Groups as Metric Spaces

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Chapter 1

Introduction

Let G be a discrete group and S a finite generating set for G. If $g \in G$ we can define a norm as follows

$$\| f \|_s = min\{n \in \mathbb{N}_0 | g = s_1^{\pm 1} s_2^{\pm 1} \dots s_n^{\pm 1}, s_i \in S\}.$$

We see from the definition that $\| g \| \geq 0$, $\| g \| = 0 \iff g = 1$, $\| gh \| \leq \| g \| + \| h \|$, and $\| g^{-1} \| = \| g \|$. Thus we can define a metric on G via the norm:

$$d(g,h) = \| g^{-1}h \| = \| h^{-1}g \| .$$

d is a distance function on G since

$d(g,h) \geq 0$

$d(g,h) = 0 \iff g = h$

$d(g,h) = d(h,g)$

$d(g,k) \leq d(g,h) + d(h,k)$

Definition 1 (Cayley Graph). *The vertices are the elements of G and two vertices g, h are joined by an edge if and only if $d(g,h) = 1$.*
We can extend the metric on G to the whole graph as follows: identify each edge with the interval $[0,1]$. The distance between two points x and y is the infimum on the lengths of paths between them:

$$d(x,y) = \inf \{l(\alpha) \,|\alpha \text{ is a path from } x \text{ to } y\}$$

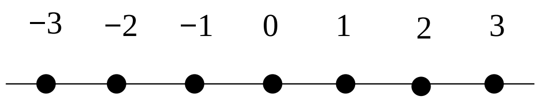
Remark. 1. *As we shall see in the examples below the Cayley graph $\mathcal{C}(G,S)$ depends on the generating set S.*

2. *Notice that this infimum is always realized since lengths of paths between vertices are always integer. Hence fixing x and y, the length of any path between them lies in a discrete set of the positive reals.*

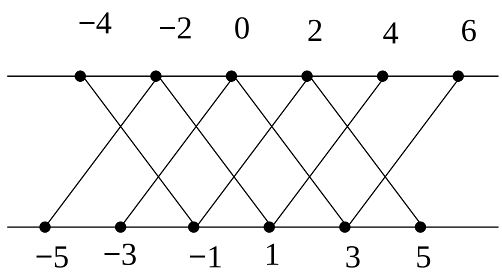
3. *Fix $g \in G$, left multiplication $L_g : G \to G$ defined by $L_g(h) = gh$ is an automorphism of (any) Cayley graph. Furthermore, it is an isometry, i.e.*

$$d(g(x), g(y)) = d(x, y)$$

Example 2. 1. *$G = (\mathbb{Z}, +)$ and $S = \{1\}$*

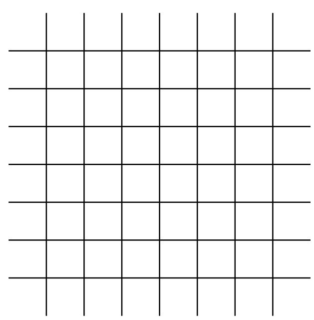


2. *$G = (Z, +)$ and $S = \{2, 3\}$*



What do these cayley graphs have in common? If we zoom out then (2) looks like a line. We are going to study properties of Cayley graphs that do not depend on the choice of generating set. These properties are "intrinsic" properties of the group.

3. *$G = \mathbb{Z} \oplus \mathbb{Z}$ and $S = \{(1, 0), (0, 1)\}$*

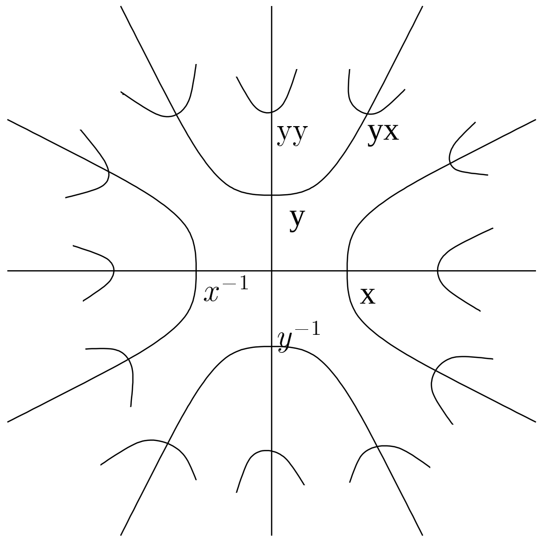


We would like at this point an equivalence relation that has the property that grid $\sim$ plane, since if we zoom out on the picture of the grid we get the plane.

4. *$G = F_2 = F_2(x, y)$ with $S = \{x, y\}$*

   Another thing that we would like are invarients to distinguish the Cayley graphs that we have so far.

5. *Infinite dihedral group D_∞. $D_\infty \subset Isom(\mathbb{R})$ where D_∞ consists of translations by even integers, and reflections in integer points.*



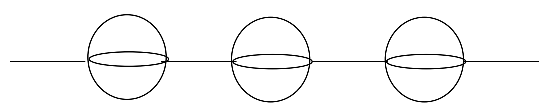
Exercise 3. *Check that D_∞ is a group.*

Let $S = \{r_0, r_1\}$ where r_i is reflection about the integer i. There is also a presentation for D_∞ which is

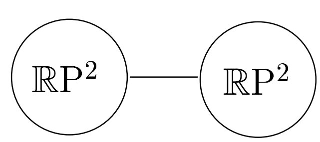
$$D_\infty =< r_0, r_1 | r_0^2 = 1, r_1^2 = 1 > .$$

Question 4. *Why are the two descriptions of D_∞ equivalent?*

Answer 5. *Replace each integer point by S^2 (could have used any simply connected space on which $\mathbb{Z}_2$ acts freely maybe S^∞ since it is contractible) and call this space $\mathcal{X}$.*

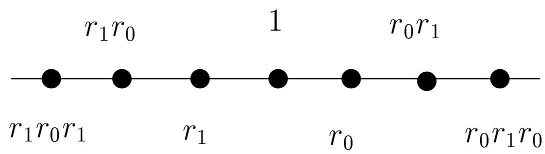
$\mathcal{X} =$ $\dots$  $\dots$

Now D_∞ acts on $\mathcal{X}$ as a covering group with the action on S^2 given by the antipotal map. The quotient is the following.

$\mathcal{X}/D_\infty =$ 

So $D_\infty \cong \pi_1(\mathcal{X}/D_\infty) = \mathbb{Z}_2 * \mathbb{Z}_2$ and thus the Cayley graph is which is the same Cayley graph as $\mathbb{Z}$.

Proposition 6. *Let S, S' be two finite generating sets for G. Then the identity map $G \to G$ is a bilipschitz homeomorphism from one metric to the other.*



Definition 7. *1. (Lipschitz) $f : M \to M'$ is Lipschitz if $\exists C > 0$ such that*

$$d_{M'}(f(x), f(y)) \leq C d_M(x, y).$$

2. (bilipschitz) f and f^{-1} are both lipschitz.

Proof. (of Proposition) We will show that $id : G_S \to G_{S'}$ is lipschitz, the rest of the proof then follows symmetrically. Let C be the largest S'-norm of any element of S. Let $g \in G$. Say $n = \parallel g \parallel_s$ where $g = s_1^{\pm 1} \dots s_n^{\pm 1}$ but $\parallel s_i \parallel_{S'} \leq C$ so $\parallel g \parallel_{S'} \leq Cn$. If $n = d_S(g, h)$ then $n = \parallel g^{-1}h \parallel_S$, so $\parallel g^{-1}h \parallel_{S'} = d_{S'}(g, h) \leq Cn$. $\square$

An example of a bilipschiz invarient would be if a group has infinite diameter.

Definition 8. *(Gromov) Let X, Y be metric spaces. A function $f : X \to Y$ is called an (L, A)-quasi-isometry $(L > 0, A \geq 0)$ if*

1. *(Coarse Lipschitz) $\forall x_1, x_2 \in X$ $d_y(f(x_1), f(x_2)) \leq L \cdot d_x(x_1, x_2) + A$. It is important to note that this allows the function to not be continuous.*
2. *(linear lower bound) $\frac{1}{L} d_x(x_1, x_2) - A \leq d_y(f(x_1), f(x_2))$*
3. *(coarse surjectivity) There exists an $R > 0$ such that every point in Y is within R of $f(X)$.*

Example 9. *1. $\mathbb{Z} \subset \mathbb{R}$ where f is the inclusion map. This is a $(1, 0)$-quasi-isometry with $R = \frac{1}{2}$.*

2. Bilipschitz homeomorphisms from $X \to Y$ are examples with

$$(L = \text{bilipschitz constant}, A = 0, R = 0).$$

3. Any f which is within bounded distance of a quasi-isometry. For example there exists a quasi-isometry $f : \mathbb{R} \to \mathbb{Z}$ given by the floor function which has $L = 1$ and $A = 1$.

Definition 10. *X, and Y are called quasi isometric if there exists a quasi-isometry $f : X \to Y$ (when suppressing the quasi-isometry constants we implicitly allow any constants).*

Theorem 11. *Being quasi-isometric is an "equivalence relation" on metric spaces.* [1]

[1] Of course, the set of metric spaces is not a set in the set theoretic sense, so we cannot formally talk of an equivalence relation. The reader should consider this as shorthand notation for the three properties satisfied by equivalence relations.

Proof. We must show that the following properties are satisfied:

- Reflexive: For any metric space X, $id : X \to X$ is a $(1, 0)$ quasi-isometry.
- Symmetric: Suppose $f : X \to Y$ is an (L, A, R)- quasi-isometry, we must find constants L', A', R' and a map $g : Y \to X$ which is a (L', A', R')-q.i. g is called a quasi-inverse for f.
  We start by defining g: For any $yinY$ choose an $x \in X$ such that $d(y, f(x)) \leq R$, property QI(3) for f guarantees such an x [2]. Define $g(y) = x$.

  - **QI(1) for g:** Given y, y' denote $x = g(y)$ and $x' = g(y')$. By definition $d(y, f(x)), d(y', f(x')) < R$, by QI(2) for f:

  $$\frac{1}{L}d(x, x') - A \leq d(f(x), f(x') \leq d(y, y') + 2R$$

  $$d(x, x') \leq Ld(y, y') + L(2R + A)$$

  $$d(g(y), g(y')) \leq Ld(y, y') + L(A + 2R)$$

  - **QI(2) for g:** Given y, y' and x, x' as above, we have

  $$d(f(x), f(x')) \leq Ld(x, x') + A$$

  by QI(1) for f. Theredfore,

  $$\begin{array}{rcl} d(y, y') & \leq & d(f(x), f(x')) + 2R \\ & \leq & Ld(x, x') + A + 2R \\ & \leq & Ld(g(y), g(y')) + A + 2R \end{array}$$

  $$\frac{1}{L}d(y, y') + \frac{A+2R}{L} \leq d(g(y), g(y'))$$

  Notice that the multiplicative QI constant for g is the same as in f but the additive constant might be quite a bit larger.

  - **QI(3) for g:** Given $x \in X$ consider $x' = g \circ f(x)$. x' is in the image of g and it was chosen so that $f(x')$ lies within a distance R from $f(x)$. By QI(2) for f we have:

  $$\frac{1}{L}d(x, x') - A \leq d(f(x), f(x')) \leq R$$

  $$d(x, x') \leq L(R + A)$$

$\square$

Question (Gromov). *Classify groups up to quasi-isometry.*

[2] We must use the countable axiom of choice but let's not worry about these technicalities

1.0.1 The Fundamental Theorem of Geometric Group Theory

Definition 12 (Geodesic metric space). *Let X be a metric space, it is a geodesic metric space if any two points $x_1, x_2 \in X$ can be joined by a path $\alpha : [0,d] \to X$ which satisfies $\forall t, t' \quad 0 \leq t < t' \leq d$: $d(\alpha(t), \alpha(t')) = t' - t$. Such a path is a geodesic parametrized according to arc length.*[3]

Example 13. *$\mathbb{R}^n$ complete Riemanninan manifolds, Cayley Graphs. A non-example is $\mathbb{R}^2 \setminus \{origin\}$ with the metric induced by the plane. The distance between $(-1,0)$ and $(1,0)$ is 2 but it is clear that there is no geodesic between them. Indeed, suppose α was a geodesic between them then there must be a t for which $\alpha(t) = (0,y)$. Since $d((-1,0),(0,y)) > 1$ then $t > 1$ and since $d((0,y),(1,0)) > 1$ we get $2 - t > 1$ so $t < 1$ and we've reached a contradiction.*

Theorem (Milnor-Švarc). *Let X be a geodesic metric space and G a finitely generated group acting on X* by isometries, *i.e. $d(g(x), g(x')) = d(x, x')$. Suppose the action also satisfies the following:*

- *The action is* cobounded*: There exists a ball $B(x_0, R)$ whose translates cover X.*
- *The action is metrically* proper*: For any $r > 0$ the ball $B(x_0, r)$ the set $Q_r = \{g | B(x_0, r) \cap g \cdot B(x_0, r)\}$ is finite.* [4]

Then G and X are quasi-isometric and in fact $\pi : G \to X$ defined by $\pi(g) = g(x_0)$ is a quasi-isometry.

Example 14.

1. *$\mathbb{Z} \sim \mathbb{R}$*
2. *$\pi_1($ orientable surface with negative Euler characteristic$) \sim \mathbb{H}^2$*
3. *Take any group G acting on X by covering translations with compact quotient K then $G \cong \pi_1(K) \sim X$. If fact Milnor was interested in estimating volume growth in Riemanninan manifolds. He noticed that it only depended on the group of isometries.*

Proof. We must show that π is a quasi-isometry.

- π **is Lipschitz**: Suppose $\|g\| = n$ then $g = \gamma_1 \cdots \gamma_n$ where $\gamma_i = s^{\pm}$ and $s \in S$. We must show that we can bound $d(g(x_0), x_0)$ linearly with respect to n. Consider the sequence

$$1, \gamma_1, \gamma_1\gamma_2, \ldots, \gamma_1\gamma_2 \cdots \gamma_n = g$$

[3] This definition differs from another commonly used one where geodesics are autoparallel curves

[4] This implies that for any ball the set of elements that take it to an overlapping ball is finite, irrespective to the center because we can translate any ball inside a $B(x_0, r)$ for a large enough r.

Note that the distance in $\mathcal{C}(G,S)$ between consecutive elements in the sequence is 1 (since left multiplication is an isometry and the gammas are generators up to sign). The corrsepnding sequence in X is

$$x_0, x_1 = \pi(\gamma_1), \ldots, x_n = \pi(g)$$

$d(x_i, x_{i+1}) = d(\gamma_1 \cdots \gamma_i(x_0), \gamma_1 \cdots \gamma_{i+1}(x_0)) = d(x_0, \gamma_{i+1}(x_0))$ now this is not necessarily bounded by one, but we must recall that S is finite so $M = \min_{s \in S}\{d(x_0, s(x_0))\}$ exists. Hence $d(x_i, x_{i+1}) < M$. By the triangle inequality $d(x_0, g(x_0)) = d(x_0, x_n) \leq n \cdot M = M \cdot \|g\|$.

- **lower bound**: Choose a geodesic α from x_0 to $g(x_0)$ whose length is d. Subdivide intp segements of length ≤ 1 to get a sequence $x_0, x_1, \ldots, x_n = g(x_0)$ where $n = \lfloor d \rfloor$. By the second property, for any x_i there's a translate $g_i(x_0)$ of x_0 within a distance R from x_i. We now have a sequence $1, g_1, g_2, \ldots g_n$ which "interpolates" between 1 and g, we would like to bound $d(g_i, g_{i+1})$ to get a linear bound on $d(1, g)$ with respect to n.

$$d(x_0, g_i^{-1} g_{i+1}(x_0)) = d(g_i(x_0), g_{i+1}(x_0)) \leq d(x_i, x_{i+1}) + 2R = 2R + 1$$

  By the properness of the action, $\|g_i^{-1} g_{i+1}\| \leq C$ [5] Hence $d(g_i, g_{i+1}) \leq C$ and $d(1, g) \leq C \cdot n = C \cdot \lfloor d \rfloor \leq C d(x_0, g(x_0)) = C d(\pi(1), \pi(g))$.

- **almost onto**: By coboundedness any x lies within R of a $g(x_0) = \pi(g)$.

□

1.0.2 Corollaries of Milnor-Švarc

Proposition 15. *If G is a finitely generated group and H is a finite-index subgroup then $G \sim_{q.i.} H$.*

Proof. H acts on the Cayley graph of G with all the required properties to apply Milnor-Švarc. □

Definition 16. *The finitely generated groups G_1 and G_2 are* commensurable *if they contain finite-index subgroups $H_i < G_i$ such that H_1 and H_2 are isomorphic.*

Note that commensurability is clearly an equivalence relation. The only property to check is transitivity. If G_i for $i = 1, 2, 3$ are groups with finite index subgroups $H_1 \cong H_2$ and $H_2' \cong H3$, consider the subgroup $J = H2 \cap H2'$. J has finite index in H_2[6]. Now pull J back to subgroups of H_1 and H_3 under the corresponding isomorphisms. Now, $J_1 \cong J_3$.

[5]Take $r = 2R + 1$ in property 3 to get that Q_{2R+1} is finite and let C be the maximum norm of elements in Q_{2R+1}

[6]The intersection of two finite index subgroups is again a finite index subgroup. Indeed, $L < G$ is finite index iff G has an action on a finite set where L is the point stabilizer. Now if G acts on S_i with point stabilizer L_i for $i = 1, 2$, consider the G action on $S_1 \times S_2$ defined by $g \cdot (s_1, s_2) = (g \cdot s_1, g \cdot s_2)$. There is a point with stabilizer $L_1 \cap L_2$.

Definition 17. *We say that a finitely generated group G is* rigid *if whenever H is a finitely generated group and $H \sim_{q.i.} G$ then H and G are commensurable.*
A class of groups $\mathcal{G}$ is rigid if whenever $H \sim_{q.i.} G$ for $G \in \mathcal{G}$, there is a $G' in \mathcal{G}$ which is commensurable to H.

