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32—X.—Tensile Tests for Cotton Yarns v.—"The Weakest Link" Theorems on the Strength of Long and of Composite Specimens

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32—X.—TENSILE TESTS FOR COTTON YARNS
V.—"THE WEAKEST LINK"
THEOREMS ON THE STRENGTH OF LONG AND OF COMPOSITE SPECIMENS

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INTRODUCTION AND SUMMARY

It is a truism, of which the mathematical implications are of no little interest, that the strength of a chain is that of its weakest link. It is equally true that the strength of a test specimen is that of its weakest element of length, whether it be a metal rod, a thread of yarn, or a cotton hair. This fact distinguishes the quantity, breaking load, from most other quantities, such as weight of which the value is determined by the average over all elements of the length. Tensile strength thus decreases with the length of specimen in a way which is definitely calculated from the distribution of strength of short specimens. The decrease in mean strength and in irregularity is directly proportional to the irregularity of the short specimens and to a factor, depending only on the multiple by which the length is increased and very simply calculated therefrom.

Variability along a specimen is shown necessarily to introduce negative skewness into all frequency curves of strength, counteracting or reinforcing any skewness that may arise from methods of production, and this must be taken into consideration when drawing conclusions from the shape of strength frequency curves. In cotton yarns, the method of production tends to produce positive skewness which is found with specimens of 3 inches. This is obliterated to yield a symmetrical curve by increasing the length to a foot or so, while leas show decided negative skewness. More generally, skewness is produced by irregularity in the frequency curves of any quantity "when the deviations of individual values are affected unequally by equal deviations of opposite sign among the constituent elements of a specimen," a criterion which fits most elastic measurements.

The relations between the strength of fibres, yarns, leas, and fabrics have often been studied empirically, but they are subject to so many disturbing factors that measurements do not lead to definite or simple conclusions, in the absence of a logical basis for comparing the results. In the present paper, five cases of specimens composed of parallel elements are analysed for a relation between the strength of the whole and of the parts. Actual specimens of all kinds of materials may reproduce the conditions more or less closely. The lea test can be brought under a simple case if modified as suggested in Paper I. of this series.

A correction is given for measurements of tendering when only a fraction of the length of a specimen is subjected to treatment, such as wear or exposure to light.

The present paper is mathematical throughout, but the conclusions are condensed into simple forms applicable to experimental results in this series or elsewhere.

VARIATION OF STRENGTH WITH LENGTH

Let the distribution of the breaking load of specimens of length l be expressed by a frequency curve, $y_1 = \varphi(f)$, where $y_1 \delta f$ is the probability that the strength of any given specimen should lie between f and $f + \delta f$, and further suppose that this function does not vary significantly from one portion to another of the whole batch of specimens.

The probability that the strength of any one thread should not be less than f is

$$\int_f^\infty \varphi(f) \,.\, df$$

that one of r lengths has a strength lying between f and $f + \delta f$ is $r\varphi(f) \,.\, df$; that none of the others has a strength less than f is

$$\left[\int_f^\infty \varphi(f) \,.\, df \right]^{r-1}$$

Hence the probability that the breaking load of any given specimen of length rl lies between f and $f + \delta f$ is $y_r \,.\, \delta f$, where

$$y_r = r\varphi(f) \left[\int_f^\infty \varphi(f) \,.\, df \right]^{r-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is a general expression independent of the form of the frequency curve provided only this be not also a function of the position of the specimen. It may be applied to any empirical frequency curve to deduce the distribution of strength in specimens of multiple lengths by giving each observation the weight $r\,(N'/N)^{r-1}$ where N' is the number of specimens of equal or greater strength, N the total number, or by the expression—

$$y_r = r\,\Sigma\,(N'/N)^{r-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

the summation extending over the y_1 specimens in the interval.

Taking as an example the measurements on 10 in. and 30 in. lengths of 32's American ring yarn, of which the results are given in Table I. of Paper I., the distribution for the 30 in. lengths deduced from that for the 10 in. lengths are compared with the observed distribution in Table I. below (last two rows).

Table I.

Comparison of Breaking Loads of 10 and 30 in. specimens—32's American Ring Yarn

Interval	130 to	140	150	160	170	180	190	200 gm.
y_1	0	0	3	10	20	18	29	24
N'	200	200	−198	−188	−168	−150	−121	−97
$3\Sigma(N'/N)^2$ ...	—	—	9	28	46	34	39	21
y_3	2	0	3	12	27	38	33	29

Interval	210 to	220	230	240	250	260	270	280 gm.
y_1	23	36	17	8	8	3	0	1
N'	−74	−38	−21	−13	12−5	4−2	1	1
$3\Sigma(N'/N)^2$ ...	13	9	1	0	0	0	0	0
y_3	24	14	10	7	1	0	0	0

The derived and observed distributions agree well enough, considering the small number of observations for the operation of the law of probability, but show a discrepancy due to a general property of yarn. The actual loss in strength and decrease in spread are less than those calculated because the average deviation between consecutive specimens is less than that between specimens widely separated or chosen at random, i.e. $\varphi(f)$ fluctuates along the length to a slight extent.

