A STATISTICAL THEORY OF THE STRENGTH OF MATERIALS

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Part I. Theoretical considerations.

According to the classical theory the ultimate strength of a material is determined by the internal stresses in a point, assuming that by a suitable combination of the three principal stresses or strains a characteristic value may be computed for the material in question. This value is supposed to be definitively decisive in judging whether the ultimate strength has been reached or not. Experimental measurements give many results which may hardly be brought to agree with this theory. Where ductile materials are concerned, reference could of course be made to the difficulties involved in a sufficiently accurate determination of the actual intensity of the stresses at the moment of rupture and due to plastic flow, or else the deficient agreement between theory and practice could be explained by the lack of homogeneity of the material, and in many cases such an explanation would naturally be perfectly justified. Nevertheless, there are tests which, as far as I know, cannot be explained by any of the existing hypotheses.

The purpose of this paper is to show that taking the elementary laws of probability as a starting-point, a theory may be developed, whose formulæ may be readily brought to agree with such measuring results inconsistent with the classical theory.

The following elementary experiment will help to elucidate our chain of reasoning. Let a rod of the length l and crossectional area A be subjected to a tensile stress and ruptured by an external force P, the internal stress σ being uniformly distributed over the sectional area. If this experiment is repeated, the value of the breaking load will not be exactly the same. The measured figures will be grouped around the computed mean value and will show some amount of dispersion, which in some cases may be quite considerable. Thus, it is not possible to indicate an absolutely exact value of the breaking load (a statement which, of course, holds true for all physical measurements), but it is possible — or at least we assume it to be possible — to indicate a definite probability of the rupture occurring at a given stress. If S is the probability, then $S = f(\sigma)$, a monotonously increasing function of the stress σ. For very low stresses $S = 0$ and for very heavy stresses $S = 1$. Let this function follow the curve S_1 represented by the unbroken line in Fig. 1. Thus, we cannot specify a definite ultimate strength, but we may state, with the aid of this given distribution curve S_1, that if, for instance, 100 rods of this kind are subjected to a tensile stress σ_1, then about 50 specimens will be ruptured and about 50 will stand the test. Let now the stressed system be composed, not of one, but of two identical rods placed beside each other. If this system is subjected to a double load $2P$, i. e. to the same tensile stress as in the case of the single rod, and if the stress reaches the value σ_1, then we know that the probability of rupture of each rod is $S = 0.5$. The probability of each rod withstanding the load is $S_i = 1—S$. According to the theory of probabilitiy, the probability S_{12} that two events having the probabilities

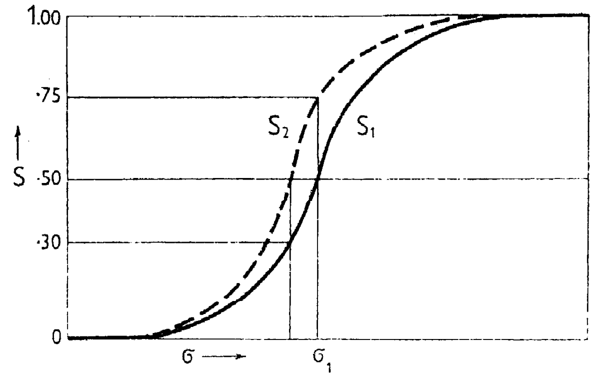


Fig. 1.

S_1 and S_2 respectively occur simultaneously is obtained by multiplying the two individual probabilities, or

$$S_{12} = S_1 \cdot S_2 \ldots\ldots\ldots\ldots \quad 1)$$

which may, of course, also be expressed in the equivalent form

$$\log S_{12} = \log S_1 + \log S_2 \ldots\ldots\ldots\ldots \quad 2)$$

It follows that in the above-mentioned example the probability that both specimens withstand the stress is S_i^2. Thus, at the stress σ_1, the probability is reduced to 0.25. Now, if one of the specimens is ruptured, then, assuming the load to be unchanged, the stress in the other specimen will rise to the double intensity 2σ, and this rod, too, will in all probability be ruptured. Consequently, if the distribution curve S_1 is given for a specimen, then the distribution curve S_2 for a system consisting of two specimens coupled in parallel may be computed from the equation

$$1—S_2 = (1—S_1)^2 \text{ or}$$
$$\log (1—S_2) = 2 \log (1—S_1) \ldots\ldots\ldots\ldots \quad 3)$$

The distribution curve S_2 is represented by the dash-line in fig. 1 and was computed from the unbroken curve S_1.

Let now the system be composed of two rods coupled in series and subjected to a load P inducing a stress σ_1 in both rods. The result will be exactly the same as in the case where the rods are coupled in parallel, namely the probability of rupture at the stress σ_1 will increase from 50 per cent to 75 per cent. It may be observed that in this case in deriving the equation 3) no other assumption is made except that the distribution curve S_1 exists. But if this assumption were not true, then it would be impossible to say anything at all about the ultimate strength. Furthermore, it is obvious that the distribution curve S_1 should not be influenced and distorted by alien factors independent of the specimen itself, as for instance method of measurement, measuring instruments, fixation of the test specimens, etc., but that it should be an expression of the strength properties of the material. In a general manner, the distribution curve S_l for any length l may be computed from the distribution curve S_1 for the unit of length according to the formula

$$1-S_l = (1-S_1)^l \text{ or}$$
$$\log (1-S_l) = l \cdot \log (1-S_1) \ldots\ldots\ldots\ldots\ldots \quad 4)$$

In this case the volume V of the stressed system is proportional to the length l, and if S_0 is the distribution curve for that length of rod which corresponds to the unit of volume, we have

$$\log (1-S) = V \cdot \log (1-S_0) \ldots\ldots\ldots\ldots\ldots\ldots \quad 5)$$

If we now put $B = -\log (1-S)$ and call B the »risk of rupture», we find that B is proportional to the volume and to $\log (1-S_0)$ which is a function of the tensile stress σ alone.

Considering now the material from a phenomenological viewpoint and assuming it to be a continuum, as is usually done in the theory of elasticity, and assuming, furthermore, the properties of the material to be such that the probability, viewed from a macroscopic standpoint, of the rupture starting at any point is equal, we say that the material is isotropic. In respect of such materials we find that the risk of rupture dB for a small volume element dv is determined by the equation

$$dB = -\log (1-S_0)\, dv \ldots\ldots\ldots\ldots\ldots\ldots \quad 6)$$

As has been mentioned in the above, $\log (1-S_0)$ is a function of σ only and is negative because $1-S_0 < 1$. Hence we have

$$dB = n(\sigma)\, dv \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 7)$$

If the distribution of stresses in the body is arbitrary, the risk of rupture is

$$B = \int n(\sigma)\, dv \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 8)$$

and the probability of rupture

$$S = 1-e^{-B} = 1 - e^{-\int n(\sigma)\, dv} \ldots\ldots\ldots\ldots\ldots \quad 9)$$

This equation expresses the fundamental law of an isotropic brittle material. In the above equation $n(\sigma)$, or the material function, is a function expressing the strength properties of the material. Inversely, we may, of course, also use 9) as a definition of an isotropic brittle material, where $n(\sigma)$ is a function which is independent both of the position of the volume element dv and the direction of the stress σ.

For an anisotropic material $n(\sigma)$ is a function, not only of the stress, but also of the coordinates and possibly of the direction of stress too. This case will not be dealt with here except in a special respect.

In fact, it is not unusual that the rupture of a brittle body starts on the surface whose properties may differ from those of the material in the interior of the body, for instance owing to the method of manufacture. In the extreme case, where all fractures start from the surface and none from the interior of the body, it may be assumed that $n(\sigma) = 0$ for each volume element in the interior, so that the above-mentioned volume integral needs only to be extended over a surface layer of the small thickness h. In this case the volume integral is replaced by a surface integral, or

$$\int^{V} n(\sigma)\, dv = \int^{A} h\, n(\sigma)\, do \quad \ldots\ldots\ldots\ldots \quad 10)$$

It is, of course, also possible that the lack of isotropy is not as marked as indicated in the above, but that we have to reckon with isotropic material in the interior of the body characterised by the material function $n_1(\sigma)$ and with another material function $n_2(\sigma)$ in the surface layer. In this case we obtain a more general formula

$$S = 1 - e^{-\int n_1(\sigma)\, dv - \int n_2(\sigma)\, do} \quad \ldots\ldots\ldots\ldots \quad 11)$$

From the preceding formulæ we shall now derive an expression for the ultimate strength σ_b. The experimental determination of the ultimate strength is carried out as follows. A number of measurements (N) is made, and the arithmetic mean value of all the measured values is computed. The higher the dispersion of the individual values, the greater is the number of measurements that must be made in order to obtain the desired accuracy of the mean value. Now, the probability that a value is included between σ and $\sigma + d\sigma$ is $\frac{dS}{d\sigma}\, d\sigma$ using the symbols as in the above. If the total number of measurements is N, then $N \frac{dS}{d\sigma}\, d\sigma$ of the values will probably be included in the range $d\sigma$ and we have

$$\sigma_b \cdot N = \int_0^\infty \sigma \cdot N \frac{dS}{d\sigma} \cdot d\sigma$$

or

$$\sigma_b = \int_{\sigma=0}^{\sigma=\infty} \sigma \cdot dS \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 12)$$

From equation 9) we obtain $\sigma_b = -\int \sigma \, de^{-\int n(\sigma)\, dv} = -/\sigma e^{-\int n(\sigma)\, dv} + + \int e^{-\int n(\sigma)\, dv} d\sigma$. As $\int n(\sigma)\, dv$ must infinitely increase with σ, it follows for large values of σ that $\int n(\sigma)\, d\sigma > \sigma_0 + k\sigma$ where σ_0 is finite and $k > 0$, whence $\sigma e^{-\int n(\sigma)\, dv} < \sigma e^{-\sigma_0 - k\sigma}$. The right side, and consequently also the left one, tends towards 0, so that we obtain the general expression for the ultimate strength

$$\sigma_b = \int_0^\infty e^{-\int n(\sigma)\, dv} \cdot d\sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 13)$$

As a criterion of uncertainty of the computed value σ_b, we may use the standard deviation a computed from the formula

$$a^2 = \int_0^\infty (\sigma - \sigma_b)^2 \, dS \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 14)$$

Since $a^2 = \int \sigma^2 \, dS - \int 2\sigma_b \cdot \sigma d\, S + \int \sigma_b^2 \, dS = \int \sigma^2 \, dS - \sigma_b^2$ and since it may be demonstrated in an analogous manner that $/_0^\infty \sigma^2 \, S = 0$, we obtain

$$a^2 = \int_0^\infty e^{-\int n(\sigma)\, dv} d\,(\sigma^2) - \sigma_b^2 \quad \ldots\ldots\ldots\ldots\ldots \quad 15)$$

In deriving the preceding formula it has been assumed that σ is a tensile stress, and that this stress is one-dimensional, i. e. that the two other principel stresses are zero. If the material is such that the risk of rupture is independent of the stress direction, then σ in the formula is a scalar quantity. If the stresses are distributed in an arbitrary way, σ is to be replaced by another scalar quantity which is definitively determined by the three principal stresses. If σ in the formulæ derived above denotes this reduced stress, the formulæ will be valid for any combination of stresses.

The formulæ derived for the ultimate strength (13) and for the standard deviation (15) include the function $n(\sigma)$ characteristic of the material. It may be of use to point out that for a special form of this function both formulæ mentioned in the above become identical with the corresponding formulæ of the classical theory of strength. Consequently, that theory is to be regarded as a special case of this statistical theory of strength. This may be readily shown if we put

$$n(\sigma) = k\left(\frac{\sigma}{\sigma_0}\right)^m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 16)$$

Now, if the exponent m increases infinitely, the distribution curve S will for $\sigma = \sigma_0$ leap from 0 to 1, and the ultimate strength becomes according to 13)

$$\sigma_b = \int_0^\infty e^{-\int n(\sigma)\, dv} \cdot d\sigma = \int_0^{\sigma_0} d\sigma = \sigma_0 \ldots\ldots\ldots\ldots\ldots \quad 17)$$

while the dispersion is according to 15)

$$a^2 = \int_0^\infty e^{-\int n(\sigma)\, dv}\, d(\sigma^2) - \sigma_b^2 = \int_0^{\sigma_0^2} d(\sigma^2) - \sigma_0^2 = 0 \ldots\ldots \quad 18)$$

Thus, σ_0 in equation 16) is identical with the ultimate strength in the classical theory, according to which rupture occurs as soon as the stress in any point of the body, irrespective of its dimensions, has reached a certain determined value.

According to this statistical conception, it is precisely $a \neq 0$ that constitutes a criterion of the invalidity of the classical theory of strength.

Upper and Lower Ultimate Strength.

In the above-mentioned formulæ it vas assumed that the distribution curve reached from ∞ to 0. It is, of course, possible that the range of distribution may be limited, i. e. that there is a finite value of stress for which the probability of rupture is unity, and another stress which does not disappear, and for which this probability is zero. We describe the former value σ_2 as upper ultimate strength and the latter value σ_1 as lower ultimate strength.

It may be useful to examine the mathematical consequences of finite values of σ_2 and σ_1. If, for a given dimension, the distribution curve follows the course indicated by the unbroken line in Fig. 2, the curve will, for very large dimensions, change into curve I and, in the case of very small dimensions, into curve II.

Now, with respect to the upper ultimate strength it is reasonable to assume that it consists of molecular strength according to SMEKAL's terminology. As, however, the latter strength is 100 to 1 000 times as high as the technical strength, the probability is so exceedingly small (apart from very small dimensions) that we are perfectly safe in assuming the upper limit of the integral to be infinite, i. e. with the exception of altogether special cases, it may be assumed that $\sigma_2 = \infty$.

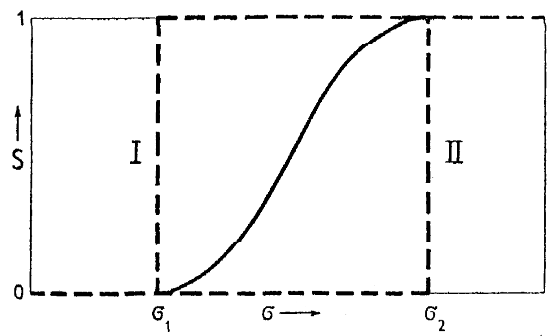


Fig. 2.

