Critical Points: Scaling and Universality

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Abstract

This talk summarizes the concepts which go into the theory of critical phenomena. The discussion is based upon three periods:

- before 1964: the ideas are constructed
- 1964-1972: revolution
- after 1972: conquest of new territory

I will focus my attention upon before and after. For "before" the main figure is Landau. For "after" the main focus will be upon the different theories which took off from the insights generated by the understanding of critical behavior.

I leave out dynamics, many other applications of critical point ideas,and a critical examination of the revolutionary period.

More on Periodization

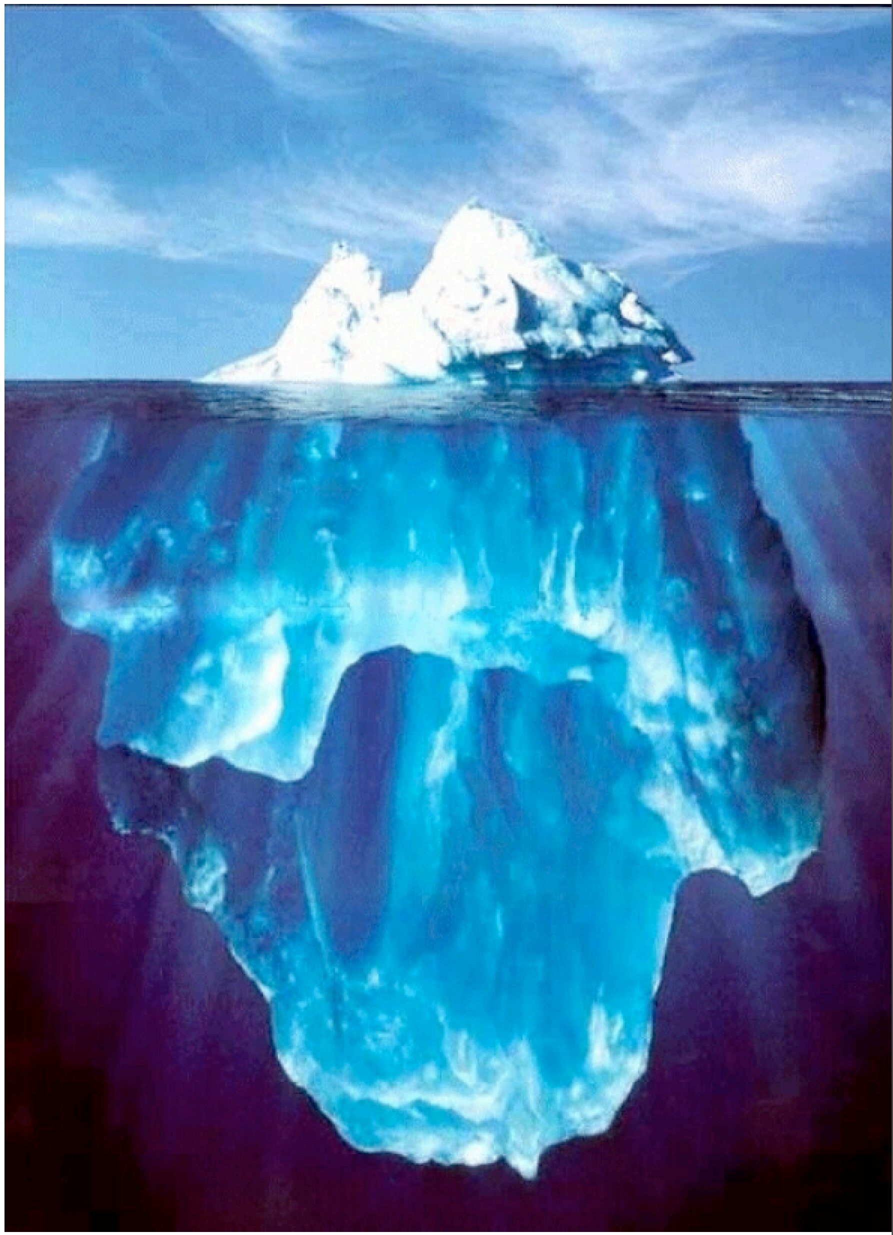
- Early period: look at the whole phase diagram, glance at critical region
- Revolutionary period: focus on critical region develop theory of critical phenomena
- Afterward: various mathematical/theoretical expressions and extensions of theory.

Issues

Before:
• For 90+ years, theory and experiment disagreed. How could this be?

During:
• Which of the basic ideas were new and which developed "before"?

After:
• What was the impact of critical point ideas?



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Many Different Phase Transitions:

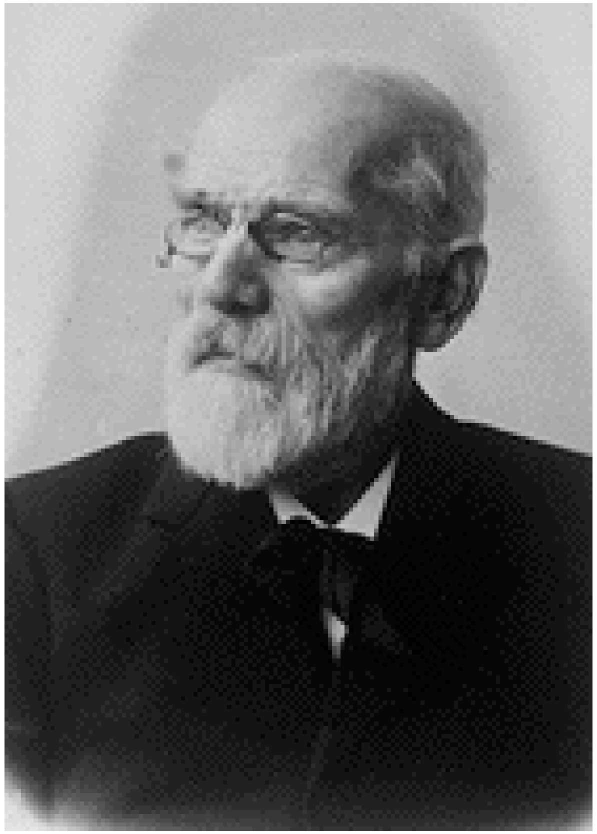
- liquid -gas
- paramagnetic to ferromagnetic
- ...

van der Waals: (1873) Different simple liquids-gas transitions have very similar thermodynamic properties. Derives mean field theory of liquid. Universal!

Curie-Weiss (1907) mean field theory of magnets.

....

But, each different phase transition calls for its own theory. not Universal!



Johannes Diderik van der Waals

theory starts from: phase transitions are singularities in free energy

J. Willard Gibbs:

gets to: phase transitions are a property of infinite systems

proof: .... consider Ising model for example

problem defined by

$$-H/(kT) = K \sum_{nn} \sigma_r \sigma_s + h \sum_r \sigma_r$$

free energy defined by

$$-F/(kT) = \ln \sum_{\{\sigma_r = \pm 1\}} \exp -H\{\sigma_r\}/(kT)$$

H is a smooth function of K and h.
Since a finite sum of exponentials of smooth functions is a positive smooth function, it follows that the free energy is smooth too.