Examples:

1. The trivial group is rigid.

2. Wall: $\mathbb{Z}$ is rigid.

3. $\mathbb{Z}^n$ is rigid (first proved using Gromov's polynomial growth theorem).

4. F_n, the free group of rank $n, n > 1$, are rigid. It is also relatively easy to see that they are commensurable to one another. For each n, F_2 has a finite index cover with fundamental group isomorphic to F_n (see figure 1.0.2)

5. If $g > 1$ then the fundamental group of S_g the closed orientable surface of genus g, is rigid (Stallings and Dunwoody). It is elementary to see that for every g, S_g covers S_2, thus their fundamental groups are commensurable to each other.

6. π_1 of a closed hyperbolic 3-manifold is not generally rigid, but the collection of such fundamental groups is rigid *as a class*.

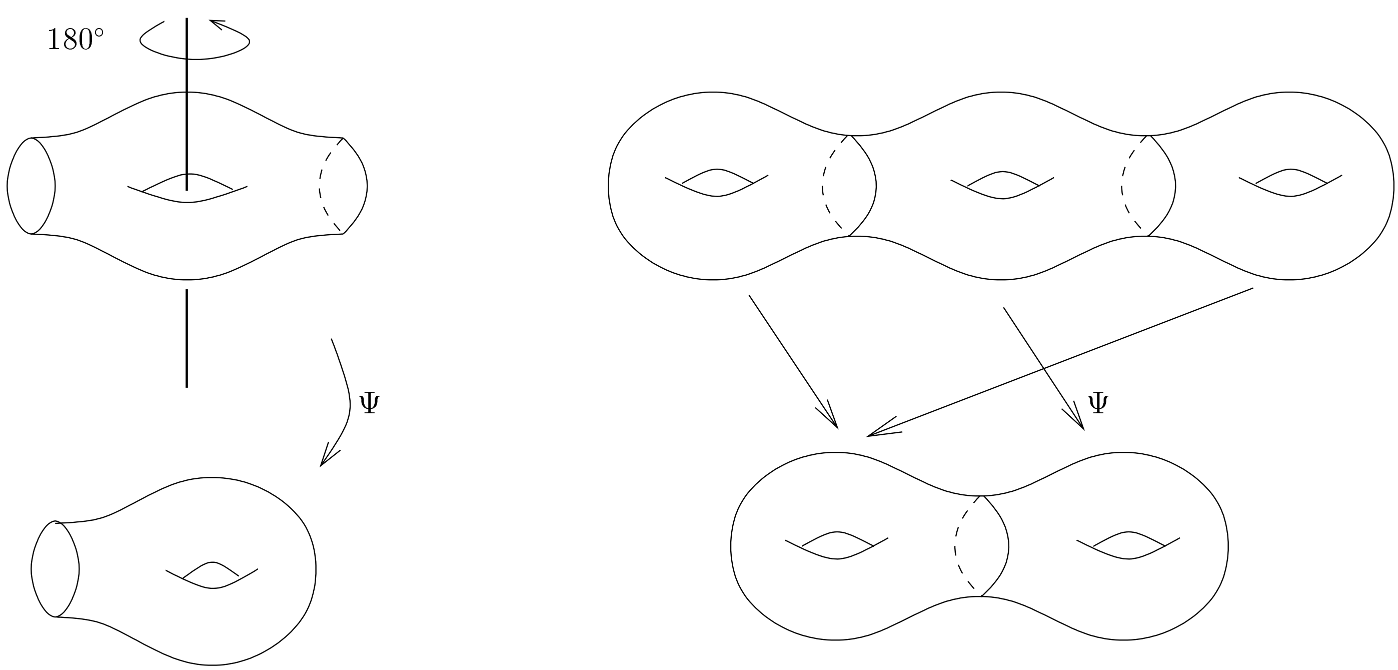


Figure 1.1: The genus 3 torus covers the genus 2 torus. Similarly, every genus g torus covers the genus 2 torus.

Definition 18. *A group G* virtually *has property P if some finite-index subgroup of G has property P.*

Definition 19. *Let X be a metric space and let $f, g : X \to X$ be quasi-isometries. We say that f and g are a bounded distance from each other, and write $f \sim g$ if there is a number $C > 0$ such that $d(f(x), g(x)) \leq C$ for each $x \in X$. Denote $QI(X) = \{[f] | f : X \to X \text{is a quasi-isometry}\}$, the quasi-isometry group of X.*

The group $\mathrm{Isom}(X)$ clearly projects into $\mathrm{QI}(X)$, and we want to understand this map.

Example 20. *If $X = \mathbb{R}$ then $Isom(X)$ consists of translations and reflections, but any two translations are equivalent to the identity in $QI(X)$. So, the only nontrivial quasi-isometry that is a projection of an isometry is the class of the reflections. So, the image of $Isom(X)$ under the projection has order 2.*

However, this doesn't exhaust the group of quasi-isometries. Let $Bilip(\mathbb{R})$ denote the group of bilipschitz homeomorphisms $\mathbb{R} \to \mathbb{R}$. Then the map $Bilip(\mathbb{R}) \to QI(R)$ is surjective (Exercise!).

Example 21. *Let X be the universal cover of ∞ (the Cayley graph of F_2). Then $Isom(X) \hookrightarrow QI(X)$ (Exercise!).*

Question 22. *How could we show that $\mathbb{Z}$ and $\mathbb{Z}^2$ are not quasi-isometric? That is, what kind of invariant could we use to distinguish them?*

Theorem 23. *If the map $Isom(X) \to QI(X)$ is an isomorphism and G is a group acting cocompactly and properly on X, and $H \sim_{q.i.} G$, then H acts on X cocompactly and properly* by isometries

$$H \hookrightarrow \mathrm{QI}(H) \cong \mathrm{QI}(G) \cong \mathrm{QI}(X) \cong \mathrm{Isom(x)}$$

The inclusion is just the H action on its Cayley graph. The first isomorphism follows from $H \sim_{q.i.} G$. The next isomorphism follows from the Milnor-Švarc theorem. The last isomorphism is just the hypothesis. We therefore get an H action on X by isometries. We still need to verify that the action is proper and cocompact.

Example 24. *If $X = SL^3(\mathbb{R})/SO_3$, Kleiner-Leeb proved that $Isom(X) \to QI(X)$, towards proving QI rigidity for the class of uniform lattices in $SL_3\mathbb{R}$.*

Papers we will look at this semester:

- Stallings' splitting theorem
- Gromov's polynomial growth paper (plus Kleiner's proof)
- Schwartz rigidity (If the fundamental groups of two noncompact hyperbolic 3-manifolds of finite volume are quasi-isometric then they are commensurable.)
- Kleiner-Leeb: Rigidity of higher-rank symmetric spaces.

Example 25. *$G_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, with A the map induced by a matrix $A \in SL_2(\mathbb{R})$. Then there is an exact sequence:*

$$1 \to \mathbb{Z}^2 \to G_A \to \mathbb{Z} \to 1.$$

For each A the group G_A falls into one of three quasi-isometry types.

1. *If A has finite order, then $G_A \sim_{q.i.} \mathbb{Z}^3$. In fact, they are commensurable because passing to A^k corresponds to a subgroup of finite index.*

2. *Let A be parabolic (trace$(A) = \pm 2$). For example, A could be $\begin{pmatrix} 1,1 \\ 0,1 \end{pmatrix}$. Then $G_A \sim_{q.i.}$ Nil, a certain 3-dimensional Lie group, and all are commensurable to each other.*

3. *Let A be hyperbolic ($|trace(A)| > 2$). That is, A has distinct real eigenvalues. For example, $A = \begin{pmatrix} 2,1 \\ 1,1 \end{pmatrix}$.*

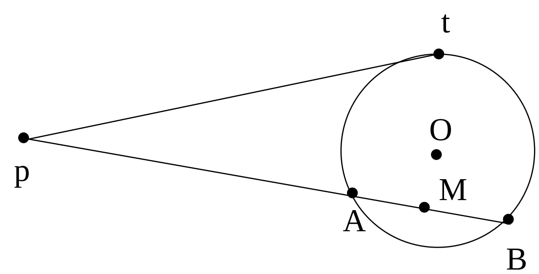
   Then $G_A \sim_{q.i.}$ Sol, a 3-dimensional solvable Lie group. Here most G_A are not commensurable to each other. For example, if $B = \begin{pmatrix} 1,1 \\ 0,1 \end{pmatrix}$ then G_B is not commensurable to the G_A for the given A.

Chapter 2

Hyperbolic Geometry

This take on Hyperbolic Geometry is from Thurston's Book

Fact 26. *(From Euclidean Geometry) $\vec{PA} \cdot \vec{PB}$ is independant of the choice of the line. In the following picture this says that $\vec{PA} \cdot \vec{PB} = \vec{Pt}^2$*



Proof.

$$\begin{aligned}
\vec{PA} = &\ \vec{PM} + \vec{MA} \\
\vec{PB} = &\ \vec{PM} + \vec{MB} \\
= &\ \vec{PM} - \vec{MA} \\
\vec{PA} \cdot \vec{PB} = &\ |PM|^2 - |MA|^2 \\
= &\ (|PO|^2 - |PM|^2) - (|OB|^2 - |OM|^2) \\
= &\ |PO|^2 - |OB|^2
\end{aligned}$$

□

Inversions in Circles

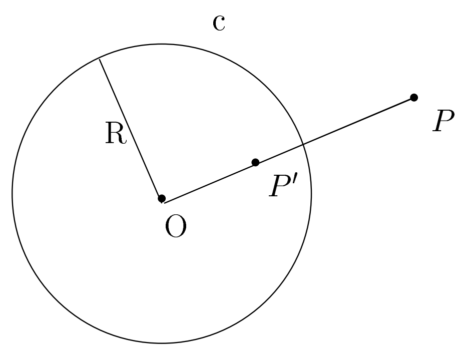
Let c be a circle such that $c \subset \mathbb{R}^2 \cup \infty = \{\text{Riemann Sphere}\}$. Define $i_c : \mathbb{R}^2 \cup \infty \to \mathbb{R}^2 \cup \infty$ via the following:

such that $\vec{OP} \cdot \vec{OP'} = R^2$. Define $i_c(P) = P'$, $i_c(0) = \infty$, $i_c(\infty) = 0$.

Properties

0 i_c is a diffeomorphism

1 $i_c|_c = id$



2 i_c interchanges the inside and outside of the circle.

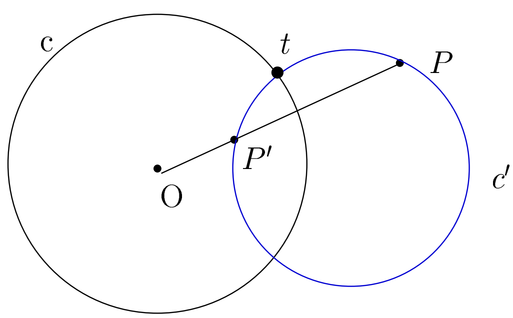
3 Lines throught the origin ($\cup\infty$) are invarient under i_c.

4 Circles orthogonal to c are i_c-invarient.

5 If $h : \mathbb{R}^2 \cup \infty \to \mathbb{R}^2 \cup \infty$ is a homothety $(x \to \lambda x)$ $(\lambda > 0)$ then $i_c h = h^{-1} i_c$.

The only property that is not obvious is number 4.

Proof. (of 4.) Need to show that $\vec{OP} \cdot \vec{OP'} = R^2$ $(= OT^2)$



□

More Properties

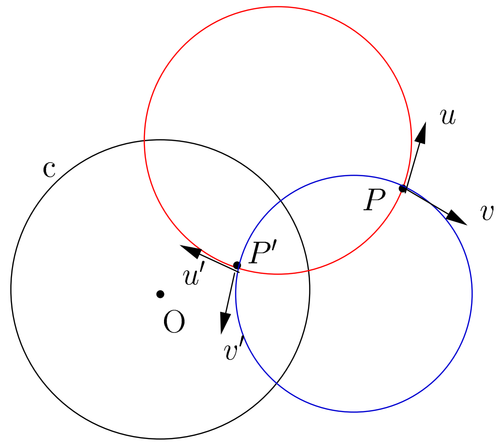
1 i_c is conformal (preserves angles).

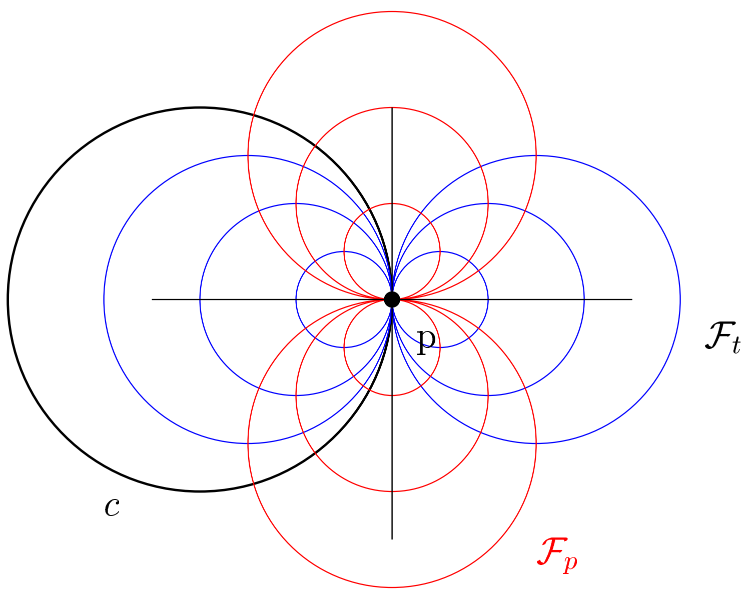
2 Circles not containing 0 go to other such circles.

3 Circles through 0 are interchanged with lines($\cup\infty$) not through 0.

Proof. Of 1. Find circles c_u and c_v that are $\perp$ c and tangent to u and v respectively.

Since i_c preserves c_v and c_u $i_c(P) = P' =$(other intersection point). $i_{c_*}(u) = u'$ and $i_{c_*}(v) = v'$ where u', v' are tangent to i_u, i_v at p'. Thus $\angle(u, v) = \angle(u', v')$.





Of 2. Start by showing that $\{circles,\ lines\} = clines$ tangent to c map to clines. Consider the family $\mathcal{F}_t$ of clines tangent to c at a point p. Also let $\mathcal{F}_p$ be the family of clines perpendicular to c at p.

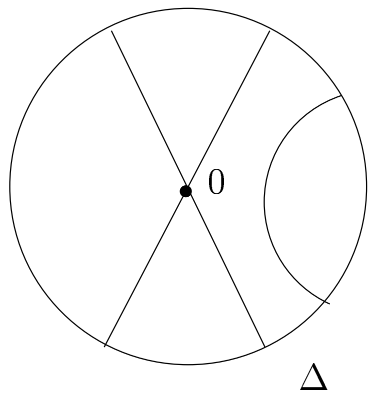
When a curve from $\mathcal{F}_t$ intersects $\mathcal{F}_p$, they meet perpendicularly. i_c preserves $\mathcal{F}_p$, so it preserves the line field tangent to $\mathcal{F}_t$. So i_c preserves the corresponding integral manifolds, and these are $\mathcal{F}_p$.

Now let c' be any circle not through 0. There is a homothety h such that $h(c')$ is tangent to c. Then $i_c(c') = i_c(h^{-1}[h(c')]) = hi_c(h(c'))$ where $i_c(h(c'))$ is a circle tangent to c.

$\square$

Poincaré Model

Let Δ be the open unit disc in $\mathbb{R}^2$. A geodesic is a diameter or an arc of a circle perpendicular to $\partial\Delta$.



A reflection $r_\gamma : \Delta \to \Delta$ in a geodesic γ is the restricion to Δ of the inversion i_c where $\gamma \subset c$ or the euclidean reflection when γ is a strait line.

An isometry of Δ is a composition of reflections.

$$Isom(\Delta) = \text{group of isometries of } \Delta$$

$$Isom_+(\Delta) = \text{ index 2 subgroup consisting of orientation preserving isometries}$$

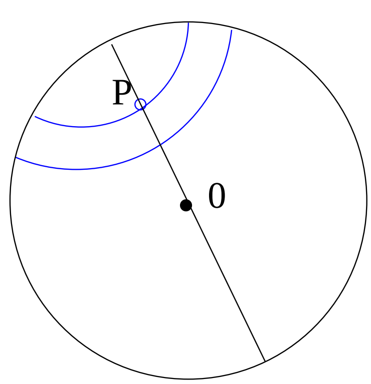
Properties. *1. $Isom_+(\Delta)$ acts transitively on Δ.*

2. Any isometry takes geodesics to geodesics and $Isom_+(\Delta)$ acts transitively on the set of all geodesics.

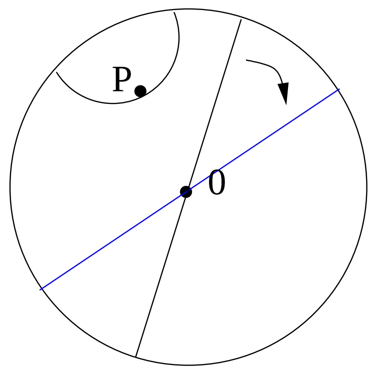
3. The stabilizer of $0 \in \Delta$ consists of the restrictions to Δ of Euclidean isometries fixing 0.

4. Any two points in Δ are on a unique geodesic.

Proof. Of 1. Easy. Use the intermediate value theorem and then compose with a reflection through 0 and P.



Of 2. The first part follows from the properties of inversions. For the second part we can assume without loss of generality that one of the points is 0 by conjugation by the isometry from the first property. Now since we can find an isometry that takes P to 0 by property one and the first part says that this is a geodesic through 0 (ie a diameter). Then rotate (composition of two reflections) to land on the desired geodesic.