The available experimental data are not sufficient for the decision whether the lower ultimate strength $\sigma_1 \neq 0$. For practical reasons this may be supposed, because if an individual extremely low value is obtained by measurement — and for reasons of probability such values will occur fairly infrequently — then it is certainly conform to the present practice to reject it as abnormal. Moreover, the lowest values are already eliminated by the preparation of test specimens, since a specimen whose lower ultimate strength were near to 0, would be ruptured before coming into the testing-machine. In general, this elimination is carried out unconsciously. An excellent example of clear and conscious elimination is provided by the preparation of glass wire tests specimens described as follows in GRIFFITH's memorable paper on »The Phenomena of Rupture and Flow in Solids«[1]:

»Fibres of diameters varying from 0.13×10^{-3} in. to 4.2×10^{-3} in., and 6 in long, were prepared by heating the glass to about 1 400° C to 1 500° C in an oxygen and coal-gas flame and drawing the fibre by hand as quickly as possible. The fibres were then put aside for about 40 hours, so that they might reach the steady state. The test specimens were prepared by breaking these fibres in tension several times until pieces about 0.5 in. long remained; these were then tested by the balance method already described. The object of this procedure was the elimination of weak places due to minute foreign bodies, local impurities and other causes.«

We shall now direct our attention to the mathematical consequences of this method of elimination. We assume that the original distribution curve is represented by the unbroken line S in Fig. 3 and that all those test specimens whose ultimate strength is $< \sigma_1$ are eliminated. If the stress σ_1 corresponds to the probability S_0 and if the original number of specimens is 1, then the number of

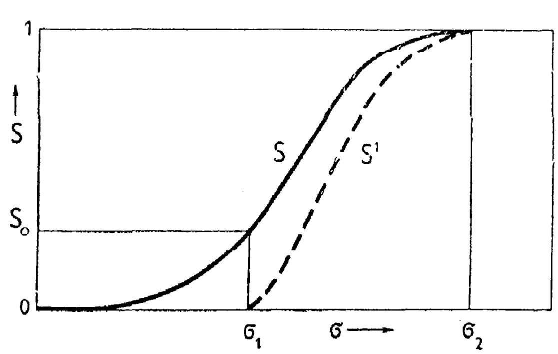


Fig. 3.

[1] GRIFFITH. Trans. Roy. Soc. 221 A, 1920—21, p. 181.

specimens remaining after the elimination will be $1—S_0$ and the relation between the distribution curve S_1 for these remaining test specimens and S will evidently be

$$S_1 = \frac{S—S_0}{1—S_0} = 1 — \frac{1}{1—S_0} \cdot e^{-B} \text{ for } \sigma > \sigma_1 \ldots\ldots\ldots \quad 19)$$

Putting $\frac{1}{1—S_0} = h$ we obtain

$$S_1 = 1 — h\, e^{-B} \text{ for } \sigma > \sigma_1 \ldots\ldots\ldots\ldots\ldots\ldots \quad 20)$$
$$\text{and } S_1 = 0 \qquad \text{for } \sigma < \sigma_1$$

With this new distribution the ultimate strength becomes

$$\sigma_b^1 = \sigma_1 + h \int_{\sigma_1}^{\infty} e^{-\int n(\sigma)\, dv} \cdot d\sigma \ldots\ldots\ldots\ldots\ldots\ldots \quad 21)$$

and the standard deviation

$$a_1^2 = \sigma_1^2 + h \int_{\sigma_1^2}^{\infty} e^{-\int n(\sigma)\, dv}\, d(\sigma^2) — \sigma_b^{1\,2} \ldots\ldots\ldots\ldots\ldots \quad 22)$$

Specialisation of the Function $n(\sigma)$.

From the experimental data available, which at present are very limited, it appears that good agreement with measuring results may frequently be obtained if we put

$$n(\sigma) = k\, \sigma^m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 23)$$

where k and m are constants. Instead of k we may introduce the constant $\sigma_0 = k^{-1/m}$. The probability of rupture for the volume V will then be

$$S = 1 — e^{-V\left(\frac{\sigma}{\sigma_0}\right)^m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 24)$$

For a volume $V = 1$ and $\sigma = \sigma_0$ $S = 0.63$ whence it follows that σ_0 is that stress which for the unit of volume gives the probability of rupture $S = 0.63$.

By introducing equation 23) into the preceding formulæ we obtain the ultimate strength according to 21)

$$\sigma_b^1 = \sigma_1 + h \int_{\sigma_1}^{\infty} e^{-\int k\, \sigma^m\, dv} \cdot d\sigma \ldots\ldots\ldots\ldots\ldots \quad 25)$$

For a uniform distribution of stresses we have

$$\int k\, \sigma^m\, dv = k\, V \sigma^m = V \left(\frac{\sigma}{\sigma_0}\right)^m$$

and

$$\sigma_b^1 = \sigma_1 + h \int_{\sigma_1}^{\infty} e^{-k\,V\,\sigma^m}\, d\,\sigma = \sigma_1 + \frac{\sigma \cdot h}{V^{1/m}} \cdot \int_{z_1}^{\infty} e^{-z^m}\, dz \quad \text{where } z_1 = \frac{\sigma_1}{\sigma_0\, V^{1/m}} \qquad 26)$$

In the special case of $\sigma_1 = 0$ we have $h = 1$ and

$$\sigma_b = \frac{\sigma_0}{V^{1/m}} \int_0^{\infty} e^{-z^m}\, dz \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 27)$$

By putting $\int_0^{\infty} e^{-z^m}\, dz = I_m$ we obtain

$$\sigma_b = \frac{I_m}{V^{1/m}} \cdot \sigma_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 28)$$

In this case the standard deviation a is determined by the formula

$$a^2 = \int_z^{\infty} e^{-V\left(\frac{\sigma}{\sigma_0}\right)^m}\, d\,(\sigma^2) - \sigma_b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 29)$$

and consequently

$$a^2 = \frac{\sigma_0^2}{V^{1/m}} \left(I_{m/2} - I_m^2\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 30)$$

whence

$$\frac{a^2}{\sigma_b^2} = \frac{I_{m/2}}{I_m^2} - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 31)$$

It follows from the equations 28) and 30) that the ultimate strength and the standard deviation increase as the volume V decreases.

It is particularly interesting to note that, as may be seen from 31), the relative dispersion is independent of the volume and is definitively determined by the exponent m.

The values of I_m and $\frac{a}{\sigma_1}$ are given in the table below for various values of m.

Table I.

m	I_m	a
1	1.000	1.00
2	0.886	0.52
3	0.896	0.36
4	0.908	0.28
8	0.940	0.16
16	0.965	0.09
∞	1.000	0.00

In the equations 21) and 25) we have supposed that the lower ultimate strength σ_1 is arrived at by rejecting measuring results inferior to σ_1. It is also possible that the material in itself has a lower ultimate strength σ_1, and assuming the distribution curve in this case to be

$$n(\sigma) = k\,(\sigma - \sigma_1)^m \quad \text{for } \sigma > \sigma_1$$
$$n(\sigma) = 0 \qquad \text{for } \sigma < \sigma_1 \quad \ldots\ldots \quad 32)$$

the ultimate strength σ_b will be determined by

$$\sigma_b = \int_{\sigma_1}^{\infty} e^{-k\, V\, (\sigma - \sigma_1)^m} \cdot d\sigma$$

whence

$$\sigma_b = \sigma_1 + \frac{\sigma_0}{V^{1/m}} \cdot \int_{0}^{\infty} e^{-z^m} \cdot dz \qquad 33)$$

or denoting the value of the integral by I_m

$$\sigma_b = \sigma_1 + \frac{\sigma_0 \cdot I_m}{V^{1/m}} \qquad 34)$$

Polydimensional stresses.

In the preceding formulæ σ is a pure tensile stress. We shall now derive the formulæ required for polydimensional stresses, but first we must analyse more closely the case of one-dimensional stress, in which the principal stress σ is located in the direction of the x-axis. The distribution of the normal stress is in rotational symmetry relatively to the x-axis. In a direction forming an angle φ with this axis, the normal stress is

$$\sigma = \sigma_x \cdot \cos^2 \varphi \qquad 35)$$

We assume now that the compressive stresses have no influence on the risk of rupture. This assumption seems good for some brittle materials. We suppose, however, that those normal stresses which are included within the small spatial angle do, contribute to the risk of rupture an amount $n_1\ (\sigma)\ do$ and that, consequently, according to equation 8), we have

$$B = \int n_1\ (\sigma)\ do$$

where the integration is extended over half the surface of the sphere. Thus, we obtain

$$B = \int_{0}^{\frac{\pi}{2}} n_1\,(\sigma) \cdot 2\pi \sin\varphi \cdot d\varphi = -\,2\pi \int_{0}^{1} n_1\ (\sigma_x \cos^2\varphi)\ d\cos\varphi$$

or if $n_1 = k_1\, \sigma^m$

$$B = \frac{2\pi}{2\,m+1} \cdot k_1\, \sigma_x^m \qquad 36)$$

Thus, if $n = k\, \sigma_x^m$ for a one-dimensional stress, we have

$$k_1 = \frac{2\,m+1}{2\,\pi} \cdot k \qquad 37)$$

If σ_x and σ_y are the principal stresses, we obtain the normal stress and the coordinates according to Fig. 4

$$\sigma = \cos^2\varphi\,(\sigma_x \cos^2\psi + \sigma_y \sin^2\psi) \quad \dots\dots \quad 38)$$

$$\text{and } do = \cos\varphi \cdot d\psi \cdot d\varphi$$

The risk of rupture is then

$$B = 2\,k_1 \int_0^{\frac{\pi}{2}} \int_{-\psi_0}^{+\psi_0} \cos^{2m+1}\varphi\,(\sigma_x \cos^2\psi + \sigma_y \sin^2\varphi)^m\, d\varphi \cdot d\psi \quad \dots\dots \quad 39)$$

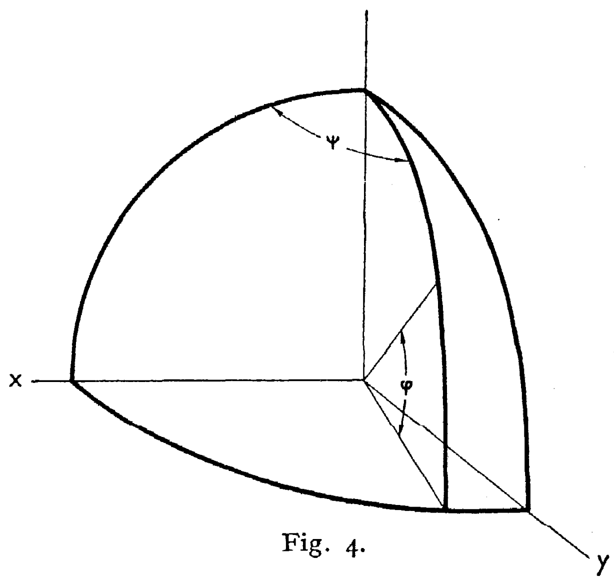


Fig. 4.

If σ_x and σ_y are tensile stresses, we have

$$\psi_0 = \frac{\pi}{2} \quad \dots\dots \quad 40)$$

If, on the other hand, σ_y is negative, i. e. a compressive stress, the integration shall only be extended over those directions in which tensile stress occurs, as we have assumed that compressive stresses have no influence on the risk of rupture. In this case we have

$$tg\,\psi_0 = \sqrt{\frac{\sigma_x}{-\sigma_y}} \quad \dots\dots \quad 41)$$

From these formulæ we obtain for any two tensile stresses

$$B = k\,(\sigma_x^2 + \sigma_y^2 + \frac{2}{3}\sigma_x \cdot \sigma_y) \text{ if } m = 2 \quad \dots\dots \quad 42)$$

and

$$B = k\left[\sigma_x^3 + \sigma_y^3 + \frac{3}{5}\sigma_x\,\sigma_y\,(\sigma_x+\sigma_y)\right] \text{ if } m = 3 \ldots\ldots\ldots \quad 43)$$

and in the special case of $\sigma_y = \sigma_x$

$$B = \frac{8}{3} k\,\sigma_x^2 \qquad \text{if } m = 2 \ldots\ldots\ldots \quad 44)$$

and

$$B = \frac{16}{5} k\,\sigma_x^3 \qquad \text{if } m = 3 \ldots\ldots\ldots \quad 45)$$

For a pure shearing stress $\sigma_x = -\sigma_y$ we have

$$B = \frac{10}{15} k\,\sigma_x^2 = \frac{10}{15} k\,\tau^2 \qquad \text{if } m = 2 \ldots\ldots\ldots \quad 46)$$

and

$$B = \frac{32}{15\,\pi} k\,\sigma_x^3 = \frac{32}{15\,\pi}\cdot \mathrm{k}\,\tau^3 \text{ if } m = 3 \ldots\ldots \quad 47)$$

For three-dimensional stresses $\sigma_x = \sigma_y = \sigma_z$ the normal stress is independent of the direction, and we obtain for the risk of rupture the formula

$$B = (2\,m + 1)\cdot k \cdot \sigma_x^m \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 48)$$

The various values of the coefficients are brought together in the table below.

Table II.

	Pure shearing stress	Tensile stress		
		One-dim.	Two-dim.	Three-dim.
$m = 2$	0.67	1	2.7	5.0
$m = 3$	0.68	1	3.2	7.0

If the ultimate strength is computed from equation 27) using the above coefficients, the following table is obtained for the ultimate strength in the case of the same volume.

Table III. Ultimate Strength for Various Combinations of Stresses.

	Pure shearing stress	Tensile stress		
		One-dim.	Two-dim.	Three-dim.
$m = 2$	1.22	1	0.61	0.45
$m = 3$	1.14	1	0.68	0.52
$m = \infty$........	1.00	1	1.00	1.00

Thus, with the aid of the above formulæ, the principal stresses, irrespective of the number of dimensions, may be reduced to an equivalent stress which coincides with the maximum tensile stress only in the case of one-dimensional stress.

Non-uniform Stresses.

