# INTEGRAL SELF-AFFINE TILES IN $\mathbb{R}^n$ I. STANDARD AND NONSTANDARD DIGIT SETS

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ABSTRACT

We investigate the measure and tiling properties of integral self-affine tiles, which are sets of positive Lebesgue measure of the form $T(\mathsf{A}, \mathscr{D}) = \{\sum_{j=1}^{\infty} \mathsf{A}^{-j} \mathbf{d}_j : \text{all } \mathbf{d}_j \in \mathscr{D}\}$, where $\mathsf{A} \in M_n(\mathbb{Z})$ is an expanding matrix with $|\det(\mathsf{A})| = m$, and $\mathscr{D} \subseteq \mathbb{Z}^n$ is a set of $m$ integer vectors. The set $\mathscr{D}$ is called a digit set, and is called *standard* if it is a complete set of residues of $\mathbb{Z}^n/\mathsf{A}(\mathbb{Z}^n)$ or arises from one by an integer affine transformation, and *nonstandard* otherwise. We prove that all sets $T(\mathsf{A}, \mathscr{D})$ have integer Lebesgue measure, and study when the measure $\mu(T(\mathsf{A}, \mathscr{D})) \neq 0$. We give a Fourier-analytic condition for $\mu(T(\mathsf{A}, \mathscr{D})) \neq 0$. We classify nonstandard digit sets in special cases, and give formulae for the measures of their associated tiles.

## 1. *Introduction*

Let $\mathsf{A}$ be an expanding integer matrix in $M_n(\mathbb{Z})$, that is, one with all eigenvalues $|\lambda_i(\mathsf{A})| > 1$, and suppose that $\mathsf{A}$ has determinant $|\det(\mathsf{A})| = m$. Let $\mathscr{D} = \{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_m\} \subseteq \mathbb{Z}^n$ be a finite set of vectors, called *digits*, of cardinality $m$. The linear maps

$$\phi_i(\mathbf{x}) = \mathsf{A}^{-1}(\mathbf{x} + \mathbf{d}_i), \quad 1 \leqslant i \leqslant m,$$

are all contractions, with respect to a suitable norm on $\mathbb{R}^n$. Associated to them is a unique compact set $T = T(\mathsf{A}, \mathscr{D})$, which is the only nonempty compact set satisfying the set-valued functional equation

$$T = \bigcup_{i=1}^{m} \phi_i(T), \tag{1.1}$$

according to a result of Hutchinson [15]. $T$ is called the *attractor* of the *iterated function system* $\mathscr{I} = \{\phi_i : 1 \leqslant i \leqslant m\}$. In fact, $T$ is given explicitly by

$$T := \left\{ \sum_{k=1}^{\infty} \mathsf{A}^{-k} \mathbf{d}_k : \text{each } \mathbf{d}_k \in \mathscr{D} \right\}. \tag{1.2}$$

An equivalent form of the functional equation (1.1) is

$$\mathsf{A}(T) = \bigcup_{i=1}^{m} (T + \mathbf{d}_i), \tag{1.3}$$

which shows that an expanded copy of $T$ is a union of translates of $T$. We call the operation of subdividing $\mathsf{A}(T)$ into translates of $T$ 'inflation'.

In the one-dimensional case, $\mathsf{A} = [b]$ with $m = |b| \geqslant 2$, (1.2) shows that $T$ is the set of allowable radix expansions to base $b$, that are zero to the left of the decimal point to base $b$ and that use the digit set $\mathscr{D}$.

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When the attractor $T(\mathsf{A},\mathscr{D})$ has positive Lebesgue measure, we call it an *integral self-affine tile*. In this case it is well known that such $T$ tile $\mathbb{R}^n$ by translations using some translation set $\mathscr{S} \subseteq \mathbb{Z}^n$; compare Theorem 1.2 of Lagarias and Wang [**17**]. The adjective '*integral*' refers to the property that the tiling set $\mathscr{S}$ is contained in the integer lattice $\mathbb{Z}^n$. A special feature of integral self-affine tiles, compared to the general self-affine tiles studied in [**17**], is the presence of the A-invariant lattice $\mathbb{Z}^n$, which permits Fourier series techniques to be applied in their study.

More generally, we define a *lattice self-affine tile* to be a set $T(\mathsf{A},\mathscr{D})$ of positive Lebesgue measure, where $\mathsf{A}$ is an expanding real matrix such that $|\det(\mathsf{A})| = m$ is an integer, and $\mathscr{D} = \{\mathbf{d}_1,\dots,\mathbf{d}_m\} \subseteq \mathbb{R}^m$ has a difference set $\Delta(\mathscr{D}) := \mathscr{D} - \mathscr{D}$ which is contained in some full rank lattice $\Lambda$ in $\mathbb{R}^m$ which is A-*invariant*, that is $\mathsf{A}(\Lambda) \subseteq \Lambda$. This includes all integral self-affine tiles, where we may take $\Lambda = \mathbb{Z}^n$. For lattice self-affine tiles, it is necessarily the case that $\mathsf{A}$ be similar to an integer matrix, and the study of such tiles can always be reduced to the study of integral self-affine tiles; see Lemma 2.1 below.

Such tiles arise in several contexts, including the study of radix expansions [**10, 11, 24, 25, 31**], the study of various non-periodic tilings and self-similar tilings of $\mathbb{R}^n$ [**1, 2, 5, 7, 8, 14, 16, 30**], the construction of fractals [**3, 27, 28**], and more recently the construction of families of orthonormal wavelets of compact support in $\mathbb{R}^n$ [**4, 12, 13, 21, 29**].

This series of papers is concerned with the problems of determining the Lebesgue measure of such tiles $T$, and of determining when such tiles give periodic tilings or lattice tilings of $\mathbb{R}^n$. Our original motivation for studying these problems arose in connection with constructing certain kinds of multidimensional wavelet bases. Gröchenig and Madych [**13**] observed that Haar-type wavelet bases always have an associated scaling function which is the characteristic function $\chi_T$ of an integral self-affine tile, and that such a tile gives a scaling function if and only if the set $T$ tiles $\mathbb{R}^n$ using the lattice $\mathbb{Z}^n$. This situation occurs exactly when the tile $T$ has Lebesgue measure $\mu(T) = 1$. For more on Haar bases of $L^2(\mathbb{R}^n)$, see [**13, 19, 20**].

In studying such tiles, we can make two simplifications. First, if we translate the digit set to

$$\mathscr{D}' = \mathscr{D} + \mathbf{v} = \{\mathbf{d}_i + \mathbf{v} : 0 \leqslant i \leqslant m-1\}, \tag{1.4}$$

then the attractor

$$T(\mathsf{A},\mathscr{D}') = T(\mathsf{A},\mathscr{D}) + \sum_{j=1}^{\infty} \mathsf{A}^{-j}\mathbf{v} \tag{1.5}$$

is itself translated. This does not affect either the measure or tiling properties of $T$, so we may always reduce to the case that $\mathbf{0} \in \mathscr{D}$. Secondly, associated to any pair $(\mathsf{A},\mathscr{D})$ is the smallest A-invariant sublattice $\mathbb{Z}[\mathsf{A},\mathscr{D}]$ of $\mathbb{Z}^n$ that contains the difference set $\Delta(\mathscr{D}) := \mathscr{D} - \mathscr{D}$, which we call the *containment lattice* of $(\mathsf{A},\mathscr{D})$. When $\mathbf{0} \in \mathscr{D}$, the containment lattice is

$$\mathbb{Z}[\mathsf{A},\mathscr{D}] := \mathbb{Z}[\mathscr{D},\mathsf{A}(\mathscr{D}),\dots,\mathsf{A}^{n-1}(\mathscr{D})], \tag{1.6}$$

as we show in (2.5) in Section 2. We call a digit set *primitive* if $\mathbb{Z}[\mathsf{A},\mathscr{D}] = \mathbb{Z}^n$, and we also call the associated tile $T(\mathsf{A},\mathscr{D})$ a *primitive tile*. In Section 2 we show that all measure and tiling questions can be reduced to the case of primitive tiles. More precisely, if the columns of a matrix $\mathsf{B}$ form a basis of $\mathbb{Z}[\mathsf{A},\mathscr{D}]$, that is, $\mathbb{Z}[\mathsf{A},\mathscr{D}] = \mathsf{B}(\mathbb{Z}^n)$, then there exists a matrix $\tilde{\mathsf{A}} \in M_n(\mathbb{Z})$ and digit set $\tilde{\mathscr{D}} \subseteq \mathbb{Z}^n$ such that $\mathbb{Z}[\tilde{\mathsf{A}},\tilde{\mathscr{D}}] = \mathbb{Z}^n$ and

$$T(\mathsf{A},\tilde{\mathscr{D}}) = \mathsf{B}(T(\tilde{\mathsf{A}},\mathscr{D})). \tag{1.7}$$

This gives
$$\mu(T(\mathsf{A},\mathscr{D})) = |\det(\mathsf{B})|\mu(T(\tilde{\mathsf{A}},\tilde{\mathscr{D}})), \tag{1.8}$$
and also shows that each tiling of $\mathbb{R}^n$ by $T(\mathsf{A},\mathscr{D})$ gives a corresponding tiling by $T(\tilde{\mathsf{A}},\tilde{\mathscr{D}})$, and vice versa.

A *standard digit set* $(\mathsf{A},\mathscr{D})$ is one such that the pair $(\tilde{\mathsf{A}},\tilde{\mathscr{D}})$ given by (1.7) has $\tilde{\mathscr{D}}$ being a complete set of coset representatives of $\mathbb{Z}^n/\tilde{\mathsf{A}}(\mathbb{Z}^n)$. (This property does not depend on the choices of $\mathsf{B}$ and $\tilde{\mathsf{A}}$ in (1.7).) In this case, the measure $\mu(T(\tilde{\mathsf{A}},\tilde{\mathscr{D}})) > 0$ by a result of Bandt **[1]**. A *nonstandard digit set* $(\mathsf{A},\mathscr{D})$ is one where $\mathbb{Z}[\mathsf{A},\mathscr{D}] = \mathsf{B}(\mathbb{Z}^n)$, $T(\mathsf{A},\mathscr{D}) = \mathsf{B}(T(\tilde{\mathsf{A}},\tilde{\mathscr{D}}))$ and $\mu(T(\tilde{\mathsf{A}},\tilde{\mathscr{D}})) > 0$, but $\tilde{\mathscr{D}}$ is not a complete set of coset representatives of $\mathbb{Z}^n/\tilde{\mathsf{A}}(\mathbb{Z}^n)$.

In this paper, we prove general facts about the Lebesgue measure of tiles, and study the differences between standard and nonstandard digit sets. For example, $\mu(T) = 1$ can occur only for standard digit sets.

In Section 2 we show the following.