Paul Ehrenfest: 1880-1933

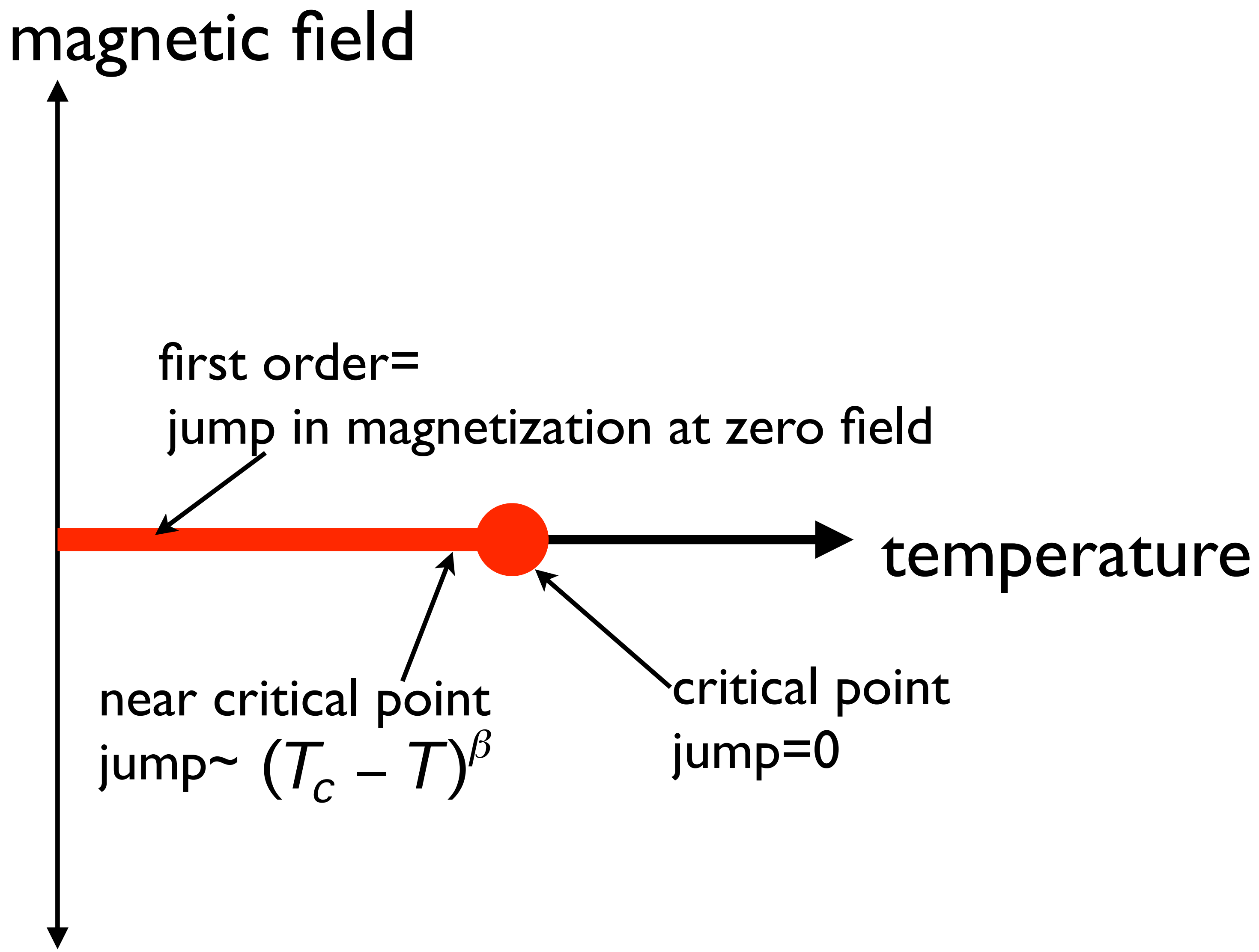
He suggested a simple classification system for phase transitions. The "order" of a phase transition is defined by which derivative of the free energy has a discontinuity. In the ferromagnetic transition M~ dF/dh jumps. Hence this is a first order transition.

In other transitions, the heat capacity, a second derivative of the free energy with respect to temperature, shows a discontinuity. This is Ehrenfest's second order transition.

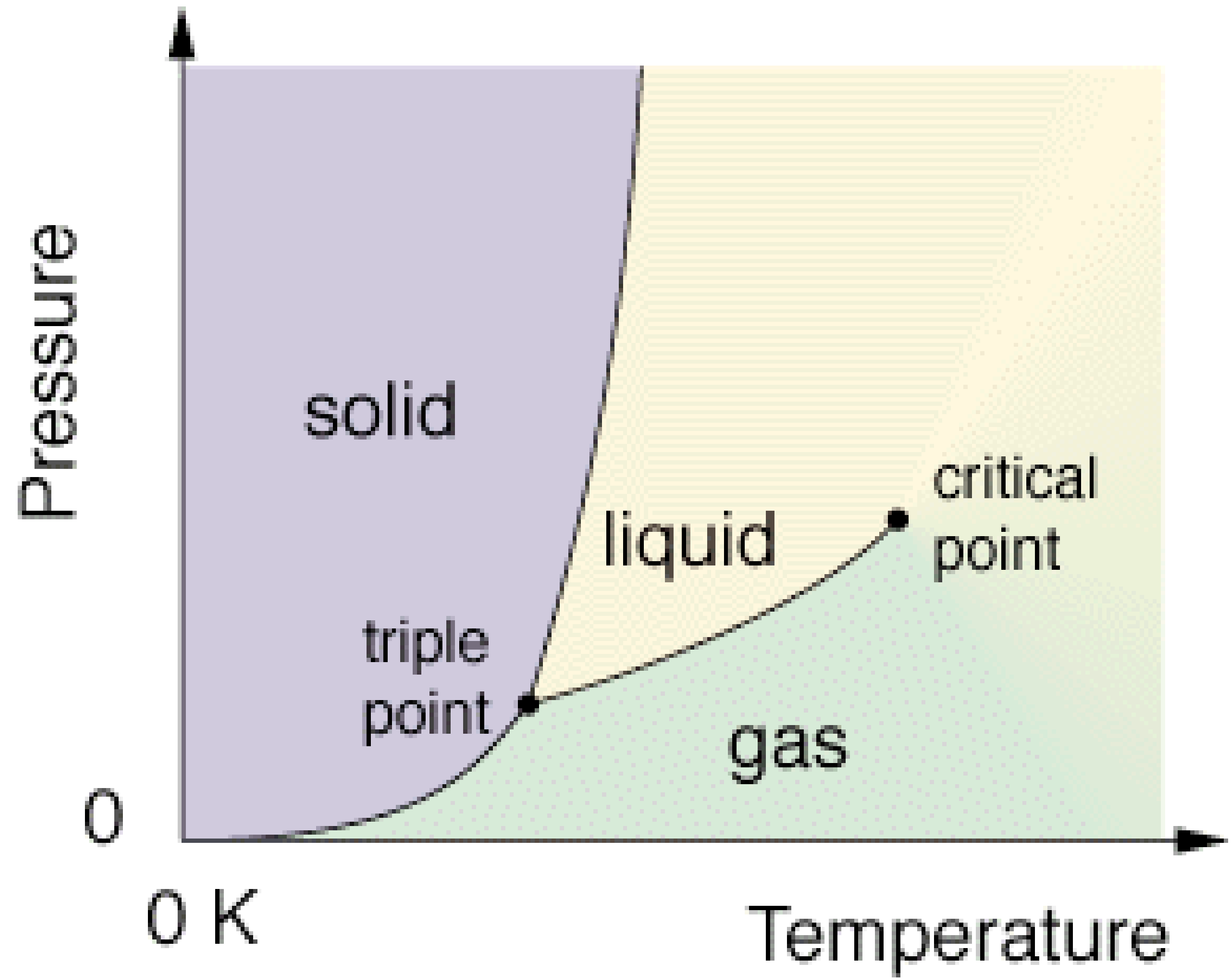
Ehrenfest's students, Leiden 1924. Left to right: Gerhard Heinrich Dieke, Samuel Abraham Goudsmit, Jan Tinbergen, Paul Ehrenfest, Ralph Kronig, and Enrico Fermi.

