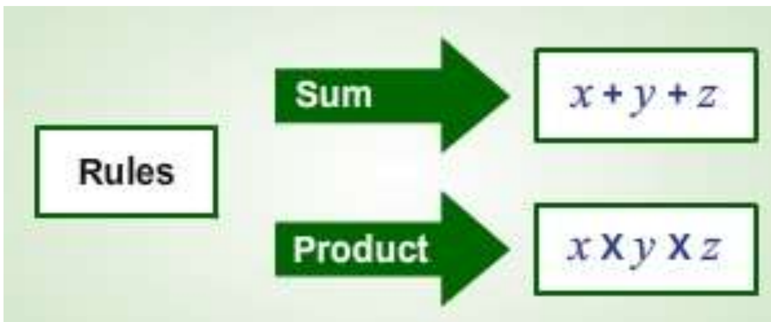
Learning Outcome

By the end of this lecture, you will be able to:

- Identify common rules of probability
- Use Bayes' Theorem for conditional probabilities

Introduction

Let x, y, *and* z be three real numbers.



Rules of operations

These rules govern how these numbers can be operated on and how the results can be derived. Similarly, there are some rules of probability that help us calculate the probability of complex events.

Dice

In this lecture, we look at the common rules of probability.

Probability Rules

Rules for possible values and sum can be given by:

Individual values	$0 \le P\ (e_i) \le 1$, for any event e_i
Sum of all values	$\sum_{i=1}^{k} P(e_i) = 1$ where: k = Number of elementary events in the finite sample space $e_i = i^{\text{th}}$ elementary event

1. Addition Rule for Elementary Events

Let E_i be an event containing k elementary events, then the probability of E_i is the sum of the probabilities of the k elementary events.

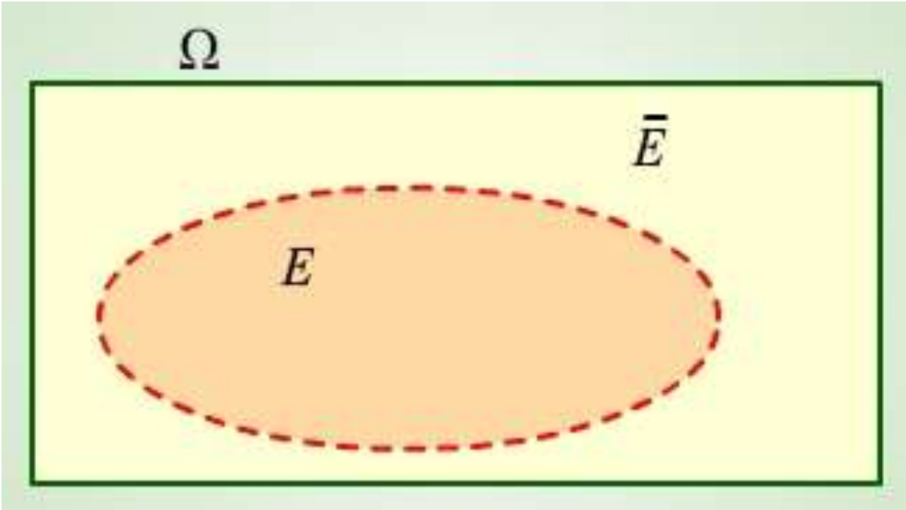
$$if: E_i = \{e_1, e_2, e_3, \ldots, e_k\}, then:$$

$$P(E_i) = P(e_1) + P(e_2) + P(e_3) + \cdots + P(e_k) \qquad (1)$$

2. Complement Rule

The complement of E denoted by $\bar{E}$ is the collection of all possible elementary events not contained in event E.The probability of the complement is given by:

