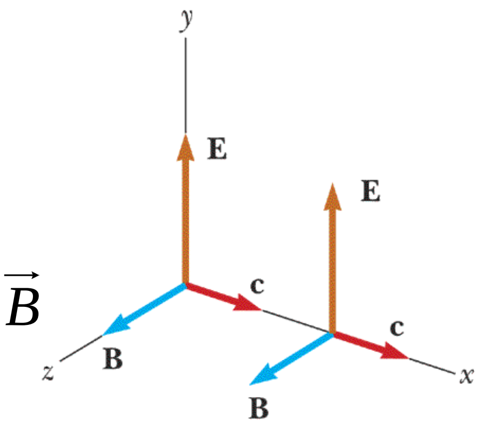


    This speed is precisely the same as the speed of light in empty space.

- The important features of electromagnetic waves.
  - The sinusoidal plane wave is the simplest solution of the wave equations

$$E = E_{\max}\cos(\omega t - kx)$$

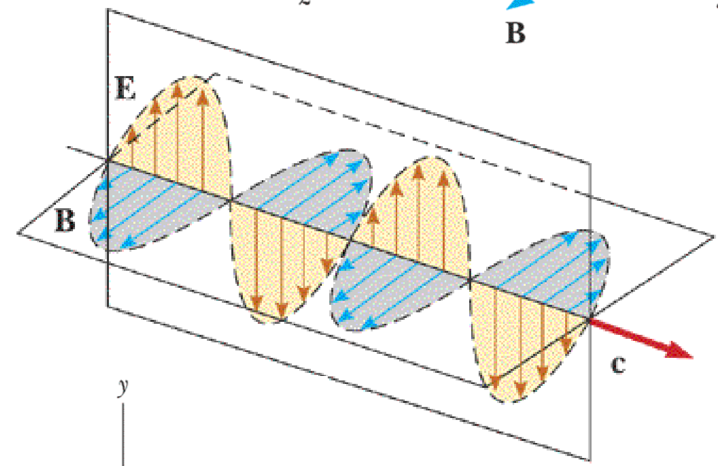
$$B = B_{\max}\cos(\omega t - kx)$$



  - The wave is transverse.

    Both $\vec{E}$ and $\vec{B}$ are perpendicular to each other, and to the direction of propagation. The direction of propagation is $\vec{E}\times\vec{B}$
  - $\vec{E}$ and $\vec{B}$ are in phase, and has a definite ratio

$$\frac{E}{B} = \frac{E_{\max}}{B_{\max}} = c, \quad E = cB, \quad \sqrt{\varepsilon_0}E = \frac{B}{\sqrt{\mu_0}}$$



  - Poynting vector: energy flow vector.

    The total energy density:

$$u = u_E + u_B = \frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0} = \frac{EB}{\mu_0 c}$$

    The energy current density:

$$S = uc = \frac{EB}{\mu_0} \qquad \boxed{\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}}$$