from Wickopedia

Magnetic Phase Diagram



Typical Fluid Phase Diagram



http://ltl.tkk.fi/research/theory/TypicalPD.gif

Major Ideas-early period-1

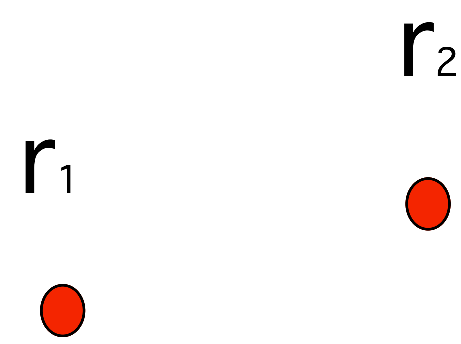
- Universality: van der Waals, Ehrenfest, Landau, King's College School: Different Phase Transitions are alike if they are described in the "right variables"

- Critical Indices: e.g. density jump ~ $(Tc-T)^{1/2}$ van der Waals

- Order parameter: Landau, many people, ...A quantity which jumps in a first order transition.

Major Ideas-early period-II

- Critical opalescence = correlations on large spatial scale Ornstein-Zernike.
- Scaling: Kolomogorov, Mandelbrot
- RG: Gell-Mann & Low, Stuckelberg & Peterman
- Effect of Fluctuations: Ginzburg



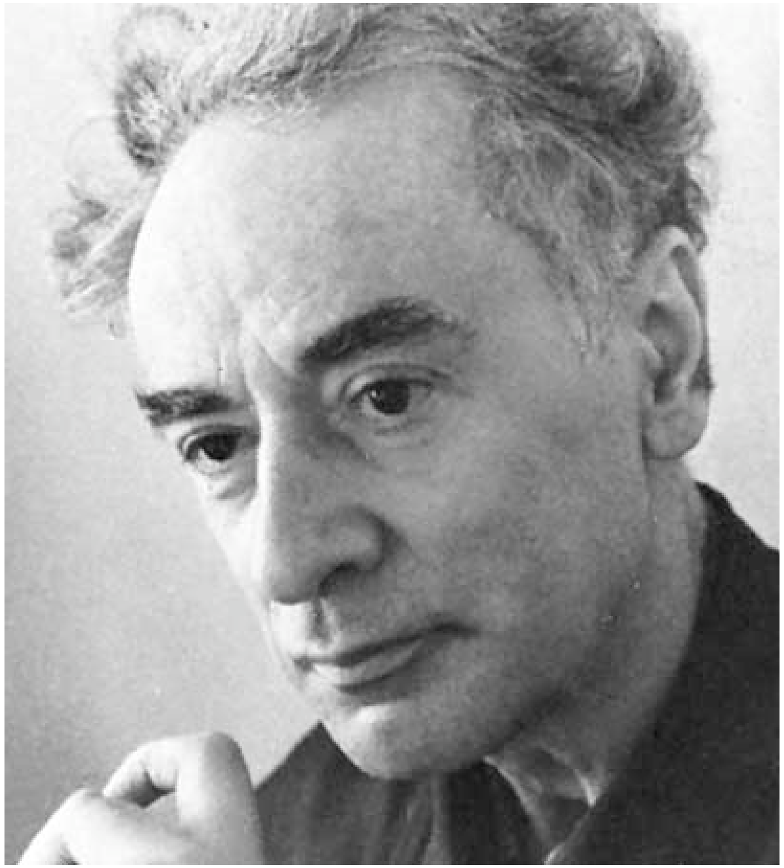
At critical point $$\langle m(r_1)m(r_2)\rangle = C\, r_{12}^{-2x}$$

(Power laws are very characteristic of problems without a natural scale of e.g. length or distance from critical temperature.)

Order Parameter, generalized

- Landau (~1937)suggested that phase transitions were manifestations of a broken symmetry, and used the order parameter to measure the extent of breaking of the symmetry.

- in ferromagnet, parameter = magnetization
- in fluid, parameter = density



Lev Landau: Sovfoto

Generalized Mean Field Schemes I

Many different mean-field schemes developed:. Each one has an order parameter, an average of a microscopic quantity. Landau generalized this by assuming an expansion in an order parameter, M=magnetization

$$F = \int d\mathbf{r}\ [a + hM + bM^2 + cM^4]$$

expansion assumes a small order parameter (works near critical point) and small fluctuations (works far away?!)

h is magnetic field

b is proportional to $(T-T_c)$

minimize F in M: result General Solution $M(h, (T-T_c))$

singularity as b,h go through zero!

singularity as h goes through zero for $T< T_c$

Generalized Mean Field Schemes II

For example for T< T_c , jump behaves as

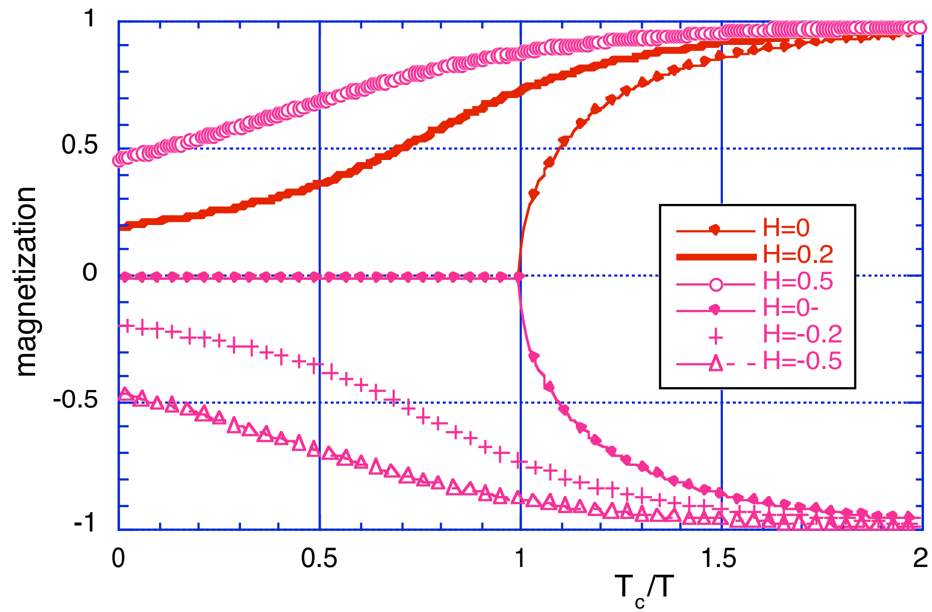
$$M \sim \sqrt{(T_c - T)}$$

This square root (β=1/2) appears to be a Universal result.