$$P(\bar{E}) = 1 - P(E) \;\; or \;\; P(E) + P(\bar{E}) = 1$$



Sample space - complement rule

5. Addition Rule for Two Events

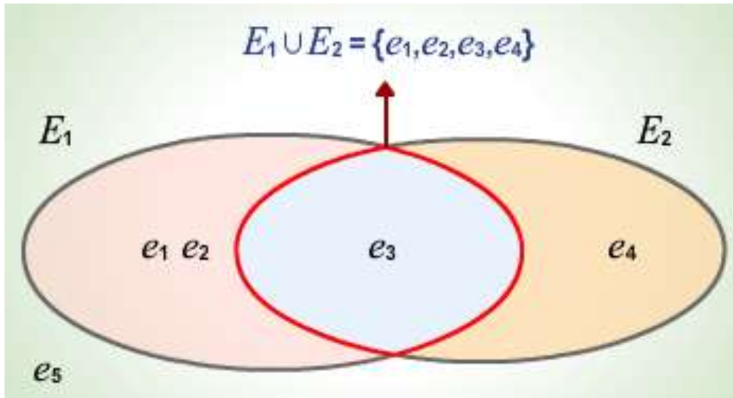
If E_1 and E_2 are two events in the sample space, then the probability of the union of the two events is the probability of all elements either in event E_1 or in event E_2 or in both without repetition.

$$\begin{aligned} P(E_1 or E_2) &= P(E_1) + P(E_2) - P(E_1 and\ E_2) \\ &= P(\{all\ elements\ either\ in\ event\ E_1 \\ &\qquad or\ in\ event\ E_2 or\ both\ without\ repetition\}) \quad (3) \end{aligned}$$

we can also write it in the form:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \qquad (4)$$

The word "or" in the rule means the union.



Sample space - addition rule for two events

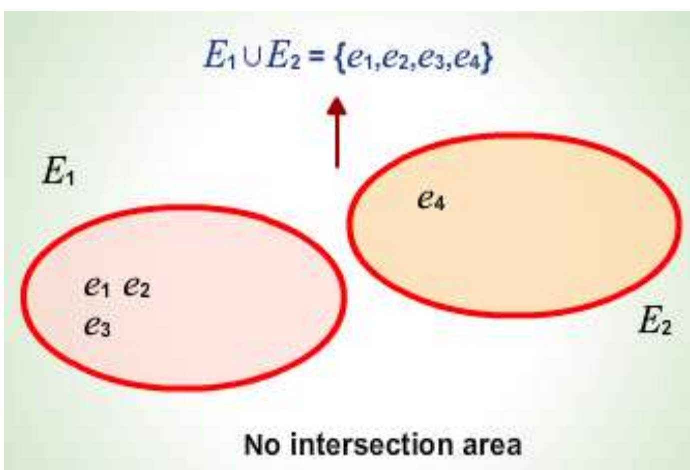
6. Addition Rule for Mutually Exclusive Events

If E_1 and E_2 are two events in the sample space, then the events are mutually exclusive if the intersection between them is empty. And the probability of the union of the two events is the probability of all elements either in event E_1 or in event E_2.

