ROOTS OF POLYNOMIALS OF BOUNDED HEIGHT

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Abstract. Let V be the set of roots of $\{-1,0,1\}$ polynomials. Each $\alpha \in V$ must be a unit which lies with its conjugates in the annulus $1/2 < |z| < 2$. We begin with an explicit example showing that this condition is not sufficient. Furthermore, for each $\varrho \in (1,2]$, we show that the set of units that lie with their conjugates in $1/\varrho < |z| < \varrho$ but do not belong to V is everywhere dense in the annulus $1/\varrho \leqslant |z| \leqslant \varrho$. This is derived from our main result claiming that the number $\alpha e^{2\pi i \ell/p}$ is not a root of a nonzero integer polynomial of height $\leqslant H$ if p is a sufficiently large prime number, $\ell < p$ is a positive integer, and $\alpha \neq 0$ is an algebraic number having at least one conjugate of modulus $\neq 1$.

1. Introduction

Recall that α is an algebraic number over the field $\mathbb{Q}$ if there is a nonzero monic polynomial $P(x) \in \mathbb{Q}[x]$ such that $P(\alpha) = 0$. Generally speaking, for any proper subset S of $\mathbb{Q}$, one may ask which algebraic numbers are roots of (not necessarily irreducible) polynomials with coefficients in S and which are not. There are many interesting problems regarding some small sets S in this context. In [24] Odlyzko and Poonen considered the set $S = \{0,1\}$. (Such polynomials are called Newman polynomials.) They have established several interesting results for the set W of roots of Newman polynomials. For instance, it was proved in [24] that the closure $\overline{W}$ is path connected. This result was recently generalized in [18]. Some interesting aspects of multiple roots of Newman polynomials have been investigated by Borwein and Mossinghoff [7], [22]. (See also [14], [19] for other problems concerning Newman polynomials.)

In this paper, we shall denote by V the set of roots of integer polynomials of height 1 with nonzero constant term, that is, $\alpha \in V$ if and only if there are $h_1, \dots, h_n \in \{-1,0,1\}$ such that $1 + h_1\alpha + \dots + h_n\alpha^n = 0$. More generally, for $H \in \mathbb{N}$, let V_H be the set of nonzero roots of integer polynomials of height $\leqslant H$. (Recall that the height of the polynomial $\sum_{j=0}^n h_j x^j$ is the maximal modulus of its coefficients $\max_{0 \leqslant j \leqslant n} |h_j|$.) Some results about the closure $\overline{V}$ similar to those about $\overline{W}$ have been established by Boush [9], Barnsley and Harrington [3] (see also [2]). According to [3], every α in the annulus $1/\sqrt{2} < |\alpha| < 1$ is a root of some $\{-1,0,1\}$ power series. Truncations of these series show that $\{z \in \mathbb{C} : 1/\sqrt{2} \leqslant |z| \leqslant 1\} \subset \overline{V}$. Since V is closed under the map $z \mapsto 1/z$, this implies that $\{z \in \mathbb{C} : \ 1/\sqrt{2} \leqslant |z| \leqslant \sqrt{2}\} \subset \overline{V}$.

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By a simple modulus consideration, it is clear that α can belong to V_H only if it is an algebraic number whose conjugates over $\mathbb{Q}$ (including α itself) all lie in the annulus $1/(H+1) < |z| < H+1$. In addition, the moduli of both extreme (leading and constant) coefficients of the minimal polynomial of α (in $\mathbb{Z}[x]$) must be $\leqslant H$. In particular, $\alpha \in V = V_1$ implies that α is a unit whose conjugates over $\mathbb{Q}$ (including α itself) all lie in the annulus $1/2 < |z| < 2$. On the other hand, Pathiaux [26] and Mignotte [21] proved that each nonzero unit whose Mahler measure $M(\alpha)$ is smaller than 2 belongs to V. (Recall that, for a unit α with conjugates $\alpha_1 = \alpha, \dots, \alpha_d$, its Mahler measure is defined by $M(\alpha) := \prod_{k=1}^{d} \max\{1, |\alpha_k|\}$.) The proof of this statement is based on Siegel's lemma. Its most general version can be found in [6].

Of course, not every unit α which lies with its conjugates in $1/2 < |z| < 2$ has the property $M(\alpha) < 2$, so there is a "gap" between necessary and sufficient conditions stated above. Let E be the "exceptional" set of units α that lie with their conjugates in $1/2 < |z| < 2$ but do not belong to V. We begin with an explicit example showing that the set E is not empty. Hence, the above mentioned necessary condition is not sufficient.

Theorem 1. *The unit $\theta := (1+\sqrt{5})(-1+i\sqrt{3})/4$ lies with its conjugates in the annulus $1/2 < |z| < 2$ and is not a root of a nonzero polynomial with $\{-1,0,1\}$ coefficients.*

The minimal polynomial of θ over $\mathbb{Q}$ is $x^4+x^3+2x^2-x+1$. Theorem 1 claims that this polynomial is not a factor of a nonzero $\{-1,0,1\}$ polynomial. Two proofs of Theorem 1 are given in Section 2. As it was observed in [24] and [18], the set $\overline{V} \cap \{z \in \mathbb{C} : \ |z| < 1\}$ is the set of zeros of power series $1 + \sum_{j=1}^{\infty} h_j z^j$, where $h_j \in \{-1,0,1\}$. Thus Theorem 1 can be easily derived from certain general theorems on the roots of power series [4] (see our first proof). Roughly speaking, with this method one can prove that $\alpha \notin V$ by showing that $\alpha' \notin \overline{V}$ for some α' which is conjugate to α. Nevertheless, we shall give another proof which works for polynomials only. Since θ is the product of the "golden mean" $(1+\sqrt{5})/2$ and the primitive cube root of unity $(-1+i\sqrt{3})/2$, it is not surprising that this proof involves Fibonacci numbers.

Our main theorem is a generalization of Theorem 1 to integer polynomials of arbitrary height and to algebraic numbers having at least one conjugate of modulus $\neq 1$. Using it, we will be able to prove that $\alpha \notin V$ although all conjugates of α lie in the interior of $\overline{V}$.

