Learning Outcome

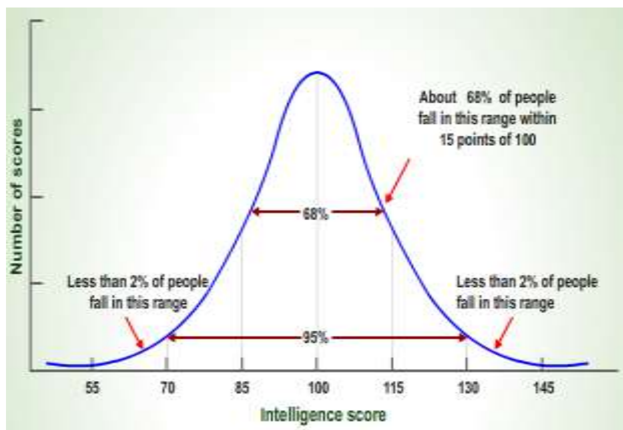
By the end of this lecture, you will be able to:

- Define Normal distribution
- Calculate the probabilities of a normal random variable

Introduction

Consider the Intelligent Quotient or IQ of a general population. In the general population, most people have average intelligence and very few people have extremely high or extremely low IQ. The table shows percentages of IQ levels of the general population. This distribution forms a curve shown here. This curve is known as the normal distribution curve.

IQ Scores	Less than 55	55-70	70-85	85-100	100-115	115-130	130-145	More than 145
Percentage	0.5	2	13.5	34	34	13.5	2	0.5



IQ scores of general population

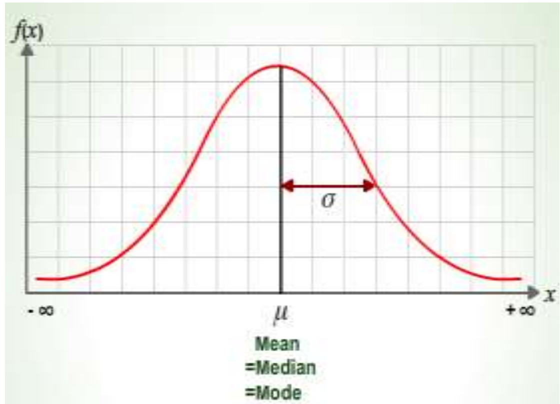
In this lecture, we will learn about normal distribution and the methods to calculate the probabilities of a normal random variable.

Definition

Normal distribution is the most common continuous distribution used in statistics. It has several important theoretical properties.

Normal distribution is:

- Unimodal, that is, the normal distribution peaks at a single value
- Symmetrical, and thus its mean = median = mode
- The random variable has an infinite theoretical range $(-\infty, +\infty)$



Normal distribution

Normal Probability Density Function

The normal probability density function is given by the expression:

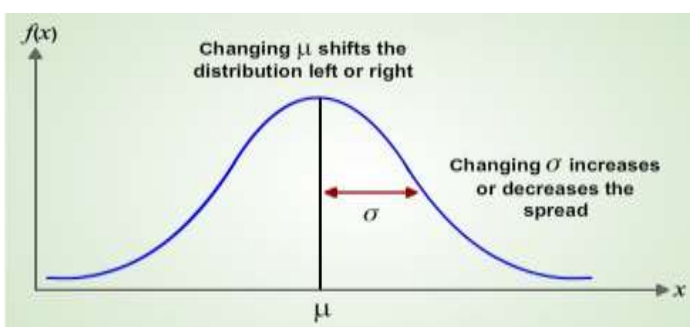
$$f(X) = \frac{1}{\sqrt{2\pi\sigma}}\, e^{\frac{-(X-\mu)^2}{2\sigma^2}} \qquad (1)$$

Where:

e = The mathematical constant approximated by 2.7 1828
π = The mathematical constant approximated by 3.14 159
μ = The mean
σ = The standard deviation
X = Any value of the continuous variable. where $-\infty < X < +\infty$

Shape of Normal Distribution

Location is determined by the mean (μ) and the Spread is determined by the standard deviation (σ).

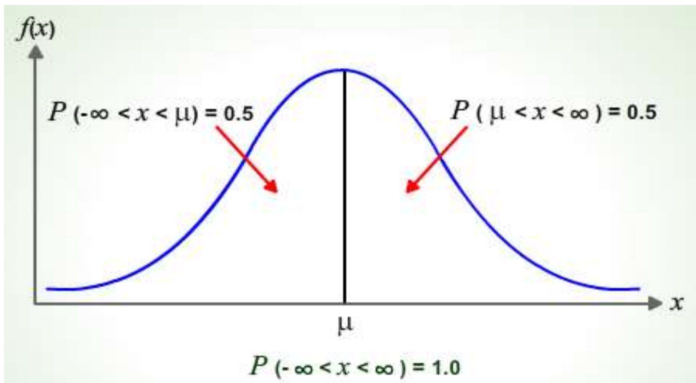


Normal distribution

By varying the parameters μ and σ, we obtain different normal distributions. Drag the respective sliders to see the change in normal distribution.