We assume that the solid is loaded by a system of external forces inducing non-uniform stresses, so that the reduced stress σ is a function of the coordinates, or $\sigma = f(x, y, z)$. Now, if all external forces increase in the same proportion, the stresses may be described, so long as the proportionality limit of the material is not exceeded, by multiplying the stress σ_p in an arbitrary point of the solid where $\sigma \neq 0$ (preferably the maximum stress), by a function $f(x, y, z)$, or

$$\sigma = \sigma_p f(x, y, z) \quad \ldots\ldots\ldots\ldots \quad 49)$$

By inserting this value into 8) we obtain for the risk of rupture

$$B = \int n\,[\sigma_p \cdot f(x, y, z)]\, d\,v \quad \ldots\ldots\ldots\ldots \quad 50)$$

Taking a given system and enlarging it in a uniform manner on a linear scale a, we obtain for the risk of rupture the expression

$$B = a^3 \int n\,[\sigma_p \cdot f(x, y, z)]\, dv \quad \ldots\ldots\ldots\ldots \quad 51)$$

This formula takes a particularly simple form if the function n has such properties that it satisfies the functional equation

$$n\,(a \cdot b) = n_1\,(a) \cdot n_2\,(b) \quad \ldots\ldots\ldots\ldots \quad 52)$$

in which case

$$B = a^3\, n_1\,(\sigma_p) \int n_2\,(f)\, dv$$

or, since $\int n_2\,(f)\, dv = c$ has a value independent of σ_p,

$$B = a^3 \cdot c\, n_1\,(\sigma_p) \quad \ldots\ldots\ldots\ldots \quad 53)$$

or

$$S = 1 - e^{-a^3\, c\, n\,(\sigma_p)} \quad \ldots\ldots\ldots\ldots \quad 54)$$

The only function which satisfies 49) is

$$n\,(x) = k \cdot x^m \quad \ldots\ldots\ldots\ldots \quad 55)$$

where k and m are constants. By inserting this into 50) and 51) we obtain

$$B = a^3\, c\, k\, \sigma_p^m \quad \ldots\ldots\ldots\ldots \quad 56)$$

and

$$S = 1 - e^{-a^3\, c\, k\, \sigma_p^m} \quad \ldots\ldots\ldots\ldots \quad 57)$$

The ultimate strength is obtained according to 13)

$$\sigma_b = \int_0^\infty e^{-a^3 c k \sigma_p^m} \cdot d\,\sigma_p = (a^3 c k)^{-\frac{1}{m}} \cdot I_m = (a^3 c)^{-\frac{1}{m}} \cdot I_m \cdot \sigma_0 \ldots\ldots 58)$$

We shall now examine three special cases where the stresses are not uniform, namely bending, torsion, and ball pressure against a plate.

Bending Strength.

In a rectangular beam of the width b and the height $2h$, subjected to a pure bending stress, the stress increases as a linear function of the distance from the neutral axis according to the formula $\sigma = \sigma_p \cdot \frac{x}{h}$.

The risk of rupture B for the length l will then be

$$B = b \cdot l \cdot k\,\sigma_p^m \int_0^h \frac{x^m}{h^m} \cdot d\,x = \frac{V}{2\,(m+1)} \cdot k\,\sigma_p^m \ldots\ldots\ldots\ldots 59)$$

In the case of pure tensile stress the risk of rupture will be $B = V \cdot k \cdot \sigma^m$. Hence it follows that the ratio of the ultimate strength in bending σ_b to the ultimate strength in tension σ_d is

$$\frac{\sigma_b}{\sigma_d} = (2\,m + 2)^{1/m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots 60)$$

For $m = 3$ we have $\frac{\sigma_b}{\sigma_d} = 2.0$ and for

$m = \infty$ we obtain $\frac{\sigma_b}{\sigma_d} = 1.0$

For a circular section with a radius r we have

$$B = \frac{2 \cdot l \cdot k \cdot \sigma_p^m}{r^m} \cdot \int_0^r x^m \sqrt{r^2 - x^2}\,dx \ldots\ldots\ldots\ldots 61)$$

which for $m = 3$ gives the result

$$B = \frac{4}{15\,\pi} \cdot V \cdot k \cdot \sigma_p^3 \ldots\ldots\ldots\ldots\ldots\ldots 62)$$

Thus, the ultimate strength in bending of circular cross-section is, as compared to tension, for $m = 3$

$$\frac{\sigma_b}{\sigma_d} = 2.28 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 63)$$

Torsional Strength.

A circular rod of the radius a and the length l is subjected to torsion, the shearing stress in the surface layer being τ_p. At the distance x from the neutral axis the stress is

$$\tau = \tau_p \cdot \frac{x}{a}$$

and, consequently, the risk of rupture

$$B = \frac{2}{m+2} V \cdot k_1 \tau_p^m \ldots\ldots\ldots\ldots\ldots\ldots \quad 64)$$

For $m = 3$ we have $B = 0.4\, k_1 V \tau_p^3$ and as $k_1 = 0.68\, k$ according to 44), we obtain

$$B = 0.27 \cdot k \cdot V \cdot \tau_p^3 \ldots\ldots\ldots\ldots\ldots \quad 65)$$

The torsional-ultimate strength of a circular cross-section is, as compared to tension, for $m = 3$

$$\frac{\sigma_t}{\sigma_d} = 1.55 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 66)$$

Compressive Strength in the Case of Contact Between Ball and Plate.

For this type of load the equation 58) cannot be used in its original form seeing that the area of the contact surface varies with the load. In order to come to a result, we assume at first that the contact surface is a constant with a radius a, and that the stress increases while the distribution of pressure is that indicated by HERTZ. The ultimate strength will then be according to 58)

$$\sigma_b = a^{-\frac{3}{m}} \cdot c^{-\frac{1}{m}} \cdot I_m \cdot \sigma_0$$

In the above formula σ_0 and I_m are material constants and c a constant determined by the distribution of stress. Thus, we have

$$\sigma_b = k_3\, a^{-\frac{3}{m}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 67)$$

This relationship between ultimate strength and radius of the contact surface can only be obtained on condition that the ball radius R has a definite value which must satisfy the condition

$$a = k_0 \cdot R \cdot \sigma_b \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 68)$$

By eliminating σ_b from 67) and 68) we obtain the relationship between a and R for the ultimate strength

$$a = k_4 \cdot R^{\frac{m}{m+3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 69)$$

According to HERTZ we have

$$a = a_1 \sqrt[3]{P \cdot R} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 70)$$

where P is the load and a_1 a constant determined by the modulus of elasticity of the material. Finally, we obtain the breaking load

$$P = k_5 R^{\frac{2m-3}{m+3}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 71)$$

It follows from this formula that if $m = \infty$, then $P = k_5 R^2$ in accordance with the requirements of the classical theory. If, on the other hand, $m = 6$, we obtain

$$P = k_5 \cdot R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 72)$$

which agrees with Auerbachs experimental tests on glass plates.

Material Function $n(\sigma)$.

In the above we have defined the function $B = \int n(\sigma)\, dv$ or $n(\sigma) = \frac{dB}{dv}$ and taking it as a starting-point, we have derived the formulæ in a purely phenomenological way. Yet an illustrative picture may be drawn of the function $n(\sigma)$ which, for brittle materials, is closely connected with GRIFFITH's amd SMEKAL's hypotheses. In fact, assuming that there are several weak places in the volume, either in the shape of cracks according to GRIFFITH, or »dislocations» in the atomic grid according to TAYLOR, so that the strength is diminished; assuming, furthermore, that all these weak places are of such nature that they give rise to rupture as soon as they fall within a volume subjected to the stress σ, and supposing, moreover, that there are n such weak places in the unit of volume and that the stress σ is concentrated within the small volume dv, then the probability of rupture will be $dS = n \cdot dv$. If, instead of this, we have p elements dv and the probability of rupture is S, then the probability that rupture will *not* occur is

$$1 - S = (1 - ds)^p = (1 - n\, dv)^p$$

or

$$S = 1 - (1 - n \cdot dv)^p$$

The total volume subjected to stress is then $p \cdot dv = V$ and

$$S = 1 - \left(1 - \frac{nv}{p}\right)^p = 1 - \left((1 - \frac{nv}{p}\right)^{\frac{p}{nv} \cdot n \cdot v}$$

Now, if p increases infinitely, while dv decreases at the same time, so that v remains constant, we obtain

$$S = 1 - \lim_{\frac{p}{n \cdot v} = \infty} \left(1 - \frac{n\,v}{p}\right)^{\frac{p}{n \cdot v} \cdot n\,v} = 1 - e^{-n \cdot v} \qquad 73)$$

It is to be noted that if $n \cdot v = 1$, i. e. if the stressed volume is equal to the mean volume disposed by each weak place, then the probability of rupture is only 63 per cent, and that in order to reach a probability of 99 per cent, the stressed volume must be 4.6 times that of the mean volume.

It may be seen that equation 70) agrees with equation 9) and that $n(\sigma)$ may thus be interpreted as that number of weak places per volume unit, which causes rupture at a stress equal to, or less than, the amount σ.

Consequently, the function $n(\sigma)$ must be a monotonously increasing function of the variable σ.

The formulæ which we have derived are of purely formal nature and do not gain any real significance unless it is possible to determine by experiment a material function $n(\sigma)$ whose form may with sufficient certainty be supposed to be valid for a given material, due regard being paid perhaps to the previous story of the material, such as working, heat-treatment, etc. In this connection it is, of course, not necessary to stipulate that the material function should be constant throughout the whole volume of the solid, just as the classical theory of strength is not able to maintain the assumption that the strength properties of all volume units are equal, irrespective of their position in the solid. Therefore, it will probably in many cases be found necessary to generalise the formulæ derived here for isotropic materials, so that they also apply to anisotropic materials. This does not present any mathematical difficulties. It may now be questioned, what advantages may be expected from the new theory, or why we should be forced to put up with the complications arising from it, or whether it should not be better to operate with variable strength properties which may be adapted to experimental results, as has been done heretofore. An answer to these questions is easy enough to find in those cases where the new theory has proved able to explain measuring results which so far have been interpreted as proofs of non-isotropy, because in this way an extensive group of phenomena may be ranged under a uniform and simple conception, so that certain conclusions may be drawn without having recourse to calculations, as, for instance, that the ultimate strength in bending is higher than in torsion, while the latter, in its turn, is higher than in pure tension, or that two and three-dimensional stresses involve a greater risk of rupture than one-dimensional stresses. If, on the other hand, the lack of isotropy must be assumed in addition to the new theory, then the answer is perhaps not quite so obvious. Yet it seems to me that it is brought forth by a discussion of a concrete case, for instance testing of cast-iron. There is no doubt that if we cast two test specimens, the one, say, 10 mm and the other 30 mm in diameter, then the properties of the material will not be the same in both cases. A tensile test will show that the ultimate strength of the thicker rod is lower than that of the thinner one. This circumstance cannot be solely explained by the new theory because by determining such properties as e. g.

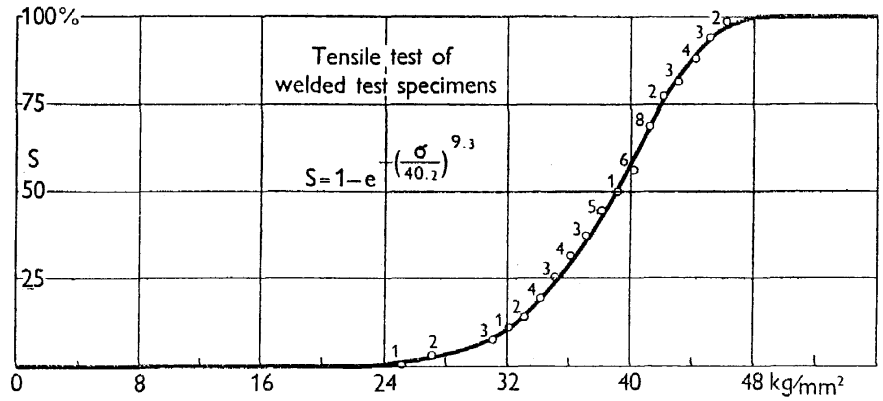


Fig. 5.

BRINELL hardness and elasticity, it may nevertheless be found that the material is not identical in these two cases. Why, then, should we not content ourselves with calculating on the basis of the old figures derived from experience? The answer is, as I hope to have demonstrated, that the volume, too, undeniably influences the measuring results. If these two effects may be segregated, so as to eliminate the volume effect, then the strong dispersion of the measuring results will be reduced, and it will be easier to investigate the laws determining the dependence of the material on the preparation and on the dimensions. An immediate consequence of this is the strict requirement that in testing materials the measurements should be made on specimens of definite length, or, at any rate, that data should be given on the length used and other dimensions which are unessential for the classical theory, but are important for the new theory. Unfortunately, this requirement is not complied with at the present time, so that only a small portion of the exceedingly abundant experimental data available may be made use of for the purpose of ascertaining the new theory. At the present moment it is therefore not possible to submit experimental results providing conclusive evidence for the practical importance of the theory. This will require extensive and lengthy work partly carried out along new lines.

Experimental Confirmation of the Theory.

The following brief report on a couple of experimental investigations does not claim to be regarded as conclusive evidence for the practical applicability of the theory.

The first problem to be examined is of practical rather than fundamental nature. In the above the general formulæ have been specialised by putting the material function $n(\sigma) = \left(\frac{\sigma}{\sigma_0}\right)^m$. For the theory in itself it is of no special significance whether the material function has this form or any other, but this particular form is mathematically simple, and the material is characterised by two constants σ_0 and m which is the least possible number, since *one* single constant, namely the ultimate strength, has proved to be insufficient.

Fig. 5 shows a distribution curve $S = 1 - e^{-k\,\sigma m}$ which is plotted as follows. If N is the number of measurements, the ordinate S between 0 per cent and 100 per cent is divided into N intervals whose centres determine the ordinates for the corresponding measuring points which have the figures of ultimate strength or breaking load respectively as abscissæ. If several measurements give the same results, only the mean value of the ordinates is plotted in the graph. The figure placed at each circle indicates the number of observations represented by the point. In this way it is easily found that

$$\int S\,d\sigma = \frac{\sum \sigma_n}{N}$$

The distribution curve in Fig. 5 is of interest because it represents the ultimate strength of a ductile material, namely the tensile tests on welded 57 test pieces, so-called »ASEA-Rods». The size of these specimens is $250 \times 34 \times 8$ mm, and they are provided with two drilled and reamed 5 mm holes located in the weld-joint at a distance of 6 mm from the edge, while the distance between the holes is 12 mm. The basic material is St. 37. The weld was made by means of 3 mm electrodes.