Theorem 2. *Let $\alpha \neq 0$ be an algebraic number having at least one conjugate of modulus $\neq 1$, and let H be a positive integer. Then there is a constant $p_0(\alpha, H)$ such that, for each prime number $p > p_0(\alpha, H)$ and $\ell \in \{1, 2, \dots, p-1\}$, the algebraic number $\alpha e^{2\pi i \ell/p}$ is not a root of a nonzero integer polynomial of height $\leqslant H$.*

The proof of Theorem 2 is given in Sections 3 and 4. Formally, we just follow our second proof of Theorem 1. However, the proof of Theorem 2 is much more subtle and uses tools from several sources (for instance, a "standard" estimate for linear forms in two logarithms and an old classical result about intersection of two cyclotomic fields). All complications come from the fact that, in general, some quotients of two conjugates of α can be roots of unity. Note that, for large p, the conjugates of $\alpha e^{2\pi i \ell/p}$ over $\mathbb{Q}$ are uniformly distributed in the angles with vertex at the origin. Usually, this reflects the fact that the height of a respective minimal polynomial is not too large (see, e.g., [15] for a converse statement). Nevertheless, Theorem 2 shows that it is large and, moreover, it cannot be reduced to a constant by multiplying the minimal polynomial of $\alpha e^{2\pi i\ell/p}$ by an arbitrary integer polynomial.

Our next theorem emphasizes that some restrictions on the size of conjugates of α cannot, in principle, guarantee that α lies in V.

Theorem 3. *For each $\varrho \in (1, 2]$, the set of units that lie with their conjugates in $1/\varrho < |z| < \varrho$ but do not belong to V is everywhere dense in the annulus $1/\varrho \leqslant |z| \leqslant \varrho$.*

In view of $\{z \in \mathbb{C} :\ 1/\sqrt{2} \leqslant |z| \leqslant \sqrt{2}\} \subset \overline{V}$, this theorem is especially interesting for $\varrho \in (1, \sqrt{2})$. Then, as we said above, each complex number α in the annulus $1/\varrho < |z| < \varrho$ is a root of certain $\{-1, 0, 1\}$ power series. However, given any $\varepsilon > 0$, by Theorem 3, there is an algebraic unit α which is not a root of unity and lies with its conjugates in a "very narrow" annulus $1/(1 + \varepsilon) < |z| < 1 + \varepsilon$ but is not a root of a $\{-1, 0, 1\}$ polynomial. The proof of Theorem 3 is based on Theorem 2 and Robinson's result stating that there are infinitely many algebraic integers lying with their conjugates in any given interval of length > 4. (See p. 51 in [23] and also [10], where a corresponding result for units was obtained.) We will prove Theorem 3 in Section 5.

Our next statement is the following zero one expansion of a number in base w:

Theorem 4. *Let $w \in [1/2, 1)$ and $r \in [0, w/(1 - w)]$ be two fixed real numbers. Then there exist $\delta_1, \delta_2, \delta_3, \dots \in \{0, 1\}$ such that*

$$r = \delta_1 w + \delta_2 w^2 + \delta_3 w^3 + \dots .$$

A simple proof of this theorem using so-called β-expansions will be given in Section 6. We will also give its corollaries concerning the density of real roots of polynomials with restricted coefficients.

In particular, Theorem 3 with $\varrho = 2$ implies that $\overline{E} = \{z \in \mathbb{C} :\ 1/2 \leqslant |z| \leqslant 2\}$, namely, that the "exceptional" set E is everywhere dense in the annulus $1/2 \leqslant |z| \leqslant 2$. Our final theorem is a generalization of this statement:

Theorem 5. *Suppose that $H \in \mathbb{N}$. Then, for each complex number z_0 satisfying $1/(H+1) \leqslant |z_0| \leqslant H + 1$ and for any $\varepsilon > 0$, there is an algebraic number α such that (i) $|\alpha - z_0| < \varepsilon$, (ii) α and its conjugates all lie in the annulus $1/(H + 1) < |z| < H + 1$, (iii) the moduli of both extreme (leading and constant) coefficients of the minimal polynomial of α (in $\mathbb{Z}[x]$) are $\leqslant H$, and (iv) α is not a root of an integer polynomial of height $\leqslant H$.*

2. Fibonacci numbers: the proof of Theorem 1

First proof of Theorem 1: Set $\alpha := (1 + \sqrt{5})/2$ and $\zeta := (-1 + i\sqrt{3})/2$. Obviously, two conjugates of $\theta = \alpha\zeta$ lie on $|z| = \alpha$ and two on $|z| = \alpha^{-1}$. Clearly, if θ is a root of a $\{-1, 0, 1\}$ polynomial, then so is $-1/\theta$. Then, obviously, $-1/\theta = (\sqrt{5} - 1)e^{\pi i/3}/2$ must be a root of certain power series $1 + \sum_{j=1}^{\infty} h_j z^j$, where $h_j \in [-1, 1]$. However, according to the computations shown in Fig. 2 of [4] (see also [5]), the modulus of a root of such power series lying on the ray $\varrho e^{\pi i/3}$, $\varrho > 0$, must be greater than 0.63, whereas $|-1/\theta| = (\sqrt{5}-1)/2 = 0.618033\dots$ is smaller, a contradiction. $\square$

Second proof of Theorem 1: Let F_n, $n = 1, 2, 3, \dots$, be the Fibonacci sequence, namely, $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for each $n \geqslant 3$. Below, we shall use the following identity

$$F_1 + F_2 + \dots + F_{n-2} = F_n - 1$$

which holds for each $n \geqslant 3$. Put $F_0 := 0$. It is well known that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) = \frac{\alpha^n - (-1/\alpha)^n}{\sqrt{5}}.$$

Therefore, $\alpha^n = F_n\alpha + F_{n-1}$ for each $n \in \mathbb{N}$. So, using $\theta^n = (\alpha\zeta)^n = (F_n\alpha + F_{n-1})\zeta^n$, $\zeta^3 = 1$ and $\zeta^2 = -\zeta - 1$, we find that

$$\theta^n = (\alpha\zeta)^n = \begin{cases} F_n\alpha + F_{n-1} & \text{if } n \equiv 0 \pmod 3, \\ F_n\alpha\zeta + F_{n-1}\zeta & \text{if } n \equiv 1 \pmod 3, \\ -F_n\alpha\zeta - F_n\alpha - F_{n-1}\zeta - F_{n-1} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

for $n \in \mathbb{N}$.

