Springer INdAM Series 23

Simon G. Chiossi
Anna Fino
Emilio Musso
Fabio Podestà
Luigi Vezzoni *Editors*

Special Metrics and Group Actions in Geometry

Springer

Springer INdAM Series

Volume 23

More information about this series at http://www.springer.com/series/10283

Simon G. Chiossi • Anna Fino • Emilio Musso •
Fabio Podestà • Luigi Vezzoni
Editors

Special Metrics and Group Actions in Geometry

Springer

Editors

Simon G. Chiossi
Departamento de Matemática Aplicada
Universidade Federal Fluminense
Niterói, RJ, Brazil

Anna Fino
Dipartimento di Mathematica "G. Peano"
Università di Torino
Torino, Italy

Emilio Musso
Dipartimento di Scienze Matematiche
Politecnico di Torino
Torino, Italy

Fabio Podestà
Dipartimento di Matematica e Informatica
Università di Firenze
Firenze, Italy

Luigi Vezzoni
Dipartimento di Mathematica "G. Peano"
Università di Torino
Torino, Italy

ISSN 2281-518X ISSN 2281-5198 (electronic)
Springer INdAM Series
ISBN 978-3-319-67518-3 ISBN 978-3-319-67519-0 (eBook)
https://doi.org/10.1007/978-3-319-67519-0

Library of Congress Control Number: 2017958448

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This volume is a follow-up to the INdAM workshop "New perspectives in differential geometry" that took place on 16–20 November 2015:

https://newperspectivesindg.wordpress.com

The editors are deeply grateful to the Istituto Nazionale di Alta Matematica "Francesco Severi" (INdAM) for generously sponsoring and hosting the event in Rome, and to the Institute's President, Giorgio Patrizio, for his steadfast support.

We are indebted to Diego Conti and Caterina Stoppato for the invaluable hard work in organising the workshop, and to Graziano Gentili and Andrew Swann for playing a key role on the scientific committee. We would also like to thank wholeheartedly the conference speakers: Ilka Agricola, Adrián Andrada, Vestislav Apostolov, John Armstrong, Fiammetta Battaglia, Roger Bielawski, Fran Burstall, Vicente Cortés, Johann Davidov, Paul Gauduchon, Marisa Fernández, Nigel Hitchin, Claude LeBrun, Thomas Madsen, Stefano Marchiafava, Vicente Muñoz, Paolo Piccinni, Uwe Semmelmann and Luis Ugarte.

On the occasion of the workshop we celebrated the 60th birthday of Simon Salamon, a worldwide leading scholar at the forefront of research in differential geometry whose extensive body of work centres around Riemannian and complex manifolds defined with reference to the action of a Lie group. The unique contributions appearing in this book focus on a variety of cutting-edge topics revolving around Salamon's interests: quaternionic and octonionic geometry, twistor spaces, almost-complex manifolds, harmonic maps, exceptional holonomy, Einstein metrics, spinors, homogeneous spaces and nilmanifolds, special geometries in dimensions 5, 6, 7 and 8, conformal geometry, moduli spaces, gauge theory, 4-manifolds, symplectic manifolds and integrable systems.

Simon Salamon is Professor of Geometry at King's College, London, and previously worked at Politecnico di Torino, Imperial College and Oxford University. The workshop was widely attended by his colleagues, friends and former students from all over the world, and this volume represents both a fitting tribute to a trailblazing force in the field and a compelling testimony to the profound and longstanding impact that Salamon has on the mathematical community.

Heartfelt thanks go to the authors who accepted the invitation to publish here: Fiammetta Battaglia, Giovanni Bazzoni, Indranil Biswas, Fran Burstall, Vicente Cortés, Andrew Dancer, Johann Davidov, Malte Dyckmanns, Marisa Fernández, Paul Gauduchon, Claude LeBrun, Andrea Loi, Jason Lotay, Thomas Madsen, Andrei Moroianu, Vicente Muñoz, Antonio Otal, Paolo Piccinni, Simon Salamon, Stefan Suhr, Andrew Swann, Aleksy Tralle, Luis Ugarte, Raquel Villacampa, Dan Zaffran and Fabio Zuddas.

We strongly believe these papers will be extremely relevant to the advancement of academic research and are certain they will serve generations to come.

Niterói, RJ, Brazil	Simon G. Chiossi
Torino, Italy	Anna Fino
Torino, Italy	Emilio Musso
Firenze, Italy	Fabio Podestà
Torino, Italy	Luigi Vezzoni

Contents

About the Editors

Prof. Simon Chiossi is a lecturer at Universidade Federal Fluminense, and previously held posts in Odense, Berlin, Torino, Marburg and Salvador. He was awarded a Ph.D. in mathematics from the University of Genoa in 2003, and his scholarly publications focus on special geometry in dimensions 4 to 8.

Prof. Anna Fino is currently a full professor at the University of Torino, where she also received her Ph.D. in Mathematics. Her research work mainly focuses on differential geometry, complex geometry, Lie groups, more specifically, Hermitian geometry, G-structures and special holonomy, and geometric flows. She has supervised three doctoral theses and she is author of 72 papers.

Prof. Emilio Musso obtained his Ph.D. in mathematics at the Washington University in St. Louisin 1987. He taught at the Universities of Florence, L'Aquila and Rome in Italy. Currently he is a professor of mathematics at the Politecnico di Torino. He has published 60 papers and 1 book on several topics in differential geometry. His research interests are in geometrical variational problems, exterior differential systems and in the interrelations between geometry, physics and integrable systems.

Prof. Fabio Podestà studied mathematics at the University of Pisa and at the Scuola Normale Superiore, where he attended the Corso di Perfezionamento in Mathematics. He is currently a full professor at the University of Florence. His research activity in the field of differential geometry mainly concerns Lie group actions preserving geometric structures. He is author of more than 50 published papers.

Prof. Luigi Vezzoni graduated in mathematics at the University of Florence in 2003, and received his Ph.D. in mathematics at the University of Pisa in 2007. He is currently an associate professor at the University of Turin. He is author of more than 40 papers in international journals and he was the main speaker at a number of international conferences including conferences in Brazil, Japan, China, Luxembourg, Germany and Bulgaria. He has also supervised several master's theses and he is currently supervising a Ph.D. thesis. His current research interests include complex geometry, special geometric structures on smooth manifolds, geometric flows and geometric analysis.

Simplicial Toric Varieties as Leaf Spaces

Fiammetta Battaglia and Dan Zaffran

Abstract We present a summary of some results from our article (Battaglia and Zaffran, Int. Math. Res. Not. IMRN 2015 no. 22 (2015), 11785–11815) and other recent results on the so-called LVMB manifolds. We emphasize some features by taking a different point of view. We present a simple variant of the Delzant construction, in which the group that is used to perform the symplectic reduction can be chosen of arbitrarily high dimension, and is always connected.

Keywords Delzant construction • Holomorphic foliations • Nonrational fans • Quasilattices • Symplectic reduction • Toric varieties

1 Introduction

At the workshop held in Rome on November 16–20, 2015, dedicated to Simon Salamon for his 60th birthday, the first author presented a joint article with Dan Zaffran [9]. In this note we make a summary of various results, including previous and subsequent literature, concerning the relation between a special class of compact foliated complex manifolds, called LVMB manifolds, toric geometry, and convex geometry; we will simultaneously treat the rational and nonrational cases. In particular we will dwell on some aspects of the article [9] that have been left aside in the published version or that were not developed at the time. Among the last, a variant of the Delzant procedure.

We are interested in the relationship between three different classes of objects; we first describe each of them briefly.

LVM manifolds originated in the context of dynamical systems. They form a large class of non-Kähler, compact, complex manifolds, introduced between 1997

F. Battaglia (✉)
Dipartimento di Matematica e Informatica U. Dini, Università di Firenze, Via S. Marta 3, 50139 Firenze, Italy
e-mail: fiammetta.battaglia@unifi.it

D. Zaffran
College of Marin, 835 College Ave, Kentfield, CA 94904, USA
e-mail: dan.zaffran@gmail.com

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_1

and 2001 in works by Lopez de Medrano, Verjovsky, and Meersseman [29, 30]. Their construction was generalized by Bosio [13], who obtained a larger class, which we will refer to as LVMB manifolds. The classical starting datum for the construction of an LVMB manifold is a set of holomorphic vector fields, inducing a $\mathbb{C}^m$-action on $\mathbb{C}^n$, together with a choice of a saturated open subset of $\mathbb{C}^n$ for which the space of orbits is a compact complex manifold, interestingly non Kähler. Each LVMB manifold N is also endowed with a holomorphic foliation $\mathcal{F}$, as shown in [16, 17] for LVM manifolds and in [7] for LVMB manifolds.

The theory of toric varieties is by now classical. Simplicial toric varieties are algebraic manifolds with at most finite quotient singularities. They can be seen as compactifications of a torus $(\mathbb{C}^*)^n$. There are several reference texts for toric varieties theory, among them [15, 20, 33]. The standard starting datum for the construction of a compact simplicial toric variety is a complete simplicial rational fan.

A fan is a set of convex polyhedral cones having certain properties. Recall that a cone in a vector space is the set of nonnegative linear combinations of a finite number of vectors, that generate the cone. Each vector generates a nonnegative half line, called a ray of the cone. A cone is simplicial if it admits a set of linearly independent generators. Let L be a lattice in the vector space $L \otimes_{\mathbb{Z}} \mathbb{R}$. A cone in $L \otimes_{\mathbb{Z}} \mathbb{R}$ is rational if each of its rays has nonempty intersection with L. A fan is simplicial if each of its cones is simplicial, and it is rational in $L \otimes_{\mathbb{Z}} \mathbb{R}$ if each of its cones is rational. In this work we will consider fans and other related convex objects.

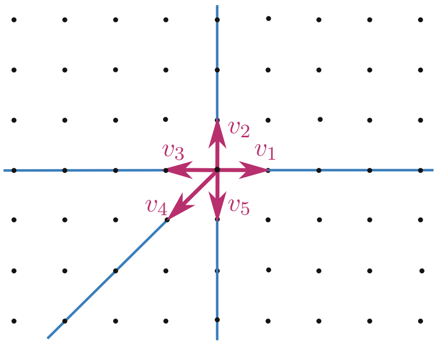
Now, is there a relation between LVMB manifolds and toric varieties? Is there a relation between certain linear $\mathbb{C}^m$-actions on $\mathbb{C}^n$ and fans? What happens when the fan is nonrational? What is a nonrational fan? Is a fan the appropriate convex object? What are the similarities and differences between LVMB manifolds and toric manifolds? In the present article we try to give an answer to these questions.

Recall that a fan in a vector space is complete if the union of its cones is the whole space; it is polytopal if it is the fan normal to a polytope. In [31] Meersseman and Verjovsky establish a precise relationship between LVM manifolds and compact simplicial *projective* toric varieties—that is, varieties associated to complete, simplicial, *polytopal* fans. They prove that the leaf space of an LVM manifold whose starting datum satisfies a further rationality condition—condition (K)—is a simplicial projective toric variety. Conversely, any simplicial projective toric variety can be obtained as leaf space of an LVM manifolds of that kind. In order to prove this last result they use Gale duality combined with symplectic reduction.

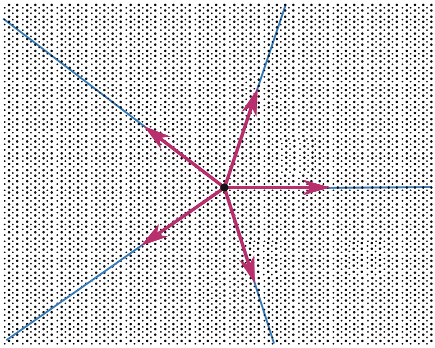
But what happens when the fan is not polytopal or nonrational?

Cupit and Zaffran in [16] establish that the class of LVM manifolds is strictly included in the LVMB family. Battisti further proves in [12] that an LVMB manifold is LVM if and only if the corresponding fan is polytopal.

How can we deal with the nonrational case? In classical toric geometry, when one considers a rational fan in $\mathbb{R}^d$, there are two data that are usually taken for granted: a lattice, and, in the lattice, a set of primitive vectors, each of which a generator of a fan ray. In order to extend this setting to the nonrational case one needs to reconsider these data. Let us illustrate how from our view-point. We introduce a convex object that allows to encode all of these data: a triangulated vector configuration. This is a pair $\{V, \mathcal{T}\}$, where $V = (v_1, \ldots, v_n)$ is a configuration (ordered and allowing repetitions) of vectors in $\mathbb{R}^d$, and $\mathcal{T}$ is a collection of subsets of $\{1, \ldots, n\}$ with suitable properties, called a triangulation of V. Consider for example a rational fan in a lattice L with h rays, and, for each ray, its primitive generator. Then, a corresponding triangulated vector configuration is a (non-unique) pair $\{V, \mathcal{T}\}$ such that: $\mathrm{Span}_{\mathbb{Z}}(V) = L$; the first h vectors in V are the selected generators of the h fan rays; the triangulation $\mathcal{T}$ carries the combinatorial information that determines the subcollections of $\{v_1, \ldots, v_n\}$ that generate all of the fan cones. Notice that $n > h$ may be needed, for example in case the set of primitive ray generators is not a generating set of L. As an example, consider the rational simplicial fan in $\mathbb{R}^2 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R}$ drawn in the picture. Its associated toric variety is $\mathbb{C}P^1 \times \mathbb{C}P^1$ blown up at one point. A corresponding triangulated vector configuration is $\{V, \mathcal{T}\}$, with $V = (\,(1,0), (0,1), (-1,0), (-1,1), (0,-1)\,)$ and $\mathcal{T}$ the triangulation whose maximal simplices are $\{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\}$; here $h = n = 5$.



We may now wonder what is preserved if each of the five rays of the above fan is rotated, so as to obtain a new fan, whose rays divide the plane into five congruent cones. The vectors of V are rotated as well, into new vectors, while we may keep the same triangulation. Thus we obtain a new triangulated vector configuration $\{V', \mathcal{T}\}$, whose vectors are generators of the new fan rays. However, the $\mathbb{Z}$-Span of V' is not a lattice but a $\mathbb{Z}$-module of higher rank, dense in $\mathbb{R}^2$. We could re-scale the vectors of V', for example into five unitary vectors. This would produce a new vector configuration $\{V'', \mathcal{T}\}$. But there is no rescaling such that the $\mathbb{Z}$-Span of V'' is a lattice. In fact the new fan is nonrational, that is, there is no lattice that has non-empty intersection with each of its rays.



Hence, the notion of triangulated vector configuration still makes sense in the nonrational case, but there is no lattice. The idea, due to Prato, is to replace the lattice with a *quasilattice*, that is the $\mathbb{Z}$-Span of a generating set of $\mathbb{R}^n$. Then the convex datum becomes a triple—which we will call Prato's datum—given by: a fan, a choice of rays generators, a choice of a quasilattice containing these generators [36]. Notice that, with such a choice, each ray intersects the quasilattice, however, the notion of primitive vector does not make sense any longer. As shown in the above examples, a Prato's datum can be naturally encoded in a single convex object: a triangulated vector configuration. Notice that there are many triangulated vector configurations that encode a given Prato's datum. The theory of vector configurations, as developed in [17], provides a precise link between triangulated vector configurations, sets of holomorphic vector fields as used in LVMB theory, fans and convex polytopes. In particular Gale duality plays an important role in connecting these different aspects. This view-point was developed in [9]. The vector configurations that we consider must satisfy two further technical conditions—already considered in [31]—they must be balanced and odd: balanced means that the sum of the vectors of V is zero, moreover we require that $n-d$ is odd. This does not mean a loss in generality. For example, let

$$\Big\{(\,(1,0),(0,1),(-1,1),(-1,0),(0,-1)\,),\mathcal{T}\Big\}$$

be the triangulated vector configuration considered above, which is not balanced. Consider the new triangulated vector configuration

$$\Big\{(\,(1,0),(0,1),(-1,1),(-1,0),(0,-1),(1,-1),(0,0)\,),\mathcal{T}\Big\}.$$

The $\mathbb{Z}$-Span of the vectors is still $\mathbb{Z}^2$ but $n = 7$, $h = 5$. The first 5 vectors are the same as before, namely the primitive generators of the fan rays. The new vectors, $v_6 = (1,-1)$ and $v_7 = (0,0)$, will be referred to as ghost vectors. The new triangulated vector configuration is balanced and odd. By Gale duality, we obtain a (non unique) $\mathbb{C}^2$-action on $\mathbb{C}^7$. This action, together with $\mathcal{T}$, in turn gives rise to an LVM manifold $(N, \mathcal{F})$ of complex dimension 4. The leaf space $N/\mathcal{F}$ is biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ blown up at one point, associated with the rational fan drawn at page 3. Consider now the triangulated vector configuration $\{V'', \mathcal{T}\}$, whose five vectors are the five roots of $z^5 = 1$ in $\mathbb{C}$. It is balanced and odd. As above this yields a (non unique) LVM manifold $(N, \mathcal{F})$ of complex dimension 3. The leaf space $N/\mathcal{F}$ is biholomorphic to the toric quasifold of complex dimension 2 described in [4, Example 2.9]. In general, in [9], for each given odd, balanced, triangulated vector configuration $\{V, \mathcal{T}\}$, not necessarily rational or polytopal, we are able to associate, via Gale duality, a (non unique) $\mathbb{C}^m$-action on $\mathbb{C}^n$. This, together with $\mathcal{T}$, in turn determines an LVMB manifold $(N, \mathcal{F})$. However, the complex leaf space $N/\mathcal{F}$, endowed with the quotient topology, only depends on the Prato's datum encoded in the triangulated vector configuration, whilst its cohomology only depends on the combinatorial type of the fan [3, 9].

More precisely, let Δ be the fan associated with $\{V, \mathcal{T}\}$, that is whose rays are generated by the first h vectors of V. When the $\mathbb{Z}$-Span of V is a lattice and the first h vectors, that is, the rays generators, are primitive, the leaf space is the simplicial toric variety associated with the rational fan Δ. This may be either a manifold or may have finite quotient singularities. If some of the first h vectors are not primitive, the leaf space is the toric variety associated with Δ, equipped with an equivariant orbifold structure [31]. The construction of these toric orbifolds was introduced by Lerman-Tolman in [27] in the symplectic set-up; they define the notion of labeled polytope to keep track of the vectors that are integer multiples of primitive rays generators. When the $\mathbb{Z}$-Span of V is a quasilattice, then the leaf space is the complex toric quasifold corresponding to the Prato's datum encoded in $\{V, \mathcal{T}\}$: we are referring here to the construction of toric quasifolds, which was introduced by Prato in the symplectic set up, together with the notion of quasifold, a generalization of orbifold [36]. Complex toric quasifolds were then defined in [4].

Conversely, a toric manifold, orbifold, or quasifold is always the leaf space of an LVMB manifold. This is proved in [31] in the polytopal case, for the toric manifold and orbifold cases. In [9] we extend this result in both directions, namely to nonpolytopal and nonrational fans. This provides also an extension of complex toric quasifolds (and therefore of complex toric orbifolds) to the nonpolytopal case.

We then focus on the polytopal case. Recall that a toric variety is projective if and only if its associated fan is polytopal. In the same spirit, for LVMB manifolds, we have the following result, due to several authors: the foliation in an LVMB manifold is transversely Kähler if and only if the manifold is LVM. Part of the inverse implication was proved by Lœb and Nicolau in [28] and then extended by Meersseman to the general LVM case. The direct implication was conjectured by Cupit and Zaffran in [16] and recently proved by Ishida in [23]. We recall these results and list a series of other characterizations of polytopality. Finally, we present a variant of the Delzant construction that naturally derives from our view point and we extend to the nonrational setting some results by Meerssemann and Verjovsky [31].

2 Preliminaries

2.1 Construction of LVMB-Manifolds

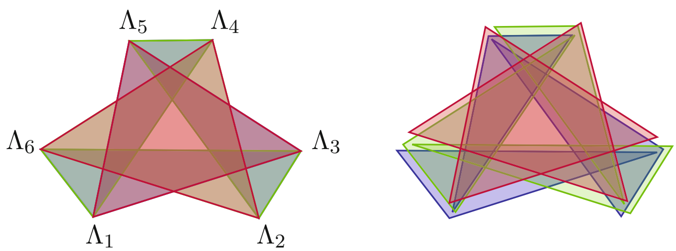
We briefly describe the constructions of LVMB manifolds. Consider a configuration of points $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ in affine space $\mathbb{C}^m$, that is, a finite ordered list. Repetitions are allowed but we assume that Λ is not contained in a proper affine subspace. We define a row vector for each $j = 1, \ldots, n$: $\Lambda_j^{\mathbb{R}} := [\text{—}\mathrm{Re}(\Lambda_j)\text{— —}\mathrm{Im}(\Lambda_j)\text{—}] \in \mathbb{R}^{2m}$. A *basis* is a subset τ^* of $\{1, \cdots, n\}$, of cardinality $2m+1$, such that the interior $\mathring{C}_\alpha$ of the convex hull of $(\Lambda_j^{\mathbb{R}})_{j\in\tau^*}$ is non empty. Now let us pair the configuration Λ with a combinatorial datum, as we do when, for a given vector configuration, we take a triangulation. A *virtual chamber* $\mathcal{T}^*$ of the configuration Λ is a collection of bases $\{\tau_\alpha^*\}_\alpha$ that satisfy Bosio's conditions [13], that is:

(i) $\mathring{C}_\alpha \cap \mathring{C}_\beta \neq \varnothing$ for every α, β;
(ii) for every $\tau_\alpha^* \in \mathcal{T}^*$ and every $i \notin \tau_\alpha^*$,
there exists $j \in \tau^*$ such that $\left(\tau_\alpha^* \setminus \{j\}\right) \cup \{i\} \in \mathcal{T}^*$.

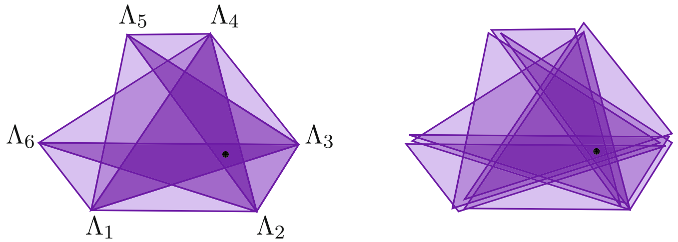
We show in the picture an example with $n = 6$ and

$$\mathcal{T}^* = \Big\{\{135\}, \{246\}, \{136\}, \{235\}, \{145\}, \{146\}, \{236\}, \{245\}\Big\},$$

to facilitate visualization, we add colors and show on the right hand side slight translations of the $\mathring{C}_\alpha$'s:



In general, $\bigcap_\alpha \mathring{C}_\alpha = \varnothing$. We say that a virtual chamber is a *chamber* when all $\mathring{C}_\alpha$'s do have a common intersection. Example with $n = 6$ and $\mathcal{T}^* = \Big\{\{124\}, \{134\}, \{135\}, \{136\}, \{235\}, \{236\}, \{245\}, \{246\}\Big\}$:



We'll see below that the special case when a virtual chamber is a chamber corresponds in the toric context to a fan being polytopal (or a toric variety being projective), and in the context of LVMB theory to a manifold being transversely Kähler.

An *LVMB datum* $\{\Lambda, \mathcal{T}^*\}$ is a configuration $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ in $\mathbb{C}^m$, with $n \geq 2m + 1$, together with a choice of a virtual chamber $\mathcal{T}^*$. We now show how to define a compact complex manifold N from an LVMB datum. From the virtual chamber $\mathcal{T}^*$ we define a subset $U(\mathcal{T}^*)$ of $\mathbb{CP}^{n-1}$ as follows: for each $\tau^* \in \mathcal{T}^*$, define $U_{\tau^*} := \left\{ [z_1 : \cdots : z_n] \in \mathbb{CP}^{n-1} \;\middle|\; \forall j \in \tau^*, z_j \neq 0 \right\}$ and take

$$U(\mathcal{T}^*) := \bigcup_{\tau^* \in \mathcal{T}^*} U_{\tau^*}.$$

We write Λ as the matrix

$$\Lambda = \begin{bmatrix} —\Lambda_1— \\ \vdots \\ —\Lambda_n— \end{bmatrix} \in \mathbb{C}^{n \times m}$$

and define the subspace $\mathfrak{h} \subset \mathbb{C}^n$ as the span of the m *columns* of Λ.

By Bosio's condition (i), $\mathfrak{h}$ has dimension m. Let $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$. The action by $\exp \mathfrak{h} \subset (\mathbb{C}^*)^n$ on $\mathbb{C}^n$, induced by the natural $(\mathbb{C}^*)^n$-action, is a $\mathbb{C}^m$-action that commutes with the diagonal $\mathbb{C}^*$-action. Then [13] the group $\exp \mathfrak{h}$ acts freely and properly on $U(\mathcal{T}^*)$, so we can define

$$N := U(\mathcal{T}^*)/\exp \mathfrak{h}.$$

Manifolds arising from this construction are called *LVMB manifolds* [13, 29, 30]. They have a very rich geometry.

Every LVMB manifold N admits a noteworthy smooth holomorphic foliation $\mathcal{F}$, which has been investigated by several authors [12, 16, 28, 30, 31, 34, 35, 37, 38]. Following [38], we can simply describe the leaves of $\mathcal{F}$ as the orbits of the action of $\exp \overline{\mathfrak{h}}$ on N, which descends from the action on $U(\mathcal{T}^*)$. There is also on N an induced action of the abelian complex group given by the quotient of $(\mathbb{C}^*)^n/\exp(\mathfrak{h})$ by $\mathrm{diag}(\mathbb{C}^*)$. This group has the same complex dimension as N, $n-1-m$, and has a dense open orbit in N. It also contains a real group isomorphic to $(S^1)^n/\mathrm{diag}(S^1)$. With respect to any metric that makes this real group act by isometries, the foliation $\mathcal{F}$ is Riemannian [9, Sect. 2.3.2].

We thus have the following

Theorem 2.1 *An LVMB datum $\{\Lambda, \mathcal{T}^*\}$ determines a compact complex manifold N of dimension $n-1-m$. This manifold is endowed with a holomorphic Riemannian foliation $\mathcal{F}$. Moreover there is an abelian complex group of same dimension acting on N with a dense open orbit. In the limiting case $n = 2m+1$, the manifold N is a compact complex torus, but whenever $n > 2m+1$, N is not Kähler.*

Another entry point of LVMB theory is that of moment-angle manifolds. This point of view was developed by Tambour and Panov-Ustinovsky [34, 37]. The latter authors and Battisti [12] are precursors to the toric methods established by Ishida [23], who also proved a remarkable group-theoretic characterization of LVMB manifolds as a large subclass of the class of complex manifolds that admit a so-called *maximal torus action*. Namely, the action of a compact torus G on a manifold M is called *maximal* when there exists $x \in M$ such that $\dim_{\mathbb{R}} G + \dim_{\mathbb{R}} G_x = \dim_{\mathbb{R}} M$, where G_x is the isotropy subgroup at x. For LVMB manifolds, the torus G is the above-mentioned real group isomorphic to $(S^1)^n/\mathrm{diag}(S^1)$. In [38] Ustinovsky shows that the family of complex manifolds presented in [34] coincides with Ishida's complex manifolds admitting a maximal torus action.

2.2 Duality LVMB-Data $\leftrightarrow$ Fans and (Quasi)Lattices

In this section we illustrate how, via Gale duality, we are able to pass from a set of data to the other. Let $W = (w_1, \dots, w_n)$ denote a vector configuration in some real, finite-dimensional vector space. We define the set of relations of W by

$$\mathrm{Rel}(W) := \left\{ (\alpha_1, \dots, \alpha_n) \;\middle|\; \sum \alpha_j w_j = 0 \right\} \subset \mathbb{R}^n.$$

We say that a vector configuration W is *balanced* when $\sum w_j = 0$; we say that it is *graded* when there exists an affine hyperplane not passing through the origin that contains all vectors of W. A graded configuration of vectors is a useful way to represent and manipulate a configuration of *points*, i.e. elements of an affine space.

Now consider an LVMB datum $\{\Lambda, \mathcal{T}^*\}$. Let A be a $2m$-dimensional affine subspace of $\mathbb{R}^{2m+1}$ not passing through the origin. Send the configuration $\Lambda^{\mathbb{R}} = (\Lambda_1^{\mathbb{R}}, \dots, \Lambda_n^{\mathbb{R}})$, defined in Sect. 2.1, in A via an affine real isomorphism from $\mathbb{R}^{2m}$ to A. This determines a graded vector configuration $W(\Lambda)$ in $\mathbb{R}^{2m+1}$. Consider a $(n-2m-1) \times n$ matrix whose rows are a basis of $\mathrm{Rel}(W(\Lambda))$. The n columns of this matrix form a vector configuration in $\mathbb{R}^{n-2m-1}$ that we denote V. This configuration is said to be in Gale duality with the configuration $\Lambda^{\mathbb{R}}$. The configuration V is balanced, and also odd since $n-(n-2m-1)$ is odd.

The collection $\mathcal{T}$ of all the complements (in $\{1, \dots, n\}$) of elements of $\mathcal{T}^*$ is a *triangulation* of V (cf. [17]). Roughly speaking, a triangulation of V is the simplicial complex determined by a simplicial fan whose ray generators form a subset of V.

Therefore, we have obtained, from $\{\Lambda, \mathcal{T}^*\}$, a triangulated vector configuration $\{V, \mathcal{T}\}$, which is unique up to an ambient real linear automorphism—for our purposes we can consider $\{V, \mathcal{T}\}$ uniquely determined by $\{\Lambda, \mathcal{T}^*\}$. Elements of $\mathcal{T}$ are called simplices. As in [30] each index in the intersection of all of the bases of $\mathcal{T}^*$ is an *indispensable index* of $\mathcal{T}^*$. We denote by k the number of such indices and we will always assume that $j \in \{1, \cdots, n\}$ is indispensable if and only if $j > n - k$. Any indispensable index i of $\mathcal{T}^*$ will not appear in any simplex of $\mathcal{T}$; it is called a *ghost index* of $\mathcal{T}$, and v_i is called a *ghost vector*. By properties of Gale duality [17, 4.1.38 (iv)], for any $\tau \in \mathcal{T}$ the vectors $(v_j)_{j \in \tau}$ are linearly independent, so they generate a simplicial cone denoted cone(τ). One can check that Bosio's conditions are equivalent to the fact that the collection of these cones for all $\tau \in \mathcal{T}$ determines a complete simplicial fan Δ [9, Prop. 2.1] ; this fan is not necessarily rational. The non necessarily closed additive subgroup Q of $\mathbb{R}^{n-2m-1}$ generated by $v_1, \dots, v_n$ is called a *quasilattice* (*cf.* [36]). We always see Q as embedded, i.e. Q is a shorthand for the pair $(\mathbb{R}^{n-2m-1}, Q)$.

In the introduction we have described how a toric datum, and, more precisely, a Prato's datum, can be encoded in a triangulated vector configuration. In more detail, consider the triple $(\Delta, \{v_1, \dots, v_h\}, Q)$ where $\Delta \subset \mathbb{R}^d$ is a complete fan, $\{v_1, \dots, v_h\}$ is a set of fan rays generators and Q is a quasilattice in $\mathbb{R}^d$ containing these generators. Then a corresponding triangulated vector configuration, balanced and odd, is a pair $\{V, \mathcal{T}\}$ with $V = (v_1, \dots, v_n)$ a vector configuration in $\mathbb{R}^d$ such that the first h vectors coincide with the given generators above, $Q = \text{Span}_{\mathbb{Z}}\{v_1, \dots, v_n\}$, $\sum_i v_i = 0$ and $n - d = 2m + 1$, with $m \in \mathbb{N}$. On the other hand $\mathcal{T}$, as we have seen, corresponds to the fan cones and it is a combinatorial datum. Remark that we may have to add vectors to the set $\{v_1, \dots, v_h\}$ in order to have all the above conditions satisfied. Notice that, in the classical toric setting, Q is a fixed lattice and the generators are taken to be primitive in Q. However, here we can take non primitive generators , and, in fact, any set of generators is allowed, as long as the quasilattice Q is chosen consistently.

Once a balanced and odd triangulated vector configuration on $\{V, \mathcal{T}\}$ is given, Gale duality yields a (non unique) LVMB datum as follows:

Consider a $(n - d = 2m + 1) \times n$ matrix whose rows are a basis of Rel(V). The n columns of this matrix form a vector configuration in $\mathbb{R}^{2m+1}$ that we denote $\Lambda^{\mathbb{R}}$, unique up to an ambient real linear automorphism. Reversing the construction of Sect. 2.1 yields the configuration Λ in $\mathbb{C}^m$, which is not unique up to an ambient complex affine automorphism (see [9, Sect. 2.2.2] for details). The complements of bases in the triangulation $\mathcal{T}$ form a virtual chamber $\mathcal{T}^*$, so we obtain an LVMB datum $\{\Lambda, \mathcal{T}^*\}$. Due to the non uniqueness of Λ, we actually obtain from $\{V, \mathcal{T}\}$ a family of (closely related) LVMB manifolds. We propose to call two members of such a family *virtually biholomorphic*. The questions posed by the non-uniqueness of the Gale duality correspondence are treated in our paper [10], in preparation.

To complete this preliminary section let us recall how to measure the nonrationality of a vector configuration V. Consider the space of linear relations $\mathrm{Rel}(V) \subset \mathbb{R}^n$, of dimension $n - d$. We say that a real subspace of $\mathbb{R}^n$ is *rational* when it admits a real basis of vectors in $\mathbb{Q}^n$ (equivalently, $\mathbb{Z}^n$). We define $a(V)$ as the dimension of the largest rational space contained in $\mathrm{Rel}(V)$, and $b(V)$ as the dimension of the smallest rational space containing $\mathrm{Rel}(V)$ (our number $a(V)$ is closely related to the number a defined in [30]). Then $0 \leq a(V) \leq n - d \leq b(V) \leq n$.

The configuration is called *rational* when $\mathrm{Rel}(V)$ is rational or, equivalently, $a(V) = n - d$, or $b(V) = n - d$. Otherwise $2 + a(V) \leq b(V)$, and all such values are possible.

Notice that $\mathrm{Span}_{\mathbb{Z}}(V)$ is an honest lattice if and only if the configuration V is rational. These numbers determine the topology of the generic leaves of the foliation $\mathcal{F}$ and their closures [9, Sect. 2.3], for example generic leaves are homeomorphic to $(S^1)^{a(V)-1} \times \mathbb{R}^{2m-a(V)+1}$. If the configuration is rational, that is $a(V) = b(V) = 2m + 1$, all leaves are closed (see [31] for a full study of this case). On the other hand there are nonrational configurations V such that $a(V) = 1$; in these cases the generic leaf is $\mathbb{C}^m$.

3 Toric and LVMB Geometry

In this section we present, in form of synthetic statements, some of the results that we proved in [9], adding some new features. For each result we outline the argument of the proof.

Theorem 3.1 *Let $\{\Lambda, \mathcal{T}^*\}$ be an LVMB-datum and let $(N, \mathcal{F})$ be the corresponding LVMB manifold. Let $\{V, \mathcal{T}\}$ be the triangulated vector configuration Gale dual to $\{\Lambda, \mathcal{T}^*\}$. Let $(\Delta, Q, \{v_1, \ldots, v_h\})$ be the fan, quasilattice and ray generators encoded in $\{V, \mathcal{T}\}$ and let X be the toric quasifold associated to this triple. Then the leaf space $N/\mathcal{F}$ is biholomorphic to X.*

First of all let us recall how we obtain X. From the Audin–Cox–Delzant construction [2, 14, 18] and its nonrational complex generalization [4], it is known that to $(\Delta, Q, \{v_1, \ldots, v_h\})$ there corresponds a geometric quotient $X = U'(\Delta)/G$, where $U'(\Delta)$ is an open subset of $\mathbb{C}^h$ that depends on the combinatorics of Δ, and $\mathrm{N}_{\mathbb{C}}$ is a complex subgroup of $(\mathbb{C}^*)^h$ that depends on Q and on the vectors $v_1, \ldots, v_h$. If the configuration is rational (resp. nonrational), then X is a complex manifold or a complex orbifold (resp. a non Hausdorff complex quasifold) of dimension d, acted on holomorphically by the torus (resp. *quasitorus*) $\mathbb{C}^d/Q$ (*cf.* [2, 4, 14, 36]); the construction in [4, Theorem 2.2] can be adapted to the nonpolytopal case. Quasifolds generalize orbifolds: the local model is a quotient of a manifold by the smooth action of a finite or countable group, non free on a closed subset of topological codimension at least 2 ([36], see also [6]). Let us describe more precisely how we construct X, and its relation with N. Let

$$\hat{U}(\mathcal{T}^*) =: \bigcup_{\tau^* \in \mathcal{T}^*} \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \forall j \in \tau^*, z_j \neq 0 \right\}. \tag{1}$$

Then $U(\mathcal{T}^*)$ is just the projectivization of $\hat{U}(\mathcal{T}^*)$. Consider $\mathbb{C}^n = \mathbb{C}^h \times \mathbb{C}^k$, then $U'(\Delta)$ is exactly the projection onto the first factor of $\hat{U}(\mathcal{T}^*)$. The manifold N is given by the quotient $U(\mathcal{T}^*)/\exp(\mathfrak{h})$. In order to pass from $\{\Lambda, \mathcal{T}^*\}$ to $\{V, \mathcal{T}\}$, let us identify $\mathbb{R}^{2m}$ with the affine space $A = \{(1, x_1, \ldots, x_{2m})\} \subset \mathbb{R}^{2m+1}$ (cf. Sect. 2.2). Then the leaf space $N/\exp(\overline{\mathfrak{h}})$ can be naturally identified with

$$\hat{U}(\mathcal{T}^*)/\exp(Rel(V)_{\mathbb{C}}).$$

On the other hand X is the orbit space $U'(\Delta)/\mathrm{N}_{\mathbb{C}}$, endowed with the quotient topology. Here

$$\mathrm{N} = \exp\left(\{\underline{a} \in \mathbb{R}^h \mid \sum_{i=1}^{h} a_i v_i \in Q\}\right) \tag{2}$$

and

$$\mathrm{N}_{\mathbb{C}} = \exp\left(\{\underline{a} \in \mathbb{C}^h \mid \sum_{i=1}^{h} a_i v_i \in Q\}\right) \tag{3}$$

are subgroups of $(S^1)^h$ and $(\mathbb{C}^*)^h$ respectively (see [4, 36] for details). When $h = n$, we have $\mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_h\} = Q$, therefore $\exp(Rel(V)_{\mathbb{C}}) = \mathrm{N}_{\mathbb{C}}$. Moreover, $\hat{U}(\mathcal{T}^*) = U'(\Delta)$. Thus the leaf space $N/\mathcal{F}$ and X are naturally identified. When $h < n$, we can define the map $f\colon N/\mathcal{F} \to X$ as follows: let $[z_1, \ldots, z_n] \in \hat{U}(\mathcal{T}^*)/(\exp(Rel(V)_{\mathbb{C}})$. Notice that, by (1), $z_j \neq 0$ for all $j = h+1, \ldots, n$. Because of the properties of $(V, \mathcal{T})$, there exists $\underline{b} \in Rel(V)_{\mathbb{C}}$ such that $e^{2\pi i b_j} \cdot z_j = 1$, for all $j = h+1, \ldots, n$. The mapping

$$f([z_1, \ldots, z_h, z_{h+1}, \ldots, z_n]) = [e^{2\pi i b_1} z_1, \ldots, e^{2\pi i b_h} z_h]$$

is well defined and identifies the leaf space $N/\mathcal{F}$ and X as complex quotients. Let us check these properties of f. Let $\underline{b}, \underline{b}' \in Rel(V)_{\mathbb{C}}$ such that $e^{2\pi i b_j} \cdot z_j = e^{2\pi i b'_j} \cdot z_j = 1$, for all $j = h+1, \ldots, n$. Then $\sum_{j=1}^{h} (b_j - b'_j) v_j \in \mathrm{Span}_{\mathbb{Z}}\{v_{h+1}, \ldots, v_n\} \subset Q$. Therefore $(e^{2\pi i(b_1 - b'_1)}, \ldots, e^{2\pi i(b_h - b'_h)}) \in \mathrm{N}_{\mathbb{C}}$, thus the map f does not depend on the choice of $\underline{b}$. Similarly it can be checked that f does not depend on the class representative $\underline{z}$ and that it is injective. Surjectivity is immediate. We can construct complex atlases for the two quotients $\hat{U}(\mathcal{T}^*)/\exp(Rel(V)_{\mathbb{C}})$ and $U'(\Delta)/\mathrm{N}_{\mathbb{C}}$ very similarly (cf. [4, Theorem 2.1] and the proof of [9, Lemma 3.2]), namely using holomorphic slices for the action of the respective groups given by $(z_1, \ldots, z_d) \to (z_1, \ldots, z_d, \underbrace{1, \ldots, 1}_{n-d})$ and $(z_1, \ldots, z_d) \to (z_1, \ldots, z_d, \underbrace{1, \ldots, 1}_{h-d})$. The map f, locally, is just the identity. We deduce that, in particular, the complex structure induced by $(N, \mathcal{F})$ on the leaf space depends only on $(\Delta, Q, \{v_1, \ldots, v_h\})$. In particular it does not depend on the choice of ghost vectors, nor on the choice of the Gale dual LVMB datum. This is consistent with [31, Theorem G, (iii)]. The (quasi)torus

$$\mathbb{C}^d/Q \simeq \mathbb{C}^h/\mathrm{N}_{\mathbb{C}} \simeq \mathbb{C}^n/\exp(Rel(V)_{\mathbb{C}})$$

acts on X holomorphically with a dense open orbit.