Of 3. $O(2) \subset Isom(\Delta)$ and $O(2) = \{\text{group of isometries fixing origin}\}$. $O(2)$ is topologically the wedge of two circles.

Let $\varphi \in Isom(\Delta)$ fix 0. Compose φ (if necessary) with an element of $O(2)$ so that φ is orientation preserving and leaves a ray from 0 invariant.

Claim 27. *$\varphi = id$*

Proof. (Of Claim) Since φ preserves angles it preserves all rays through 0. This implies that all geodesics are preserved. Now given any point represent it as the intersection of two geodesics so that every point is fixed. □

The claim implies that $Stab = O(2)$.

□

Definition 28. *Let ρ_1 and ρ_2 be two Riemannian metrics on X. We say that ρ_1 is* conformally equivalent *to ρ_2 if there exists a continuous map $\phi : X \to \mathbb{R}^+$ such that for all $x \in X$:*

$$< v_x, w_x >_{\rho_1} = \phi(x) < v_x, w_x >_{\rho_2}$$

This is equivalent to the condition that the angles measured with respect to the two metrics are equal.

Corollary 29. *Up to scale, there is a unique Riemannian metric on Δ which is invariant under the isometry group. This metric is conformally equivalent to the Euclidean metric. Geodesics in the Riemannian metric are Euclidean lines and circles perpendicular to $\partial\Delta$ intersected with δ. Circles in the Riemannian metric (i.e., the locus of points equidistant from the center) are Euclidean circles.*

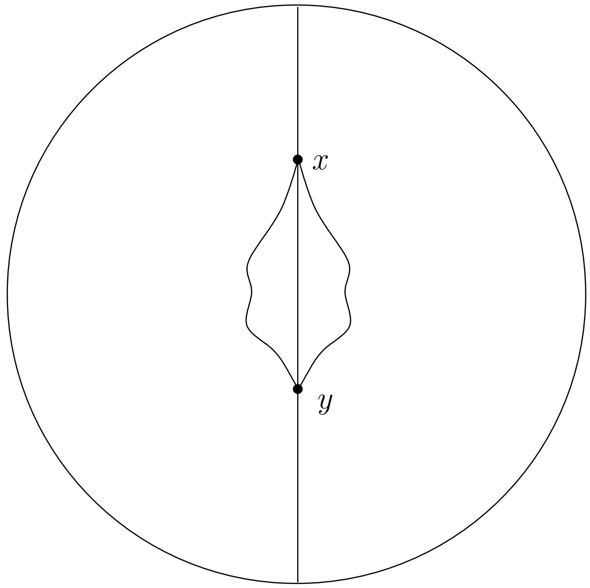


Figure 2.1: Because in a Riemannian metric space local geodesics are unique, a geodesic must be contained in the fixed point set of an isometry that fixes x and y.

Proof. Let ρ be a Riemannian metric invariant under the isometry group. Notice that ρ is uniquely determined by its value at a base point (say $0 \in \Delta$) since the group acts transitively on X. After rescaling we may assume that ρ is equal the Euclidean Reimannian metric at that base point. Now at 0, Euclidean angles and ρ angles coincide. Since inversions preserve Euclidean angles, ρ-angles and Euclidean angles agree at every point. Indeed, if $v, w \in T_x\Delta$ let g be an isometry taking x to 0, then $\angle_\rho(v, w) = \angle_\rho\left(d_x g(v), d_x g(w)\right)$ where the later two vectors lie in $T_0\Delta$ hence

$$\begin{aligned}&\angle_\rho\left(d_x g(v), d_x g(w)\right) = \angle_{\text{Euc}}\left(d_x g(v), d_x g(w)\right) =\\ &\angle_{\text{Euc}}\left(d_0 g^{-1}(d_x g(v)), d_0 g^{-1}(d_x g(w))\right) =\\ &\angle_{\text{Euc}}\left(v, w\right)\end{aligned}$$

Thus, ρ and Euc are conformally equivalent.

Since rotations about the origin are isometries for both the Euclidean and the hyperbolic metrics on Δ, Euclidean circles about the origin are hyperbolic circles. Since Euclidean circles are preserved by hyperbolic isometries, they are hyperbolic circles even when they are no longer centered at the origin (however, the Euclidean and hyperblic centers of the circle do not coincide).

Let γ be a hyperbolic geodesic from x to y. Then x and y either lie on a circle perpendicular to $\partial\Delta$ or on a diagonal of Δ. Let C be the diagonal or arc of the circle intersected with Δ. Then $g = i_C$ is an isometry fixing x and y. Therefore, it takes γ to another hyperbolic geodesic from x to y. In a Riemannian metric, geodesics are locally unique. Therefore, if x and y are sufficiently close, $g(\gamma) = \gamma$ so γ must be the subarc of C between x and y (see Figure 2). Now even if x and y aren't close, γ must be a local geodesic, so at every point it must coincide with a subarc of a diameter or a circle perpendicular to $\partial\Delta$. But two such distinct curves cannot be tangent at

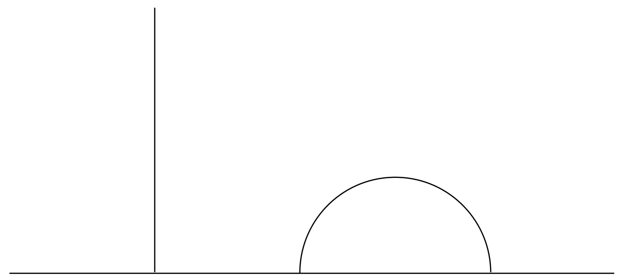


Figure 2.2: Geodesics in the upper half space model are semicircles centered on the real line, and vertical lines.

a point in Δ and so a hyperbolic geodesic must be a subarc of a diameter of Δ or a circle perpendicular to $\partial\Delta$. ☐

Question 30. *Write down an explicit expression for the metric.*

2.1 The Upper Half Space Model

$$U = \{z \in \mathbb{C} | \mathrm{Im} z > 0\}$$

One can apply an inversion to Δ to obtain U (for example: $\phi(z) = \frac{1-z}{z-i}$).
The isometry group of U consists of reflections about these geodesics.

Theorem 31. *The following map is an isomorphism:*

$$\Phi = \begin{cases} PSL_2\mathbb{R} & \longrightarrow \quad Isom_+(U) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longrightarrow \quad z \to \frac{az+b}{cz+d} \end{cases}$$

Proof. First consider the above map with $\mathrm{SL}_2(\mathbb{R})$ as its domain. We must show (among other things) that it lands where it's supposed to - orientation preserving isometries. Consider the images of the following matrices:

- $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \longrightarrow (z \to z+r)$ this is a composition of two reflections in geodesic lines perpendicular to $\mathbb{R}$. Thus it is an orientation preserving isometry.

- $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longrightarrow (z \to \frac{1}{-z})$ this is a composition of reflection in the unit circle: $\frac{1}{z}$ and in the y-axis $-z$. Thus it is an orientation preserving isometry.

Notice that $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ therefore lower triangular matrices also land in orientation preserving isometries. In addition, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ is sent to the map $z \to \lambda^2 z$ which is a composition of two inversions centered at the origin

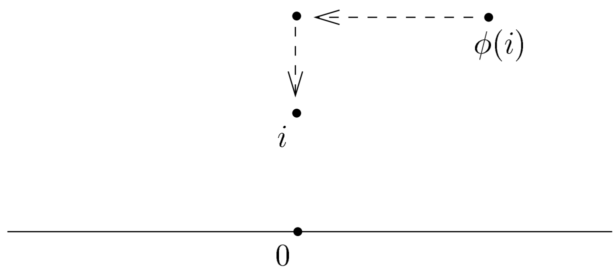


Figure 2.3: If ϕ doesn't stabilize i compose with maps in the image of Φ to get one which does.

(much like the composition of two reflections in perpendicular lines is a translation). Finally, by Gaussian elimination, every matrix in $\mathrm{SL}_2\mathbb{R}$ can be written as a product of $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ hence the map on $\mathrm{SL}_2(\mathbb{R})$ is well defined. Next, a matrix is in the kernel of this map if $b = c = 0$ and $a = d$. Thus this map descends to a monomorphism from $\mathrm{PSL}_2(\mathbb{R})$.

We wish to compute which matrices land in $\mathrm{Stab}(i)$. Suppose $\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Stab}(i)$, then $\frac{ai+b}{ci+d} = i$ which implies $a = d, b = -c$. Since the determinant equals 1, get $a^2 + b^2 = 1$ and therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSO(2) = SO(2)$. Therefore, Φ maps $SO(2)$ onto the stabilizer of i in $\mathrm{Isom}_+(U)$. Now let $\phi : U \to U$ be any orientation preserving isometry. After composing ϕ with a translation and a dialation (which are both in the image of Φ) we can assume that ϕ fixes i, and we've already shown that such a map is in the image. $\square$

Now we are in a better position to figure out what the invariant Riemannian metric should be. At i, choose the metric to coincide with the standard inner product. Since horizontal translations are isometries, the metric should only be a function of y. Since dialation by λ is an isometry the vector v at i and the vector λv at λi should have the same norm. Thus we obtain the metric $\boxed{ds^2 = \frac{1}{y^2} dt^2}$ where dt^2 is the Euclidean metric.

Exercise 32. *In Δ, the hyperbolic metric is: $ds^2 = \frac{4}{(1-r^2)^2} dt^2$*

Remark. *By the formula:*

$$4 < v, w >= \|v + w\|^2 - \|v - w\|^2$$

The norm determines the inner product when it exists.

2.1.1 Hyperbolic Triangles

Proposition 33. *All ideal triangles are congruent and have area equal to π.*

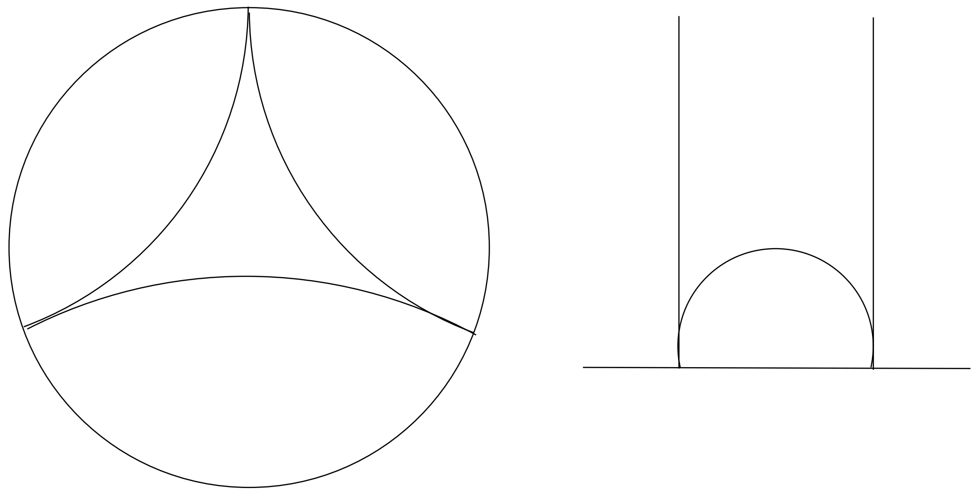


Figure 2.4: Ideal triangles in the disc model and in the upper half space model.

Proof. Here it is best to work in U. Consider the ideal triangle with vertices at $0, 1$ and ∞.

$$\text{Area} = \int_{-1}^{1} dx \int_{\infty}^{\sqrt{1-x^2}} \frac{1}{y^2} dy = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{cos\theta} cos\theta d\theta = \pi$$

The triangle with vertices at r, s, t will be taken to the one above by the isometry $\frac{z-r}{z-t}\frac{s-r}{s-t}$ which takes: $r \to 0, s \to 1, t \to \infty$. □

Proposition 34. *All $\frac{2}{3}$ ideal triangles (i.e. two of whose vertices are on the boundary) with a given exterior angle θ are congruent, and its area is θ.*

Proof. For any $x \in \Delta$ and vectors $v, u \in T_x\Delta$ with $\angle(v, u) = \theta$ there is an isometry g that takes x to 0, v to $(1, 0)$, and u to the vector at an angle θ from the x axis.
Claim: (Gauss) $A(\theta_1 + \theta_2) = A(\theta_1) + A(\theta_2)$
See figure 2.1.1 for proof.
To finish the proof, recall that if $A : [0, \pi] \to \mathbb{R}$ is continuous, $\mathbb{Q}$-linear with $A(0) = 0$ and $A(\pi) = \pi$, then $A(\theta) = \theta$. □

Proposition 35. *The area of any geodesic triangle with angles α, β, γ is $\pi - (\alpha + \beta + \gamma)$.*

Corollary 36. *The curvature of the hyperbolic metric is -1.*

Proof. A regular right hexagon has area π because it consists of six congruent triangles with angles: $\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{4}$ and so their areas are $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$ each. Double the all-right hexagon along alternating edges to get a pair of pants with area 2π. Double again along the boundary to get a genus 2 handle body S. By Gauss-Bonnet:

$$\int_S \kappa dA = 2\pi\chi(S)$$

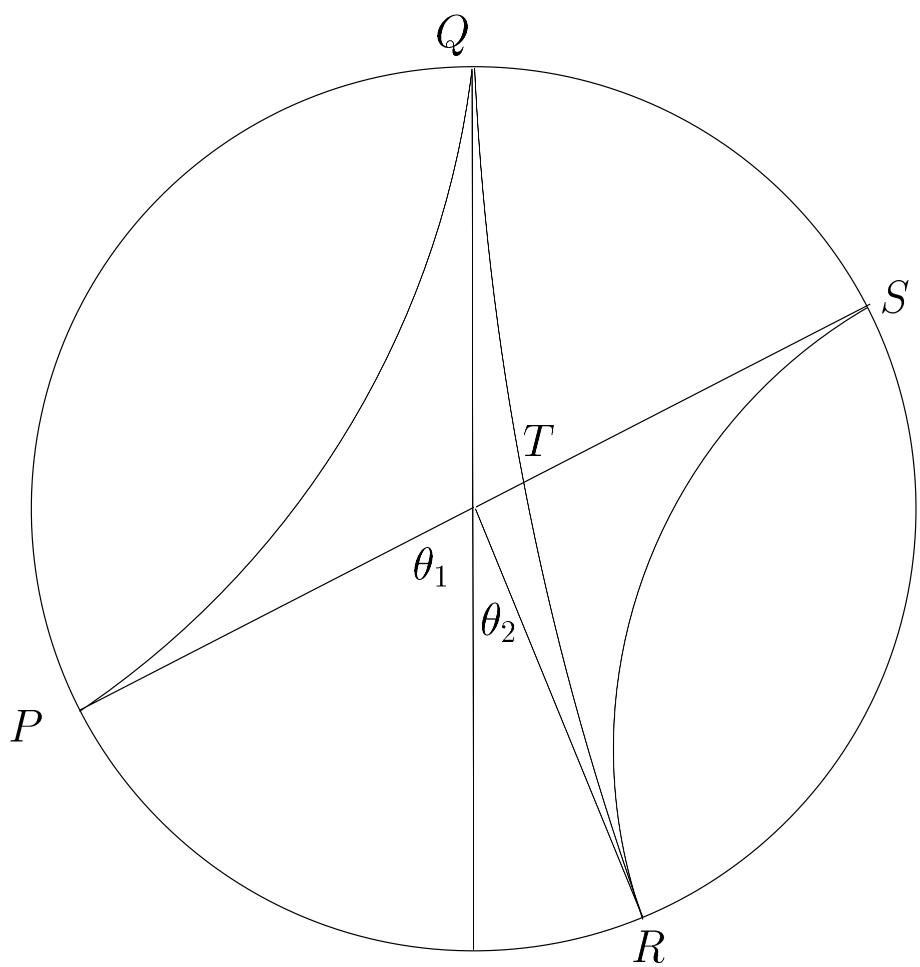


Figure 2.5: $A(\theta_1 + \theta_2)$ is the area of the triangle SOR. $A(\theta_1) = \text{Area}(QOP)$ and $A(\theta_2) = \text{Area}(QOR)$. Rotating by π about T, gives us a congruence of ΔSTR and ΔPTQ. So Area(SOR)= Area(STR)+ Area(TOR) = Area(PTQ)+Area(TOR) = $A(\theta_1) + A(\theta_2)$.

By the homogeneity of the hyperbolic plane, κ is constant. Thus,

$$4\pi \cdot \kappa = 2\pi(-2)$$

Hence $\kappa = -1$. $\square$

Remark. *An orbit of a one parameter family of isomorphisms is a geometrically significant objects. For example:*

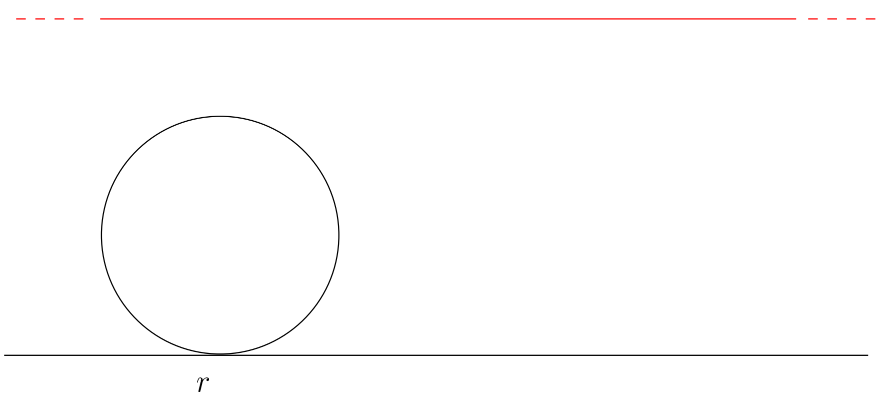


Figure 2.6: The red line describes horocycle based at infinity and the black circle is a horocycle based at $r \in \mathbb{R}$.