The frequency curves from the tests on 600 specimens of 9 in. and 27 in. lengths are shown in Fig. 1. The observed curve for the longer length (II)

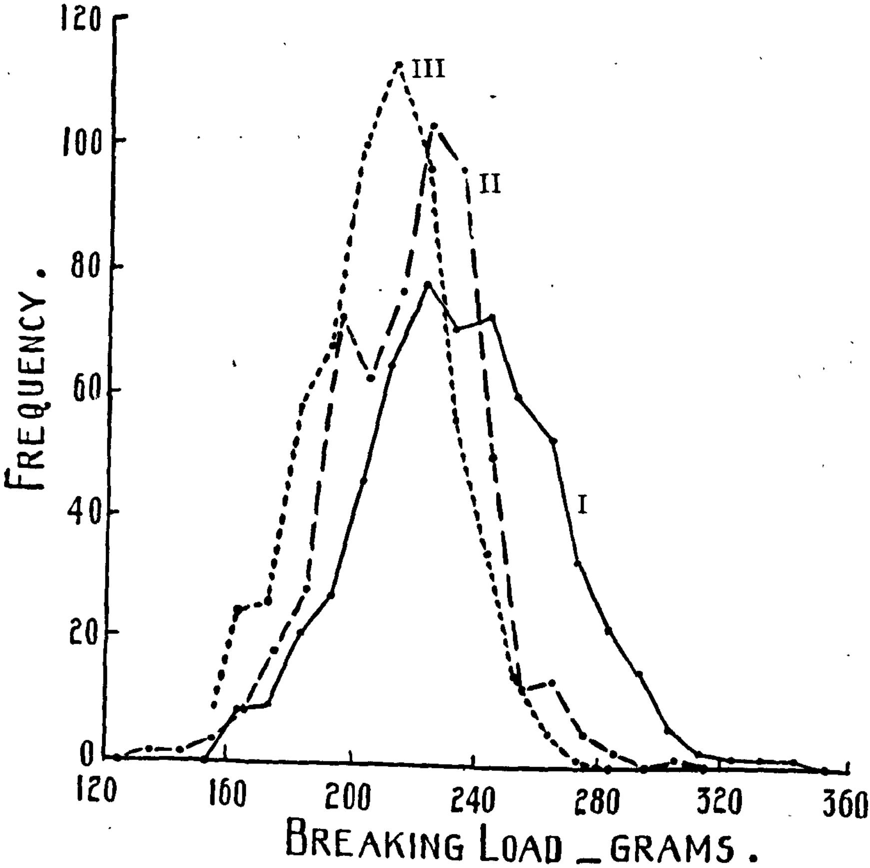

FIG. 1

shows a marked decrease in mean strength and in standard deviation, but the change is less than that given by the transformation (III) of the curve for the short specimens (I).

To obtain an analytical expression for the variation with length, it is necessary to assume one for the original frequency curve. Though observed frequency arrays are commonly irregular and may be skew, the normal curve is the most convenient general form and gives a sufficiently close approximation for the present purpose. The distribution of strength for a multiple length is then given by writing in Equation (1)—

$$\varphi(f) = \frac{h}{\sqrt{\pi}} e^{-h^2 (f-a)^2} \qquad (2)$$

where a is the mean and h the "modulus of precision" for lengths l, the latter being equal to $1/\sqrt{2}\sigma$ (σ being the Standard Deviation). Then, writing $f - a = x$

$$y_r = r \cdot \frac{h}{\sqrt{\pi}} \cdot e^{-h^2x^2} \left[\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-h^2x^2} \cdot hdx - \frac{1}{\sqrt{\pi}} \int_0^x e^{-h^2x^2} \cdot hdx \right]^{r-1}$$

$$= \frac{h}{\sqrt{\pi}} \cdot e^{-h^2x^2} \cdot r \left[\frac{1 - \Phi(hx)}{2} \right]^{r-1}, \text{ where } \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} \cdot dx \qquad \text{(1b)}$$