Remark 3.2 If Q is a lattice then X is a toric orbifold, if Q is a lattice and the v_i are primitive then X is the toric variety associated to Δ, with no additional orbifold structure.

Remark 3.3 Remark that the action of the group $\mathrm{N}_{\mathbb{C}}$ does induce a holomorphic foliation on $U'(\Delta)$. However, since $\mathrm{N}_{\mathbb{C}}$ is, in general, for rational and nonrational configurations, not connected, the leaf space is *not* X (see for a nonrational example [5] and for a rational example [9, Example 2.4.2]). Following [31], this problem is overcome in our construction by "increasing the dimension" (cf. [9, Example 2.4.2]).

We will see this same phenomenon in the symplectic setting with the variant of the Delzant construction introduced in Sect. 5.

Theorem 3.4 *Let $(\Delta, Q, \{v_1, \ldots, v_h\})$ be a complete fan, a quasilattice and ray generators in Q. Let X be the toric quasifold associated to this triple. Then there exists an LVMB manifold $(N, \mathcal{F})$ whose leaf space is biholomorphic to X; there are infinitely many such LVMB manifolds.*

By Sect. 2.2 it is sufficient to consider a triangulated vector configuration $\{V, \mathcal{T}\}$ encoding $(\Delta, Q, \{v_1, \ldots, v_h\})$. We then take any Gale dual LVMB-datum $\{\Lambda, \mathcal{T}^*\}$. This determines an LVMB manifold N, whose leaf space $N/\mathcal{F}$ can be identified with X by Theorem 3.1. The next two corollaries are proved in the polytopal case in [31, Theorem G]; the first is proved, in this generality, in [16].

Corollary 3.5 *Let $(\Delta, L, \{v_1, \ldots, v_h\})$ be a complete fan, lattice and primitive ray generators in L and let X be the toric variety determined by this triple. Then there exists an infinite family of LVMB manifold $(N, \mathcal{F})$ whose leaf space is biholomorphic to X.*

Corollary 3.6 *Let $(\Delta, L, \{v_1, \ldots, v_h\})$ be a complete fan, lattice and ray generators in L and let X be the toric variety obtained from this triple, with the orbifold structure determined by the ray generators multiplicities. Then there exists an infinite family of LVMB manifolds $(N, \mathcal{F})$ whose leaf space is biholomorphic to X, with its orbifold structure.*

In conclusion, LVMB manifolds, through Gale duality, model rational and nonrational complete simplicial varieties, not necessarily projective, and avoid all singularities [9]. We note also the interesting announcement of Katzarkov–Lupercio–Meersseman–Verjovsky [24] that the nonrational LVM manifolds can also serve as a model for a notion of noncommutative (projective, simplicial) toric varieties.

4 The Polytopal Case

In this section we specialize to polytopal fans: we say that a fan Δ is polytopal when it is the *dual* (or *normal*) fan to some polytope P. Loosely speaking, this happens when there exists a polytope P such that the fan rays are normal to the polytope facets, and the other cones of Δ are related to the faces of P by an inclusion-reversing bijection. This implies important properties for the corresponding spaces. For example it is well known that a toric variety is projective if and only if its fan is polytopal. An analogous result, whose proof was completed by Ishida in 2015, holds at the level of LVMB manifolds. First we need to recall a definition: a complex foliated manifold $(N, \mathcal{F})$ is *transversely Kähler* when there exists a closed $(1, 1)$ form ω on N such that $\omega_x(X, JX) \geq 0$ for all $x \in X$ and for all $X \in T_xX$, with equality if and only if X is tangent to $\mathcal{F}$. We then have:

Theorem 4.1 *Let $(\Lambda_1, \dots, \Lambda_n)$ be an affine point configuration in $\mathbb{C}^m$, together with the choice of a virtual chamber $\mathcal{T}^*$, satisfying Bosio's conditions. Let Δ be the associated fan. Then the LVMB manifold $(N, \mathcal{F})$ corresponding to $(\Lambda_1, \dots, \Lambda_n)$ is transversely Kähler if and only if Δ is polytopal.*

Proof of the inverse implication by Loeb and Nicolau [28] (for $m = 1$) and Meersseman [30] (for $m \geq 2$); proof of the direct implication by Cupit and Zaffran [16] under condition (K). The general case is due to Ishida [23, Theorem 5.7].

Polytopality has several further characterizations, depending on the view-point we want to stress. Here is a list:

1. the fan Δ is *polytopal*;

$\Leftrightarrow$ 2. the foliation $\mathcal{F}$ is transversely Kähler;

$\Leftrightarrow$ 3. The triangulation is *regular*

$\Leftrightarrow$ 4. There exists a *height function* on V that induces $\mathcal{T}$

$\Leftrightarrow$ 5. The virtual chamber defines a nonempty *chamber*, i.e., $\bigcap_\alpha \mathring{C}_\alpha \neq \varnothing$

$\Leftrightarrow$ 6. There exists $\nu \in \mathbb{R}^{2m}$ such that $\forall \tau \subset \{1 \dots n\}$, $\tau \in \mathcal{T}$ if and only if ν is in the interior of the convex hull of $\left\{ \Lambda_j^{\mathbb{R}} \mid j \in \tau^c \right\}$

Note that 4. is the explicit definition of 3. We refer to [17] for details on triangulations and the meaning of regularity.

Let us mention, in conclusion, some result about the topology of the leaf space. The Betti numbers of a complex toric quasifold are computed in [3, 3.3], under the polytopality assumption: odd Betti numbers are zero, even Betti numbers give the h-vector of the corresponding fan. This result is proved for the larger class of simplicial shellable fans in [9, Theorem 3.1], where the basic cohomology of the foliated manifold $(N, \mathcal{F})$ yields the cohomology of the leaf space. We also prove, by adding back polytopality, that the basic cohomology is generated in degree two. Furthermore El Kacimi's theorem [19, 3.4.7] applies and gives the Hard Lefschetz theorem for the basic cohomology of LVM manifolds.

5 A Variant of the Delzant Construction

The construction introduced by Delzant in [18] allows to explicitly construct, via symplectic reduction, a symplectic toric manifold from a Delzant polytope. This is a convex polytope of full dimension in $(\mathbb{R}^d)^*$ that is simple, rational and satisfies a certain integrality condition. Roughly speaking a Delzant polytope P, at each vertex, looks like an orthant at the origin. Simple means that, for each vertex ν, there are exactly d facets of P meeting at ν. The fan Δ normal to P is the fan $\Delta \subset \mathbb{R}^d$ whose rays are normal to the polytope facets and have inward pointing directions. The polytope P is simple if and only if the fan Δ is simplicial. The polytope P is rational if there exists a lattice L in $\mathbb{R}^d$ with respect to which Δ is rational. Duality yields a bijective correspondence between the polytope vertices and the fan maximal cones. The Delzant's integrality condition is satisfied if, for each vertex ν, the primitive ray generators of its corresponding maximal cone give a basis of L. On the other hand a symplectic toric manifold (M, ω) is a compact, connected, symplectic manifold equipped with the effective Hamiltonian action of a torus T such that $\dim M = 2 \dim T$. By torus in this section we mean a compact torus, that is a torus isomorphic to $(S^1)^r$, for some $r \in \mathbb{N}_{>0}$. The convexity theorem of Atiyah [1] and Guillemin–Stenberg [21] asserts that, if (M, ω) is a compact, connected, symplectic manifold, endowed with the Hamiltonian action of a torus T, with group lattice L and Lie algebra $\mathfrak{t} = L \otimes_{\mathbb{Z}} \mathbb{R}$, then the image of the corresponding moment mapping Φ is a rational convex polytope in $\mathfrak{t}^*$, called moment polytope. A very important application of the convexity theorem is the Delzant's theorem, that completely classifies symplectic toric manifolds: the moment polytope of a symplectic toric manifold (M, T, ω, Φ) is a Delzant polytope P; in turn, P uniquely determines (M, T, ω, Φ) up to equivariant symplectomorphisms. A key point in the proof of this theorem is the above-mentioned procedure, that allows to construct, from a given Delzant polytope, a symplectic toric manifold. This has proved to be an extremely fruitful tool, in symplectic and contact geometry, with a great variety of applications.

We present here a simple variant of this construction, that, as we will see, also applies to the generalizations of the Delzant procedure introduced by Lerman-Tolman [27] and Prato [36]. Furthermore, our variant is strictly related to the LVM manifolds described in the previous section. But let us recall first the classical Delzant construction. Let L be a lattice of rank d in $\mathbb{R}^d$. A convex polytope P in $(\mathbb{R}^d)^*$ can be always written as intersection of closed half spaces. When these closed half spaces are in bijective correspondence with the polytope facets, this intersection is said to be minimal:

$$P = \cap_{j=1}^{h} \{\mu \in (\mathbb{R}^d)^* \mid \langle \mu, v_j \rangle \geq l_j\}.$$

Here the v_i are taken to be primitive and inward pointing, the l_i's are real coefficients determined by the v_i. Denote $V = (v_1, \ldots, v_h)$. Let N be the subtorus of $(S^1)^h$ defined in (2). The integrality condition implies that $\mathrm{N} = \exp(Rel(V))$. The induced action of this group on $\mathbb{C}^h$ is Hamiltonian with respect to the standard Kähler form of $\mathbb{C}^h$. Let $\Psi\colon \mathbb{C}^h \to (Rel(V))^*$ be the corresponding moment mapping (the choice of the constant is such that $\Psi(0) = i^*(\sum_{j=1}^{h} l_j e_j^*)$, where $i\colon Rel(V) \hookrightarrow \mathbb{R}^h$ is the inclusion). Then the reduced space $\Psi^{-1}(0)/\mathrm{N}$ is the toric symplectic manifold $(M_P, \mathbb{R}^d/L, \omega, \Phi)$ of dimension d, with moment polytope P. Notice that $\mathbb{R}^d/L \simeq (S^1)^h/\mathrm{N}$.

The question that we want to pose is: what happens if we consider an intersection of half spaces that is not minimal? And what is the relation of this set-up with LVM manifolds and the previous sections? Let us start with the following

Proposition 5.1 *Consider $n \geq h$. Let L be a lattice in $\mathbb{R}^d$ and let P be the Delzant polytope in $(\mathbb{R}^d)^*$ defined by a not necessarily minimal intersection*

$$P = \cap_{j=1}^{n} \{\mu \in (\mathbb{R}^d)^* \mid \langle \mu, v_j \rangle \geq l_j\},$$

where

- *$v_1 \ldots, v_h$ are primitive in L and generate the h rays of the normal fan to P;*

and, if $n > h$,

- *for each $j = h+1, \ldots, n$, $v_j \in L$*
- *$P \subset \{\mu \in (\mathbb{R}^d)^* \mid \langle \mu, v_j \rangle > l_j\}$, with $j = h+1, \ldots, n$*

Consider $V = (v_1, \ldots, v_n)$ and take the subgroup $\mathrm{N} = \exp(Rel(V)) \subset (S^1)^n$. Let Ψ be the moment mapping with respect to the induced action of N on $\mathbb{C}^n$. Then the reduced space $M = \Psi^{-1}(0)/\mathrm{N}$ is a compact symplectic manifold of dimension $2d$, endowed with the effective Hamiltonian action of the torus $\mathbb{R}^d/L$ such that the image of the corresponding moment mapping is exactly P. Namely M is the symplectic toric manifold $(M_P, \mathbb{R}^d/L, \omega, \Phi)$ corresponding to P.

Notice that there are no conditions on the vectors $v_j, j = h+1, \ldots, n$. They only have to lie in the lattice L. As in the previous sections, repetitions are allowed, and even zero vectors. These last yield degenerate half-spaces, coinciding with the whole space $(\mathbb{R}^d)^*$. We are allowed to add as many half spaces as we want. The proof of the Delzant procedure applies with no substantial changes. The key observation is that N acts freely on $\Psi^{-1}(0)$. We will later need the explicit expression of Ψ: let $M \in M_{n\times(n-d)}(\mathbb{R})$ be a matrix whose columns give a basis of $Rel(V)$. Then, expressed in components with respect the basis of $(Rel(V))^*$ dual to that basis, we have:

$$\Psi(\underline{z}) = (|z_1|^2 + l_1, \dots, |z_n|^2 + l_n)M. \tag{4}$$

In particular, when the configuration is balanced, we can always take the first column to be $(1, \dots, 1)$, so that one of the components of Ψ is

$$(\sum_{i=1}^{n}(|z_i|^2 + l), \tag{5}$$

with $l = \sum_{i=1}^{n} l_i$. Therefore $\Psi^{-1}(0)$ is contained in the sphere S_r^{2n-1} of radius $r = -l$.

Remark 5.2 If, for some indices $\{j_1, \dots, j_r\} \subset \{h+1, \dots, n\}$, the third requirement is dropped, that is the hyperplanes $\{\mu \in (\mathbb{R}^d)^* \mid \langle \mu, v_{j_k} \rangle = l_{j_k}\}$ may intersect P, the construction can still be followed step by step. But, in this case, 0 is not a regular value of Ψ and the level set $\Psi^{-1}(0)$ is not smooth. However, the reduced space continues to be smooth. This phenomenon was observed and thoroughly investigated by Guillemin-Sternberg in [22]. Since the Delzant condition ensures that for each $\underline{z} \in \Psi^{-1}(0)$ the isotropy group $N_{\underline{z}}$ is connected, there are no orbifold singularities in the quotient $\Psi^{-1}(0)/N$.

Recall that the Delzant procedure was generalized by Lerman-Tolman [27] to the cases in which P is a simple convex polytope, rational with respect to a lattice L, and the vectors v_i are not necessarily primitive in L. Prato further generalized the Delzant construction, so as to include the nonrational setting: let P be a simple convex polytope. A triple $(P, Q, \{v_1, \dots, v_h\})$, with Q a quasilattice and $v_i \in Q$ generators of the normal fan to P, will be called a symplectic Prato's datum. The resulting reduced space is a symplectic toric quasifold M, of dimension $2d$, determined by the triple $(P, Q, \{v_1, \dots, v_h\})$. When $Q = L$ is a lattice, the Lerman-Tolman case is recovered, the resulting reduced space is the symplectic toric orbifold determined by $(P, Q, \{v_1, \dots, v_h\})$. When $Q = L$ is a lattice, the vectors are primitive and the polytope is Delzant, the resulting reduced space is the symplectic toric manifold M_P.

We now state a proposition similar to Proposition 5.1 for the above described cases. This variant may be useful for all cases such that

$$\mathrm{Span}_{\mathbb{Z}}\{v_1, \dots, v_h\} \subsetneqq Q,$$

where Q is a (quasi)lattice. This can happen when the polytope is not Delzant or when the first h vectors are not primitive ($Q = L$ is a lattice) or in the nonrational setting. We then have $\exp(Rel(v_1, \dots, v_h)) \subsetneq \mathrm{N}$. That is N is not connected.

By extending Proposition 5.1 we obtain the same reduced spaces resulting from the generalized Delzant procedure, however, the group N with respect to which we perform the symplectic reduction is connected. It is enough to increase the number of half-spaces, exactly as observed in Sect. 3, in the complex set-up.

Proposition 5.3 *Let $(P, Q, \{v_1, \dots, v_h\})$ a symplectic Prato's datum. Let $n \geq h$ and let P be given by*

$$P = \cap_{j=1}^{n} \{\mu \in (\mathbb{R}^d)^* \mid \langle \mu, v_j \rangle \geq l_j\},$$

where:

- *$Span_{\mathbb{Z}}\{v_1, \dots, v_n\} = Q$;*

and, if $n > h$

- *$P \subset \{\mu \in (\mathbb{R}^d)^* \mid \langle \mu, v_j \rangle > l_j\}$, with $j = h+1, \dots, n$.*

Let $V = (v_1, \dots, v_n)$ and consider the subgroup $\mathrm{N} = \exp(Rel(V))$ in $(S^1)^n$. Let Ψ be the moment mapping with respect to the induced action of N on $\mathbb{C}^n$. Then the reduced space $M = \Psi^{-1}(0)/\mathrm{N}$ is endowed with the effective Hamiltonian action of the quasitorus $\mathbb{R}^d/Q$. The image of the corresponding moment mapping Φ is exactly P. Moreover M is equivariantly symplectomorphic to the symplectic quasifold (orbifold) determined by $(P, Q, \{v_1, \dots, v_h\})$.

The generalized Delzant procedure [36] applies with no essential changes. We only have to check that M can be identified with the symplectic quasifold M' determined by $(P, Q, \{v_1, \dots, v_h\})$. We outline the argument: let $M' = (\Psi')^{-1}(0)/\mathrm{N}'$ be the symplectic quasifold corresponding to $(P, Q, \{v_1, \dots, v_h\})$. Let Φ' be the moment mapping with respect to action of the quasitorus $\mathbb{R}^d/Q$ such that $\Phi'(M') = P$. Consider the natural inclusion $\mathbb{R}^h \subset \mathbb{R}^n$. We may view $Rel(v_1, \dots, v_h)$ as a subset of $Rel(V)$. This allows to write explicitly an equivariant map $(\Psi')^{-1}(0) \xrightarrow{i} \Psi^{-1}(0)$ that induces a symplectomorphism $M' \to M$. The key point is to verify that N' at $\underline{z}$ and N at $i(\underline{z})$ have the same stabilizers, where N' and N are defined in (2) and, in particular, $\mathrm{N} = \exp(Rel(V))$.

Finally notice that, as in Remark 5.2, we can consider the cases in which the condition $P \subset \{\mu \in (\mathbb{R}^d)^* \mid \langle \mu, v_j \rangle > l_j\}$ is dropped for some indices in $\{h+1, \dots, n\}$. Although the level set is singular, the resulting reduced space is a manifold, orbifold or quasifold. These degenerate cases turn out to be relevant in various instances, see for example [11], [7, Remark 2.6] and [8].

Remark 5.4 Consider the datum $(P, Q, \{v_1, \dots, v_h\})$ and a non minimal, non degenerate presentation of P. By applying the generalized Delzant procedure to this presentation, as in Proposition 5.3, we obtain the reduced space corresponding to $(P, Q, \{v_1, \dots, v_h\})$. However, the group that we use for the reduction is always connected and of arbitrarily high dimension. In the Delzant case, by Remark 5.2, non degeneracy can be dropped.

Symplectic and complex quotients can often be identified via a natural homeomorphism or diffeomorphism. These kind of results can be dated back to the work by Kempf and Ness [25] and Kirwan [26], and apply to a number of settings, of finite and infinite dimension. A model example is given by the standard actions of S^1 and $(S^1)_{\mathbb{C}} = \mathbb{C}^*$ on $\mathbb{C}^n$. It is a toric example: let the polytope P be the simplex in $\mathbb{R}^{n-1}$, with vertices the origin and $re_1, \ldots, re_{n-1}$, with $r = -l$. The fan Δ is its normal fan. By (5) the moment mapping with respect to the S^1-action is $\Psi(\underline{z}) = \sum_{i=1}^n |z_i|^2 - r$. The zero level set is therefore the sphere S_r^{2n-1}. While $U'(\Delta) = U(\mathcal{T}) = \mathbb{C}^n \setminus \{0\}$. We have the following diagram:

$$\begin{array}{ccc} S_r^{2n-1} & \hookrightarrow & \mathbb{C}^n \setminus \{0\} \\ \downarrow & & \downarrow \\ S_r^{2n-1}/S^1 & \xrightarrow{\chi} & \mathbb{C}P^{n-1} \end{array}$$

The inclusion of the zero level set in $U'(\Delta)$ induces a diffeomorphism at the level of the quotients, moreover the symplectic structure on S_r^{2n-1}/S^1 is compatible with the complex structure of $\mathbb{C}P^{n-1}$ and induces a Kähler structure on the projective space, multiple of the Fubini-Study metric. This principle holds also in our case. Assume from now on the hypotheses of Proposition 5.3 and take $\{V, \mathcal{T}\}$ to be the corresponding triangulated vector configuration, namely $V = \{v_1, \ldots, v_n\}$ and $\mathcal{T}$ the triangulation determined by the fan Δ. A simple adaptation of [4, Theorem 3.2] yields the following

Proposition 5.5 *The level set* $\Psi^{-1}(0)$ *is contained in* $\hat{U}(\mathcal{T}^*)$. *This inclusion induces an equivariant diffeomorphism*

$$\chi\colon \Psi^{-1}(0)/\exp(Rel(V)) \to \hat{U}(\mathcal{T}^*)/\exp(Rel(V)_{\mathbb{C}})$$

with respect to the actions of the quasitorus $\mathbb{R}^d/Q$ *and the complex quasitorus* $\mathbb{C}^d/Q$. *Moreover the induced symplectic form on the complex quasifold* $\hat{U}(\mathcal{T}^*)/\exp(Rel(V)_{\mathbb{C}}$ *is Kähler.*

We now insert our foliated complex manifold $(N, \mathcal{F})$ in this picture. Let $\{\Lambda, \mathcal{T}^*\}$ an LVMB datum Gale dual to $\{V, \mathcal{T}\}$. Then the columns of the matrix

$$M = \begin{bmatrix} 1 — \Lambda_1^{\mathbb{R}} — \\ \vdots \\ 1 — \Lambda_n^{\mathbb{R}} — \end{bmatrix}$$

are a basis of $Rel(V)$. Recall formulae (4) and (5) and consider the following commutative diagram:

$$\begin{tikzcd}
S_r^{2n-1} \supset \Psi^{-1}(0) \arrow[r, hook, "\hat{i}"] \arrow[d, "S^1"'] & \hat{U}(\mathcal{T}^*) \subset \mathbb{C}^n \setminus \{0\} \arrow[d, "\mathbb{C}^*"] \\
S_r^{2n-1}/S^1 \supset \Psi^{-1}(0)/S^1 \arrow[r, hook, "i"] \arrow[dd, "\exp(Rel(V))/S^1"'] \arrow[dr, "f"] & U(\mathcal{T}^*) \subset \mathbb{CP}^{n-1} \arrow[d, "\exp(\mathfrak{h})"] \\
& N \arrow[d, "\exp(\overline{\mathfrak{h}})"] \\
M \arrow[r, "\chi"] & X
\end{tikzcd} .$$

The composition of the vertical maps on the left-hand side give the quotient map by the group $\exp(Rel(V))$, while the composition of the vertical maps on the right-hand side give the quotient map by the group $\exp(Rel(V)_{\mathbb{C}})$.

From Proposition 5.5 and the proof of [4, Theorem 3.2] we know that, for each $\underline{z} \in \Psi^{-1}(0) \subset \hat{U}(\mathcal{T}^*)$, the orbit $\exp(Rel(V)_{\mathbb{C}}) \cdot \underline{z}$ intersects $\Psi^{-1}(0)$ exactly in the orbit $\exp(Rel(V)) \cdot \underline{z}$. Moreover the orbit $\exp(iRel(V)) \cdot \underline{z}$ intersects $\Psi^{-1}(0)$ transversally, only in $\underline{z}$. This still holds for $\Psi^{-1}(0)/S^1$ and the orbits $\exp(Rel(V)_{\mathbb{C}}/\mathbb{C}) \cdot \underline{z}$ and $\exp(iRel(V)/i\mathbb{R}) \cdot \underline{z}$. Since $\exp(\mathfrak{h}) \cap (S^1)^n = \{1\}$ we have that the orbit $\exp(\mathfrak{h}) \cdot \underline{z}$ intersects $\Psi^{-1}(0)/S^1$ transversally, only in $\underline{z}$. Therefore f is a diffeomorphism. This is similar to [30, p. 83]. Furthermore, f sends $\exp(Rel(V))/S^1$-orbits in $\Psi^{-1}(0)/S^1$ onto $\exp(\overline{h})$-orbits in N. Now let ω be the presymplectic form on $\Psi^{-1}(0)/S^1$ induced by the standard Kähler form in $\mathbb{C}^n$ (cf. [31, Proposition D]). Then, by the symplectic reduction properties, at each point $p \in \Psi^{-1}(0)/S^1$, the kernel of ω is the tangent space to the $\exp(Rel(V))/S^1$-orbit through p. Thus the kernel of $(f^{-1})^*(\omega)$ is the tangent space to the leaf $\exp(\overline{h}) \cdot f(p)$. Moreover, $(f^{-1})^*(\omega)$ is clearly compatible with the complex structure on N. Therefore the form $(f^{-1})^*(\omega)$ endows $(N, \mathcal{F})$ with a transversely Kähler structure. This is of course, as we have seen in Sect. 4, a well known result, that we recover in our variant of the Delzant construction picture. This responds to a suggestion of [31, p. 74].

We conclude by recalling some recent results by Ishida [23], where he adopts symplectic methods to the study of complex manifolds admitting a maximal torus action. He proves a theorem analogous to the Atiyah and Guillemin-Sternberg convexity theorem, in the context of foliated manifolds endowed with a transversely symplectic form. Using this theorem he proves the direct implication of Theorem 4.1, conjectured in [16].

Finally, in connection with this symplectic view-point, let us mention the very recent preprint by Nguyen and Ratiu [32].

Acknowledgements The authors would like to thank Leonor Godinho for pointing out the relevance of [22].

References

1. M. Atiyah, Convexity and commuting Hamiltonians. Bull. Lond. Math. Soc. **14**, 1–15 (1982)
2. M. Audin, *The Topology of Torus Actions on Symplectic Manifolds*. Progress in Mathematics, vol. 93 (Birkhäuser, Basel, 1991)
3. F. Battaglia, Betti numbers of the geometric spaces associated to nonrational simple convex polytopes. Proc. Am. Math. Soc. **139**, 2309–2315 (2011)
4. F. Battaglia, E. Prato, Generalized toric varieties for simple nonrational convex polytopes. Intern. Math. Res. Not. **24**, 1315–1337 (2001)
5. F. Battaglia, E. Prato, The symplectic geometry of Penrose rhombus tilings. J. Symplectic Geom. **6**, 139–158 (2008)
6. F. Battaglia, E. Prato, The symplectic Penrose kite. Commun. Math. Phys. **299**, 577–601 (2010)
7. F. Battaglia, E. Prato, Nonrational symplectic toric cuts. Preprint (2016). arXiv:1606.00610 [math.SG]
8. F. Battaglia, E. Prato, Nonrational symplectic toric reduction, in preparation.
9. F. Battaglia, D. Zaffran, Foliations modeling nonrational simplicial toric varieties. Int. Math. Res. Not. IMRN **2015**(22), 11785–11815 (2015)
10. F. Battaglia, D. Zaffran, LVMB-manifolds as equivariant group compatifications, in preparation
11. F. Battaglia, L. Godinho, A. Mandini, Contact-symplectic toric spaces, in preparation
12. L. Battisti, LVMB manifolds and quotients of toric varieties. Math. Z. **275**(1–2), 549–568 (2013)
13. F. Bosio, Variétés complexes compactes: une généralisation de la construction de Meersseman et de López de Medrano-Verjovsky. Ann. Inst. Fourier **51**(5), 1259–1297 (2001)
14. D. Cox, The homogeneous coordinate ring of a toric variety. J. Algebraic Geom. **4**(1), 17–50 (1995)
15. D.A. Cox, J.B. Little, H.K. Schenck, *Toric Varieties, Graduate Studies in Mathematics*, vol. 124 (American Mathematical Society, Providence, 2011)
16. S. Cupit, D. Zaffran, Non-Kähler manifolds and GIT-quotients. Math. Z. **257**(4), 783–797 (2007)
17. J.A. De Loera, J. Rambau, F. Santos, *Triangulations: Structures for Algorithms and Applications*. Algorithms and Computation in Mathematics, vol. 25 (Springer, Berlin, 2010), 539 pp.
18. T. Delzant, Hamiltoniens periodiques et images convexes de l'application moment. Bull. Soc. Math. France **116**, 315–339 (1988)
19. A. El Kacimi Alaoui, Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications. Compos. Math. **73**, 57–106 (1990)
20. W. Fulton, *Introduction to Toric Varieties* (Princeton University Press, Princeton, 1993)
21. V. Guillemin, S. Sternberg, Convexity properties of the moment mapping. Invent. Math. **67**, 491–513 (1982)
22. V. Guillemin, S. Sternberg, Birational equivalence in the symplectic category. Invent. Math. **97**, 485–522 (1989)
23. H. Ishida, Torus invariant transverse Kähler foliations. Trans. Am. Math. Soc. **369**, 5137–515 (2017)
24. L. Katzarkov, E. Lupercio, L. Meersseman, A. Verjovsky, The definition of a non-commutative toric variety. Contemp. Math. **620**, 223–250 (2014)

25. G. Kempf, L. Ness, The length of vectors in representation spaces, in *Algebraic Geometry, Summer Meeting, Copenhagen, August 7–12, 1978*. Lecture Notes in Mathematics, vol. 732 (1979), pp. 233–243
26. F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*. Mathematical Notes, vol. 31 (Princeton University Press, Princeton, 1984)
27. E. Lerman, S. Tolman, Hamiltonian torus actions on symplectic orbifolds and toric varieties. Trans. Am. Math. Soc. **349**(10), 4201–4230 (1997)
28. J. Loeb, M. Nicolau, On the complex geometry of a class of non Kählerian manifolds. Isr. J. Math. **110**, 371–379 (1999)
29. S. López de Medrano, A. Verjovsky, A new family of complex, compact, non symplectic manifolds. Bull. Braz. Math. Soc. **28**, 253–269 (1997)
30. L. Meersseman, A new geometric construction of compact complex manifolds in any dimension. Math. Ann. **317**, 79–115 (2000)
31. L. Meerssemann, A. Verjovsky, Holomorphic principal bundles over projective toric varieties. J. Reine Angew. Math. 572, 57–96 (2004)
32. T.Z. Nguyen, T. Ratiu, Presymplectic convexity and (ir)rational polytopes. Preprint arXiv:1705.11110 [math.SG]
33. T. Oda, *Convex Bodies and Algebraic Geometry* (Springer, Berlin, 1988)
34. T. Panov, Y. Ustinovsky, Complex-analytic structures on moment-angle manifolds. Mosc. Math. J. **12**(1), 149–172 (2012)
35. T. Panov, Y. Ustinovski, M. Verbitsky, Complex geometry of moment-angle manifolds. Math. Z. **284**(1), 309–333 (2016)
36. E. Prato, Simple non-rational convex polytopes via symplectic geometry. Topology **40**, 961–975 (2001)
37. J. Tambour, LVMB manifolds and simplicial spheres. Ann. Inst. Fourier **62**(4), 1289–1317 (2012)
38. Y. Ustinovsky, Geometry of compact complex manifolds with maximal torus action. Proc. Steklov Inst. Math. **286**(1), 198–208 (2014)

Homotopic Properties of Kähler Orbifolds

Giovanni Bazzoni, Indranil Biswas, Marisa Fernández, Vicente Muñoz, and Aleksy Tralle

To Simon Salamon on the occasion of his 60th birthday

Abstract We prove the formality and the evenness of odd-degree Betti numbers for compact Kähler orbifolds, by adapting the classical proofs for Kähler manifolds. As a consequence, we obtain examples of symplectic orbifolds not admitting any Kähler orbifold structure. We also review the known examples of non-formal simply connected Sasakian manifolds, and produce an example of a non-formal quasi-regular Sasakian manifold with Betti numbers $b_1 = 0$ and $b_2 > 1$.

G. Bazzoni
FB Mathematik & Informatik, Philipps-Universität Marburg, Hans-Meerwein-Str. 6, Campus Lahnberge, 35032 Marburg, Germany
e-mail: bazzoni@mathematik.uni-marburg.de

I. Biswas
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India
e-mail: indranil@math.tifr.res.in

M. Fernández (✉)
Departamento de Matemáticas, Facultad de Ciencia y Tecnología, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain
e-mail: marisa.fernandez@ehu.es

V. Muñoz
Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain
e-mail: vicente.munoz@mat.ucm.es

A. Tralle
Department of Mathematics and Computer Science, University of Warmia and Mazury, Słoneczna 54, 10-710 Olsztyn, Poland
e-mail: tralle@matman.uwm.edu.pl

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_2

Keywords Formality • Hard Lefschetz theorem • Kähler orbifolds • Massey products • Sasakian manifolds • Symplectic orbifolds

2010 *Mathematical Subject Classification.* 57R18, 55S30

1 Introduction

A Kähler manifold M is a complex manifold, admitting a Hermitian metric h, such that the $(1,1)$-form $\omega = \operatorname{Im} h$ is closed, and so symplectic, where $\operatorname{Im} h$ is the imaginary part of h. The real part $g = \operatorname{Re} h$ of h is a Riemannian metric which is called the *Kähler metric* associated to ω. If a compact manifold admits a Kähler metric, then it inherits some very striking topological properties, for example: theory of Kähler groups, evenness of odd-degree Betti numbers, hard Lefschetz theorem, formality of the rational homotopy type (see [10, 40]).

Kähler metrics can be also defined on *orbifolds*. A smooth orbifold X, of dimension n, is a Hausdorff topological space admitting an open cover $\{U_i\}_{i\in I}$, such that each U_i is homeomorphic to a quotient $\Gamma_i\backslash\widetilde{U}_i$, where $\widetilde{U}_i \subset \mathbb{R}^n$ is an open subset, $\Gamma_i \subset \mathrm{GL}(n,\mathbb{R})$ a finite group acting on $\widetilde{U}_i$, and there is a Γ_i-invariant continuous map $\varphi_i\colon \widetilde{U}_i \longrightarrow U_i$ inducing a homeomorphism from $\Gamma_i\backslash\widetilde{U}_i$ onto U_i. Moreover, the gluing maps are required to be smooth and compatible with the group action (see Sect. 3 for the details).

The orbifold differential forms on a smooth orbifold are defined in local charts as Γ_i-invariant differential forms on each open set $\widetilde{U}_i$, which are compatible with the gluing maps. The de Rham complex is defined in the same way as for smooth manifolds, and the de Rham cohomology is equal to the singular cohomology. This result and Poincaré duality theorem were first proved by Satake, who introduced the notion of orbifold under the name "V-manifold" [33]. Since Satake, various index theorems were generalized by Kawasaki to the category of V-manifolds (see [20–22] and the book by Atiyah [2]). In the late 1970s, Thurston [36] rediscovered the concept of V-manifold, under the name of orbifold, in his study of the geometry of 3-manifolds, and defined the orbifold fundamental group. Even though orbifolds were already very important objects in mathematics, with the work of Dixon, Harvey, Vafa and Witten on conformal field theory [11], the interest on orbifolds dramatically increased, due to their role in string theory (see [1] and the references therein).

A complex orbifold, of complex dimension n, is an orbifold X with charts $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ as above satisfying the conditions that $\widetilde{U}_i \subset \mathbb{C}^n$, $\Gamma_i \subset \mathrm{GL}(n,\mathbb{C})$, and all the gluing maps are given by biholomorphisms. One can also define *orbifold complex forms* and *orbifold Hermitian metrics* on X (see Sect. 5 for the details). A complex orbifold X is said to be *Kähler* if X admits an orbifold Hermitian metric such that the associated orbifold Kähler form is closed. The notion of complex orbifold was introduced, under the name of *complex V-manifold*, by Baily [3] who generalized the Hodge decomposition theorem to Riemannian V-manifolds.

Although compact Kähler orbifolds are not smooth manifolds in general, they continue to possess some topological properties of Kähler manifolds. There are two possible points of view to look at topological properties of orbifolds. One is to look at the topological properties of the underlying topological space, and the other is to look at specific orbifold invariants such as the orbifold fundamental group or the orbifold cohomology. We shall focus on the former, since the latter is more adequate for the interplay between the topological space and the subspaces defining the orbifold ramification locus. So when we talk of the fundamental group or the homology or cohomology of the orbifold, we refer to those of the underlying topological spaces.

A compact Kähler orbifold is the leaf space of a foliation on a compact manifold Y [17, Proposition 4.1], and such a foliation is transversely Kähler [39, Proposition 1.4]. Moreover, the basic cohomology of Y is isomorphic to the singular cohomology of the orbifold over $\mathbb{C}$ [32, 5.3]. In [39] it is proved that any compact Kähler orbifold satisfies the hard Lefschetz property. This is done by using a result of El Kacimi-Alaoui [12] which says that the basic cohomology of a transversely Kähler foliation on a compact manifold satisfies the hard Lefschetz property. On the other hand, the dd^c-lemma for the algebra of the basic forms of a transversely Kähler foliation was shown in [9]. Also in [12] it is proved that the basic Dolbeault cohomology of a transversely Kähler foliation on a compact manifold has the same properties as the Dolbeault cohomology of a compact Kähler manifold. So, compact Kähler orbifolds possess the earlier mentioned topological properties of Kähler manifolds. Regarding the fundamental group of a Kähler orbifold, the fundamental group of the topological space underlying the orbifold actually coincides with the fundamental group of a resolution [24, Theorem 7.8.1]. Therefore, these fundamental groups of Kähler orbifolds satisfy the same restrictions as the fundamental groups of compact Kähler manifolds.

The main purpose of this paper is to prove that compact Kähler orbifolds are formal. This is achieved by adapting the proof of formality for Kähler manifolds given in [10]. The machinery used is described in Sects. 2–4. In Sects. 2 and 3 we focus on the formality of smooth manifolds and orbifolds, respectively, and in Sect. 4 we study elliptic operators on complex orbifolds following [40], but it was first developed by Baily in the aforementioned paper [3]. Then, in Sect. 5 the orbifold Dolbeault cohomology of a complex orbifold is defined, and the $\partial\overline{\partial}$-lemma for compact Kähler orbifolds is proved (Lemma 5.4). The formality of compact Kähler orbifolds is deduced using this (Theorem 5.5). Moreover, in Proposition 5.2 we prove that the orbifold Dolbeault cohomology is equipped with an analogue of the Hodge decomposition for Kähler manifolds. Consequently, the odd Betti numbers of compact Kähler orbifolds are even. (A Hodge decomposition theory for nearly Kähler manifolds was developed by Verbitsky in [38], where it is noted that this theory works also for nearly Kähler orbifolds.) In Sect. 6, we produce examples of symplectic orbifolds which do not admit any Kähler orbifold metric (as they are non-formal or they do not possess the hard Lefschetz property).

Closely related to Kähler orbifolds are Sasakian manifolds. Such a manifold is a Riemannian manifold (N, g), of dimension $2n + 1$, such that its cone

$(N\times\mathbb{R}^+,\ g^c = t^2 g + dt^2)$ is Kähler, and so the holonomy group for g^c is a subgroup of $\mathrm{U}(n+1)$. The Kähler structure on the cone induces a Sasakian structure on the base of the cone. In particular, the complex structure on the cone gives rise to a Reeb vector field.

If N admits a Sasakian structure, then in [31] it is proved that N also admits a quasi-regular Sasakian structure. The space X of leaves of a quasi-regular Sasakian structure is a Kähler orbifold with cyclic quotient singularities, and there is an orbifold circle bundle $S^1 \hookrightarrow N \xrightarrow{\pi} X$ such that the contact form η satisfies the equation $d\eta = \pi^*\omega$, where ω is the orbifold Kähler form. If X is a Kähler manifold, then the Sasakian structure on N is regular.

However, opposed to Kähler orbifolds, formality is not an obstruction to the existence of a Sasakian structure even on simply connected manifolds [5]. On the other hand, all quadruple and higher order Massey products are trivial on any Sasakian manifold. In fact, in [5] it is proved that, for any $n \geq 3$, there exists a simply connected compact regular Sasakian manifold, of dimension $2n+1$, which is non-formal, in fact not 3-formal, in the sense of Definition 2.2. (Note that simply connected compact manifolds of dimension at most 6 are formal [14, 29].) In Sect. 7 we review these examples and show that they have a non-trivial (triple) Massey product, which implies that they are not formal.

Regarding the simply connected compact regular Sasakian manifolds that are formal, the odd-dimensional sphere S^{2n+1} is the most basic example of them. By Theorem 2.3 we know that any 7-dimensional simply connected compact manifold (Sasakian or not) with $b_2 \leq 1$ is formal. In [16], examples are given of simply connected formal compact regular Sasakian manifolds, of dimension 7, with second Betti number $b_2 \geq 2$. This result and Proposition 7.1 (Sect. 7) show that, for every $n \geq 3$, there exists a simply connected compact regular Sasakian manifold, of dimension $2n+1 \geq 7$, which is formal and has $b_2 \neq 0$. We end up with an example of a quasi-regular (non-regular) Sasakian manifold with $b_1 = 0$ which is non-formal.

2 Formality of Manifolds

In this section some definitions and results about minimal models and Massey products on smooth manifolds are reviewed; see [10, 13] for more details.

We work with the *differential graded commutative algebras*, or DGAs, over the field $\mathbb{R}$ of real numbers. The degree of an element a of a DGA is denoted by $|a|$. A DGA $(\mathcal{A}, d)$ is *minimal* if:

(1) $\mathcal{A}$ is free as an algebra, that is $\mathcal{A}$ is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus_i V^i$, and
(2) there is a collection of generators $\{a_\tau\}_{\tau\in I}$ indexed by some well ordered set I, such that $|a_\mu| \leq |a_\tau|$ if $\mu < \tau$ and each da_τ is expressed in terms of the previous a_μ, $\mu < \tau$. This implies that da_τ does not have a linear part.

In our context, the main example of DGA is the de Rham complex $(\Omega^*(M), d)$ of a smooth manifold M, where d is the exterior differential.