It would hardly be possible to obtain a better agreement between the measuring-values and the computed curve by means of any other probability function.

In a very careful investigation KUNTZE[1] has determined the compressive, tensile, torsional, and bending strength of 4×25 specimens made of chemically pure stearic acid and fine-sifted plaster-of-Paris mixed in the proportion 72 to 28. After heating and thorough mixing, the vessel containing the melt

[1] KUNTZE, Ein Beitrag zur Festigkeitslehre, Ann. Phys. 11, 1903, p. 1020.

heated to 120° C was evacuated during a considerable period of time in order to eliminate air-bubbles. After solidifying the material was hard and brittle and showed a fine-grained fracture surface. With the exception of the specimens used for the compressive tests, all rods were 50 cm in length. With a view to obtaining the rupture in the middle of the specimen, the rods were turned to a somewhat smaller diameter in the middle than at the ends. The specimens employed for the torsional tests had a cylindrical middle part of some length. Thus, the volume of these specimens was slightly less than that of the other rods. All 100 measured values are reported. Here we reproduce only the complete tensile test series (Table IV) and the figures for the ultimate strength in tension, torsion, and bending. As regards the other details, reference is made to the original paper.

Tab. IV. Tensile Tests on Specimens made of Stearic Acid and Plaster-of-Paris.

No.	Area mm²	σd kg/mm²	No.	Area mm²	σd kg/mm²
1	21.50	0.61	14	23.10	0.58
2	22.31	0.60	15	21.91	0.62
3	23.76	0.50	16	23.23	0.50
4	15.18	0.63	17	25.80	0.50
5	22.03	0.48	18	20.68	0.52
6	22.79	0.60	19	15.90	0.59
7	28.88	0.56	20	16.47	0.50
8	17.79	0.59	21	18.75	0.54
9	23.60	0.60	22	17.91	0.55
10	28.51	0.52	23	25.55	0.50
11	25.93	0.57	24	22.03	0.48
12	22.97	0.55	25	20.96	0.47
13	21.03	0.59			

Compression $\sigma_{tr} = 3250 \cdot 10^5 \frac{\text{gr.}}{\text{cm sek.}^2}$

Tension $\sigma_d = 540 \cdot 10^5$ »

Torsion $\sigma_t = 793 \cdot 10^5$ »

Bending $\sigma_b = 986 \cdot 10^5$ »

All tensile fractures were located at the smallest diameter of the rod and showed a smooth fracture surface at right angles to the tensile force. Table IV shows that the cross-sectional area varied from 15.18 to 28.88 mm². It is therefore possible, to compute the relationship between ultimate strength and cross-sectional area and hence also the coefficient m.

From equation 34) it follows that

$$\sigma = k \cdot V^{-\frac{1}{m}} \qquad 74)$$

By computing k and m according to the method of least squares, we obtain

$$\frac{1}{m} = \frac{\sum \log V_v \sum \log \sigma_v - n \sum \log V_v \cdot \log \sigma_v}{n \sum \log^2 V_v - (\sum \log V_v)^2} \quad \ldots\ldots\ldots \quad 75)$$

By inserting the 25 figures from Table IV into equation 75) we have $m =$ $= 5.14$.

For $m = 5$ we obtain from 64) $\frac{\sigma_t}{\sigma_d} = 1.39$ and from 61) $\frac{\sigma_b}{\sigma_d} = 1.83$.

From σ_d we compute σ_t and σ_b. In Table V they are compared with the values measured by KUNTZE. The agreement is excellent and would be still better if regard had been paid to the smaller volume of the torsional specimens, which would result in a slight increase of the figure 750. It is to be noted that the purpose of KUNTZE's investigation was precisely the checking of the various theories of strength, but he found that none of them agreed with the observations, and that it was in general impossible to find any linear combination of the principal stresses or strains, which at the ultimate strength would reach an individual limit characteristic of the material under test.

Table V.

	Ultimate Strength	
	Observed	Calculated
σ_d	540	540
σ_t	793	750
σ_b	986	990

It may be observed, however, that there is a lack of agreement between the theory and the measuring results in respect of the compressive strength. While the measured figure is much higher than the tensile strength (6 to 1), yet according to the new theory, the compressive strength ought to have been infinitely high. The fact of the matter is that on account of the difference in the elasticity of the testing-piston and the test specimens and owing to the tangential forces due to this difference, tensile stresses are produced in the test specimen, which may be the direct cause of rupture. In the case of other brittle materials the measured ratio of compressive strength to tensile strength is within the limits of 9 to 25. In one isolated case[1] even a figure of 50 was measured. It is not improbable that the high values occur in those cases where the elimination of the tensile stresses was more complete, and that pure compressive stresses are without influence.

[1] BRIDGMAN, Phil. Mag. 24, 1912, p. 63.

Mention should also be made of a study by PETERSON[1] which speaks in favour of the theory. This investigator thought that the maximum stresses in hollow chamfers, threads, etc. could be determined by means of plaster-of-Paris specimens. The observed results showed deficient agreement with the calculations made according to the formulæ of the theory of elasticity. Here we shall only reproduce the results of the tensile tests on flat specimens provided with holes. It follows from the theory of elasticity that if the diameter of the hole is small in comparison with the width of the specimen, then the

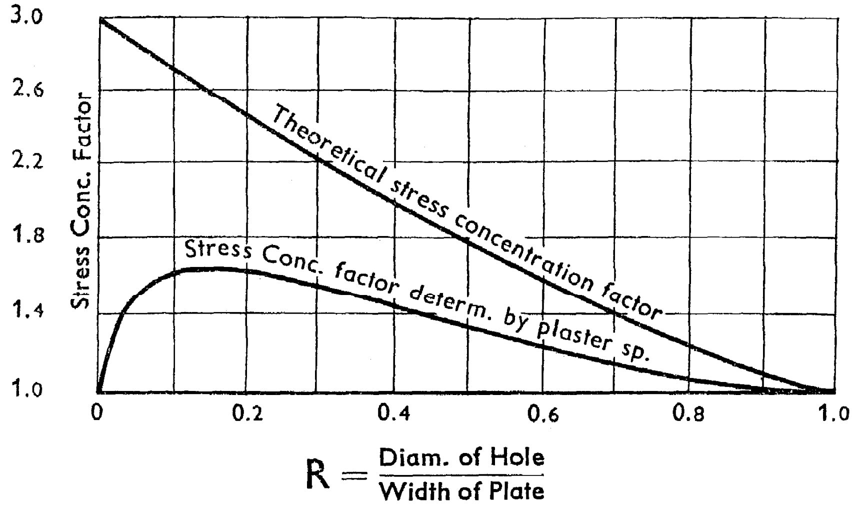


Fig. 6.

maximum stress in the chamfer edge is 3 times the mean stress in the whole sectional area. The test results are shown in the curves of Fig. 6. A calculation of the risk of rupture for this very intricate distribution of stresses would be rather tiresome, but it is not difficult to see that the new theory would give as a result a curve which, on the whole, would agree with the measuring results seeing that the greater the reduction of the hole in relation to the width of the specimen, the less will be the over-stressed volume, so that a very small hole would exert no influence upon the ultimate strength in spite of the fact that the stresses in the chamfer edge may in reality be thrice as high as the mean stress, as is shown by the elasticity calculations.

It seems to follow from the two above-mentioned examples that, at least for some materials, the new theory gives results which agree well with actual observations.

[1] PETERSON, An Investigation of Stress Concentration by Means of Plaster-of-Paris Specimens. Mech. Engng. 48, 1926, p. 1449.

Part II. Empirical Data.

The formula set up in Part I are based on purely theoretical speculations. The experimental data available at that time in a form in which they could be used to verify the formulae, were not sufficiently comprehensive. The test results dealt with in Part II are primarily intended to demonstrate the existence of a distribution function characteristic of each individual material and are used to establish the mathematical form of that function, whereas no attempt is made to advance experimental evidence for the correctness of the strength formular developed in Part I.

The theoretical formulae of static strength are based on the assumption that if the formation of cracks begins in any of the volume elements of the body, the cracks will spread over the whole sectional surface or, in other words, that an event taking place in an element is identified with the same event taking place in the body as a whole. This assumption, and hence also the formulae previously derived, may therefore be applied to the study of many other physical phenomena, among which mention may be made of electrical discharge in gases, liquids, or solids, fatigue phenomena, and transition from upper to lower yield point in the case of steel, etc.

In what follows the material function has been computed from the experimental figures by graphic methods only. This procedure has been adopted for the reason that even if the constants may readily be determined by arithmetical methods, these methods do not allow of the same clear survey of the question as the graphic methods. In the case of distribution curves for Series 2 and 10, for instance, an arithmetical computation of the constants would result in a compromise between the constituent elements of the curves, which appear clearly in the diagrams. It is, therefore, always convenient to interpret the measuring series at first graphically, with a view to obtaining a general notion of the character of the curve, and to ascertain that the test results are statistically homogeneous. This should always be done before using some arithmetical method for the computation of the constants, if desired, for instance the moment method. It is, however, inadvisable to compute the constants with an accuracy which is not supported by any physical reality, as is shown in fig. 3.

The system of coordinates which is best suited to the computation of distribution curves follows from, and is motivated by, equation 9) according to which the probability of rupture is $S = 1 - e^{-\int n(\sigma)\,dv}$. For a uniform distribution of stresses throughout the volume we have $S = 1 - e^{-V \cdot n(\sigma)}$ and, consequently,

$$\log \log \frac{1}{1-S} = \log n(\sigma) + \log V \quad \ldots\ldots\ldots\ldots\ldots \quad 76)$$

Hence it follows that if loglog $\frac{1}{1-S}$ is plotted as an ordinate and an arbitrary function $f(\sigma)$ as an abscissa in a system of rectangular coordinates, then a variation of the volume V of the test specimen will only imply a parallel displacement, but no deformation of the distribution function. This circumstance, of course, facilitates the study, especially if the material function assumes the form

$$n(\sigma) = \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m \quad \dots\dots\dots\dots\dots\dots\dots \quad 77)$$

and if we take $f(\sigma) = \log(\sigma - \sigma_u)$ as abscissa, since in that case we obtain

$$\log\log\frac{1}{1-S} = m\log(\sigma - \sigma_u) - m\log\sigma_0 + V \quad \dots\dots\dots \quad 78)$$

and the distribution function will be linear.

The application of this method presupposes the knowledge of the constant σ_u which, as a rule, is not known. In this case it is advisable to plot the test figures at first in the system of coordinates loglog $\frac{1-S}{1}$ and $\log\sigma$. If the figures happen to follow a straight line, as for example in Series 3a, this implies obviosuly that $\sigma_u = 0$. If, on the other hand, the test figures are located on a curved line (as in Series ld), it follows that $\sigma_u = 0$, and a tentative value is taken for σ_u. If this value is too high, the curve will be bent in the opposite direction (concave upwards), and further tentative values must be taken, until the curve approximates a straight line as closely as possible. It will be found that after some experience a few attempts according to this method suffice to bring forth results and that the curvature of the curve is very sensitive to variations of σ_u, provided that the test series is passably large, so that the value of σ_u may be determined with a degree of accuracy corresponding to the scope of the test series. Thereafter, the two remaining constants m and σ_u may be readily computed from the curve rectified in the above-described manner. This method has been applied to the test data dealt with below.

A brief description of the method used for the determination of S may also be necessary. The distribution curve shown in fig. 5 has been obtained by means of method which, as a rule, is employed in statistics, and has been described on Page 21. Upon closer examination it will be found that uniform distribution of the test figures over the S-axis is not self-evident or, in other words, that in a series of $N = 100$ the first value will have an ordinate $S = 0.5$ per cent, and the hundredth $S = 99.5$ per cent. To elucidate this question, it is necessary to determine the distribution functions for the individual values n in a series of N values taken by random sampling from a large universe. Then, the medians may be easily computed from these curves. This method has also been applied to the following curves whose ordinate is denoted by loglog $\frac{1}{1-S}$. These values of S must be computed for each value of n and N. This procedure is very slow if the values of N are high. Now, it appears as much justified to take the arithmetical mean value instead of the median. From a practical point of view, the arithmetical mean offers at any rate an important advantage, namely that the work of calculation is materially reduced because the arithmetical mean S_n for the n-th value follows the simple formula

$$S_n = \frac{n}{N+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 79)$$

and thus we have

$$\frac{1}{1-S} = \frac{N+1}{N+1-n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 80)$$

This formula was used in all those cases where the ordinate is denoted by loglog $\frac{N+1}{N+1-n}$.

The graphic method used in this investigation may be summarised as follows. The N values of a series are arranged according to the increasing values of σ. Assuming that o denotes the rank and σ_n the n-th measured value, so that n passes through all integral numbers from 1 to N, loglog $\frac{N+1}{N+1-n}$ is plotted as an ordinate of the n-th value and $\log(\sigma_n - \sigma_u)$ as its abscissa in a system of rectangular coordinates, σ_u being a constant chosen in such a way that the N points are located as closely as possible to a straight line. The two remaining constants σ_0 and m are computed from this straight line.

In the case of small or moderately large series ($N \leqq 150$) and where the accuracy of measurement was sufficient, all measuring points were plotted in the diagram. Where large series are concerned, the data must be divided into suitable class intervals. For such grouped data it does not matter whether the mid-value or the upper limit of a class interval is taken to be the value of σ_n. Any variation in this respect merely results in a variation of σ_u by a half class interval. This may be seen from Series 3 where the values in 3a) are mid-values and in 3b) upper limits of the interval. In this case the class-interval was 1 gr, so that the theoretical difference should have been 0.5 gr. while the graphic method actually gave 0.46 gr.

Detailed Data concerning the Various Measuring Series.

Series 1. Bending Strength of Cylindrical Porcelain Rods.
a) White glazing. $N = 102$. Mean diam. 19.0 mm.
b) Green glazing. $N = 102$. Mean diam. 19.2 mm.
c) Unglazed. $N = 102$. Mean diam. 18.6 mm
d) Brown glazing. $N = 106$. Mean diam. 19.1 mm.
e) Comparison of the four curves.

All these tests were made at the AB Iföverken with the aid of an ordinary testing machine. The distance between the outer points of support was 100 mm. The load was applied in the centre. The porcelain and the glazings were those commonly used in the factory. The temperature of the oven was determined by means of Seger cones nos. 13 and 14.