Mean Field Theory predicts all near-critical behavior and defines a bunch of critical indices and correlation functions.

$$< m(r_1)m(r_2) >= C\, r_{12}^{-2x}$$

mean field theory gives x=d/2 -1



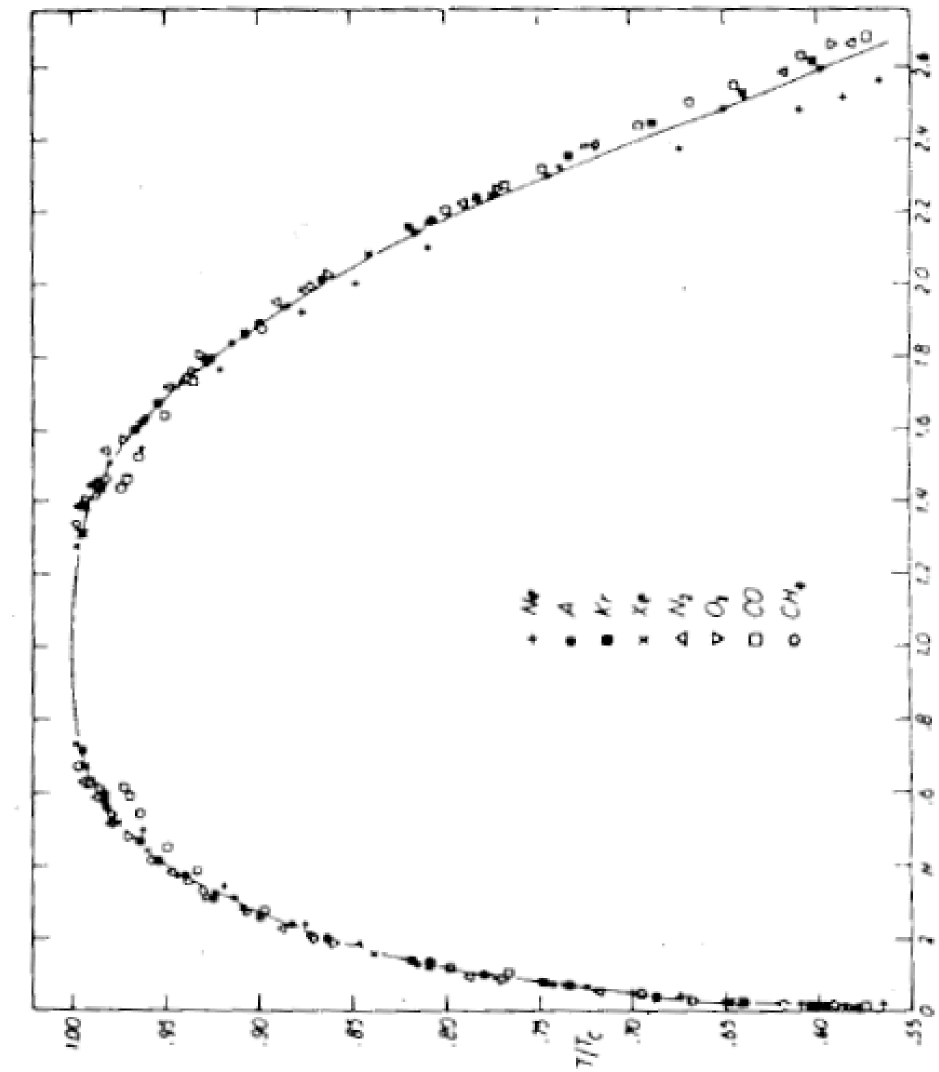
order parameter in mean field transition

A worry?

Mean field theory gives $M \sim (T_c - T)^{\beta}$
and $\beta = 1/2$
This power is, however, wrong. Experiments show result closer to

$$M \sim (T_c - T)^{1/3} \quad \text{in 3-D}$$

1880-1960: No one worries much about discrepancies



order parameter: density versus Temperature in liquid gas phase transition. After E.A. Guggenheim J. Chem. Phys. **13** 253 (1945)