The cohomology of a differential graded commutative algebra $(\mathcal{A}, d)$ is denoted by $H^*(\mathcal{A})$. $H^*(\mathcal{A})$ is naturally a DGA with the product inherited from that on $\mathcal{A}$ while the differential on $H^*(\mathcal{A})$ is identically zero.

A DGA $(\mathcal{A}, d)$ is called *connected* if $H^0(\mathcal{A}) = \mathbb{R}$, and it is called 1-*connected* if, in addition, $H^1(\mathcal{A}) = 0$.

Morphisms between DGAs are required to preserve the degree and to commute with the differential. We shall say that $(\bigwedge V, d)$ is a *minimal model* of a differential graded commutative algebra $(\mathcal{A}, d)$ if $(\bigwedge V, d)$ is minimal and there exists a morphism of differential graded algebras

$$\rho\colon (\bigwedge V, d) \longrightarrow (\mathcal{A}, d)$$

inducing an isomorphism $\rho^* : H^*(\bigwedge V) \xrightarrow{\sim} H^*(\mathcal{A})$ of cohomologies. In [19], Halperin proved that any connected differential graded algebra $(\mathcal{A}, d)$ has a minimal model unique up to isomorphism. For 1-connected differential algebras, a similar result was proved by Deligne et al. [10], Griffiths and Morgan [18], and Sullivan [35].

A *minimal model* of a connected smooth manifold M is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega^*(M), d)$ of differential forms on M. If M is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to the space V^i of generators in degree i, for any i. The latter also happens when $i > 1$ and M is nilpotent, that is, the fundamental group $\pi_1(M)$ is nilpotent and its action on $\pi_j(M)$ is nilpotent for all $j > 1$ (see [10]).

We say that a DGA $(\mathcal{A}, d)$ is a *model* of a manifold M if $(\mathcal{A}, d)$ and M have the same minimal model. Thus, if $(\bigwedge V, d)$ is the minimal model of M, we have

$$(\mathcal{A}, d) \xleftarrow{\nu} (\bigwedge V, d) \xrightarrow{\rho} (\Omega^*(M), d),$$

where ρ and ν are quasi-isomorphisms, meaning morphisms of DGAs such that the induced homomorphisms in cohomology are isomorphisms.

Recall that a minimal algebra $(\bigwedge V, d)$ is called *formal* if there exists a morphism of differential algebras $\psi : (\bigwedge V, d) \longrightarrow (H^*(\bigwedge V), 0)$ inducing the identity map on cohomology. A DGA $(\mathcal{A}, d)$ is formal if its minimal model is formal.

A smooth manifold M is called *formal* if its minimal model is formal. Many examples of formal manifolds are known: spheres, projective spaces, compact Lie groups, symmetric spaces, flag manifolds, and compact Kähler manifolds.

The formality property of a minimal algebra is characterized as follows.

Proposition 2.1 ([10]) *A minimal algebra $(\bigwedge V, d)$ is formal if and only if the space V can be decomposed into a direct sum $V = C \oplus N$ with $d(C) = 0$ and d injective on N, such that every closed element in the ideal $I(N)$ in $\bigwedge V$ generated by N is exact.*

This characterization of formality can be weakened using the concept of s-formality introduced in [14].

Definition 2.2 A minimal algebra $(\bigwedge V, d)$ is s-formal ($s > 0$) if for each $i \leq s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the following conditions:

(1) $d(C^i) = 0$,
(2) the differential map $d\colon N^i \longrightarrow \bigwedge V$ is injective, and
(3) any closed element in the ideal $I_s = I(\bigoplus_{i\leq s} N^i)$, generated by the space $\bigoplus_{i\leq s} N^i$ in the free algebra $\bigwedge(\bigoplus_{i\leq s} V^i)$, is exact in $\bigwedge V$.

A smooth manifold M is s-formal if its minimal model is s-formal. Clearly, if M is formal then M is s-formal for every $s > 0$. The main result of [14] shows that sometimes the weaker condition of s-formality implies formality.

Theorem 2.3 ([14]) *Let M be a connected and orientable compact differentiable manifold of dimension $2n$ or $(2n-1)$. Then M is formal if and only if it is $(n-1)$-formal.*

One can check that any simply connected compact manifold is 2-formal. Therefore, Theorem 2.3 implies that any simply connected compact manifold of dimension at most six is formal. (This result was proved earlier in [29].)

In order to detect non-formality, instead of computing the minimal model, which is usually a lengthy process, one can use Massey products, which are obstructions to formality. The simplest type of Massey product is the triple (also known as ordinary) Massey product. This will be defined next.

Let $(\mathcal{A}, d)$ be a DGA (in particular, it can be the de Rham complex of differential forms on a smooth manifold). Suppose that there are cohomology classes $[a_i] \in H^{p_i}(\mathcal{A}), p_i > 0, 1 \leq i \leq 3$, such that $a_1 \cdot a_2$ and $a_2 \cdot a_3$ are exact. Write $a_1 \cdot a_2 = da_{1,2}$ and $a_2 \cdot a_3 = da_{2,3}$. The *(triple) Massey product* of the classes $[a_i]$ is defined as

$$\langle [a_1], [a_2], [a_3] \rangle = [a_1 \cdot a_{2,3} + (-1)^{p_1+1} a_{1,2} \cdot a_3] \in \frac{H^{p_1+p_2+p_3-1}(\mathcal{A})}{[a_1] \cdot H^{p_2+p_3-1}(\mathcal{A}) + [a_3] \cdot H^{p_1+p_2-1}(\mathcal{A})}.$$

Note that a Massey product $\langle [a_1], [a_2], [a_3] \rangle$ on $(\mathcal{A}, d_{\mathcal{A}})$ is zero (or trivial) if and only if there exist $\widetilde{x}, \widetilde{y} \in \mathcal{A}$ such that $a_1 \cdot a_2 = d_{\mathcal{A}} \widetilde{x}$, $a_2 \cdot a_3 = d_{\mathcal{A}} \widetilde{y}$ and

$$0 = [a_1 \cdot \widetilde{y} + (-1)^{p_1+1} \widetilde{x} \cdot a_3] \in H^{p_1+p_2+p_3-1}(\mathcal{A}).$$

We will use also the following property.

Lemma 2.4 *Let M be a connected smooth manifold. Then, Massey products on M can be calculated by using any model of M.*

Proof It is enough to prove the following: $\varphi\colon (\mathcal{A}, d_{\mathcal{A}}) \longrightarrow (\mathcal{B}, d_{\mathcal{B}})$ is a quasi-isomorphism, then

$$\varphi^*(\langle [a_1], [a_2], [a_3]\rangle) = \langle [a_1'], [a_2'], [a_3']\rangle$$

for $[a_j'] = \varphi^*([a_j])$. But this is clear; indeed, take $a_1 \cdot a_2 = d_{\mathcal{A}}x, \ \ a_2 \cdot a_3 = d_{\mathcal{A}}y$ and let

$$f = [a_1 \cdot y + (-1)^{p_1+1} x \cdot a_3] \in \frac{H^{p_1+p_2+p_3-1}(\mathcal{A})}{[a_1] \cdot H^{p_2+p_3-1}(\mathcal{A}) + [a_3] \cdot H^{p_1+p_2-1}(\mathcal{A})}$$

be its Massey product $\langle [a_1], [a_2], [a_3]\rangle$. Then the elements $a_j' = \varphi(a_j)$ satisfy $a_1' \cdot a_2' = d_{\mathcal{B}}x', \ \ a_2' \cdot a_3' = d_{\mathcal{B}}y'$, where $x' = \varphi(x)$, $y' = \varphi(y)$. Therefore,

$$f' = [a_1' {\cdot} y' {+} (-1)^{p_1+1} x' {\cdot} a_3'] = \varphi^*(f) \in \frac{H^{p_1+p_2+p_3-1}(\mathcal{B})}{[a_1'] \cdot H^{p_2+p_3-1}(\mathcal{B}) + [a_3'] \cdot H^{p_1+p_2-1}(\mathcal{B})}$$

is the Massey product $\langle [a_1'], [a_2'], [a_3']\rangle$. □

Now we move to the definition of higher Massey products (see [37]). Given

$$[a_i] \in H^*(\mathcal{A}), \ \ 1 \leq i \leq t, \ \ t \geq 3\,,$$

the Massey product $\langle [a_1], [a_2], \cdots, [a_t]\rangle$, is defined if there are elements $a_{i,j}$ on $\mathcal{A}$, with $1 \leq i \leq j \leq t$ and $(i, j) \neq (1, t)$, such that

$$\begin{aligned} a_{i,i} &= a_i\,, \\ d\, a_{i,j} &= \sum_{k=i}^{j-1} (-1)^{|a_{i,k}|} a_{i,k} \cdot a_{k+1,j}\,. \end{aligned} \tag{1}$$

Then the *Massey product* is the set of all cohomology classes

$$\langle [a_1], [a_2], \cdots, [a_t]\rangle$$
$$= \left\{ \left[\sum_{k=1}^{t-1} (-1)^{|a_{1,k}|} a_{1,k} \cdot a_{k+1,t} \right] \mid a_{i,j} \text{ as in (1)} \right\} \subset H^{|a_1|+\cdots+|a_t|-(t-2)}(\mathcal{A})\,.$$

We say that the Massey product is *zero* if

$$0 \in \langle [a_1], [a_2], \cdots, [a_t]\rangle\,.$$

Note that the higher order Massey product $\langle [a_1], [a_2], \cdots, [a_t]\rangle$ of order $t \geq 4$ is defined if all the Massey products $\langle [a_i], \cdots, [a_{i+p-1}]\rangle$ of order p, where $3 \leq p \leq t-1$ and $1 \leq i \leq t-p+1$, are defined and trivial.

Massey products are related to formality by the following well-known result.

Theorem 2.5 ([10, 37]) *A DGA which has a non-zero Massey product is not formal.*

Another obstruction to the formality is given by the *a-Massey products* introduced in [8], which generalize the triple Massey product and have the advantage of being simpler to compute compared to the higher order Massey products. They are defined as follows. Let $(\mathcal{A}, d)$ be a DGA, and let $a, b_1, \ldots, b_n \in \mathcal{A}$ be closed elements such that the degree $|a|$ of a is even and $a \cdot b_i$ is exact for all i. Let ξ_i be any form such that $d\xi_i = a \cdot b_i$. Then the nth *order a-Massey product* of the b_i is the subset

$$\langle a; b_1, \ldots, b_n \rangle$$

$$:= \left\{ \left[\sum_i (-1)^{|\xi_1|+\cdots+|\xi_{i-1}|} \xi_1 \cdot \ldots \cdot \xi_{i-1} \cdot b_i \cdot \xi_{i+1} \cdot \ldots \cdot \xi_n \right] \mid d\xi_i = a \cdot b_i \right\} \subset H^*(\mathcal{A}).$$

We say that the a-Massey product is *zero* if $0 \in \langle a; b_1, \ldots, b_n \rangle$.

Theorem 2.6 ([8]) *A DGA which has a non-zero a-Massey product is not formal.*

3 Orbifolds

In this section, we collect some results about smooth orbifolds and formality of these spaces (see [1, 6, 18, 23, 33, 34, 36]).

Let X be a topological space. Fix an integer $n > 0$. An *orbifold chart* $(U, \widetilde{U}, \Gamma, \varphi)$ on X consists of an open set $U \subset X$, a connected and open set $\widetilde{U} \subset \mathbb{R}^n$, a finite group $\Gamma \subset \mathrm{GL}(n, \mathbb{R})$ acting smoothly and effectively on $\widetilde{U}$, and a continuous map

$$\varphi \colon \widetilde{U} \longrightarrow U,$$

which is Γ-invariant (that is $\varphi = \varphi \circ \gamma$, for all $\gamma \in \Gamma$) and such that it induces a homeomorphism

$$\Gamma \backslash \widetilde{U} \xrightarrow{\cong} U$$

from the quotient space $\Gamma \backslash \widetilde{U}$ onto U.

Definition 3.1 A smooth orbifold X, of dimension n, is a Hausdorff, paracompact topological space endowed with an *orbifold atlas* $\mathcal{A} = \{(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)\}_{i \in I}$, that is $\mathcal{A}$ is a family of orbifold charts which satisfy the following conditions:

i) $\{U_i\}_{i \in I}$ is an open cover of X;

ii) If $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(U_j, \widetilde{U}_j, \Gamma_j, \varphi_j)$, $i, j \in I$, are two orbifold charts, with $U_i \cap U_j \neq \emptyset$, then for each point $p \in U_i \cap U_j$ there exists an orbifold chart $(U_k, \widetilde{U}_k, \Gamma_k, \varphi_k)$ $(k \in I)$ such that $p \in U_k \subset U_i \cap U_j$;

iii) If $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(U_j, \widetilde{U}_j, \Gamma_j, \varphi_j)$, $i, j \in I$, are two orbifold charts, with $U_i \subset U_j$, then there exist a smooth embedding, called *change of charts* (or *embedding* or *gluing map*)

$$\rho_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$$

(so that $\widetilde{U}_i$ and $\rho_{ij}(\widetilde{U}_i)$ are diffeomorphic) such that

$$\varphi_i = \varphi_j \circ \rho_{ij}.$$

Note that, in most references, the definition given of orbifold chart $(U, \widetilde{U}, \Gamma, \varphi)$ does not explicitly require the condition that the finite group Γ is such that $\Gamma \subset \mathrm{GL}(n, \mathbb{R})$. But since smooth actions are locally linearizable (see [7, p. 308]), any orbifold has an atlas consisting of *linear charts*, that is charts of the form $(U_i, \mathbb{R}^n, \Gamma_i, \varphi_i)$ where Γ_i acts on $\mathbb{R}^n$ via an orthogonal representation $\Gamma_i \subset \mathrm{O}(n)$. Since Γ_i is finite, we can consider an orbifold atlas on a topological space X as given in Definition 3.1.

As with smooth manifolds, two orbifold atlases $\mathcal{A}$ and $\mathcal{A}'$ on X are said to be *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is also an orbifold atlas. Equivalent atlases on X are regarded as defining the same orbifold structure on X. Every orbifold atlas for X is contained in a unique maximal one, and two orbifold atlases are equivalent if and only if they are contained in the same maximal one.

Now, we consider some important points about Definition 3.1. Suppose that X is a smooth orbifold, with two orbifold charts $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(U_j, \widetilde{U}_j, \Gamma_j, \varphi_j)$, such that $U_i \subset U_j$. Let $\rho_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ be a change of charts (in the sense of Definition 3.1). Note that $\rho_{ij} \circ \gamma : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ is also a change of charts, for all $\gamma \in \Gamma_i$. We will see that, for $\gamma \in \Gamma_i$, there is an element $\widetilde{\gamma} \in \Gamma_j$ such that $\rho_{ij} \circ \gamma = \widetilde{\gamma} \circ \rho_{ij}$. In [26] it is proved the following result, which was proved by Satake in [33] under the added assumption that the fixed point set has codimension at least two.

Proposition 3.2 ([26, Proposition A.1]) *Let $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(U_j, \widetilde{U}_j, \Gamma_j, \varphi_j)$ be two orbifold charts on X, with $U_i \subset U_j$. If ρ_{ij}, $\mu_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ are two change of charts, then there exists a unique $\gamma_j \in \Gamma_j$ such that $\mu_{ij} = \gamma_j \circ \rho_{ij}$.*

As a consequence of Proposition 3.2, a change of orbifold charts $\rho_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ induces an injective homomorphism $f_{ij} : \Gamma_i \longrightarrow \Gamma_j$ which is given by

$$\rho_{ij} \circ \gamma = f_{ij}(\gamma) \circ \rho_{ij}, \tag{2}$$

that is $\rho_{ij}(\gamma \cdot x) = f_{ij}(\gamma) \cdot \rho_{ij}(x)$, for all $\gamma \in \Gamma_i$ and $x \in \widetilde{U}_i$.

Also in [26] it is proved the following.

Lemma 3.3 ([26, Lemma A.2]) *Let $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(U_j, \widetilde{U}_j, \Gamma_j, \varphi_j)$ be two orbifold charts on X, with $U_i \subset U_j$. Consider $\rho_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ a change of charts which is equivariant with respect to the injective homomorphism $f_{ij} : \Gamma_i \longrightarrow \Gamma_j$. If there exists an element $\gamma_j \in \Gamma_j$ such that $\rho_{ij}(\widetilde{U}_i) \cap \gamma_j \cdot \rho_{ij}(\widetilde{U}_i) \neq \emptyset$, then $\gamma_j \in \mathrm{Im}(f_{ij})$, and so $\rho_{ij}(\widetilde{U}_i) = \gamma_j \cdot \rho_{ij}(\widetilde{U}_i)$.*

Let X be a smooth orbifold, with an atlas $\{(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)\}$, and let $p \in X$. Consider $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ an orbifold chart around p, that is $p = \varphi_i(x) \in U_i$ with $x \in \widetilde{U}_i$, and denote by $\Gamma_i(x) \subset \Gamma_i$ the isotropy subgroup for the point x. Note that, up to conjugation, the group $\Gamma_i(x)$ does not depend on the choice of the orbifold chart around p. In fact, if $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ is an orbifold chart around p and $p = \varphi_i(x) = \varphi_i(x') \in U_i$ with $x, x' \in \widetilde{U}_i$, then $\Gamma_i(x')$ is conjugate to $\Gamma_i(x)$. (Indeed, there is a group isomorphism $L_a : \Gamma_i(x) \longrightarrow \Gamma_i(x')$ such that, for $\gamma \in \Gamma_i(x)$, $L_a(\gamma) = a\gamma a^{-1}$ with $a \in \mathrm{GL}(n, \mathbb{R})$.) Moreover, if $(U_j, \widetilde{U}_j, \Gamma_j, \varphi_j)$ is other orbifold chart with $p = \varphi_j(y) \in U_j$, then we have a third orbifold chart $(U_k, \widetilde{U}_k, \Gamma_k, \varphi_k)$ around $p = \varphi_k(z) \in U_k$, together with smooth embeddings

$$\rho_{ki} : \widetilde{U}_k \longrightarrow \widetilde{U}_i, \qquad \rho_{kj} : \widetilde{U}_k \longrightarrow \widetilde{U}_j,$$

and injective homomorphisms $f_{ki} : \Gamma_k \longrightarrow \Gamma_i$, $f_{kj} : \Gamma_k \longrightarrow \Gamma_j$ such that ρ_{ki} and ρ_{kj} satisfy (2) with respect to f_{ki} and f_{kj}, respectively. Thus, f_{ki} and f_{kj} define monomorphisms $\Gamma_k(z) \hookrightarrow \Gamma_i(x)$ and $\Gamma_k(z) \hookrightarrow \Gamma_j(y)$. But these monomorphisms must be also onto by Lemma 3.3. So,

$$\Gamma_k(z) \cong \Gamma_j(y) \cong \Gamma_i(x).$$

This justifies that the group $\Gamma_i(x)$ is called the *(local) isotropy group* of p, and it is denoted Γ_p. When $\Gamma_p \neq \mathrm{Id}$, the point p is said to be a *singular* point of the orbifold X. The points p with $\Gamma_p = \mathrm{Id}$ are called *regular* points. The set of singular points

$$S = \{p \in X \mid \Gamma_p \neq \mathrm{Id}\}$$

is called the *singular locus of the orbifold* X (or *orbifold singular set*). Then $X - S$ is a smooth n-dimensional manifold.

The singular locus can be stratified according to the isotropy groups. For each group H, we have a subset $S_H = \{p \in X \mid \Gamma_p = H\}$. It is easily seen that the connected components of S_H are locally closed smooth submanifolds of X. Moreover, the closure $\overline{S}_H$ contains components of other $S_{H'}$, with $H < H'$. This is an immediate consequence of the fact that it holds on every orbifold chart (in an orbifold chart $(U, \widetilde{U}, \Gamma, \varphi)$, the sets S_H are linear subsets of $\widetilde{U}$). As a consequence, we can give a CW-structure to X compatible with the stratification, that is, such that the subsets $\overline{S}_H$ are CW-subcomplexes. For basics on stratified spaces the reader can see [18].

Any smooth manifold is a smooth orbifold for which each of the finite groups Γ_i is the trivial group, so that we get $\widetilde{U}_i$ homeomorphic to U_i. The most natural examples of orbifolds appear when we take the quotient space $X = M/\Gamma$ of a smooth manifold M by a finite group Γ acting smoothly and effectively on M. Let $\pi: M \to X$ be the natural projection. Note that given un point $p \in M$ with isotropy group $\Gamma_p \subset \Gamma$, then there is a chart $(U, \widetilde{U}, \phi)$ of $p = \phi(x) \in U$ in M, with $U = \phi(\widetilde{U})$, such that U is Γ_p-invariant. Then, an orbifold chart of $\pi(p) \in X$ is $(\pi(U), \widetilde{U}, \Gamma_p, \pi \circ \phi)$, the change of charts ρ_{ij} are the change of coordinates on the manifold M, and the monomorphisms f_{ij} are the identity map of Γ_p. Such an orbifold

$$X = M/\Gamma$$

is called *effective global quotient* orbifold [1, Definition 1.8].

Moreover, if M is oriented and the action of Γ preserves the orientation, then X is an oriented orbifold. In general, an orbifold X, with atlas $\{(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)\}$, is *oriented* if each $\widetilde{U}_i$ is oriented, the action of Γ_i is orientation-preserving, and all the change of charts $\rho_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ are orientation-preserving.

Definition 3.4 ([6]) Let X and Y be two orbifolds (not necessarily of the same dimension) and let $\{(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)\}$ and $\{(V_j, \widetilde{V}_j, \Upsilon_j, \psi_j)\}$ be atlases for X and Y, respectively. A map $f : X \longrightarrow Y$ is said to be an *orbifold map* (or *smooth map*) if f is a continuous map between the underlying topological spaces, and for every point $p \in X$ there are orbifold charts $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(V_i, \widetilde{V}_i, \Upsilon_i, \psi_i)$ for p and $f(p)$, respectively, with $f(U_i) \subset V_i$, a differentiable map $\widetilde{f}_i : \widetilde{U}_i \longrightarrow \widetilde{V}_i$, and a homomorphism $\varpi_i : \Gamma_i \to \Upsilon_i$ such that $\widetilde{f}_i \circ \gamma = \varpi_i(\gamma) \circ \widetilde{f}_i$ for all $\gamma \in \Gamma_i$, and

$$f_{|U_i} \circ \varphi_i = \psi_i \circ \widetilde{f}_i.$$

Moreover, f is said to be *good* if every map $\widetilde{f}_i$ is *compatible with the changes of charts* in the following sense:

i) if $\rho_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ is a change of charts for p, then there is a change of charts $\mu(\rho_{ij}) : \widetilde{V}_i \longrightarrow \widetilde{V}_j$ for $f(p)$ such that

$$\widetilde{f}_j \circ \rho_{ij} = \mu(\rho_{ij}) \circ \widetilde{f}_i;$$

ii) if $\rho_{ki} : \widetilde{U}_k \longrightarrow \widetilde{U}_i$ is a change of charts for p, then

$$\mu(\rho_{ij} \circ \rho_{ki}) = \mu(\rho_{ij}) \circ \mu(\rho_{ki}).$$

Therefore, an orbifold map $f : X \longrightarrow Y$ is determined by a smooth map $\widetilde{f}_i : \widetilde{U}_i \longrightarrow \widetilde{V}_i$, for every orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ on X, such that every $\widetilde{f}_i$ is Γ_i-equivariant and compatible with the change of orbifold charts.

Note that conditions i) and ii) in Definition 3.4 imply that the composition of orbifold maps is an orbifold map. Moreover, if $f : X \longrightarrow Y$ is an orbifold map, then there exists an induced homomorphism from Γ_p to $\Upsilon_{f(p)}$. Also, considering $\mathbb{R}$

as an orbifold, we can define *orbifold functions* on an orbifold X as orbifold maps $f\colon X \longrightarrow \mathbb{R}$.

Two orbifolds X and Y are said to be *diffeomorphic* if there exist orbifold maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$, where 1_X and 1_Y are the respective identity maps. Note that a diffeomorphism between orbifolds gives a homeomorphism between the underlying topological spaces.

Many of the usual differential geometric concepts that hold for smooth manifolds also hold for smooth orbifolds; in particular, the notion of vector bundle [6, Definition 4.2.7]. Using transition maps, orbifold vector bundles can be defined as follows [34].

Definition 3.5 Let X be a smooth orbifold, of dimension n, and let $\{(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)\}_{i\in I}$ be an atlas on X. An *orbifold vector bundle over X and fiber* $\mathbb{R}^m$ consists of a smooth orbifold E, of dimension $m + n$, and an orbifold map

$$\pi : E \longrightarrow X,$$

called *projection*, satisfying the following conditions:

i) For every orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ on X, there exists a homomorphism

$$\rho_i\colon \Gamma_i \longrightarrow \mathrm{GL}(\mathbb{R}^m)$$

and an orbifold chart $(V_i, \widetilde{V}_i, \Gamma_i, \Psi_i)$ on E, such that $V_i = \pi^{-1}(U_i)$, $\widetilde{V}_i = \widetilde{U}_i \times \mathbb{R}^m$, the action of Γ_i on $\widetilde{U}_i \times \mathbb{R}^m$ is the diagonal action (i.e. $\gamma \cdot (x, u) = (\gamma \cdot x, \rho_i(\gamma)(u))$, for $\gamma \in \Gamma_i$, $x \in \widetilde{U}_i$ and for $u \in \mathbb{R}^m$), and the map

$$\Psi_i\colon \widetilde{V}_i = \widetilde{U}_i \times \mathbb{R}^m \longrightarrow E_{|U_i} := \pi^{-1}(U_i)$$

is such that $\pi_{|V_i} \circ \Psi_i = \varphi_i \circ \mathrm{pr}_1$, where $\mathrm{pr}_1\colon \widetilde{U}_i \times \mathbb{R}^m \longrightarrow \widetilde{U}_i$ is the natural projection, Ψ_i is Γ_i-invariant for the action of Γ_i on $\widetilde{U}_i \times \mathbb{R}^m$, and it induces a homeomorphism

$$\Gamma_i\backslash(\widetilde{U}_i \times \mathbb{R}^m) \cong E_{|U_i}$$

ii) If $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(U_j, \widetilde{U}_j, \Gamma_j, \varphi_j)$ are two orbifold charts on X, with $U_i \subset U_j$, and $\rho_{ij}\colon \widetilde{U}_i \longrightarrow \widetilde{U}_j$ is a change of charts, then there exists a differentiable map, called *transition map*

$$g_{ij}\colon \widetilde{U}_i \longrightarrow \mathrm{GL}(\mathbb{R}^m),$$

and a change of charts $\lambda_{ij}\colon \widetilde{V}_i = \widetilde{U}_i \times \mathbb{R}^m \longrightarrow \widetilde{V}_j = \widetilde{U}_j \times \mathbb{R}^m$ on E, such that

$$\lambda_{ij}(x, u) = \big(\rho_{ij}(x), g_{ij}(x)(u)\big),$$

for all $(x, u) \in \widetilde{U}_i \times \mathbb{R}^m$.

Note that if $\pi : E \longrightarrow X$ is an orbifold vector bundle, and $p \in X$, then the *fiber* $\pi^{-1}(p)$ is not always a vector space. In fact, if $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ is an orbifold chart on X around $p = \varphi_i(x) \in X$, then

$$\pi^{-1}(p) \cong \Gamma_p \backslash (x \times \mathbb{R}^m) \cong \Gamma_p \backslash \mathbb{R}^m,$$

where $\Gamma_p = \Gamma_i(x)$ is the isotropy group of p. Thus, $\pi^{-1}(p) \cong \mathbb{R}^m$ if p is a regular point ($\Gamma_p = \mathrm{Id}$) of X, but $\pi^{-1}(p)$ is not a vector space when p is a singular point.

Definition 3.6 A *section* (or *orbifold smooth section*) of an orbifold vector bundle $\pi : E \longrightarrow X$ is an orbifold map $s : X \longrightarrow E$ such that $\pi \circ s = 1_X$. Therefore, if $\{(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)\}$ is an atlas on X, then s consists of a family of smooth maps $\{s_i : \widetilde{U}_i \longrightarrow \mathbb{R}^m\}$, such that every s_i is Γ_i-equivariant and compatible with the changes of charts on X (in the sense of Definition 3.4). We denote the space of (orbifold smooth) sections of E by $\mathcal{C}^\infty(E)$.

To construct the *orbifold tangent bundle* TX of an orbifold X, of dimension n, we continue to use the notation of Definition 3.5. We define the orbifold charts and the transition maps for TX as follows. For each orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ of X, we consider the tangent bundle $T\widetilde{U}_i$ over $\widetilde{U}_i$, so $T\widetilde{U}_i \cong \widetilde{U}_i \times \mathbb{R}^n$. Take $\rho_i : \Gamma_i \longrightarrow \mathrm{GL}(\mathbb{R}^n)$ the homomorphism given by the action of Γ_i on $\mathbb{R}^n$. Then $(E_{|U_i}, \widetilde{U}_i \times \mathbb{R}^n, \Gamma_i, \rho_i, \Psi_i)$ is an orbifold chart for TX, where $E_{|U_i} = \Gamma_i \backslash T\widetilde{U}_i$. Moreover, if $\rho_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ is a change of charts for X, the transition map

$$g_{ij} : \widetilde{U}_i \longrightarrow \mathrm{GL}(\mathbb{R}^n)$$

for TX is such that $g_{ij}(x)$ is the Jacobian matrix of the map ρ_{ij} at the point $x \in \widetilde{U}_i$. Therefore TX is a $2n$-dimensional orbifold, and the natural projection $\pi : TX \longrightarrow X$ defines a smooth map of orbifolds, with fibers $\pi^{-1}(p) \cong \Gamma_p \backslash (x \times \mathbb{R}^n) \cong \Gamma_p \backslash \mathbb{R}^n$, for $p \in X$. Therefore, one can consider tangent vectors to X at the point $p \in X$ if p is a regular point.

The orbifold cotangent bundle T^*X and the orbifold tensor bundles are constructed similarly. Thus, one can consider Riemannian metrics, almost complex structures, orbifold forms, connections, etc.

An (orbifold) *Riemannian metric* g on X is a positive definite symmetric tensor in $T^*X \otimes T^*X$. This is equivalent to have, for each orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ on X, a Riemannian metric g_i on the open set $\widetilde{U}_i$ that is invariant under the action of Γ_i on $\widetilde{U}_i$ (Γ_i acts on $\widetilde{U}_i$ by isometries), and the change of charts $\rho_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ are isometries, that is $\rho_{ij}^*\big(g_j|_{\rho_{ij}(\widetilde{U}_i)}\big) = g_i$.

An (orbifold) *almost complex structure* J on X is an endomorphism $J : TX \longrightarrow TX$ such that $J^2 = -\mathrm{Id}$. Thus, J is determined by an almost complex structure J_i on $\widetilde{U}_i$, for every orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ on X, such that the action of Γ_i on $\widetilde{U}_i$ is by biholomorphic maps, and any change of charts $\rho_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ is a holomorphic embedding.

An *orbifold p-form* α on X is a section of $\bigwedge^p T^*X$. This means that, for each orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ on X, we have a differential p-form α_i on the open set

$\widetilde{U}_i$, such that every α_i is Γ_i-invariant (i.e. $\gamma_i^*(\alpha_i) = \alpha_i$, for $\gamma_i \in \Gamma_i$), and any change of charts $\rho_{ij} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ satisfies $\rho_{ij}^*(\alpha_j) = \alpha_i$.

The space of p-forms on X is denoted by $\Omega^p_{orb}(X)$. The wedge product of orbifold forms and the exterior differential d on X are well defined. Thus, we have

$$d\colon \Omega^p_{orb}(X) \longrightarrow \Omega^{p+1}_{orb}(X)\,.$$

The constant sheaf $\mathbb{R}$ has a resolution

$$0 \longrightarrow \mathbb{R} \longrightarrow \boldsymbol{\Omega}^0_{orb} \longrightarrow \boldsymbol{\Omega}^1_{orb} \longrightarrow \dots , \tag{3}$$

where $\boldsymbol{\Omega}^p_{orb}$ is the sheaf of smooth sections of $\bigwedge^p T^*X$. To prove that this is a resolution, it is enough to prove that it is exact over any neighborhood of the form $U = \widetilde{U}/\Gamma$. As the group Γ is finite, it is conjugate to a subgroup of $\mathrm{O}(n)$, so we can assume that $\Gamma \subset \mathrm{O}(n)$. We take $\widetilde{U} = B_\epsilon(0)$ (the ball in $\mathbb{R}^n$ of radius ϵ around the origin). Then

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0(\widetilde{U}) \longrightarrow \Omega^1(\widetilde{U}) \longrightarrow \dots \tag{4}$$

is exact. The functor $V \mapsto V^\Gamma$ that sends any vector space V with a Γ-action, to its Γ-invariant part, is an exact functor. This is true because Γ is finite, so the map $\pi_{V,\Gamma} : V \to V$, $\pi_{V,\Gamma}(v) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} g\,v$, is the projection over V^Γ, and $V = V^\Gamma \oplus \ker \pi_{V,\Gamma}$. If $V_1 \to V_2 \to V_3$ is an exact sequence, then we can decompose it into two exact sequences $V_1^\Gamma \to V_2^\Gamma \to V_3^\Gamma$ and $\ker \pi_{V_1,\Gamma} \to \ker \pi_{V_2,\Gamma} \to \ker \pi_{V_3,\Gamma}$. Therefore taking the Γ-invariant forms of (4), we get an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0(\widetilde{U})^\Gamma \longrightarrow \Omega^1(\widetilde{U})^\Gamma \longrightarrow \dots$$

meaning that sequence of sheaves (3) is exact. Since (3) is exact, the cohomology of the complex $(\Omega^*_{orb}(X), d)$ is isomorphic to the singular cohomology $H^*(X, \mathbb{R})$ (cf. [40, Example 2.11]).

We can see more explicitly this isomorphism with duality by pairing with homology classes in singular homology $H_*(X, \mathbb{R})$. Recall that we have a CW-complex structure for X such that the singular sets $S_H = \{p \in X \,|\, \Gamma_p = H\}$ are CW-subcomplexes. Then for an orbifold k-form α on X and a k-cell $D \subset X$, we have an integration map $\int_D \alpha$. This is defined as follows: we can assume that D is inside an orbifold chart $(U, \widetilde{U}, \Gamma, \varphi)$. Let $D \subset \overline{S}_H$, where H is some isotropy group, and assume that the interior of D lies in S_H. Under the quotient map $\pi : \widetilde{U} \to U$, the preimage of $\pi^{-1}(S_H \cap U)$ is contained in a linear subspace, and the map $\pi : \pi^{-1}(S_H \cap U) \to S_H \cap U$ is $|H| : 1$. We define $\int_D \alpha = \frac{|H|}{|\Gamma|}\int_{\pi^{-1}(D)} \widetilde{\alpha}$, where $\widetilde{\alpha} \in \Omega^k(\widetilde{U})$ is the representative of α in the orbifold chart. It is easily seen that this is compatible with the orbifold changes of charts, and that it satisfies an orbifold version of Stokes theorem.

Remark 3.7 Suppose that $X = M/\Gamma$ is an oriented effective global orbifold, that is X is the quotient of a smooth manifold M by a finite group Γ acting smoothly and effectively on M. Then, the definition of orbifold forms implies that any Γ-invariant differential k-form α on M defines an orbifold k-form $\widehat{\alpha}$ on X, and vice-versa. Moreover, it is straightforward to check that the exterior derivative on M preserves Γ-invariance. Thus, if $\left(\Omega^k(M)\right)^\Gamma$ denotes the space of the Γ-invariant differential k-form on M, and $H^k(M, \mathbb{R})^\Gamma \subset H^k(M, \mathbb{R})$ is the subspace of the cohomology classes of degree k on M such that each of these classes has a representative that is a Γ-invariant differential k-form, then we have

$$\Omega^k_{orb}(X) = \left(\Omega^k(M)\right)^\Gamma, \qquad H^k(X, \mathbb{R}) = H^k(M, \mathbb{R})^\Gamma. \tag{5}$$

The last formula follows by the exactness of the functor $V \mapsto V^\Gamma$, so that taking cohomology commutes with taking Γ-invariant part.

For any compact supported orbifold n-form $\widehat{\alpha}$ on X, which is by definition a Γ-invariant compact supported differential n-form α on M, the integration of $\widehat{\alpha}$ on X is defined by

$$\int_X \widehat{\alpha} = |\Gamma| \int_M \alpha, \tag{6}$$

where $|\Gamma|$ is the order of the group Γ. More generally, one can extend the notion of integration to arbitrary orbifolds by working in orbifold charts via a partition of unity ([1, p. 34], [33]).

Definition 3.8 Let X be an orbifold. A *minimal model* for X is a minimal model $(\bigwedge V, d)$ for the DGA $(\Omega^*_{orb}(X), d)$. The orbifold X is *formal* if its minimal model is formal (see Sect. 2).

Proposition 3.9 *Let* $(\bigwedge V, d)$ *be the minimal model of an orbifold* X. *Then* $H^*(\bigwedge V) = H^*(X, \mathbb{R})$, *where the latter means singular cohomology with real coefficients.*

For a simply connected orbifold X, the dual of the real homotopy vector space $\pi_i(X) \otimes \mathbb{R}$ is isomorphic to the space V^i of generators in degree i, for any i, where $\pi_i(X)$ is the homotopy group of order i of the underlying topological space in X. In fact, the proof given in [10] for simply connected manifolds, also works for simply connected orbifolds (that is, orbifolds for which the topological space X is simply connected).

Moreover, the proof of Theorem 2.3 given in [14] only uses that the cohomology $H^*(M)$ is a Poincaré duality algebra. By Satake [33], we know that the singular cohomology of a compact oriented orbifold also satisfies a Poincaré duality. Thus, Theorem 2.3 also holds for compact connected orientable orbifolds. Hence, we have the following lemma.

Lemma 3.10 *Any simply connected compact orbifold of dimension at most* 6 *is formal.*

The notion of formality is also defined for CW-complexes which have a minimal model $(\bigwedge V, d)$. Such a minimal model is constructed as the minimal model associated to the differential complex of piecewise-linear polynomial forms [13, 18]. In particular, we have a minimal model $(\bigwedge V, d)$ for orbifolds.

4 Elliptic Differential Operators on Orbifolds

Here we study elliptic differential operators on complex orbifolds by adapting to these spaces the elliptic operator theory on complex manifolds [40, Chap. IV].

A *complex orbifold*, of complex dimension n, is an orbifold X whose orbifold charts are of the form $\{(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)\}$, where $\widetilde{U}_i \subset \mathbb{C}^n$, $\Gamma_i \subset \mathrm{GL}(n, \mathbb{C})$ is a finite group acting on $\widetilde{U}_i$ by biholomorphisms, and all the changes of charts $\rho_{ij}: \widetilde{U}_i \longrightarrow \widetilde{U}_j$ are holomorphic embeddings. Thus, any complex orbifold has associated an almost complex structure J.

If X and Y are complex orbifolds, a map $f: X \longrightarrow Y$ is said to be an *orbifold holomorphic map* (or *holomorphic map*) if f is a continuous map between the underlying topological spaces, and for every point $p \in X$ there are orbifold charts $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(V_i, \widetilde{V}_i, \Upsilon_i, \psi_i)$ for p and $f(p)$, respectively, with $f(U_i) \subset V_i$, and a holomorphic map $\widetilde{f}_i: \widetilde{U}_i \longrightarrow \widetilde{V}_i$ such that $\widetilde{f}_i$ is Γ_i-*equivariant* and compatible with changes of charts (in the sense of Definition 3.4).

Similarly to orbifold vector bundles, one can define *complex orbifold vector bundles*. Let X be a complex orbifold, of complex dimension n. A *complex orbifold vector bundle* over X and fiber $\mathbb{C}^m$ consists of a complex orbifold E, of complex dimension $m + n$, and a holomorphic orbifold map

$$\pi: E \longrightarrow X,$$

such that the atlas on E has charts of the type $(E_{|U_i}, \widetilde{U}_i \times \mathbb{C}^m, \Gamma_i, \rho_i, \Psi_i)$, where $\rho_i: \Gamma_i \longrightarrow \mathrm{GL}(\mathbb{C}^m)$ is a homomorphism, and

$$\Psi_i: \widetilde{U}_i \times \mathbb{C}^m \longrightarrow E_{|U_i} := \pi^{-1}(U_i)$$

is a Γ_i-invariant map, for the diagonal action of Γ_i on $\widetilde{U}_i \times \mathbb{C}^m$ (the group Γ_i acts on $\mathbb{C}^m$ via ρ_i), with $\Gamma_i \backslash (\widetilde{U}_i \times \mathbb{C}^m) \cong E_{|U_i}$.

A *Hermitian metric* h on X is a collection $\{h_i\}$, where each h_i is a Hermitian metric on the open set $\widetilde{U}_i$ of the (complex) orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ on X, such that every h_i is Γ_i-invariant, and all the changes of charts $\rho_{ij}: \widetilde{U}_i \longrightarrow \widetilde{U}_j$ are given by holomorphic and isometric embeddings. A slight modification of the usual partition of unity argument shows that every complex orbifold has a Hermitian metric [25].

Complex orbifold forms on a complex orbifold and the orbifold Dolbeault cohomology will be considered in Sect. 5.