Series 2. Tensile Strength of Limhamn Standard Portland Cement.
$N = 680 + 1\,082 + 1\,106$.

These tests are daily routine tests made for about a year at the Limhamn Factory of the Skånska Cementaktiebolaget. They were made according to Swedish standard rules using German standard sand (1 part of cement to 3 parts of sand). The test specimens were stored in water for 2, 7, and 28 days respectively.

Series 3. Fibre Strength of Indian Cotton.
a) Strength of single fibres of Surat 1 027 A. L. F. N = 1 000.
b) Ditto. $N = 3\,000$.

The figures are extracted from a paper by Koshal and Turner »Studies in the Sampling of Cotton for the Determination of Fibre-Properties» published in Journ. of Text. Inst., Vol. XXI, 1930, pp. 325—370. The apparatus used for obtaining the breaking load was Baratt's Fibre Balance. Each hair was mounted on two eyelets by means of a mixture of sealing wax and paraffin wax, a length of about 2 mm being embedded in the wax at each end; except for this, the full length of the fibre was used in each test. The fibre length was distributed as shown in Series 4. The figures in 3a for Pearson's curve type 1 are reprinted from the above-mentioned paper.

Series 4. Fibre Length of Indian Cotton.
$N = 3\,000$. The figures are taken from the paper by Koshal and Turner (see Series 3). The length was measured by means of a telescope with a micrometer eyepiece.

Series 5. Breaking Strength of Cotton Yarn.
$N = 1\,914$. The figures are taken from a book by Pearson »The Application of Statistical Methods to Industrial Standardisation and Quality Control». Brit. Standard Inst. no. 600—1935, p 125.

Series 6. Breaking Strength of Cotton Fabric Strips.
$N = 128$. The figures are reprinted from a paper by Turner »Random and Systematic Selections of Warp Specimens in Cloth Sampling», Journ. of Text. Inst., Vol. XXII, 1930, p. T 77.

Series 7. Strength of Clear Wood, Green Sitka Spruce.
$N = 1\,295$. The figures are drawn from Newlin, »Unit Stresses in Timber», Proc. Am. Soc. Civil Eng., 52, 1926, pp. 1436—1443. Results of modulus of rupture tests on small specimens of Sitka Spruce, a material extensively used for aeroplane propellers during the War. The moisture content varied during the tests. This circumstance may be assumed to account for the irregularities of the curve.

Series 8. Ultimate Strength of Stearic Acid and Plaster-of-Paris. $N = 3 \times 25$. Figures from Kuntze, "Ein Beitrag zur Festigkeitslehre", Ann. Phys., 11, 1903, p. 1 020.
a) Curves based on Kuntze's figures.
b) As has been shown in Part I, p. 23, the coefficient m may be computed from the tensile test series thanks to the correlation between volume and tensile strength. This gives $m = 5.12$. The same value of m is obtained, and the curves become rectilinear according to 12b, if it is assumed that 25 per cent of the lowest values and 25 per cent of the highest ones were rejected in the tests. Regarding the curves which were plotted as if the 12 values below the given series of 25 and the 13 values above that series were rejected, these are rectilinear, they have a slope indicated by the measured mean values, and are displaced in relation to one another as required by the theory developed in Part I.

Series 9. Tensile Strength of Aluminium Die Castings.
$N = 60$. The figures are drawn from Shewhart's »Economic Control of Quality of Manufactured, Product». 1931. Table 3, p. 42

Series 10. Tensile Strength of Malleable Iron Castings.
$N = 15 + 60$. Figures from Pearson, Brit. Stand. Inst., no. 600, 1935. The original series comprised 75 figures resulting in an irregular curve. It was found that the test specimens had come from two separate factories, 15 from the one and 60 from the other. After division two statistically homogeneous curves were obtained.

Series 11. Tensile Strength of Drilled Test Specimens.
$N = 57 + 20 + 20$. The welded test specimens, 57 in number, were made at the Welding School of the ASEA Co. by 19 pupils. The two further series were made by the AGA Co. and comprise non-welded specimens made of St 37. The specimens of the one series had the same size as in the previously

mentioned case, while the other was half-scale size. In the last-mentioned case the volume is 1 to 8 of the normal volume, so that according to our theory this curve should be located below the foregoing curve at a distance of log 8, as is shown in the figure. Circles indicate measured values.

Series 12. Spark Voltage Between Spherical Electrodes.

The measurements were made at the ASEA Laboratory, Västerås. Diameter of the spheres = 10 mm. Distance between spheres = 1 mm. The figures represent the spark voltage for sinusoidal alternating current.

a)) $N = 30$. In these tests the spark gap was immersed in a tank containing about 1 m^3 of oil thoroughly stirred in the intervals between the subsequent tests. Nevertheless, a certain rank correlation is noticeable in the figures, probably due to contamination by soot.

b) $N = 30$. In these tests the spark gap was in air. Sinusoidal alternating current was used.

Series 13. Spark Voltage Between Spherical Electrodes.

$N = 30$. The measurements were made at the Electrical Laboratories of the Royal Technical University, Stockholm. The figures represent spark voltages for sinusoidal alternating current. The lowest figure is the first in the measuring series, so that its value has probably been influenced by some foreign body (dust grains etc.). The two other low values are likewise in the beginning of the measuring series.

Series 14. Spark Gap Voltage.

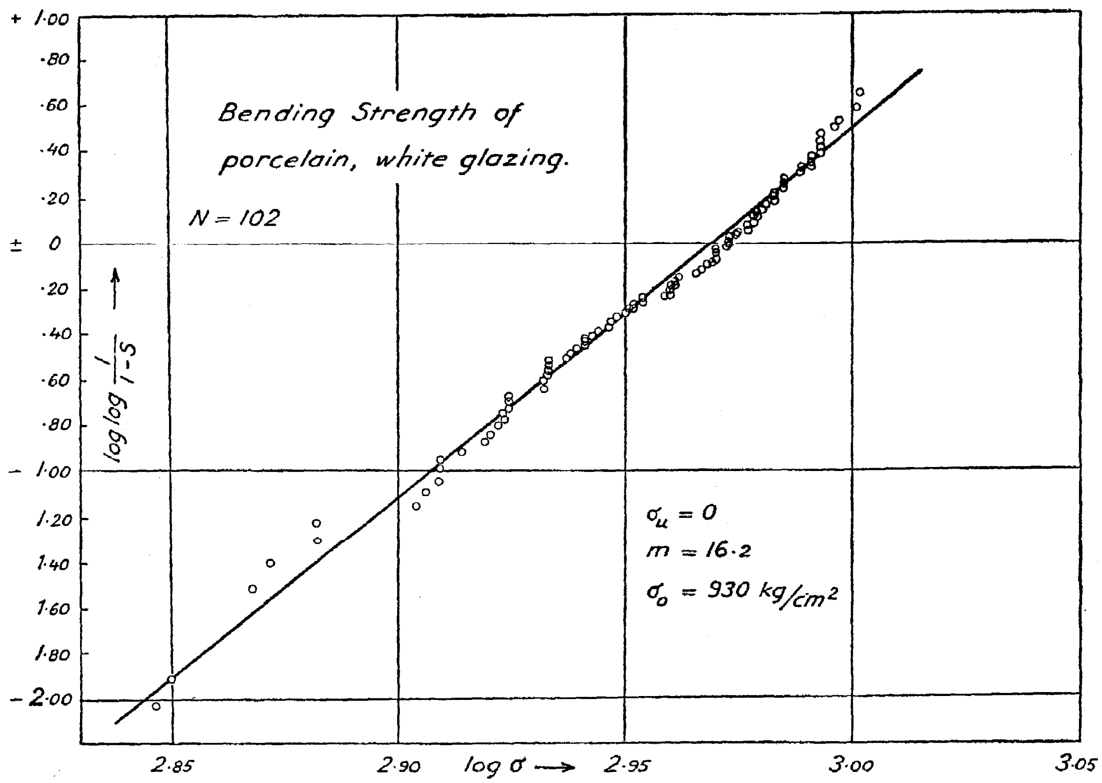
The data on spark voltage for sphere diameters from 20 up to 2 000 mm and on the distance between spheres, which is half the sphere diameter, are taken from the »Förslag till Normer för Spänningsprov» published in May 1938 by Svenska Teknologföreningen (the Swedish Association of Engineers and Architects). Accordingly, the values correspond to the same geometrical uniform and after applying to this phenomenon the theory set up in Part I, it is found that if a distribution function according to 24) is used, then $\frac{E}{l}$ is proportional to $l^{-\frac{3}{m}}$ in accordance with equation 28). E is the spark voltage in kilovolts and l the distance between spheres in cm. The curve shows good agreement with the exception of the smallest distance between electrodes. This is probably due to the fact that the spark voltage has a lower limit, so that equations 32) and 34) ought to be used. If σ_1 is small, it does not exert any influence at higher voltages, but it can manifest itself at the comparatively low voltage which is required for flashover in the case of a sphere 20 mm in diameter, the distance between electrodes being only 10 mm.

Series 15. Fatigue tests on valve-spring wire.

$N = 30 + 30$. The tests were made at the SKF Laboratory, Göteborg, with the aid of a rotating-wire fatigue machine. Taking Part I of this paper as a starting-point, a fatigue theory may be devoloped according to which the system of coordinates used in the figure will give a linear distribution function. In this case σ_1 is a determinate function of the alternating tension applied to the specimen, while T denotes the endurance of the specimen. An increase in volume of the test specimen results in an upward parallel displacement of the curve.

The experimental data dealt with here have for the most part been obtained tanks to the kindness of the following companies, viz., AB Svenska Kullagerfabriken, Allmänna Svenska Elektriska AB, AB Gasaccumulator, Skånska Cementaktiebolaget, AB Ifőverken. I wish to express my gratitude to many prominent engineers of these companies for their friendly interest and assistance in collecting experimental results. I would like especially to mention Chief-engineer OLOF LUND and Dr.-Ing. LENNART FORSÉN in this connection. I also wish to thank Miss Wiwica Weibull for the very comprehensive work performed in computing the necessary tables and in the mathematical treatment of the test results.

Series 1 a



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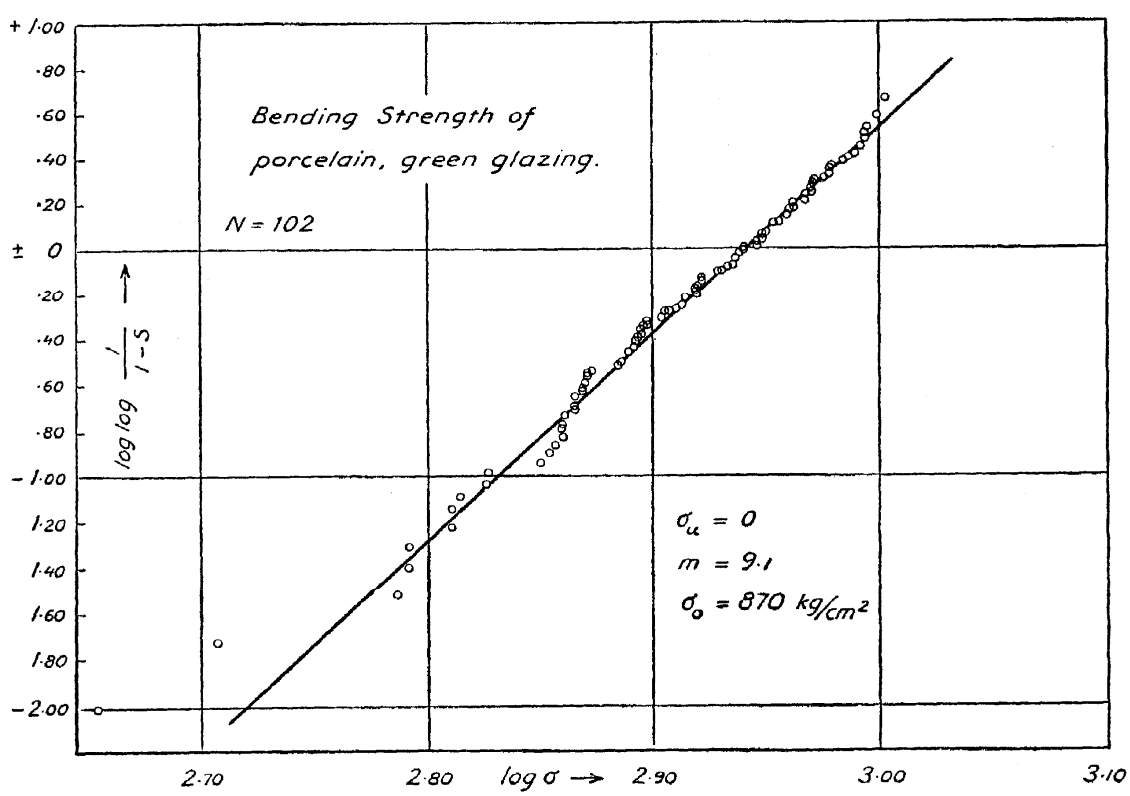

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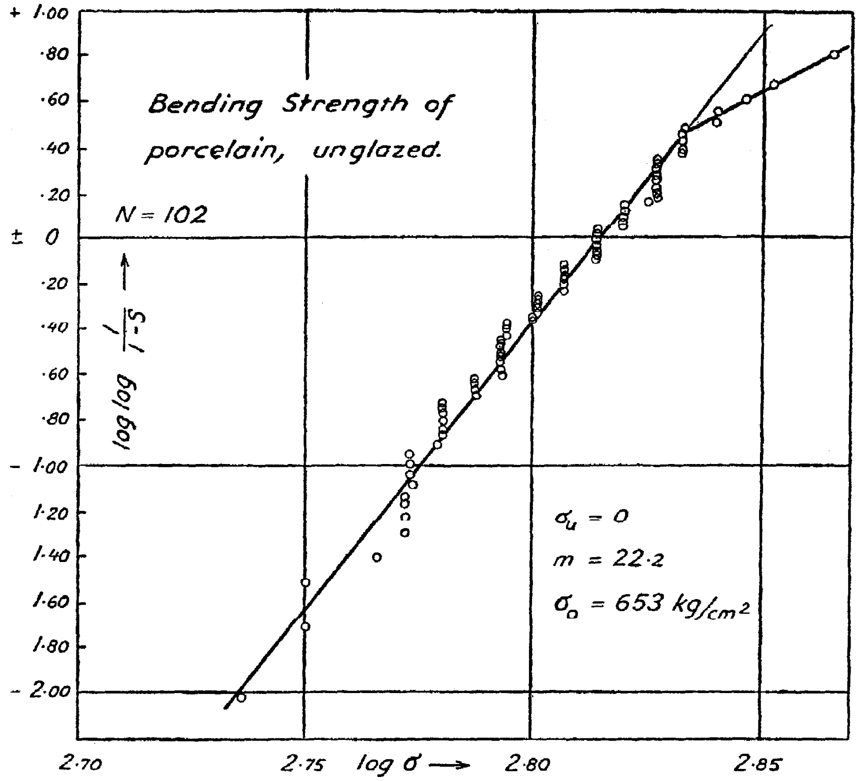

