# Extensions of Abelian Automata Groups

Chris Grossack (advised by Klaus Sutner)

Carnegie Mellon University

## Mealy Automata

A **Mealy Automaton** $\mathcal{A}$ is a finite state machine which encodes a family of continuous functions from Cantor Space to itself. For us, these continuous functions will always be homeomorphisms, and thus we may associate to a machine $\mathcal{A}$ a subgroup $\mathcal{G}(\mathcal{A})$ of the automorphisms of Cantor Space. These are known as **Automata Groups** in the literature.

Classifying all groups generated by even 3-state machines is still and open problem, so we will focus attention on those which generate abelain groups.

We can define two operations from $\mathcal{G}(\mathcal{A})$ to itself called **Residuation**, where the 0-residual of $f$ is defined to be the unique function $\partial_0 f$ so that

$$f(0s) = f(0)(\partial_0 f)(s).$$

The 1-residual $\partial_1 f$ is defined analogously.

A function $f$ is called **even** if it copies its first input bit, that is $f(as) = a\partial_a f(s)$, and is called **odd** otherwise.

It is a theorem of Sutner that, in the abelian case, $f$ is even if and only if $\partial_0 f = \partial_1 f$.

## Past Results

In their paper "Automorphisms of the binary tree: State-closed subgroups and dynamics of 1/2-endomorphisms", Nerkashevych and Sidki show that abelian automata groups are isomorphic to integer lattices, and moreover, there is a "1/2-integral" matrix $\mathbf{A}_\mathcal{A}$ of irreducible character so that residuation lifts to an affine map. Succinctly, for some $\varphi$:

$$\varphi : \mathcal{G}(\mathcal{A}) \cong \mathbb{Z}^m$$

$$\varphi(\partial_0 f) = \begin{cases} \mathbf{A}\varphi(f) & f \text{ even} \\ \mathbf{A}(\varphi(f) - \overline{e}) & f \text{ odd} \end{cases}$$

$$\varphi(\partial_1 f) = \begin{cases} \mathbf{A}\varphi(f) & f \text{ even} \\ \mathbf{A}(\varphi(f) + \overline{e}) & f \text{ odd} \end{cases}$$

Moreover, $\varphi$ can be chosen so that the first component of $\varphi(f)$ is even iff $f$ is even. Under this additional constraint, $\overline{e}$ must be odd, and we can put $\mathbf{A}$ into rational canonical form ($a_i \in \mathbb{Z}$):

$$\begin{pmatrix} \frac{a_1}{2} & 1 & 0 & \dots & 0 \\ \frac{a_2}{2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n-1}}{2} & 0 & 0 & \dots & 1 \\ \frac{a_n}{2} & 0 & 0 & \dots & 0 \end{pmatrix}$$

$\chi_\mathbf{A}$is $\mathbb{Q}$-irreducible

Finally, our machines are finite if and only if the spectral radius of $\mathbf{A}$ is $\rho(\mathbf{A}) < 1$.

## The Questions

By choosing a 1/2-integral matrix $\mathbf{A}$ and an odd residuation vector $\overline{e}$, we can view $\mathbb{Z}^m$ as a Mealy Automaton with countably many states $\mathfrak{C}(\mathbf{A}, \overline{e})$. It is then natural to ask the following questions:

- For which choices of $\mathbf{A}$, $\overline{e}$ can we find a machine $\mathcal{A}$ as a subautomaton of $\mathfrak{C}(\mathbf{A}, \overline{e})$?
- For which $\overline{e}$ is $\mathcal{G}(\mathcal{A}) \cong \mathfrak{C}(\mathbf{A}, \overline{e})$ as an automaton?
- When is $\mathcal{G}(\mathcal{A}_1)$ a subgroup of $\mathcal{G}(\mathcal{A}_2)$?
- If $\mathcal{A}$ is a subautomaton of $\mathfrak{C}(\mathbf{A}, \overline{e})$, then where is it located? That is, for a state $\alpha \in \mathcal{A}$, what is $\varphi(\alpha) \in \mathbb{Z}^m$?
- Is there a way to remove this parameter $\overline{e}$?