THEOREM 1.1. *Let* $\mathsf{A}$ *be an expanding integer matrix, and* $\mathscr{D} \subseteq \mathbb{Z}^n$ *be a digit set with* $|\mathscr{D}| = |\det(\mathsf{A})|$. *Then* $\mu(T(\mathsf{A},\mathscr{D}))$ *is an integer. In particular,*
$$\mu(T(\mathsf{A},\mathscr{D})) = \gamma(\mathsf{A},\mathscr{D})[\mathbb{Z}^n:\mathbb{Z}[\mathsf{A}:\mathscr{D}]] \tag{1.9}$$
*for some integer* $\gamma(\mathsf{A},\mathscr{D})$.

It seems a nontrivial problem to compute explicitly the Lebesgue measure, and the functional equation (1.3) gives no direct information about it. We show in Part II that if $\mathscr{D}$ is a standard digit set for $\mathsf{A}$, and $\mathsf{A}$ satisfies some extra conditions, then $\gamma(\mathsf{A},\mathscr{D}) = 1$.

Theorem 1.1 has several consequences.

COROLLARY 1.1a. *If* $\mu(T(\mathsf{A},\mathscr{D})) = 1$, *then necessarily* $\mathbb{Z}[\mathsf{A},\mathscr{D}] = \mathbb{Z}^n$ *and* $\mathscr{D}$ *must be a complete set of coset representatives of* $\mathbb{Z}^n/\mathsf{A}(\mathbb{Z}^n)$. *That is,* $\mathscr{D}$ *must be a standard digit set.*

The converse is not true, as the tile with $\mathsf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\mathscr{D} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ analysed in **[17]** shows; it has $\mu(T(\mathsf{A},\mathscr{D})) = 3$.

To state another consequence, we say that a region $T$ and a translation set $\mathscr{T}$ give a *perfect multiple covering of* $\mathbb{R}^n$ *of multiplicity* $k$ if the family $\{T+\mathbf{t}:\mathbf{t}\in\mathscr{T}\}$ covers every point of $\mathbb{R}^n$ at least $k$ times, and covers almost every point of $\mathbb{R}^n$ exactly $k$ times.

COROLLARY 1.1b. *Suppose that* $\mathsf{A}$ *is an expanding integer matrix, and* $\mathscr{D} \subseteq \mathbb{Z}^n$ *is a digit set with* $|\mathscr{D}| = |\det(\mathsf{A})|$, *with* $\mu(T(\mathsf{A},\mathscr{D})) > 0$. *Then* $T(\mathsf{A},\mathscr{D}) + \mathbb{Z}^n$ *is a perfect multiple covering of* $\mathbb{R}^n$ *with multiplicity* $\mu(T(\mathsf{A},\mathscr{D}))$.

In Section 3 we construct examples of nonstandard digit sets $T(\mathsf{A},\mathscr{D})$ which have a product form suggested by the one-dimensional nonstandard digit sets constructed in Odlyzko **[25]**. Suppose that $\mathscr{E}$ contains $\mathbf{0}$ and gives a complete set of coset representatives of $\mathbb{Z}^n/\mathsf{A}(\mathbb{Z}^n)$, and suppose that it has a factorization
$$\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2 + \ldots + \mathscr{E}_r, \tag{1.10a}$$
$$|\mathscr{E}| = |\mathscr{E}_1|\,|\mathscr{E}_2| \ldots |\mathscr{E}_r| = |\det(\mathsf{A})|, \tag{1.10b}$$
where $\mathbf{0} \in \mathscr{E}_i$ for all $i$, and each $|\mathscr{E}_i| \geqslant 2$. Then for any integers $0 \leqslant f(1) \leqslant f(2) \leqslant \ldots \leqslant f(r)$, set
$$\mathscr{D} := \mathsf{A}^{f(1)}(\mathscr{E}_1) + \mathsf{A}^{f(2)}(\mathscr{E}_2) + \ldots + \mathsf{A}^{f(r)}(\mathscr{E}_r). \tag{1.11}$$

We call any such $\mathscr{D}$ a *product-form digit set*. If some $f(i) > 0$, then this is a nonstandard digit set. For these, we have the following.

THEOREM 1.2. *A product-form digit set tile* $T(\mathsf{A}, \mathscr{D})$ *is a measure-disjoint union of translates of* $T(\mathsf{A}, \mathscr{E})$, *and has measure*

$$\mu(T(\mathsf{A}, \mathscr{D})) = \mu(T(\mathsf{A}, \mathscr{E})) \prod_{i=1}^{r} |\mathscr{E}_i|^{f(i)}. \tag{1.12}$$

*Furthermore,* $T(\mathsf{A}, \mathscr{D})$ *always tiles* $\mathbb{R}^n$ *with a periodic tiling whenever* $T(\mathsf{A}, \mathscr{E})$ *tiles* $\mathbb{R}^n$ *with a periodic tiling.*

In Part II, we prove that for all standard digit sets $\mathscr{E}$, the tile $T(\mathsf{A}, \mathscr{E})$ actually gives a lattice tiling of $\mathbb{R}^n$; using this result, we may remove the extra hypothesis appearing in the final part of this theorem, and conclude that a product-form digit set tile $T(\mathsf{A}, \mathscr{D})$ always tiles $\mathbb{R}^n$ with a periodic tiling.

The product-form construction above can be used to produce nonstandard digit sets for all expanding $\mathsf{A}$ such that $|\det(\mathsf{A})|$ is not prime. In Section 4 we treat the remaining case when $|\det(\mathsf{A})| = p$ is prime, and prove that nonstandard digit sets do not occur for 'most' such $\mathsf{A}$ (Theorem 4.1). This includes all one-dimensional $\mathsf{A} = \pm p$, a result previously obtained by Kenyon **[16]**.

The classification of nonstandard digit sets seems a quite difficult problem, even in the one-dimensional case. In Section 5 we classify all nonstandard digit sets in one dimension when $\det(\mathsf{A})$ is a prime power (Theorem 5.1). In this case, there exist nonstandard digit sets not of product-form, for example, $\mathsf{A} = [4]$ and $\mathscr{D} = \{0, 1, 8, 25\}$. However, all one-dimensional nonstandard digit sets with $|\det(\mathsf{A})| = p^n$ retain a weaker structure resembling product-form. This classification uses properties of the semigroup of nonnegative integer relations among the $p^n$th roots of unity. Extending this classification to cases where $|\det(\mathsf{A})| = m$ is not a prime power requires understanding the structure of the semigroup of positive integer relations among the $m^k$th roots of unity, and this apparently is a complicated problem; see **[23]**. Lenstra **[22]** gives a survey of results concerning integer relations between roots of unity.

Concerning the structure of tilings, Part II shows that standard digit set tiles always give at least one lattice tiling. This is not true for all nonstandard digit sets; see Example 3.1 in Section 3. To complete the theory on the structure of such tilings, we propose the following conjecture.

CONJECTURE 1.1 (Periodic Tiling Conjecture). *Let* $\mathsf{A} \subseteq M_n(\mathbb{Z})$ *be an expanding integer matrix, and* $\mathscr{D} \subseteq \mathbb{Z}^n$ *a digit set of cardinality* $|\det(\mathsf{A})|$ *such that* $\mu(T(\mathsf{A}, \mathscr{D})) > 0$. *Then there is a sublattice* $\Lambda$ *of* $\mathbb{Z}^n$ *such that* $T(\mathsf{A}, \mathscr{D})$ *tiles* $\mathbb{R}^n$ *by a finite number of cosets of* $\Lambda$.

It is fairly easy to prove the Periodic Tiling Conjecture in the one-dimensional case.

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## 2. *Lebesgue measure of attractors*

We first show that for studying measure and tiling properties of lattice self-affine tiles, we can always reduce by an affine transformation to the case of integral self-affine tiles $T(\mathsf{A},\mathscr{D})$ such that $(\mathsf{A},\mathscr{D})$ is primitive and $\mathbf{0}\in\mathscr{D}$.

LEMMA 2.1. *Let $T(\mathsf{A},\mathscr{D})$ be a lattice self-affine tile in $\mathbb{R}^n$. Then $\mathsf{A}$ is similar to an integer matrix $\tilde{\mathsf{A}}\in M_n(\mathbb{Z})$. There is an invertible affine transformation $\mathsf{L}:\mathbb{R}^n\to\mathbb{R}^n$ such that $\mathsf{L}(T(\mathsf{A},\mathscr{D}))=T(\tilde{\mathsf{A}},\tilde{\mathscr{D}})$, in which $T(\tilde{\mathsf{A}},\tilde{\mathscr{D}})$ is an integral self-affine tile such that $(\tilde{\mathsf{A}},\mathscr{D})$ is primitive, that is, $\mathbb{Z}[\tilde{\mathsf{A}},\tilde{\mathscr{D}}]=\mathbb{Z}^n$, and $\mathbf{0}\in\mathscr{D}$.*

*Proof.* By hypothesis, there is a full rank $\mathsf{A}$-invariant lattice $\Lambda$ containing $\Delta(\mathscr{D})=\mathscr{D}-\mathscr{D}$. The smallest such $\mathsf{A}$-invariant lattice is denoted $\mathbb{Z}[\mathsf{A},\mathscr{D}]$, and is

$$\mathbb{Z}[\mathsf{A},\mathscr{D}]:=\mathbb{Z}[\Delta(\mathscr{D}),\mathsf{A}(\Delta(\mathscr{D})),\mathsf{A}^2(\Delta(\mathscr{D})),\ldots]. \tag{2.1}$$

We first reduce to the case that $\mathbf{0}\in\mathscr{D}$ by a translation. Set $\mathscr{D}'=\mathscr{D}-\mathbf{v}$. This has the effect of translating the tile, namely

$$T(\mathsf{A},\mathscr{D}')=T(\mathsf{A},\mathscr{D})-\mathbf{v}^*,$$

where $\mathbf{v}^*=\sum_{k=1}^{\infty}\mathsf{A}^{-k}\mathbf{v}$, using (1.2), which is still valid for lattice self-affine tiles. Note that $\Delta(\mathscr{D}')=\Delta(\mathscr{D})$, hence $\mathbb{Z}[\mathsf{A},\mathscr{D}']=\mathbb{Z}[\mathsf{A},\mathscr{D}]$. We choose $\mathbf{v}\in\mathscr{D}$ to obtain $\mathbf{0}\in\mathscr{D}'$.