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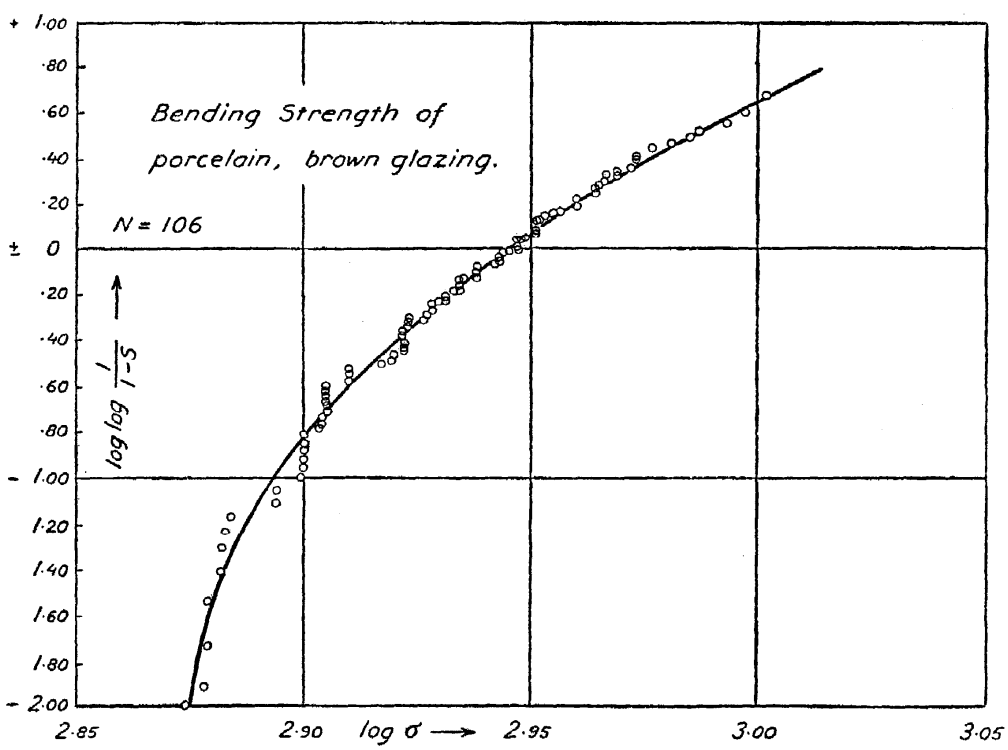

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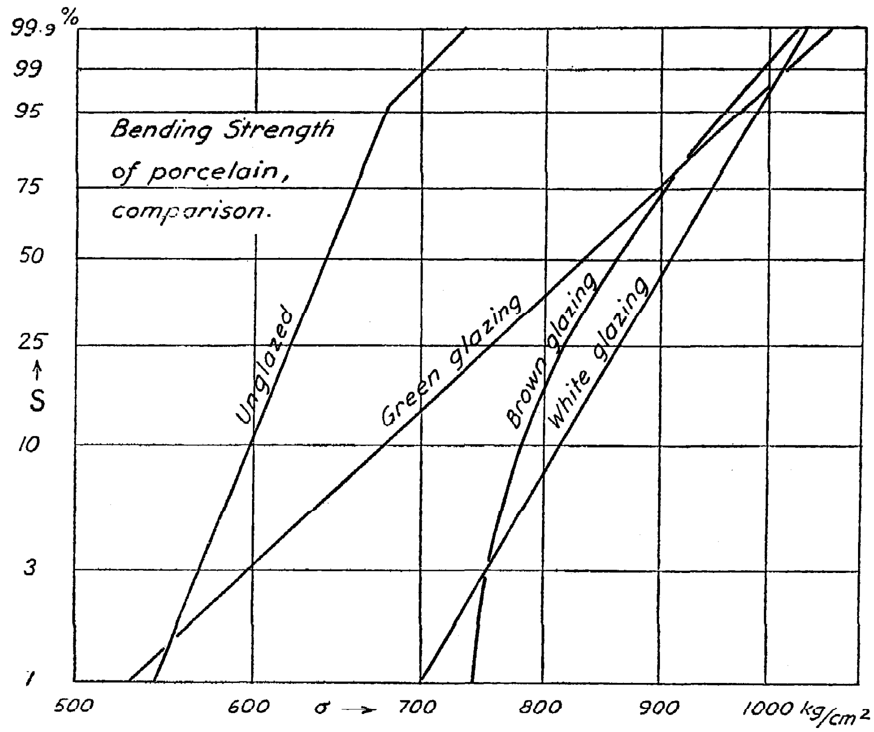

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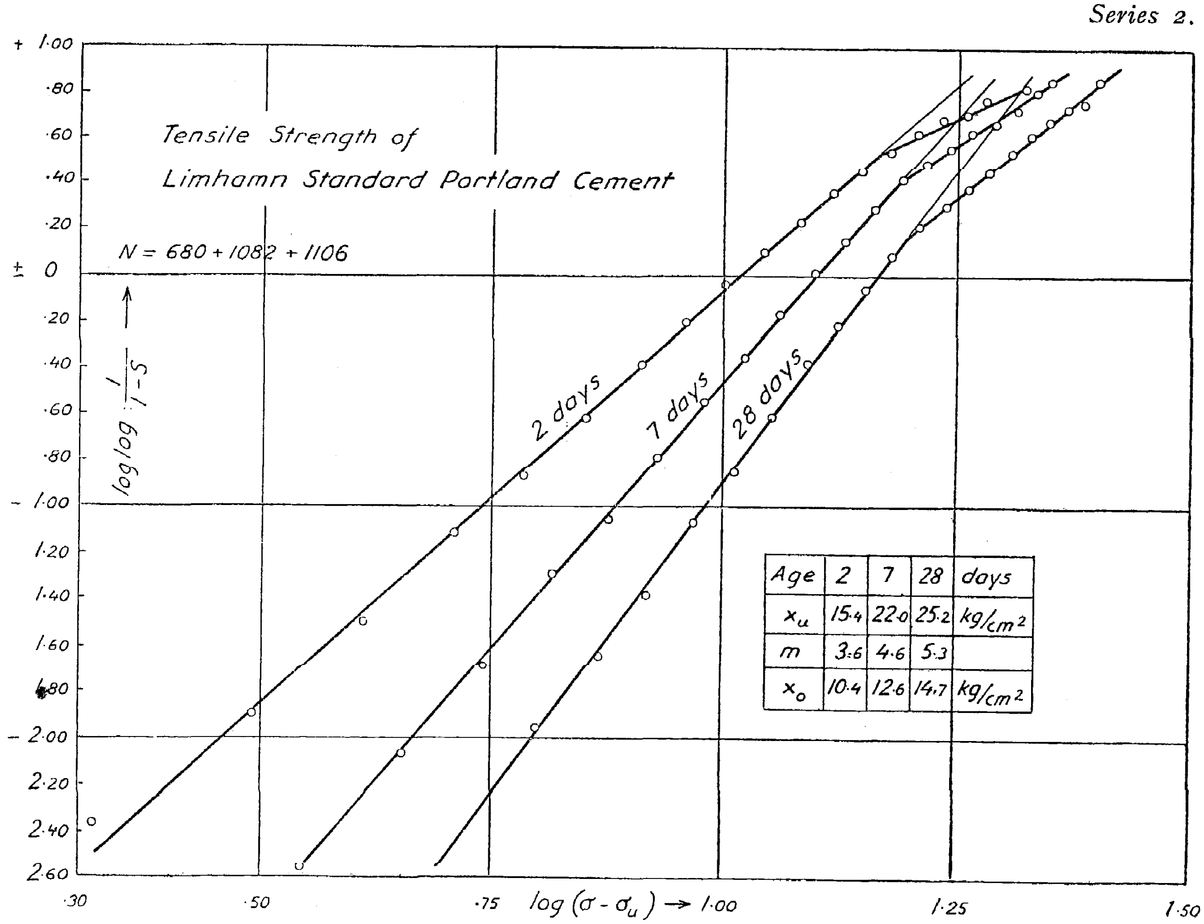

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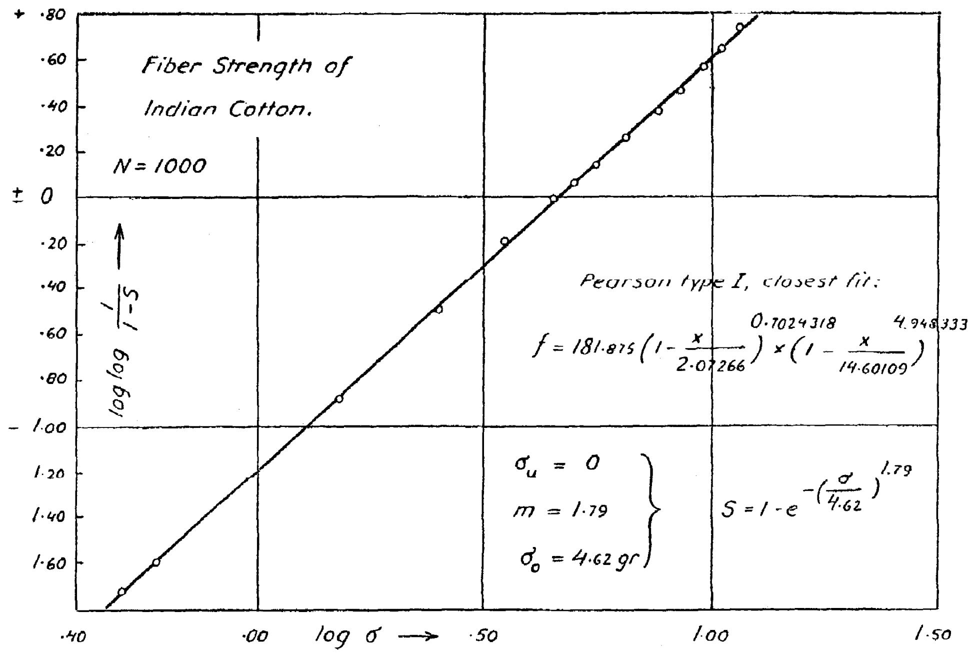

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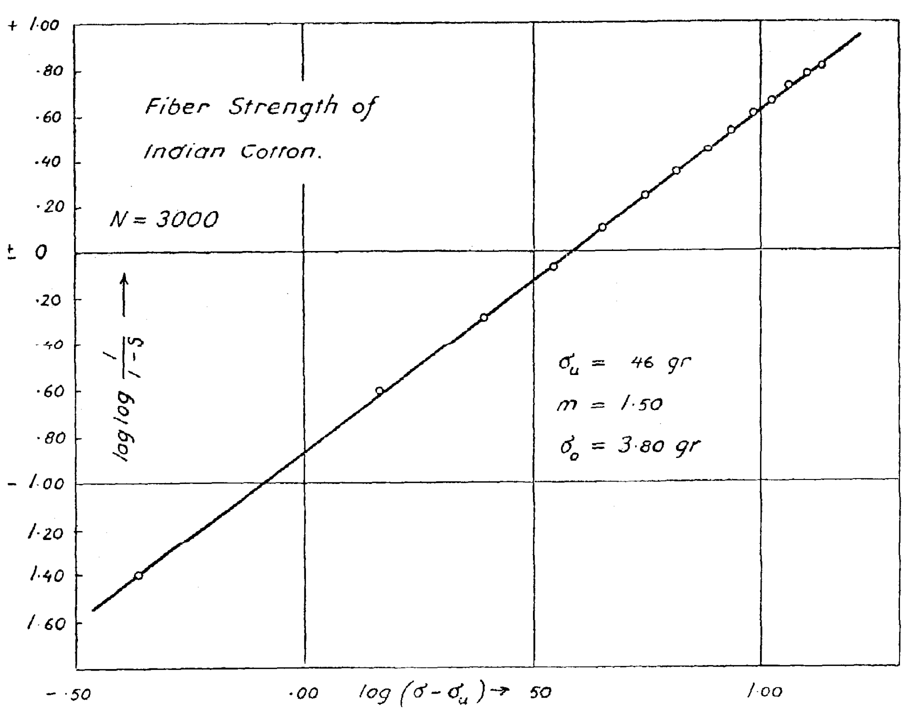

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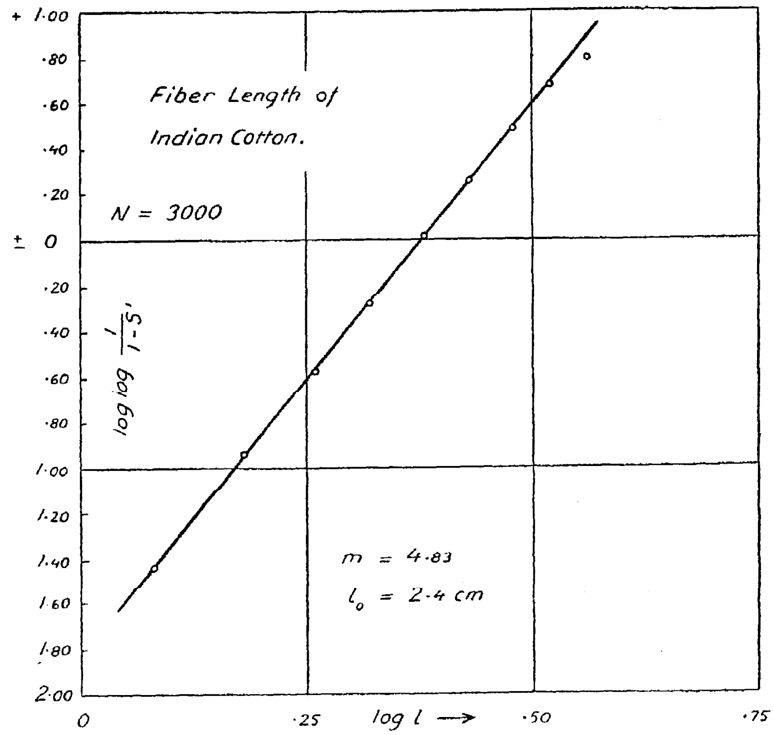