Let $E \rightarrow X$ be a complex orbifold vector bundle endowed with a Hermitian metric. A *Hermitian connection* ∇ on E is defined to be a collection $\{\nabla_i\}$, where each ∇_i is a Γ_i-equivariant Hermitian connection on $\widetilde{U}_i$, for every complex orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ on X, and such that ∇_i is compatible with changes of charts. Using ∇, we can define *Sobolev norms on sections* of E. For a section s supported on a chart U_i, define

$$||s||_{W^m(E|_{U_i})} := \frac{1}{|\Gamma_i|}||s||_{W^m(\widetilde{U}_i)} ,$$

where W^m denotes the usual Sobolev m-norm. That is, $||s||_{W^m(\widetilde{U}_i)} = \sum_{k=0}^m ||\nabla_i^k s||_{L^2}$. For orbifold sections s of E, we define $||s||_{W^m(E)} = \sum_i ||\rho_i s||_{W^m(E|_{U_i})}$, where $\{U_i\}$ is a covering of X by orbifold charts, and $\{\rho_i\}$ a subordinated (orbifold) partition of unity. The space $W^m(E)$ is the completion with respect to the W^m-norm of the space of (orbifold smooth) sections. In particular, $W^0(E) = L^2(E)$. The Sobolev embedding theorem and Rellich's lemma hold for orbifolds (the proof in [40, Chap. IV.1] can be extended to orbifolds verbatim).

A *differential operator* $L \in \mathrm{Diff}_k(E, F)$ *of order* k between complex vector bundles E and F is a linear operator which is on an orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ of the form

$$L = \sum_{|\sigma|\leq k} a_\sigma(x)\frac{D^{|\sigma|}}{D^\sigma x} , \tag{7}$$

where $a_\sigma(x) \in \mathrm{Hom}(E, F)$ is defined on each $\widetilde{U}_i$ and it is Γ_i-equivariant. The *symbol* of L is defined as

$$\sigma_k(L)(x, \xi) = \sum_{|\sigma|=k} a_\sigma(x)\xi^\sigma ,$$

for $x \in \widetilde{U}_i$, $\xi \in \mathbb{R}^n$. It is easily seen that this defines a symbol $\sigma_k(L)(x, \xi)$, for $x \in \widetilde{U}_i$ and $\xi \in T_x^*\widetilde{U}_i$, which is Γ_i-equivariant, that is, an orbifold section of the orbifold bundle $\mathrm{Hom}(E, F) \otimes (T^*X)^{\otimes k}$. We say that L is an *elliptic operator* if the symbol of L is an isomorphism for any $\xi \neq 0$.

The *adjoint* L^* *of a differential operator* $L \in \mathrm{Diff}_k(E, F)$ is the operator defined by:

$$\langle L(s)\, , t\rangle = \langle s\, , L^*(t)\rangle , \tag{8}$$

for any orbifold sections s, t of E, F, respectively. It turns out that $L^* \in \mathrm{Diff}_k(F, E)$. For checking this, we go to an orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$. Then L is written as (7). Then the equality (8), for compactly supported Γ_i-equivariant sections on $\widetilde{U}_i$, shows that L^* has the form (7) for suitable coefficients $a_\sigma(x) \in \mathrm{Hom}(F, E)$, also Γ_i-equivariant. An operator $L \in \mathrm{Diff}_k(E) := \mathrm{Diff}_k(E, E)$ is called *self-adjoint* if $L^* = L$.

Theorem 4.1 *Let $L \in \mathrm{Diff}_k(E)$ be self-adjoint and elliptic. Let*

$$\mathcal{H}_L(E) = \{v \in \mathcal{C}^\infty(E) \mid L(v) = 0\}.$$

Then there exist linear mappings H_L, $G_L\colon \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(E)$ such that

(1) $H_L(\mathcal{C}^\infty(E)) = \mathcal{H}_L(E)$ *and* $\dim \mathcal{H}_L(E) < \infty$,
(2) $L \circ G_L + H_L = G_L \circ L + H_L = \mathrm{Id}$,
(3) H_L, G_L *extend to bounded operators in $L^2(E)$, and*
(4) $\mathcal{C}^\infty(E) = \mathcal{H}_L(E) \oplus G_L \circ L(\mathcal{C}^\infty(E)) = \mathcal{H}_L(E) \oplus L \circ G_L(\mathcal{C}^\infty(E))$, *with the decomposition being orthogonal with respect to the L^2-metric.*

Proof The theory in Chap. VI.3 of [40] works for orbifolds. A pseudo-differential operator is a linear operator L which is locally of the form

$$u(x) \longmapsto L(p)u(x) = \int p(x,\xi)\widehat{u}(\xi)e^{i\langle x,\xi\rangle}d\xi$$

for compactly supported $u(x)$, where $p(x,\xi)$ is a Γ-invariant function on $T^*\widetilde{U} = \widetilde{U}\times\mathbb{R}^n$ such that the growth conditions in Definition 3.1 of [40, Chap. VI] hold. Note that $L(p)$ takes Γ-equivariant sections to Γ-equivariant sections. If we decompose $\mathcal{C}^\infty(\widetilde{U}) = \mathcal{C}^\infty(\widetilde{U})^\Gamma \oplus D$, where $D = \{s \mid \sum_{g\in\Gamma} g^*s = 0\}$, then $L(p)$ maps D to D.

A pseudo-differential operator

$$L\colon \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(E)$$

is of order k if it extends continuously to $L\colon W^m(E) \longrightarrow W^{m+k}(E)$ for every m. Note that locally, L maps Γ-equivariant sections of $W^m(\widetilde{U})$ to Γ-equivariant sections of $W^{m+k}(\widetilde{U})$. In particular, a differential operator of order m is a pseudo-differential operator of order m.

First, using the ellipticity of L, one constructs a pseudo-differential operator $\widetilde{L}$, such that $L \circ \widetilde{L} - \mathrm{Id}$ and $\widetilde{L} \circ L - \mathrm{Id}$ are of order -1. With this, one can check the regularity of the solutions of the equation $Lv = 0$, that is

$$\mathcal{H}_L(E)_m = \{v \in W^m(E) \mid Lv = 0\} \subset \mathcal{C}^\infty(E),$$

so that $\mathcal{H}_L(E) = \mathcal{H}_L(E)_m$ for all m. Using Rellich's lemma, this proves that $\mathcal{H}_L(E)$ is of finite dimension. Now H_L is defined as projection onto $\mathcal{H}_L(E)$, and G_L is defined as the inverse of L on the orthogonal complement to $\mathcal{H}_L(E)$ and zero on $\mathcal{H}_L(E)$. With this, it turns out that G_L is an operator of negative order. The rest of the assertions are now straightforward. □

Let $E_0, E_1, \ldots, E_N$ be a collection of complex orbifold vector bundles over X. A sequence of differential operators

$$\mathcal{C}^\infty(E_0) \xrightarrow{L_0} \mathcal{C}^\infty(E_1) \xrightarrow{L_1} \mathcal{C}^\infty(E_2) \xrightarrow{L_2} \ldots \xrightarrow{L_{N-1}} \mathcal{C}^\infty(E_N)$$

is an *elliptic complex* if $L_i \circ L_{i-1} = 0$, $i = 1, \cdots, N-1$, and the sequence of symbols

$$0 \longrightarrow (E_0)_x \xrightarrow{\sigma(L_0)} (E_1)_x \xrightarrow{\sigma(L_1)} (E_2)_x \xrightarrow{\sigma(L_2)} \dots \xrightarrow{\sigma(L_{N-1})} (E_N)_x \longrightarrow 0$$

is exact for all $x \in X$, $\xi \neq 0$. The cohomology groups of the complex are defined to be

$$H^q(E) := \frac{\ker L_q}{\operatorname{im} L_{q-1}} .$$

Writing $E = \bigoplus_{i=1}^N E_i$, $L = \sum_{i=1}^{N-1} L_i$, and

$$\Delta = L^* L + L L^*$$

with respect to some fixed Hermitian metric on every E_i, $0 \le i \le N$, we have an elliptic operator $\Delta : \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(E)$. Note that $\Delta : \mathcal{C}^\infty(E_i) \longrightarrow \mathcal{C}^\infty(E_i)$, for all $i = 0, 1, \dots, N$. We denote

$$\mathcal{H}^j(E) = \ker(\Delta|_{E_j}) .$$

The following is an analogue of Theorem 5.2 in [40, Chap. V].

Theorem 4.2 *Let $(\mathcal{C}^\infty(E), L)$ be an elliptic complex equipped with an inner product. Then the following statements hold:*

(1) *There is an orthogonal decomposition*

$$\mathcal{C}^\infty(E) = \mathcal{H}(E) \oplus LL^* G(\mathcal{C}^\infty(E)) \oplus L^* LG(\mathcal{C}^\infty(E)) .$$

(2) $\mathrm{Id} = H + \Delta G = H + G\Delta$, $HG = GH = H\Delta = \Delta H = 0$, $L\Delta = \Delta L$, $L^*\Delta = \Delta L^*$, $LG = GL$, $L^*G = GL^*$, $LH = HL = L^*H = HL^* = 0$.
(3) $\dim \mathcal{H}^j(E) < \infty$, *and there is a canonical isomorphism* $\mathcal{H}^j(E) \cong H^j(E)$.
(4) $\Delta v = 0 \iff Lv = L^*v = 0$ *for all* $v \in \mathcal{C}^\infty(E)$.

The complex

$$\Omega^0_{orb}(X) \xrightarrow{d} \Omega^1_{orb}(X) \xrightarrow{d} \Omega^2_{orb}(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{orb}(X)$$

is elliptic. Hence Theorem 4.2 implies that

$$H^k(X) \cong \mathcal{H}^k(X) = \ker(\Delta : \Omega^k_{orb}(X) \longrightarrow \Omega^k_{orb}(X)) , \tag{9}$$

where $\Delta = dd^* + d^*d$.

5 Kähler Orbifolds

Let X be a complex orbifold, of complex dimension n, with an atlas $\{(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)\}$. As for complex manifolds, we can consider *orbifold complex forms* on X. An *orbifold complex k-form* α on X is given by a complex k-form α_i on the open set $\widetilde{U}_i$, for each orbifold chart $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$, and such that every α_i is Γ_i-invariant and preserved by all the change of charts. We say that α is *bigraded* of type (p, q), with $k = p + q$, if each α_i is a (p, q)-form on $\widetilde{U}_i$. Denote by $\Omega^{p,q}_{orb}(X)$ the space of orbifold (p, q)-forms on X. Then, we have the type decomposition of the exterior derivative $d = \partial + \overline{\partial}$, where

$$\partial\colon \Omega^{p,q}_{orb}(X) \longrightarrow \Omega^{p+1,q}_{orb}(X) \quad \text{and} \quad \overline{\partial}\colon \Omega^{p,q}_{orb}(X) \longrightarrow \Omega^{p,q+1}_{orb}(X)\,.$$

The (orbifold) Dolbeault cohomology of X is defined to be

$$H^{p,q}(X) := \frac{\ker(\overline{\partial}\colon \Omega^{p,q}_{orb}(X) \longrightarrow \Omega^{p,q+1}_{orb}(X))}{\overline{\partial}(\Omega^{p,q-1}_{orb}(X))}\,.$$

Fix an orbifold Hermitian metric on X. For any $p \geq 0$, the complex

$$0 \longrightarrow \Omega^{p,0}_{orb}(X) \xrightarrow{\overline{\partial}} \Omega^{p,1}_{orb}(X) \xrightarrow{\overline{\partial}} \Omega^{p,2}_{orb}(X) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \Omega^{p,n}_{orb}(X) \longrightarrow 0$$

is elliptic, where n is the complex dimension of X. Hence Theorem 4.2 implies that

$$H^{p,q}(X) \cong \mathcal{H}^{p,q}(X) = \ker(\Delta_{\overline{\partial}}\colon \Omega^{p,q}_{orb}(X) \longrightarrow \Omega^{p,q}_{orb}(X))\,.$$

where $\Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$.

Let (X, J, h) be a complex Hermitian orbifold, with orbifold complex structure J and Hermitian metric h. Thus, we have an orbifold Riemannian metric $g = \operatorname{Re} h$ and an orbifold 2-form $\omega \in \Omega^{1,1}_{orb}(X)$ defined by

$$\omega = \operatorname{Im} h\,.$$

Then, $\omega^n \neq 0$, where n is the complex dimension of X.

Definition 5.1 A complex Hermitian orbifold (X, h) is called *Kähler orbifold* if the associated fundamental form ω is closed, that is $d\omega = 0$.

Proposition 5.2 *For a compact Kähler orbifold,*

$$\Delta = 2\Delta_{\overline{\partial}}\,.$$

Therefore $\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$.

Proof This is true on the dense open subset of non-singular points of X by Theorem 4.7 of [40, Chap. V]. So it holds everywhere on X. More specifically, at a singular point $p \in X$, we take an orbifold chart $(U, \widetilde{U}, \Gamma, \varphi)$. Then the equality $\Delta = 2\Delta_{\overline{\partial}}$ holds on $\widetilde{U} - S$, where S is a collection of linear subspaces (where Γ acts with non-trivial isotropy). As this is an equality of two smooth differential operators on $\widetilde{U}$, they coincide everywhere on $\widetilde{U}$. Note that this is the meaning of being equal as *orbifold* differential operators. □

Corollary 5.3 *For a compact Kähler orbifold, $b_k(X)$ is even for k odd.*

Proof Clearly, conjugation gives a map $\Omega^{p,q}_{orb}(X) \longrightarrow \Omega^{q,p}_{orb}(X)$ that commutes with Δ (as this is a real operator). Therefore, the induced map $\mathcal{H}^{p,q}(X) \longrightarrow \mathcal{H}^{q,p}(X)$ is an isomorphism. In particular, $h^{p,q}(X) = h^{q,p}(X)$, where $h^{p,q}(X) = \dim H^{p,q}(X)$. Thus, $b_k(X) = \sum_{k=p+q} h^{p,q}(X)$ is even for k odd. □

Lemma 5.4

(1) *Take $\alpha \in \Omega^{p,q}_{orb}(X)$ with $\partial\alpha = 0$. If $\alpha = \overline{\partial}\beta$ for some β, then there exists ψ such that $\alpha = \partial\overline{\partial}\psi$.*
(2) *Take $\alpha \in \Omega^{p,q}_{orb}(X)$ with $\overline{\partial}\alpha = 0$. If $\alpha = \partial\beta$ for some β, then there exists ψ such that $\alpha = \partial\overline{\partial}\psi$.*

Proof Using Theorem 4.2,

$$\alpha = H\alpha + \Delta_{\overline{\partial}}G\alpha = H\alpha + \overline{\partial}\,\overline{\partial}^*G\alpha + \overline{\partial}^*\overline{\partial}G\alpha\,,$$

where $G = G_{\overline{\partial}}$ is the Green's operator associated to $\overline{\partial}$. As $\alpha = \overline{\partial}\beta$, the cohomology class represented by α vanishes, so $H\alpha = 0$. Then, since G commutes with $\overline{\partial}$, we have $\overline{\partial}G\alpha = G\overline{\partial}\alpha = 0$. Hence $\alpha = \overline{\partial}\,\overline{\partial}^*G\alpha = \overline{\partial}G(\overline{\partial}^*\alpha)$.

Now $\overline{\partial}^* = \sqrt{-1}[\Lambda, \partial]$, where $\Lambda = L^*_\omega$ and $L_\omega(\beta) = \omega \wedge \beta$. So $\overline{\partial}^*\alpha = -\sqrt{-1}\partial\Lambda\alpha$, because $\partial\alpha = 0$. Hence $\alpha = \overline{\partial}G(-\sqrt{-1}\partial\Lambda\alpha) = -\sqrt{-1}\,\overline{\partial}\partial(G\Lambda\alpha)$. Therefore, taking $\psi = -\sqrt{-1}G\Lambda\alpha$, we conclude the proof of the first part.

The proof of the second part is identical. □

Theorem 5.5 *Let X be a compact Kähler orbifold. Then X is formal.*

Proof We have to show that $(\Omega^*_{orb}(X), d)$ and $(H^*(X), 0)$ are quasi-isomorphic differential graded commutative algebras (DGA).

Consider the DGA $(\ker\partial, \overline{\partial})$. We will show that

$$\imath\colon (\ker\partial, \overline{\partial}) \hookrightarrow (\Omega^*_{orb}(X), d)$$

is a quasi-isomorphism. To prove surjectivity, we can take a (p, q)-form α which is d-closed (see Proposition 5.2). If $d\alpha = 0$, then $\partial\alpha = 0$ and $\overline{\partial}\alpha = 0$. So $\alpha \in \ker\partial$ and $\imath^*[\alpha] = [\alpha]$. For injectivity, take $\alpha \in \ker\partial$ such that $\imath^*[\alpha] = 0$. Then $\overline{\partial}\alpha = 0$ and $\alpha = d\beta$, for some form β. Therefore, $\alpha = \partial\beta + \overline{\partial}\beta$. Thus we have $\overline{\partial}(\partial\beta) = 0$. By Lemma 5.4, we have that $\partial\beta = \partial\overline{\partial}\psi$ for some ψ. Hence

$\alpha = \bar{\partial}\beta + \partial\bar{\partial}\psi = \bar{\partial}(\beta - \partial\psi - \bar{\partial}\psi)$. Note that $\partial(\beta - \partial\psi - \bar{\partial}\psi) = \partial\beta - \partial\bar{\partial}\psi = 0$, so $\beta - \partial\psi - \bar{\partial}\psi \in \ker\partial$.

Next we will show that the projection given by

$$H\colon (\ker\partial, \bar{\partial}) \longrightarrow (\mathcal{H}^*_{\bar{\partial}}(X), 0)$$

is a quasi-isomorphism. Here $\mathcal{H}^*_{\bar{\partial}}(X)$ is given the algebra structure given by the natural isomorphism $\rho : \mathcal{H}^*_{\bar{\partial}}(X) \longrightarrow H^*(X, \mathbb{C})$, $\rho(\alpha) = [\alpha]$.

Let $\alpha \in \ker\partial \cap \ker\bar{\partial}$. Then $\bar{\partial}^*\alpha = \sqrt{-1}[\Lambda, \partial]\alpha = -\sqrt{-1}\partial(\Lambda\alpha)$. So

$$\alpha = H\alpha + G(\bar{\partial}\,\bar{\partial}^*\alpha + \bar{\partial}^*\bar{\partial}\alpha) = H\alpha - \sqrt{-1}G\bar{\partial}\partial(\Lambda\alpha)\,,$$

that is $\alpha = H\alpha + \partial\bar{\partial}\psi$, for some ψ. Therefore, if $H\alpha = 0$, then $\alpha = -\bar{\partial}(\partial\psi)$, with $\partial\psi \in \ker\partial$. This proves injectivity.

Now suppose $\alpha = H\alpha + \partial\bar{\partial}\psi$ and $\beta = H\beta + \partial\bar{\partial}\phi$. So

$$\alpha \wedge \beta = H\alpha \wedge H\beta + \partial\bar{\partial}\Phi$$

for some Φ, hence $[H(\alpha \wedge \beta)] = [H\alpha \wedge H\beta] = [H\alpha] \wedge [H\beta]$. This implies that H is a DGA map, where $\mathcal{H}^*_{\bar{\partial}}(X)$ has the algebra structure given by the isomorphism $\mathcal{H}^*_{\bar{\partial}}(X) \cong H^*(X, \mathbb{C})$.

Finally, let us show surjectivity of H. Take α to be harmonic. Then $\bar{\partial}\alpha = 0$ and $\bar{\partial}^*\alpha = 0$. Since $\Delta = 2\Delta_{\bar{\partial}}$, we also have $d\alpha = 0$ and $\partial\alpha = 0$. So $H([\alpha]) = \alpha$. □

The hard Lefschetz property is proved in [39], but we shall give a proof with the current techniques for completeness.

Theorem 5.6 *Let (X, ω) be a compact Kähler orbifold. Then the map*

$$L^{n-k}_\omega\colon H^k(X) \longrightarrow H^{2n-k}(X) \tag{10}$$

is an isomorphism for $0 \le k \le n$.

Proof It is enough to see that (10) is onto, since by Poincaré duality both spaces have the same dimension. As $[L_\omega, \Delta] = 0$, then L_ω sends harmonic forms to harmonic forms. Therefore we have to see that

$$L^{n-k}_\omega\colon \mathcal{H}^k(X) \longrightarrow \mathcal{H}^{2n-k}(X)$$

is surjective. We shall prove this by induction on $k = 0, 1, \ldots, n$. Take a harmonic $(2n-k)$-form a. By induction on k applied to $L_\omega(a)$, we have that $L_\omega(a) = L^{n-k+2}_\omega(c)$ for a $(k-2)$-form c. Therefore $a' = a - L^{n-k+1}_\omega(c)$ is *primitive*, $L_\omega(a') = 0$. Let us see that the Lefschetz map is surjective for a primitive a'.

We have that $[\Lambda, L_\omega] = n - p$ on p-forms. As $\Lambda a = 0$, we have $L_\omega(\Lambda a') = (k-n)a'$, so $(L_\omega(\Lambda a'))^{n-k} = c\,a'$, for a constant c. Using repeatedly that $L_\omega \Lambda = \Lambda L_\omega + c\mathrm{Id}$, for (different constants c's), we get that $L_\omega^{n-k}\Lambda^{n-k}a' = c\,a'$, for another constant c. The map

$$L_\omega^{n-k}\colon \Omega^k_{orb}(X) \longrightarrow \Omega^{2n-k}_{orb}(X)$$

is an isomorphism (it is a bundle isomorphism). So the above constant c is nonzero. Therefore, $a' = L_\omega^{n-k}(b)$ with $b = \frac{1}{c}\Lambda^{n-k}(a')$ a harmonic k-form (since Λ also sends harmonic forms to harmonic forms). This finishes the proof of the theorem.

□

6 Symplectic Orbifolds with No Kähler Orbifold Structure

We shall include two examples of symplectic orbifolds, of dimensions 6 and 8, taken from the constructions in [4] and [15], which cannot admit the structure of an orbifold Kähler manifold. The first one because it does not satisfy the hard Lefschetz property, and the second one because it is non-formal. Both admit complex and symplectic (orbifold) structures.

Before going to those examples, let us recall the definition of a symplectic orbifold.

Definition 6.1 A symplectic orbifold (X, ω) consists of a $2n$-dimensional orbifold X and an orbifold 2-form ω such that $d\omega = 0$ and $\omega^n > 0$ everywhere.

Note that if (M, Ω) is a symplectic manifold, with symplectic form Ω, and Γ is a finite group acting effectively on M and preserving Ω, then $X = M/\Gamma$ is a symplectic orbifold. In fact, by Remark 3.7, $X = M/\Gamma$ is an orbifold, and the symplectic form Ω descends to X via the natural projection $\pi : M \to X$. The map π is differentiable in the orbifold sense (actually it is a submersion).

6.1 6-Dimensional Example

Consider the complex Heisenberg group $H_{\mathbb{C}}$, that is the complex nilpotent Lie group of (complex) dimension 3 consisting of matrices of the form

$$\begin{pmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In terms of the natural (complex) coordinate functions (u_1, u_2, u_3) on $H_{\mathbb{C}}$, we have that the complex 1-forms $\mu = du_1$, $\nu = du_2$ and $\theta = du_3 - u_2\,du_1$ are left invariant, and

$$d\mu = d\nu = 0, \quad d\theta = \mu \wedge \nu\,.$$

Let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi i/6}$, and consider the discrete subgroup $\Gamma \subset H_{\mathbb{C}}$ formed by the matrices in which $u_1, u_2, u_3 \in \Lambda$. We define the compact (parallelizable) nilmanifold

$$M = \Gamma\backslash H_{\mathbb{C}}\,.$$

We can describe M as a principal torus bundle

$$T^2 = \mathbb{C}/\Lambda \longrightarrow M \longrightarrow T^4 = (\mathbb{C}/\Lambda)^2,$$

by the projection $(u_1,\, u_2,\, u_3) \longmapsto (u_1,\, u_2)$.

Consider the action of the finite group $\mathbb{Z}_6$ on $H_{\mathbb{C}}$ given by the generator

$$\begin{aligned} \rho\colon H_{\mathbb{C}} &\longrightarrow H_{\mathbb{C}} \\ (u_1, u_2, u_3) &\longmapsto (\zeta^4\, u_1, \zeta\, u_2, \zeta^5\, u_3). \end{aligned}$$

For this action, clearly $\rho(p\cdot q) = \rho(p)\cdot\rho(q)$, for all $p, q \in H_{\mathbb{C}}$, where $\cdot$ denotes the natural group structure of $H_{\mathbb{C}}$. Moreover, we have $\rho(\Gamma) = \Gamma$. Thus, ρ induces an action on the quotient $M = \Gamma\backslash H_{\mathbb{C}}$. Let $\rho\colon M \longrightarrow M$ be the $\mathbb{Z}_6$-action. The action on 1-forms is given by

$$\rho^*\mu = \zeta^4\,\mu, \quad \rho^*\nu = \zeta\,\nu, \quad \rho^*\theta = \zeta^5\,\theta.$$

Proposition 6.2 *$X = M/\mathbb{Z}_6$ is a simply connected, compact, formal* 6*-orbifold admitting complex and symplectic structures.*

Proof Since the $\mathbb{Z}_6$-action on M is effective, the quotient space $X = M/\mathbb{Z}_6$ is an orbifold. (The singular points of X are determined in [4, Sect. 4].) Clearly X is compact since M is compact. In [4, Proposition 6.1], it is proved that the 6-orbifold X (denoted by $\widehat{M}$ in [4]) is simply connected. Then, X is formal because any simply connected compact orbifold of dimension 6 is formal by Lemma 3.10.

The orbifold X has a complex orbifold structure, as in Proposition 6.4. We define the complex 2-form ω on M by

$$\omega = -\sqrt{-1}\,\mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta}\,. \tag{11}$$

Clearly, ω is a real closed 2-form on M such that $\omega^3 > 0$, so ω is a symplectic form on M. Moreover, the form ω is $\mathbb{Z}_6$-invariant. Indeed, $\rho^*\omega = -i\,\mu \wedge \bar{\mu} + \zeta^6\nu \wedge \theta + \zeta^{-6}\bar{\nu} \wedge \bar{\theta} = \omega$. Therefore X is a symplectic 6-orbifold, with the symplectic form $\widehat{\omega}$ induced by ω. □

In order to prove that X does not admit any Kähler structure, we are going to check that it does not satisfy the hard Lefschetz property for any symplectic form. We compute the cohomology of X. By a theorem of Nomizu [30], the cohomology of the nilmanifold M is:

$$
\begin{aligned}
H^0(M,\mathbb{C}) &= \langle 1\rangle,\\
H^1(M,\mathbb{C}) &= \langle [\mu],[\bar\mu],[\nu],[\bar\nu]\rangle,\\
H^2(M,\mathbb{C}) &= \langle [\mu\wedge\bar\mu],[\mu\wedge\bar\nu],[\bar\mu\wedge\nu],[\nu\wedge\bar\nu],[\mu\wedge\theta],[\bar\mu\wedge\bar\theta],[\nu\wedge\theta],[\bar\nu\wedge\bar\theta]\rangle,\\
H^3(M,\mathbb{C}) &= \langle [\mu\wedge\bar\mu\wedge\theta],[\mu\wedge\bar\mu\wedge\bar\theta],[\nu\wedge\bar\nu\wedge\theta],[\nu\wedge\bar\nu\wedge\bar\theta],[\mu\wedge\nu\wedge\theta],[\bar\mu\wedge\bar\nu\wedge\bar\theta]\\
&\qquad [\mu\wedge\bar\nu\wedge\theta],[\mu\wedge\bar\nu\wedge\bar\theta],[\bar\mu\wedge\nu\wedge\theta],[\bar\mu\wedge\nu\wedge\bar\theta]\rangle,\\
H^4(M,\mathbb{C}) &= \langle [\mu\wedge\bar\mu\wedge\nu\wedge\theta],[\mu\wedge\bar\mu\wedge\bar\nu\wedge\bar\theta],[\bar\mu\wedge\nu\wedge\bar\nu\wedge\bar\theta],[\mu\wedge\nu\wedge\bar\nu\wedge\theta],\\
&\qquad [\mu\wedge\bar\mu\wedge\theta\wedge\bar\theta],[\nu\wedge\bar\nu\wedge\theta\wedge\bar\theta],[\mu\wedge\bar\nu\wedge\theta\wedge\bar\theta],[\bar\mu\wedge\nu\wedge\theta\wedge\bar\theta]\rangle,\\
H^5(M,\mathbb{C}) &= \langle [\mu\wedge\bar\mu\wedge\nu\wedge\theta\wedge\bar\theta],[\mu\wedge\bar\mu\wedge\bar\nu\wedge\theta\wedge\bar\theta],[\mu\wedge\nu\wedge\bar\nu\wedge\theta\wedge\bar\theta],\\
&\qquad [\bar\mu\wedge\nu\wedge\bar\nu\wedge\theta\wedge\bar\theta]\rangle,\\
H^6(M,\mathbb{C}) &= \langle [\mu\wedge\bar\mu\wedge\nu\wedge\bar\nu\wedge\theta\wedge\bar\theta]\rangle.
\end{aligned}
$$

According with (5), any $\mathbb{Z}_6$-invariant k-form on M defines an orbifold k-form on X, and vice-versa. Moreover, the cohomology $H^*(X) = H^*(M)^{\mathbb{Z}_6}$ is:

$$
\begin{aligned}
H^0(X,\mathbb{C}) &= \langle 1\rangle,\\
H^1(X,\mathbb{C}) &= 0,\\
H^2(X,\mathbb{C}) &= \langle [\mu\wedge\bar\mu],[\nu\wedge\bar\nu],[\nu\wedge\theta],[\bar\nu\wedge\bar\theta]\rangle,\\
H^3(X,\mathbb{C}) &= 0,\\
H^4(X,\mathbb{C}) &= \langle [\mu\wedge\bar\mu\wedge\nu\wedge\theta],[\mu\wedge\bar\mu\wedge\bar\nu\wedge\bar\theta],[\mu\wedge\bar\mu\wedge\theta\wedge\bar\theta],[\nu\wedge\bar\nu\wedge\theta\wedge\bar\theta]\rangle,\\
H^5(X,\mathbb{C}) &= 0,\\
H^6(X,\mathbb{C}) &= \langle [\mu\wedge\bar\mu\wedge\nu\wedge\bar\nu\wedge\theta\wedge\bar\theta]\rangle,
\end{aligned}
$$

where we use the same notation for the $\mathbb{Z}_6$-invariant forms on M and those induced on the orbifold X.

The cohomology class

$$[\beta] = [\nu\wedge\bar\nu] \in H^2(X)$$

satisfies the equation $[\beta]\wedge[\alpha_1]\wedge[\alpha_2] = 0$ for any $[\alpha_1],\ [\alpha_2] \in H^2(X)$. Therefore this class is always in the kernel of

$$L_{\omega'}\colon H^2(X) \longrightarrow H^4(X),$$

for any (orbifold) symplectic form ω'. So we have the following:

Proposition 6.3 *The orbifold X does not admit an orbifold Kähler structure since it does not satisfy the hard Lefschetz property for any orbifold symplectic form.*

6.2 8-Dimensional Example

Consider again the complex Heisenberg group $H_{\mathbb{C}}$ and set $G = H_{\mathbb{C}} \times \mathbb{C}$, where $\mathbb{C}$ is the additive group of complex numbers. We denote by u_4 the coordinate function corresponding to this extra factor. In terms of the natural (complex) coordinate functions (u_1, u_2, u_3, u_4) on G, the complex 1-forms $\mu = du_1$, $\nu = du_2$, $\theta = du_3 - u_2\, du_1$ and $\eta = du_4$ are left invariant, and

$$d\mu = d\nu = d\eta = 0, \quad d\theta = \mu \wedge \nu\,.$$

Let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi\sqrt{-1}/3}$, and consider the discrete subgroup $\Gamma \subset G$ formed by the matrices in which $u_1, u_2, u_3, u_4 \in \Lambda$. We define the compact (parallelizable) nilmanifold

$$M = \Gamma\backslash G\,.$$

We can describe M as a principal torus bundle

$$T^2 = \mathbb{C}/\Lambda \longrightarrow M \longrightarrow T^6 = (\mathbb{C}/\Lambda)^3\,,$$

by the projection $(u_1,\, u_2,\, u_3,\, u_4) \longmapsto (u_1,\, u_2,\, u_4)$.

Now introduce the following action of the finite group $\mathbb{Z}_3$

$$\begin{aligned} \rho\colon G &\longrightarrow G \\ (u_1, u_2, u_3, u_4) &\longmapsto (\zeta\, u_1, \zeta\, u_2, \zeta^2\, u_3, \zeta\, u_4). \end{aligned}$$

Note that $\rho(p \cdot q) = \rho(p) \cdot \rho(q)$, for $p, q \in G$, where the dot denotes the natural group structure of G. The map ρ is a particular case of a homothetic transformation (by ζ in this case) which is well defined for all nilpotent simply connected Lie groups with graded Lie algebra. Moreover $\rho(\Gamma) = \Gamma$, therefore ρ induces an action on the quotient $M = \Gamma\backslash G$. This action is free away from 3^4 fixed points corresponding to $u_i = n/(1-\zeta)$, for $n = 0, 1$ and 2.

The action on the forms is given by

$$\rho^*\mu = \zeta\,\mu, \quad \rho^*\nu = \zeta\,\nu, \quad \rho^*\theta = \zeta^2\,\theta, \quad \rho^*\eta = \zeta\,\eta.$$

Proposition 6.4 $X = M/\mathbb{Z}_3$ *is an* 8*-orbifold admitting complex and symplectic structures.*

Proof Just as in Proposition 6.2, it turns out that X is an 8-orbifold since the $\mathbb{Z}_3$-action on M is effective. The nilmanifold M is a complex manifold whose complex structure J coincides with the multiplication by $\sqrt{-1}$ on each tangent space T_pM, $p \in M$. Then one can check that J commutes with the $\mathbb{Z}_3$-action ρ on M, that is $(\rho_*)_p \circ J_p = J_{\rho(p)} \circ (\rho_*)_p$, for any point $p \in M$. Hence, J induces a complex structure on the quotient $X = M/\mathbb{Z}_3$.

The complex 2-form

$$\omega = \sqrt{-1}\,\mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta} + \sqrt{-1}\,\eta \wedge \bar{\eta}$$

is actually a real form which is clearly closed and which has the property that $\omega^4 \neq 0$. Thus ω is a symplectic form on M. Moreover, ω is $\mathbb{Z}_3$-invariant. Hence the space $X = M/\mathbb{Z}_3$ is a symplectic orbifold, with the symplectic form $\widehat{\omega}$ induced by ω. □

The orbifold X does not admit a Kähler orbifold structure because it is non-formal, as shown in the following theorem.

Theorem 6.5 *The orbifold X is non-formal.*

Proof We start by considering the nilmanifold M. Consider the following closed forms:

$$\alpha = \mu \wedge \bar{\mu}, \quad \beta_1 = \nu \wedge \bar{\nu}, \quad \beta_2 = \nu \wedge \bar{\eta}, \quad \beta_3 = \bar{\nu} \wedge \eta.$$

Then

$$\alpha \wedge \beta_1 = d(-\theta \wedge \bar{\mu} \wedge \bar{\nu}), \quad \alpha \wedge \beta_2 = d(-\theta \wedge \bar{\mu} \wedge \bar{\eta}), \quad \alpha \wedge \beta_3 = d(\bar{\theta} \wedge \mu \wedge \eta).$$

All the forms $\alpha, \beta_1, \beta_2, \beta_3, \xi_1 = -\theta \wedge \bar{\mu} \wedge \bar{\nu}, \xi_2 = -\theta \wedge \bar{\mu} \wedge \bar{\eta}$ and $\xi_3 = \bar{\theta} \wedge \mu \wedge \eta$ are $\mathbb{Z}_3$-invariant. Hence by (5) they descend to orbifold forms (denoted with a $\tilde{}$) on the quotient $X = M/\mathbb{Z}_3$.

We consider the a-Massey product

$$\langle a; b_1, b_2, b_3 \rangle,$$

for $a = [\tilde{\alpha}], b_i = [\tilde{\beta_i}] \in H^2(X)$, $i = 1, 2, 3$. By Nomizu's theorem mentioned earlier, the cohomology of M up to degree 3 is

$$\begin{aligned}
H^0(M, \mathbb{C}) &= \langle 1 \rangle, \\
H^1(M, \mathbb{C}) &= \langle [\mu], [\bar{\mu}], [\nu], [\bar{\nu}], [\eta], [\bar{\eta}] \rangle, \\
H^2(M, \mathbb{C}) &= \langle [\mu \wedge \bar{\mu}], [\mu \wedge \bar{\nu}], [\mu \wedge \theta], [\mu \wedge \eta], [\mu \wedge \bar{\eta}], [\bar{\mu} \wedge \nu], [\bar{\mu} \wedge \bar{\theta}], [\bar{\mu} \wedge \eta], \\
&\quad [\bar{\mu} \wedge \bar{\eta}], [\nu \wedge \bar{\nu}], \\
&\quad [\nu \wedge \theta], [\nu \wedge \eta], [\nu \wedge \bar{\eta}], [\bar{\nu} \wedge \bar{\theta}], [\bar{\nu} \wedge \eta], [\bar{\nu} \wedge \bar{\eta}], [\eta \wedge \bar{\eta}] \rangle, \\
H^3(M, \mathbb{C}) &= A \oplus \bar{A}\,,
\end{aligned}$$

where

$$
\begin{aligned}
A =& \langle [\mu \wedge \bar{\mu} \wedge \bar{\theta}], [\mu \wedge \bar{\mu} \wedge \eta], [\mu \wedge \nu \wedge \theta], [\mu \wedge \bar{\nu} \wedge \bar{\theta}], [\mu \wedge \bar{\nu} \wedge \eta], [\mu \wedge \theta \wedge \eta], [\mu \wedge \eta \wedge \bar{\eta}], \\
& [\bar{\mu} \wedge \nu \wedge \bar{\theta}], [\bar{\mu} \wedge \nu \wedge \eta], [\bar{\mu} \wedge \bar{\theta} \wedge \eta], [\nu \wedge \bar{\nu} \wedge \bar{\theta}], [\nu \wedge \bar{\nu} \wedge \eta], [\nu \wedge \theta \wedge \eta], [\nu \wedge \eta \wedge \bar{\eta}], \\
& [\bar{\nu} \wedge \bar{\theta} \wedge \eta]\rangle.
\end{aligned}
$$

Now $\mathbb{Z}_3$ acts on A by multiplication with ζ and on $\bar{A}$ by multiplication with $\bar{\zeta}$, hence $H^3(X) = H^3(M)^{\mathbb{Z}_3} = 0$. By Cavalcanti et al. [8, Proposition 2.7], the a-Massey product $\langle a; b_1, b_2, b_3\rangle$ has no indeterminacy.

We denote by q the projection $M \longrightarrow X$, and compute

$$
\begin{aligned}
\langle a; b_1, b_2, b_3\rangle &= [\tilde{\xi}_1 \wedge \tilde{\xi}_2 \wedge \tilde{\beta}_3 + \tilde{\xi}_2 \wedge \tilde{\xi}_3 \wedge \tilde{\beta}_1 + \tilde{\xi}_3 \wedge \tilde{\xi}_1 \wedge \tilde{\beta}_2] = \\
&= q_*[\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2] = \\
&= 2q_*[\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}]
\end{aligned}
$$

which is non-zero, since by (6) we have

$$
\begin{aligned}
\int_X \langle a; b_1, b_2, b_3\rangle &= 2\int_X q_*[\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}] = \\
&= 6\int_M [\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}] \neq 0.
\end{aligned}
$$

By Theorem 2.6 and Definition 3.8, the orbifold X is non-formal. □

7 Simply Connected Sasakian Manifolds

First, we recall some definitions and results on Sasakian manifolds (see [6] for more details).

An odd-dimensional Riemannian manifold (N, g) is *Sasakian* if its cone $(N \times \mathbb{R}^+, g^c = t^2 g + dt^2)$ is Kähler, that is the cone metric $g^c = t^2 g + dt^2$ admits a compatible integrable almost complex structure J so that $(N \times \mathbb{R}^+, g^c = t^2 g + dt^2, J)$ is a Kähler manifold. In this case the Reeb vector field $\xi = J\partial_t$ is a Killing vector field of unit length. The corresponding 1-form η defined by $\eta(X) = g(\xi, X)$, for any vector field X on N, is a contact form, meaning $\eta \wedge (d\eta)^n \neq 0$ at every point of N, where $\dim N = 2n + 1$.

A Sasakian structure on N is called *quasi-regular* if there is a positive integer δ satisfying the condition that each point of N has a coordinate chart (U, t) with respect to ξ (the coordinate t is in the direction of ξ) such that each leaf of ξ passes through U at most δ times. If $\delta = 1$, then the Sasakian structure is called *regular*. (See [6, p. 188].) A result of [31] says that if N admits a Sasakian structure, then it admits also a quasi-regular Sasakian structure.

If M is a Kähler manifold whose Kähler form ω defines an integral cohomology class, then the total space of the circle bundle $S^1 \hookrightarrow N \xrightarrow{\pi} M$ with Euler class $[\omega] \in H^2(M, \mathbb{Z})$ is a regular Sasakian manifold with contact form η such that $d\eta = \pi^*(\omega)$. The converse also holds: if N is a regular Sasakian structure then the space of leaves X is a Kähler manifold, and we have a circle bundle $S^1 \hookrightarrow N \to M$ as above. If N has a quasi-regular Sasakian structure, then the space of leaves X is a Kähler orbifold with cyclic quotient singularities, and there is an orbifold circle bundle $S^1 \hookrightarrow N \to X$ such that the contact form η satisfies $d\eta = \pi^*(\omega)$, where ω is the orbifold Kähler form. Note that the map π is an orbifold submersion, so that $\pi^*(\omega)$ is a well-defined (smooth) 2-form on the total space N, which is a smooth manifold. This defines a Sasakian structure on N by Muñoz et al. [28, Theorem 20].