- *The y axis, which is a geodesic and its equidistant lines are given by orbits of the family* $\left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \middle| t \in \mathbb{R} \right\}$
- *Circles centered at the origin are given by orbits of the family* $\left\{ \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \middle| t \in \mathbb{R} \right\}$
- *Horocircles centered at infinity are given by orbits of the family* $\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$

Proposition 37. *A bounded subset B of $\mathbb{H}^2$ is contained in a unique closed disc of smallest radius.*

Proof. The existence of this disc follows by compactness. Indeed let $R = \inf\{r | B \subseteq D(c, r)\}$. Let $D(c_i, r_i)$ be a sequence where $r_i \to R$. After passing to a subsequence $c_i \to c_0$ and so $B \subseteq D(c_0, R)$ (because this disc is closed).
As for uniqueness, suppose that $B \subseteq D(c_1, R) \cap D(c_2, R)$, where $c_1 \neq c_2$. Let c_0 be the point between c_1 and c_2, and T is a point of intersection of the circles which bound the aforementioned discs.

Claim 38. *1* $d(c_0, T) < d(c_1, T) = R$

2 If $P \in D(c_1, R) \cap D(c_2, R)$ then $d(c_0, P) \leq d(c_1, P)$.

This claim is an example of how we sometimes need to switch from one model of the hyperbolic plane to the other in order to explicitly see certain properties.
For the proof of the first statement consider figure 2.1.1
For the second statement place c_0 at the origin of Δ. ??? □

Corollary 39. *If $K \subseteq Isom+(\mathbb{H}^2) = PSL_2\mathbb{R}$ is a compact subgroup then K fixes a point in $\mathbb{H}^2$ and is conjugate into $SO(2)$. Therefore, $SO(2)$ is the maximal compact subgroup of $PSL_2\mathbb{R}$.*

Proof. Let B be the orbit of a point under K. Then B is compact, hence bounded. Let D be the unique smallest closed disc which contains it. Since B is invariant under K, so is D. Therefore, its center c is fixed by all elements in K. Therefore K is contained in the stabilizer of the center which is conjugate to $SO(2)$. □

Remark.

$$SO(2) \to PSL_2(\mathbb{C}) \xrightarrow[\phi \to \phi(origin)]{ev} \mathbb{H}^2$$

This is a locally trivial fiber bundle. Any locally trivial fiber bundle with a contractible base is the trivial bundle i.e. a product. Thus

$$PSL_2(\mathbb{R}) \cong \mathbb{H}^2 \times S^1$$

This proof generalizes to any Lie group of non-compact type.

2.2 The Nil Geometry

The following is a classification of low dimensional connected Lie groups up to covering space equivalence.

dim = 1: $\mathbb{R}$

dim = 2: $\mathbb{R}^2, \mathbb{R} \rtimes \mathbb{R} = \{x \to ax + b | a > 0 b \in \mathbb{R}\}$

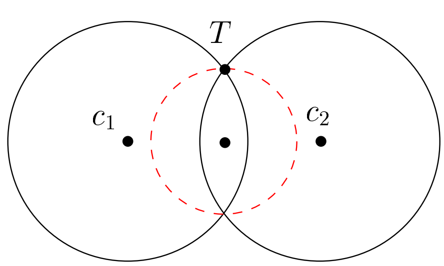


Figure 2.7: The intersection of two circles, which doesn't contain either center, is contained in a circle of strictly smaller radius.

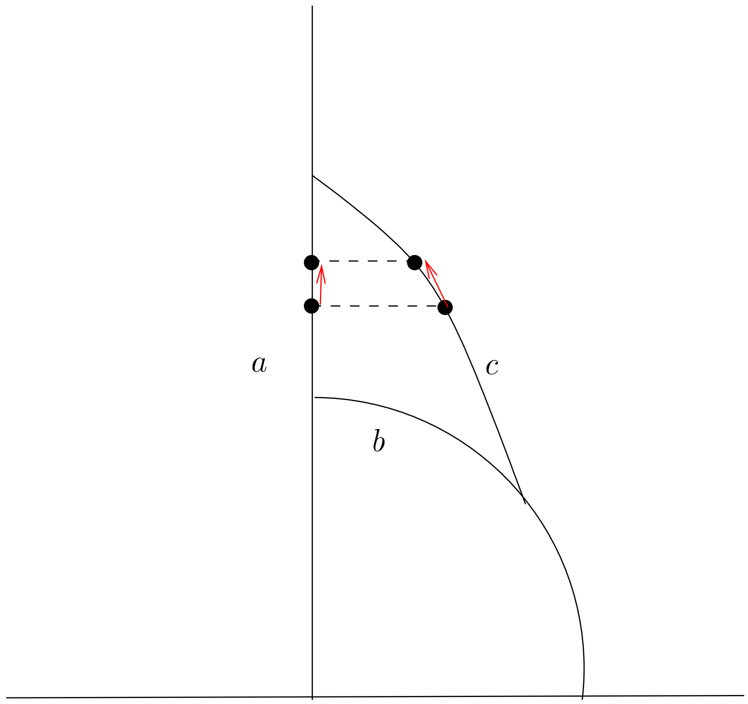


Figure 2.8: We must show that the hypothenus of a right angled triangle is longer than each side. This follows from the fact that projection onto geodesics strictly decreases distences.

dim = 3: Bianchi proved that they are $\mathrm{SL}_2\mathbb{R}, \mathrm{SO}(3)$ and the solvable ones are $\mathbb{R}^2 \rtimes \mathbb{R}$. There are nine types of such groups where two types are infinite families.

Definition 40 (The Heisenberg Group).

$$Nil = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$$

There is a short exact sequence

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{R}^2 & \to & \mathrm{Nil} & \to & \mathbb{R} & \to & 1 \\ 1 & \to & \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} & \to & \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} & \to & x & \to & 1 \end{array}$$

This sequence splits and so the group is a semidirect product. $\mathrm{Nil} = \mathbb{R}^2 \rtimes_{A_t} \mathbb{R}$ where $A_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

The center of Nil is $\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cong \mathbb{Z}$ This group is 2-step nilpotent as can be seen from the following short exact sequence (which doesn't split):

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{R} & \to & \text{Nil} & \to & \mathbb{R}^2 & \to & 1 \\ 1 & \to & \text{center} & \to & \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} & \to & (x,y) & \to & 1 \end{array}$$

2.2.1 A model for Nil

Let $\mathcal{G}$ be the set of equivalence classes of PL paths starting at the origin, where two paths are equivalent if they cobound 0 area. Where the area is defined by $\int_\Omega dxdy = \int_{\partial\Omega} xdy$. Therefore $\alpha \sim \beta$ if $\int_\alpha xdy = \int_\beta xdy$
There is a group operation on $\mathcal{G}$ namely $\alpha \cdot \beta$ is the concatenation of α with the translate of β which begins where α ends.

Proposition 41. *$\mathcal{G} \cong Nil$ where the isomorphism takes $[\alpha]$ to* $\begin{pmatrix} 1 & \alpha(1)_x & \int_\alpha xdy \\ 0 & 1 & \alpha(1)_y \\ 0 & 0 & 1 \end{pmatrix}$

Proof. This map is clearly well defined one to one and onto. One must check that it is a homomorphism. Clearly $\alpha\beta$ ends at $(\alpha(1)_x + \beta(1)_x, \alpha(1)_y + \beta(1)_y)$. The area bounded by $\alpha\beta$ is area(α) + area$(\beta) + \alpha(1)_y\beta(1)_x$. ☐

There are things that are easier to see in the geometric model. For instance the fact that Nil is nilpotent. Consider the short exact sequence:

$$1 \to \mathbb{R} \to \mathcal{G} \xrightarrow{\phi} \mathbb{R}^2 \to 1$$

Where $\phi([\alpha]) = (\alpha(1)_x, \alpha(1)_y)$ (notice that two equivalent paths have the same endpoint). The kernel is the subgroup of closed loops.

Claim 42. *$ker(\phi)$ is the center of $\mathcal{G}$*

Proof. We must show that $\alpha\gamma \sim \gamma\alpha$ for every α and every closed γ. Let γ' the $\alpha(1)$ translate of γ. We need to show $\int_\gamma xdy = \int_{\gamma'} xdy$. Let γ'' be the $\alpha(1)_y$ translate of γ. Then clearly $\int_\gamma xdy = \int_{\gamma''} xdy$. But the form $xdy - ydx$ is exact so it vanished on closed loops. Hence $\int_{\gamma''} xdy = \int_{\gamma''} ydx = \int_{\gamma'} ydx = \int_{\gamma'} xdy$. And we get our claim. ☐

¿From the geometric model one easily sees that $[\alpha, \beta]$ is a closed loop and therefore central. so for any γ: $[\gamma, [\alpha, \beta]]$ is trivial. Thus $\mathcal{G} \cong$ Nil is 2-step nilpotent.

The discrete version of Nil is the Hisenberg Group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$$

It is important to note that $H \trianglelefteq Nil$ and $H \hookrightarrow Nil$.

Proposition 43. *H has the following presentation.*

$$H \cong \langle X, Y, Z | Z \text{ is central}, \ [x, y] = z \rangle$$

Proof. Construct a map $\varphi :< X, Y, Z | Z$ is central, $[X, Y] = Z > \to H$ via

$$X \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ Y \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ \text{and } Z \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The issue is if the map is $1:1$.

Claim 44. *φ is $1:1$.*

Proof. Take some word W in X, Y, Z and assume that $\varphi(W) = 1$. Put W into the "normal form" $X^m Y^n Z^k$. We can push all the Z's to the end using the fact that Z is central. If Y appears before X use

$$\begin{aligned} &YX = XYZ^{-1} \\ &Y^{-1}X^{-1} = X^{-1}Y^{-1}Z^{-1} \\ &Y^{-1}X = XY^{-1}Z \\ &YX^{-1} = X^{-1}YZ \end{aligned}$$

□

Now

$$\varphi(X^m Y^n Z^k) = \begin{pmatrix} 1 & m & mn + k \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\varphi(w) = 1$ then $m, n = 0$ which implies $k = 0$ and thus $w = 1$

□

Important Properties

1. The center of H is "distorted". The reason is that

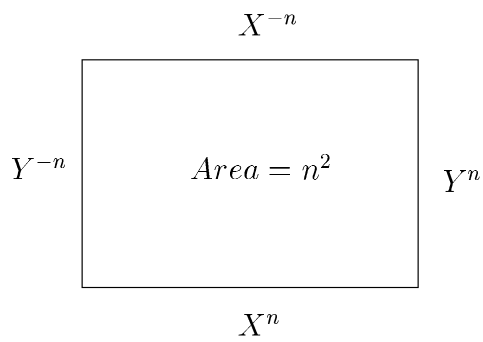
$$\begin{aligned} [X^n, Y^n] = & \ X^n Y^n X^{-n} Y^{-n} \\ = & \ X^n X^{-n} Y^n Y^{-n} Z^{n^2} \\ = & \ Z^{n^2} \end{aligned}.$$

This implies that the shortest path from $\mathbb{Z}^{n^2}$ to 0 is not along the Z-axis.

Definition 45. *Let G be a finitely generated group and $H < G$ a finitely generated subgroup. We say that H is undistorted in G if there exists $C > 0$ such that for every $h \in H$*

$$\| h \|_H \leq C \| h \|_G$$

with respect to fixed generating sets of G and H. Otherwise H is "distorted".



Example 46. *The center of the Heisenberg group is distorted.*

$$\| Z^{n^2} \|_{center} = n^2, \text{ but } \| Z^{n^2} \|_H \leq 4n$$

2. $M = Nil/H$ is a 3-manifold. Why?

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{R}^2 & \to & Nil & \to & \mathbb{R} & \to & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & \mathbb{Z}^2 & \to & H & \to & \mathbb{Z} & \to & 1 \end{array}$$

gives a fiber bundle

$$\begin{array}{ccccc} \mathbb{R}^2/\mathbb{Z}^2 & \hookrightarrow & Nil/H & \hookrightarrow & \mathbb{R}/\mathbb{Z} \\ \| & & \| & & \| \\ T^2 & & M^3 & & S^1 \end{array}$$

In particular M^3 is the mapping torus of a diffeomorphism of T^2 given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot T^2 \to T^2$$

, where

$$M^3 = T^2 \times [0,1]/(x,1) \sim (\varphi(x), 0)$$

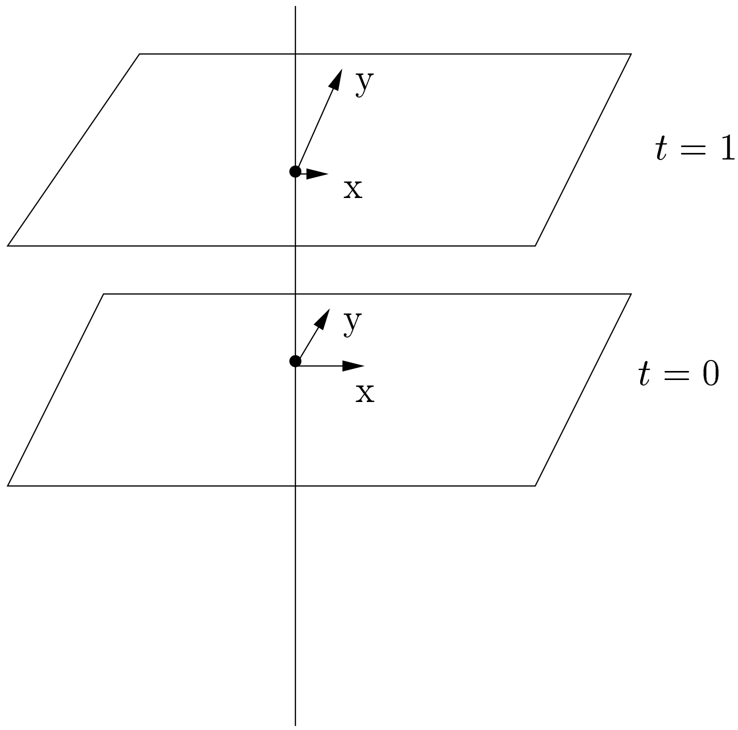
Also, M^3 is a circle bundle over T^2. It turns out that all 3-manifolds with the Nil geometry are covering spaces of this one since every lattice in Nil is commensurable with H.

2.3 Sol

Definition 47. $Sol = \mathbb{R}^2 \rtimes_{A_t} \mathbb{R}$ *with*

$$A_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

which is a 1-parameter subgroup of $GL_2(\mathbb{R})$.



Consider the following metric on $\mathbb{R}^3$:

$$ds^2 = e^{-2t}dx^2 + e^{2t}dy^2 + dt^2.$$

When $t = 0$ you get the euclidean metric.

$\mathbb{R}^2$ acts by translations

$$T_{u,v}(x, y, t) = (x + u, y + v, t).$$

$\mathbb{R}$ also acts via: if $s \in \mathbb{R}$ let

$$\varphi_s : (x, y, t) \mapsto (e^s x, e^{-s} y, t + s).$$

These are all isometries. Now φ_s acts on translations:

$$\begin{aligned} \varphi_s \circ T_{u,v} \circ \varphi_s^{-1}(x, y, z) &= \varphi_s \circ T_{u,v}(e^{-s}x, e^s y, t - s) \\ &= \varphi_s(e^{-s}x + u, e^s y + v, t - s) \\ &= (x + e^s u, y + e^{-s} v, t) \\ &= T_{e^s u, e^{-s} v}(x, y, t). \end{aligned}$$

Thus $\{\varphi_s\}$ and $\{T_{u,v}\}$ generate Sol.

2.4 Bianchi classification of 3-dimensional Lie groups

If A_t is a 1-parameter subgroup of $GL_2(\mathbb{R})$, then we can consider $G = \mathbb{R}^2 \rtimes_{A_t} \mathbb{R}$.
If B_t is another 1-parameter subgroup, then

$$\mathbb{R}^2 \rtimes_{A_t} \mathbb{R} \cong \mathbb{R}^2 \rtimes_{B_t} \mathbb{R}$$

if one of the following holds

- $B_t = CA_tC^{-1}$ for some $C \in GL_2(\mathbb{R})$;
- $B_t = A_{ct}$ for some $c \in \mathbb{R}$, $c \neq 0$.

Theorem 48 (Bianchi's Theorem). *Every connected 3-dimensional Lie group is, up to covering spaces, isomorphic to one of the following:*

- $SL_2(\mathbb{R})$ *(semisimple and unimodular);*
- SO_3 *(semisimple and unimodular);*
- $\mathbb{R}^2 \rtimes_{A_t} \mathbb{R}$ *which breaks up into 9 families:*
    1. $\mathbb{R}^3$, *if* $A_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *(abelian and unimodular);*
    2. *Nil, if* $A_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ *(nilpotent and unimodular);*
    3. *Sol, if* $A_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ *(solvable and not unimodular);*
    4. $Isom_+(\mathbb{R}^2)$, *if* $A_t = \begin{pmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{pmatrix}$ *(solvable and unimodular);*
    5. *Solvable and not unimodular. This algebra is a product of* $\mathbb{R}$ *and the 2-dimensional non-abelian Lie algebra. The simply connected group has center* $\mathbb{R}$ *and outer automorphism group the group of non-zero real numbers. The matrix* A_1 *has one zero and one non-zero eigenvalue.*
    6. *Solvable and not unimodular.* $[y, z] = 0$, $[x, y] = y$, $[x, z] = y + z$. *The simply connected group has trivial center and outer automorphism group the product of the reals and a group of order 2. The matrix* A_1 *has two equal non-zero eigenvalues, but is not semisimple.*
    7. *Solvable and not unimodular.* $[y, z] = 0$, $[x, y] = y$, $[x, z] = z$. *The simply connected group has trivial center and outer automorphism group the elements of* $GL_2(\mathbb{R})$ *of determinant* $+1$ *or* -1. *The matrix* A_1 *has two equal eigenvalues, and is semisimple.*
    8. *Solvable and unimodular. This Lie algebra is the semidirect product of* $\mathbb{R}^2$ *by* $\mathbb{R}$, *with* $\mathbb{R}$ *where the matrix* A_1 *has non-zero distinct real eigenvalues with zero sum. It is the Lie algebra of the group of isometries of 2-dimensional Minkowski space. The simply connected group has trivial center and outer automporphism group the product of the positive real numbers with the dihedral group of order 8.*
    9. *Solvable and not unimodular. An infinite family. Semidirect products of* $\mathbb{R}^2$ *by* $\mathbb{R}$, *where the matrix* A_1 *has non-real and non-imaginary eigenvalues. The simply connected group has trivial center and outer automorphism group the non-zero reals.*

Constructing lattices

Let $A \in SL_2(\mathbb{Z})$. Let A_t be a 1-parameter subgroup such that $A_1 = A$. Then

$$\begin{array}{ccccc} \Gamma & = & \mathbb{Z}^2 \rtimes_A \mathbb{Z} & \hookrightarrow & \mathbb{R}^2 \rtimes_{A_t} \mathbb{R} \\ & & A & \mapsto & A_t \end{array}$$

is a discrete subgroup and the quotient G/Γ is a compact 3-manifold.