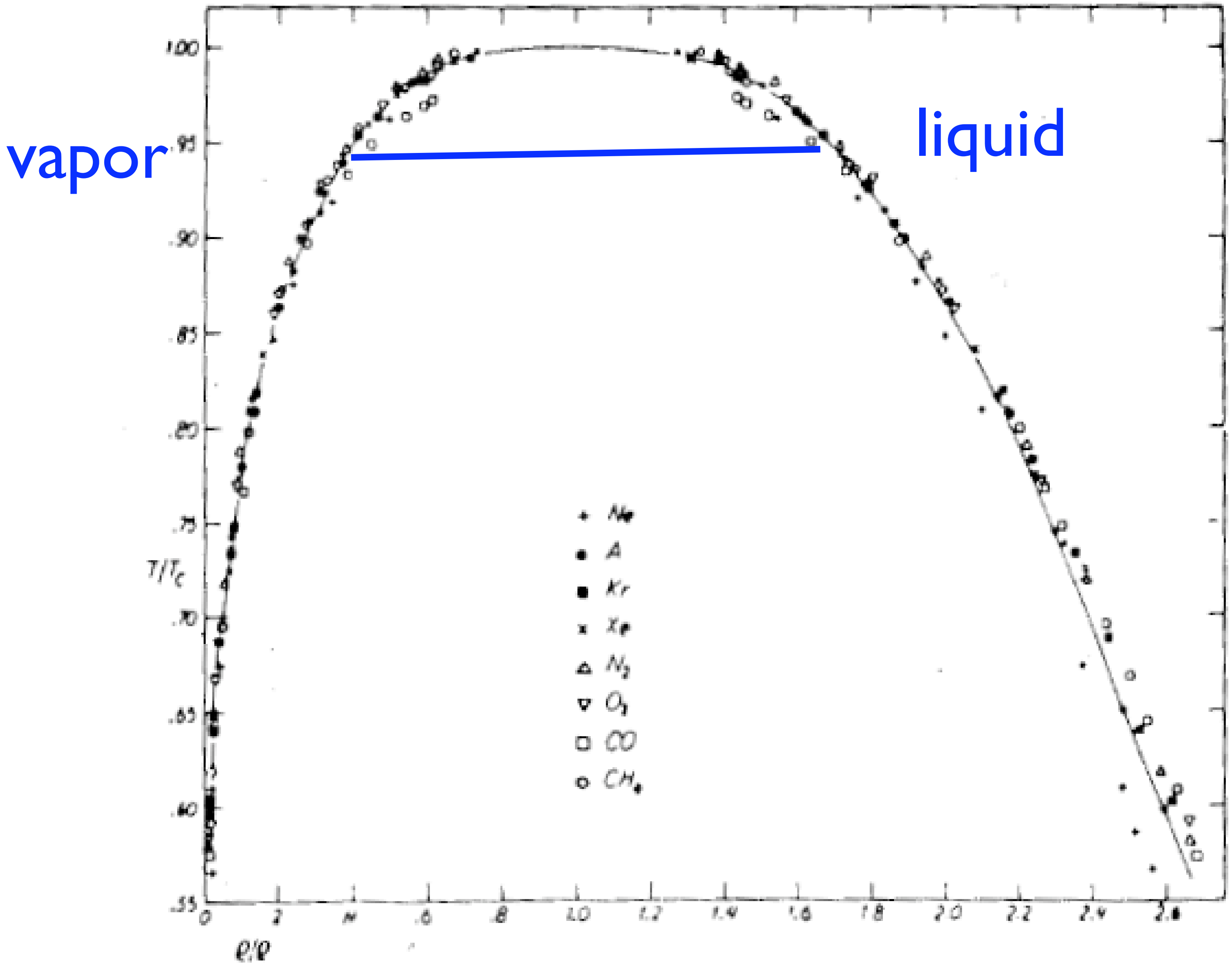


Figure 1.6 Reduced densities of coexisting liquid and gas phases for a number of simple molecular fluids (Guggenheim 1945). The experimental points support a law of corresponding states, but the universal curve is cubic rather than quadratic as required by van der Waals' theory.

Theoretical work

Onsager solution (1943) for 2D Ising model gives infinity in specific heat and correlation function index $x = \beta = 1/8$(Yang)--contradicts Landau theory which has $x=0$, $\beta=1/2$

L. Onsager, Phys. Rev. **65** 117 (1944)

C.N, Yang, Phys. Rev. **85** 808 (1952)

Kings' College school (Domb, Fisher, ... (1949-)) calculates indices using series expansion method. Gets values close to $\beta=1/8$ in two dimensions and $\beta=1/3$ in three and not the Landau\van der Waals value, $\beta=1/2$.

Still Landau's no-fluctuation theory of phase transitions stands

Turbulence Work

Kolmogorov theory (1941) uses a mean field argument to predict velocity in cascade of energy toward small scales.

Result: velocity difference at scale r behaves as

$$\delta v(r) \sim (r)^{1/3}$$

(N.B. First Scaling theory)

Later Landau criticizes K's work for leaving out fluctuations. Kolmogorov modifies theory (1953) by assuming rather strong fluctuations in velocity.

Still Landau's no-fluctuation theory of phase transitions stands

Pre-revolutionary era: Summary

Set our qualitative nature of the phase diagram.

Thought up many of the important theoretical concepts.

Did not produce a satisfactory theoretical synthesis.

Experimenters and series expanders measured many of the critical indices.

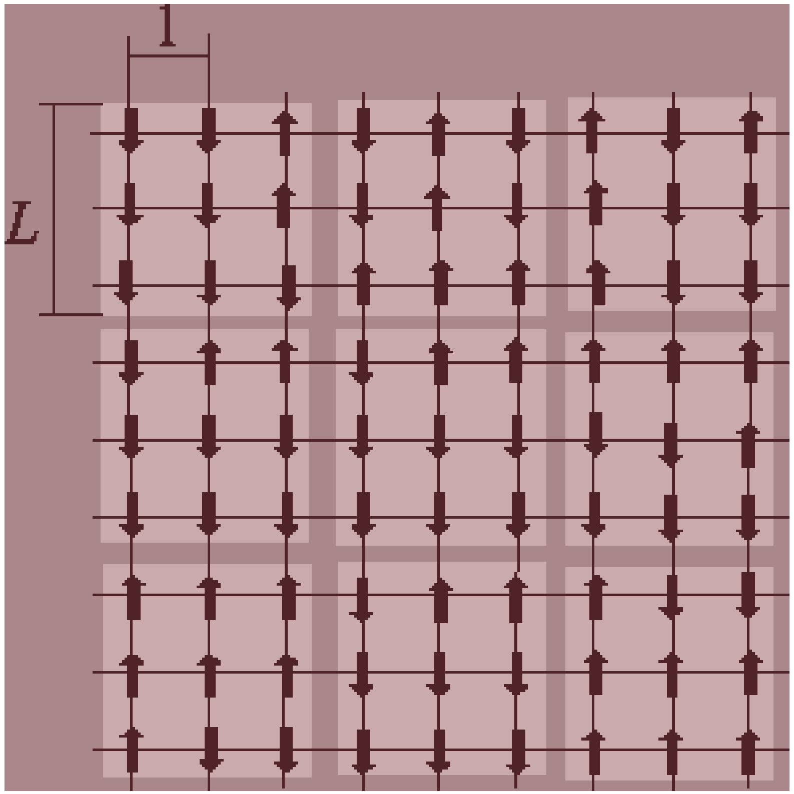
Theory got wrong values.

Explained phase transition as a symmetry breaking.

Was not much concerned about how information on "which phase" was transfered from one part of material to another.

A Revolutionary Period:
1960s and early 70s
I. New Phenomenology

Recognize that 'critical phenomena' is a subject US NBS conference 1965.



• Don't look at the entire phase diagram, focus on the region near the critical point.

Get a whole host of new experiments, embrace a new phenomenology,
Ben Widom: scaling
Pokrovskii Patashinskii; correlations
Kadanoff: Block renormalization
[I implicitly used Landau's ideas of quasi-particle and effective Hamiltonians.]

A review paper

L.P. Kadanoff, W. Gotze, D. Hamblen, R. Hecht, E.A.S. Lewis, V.V. Palciauskas, M. Rayl, J. Swift, D. Aspnes, and J.W. Kane, Static Phenomena Near Critical Points: Theory and Experiment, Rev. Mod Phys. 39 395 (1967)

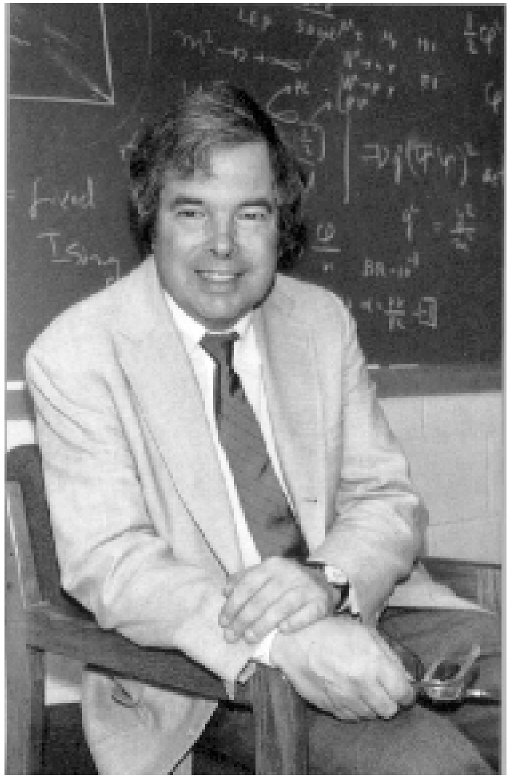
Shortly after the new phenomenology was introduced we (see above) wrote a review in which we look for new used everything, including all the existing experiments and series expansions to see whether the new phenomenology was working.