7.1 A Simply Connected Non-formal Sasakian Manifold

Examples of simply connected non-formal Sasakian manifolds, of dimension $2n + 1 \geq 7$, are given in [5]. There it is proved that those examples are non-formal because they are not 3-formal, in the sense of Definition 2.2. Here we show the non-formality proving that they have a non-trivial triple Massey product.

Note that if N is a simply connected, compact and non-formal manifold (not necessarily Sasakian), then $\dim N \geq 7$. Indeed, Theorem 2.3 gives that simply connected compact manifolds of dimension at most 6 are formal [14, 29]. Moreover, a 7-dimensional simply connected Sasakian manifold is formal if and only if all the triple Massey products are trivial [27].

To construct a simply connected non-formal Sasakian 7-manifold, we consider the Kähler manifold $M = S^2 \times S^2 \times S^2$ with Kähler form

$$\omega = \omega_1 + \omega_2 + \omega_3 \,,$$

where ω_1, ω_2 and ω_3 are the generators of the integral cohomology group of each of the S^2-factors on $S^2 \times S^2 \times S^2$. Let N be the total space N of the principal S^1-bundle

$$S^1 \hookrightarrow N \longrightarrow M = S^2 \times S^2 \times S^2 \,,$$

with Euler class $[\omega] \in H^2(M, \mathbb{Z})$. Then, N is a simply connected compact (regular) Sasakian manifold, with contact form η such that $d\eta = \pi^*(\omega)$.

From now on, we write $a_i = [\omega_i] \in H^2(S^2)$. Since $M = S^2 \times S^2 \times S^2$ is formal, a model of M is $(H^*(S^2 \times S^2 \times S^2), 0)$, where $H^*(S^2 \times S^2 \times S^2)$ is the de Rham cohomology algebra of $S^2 \times S^2 \times S^2$, that is

$$\begin{aligned}
&H^0(M) = \langle 1 \rangle, \\
&H^1(M) = H^3(M) = H^5(M) = 0 \,, \\
&H^2(M) = \langle a_1, a_2, a_3 \rangle,
\end{aligned}$$

$$H^4(M) = \langle a_1 \cdot a_2,\, a_1 \cdot a_3,\, a_2 \cdot a_3 \rangle,$$

$$H^6(M) = \langle a_1 \cdot a_2 \cdot a_3 \rangle.$$

Therefore, a model of N is the DGA $\Big(H^*(M) \otimes \bigwedge(x),\, d\Big)$, where $|x| = 1$, $d(H^*(M)) = 0$ and $dx = a_1 + a_2 + a_3$. By Lemma 2.4, we know that Massey products on a manifold can be computed by using any model for the manifold. Since $a_1 \cdot a_1 = 0$ and $a_1 \cdot a_2 = \frac{1}{2}d(a_1 \cdot x + a_2 \cdot x - a_3 \cdot x)$, we have that the (triple) Massey product $\langle a_1, a_1, a_2\rangle = \frac{1}{2}[(a_1 \cdot a_2 - a_1 \cdot a_3) \cdot x]$ is defined and it is non-trivial. Note that there is no indeterminacy of the Massey product, since it lives in $a_1 \cdot H^3(N) + a_2 \cdot H^3(N)$, but $H^3(N) = 0$, since by the Gysin sequence, it equals the kernel of $[\omega] : H^2(M) \to H^4(M)$, which is an isomorphism. So N is non-formal.

The case $n > 3$ is similar and it is deduced as follows. Consider $B = S^2 \times \overset{(n)}{\cdots} \times S^2$. Let $a_1, \dots, a_n \in H^2(B)$ be the cohomology classes given by each of the S^2-factors. Then the Kähler class is given by $[\omega] = a_1 + \cdots + a_n$. Consider the circle bundle

$$S^1 \hookrightarrow N \longrightarrow B$$

with first Chern class equal to $[\omega]$.

Using again Lemma 2.4, we know that Massey products on N can be computed by using any model for N. Since B is formal, a model of B is the DGA $\Big(H^*(B),\, 0\Big)$. Thus, a model of N is the DGA $\Big(H^*(B) \otimes \bigwedge(x),\, d\Big)$, where $|x| = 1$, $d(H^*(B)) = 0$ and $dx = a_1 + a_2 + \ldots + a_n$. Now, one can check that $a_1 \cdot a_1 = 0$ and

$$a_1 \cdot a_2 \ldots a_{n-2} \cdot a_{n-1} = \frac{1}{2} d\Big((a_1 \cdot a_2 \ldots a_{n-2} + a_2 \cdot a_3 \ldots a_{n-2} \cdot a_{n-1} - a_2 \cdot a_3 \ldots a_{n-2} \cdot a_n) \cdot x\Big).$$

Thus the Massey product $\langle a_1,\, a_1,\, a_2 \cdot a_3 \ldots a_{n-2} \cdot a_{n-1}\rangle$ is defined and a representative is $[(a_1 \cdot a_2 \ldots a_{n-2} \cdot a_{n-1} - a_1 \cdot a_2 \ldots a_{n-2} \cdot a_n) \cdot x]$ which is non-trivial. Hence, we conclude that N is non-formal.

7.2 Simply Connected Formal Sasakian Manifolds with $b_2 \neq 0$

The most basic example of a simply connected compact regular Sasakian manifold is the odd-dimensional sphere S^{2n+1} considered as the total space of the Hopf fibration $S^{2n+1} \hookrightarrow \mathbb{CP}^n$. It is well-known that S^{2n+1} is formal. In this section, we show examples of simply connected compact Sasakian manifolds, with second Betti number $b_2 \neq 0$, which are formal.

Note that Theorem 2.3 implies that any simply connected compact manifold (Sasakian or not) of dimension ≤ 7 and with $b_2 \leq 1$, is formal. Examples of 7-dimensional simply connected compact Sasakian manifolds, with $b_2 \geq 2$, which are formal are given in [16].

To show examples of simply connected formal Sasakian manifolds, of dimension ≥ 9 and with $b_2 \neq 0$, we consider the Kähler manifold

$$M = \mathbb{CP}^{n-1} \times S^2,$$

with Kähler form

$$\omega = \omega_1 + \omega_2,$$

where ω_1 and ω_2 are the generators of the integral cohomology groups $H^2(\mathbb{CP}^{n-1}, \mathbb{Z})$ and $H^2(S^2, \mathbb{Z})$, respectively. Let N be the total space N of the principal S^1-bundle

$$S^1 \hookrightarrow N \longrightarrow M = \mathbb{CP}^{n-1} \times S^2,$$

with Euler class $[\omega] \in H^2(M, \mathbb{Z})$. Then, N is a simply connected compact (regular) Sasakian manifold, of dimension $2n+1$, with contact form η such that $d\eta = \pi^*(\omega)$.

Proposition 7.1 *The total space N of the circle bundle $S^1 \hookrightarrow N \longrightarrow M = \mathbb{CP}^{n-1} \times S^2$, with Euler class $[\omega]$, is a simply connected compact Sasakian manifold, with second Betti number $b_2 = 1$, which is formal.*

Proof Suppose $n \geq 4$. We will determine a minimal model of the $(2n+1)$-manifold N.

Clearly $M = \mathbb{CP}^{n-1} \times S^2$ is formal because M is Kähler. Hence, a (non-minimal) model of M is the DGA $(H^*(M), 0)$, where $H^*(M)$ is the de Rham cohomology algebra of M. Thus, a (non-minimal) model of N is the differential algebra $(\mathcal{A}, d)$, where

$$\mathcal{A} = H^*(M) \otimes \bigwedge(x), \quad |x| = 1, \quad d(H^*(M)) = 0, \quad dx = a_1 + a_2,$$

where a_1 is the integral cohomology class defined by the Kähler form ω_1 on $\mathbb{CP}^{n-1}$, and a_2 is the integral cohomology class defined by the Kähler form ω_2 on S^2. Then, the minimal model associated to this model of N is

$$(\mathcal{M}, D) = (\bigwedge(a, b, z), D),$$

where $|a| = 2$, $|b| = 3$ and $|z| = 2n-1$, while the differential D is given by $Da = Db = 0$ and $Dz = a^n$. Therefore, we get

$$N^i = 0,$$

for $1 \leq i \leq n$. Then, Theorem 2.3 implies that N is formal because it is n-formal. □

7.3 Non-formal Quasi-Regular Sasakian Manifolds with $b_1 = 0$

The previous examples can be tweaked to obtain also examples of quasi-regular Sasakian manifolds P, where the base of the (orbifold) circle bundle $S^1 \hookrightarrow P \to X$ is an honest orbifold Kähler manifold X. Obtaining simply connected manifolds P in this way is a delicate matter, since the fundamental group of P relates to the *orbifold fundamental group* of X, and not its fundamental group (see [24] and [28] for discussions on these issues). Therefore we content ourselves with writing down examples with $H_1(P, \mathbb{Z}) = 0$.

Consider a complex 3-torus $T^3 = \mathbb{C}^3/\Gamma$, where Γ is the discrete subgroup of $\mathbb{C}^3$ consisting of the elements $(z_1, z_2, z_3) \in \mathbb{C}^3$ whose components z_1, z_2 and z_3 are Gaussian integers. Now consider the action of the finite group $\mathbb{Z}_2$ on $\mathbb{C}^3$ given by

$$\begin{aligned} \varphi\colon \mathbb{C}^3 &\to \mathbb{C}^3 \\ (z_1, z_2, z_3) &\mapsto (-z_1, -z_2, -z_3), \end{aligned}$$

where φ is the generator of $\mathbb{Z}_2$. This action satisfies that $\varphi(z + z') = \varphi(z) + \varphi(z')$, for $z, z' \in \mathbb{C}^3$. Moreover, $\varphi(\Gamma) = \Gamma$. Therefore, φ induces an action on $T^3 = \mathbb{C}^3/\Gamma$ with 2^6 fixed points corresponding to $(z_1 = u_1 + i\,u_2, z_2 = u_3 + i\,u_4, z_3 = u_5 + i\,u_6)$ with $u_i = 0, \frac{1}{2}$. Thus, the quotient space

$$X = T^3/\mathbb{Z}_2$$

is a Kähler orbifold of (real) dimension 6 with 2^6 isolated orbifold singularities of order 2. In fact, one can check that the standard complex structure J on T^3 commutes with the $\mathbb{Z}_2$-action, that is $(\varphi_*)_z \circ J_z = J_{\varphi(z)} \circ (\varphi_*)_z$, for any point $z \in T^3$. Moreover, the standard Hermitian metric and the Kähler form ω' on T^3 are $\mathbb{Z}_2$-invariant, and so they induce an orbifold Hermitian metric and an orbifold Kähler form ω on X, respectively.

By (5), the cohomology of X is given by $H^1(X, \mathbb{Z}) = H^1(T^3, \mathbb{Z})^{\mathbb{Z}_2} = 0$, hence $b_1(X) = 0$. Now consider the orbifold circle bundle

$$S^1 \hookrightarrow P \xrightarrow{\pi} X,$$

given by $c_1(P) = [\omega]$. We have the following:

Proposition 7.2 *The manifold P is a 7-dimensional quasi-regular Sasakian manifold with $b_1 = 0$ which is non-formal.*

Proof The total space of the orbifold circle bundle P has a Sasakian structure with contact form η such that $d\eta = \pi^*(\omega)$, by Muñoz et al. [28, Theorem 20] (the proof of this result is given in the K-contact case but it works also for the Sasakian case). The Leray spectral sequence gives that $b_1(P) = 0$.

Let us see that P is non-formal. First note that the cohomology of T^3 is the exterior algebra $\bigwedge^*(x_1,\dots,x_6)$, with $|x_i| = 1, 1 \le i \le 6$. Then $H^*(X) = \bigwedge^{\text{even}}(x_1,\dots,x_6)$. Let $a_1 = x_1x_2$, $a_2 = x_3x_4$, $a_3 = x_5x_6$, so that $[\omega] = a_1 + a_2 + a_3$. As in Sect. 7.1, there is non-trivial (triple) Massey product in P. Indeed, $a_1 \cdot a_1 = 0$ and $a_1 \cdot a_2 = \frac{1}{2} d\big((a_1 + a_2 - a_3)\cdot\eta\big)$. Then,

$$\langle a_1, a_1, a_2\rangle = \frac{1}{2}[(a_1\cdot a_2 - a_1\cdot a_3)\cdot\eta],$$

where $d\eta = \pi^*(\omega)$. So P is non-formal.

There is a geometrical explanation of the above Massey product. If $T = \mathbb{C}/\mathbb{Z}^2$ is the 2-torus, then the quotient $T/\mathbb{Z}_2 \cong S^2$, as a topological manifold. Thus

$$T^3/(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2) = (T/\mathbb{Z}_2)\times(T/\mathbb{Z}_2)\times(T/\mathbb{Z}_2) \cong S^2\times S^2\times S^2 = M\,,$$

where each of the factors of $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ acts on each of the three factors of $T^3 = T\times T\times T$, respectively, and M is the 6-manifold of Sect. 7.1. Therefore, the orbifold X sits in the middle of two quotient maps

$$T^3 \to X = T^3/\mathbb{Z}_2 \to M \cong T^3/(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2).$$

Then there is a diagram

$$\begin{array}{ccccc} S^1 & \hookrightarrow & P & \longrightarrow & X \\ || & & \downarrow & & \downarrow \\ S^1 & \hookrightarrow & N & \longrightarrow & M \end{array}$$

where N is the 7-manifold of Sect. 7.1. So, P and N are the same topological manifold. Then the non-zero Massey product of N produces the non-zero Massey product for P, giving the non-formality of P. ☐

Acknowledgements We are grateful to the referees for their helpful comments. The first author is supported by a Post-Doc grant at Philipps-Universität Marburg. The second author is supported by the J. C. Bose Fellowship. The third author is partially supported through Project MINECO (Spain) MTM2014-54804-P and Basque Government Project IT1094-16. The fourth author is partially supported by Project MINECO (Spain) MTM2015-63612-P.

References

1. A. Adem, J. Leida, Y. Ruan, *Orbifolds and String Theory* (Cambridge University Press, Cambridge, 2007)
2. M. Atiyah, *Elliptic Operators and Compact Groups*. Lecture Notes in Mathematics, vol. 401 (Springer, Berlin, 1974)

3. W. Baily, The decomposition theorem for V-manifolds. Am. J. Math. **78**, 862–888 (1956)
4. G. Bazzoni, M. Fernández, V. Muñoz, A 6-dimensional simply connected complex and symplectic manifold with no Kähler metric. J. Symplectic Geom. (to appear). arxiv:1410.6045
5. I. Biswas, M. Fernández, V. Muñoz, A. Tralle, On formality of Sasakian manifolds. J. Topol. **9**, 161–180 (2016)
6. C. Boyer, K. Galicki, *Sasakian Geometry* (Oxford University Press, Oxford, 2007)
7. G.E. Bredon, *Introduction to Compact Transformation Groups*. Pure and Applied Mathematics, vol. 46 (Academic Press, New York, 1972)
8. G. Cavalcanti, M. Fernández, V. Muñoz, Symplectic resolutions, Lefschetz property and formality. Adv. Math. **218**, 576–599 (2008)
9. L. Cordero, R. Wolak, Properties of the basic cohomology of transversely Kähler foliations. Rend. Circolo Mat. Palermo **40**, 177–188 (1991)
10. P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, Real homotopy theory of Kähler manifolds. Invent. Math. **29**, 245–274 (1975)
11. L.J. Dixon, J.A. Harvey, C. Vafa, E. Witten, Strings on orbifolds I. Nucl. Phys. B **261**, 678–686 (1985)
12. A. El Kacimi Alaoui, Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications. Compos. Math. **73**, 57–106 (1990)
13. Y. Felix, S. Halperin, J.-C. Thomas, *Rational Homotopy Theory* (Springer, Berlin, 2002)
14. M. Fernández, V. Muñoz, Formality of Donaldson submanifolds. Math. Z. **250**, 149–175 (2005)
15. M. Fernández, V. Muñoz, An 8-dimensional non-formal simply connected symplectic manifold. Ann. Math. **167**, 1045–1054 (2008)
16. M. Fernández, S. Ivanov, V. Muñoz, Formality of 7-dimensional 3-Sasakian manifolds. Ann. Sc. Norm. Super. Pisa Cl. Sci (to appear). arxiv:1511.08930
17. J. Girbau, A. Haefliger, D. Sundararaman, On deformations of transversely holomorphic foliations. J. Reine Angew Math. **345**, 122–147 (1983)
18. P. Griffiths, J. Morgan, *Rational Homotopy Theory and Differential Forms*. Progress in Mathematics, vol. 16 (Birkhäuser, Basel, 1981)
19. S. Halperin, Lectures on minimal models. Mém. Soc. Math. France **230**, 261 (1983)
20. T. Kawasaki, The signature theorem for V-manifolds. Topology **17**, 75–83 (1978)
21. T. Kawasaki, The Riemann-Roch theorem for complex V-manifolds. Osaka J. Math. **16**, 151–159 (1979)
22. T. Kawasaki, The index of elliptic operators over V-manifolds. Nagoya Math. J. **84**, 135–157 (1981)
23. B. Kleiner, J. Lott, Geometrization of three-dimensional orbifolds via Ricci flow. Astérisque **365**, 101–177 (2014)
24. J. Kollár, Shafarevich maps and plurigenera of algebraic varieties. Invent. Math. **113**, 177–216 (1993)
25. I. Moerdijk, J. Mrcun, *Introduction to Foliations and Lie Groupoids*. Cambridge Studies in Advanced Mathematics, vol. 91 (Cambridge University Press, Cambridge, 2003)
26. I. Moerdijk, D.A. Pronk, Orbifolds, sheaves and groupoids. K-Theory **12**, 3–21 (1997)
27. V. Muñoz, A. Tralle, Simply connected K-contact and Sasakian manifolds of dimension 7. Math. Z. **281**, 457–470 (2015)
28. V. Muñoz, J.A. Rojo, A. Tralle, Homology Smale-Barden manifolds with K-contact and Sasakian structures. arxiv:1601.06136
29. J. Neisendorfer, T. Miller, Formal and coformal spaces. Ill. J. Math. **22**, 565–580 (1978)
30. K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups. Ann. Math. **59**, 531–538 (1954)
31. L. Ornea, M. Verbitsky, Sasakian structures on CR-manifolds. Geom. Dedicata **125**, 159–173 (2007)
32. M. Pflaum, *Analytic and Geometric Study of Stratified Spaces*. Lecture Notes in Mathematics, vol. 1768 (Springer, Berlin, 2001)
33. I. Satake, On a generalization of the notion of manifold. Proc. Natl. Acad. Sci. USA **42**, 359–363 (1956)

34. I. Satake, The Gauss-Bonnet theorem for V-manifolds. J. Math. Soc. Jpn. **9**, 464–492 (1957)
35. D. Sullivan, Infinitesimal computations in topology. Inst. Hautes Études Sci. Publ. Math. **47**, 269–331 (1978)
36. W.P. Thurston, *The Geometry and Topology of 3-Manifolds*. Mimeographed Notes (Princeton University, Princeton, 1979)
37. A. Tralle, J. Oprea, *Symplectic Manifolds with no Kähler Structure*. Lecture Notes in Mathematics, vol. 1661 (Springer, Berlin, 1997)
38. M. Verbitsky, Hodge theory on nearly Kähler manifolds. Geom. Topol. **15**, 2111–2133 (2011)
39. Z. Wang, D. Zaffran, A remark on the Hard Lefschetz theorem for Kähler orbifolds. Proc. Am. Math. Soc. **137**, 2497–2501 (2009)
40. R. Wells, *Differential Analysis on Complex Manifolds*. Graduate Texts in Mathematics, vol. 65 (Springer, New York/Berlin, 1980)

Notes on Transformations in Integrable Geometry

Fran Burstall

Abstract We describe the gauge-theoretic approach to transformations in integrable geometry through discussion of two classical examples: surface of constant negative Gauss curvature and isothermic surfaces.

Keywords Baecklund transformation • Bianchi permutability • Darboux transformation • Integrable system • Isothermic surface • K-surface

1 Prospectus

Roughly speaking, a differential-geometric system, be it smooth, discrete or semi-discrete, is integrable if it has some or all of the following properties:

1. an infinite-dimensional symmetry group.
2. explicit solutions.
3. algebro-geometric solutions via spectral curves and/or theta functions.

In these notes, I shall focus on a manifestation of the first item: transformations whereby new solutions are constructed from old. The theory applies in many situations including:

- surfaces in $\mathbb{R}^3$ with constant mean or Gauss curvature [1, 2, 24] or, more generally, linear Weingarten surfaces in 3-dimensional spaces forms.
- (constrained) Willmore surfaces in S^n [13].
- projective minimal and Lie minimal surfaces in $\mathbb{P}^3$ and S^3 respectively [12, 18].
- affine spheres [7, 23].
- harmonic maps of a surface into a pseudo-Riemannian symmetric space [29, 30]: this includes many of the preceding examples via some form of Gauss map construction.
- isothermic surfaces in S^n [4, 5, 9, 16, 21, 26] or, more generally, isothermic submanifolds in symmetric R-spaces [15].

F. Burstall (✉)
Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
e-mail: feb@maths.bath.ac.uk

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*,
Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_3

- Möbius flat submanifolds of S^n [10, 11]: these include Guichard surfaces and conformally flat submanifolds with flat normal bundles, in particular, conformally flat hypersurfaces.
- omega surfaces in Lie sphere geometry [25].
- curved flats in pseudo-Riemannian symmetric spaces: these are related to the last four items.
- self-dual Yang–Mills fields [19]: many of our low-dimensional examples are dimensional reductions of these [31].

I shall discuss this theory via two examples: *K-surfaces* (surfaces in $\mathbb{R}^3$ of constant Gauss curvature) and *isothermic surfaces*. In both cases, I will emphasise:

- the geometry of transformations
- a gauge-theoretic approach of wide applicability.

An alternative take on these matters can be found in the more extensive survey of Terng [28].

2 *K*-Surfaces

2.1 Classical Surface Geometry

Let $f : \Sigma^2 \to \mathbb{R}^3$ be an immersion with Gauss map $N{:}\Sigma \to S^2$. Thus:

$$N \cdot \mathrm{d}f = 0.$$

These yield three invariant quadratic forms:

$$\begin{aligned} \mathrm{I} &:= \mathrm{d}f \cdot \mathrm{d}f \\ \mathrm{II} &:= -\mathrm{d}f \cdot \mathrm{d}N \\ \mathrm{III} &:= \mathrm{d}N \cdot \mathrm{d}N \end{aligned}$$

and the famous theorem of Bonnet says that the first two determine f up to a rigid motion.

Lowering an index on II gives the *shape operator* $S := -(\mathrm{d}f)^{-1} \circ \mathrm{d}N$, a symmetric (with respect to I) endomorphism on $\mathrm{T}\Sigma$. The shape operator has eigenvalues κ_1, κ_2, the *principal curvatures* from which the *mean curvature* H and the *Gauss curvature* K are given by

$$\begin{aligned} H &:= \tfrac{1}{2}(\kappa_1 + \kappa_2) \\ K &:= \kappa_1 \kappa_2. \end{aligned}$$

Further, the Cayley–Hamilton theorem applied to S gives:

$$\mathrm{III} - 2H\mathrm{II} + K\mathrm{I} = 0. \tag{1}$$

2.2 Lelieuvre's Formula

Let us suppose that $K < 0$ and write $K = -1/\rho^2$. In this case, f admits asymptotic coordinates ξ, η, thus:

$$N_\xi \cdot f_\xi = 0 = N_\eta \cdot f_\eta.$$

It follows at once that there are functions a, b so that

$$a(N \times N_\xi) = f_\xi \qquad b(N \times N_\eta) = f_\eta.$$

Now the symmetry of II: $N_\xi \cdot f_\eta = N_\eta \cdot f_\xi$, rapidly yields $a = -b$ while (1) evaluated on ∂_ξ gives $a^2 = \rho^2$ so that we have *Lelieuvre's Formula*:

$$\begin{aligned} \rho(N \times N_\xi) &= f_\xi \\ \rho(N \times N_\eta) &= -f_\eta. \end{aligned} \tag{2}$$

Cross-differentiating (2) gives us two formulae for the tangential component of $f_{\xi\eta}$:

$$f_{\xi\eta}{}^T = \rho_\eta N \times N_\xi + \rho N \times N_{\xi\eta} = \rho_\xi N \times N_\eta - \rho N \times N_{\xi\eta}.$$

From this and the linear independence of $N \times N_\xi$ and $N \times N_\eta$, we easily see that $f_{\xi\eta}{}^T = 0$ if and only if ρ is constant if and only if $N \times N_{\xi\eta} = 0$, that is, $N : (\Sigma, \mathrm{II}) \to S^2$ is a harmonic map. Finally, $f_{\xi\eta}{}^T = 0$ if and only if

$$(f_\xi \cdot f_\xi)_\eta = 0 = (f_\eta \cdot f_\eta)_\xi.$$

We conclude:

Theorem 2.1 *The following are equivalent:*
- *K is constant.*
- *$N : (\Sigma, \mathrm{II}) \to S^2$ is harmonic.*
- *Asymptotic coordinates can be chosen so that $\|f_\xi\| = 1 = \|f_\eta\|$. We say such coordinates are* Tchebyshev *for f.*

2.3 Geometry of K-Surfaces

Definition $f : \Sigma \to \mathbb{R}^3$ is a *K-surface* if it has constant, negative Gauss curvature.

From Theorem 2.1, we see that a K-surface admits Tchebyshev coordinates ξ, η with respect to which we have

$$\mathrm{I} = \mathrm{d}\xi^2 + 2\cos\omega \mathrm{d}\xi \mathrm{d}\eta + \mathrm{d}\eta^2$$

$$\mathrm{II} = \frac{2}{\rho} \sin\omega \mathrm{d}\xi \mathrm{d}\eta,$$

where ω is the angle between the coordinate directions.

For such I, II, the Codazzi equations are vacuous while the Gauss equation reads

$$\omega_{\xi\eta} = \frac{1}{\rho^2} \sin \omega. \tag{3}$$

Thus any solution of (3) gives rise to a K-surface.

Let us now turn to the symmetries of the situation:

2.3.1 Bäcklund Transformations

Let f be a K-surface and, following Bianchi [2] and Bäcklund [1], seek $\hat{f} : \Sigma \to \mathbb{R}^3$ such that:

- $\hat{f} - f$ is tangent to both f and $\hat{f}$.
- $\|\hat{f} - f\|$ is constant.
- $\hat{N} \cdot N$ is constant. (Bianchi considered the case $\hat{N} \cdot N = 0$.)

Then:

1. $\hat{f}$ exists if and only if f is a K-surface (and then, by symmetry, $\hat{f}$ is a K-surface too, in fact with the same value of K).
2. Given $a > 0$, $p_0 \in \Sigma$ and a ray $\ell_0 \subset T_{p_0}\Sigma$, one solves commuting ODE to get (locally) a unique $\hat{f}$ with

$$\|\hat{f} - f\| = \frac{2\rho}{a + a^{-1}}$$

$$\hat{N} \cdot N = \frac{a^{-1} - a}{a^{-1} + a}$$

$$\hat{f}(p_0) \in \ell_0.$$

   Thus, if $\hat{N} \cdot N = \cos\theta$, $\|\hat{f} - f\| = \rho \sin\theta$.

   We write $\hat{f} = f_a$ and say that f_a is a *Bäcklund transform* of f,
3. ξ, η are asymptotic, in fact Tchebyshev, for $\hat{f}$ too. In classical terminology, f and $\hat{f}$ are the focal surfaces of a *W-congruence*.
4. Permutability (Bianchi [3]): given a K-surface f and two Bäcklund transforms f_a, f_b with $a \neq b$, one can choose initial conditions so that there is a fourth K-surface $\hat{f}$ with

$$\hat{f} = (f_a)_b = (f_b)_a.$$

   Exercise $\hat{f}$ is algebraic in f, f_a, f_b.

   One can iterate the procedure and so build up a quad-graph of K-surfaces. At each point $p \in \Sigma$, the corresponding quad-graph of points in $\mathbb{R}^3$ is a discrete K-surface in the sense of Bobenko–Pinkall.

2.3.2 Lie Transform

If ξ, η are Tchebyshev coordinates for a K-surface f, we have seen that the Gauss–Codazzi equations reduce to the sine-Gordon equation (3) for the angle ω between coordinate directions.

However, for $\mu \in \mathbb{R}^\times$, we observe that

$$\omega^\mu(\xi, \eta) := \omega(\mu^{-1}\xi, \mu\eta)$$

also solves (3) and so gives rise to a new K-surface f^μ with

$$\mathrm{I}_{f^\mu} = \mathrm{d}\xi^2 + 2\cos\omega^\mu \mathrm{d}\xi \mathrm{d}\eta + \mathrm{d}\eta^2.$$

Such an f^μ is a *Lie transform* of f.

2.4 Harmonic Maps and Flat Connections

We have seen that K-surfaces give rise to harmonic maps $(\Sigma, \mathrm{II}) \to S^2$. The converse is also true: let c be a conformal structure on Σ of signature $(1, 1)$, $*$ the Hodge-star of c and ξ, η null coordinates. Let $N : (\Sigma, c) \to S^2$ so that

$$*\mathrm{d}N = N_\xi \mathrm{d}\xi - N_\eta \mathrm{d}\eta.$$

It is easy to see that N is harmonic if and only if

$$\mathrm{d}(N \times *\mathrm{d}N) = 0$$

in which case we can locally find $f : \Sigma \to \mathbb{R}^3$ such that $N \times *\mathrm{d}N = \mathrm{d}f$, that is,

$$\begin{aligned} N \times N_\xi &= f_\xi \\ N \times N_\eta &= -f_\eta. \end{aligned}$$

It follows at once that, whenever f, equivalently N, immerses,

1. $N \perp f_\xi, f_\eta$ so that N is the Gauss map of f.
2. $N_\xi \cdot f_\xi = 0 = N_\eta \cdot f_\eta$ so that ξ, η are asymptotic for f whence $c = \langle \mathrm{II} \rangle$.
3. $K = -1$ after recourse to (1).

2.4.1 Flat Connections

The basic observation for all that follows is that harmonic maps give rise to a holomorphic family of flat connections on the trivial bundle $\underline{\mathbb{R}}^3 := \Sigma \times \mathbb{R}^3$. We rehearse this construction in such a way as to indicate how it generalises to any (pseudo)-Riemannian symmetric target.

So let $N : (\Sigma, c) \to S^2$ and $\rho^N : \Sigma \to \mathrm{O}(3)$ be the reflection across $N^\perp$. The orthogonal decomposition

$$\underline{\mathbb{R}}^3 = \langle N \rangle \oplus N^\perp$$

induces a decomposition of the flat connection d:

$$\mathrm{d} = \mathcal{D} + \mathcal{N}$$

where N, ρ^N are $\mathcal{D}$-parallel and $\mathcal{N} \in \Omega^1(\mathfrak{so}(3))$ anti-commutes with ρ^N.

We shall several times have recourse to the identification $\mathbb{R}^3 \cong \mathfrak{so}(3)$ given by

$$v \mapsto (u \mapsto v \times u) \tag{4}$$

under which $\mathcal{N}$ is identified with $N \times \mathrm{d}N$.

The structure equations of the situation express the flatness of d and read:

$$R^{\mathcal{D}} + \tfrac{1}{2}[\mathcal{N} \wedge \mathcal{N}] = 0$$
$$\mathrm{d}^{\mathcal{D}}\mathcal{N} = 0,$$

while N is harmonic if and only if $\mathrm{d} * \mathcal{N} = 0$, or, equivalently, $\mathrm{d}^{\mathcal{D}} * \mathcal{N} = 0$ since $[*\mathcal{N} \wedge \mathcal{N}]$ is always zero.

Now write

$$\mathcal{N} = \mathcal{N}^+ + \mathcal{N}^-$$

where $*\mathcal{N}^\pm = \pm\mathcal{N}^\pm$. Then we have

$$\mathrm{d}^{\mathcal{D}}\mathcal{N} = \mathrm{d}^{\mathcal{D}}\mathcal{N}^+ + \mathrm{d}^{\mathcal{D}}\mathcal{N}^- = 0$$
$$\mathrm{d}^{\mathcal{D}} * \mathcal{N} = \mathrm{d}^{\mathcal{D}}\mathcal{N}^+ - \mathrm{d}^{\mathcal{D}}\mathcal{N}^-$$

so that N is harmonic if and only if $\mathrm{d}^{\mathcal{D}}\mathcal{N}^\pm = 0$.

Let $\lambda \in \mathbb{C}^\times$ and define a connection d_λ on $\underline{\mathbb{C}}^3$ by

$$\mathrm{d}_\lambda = \mathcal{D} + \lambda\mathcal{N}^+ + \lambda^{-1}\mathcal{N}^-.$$

Then, comparing coefficients of λ in R^{d_λ}, we have:

Proposition 2.2 *N is harmonic if and only if* d_λ *is flat for all* $\lambda \in \mathbb{C}^\times$.
We note that d_λ has the following four properties:

(i) $\lambda \mapsto d_\lambda^+$ is holomorphic on $\mathbb{C}$ with a simple pole at ∞ while $\lambda \mapsto \mathrm{d}_\lambda^-$ is holomorphic on $\mathbb{C}^\times \cup \{\infty\}$ with a simple pole at 0.

(ii) $\rho^N \cdot \mathrm{d}_\lambda = \mathrm{d}_{-\lambda}$. Here, and below, for connection D and gauge transformation $g : \Sigma \to \mathrm{O}(3, \mathbb{C})$, $g \cdot D = g \circ D \circ g^{-1}$, the usual action of gauge transformations on connections.
(iii) $\mathrm{d}_{\bar{\lambda}} = \overline{\mathrm{d}_\lambda}$.
(iv) $\mathrm{d}_1 = \mathrm{d}$.

Exercise These properties uniquely determine d_λ.
Thus:

Proposition 2.3 *N is harmonic if and only if there is a family $\lambda \mapsto \mathrm{d}_\lambda$ of flat connections with properties (i)–(iv).*

2.5 Spectral Deformation

Let N be harmonic with flat connections d_λ. Since d_λ is flat, there is a locally a trivialising gauge $T_\lambda : \Sigma \to \mathrm{SO}(3, \mathbb{C})$, that is,

$$T_\lambda \cdot \mathrm{d}_\lambda = \mathrm{d}.$$

Now fix $\mu \in \mathbb{R}^\times$ and set $\mathrm{d}^\mu_\lambda := \mathrm{d}_{\lambda\mu}$. We notice that $\lambda \mapsto \mathrm{d}^\mu_\lambda$ has properties (i)–(iii) but $\mathrm{d}^\mu_1 = \mathrm{d}_\mu$. It follows then that $\lambda \mapsto T_\mu \cdot \mathrm{d}^\mu_\lambda$ has (i)–(iv) with respect to $N^\mu := T_\mu N : \Sigma \to S^2$. We therefore conclude

Theorem 2.4 *$N^\mu : (\Sigma, c) \to S^2$ is harmonic and so gives rise to a K-surface f^μ.*
Moreover,

$$\mathrm{d}N^\mu = T_\mu(\mathrm{d}_\mu N) = T_\mu(\mu \mathrm{d}^+ N + \mu^{-1}\mathrm{d}^- N).$$

If ξ, η are Tchebyshev for f, it follows from this that f^μ has first fundamental form

$$I_{f^\mu} = \mu^2 \mathrm{d}\xi^2 + 2\cos\omega \mathrm{d}\xi \mathrm{d}\eta + \mu^{-2}\mathrm{d}\eta^2.$$

Thus $\hat{\xi} = \mu\xi$ and $\hat{\eta} = \mu^{-1}\eta$ are Tchebyshev for f^μ and the corresponding sine-Gordon solution is $\omega(\mu^{-1}\hat{\xi}, \mu\hat{\eta})$. Otherwise said, f^μ is a Lie transform of f.

2.6 Sym Formula

Knowing the trivialising gauges T_λ allows us to compute Lie transforms without integrations. Indeed, by definition,

$$T_\lambda \circ \mathrm{d}_\lambda = \mathrm{d} \circ T_\lambda$$

and differentiating this with respect to λ at μ yields:

$$\mathrm{Ad}_{T_\mu}(\partial \mathrm{d}_\lambda/\partial\lambda|_\mu) = \mathrm{d}(\partial T_\lambda/\partial\lambda|_\mu T_\mu^{-1}).$$

The left side of this reads

$$\mathrm{Ad}_{T_\mu}(\mathcal{N}^+ - \mathcal{N}^-/\mu^2) = \frac{1}{\mu} * \mathcal{N}^\mu,$$

or, using the identification (4),

$$N^\mu \times *\mathrm{d}N^\mu = \mu \mathrm{d}(\partial T_\lambda/\partial\lambda|_\mu T_\mu^{-1}).$$

Thus we have the *Sym formula* [27]:

$$f^\mu = \mu\partial T_\lambda/\partial\lambda|_\mu T_\mu^{-1}, \tag{5}$$

where we use (4) to view the right side as a map $\Sigma \to \mathbb{R}^3$. In particular, taking $\mu = 1$ and assuming, without loss of generality that $T_1 = 1$, we recover our original K-surface:

$$f = \partial T_\lambda/\partial\lambda|_{\lambda=1}. \tag{6}$$

2.7 Parallel Sections and Bäcklund Transformations

Again we start with a harmonic $N : (\Sigma, c) \to S^2$ and its family of flat connections d_λ. We seek to construct a holomorphic family of gauge transformations $r(\lambda) : \Sigma \to \mathrm{SO}(3,\mathbb{C})$ so that the connections $r(\lambda)\cdot d_\lambda$ have properties (i)–(iv) with respect to a new map $\hat{N} : \Sigma \to S^2$. Since these gauged connections are flat, $\hat{N}$ will be harmonic.

We will build our gauge transformations from d_λ-parallel subbundles of $\underline{\mathbb{C}}^3$ using an avatar of a construction of Terng–Uhlenbeck [29]. First the algebra: for null line subbundles $L, L^* \leq \underline{\mathbb{C}}^3$ with $L \cap L^* = \{0\}$, define

$$\Gamma_{L^*}^L(\lambda) = \begin{cases} \lambda & \text{on } L \\ 1 & \text{on } (L \oplus L^*)^\perp \\ \lambda^{-1} & \text{on } L^* \end{cases} : \Sigma \to \mathrm{SO}(3,\mathbb{C}).$$

The decisive properties of $\Gamma_{L^*}^L$ is that it takes values in semisimple homomorphisms $\mathbb{C}^\times \to \mathrm{SO}(3,\mathbb{C})$ and that $\mathrm{Ad}\Gamma_{L^*}^L$ has only simple poles at 0 and ∞.

Now fix $a > 0$ and choose L so that:

1. L is d_{ia}-parallel.
2. $\rho^N L = \bar{L}$. Denote this bundle by L^*.
3. $L \cap L^* = \{0\}$.

Remarks

(1) The last two conditions amount to demanding that $(L \oplus L^*)^\perp$ is a real line tangent to N or, equivalently, f. It is on this line that our Bäcklund transform will eventually lie.

(2) The conditions are compatible: both $\rho^N L$ and $\bar{L}$ are d_{-ia}-parallel and so coincide as soon as they do so at an initial point.

(3) In fact, L is completely determined by the data of a unit tangent vector t at a single point $p_0 \in \Sigma$. We take for L_{p_0} and $L^*_{p_0}$ the $\pm i$-eigenspaces of $v \mapsto t \times v$ and then define L and L^* by parallel transport. Of course, condition (3) may fail eventually.

Thanks to condition (2), we have

$$\begin{aligned} \rho^N \Gamma^L_{L^*}(\lambda) &= \Gamma^L_{L^*}(\lambda^{-1}) \\ \overline{\Gamma^L_{L^*}(\lambda)} &= \Gamma^L_{L^*}(1/\bar{\lambda}). \end{aligned} \tag{7}$$

After all this preparation, we finally set:

$$r(\lambda) = \Gamma^L_{L^*}\left((\frac{1+ia}{1-ia})(\frac{\lambda - ia}{\lambda + ia}) \right).$$

We have:

- $\lambda \mapsto r(\lambda)$ is holomorphic on $\mathbb{P}^1 \setminus \{\pm ia\}$.
- $\overline{r(\lambda)} = r(\bar{\lambda})$ so that, in particular, $r(\lambda)$ takes values in $\mathrm{SO}(3)$ for $\lambda \in \mathbb{RP}^1$.
- $r(-\lambda) \circ \rho^N \circ r(\lambda)^{-1}$ is independent of λ and so coincides with $r(\infty)\rho^N r(\infty)^{-1} = \rho^{r(\infty)N}$. Thus

$$r(-\lambda) \circ \rho^N = \rho^{r(\infty)N} \circ r(\lambda). \tag{8}$$

- $r(1) = 1$.

We therefore set:

$$\hat{N} = r(\infty)N$$

$$\hat{\mathrm{d}}_\lambda = r(\lambda) \cdot \mathrm{d}_\lambda.$$

Proposition 2.5 $\hat{\mathrm{d}}_\lambda$ *has properties (i)–(iv) with respect to* $\hat{N}$.