Example 49. *If we consider*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

then we have $\Gamma = H \hookrightarrow Nil$.

Example 50. *If we consider*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = C \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} C^{-1}, \quad A_t = C \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix} C^{-1}$$

then we have

$$\Gamma = \mathbb{Z}^2 \rtimes_A \mathbb{Z} \hookrightarrow \mathbb{R}^2 \rtimes_{A_t} \mathbb{R} \cong \mathbb{R}^2 \rtimes_{M_t} \mathbb{R} = Sol$$

where

$$M_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

We remember what is a lattice.

Definition 51. *A lattice Γ of a Lie group G is a discrete subgroup of G with compact quotient G/Γ.*

Fact 52. *If Γ_1 and Γ_2 are lattices in Nil, then some subgroup of Γ_1 of finite index is conjugate to some subgroup of Γ_2 of finite index (i.e. all the lattices in Nil are commensurable).*

This fact is not true if we consider the lattices in Sol.

Theorem 53. *There are many lattices in Sol that are not commensurable to each other.*

Moreover, we have the following important theorem.

Theorem 54 (Eskin-Fisher-Whyte). *Every finite generated group which is QI to Sol is isomorphic to a lattice in Sol up to finite kernels and subgroups of finite index, i.e. if $\Gamma \sim_{QI} Sol$, then there exists a finite subgroup $\Gamma' < \Gamma$ and there exists a homomorphism $\varphi : \Gamma' \to Sol$ such that $ker(\varphi)$ is finite and $Im(\varphi)$ is a lattice.*

Let

$$A_1 := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

Now, we want to prove that $\Gamma_1 = \mathbb{Z}^2 \rtimes_{A_1} \mathbb{Z}$ and $\Gamma_2 = \mathbb{Z}^2 \rtimes_{A_2} \mathbb{Z}$ are not commensurable. In order to prove this result, we need the following fact.

Fact 55. *Every subgroup of Γ_i isomorphic to $\mathbb{Z}^2$ is a subgroup of the kernel to $\mathbb{Z}$.*

We fix the following notation:

$$\mathbb{Z}^2 \rtimes \mathbb{Z} := \{(v, t^n) \mid v \in \mathbb{Z}^2,\, n \in \mathbb{Z},\, (v, t^n) \cdot (w, t^m) = (v + A^n(w), t^{n+m})\}.$$

Proof. We consider a subgroup $H \cong \mathbb{Z}^2 < \Gamma_i$. We have

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}^2 & \to & \mathbb{Z}^2 \rtimes \mathbb{Z} & \overset{\pi}{-->} & \mathbb{Z} & \to & 1 \\ & & & & \uparrow & & & & \\ & & & & H & & & & \end{array}$$

Let assume $\pi(H) \neq 0$ and say $(v, t^n) \in H$ maps nontrivially, $n \neq 0$. Say $(w, t^0) = (w, 1) \in H \cap ker(\pi)$, $w \neq 0$.
Clearly, $(v, t^n) \cdot (w, 1) = (v + A^n(w), t^n)$ is equal to $(w, 1) \cdot (v, t^n) = (v + w, t^n)$ if and only if $A^n(w) = w$. This implies that $w = 0$ and this is a contradiction. $\square$

Now, we choose a nontrivial element $v \in \mathbb{Z}^2$ and some $\varphi = (w, t^k) \notin \mathbb{Z}^2$. We compute the limit

$$lim_{n\to\infty} \frac{log \parallel \varphi^n \cdot v \cdot \varphi^{-n} \parallel_{\mathbb{Z}^2}}{n} = lim_{n\to\infty} \frac{log \parallel A^{kn}(v) \parallel_{\mathbb{Z}^2}}{n} =$$

$$= lim_{n\to\infty} \frac{log(c\lambda^{kn})}{n} = klog(\lambda) = log(\lambda^k).$$

Since we have

$$\lambda_{A_1} = \frac{3+\sqrt{5}}{2}$$

$$\lambda_{A_2} = 2 + \sqrt{2}$$

then

$$\lambda_{A_1}^k \neq \lambda_{A_2}^l$$

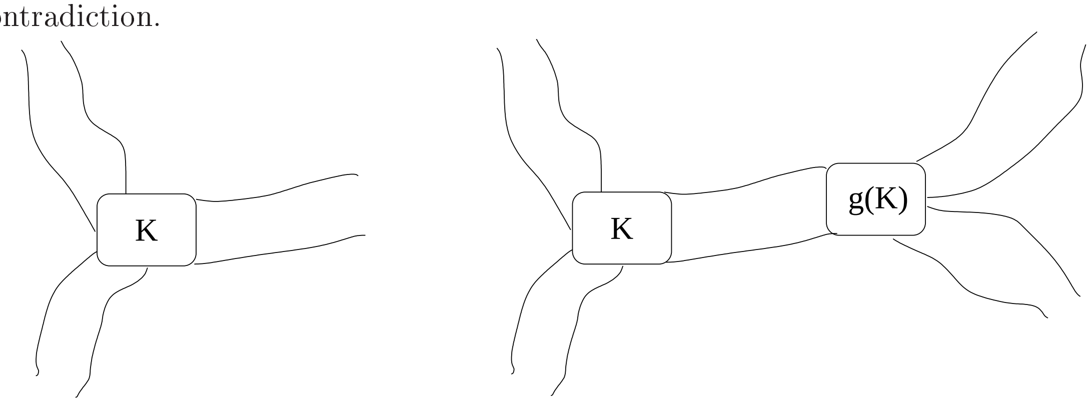
and hence Γ_1 and Γ_2 are not commensurable.

Chapter 3

Ends

Theorem 56. *(Hopf) A finitely generated group G can have $0, 1, 2$ or infinitely many ends.*

Proof. Let X be the Cayley graph of G. Suppose X has n ends where $n > 2$ and $n < \infty$. By definition there is a connected compact subset K so that $X - K$ contains n components with non-compact closure denoted U_i. Then choose $g \in G$ such that $g(K) \subset U_i$ for some i. Since left multiplication by elements of G is an isometry $g(K)$ has n components with non-compact closure. Thus $X - \{K \cup g(K)\}$ has at least $2(n-1)$ components with non-compact closure. Thus $2(n-1) > n$ which provides a contradiction.



□

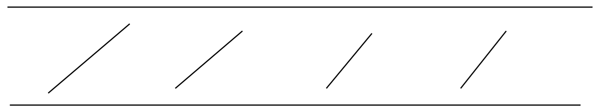
We would now like to classify all groups corresponding to how many ends they have.

Observation 57. *A group has 0 ends if and only if G is finite*

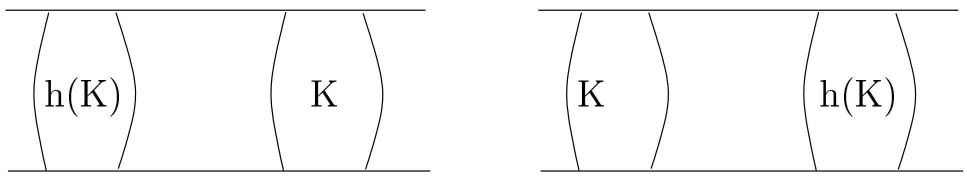
Theorem 58. *(Wall) If G has 2 ends then G contains a subgroup of finite index isomorphic to $\mathbb{Z}$.*

Proof. (Stallings) We will show that G contains a subgroup H of index 1 or 2 and H admits an epimorphism $H \twoheadrightarrow \mathbb{Z}$ with finite kernal.

Now G acts on the set of ends of X. Let H be the subgroup of G consisting of the elements that fix both ends (has index either 1 or 2). Define the flux homomorphism



$X = Cay(G)$



$\varphi(h) < 0$ $\varphi(h) > 0$

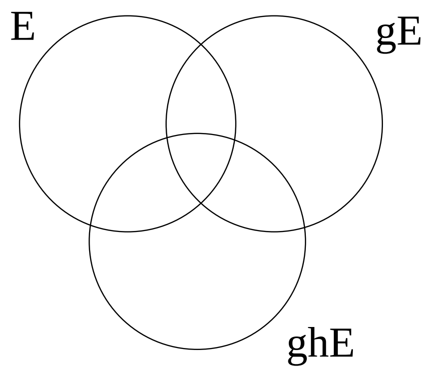
$\varphi : H \to \mathbb{Z}$ via

$$\varphi(h) = |h(E) \cap E^*| - |K \cap h(E^*)| \in \mathbb{Z}$$

where $|K|$ of some region K is the number of vertices in K and E is a fixed infinite set of vertices of X such that the complement E^* is also infinite and so that there are only finitely many edges in X with one vertex in E and one in E^*.

Claim 59. *φ is a homomorphism.*

It might help to keep the following Venn diagram in mind.



$$\begin{aligned}
\varphi(g) = \; & |E^* \cap gE| - |E \cap gE^*| \\
= \; & |E^* \cap gE \cap ghE| + |E^* \cap gE \cap ghE^*| - |E \cap gE^* \cap ghE| - |E \cap gE^* \cap ghE^*| \\
\\
\varphi(h) = \; & |E^* \cap hE| - |E \cap hE^*| \\
= \; & |gE^* \cap ghE| - |gE \cap ghE^*| \\
= \; & |E \cap gE^* \cap ghE| + |E^* \cap gE^* \cap ghE| - |E \cap gE \cap ghE^*| - |E^* \cap gE \cap ghE^*| \\
\\
\varphi(g) + \varphi(h) = \; & |E^* \cap gE^* \cap ghE| - |E \cap gE \cap ghE^*| + |E^* \cap gE \cap ghE| - |E \cap gE^* \cap ghE^*| \\
= \; & |E^* \cap ghE| - |E \cap ghE^*| \\
= \; & \varphi(gh)
\end{aligned}$$

Claim 60. *The kernal of φ is finite.*

We'll prove this for E of the following kind. Let K be a finite subcomplex of X such that $X - K$ has 2 components with noncompact closure and let E be the set of vertices in one of these components. Also assume that K is connected and that $X - K$ does not have any components with compact closure. Observe that if $h \in Ker(\varphi)$ then $h(K) \cap K \neq \emptyset$ so there are only finitely many such h.

$\square$

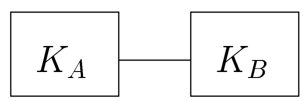
Quiz 61. *How many ends does*

$$\mathbb{Z}^2 \rtimes_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z} \cong SOL$$

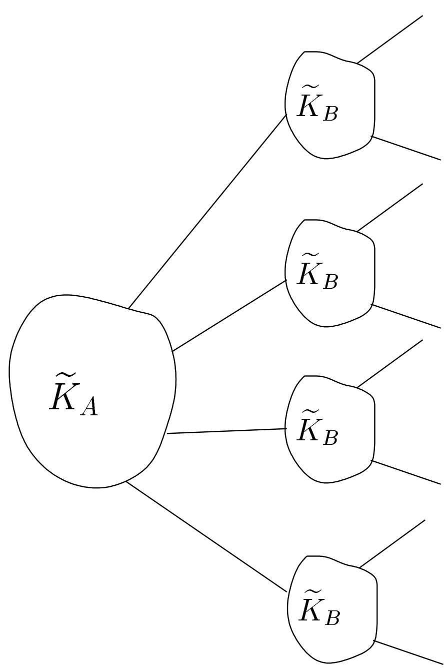
have? (answer is 1).

Example 62. *If $G = A *_F B$ where F is a finite group, $|A : F| \geq 3, |B : F| \geq 2$ then G has infinitely many ends (A and B are both finitely generated).*

Say A, B are in fact finitely presented. So $A = \pi_1(K_A)$ and $B = \pi_1(K_B)$ where K_A and K_B are finite complexes. Say $F = \{1\}$ then $G = \pi_1(K_G)$ where K_G is as below.



Construct the universal cover of K_G.



By Milnor-Svarc, G is quasi-isometric to the above space and clearly has infinitely many ends.

Theorem 63. *Suppose a finitely generated group G has infinitely many ends. Then G "splits over a finite subgroup". IE $G = A *_F B$ or $G = A*_F$ with F finite. In the first case $|A : F| \geq 3$ and $|B : F| \geq 2$. Note that $1*_1 = \mathbb{Z}$. In the second case $|A : F| \geq 2$.*

Theorem 64. *If G is two ended then G is virtually $\mathbb{Z}$*

Corollary 65. *If G is q.i. to $\mathbb{Z}$ then G is virtually $\mathbb{Z}$.*

Alternative proof of the corollary. The outline was given in class and the details filled in by the students. Consider $X = \mathrm{Cay}(G)$ we first find an embedded geodesic line $l \hookrightarrow X$ which is q.i. to X. We use this line to prove the existence of an element $g \in G$ such that $d(1, g^2) \geq 2d(1, g) - const$. This element cannot have finite order and therefore $< g >\cong \mathbb{Z}$. Finally we prove that $< g >$ has finite index in X.

Step 1

Claim 66. *There is a geodesic line $l : \mathbb{R} \hookrightarrow X$*

Let x_i and y_i be sequences of points going out the two ends and let $[x_i, y_i]$ denote a geodesic from x_i to y_i. There is an R such that $X \setminus B(1_G, R)$ has two components, one containing all but finitely many x_is and the other containing all but finitely many y_is. By passing to a subsequence we may assume that $[x_i, y_i]$ intersects $B(1_G, R)$ for all $i > 1$. By passing to a further subsequence we may assume that $[x_i, y_i]$ coinside in $B(1, R)$ for all i (Because $B(1, R)$ is finite). Continuing this way, by passing to further subsequences we may assume that for $i > k$ $[x_i, y_i]$ coincide in $B(1, kR)$. So $[x_i, y_i]$ converges to a geodesic $l : \mathbb{R} \hookrightarrow X$. If $g \in \mathrm{Im}(l)$ then $l' = g \circ l$ is a geodesic whose image contains the identity. Therefore we may assume that $\mathrm{Im}(l)$ contains 1_G.

Step 2 The inclusion $l : \mathbb{R} \hookrightarrow X$ is a quasi-isometry.
Since l is a geodesic, for any two vertices v, w in l, $d_G(v, w) = d_l(v, w)$. therefore it is a $(1, 0)$-quasi isometric embedding. We must show that it is a quasi-isometry. Let $f : X \to \mathbb{R}$ be the quasi-isometry which we assume exists. It is enough to show that $\mathbb{R} \xrightarrow{l} X \xrightarrow{f} \mathbb{R}$ is a quasi-isometry.

Claim 67. *If $h : \mathbb{R} \to \mathbb{R}$ is a quasi-isometric embedding then h is a quasi-isometry.*

We remark that this claim is false in general, see figure 3 for an example.

Proof. If h is an a, b quasi-isometric-embedding then

$$\frac{1}{a}|x - y| - b \leq |f(x) - f(y)| \leq a|x - y| + b$$

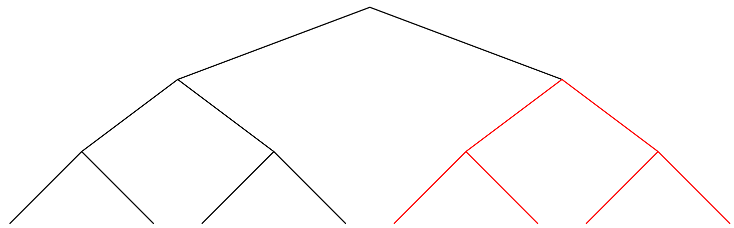


Figure 3.1: Consider the map that takes the whole tree to the red subtree. It is obviously a q.i embedding but the image is not almost surjective.

We claim that for any $z \in \mathbb{R}$ there is an x such that $|z - f(x)| < \frac{a+b}{2}$. Suppose not, denote $s = \sup\{\mathrm{Im}(f) \cap (-\infty, z)\}$ and $i = \inf\{\mathrm{Im}(f) \cap (z, \infty)\}$. Let x_1 be a point such that $f(x_1) < z$ and $f(x_1) > s - 1$ and x_2 is such that $f(x_2) > z$ and $f(x_2) < i + 1$ see figure 3. If $f : X \to Y$ is quasi isometric embedding then $f_* : \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$ is an injection, so both x_1 and x_2 must exist. By assumption $f(x_2) - f(x_1) \geq a + b$. Without loss of generality suppose that $x_1 < x_2$. Since f is a q.i. embedding, if $x \in [x_1, x_1+1]$ then $|f(x) - f(x_1)| \leq a+b$ hence $f(x) < s$. But the same is true for $x \in [x_1 + 1, x_1 + 2]$ and by induction for $x > x_1$, contradicting the existence of x_2. (The same proof can be made to show that there is no gap in the image of length greater than b). $\square$

Step 3

Claim 68. *Every $g \in G$ acts on Im(l) by (1,2R)-quasi-isometries where R is the sujectivity constant for $l : \mathbb{R} \hookrightarrow X$.*

For $g \in G$ let us define its action on l. The problem that might arise is that for some $v \in l$, $g(v)$ is not on l. Since l is almost surjective, there is an R and a w on l such that $d(w, g(v)) < R$. Since l is a geodesic, there are no more than R such vertices, so pick one and call it $g_{\#}(v)$. $g_{\#} : l \to l$ is a $(1, 2R)$-quasi-isometry. Indeed

$$d(g_{\#}(v), g_{\#}(u)) < d(g(v), g(u)) + 2R = d(v, u) + 2R$$

and

$$d(g_{\#}(v), g_{\#}(u)) > d(g(v), g(u)) - 2R = d(v, u) - 2R$$

By the previous step it is a quasi-isometry.