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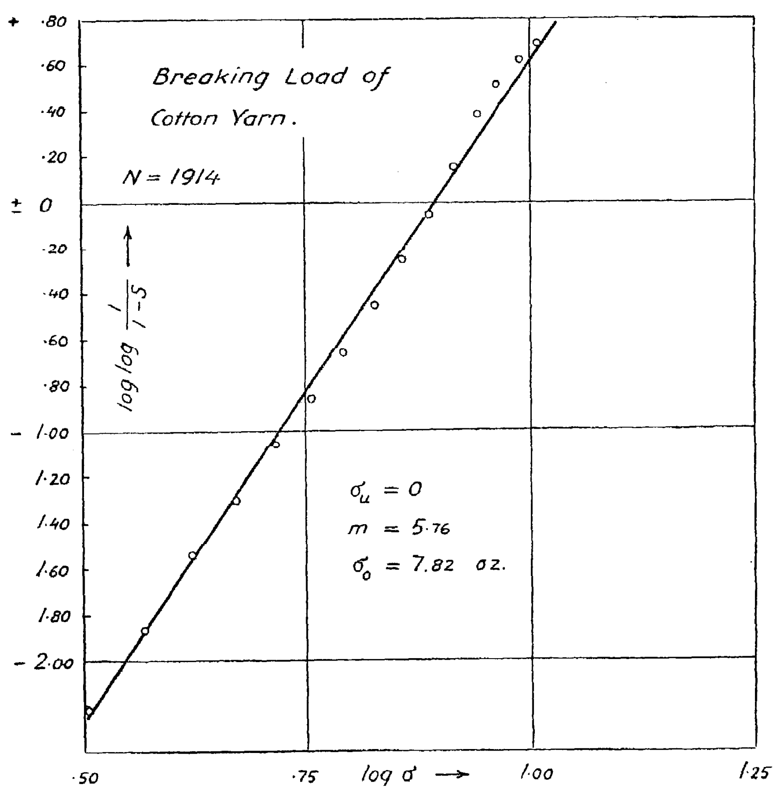

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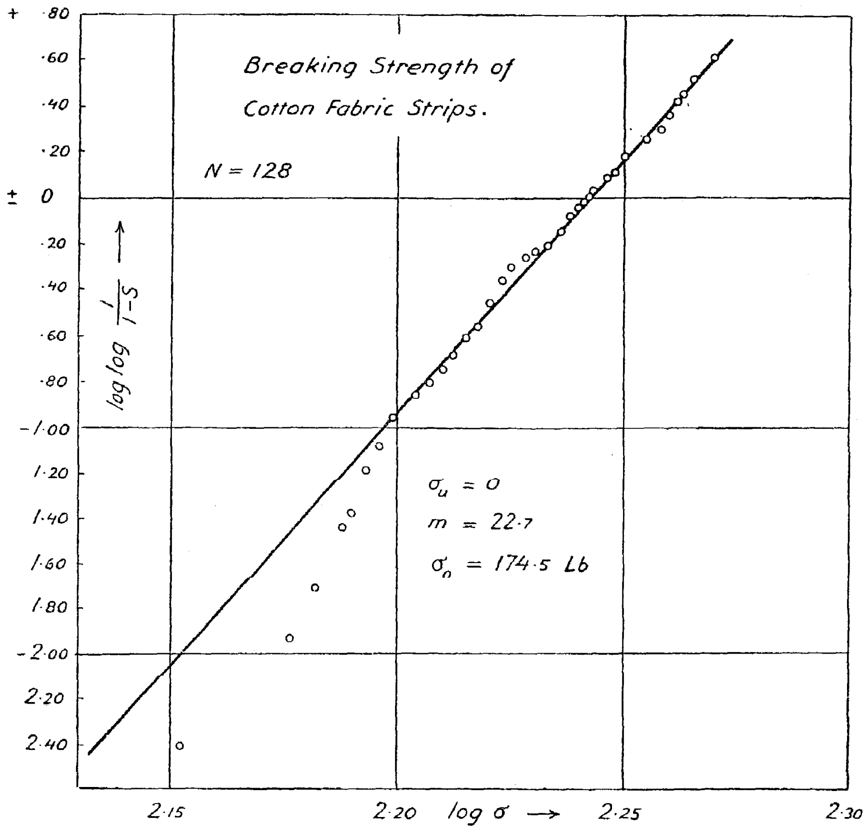

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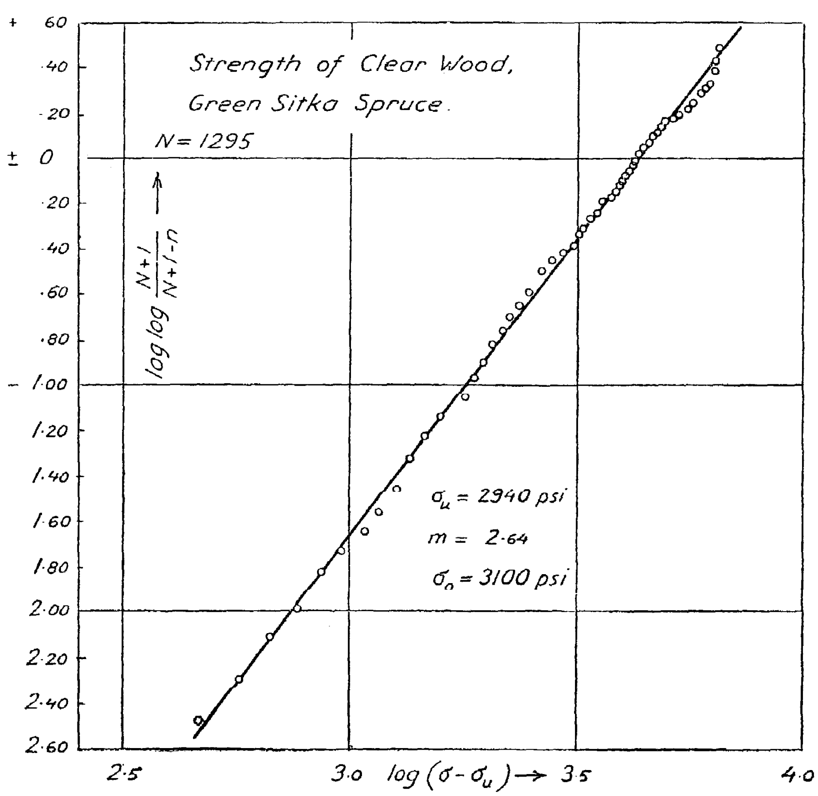

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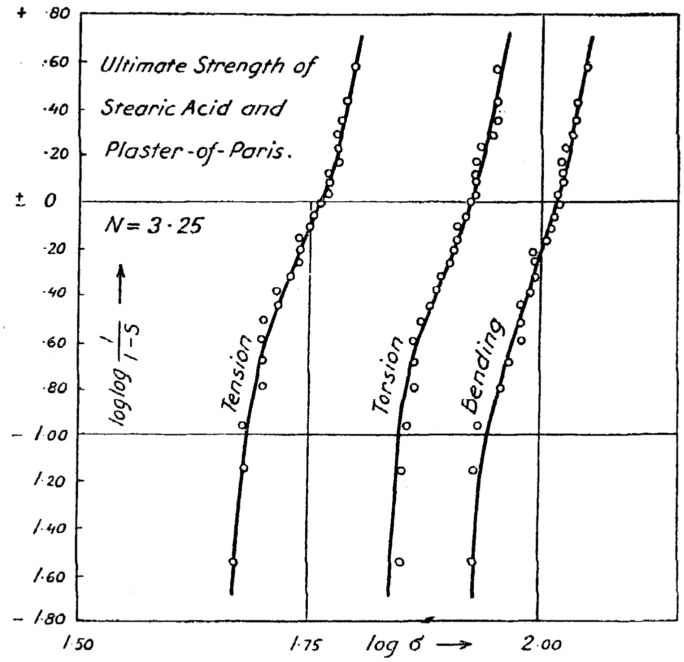

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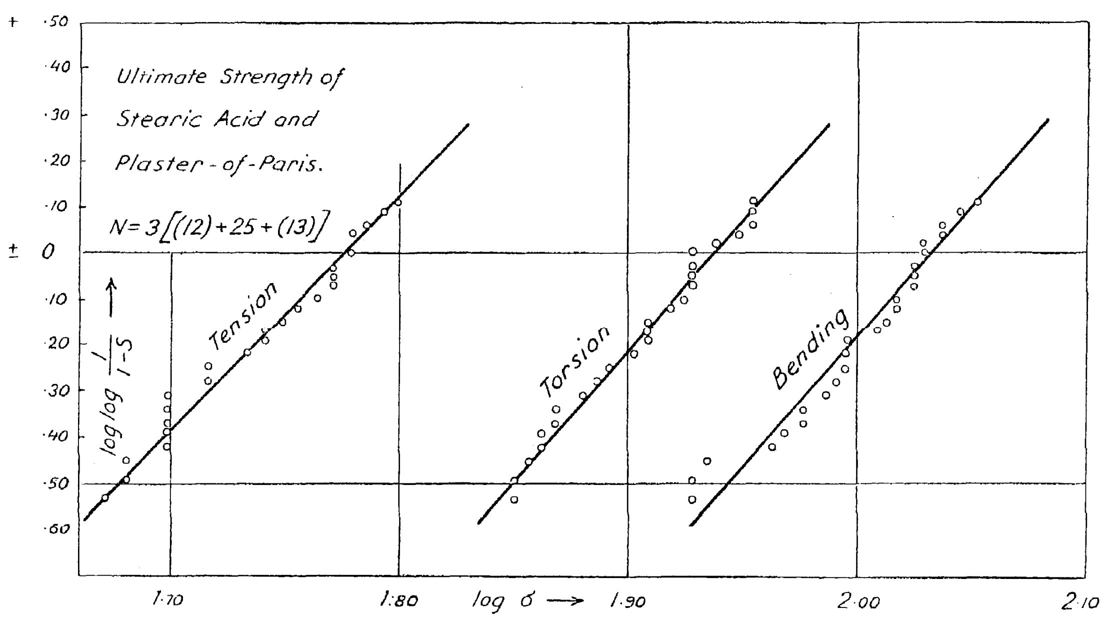

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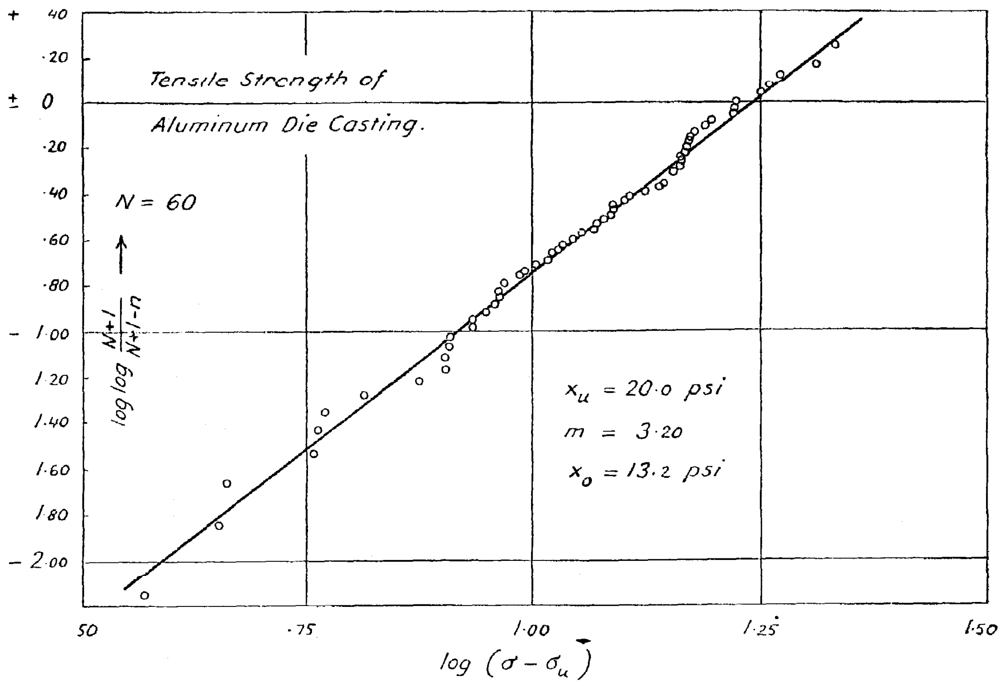