Proof This is all a straightforward verification except for item (i). Since $r(\lambda)$ is holomorphic near zero and infinity, $\hat{\mathrm{d}}_\lambda$ has the same poles there as d_λ so the main issue is to see that $\lambda \mapsto \hat{\mathrm{d}}_\lambda$ is holomorphic at $\pm ia$. This is where the fact that L and L^* are parallel comes in and is an immediate consequence of the following

Lemma 2.6 *Let L, L^* be null line subbundles, $\lambda \mapsto \mathrm{d}_\lambda$ any holomorphic family of connections and ψ^α_β any linear fractional transformation with a zero at α and a pole at β.*

Then $\lambda \mapsto \Gamma^L_{L^*}(\psi^\alpha_\beta(\lambda)) \cdot \mathrm{d}_\lambda$ is holomorphic at α if and only if L is d^α-parallel and holomorphic at β if and only if L^* is d_β-parallel. □

We therefore conclude:

Theorem 2.7 *$\hat{N} : (\Sigma, c) \to S^2$ is harmonic with associated flat connections $\hat{\mathrm{d}}_\lambda = r(\lambda) \cdot \mathrm{d}_\lambda$.*

It is important that we have control on $\hat{\mathrm{d}}_\lambda$ since this allows us to iterate the construction as we shall see below.

Now let us turn to the geometry of the situation. N and $\hat{N}$ are the Gauss maps of K-surfaces f and $\hat{f}$, both with $K = -1$. We compute $\hat{f}$ via the Sym formula: if T_λ is a trivialising gauge for d_λ with $T_1 = 1$, then $T_\lambda r(\lambda)^{-1}$ is a trivialising gauge for $\hat{\mathrm{d}}_\lambda$ so that (6) yields

$$\hat{f} = f - \partial r/\partial\lambda|_{\lambda=1}.$$

The chain rule gives

$$\partial r/\partial\lambda|_{\lambda=1} = \frac{2ia}{1+a^2}(\Gamma^L_{L^*})'(1)$$

which, under the identification (4), is

$$\frac{2a}{1+a^2}t$$

for t a real unit length section of $(L \oplus L^*)^\perp \leq \mathrm{d}f(T\Sigma)$.

We therefore conclude:

- $\hat{f} - f = -2t/(a + a^{-1})$ and so is tangent to f and of constant length.
- Since $r(\infty)$ is rotation about t through angle θ for

  $$e^{i\theta} = \frac{1+ia}{1-ia},$$

  $\hat{N} = r(\infty)N$ is orthogonal to t (so that $\hat{f} - f$ is tangent to $\hat{f}$ as well) and

  $$\hat{N} \cdot N = \mathrm{Re}\frac{1+ia}{1-ia} = \frac{a^{-1} - a}{a^{-1} + a}.$$

  Thus $\hat{f} = f_a$, a Bäcklund transform of f.
- The conformal structures of $\mathrm{I\!I}$ and $\hat{\mathrm{I\!I}}$ coincide (they are both c). Otherwise said, the asymptotic lines of f and $\hat{f}$ coincide.

2.8 Permutability

Suppose that we have a K-surface and two Bäcklund transforms f_a and f_b produced from d_{ia}-parallel L_a and d_{ib}-parallel L_b, respectively. We now seek a fourth K-surface

$$f_{ab} = (f_a)_b = (f_b)_a.$$

For this we will need a d^a_{ib}-parallel $\hat{L}_b$ and a d^b_{ia}-parallel $\hat{L}_a$. However, since $\mathrm{d}^a_{ib} = r_a(ib) \cdot \mathrm{d}_{ib}$ etc, we have natural candidates in

$$\begin{aligned} \hat{L}_b &:= r_a(ib)L_b \\ \hat{L}_a &:= r_b(ia)L_a. \end{aligned} \tag{9}$$

Exercise Check that $\hat{L}_a, \hat{L}_b$ so defined satisfy

$$\rho^a \hat{L}_b = \overline{\hat{L}_b} \qquad \rho^b \hat{L}_a = \overline{\hat{L}_a}.$$

We therefore have K-surfaces $(f_a)_b$ and $(f_b)_a$ with associated connections $\hat{r}_b(\lambda) \cdot \mathrm{d}^a_\lambda$ and $\hat{r}_a(\lambda) \cdot \mathrm{d}^b_\lambda$. The key to showing these coincide is the following

Proposition 2.8 $\hat{r}_b r_a = \hat{r}_a r_b$.
For this we need a lemma which is a discrete version of Lemma 2.6:

Lemma 2.9 *Let $\ell^\pm, \hat{\ell}^\pm$ be two pairs of distinct null lines, ψ^α_β a linear fraction transformation with a zero at α and a pole at β and $\lambda \mapsto E(\lambda)$ holomorphic near α and β. Then*

$$\lambda \mapsto \Gamma^{\hat{\ell}+}_{\hat{\ell}_-}(\psi^\alpha_\beta(\lambda))E(\lambda)(\Gamma^{\ell+}_{\ell-}(\psi^\alpha_\beta(\lambda)))^{-1}$$

is holomorphic at α if and only if $E(\alpha)\ell^+ = \hat{\ell}^+$ and holomorphic at β if and only if $E(\beta)\ell^- = \hat{\ell}^-$. □

Proof of Proposition 2.8 We show that $R := \hat{r}_b r_a r_b^{-1} \hat{r}_a^{-1}$ is identically 1. Note R is holomorphic on $\mathbb{P}^1 \setminus \{\pm ia, \pm ib\}$ and $R(1) = 1$. Now Lemma 2.9 together with (9) shows that $\hat{r}_b r_a r_b^{-1}$ is holomorphic at $\pm ib$ and that $r_a r_b^{-1} \hat{r}_a^{-1}$ is holomorphic at $\pm ia$ so that R is holomorphic on $\mathbb{P}^1$ and so is constant. □

In particular,

$$(f_a)_b = f - \partial(\hat{r}_b r_a)/\partial\lambda|_{\lambda=1} = f - \partial(\hat{r}_a r_b)/\partial\lambda|_{\lambda=1} = (f_b)_a$$

and we have established Bianchi permutability.

3 Isothermic Surfaces

3.1 Classical Theory

First studied by Bour [8] in 1862, a surface $f : \Sigma \to \mathbb{R}^3$ is *isothermic* if it admits conformal curvature line coordinates x, y so that

$$\mathrm{I} := e^{2u}(\mathrm{d}x^2 + \mathrm{d}y^2)$$
$$\mathrm{II} = e^{2u}(\kappa_1 \mathrm{d}x^2 + \kappa_2 \mathrm{d}y^2).$$

A more invariant formulation is that there should exist a non-zero holomorphic quadratic differential q on Σ such that

$$[q, \mathrm{II}] = 0,$$

or, more explicitly, $[S, Q] = 0$ where Q is the symmetric endomorphism with $q = I(Q\,,\,)$. The relationship between the two formulations is given by setting $z = x + iy$ and then $q = \mathrm{d}z^2$.

Examples

- cones, cylinders and surfaces of revolution are isothermic: for the last, parametrise the profile curve in the upper half plane by hyperbolic arc length to get conformal curvature line coordinates.

  In particular, we see that isothermic surfaces have no regularity.
- (Stereo-images of) surfaces of constant H in 3-dimensional space-forms. Here we take q to be the Hopf differential $\mathrm{II}^{2,0}$.
- quadrics. Sadly, I know no short or conceptual argument for this.

  Isothermic surfaces have many symmetries:

1. Conformal invariance: if $\Phi : \mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$ is a conformal diffeomorphism and f is isothermic, then $\Phi \circ f$ is isothermic too. This is because, while II is certainly not conformally invariant, its trace-free part II_0 is and $[q, \mathrm{II}] = [q, \mathrm{II}_0]$.
2. In 1867, Christoffel [17] showed that f is isothermic if and only if there is (locally) a *dual surface* $f^c : \Sigma \to \mathbb{R}^3$ such that

   - The metrics I and I^c are in the same conformal class.
   - f and f^c have parallel tangent planes: $\mathrm{d}f(T\Sigma) = \mathrm{d}f^c(T\Sigma)$.
   - $\det(\mathrm{d}f^{-1} \circ \mathrm{d}f^c) < 0$.

   Of course, the symmetry of the conditions means that f^c is isothermic also and that $(f^c)^c = f$.

   Examples

   – When f has constant mean curvature $H \neq 0$, $f^c = f + N/H$ which has the same constant mean curvature.

   – When f is minimal, $f^c = N$, the Gauss map.

Conversely, any conformal map $N : \Sigma \to S^2$ is isothermic with respect to any holomorphic quadratic differential q. Fixing such a q, we obtain a minimal surface $N^c : \Sigma \to \mathbb{R}^3$: this is the celebrated Weierstrass–Enneper formula!

3. After Darboux [21], we seek a surface $\hat{f} : \Sigma \to \mathbb{R}^3 \cup \{\infty\} = S^3$ such that

- f and $\hat{f}$ induce the same conformal structure on Σ.
- f and $\hat{f}$ have the same curvature lines.
- For each $p \in \Sigma$, there is a 2-sphere $S(p) \subset S^3$ to which both f and $\hat{f}$ are tangent at p.

In classical terminology, f and $\hat{f}$ are the enveloping surfaces of a *conformal Ribaucour sphere congruence*.

Here are the facts:

(a) $\hat{f}$ exists if and only if f is isothermic so that, by symmetry, $\hat{f}$ is isothermic also.

(b) For $a \in \mathbb{R}^\times$ and initial point $y_0 \in S^3 \setminus \{f(p_0)\}$, we can find a unique such $\hat{f}$ with $\hat{f}(p_0) = y_0$ by solving a completely integrable 5×5 system of linear differential equations with a quadratic constraint (thus, to anticipate, finding a parallel section of a metric connection!).

We write $\hat{f} = f_a$ and call it a *Darboux transformation of f with parameter a*.

(c) Permutability (Bianchi [4]): Given isothermic f and two Darboux transforms f_a and f_b with $a \neq b$, there is a fourth isothermic surface $f_{ab} = (f_a)_b = (f_b)_a$ which is algebraically determined by f, f_a, f_b. Indeed, Demoulin [22] shows that f, f_a, f_b, f_{ab} are pointwise concircular with constant cross-ratio $(f_b, f_a; f, f_{ab}) = a/b$! We call a quadruple of surfaces related in this way a *Bianchi quadrilateral*.

One can iterate this procedure to construct a quad-graph of isothermic surfaces. At each point, the corresponding quad-graph of points in S^3 with concircular elementary quadrilaterals of prescribed cross-ratio gives a discrete isothermic surface in the sense of Bobenko–Pinkall [6] as we shall see in Sect. 3.8.

Moreover, if we now add a third surface f_c, we can apply this result to obtain f_{ab}, f_{ac}, f_{bc} and then obtain an eighth surface f_{abc} such that

$$f_a, f_{ab}, f_{ac}, f_{abc}$$
$$f_b, f_{ab}, f_{bc}, f_{abc}$$
$$f_c, f_{ac}, f_{bc}, f_{abc}$$

are *all* Bianchi quadrilaterals.

This "cube theorem", also due to Bianchi (and also available for Bäcklund transformations of K-surfaces), can be viewed as the construction of a Darboux transform for discrete isothermic surfaces, see Sect. 3.10.

4. Spectral deformation (Calapso [16], Bianchi [5]): Given f isothermic, there is a 1-parameter family f_t of isothermic surfaces with $f = f_0$ inducing the same conformal structure on Σ and having the same II_0. The f_t are called *T-transforms* of f.

   Aside: We know that I and II determine a surface in $\mathbb{R}^3$ up to rigid motions. It is therefore natural to ask if the conformal invariants $\langle \mathrm{I} \rangle$ and II_0 determine a surface in S^3 up to conformal diffeomorphism. The T-transforms of an isothermic surface show that the answer is no but, according to Cartan,[1] are the only such witnesses: if f is not isothermic it is determined up to conformal diffeomorphism by $\langle \mathrm{I} \rangle$ and II_0.

In this story, we recognise some familiar features: solutions in 1-parameter families and new solutions from commuting ODE. We will see how our gauge theoretic formalism applies in this situation.

3.2 Conformal Geometry (Rapid Introduction)

The conformal invariance of isothermic surfaces suggests that we should work on the conformal compactification $\mathbb{R}^3 \cup \{\infty\} = S^3$ of $\mathbb{R}^3$. For this, Darboux [20] offers a convenient model which essentially linearises the situation.

Let $\mathbb{R}^{4,1}$ be a 5-dimensional Minkowski space with a metric $(\,,\,)$ of signature $++++-$ and let $\mathcal{L} \subset \mathbb{R}^{4,1}$ be the light-cone:

$$\mathcal{L} = \{v \in \mathbb{R}^{4,1} \colon (v, v) = 0\}.$$

The collection of lines through zero in $\mathcal{L}$ is the projective light-cone $\mathbb{P}(\mathcal{L})$ which is a smooth quadric in $\mathbb{P}(\mathbb{R}^{4,1})$ diffeomorphic to S^3.

$\mathbb{P}(\mathcal{L})$ has a conformal structure: any section $\sigma : \mathbb{P}(\mathcal{L}) \to \mathcal{L}^\times$ of the projection $\mathcal{L}^\times \to \mathbb{P}(\mathcal{L})$ gives rises to a positive definite metric

$$g_\sigma(X, Y) := (\mathrm{d}_X\sigma, \mathrm{d}_Y\sigma)$$

and it is easy to see that $g_{e^u\sigma} = e^{2u} g_\sigma$. Conic sections give constant curvature metrics: more explicitly, let $t_0 \in \mathbb{R}^{4,1}$ have $(t_0, t_0) = -1$ and write $\mathbb{R}^{4,1} = \mathbb{R}^4 \oplus \langle t_0 \rangle$. Then the map $x \mapsto x + t_0 : S^3 \to \mathcal{L}$ from the unit sphere of $\mathbb{R}^4$ induces a conformal diffeomorphism onto $\mathbb{P}(\mathcal{L})$. Thus $\mathbb{P}(\mathcal{L}) \cong S^3$ as conformal manifolds.

k-spheres in S^3 are linear objects in this picture: they are the subsets $\mathbb{P}(W \cap \mathcal{L}) \subset \mathbb{P}(\mathcal{L})$ where $W \leq \mathbb{R}^{4,1}$ is a linear subspace of signature $(k+1, 1)$. For example, we obtain the circle through three distinct points in $\mathbb{P}(\mathcal{L})$ by taking W to be the $(2, 1)$-plane they span.

---

[1] See [10, 14] for modern treatments.

Exercise Let W be a $(3,1)$-plane. Show that reflection across W induces the inversion of $\mathbb{P}(\mathcal{L}) = S^3$ in the corresponding 2-sphere.
More generally, the subgroup $O_+(4,1)$ of the orthogonal group that preserves the components of the light cone acts effectively by conformal diffeomorphisms on $\mathbb{P}(\mathcal{L})$ and, by the exercise, has all inversions in its image. It follows from a theorem of Liouville that $O_+(4,1)$ is the conformal diffeomorphism group of S^3.

In the light of all this, we henceforth treat maps $f : \Sigma \to \mathbb{P}(\mathcal{L})$ and identify such maps with null line subbundles $f \leq \underline{\mathbb{R}}^{4,1}$ via $f(x) = f_x$.

3.3 Isothermic Surfaces Reformulated

We give a third and final reformulation of the isothermic condition by exploiting the structure of S^3 as a homogeneous space for $G := O_+(4,1)$.

Since G acts transitively, we have, for $x \in S^3$ an isomorphism

$$T_x S^3 \cong \mathfrak{g}/\mathfrak{p}_x,$$

where $\mathfrak{p}_x$ is the infinitesimal stabiliser of x. The key algebraic ingredient in what follows is that $\mathfrak{p}_x$ is a *parabolic* subalgebra with abelian nilradical: this means that the polar $\mathfrak{p}_x^\perp$ with respect to the Killing form is an ad-nilpotent abelian subalgebra (in fact, it is the algebra of infinitesimal translations on the $\mathbb{R}^3$ obtained by stereoprojecting away from x).

Remark We identify $\mathfrak{g} = \mathfrak{so}(4,1)$ with $\wedge^2\mathbb{R}^{4,1}$ via

$$(u \wedge v)w = (u,w)v - (v,w)u$$

and then $\mathfrak{p}_x^\perp = x \wedge x^\perp$.

Now $T_x^* S^3 \cong (\mathfrak{g}/\mathfrak{p}_x)^*$ which is isomorphic to $\mathfrak{p}_x^\perp \leq \mathfrak{g}$ via the Killing form. We have therefore identified T^*S^3 with a bundle of abelian subalgebras of $\mathfrak{g}$.

With this in hand, let q be a symmetric $(2,0)$-form on Σ and $f : \Sigma \to S^3$ an immersion. Then $q + \bar{q}$ is a section of $S^2T^*\Sigma$ and so may be viewed as a 1-form with values in $T^*\Sigma$. Moreover, $\mathrm{d}f$ and the conformal structure of S^3 allow us to view $T^*\Sigma$ as a subbundle of $f^{-1}T^*S^3$ and so as a subbundle of $\underline{\mathfrak{g}}$. Chaining all this together, we see that $q + \bar{q}$ gives rise to a $\mathfrak{g}$-valued 1-form η taking values in the bundle of abelian subalgebras $\mathfrak{p}_f^\perp = f \wedge f^\perp$.

The crucial fact is now:

Proposition 3.1 *q is a holomorphic quadratic differential with $[q, \mathrm{II}_0] = 0$ if and only if $\mathrm{d}\eta = 0$.*
The converse is also true and we arrive at our final formulation of the isothermic condition:

Theorem 3.2 *f is isothermic if and only if there is a non-zero $\eta \in \Omega^1(\mathfrak{g})$ with*

(1) $\mathrm{d}\eta = 0$.
(2) *η takes values in $f \wedge f^\perp$.*

3.4 Flat Connections

For f an isothermic surface with closed form η, we define, for each $t \in \mathbb{R}$, a metric connection $\mathrm{d}_t = \mathrm{d} + t\eta$ on $\underline{\mathbb{R}}^{4,1}$ and note that,

$$R^{\mathrm{d}_t} = R^{\mathrm{d}} + \mathrm{d}\eta + \tfrac{1}{2}[\eta \wedge \eta] = 0$$

since each summand vanishes separately: $[\eta \wedge \eta] = 0$ since η takes values in the abelian subalgebra $f \wedge f^{\perp}$.

Thus we once again have a family of flat connections which we now exploit.

3.5 Spectral Deformation

Since each d_t is flat, we may locally find a trivialising gauge $T_t : \Sigma \to \mathrm{SO}(4,1)$ with $T_t \cdot \mathrm{d}_t = \mathrm{d}$.

For $s \in \mathbb{R}^{\times}$, we have $\mathrm{d}_{s+t} = \mathrm{d}_s + t\eta$ so that

$$\mathrm{d}^s_t := T_s \cdot \mathrm{d}_{s+t} = \mathrm{d} + \mathrm{Ad}_{T_s}\eta$$

is flat for all t. Set $f^s = T_s f$ and $\eta_s := \mathrm{Ad}_{T_s}\eta$ which takes values in the bundle of abelian subalgebras $f_s \wedge f_s^{\perp}$ so that $[\eta_s \wedge \eta_s] = 0$. The flatness of d^s_t now tells us that $\mathrm{d}\eta_s = 0$ so that f_s is isothermic.

In fact, the f_s, $s \in \mathbb{R}^{\times}$, are the T-transforms of Bianchi and Calapso.

3.6 Parallel Sections and Darboux Transforms

Here the analysis is much easier than for K-surfaces: there we needed a slightly elaborate construction to arrive at a new surface from parallel line subbundles. For isothermic surfaces, the new surfaces *are* the parallel line subbundles!

Indeed, for f isothermic with connections d_t and $a \in \mathbb{R}^{\times}$, choose a null line subbundle $\hat{f} \leq \underline{\mathbb{R}}^{4,1}$ such that

1. $\hat{f}$ is d_a-parallel.
2. $f \cap \hat{f} = \{0\}$ (this condition may eventually fail far from the initial condition and this will introduce singularities into our transform).

Then:

- $\hat{f}$ is isothermic.
- $\hat{f}$ is a Darboux transform of f with parameter a.
- $\hat{\mathrm{d}}_t = \Gamma^{\hat{f}}_f(1 - t/a) \cdot \mathrm{d}_t$.

3.7 Permutability

Suppose now that we have isothermic f and two Darboux transforms f_a and f_b with connections d_t^a and d_t^b. We seek a fourth isothermic surface f_{ab} which is a simultaneous Darboux transform of f_a and f_b:

$$f_{ab} = (f_a)_b = (f_b)_a.$$

Thus we need $(f_a)_b$ to be d_b^a-parallel and $(f_b)_a$ to be d_a^b-parallel. The obvious candidates are:

$$\begin{aligned}(f_a)_b &= \Gamma_f^{f_a}(1 - b/a)f_b\\ (f_b)_a &= \Gamma_f^{f_b}(1 - a/b)f_a.\end{aligned}$$

We shall give two arguments that these coincide.

First note that, for $x, y, z \in \mathbb{P}(\mathcal{L})$,

$$t \mapsto \Gamma_y^x(t)z$$

is a rational parametrisation of the circle through x, y, z by $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ with

$$\begin{aligned}\infty &\mapsto x\\ 0 &\mapsto y\\ 1 &\mapsto z\end{aligned}$$

so that $\Gamma_y^x(t)z$ has cross-ratio t with x, y, z:

$$(x, y; z, \Gamma_y^x(t)z) = t.$$

We therefore conclude that

$$\begin{aligned}(f_a, f; f_b, (f_a)_b) &= 1 - b/a\\ (f_b, f, f_a, (f_b)_a) &= 1 - a/b\end{aligned} \tag{10}$$

whence a symmetry of the cross-ratio yields

$$(f_a, f; f_b, (f_a)_b) = (f_a, f; f_b, (f_b)_a)$$

so that $(f_a)_b = (f_b)_a$.

Our second argument extracts more. We prove

$$\Gamma_{f_a}^{(f_a)_b}(1 - t/b)\Gamma_f^{f_a}(1 - t/a) = \Gamma_{f_a}^{f_b}\left(\tfrac{1-t/b}{1-t/a}\right) = \Gamma_{f_b}^{(f_b)_a}(1 - t/a)\Gamma_f^{f_b}(1 - t/b). \tag{11}$$

Proof of (11) Let L and M denote the left and middle members of (11) respectively. It suffices to prove $L = M$ as the remaining equality follows by swopping the roles of a and b. For $L = M$, we note that L and M agree on f_a and $f_a^\perp/f_a$ so, since both are orthogonal, it is enough to show that $Lf_b = f_b$ or, equivalently,

$$\Gamma_f^{f_a}(1 - t/a)f_b = \Gamma_{(f_a)_b}^{f_a}(1 - t/b)f_b.$$

However, these are rational parametrisations of the same circle that agree at $\infty, 0, b$ and so everywhere. □

We now evaluate (11) on f to get

$$\Gamma_{f_a}^{(f_a)_b}(1 - t/b)f = \Gamma_{f_a}^{f_b}(\tfrac{1-t/b}{1-t/a})f = \Gamma_{f_b}^{(f_b)_a}(1 - t/a)f$$

and then take $t = \infty$ to conclude

$$(f_a)_b = \Gamma_{f_a}^{f_b}(a/b)f = (f_b)_a.$$

As a consequence $(f_b,f_a;f,f_{ab}) = a/b$ which is another version of (10).

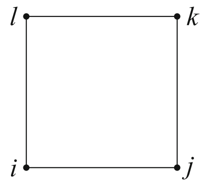
The same argument proves the cube theorem: introduce a third Darboux transform f_c and so three surfaces f_{ab},f_{bc},f_{ac}. This gives rise to three new Bianchi quadrilaterals which we show share a common surface. To do this, evaluate (11) on f_c at $t = c$ to get

$$\Gamma_{f_a}^{f_{ab}}(1 - c/b)f_{ac} = \Gamma_{f_a}^{f_b}(\tfrac{1-c/b}{1-c/a})f_c = \Gamma_{f_b}^{f_{ab}}(1 - c/a)f_{bc}.$$

Here the left member is the simultaneous Darboux transform of f_{ab} and f_{ac} and the right is the simultaneous transform of f_{ab} and f_{bc}. The remaining equality follows by symmetry and the theorem is proved. As a bonus, we see that f_a,f_b,f_c,f_{abc} are also concircular with cross ratio $\frac{1-c/b}{1-c/a}$.

3.8 Discrete Isothermic Surfaces

View $\mathbb{Z}^2$ as the vertices of a combinatorial structure with edges between adjacent vertices and quadrilateral faces. We denote the directed edge from i to j by (j, i) and label the faces of an elementary quadrilateral as follows:



According to Bobenko–Pinkall [6], a discrete isothermic surface is a map $f : \mathbb{Z}^2 \to S^3 = \mathbb{P}(\mathcal{L})$ along with a *factorising function* a from undirected edges to $\mathbb{R}^\times$ such that

- a is equal on opposite edges: $a(i,j) = a(l,k)$ and $a(i,l) = a(j,k)$.
- f has concircular values on each elementary quadrilateral with cross-ratio given by

$$(f(l),f(j);f(i),f(k)) = a(i,j)/a(i,l).$$

Thus the geometry is the pointwise geometry of Bianchi quadrilaterals of smooth isothermic surfaces.

3.9 Discrete Gauge Theory

The idea of discrete gauge theory is to replace connections by parallel transport and vanishing curvature by trivial holonomy. Here are the main ingredients:

- A *discrete vector bundle* V of rank n assigns a n-dimensional vector space V_i to each $i \in \mathbb{Z}^2$.

  For example, the trivial bundle $\underline{\mathbb{R}}^{4,1}$ has $\mathbb{R}^{4,1}_i = \mathbb{R}^{4,1}$ for each i.
- A *section* of V is a map $\sigma : \mathbb{Z}^2 \to \bigsqcup_i V_i$ such that $\sigma(i) \in V_i$, for all i.
- A *discrete connection* Γ on V, assigns to each directed edge (j,i) a linear isomorphism $\Gamma_{ji} : V_i \to V_j$ such that

$$\Gamma_{ij} = \Gamma_{ji}^{-1}.$$

  Example: the trivial connection 1 on $\underline{\mathbb{R}}^{4,1}$ has $1_{ji} = 1$, for all edges (j,i).
- A section of V is *parallel* for Γ if $\sigma(j) = \Gamma_{ji}\sigma(i)$, for all edges (j,i).
- A *discrete gauge transformation* assigns to each i a linear isomorphism $g(i) : V_i \to V_i$.

  These act on connections by $(g \cdot \Gamma)_{ji} = g(j) \circ \Gamma_{ji} \circ g(i)^{-1}$.
- A connection Γ is *flat* if, on every elementary quadrilateral we have

$$\Gamma_{il}\Gamma_{lk}\Gamma_{kj}\Gamma_{ji} = 1$$

  or, equivalently,

$$\Gamma_{kl}\Gamma_{li} = \Gamma_{kj}\Gamma_{ji}.$$

  In this case, we can find a *trivialising gauge* $T : V \to \underline{\mathbb{R}}^n$ such that

$$T \cdot \Gamma = 1,$$

that is, $T(i) : V_i \cong \mathbb{R}^n$ with

$$\Gamma_{ji} = T(j)^{-1} T(i).$$

We now have parallel sections through any point of V via $\sigma = T^{-1} x_0$ for constant $x_0 \in \mathbb{R}^n$.

3.10 Gauge Theory of Discrete Isothermic Surfaces

Given $f : \mathbb{Z}^2 \to S^3 = \mathbb{P}(\mathcal{L})$ and a factorising function a on edges, equal on opposite edges, we define a family of connections Γ^t on $\underline{\mathbb{R}}^{4,1}$ by:

$$\Gamma^t_{ji} = \Gamma^{f(j)}_{f(i)}(1 - t/a(i,j)).$$

The arguments of Sect. 3.7 and especially (11) essentially establish the following result:

Theorem 3.3 *f is discrete isothermic with factorising function a if and only if Γ^t is flat for all $t \in \mathbb{R}$.*

We may now apply all our previous gauge theoretic arguments in this new setting! We give just one example: a Darboux transform of a discrete isothermic surface should be given by a parallel null line subbundle. To verify this, fix $\hat{a} \in \mathbb{R}^\times$ and let $\hat{f} \leq \underline{\mathbb{R}}^{4,1}$ be a null line subbundle such that

1. $\hat{f}$ is $\Gamma^{\hat{a}}$-parallel.
2. $f(i) \cap \hat{f} = \{0\}$, for all i.

Spelling out the parallel condition gives

$$\Gamma^{f(j)}_{f(i)}(1 - \hat{a}/a(i,j))\hat{f}(i) = \hat{f}(j)$$

so that $f(i), f(j), \hat{f}(i), \hat{f}(j)$ are concircular with fixed cross ratio $a(i,j)/\hat{a}$. Relabelling (11) to describe this quadrilateral gives

$$\Gamma^{\hat{f}(j)}_{f(j)}(1 - t/\hat{a})\Gamma^{f(j)}_{f(i)}(1 - t/a(i,j)) = \Gamma^{\hat{f}(j)}_{\hat{f}(i)}(1 - t/a(i,j))\Gamma^{f(i)}_{f(i)}(1 - t/\hat{a}).$$

Otherwise said:

$$\Gamma^{\hat{f}}_{f}(1 - t/\hat{a}) \cdot \Gamma^t = \hat{\Gamma}^t.$$

Now $\hat{\Gamma}^t$ is flat for all t being a gauge of flat Γ^t so that $\hat{f}$ is indeed isothermic with the same factorising function as f.

We remark that we can iterate this construction and so build up a map $F : \mathbb{Z}^3 \to S^3$ whose restrictions to level sets $\{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_3 = m\}$ are our iterated Darboux transforms. Moreover, the two other families of level sets obtained by holding n_1 or n_2 fixed also consist (as we have just seen) of concircular quadrilaterals with factorising cross-ratios and so are isothermic also. We therefore have a triple system of discrete isothermic surfaces!

Exercise Find a spectral deformation for discrete isothermic surfaces.

References

1. A.-V. Bäcklund, *Om ytor med kostant negativ krökning*, Lunds Universitets Årsskrift **XIX** (1883)
2. L. Bianchi, Ricerche sulle superficie elicoidali e sulle superficie a curvatura costante. Ann. Sc. Norm. Super. Pisa Cl. Sci. **2**, 285–341 (1879). MR1556595
3. L. Bianchi, Sulle trasformazione di bäcklund per le superficie pseudosferiche. Rend. Lincei **5**, 3–12 (1892)
4. L. Bianchi, Ricerche sulle superficie isoterme e sulla deformazione delle quadriche. Ann. di Mat. **11**, 93–157 (1905)
5. L. Bianchi, Complementi alle ricerche sulle superficie isoterme. Ann. di Mat. **12**, 19–54 (1905)
6. A. Bobenko, U. Pinkall, Discrete isothermic surfaces. J. Reine Angew. Math. **475**, 187–208 (1996). MR1396732 (97f:53004)
7. A.I. Bobenko, W.K. Schief, Discrete indefinite affine spheres, in *Discrete Integrable Geometry and Physics (Vienna, 1996)* (1999), pp. 113–138. MR1676596 (2001e:53012)
8. E. Bour, Théorie de la déformation des surfaces, J. L'École Impériale Polytechnique, **XXXIX**, 1–148 (1862)
9. F.E. Burstall, Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems, in *Integrable Systems, Geometry, and Topology* (2006), pp. 1–82. MR2222512 (2008b:53006)
10. F.E. Burstall, D.M.J. Calderbank, Conformal submanifold geometry I–III (2010). Preprint, arXiv:1006.5700 [math.DG]
11. F.E. Burstall, D.M.J. Calderbank, Conformal submanifold geometry iv–v (in preparation)
12. F. Burstall, U. Hertrich-Jeromin, Harmonic maps in unfashionable geometries. Manuscripta Math. **108** (2), 171–189 (2002). MR1918585 (2003f:53114)
13. F.E. Burstall, Á.C. Quintino, Dressing transformations of constrained Willmore surfaces. Commun. Anal. Geom. **22** (3), 469–518 (2014). MR3228303
14. F. Burstall, F. Pedit, U. Pinkall, Schwarzian derivatives and flows of surfaces, in *Differential Geometry and Integrable Systems (Tokyo, 2000)* (2002), pp. 39–61. MR1955628 (2004f:53010)
15. F.E. Burstall, N.M. Donaldson, F. Pedit, U. Pinkall, Isothermic submanifolds of symmetric R-spaces. J. Reine Angew. Math. **660**, 191–243 (2011). MR2855825
16. P. Calapso, Sulle superficie a linee di curvatura isoterme. Rendiconti Circolo Matematico di Palermo **17**, 275–286 (1903)
17. E. Christoffel, Ueber einige allgemeine Eigenshaften der Minimumsflächen. Crelle's J. **67**, 218–228 (1867)
18. D.J. Clarke, Integrability in submanifold geometry, Ph.D. Thesis, 2012
19. L. Crane, Action of the loop group on the self-dual Yang-Mills equation. Commun. Math. Phys. **110** (3), 391–414 (1987). MR891944 (88i:58167)
20. G. Darboux, Leçcons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. Parts 1 and 2, Gauthier-Villars, Paris, 1887

21. G. Darboux, Sur les surfaces isothermiques. C.R. Acad. Sci. Paris **128**, 1299–1305, 1538 (1899)
22. A. Demoulin, Sur les systèmes et les congruences K. C. R. Acad. Sci. Paris Sér. I Math. **150**, 150, 156–159, 310–312 (1910)
23. E.V. Ferapontov, W.K. Schief, Surfaces of Demoulin: differential geometry, Bäcklund transformation and integrability. J. Geom. Phys. **30**(4), 343–363 (1999). MR1700564 (2001i:53018)
24. S. Lie, Ueber flächen, deren krümmungsradien durch eine relation verknüpft sind. Archiv for Mathematik og Naturvidenskab **IV**, 507–512 (1879)
25. M. Pember, Special surface classes, Ph.D. Thesis, 2015
26. W.K. Schief, Isothermic surfaces in spaces of arbitrary dimension: integrability, discretization, and Bäcklund transformations—a discrete Calapso equation. Stud. Appl. Math. **106**(1), 85–137 (2001). MR1805487 (2002k:37140)
27. A. Sym, Soliton surfaces and their applications (soliton geometry from spectral problems), in *Geometric Aspects of the Einstein Equations and Integrable Systems (Scheveningen, 1984)* (1985), pp. 154–231. MR828048 (87g:58056)
28. C.-L. Terng, Geometric transformations and soliton equations, in *Handbook of Geometric Analysis*, vol. 2 (Int. Press, Somerville, 2010), pp. 301–358. MR2743444
29. C.-L. Terng, K. Uhlenbeck, Bäcklund transformations and loop group actions. Commun. Pure Appl. Math. **53**(1), 1–75 (2000). MR1715533 (2000k:37116)
30. K. Uhlenbeck, Harmonic maps into Lie groups: classical solutions of the chiral model. J. Differ. Geom. **30**(1), 1–50 (1989). MR1001271 (90g:58028)
31. R.S. Ward, Integrable and solvable systems, and relations among them. Philos. Trans. R. Soc. London Ser. A **315**(1533), 451–457 (1985). With discussion, New developments in the theory and application of solitons. MR836745 (87e:58105)

Completeness of Projective Special Kähler and Quaternionic Kähler Manifolds

Vicente Cortés, Malte Dyckmanns, and Stefan Suhr

Dedicated to Simon Salamon on the occasion of his 60th birthday

Abstract We prove that every projective special Kähler manifold with *regular boundary behaviour* is complete and defines a family of complete quaternionic Kähler manifolds depending on a parameter $c \geq 0$. We also show that, irrespective of its boundary behaviour, every complete projective special Kähler manifold with *cubic prepotential* gives rise to such a family. Examples include non-trivial deformations of non-compact symmetric quaternionic Kähler manifolds.

Keywords C-map • Completeness • Ferrara-Sabharwal metric • One-loop deformation • Quaternionic Kähler manifolds • Special Kähler manifolds

MSC Classification: 53C26

1 Introduction

Quaternionic Kähler manifolds constitute a much studied class of Einstein manifolds of special holonomy [5]. All known complete examples of positive scalar curvature are symmetric of compact type (Wolf spaces) and it has been conjectured that there are no more complete quaternionic Kähler manifolds of positive scalar curvature [16]. Besides the noncompact duals of the Wolf spaces, there exist also

V. Cortés (✉) • M. Dyckmanns
Department of Mathematics and Center for Mathematical Physics, University of Hamburg, Bundesstraße 55, 20146 Hamburg, Germany
e-mail: vicente.cortes@uni-hamburg.de

S. Suhr
Départment de mathématiques et applications, École normale supérieure, 45 rue d'Ulm, 75005 Paris, France
e-mail: stefan.suhr@ens.fr

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_4

nonsymmetric complete examples of negative scalar curvature including locally symmetric spaces, nonsymmetric homogeneous spaces (Alekseevsky spaces) and deformations of quaternionic hyperbolic space [15]. Our work is motivated by the desire to obtain further complete examples of quaternionic Kähler manifolds using ideas from supergravity and string theory.

Based on general supersymmetry arguments [4] and dimensional reduction in field theory it has been known for a long time in the physics community that projective special Kähler manifolds (see Definition 3) are related to quaternionic Kähler manifolds of negative scalar curvature. This correspondence, known as the supergravity c-map, was established by Ferrara and Sabharwal [12] who explicitly associated a quaternionic Kähler metric with every projective special Kähler domain (see Definition 5), cf. [14] for another proof. It was shown in [7] that the supergravity c-map maps every complete projective special Kähler manifold to a complete quaternionic Kähler manifold.

Motivated by the fact that in the low energy limit string theory is described by supergravity, Robles Llana, Saueressig and Vandoren [17] proposed a deformation of the Ferrara-Sabharwal metric (or supergravity c-map metric) depending on a real parameter. This deformation, know as the one-loop deformation, is interpreted as the full perturbative quantum correction (with no higher loop corrections) of supergravity when embedded into string theory. It was proven in [3] using an indefinite version of the HK/QK correspondence [2] that the one-loop deformation of the Ferrara-Sabharwal metric is indeed quaternionic Kähler on its domain of positivity. As a corollary, one obtains a new proof of the quaternionic Kähler property for the (undeformed) Ferrara-Sabharwal metric. It was also found that the completeness of the metric depends on the sign of the deformation parameter. In particular, it was shown that the one-loop deformation of the complex hyperbolic plane is complete for positive deformation parameter and incomplete for negative deformation parameter.

The purpose of this paper is to give general completeness results for projective special Kähler manifolds and one-loop deformations of Ferrara-Sabharwal metrics. These results make it possible to construct many new explicit complete quaternionic Kähler manifolds of negative scalar curvature by the supergravity c-map and its one-loop quantum correction.

After reviewing some basic definitions and facts concerning special Kähler manifolds in the first section, we introduce the notion of regular boundary behaviour for special Kähler manifolds in the second section. The main result of that section is that every projective special Kähler manifold with regular boundary behaviour is complete, see Theorem 7 and its Corollary 8 for projective special Kähler domains.

In the third section we study the one-loop deformation of Ferrara-Sabharwal metrics for nonnegative deformation parameter. We show that the one-loop deformation is not only defined in the case of projective special Kähler domains but is a globally defined one-parameter family of quaternionic Kähler metrics for every projective special Kähler manifold, see Theorem 12. Moreover, we show that the resulting quaternionic Kähler manifolds carry a globally defined integrable complex structure subordinate to the quaternionic structure.

In the fourth section we prove the completeness of the one-loop deformation for nonnegative deformation parameter under the assumption that the initial projective special Kähler manifold has either regular boundary behaviour (see Theorem 13) or is complete with cubic prepotential (see Theorem 27). The latter projective special Kähler manifolds are precisely those which can be obtained by dimensional reduction from five-dimensional supergravity [11] with complete scalar geometry [7]. The corresponding construction is known as the supergravity r-map, which maps projective special real manifolds to projective special Kähler domains.

As the simplest[1] application of Theorem 27 (see Example 28) we discuss a one-parameter deformation of the metric of the noncompact symmetric space $G_2^*/SO(4)$ by locally inhomogeneous complete quaternionic Kähler metrics, where G_2^* denotes the noncompact real form of the complex Lie group of type G_2. In fact, Theorem 27 implies the completeness of the one-loop deformation for all the symmetric quaternionic Kähler manifolds of noncompact type with exception of the quaternionic hyperbolic spaces (which are not in the image of the supergravity c-map) and the spaces $\tilde{X}(n+1) = \frac{SU(n+1,2)}{S[U(n+1)\times U(2)]}$.

Similarly, applying Theorem 13 to the complex hyperbolic space (which is a projective special Kähler domain with regular boundary behaviour) we obtain the completeness of the one-parameter deformation of the remaining symmetric spaces $\tilde{X}(n+1)$, see Example 14.

Based on the effective necessary and sufficient completeness criterion for projective special real manifolds provided in [10, Thm. 2.6], it is easy to construct many more examples of complete projective special Kähler domains with cubic prepotential (see for example [9] and work in progress by Jüngling, Lindemann and the first two authors) and corresponding one-loop deformed quaternionic Kähler manifolds by Theorem 27.

2 Preliminaries

2.1 Conical and Projective Special Kähler Manifolds

First we recall some basic facts and definitions of special Kähler geometry [1, 6].