$$\text{If } E_1 \text{ and } E_2 \text{ are mutually exclusive, then:}$$

$$1.\ P(E_1 \text{ and } E_2) = 0 \qquad (5)$$

$$2.\ P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) \qquad (6)$$

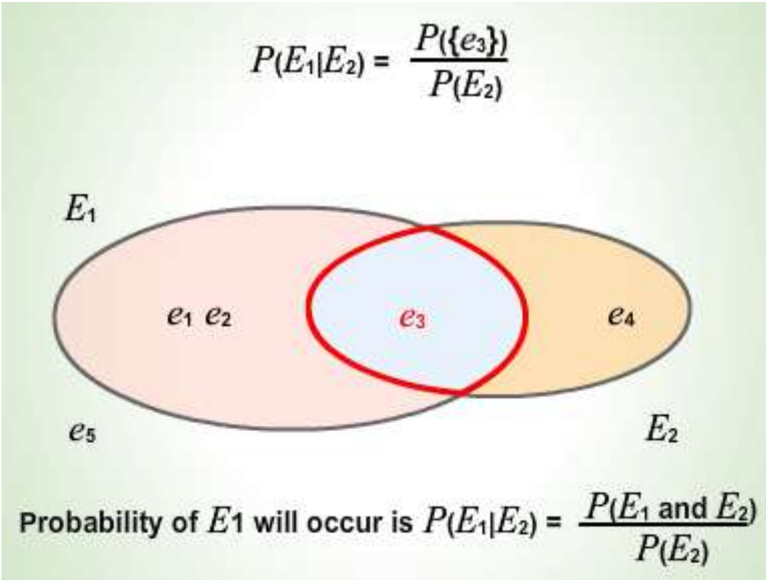


Sample space - addition rule for mutually exclusive events

5. Conditional Probability for any Two Events

Let E_1 and E_2 be any two events, then if E_2 occurred, then the probability of E_1 is given by:

$$P(E_1|E_2) = \frac{P(E_1 \; and \; E_2)}{P(E_2)} \quad where \quad P(E_2) > 0 \qquad (7)$$



Sample space - conditional probability

6. Independent Events

If E_1 and E_2 are two independent events, then the conditional probability for E_1 given E_2 and the probability of E_2 given E_1 are given by:

$$P(E_1|E_2) = P(E_1) \; where \; P(E_2) > 0$$

$$P(E_2|E_1) = P(E_2) \; where \; P(E_1) > 0$$

Note:

1. If $P(E_i \cap E_j) = P(E_i)P(E_j)$, then the two events are independent.
2. If E_1, E_2 are independent, then the following events are independent:
   a) $\bar{E}_1$, E_2
   b) $E_{1,}\, \bar{E}_2$
   c) $\bar{E}_{1,}\, \bar{E}_2$

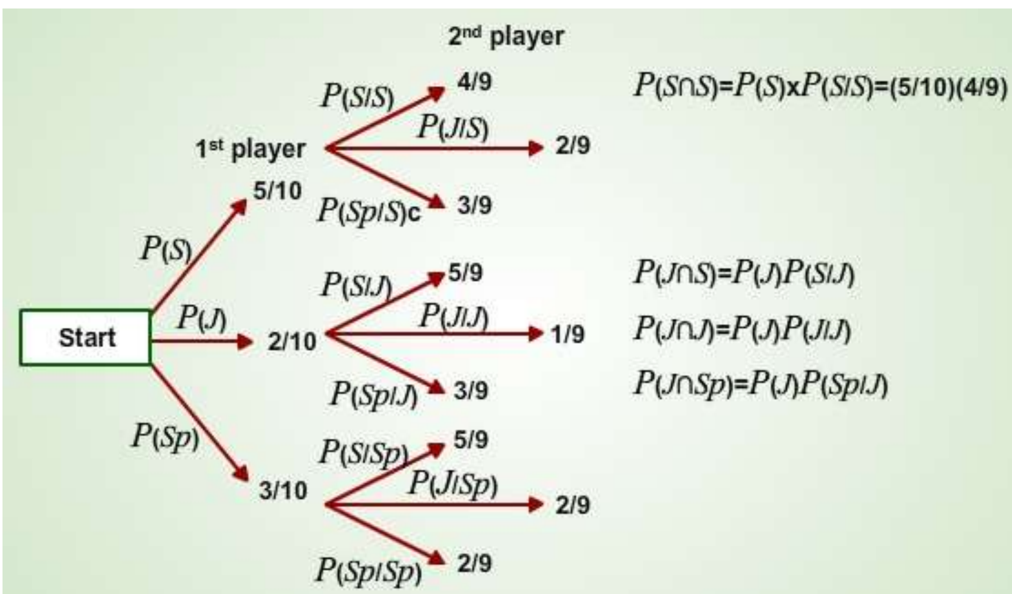
Example

A basketball team has ten players. Five are seniors (S), two are juniors $(J)$and three are sophomores (Sp). Two players are randomly selected to serve as captains for the next game.

1. What is the probability that both players selected are seniors?
2. What is the probability that the first is junior?

Solution:

Use the tree diagram to simplify the problem.