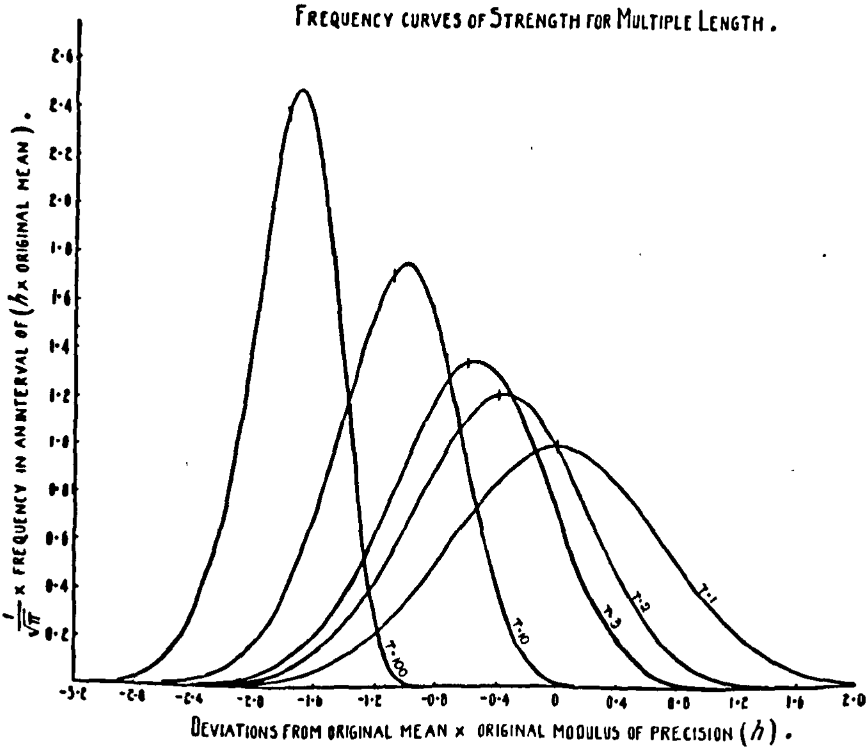

FIG. 2

The error function, $\Phi(x)$, cannot be obtained in the form of an integrated algebraic expression, and in consequence no such form can be obtained for the frequency function, y_r, the mean or standard deviation of the breaking loads of specimens of length rl. The error function can, however, be integrated by series and tables of its values are published. From these, curves have been drawn (Fig. 2) showing a normal frequency curve for a length l and the derived curves for lengths $2l$, $3l$, $10l$, and $100l$. For simple and general application, these have been plotted against multiples of hx for abscissa and $1\sqrt{\pi}$ for ordinate.

It will be seen that the mean strength decreases continuously with length, and the height of the peaks show that the standard deviation also decreases. In addition the curves become negatively skew so that the mean, median, and mode are no longer identical. Of these, the last two can be obtained without integrating y_r.

The mode is given by the condition—

$$\frac{d}{df}\cdot y_r = 0$$

or

$$\varphi(f)\cdot\left[\int_f^{\infty}\varphi(f)\cdot df\right]^{r-2}\cdot\left\{(r-1)\,\varphi(f)+2h^2(f-a)\int_f^{\infty}\varphi(f)\cdot df\right\}=0$$

The first two factors are zero when $f = \pm\infty$, the mode being given by the condition that the expression in curly brackets be zero, or—

$$\frac{h^2(a-f)}{\varphi(f)}\cdot\int_f^{\infty}\varphi(f)\,.\,df = \frac{r-1}{2}$$

or

$$\frac{hx}{e^{-h^2x^2}}\left[\frac{1-\Phi\,(hx)}{2}\right] = -\frac{r-1}{2\sqrt{\pi}} \quad . \quad . \quad . \quad (2a)$$

The median is given by the condition that one half the specimens contain a length l of strength less than the median, i.e. the probability that 1 in r lengths has a strength less than the median is $\frac{1}{2}$. The probability that any one length deviates below the mean by more than hx is

$$\frac{1-\Phi\,(hx)}{2}$$

that one or more of r lengths shows such a deviation is

$$1-\left[1-\frac{1-\Phi\,(hx)}{2}\right]^r = 1-\left[\frac{1+\Phi\,(hx)}{2}\right]^r$$

which probability is $\frac{1}{2}$ for strengths below the median of lengths rl. Hence, if hx be the deviation from the original mean of this median,

$$\left[\frac{1-\Phi\,(hx)}{2}\right]^r = \frac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$\frac{1-\Phi\,(hx)}{2} = 2^{-\frac{1}{r}}$$

The mean strength of the multiple lengths, a_r, is given by

$$a_r = \int_{-\infty}^{+\infty} y_r\,.\,f\,.\,df \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which can in general only be evaluated by calculations from plotted curves.

It is a simple matter to plot the functions of (hx) on the left hand side of Equations (2a) and (3), and read off the values which satisfy these equations for any value of r. A number of such values are shown in Table II. and the curves for the decrease in median and mode are given in Fig. 3. It is generally approximately true of a skew curve that the difference between the mean and mode is three times that between mean and median. The values of the mean obtained on this assumption are also shown.