Next let $\mathsf{B}$ be an (invertible) $n\times n$ matrix whose column vectors are a basis for the lattice $\mathbb{Z}[\mathsf{A},\mathscr{D}]$. Let $[\mathbf{e}_1,\ldots,\mathbf{e}_n]$ be the standard basis of column vectors, where $\mathbf{e}_i$ has a 1 in the $i$th coordinate and 0s elsewhere. The vectors $\mathsf{A}\mathsf{B}\mathbf{e}_i$ are all contained in $\mathbb{Z}[\mathsf{A},\mathscr{D}]$ because it is $\mathsf{A}$-invariant, hence $\mathsf{A}\mathsf{B}\mathbf{e}_j=\mathsf{B}\tilde{\mathbf{a}}_j$ for some $\tilde{\mathbf{a}}_j\in\mathbb{Z}^n$. Thus we have

$$\mathsf{A}\mathsf{B}=\mathsf{B}\tilde{\mathsf{A}}, \tag{2.2}$$

with $\tilde{\mathsf{A}}=[\tilde{\mathbf{a}}_1,\ldots,\tilde{\mathbf{a}}_n]\in M_n(\mathbb{Z})$, so $\mathsf{A}$ is similar to the integral matrix $\tilde{\mathsf{A}}$.

Now define $\tilde{\mathscr{D}}=\{\tilde{\mathbf{d}}_i\colon 1\leqslant i\leqslant m\}$ by $\mathsf{B}\tilde{\mathbf{d}}_i=\mathbf{d}_i'$. Since $\mathbf{d}_i'=\mathbf{d}_i'-\mathbf{0}\in\Delta(\mathscr{D})$, we have $\mathbf{d}_i'\in\mathbb{Z}[\mathsf{A},\mathscr{D}]$, and hence $\tilde{\mathbf{d}}_i\in\mathbb{Z}^n$. Thus $\tilde{\mathscr{D}}\subseteq\mathbb{Z}^n$ is an integer digit set with $\mathbf{0}\in\tilde{\mathscr{D}}$, and

$$\mathscr{D}'=\mathsf{B}(\tilde{\mathscr{D}})=\{\mathsf{B}\tilde{\mathbf{d}}_i\colon 1\leqslant i\leqslant m\}.$$

The relation (2.2) implies that

$$T(\mathsf{A},\mathscr{D}')=\mathsf{B}(T(\tilde{\mathsf{A}},\mathscr{D})).$$

This fact follows easily from the characterization (1.2) of the attractor $T(\mathsf{A},\mathscr{D}')$, since

$$\begin{aligned}\sum_{k=1}^{\infty}\mathsf{A}^{-k}\mathbf{d}_{i_k}' &= \sum_{k=1}^{\infty}\mathsf{A}^{-k}\mathsf{B}\tilde{\mathbf{d}}_{i_k}\\ &= \mathsf{B}\left(\sum_{k=1}^{\infty}\tilde{\mathsf{A}}^{-k}\tilde{\mathbf{d}}_{i_k}\right).\end{aligned} \tag{2.3}$$

Thus

$$T(\mathsf{A},\mathscr{D})=\mathsf{B}(T(\tilde{\mathsf{A}},\tilde{\mathscr{D}}))+\mathbf{v}^*, \tag{2.4}$$

which gives $T(\tilde{\mathsf{A}},\tilde{\mathscr{D}})=\mathsf{L}(T(\mathsf{A},\mathscr{D}))$ with $\mathsf{L}(\mathbf{x})=\mathsf{B}^{-1}\mathbf{x}-\mathsf{B}^{-1}\mathbf{v}^*$. Since $(\tilde{\mathsf{A}},\tilde{\mathscr{D}})$ is integral, its containment lattice is given by

$$\mathbb{Z}[\tilde{\mathsf{A}},\tilde{\mathscr{D}}]=\mathbb{Z}[\Delta(\tilde{\mathscr{D}}),\tilde{\mathsf{A}}(\Delta(\tilde{\mathscr{D}})),\ldots,\tilde{\mathsf{A}}^{n-1}(\Delta(\tilde{\mathscr{D}}))],$$

because $\tilde{\mathsf{A}}^n$ is an integral linear combination of $\{\tilde{\mathsf{A}}^i\colon 0\leqslant i\leqslant n-1\}$. Also, since $\mathbf{0}\in\tilde{\mathscr{D}}$, the set $\tilde{\mathscr{D}}$ generates $\Delta(\tilde{\mathscr{D}})$, hence

$$\mathbb{Z}[\tilde{\mathsf{A}},\tilde{\mathscr{D}}]=\mathbb{Z}[\tilde{\mathscr{D}},\tilde{\mathsf{A}}(\tilde{\mathscr{D}}),\ldots,\tilde{\mathsf{A}}^{n-1}(\tilde{\mathscr{D}})]. \tag{2.5}$$

The identity (2.3) also gives

$$\sum_{k=1}^{m} \mathsf{A}^k \mathbf{d}'_{i_k} = \mathsf{B}\left(\sum_{k=1}^{m} \tilde{\mathsf{A}}^k \tilde{\mathbf{d}}_{i_k}\right),$$

which shows that

$$\mathbb{Z}[\mathsf{A}, \mathscr{D}] = \mathbb{Z}[\mathsf{A}, \mathscr{D}'] = \mathsf{B}(\mathbb{Z}[\tilde{\mathsf{A}}, \tilde{\mathscr{D}}]).$$

Since $\mathsf{B}$ is a basis of $\mathbb{Z}[\mathsf{A}, \mathscr{D}]$, this forces

$$\mathbb{Z}[\tilde{\mathsf{A}}, \tilde{\mathscr{D}}] = \mathbb{Z}^n,$$

hence $\tilde{\mathscr{D}}$ is a primitive digit set for $\tilde{\mathsf{A}}$.

Lemma 2.1 preserves information on the measure of tiles, since (2.4) gives

$$\mu(T(\mathsf{A}, \mathscr{D})) = |\det(\mathsf{B})| \mu(T(\tilde{\mathsf{A}}, \tilde{\mathscr{D}})). \tag{2.6}$$

We can also apply Lemma 2.1 when $T(\mathsf{A}, \mathscr{D})$ is an integral self-affine tile but is not primitive. Then $\mathbb{Z}[\mathsf{A}, \mathscr{D}] \subseteq \mathbb{Z}^n$, and

$$|\det(\mathsf{B})| = [\mathbb{Z}^n : \mathbb{Z}[\mathsf{A}, \mathscr{D}]]. \tag{2.7}$$

We can reverse this process. Given $(\tilde{\mathsf{A}}, \tilde{\mathscr{D}})$ and any invertible matrix $\mathsf{B} \in M_n(\mathbb{Z})$ for which the matrix $\mathsf{A} := \mathsf{B}\tilde{\mathsf{A}}\mathsf{B}^{-1}$ lies in $M_n(\mathbb{Z})$, then $(\mathsf{A}, \mathscr{D})$ with $\mathscr{D} = \mathsf{B}(\tilde{\mathscr{D}}) \subseteq \mathbb{Z}^n$ and (2.7) holds. A price is paid, in that the expanding integer matrix $\tilde{\mathsf{A}}$ is changed to a different integer matrix $\mathsf{A}$, which is similar to $\tilde{\mathsf{A}}$ over $\mathbb{Q}$ but not necessarily similar to $\mathsf{A}$ over $\mathbb{Z}$.

We now show that the Lebesgue measure of any integral self-affine tile is always an integer, possibly zero.

*Proof of Theorem* 1.1. Suppose that $T = T(\mathsf{A}, \mathscr{D})$ with $|\mathscr{D}| = |\det(\mathsf{A})| = m$ is given, and that $\mu(T) > 0$. This implies that $T$ contains an open set and is the closure of its interior, and that the Lebesgue measure of its boundary is 0; see [**17**, Theorem 1.1]. Consider the projection $\pi_n : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, that is,

$$\pi_n(\mathbf{x}) \equiv \mathbf{x} \,(\mathrm{mod}\, 1). \tag{2.8}$$

Let $\bar{\nu}$ denote the measure on $\mathbb{R}^n/\mathbb{Z}^n$ induced by this projection from Lebesgue measure on $T$, that is, $d\bar{\nu}$ is the projection of the density $\chi_T(\mathbf{x})\, d\mu(\mathbf{x})$. Let $\bar{\mu}$ denote Haar measure on $\mathbb{R}^n/\mathbb{Z}^n$, and we have

$$d\bar{\nu}(\mathbf{y}) = \phi(\mathbf{y})\, d\bar{\mu}(\mathbf{y}), \quad \text{for } \mathbf{y} \in \mathbb{R}^n/\mathbb{Z}^n, \tag{2.9}$$

where $\phi : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}$ is the integer-valued function

$$\phi(\mathbf{y}) := \#\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in \pi_n^{-1}(\mathbf{y}) \cap T\}, \tag{2.10}$$

which counts the number of preimages of $\mathbf{y}$ lying in $T$. Also, $\phi$ is positive on a set of positive measure since $\mu(T) > 0$, and it is finite everywhere since $T$ is compact. It is locally constant off the measure zero set $\pi_n(\partial T)$, and in particular $\bar{\nu}$ is absolutely continuous with respect to Haar measure $\bar{\mu}$.

We shall show that $\bar{\nu}$ is an invariant measure for the endomorphism of $\mathbb{R}^n/\mathbb{Z}^n$ induced by $\mathsf{A}$. Since $\bar{\nu}$ is absolutely continuous, we need only show that

$$\bar{\nu}(S) = \bar{\nu}(\mathsf{A}^{-1}(S)), \quad S \text{ any open set}. \tag{2.11}$$

To show this, since $\phi(\mathbf{x})$ is locally constant, it suffices to prove that

$$\phi(\mathbf{y}) = \frac{1}{m}\Bigg(\sum_{\substack{\mathsf{A}\mathbf{z}=\mathbf{y}\\ \mathbf{z}\in\mathbb{R}^n/\mathbb{Z}^n}} \phi(\mathbf{z})\Bigg). \tag{2.12}$$

So suppose $\mathbf{y} = \mathsf{A}\mathbf{x}$. The inflation identity

$$\mathsf{A}(T) = \bigcup_{\mathbf{d}\in\mathscr{D}} (T+\mathbf{d})$$

yields

$$\#\{\mathbf{x}\in\mathbb{R}^n : \mathbf{x}\in\pi_n^{-1}(\mathbf{y}) \cap \mathsf{A}(T)\} = m\#\{\mathbf{x}\in\mathbb{R}^n : \mathbf{x}\in\pi_n^{-1}(\mathbf{y}) \cap T\}. \tag{2.13}$$

However, since $\mathbf{x} \to \mathsf{A}\mathbf{x}$ is one-to-one on $\mathbb{R}^n$,

$$\#\{\mathbf{x}\in\mathbb{R}^n : \mathbf{x}\in\pi_n^{-1}(\mathbf{z}) \cap T\} = \#\{\mathbf{x}\in\mathbb{R}^n : \mathbf{x}\in \mathsf{A}(\pi_n^{-1}(\mathbf{z})) \cap \mathsf{A}(T)\},$$

and this implies

$$\#\{\mathbf{x}\in\mathbb{R}^n : \mathbf{x}\in\pi_n^{-1}(\mathbf{y}) \cap \mathsf{A}(T)\} = \sum_{\substack{\mathsf{A}\mathbf{z}=\mathbf{y}\\ \mathbf{z}\in\mathbb{R}^n/\mathbb{Z}^n}} \#\{\mathbf{x}\in\mathbb{R}^n : \mathbf{x}\in\pi_n^{-1}(\mathbf{z}) \cap T\}.$$

Combining this with (2.13) yields (2.12).