Tree diagram

1. The probability that both players selected are seniors.

$$P(both\ seniors) = P(1^{st}\ senior\ and\ 2^{nd}\ senior)$$
$$= P\ (1^{st}\ senior)P(2^{nd}\ senior|1^{st}\ senior)$$
$$= \frac{5}{10}\left(\frac{4}{9}\right) = \frac{20}{90} = \frac{2}{9}$$

2. The probability that the first is junior.

$$P(1^{st}\ junior) = P(1^{st}\ junior\ \ and\ \ 2^{nd}\ junior)$$
$$+P\ (1^{st}\ junior\ and\ \ 2^{nd}\ senior)$$
$$+P\ (1^{st}\ junior\ and\ \ 2^{nd}\ sophomore)$$
$$= \frac{2}{10}\frac{1}{9} + \frac{2}{10}\frac{5}{9} + \frac{2}{10}\frac{3}{9} = \frac{18}{90}$$

Example

Assume, 70% of the cars have airbag (A) and 40% have a CD player (CD). 20% of the cars have both.

1. What is the probability that a car has a CD player, given that it has an airbag?

Solution:

The probability that a car has a CD player, given that it has an airbag.

$$P(CD|A) = \frac{P(CD \cap A)}{P(A)} = \frac{0.2}{0.7} = \frac{2}{7}$$

2. What is the probability that a car has no CD and no airbag?

Solution:

The probability that a car with no CD and no airbag.

$$P(\overline{CD} \cap \bar{A}) = P(\overline{CD \cup A})$$
$$= 1 - P(CD \cup A) = 1 - \big(P(CD) + P(A) - P(CD \cap A)\big) = 1 - 0.9 = 0.1$$

3. What is the probability of a car with no CD but with an airbag?

Solution:

The probability of a car with no CD but with an airbag.

$$P(\overline{CD} \cap A) = P(A) - P(CD \cap A) = 0.7 - 0.2 = 0.5$$

4. What is the probability of a car with no CD if you know that it has no airbag?

Solution:

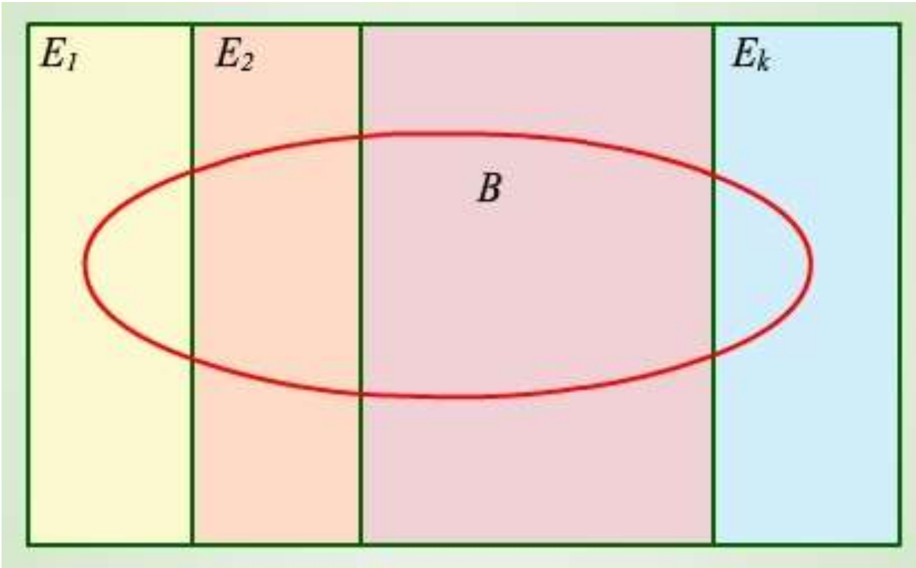
The probability of a car with no CD if you know that it has no airbag.

$$P(\overline{CD}|\bar{A}) = \frac{P(\overline{CD} \cap \bar{A})}{P(\bar{A})} = \frac{0.1}{0.3} = \frac{1}{3}$$

Bayes' Rule

Bayes theorem is used to revise previously calculated probabilities based on new information.