To obtain a contradiction, suppose that θ is a root of a nonzero $\{-1, 0, 1\}$ polynomial. Then $b_q\theta^q + \dots + b_0 = 0$ with $b_q = 1$, $b_{q-1}, \dots, b_0 \in \{-1, 0, 1\}$, $q > 4$. Selecting $\delta \in \{0, 1, 2\}$ such that $q + \delta = 3k + 1$, where $k \in \mathbb{N}$, we get $\theta^{3k+1} + b_{3k-\delta}\theta^{3k} + \dots + b_0\theta^\delta = 0$. Hence, there are $c_{3k}, \dots, c_1, c_0 \in \{-1, 0, 1\}$ such that

$$\theta^{3k+1} + c_{3k}\theta^{3k} + \dots + c_1\theta + c_0 = 0.$$

Evidently, $1, \zeta, \alpha, \alpha\zeta$ is a basis of the field $\mathbb{Q}(\alpha, \zeta) = \mathbb{Q}(\sqrt{5}, i\sqrt{3})$. Expressing each θ^n, where $n = 0, 1, \dots, 3k+1$, in this basis by the above formulas we obtain that the coefficient for $\alpha\zeta$ in the expression $\theta^{3k+1} + c_{3k}\theta^{3k} + \dots + c_0$ is equal to $F_{3k+1} + \sum_{j=0}^{k-1} c_{3j+1}F_{3j+1} - \sum_{j=0}^{k-1} c_{3j+2}F_{3j+2}$. It must be equal to zero, so

$$F_{3k+1} = \sum_{j=0}^{k-1} c_{3j+2}F_{3j+2} - \sum_{j=0}^{k-1} c_{3j+1}F_{3j+1}.$$

However, using $c_{3j+2} \leqslant 1$ and $-c_{3j+1} \leqslant 1$, we derive that the right-hand side of this equality is smaller than or equal to

$$\sum_{j=0}^{k-1} F_{3j+2} + \sum_{j=0}^{k-1} F_{3j+1} \leqslant F_1 + F_2 + \dots + F_{3k-1} = F_{3k+1} - 1 < F_{3k+1},$$

a contradiction. $\square$

3. Auxiliary results

In Section 2 we used the fact that the numbers $1, i\sqrt{3}, \sqrt{5}, i\sqrt{15}$ are linearly independent over $\mathbb{Q}$. Below, this observation will be replaced by the following lemma:

Lemma 6. *Let α be an algebraic number of degree d. Then, for each prime number $p > p(\alpha)$, the $d(p-1)$ numbers $\zeta_p^j \alpha^k$, where $\zeta_p = e^{2\pi i/p}$, $0 \leqslant j \leqslant p-2$, $0 \leqslant k \leqslant d-1$, form a $\mathbb{Q}$-basis of the field $\mathbb{Q}(\alpha, \zeta_p)$.*

Proof of Lemma 6: Since $d = \deg(\alpha)$ and $p - 1 = \deg(\zeta_p)$, it suffices to show that the numbers $\zeta_p^j \alpha^k$, where $0 \leqslant j \leqslant p-2$ and $0 \leqslant k \leqslant d-1$, are linearly independent over $\mathbb{Q}$. To obtain a contradiction, suppose that they are linearly dependent over $\mathbb{Q}$. Then the minimal polynomial $P(x) = (x - \alpha_1) \dots (x - \alpha_d)$ of α over $\mathbb{Q}$ (in $\mathbb{Q}[x]$) is reducible in $\mathbb{Q}(\zeta_p)[x]$. If this happens for each $p \in \mathcal{P}_1$, where $\mathcal{P}_1$ is an infinite set of prime numbers, then there is a nonempty proper subset J of $\{1, 2, \dots, d\}$ and an infinite subset $\mathcal{P}_2$ of $\mathcal{P}_1$ such that the polynomial $P_J(x) = \prod_{j \in J}(x - \alpha_j)$ has coefficients in $\mathbb{Q}(\zeta_p)$ for each $p \in \mathcal{P}_2$. Since $P_J(x) \notin \mathbb{Q}[x]$, at least one coefficient of $P_J(x)$ is irrational. Thus, there is an irrational number $f \in \mathbb{Q}(\alpha_1, \dots, \alpha_d)$ which belongs to infinitely many fields $\mathbb{Q}(\zeta_p)$. However, $\mathbb{Q}(\zeta_p) \cap \mathbb{Q}(\zeta_{p'}) = \mathbb{Q}$ for any prime numbers $p \neq p'$ (see, e.g., [20]), a contradiction. $\square$

In the remaining part of this section we shall prove the following key lemma:

Lemma 7. *Let $\alpha \neq 0$ be an algebraic number, and let $\xi \neq 0$ be a fixed number lying in the field $\mathbb{Q}(\alpha)$. Suppose that p is a sufficiently large prime number and $p_0 \in \{0, 1, \dots, p-1\}$. Then there is a positive constant $\lambda = \lambda(\alpha, \xi)$ and infinitely many $m \in \mathbb{N}$ for which $|\mathrm{Trace}(\xi \alpha^{pm+p_0})| > p^{-\lambda} \overline{|\alpha|}^{pm+p_0}$.*

Here and below, $\overline{|\alpha|}$ is the maximal modulus of conjugates of α over $\mathbb{Q}$ and $\mathrm{Trace}(\beta)$ is the sum of conjugates of β over $\mathbb{Q}$.