It turns out that all of these questions are deeply connected, and understanding the answer requires understanding these lattices not at the level of abelian groups, but at the level of Modules.

## The Key

One (perhaps surprising) fact that prompted this investigation is the presence of a machine, called the **Principal Automaton** $\mathfrak{A}(\mathbf{A})$ living inside every $\mathfrak{C}(\mathbf{A}, \overline{e})$. Indeed, we present the following facts (which are partial answers to the above questions):

- $\mathfrak{A}(\mathbf{A})$ can be found in $\mathfrak{C}(\mathbf{A}, \overline{e})$ for every $\overline{e}$
- $\mathcal{G}(\mathfrak{A})$ is a subgroup of $\mathcal{G}(\mathcal{A})$ whenever $\mathcal{A}$ and $\mathfrak{A}$ share a matrix.
- There is a state $\delta \in \mathfrak{A}$ such that $\varphi(\delta) = \overline{e}$ in $\mathfrak{C}(\mathbf{A}, \overline{e})$

## The Technique

While $\mathbf{A} : 2\mathbb{Z} \oplus \mathbb{Z}^{m-1} \to \mathbb{Z}^m$ needs an even vector as input to ensure an integer vector output, $\mathbf{A}^{-1}$ can take in any integer and output an integer vector. With this in mind, we can (almost) make $\mathfrak{C}(\mathbf{A}, \overline{e})$ into a $\mathbb{Z}[x]$-module, where the action $x \cdot \overline{v}$ is given by $\mathbf{A}^{-1}\overline{v}$. In actuality, the action is only defined for polynomials with odd constant term.

Since $\chi_\mathbf{A}$ is irreducible, we see that this module is cyclic, indeed it is generated by $\overline{e}_1$, the first standard basis vector. We define for a polynomial (with odd coefficient) $p$

$$p \cdot \mathcal{G} = \mathfrak{C}(\mathbf{A}, p \cdot \overline{e}_1)$$

Then if $pq = r$, we have $p \cdot \mathcal{G} \hookrightarrow r \cdot \mathcal{G}$ by the canonical inclusion $\overline{v} \mapsto q \cdot \overline{v}$, which preserves both the group structure and the residuation structure.

## The Answers

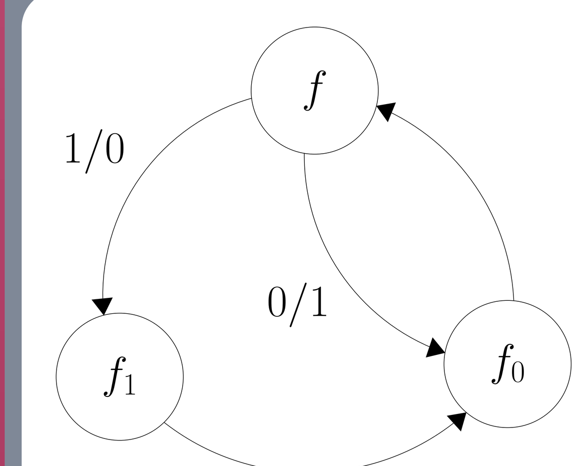
From a given automaton $\mathcal{A}$ with an odd state $\alpha$, one can recover an equation in $\overline{v}$ and $\overline{e}$. This lets us solve for $\overline{e}$ in terms of $\overline{v}$, and we see that, for any odd $\overline{v}$, we can find an $\overline{e}$ so $\mathcal{A}$ appears as a subautomaton of $\mathfrak{C}(\mathbf{A}, \overline{e})$ with $\alpha$ positioned at $\overline{v}$.

Coupled with the module theory above, we see that $3\alpha = 3e_1$ in $\overline{e} = 3e_1$ should compute the same function as $\delta = e_1$ in $\overline{e} = e_1$. That is, $\alpha \circ \alpha \circ \alpha = \delta$, and informally, $\alpha = \frac{1}{3}\delta$. Thus, large values of $\overline{e}$ correspond to the presence of many "fractional elements" of $\mathcal{G}(\mathfrak{A})$. Scaling up our group by a polynomial $p$ has the effect of making elements $p^{-1} \cdot \overline{v}$ integral, which is why new automata appear.