We concluded that it agreed with all the data in a satisfactory fashion.

A Revolutionary Period:

1960s and early 70s

II. Wilson's New Theory



Kenneth G. Wilson
synthesizes new theory

Wilson's big changes:

- use new phenomenology
- replace mean fields by fluctuations,
- consider many different kinds of fluctuations at the same time
- new idea: the "fixed point"

new theory matches experiments

everything seems to be explained

The Crucial Ideas-for the revolution

Ideas:

- Criticality: recognize a subject in itself
- Scaling: Behavior has invariance as length scale is changed
- Universality: Expect that critical phenomena problems can be divided into different "universality classes"
- Running Couplings: Depend on scale. Cf. standard model based on effective couplings of Landau & others.
- Fixed Point: Singularities when couplings stop running. K. Wilson
- Renormalization Group: K. Wilson (1971), calculational method based on ideas above.

Each Item, except 'fixed points', is a "consensus" product of many minds. The brilliant synthesis is due to Wilson.

The Outcome of Revolution

Excellent quantitative and qualitative understanding of phase transitions in all dimensions. Information about

- Universality Classes

All problems divided into "Universality Classes" based upon dimension, symmetry of order parameter, ...

Different Universality Classes have different critical behavior

e.g. Ising model, ferromagnet, liquid-gas are in same class

XYZ model, with a 3-component spin, is in different class

- Phase diagram near the critical point

- Approximate form of thermodynamic functions near the critical point

- Approximate values of critical Indices

Conceptual Advances

First order phase transition represent a choice among several available states or phases. This choice is made by the entire thermodynamic system.

Critical phenomena are the vacillations in decision making as the system chooses its phase.

Information is transferred from place to place via local values of the order parameter.

There are natural thermodynamic variables to describe the process. The system is best described using these variable.

Each variable obeys a simple scaling.

Next: After the Revolution:

The RG point of view is Fully Absorbed into Particle Physics

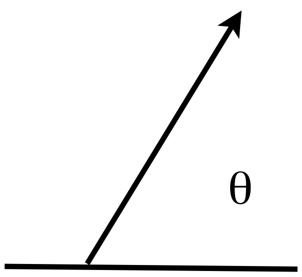
Running coupling constants help define the standard model.

The model is extrapolated back to when weak, electromagnetic and strong interactions are all equal.

Asymptotic freedom--weakening of strong interactions at small distances--permits high energy calculations. Gross, Wilczek, Politzer (1973).

2D XY model-
Kosterlitz & Thouless (1973)

S= (cos θ, sin θ),



The XY model is a set of two-dimensional spins, described by an angle θ and a nearest neighbor coupling

K cos ($\theta - \theta'$).

There is an elegant description in terms of free charges and monopoles with electromagnetic interactions between them.

This work is also important because it is a successful calculation involving topological excitations. Milestone.

Coulomb gas: B. Nienhuis (~1985)

Nienhuis extended this approach to give a description of correlation functions in terms of a screened coulomb gas (and magnetic monopoles) for many kinds of critical phenomena problems in two dimensions. The list includes:

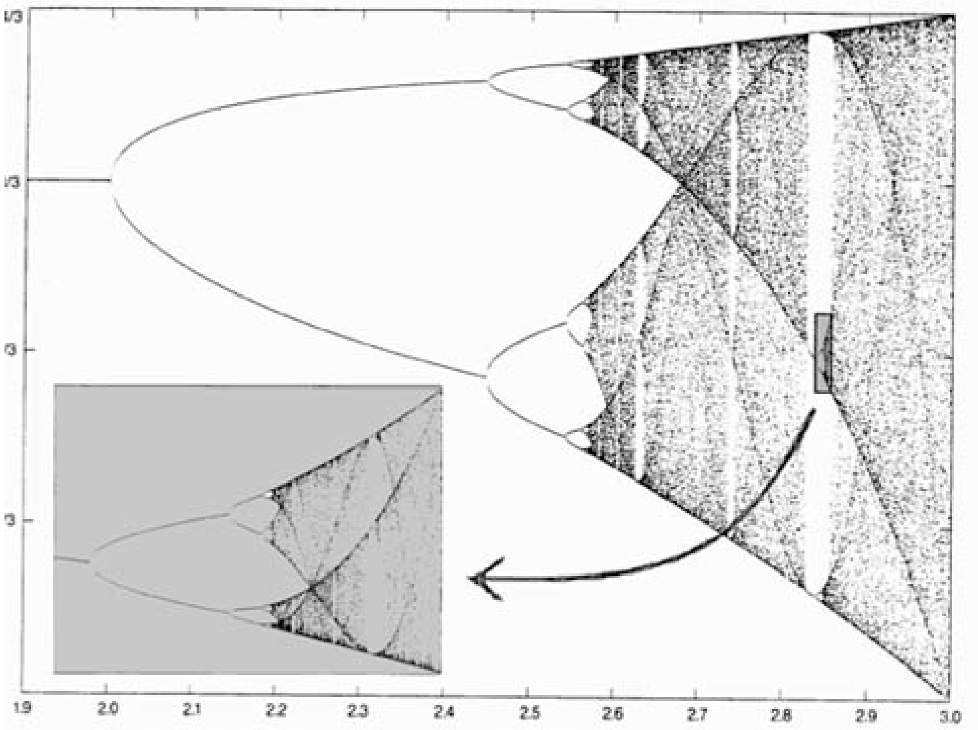
- q-state Potts models (spin takes on q values)
- O_n models (n-component vectors)

These models permit generalizations in which behavior varies continuously with q or n.

Feigenbaum: Analyzes Route to Chaos (1978)

Feigenbaum used RG, Universality, and Scaling concepts to investigate the period doubling route to chaos via the discrete equation

$x(t+1) = r\,x(t)\,(1-x(t))$



long-term x-values versus r

Field Theory

Pre-Revolutionary Period:
Onsager solution of 2-D Ising model is free fermion theory

Revolutionary Period:
The connection: Wilson & Fisher (1972) use Ginzburg Landau free energy formulation plus path integral formulation of quantum mechanics to describe critical phenomena theory as a close relative of quantum field theory.

After the Revolution: Polyakov and others reformulate critical point theory as Conformal Field Theory, a field theory for situations invariant under scale transformations but not shear-type distortions

Conformal Field Theory: I

A. Polyakov emphasized that there is a special form of field theory which holds at critical points, i.e. places in which there is full scale invariance. In two dimensions this is super-special because the invariance includes all kinds of conformal (angle preserving) transformations which can then be studied through the use of complex variable methods.

Space distortions are based upon stress tensor operators, with the Virasoro algebra being the algebra of local stress tensor densities. Just as spinors, vectors and tensors are derived as representations of the rotation group algebra, equally the local operators of critical phenomena have properties, including critical indices, derivable from the fact that they are representations of the Virasoro algebra.

Continuously varying families of solutions are generated in this fashion

Conformal Field Theory: II

Friedan Qiu & Shenker (1984) showed that unitarity, a quality of all field theories representing possible quantum processes, limits the domain of field theories to include all the known critical 2-D models, e.g. q=2,3, 4 Potts models, but not others (e.g q=1.5).