The next statement is a simple application of the Vandermonde determinant. See, for instance, p. 134 in [11] or [12] for an almost "one line proof".

Lemma 8. *Let $s \in \mathbb{N}$, and let $\omega_1, \dots, \omega_s$ be distinct complex numbers. Suppose that $Y_0, \dots, Y_{s-1} \in \mathbb{C}$. Then the linear system $X_1 \omega_1^k + \dots + X_s \omega_s^k = Y_k$, where $k = 0, 1, \dots, s-1$, has a unique solution*

$$X_t = \frac{\sum_{k=0}^{s-1} (\omega_t^{s-k-1} + \sigma_1 \omega_t^{s-k-2} + \dots + \sigma_{s-k-2}\omega_t + \sigma_{s-k-1}) Y_k}{Q'(\omega_t)},$$

$t = 1, 2, \dots, s$, where $Q(x) = (x - \omega_1) \dots (x - \omega_s) := x^s + \sigma_1 x^{s-1} + \dots + \sigma_{s-1} x + \sigma_s$.

The next lemma will be derived from the theory of linear forms in two logarithms. Here, $||x||$ stands for the distance from a real number x to the nearest integer.

Lemma 9. *Suppose that $\alpha = \varrho e^{i\varphi}$ and $\alpha' = \varrho e^{i\varphi'}$ are two distinct algebraic numbers of equal moduli. Then there is a positive constant $\lambda_1 = \lambda_1(\alpha, \alpha')$ such that the inequality $||p(\varphi - \varphi')/2\pi|| > p^{-\lambda_1}$ holds for each prime number $p > p_1(\alpha, \alpha')$.*

Proof of Lemma 9: There is no loss of generality in assuming that $0 \leqslant \varphi' < \varphi < 2\pi$. If $(\varphi - \varphi')/2\pi$ is a rational number, say, $(\varphi - \varphi')/2\pi = u/v$ with positive integers $u < v$ (where $v \geqslant 2$) then $||pu/v|| \geqslant 1/v$ for each prime number $p > v$. This proves the required inequality in this case. Now suppose that $(\varphi - \varphi')/2\pi \notin \mathbb{Q}$. Then, as the logarithms of algebraic numbers $\log(\alpha/\alpha') = (\varphi - \varphi')i$ and $\log(-1) = \pi i$ are linearly independent over $\mathbb{Q}$, the theory of linear forms in logarithms (see, e.g., [1]) says that there is a constant $\lambda_1 = \lambda_1(\alpha, \alpha')$ such that $|p \log(\alpha/\alpha') - 2m \log(-1)| > p^{-\lambda_1}$ for all integer $p \geqslant 2$ and $m \in \mathbb{Z}$. On dividing both sides by $2\pi = |2\pi i|$ we obtain the required inequality. $\square$

Lemma 10. *Suppose that $\alpha_1 = \varrho e^{\varphi_1 i}, \dots, \alpha_s = \varrho e^{\varphi_s i}$ are distinct algebraic numbers of equal moduli and $\eta_1, \dots, \eta_s \in \mathbb{C}$. Let p be a prime number. Set $D_m := \eta_1 e^{mp\varphi_1 i} + \dots + \eta_s e^{mp\varphi_s i}$ for $m \in \mathbb{N}$. Then there are positive constants $\lambda_2 = \lambda_2(\alpha_1, \dots, \alpha_s, s)$ and $p_2 = p_2(\alpha_1, \dots, \alpha_s)$ such that*

$$\max\{|D_m|, |D_{m+1}|, \dots, |D_{m+s-1}|\} \geqslant |\eta_1| p^{-\lambda_2}$$

for each prime number $p > p_2$ and each $m \in \mathbb{N}$.

Proof of Lemma 10: We can certainly assume that $\eta_1 \neq 0$. Fix $m \in \mathbb{N}$. Set $X_t := \eta_t e^{mp\varphi_t i}$ and $\omega_t := e^{p\varphi_t i}$ for $t = 1, \dots, s$ and consider the linear system $X_1 \omega_1^k + \dots + X_s \omega_s^k = D_{m+k}$, where $k = 0, 1, \dots, s-1$. Using $|\omega_t| = 1$ for $t \in \{1, \dots, s\}$, we deduce that $|\sigma_j| = |\omega_1 \dots \omega_j + \dots + \omega_{s-j+1} \dots \omega_s| \leqslant \binom{s}{j}$. Since $Q'(\omega_1) = \prod_{j=2}^{s} (\omega_1 - \omega_j)$, Lemma 8 implies that $|X_1| \prod_{j=2}^{s} |\omega_1 - \omega_j| \leqslant 2^s (|D_m| + \dots + |D_{m+s-1}|)$. Note that

$$|\omega_1 - \omega_j| = 2|\sin(p(\varphi_1 - \varphi_j)/2)| = 2|\sin(\pi ||p(\varphi_1 - \varphi_j)/2\pi||)| \geqslant 4||p(\varphi_1 - \varphi_j)/2\pi||.$$

(Here, the last inequality follows from $\sin x \geqslant 2x/\pi$, where $x \in [0, \pi/2]$.) Hence, by Lemma 9, $|\omega_1 - \omega_j| > 4p^{-\lambda_1}$ for each $j \in \{2, \dots, s\}$.

Next, using $|\eta_1| = X_1$ and $\prod_{j=2}^{s} |\omega_1 - \omega_j| > (4p^{-\lambda_1})^{s-1}$, we find that at least one of the numbers $|D_m|, \dots, |D_{m+s-1}|$ is greater than $|\eta_1| 4^{s-1} p^{-(s-1)\lambda_1} / s2^s = |\eta_1| 2^{s-2} p^{-(s-1)\lambda_1} / s$. This is greater than $|\eta_1| p^{-\lambda_2}$ with $\lambda_2 = (s-1)\lambda_1 + 1$ for each $p \geqslant 2$. $\square$

We can now prove Lemma 7.