Claim 69. *For any g either $d(1, g^2) < 4R$ or $d_X(1, g^2) \geq 2d(1, g) + 6R$.*

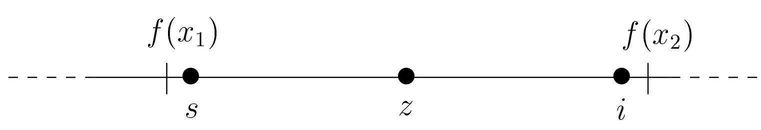


Figure 3.2: z is a point in the gap of the image of f. s and i are the closest points to z in the closure of Im(l).

We know that

$$d(1, g_\#(1)) - 2R \leq d(g_\#(1), g_\#^2(1)) \leq d(1, g_\#(1)) + 2R$$

Since l is a line

$$d(g_\#^2(1), 1) \leq 2R \text{ or } d_X(1, g_\#^2(1)) \geq 2d(1, g_\#(1)) - 2R$$

This is almost but not quite what we want: we need to replace $g_\#$ with g. By definition $d(g_\#(1), g) < R$ and since g acts by isometries $d(g(g_\#(1)), g^2) < R$. Again by the definition of $g_\#$, $d(g_\#(g_\#(1)), g(g_\#(1))) < R$ and thus we get $d(g_\#^2(1), g^2) < 2R$. So if $d(g_\#^2(1), 1) < 2R$ then

$$d(g^2, 1) \leq d(g^2, g_\#^2(1)) + d(g_\#^2(1), 1) < 4R$$

If $d_X(1, g_\#^2(1)) \geq 2d(1, g_\#(1)) - 2R$ then

$$d(1, g^2) \geq d(1, g_\#^2(1)) - d(g_\#^2(1), g^2) > 2d(1, g_\#(1)) - 2R - 2R > 2d(1, g) - 6R$$

Step 4

Claim 70. *If $g \in Im(l)$ such that $d(1, g) > 6R$ and $d(1, g^2) < 4R$ then $g_\#$ acts as a transposition on the ends of $Im(l)$*

Proof. Take $x > g$ on l. Then

$$\begin{aligned} d(g_\#(x), g) \geq d(g_\#(x), g_\#(1)) - d(g_\#(1), g) &\geq d(g_\#(x), g_\#(1)) - R \\ &\geq d(x, 1) - 2R - R \geq d(x, 1) - 3R \end{aligned}$$

And

$$d(1, g_\#(x)) \leq d(1, g^2) + d(g^2, g_\#(g)) + d(g_\#(g), g_\#(x)) \leq d(g, x) + 7R$$

Now since we are on a line, $d(g, x) = d(1, x) - d(x, g) \leq d(1, x) - 10R$ therefore, $d(1, g_\#(x)) \leq d(g, x) + 7R \leq d(1, x) - 3R \leq d(g, g_\#(x))$ We get that $g_\#(x)$ is closer to 1 than to g. So the end $x > g$ is taken to the end $x < g$. □

The subgroup that switches the ends is an index 2 subgroup of G. Therefore there exists a g such that $d(1, g^2) > 2d(1, g) - 6R$. Consider the group $< g > \cong \mathbb{Z}$. By Step 2, it is quasi-isometric to X. Therefore, it has finite index in G. Thus we conclude that G is virtually $\mathbb{Z}$.

□

Theorem 71 (Stallings). *If G is finitely generated with infinitely many ends, then $G \cong A *_F B$ where F is finite and $[A : F] \geq 2$, and $[B : F] \geq 3$ and $A \neq G, B \neq G$ or $A*_F$ with F finite.*

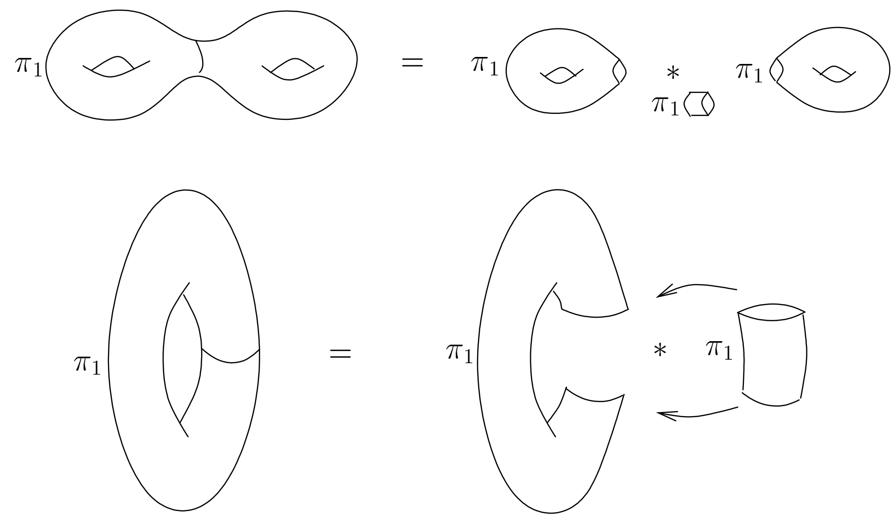


Figure 3.3: The figure on the top shows a group splitting as an amalgamated product. The bottom figure shows an HNN extention splitting

Definition 72 (HNN Extention). *Let $i_1, i_2 : C \to A$ be two embeddings then the HNN-extention*

$$A*_C =< \ A, t \mid t^{-1} i_1(c) t = i_2(c) \ \forall c \in C \ >$$

HNN stands for Higman-Neumann-Neumann. It is related to the amalgamated product via Van Kampen's theorem.

The strategy of the proof will be to find a G-action on a tree, and then conclude that the group splits. Figure 3 illustrates how a splitting induces a G-action on a tree.

Corollary 73. *If $G \cong F_k$ where $k > 1$ then G is virtually free.*

Special case. We will prove this in the case that G is torsion free. G has finitely many ends, thus by Stalling's theorem $G \cong A *_F B$ or $A*_F$. Torsion free implies $F = \{1\}$. Now we appeal to Grusko's theorem: $n(A * B) = n(A) + n(B)$ where n denotes the least number of generators. A and B must also be q.i. to free groups and so by induction they are free (no torsion) therefore, so is G. □

3.1 Tracks

Let K be a simplicial complex of dimension ≤ 2. A track in K is a connected subset $\tau \subset K$ such that:

1. $\tau \cap K^{(0)} = \phi$
2. If e is an edge then $\tau \cap e$ is finite

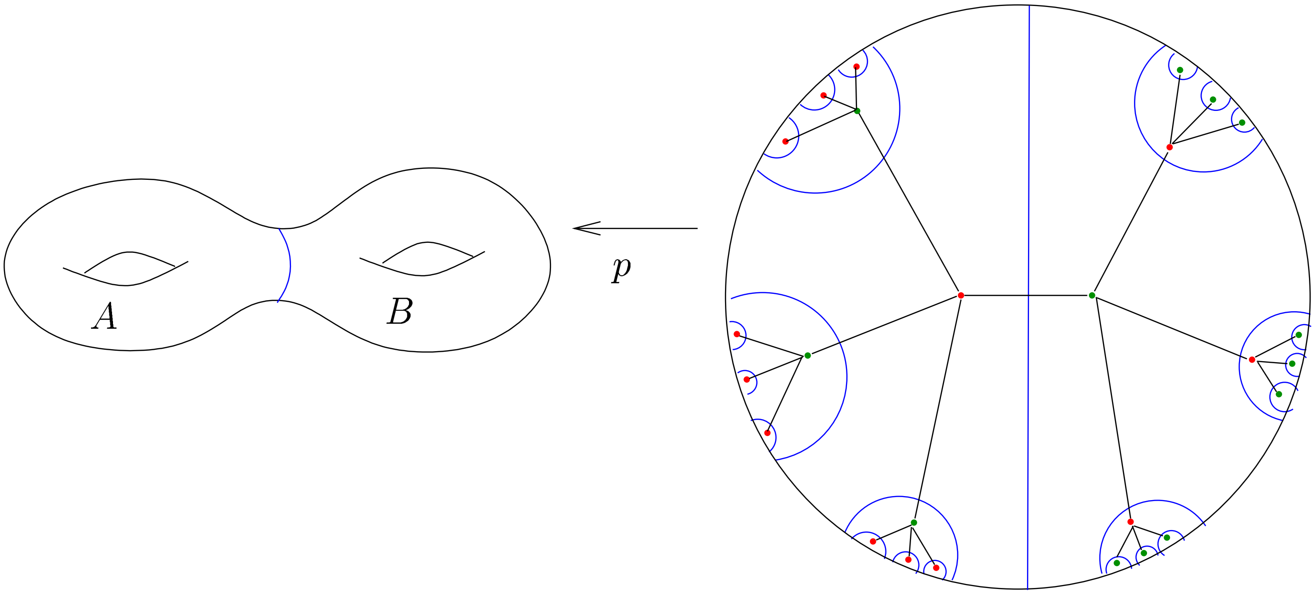


Figure 3.4: The separating curve determines a splitting of the fundamental group of the surface G. This curve lifts to $\mathbb{H}^2$ and one can construct the dual tree. G acts on this tree by deck trasformations where the stabilizer of an edge is conjugate to the group generated by the deck transformation of the separating curve. The stabilizer of a red vertex is conjugate to the fundamental group of A and the stabilizer of a green vertex is conjugate to the fundamental group of B.

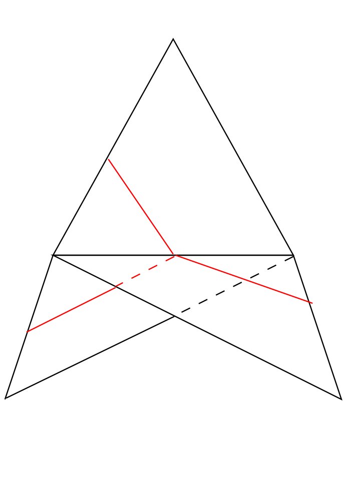


Figure 3.5: The figure on the left is a track the ones in the middle and right are not tracks.

3. If σ is a two simplex in K then $\tau \cap e$ is a finite union of pairwise disjoint straight arcs with endpoints in $\partial\sigma$.

Theorem 74 (Stallings). *If G is (almost) finitely presentable, with more than one end then G splits non-trivially*[1] *as an amalgamated product $G \cong A *_C B$ or as an HNN-extension $G \cong A*_C$ over a finite group C.*

Proof. (Dunwoody) Since G is finitely presented, it acts freely and cocompactly on a simply connected 2-complex K. By Milnor-Svarc K has two or more ends. It follows that there exists an essential track $\tau \subseteq K$ such that $g(\tau) \cap \tau$ is either empty or equal to τ for all $g \in G$. Consider the orbit of τ and construct the dual tree, i.e. vertices are components of $K \setminus \bigcup_{g\in G} g(\tau)$ and two vertices are joined by an edge if the corresponding components share a translate of τ in their frontier. Since K is connected so is T. Since every K translate separates, every edge separates so T is a tree. G acts on T, with one edge orbit (edges correspond to τ translates). The stabilizer of an edge is finite because τ is finite and the action is proper. The stabilizer of a vertex is a proper subgroup since elements that take its frontier far away will also move that component. Now we apply Bass Serre theory to get the splitting. □

What we actually need to finish off the previous proof is the following proposition:

Proposition 75. *Suppose G acts on a tree T:*

1. *There is one orbit of edges.*
2. *There are no global fixed points, i.e. no vertex is fixed under the whole group.*
3. *The action is without inversions*[2]*, i.e. if e is an edge such that $g(e) = e$ then $g|_e = id$ (the other option is that g flips e that is interchanges its endpoints)*

Then G splits as an amalgamated product or an HNN extension.

Proof. Choose a simplicial complex E which is simply connected and admits a free action of G (for example, the universal cover of a $K(G,1)$).
<u>Borel's construction</u>: Consider the diagonal action of G on $E \times T$. This action is free and induces a map $E \times T/G \xrightarrow{\pi} T/G$. T/G is a graph with one edge, it has either one or two vertices.

We assume that the quotient is an edge e with two distinct endpoints (the other case is similar). Let $X_1 = \pi^{-1}$(right two thirds of e), $X_2 = \pi^{-1}$(left two thirds of e), and $X_0 = \pi^{-1}$(midpoint of e).

$$X_0 = \coprod E \times \text{midpoints of edges in } T \Big/ G = E \times \{e\}/\mathrm{Stab}_G(e)$$

Since E is simply connected and the action is a covering space action we have $C := \pi_1(X_0) = \mathrm{Stab}_G(e)$ Likewise $A := \pi_1(X_1)$ and $B := \pi_1(X_2)$. By the Van-Kampen theorem we get $\pi_1(X) = A *_C B$. □

[1] $A *_C B$ is non-trivial if both A and B are not isomorphic to G.

[2] If the action has inversions one could get rid of them by subdividing the edges - notice that there is still only one orbit of edges.

Figure 3.6: The possible quotiens of T/G labeled by the stabilizers.

Remarks:

1. Stalling's original paper proves the theorem for finitely generated groups using essentially the same argument except that tracks are replaced by "bipolar structures" (U, U^*)

2. This raises a problem called accessibility. Say a finitely generated group G has more than one end. By Stallings we can write $G \cong A *_C B$. If A or B have more than one end then we can continue: $A \cong D *_F E$. We can get a decomposition of G into $D *_E F *_C B$by constructing the following G tree. Blow up the vertices of T_G stabilized by A, into the tree T_A which corresponds to A's decomposition. C fixes a vertex in T_A and that is where we attach the edges of the enveloping tree T_G to T_A. The result is a G tree which produces the decomposition: $G \cong D *_E F *_C B$.
What's the problem then? Does this process have to stop resulting in an amalgamated product of finite or 1-ended groups over finite groups? This is the accessibility question. While in general the answer is no, Dunwoody answered this question affirmatively for finitely presented groups. He showed that subsequent splittings can be realized in K by essential finite tracks $\tau_1, \tau_2, \tau_3, \ldots$ which are pairwise disjoint. We may also assume that they are contained in some (finite) fundamental domain. But, using an old argument of Kneser, we can prove that for every finite 2-complex L there is a number N such that in any N pairwise disjoint tracks in L there are two which are parallel.

3.

Theorem 76. *If G is quasi-isometric F_k where $k \geq 2$ then G is virtually free.*

Proof. Ingredients:

a. If G is quasi isometric to H and H is finitely presented then so is G. (We need this to know that G is accessible.

b. Any finitely presented group can be represented as a finite graph of groups with finite edge groups and finite or 1-ended vertex groups (Dunwoody's theorem)

c. No infinite group is q.i. to a 1-ended tree.
In Dunwoody's graph, if one of the vertex groups were one ended then the quasi-isometry would map it onto a subgroup of F_k which is free and

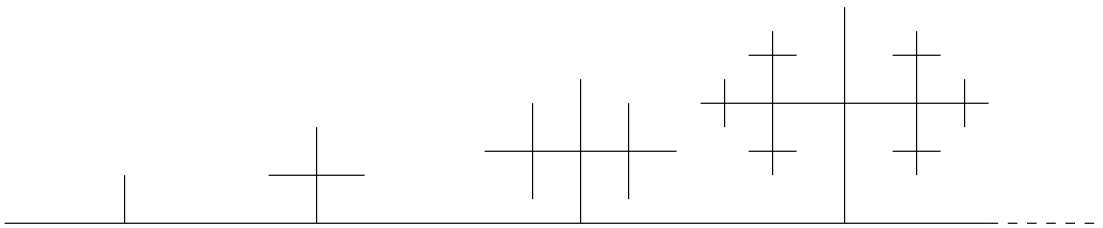


Figure 3.7: This is an example of a 1-ended tree. No group is q.i. to it.

1-ended thus q.i. to a 1-ended tree which is impossible by c. Therefore, the graph of groups for G all vertex groups are all finite.