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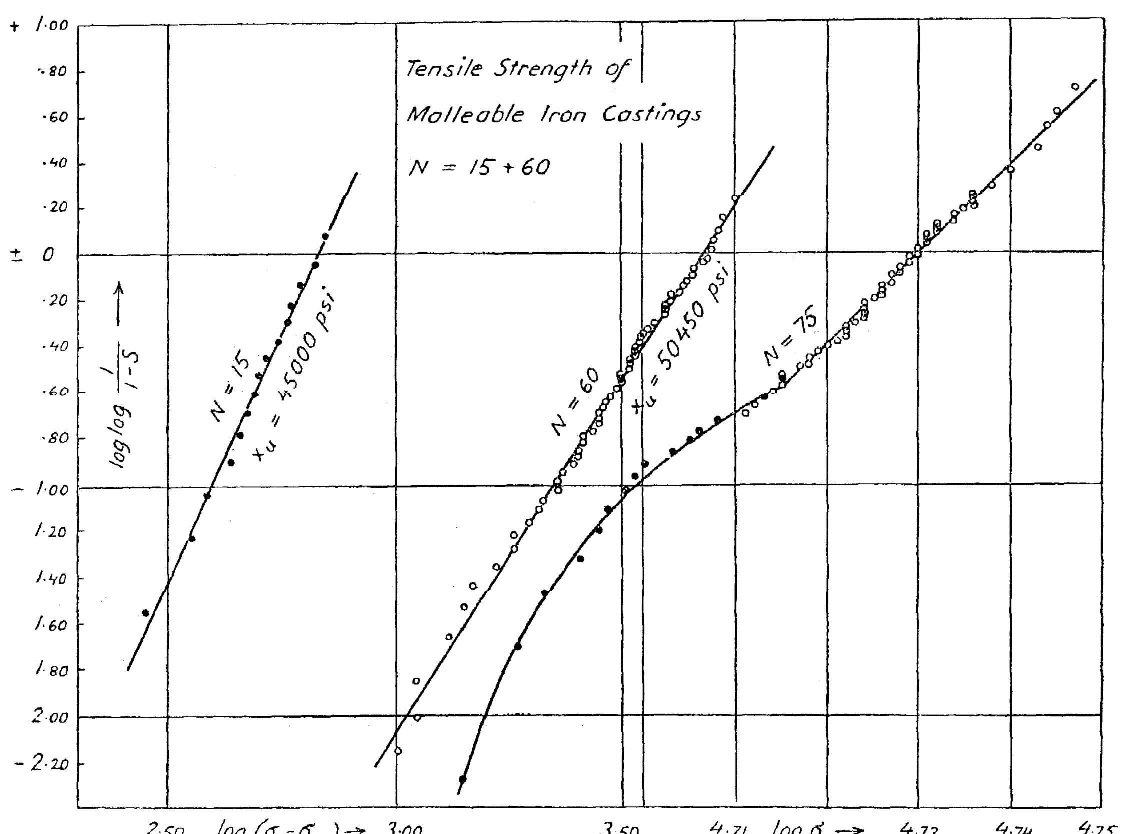

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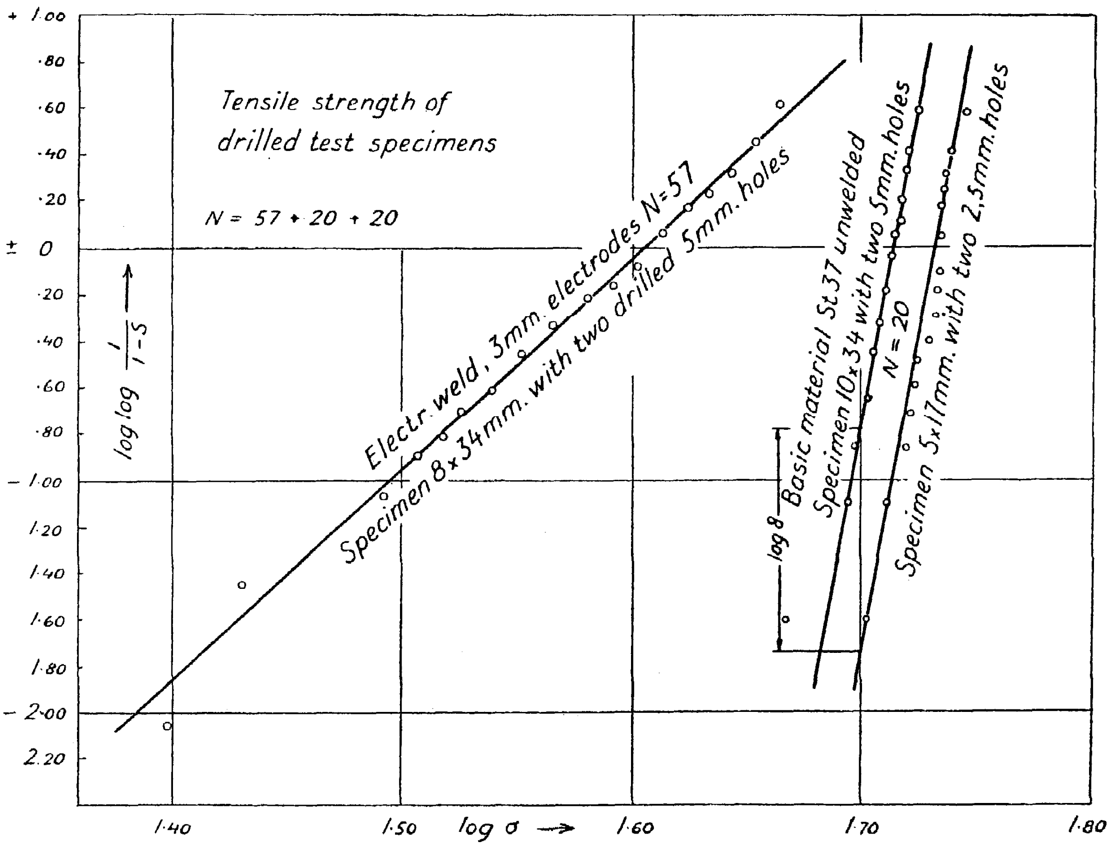

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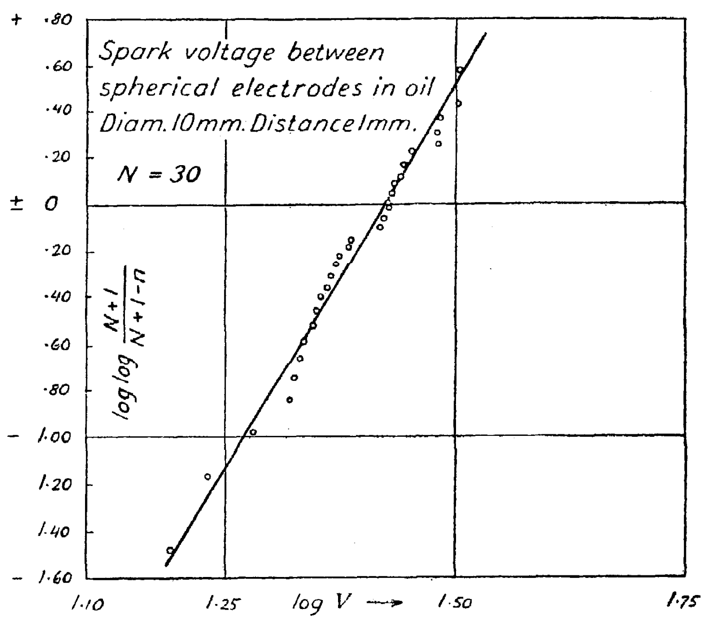

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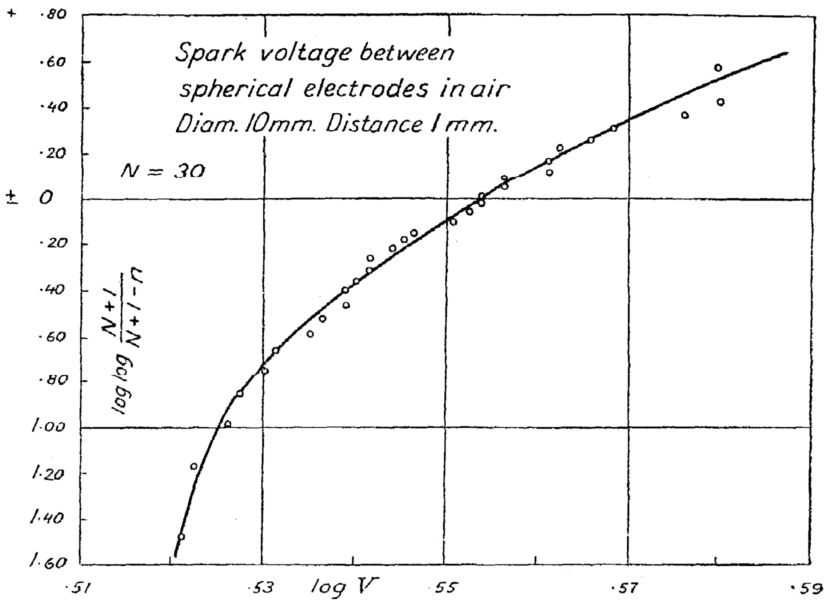

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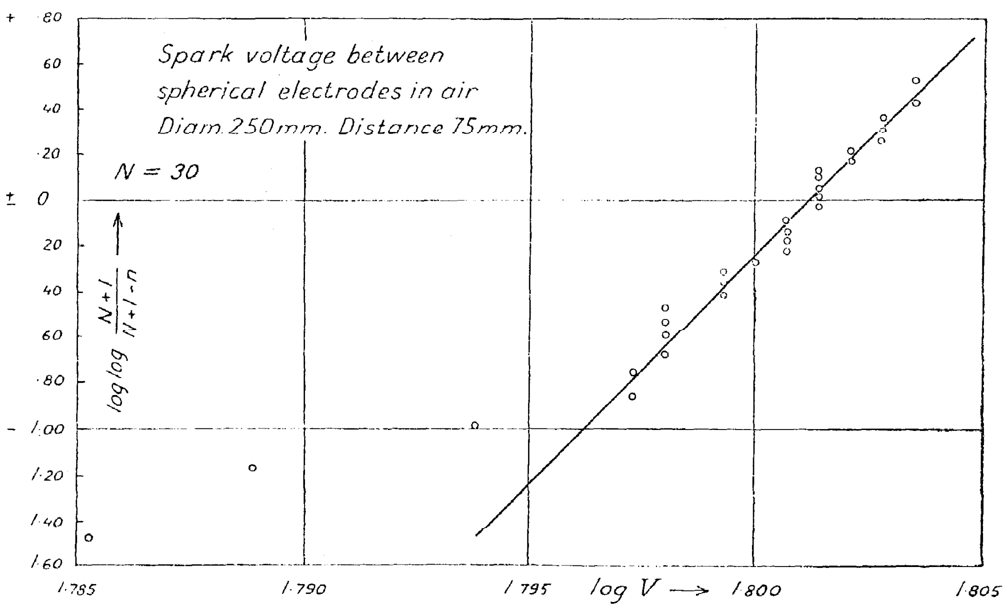

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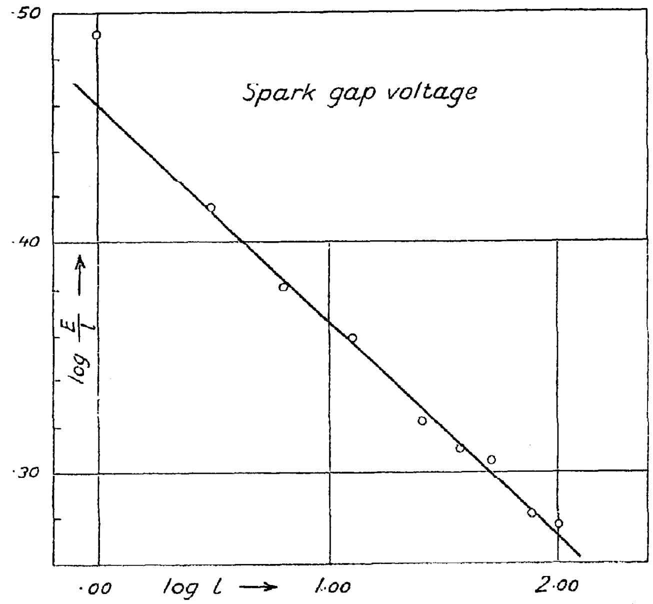

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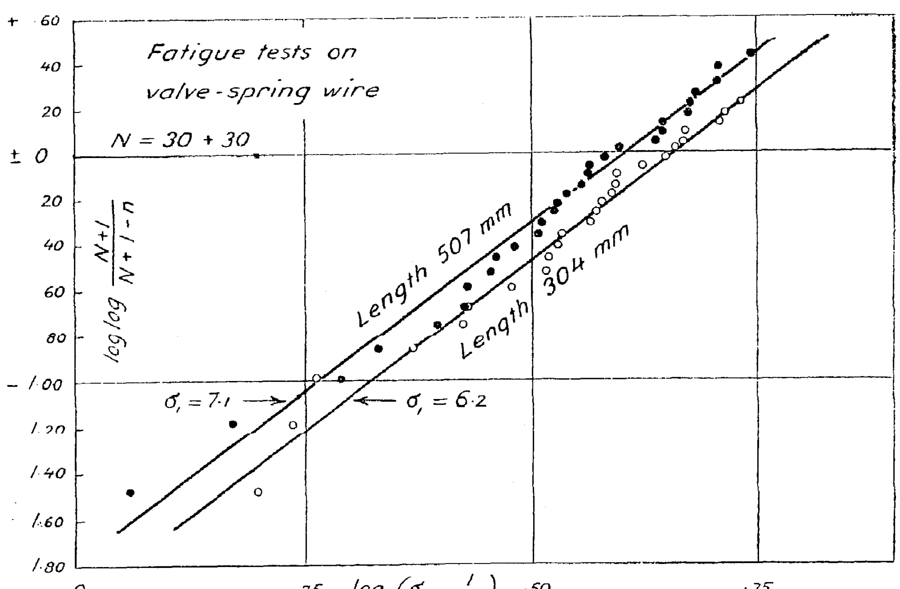

Summary.

Part I.

The classical theory of strength is obviously incompatible with numerous results of experimental research. This discrepancy may be bridged over by considering as an essential element of the problem the dispersion obtained in experimental measuring of the ultimate strength. Viewed from this standpoint, the ultimate strength of a material can not be expressed by a single numerical value, as has been done heretofore, and a statistical distribution function will be required for this purpose. The application of the calculus of probability leads to the fundamental law of the theory, viz., that the probability of rupture (S) at any given distributionof stresses (σ) over a volume (V) is determined by the equation.

$$\log\ (1 - S) = -\int_V n(\sigma)\,dv$$

where $n\ (\sigma)$ is a function characteristic of each particular material. This fundamental formula allows to compute the influence of the volume on the ultimate strength, the relation between tensile, bending, and torsional strength, etc.

An experimental substantiation of the theory is provided by observations obtained from tensile, bending, and torsional tests on rods made of stearic acid and plaster-of-Paris.

Part II.

In this part of the paper a description is given of the graphic method used for the statistical treatment of the observations and of som measuring series relating to strength of materials under the action of mechanical and electrical forces. It is shown that the material function may be expressed by the formula $n(\sigma) = \left(\dfrac{\sigma - \sigma_u}{\sigma_0}\right)^m$ where σ_u, σ_0 and m are constants characteristic of the material. This formula applies to statistically homogenous matetials.