Proof of Lemma 7: Set $S_n := \xi_1 \alpha_1^n + \dots + \xi_d \alpha_d^n$. Here, $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ are the conjugates of α over $\mathbb{Q}$ and $\xi_j = g(\alpha_j)$, where $\xi = \xi_1$ is written as $\xi = g(\alpha)$ with $g(x) \in \mathbb{Q}[x]$. Evidently, $S_n = d_0 \mathrm{Trace}(\xi \alpha^n)$, where d_0 is a positive integer divisor of d. What is left is to show that $|S_{pm+p_0}| > d\overline{|\alpha|}^{pm+p_0} p^{-\lambda}$ for infinitely many $m \in \mathbb{N}$.

Suppose that $\alpha_1 = \varrho e^{\varphi_1 i}, \dots, \alpha_s = \varrho e^{\varphi_s i}$ are the conjugates of α lying on the circle $|z| = \varrho = \overline{|\alpha|}$, whereas all remaining conjugates of α lie in the disc $|z| \leqslant \varrho_1 < \varrho$. Set $D_m := \xi_1 e^{p_0 \varphi_1 i} e^{mp\varphi_1 i} + \dots + \xi_s e^{p_0 \varphi_s i} e^{mp\varphi_s i}$. Then

$$S_{pm+p_0} = \varrho^{pm+p_0} D_m + \sum_{j=s+1}^{d} \xi_j \alpha_j^{pm+p_0}.$$

Take $p > p_2(\alpha)$ and one of $m \in \mathbb{N}$ that satisfy the condition of Lemma 10, more precisely, $m \in \mathbb{N}$ such that $|D_m| \geqslant |\xi_1| p^{-\lambda_2}$. Suppose, in addition, that m is so large that

$$(d-s)\varrho_1^{pm+p_0} \max_{|z|\leqslant\overline{|\xi|}} |g(z)| < |\xi_1| \varrho^{pm+p_0} p^{-\lambda_2}/2.$$

Then $|S_{pm+p_0}|\varrho^{-pm-p_0} > |D_m| - (\varrho_1/\varrho)^{pm+p_0}(d-s)\max_{1\leqslant j\leqslant d}|\xi_j| > |\xi_1|p^{-\lambda_2}/2$. This is greater than $dp^{-\lambda}$ for some $\lambda = \lambda(\alpha, \xi)$ and $p > p_3(\alpha)$. Thus $|S_{pm+p_0}|\varrho^{-pm-p_0} > dp^{-\lambda}$ for infinitely many $m \in \mathbb{N}$. $\square$

As a special case of Lemma 7, we state the following corollary:

Corollary 11. *Let $\alpha \neq 0$ be an algebraic number of degree d with minimal polynomial $P(x) = (x-\alpha_1)\dots(x-\alpha_d) \in \mathbb{Q}[x]$. Then there are positive constants $\lambda_3 = \lambda_3(\alpha)$, $p_4(\alpha)$ such that for each prime number $p > p_4(\alpha)$ the inequality*

$$\left| \frac{\alpha_1^{pm+p-2}}{P'(\alpha_1)} + \dots + \frac{\alpha_d^{pm+p-2}}{P'(\alpha_d)} \right| > p^{-\lambda_3} \overline{|\alpha|}^{pm+p-2}$$

holds for infinitely many $m \in \mathbb{N}$.

4. On numbers which are multiples of roots of unity

Proof of Theorem 2: Assume that $\alpha\zeta_p^\ell$ (where $\zeta_p = e^{2\pi i/p}$ and $\alpha \neq 0$) is a root of a nonzero integer polynomial of height $\leqslant H$. Evidently, $\alpha^{-1}\zeta_p^{-\ell}$ is also a root of a nonzero integer polynomial of height $\leqslant H$. Take the number $\beta \in \{\alpha_1, \dots, \alpha_d, \alpha_1^{-1}, \dots, \alpha_d^{-1}\}$ of largest modulus. Then there is an index $j \in \{1, \dots, d\}$ such that either $\beta = \alpha_j$ or $\beta = \alpha_j^{-1}$. In the first case, there is a $k \in \{1, \dots, p-1\}$ such that $\alpha_j\zeta_p^k$ is a conjugate of $\alpha\zeta_p^\ell$. In the second case, there is a $k \in \{1, \dots, p-1\}$ such that $\alpha_j^{-1}\zeta_p^k$ is a conjugate of $\alpha^{-1}\zeta_p^{-\ell}$. Hence, in both cases, $\beta\zeta_p^k$ is a root of a nonzero integer polynomial of height $\leqslant H$ and $|\beta| = \overline{|\beta|} > 1$.

Thus, without loss of generality we can assume that $\beta = \alpha$. Furthermore, we can label the conjugates of α so that $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_s$ will be the conjugates of α on the circle $|z| = \overline{|\alpha|} > 1$. By our assumption, $\alpha\zeta$, where $\zeta := \zeta_p^k$, is a root of a nonzero integer polynomial of height $\leqslant H$. Hence there are integers $b_q \neq 0, b_{q-1}, \dots, b_0 \in \{-H, \dots, H-1, H\}$ such that $b_q(\alpha\zeta)^q + \dots + b_1\alpha\zeta + b_0 = 0$. Multiplying this equality by a proper power of $\alpha\zeta$ we can assume that $q = pm + p - 2$ with $m \in \mathbb{N}$. Dividing by b_q yields that there are rational numbers $c_{pm+p-3}, \dots, c_0$ satisfying $|c_{pm+p-3}| \leqslant H, \dots, |c_0| \leqslant H$ such that

$$(\alpha\zeta)^{pm+p-2} = c_{pm+p-3}(\alpha\zeta)^{pm+p-3} + \dots + c_1\alpha\zeta + c_0.$$