Definition 1 A conical affine special Kähler manifold (M, J, g, ∇, ξ) is a pseudo-Kähler manifold (M, J, g) endowed with a flat torsion-free connection ∇ and a vector field ξ such that

1. $\nabla\omega = 0$, where $\omega = g(J., .)$ is the Kähler form,
2. $d^\nabla J = 0$, where J is considered as a 1-form with values in TM,
3. $\nabla\xi = D\xi = \mathrm{Id}$, where D is the Levi-Civita connection and
4. g is positive definite on the distribution $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ and negative definite on $\mathcal{D}^\perp$.

[1] The corresponding projective special real manifold is a point.

Note that the affine special Kähler metric g has the global Kähler potential $f = g(\xi, \xi)$ in the sense that

$$\frac{i}{2}\partial\bar{\partial} f = \omega.$$

Furthermore the vector fields ξ and $J\xi$ generate a holomorphic homothetic action of a 2-dimensional Abelian[2] Lie algebra and $J\xi$ is a Killing vector field.

Proposition 2 *Let (M, J, g, ∇, ξ) be a conical affine special Kähler manifold such that the vector fields ξ and $J\xi$ generate a principal $\mathbb{C}^*$-action. Then the degenerate symmetric tensor field*

$$g' := -\frac{g}{f} + \frac{\alpha^2 + (J^*\alpha)^2}{f^2}, \tag{1}$$

where $\alpha := g(\xi, \cdot) = \frac{1}{2}df$, induces a Kähler metric $\bar{g}$ on the quotient (complex) manifold $\bar{M}$.

Proof It suffices to check that the kernel of g' is exactly $\mathcal{D}$, the distribution tangent to the $\mathbb{C}^*$-orbits, and that g' is invariant under the $\mathbb{C}^*$-action. □

Definition 3 A projective special Kähler manifold $(\bar{M}, \bar{g})$ is a quotient as in the previous proposition with canonical projection $\pi: M \to \bar{M}$.

Notice that the projective special Kähler metric is related to the tensor field (1) by $g' = \pi^*\bar{g}$.

2.2 Conical and Projective Special Kähler Domains

In this section we describe an important class of special Kähler manifolds, the so-called special Kähler domains. It is known that every special Kähler manifold is locally isomorphic to a special Kähler domain [1].

Let $F : M \to \mathbb{C}$ be a holomorphic function on a $\mathbb{C}^*$-invariant domain $M \subset \mathbb{C}^{n+1} \setminus \{0\}$ such that

1. F is homogeneous of degree 2, that is $F(az) = a^2F(z)$ for all $z \in M$, $a \in \mathbb{C}^*$,
2. the real matrix $(N_{IJ}(z))_{I,J=0,\dots,n}$, defined by

$$N_{IJ}(z) := 2\mathrm{Im}\, F_{IJ}(z) = -i(F_{IJ}(z) - \overline{F_{IJ}(z)}),$$

   is of signature $(1, n)$ for all $z \in M$, where $F_I := \frac{\partial F}{\partial z^I}$, $F_{IJ} := \frac{\partial^2 F}{\partial z^I \partial z^J}$ etc.,
3. $f(z) := \sum N_{IJ}(z) z^I \bar{z}^J > 0$ for all $z \in M$.

[2] Note that a (real) holomorphic vector field X always commutes with JX: $\mathcal{L}_X(JX) = (\mathcal{L}_X J)X = 0$.

Definition 4 A conical special Kähler domain (M, g, F) is a $\mathbb{C}^*$-invariant domain $M \subset \mathbb{C}^{n+1} \setminus \{0\}$ endowed with a holomorphic function F (called holomorphic prepotential) as above and with the pseudo-Riemannian metric

$$g = \sum N_{IJ} dz^I d\bar{z}^J.$$

Notice that g has signature $(2, 2n)$ and is pseudo-Kähler with the Kähler potential f. A conical special Kähler domain becomes a conical special Kähler manifold if we endow it with the complex structure J and the position vector field ξ induced from the ambient space $\mathbb{C}^{n+1}$. The flat connection ∇ is induced by the standard flat connection on $\mathbb{R}^{2n+2}$ via the immersion $M \ni (z^0, \dots, z^n) \mapsto \mathrm{Re}(z^0, \dots, z^n, F_0, \dots, F_n)$.

Next we consider the domain $\bar{M} = \pi(M) \subset \mathbb{C}P^n$ which is the image of M under the projection

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n.$$

The quotient manifold $\bar{M}$ inherits a (positive definite) Kähler metric $\bar{g}$ uniquely determined by

$$\pi^* \bar{g} = -\frac{g}{f} + \frac{\alpha^2 + (J^*\alpha)^2}{f^2}, \tag{2}$$

where $\alpha := g(\xi, \cdot) = \frac{1}{2} df$.

Definition 5 A projective special Kähler domain $(\bar{M}, \bar{g})$ is the quotient $\bar{M}$ of a conical special Kähler domain M by the natural $\mathbb{C}^*$-action, endowed with its canonical Kähler metric $\bar{g}$.

Now we describe a local Kähler potential for the projective special Kähler metric $\bar{g}$ in a neighborhood of a point $p \in \bar{M}$. This yields a local Kähler potential $\mathcal{K}$ for projective special Kähler manifolds. Let λ be any linear function on $\mathbb{C}^{n+1}$ such that p lies in the affine chart $\{\lambda \neq 0\} \subset \mathbb{C}P^n$. The function $\frac{f}{\lambda\bar{\lambda}}$ is homogeneous of degree 0 on $M \cap \{\lambda \neq 0\}$ and therefore well defined on $\pi(M \cap \{\lambda \neq 0\}) = \bar{M} \cap \{\lambda \neq 0\}$. Then

$$\mathcal{K} := -\log\left(\frac{f}{\lambda\bar{\lambda}}\right)$$

is a Kähler potential for the metric $\bar{g}$ on the open subset $\bar{M} \cap \{\lambda \neq 0\}$. By an appropriate choice of linear coordinates $(z^0, \dots, z^n)$ on $\mathbb{C}^{n+1}$ we can assume that $\lambda = z^0$.

3 Special Kähler Manifolds with Regular Boundary Behaviour

Now we consider certain compactifications of projective special Kähler manifolds by adding a boundary. As a first step we consider conical affine special Kähler manifolds with boundary.

Definition 6 A conical affine special Kähler manifold with regular boundary behaviour is a conical affine special Kähler manifold (M, J, g, ∇, ξ) which admits an embedding $i: M \to \mathcal{M}$ into a manifold with boundary $\mathcal{M}$ such that $i(M) = \operatorname{int}\mathcal{M} := \mathcal{M} \setminus \partial\mathcal{M}$ and the tensor fields (J, g, ξ) smoothly extend to $\mathcal{M}$ such that, for all boundary points $p \in \partial\mathcal{M}$, $f(p) = 0$, $df_p \neq 0$ and g_p is negative semi-definite on $\mathcal{H}_p := T_p\partial\mathcal{M} \cap J(T_p\partial\mathcal{M})$ with kernel $\operatorname{span}\{\xi_p, J\xi_p\}$, where $f = g(\xi, \xi)$.

Note that for the smooth extendability of the metric g it is sufficient to assume that J and f smoothly extend to the boundary. Indeed this follows from the fact that f is a Kähler potential for g.

As in the case of empty boundary, we will assume that ξ and $J\xi$ generate a principal $\mathbb{C}^*$-action on the manifold $\mathcal{M}$. Then $\bar{\mathcal{M}} = \mathcal{M}/\mathbb{C}^*$ is a manifold with boundary and its interior $\bar{M} = M/\mathbb{C}^*$ is a projective special Kähler manifold with projective special Kähler metric $\bar{g}$. If the manifold $\bar{\mathcal{M}}$ with boundary is compact, then we will call $(\bar{M}, \bar{g})$ a projective special Kähler manifold with regular boundary behaviour.

The projective special Kähler domains considered in Remark 1 below, are examples of projective special Kähler manifolds with regular boundary behaviour.

Theorem 7 *Every projective special Kähler manifold with regular boundary behaviour is complete.*

Proof Consider the underlying conical affine special Kähler manifold (M, J, g, ∇, ξ) with regular boundary behaviour. We first show that g_p is nondegenerate for every point $p \in \partial\mathcal{M}$. By definition of regular boundary behavior we have $g|_{\mathcal{H}_p \times \mathcal{H}_p} \leq 0$ with kernel $\operatorname{span}\{\xi_p, J\xi_p\}$. Let $\mathcal{H}'_p \subset \mathcal{H}_p$ be a complex hyperplane not containing ξ_p. Then g_p is negative definite on $\mathcal{H}'_p$. For dimensional reasons $\mathcal{H}_p$ is a real codimension one subspace of $T_p\partial\mathcal{M}$. Let w be a vector in the complement of $\mathcal{H}_p$ in $T_p\partial\mathcal{M}$. By applying the Gram-Schmidt procedure we can assume that w is g_p-orthogonal to $\mathcal{H}'_p$ in $T_p\partial\mathcal{M}$. Then $\operatorname{span}\{w, Jw\}$ is g_p-orthogonal to $\mathcal{H}'_p$ by the J-invariance of g_p. Since the real 4-dimensional vector space $\operatorname{span}\{\xi_p, J\xi_p, w, Jw\}$ is g_p-orthogonal to $\mathcal{H}'_p$ in $T_p\mathcal{M}$ it suffices to show that g_p is nondegenerate on $\operatorname{span}\{\xi_p, J\xi_p, w, Jw\}$. By continuity of df and ξ we know that

$$2g_p(\xi_p, .) = df_p.$$

Since $Jw \notin T_p\partial\mathcal{M}$ and $w \in T_p\partial\mathcal{M}$ we have

$$0 \neq df_p(Jw) = 2g_p(\xi_p, Jw) = -2g_p(J\xi_p, w) \text{ and } 0 = df_p(w) = 2g_p(\xi_p, w) = 2g_p(J\xi_p, Jw).$$

Now by considering the representing matrix of g_p on $\mathrm{span}\{\xi_p, J\xi_p, w, Jw\}$ and using that g_p vanishes on $\mathrm{span}\{\xi_p, J\xi_p\}$ we see that g_p is nondegenerate. This proves that g_p is nondegenerate and, therefore, of signature $(2, 2n)$ by continuity.

Let $\gamma : I \to \bar{M}$, $I = [0, b)$, $0 < b \leq \infty$, be a curve which is not contained in any compact subset of $\bar{M}$. We will show that γ has infinite length under the assumption of regular boundary behaviour. Call a point $p \in \bar{\mathcal{M}}$ an accumulation point of γ if there exists a sequence $t_i \in I$ such that $\lim t_i = b$ and $\lim \gamma(t_i) = p$. By our assumption, γ has at least one accumulation point $\bar{p}_0$ on the boundary. We distinguish two cases:

1st case: γ has exactly one accumulation point $\bar{p}_0$ which necessarily lies on the boundary. Under this hypothesis, for every neighborhood of $\bar{p}_0$ we can find $a \in I$ such that $\gamma([a, b))$ is fully contained in that neighborhood.

Choose a point $p_0 \in \pi^{-1}(\bar{p}_0) \subseteq \partial\mathcal{M}$. Since the signature of g_{p_0} is $(2, 2n)$, there exists a complex hyperplane $E \subset T_{p_0}\mathcal{M}$ on which $-g$ is positive definite. Let M' denote a complex hypersurface through p_0 tangent to E such that $-g|_{TM' \times TM'}$ is positive definite.

The pullback of the projective special Kähler metric can be estimated on $N = \mathrm{int}(M')$ as follows

$$(\pi^*\bar{g})|_N = -\left.\frac{g}{f}\right|_N + \left.\frac{\alpha^2 + (J^*\alpha)^2}{f^2}\right|_N \geq \left.\frac{\alpha^2}{f^2}\right|_N = \frac{df^2}{4f^2}. \tag{3}$$

Now we show how this implies that γ has infinite length. We can assume by shifting the initial point of the interval I that γ is fully contained in $\pi(N) \subset \bar{M}$. Let $\gamma_N : I \to N$ be the curve which projects to γ under $\pi|_N$. Then there exists a sequence $t_i \in [0, b)$ such that $f(\gamma_N(t_i)) \to 0$ and $\gamma_N([0, t_i]) \subset \gamma_N(I) \subset N$. In view of (3), we have

$$\begin{aligned} L(\gamma) \geq L\big(\gamma|_{[0,t_i]}\big) = L^{\pi^*\bar{g}}(\gamma_N|_{[0,t_i]}) &\geq \frac{1}{2}\int_0^{t_i} \left|\frac{d}{dt}\log f \circ \gamma_N\right| dt \\ &\geq -\frac{1}{2}\int_0^{t_i} \frac{d}{dt}\log f \circ \gamma_N \, dt = \frac{1}{2}\big(\log f(\gamma_N(0)) - \log f(\gamma_N(t_i))\big) \to \infty. \end{aligned}$$

This shows that γ has infinite length.

2nd case: γ has at least two accumulation points. Let $\bar{p}_0 \neq \bar{p}_1$ be such accumulation points. We know that at least one accumulation point, e.g. $\bar{p}_0$, lies in the boundary. Under the assumption that there exists a second accumulation point, we now show that the second accumulation point can be taken arbitrarily near to $\bar{p}_0$. In other words, we claim that for every given neighborhood U of $\bar{p}_0$ there exists an accumulation point $\bar{p}_2 \in U \setminus \{\bar{p}_0\}$. Indeed let us denote by $B_r^{aux}(\bar{p}_0)$ the ball of radius $r > 0$ centered at $\bar{p}_0$ with respect to an auxiliary Riemannian metric on $\bar{\mathcal{M}}$. Choose $r > 0$ such that $\overline{B_r^{aux}(\bar{p}_0)} \subset U$. If $\bar{p}_1 \in U$ there is nothing to prove. If $\bar{p}_1 \notin U$ choose sequences $s_i < t_i < s_{i+1}$ such that $\lim_{i\to\infty} \gamma(s_i) = \bar{p}_0$ and $\lim_{i\to\infty} \gamma(t_i) = \bar{p}_1$. We can assume that $\gamma(s_i) \in B_{r/2}^{aux}(\bar{p}_0)$ and $\gamma(t_i) \notin \overline{B_r^{aux}(\bar{p}_0)}$ for all i. Then there exists a sequence $u_i \in (s_i, t_i)$ with $\gamma(u_i) \in B_r^{aux}(\bar{p}_0) \setminus B_{r/2}^{aux}(\bar{p}_0)$. The sequence $\gamma(u_i)$

has an accumulation point $\bar{p}_2 \in \overline{B_r^{aux}(\bar{p}_0)} \subset U$. We will continue to denote this accumulation point arbitrarily close to $\overline{p}_0$ by $\bar{p}_1$.

If $\overline{p}_1 \in \overline{M}$ it is easy to see that γ has infinite length. In fact consider a geodesically convex ball $B_\delta(\bar{p}_1)$ of radius $\delta > 0$ centered at $\bar{p}_1$ with respect to $\bar{g}$. We take δ sufficiently small such that $B_\delta(\bar{p}_1)$ is relatively compact in $\bar{M}$. Since the curve γ intersects the ball $B_{\delta/2}(\bar{p}_1)$ an arbitrarily large number of times k, the length of γ is larger or equal than $k\delta \to \infty$.

Thus we can assume that $\bar{p}_1$ lies in the boundary as well. By restricting U we can assume that U is in the image of a complex hypersurface $M' \subset \overline{\mathcal{M}}$ as above. We can further assume that $f \leq \epsilon$ on M'. Since $g' = \pi^*\bar{g}$ is given by (1) the Riemannian metric $\pi^*\bar{g}|_N$ on $N = \mathrm{int}(M')$ is bounded from below by the Riemannian metric

$$-\left.\frac{g}{f}\right|_N \geq -\frac{1}{\epsilon} g|_N. \tag{4}$$

Let us denote by $B'_r(p)$ the ball centered at $p \in M'$ of radius $r > 0$ with respect to the Riemannian metric $-g|_{M'}$ on M'. We choose $\delta > 0$ such that $B'_\delta(p_0)$ is relatively compact in M'. Then every curve in $B'_\delta(p_0)$ from $B'_{\delta/2}(p_0) \subset M'$ ($p_0 := (\pi|_{M'})^{-1}(\bar{p}_0)$) which leaves $B'_\delta(p_0)$ has length with respect to $-g|_{M'}$ bounded from below by some positive constant c (in fact $c = \delta/2$). Since we can assume that $p_1 := (\pi|_{M'})^{-1}(\bar{p}_1)$ is arbitrarily close to p_0 we can assume that $p_1 \in B'_\delta(p_0)$ and there exist disjoint balls $B'_{\delta'}(p_0), B'_{\delta'}(p_1) \subset B'_\delta(p_0)$ which have distance with respect to $-g|_{M'}$ bounded from below by some positive constant. By reducing the above constant c, if necessary, we can assume that this constant is again c. Then we can conclude that every curve which connects a point in $B'_{\delta'}(p_0)$ with a point in $B'_{\delta'}(p_1)$ has length with respect to $-g|_{M'}$ bounded from below by c. Since p_0 and p_1 are accumulation points of γ either γ leaves the set $\pi(N)$ infinitely often, in which case γ has infinite length, or γ stays eventually inside $\pi(N)$, in which case it can be eventually identified with a curve γ_N in N by the projection $\pi|_N$. Since $\bar{p}_0$ and $\bar{p}_1$ are accumulation points of γ_N there exists an infinite number of arcs of γ_N in N connecting $B'_{\delta'}(p_0)$ with $B'_{\delta'}(p_1)$. Again the length is infinite. In both cases we used the estimate (4) together with the lower bound c on the length of arcs with respect to $-g|_N$. $\square$

Remark 1 In the case of conical affine special Kähler domains the description of regular boundary behaviour simplifies as follows. Let $(\bar{M}, \bar{g})$ be a projective special Kähler domain with underlying conical special Kähler domain (M, g, F). Suppose that the affine Kähler potential f extends to a smooth function (denoted again by f) on some neighborhood of $\mathrm{cl}(M) \setminus \{0\}$, where $\mathrm{cl}(M)$ denotes the closure of M, such that $f(p) = 0$, $df_p \neq 0$, and that g_p is negative semi-definite on $T_p\partial M \cap J(T_p\partial M)$ with kernel $\mathbb{C}\xi_p = \mathbb{C}p$ for all boundary points $p \in \partial M \setminus \{0\}$. Then (M, g, F) is an example of a conical affine special Kähler manifold with regular boundary behaviour and $(\bar{M}, \bar{g})$ an example of a projective special Kähler manifold with regular boundary behaviour.

The following result is an immediate consequence of Theorem 7.

Corollary 8 *Under the above assumptions on the boundary behaviour of the affine Kähler potential f in Remark 1, the Riemannian manifold $(\bar{M}, \bar{g})$ is complete.*

4 One-Loop Deformed Ferrara-Sabharwal Metric

In this section we will recall the definition of the one-loop (quantum) deformation of the Ferrara-Sabharwal metric which is a one-parameter family of quaternionic Kähler metrics associated with a projective special Kähler domain [3, 17]. The fact that the metric is quaternionic Kähler was proven in [3] with the help of an indefinite version of Haydys' HK/QK correspondence [13] developed in [2]. This implies that the reduced scalar curvature $\nu = \frac{\text{scal}}{4m(m+2)}$ is negative and more precisely given by $\nu = -2$ with the present normalizations. Here m is the quaternionic dimension of the quaternionic Kähler manifold. In the special case of the (undeformed) Ferrara-Sabharwal metric the quaternionic Kähler property was obtained by different methods in [12, 14].

Every projective special Kähler manifold admits a covering by projective special Kähler domains and we will show that the one-loop deformed Ferrara-Sabharwal metrics associated with the domains can be consistently glued to a globally defined (quaternionic Kähler; to be shown) metric. This generalizes the result that the Ferrara-Sabharwal metric, which was originally defined for special Kähler domains [12], is globally defined for every projective special Kähler manifold [7]. We will also show that the above quantum deformed quaternionic Kähler manifolds admit a globally defined integrable complex structure J_1 subordinate to the quaternionic structure, generalizing results of [8] for the Ferrara-Sabharwal metric.

4.1 The Supergravity C-Map

Let $(\bar{M}, \bar{g})$ be a projective special Kähler domain of complex dimension n. The **supergravity c-map** [12] associates with $(\bar{M}, \bar{g})$ a quaternionic Kähler manifold $(\bar{N}, g_{\bar{N}})$ of dimension $4n+4$. Following the conventions of [7], we have $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ and

$$\begin{aligned}
g_{\bar{N}} &= \bar{g} + g_G, \\
g_G &= \frac{1}{4\rho^2} d\rho^2 + \frac{1}{4\rho^2}\left(d\tilde{\phi} + \sum \left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right)^2 + \frac{1}{2\rho}\sum \mathcal{I}_{IJ}(m) d\zeta^I d\zeta^J \\
&\quad + \frac{1}{2\rho}\sum \mathcal{I}^{IJ}(m)(d\tilde{\zeta}_I + \mathcal{R}_{IK}(m) d\zeta^K)(d\tilde{\zeta}_J + \mathcal{R}_{JL}(m) d\zeta^L),
\end{aligned}$$

where $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I)$, $I = 0, 1, \ldots, n$, are standard coordinates on $\mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$. The real-valued matrices $\mathcal{I}(m) := (\mathcal{I}_{IJ}(m))$ and $\mathcal{R}(m) := (\mathcal{R}_{IJ}(m))$ depend only on

$m \in \bar{M}$ and $\mathcal{I}(m)$ is invertible with the inverse $\mathcal{I}^{-1}(m) =: (\mathcal{I}^{IJ}(m))$. More precisely,

$$\mathcal{N}_{IJ} := \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} := \bar{F}_{IJ} + i\frac{\sum_K N_{IK}z^K \sum_L N_{JL}z^L}{\sum_{IJ} N_{IJ}z^I z^J}, \quad N_{IJ} := 2\mathrm{Im}F_{IJ}, \tag{5}$$

where F is the holomorphic prepotential with respect to some system of special holomorphic coordinates (z^I) on the underlying conical special Kähler domain $M \to \bar{M}$. Notice that the expressions are homogeneous of degree zero and, hence, well-defined functions on $\bar{M}$. It is shown in [7, Cor. 5] that the matrix $\mathcal{I}(m)$ is positive definite and hence invertible and that the metric $g_{\bar{N}}$ does not depend on the choice of special coordinates [7, Thm. 9]. It is also shown that $(\bar{N}, g_{\bar{N}})$ is complete if and only if $(\bar{M}, \bar{g})$ is complete [7, Thm. 5].Using $(p_a)_{a=1,\dots,2n+2} := (\tilde{\zeta}_I, \zeta^J)_{IJ=0,\dots,n}$ and the positive definite matrix [7]

$$(\hat{H}^{ab}) := \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix},$$

we can combine the last two terms of g_G into $\frac{1}{2\rho}\sum dp_a\hat{H}^{ab}dp_b$, i.e. the quaternionic Kähler metric is given by

$$g_{FS} := g_{\bar{N}} = \bar{g} + \frac{1}{4\rho^2}d\rho^2 + \frac{1}{4\rho^2}\left(d\tilde{\phi} + \sum\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right)^2 + \frac{1}{2\rho}\sum dp_a\hat{H}^{ab}dp_b. \tag{6}$$

This metric is known as the **Ferrara-Sabharwal** metric.

4.2 The One-Loop Deformation

Now we consider a family of metrics g^c_{FS} depending on a real parameter c such that $g^0_{FS} = g_{FS}$. To define this family we assume for the moment that $z^0 \neq 0$ on the conical affine special Kähler domain $M \subset \mathbb{C}^{n+1}$. Under this assumption we can consider the projective special Kähler domain as a subset $\bar{M} \subset \mathbb{C}^n \subset \mathbb{C}P^n$.

Definition 9 For any $c \in \mathbb{R}$, the metric

$$\begin{aligned} g^c_{FS} = \frac{\rho+c}{\rho}\bar{g} + \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}d\rho^2 + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}(d\tilde{\phi} + \sum_{I=0}^{n}(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + cd^c\mathcal{K})^2 \\ + \frac{1}{2\rho}\sum_{a,b=1}^{2n+2} dp_a\hat{H}^{ab}dp_b + \frac{2c}{\rho^2}e^{\mathcal{K}}\left|\sum_{I=0}^{n}(X^I d\tilde{\zeta}_I + F_I(X)d\zeta^I)\right|^2 \end{aligned} \tag{7}$$

is defined on the domains

$$\begin{aligned} N'_{(4n+4,\,0)} &:= \{\rho > -2c,\ \rho > 0\} \subset \bar{N}, \\ N'_{(4n,\,4)} &:= \{-c < \rho < -2c\} \subset \bar{N}, \\ N'_{(4,\,4n)} &:= \bar{M} \times \{-c < \rho < 0\} \times \mathbb{R}^{2n+3} \subset \bar{M} \times \mathbb{R}^{<0} \times \mathbb{R}^{2n+3} \end{aligned} \tag{8}$$

for any projective special Kähler domain $\bar{M}$ defined by a holomorphic function F on the underlying conical affine special Kähler domain M, where $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$, $(X^\mu)_{\mu=1,\ldots,n}$ are standard inhomogeneous holomorphic coordinates on $\bar{M} \subset \mathbb{C}^n$, $X^0 := 1$, the real coordinate ρ corresponds to the second factor, $(\tilde{\phi}, \tilde{\zeta}_I, \zeta^I)_{I=0,\ldots,n}$ are standard real coordinates on $\mathbb{R}^{2n+3}$, and $\mathcal{K} := -\log \sum_{I,J=0}^n X^I N_{IJ}(X)\bar{X}^J$ is the Kähler potential for $\bar{g}$. The metric g^c_{FS} is called the **one-loop deformed Ferrara-Sabharwal metric**.

Proposition 10 *Let $\bar{M} \subset \mathbb{C}^n \subset \mathbb{C}P^n$ be a projective special Kähler domain and g^c_{FS}, $g^{c'}_{FS}$ one-loop deformed Ferrara-Sabharwal metrics for positive deformation parameters c, $c' \in \mathbb{R}^{>0}$ defined on $\bar{N} = N'_{(4n+4,\,0)}$. Then $(\bar{N}, g^c_{FS})$ and $(\bar{N}, g^{c'}_{FS})$ are isometric.*

Proof Any $e^\lambda \in \mathbb{R}^{>0}$ acts diffeomorphically on $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ as follows:

$$\bar{N} \to \bar{N}, \quad (m, \rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I)_{I=0,\ldots,n} \mapsto (m, e^\lambda \rho, e^\lambda \tilde{\phi}, e^{\lambda/2}\tilde{\zeta}_I, e^{\lambda/2}\zeta^I)_{I=0,\ldots,n}.$$

Under this action, $g^c_{FS} \mapsto g^{e^{-\lambda}c}_{FS}$. Choosing $e^\lambda = c/c'$, this shows that $(\bar{N}, g^c_{FS})$ and $(\bar{N}, g^{c'}_{FS})$ are isometric. □

4.3 Globalization of the One-Loop Deformed Metric

Let $(\bar{M}, \bar{g})$ be a projective special Kähler manifold with underlying conical affine special Kähler manifold (M, J, g, ∇, ξ). Consider a covering of $\bar{M}$ by open subsets $\bar{M}_\alpha$ isomorphic to projective special Kähler domains. Over the preimage $M_\alpha := \pi^{-1}(\bar{M}_\alpha)$ we have a system of so-called conical affine special coordinates $(z^I)_{0\leq I\leq n}$ which correspond to the natural coordinates in the underlying conical affine special Kähler domain equipped with the holomorphic prepotential F. Notice that the map $\phi_\alpha\colon M_\alpha \to \mathbb{C}^{2n+2}$, $p \mapsto (z^I, F_I)|_p$, where F_I denotes the I-th partial derivative at the point $z = (z^0, \ldots, z^n)$, is a conical nondegenerate Lagrangian immersion in the sense of [6]. Further note that the coordinates as well as the prepotential depend on α. To indicate this dependence we will write z^I_α, F_α etc. Since any pair of conical nondegenerate Lagrangian immersions is related by a real linear symplectic transformation [1, 6] there exists an element

$$\mathcal{O} = \mathcal{O}_{\beta,\alpha} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(\mathbb{R}^{2n+2})$$

such that $\phi_\beta = \mathcal{O} \circ \phi_\alpha$ on $M_\alpha \cap M_\beta$.

Define $\bar{N}_\alpha := \bar{M}_\alpha \times \mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$ and $N_\alpha := M_\alpha \times \mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$, where $\mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ is endowed with the standard coordinate system $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^J) = (\rho_\alpha, \tilde{\phi}_\alpha, \tilde{\zeta}_{I,\alpha}, \zeta^J_\alpha) =: (\rho_\alpha, \tilde{\phi}_\alpha, v_\alpha)$ and $S^1_c := \mathbb{R}/2\pi c\mathbb{Z}$. Notice that S^1_c can be canonically identified with $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ by $[x] \mapsto [cx]$ if $c \neq 0$ and that $S^1_0 = \mathbb{R}$.

Next we define an equivalence relation on the disjoint union of the $\bar{N}_\alpha$ (and similarly on the disjoint union of the N_α)

$$(m_\alpha, \rho_\alpha, \tilde{\phi}_\alpha, v_\alpha) \sim (m_\beta, \rho_\beta, \tilde{\phi}_\beta, v_\beta)$$

$$:\Leftrightarrow m_\alpha = m_\beta,\ \rho_\alpha = \rho_\beta,\ \tilde{\phi}_\beta = \tilde{\phi}_\alpha - ic \log\left(\frac{z^0_\alpha \bar{z}^0_\beta}{z^0_\beta \bar{z}^0_\alpha}\right),\ v_\beta = (\mathcal{O}^t_{\beta,\alpha})^{-1} v_\alpha.$$

Proposition 11 *The quotient $\bar{N} := \cup_\alpha \bar{N}_\alpha / \sim$ is a smooth manifold of real dimension $4n+4$ fibering over the projective special Kähler manifold $\bar{M}$ as a bundle of flat symplectic manifolds modeled on the quotient of a symplectic vector space $\mathbb{R}^{2n+2}$ by a cyclic group of translations (the cyclic group is trivial for $c = 0$). By π we denote the induced natural projection $\bar{N} \to \bar{M}$. Similarly, the quotient $N := \cup_\alpha N_\alpha / \sim$ is a bundle over the conical affine special Kähler manifold M with flat symplectic fibers.*

Proof It is clear that $\bar{N}$ is a fibre bundle with standard fibre $\mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$. By taking the logarithm of ρ one can identify the standard fibre with the quotient $\mathbb{R} \times S^1_c \times \mathbb{R}^{2n+2}$ of $\mathbb{R}^{2n+4}$ by the group of translations $2\pi c\mathbb{Z}$ acting on the second coordinate. Since the transition functions take values in the group of affine symplectic transformations of $\mathbb{R} \times S^1_c \times \mathbb{R}^{2n+2}$, the fibers of the resulting bundle naturally carry a flat symplectic structure. In fact, the linear part of the transition functions takes values in the subgroup $\{\mathrm{Id}_{\mathbb{R}^2}\} \times \mathrm{Sp}(\mathbb{R}^{2n+2}) \subset \mathrm{Sp}(\mathbb{R}^{2n+4})$. □

To avoid a parameter-dependence of the domain of definition of the metric we will assume from now on for simplicity that the one-loop parameter $c > 0$.

Theorem 12 *The quaternionic Kähler metrics $g^c_{FS,\alpha}$, $c > 0$, given by (7) on each coordinate domain $\bar{N}_\alpha$ of $\bar{N}$ using the coordinates $(X^\mu, \rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^J) = (X^\mu_\alpha, \rho_\alpha, \tilde{\phi}_\alpha, \tilde{\zeta}_{I,\alpha}, \zeta^J_\alpha)$ induce a well-defined quaternionic Kähler metric g^c_{FS} on $\bar{N}$. Furthermore there exists a globally defined integrable complex structure J_1 subordinate to the parallel skew-symmetric quaternionic structure Q of $(\bar{N}, g^c_{FS})$.*

Proof First we show that the quaternionic Kähler metrics defined on the domains $\bar{N}_\alpha$ are consistent. The terms $\frac{\rho+c}{\rho}\bar{g}$ and $\frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}d\rho^2$ in (7) are manifestly coordinate independent, since the transition functions do not act on ρ. The one-form $\eta_{can} := \sum_{I=0}^n (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)$ is obviously invariant under linear symplectic transformations and therefore also coordinate independent. The invariance of the term $\sum_{a,b=1}^{2n+2} dp_a\hat{H}^{ab}dp_b$ was shown in [7, Lemma 4]. Next we show the invariance of $d\tilde{\phi} + cd^c\mathcal{K}$. Since

$$\sum_{I,J} X^I N_{IJ}\bar{X}^J = \frac{f}{z^0\bar{z}^0},$$

where $f = g(\xi, \xi) = \sum_{I,J} z^I N_{IJ}\bar{z}^J$ is coordinate independent (but defined on N, not on $\bar{N}$), we see that

$$cd^c\mathcal{K}_\beta - cd^c\mathcal{K}_\alpha = cd^c \log\left(\frac{z^0_\beta\bar{z}^0_\beta}{z^0_\alpha\bar{z}^0_\alpha}\right) = icd\log\left(\frac{z^0_\alpha\bar{z}^0_\beta}{z^0_\beta\bar{z}^0_\alpha}\right),$$

where we have used that $d^c = -J^*d$ on functions. By the transition rule for $\tilde{\phi}$ we have

$$d\tilde{\phi}_\beta = d\tilde{\phi}_\alpha - icd\log\left(\frac{z^0_\alpha\bar{z}^0_\beta}{z^0_\beta\bar{z}^0_\alpha}\right).$$

This shows the invariance of $d\tilde{\phi} + cd^c\mathcal{K}$.

Finally we show the invariance of $e^{\mathcal{K}}\left|\sum_{I=0}^n (X^I d\tilde{\zeta}_I + F_I(X)d\zeta^I)\right|^2$. By rewriting this as

$$\begin{aligned}\frac{1}{\sum X^I N_{IJ}(X)\bar{X}^J}\left|\sum_{I=0}^n (X^I d\tilde{\zeta}_I + F_I(X)d\zeta^I)\right|^2 &= \tfrac{z^0\bar{z}^0}{f}\left|\textstyle\sum_{I=0}^n \frac{z^I}{z^0}d\tilde{\zeta}_I + F_I(\frac{z}{z^0})d\zeta^I\right|^2\\ &= \tfrac{1}{f}\left|\textstyle\sum_{I=0}^n z^I d\tilde{\zeta}_I + F_I(z)d\zeta^I\right|^2\end{aligned}$$

we see that the term is coordinate independent. In fact, the sum $\sum_{I=0}^n z^I d\tilde{\zeta}_I + F_I(z)d\zeta^I$ is obtained from the natural pairing between $\mathbb{C}^{2n+2}$ and $(\mathbb{C}^{2n+2})^* \supset (\mathbb{R}^{2n+2})^*$ which is, in particular, invariant under linear symplectic transformations. Summarizing we have shown that the metric g^c_{FS} is well defined on $\bar{N}$.

Since g^c_{FS} is quaternionic Kähler (of negative scalar curvature) on each of the domains $\bar{N}_\alpha$ it follows that g^c_{FS} is a quaternionic Kähler metric. In fact, the locally defined parallel skew-symmetric quaternionic structures on the domains $\bar{N}_\alpha$ are uniquely determined by the Lie algebra of the holonomy group of $g^c_{FS}|_{\bar{N}_\alpha}$ and therefore extend to a globally defined quaternionic structure Q. It can be also checked by direct calculations (see below) that the locally defined quaternionic structures Q_α on $\bar{N}_\alpha$ are consistent. In fact, the description of the quaternionic Kähler structure on $\bar{N}_\alpha$ in terms of the HK/QK-correspondence [3] yields an almost hypercomplex structure (J_1, J_2, J_3) on $\bar{N}_\alpha$ which defines the quaternionic structure

Q_α. The structure is defined by the three Kähler forms $\omega_i = g^c_{FS}(J_i\cdot, \cdot)$, $i = 1, 2, 3$. These are given by

$$\omega_i = -d\theta_i + 2\theta_j \wedge \theta_k,$$

where (i, j, k) is a cyclic permutation of $\{1, 2, 3\}$ and the one-forms θ_i on $\bar{N}_\alpha$ are defined by

$$\theta_1 = -\frac{1}{4\rho}\left(d\tilde{\phi} + (\rho + c)d^c\mathcal{K} - \eta_{can}\right)$$

$$\theta_2 + i\theta_3 = i\frac{\sqrt{\rho + c}}{\rho}e^{\mathcal{K}/2}\sum_{I=0}^{n} X^I A_I, \quad A_I := d\tilde{\zeta}_I + \sum_J F_{IJ}d\zeta^J.$$

Next we prove that Q admits a global section J_1 by showing that the Kähler form ω_1 is invariantly defined, i.e. coordinate independent. First we remark that θ_1 can be decomposed as

$$\theta_1 = -\frac{1}{4\rho}\left(d\tilde{\phi} + cd^c\mathcal{K} - \eta_{can}\right) - \frac{1}{4}d^c\mathcal{K},$$

where the first was already shown to be invariant. Using that $\mathcal{K} = -\log\frac{f}{(r^0)^2}$, where $z^0 = r^0 e^{i\varphi^0}$, the second term can be decomposed as

$$-\frac{1}{4}d^c\mathcal{K} = \frac{1}{4}d^c\log f + \frac{1}{2}d^c\log r^0 = \frac{1}{4}d^c\log f - \frac{1}{2}d\varphi^0.$$

Since the first term on the right-hand side is invariant we see that

$$\theta_1 = \theta_1^{inv} - \frac{1}{2}d\varphi^0,$$

where θ_1^{inv} is coordinate independent. This implies that $d\theta_1$ is invariant. Now we observe that

$$\sum z^I A_I$$

is invariant (defined on N). This follows from

$$\sum z^I A_I = \sum z^I d\tilde{\zeta}_I + F_I(z)d\zeta^I,$$

where the right-hand side was already observed to be invariant. As a consequence, the two-form

$$\theta_2 \wedge \theta_3 = -\frac{1}{2i}(\theta_2 + i\theta_3) \wedge (\theta_2 - i\theta_3)$$

is also invariant, since

$$\theta_2 + i\theta_3 = \frac{i}{\rho}\left(\frac{\rho + c}{f}\right)^{\frac{1}{2}} e^{-i\varphi^0} \sum z^I A_I,$$

which implies that $e^{i\varphi^0}(\theta_2 + i\theta_3)$ is a well defined one-form on N.

Combining these results we have shown that $\omega_1 = -d\theta_1 + 2\theta_2 \wedge \theta_3$ is invariant. By similar calculations it is easy to show that a conformal multiple $e^{i\varphi^0}\omega$ of the $(2,0)$-form

$$\omega = \omega_2 + i\omega_3$$

with respect to J_1 is invariantly defined on N (and horizontal with respect to the projection $N \to \bar{N}$ induced by $M \to \bar{M}$). This implies that the complex plane spanned by ω and $\bar{\omega}$ is invariantly defined on $\bar{N}$ and therefore the real plane spanned by ω_2 and ω_3. This reproves the fact that the quaternionic structure is well-defined.

Now we prove the integrability of J_1. It is sufficient to check this on $\bar{N}_\alpha$. In the case $c = 0$ this was previously shown in [8]. With the definition of ω_1 above we compute

$$\begin{aligned}
\omega_1 &= \frac{1}{4\rho}\left(d\rho \wedge d^c\mathcal{K} + (\rho + c)\, dd^c\mathcal{K} - 2\sum_{I=0}^{n} d\tilde{\zeta}_I \wedge d\zeta^I\right) + \frac{1}{\rho} d\rho \wedge \theta_1 \\
&\quad + \frac{\rho + c}{\rho^2} e^{\mathcal{K}} i(\sum_I X^I A_I) \wedge (\sum_J \bar{X}^J \bar{A}_J) \\
&= \frac{\rho + c}{\rho}\frac{1}{4} dd^c\mathcal{K} + \frac{i}{2}\frac{1}{4\rho^2}\frac{\rho + c}{\rho + 2c}\tau \wedge \bar{\tau} - \frac{i}{2}\frac{1}{\rho}\sum_{I,J=0}^{n} N^{IJ} A_I \wedge \bar{A}_J \\
&\quad + \frac{i}{2}\frac{2\rho + 2c}{\rho^2} e^{\mathcal{K}} (\sum_I X^I A_I) \wedge (\sum_J \bar{X}^J \bar{A}_J),
\end{aligned} \tag{9}$$

where

$$\tau := d\tilde{\phi} + \eta_{can} + c d^c\mathcal{K} + i\frac{\rho + 2c}{\rho + c} d\rho$$

and we used that

$$\sum_{I,J=0}^{n} iN^{IJ} A_I \wedge \bar{A}_J = \sum_{I,J,\,K=0}^{n} iN^{IJ}(F_{IK} - \bar{F}_{IK}) d\zeta^K \wedge \tilde{\zeta}_J = \sum_{I=0}^{n} d\tilde{\zeta}_I \wedge d\zeta^I.$$

Together with the expression

$$g^c_{FS} = \frac{\rho+c}{\rho}\bar{g} + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}|\tau|^2 - \frac{1}{\rho}\sum_{I,J=0}^{n} N^{IJ}A_I\bar{A}_J + \frac{2\rho+2c}{\rho^2}e^{\mathcal{K}}\Big|\sum_{I=0}^{n} X^I A_I\Big|^2$$

for the deformed Ferrara-Sabharwal metric, which can be proven using [3, Lemma 3], (9) shows that

$$(\tau,\, dX^\mu,\, A_I)^{\mu=1,\dots,n}_{I=0,\dots,n}$$

is a coframe of holomorphic one-forms with respect to J_1. This can be linearly combined into the coframe

$$\Big(\tau + 2ic\partial\mathcal{K} - 2\sum_{I=0}^{n}\zeta^I A_I - \sum_{I,J,K=0}^{n}\zeta^I F_{IJK}(X)\zeta^J dX^K,$$
$$dX^\mu,\ \frac{1}{2}(A_I - \sum_{J,K=0}^{n} F_{IJK}(X)\zeta^J dX^K)\Big)$$

of closed holomorphic one-forms which corresponds to the J_1-holomorphic coordinate system

$$(\chi,\ X^\mu,\ w_I = \frac{1}{2}(\tilde{\zeta}_I + \sum_{J=0}^{n} F_{IJ}(X)\zeta^J))^{\mu=1,\dots,n}_{I=0,\dots,n},$$

where

$$\chi := \tilde{\phi} + i(\rho + c(\mathcal{K} + \log(\rho+c))) - \sum_{I=0}^{n}\zeta^I\tilde{\zeta}_I - \sum_{I,J=0}^{n}\zeta^I F_{IJ}(X)\zeta^J.$$

This proves the integrability of J_1. □

5 Completeness of the One-Loop Deformation

5.1 *Completeness of the One-Loop Deformation for Projective Special Kähler Manifolds with Regular Boundary Behaviour*

In this and the next section, we prove under two different types of natural assumptions the completeness of the one-loop deformed Ferrara-Sabharwal metric g^c_{FS} (see Definition 9 and Theorem 12) on $\bar{N}$ for $c \geq 0$. For $c < 0$ and the case of projective special Kähler domains, $(N'_{(4n+4,\,0)}, g^c_{FS})$ is known to be incomplete [3, Rem. 9].