Table II.*
The Strength of Multiple Lengths

r	Mode	Median	Mean	S.D. $\times\sqrt{2}$	—Sk.	$r^{-\frac{1}{4}}$	$3(1-r^{-\frac{1}{4}})$
1	0	0	0	1	0	1	0
2	·360	·386	·399	·82	·067	·87	·388
3	·543	·580	·598	·74	·106	·80	·592
4	·663	·705	·726	·70	·128	·76	·727
5	·751	·797	·820	·66	·147	·72	·826
10	1·006	1·062	1·09	·58	·205	·63	1·11
30	1·351	1·414	1·45	·49	·271	·51	1·48
40	1·434	1·496	1·53	·48	·276	·48	1·57
80	1·622	1·688	1·72	·44	·316	·42	1·75
100	1·679	1·743	1·77	·42	·324	·40	1·81
160	1·796	1·860	1·89	·41	·334	·36	1·91
275	1·922	1·988	2·02	·39	·358	·33	2·02
1000	2·202	2·266	2·30	·35	·388	·25	2·25

*The Skewness, Sk $= \dfrac{\text{Mean-Mode}}{\text{S.D.}}$. The other quantities are given as the deviation below the original mean multiplied by the modulus of precision h of the normal curve for $r = 1$. As mean and mode are found from plotted curves, the last figure is uncertain. To express these values as ratios of the S.D. of the original curve, multiply by $\sqrt{2}$.

The value for S.D. $\times \sqrt{2}$ is the inverse of the height of the curve, in Fig. 1, at the mean.

The last two columns give a general approximate formula for S.D. and mean.

As the difference between mode and median rapidly approaches a constant value 0·065, the skewness for large multiples is given approximately by $-0{\cdot}138\, r^{-\frac{1}{4}}$.

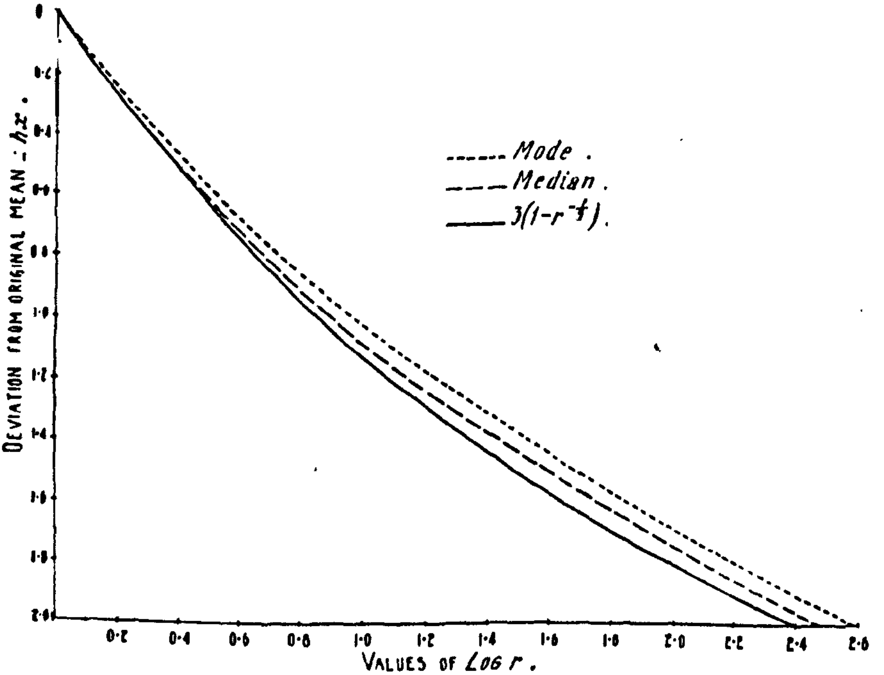

Fig. 3

*Since this was written, a paper[4] has been published with a very detailed table of the range between the extreme individuals of samples taken from a normal population. From Table X., *loc. cit.*,[4] the values given in this table can easily be calculated to six significant figures. The tables agree closely though obtained independently by very different methods. Table I.[4] also shows that the approximation to the S.D. is very satisfactory.

If the curves remained normal, the standard deviation would be inversely proportional to the height of the curve. Owing to the introduction of skewness the spread is somewhat greater, and it is better and sufficiently accurate to take the ordinate at the mean as the measure of S.D. The skewness is then given by—

$$\mathrm{Sk} = \frac{3}{\sqrt{2}} \cdot \frac{\text{median} - \text{mode}}{y_r \text{ at mean}}$$

the median and mode being given by deviations below the original mean in terms of $1/h$.

From the curves $r = 10$ and $r = 100$, the mean and S.D. were calculated from the ordinates at intervals $hx = 0{\cdot}05$, as from an experimental curve, giving values, expressed as in Table II., for the mean 1·087 and 1·769, for $\sqrt{2} \times$ S.D. 0·605 and 0·441, and for skewness $-$ 0·189 and $-$ 0·288, respectively.