Next, we take a prime number p and $m \in \mathbb{N}$ so large that Lemma 6 and Corollary 11 are true and also so large that $\overline{|\alpha|}^p > 1 + 2T\overline{|\alpha|}^2 Hp^{\lambda_3}$, where

$$T := d / \min_{1\leqslant k\leqslant d} |P'(\alpha_k)|$$

and where λ_3 is the constant of Corollary 11. Here,

$$P(x) = (x-\alpha_1)\dots(x-\alpha_d) = x^d + a_{d-1}x^{d-1} + \dots + a_0 \in \mathbb{Q}[x]$$

is the minimal polynomial of α over $\mathbb{Q}$ (in $\mathbb{Q}[x]$). The proof of the theorem will be completed as soon as we derive the opposite inequality

$$\overline{|\alpha|}^p < 1 + 2T\overline{|\alpha|}^2 Hp^{\lambda_3}.$$

Write each integer power of α as a linear form with rational coefficients in the basis $1, \alpha, \dots, \alpha^{d-1}$:

$$\alpha^n := A_{n,d-1}\alpha^{d-1} + A_{n,d-2}\alpha^{d-2} + \dots + A_{n,0}.$$

Equality $\alpha^{n+d}+a_{d-1}\alpha^{n+d-1}+\dots+a_0\alpha^n = \alpha^n P(\alpha) = 0$ shows that, for each $j \in \{0, 1, \dots, d-1\}$, the sequence of rational numbers $A_{n,j}$, $n = 0, 1, 2, \dots$, satisfies the linear recurrence relation

$$A_{n+d,j} + a_{d-1}A_{n+d-1,j} + \dots + a_0 A_{n,j} = 0,$$

where $A_{0,j} = \dots = A_{j-1,j} = A_{j+1,j} = \dots = A_{d-1,j} = 0$ and $A_{j,j} = 1$. Hence, there exist certain complex numbers $\xi_{1,j}, \dots, \xi_{d,j}$ such that $A_{n,j} = \xi_{1,j}\alpha_1^n + \dots + \xi_{d,j}\alpha_d^n$ for each integer $n \geqslant 0$. On applying Lemma 8 to $X_t = \xi_{t,j}$ (where $1 \leqslant t \leqslant d$ and $n = 0, 1, \dots, d-1$), we obtain

$$\xi_{t,j} = (\alpha_t^{d-j-1} + a_{d-1}\alpha_t^{d-j-2} + \dots + a_{j+2}\alpha_t + a_{j+1})/P'(\alpha_t).$$

In particular, this implies that $\xi_{1,j} \in \mathbb{Q}(\alpha_1), \dots, \xi_{d,j} \in \mathbb{Q}(\alpha_d)$ are conjugate algebraic numbers. Furthermore, setting $j = d-1$, we derive that

$$A_{n,d-1} = \frac{\alpha_1^n}{P'(\alpha_1)} + \frac{\alpha_2^n}{P'(\alpha_2)} + \dots + \frac{\alpha_d^n}{P'(\alpha_d)}$$

for $n = 0, 1, 2, \dots$. Since $T = d/\min_{1 \leqslant k \leqslant d} |P'(\alpha_k)|$, we obtain $|A_{n,d-1}| \leqslant T\overline{|\alpha|}^n$ for every $n \in \mathbb{N}$.

Next, since $\zeta^{p-1} = -\zeta^{p-2} - \dots - \zeta - 1$, setting $n_p \equiv n \pmod{p} \in \{0, 1, \dots, p-1\}$, we can express nth power of $\alpha\zeta$ as

$$(\alpha\zeta)^n = \begin{cases} A_{n,d-1}\alpha^{d-1}\zeta^{n_p} + A_{n,d-2}\alpha^{d-2}\zeta^{n_p} + \dots + A_{n,0}\zeta^{n_p} & \text{if } n_p \leqslant p-2, \\ A_{n,d-1}\alpha^{d-1}(-\zeta^{p-2} - \dots - 1) + \dots + A_{n,0}(-\zeta^{p-2} - \dots - 1) & \text{if } n_p = p-1. \end{cases}$$

Let us write each power of $\alpha\zeta$ as above and then collect the coefficient for $\alpha^{d-1}\zeta^{p-2}$ in the expression $(\alpha\zeta)^{pm+p-2} - c_{pm+p-3}(\alpha\zeta)^{pm+p-3} - \dots - c_1\alpha\zeta - c_0 = 0$. By Lemma 6, we obtain that

$$A_{pm+p-2,d-1} = -\sum_{j=1}^{m} c_{pj-1}A_{pj-1,d-1} + \sum_{j=1}^{m} c_{pj-2}A_{pj-2,d-1}.$$

Since $|A_{n,d-1}| \leqslant T\overline{|\alpha|}^n$ and $|c_n| \leqslant H$, the modulus of the right-hand side is smaller than or equal to

$$H\sum_{j=1}^{m} T\overline{|\alpha|}^{pj-1} + H\sum_{j=1}^{m} T\overline{|\alpha|}^{pj-2} \leqslant 2TH\sum_{j=1}^{m} \overline{|\alpha|}^{pj} < 2TH\overline{|\alpha|}^{pm+p}/(\overline{|\alpha|}^p - 1).$$

Recall that we took $m \in \mathbb{N}$ for which Corollary 11 holds. Therefore, the modulus of the left-hand side, $|A_{pm+p-2,d-1}|$, is greater than $\overline{|\alpha|}^{pm+p-2}p^{-\lambda_3}$. Cancelling $\overline{|\alpha|}^{pm+p-2}$ in

$\overline{|\alpha|}^{pm+p-2}p^{-\lambda_3} < 2TH\overline{|\alpha|}^{pm+p}/(\overline{|\alpha|}^p - 1)$ produces the required inequality $\overline{|\alpha|}^p < 1 + 2T\overline{|\alpha|}^2 Hp^{\lambda_3}$. $\square$