To each automaton $\mathcal{A}$, we can associate a unique polynomial $p$ (up to units) which is minimal and renders $\mathcal{A}$ integral (by taking $\overline{v} = e_1$ in the above construction). Then $\mathcal{G}(\mathcal{A}) \cong \mathfrak{C}(\mathbf{A}, p \cdot e_1)$ as groups and as automata. We also see $\mathcal{G}(\mathcal{A}) \leq \mathcal{G}(\mathcal{B})$ if and only if $p_\mathcal{A} \mid p_\mathcal{B}$.

Finally, we can remove this parameter $\overline{e}$ entirely by working in a universal "limit automaton". Indeed, we have a category with objects $\mathfrak{C}(\mathbf{A}, \overline{e})$ varying over $\overline{e}$ and morphisms $\overline{v} \mapsto p \cdot \overline{v}$. It is easy to see that this category admits a Fraïssé limit, in which every automaton with matrix $\mathbf{A}$ shows up exactly once (One could also take the direct limit along the poset of polynomials ordered by division to recover the same limiting structure).

## An Important Example: $\mathcal{A}_2^3$




$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$$

$\partial_0 f = f_0$

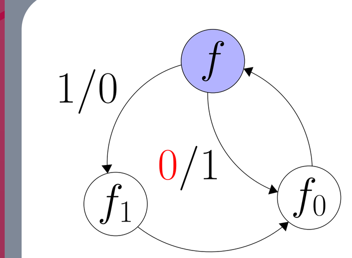
$\partial_1 f = f_1$

$\partial_0 f_0 = \partial_1 f_0 = f$

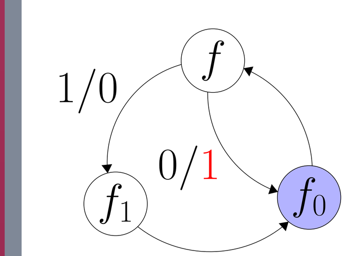
$\partial_0 f_1 = \partial_1 f_1 = f_0$

(Unlabeled edges correspond to both 0/0 and 1/1 edges)

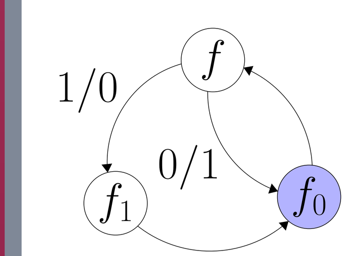
## Computing $f(0110\dots)$



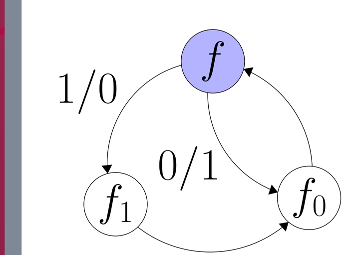

$f(0110\dots)$



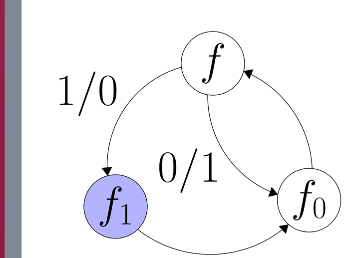

$1f_0(110\dots)$




$1f_0(110\dots)$




$11f(10\dots)$




$110f_1(0\dots)$

## An Embedding

Say $\varphi(f) = \overline{v}$. Then,

$$\mathbf{A}(\mathbf{A}(\overline{v} - \overline{e})) = \overline{v}.$$

Thus

$$\overline{e} = \mathbf{A}^{-1}(\mathbf{A}^2 - I)\overline{v}.$$

Substituting, say $\overline{v} = e_1 = (1, 0)$ gives $\overline{e} = (-3, -2)$.

One can check that $f = (1, 0)$, $f_0 = (0, 1)$, $f_1 = (-2, -2)$ actually witnesses $\mathcal{A}_2^3$ as a subautomaton of $\mathfrak{C}(\mathbf{A}, (-3, -2))$.