The mean for the curve $r = 2$, may be obtained by an independent argument, assuming normal distributions. Suppose a length $2nl$ be tested in $2n$ lengths of l, and alternatively in n lengths of $2l$. Let f be the strength of the weaker of a pair of lengths l, f' of the stronger.

$$\text{The mean of the } 2n \text{ lengths} = \frac{\Sigma f + \Sigma f'}{2n},$$

$$\text{of the } n \text{ double lengths} = \frac{\Sigma f}{n} = \frac{2\Sigma f}{2n}.$$

$$\text{The difference of the means} = \frac{\Sigma(f' - f)}{2n}$$

$$\text{In a normal distribution } \sqrt{\frac{\Sigma(f' - f)^2}{n}} = \sqrt{2} \cdot \sigma$$

where σ is the S.D. for the single lengths, and f and f' are any two values taken at random, also the mean absolute difference is $\sqrt{2/\pi}$ times the root mean square difference, or

$$\frac{\Sigma(f' - f)}{n} = \frac{2\sigma}{\sqrt{\pi}}$$

$$\therefore \text{The decrease in mean strength} = \frac{\sigma}{\sqrt{\pi}} = 0{\cdot}565\,\sigma = 0{\cdot}399/h.$$

The above values for the mean when $r = 2$, 10, and 100, agree within a fraction of 1% with those calculated from the mode and median, which may be accepted for all the values of r. The argument for double lengths shows most simply the fact that the loss in strength is determined by the differences within the multiple length, the "internal" variability of long specimens, and that the changes in strength and regularity will be less than those calculated from the variability of a large batch of short specimens, when, as in cotton yarn, consecutive lengths tend to be more similar.

In practice it is most convenient to express the changes in terms of the mean and standard deviation. If a_l be the mean from lengths l, σ_l the standard deviation, then—

$$a_l - a_{rl} = v_r \cdot \sigma_l \qquad (5)$$

$$\text{and} \quad \sigma_{rl}/\sigma_l = u_r \qquad (6)$$

where v_r and u_r are independent of the variability and are known functions of the multiple r, if the curve for lengths l be normal. It is not practicable, from the irregular frequency curves actually found from yarn, to determine whether the shorter length gives a normal curve nor to allow for skewness if present. To obtain a general expression for the effect of length on strength, small changes in v_r and u_r due to skewness must be ignored and simple relations between them then appear. For, from (5) and (6)

$$a_l - a_{rsl} = v_{rs} \, . \, \sigma_l \qquad (5a)$$
and $$\sigma_{rsl}/\sigma_l = u_{rs} \qquad (6a)$$
Also $$a_{rl} - a_{rsl} = v_s \, . \, \sigma_{rl} \qquad (5b)$$
and $$\sigma_{rsl}/\sigma_{rl} = u_s \qquad (6b)$$
$$\therefore \quad a_{rl} - a_{rsl} = (v_{rs} - v_r)\,\sigma_l$$
and $$\sigma_{rsl}/\sigma_{rl} = u_{rs}/u_r$$

Whence $$v_s\sigma_{rl} = (v_{rs} - v_r)\sigma_l \quad \text{and} \quad \frac{\sigma_{rsl}}{\sigma_{rl}} = \frac{u_{rs}}{u_r}$$

or $$v_s = \frac{v_{rs} - v_r}{u_r}$$

and $$v_{rs} = v_r + v_s \, . \, u_r = v_s + v_r \, . \, u_s$$

$$\therefore \quad \frac{v_r}{1 - u_r} = \frac{v_s}{1 - u_s} = \text{constant} = c$$

or $$v_r = c\,(1 - u) \qquad (7)$$

But $u_{rs} = u_r \, . \, u_s$, whence by continuing to factorise r and s comes the relation that u for any value of r is equal to the product of u for the factors of r, or to the nth power of u for the nth root of r.

$$\therefore \quad u_r = u_e^{\log_e r} = r^b \text{ where } b = \log_e u_e \qquad (8)$$
and $$v_r = c(1 - r^b); \quad u_r = r^b \qquad (9)$$

The values of c and b from the values given in Table II. change slowly with r concurrently with the increase of skewness, but a sufficiently good approximation is given by the values 3 and $-\frac{1}{8}$ respectively, when the change of mean is expressed in units $1/h$, or 4·2 and $-\frac{1}{8}$ in Equations (5) and (6), and 5·3 and $-\frac{1}{8}$ when calculating the effect from the mean deviation of the shorter lengths. The values given by this general formula, *vide* Table II., are as close as the assumption of normal distribution for an empirical frequency array warrants.

SKEWNESS

It has been shown that if the strength of specimens of any given length varies according to the normal law, then the frequency curve for longer specimens is negatively skew or, in general, strength distributions cannot be expressed by a normal or symmetrical frequency curve. The argument remains unaltered for any quantity of which the value for a specimen is determined by the minimum value occurring among its elements. By simply subtracting each value from a large fixed value, it is seen that positive skewness is necessarily introduced into the distribution of any quantity determined by a maximum value among the elements of the specimen, e.g., the thickness of cloth measured between plates without compression. To a less degree, but sufficient to make the normal law theoretically inapplicable, skew distributions will be found whenever the deviations of individual values are affected unequally by equal deviations of opposite sign among the constituent elements of a specimen, e.g. the torsional rigidity of rods or the bending moments in bars.