5. Full sets of conjugates in an interval: the proof of Theorem 3

Proof of Theorem 3: Fix $\varrho \in (1, 2]$ and $z_0 = we^{i\varphi}$, where $w \in [1/\varrho, \varrho]$. We will show that, for any $\varepsilon > 0$, there is a unit α which lies with its conjugates in $1/\varrho < |z| < \varrho$, satisfies $|\alpha - z_0| < 2\sqrt{\varepsilon}$ and is not a root of a $\{-1, 0, 1\}$ polynomial. We remark that it suffices to prove this for $w = \varrho$. Indeed, if $w \in (1, \varrho)$ then we can consider the narrower annulus $1/w < |z| < w$ for the conjugates of α. For $w \in [1/\varrho, 1)$, we can replace α by α^{-1} and w by w^{-1}. Finally, for $w = 1$, namely, $z_0 = e^{i\varphi}$ we can replace z_0 by $(1 + 2\sqrt{\varepsilon})z_0$ and set $\varrho = 1 + 2\sqrt{\varepsilon}$. Then $|\alpha - (1 + 2\sqrt{\varepsilon})e^{i\varphi}| < 2\sqrt{\varepsilon}$ implies that $|\alpha - e^{i\varphi}| < 4\sqrt{\varepsilon}$ and the inequality $\varepsilon < (1 - 1/\varrho)^2$ below holds. So there is no loss of generality in assuming that $z_0 = \varrho e^{i\varphi}$.

Fix $\varepsilon > 0$ so small that $\varepsilon < (1 - 1/\varrho)^2$. Then $2 + \varepsilon < \varrho + \varrho^{-1}$. Consider the interval $[\varrho + \varrho^{-1} - 4 - 2\varepsilon/3, \varrho + \varrho^{-1} - \varepsilon/2]$. Since its length $4 + \varepsilon/6$ is strictly greater than 4, the interval contains infinitely many full sets of conjugate algebraic integers (see p. 51 in [23]). Similarly (see [23]), the interval $[\varrho + \varrho^{-1} - 4 - 2\varepsilon/3, \varrho + \varrho^{-1} - \varepsilon]$ of length $4 - \varepsilon/3 < 4$ contains only finitely many full sets of conjugate algebraic integers. Thus, there is an algebraic integer $\alpha_0 \in [\varrho + \varrho^{-1} - \varepsilon, \varrho + \varrho^{-1} - \varepsilon/2]$ whose conjugates all lie in the interval $[\varrho + \varrho^{-1} - 4 - 2\varepsilon/3, \varrho + \varrho^{-1} - \varepsilon/2]$. Since $\alpha_0 > 2$, there is a $\beta > 1$ that satisfies $\alpha_0 = \beta + \beta^{-1}$. Clearly, β is a real algebraic unit. Note that $-2 < \varrho + \varrho^{-1} - 4 - 2\varepsilon/3$. Thus, setting

$$u := \frac{\varrho + \varrho^{-1} - \varepsilon + \sqrt{(\varrho + \varrho^{-1} - \varepsilon)^2 - 4}}{2}, \ v := \frac{\varrho + \varrho^{-1} - \varepsilon/2 + \sqrt{(\varrho + \varrho^{-1} - \varepsilon/2)^2 - 4}}{2},$$

we derive that each conjugate of $\beta \in [u, v]$ lies either in the interval $[1/v, v]$ or on the unit circle. So β lies with its conjugates in the annulus $1/v \leqslant |z| \leqslant v$. It is easy to check that $v < \varrho$ and, using $\varepsilon < (1 - 1/\varrho)^2$, that $u > \varrho - \sqrt{\varepsilon}$.

Summarizing, we see that there is a real algebraic unit β satisfying $|\beta - \varrho| < \sqrt{\varepsilon}$ whose conjugates all lie in $1/\varrho < |z| < \varrho$. Since $|z_0| = \varrho$, by taking a sufficiently large prime number p we can find $\ell \in \{1, \ldots, p - 1\}$ such that $|\beta e^{2\pi i\ell/p} - z_0| < 2\sqrt{\varepsilon}$. Evidently, the conjugates of $\alpha := \beta e^{2\pi i\ell/p}$ are all lying in the annulus $1/\varrho < |z| < \varrho$ and the unit α is not a root of a $\{-1, 0, 1\}$ polynomial by Theorem 2. $\square$

6. Zero one expansions

Proof of Theorem 4: The theorem is trivial for $r = 0$, $r = w/(1 - w)$ and for $w = 1/2$ (expansion of r in base 2). Set $\beta := w^{-1}$ and fix $r \in (0, 1/(\beta - 1))$ and $\beta \in (1, 2)$. Note that $1/(\beta - 1) = \beta^{-1} + \beta^{-2} + \ldots$. The theorem then follows from the fact that the number $r \in (0, 1/(\beta - 1))$ has the β-expansion $r = \sum_{j=1}^{\infty} r_{-j}\beta^{-j}$, where $r_{-j} \in \mathbb{Z}$, $0 \leqslant r_{-j} < \beta$ (see [27]). Thus we can set $\delta_j := r_{-j} \in \{0, 1\}$ for each $j \geqslant 1$. The "digits" δ_j can be computed by the "greedy" algorithm. (See [25], [27] and, for instance, [16], [17] for more about complexity and arithmetics of β-expansions.) $\square$

Similarly, for any given $\beta \in (1, H+1)$, by expanding 1 as $1 = \sum_{j=1}^{\infty} r_{-j}\beta^{-j}$, where $r_{-j} \in \{0, 1, \dots, H\}$, we obtain that $1 = \delta_1 w + \delta_2 w^2 \dots$, where $w := \beta^{-1} \in (1/(H+1), 1)$ and $\delta_1, \delta_2, \dots \in \{0, 1, \dots, H\}$. Set $R_n(x) := \delta_n x^n + \dots + \delta_1 x - 1$. It is clear that, for each fixed number $w \in (1/(H+1), 1)$ and any $\varepsilon > 0$, there is an $n \in \mathbb{N}$ so large that $R_n(x)$ has a real root α which lies in the interval $[w, w+\varepsilon)$. (See [8] for the rate of approximation in the case $H = 1$.) Similarly, by considering the polynomials $R_n(-x)$ and $x^n R_n(1/x)$ (they all are integer polynomials of height $\leqslant H$), we derive that $[-H-1, -1/(H+1)] \cup [1/(H+1), H+1] \in \overline{V_H}$.