Finding Normal Probabilities

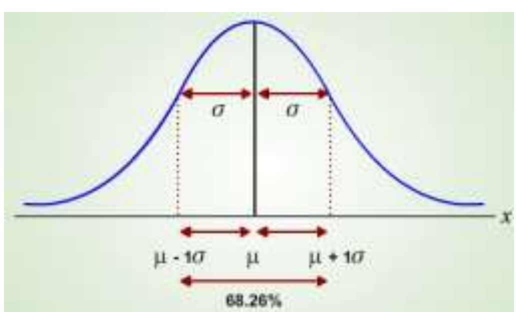
Probability is measured by the area under the curve. The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below.



Probability

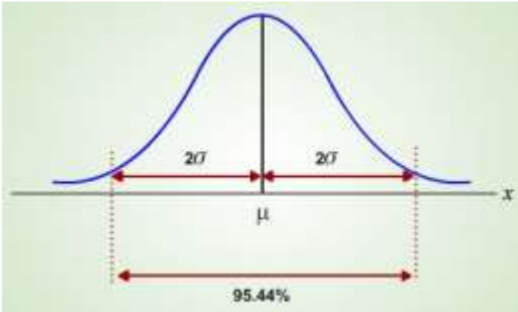
Empirical Rule

The distribution of values around the mean depends on the empirical rule. As per the empirical rule, μ ± 1 sigma encloses about 68% of x's, μ ± 2 sigma covers about 95% of x's, and μ ± 3sigma covers about 99.7% of x's.



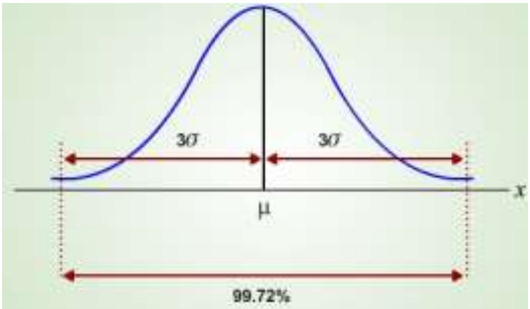
$\mu \pm 1\sigma$ encloses about 68% of x ' s

1. On clicking the 2nd one, display the one with 95.44% written below,



$\mu \pm 2\sigma$ covers about 95% of x's

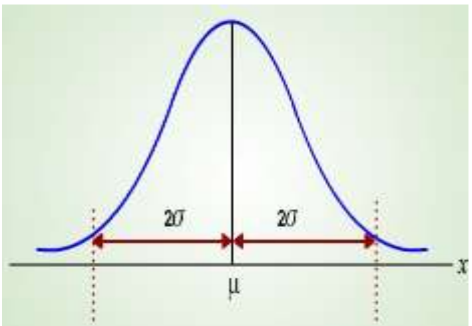
2. On clicking the 3rd one, display the one with 99.72% written below.



$\mu \pm 3\sigma$ covers about 99.7% of x's

Importance of the Empirical Rule

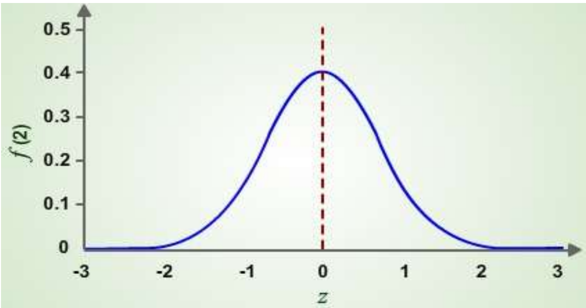
If a value is about 2 or more standard deviations away from the mean in a normal distribution, then it is far from the mean. The chance that a value that far or farther away from the mean is highly unlikely, given that particular mean and standard deviation.



Normal distribution

Translation to the Standard Normal Distribution

Any normal distribution (with any mean and standard deviation combination) can be transformed into the standard normal distribution (the "z" distribution) with mean 0 and standard deviation 1



Density function of the standard normal distribution

So we need to transform x units into z units. Translate from x to the standard normal (the "z" distribution) by subtracting the mean of x and dividing by its standard deviation:

$$z = \frac{x - \mu}{\sigma} \qquad (2)$$

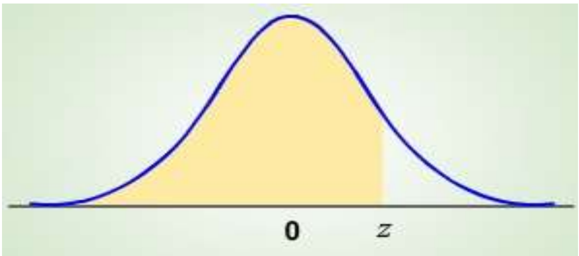
Any set of normally distributed values can be converted to its standardized form. Then we can determine the probabilities by using the cumulative standard normal distribution table.