Now recall that Haar measure $\bar{\mu}$ is an invariant measure for the endomorphism $\mathsf{A}$ of $\mathbb{R}^n/\mathbb{Z}^n$, and $\mathsf{A}$ is ergodic with respect to $\bar{\mu}$; compare [**32**, Corollary 1.10.1]. This immediately forces $\bar{\nu} = c\bar{\mu}$ for some constant $c > 0$. To see this, note that the pointwise ergodic theorem gives, for any characteristic function $\chi_S$ of a set $S$,

$$\frac{1}{n}\sum_{j=1}^{n} \chi_S(\mathsf{A}^j\mathbf{x}) \longrightarrow \bar{\mu}(S) \quad (\text{a.e. } \bar{\mu}).$$

The $\mathsf{A}$-invariance of $\bar{\nu}$ gives

$$\bar{\nu}(S) = \int_{\mathbb{R}^n/\mathbb{Z}^n} \chi_S(\mathbf{x})\, d\bar{\nu} = \int_{\mathbb{R}^n/\mathbb{Z}^n} \chi_S(\mathsf{A}^n\mathbf{x})\, d\bar{\nu}.$$

Consequently,

$$\begin{aligned}\bar{\nu}(S) &= \int_{\mathbb{R}^n/\mathbb{Z}^n} \left(\frac{1}{n}\sum_{j=1}^{n} \chi_S(\mathsf{A}^j\mathbf{x})\right) d\bar{\nu} \\ &= \bar{\mu}(S)\left(\int_{\mathbb{R}^n/\mathbb{Z}^n} d\bar{\nu}\right),\end{aligned}$$

as follows by letting $n \to \infty$ and using the absolute continuity of $\bar{\nu}$ with respect to $\bar{\mu}$. Thus $\phi(\mathbf{x})$ is constant almost everywhere with respect to $\bar{\mu}$, with value $k_0 \in \mathbb{Z}^+$, so that

$$\mu(T) = \int_{\mathbb{R}^n/\mathbb{Z}^n} d\bar{\nu} = \int_{\mathbb{R}^n/\mathbb{Z}^n} \phi(\mathbf{x})\, d\bar{\mu} = k_0$$

is an integer.

Finally, to obtain (1.9), apply this result to the associated primitive tile $T(\tilde{\mathsf{A}}, \tilde{\mathscr{D}})$ in (2.4), and use (2.6) and (2.7).

*Proof of Corollary* 1.1a. Suppose $\mu(T(\mathsf{A}, \mathscr{D})) = 1$. Then $\mathbb{Z}[\mathsf{A}, \mathscr{D}] = \mathbb{Z}^n$ by (1.9). For the second part, we show the contrapositive. Suppose that $\mathscr{D}$ is not a complete set of coset representatives, so there exist $\mathbf{d}, \mathbf{d}' \in \mathscr{D}$ with $\mathbf{d} - \mathbf{d}' \in \mathsf{A}(\mathbb{Z}^n)$. Then the tile $T$ contains as subsets $\mathsf{A}^{-1}(T) + \mathsf{A}^{-1}\mathbf{d}$ and $\mathsf{A}^{-1}(T) + \mathsf{A}^{-1}\mathbf{d}'$, that is, the sets of all expansions (1.2) with $\mathbf{d}_1 = \mathbf{d}$ or $\mathbf{d}'$, respectively. These sets both have positive measure $\mu(\mathsf{A}^{-1}(T))$, and since $\mathsf{A}^{-1}\mathbf{d} \equiv \mathsf{A}^{-1}\mathbf{d}' \pmod 1$, each point $\mathbf{y} \in \mathbb{R}^n/\mathbb{Z}^n$ with $\mathbf{y} \in \pi_n(\mathsf{A}^{-1}(T) + \mathsf{A}^{-1}\mathbf{d})$ has multiplicity $\phi(\mathbf{y})$, with $\phi(\mathbf{y}) \geqslant 2$. The proof of Theorem 1.1 showed that $\phi(\mathbf{y}) = \mu(T(\mathsf{A}, \mathscr{D}))$ on a set of Haar measure one, whence $\mu(T(\mathsf{A}, \mathscr{D})) \geqslant 2$.

*Proof of Corollary* 1.1b. The proof of Theorem 1.1 showed that for $\mathbf{y} \in \mathbb{R}^n/\mathbb{Z}^n$, the function $\phi(\mathbf{y})$ defined by (2.12) equals $\mu(T(\mathsf{A}, \mathscr{D}))$ a.e. The fact that every point of $\mathbb{R}^n$ is covered at least $\mu(T(\mathsf{A}, \mathscr{D}))$ times follows because $T(\mathsf{A}, \mathscr{D})$ is a closed set, using the fact that the set of points of multiplicity $\mu(T(\mathsf{A}, \mathscr{D}))$ is dense in $\mathbb{R}^n$.

We next derive conditions for an integral self-affine attractor $T = T(\mathsf{A}, \mathscr{D})$ to have Lebesgue measure $\mu(T) > 0$. These conditions are essentially due to Kenyon **[16]**, except that **[16]** proves (iii) only in the one-dimensional case.

THEOREM 2.1. *For any expanding matrix* $\mathsf{A} \in M_n(\mathbb{Z})$ *and a digit set* $\mathscr{D} \subseteq \mathbb{Z}^n$ *with* $|\mathscr{D}| = |\det(\mathsf{A})| = m$, *the following three conditions are equivalent.*

(i) *The measure* $\mu(T(\mathsf{A}, \mathscr{D})) > 0$.

(ii) *For each* $k \geqslant 1$, *the set*

$$\mathscr{D}_{\mathsf{A},k} := \left\{ \sum_{j=0}^{k-1} \mathsf{A}^j \mathbf{d}_j : \text{each } \mathbf{d}_j \in \mathscr{D} \right\} \tag{2.14}$$

*contains* $|\det(\mathsf{A})|^k$ *distinct elements.*

(iii) *For each* $\mathbf{m} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$, *there exists a nonnegative integer* $k = k(\mathbf{m})$ *such that the function*

$$\hat{h}_k(\mathbf{x}) := \frac{1}{m} \sum_{\mathbf{d} \in \mathscr{D}} \exp(2\pi i \langle \mathsf{A}^{-k}\mathbf{d}, \mathbf{x} \rangle) \tag{2.15}$$

*has* $\hat{h}_k(\mathbf{m}) = 0$.

*Proof.* The equivalence (i)$\Leftrightarrow$(ii) is **[16**, Theorem 10]. A detailed proof appears in **[17**, Theorem 1.1].

To prove (i)$\Leftrightarrow$(iii), we generalize the proof of **[16**, Theorem 15]. Consider the measures

$$h_k(\mathbf{y}) = \frac{1}{m} \sum_{\mathbf{d} \in \mathscr{D}} \delta(\mathbf{y} - \mathsf{A}^k\mathbf{d} \pmod 1), \quad k = 1, 2, \ldots, \tag{2.16}$$

on $\mathbb{R}^n/\mathbb{Z}^n$, where $\delta(\mathbf{y} - \mathbf{y}_0)$ denotes a Dirac delta function (point measure) at $\mathbf{y}_0$. The Fourier transform $\hat{h}_k(\mathbf{m})$ for $\mathbf{m} \in \mathbb{Z}^n$ is

$$\begin{aligned} \hat{h}_k(\mathbf{m}) &= \int_{\mathbb{R}^n/\mathbb{Z}^n} h_k(\mathbf{y}) \exp(2\pi i \langle \mathbf{y}, \mathbf{m} \rangle)\, dy \\ &= \frac{1}{m} \sum_{\mathbf{d} \in \mathscr{D}} \exp(2\pi i \langle \mathsf{A}^{-k}\mathbf{d}, \mathbf{m} \rangle). \end{aligned}$$

For each $k \geqslant 1$, form the convolved measure

$$\nu_k(\mathbf{y}) = h_1 * h_2 * \ldots * h_k. \tag{2.17}$$

The Fourier transform $\{\hat{\nu}_k(\mathbf{m})\colon \mathbf{m}\in\mathbb{Z}^n\}$ of $\nu_k$ is

$$\hat{\nu}_k(\mathbf{m}) = \prod_{j=1}^{k}\left(\frac{1}{m}\sum_{\mathbf{d}\in\mathcal{D}}\exp(2\pi i\langle \mathsf{A}^{-j}\mathbf{d},\mathbf{m}\rangle)\right). \tag{2.18}$$

Now consider the infinite product

$$\hat{\nu}_\infty(\mathbf{x}) := \prod_{k=1}^{\infty}\hat{h}_k(\mathbf{x}) = \prod_{k=1}^{\infty}\left(\frac{1}{m}\sum_{\mathbf{d}\in\mathcal{D}}\exp(2\pi i\langle \mathsf{A}^{-k}\mathbf{d},\mathbf{x}\rangle)\right). \tag{2.19}$$

This infinite product converges absolutely for all $\mathbf{x}\in\mathbb{R}^n$, because $\|\mathsf{A}^{-k}\|_2 = O(|\lambda_n|^{-k})$ as $k\to\infty$, where $|\lambda_n|>1$ is the smallest modulus eigenvalue of the expanding matrix $\mathsf{A}$, so that the $k$th term in the product is $1+O(|\lambda_n|^{-k}\|\mathbf{x}\|)$ for large enough $k$. In particular, the values $\hat{\nu}_k(\mathbf{m})$ converge pointwise to $\hat{\nu}_\infty(\mathbf{m})$, as $k\to\infty$. This implies that the measures $\nu_k$ weakly converge to a *unique* limit measure $\nu_\infty$ whose Fourier transform is (2.19).

Now suppose that (i) holds. It suffices to prove that the measures $\nu_k$ converge to a nonzero constant $c$ times Haar measure $\bar{\mu}$ on $\mathbb{R}^n/\mathbb{Z}^n$. (The fact that $\hat{\nu}_\infty(\mathbf{0}) = 1$ then implies that the constant $c = 1$.) Now $\nu_\infty = c\bar{\mu}$, whence

$$\hat{\nu}_\infty(\mathbf{m}) = 0 \quad \text{for } \mathbf{m}\in\mathbb{Z}^n\backslash\{\mathbf{0}\}.$$

The absolute convergence of the infinite product (2.19) then forces $\hat{h}_k(\mathbf{m}) = 0$ for some $k\geqslant 1$, which is (iii).