This class of quantity is so large and important that it is highly desirable to reproduce (1b) with an algebraic type of curve. As it is derived from the normal curve, the type to suggest itself is

$$y = e^{-h^2x^2} (a + bx + cx^2 + \ . \ . \ .). \quad . \quad . \quad . \quad (10)$$

Some attempts were made to obtain analytical expressions for the constants of such a formula by fitting it at the median, mode, and original mean, but even with one or two terms only, the result is a complicated implicit function for the decrease of mean. The curve for long specimens, say $r = 100$, is closely imitated by

$$y = y_0 x^{-p} \, . \, e^{-r/x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Elderton[2] gives this curve fitted to mortality-age figures and it is not unreasonable to connect up such statistics with the above analysis by supposing death to result from the failure of the weakest of a number of bodily functions, the potential life of each varying in a normal manner. There does not seem, unfortunately, to be any algebraic expression for curves of type (1b) comparable in simplicity with the normal law.

The skewness, like the drop in strength, depends on the "internal" variability. If this be small compared with the variations between individuals due to their different origin, the skewness will be negligible or determined by the nature of the production. The distribution of strength of cotton hairs from a commercial sample does not show systematic skewness, being governed by the large differences between individuals due to differences in growth, these fitting into the criterion for normal distribution, that "the deviations be constituted by the summation of a very large number of independent deviations."[5]

In the case of cotton yarns, the element of length is that which is just too great to be spanned by a cotton hair. For specimens a few inches in length, the strength distribution should be unaffected by the skewness due to "internal variability" and be governed by the nature of production. This does not obey the normal criterion as the number of hairs per section gives a positively skew array, owing to the tendency of bunches of hairs to remain together in drafting. Moreover, twist runs into thin more than thick places, so that the strength increases with fibre number to a power lower than unity, which also tends to introduce positive skewness.[3] Very short specimens should therefore give more or less positively skew distributions. Andrews and Oxley[1] have found the skewnesses $+ 0{\cdot}085$ and $+ 0{\cdot}173$ from two sets of 5,000 observations on 3 in. lengths. With longer specimens this skewness should decrease to zero and become negative. Specimens of 12 in., tested by pendulum and Moscrop testers, show no systematic skewness which agrees with theory, the skewness for $r = 4$ being $- 0{\cdot}128$.

LOCAL TENDERING

In tests of tendering by wear and by light, a portion of each specimen is often exposed, the remainder being unaltered. The effect cannot then be strictly estimated by comparing the strength of untreated and treated specimens as the original strength of the tendered portions would be that of the partial, not the whole, length.

Let the total length of specimen be rl, the length tendered l, $\varphi(f)$ the frequency function of strength for untendered lengths l, $\varphi'(f)$ of tendered lengths. The strength distribution of untendered lengths rl is as before given by Equation 1. For lengths rl, of which a portion l has been tendered,

$$y'_r = \varphi'(f)\left[\int_f^\infty \varphi(f)\,.\,df\right]^{r-1} + (r-1)\,\varphi(f)\left[\int_f^\infty \varphi(f)\,.\,df\right]^{r-2}\int_f^\infty \varphi'(f)\,.\,df \qquad (12)$$

This is a rigorous and general expression which may be greatly simplified for practical purposes if the tendering is large enough to localise the breakages. In this case the apparent tendering is $(a_r - a_1')$ and the real tendering—

$$\begin{aligned}(a_1 - a_1') &= (a_r - a_1') + (a_1 - a_r)\\ &= (a_r - a_1') + v\sigma_1\\ &= (a_r - a_1') + \frac{v}{u}\cdot\sigma_r \qquad (13)\end{aligned}$$

As the effective value of σ in yarn may be rather less than that calculated in the usual way over a large batch, the value of r should be kept as small as possible to minimise the correction. A practical and simple correction of $(+ v\sigma_r)$ would be in accord with the theory and with the experimental results shown in Table I., Paper I.

COMPOSITE SPECIMENS

Actual specimens are never mathematical units of homogeneous properties, but may be regarded as built up of elements both of length and cross-section. The relation between the strength of elements of length and of long specimens has been derived; that between elements of cross-section and the composite specimen is of a more complicated nature. The problem is involved in the strength of leas of yarn, of a yarn in terms of fibre strength, of a fabric in terms of yarn strength, of electric wire flex, of silk threads, and even of a wooden or metal bar if the cross-section be regarded as constituted of parallel elements.

The relation between the strength of the elements and of the composite specimen depends on the variations of tension and of limiting extension among the strands. Actual conditions may be approximated to by one of several mathematically definite cases.

Case (*a*)—Strands gripped at the ends, of equal original length and uniform breaking extension. However variable the breaking load of single strands, the maximum load on the specimen will be taken just before any strand breaks and will be the sum of the breaking loads of all the strands. As a lea test, this would give a constant 100% "lea ratio," but the conditions are not nearly realised in practice, depending on a very unlikely characteristic of the threads, and this ideal furnishes a most unsound basis for analysis of the test.