Observe the equation derived from the given diagram.



Bayes' rule

$$P(E_j|B) = \frac{P(E_j \cap B)}{P(B)} = \frac{P(B|E_j)P(E_j)}{\sum_{i=1}^{k} P(B|E_i)P(E_i)} \qquad (9)$$

Where:

E_i = i^{th} event of k mutually exclusive and collectively exhaustive events

B = New event that might impact $P(E_i)$

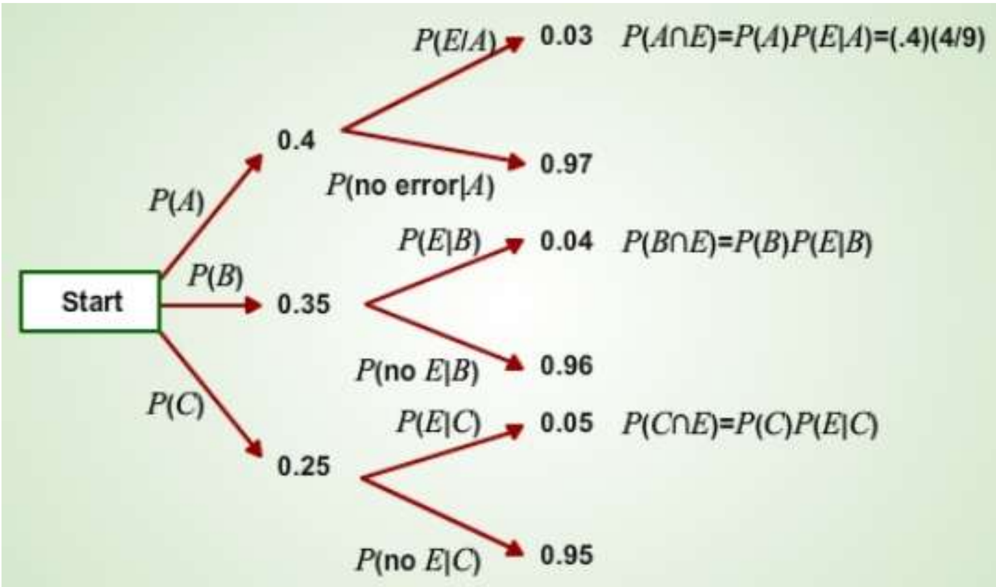
Example

In a small branch for a certain bank there are three tellers *A*, *B*, and *C*. The work load is distributed among them as 40% for teller *A*, 35% for teller *B*, and 25% for teller *C*. The branch manager found errors in 3% of the jobs done by teller *A*, in 4% of the jobs done by teller *B*, and in 5% of the jobs done by teller *C*.

1. If the branch manager selects a job from the branch randomly, find the probability that it has an error in it.
2. If the branch manager selects a job randomly and found that it has an error in it, find the probability that it was done by teller *B*.

Solution:

Use the tree diagram to simplify the problem. Let *E* denote the error.



Tree diagram

1. If the branch manager selects a job from the branch randomly, the probability that it has an error in it.

$$
\begin{aligned}
P(E) &= P(A \cap E) + P(B \cap E) + P(C \cap E) \\
&= P(A)\, P(E|A) + P(B) P(E|B) + P(C) P(E|C) \\
&= (0.4)(0.03) + (0.35)(0.04) + (0.25)(0.05) \\
&= 0.0385
\end{aligned}
$$

2. If the branch manager selects a job randomly and found that it has an error in it, the probability that it was done by teller B.

$$
\begin{aligned}
P(B|E) &= \frac{P(B \cap E)}{P(E)} \\
&= \frac{(0.35)(0.04)}{0.0385} = 0.363636
\end{aligned}
$$

Recap

In this lecture, you have learned that:

- The key rules of probability are:
  - The condition on individual values
  - The condition on the sum of all values
  - Addition rule for elementary events
  - Addition rule for two events
  - Addition rule for mutually exclusive events
  - Complement rule
  - Conditional probability
  - Conditional probability for independent events
  - Bayes' rule