Each measure $\nu_k$ arises as the projection under the map $\pi_n\colon \mathbb{R}^n\to\mathbb{R}^n/\mathbb{Z}^n$ of the discrete measure

$$\omega_k := \frac{1}{m^k}\sum_{\mathbf{d}\in\mathcal{D}_{\mathsf{A},k}}\delta(\mathbf{x}-\mathsf{A}^{-k}\mathbf{d}).$$

Here the set $\mathsf{A}^{-k}(\mathcal{D}_{\mathsf{A},k})$ is the set of $m^k$ centres of translates of the shrunken tile $\mathsf{A}^{-k}(T)$ which together form the tiling of $T$ obtained by iterating the mapping (1.1) $k$ times. To show that $\nu_k$ converges weakly to $c\bar{\mu}$, it suffices to show that the measures $\omega_k$ converge weakly to a nonzero constant times Lebesgue measure on $T$, for example, to the measure $\omega_\infty$ having density

$$d\omega_\infty(\mathbf{x}) := \frac{1}{\mu(T)}\chi_T(\mathbf{x})\,d\mu(\mathbf{x}).$$

To establish this, observe that each $\omega_k$ is a Radon measure, so to show weak convergence to $\omega_\infty$, by [**9**, §1.9] it suffices to establish that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} f(\mathbf{x})\,d\omega_k(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x})\,d\omega_\infty(\mathbf{x}) \tag{2.20}$$

holds for all $f(\mathbf{x})\in C_c(\mathbb{R}^n)$, the set of compactly supported continuous functions on $\mathbb{R}^n$. To do this, we use the fact that

$$\operatorname{diam}(\mathsf{A}^{-k}(T))\to 0 \quad \text{as } k\to\infty,$$

which follows since $\mathsf{A}$ is an expanding matrix, and the fact that the boundary of $T$ has measure zero. Take a fixed $f(\mathbf{x})\in C_c(\mathbb{R}^n)$, and by definition

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,d\omega_k(\mathbf{x}) = \frac{1}{m^k}\sum_{\mathbf{d}\in\mathcal{D}_{\mathsf{A},k}} f(\mathsf{A}^{-k}\mathbf{d}).$$

Now any such $f$ is uniformly continuous, that is, for each $\delta > 0$ there is an $\varepsilon > 0$ such that $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| \leqslant \delta$ whenever $\|\mathbf{x}_1 - \mathbf{x}_2\| < \varepsilon$. Thus if $\operatorname{diam}(\mathsf{A}^{-k}(T)) < 3$, then

$$\left| \int_{\mathsf{A}^{-k}(T+\mathbf{d})} f(\mathbf{x})\, d\mu(\mathbf{x}) - \mu(\mathsf{A}^{-k}(T)) f(\mathsf{A}^{-k}\mathbf{d}) \right| < \mu(\mathsf{A}^{-k}(T))\, \delta. \tag{2.21}$$

However, $\mu(\mathsf{A}^{-k}(T)) = \mu(T)/m^k$, so summing (2.21) over all $\mathbf{d} \in \mathscr{D}_{\mathsf{A},k}$ gives

$$\left| \int_T f(\mathbf{x})\, d\mu - \mu(T) \int_{\mathbb{R}^n} f(\mathbf{x})\, d\omega_k(\mathbf{x}) \right| \leqslant \mu(T)\, \delta,$$

and this holds for all sufficiently large $k$. Letting $\delta \to 0$ gives (2.20), which proves weak convergence, so (iii) follows.

Conversely, suppose that (iii) holds. Then $\hat{v}_\infty(\mathbf{0}) = 1$ and $\hat{v}_\infty(\mathbf{m}) = 0$ if $\mathbf{m} \neq \mathbf{0}$, so that $v_\infty$ is Haar measure $\bar{\mu}$ on $\mathbb{R}^n/\mathbb{Z}^n$. The discrete measures $v_k$ weakly converge to $\bar{\mu}$, whence each $\mathbf{x} \in \mathbb{R}^n/\mathbb{Z}^n$ must be a limit point (mod 1) of some sequence of points

$$\mathbf{y}_k = \sum_{j=1}^{n_k} \mathsf{A}^{-j} \mathbf{d}_{i_j}$$

in $T$. However, $T$ is compact, lying inside some box $|\mathbf{x}| \leqslant B$, hence

$$[0,1]^n \subseteq \bigcup_{\substack{\mathbf{m} \in \mathbb{Z}^n \\ \|\mathbf{m}\| \leqslant B}} (T + \mathbf{m}).$$

Since this is a finite union, some set $T + \mathbf{m}$ contains an open set, whence $T$ contains an open set and $\mu(T) > 0$, which is (i).

## 3. *Product-form digit sets*

Recall that a digit set $(\mathsf{A}, \mathscr{D})$ is a *product-form digit set* if $\mathscr{D}$ has an additive factorization

$$\mathscr{D} = \mathsf{A}^{f(1)}(\mathscr{E}_1) + \mathsf{A}^{f(2)}(\mathscr{E}_2) + \ldots + \mathsf{A}^{f(r)}(\mathscr{E}_r),$$

in which $r \geqslant 2$, and $0 \leqslant f(1) \leqslant f(2) \leqslant \ldots \leqslant f(r)$, and where $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_r \subseteq \mathbb{Z}^n$ each have $\mathbf{0} \in \mathscr{E}_i$ and $|\mathscr{E}_i| \geqslant 2$, and

$$\mathscr{E} := \mathscr{E}_1 + \mathscr{E}_2 + \ldots + \mathscr{E}_r \tag{3.1}$$

is a complete set of coset representatives of $\mathbb{Z}^n/\mathsf{A}(\mathbb{Z}^n)$, with all sums distinct, that is,

$$|\mathscr{E}| = |\mathscr{E}_1|\,|\mathscr{E}_2| \ldots |\mathscr{E}_r| = |\det(\mathsf{A})|. \tag{3.2}$$

*Proof of Theorem* 1.2. The tile $T(\mathsf{A}, \mathscr{E})$ has positive measure since $\mathscr{E}$ is a standard digit set, and

$$T(\mathsf{A}, \mathscr{E}) = \left\{ \sum_{j=1}^{\infty} \mathsf{A}^{-j} \mathbf{e}_j : \text{all } \mathbf{e}_j \in \mathscr{E} \right\}.$$

According to (3.2), each $\mathbf{e}_j \in \mathscr{E}$ can be uniquely decomposed as

$$\mathbf{e}_j = \mathbf{e}_{j,1} + \mathbf{e}_{j,2} + \ldots + \mathbf{e}_{j,r}, \quad \text{with } \mathbf{e}_{j,i} \in \mathscr{E}_i,\ 1 \leqslant i \leqslant r. \tag{3.3}$$

Thus if $E_i$ denotes the set

$$E_i := \left\{ \sum_{j=1}^{\infty} \mathsf{A}^{-j} \mathbf{e}_{j,i} : \text{all } \mathbf{e}_{j,i} \in \mathscr{E}_i \right\}, \tag{3.4}$$

then

$$\begin{aligned} T(\mathsf{A}, \mathscr{E}) &= \left\{ \sum_{j=1}^{\infty} \mathsf{A}^{-j}(\mathbf{e}_{j,1} + \ldots + \mathbf{e}_{j,r}) : \text{all } \mathbf{e}_{j,i} \in \mathscr{E}_i \right\} \\ &= E_1 + E_2 + \ldots + E_r, \end{aligned} \tag{3.5}$$

where $+$ denotes set addition. Each $E_i$ is an integral self-affine attractor $E_i = T(\mathsf{A}, \mathscr{E}_i)$, which has Lebesgue measure zero because $\mathscr{E}_i$ has less than $|\det(\mathsf{A})|$ digits. Now the conditions (3.1), (3.2) and $0 \leqslant f(1) \leqslant f(2) \leqslant \ldots \leqslant f(r)$ imply that $|\mathscr{D}| = |\det(\mathsf{A})|$, and that each $\mathbf{d}_j \in \mathscr{D}$ has a unique decomposition

$$\mathbf{d}_j = \mathsf{A}^{f(1)}(\mathbf{e}_{j,1}) + \mathsf{A}^{f(2)}(\mathbf{e}_{j,2}) + \ldots + \mathsf{A}^{f(r)}(\mathbf{e}_{j,r}) \tag{3.6}$$

so, just as above,

$$\begin{aligned} T(\mathsf{A}, \mathscr{D}) &= \left\{ \sum_{j=1}^{\infty} \mathsf{A}^{-j} \mathbf{d}_j : \text{all } \mathbf{d}_j \in \mathscr{D} \right\} \\ &= \mathsf{A}^{f(1)}(E_1) + \mathsf{A}^{f(2)}(E_2) + \ldots + \mathsf{A}^{f(r)}(E_r). \end{aligned} \tag{3.7}$$

Next note that

$$\begin{aligned} \mathsf{A}^k(E_i) &= \left\{ \mathsf{A}^k \left( \sum_{j=1}^{\infty} \mathsf{A}^{-j} \mathbf{e}_{j,i} \right) : \text{all } \mathbf{e}_{j,i} \in \mathscr{E}_i \right\} \\ &= E_i + \mathscr{A}_{i,k}, \end{aligned} \tag{3.8}$$

where $\mathscr{A}_{i,k} \subseteq \mathbb{Z}^n$ is the set

$$\mathscr{A}_{i,k} := \left\{ \sum_{j=0}^{k-1} \mathsf{A}^j \mathbf{e}_{j,i} : \text{all } \mathbf{e}_{j,i} \in \mathscr{E}_i \right\}, \tag{3.9}$$

with $\mathscr{A}_{i,k} = \{\mathbf{0}\}$ if $k = 0$. Thus

$$\begin{aligned} T(\mathsf{A}, \mathscr{D}) &= (E_1 + \mathscr{A}_{1,f(1)}) + \ldots + (E_r + \mathscr{A}_{r,f(r)}) \\ &= (E_1 + E_2 + \ldots + E_r) + (\mathscr{A}_{1,f(1)} + \ldots + \mathscr{A}_{r,f(r)}) \\ &= T(\mathsf{A}, \mathscr{E}) + (\mathscr{A}_{1,f(1)} + \ldots + \mathscr{A}_{r,f(r)}). \end{aligned} \tag{3.10}$$