Case (*b*)—Strands gripped at the ends and under uniform tension. This case is approximated to in a lea test if the ratio of load to extension is very regular, particularly if the lea is wound under a high uniform tension. The load on the specimen F is $(N - n)f$, where N is the total number of strands, n the number broken, f the tension on each strand. If the frequency curve for single-strand breaking load be $y = \varphi(f)$ and N be large,

$$\frac{N-n}{N} = \int_f^\infty \varphi(f).\ df \qquad (14)$$

and $$\frac{F}{N} = f\int_f^\infty \varphi(f).\ df \qquad (15)$$

F is a maximum when $\frac{dF}{df} = 0$, or

$$\frac{1}{N} \cdot \frac{dF}{df} = \int_f^\infty \varphi(f).\, df - f\,\varphi(f) = 0 \qquad (16)$$

The tension on each unbroken strand at the maximum load is f or $(a + x)$ where—

$$f = \frac{1}{\varphi(f)} \int_f^\infty \varphi(f)\,.\,df = \frac{N-n}{Ny} \qquad (16a)$$

for an empirical frequency curve, or

$$h(a+x) = \frac{1 - \Phi(hx)}{2} \cdot \frac{\sqrt{\pi}}{e^{-h^2x^2}} \qquad (16b)$$

for the normal curve. The ratio of the strength per strand to the mean single-strand strength is then—

$$\frac{F}{Na} = \frac{f^2\,\varphi(f)}{a} = \frac{N-n}{N} \cdot \frac{f}{a} \qquad (15a)$$

for an empirical curve,

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{h^2(a+x)^2}{ha} \cdot e^{-h^2x^2} \qquad (15b)$$

for the normal curve. The proportion of unbroken threads is—

$$\frac{N-n}{N} = f\varphi(f) = f\,.\,y \qquad (14a)$$

for an empirical curve,

$$= \frac{h(a+x)}{\sqrt{\pi}} \cdot e^{-h^2x^2} \qquad (14b)$$

for the normal curve.

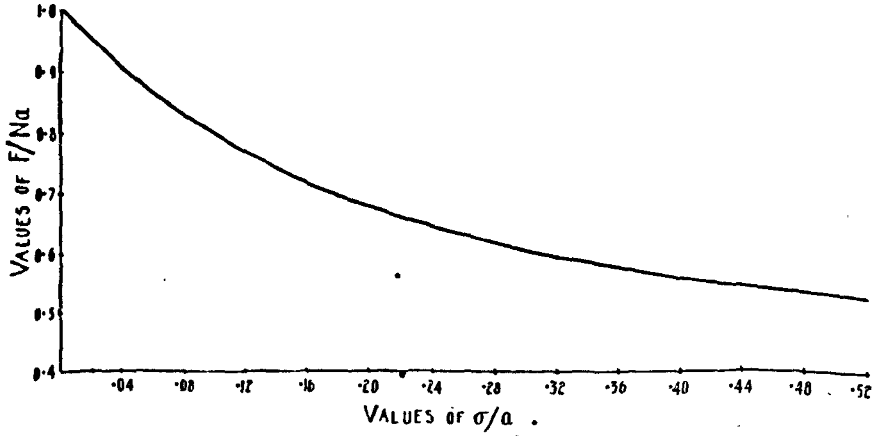

FIG. 4

These equations apply to a continuous frequency curve strictly only when all strength intervals are equally represented in the specimen. The mean breaking load of a number of specimens of finite size will vary about a mean somewhat greater than that given by (15b). This relation, expressed in terms of the standard deviation, is shown graphically in Fig. 4. Slight differences of tension among the strands may be allowed for by corresponding changes in breaking load, i.e. by adjusting the value of h.

Case (c)—Gripped strands originally of the same length. The extension e of the strands at any moment is uniform, the number of unbroken ends is

$$\frac{N-n}{N} = \int_e^\infty \varphi(e).\,de \qquad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where $\varphi(e)$ is the frequency distribution of breaking extension.

The force on the specimen will be affected by any correlation between breaking extension and the load-extension ratio for single strands. Tests made on the Shorter instrument on 45 threads of 36's Sakel yarn (Paper III.) showed that the mean ratios for the survivors after no breaks, the first five lowest extensions, the second five, &c., were ·773, ·770, ·764, ·761, ·760, ·756, ·762, ·769, ·755. It is accurate enough to use an unvarying mean ratio of load to extension, $E = f/e$, when—

$$\frac{F}{N} = E.\,e.\int_e^\infty \varphi(e).\,de \qquad . \quad . \quad . \quad . \quad . \quad (18)$$

Then F is a maximum when—

$$e = \frac{1}{\varphi(e)}\cdot\int_e^\infty \varphi(e)\,de \qquad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and the ratio of strength per strand to mean single-strand strength is given by substituting this value in the formula.

$$\frac{F}{Na} = \frac{E}{a}\cdot e^2\varphi(e) = \frac{e^2}{\bar{e}}\cdot\varphi(e) \qquad . \quad . \quad . \quad . \quad . \quad (18a)$$

This result is identical with the foregoing, except that the frequency curve of breaking extension is substituted for that of breaking load. The decrease in strength per thread for cases (b) and (c) can be found from Fig. 3. Case (c) probably gives the nearest simple approximation to a large number of physical cases, including the lea test, when the threads are gripped so tightly that they do not slip when neighbouring threads break.