Corollary 12. *Suppose that $H \in \mathbb{N}$. Then, for any $w \in [-H+1, -1/(H+1)] \cup [1/(H+1), H+1]$ and $\varepsilon > 0$, there is a nonzero integer polynomial of height $\leqslant H$ whose real root α satisfies $|\alpha - w| < \varepsilon$.*

For example, let us take the first nine nonzero terms of β-expansions of 1 in bases $\beta = 3/2$ and $\beta = \pi/2$. This produces the polynomials

$$x^{39} + x^{34} + x^{27} + x^{17} + x^{15} + x^{12} + x^9 + x^3 + x - 1$$

and

$$x^{37} + x^{34} + x^{28} + x^{24} + x^{22} + x^{16} + x^5 + x^3 + x - 1$$

having roots $0.66666667\dots$ and $0.636619774\dots$ close to $2/3$ and to $2/\pi = 0.636619772\dots$, respectively.

Next, we will combine this corollary with Theorem 2.

Proof of Theorem 5: Corollary 12 implies that for each $w \in [1/(H+1), H+1]$ and $\varepsilon > 0$ there is a real root α_0 of a nonzero polynomial of height $\leqslant H$ such that $|\alpha_0 - w| < \varepsilon/2$. Clearly, α_0 can be chosen different from ± 1, so α_0 has at least one conjugate of modulus $\neq 1$. Moreover, α_0 lies with its conjugates in the annulus $1/(H+1) < |z| < H+1$. Thus, for each complex number $z_0 = we^{i\varphi}$ in the annulus $1/(H+1) \leqslant |z| \leqslant H+1$ and any positive number ε, there is a large prime number p and $\ell \in \{1, 2, \dots, p-1\}$ such that $|\alpha_0 e^{2\pi i \ell/p} - z_0| < \varepsilon$. Clearly, $\alpha = \alpha_0 e^{2\pi i \ell/p}$ is an algebraic number which lies with its conjugates in the annulus $1/(H+1) < |z| < H+1$. Furthermore, the moduli of both extreme (leading and constant) coefficients of the minimal polynomial of α (in $\mathbb{Z}[x]$) are $\leqslant H$. Finally, by Theorem 2, α is not a root of a nonzero integer polynomial of height $\leqslant H$, if p is sufficiently large. This completes the proof of the theorem. $\square$

Selecting $r := (1 + w/(1-w))/2$ in Theorem 4 we derive that, for each $w \in [1/2, 1)$, the number $(1 + w/(1-w))/2 = (1 + w + w^2 + \dots)/2$ is equal to $\delta_1 w + \delta_2 w^2 + \dots$, where $\delta_1, \delta_2, \dots \in \{0, 1\}$. Thus $\sum_{j=1}^{\infty}(2\delta_j - 1)w^j - 1 = 0$. Note that $2\delta_j - 1 \in \{-1, 1\}$. Next, let us consider the following partial sums $R_n(x) = \sum_{j=1}^{n}(2\delta_j - 1)x^j - 1$ which are polynomials of degree n. Fix $w \in [1/2, 1)$. Let α_n be the root of $R_n(x)$ closest to w. By continuity, we see that $\alpha_n \to w$ as $n \to \infty$. Thus, for each $w \in [1/2, 1]$, there is a polynomial with $\{-1, 1\}$ coefficients, whose root α satisfies $|\alpha - w| < \varepsilon$. Similarly, by considering $R_n(-x)$ and $x^n R_n(1/x)$, we obtain the following statement which is more precise than Corollary 12 for $H = 1$:

Corollary 13. *For any $w \in [-2,-1/2] \cup [1/2,2]$ and $\varepsilon > 0$, there is a nonzero polynomial with $\{-1,1\}$ coefficients whose root α satisfies $|\alpha - w| < \varepsilon$.*

In [18], Hare, Mohammadzadeh and Trujillo considered the set of polynomials with cyclotomic coefficients. Let $\mathcal{C}$ be the set of their roots, namely, $\alpha \in \mathcal{C}$ if and only if there are $n \in \mathbb{N}$ and certain roots of unity $h_1, \dots, h_n$ such that $1 + h_1\alpha + \dots + h_n\alpha^n = 0$. Observe that then, for each $r \in \mathbb{Q}$, the number $\alpha e^{2\pi i r}$ is a root of the polynomial $1 + h_1 e^{-2\pi i r} z + \dots + h_n e^{-2\pi i n r} z^n$ with cyclotomic coefficients. Combined with Corollary 13 this gives the following complete description of $\overline{\mathcal{C}}$:

Corollary 14. *We have $\overline{\mathcal{C}} = \{z \in \mathbb{C} : \ 1/2 \leqslant |z| \leqslant 2\}$.*

Note that Theorem 3 yields that for each $\varrho \in (1, (1+\sqrt{5})/2]$ and any $\varepsilon > 0$ there is a unit α which lies with its conjugates in the annulus $1/\varrho < |z| < \varrho$, satisfies $|\alpha - z_0| < \varepsilon$ and is not a root of a Newman polynomial. In addition, α can be chosen so that it has no real conjugates (see the proof of Theorem 3). It is known that $\overline{W} \subset \{z \in \mathbb{C} : (\sqrt{5}-1)/2 \leqslant |z| \leqslant (\sqrt{5}+1)/2\}$ (see [28]). On the other hand, there is a $\delta > 0$ such that $\{z \in \mathbb{C} : \ 1-\delta \leqslant |z| \leqslant 1+\delta, \ |\Re(z)| \geqslant \delta\} \subset \overline{W}$ (see [24]). Once again, as in case of V, there are no restrictions on the size of conjugates of α which can guarantee that $\alpha \in W$, although all conjugates of α lie in the interior of $\overline{W}$.

Finally, we remark that the condition (ii) of Theorem 5 can be replaced by a more restrictive condition. One can use, for instance, Corollary 2 of [13], where an irreducible polynomial with roots lying "close" to two circles $|z| = R > 1$ and $|z| = r < 1$ is given. The construction is based on Rouché's theorem.

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