$T(\mathsf{A}, \mathscr{E})$ is a standard digit set, with $\mu(T(\mathsf{A}, \mathscr{E})) > 0$, and Theorem 2.1(ii) asserts that

$$\mathscr{E}_{\mathsf{A},m} := \left\{ \sum_{j=0}^{m-1} \mathsf{A}^j \mathbf{e}_j : \text{all } \mathbf{e}_j \in \mathscr{E} \right\}$$

has $|\det(\mathsf{A})|^m$ distinct elements, so each $\mathscr{A}_{i,f(i)} \subseteq \mathscr{E}_{\mathsf{A},f(i)}$ has $|\mathscr{E}_i|^{f(i)}$ distinct elements. Furthermore, we have

$$\mathscr{A}_{1,f(1)} + \ldots + \mathscr{A}_{r,f(r)} \subseteq \mathscr{E}_{\mathsf{A},f(r)}, \tag{3.11}$$

and the uniqueness of the representation (3.3) shows that

$$\begin{aligned} |\mathscr{A}_{1,f(1)} + \ldots + \mathscr{A}_{r,f(r)}| &= \prod_{i=1}^{r} |\mathscr{A}_{i,f(i)}| \\ &= \prod_{i=1}^{r} |\mathscr{E}_i|^{f(i)}. \end{aligned} \tag{3.12}$$

Next we recall the identity

$$\mathsf{A}^{f(r)}(T(\mathsf{A},\mathscr{E})) = T(\mathsf{A},\mathscr{E}) + \mathscr{E}_{\mathsf{A},f(r)}. \tag{3.13}$$

The inclusion (3.11) now implies that the right-hand side of (3.10) is a measure-disjoint subset of the tiling (3.13) of $\mathsf{A}^{f(r)}(T(\mathsf{A},\mathscr{E}))$ by translates of $T(\mathsf{A},\mathscr{E})$, and (3.10) and (3.12) together yield

$$\mu(T(\mathsf{A},\mathscr{D})) = \mu(T(\mathsf{A},\mathscr{E}))\prod_{i=1}^{r}|\mathscr{E}_i|^{f(i)}.$$

To complete the proof, we shall show that the inflated tile $\mathsf{A}^{f(r)}(T(\mathsf{A},\mathscr{E}))$ can be perfectly tiled by translates of $T(\mathsf{A},\mathscr{D})$. In view of (3.10), it suffices to show that there is a set $\mathscr{B}$ of translations such that

$$\mathscr{E}_{\mathsf{A},f(r)} = (\mathscr{A}_{1,f(1)} + \mathscr{A}_{2,f(2)} + \dots + \mathscr{A}_{r,f(r)}) + \mathscr{B},$$

with all sums distinct. If $0 \leqslant f(1) \leqslant \dots \leqslant f(s) < f(s+1) = f(s+2) = \dots = f(r)$, the uniqueness of the decomposition (3.3) immediately implies that we need only take

$$\mathscr{B} = \left\{\sum_{j=f(1)}^{f(r)-1} \mathsf{A}^j\mathbf{e}_{j,1} : \text{all } \mathbf{e}_{j,1}\in\mathscr{E}_1\right\} + \dots + \left\{\sum_{j=f(s)}^{f(r)-1} \mathsf{A}^j\mathbf{e}_{j,s} : \text{all } \mathbf{e}_{j,s}\in\mathscr{E}_s\right\}.$$

Thus if $T(\mathsf{A},\mathscr{E})$ gives a periodic tiling of $\mathbb{R}^n$, so does $T(\mathsf{A},\mathscr{D})$, using the periodic tiling by $\mathsf{A}^{f(r)}(T(\mathsf{A},\mathscr{E}))$.

The following example shows that even if $T(\mathsf{A},\mathscr{E})$ gives a lattice tiling of $\mathbb{R}^n$, we may only conclude that the product-form tile $T(\mathsf{A},\mathscr{D})$ gives a periodic tiling of $\mathbb{R}^n$.

EXAMPLE 3.1. Let $\mathsf{A} = [4]$ and $\mathscr{D} = \{0,1,8,9\}$. This is a product-form digit set with $\mathscr{E}_1 = \{0,1\}$, $\mathscr{E}_2 = \{0,2\}$ and $f(1) = 0$, $f(2) = 1$. Then $\mathscr{E} = \{0,1,2,3\}$, so $T(\mathsf{A},\mathscr{E}) = [0,1]$, which certainly lattice tiles $\mathbb{R}$. Now (3.10) yields $T(\mathsf{A},\mathscr{D}) = [0,1]\cup[2,3]$. It is easy to see that $T(\mathsf{A},\mathscr{D})$ cannot lattice tile $\mathbb{R}$, but it has two different periodic tilings of $\mathbb{R}$ with period lattice $4\mathbb{Z}$, namely

$$\mathscr{S} = \{j+4\mathbb{Z} : j = 0,1\} \quad\text{and}\quad \mathscr{S}' = \{j+4\mathbb{Z} : j = 0,3\}.$$

## 4. *Classifying digit sets: prime determinant case*

The classification of all digit sets $\mathscr{D}$ such that $T(\mathsf{A},\mathscr{D})$ has positive measure appears to be a difficult question. The Fourier-analytic condition for $\mu(T(\mathsf{A},\mathscr{D})) > 0$ given in Theorem 2.1 states that certain sums of roots of unity must be zero. Thus these questions are closely tied to the structure of the semigroup of nonnegative integer relations among the $m^k$th roots of unity, where $m = |\det(\mathsf{A})|$. The structure of these semigroups is complicated for general $m$, but when $m$ is a prime power, these semigroups have a simple structure; see Lemma 4.1 below. We apply this lemma to characterize those expanding matrices $\mathsf{A}$ with $\det(\mathsf{A}) = p$ which have the property that all digit sets with $\mu(T(\mathsf{A},\mathscr{D})) > 0$ are standard digit sets.

We study $m$th roots of unity, and use the notation

$$\mathscr{S}_m := \exp\left(\frac{2\pi i}{m}\right).$$

By a *lattice* we mean a discrete additive subgroup of $\mathbb{R}^n$, and its *rank* is the dimension of the $\mathbb{R}$-vector space it spans. Let $\Lambda_m$ denote the lattice of integer relations among the $m$th roots of unity, that is,

$$\Lambda_m := \left\{(a_1, \ldots, a_m) \in \mathbb{Z}^m : \sum_{k=0}^{m-1} a_k \mathscr{S}_m^k = 0\right\}. \tag{4.1}$$

Let $\mathbb{Z}_+^m$ denote the set of nonnegative vectors in $\mathbb{Z}^m$, and let $\Lambda_m^+$ denote the semigroup of nonnegative integer relations among the roots of unity, that is,

$$\Lambda_m^+ = \Lambda_m \cap \mathbb{Z}_+^m. \tag{4.2}$$

The fact that the field $\mathbb{Q}(\mathscr{S}_m)$ has degree $\phi(m)$ over $\mathbb{Q}$, where $\phi(m)$ is Euler's totient function, implies immediately that

$$\operatorname{rank}(\Lambda_m) := m - \phi(m). \tag{4.3}$$

In particular, for a prime $p$, $\operatorname{rank}(\Lambda_p) = 1$ with generator $\sum_{k=0}^{p-1} \mathscr{S}_p^k = 0$. de Bruijn [6] and Schoenberg [26] found a (redundant) set of generators for $\Lambda_m$, namely the set

$$(\mathscr{S}_m)^j \left(\sum_{k=0}^{p-1} \mathscr{S}_p^k\right) = 0, \quad \text{for } 0 \leqslant j < \frac{m}{p},$$

in which $p$ runs over all primes dividing $m$. Mann [23] re-proved this result, and also proved that $\Lambda_m^+$ is a finitely generated semigroup. Certainly, $\Lambda_m^+$ requires at least $\operatorname{rank}(\Lambda_m)$ generators, and we now show that equality holds when $m$ is a prime power.

LEMMA 4.1. *Let $q = p^n$ be a power of a prime $p$. Then* $\operatorname{rank}(\Lambda_q) = p^{n-1}$, *and the semigroup $\Lambda_q^+$ is generated by the $p^{n-1}$ integer relations*

$$(\mathscr{S}_{p^n})^j \left(\sum_{k=0}^{p-1} \mathscr{S}_p^k\right) = 0, \quad \textit{for } 0 \leqslant j \leqslant p^{n-1} - 1. \tag{4.4}$$

*Proof.* The theorem of Schoenberg shows that every element in $\Lambda_q$ is a (positive or negative) integer combination of the relations (4.4), which span $\Lambda_q$. However, all these relations (4.4) are in $\Lambda_q^+$. Let $\mathbf{v}_j \in \mathbb{Z}_+^q$ denote the vectors associated to (4.4) for $0 \leqslant j < p^{n-1}$. The *support* of a vector $\mathbf{v}$ is the set of coordinates where it does not vanish. Now all $\mathbf{v}_i$ above have disjoint supports. Thus if $\mathbf{w} = \sum n_j \mathbf{v}_j$ for $n_j \in \mathbb{Z}$ and some $n_j < 0$, then $\mathbf{w}$ has some negative coordinate so $\mathbf{w} \notin \Lambda_q^+$. Thus $\Lambda_q^+ = \mathbb{Z}^+[\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{p^n-1}]$, completing the proof.

The 'disjoint support' property used in the proof of this lemma fails when $m$ is not a prime power, and when $m$ is not a prime power $\Lambda_m^+$ always requires more than $m - \phi(m)$ generators.

We apply this result to show that nonstandard digit sets do not exist for many $\mathsf{A}$ such that $|\det(\mathsf{A})| = p$ is prime.

THEOREM 4.1. *Let $\mathsf{A} \in M_n(\mathbb{Z})$ be an expanding integer matrix such that $|\det(\mathsf{A})| = p$ is prime, and suppose that*

$$p\mathbb{Z}^n \not\subseteq \mathsf{A}^2(\mathbb{Z}^n). \tag{4.5}$$

*If $\mathscr{D} \subseteq \mathbb{Z}^n$ is a digit set with $|\mathscr{D}| = p$, then $\mu(T(\mathsf{A}, \mathscr{D})) > 0$ if and only if $\mathscr{D}$ is a standard digit set.*

The condition (4.5) is always satisfied in the one-dimensional case, where $\mathsf{A} = \pm p$. This case was previously settled by Kenyon [**16**, p. 262]. It may well be true that the conclusion of Theorem 4.1 holds for all expanding $\mathsf{A}$ with $|\det(\mathsf{A})| = p$.

The condition (4.5) arises in connection with the adjoint matrix $\mathsf{A}^{\dagger} \in M_n(\mathbb{Z})$, defined by $\mathsf{A}^{\dagger} := p\mathsf{A}^{-1}$, and is equivalent to the condition

$$\mathsf{A}^{\dagger}(\mathbb{Z}^n) \nsubseteq \mathsf{A}(\mathbb{Z}^n), \tag{4.6}$$

as may be seen by multiplying both sides of (4.5) by $\mathsf{A}^{-1}$. Note that (4.5) is invariant under multiplication by $\mathsf{Q} \in GL(n, \mathbb{Z})$ since $\mathsf{Q}(\mathbb{Z}^n) = \mathbb{Z}^n$, hence it depends only on the Hermite normal form of $\mathsf{A}^2$. In fact, $p\mathbb{Z}^n \subseteq \mathsf{A}^2(\mathbb{Z}^n)$ holds if and only if the (lower-triangular) Hermite normal form $\mathsf{H} = [h_{ij}]$ of $\mathsf{A}^2$ has some $i < j$ with $h_{ii} = h_{jj} = p$ and $h_{ij} = 0$, in which case $h_{kk} = 1$ for $k \neq i, j$.