Further we get a quite different algebra for each model.

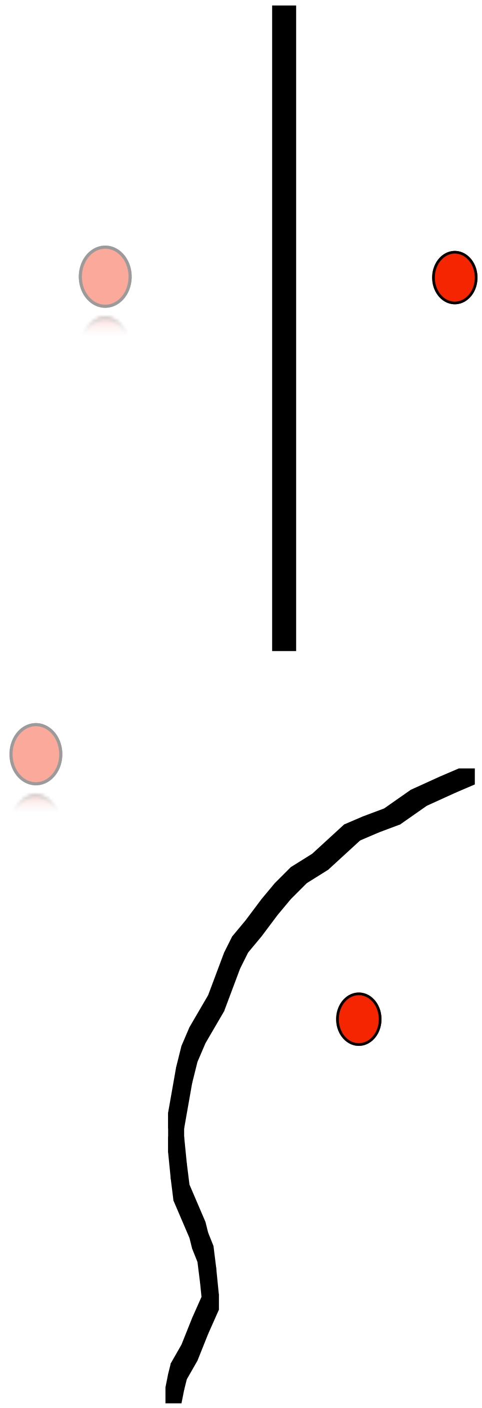
Friedan Qiu & Shenker PRL 52 1575 (1984)

For monograph material see Di Francesco, Mathieu, Senechal Conformal Field Theory, Springer, 1997

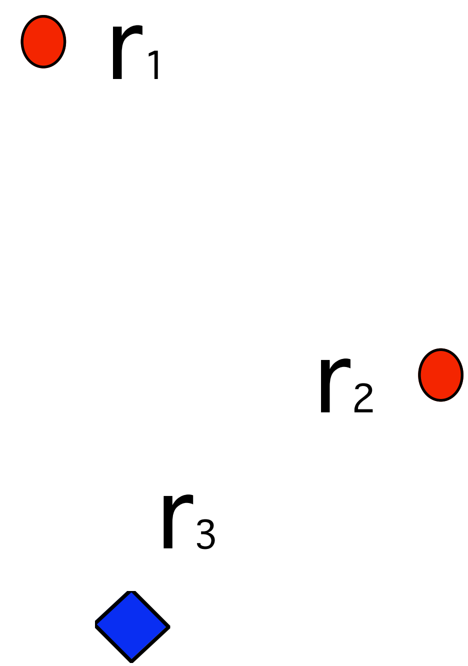
Conformal Field Theory: III

Compared to general situations, all the familiar statistical mechanical models are all degenerate and truncated, and permit considerable calculation of correlation functions.

Further conformal transformation permit the calculation of correlations in all kinds of shapes from just a few shapes: plane, half-plane, interior of circle.



Many Exact Calculations



find correlation of two spins
(local magnetization densities)
and a local energy density

$$< \sigma(r_1)\sigma(r_2)\epsilon(r_3) > \quad = C{r_{12}}^{x_\epsilon - 2x}(r_{23}r_{13})^{-x_\epsilon}$$

For Ising model, $x=1/8$ and $x_\varepsilon = 1$

Quantum Gravity:

The next step beyond scale invariance is no scale at all. One can formulate "quantum gravity theory" as a classical theory in which one sums over all possible metrics on a given space.

In two dimensions, there are no physical degrees of freedom in gravity theory, and the summation can be carried out exactly. (Gross & Migdal, Douglas & Shenker, Brezin & Kazakov.) In addition to being a solvable gravity model, this approach offers a good start for critical problems. For example B. Duplantier carried out a calculation in which he calculated the spectrum of electric fields in the neighborhood

SLE=Schramm-Loewner-Evolution

Conformal field theory took us to a point at which we could formulate critical phenomena problems on surfaces of various shapes. The most recent area of progress arises from work of Oded Schramm, who combined the complex analytic techniques of Loewner with methods of modern mathematical probability theory to gain new insight into the shapes which arise in critical phenomena.