Cumulative Standard Normal Distribution Table

Given here are cumulative standard normal distribution tables

- **Cumulative Standard Normal Distribution Table - 1**

The Cumulative Standard Normal Distribution

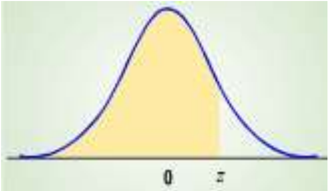


Entry represented area under the cumulative standardized normal distribution from - ∞ to Z

Cumulative Probabilities										
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

- **Cumulative Standard Normal Distribution Table - 2**

The Cumulative Standard Normal Distribution



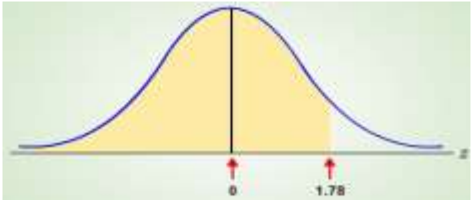
Entry represented area under the cumulative standardized normal distribution from - ∞ to

Cumulative Probabilities										
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Example

Let Z has a standard Normal distribution, find:
$P(Z < 1.78)$

Solution:



Normal distribution

Since the shaded area in the left of 1.78, scan down the Z column from the cumulative standard normal distribution table until you locate the Z value of interest which is 1.7 (**The column gives the value of z to the second decimal point**) and scan left the Z row from the cumulative standard normal distribution table until you locate the Z value of interest which is 0.08 (**The row shows the value of z to the first decimal point**) then you look up the Z value of 1.78 by matching the appropriate Z column 1.7 with the appropriate Z row 0.08 as shown in the table, the resulting probability (area under the curve) less than 1.78 standard deviations below the mean is 0.9625.

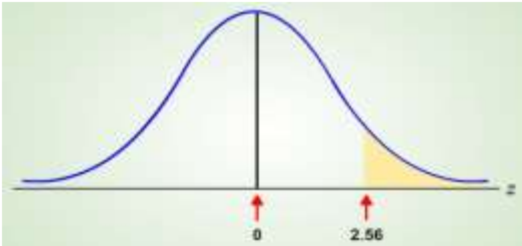
Cumulative Probabilities - tab										
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706

Example

Let Z has a standard normal distribution, find: $P(Z < 2.56)$.

Solution:

To find the probability (the area under the curve) greater than 1.78, draw the bell shape



$P(Z > 2.56)$

Since the shaded area to the right of the Z value, we must use the complement rule since the cumulative standard normal distribution table gives us the probability (area under the curve) to the left the Z value. thus, the probability that the z value more than 2.56 is the complement of less than 2.56

$$\begin{aligned} P(Z > 2.56) &= 1 - P(Z < 2.56) \\ &= 1 - 0.9948 = 0.0052 \end{aligned}$$

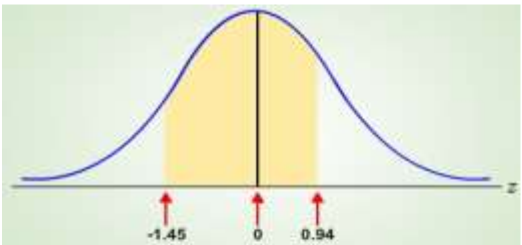
Example

Let Z has a standard Normal distribution, find:

$P(- 1.45 < Z < 0.94)$

Solution:

find the probability (the area under the curve) between – 1.45 and 0.94, draw the bell shape



$P(-1.45 < Z < 0.94)$

From the figure you can see that the area of interest is located between two values and the cumulative standard normal distribution table give us the probability (area under the curve) to the left the Z value, so we can find the probability less than the largest Z value and the area less than the smallest Z value then subtract the two areas to find the probability between – 1.45 and 0.94

$$P(-1.45 < Z < 0.94) = P(Z < 0.94) - P(Z < -1.45)$$
$$= 0.8264 - 0.0735 = 0.7529$$

Example

Let Z has a standard Normal distribution, find:

Find ‘a' such that the $P(Z < a) = 0.9$

Solution :

We look for a value such that the area under the normal curve less than this value is 90%. Using the body of the cumulative standard normal distribution table, search for the area (probability) of 0.90. the closest result is 0.8997 as shown in the table below, working from this area to the margins of the table, the Z value corresponding to the particular Z row (1.2) and Z column (0.08) is 1.28

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
.	.	.	.	.	.	.	.	.	.	.
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
.	.	.	.	.	.	.	.	.	.	.