*Proof.* Suppose that $\mu(T(\mathsf{A}, \mathscr{D})) > 0$. Without loss of generality, we can replace any pair $(\mathsf{A}, \mathscr{D})$ with $(\tilde{\mathsf{A}}, \tilde{\mathscr{D}})$ satisfying $\mathbb{Z}[\tilde{\mathsf{A}}, \tilde{\mathscr{D}}] = \mathbb{Z}^n$ as at the beginning of Section 2, such that $T(\mathsf{A}, \mathscr{D}) = \mathsf{B}(T(\tilde{\mathsf{A}}, \tilde{\mathscr{D}}))$, and $\tilde{\mathsf{A}} = \mathsf{B}^{-1}\mathsf{A}\mathsf{B} \in M_n(\mathbb{Z})$. We still have $|\det(\tilde{\mathsf{A}})| = p$. It now suffices to prove that $\tilde{\mathscr{D}}$ is a complete set of coset representatives of $\mathbb{Z}^n/\tilde{\mathsf{A}}(\mathbb{Z}^n)$, because the converse assertion that $\mu(T(\tilde{\mathsf{A}}, \tilde{\mathscr{D}})) > 0$ for any complete set of coset representatives $\tilde{\mathscr{D}}$ is a theorem of Bandt [**1**], and also follows via Theorem 2.1(iii).

Thus we suppose $\mathbb{Z}[\mathsf{A}, \mathscr{D}] = \mathbb{Z}^n$, and without loss of generality we further obtain $\mathbf{0} \in \mathscr{D}$, by a translation. We use Theorem 2.1(iii), which asserts that for each $\mathbf{m} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$, there is some $k = k(\mathbf{m}) \geqslant 1$ with $\hat{h}_k(\mathbf{m}) = 0$, where

$$\hat{h}_k(\mathbf{m}) := \sum_{\mathbf{d} \in \mathscr{D}} \exp(2\pi i \langle \mathsf{A}^{-k}\mathbf{d}, \mathbf{m} \rangle).$$

Now the adjoint matrix

$$\mathsf{A}^{\dagger} := p\mathsf{A}^{-1}$$

lies in $M_n(\mathbb{Z})$, hence

$$\hat{h}_k(\mathbf{m}) = \sum_{\mathbf{d} \in \mathscr{D}} \exp\left(\frac{2\pi i}{p^k} \langle (\mathsf{A}^{\dagger})^k \mathbf{d}, \mathbf{m} \rangle\right) = 0 \tag{4.7}$$

is a sum of $p^k$th roots of unity. Applying Lemma 4.1 shows that the right-hand side of (4.7) must have the form

$$(\mathscr{S}_{p^k})^a \left( \sum_{i=0}^{p-1} \mathscr{S}_p^i \right) = 0 \tag{4.8}$$

for some $0 \leqslant a < p^{k-1}$. However, $\mathbf{0} \in \mathscr{D}$ implies that one term in (4.7) is 1, and this forces $a = 0$, that is, all terms in (4.7) must actually be distinct $p$th roots of unity. In particular, for $k = k(\mathbf{m})$, we must have

$$p^{k-1} \parallel \langle (\mathsf{A}^{\dagger})^k \mathbf{d}, \mathbf{m} \rangle, \quad \text{all } \mathbf{d} \in \mathscr{D} \backslash \{\mathbf{0}\}, \tag{4.9}$$

where $p^{k-1} \parallel m$ means $p^{k-1} \mid m$ and $p^k \nmid m$.

Now suppose that there is some $\mathbf{m} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ with $k(\mathbf{m}) = 1$. Then $\{\langle \mathsf{A}^{\dagger}\mathbf{d}, \mathbf{m} \rangle : \mathbf{d} \in \mathscr{D}\}$ must give $p$ distinct residue classes $(\text{mod}\, p)$. If, however, two digits $\mathbf{d}_1 \equiv \mathbf{d}_2$ $(\text{mod}\, \mathsf{A}(\mathbb{Z}^n))$, then

$$\langle \mathsf{A}^{\dagger}(\mathbf{d}_1 - \mathbf{d}_2), \mathbf{m} \rangle \in \langle \mathsf{A}^{\dagger}(\mathsf{A}(\mathbb{Z}^n)), \mathbb{Z}^n \rangle = \langle p\mathbb{Z}^n, \mathbb{Z}^n \rangle = p\mathbb{Z},$$

a contradiction. Thus $\mathscr{D}$ is a complete set of residues of $\mathbb{Z}^n/\mathsf{A}(\mathbb{Z}^n)$ in this case.

To complete the proof, we show there always exists $\mathbf{m} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ with $k(\mathbf{m}) = 1$. Choose any $\mathbf{m} \in \mathbb{Z}^n \backslash \mathsf{A}^T(\mathbb{Z}^n)$. Then $(\mathsf{A}^\dagger)^T\mathbf{m} \in (\mathsf{A}^\dagger)^T(\mathbb{Z}^n) \backslash p\mathbb{Z}^n$, hence there exists some $\mathbf{w} \in \mathbb{Z}^n$ with

$$\langle (\mathsf{A}^\dagger)\mathbf{w}, \mathbf{m}\rangle = \mathbf{w}^T(\mathsf{A}^\dagger)^T\mathbf{m} \not\equiv 0 \pmod{p}. \tag{4.10}$$

Since $\mathbb{Z}[\mathsf{A}, \mathscr{D}] = \mathbb{Z}^n$ and $0 \in \mathscr{D}$, we may write

$$\mathbf{w} = \sum_{k=0}^{L} \left\{ \sum_{\mathbf{d} \in \mathscr{D}} n_k(\mathbf{d})\, \mathsf{A}^k \mathbf{d} \right\}, \quad \text{for some } L \geqslant 0,$$

in which $\{n_k(\mathbf{d})\}$ are integers. Substituting this in (4.10) and using $\mathsf{A}^T(\mathsf{A}^\dagger)^T = p\mathsf{I}$ yields

$$\sum_{\mathbf{d} \in \mathscr{D}} n_0(\mathbf{d}) \langle \mathsf{A}^\dagger \mathbf{d}, \mathbf{m}\rangle \not\equiv 0 \pmod{p}.$$

Thus there must exist $\mathbf{d} \in \mathscr{D}$ with

$$p \nmid \langle \mathsf{A}^\dagger \mathbf{d}, \mathbf{m}\rangle. \tag{4.11}$$

Now we suppose that (4.5) holds, in the equivalent form $\mathsf{A}^\dagger(\mathbb{Z}^N) \nsubseteq \mathsf{A}(\mathbb{Z}^n)$ given by (4.6), and we show that this implies

$$p \nmid \langle (\mathsf{A}^\dagger)^k \mathbf{d}, \mathbf{m}\rangle, \quad \text{all}\, k \geqslant 1. \tag{4.12}$$

The group $\mathbb{Z}^n/\mathsf{A}(\mathbb{Z}^n)$ is cyclic of order $p$, and $\mathbf{d}$ generates this group, for if $\mathbf{d} \in \mathsf{A}(\mathbb{Z}^n)$ then $\mathsf{A}^\dagger\mathbf{d} \in \mathsf{A}^\dagger\mathsf{A}(\mathbb{Z}^n) = p\mathbb{Z}^n$, which contradicts (4.11). Thus $\{j\mathbf{d} : 0 \leqslant j < p\}$ is a complete set of coset representatives of $\mathbb{Z}^n/\mathsf{A}(\mathbb{Z}^n)$, hence each $\mathbf{w} \in \mathbb{Z}^n$ has a unique expansion

$$\mathbf{w} = \sum_{k=0}^{L} j_k \mathsf{A}^k \mathbf{d}, \quad 0 \leqslant j_k < p. \tag{4.13}$$

Now write

$$\mathsf{A}^\dagger \mathbf{d} \equiv j\mathbf{d} \ (\mathrm{mod}\ \mathsf{A}(\mathbb{Z}^n)), \quad 0 \leqslant j < p. \tag{4.14}$$

The condition (4.6) implies that $j \neq 0$, for if $\mathsf{A}^\dagger\mathbf{d} \in \mathsf{A}(\mathbb{Z}^n)$, then applying $\mathsf{A}^\dagger$ to both sides of (4.13) would give $\mathsf{A}^\dagger\mathbf{w} \in \mathsf{A}(\mathbb{Z}^n)$ for all $\mathbf{w} \in \mathbb{Z}^n$, which contradicts (4.6). Now (4.14) gives

$$(\mathsf{A}^\dagger)^k \mathbf{d} \equiv j^k \mathbf{d} \ (\mathrm{mod}\ \mathsf{A}(\mathbb{Z}^n)),$$

hence

$$\begin{aligned} \langle (\mathsf{A}^\dagger)^k \mathbf{d}, \mathbf{m}\rangle &= \langle \mathsf{A}^\dagger(j^{k-1}\mathbf{d} + \mathsf{A}\mathbf{w}_k), \mathbf{m}\rangle \\ &\equiv j^{k-1} \langle \mathsf{A}^\dagger \mathbf{d}, \mathbf{m}\rangle \pmod{p} \\ &\not\equiv 0 \pmod{p}, \end{aligned}$$

which proves (4.12).

Comparing (4.12) with (4.9) shows that $k(\mathbf{m}) = 1$, and this completes the proof.

## 5. *Classification of digit sets: one-dimensional case*

Even in the one-dimensional case $\mathsf{A} = [b]$, a classification of all digit sets $\mathscr{D}$ with $\mu(T(b, \mathscr{D})) > 0$ appears complicated. In this section we give a complete classification when $b = p^l$ is a prime power. In particular, there exist nonstandard digit sets which are not of product-form. At the end of the section, we show that results of Odlyzko **[25]** suffice to classify all one-dimensional digit sets $\mathscr{D}$ such that $b \geqslant 2$ and $T(b, \mathscr{D})$ is a finite union of intervals.