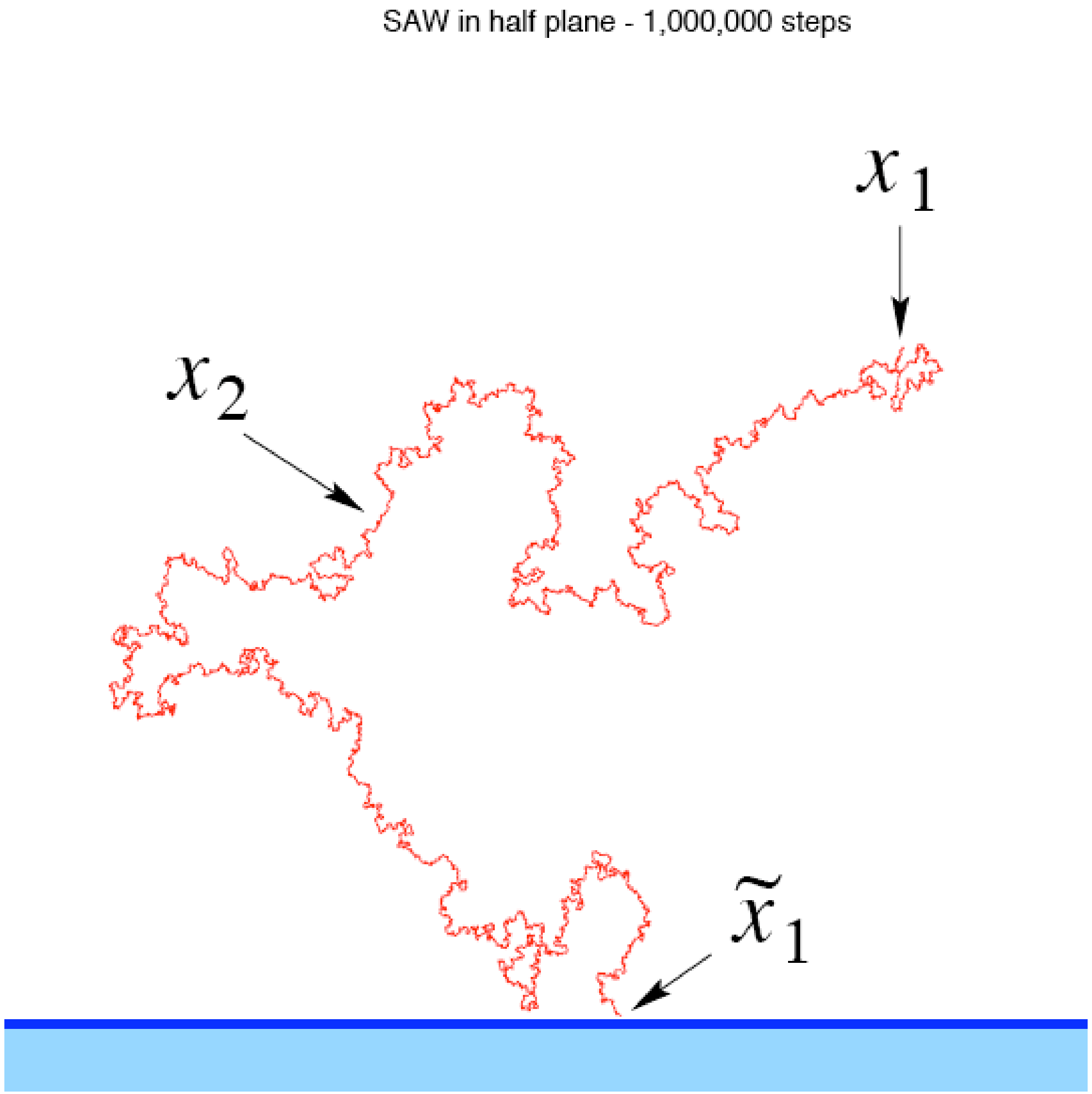
Oded Schramm

O. Shramm, Israel J. Math. 118 (2000), 221--288.

From Schramm to Critical Shapes

The work of the 20th. Century on critical phenomena was centered on thermodynamics, and correlation functions.

We depicted but did not calculate the shapes of the correlated fractal clusters arising in critical situations. Schramm provided a constructive technique, involving a differential equation, for making the ensemble of such clusters at critical points.

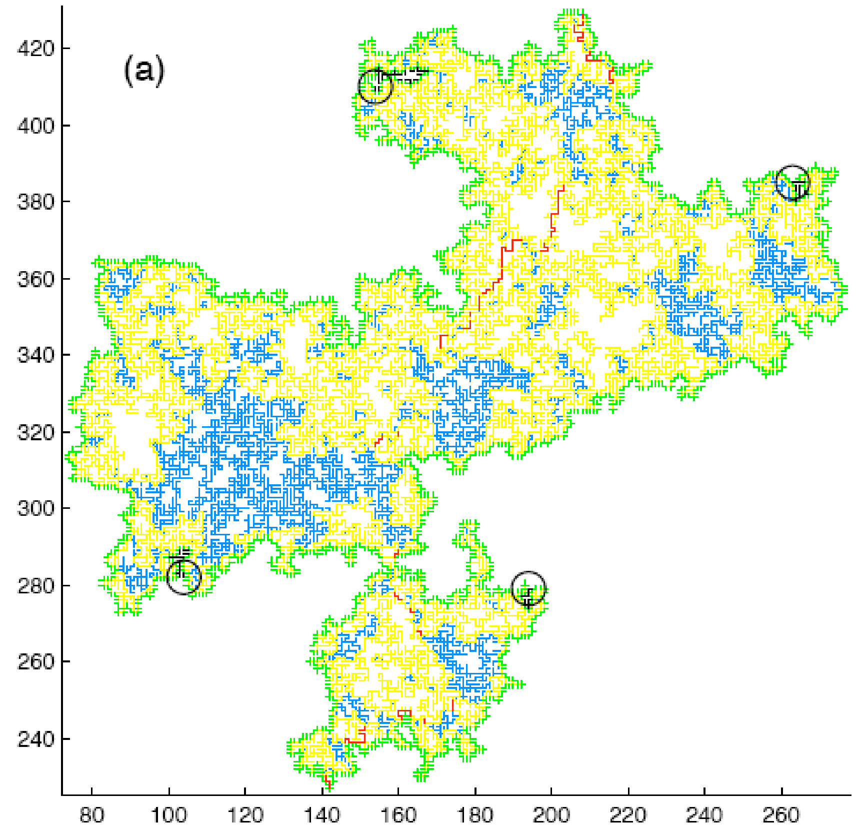


after Duplantier

SLE-II

Define a critical cluster as the shape of cluster of spins pointed in the same direction in an Ising model, or of a connected set of occupied sites in a percolation problem.

Problem: Define the ensemble of cluster shapes for any critical situation. Before Schramm's work we ignored this aspect of critical problems.



(J. Asikainen et al., 2003)

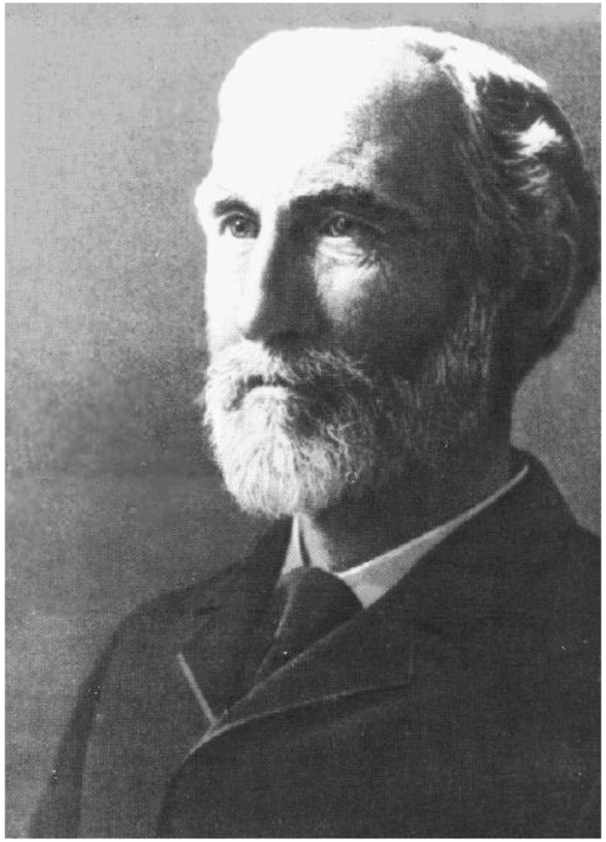
percolation cluster after Duplantier

Summary

Critical behavior occurs at but one point of the phase diagram of a typical system. It is anomalous in that it is usually dominated by fluctuations rather than average values. These two facts provide a partial explanation of why it took until the 1960s before it became a major scientific concern. Nonetheless most of the ideas used in the eventual theoretical synthesis were generated in this early period.

Around 1970, these concepts were combined with experimental and numerical results to produce a complete and beautiful theory of critical point behavior.

In the subsequent period the "revolutionary synthesis" radiated outward to (further) inform particle physics, mathematical statistics, various dynamical theories....



JW Gibbs

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