Case (d)—Strands maintained at uniform tension and all slip when one breaks. This is realised when a long continuous thread is wound over hooks or pulleys incapable of maintaining differences of tension. The case is fully covered by the formulæ for long specimens.

Case (e)—Strands of uniform original length uniformly extended (gripped) till one breaks, when all slip. This bears the same relation to (d) that (c) does to (b).

The breaking extension is the minimum value among the individual strands and is given by formulæ mathematically identical with those for the breaking load of a long specimen.

If e_r be the mean breaking extension of a specimen of r strands, S_r the standard deviation,

$$e_r = e_1 - vS_1 \text{ and } \frac{S_r}{S_1} = u \qquad . \quad . \quad . \quad (20)$$

The mean ratio of load to extension E may be taken to a first approximation as constant for its variability decreases as $r^{\frac{1}{2}}$ against $r^{\frac{1}{4}}$ for the breaking extension of the whole specimens. Then—

$$E\,e_r = E\,e_1 - rES_1$$

$$\text{or} \quad a_r = a_1\left(1 - v \cdot \frac{S_1}{e_1}\right) \quad \ldots\ldots \quad (21)$$

$$\text{and} \quad \frac{\sigma_r}{a_r} = \frac{S_r}{e_r} = \frac{S_1 u}{e_r} = u \cdot \frac{e_1}{e_r} \cdot \frac{S_1}{e_1}$$

$$\text{or} \quad \sigma_r = u \,.\, E \,.\, S_1 \quad \ldots\ldots \quad (22)$$

Equation (22) is approximate in so far as it neglects the variability of E which may be appreciable on specimens of a small number of strands. To take it into account would involve consideration of the correlation between E and e, and the complication is not justified as the equation is nearly true even for $r = 1$, the variability of load being of the same order as that of extension.

THE LEA TEST

When a lea of yarn is wound and tested without particular care, a rough approximation is made to case (*c*) and case (*e*), varying between them according to adventitious circumstances. If the ends be clamped the approximation to case (*c*) is improved, but the difficulty of obtaining uniform initial tension and of avoiding tearing is so great that actually the results are lower than without clamping. The case is, moreover, much less simple than case (*e*) in correlating with single-thread properties.

If a thread is led over a number of light well-mounted pulleys, a close approximation to case (*d*) is obtained which gives the simplest possible relations, but the method could hardly be applied to more than a few turns. If wound over the usual hooks to a fewer number of turns than a whole lea, and carefully placed so that the threads do not interfere with each other, and if loaded so slowly that the threads can slip without increase of load at the first break, then the conditions are very accurately those of case (*e*), and the relations simple and of useful significance.

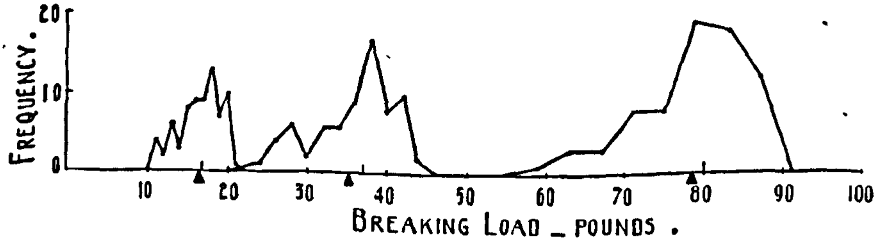

FIG. 5

The friction between threads and hook allows a ratio of tensions of 1·6 between the two sides (Paper I., Appendix 1). This does not allow any material differences of tension at the beginning of loading and the threads slip slightly to adjust any unevenness. The tension developed in stretching varies according to the ratio of load to extension. From the Shorter diagrams of 36's Sakel yarn referred to in case (*c*), the standard deviation of this ratio is 6·4% of the mean. The probable difference of tension between two adjacent threads is then 6·1%, and the probability of differences of 60%

over the tension developed in even the most extensible threads is vanishingly small. The suggested modification of the lea test is therefore a very close realisation of case (*e*).

Negative skewness is introduced by the conditions of this case to the same extent as for long specimens, that introduced by a multiple of 160 being 0·334. The frequency polygons for the strength of 72 leas, half-leas, and quarter leas of 60's West Indian yarn wound off a cheese are shown in Fig. 5. The skewnesses, calculated from the mean and median, are respectively − 0·565, − 0·906, and − 0·463, and, furthermore, they approximate to a smooth curve of the type derived analytically (Fig. 1).

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