Example

A company produces light bulbs whose lifetimes follow a normal distribution with mean 500 hours and standard deviation 50 hours. If a light bulb is chosen randomly from the company’s output:

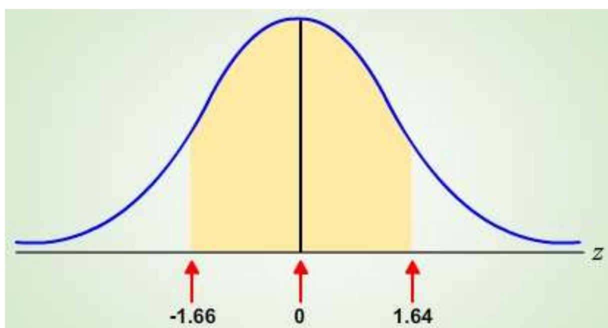
1. What is the percentage of the bulbs that has lifetime between 417 and 582 hours?

Solution:

Define the random variable X:
The lifetimes of the light bulbs which are normally distributed with mean 500 hours and standard deviation 50 hours. To find the probability that the lifetime between 417 and 582 hours, convert the values to standard normal with mean equal to 0 hour and standard deviation 1 hour.

$$
\begin{aligned}
P(417 < x > 582) &= P\left(\frac{417-500}{50} < z < \frac{582-500}{50}\right) \\
&= P(-1.66 < z > 1.64) \\
&= P(z < 1.66) - (z < -1.64)
\end{aligned}
$$

Draw the bell shape. You can see that the area of interest is located between two values – 1.66 and 1.64.
Find the area to the left of 1.64 and the area to the left of – 1.66 then subtract the two areas to find the probability between 1.66 and -1.64



Normal distribution

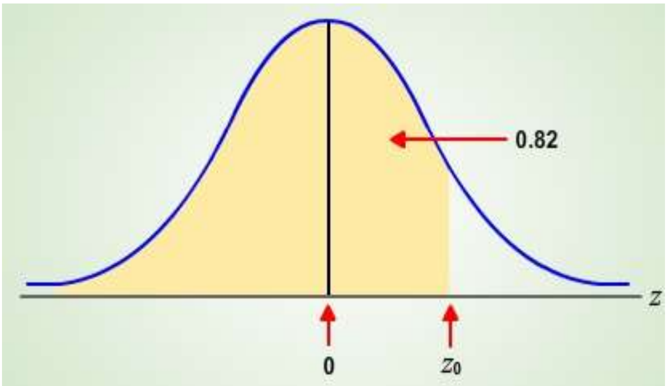
$$
\begin{aligned}
P(417 < x > 582) &= P\left(\frac{417-500}{50} < z < \frac{582-500}{50}\right) \\
&= P(-1.66 < z > 1.64) \\
&= P(z < 1.66) - (z < -1.64)
\end{aligned}
$$

90.1 % of the bulbs has lifetime between 417 and 582 hours

2. Find x where 82% of the bulbs have lifetime of at most x hours.

Solution :

After drawing the bell shape, we need to find a value x such that the area to the left it this value equal to 82%. Convert the values to standard normal with mean equal to 0 hour and standard deviation 1 hour. Look for a value such that the area under the normal curve less than this value is 82%. Using the body of the cumulative standard normal distribution table, search for the area (probability) of 0.82. the closest result is 0.8212, working from this area to the margins of the table, the Z value corresponding to the particular Z row (0.9) and Z column (0.02) is 0.92



Normal distribution

$$P(X < x) = 0.82$$

$$P\left(z < \frac{x - 500}{50}\right) = 0.82$$

$$P(z < z_0) = 0.82$$

So, from standard normal table once you find Z value, you use the transformation formula to determine the X value

$$z_0 = \frac{x - 500}{50} = 0.92$$

$$x = 50(0.92) + 500$$

$$x = 546$$

Exercise

The scores for the students in Stat 211 are normally distributed with mean 56 and standard deviation 12. If a student is randomly selected :

- Find the probability that the student scores more than 85
- Find the probability that the student scores more than 65 but less than 80
- If the instructor will give the highest 10% A+, find the minimum score to get A+
- If a sample of size 10 students randomly selected, find the probability that 3 of them will score more than 85.

Recap

In this lecture, you have learned that:

- Normal distribution is the most common continuous distribution used in statistics. It has several important theoretical properties

- Normal distribution is unimodal, that is, the normal distribution peaks at a single value

- It is symmetrical, and thus its mean is equal to median, which is equal to mode

- Probability is measured by the area under the curve. The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below