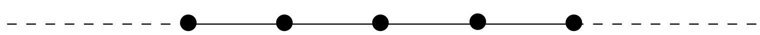
d. (Baumslag) A finite graph of groups of finite groups is virtually free

□

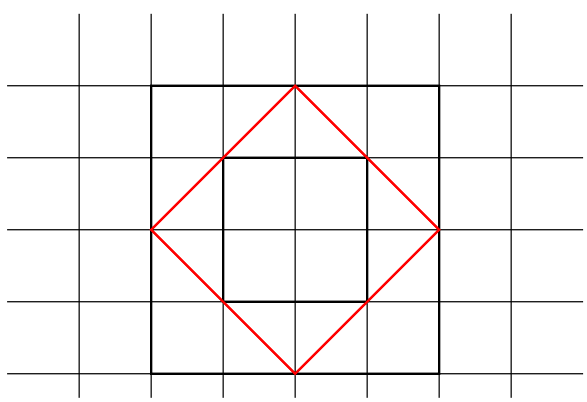
3.2 Growth of Groups

Definition 77. *Let X be the Cayley graph of a finitely generated group G. Denote by $b(R)$ the number of vertices of X in the metric ball of radius R centered at 1_G.*

Example 78. *1. If $G = \mathbb{Z}$ then $b(R) = 2R + 1$ which has linear growth.*

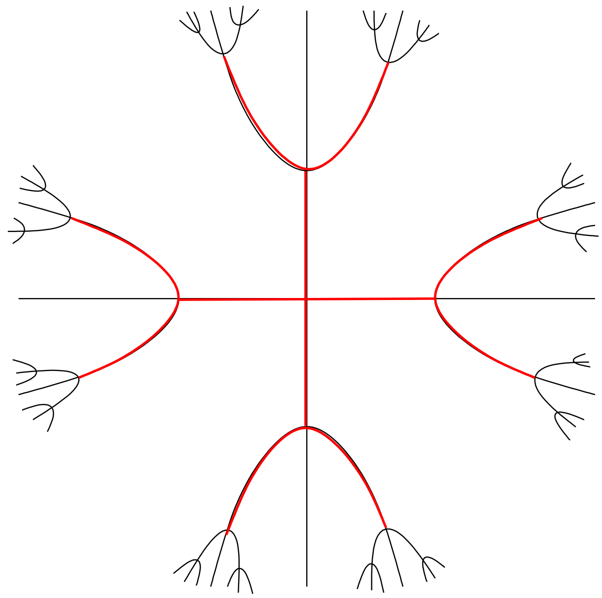
$X =$ 

2. If $G = \mathbb{Z}^2$ then $(\frac{R}{2})^2 < b(R) < (2R)^2$ which has quadratic growth.

$X =$ 

3. If $G = F_2$ then $b(R) = 4 \cdot 3^{R-1} + 1$ which has exponential growth.

Question 79. *How does $b(R)$ change under change of generators? How does it change under a general quasi-isometry?*

$X =$ 

Definition 80. *For maps $\alpha, \beta : \mathbb{N} \to \mathbb{R}_+$ we write $\beta \prec \alpha$ (asymptotic inequality) if $\exists c_1, c_2, R_0 > 0$ such that $\beta(R) \leq c_1\alpha(c_2R)$ for $R > R_0$. $\alpha \sim \beta$ if $\alpha \prec \beta$ and $\beta \prec \alpha$*

Example 81. $2^R \sim 3^R$ *(since $3^R < 2^{2R}$).*
$4R^2 \sim \frac{1}{4}R^2$

Proposition 82. *If X, X' are q.i. Cayley graphs then $b \sim b'$.*

Proof. Let $f : X \to X'$ be a q.i. We may assume that verices are mapped to vertices. There is an upper bound on the number of vertices m of X that are mapped to the same vertex in X'. Indeed:

$$\frac{1}{L}d(x_1, x_2) - A \leq d(x', x') = 0$$

Which implies $d(x_1, x_2) \leq LA$, so $m \leq b(LA)$. Moreover, $f(B_X(R)) \subseteq B_{X'}(LR+A) \subseteq B_{X'}((L+A)R)$ when $R > 1$. Therefore, $b(R) \leq b'((L+A)R) \cdot m \leq LAb'((L+A)R)$ therefore $b \prec b'$, and by symmetry $b \sim b'$. □

Remark. *If the growth function fo G is sub-polynomial then Gromov proves that G is virtually nilpotent and so the growth function is polynomial on the nose.*

Exercise 83. *If $b(R)$ is a growth function then $b(R) \prec 2^R$*

Definition 84. *G grows exponentially if $b \sim 2^R$ and subexponentially otherwise. G grows poynomially if $\exists n$ such that $b(R) < R^n$ for some n (the least such n is well defined).*

Example 85. *$G = Z^n$ then $b(R) \sim R^n$. Therefore Z^n is q.i. to Z^m if and only if $n = m$.*

Definition 86. *If G has subexponential growth and non-polynomial growth then G has intermediate growth.*

Question 87 (Milnor). *Do intermediate growth groups exist?*

There are f.g. groups whose growth is a rational function but no known examples of finitely presented groups exist.

Exercise 88. *1. If G acts cocompactly and properly discontinuously on a connected simplicial complex X then $b_G \sim b_X$ (where b_X denotes the edge paths of length R starting at the identity).*

2. Let M be a closed Riemannian manifold $\widetilde{M}$ is its universal cover. Let $V_R =$ the volume of the R ball centered at the identity in $\widetilde{M}$ then $b_{\pi_1 X}(R) \sim V_R$. Actuall, more is known to be true:

$$vol_{\widetilde{M}}(B_{x_0}(\varepsilon)) \sim (1 - k_n s_{x_0} \varepsilon^2) vol_{\mathbb{R}^n}(\varepsilon)$$

where k_n is a universal constant which depends on on the dimension of $\widetilde{(M)}$ and s_{x_0} is the scalar curvature at x_0. If $s_{x_0} > 0$ then the volume is smaller then the corresponding volume in $\mathbb{R}^n$. Milnor noticed that the geometric assuption $s > 0$ has algebraic consequences on $\pi_1(M)$. Apealing to Gromov's theorem, you can actually conclude that $\pi_1(M)$ is virtually nilpotent.

Theorem 89. *If G is virtually nilpotent then $b_G(R) \prec R^n$ for some n.*

Exercise 90. *The Heisenberg group has growth R^4. This is surprising since the group is three dimensional, however considering the short exact sequence:*

$$1 \to \mathbb{Z} \to H \to \mathbb{Z}^2 \to 1$$

There are R^2 vertices in the ball of radius R about 1 in $\mathbb{Z}^2$ and R^2 points in $\mathbb{Z}$ mapping into this ball (since $z^{R^2} = [x^R, y^R] \in B(4R)$). Therefore, the growth is $R^2 \cdot R^2 = R^4$, Notice that this is just a heuristic argument and one should be more careful c.f. Bass' theorem.

Remark. *Bass showed that if G is a nilpotent group with lower central series $G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$ the growth function has degree d where $d = \sum i d_i$, $d_i = rank(G_i/G_{i+1})$.*

Theorem 91 (Gromov). [3] *If G has polynomial growth then it is nilpotent.*

Corollary 92. *The class of virtually nilpotent groups is rigid (while the class of virtually solvable groups is not).*

[3]The big bang of GGT

Chapter 4

Gromov's Polynomial Growth Theorem

Theorem 93 (Gromov). *If Γ is finitely generated and has polynomial growth, then Γ is virtually nilpotent.*

Ingredients:

1. **Wolf**: If Γ is finitely generated and (virtually) nilpotent, then Γ has polynomial growth.

2. **Milnor-Wolf**: If Γ is finitely generated solvable, then Γ either grows exponentially or it is virtually nilpotent.

3. **Lemma A**: Suppose Γ is finitely generated and has subexponential growth. If

$$1 \to K \to \Gamma \to \mathbb{Z} \to 1$$

   is a short exact sequence, then K is finitely generated. If, in addition, Γ has growth $\leq R^d$, then K has growth $\leq R^{d-1}$.

4. **Gromov-Hausdorff limits**: Suppose Γ is finitely generated and has polynomial growth. Then there is a metric space Y and a homomorphism $l : \Gamma \to Isom(Y)$ such that

   (a) Y is homogeneous (i.e. $\forall y_1, y_2 \in Y$ there exists a isometry of Y which sends $y_1 \mapsto y_2$);

   (b) Y is connected, locally connected;

   (c) Y is complete;

   (d) Y is locally compact and $dim\, Y < \infty$;

   (e) if $l(\Gamma)$ is finite and Γ is not virtually abelian, then $\Gamma' = ker(l)$ has, $\forall$ neighborhood U of $1_Y \in Isom(Y)$, a representation $\rho : \Gamma' \to Isom(Y)$ such that $\rho(\Gamma') \cap (U \setminus \{1_Y\}) \neq \emptyset$.

5. **Hilbert 5^{th} Problem** (Montgomery-Zippin-Gleason): If Y satisfies 1)-4), then $Isom(Y)$ is a Lie group with finitely many components.

6. **Tits Alternative**: A finitely generated subgroup of $GL_n(\mathbb{R})$ is either virtually solvable or contains F_2.

7. **Jordan's Theorem**: $\forall n\ \exists q(n) \in \mathbb{N}$ such that every finite subgroup of $GL_n(\mathbb{R})$ contains an abelian subgroup of index $\leq q$.

We observe that Tits Alternative is true if $GL_n(\mathbb{R})$ is replaced by a Lie group L with finitely many components as follows:

1. Step 1): We may assume that L is connected otherwise we replace Γ by $\Gamma \cap L_0$, where L_0 is the component of 1.

2. Step 2): We consider $Ad : L \to GL_n(\mathbb{R})$. If L has a trivial center, Ad is injective and the statement follows from the $GL_n(\mathbb{R})$-version.
   In general, the center is in the kernel, and we get
   $$1 \to Z \hookrightarrow \Gamma \twoheadrightarrow \Gamma' \subset GL_n(\mathbb{R}).$$
   Γ' satisfies Tits. If Γ' is virtually solvable, so is Γ. If $\Gamma' \supset F_2$, so does Γ.

Now we prove the Gamov Theorem.

Proof. By Lemma A, the Milnor-Wolf Theorem and the induction on d, it suffices to construct an epimorphism $\Gamma \subset \Gamma' \to \mathbb{Z}$, where $\Gamma' \subset \Gamma$ has finite index.
We have $l : \Gamma \to Isom(Y)$.
If $l(\Gamma)$ is infinite, it is virtually solvable by Tits. By the Milnor-Wolf Theorem, in fact $l(\Gamma)$ is virtually nilpotent.
An infinite virtually nilpotent group N has a finite index subgroup that maps onto $\mathbb{Z}$.
Indeed, we replace N by a nilpotent finite index subgroup also called N. If $N/[N, N]$ is infinite, then $N/[N, N] \cong \mathbb{Z}^k \times P^{\text{finite}}$ $(k > 0)$ and we can project to $\mathbb{Z}$. Otherwise, we replace N by $[N, N]$ and we use induction on the length of the lower central series.
Now, let suppose $l(\Gamma)$ is finite and use the number 5) above.
Let $L :=$ the component of 1_Y of $Isom(Y)$.
We claim that there exists a subgroup $\Delta \subset \Gamma$ of finite index such that Δ admits homomorphisms to L with arbitrarily large images.
The key idea to prove this claim is to consider the exponential map $exp : \mathfrak{l} \to G$. Then $\forall n$ there exists a neighborhood U of $1 \in L$ such that a nontrivial element in U has order $\leq n$.
Thus, by the number 5) above, there are homomorphisms $\Gamma' \to Isom(Y)$ with arbitrarily large images.
Some subgroup Δ of Γ' of index $\leq$ the number of components of $Isom(Y)$ maps to L infinitely many often.
If one of these maps has infinite image, we proceed as before. Now, suppose they are

all finite.
By Jordan theorem, there is a further subgroup $\Delta' \subset \Delta$ of index $\leq q$ such that under infinitely many of these representations, the image is abelian.
Therefore Δ' has infinite abelianization, so it maps to $\mathbb{Z}$ and we are done. $\square$

4.1 Gromov-Hausdorff convergence

Suppose X is a compact metric spac. Let 2^X denote the set $\{A \subseteq X | A \neq \phi \text{ is closed }\}$ and define a metric on it as follows:

$$d_{2^X}(A, B) = \inf\{\epsilon | A \subseteq N_\epsilon(B), B \subseteq N_\epsilon(A)\}$$

This is called the Hausdorff metric.

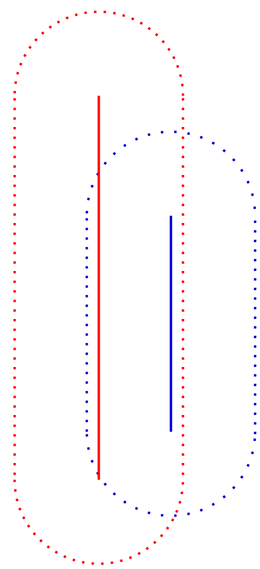


Figure 4.1: If $A \subseteq N_\epsilon(B)$ then B need not be contained in $N_\epsilon(A)$

Remark. *$A \subseteq N_\epsilon(B)$ doesn't imply $B \subseteq N_\epsilon(A)$. For example $X = [0, 1]$, $A = \{0, 1\}$, $B = \{\frac{1}{4}\}$, then $\epsilon = \frac{1}{3}$ works one way but not the other.*

Lemma 94. *2^X is compact.*

Proof. Let $\{A_i\}$ be a sequence of closed subsets in X. We must show that there's a convergent subsequence. Fix a countable basis for the topology of X: $U_1, U_2, U_3, \ldots$. After passing to the subsequence $A^1_j = A_{i_j}$ we may assume that either $A^1_j \cap U_1$ is empty for all j or non-empty for all j. Let A^2_j be a subsequence of A^1_j such that either A^2_j intersects U_2 for all j or is disjoint from U_2 for all j. Continue to form the sequence A^k_j. The sequence A^j_j has the property that for all U_k either A^j_j is eventually disjoint from U_k or persistently intersects it from some point onwards. Define $B = \{b \in B |$ every neighborhood of b eventually intersects the A_i's$\}$. Then $B = \lim_{i\to\infty} A_i$ $\square$

Lemma 95. *If $A_i \to A$ and all of the A_i's are isometric to each other, then A is isometric to each of them.*

Proof. Fix isometries $\phi_i : A_1 \to A_i$. Choose a dense subset $X = \{x_1, x_2, \dots\}$ of A_1. There is a subsequence of $\{\phi_i(x_1)\}$ which converges to some point $\overline{x_1} \in A$ by a diagonal argument similar to the one in the previous lemma, after passing to a subsequence $\phi_i(x_j) \to \overline{x_j}$ for all j. The map $\phi : x_j \to \overline{x_j}$ is an isometric embedding of X into A. To prove that this map extends to an isometry $A_1 \to A$ we must show that it is onto. For $a \in A$ choose a subsequence $a_i \in A_i$ so that $a_i \to a$. For any k, i there is a j, depending on i and k such that $d_{A_i}(\phi_i(x_j), a_i)) < \frac{1}{k}$. For a single k, choose a convergent sequence of x_js which converges to x_k. Then $d_A(\phi(x_k), a) < \frac{1}{k}$. A limit point of $\{x_k\}$ will be taken by ϕ to a. $\square$

Definition 96 (Gromov-Haudorff distance). *Let A, B be compact metric spaces. The G-H distance is:*

$$D(A, B) = \inf\{\epsilon | \exists \text{ isometric embeddings } A \to Z, B \to Z \text{ with } d_{2^Z}(A, B) < \epsilon\}$$

There is also a pointed version of this construction: A, B are equipped with basepoints a_0, b_0 and we insist $d(a_0, b_0) < \epsilon$.

Proposition 97.
1. $D(A, B) < \infty$
2. $D(A, B) = D(B, A)$
3. $D(A, C) < D(A, B) + D(B, C)$
4. $D(A, B) = 0 \implies A \underset{isometric}{\cong} B$

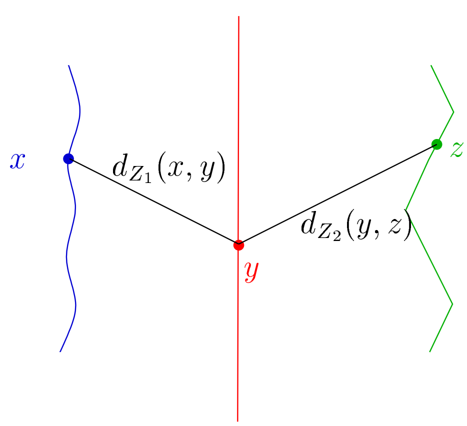


Figure 4.2: proof of property 2 in for the Gromov-Hausdorff metric.

Proof. 1. We need to show that both A and B isometrically embed in some Z. Suppose $\operatorname{diam}(A), \operatorname{diam}(B) < 2D$, take $Z = A \coprod B$ with $d(a, b) = D$ for all $a \in A$ and $b \in B$.

2. This is obvious.

3. Take $Z_1 \supseteq A \cup B$ and $Z_2 \supseteq B \cup C$ such that $d(A, B) < D(A, B) + \epsilon$ and $d(B, C) < D(B, C) + \epsilon$. We can assume A, B are disjoint in Z_1 and B, C are disjoint in Z_2. Define a metric on $A \coprod C$ via:

$$d(a, c) = \inf_{b \in B} \{d_{Z_1}(a, b) + d_{Z_2}(b, c)\}$$

   Then $d(A, C) < D(A, B) + D(B, C) + 2\epsilon$

4. Suppose $D(A, B) = 0$ then $inf_{A,B \hookrightarrow Z}\{d_Z(A, B)\} = 0$. There is a sequence A_i and $Z_i \supseteq A_i, B$ so that $d_{Z_i}(A_i, B) < \frac{1}{i}$. Define a metric on $A = (\coprod A_i) \coprod B$ by $d(a_i, a_j) = \inf_{b \in B}\{d(a_1, b) + d(b, a_j)\}$.
   Check that Z is a compact metric space and that $A_i \to B$ in 2^Z. So by a previous lemma $A \cong B$.

□

4.1.1 Gromov's Compactness Criterion

$\{X_i\}$ is a sequence of compact metric spaces.

Theorem 98 (compactness criterion). *If*

1. *Uniformly bounded diameter - There exists D such that $diam(X_i) < D$ for all i.*

2. *Uniform compactness - $\forall \epsilon > 0$, $\exists N$ such that each X_i can be covered by N balls.*

Then some subsequence of X_i converges to a compact metric space X. The same holds in the pointed category.