THEOREM 5.1. *Let $p^l$ be a prime power, and suppose that $\mathscr{D} \subseteq \mathbb{Z}$ has $|\mathscr{D}| = p^l$. The following are equivalent.*

(1) *The measure $\mu(T(p^l, \mathscr{D})) > 0$.*

(2) *There are exponents $1 \leqslant e(1) < e(2) < \ldots < e(l)$ such that $\{e(1), \ldots, e(l)\}$ is a complete set of residue classes* $(\bmod\, l)$, *and $\mathscr{D} = \{d(j_1, \ldots, j_l)\colon 0 \leqslant j_k \leqslant p-1\}$ with*

$$d(j_1, \ldots, j_l) = c_0 + \sum_{k=1}^{l} \{j_k p^{e(k-1)} + c(j_1, j_2, \ldots, j_k) p^{e(k)}\}, \tag{5.1}$$

*where $e(0) = 0$ and $c_0$ and all $c(j_1, \ldots, j_k)$ are integers satisfying*

$$0 \leqslant c_0 < p^{e(1)-1}, \tag{5.2a}$$

$$0 \leqslant c(j_1, \ldots, j_k) < p^{e(k+1)-e(k)-1}, \quad 1 \leqslant k \leqslant l-1, \tag{5.2b}$$

*and the $c(j_1, \ldots, j_l)$ are unrestricted integers.*

*Proof.* We use the Fourier characterization (iii) of Theorem 2.1, which involves the quantities

$$\hat{h}_k(m) := \frac{1}{p^l}\left(\sum_{d \in \mathscr{D}} \exp\left(\frac{2\pi i m d}{p^{lk}}\right)\right). \tag{5.3}$$

Theorem 2.1(iii) says that $\mu(T(p^l, \mathscr{D})) > 0$ if and only if for each $m \in \mathbb{Z}\backslash\{0\}$, there is some $k = k(m) \geqslant 1$ with $\hat{h}_k(m) = 0$.

(1) $\Rightarrow$ (2). We have $\mu(T(p^l, \mathscr{D})) > 0$, so taking $m = p^{i-1}$ in (5.3) for $1 \leqslant i \leqslant l$, there exists a minimal positive integer $b(i)$ with

$$\hat{h}_{b(i)}(p^{i-1}) = 0, \quad 1 \leqslant i \leqslant l. \tag{5.4}$$

According to (5.3), the left-hand side of (5.4) is a sum of $p^{g(i)}$th roots of unity, where

$$g(i) := lb(i) - i + 1 \equiv 1 - i \pmod{l}. \tag{5.5}$$

Let $\{e(i)\colon 1 \leqslant i \leqslant n\}$ denote the $g(i)$ permuted to be in increasing order, and write $e(i) = g(\sigma(i))$. Now (5.5) shows that the $\{e(i)\}$ form a complete residue system $(\bmod\, l)$, hence we may write

$$1 \leqslant e(1) < e(2) < \ldots < e(l),$$

all terms being distinct. Also, (5.5) shows that one can recover $\sigma(i)$ from $e(i)$ via

$$i \equiv -\sigma(i) + 1 \pmod{l}. \tag{5.6}$$

Thus one recovers the exponents $\{b(i)\}$ from $\{e(i)\}$ via

$$b(\sigma(i)) = \frac{1}{l}(e(i) + \sigma(i) - 1). \tag{5.7}$$

At this point we apply Lemma 4.1, which implies that each relation $p^l \hat{h}_{b(i)}(p^{i-1}) = 0$, which itself consists of a sum of $p^{g(i)}$th roots of unity, must additively split into $p^{l-1}$ sums, each of which separately add to zero, and each of which contains exactly $p$ elements. Furthermore, each such sum has the form

$$(\mathscr{S}_{p^{g(i)}})^j \left(\sum_{k=0}^{p-1} \mathscr{S}_p^k\right) = 0. \tag{5.8}$$

We now consider the relations (5.4) successively in order of *decreasing* exponent $e(i)$, using (5.7), so we begin with the additive splitting for $i = \sigma(l)$, where $p^l \hat{h}_{b(\sigma(l))}(p^{\sigma(l)-1})$

is a sum of $p^{e(l)}$th roots of unity. The set $\mathscr{D}$ must therefore divide up into $p^{l-1}$ groups of $p$ elements each, corresponding to the relations (5.8). We index the groups by $0 \leqslant J < p^{l-1}$, with

$$J = j_1 + j_2 p + \ldots + j_{l-1} p^{l-2}, \quad 0 \leqslant j_i < p,$$

and the $p$ elements in each group are indexed by $0 \leqslant j_l < p$. The group relation (5.8) forces the elements of $\mathscr{D}$ to have the form

$$d(J, j_l) = a_1(J) + j_l p^{e(l)-1} + c(j_1, \ldots, j_l) p^{e(l)}, \tag{5.9}$$

in which

$$0 \leqslant a_1(J) < p^{e(l)-1} \tag{5.10}$$

is constant on the $J$th group, and the integers $c(j_1, j_2, \ldots, j_l)$ are arbitrary. (Here the exponent $j$ in (5.8) corresponds to $j_l$, while the index $k$ corresponds to $a_1(J)$.)

Next we consider the additive splitting for the sum of $p^{e(l-1)}$th roots of unity in $p^l \hat{h}_{b(\sigma(l-1))}(p^{\sigma(l-1)-1})$. These roots of unity are *completely determined* by the coefficients $a_1(J)$ in (5.9), since $e(l-1) \leqslant e(l)-1$. Thus the additive splitting of $\hat{h}_{b(\sigma(l-1))}(\cdot)$ says that the $p^{l-1}$ coefficients $a_1(J)$ split up onto $p^{l-2}$ groups (5.8) of $p$ elements each. We index the groups by $0 \leqslant J' < p^{l-2}$ with

$$J' = j_1 + j_2 p + \ldots + j_{l-2} p^{l-3}, \quad 0 \leqslant j_i < p,$$

and index the $p$ elements in each group by $0 \leqslant j_{l-1} < p$. The group relations (5.8) force

$$a_1(J) = a_2(J') + j_{l-1} p^{e(l-1)-1} + c(j_1, \ldots, j_{l-2}) p^{e(l-1)}, \tag{5.11}$$

in which

$$0 \leqslant a_2(J') < p^{e(l-1)} \tag{5.12}$$

and

$$0 \leqslant c(j_1, \ldots, j_{l-1}) < p^{e(l)-e(l-1)-1}, \tag{5.13}$$

the last bound being forced by (5.10). Now we proceed by induction on decreasing $e(j)$ to obtain the decomposition (5.1) with side conditions (5.2). This proves (2).

(2) $\Rightarrow$ (1). We check the criterion of Theorem 2.1(iii). For each $m \in \mathbb{Z} \backslash \{0\}$, we must produce some $\hat{h}_k(m) = 0$. Condition (2) guarantees that

$$\hat{h}_{b(i)}(p^{i-1}) = 0, \quad 1 \leqslant i < l,$$

where $b(i)$ is computed from $e(i)$ via (5.6) and (5.7). Furthermore, if $(\tilde{m}, p) = 1$, then

$$\hat{h}_{b(i)}(p^{i-1}\tilde{m}) = 0, \tag{5.14}$$

since the map $\mathscr{S}_{p^g} \rightarrow (\mathscr{S}_{p^g})^{\tilde{m}}$ is an automorphism of the field of $p^g$th roots of unity, and 0 goes to 0 under any automorphism. Finally, one has the identity

$$\hat{h}_k(m) \equiv \hat{h}_{k+j}(p^{jl} m),$$

valid for all $m \in \mathbb{Z}$, by inspection of (5.3). Thus (5.14) gives

$$\hat{h}_{b(i)+j}(p^{jl+i-1}\tilde{m}) = 0, \quad 1 \leqslant i \leqslant l, \ j \geqslant 0. \tag{5.15}$$

However, every $m \in \mathbb{Z} \backslash \{0\}$ is of the form $m = p^{jl+i-1}\tilde{m}$ with $(p, \tilde{m}) = 1$, $1 \leqslant i \leqslant l$, $j \geqslant 0$, so the criterion of Theorem 2.1(iii) is satisfied.

The classification of Theorem 5.1 does not make it clear which digit sets are primitive. Recall that $\mathscr{D}$ is primitive if $\mathbb{Z}[\mathsf{A}, \mathscr{D}] = \mathbb{Z}$, which is equivalent to

$$\gcd(d_i - d_j : i \neq j) = 1,$$

using (2.1). A necessary condition for primitivity is that $e(1) = 1$, and a sufficient condition is that $0, 1 \in \mathscr{D}$.

As a nontrivial example of the classification of Theorem 5.1, take $q = 4$ and $\mathscr{D} = \{0, 1, 8, 25\}$, which has $b(1) = 2$, $b(2) = 1$, so that $e(1) = 2b(2) - 1 = 1$ and $e(2) = 2b(1) = 4$. It is easily checked that $\mathscr{D}$ is not of product-form (1.11), nor is it a translate of any product-form digit set.

Odlyzko [25] classifies all one-dimensional nonnegative digit sets with $A = [b]$ for $b \geqslant 2$ whose associated tile $T(b, \mathscr{D})$ tiles $\mathbb{R}^+$ by repeated inflation by $b$. Call a digit set $\mathscr{D} \subseteq \mathbb{Z}$ of *strict product-form* if it is of product-form (1.11) with the extra condition that $\mathscr{E} = \{0, 1, 2, \ldots, b-1\}$. Odlyzko shows that $T(b, \mathscr{D})$ tiles $\mathbb{R}^+$ by repeated inflation by $b$ if and only if $\mathscr{D} = j\mathscr{D}'$, where $j \geqslant 1$ is an integer and $\mathscr{D}'$ is of strict product-form. His results have a geometric reformulation, as follows.

THEOREM 5.2. *Suppose that $\mathscr{D} \subseteq \mathbb{Z}$ is a set of $b \geqslant 2$ nonnegative integers, and that $0 \in \mathscr{D}$. The following are equivalent.*

(1) *$T(b, \mathscr{D})$ is a finite union of intervals.*

(2) *$\mathscr{D} = j\mathscr{D}'$, where $j \geqslant 1$ is an integer and $\mathscr{D}'$ is a strict product-form digit set.*

*Proof.* (1) $\Rightarrow$ (2). If $T(b, \mathscr{D})$ is a finite union of intervals, then since $0 \in T$, it includes some interval $[0, \delta]$. Then $T(b, \mathscr{D})$ tiles $\mathbb{R}^+$ by inflation, that is, every nonnegative real number $\theta$ has at least one expansion

$$\theta = \sum_{j=-k}^{\infty} d_j b^{-j}.$$

Now (2) follows by the main theorem of [25].

(2) $\Rightarrow$ (1). The proof of [25, Lemma 5] shows that for all strict product-form digit sets, $T(b, \mathscr{D}')$ is a finite union of intervals.

COROLLARY 5.2a. *For $b \geqslant 2$, a one-dimensional self-affine tile $T(b, \mathscr{D})$ is a finite union of intervals if and only if there are integers $j_1 \geqslant 1$ and $j_2$ such that $\mathscr{D} = j_1 \mathscr{D}' + j_2$ where $\mathscr{D}'$ is a strict product-form digit set.*

In particular, by the earlier remarks, the attractor $T(4, \mathscr{D})$ for $\mathscr{D} = \{0, 1, 8, 25\}$ must consist of an infinite number of intervals.

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