Proof. Fix $X = X_i$. There is a standard construction of embedding X in $\mathbb{R}^X$ - the space of continuous functions from X to $\mathbb{R}$, via the map that sends x to the function $y \to d(x, y)$. Clearly this function is 1-1 since each function in the image has a different zero. The problem is that $\mathbb{R}^X$ is not a compact space. Moreover even if $\text{Diam}(X) < D$ and we embed into $[0, D]^X$ we don't improve the situation.[1].
We want to modify this construction by replacing $\mathbb{R}^X$ with a compact space. Fix a sequence $\epsilon_i \to 0$ and let $a_1, a_2, \cdots a_{N_1}$ be an ϵ_1-net in X (notice that N_1 does not depend on the X_i that we've chosen in the beginning). Let $\{a_{i,j}\}_{1,\ldots N_2}$ be an ϵ_2-net in the ball $B(a_i, \epsilon_1)$. Continuing this way, we've constructed $\mathbf{F} = \{a_1, \ldots a_{N_1}, a_{i,j}, a_{i,j,k}, \ldots\}$ which is a countable set. Let $\mathbf{Z}$ be the space of continuous functions $f : \mathbf{F} \to \mathbb{R}$ such that:

1. $0 \leq f(a_i) \leq D$ where $D > \text{diam}(X)$

2. $|f(a_{i,j} - f(a_i)| < \epsilon_1$

[1] For example if the space $[0, 1]^{[0,1]}$ is not compact since the sequence x^n doesn't converge to a function in the space (it converges pointwise to a non-continuous function)

3. $|f(a_{i,j,k} - f(a_i, j)| < \epsilon_2$

etc...

With respect to the sup metric this space is compact. Indeed, for any Cauchy sequence there exists a pointwise limit which automatically satisfies all of the inequalities above. X embeds into Z via the map that send x to the function $a_{i,\dots,l} \to d(x, a_{i\dots,l}$. Now let i vary. By the uniform compactness criterion, we can embed every X_i in the same space Z. Now we use the previous lemma that 2^Z is compact to conclude that X_i must have a convergent subsequence. □

Remark. *The compactness criterion in the previous theorem is also a necessary one. If X_i is a convergent sequence of compact spaces where the limit is a compact metric space X. Then for any $\epsilon > 0$, Let M_ϵ be the number of $\frac{\epsilon}{3}$-balls covering X. Let i_0 be the index such that for all $i > i_0$, $D(X_i, X) < \frac{\epsilon}{4}$. For $i > i_0$ we show that M_ϵ ϵ-balls suffice to cover X_i. There is a Z_i such that $d_{2^{Z_i}}(X_i, X) < \frac{\epsilon}{3}$. Cover X in Z_i by M $\frac{\epsilon}{3}$-balls: $B(t_1, \frac{\epsilon}{3}), \dots B(t_M, \frac{\epsilon}{3})$. For each j pick a point y_j in X_i, a distence at most $\frac{\epsilon}{3}$ from t_j. $\cup_{j=1}^M B(y_j, \epsilon)$ contains X_i. Indeed, Take $y \in X_i$ then there is a $t \in X$ s.t. $d(y,t) < \frac{\epsilon}{3}$. There is a j such that $d(t_j, x) < \frac{\epsilon}{3}$, so $d(y, x_j) \leq d(y,t) + d(t, t_j) + d(t_j, y_j) < \epsilon$. Therefore, the union of the ϵ balls about the y_j's cover all of X_i. The uniform N in the lemma is $\max\{N_1, \dots N_{i_0}, M\}$ where N_i is the number of ϵ balls required to cover X_i. Similarly one can show that X_i are uniformly bounded.*

Definition 99. *Let (X_i, x_i) be proper (not necessarily compact) metric spaces. Then (X_i, x_i) converges to (X, x) if for every radius R, $B_{X_i}(x_i, R)$ converges to $B_X(x, R)$.*

Proposition 100. *Let (X_i, x_i) be proper metric spaces such that $B(x_i, R)$ is uniformly compact. Then some subsequence of (X_i, x_i) converges to a proper metric space.*

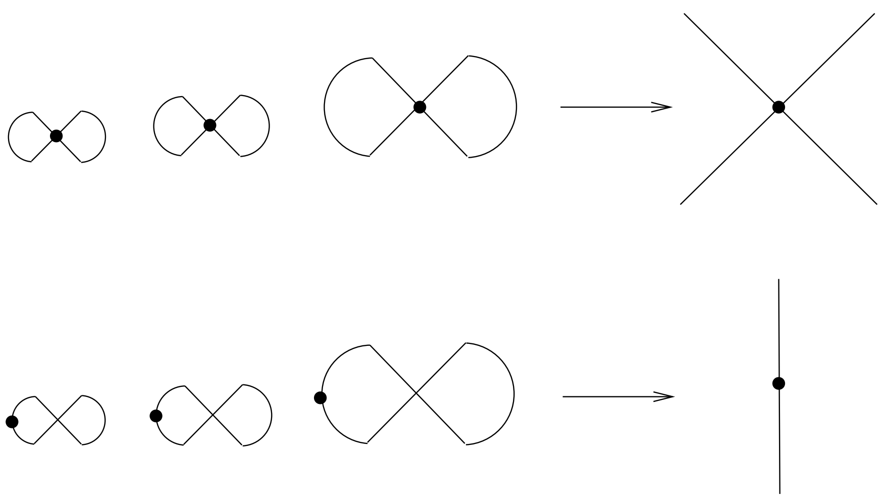


Figure 4.3: Convergence of proper metric spaces. Notice that the limit might depend on the choice of basepoints x_i.

Proof. Use gromov's criterion for compact spaces to prove that $B(x_i, R)$ converges for every R. □

Example 101. *1. $X_i = \frac{1}{i}\mathbb{R}^n$. Notice that X_i are isometric to $\mathbb{R}^n$ so it is straightforward to conclude that the limit is $\mathbb{R}^n$.*

2. *Consider $X_i = \frac{1}{i}\mathbb{Z}^n$. The limit is $\mathbb{R}^n$ with the l_1 metric. (exercise)*

3. *$X_i =$ complete metric on $\mathbb{R}^2$ with curvature bounded below by χ_0. Some subsequence of the X_is converges (check Gromov's condition).*

4. *$X_i = \frac{1}{i}\mathbb{H}^2$. No subsequence converges.*
   Suppose some subsequence converged to X and pass to that subsequence. Consider the ball B of radius 2 about some point in X. Because X is proper, it can be covered by a finite number N, of 1-balls. This implies that the ball B_i of radius 2 about some point in X_i can be covered by M balls of radius 1 where M might be larger than N but still finite and independent of i. Since X_i are homogeneous spaces we may assume that B_i is centered at the origin of X_i. Therefore, it follows that the ball $B_o(2i)$ about the origin in $\mathbb{H}^2$ can be covered by M $B(i)$ balls. But this contradicts basic facts about the area of balls in $\mathbb{H}^2$.

$$M \geq \frac{Area_{\mathbb{H}^2}(B(2i))}{Area_{\mathbb{H}^2}(B(i))} \sim \frac{e^{2i}}{e^i} = e^i \xrightarrow[i\to\infty]{} \infty$$

5. *Gromov shows that if Γ has polynomial growth then there is some sequence of $d_i \xrightarrow[i\to\infty]{} \infty$ such that $X_i = \frac{1}{d_i}\Gamma$ converges.*

6. *Let S be a surface with a Reimannian metric, and TS its tangent bundle. There is a natural Reimannian metric on TS. On the vertical part of TS (the fiber) put a standard Euclidean metric, on the horizontal part pull back the metric on S. Consider T_ε the submanifold of TS containing vectors of length ε, T_ε stabilizes for small ε. Denote the resulting space T_1. Then $\frac{1}{i}T_1 \to S$. In this way you can construct an example of a sequence of manifolds converging to a non-manifold (???)*

Definition 102. *Suppose $(X_i, x_i) \to (X, x)$ then there exists a choice of metrics on $X_j \coprod X$ for $j = 1, 2, 3, \ldots$ so that for all $r > 0$ there is an $\varepsilon > 0$ such that the following holds for $j > j_0$:*

$$d(x_j, x) \leq \varepsilon, \quad B_{x_j}(r) \subseteq N_\varepsilon(x), \quad B_x(r) \subseteq N_\varepsilon(x_j)$$

Fix such a choice and write $(X, x_i) \implies (X, x)$. It makes sense to say that $x'_j \to x'$ if $d(x'_j, x') \to 0$.
If $(X_i, x_i) \implies (X, x)$ and $f_i : X_i \to X_i$ then we say that f_i definitely converges to $f : X \to X$ if whenever $x'_j \to x'$ then $f_j(x'_j) \to f(x')$.

Lemma 103 (Isometry Lemma). *Suppose $(x_i, x_i)implies(X, x)$ and $f_i : X_i \to X_i$ are isometries with $d(f(x_i), x_i) \leq C$ then after passing to a subsequence $f_i \implies f : X \to X$ and f is an isometry.*

Proof. Let $z \in X$, we define $f(z)$. Choose a sequence $z_i \in X_i$ such that $z_i \to z$. After passing to a subsequence $f(z_i) \to w$ some point in X. Let $f(z) = w$. By a diagonal argument we can make the same subsequence of maps work for a dense countable subset of X. On this subset, f preserves distances. f extends to an isometry $f : X \to X$. Need to worry about why f is onto. $\square$

We've proved that if Γ grows polynomially then there exists $r_i \to \infty$ such that $\frac{1}{r_i}\Gamma \xrightarrow[i\to\infty]{} Y$. Next we will prove:

Proposition 104. *Y satisfies the following:*

1. *it is homogeneous.*
2. *it is connected, and locally connected.*
3. *it is complete.*
4. *it is locally compact.*
5. *$dim(Y) < \infty$*

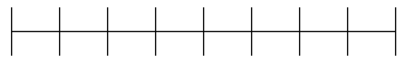
Proof. Since $\frac{1}{r_i}\Gamma$ is compact their limit is a proper metric space and therefore locally compact and complete.
Y is homogenious: for points $y_1, y_2 \in Y$ choose sequencese $x_i, x_i' \in X_i = \frac{1}{r_i}\Gamma$ which are vertices and $x_i \implies y$, $x_i' \implies y$. One can consider a group element g_i which is an isometry of X_i which takes x_i to x_i'. By the isometry lemma we obtain a limiting isometry $f : X \to X$ which takes y_1 to y_2.
Y is not only path connected but it is a geodesic metric space: It is enough to shoe that for any two points y_0 and y_1 one can find a midpoint z i.e. a point for which $d(y_0, z) + d(y_1, z) = d(y_0, y_1) = 2d(x_0, z)$ (because if we have this then we know where to send the diadic rationals, and then we extend the map continuously to the rest of the interval). We use definite convergence: let $x_i \implies y_0$ and $x_i' \implies y_1$. Take z_i to be a vetrex at a minimal distence from the midpoint between x_i and x_i'. The errors in the equality above are smaller than $\frac{1}{2r_i}$ so the equality holds in the limit after passing to a convergent subequence $z_i \to z \in Y$.
$\dim(Y) < \infty$ will be proven in the following subsection. $\square$

4.1.2 Hausdorff Dimension

As a warm up let us introduce Box dimension:
If E is a metric space define $H^s_\delta(E) = \inf\{\Sigma \mathrm{diam}(A_i)^s | \mathrm{diam} A_i \leq \delta, \cup A_i \supseteq E\}$ for $s, \delta > 0$ real numbers. For $\delta_1 < \delta_2$ then $H^s_{\delta_1}(E) \geq H^s_{\delta_2}(E)$ define $H^s = \lim_{\delta\to 0} H^s_\delta(E)$.

Let I^n be the unit n-cube. Then:

$$H^s(I^n) = \begin{cases} 0 & s > n \\ \text{finite} \neq 0 & s = n \\ \infty & s < n \end{cases}$$



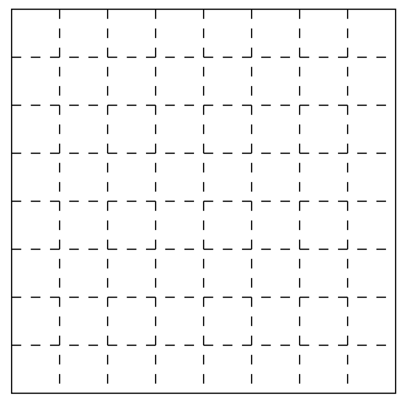


Figure 4.4: One needs $\frac{1}{\epsilon}$ subintervals to cover a unit inteval by subintervals of length ϵ, whereas one needs approximately $\frac{1}{\epsilon^2}$ subsquares of diameter ϵ to cover a unit square. The first exponent is 1 and the second is 2. These are the box dimensions of the interval and square respectively.

Definition 105. *There exists a critical constant $s = s_0$ such that for $s < s_0$ $H^s(E) = \infty$ and for $s > s_0$ $H^s(E) = 0$. The Hausdorff dimension of E is s_0.*

Definition 106. *The following gives a recursive defintion of dimension:*

1. $\dim(X) = -1$ *iff* $X = \Phi$

2. $\dim(X) \leq n$ *iff* $\forall x \in X$ *has an arbitrarily small neighborhood* U *such that* $\dim Fr(U) \leq n - 1$.

Fubini: If $H^s(E) = 0$ then for almost all $r > 0$ we have $H^{s-1}(S_{x_0}(r) = 0$. In particular:

Corollary 107. $\dim E \leq$ *HausDimE.*

Proposition 108. *HausDim*$(Y) \leq d + 1$ *where d is the degree of the polynomial growth of the group.*

Proof. By construction of the sequence of r_i, every $\frac{1}{2}$-ball can be covered by $2^{j(d+1)} = (2^j)^{d+1}$ balls of radius $\frac{1}{2^j}$. ☐

Example 109. *$X = \mathbb{Z}$ and $d(i,j) = \sqrt{|i-j|}$ what is $\lim_{n\to\infty} \frac{1}{n}X$? Consider $Y = \mathbb{R}$ with $d(x,y) = \sqrt{|x-y|}$ notice that $\phi_n : Y \to Y$ that takes $x \to n^2x$ scales the metric by n. Therefore Y and $\frac{1}{n}Y$ are isometric, and $\frac{1}{n}Y \to Y$. What is HausDim(Y)? Notice that the diameter of $[0,\epsilon]$ is $\sqrt{\epsilon}$ therefore we need $\frac{1}{\epsilon^2}$ intervals of diam ϵ to cover the unit interval (the diameter of $[0,\epsilon^2]$ is ϵ). So HausDim$(Y) = 2$.*

4.1.3 Representations of Γ into Isom(Y)

Γ acts on $\frac{1}{r_i}\Gamma$ by left translations. Any fixed γ moves some point by $\frac{\|\gamma\|}{r_i} \to 0$. After passing to a subsequence, the isometries represented by γ converge to an isometry of Y which fixes the basepoint. Thus we get a Γ action on Y by isometries $l : \Gamma \to \mathrm{Isom}(Y)$.

Claim 110. *If $l(\gamma)$ is finite and Γ isn't virtually abelian then $\Gamma' = ker(l)$ satisfies: for any neighborhood U of $1_Y \in Isom(Y)$ there is a representation $\rho : \Gamma' \to Isom(Y)$ such that $Im(\rho) \cap (U \setminus \{1_Y\}) \neq \Phi$.*

Example 111. *If $\Gamma = \mathbb{Z}^n$ the action on Y we get by this Gromov limit is the trivial action on $Y = \mathbb{R}^n$. Since $\mathbb{Z}^n$ is abelian we cannot apply the previous claim.*

Lemma 112. *Suppose $\exists C > 0$ such that every generator $\gamma \in \Gamma$ moves every $x \in \Gamma$ by less than C, i.e. $d(x, \gamma \cdot x) \leq C$ then Γ is virtually abelian.*

Proof. For all $x \in \Gamma$, $\|x\gamma x^{-1}\| \leq C$ therefore γ's conjugacy class is finite. There is a finite index subgroup $\Gamma_1 < \Gamma$ which acts trivially on γ's conjugacy class, i.e. all elments of Γ_1 commute with γ. Do this for every generator and intersect $\Gamma_1, \ldots, \Gamma_n$ then we get a finite index subgroup whose elements commute with the generators, hence with every element of Γ. In particular, this finite index subgroup is abelian. $\square$

Proof of Claim **??**. First let us introduce the notation: $D(\Gamma, r) = \max\{d(\gamma\beta, \beta) | \beta \in B(r), \gamma \text{ is a generator }\}$ is the maximum distence a point in the r-ball is translated by a generator.
Suppose $l(\Gamma)$ is finite. Pass to $\Gamma' = \ker(l)$ a finite index subgoup on which the representation is trivial. For any generator $\gamma \in \Gamma'$, $l(\gamma)$ is the identity. So for any convergent sequence $x_i \to y \in Y$ we have $d_{X_i}(x_i, \gamma x_i) = 0$. In particular, for $x_i = \beta \in \Gamma$ we have $d_{X_i}(\gamma\beta, \beta) = \frac{1}{r_i} d(\gamma\beta, \beta) \to 0$. In particular, $\frac{1}{r_i} D(\Gamma, r_i) \to 0$. If $D(\Gamma, r_i) \leq C$ for some global constant C then by the previous lemma Γ' is virtually abelian and so is Γ.
Therefore suppose that $D(\Gamma, r_i) \to \infty$ sublinearly.

Claim 113 (Displacement Lemma). *For each $\epsilon > 0$ there is a sequence $\alpha_i \in \Gamma$ such that $\lim_{i\to\infty} D(\alpha_i^{-1}\Gamma\alpha_i, r_i) = \epsilon$.*

By unboundedness of $D(\Gamma, r_i)$: there is a point x_i such that $d(\gamma x_i, x_i) > \epsilon r_i$ for some generator γ. Hence $d(x_i^{-1}\gamma x_i, 1) > \epsilon r_i$ and $D(x_i^{-1}\Gamma x_i, r_i) > \epsilon r_i$. Consider a path from 1 to x_i in the cayley graph of Γ. It is elementary to see that if y and z are consequtive points on this path then $|D(y^{-1}\Gamma y, r_i) - D(z^{-1}\Gamma z, r_i)| \leq 2$. Together with: $D(\Gamma, r_i) < r_i\epsilon$ and $D(x_i^{-1}\Gamma x_i, r_i) > \epsilon r_i$ we get that for some α_i on this path $|D(\alpha_i^{-1}\Gamma\alpha_i, r_i) - \epsilon r_i| \leq 2$. This proves the subclaim and therefore the claim itself. $\square$