Theorem 13 *Let $(\bar{M}, \bar{g})$ be a projective special Kähler manifold with regular boundary behaviour and $(\bar{N}, g^c_{FS})$ the one-loop deformed Ferrara-Sabharwal (quaternionic Kähler) manifold associated to $(\bar{M}, \bar{g})$. Then $(\bar{N}, g^c_{FS})$ is complete for all $c \geq 0$.*

Example 14 The projective special Kähler manifold $\mathbb{C}H^n$ with quadratic holomorphic prepotential $F = \frac{i}{2}((z^0)^2 - \sum_{\mu=1}^n (z^\mu)^2)$ on the conical affine special Kähler domain $M := \{|z^0|^2 > \sum_{\mu=1}^n |z^\mu|^2\}$ has regular boundary behaviour in the sense of Definition 6. Thus Corollary 8 implies the completeness of the projective special Kähler domain $\mathbb{C}H^n$.

We know that $(\bar{N}, g_{FS})$ is isometric to the series of Wolf spaces

$$\tilde{X}(n+1) = \frac{SU(n+1,\,2)}{S[U(n+1) \times U(2)]} \tag{10}$$

of non-compact type, see e.g. [11].

Corollary 15 *For any $n \in \mathbb{N}_0$ and $c \in \mathbb{R}^{\geq 0}$, the deformed Ferrara-Sabharwal metric*

$$\begin{aligned}
g^c_{FS} = {} & \frac{\rho + c}{\rho} \frac{1}{1 - \|X\|^2} \Big(\sum_{\mu=1}^n dX^\mu d\bar{X}^\mu + \frac{1}{1 - \|X\|^2} \Big| \sum_{\mu=1}^n \bar{X}^\mu dX^\mu \Big|^2 \Big) \\
& + \frac{1}{4\rho^2} \frac{\rho + 2c}{\rho + c} d\rho^2 - \frac{2}{\rho} (dw_0 d\bar{w}_0 - \sum_{\mu=1}^n dw_\mu d\bar{w}_\mu) \\
& + \frac{\rho + c}{\rho^2} \frac{4}{1 - \|X\|^2} \Big| dw_0 + \sum_{\mu=1}^n X^\mu dw_\mu \Big|^2 \\
& + \frac{1}{4\rho^2} \frac{\rho + c}{\rho + 2c} \Big(d\tilde{\phi} - 4Im\big(\bar{w}_0 dw_0 - \sum_{\mu=1}^n \bar{w}_\mu dw_\mu\big) + \frac{2c}{1 - \|X\|^2} Im \sum_{\mu=1}^n \bar{X}^\mu dX^\mu \Big)^2
\end{aligned}$$

with $w_0 := \frac{1}{2}(\tilde{\zeta}_0 + i\zeta^0)$, $w_\mu := \frac{1}{2}(\tilde{\zeta}_\mu - i\zeta^\mu)$, $\mu = 1, \ldots, n$, on[3]

$$\bar{N} = \{(X,\, \rho,\, \tilde{\phi},\, w) \in \mathbb{C}^n \times \mathbb{R}^{>0} \times \mathbb{R} \times \mathbb{C}^{n+1} \mid \|X\|^2 < 1\}$$

[3]In the case of a projective special Kähler domain $\bar{M}$ we consider $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ as in Definition 9, rather than its cyclic quotient $\bar{M} \times \mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$ on which the metric is also defined.

defined by the holomorphic function

$$F = \frac{i}{2}\left((z^0)^2 - \sum_{\mu=1}^{n}(z^\mu)^2\right) \text{ on } M := \left\{|z^0|^2 > \sum_{\mu=1}^{n}|z^\mu|^2\right\}$$

is a complete quaternionic Kähler metric. Furthermore $(\bar{N}, g_{FS})$ *is isometric to the symmetric space* $\tilde{X}(n+1) = \frac{SU(n+1,2)}{S[U(n+1)\times U(2)]}$.

Proof of Theorem 13 Let $\gamma\colon [0, b) \to \bar{N}$ be a smooth curve which leaves every compact subset of $\bar{N}$, $b \in (0, \infty]$. We have to show that γ has infinite length. By Theorem 7 we know that $(\bar{M}, \bar{g})$ is complete.

Lemma 16 *For every complete Riemannian manifold* (M, g) *and* $c \geq 0$ *the Riemannian manifold*

$$\left(M \times \mathbb{R}^{>0}, \frac{\rho + c}{\rho} g + \frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c} d\rho^2\right)$$

is complete. Here ρ *denotes the* $\mathbb{R}^{>0}$*-coordinate.*

Proof This follows from the estimate

$$\frac{\rho + c}{\rho} g + \frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c} d\rho^2 \geq g + \frac{1}{4}(d\log\rho)^2.$$

□

We consider the projection

$$\bar{N} \to \bar{M} \times \mathbb{R}^{>0}, \ p \mapsto (\pi(p), \rho(p)),$$

where $\pi\colon \bar{N} \to \bar{M}$ is the fibre bundle projection introduced in Proposition 11. Since the metric

$$\frac{\rho + c}{\rho} g + \frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c} d\rho^2$$

on the base $\bar{M} \times \mathbb{R}^{>0}$ is complete by the previous lemma, the projection $\bar{\gamma}$ of γ to $\bar{M} \times \mathbb{R}^{>0}$ either stays in a compact set or has infinite length. In the latter case γ has infinite length. So we can assume that $\bar{\gamma}$ stays in a compact set.

Using similar arguments as in the proof of Theorem 7 we can assume that $\bar{\gamma}$ has a unique accumulation point $(\bar{p}_0, \rho_0)$. In fact, the existence of two different accumulation points implies that $\bar{\gamma}$ and, hence, γ have infinite length. There exists a sequence $t_i \to b$ with $\bar{\gamma}(t_i) \to (\bar{p}_0, \rho_0) \in \bar{M} \times \mathbb{R}^{>0}$ and $\gamma(t_i)$ leaves every compact subset of $\bar{N}_\alpha \cong \bar{M}_\alpha \times \mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$, where $\bar{M}_\alpha$ is a projective special Kähler domain containing $(\bar{p}_0, \rho_0)$ and $\bar{N}_\alpha$ is the corresponding trivial fibre bundle endowed with the one-loop deformed Ferrara-Sabharwal metric associated to the projective special Kähler domain $\bar{M}_\alpha$. Note that $\pi_{\mathbb{R}^{2n+2}}(\gamma(t_i)) \in \mathbb{R}^{2n+2}$ is unbounded.

Lemma 17 *For $\varepsilon > 0$ and sufficiently small relatively compact $\bar{M}_\alpha \subset \bar{M}$ we have*[4] $g^c_{FS} \geq \delta \cdot g_{FS}$ *on $\bar{N}_\alpha \cap \{\rho > \varepsilon\}$ for some $\delta = \delta(\alpha, \varepsilon) > 0$.*

Proof Choose linear coordinates $(z^0, \dots, z^n)$ for the underlying conical affine special Kähler domain M_α such that g_α restricted to the $(z^1, \dots, z^n)$-plane is positive definite. This can always be achieved by restricting the coordinate domain. Then it follows from (1) that $\bar{g}_\alpha \geq \frac{k}{4}(d^c\mathcal{K})^2$ for some $k > 0$. Let $\varepsilon > 0$ be given. We claim that

$$g^c_{FS} \geq \frac{1}{2}\frac{k\varepsilon}{k\varepsilon + c} g_{FS}$$

on $\bar{N}_\alpha \cap \{\rho > \varepsilon\}$. Note first that

$$\bar{g} + \frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c} d\rho^2 \geq \frac{1}{2}\frac{k\varepsilon}{k\varepsilon + c}\left(\bar{g} + \frac{1}{4\rho^2} d\rho^2\right).$$

Next the last two expressions in the definition of g^c_{FS} can be estimated from below

$$\frac{1}{2\rho}\sum dp_a \hat{H}^{ab} dp_b + \frac{2c}{\rho^2} e^{\mathcal{K}} \left|\sum (X^I d\tilde{\zeta}_I + F_I(X) d\zeta^I)\right|^2 \geq \frac{1}{2}\frac{k\epsilon}{k\epsilon + c}\frac{1}{2\rho}\sum dp_a \hat{H}^{ab} dp_b,$$

since $\frac{k\epsilon}{k\epsilon + c} \leq 1$ and $(\hat{H}^{ab})$ is positive definite. Last setting

$$\theta_0 := d\tilde{\phi} + \sum (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I),$$

we conclude

$$\begin{aligned}
& \frac{c}{\rho}\bar{g} + \frac{1}{4\rho^2}\frac{\rho + c}{\rho + 2c}(\theta_0 + c d^c\mathcal{K})^2 \\
\geq\; & \frac{kc}{4\rho}(d^c\mathcal{K})^2 + \frac{1}{4\rho^2}\underbrace{\frac{\rho + c}{\rho + 2c}}_{\frac{1}{2} \leq \dots \leq 1}\left(\underbrace{\frac{c}{k\varepsilon + c}(\theta_0 + (k\varepsilon + c)d^c\mathcal{K})^2}_{\geq 0} + \frac{k\varepsilon}{k\varepsilon + c}\theta_0^2 - kc\varepsilon(d^c\mathcal{K})^2\right) \\
\geq\; & \frac{1}{2}\frac{k\varepsilon}{k\varepsilon + c}\frac{1}{4\rho^2}\theta_0^2 + \frac{ck}{4\rho^2}(\rho - \varepsilon)(d^c\mathcal{K})^2 \\
\geq\; & \frac{1}{2}\frac{k\varepsilon}{k\varepsilon + c}\frac{1}{4\rho^2}\left(d\tilde{\phi} + \sum (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)\right)^2,
\end{aligned}$$

[4]Here g_{FS} denotes the metric on $\bar{N}_\alpha = \bar{M}_\alpha \times \mathbb{R}^{>0} \times S^1_c \times \mathbb{R}^{2n+2}$ induced by the metric g_{FS} on $\bar{M}_\alpha \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$. Alternatively one can compare the metrics by pulling back g^c_{FS} to the cyclic covering $\bar{M}_\alpha \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3} \to \bar{N}_\alpha$.

where the last inequality follows from $\rho > \varepsilon$. Combining these three inequalities, we have shown that

$$g^c_{FS} \geq \frac{1}{2}\frac{k\varepsilon}{k\varepsilon + c} g_{FS}$$

on $\bar{N}_\alpha \cap \{\rho > \varepsilon\}$. □

Choose $\varepsilon > 0$ such that $\rho_0 \geq 2\varepsilon$. For the undeformed metric g_{FS} on $\bar{N}_\alpha$ we have $g_{FS} = \bar{g}|_{\bar{M}_\alpha} + g_G$, where g_G is a family of left invariant metrics on $G = \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ endowed with the Lie group structure defined in [7].

Since $\bar{M}_\alpha \subset \bar{M}$ is relatively compact, we can estimate $g_G \geq \mathrm{const} g^0_G$ for some left invariant metric g^0_G on the group fibre G. This implies that the curve γ has infinite length, since every homogenous Riemannian metric is complete and the length of γ can be estimated by the length of its projection to G. □

5.2 Completeness of the One-Loop Deformation for Complete Projective Special Kähler Manifolds with Cubic Prepotential

In this section, we prove completeness of the one-loop deformation g^c_{FS} in the case of complete projective special Kähler manifolds in the image of the supergravity r-map. We will recall the definition of the latter manifolds below. They are also know as *projective special Kähler manifolds with cubic prepotential* or *projective very special Kähler manifolds*.

In Sect. 5.2.1, we introduce projective special real geometry and the supergravity r-map. The latter assigns a complete projective special Kähler manifold to each complete projective special real manifold. In Sect. 5.2.2, we derive a sufficient condition for the completeness of $(N'_{(4n+4,0)}, g^c_{FS})$ for $c \in \mathbb{R}^{\geq 0}$. Recall that we construct $(N'_{(4n+4,0)}, g^c_{FS})$ from a projective special Kähler manifold. We prove the completeness of $(N'_{(4n+4,0)}, g^c_{FS})$ in the case that the projective special Kähler manifold is obtained from a complete projective special real manifold via the supergravity r-map and in the case of $\mathbb{C}H^n$.

As a corollary, we obtain deformations by complete quaternionic Kähler metrics of all known homogeneous quaternionic Kähler manifolds of negative scalar curvature (including symmetric spaces), except for quaternionic hyperbolic space. In the case of the series $\tilde{X}(n+1) = \frac{SU(n+1,2)}{S[U(n+1)\times U(2)]}$, which corresponds to the projective special Kähler domains $\mathbb{C}H^n$ with quadratic prepotential, we already gave a simple and explicit expression for the deformed metric in Corollary 15.

In this chapter, we only discuss positive definite quaternionic Kähler metrics.

5.2.1 Projective Special Real Geometry and the Supergravity R-Map

Definition 18 Let h be a homogeneous cubic polynomial in n variables with real coefficients and let $U \subset \mathbb{R}^n \setminus \{0\}$ be an $\mathbb{R}^{>0}$-invariant domain such that $h|_U > 0$ and such that $g_{\mathcal{H}} := -\partial^2 h\big|_{\mathcal{H}}$ is a Riemannian metric on the hypersurface $\mathcal{H} := \{x \in U \mid h(x) = 1\} \subset U$. Then $(\mathcal{H}, g_{\mathcal{H}})$ is called a **projective special real** (PSR) manifold.

Define $\bar{M} := \mathbb{R}^n + iU \subset \mathbb{C}^n$. We endow $\bar{M}$ with the standard complex structure and use holomorphic coordinates $(X^\mu = y^\mu + ix^\mu)_{\mu=1,\dots,n} \in \mathbb{R}^n + iU$. We define a Kähler metric

$$\begin{aligned}\bar{g} = 2\sum_{\mu,\nu=1}^{n} g_{\mu\bar{\nu}} dX^\mu d\bar{X}^\nu &:= \sum_{\mu,\nu=1}^{n} \frac{\partial^2 \mathcal{K}}{\partial X^\mu \partial \bar{X}^\nu} dX^\mu d\bar{X}^\nu \\ &= \frac{1}{2}\sum_{\mu,\nu=1}^{n} \frac{\partial^2 \mathcal{K}}{\partial X^\mu \partial \bar{X}^\nu}(dX^\mu \otimes d\bar{X}^\nu + d\bar{X}^\nu \otimes dX^\mu)\end{aligned}$$

on $\bar{M}$ with Kähler potential

$$\mathcal{K}(X, \bar{X}) := -\log 8h(x) = -\log h\left(i(\bar{X} - X)\right). \tag{11}$$

Definition 19 The correspondence $(\mathcal{H}, g_{\mathcal{H}}) \mapsto (\bar{M}, \bar{g})$ is called the **supergravity r-map**.

Remark 2 With $\frac{\partial}{\partial X^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial y^\mu} - i\frac{\partial}{\partial x^\mu}\right)$, we have

$$\begin{aligned}2\bar{g}\left(\frac{\partial}{\partial X^\mu}, \frac{\partial}{\partial \bar{X}^\nu}\right) = 2g_{\mu\bar{\nu}} &= \frac{\partial^2 \mathcal{K}(X, \bar{X})}{\partial X^\mu \partial \bar{X}^\nu} =: \mathcal{K}_{\mu\bar{\nu}} \\ &= -\frac{1}{4}\frac{\partial^2 \log h(x)}{\partial x^\mu \partial x^\nu} = -\frac{h_{\mu\nu}(x)}{4h(x)} + \frac{h_\mu(x) h_\nu(x)}{4h^2(x)},\end{aligned} \tag{12}$$

where $h_\mu(x) := \frac{\partial h(x)}{\partial x^\mu}$, $h_{\mu\nu}(x) := \frac{\partial^2 h(x)}{\partial x^\mu \partial x^\nu}$, etc., for $\mu, \nu = 1, \dots, n$.

The inverse $(\mathcal{K}^{\bar{\nu}\lambda})_{\nu,\lambda=1,\dots,n}$ of $(\mathcal{K}_{\mu\bar{\nu}})_{\mu,\nu=1,\dots,n}$ is given by

$$\mathcal{K}^{\bar{\nu}\lambda} = -4h(x)h^{\nu\lambda}(x) + 2x^\nu x^\lambda. \tag{13}$$

This can be shown using the fact that h is a homogeneous polynomial of degree three:

$$\begin{aligned}&\sum_{\mu=1}^{n} h_\mu(x) x^\mu = 3h(x), \quad \sum_{\nu=1}^{n} h_{\mu\nu}(x) x^\nu = 2h_\mu(x), \\ &\sum_{\rho=1}^{n} h_{\mu\nu\rho}(x) x^\rho = h_{\mu\nu}, \quad h_{\mu\nu\rho\sigma} = 0.\end{aligned} \tag{14}$$

Remark 3 Note that any manifold $(\bar{M}, \bar{g})$ in the image of the supergravity r-map is a projective special Kähler domain. The corresponding conical affine special Kähler domain is the trivial $\mathbb{C}^*$-bundle

$$M := \{z = z^0 \cdot (1, X) \in \mathbb{C}^{n+1} \mid z^0 \in \mathbb{C}^*,\ X \in \bar{M} = \mathbb{R}^n + iU\} \to \bar{M}$$

endowed with the standard complex structure J and the metric g_M defined by the holomorphic function

$$F : M \to \mathbb{C}, \quad F(z^0, \dots, z^n) = \frac{h(z^1, \cdots, z^n)}{z^0}.$$

Note that in general, the flat connection[5] ∇ on M is not the standard one induced from $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$. The homothetic vector field ξ is given by $\xi = \sum_{I=0}^{n}(z^I \frac{\partial}{\partial z^I} + \bar{z}^I \frac{\partial}{\partial \bar{z}^I})$. To check that $\bar{g}$ is the corresponding projective special Kähler metric, one uses the fact that

$$8|z^0|^2 h(x) = \sum_{I,J=0}^{n} z^I N_{IJ}(z, \bar{z})\bar{z}^J, \tag{15}$$

where as above, $x = (\operatorname{Im} X^1, \dots, \operatorname{Im} X^n) = (\operatorname{Im} \frac{z^1}{z^0}, \dots, \operatorname{Im} \frac{z^n}{z^0}) \in U$ (see [7]).

Definition 20 A Kähler manifold $(\bar{M}, \bar{g})$ in the image of the supergravity r-map is called a **projective very special Kähler manifold**.

Due to the following two results, projective special real geometry constitutes a powerful tool for the construction of complete projective special Kähler manifolds.

Theorem 21 ([7]) *The supergravity r-map preserves completeness, i.e. it assigns a complete projective special Kähler manifold to each complete projective special real manifold.*

The question of completeness for a projective special real manifold $(\mathcal{H}, g_{\mathcal{H}})$ reduces to a simple topological question for the hypersurface $\mathcal{H} \subset \mathbb{R}^n$:

Theorem 22 ([10, Thm. 2.6.]) *Let $(\mathcal{H}, g_{\mathcal{H}})$ be a projective special real manifold of dimension $n-1$. If $\mathcal{H} \subset \mathbb{R}^n$ is closed, then $(\mathcal{H}, g_{\mathcal{H}})$ is complete.*

Remark 4 In low dimensions, it is possible to classify all complete projective special real manifolds up to linear isomorphisms of the ambient space. In the case of curves, there are exactly two examples [7]. In the case of surfaces, there exist precisely five discrete examples and a one-parameter family [9].

[5] ∇ is defined by $x^I = \operatorname{Re} z^I$ and $y_I = \operatorname{Re} F_I(z)$ being flat, $I = 0, \dots, n$ (see [1]).

5.2.2 The Completeness Theorem

Definition 23 The **q-map** is the composition of the supergravity r- and c-map. It assigns a $(4n + 4)$-dimensional quaternionic Kähler manifold to each $(n - 1)$-dimensional projective special real manifold.

Remark 5 Except for quaternionic hyperbolic space $\mathbb{H}H^{n+1}$, all Wolf spaces of non-compact type and all known homogeneous, non-symmetric quaternionic Kähler manifolds (called normal quaternionic Kähler manifolds or Alekseevsky spaces) are in the image of the supergravity c-map. While the series $\tilde{X}(n+1) = Gr_{0,2}(\mathbb{C}^{n+1,2})$ of non-compact Wolf spaces can be obtained via the supergravity c-map from the projective special Kähler manifold $\mathbb{C}H^n$ (with holomorphic prepotential $F = \frac{i}{2}((z^0)^2 - \sum_{\mu=1}^{n}(z^\mu)^2))$, which is not in the image of the supergravity r-map, all the other manifolds mentioned above are in the image of the q-map.

Below, we prove the completeness of the one-loop deformation of the Ferrara-Sabharwal metric with positive deformation parameter $c \in \mathbb{R}^{\geq 0}$ for all manifolds in the image of the q-map.

Due to the following result, both the supergravity c-map and the q-map preserve completeness:

Theorem 24 ([7]) *The supergravity c-map assigns a complete quaternionic Kähler manifold of dimension* $4n + 4$ *to each complete projective special Kähler manifold of dimension* $2n$*.*

Let $(\bar{M}, \bar{g})$ be a projective special Kähler domain with underlying conical special Kähler domain (M, g, F). As in Sect. 4.2, we assume that $M \subset \{z^0 \neq 0\} \subset \mathbb{C}^{n+1}$ and identify $\bar{M}$ with $M \cap \{z^0 = 1\}$. Then, by restricting the tensor field $\frac{g}{f}$ to $\bar{M} \subset M$, we can write

$$\bar{g} = -\frac{g}{f} + (\partial\mathcal{K})(\bar{\partial}\mathcal{K}) = -\frac{g}{f} + \frac{1}{4}(d\mathcal{K})^2 + \frac{1}{4}(d^c\mathcal{K})^2. \tag{16}$$

We consider the one-loop deformed Ferrara-Sabharwal metric (see Eq. (7))

$$\begin{aligned} g^c_{FS} = \frac{\rho + c}{\rho}\bar{g} + \frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c}d\rho^2 + \frac{1}{4\rho^2}\frac{\rho + c}{\rho + 2c}\Big(d\tilde{\phi} + \sum_{I=0}^{n}(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + c d^c\mathcal{K}\Big)^2 \\ + \frac{1}{2\rho}\sum_{a,b=1}^{2n+2} dp_a \hat{H}^{ab} dp_b + \frac{2c}{\rho^2}e^{\mathcal{K}}\left|\sum_{I=0}^{n}(X^I d\tilde{\zeta}_I + F_I(X) d\zeta^I)\right|^2 \end{aligned} \tag{17}$$

for $c \in \mathbb{R}^{\geq 0}$ defined on $N'_{(4n+4,0)} = \bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ endowed with global coordinates

$$(X^\mu,\ \rho,\ \tilde{\phi},\ \tilde{\zeta}_I,\ \zeta^I)^{\mu=1,\dots,n}_{I=0,\dots,n}.$$

Proposition 25 *If* $(\bar{M}, \bar{g})$ *is complete and* $\bar{g} \geq \frac{k}{4}(d^c\mathcal{K})^2$*, for some* $k \in \mathbb{R}^{>0}$*, then* $(\bar{N}, g^c_{FS})$ *is complete for every* $c \in \mathbb{R}^{\geq 0}$.

Proof $(\bar{N}, g^0_{FS})$ is complete by Theorem 24. Since every curve on $(\bar{N}, g^c_{FS})$ approaching $\rho = 0$ has infinite length, we can restrict to $\{\rho > \epsilon\} \subset \bar{N}$ for some $\epsilon > 0$. With the same argument as in Lemma 17 one shows

$$g^c_{FS} \geq \frac{1}{2}\frac{k\epsilon}{k\epsilon + c} g^0_{FS}$$

using that $\bar{g} \geq \frac{k}{4}(d^c\mathcal{K})^2$. Since $(\bar{N}, g^0_{FS})$ is complete, this shows that $(\bar{N}, g^c_{FS})$ is complete as well for $c \in \mathbb{R}^{\geq 0}$. □

For quaternionic Kähler manifolds in the image of the q-map, the prepotential is $F(z) = \frac{h(z^1, \ldots, z^n)}{z^0}$.

Lemma 26 *For projective special Kähler manifolds in the image of the supergravity r-map we have*

$$\bar{g} \geq \frac{1}{12}(d^c\mathcal{K})^2.$$

Proof First, we show that

$$\tilde{g} := -\sum_{\mu,\nu=1}^{n} \frac{h_{\mu\nu}(x)}{h(x)} dy^\mu dy^\nu \geq -\frac{2}{3}(d^c\mathcal{K})^2. \tag{18}$$

Considering $\tilde{g}$ as a family of pseudo-Riemannian metrics on $\mathbb{R}^n$ depending on a parameter $x \in U$, the left hand side is positive definite on the orthogonal complement $Y^{\perp_{\tilde{g}}}$ of $Y := \sum_{\mu=1}^n x^\mu \partial_{y^\mu}$, while the right hand side is zero, since $\tilde{g}(Y, \cdot) = 2d^c\mathcal{K}$. In the direction of Y, we have $\tilde{g}(Y, Y) = -6 = -\frac{2}{3}(d^c\mathcal{K})^2(Y, Y)$.

Equation (18) implies

$$\bar{g} \geq \frac{1}{4h(x)} \sum_{\mu,\nu=1}^{n} \left(-h_{\mu\nu}(x) + \frac{h_\mu(x)h_\nu(x)}{h(x)} \right) dy^\mu dy^n \geq -\frac{1}{6}(d^c\mathcal{K})^2 + \frac{1}{4}(d^c\mathcal{K})^2 = \frac{1}{12}(d^c\mathcal{K})^2.$$

□

This shows that the assumption of Proposition 25 is fulfilled with $k = 1/3$ for projective special Kähler manifolds in the image of the supergravity r-map and proves the following theorem.

Theorem 27 *Let* $(\mathcal{H}, g_{\mathcal{H}})$ *be a complete projective special real manifold of dimension* $n-1$ *and* g^c_{FS}*,* $c \in \mathbb{R}^{\geq 0}$*, the one-loop deformed Ferrara-Sabharwal metric on* $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ *defined by the projective special Kähler domain* $(\bar{M}, \bar{g})$ *obtained from* $(\mathcal{H}, g_{\mathcal{H}})$ *via the supergravity r-map. Then* $(\bar{N}, g^c_{FS})$

is a complete quaternionic Kähler manifold. $(\bar{N}, g^0_{FS})$ *is the complete quaternionic Kähler manifold obtained from* $(\mathcal{H}, g_{\mathcal{H}})$ *via the q-map.*

Example 28 For the case $n = 1$ $(h = x^3)$, $(\bar{N}, g^0_{FS})$ is isometric to the symmetric space $G_2^*/SO(4)$. In this case we checked using computer algebra software that the squared pointwise norm of the Riemann tensor with respect to the metric is

$$\sum_{i,j,k,l,\tilde{i},\tilde{j},\tilde{k},\tilde{l}=1}^{8} R_{ijkl} g^{i\tilde{i}} g^{j\tilde{j}} g^{k\tilde{k}} g^{l\tilde{l}} R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}}$$

$$= \frac{128\begin{pmatrix} 528c^7 + 2112c^6\rho + 3664c^5\rho^2 + 3568c^4\rho^3 \\ \qquad + 2110c^3\rho^4 + 764c^2\rho^5 + 161c\rho^6 + 17\rho^7 \end{pmatrix}}{3(c+\rho)(2c+\rho)^6}.$$

For $c > 0$, this function is non-constant, which shows that $(\bar{N}, g^c_{FS})$ is not locally homogeneous for $c > 0$.

Acknowledgements The research leading to these results has received funding from the German Science Foundation (DFG) under the Research Training Group 1670 "Mathematics inspired by String Theory" and from the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013)/ERC Grant Agreement 307062.

V.C. thanks the École Normale Supérieure for hospitality and support in Paris.

References

1. D.V. Alekseevsky, V. Cortés, C. Devchand, Special complex manifolds. J. Geom. Phys. **42**(1–2), 85–105 (2002)
2. D.V. Alekseevsky, V. Cortés, T. Mohaupt, Conification of Kähler and hyper-Kähler manifolds. Commun. Math. Phys. **324**(2), 637–655 (2013)
3. D.V. Alekseevsky, V. Cortés, M. Dyckmanns, T. Mohaupt, Quaternionic Kähler metrics associated with special Kähler manifolds. J. Geom. Phys. **92**, 271–287 (2015)
4. J. Bagger, E. Witten, Matter couplings in N=2 supergravity. Nuclear Phys. **B 222**(1), 1–10 (1983)
5. A.L. Besse, *Einstein Manifolds* (Springer, Berlin, 1987)
6. V. Cortés, T. Mohaupt, Special geometry of Euclidean supersymmetry III: the local r-map, instantons and black holes. J. High Energy Phys. **0907**, 066 (2009)
7. V. Cortés, X. Han, T. Mohaupt, Completeness in supergravity constructions. Commun. Math. Phys. **311**(1), 191–213 (2012)
8. V. Cortés, J. Louis, P. Smyth, H. Triendl, On certain Kähler quotients of quaternionic Kähler manifolds. Commun. Math. Phys. **317**(3), 787–816 (2013)
9. V. Cortés, M. Dyckmanns, D. Lindemann, Classification of complete projective special real surfaces. Proc. Lond. Math. Soc. (3) **109**(2), 423–445 (2014)
10. V. Cortés, M. Nardmann, S. Suhr, Completeness of hyperbolic centroaffine hypersurfaces. Commun. Anal. Geom. **24**(1), 59–92 (2016)
11. B. de Wit, A. Van Proeyen, Special geometry, cubic polynomials and homogeneous quaternionic spaces. Commun. Math. Phys. **149**(2), 307–333 (1992)

12. S. Ferrara, S. Sabharwal, Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces. Nucl. Phys. **B332**(2), 317–332 (1990)
13. A. Haydys, Hyper-Kähler and quaternionic Kähler manifolds with S^1*-symmetries*. J. Geom. Phys. **58**(3), 293–306 (2008)
14. N. Hitchin, Quaternionic Kähler moduli spaces, in *Riemannian Topology and Geometric Structures on Manifolds*, ed. by K. Galicki, S. Simanca. Progress in Mathematics, vol. 271 (Birkhäuser, Boston, 2009), pp. 49–61
15. C. LeBrun, On complete quaternionic-Kähler manifolds. Duke Math. J. **63**(3), 723–743 (1991)
16. C. LeBrun, S. Salamon, Strong rigidity of positive quaternion-Kähler manifolds. Invent. Math. **118**(1), 109–132 (1994)
17. D. Robles-Llana, F. Saueressig, S. Vandoren, String loop corrected hypermultiplet moduli spaces. J. High Energy Phys. **0603**, 081 (2006).

Hypertoric Manifolds and HyperKähler Moment Maps

Andrew Dancer and Andrew Swann

To Simon Salamon on the occasion of his 60th birthday

Abstract We discuss various aspects of moment map geometry in symplectic and hyperKähler geometry. In particular, we classify complete hyperKähler manifolds of dimension $4n$ with a tri-Hamiltonian action of a torus of dimension n, without any assumption on the finiteness of the Betti numbers. As a result we find that the hyperKähler moment in these cases has connected fibres, a property that is true for symplectic moment maps, and is surjective. New examples of hypertoric manifolds of infinite topological type are produced. We provide examples of non-Abelian tri-Hamiltonian group actions of connected groups on complete hyperKähler manifolds such that the hyperKähler moment map is not surjective and has some fibres that are not connected. We also discuss relationships to symplectic cuts, hyperKähler modifications and implosion constructions.

Keywords Complete metric • Disconnected fibres • HyperKähler manifold • Infinite topology • Moment map • Non-surjectivity • Toric manifold

1 Introduction

A symplectic structure on a (necessarily even-dimensional) manifold is a closed non-degenerate two-form. Several Riemannian and pseudo-Riemannian geometries have been developed over the years which give rise to a symplectic structure as part of their data. The most famous example is that of a *hyperKähler structure*,

A. Dancer
Jesus College, Oxford OX1 3DW, UK
e-mail: dancer@maths.ox.ac.uk

A. Swann (✉)
Department of Mathematics, Aarhus University, Ny Munkegade 118, Bldg 1530, DK-8000 Aarhus C, Denmark
e-mail: swann@math.au.dk

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_5

where we have a Riemannian metric g and complex structures I, J, K obeying the quaternionic multiplication relations, and such that g is Kähler with respect to I, J, K. We therefore obtain a triple $(\omega_I, \omega_J, \omega_K)$ of symplectic forms.

One of the foundational results of symplectic geometry is the Darboux Theorem, which says that locally a symplectic structure can be put into a standard form $\omega = \sum_{i=1}^n dp_i \wedge dq_i$. Many of the interesting questions in symplectic geometry are therefore global in nature, giving the subject a more topological flavour.

Geometries involving a metric of course do not have a Darboux-type theorem, because the metric contains local information through its curvature tensor. However, there is one area of symplectic geometry, that concerning *moment maps* where a rich theory has been developed for other geometries by analogy with the symplectic situation. In this paper we shall discuss some aspects of this, especially related to hypertoric manifolds, cutting and implosion.

2 Hypertoric Manifolds

Let M be a hyperKähler manifold M of dimension $4n$. We say that an action of a group G on M is *tri-symplectic* if it preserves each of the symplectic forms ω_I, ω_J and ω_K. This is equivalent to G preserving both the metric g and each of the associated complex structures I, J and K; so the action is isometric and *tri-holomorphic*. We will usually assume that G is connected and that the action is effective.

Because hyperKähler metrics are Ricci-flat, we have that if M is compact, then any Killing field X is parallel and so G is Abelian. As the complex structures are also parallel the distribution $\mathbb{H}X = \mathrm{Span}_{\mathbb{R}}\{X, IX, JX, KX\}$ is integrable and flat. Up to finite covers, such an M is a product $T^{4m} \times M_0$, with G acting trivially on M_0.

Thus the interesting cases are when M is non-compact. From the Riemannian perspective it is now natural to consider complete metrics. Note that by Alekseevskiĭ and Kimel'fel'd [1], any homogeneous hyperKähler manifold is flat; such a manifold is necessarily complete, so its universal cover is $\mathbb{R}^{4n}$ with the flat metric. Thus one should consider actions on M with orbits of dimension strictly less than $4n$.

One says that a tri-holomorphic action of G on M is *tri-Hamiltonian* if it is Hamiltonian for each symplectic structure, meaning that there are equivariant moment maps

$$\mu_I, \mu_J, \mu_K \colon M \to \mathfrak{g}^*, \tag{1}$$

$$d\mu_A^X = X \lrcorner\, \omega_A, \tag{2}$$

where $\mu_A^X = \langle \mu_A, X \rangle$. Here we write X both for the element of $\mathfrak{g}$ and the corresponding vector field $x \mapsto X_x$ on M. Also $\langle \alpha, X \rangle = \alpha(X)$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

As each $X \in \mathfrak{g}$ preserves ω_A, we have $0 = L_X\omega_A = X \lrcorner\, d\omega_A + d(X \lrcorner\, \omega_A) = d(X \lrcorner\, \omega_A)$, so $X \lrcorner\, \omega_A$ is exact. Thus if M is simply-connected then, Eq. (2) has a solution $\mu_A^X \in C^\infty(M)$ that is unique up to an additive constant.

2.1 Abelian Actions

For an Abelian group G, equivariance of μ_A is the just the condition $L_X\mu_A^Y = 0$ for each $X, Y \in \mathfrak{g}$. But $L_X\mu_A^Y = X \lrcorner d\mu_A^Y = -\omega_A(X, Y) = -g(AX, Y)$ and $d(\omega_A(X, Y)) = L_Y(X \lrcorner \omega_A) = 0$. So $L_X\mu_A^Y$ is constant and the action is tri-Hamiltonian only if for each A we have $\mathcal{G} \perp A\mathcal{G}$, where $\mathcal{G}_x = \{X_x \mid X \in \mathfrak{g}\} \subset T_xM$. This last condition is equivalent to $\dim \mathbb{H}\mathcal{G}_x = 4 \dim \mathcal{G}_x$ for each $x \in M$.

Proposition 1 *Suppose G is a connected Abelian group that has an effective tri-Hamiltonian action on a connected hyperKähler manifold M of dimension $4n$. Then the dimension of G is at most n.*

Proof For each $x \in M$, the discussion above shows that the tri-Hamiltonian condition gives $\dim \mathcal{G}_x \leqslant n$. We thus need to show that there is some $x \in M$ such that the map $\mathfrak{g} \to \mathcal{G}_x$, $X \mapsto X_x$, is injective.

Fix a point $x \in M$ such that $\dim \operatorname{stab}_G(x)$ is the least possible. Note that $H = \operatorname{stab}_G(x)$ is a compact subgroup of $\mathrm{Sp}(n) \leqslant \mathrm{SO}(4n)$. We may therefore H-invariantly write $T_xM = T_x(G \cdot x) \oplus W$ as an orthogonal direct sum of the tangent space to the orbit through x and its orthogonal complement W. Now consider the map $F \colon G \times W \to M$ given by

$$F(g, w) = g \cdot (\exp_x w) = \exp_{gx} g_* w.$$

At $(e, 0) \in G \times W$ this has differential $(F_*)_{(e,0)}(X, w) = X_x + w$ and $F(gh, (h_*)^{-1}w) = F(g, w)$ for each $h \in H$. Thus F descends to a diffeomorphism from a neighbourhood of $(e, 0) \in G \times_H W$ to a neighbourhood U of $x \in M$ which is equivariant for the action of $\mathfrak{g}$. In particular $\operatorname{stab}_G F(e, w) \subset H$ when $F(e, w) \in U$.

As G acts effectively, we have for each $X \in \mathfrak{g} \setminus \{0\}$ there is some point y with $X_y \neq 0$. But M is Ricci-flat, so the Killing vector field X is analytic, thus the set $\{y \in M \mid X_y \neq 0\}$ is open and dense.

If $\dim H = \dim \operatorname{stab}_G(x)$ is non-zero, then there is a non-zero element $X \in \mathfrak{h}$. Now X is non-zero at some point $y = F(g, w)$ of U, and $z = g^{-1}y = F(e, w)$ has $X_z = (g_*)^{-1}X_y \neq 0$ too, since G is Abelian. So $\operatorname{Lie} \operatorname{stab}_G(z)$ is a subspace of $\mathfrak{h}$ not containing X. It follows that $\dim \operatorname{stab}_G(z) < \dim \operatorname{stab}_G(x)$, contradicting our choice of x.

We conclude that $\dim \mathfrak{h} = 0$, so $\operatorname{stab}_G(x)$ is finite and $\mathfrak{g} \mapsto \mathcal{G}_x$ is a bijection. Thus $\dim \mathfrak{g} \leqslant n$. □

Any connected Abelian group of finite dimension is of the form $G = \mathbb{R}^m \times T^k$ for some $m, k \geqslant 0$. If M is simply-connected then the tri-symplectic T^k-action is necessarily tri-Hamiltonian: each μ_A^Y obtains its maximum on each T^k-orbit, and so $L_X\mu_A^Y = X \lrcorner d\mu_A^Y$ is zero at these points, and hence on all of M. If $\dim G = n$ and the G-action is tri-Hamiltonian, Bielawski [5] proves that the $\mathbb{R}^m$ factor acts freely and any discrete subgroup of $\mathbb{R}^m$ acts properly discontinuously, so a discrete quotient of M has a tri-Hamiltonian T^n-action. In general, a hyperKähler manifold of dimension $4n$ with a tri-Hamiltonian T^n action is called *hypertoric*.

Bielawski [5] classified the hypertoric manifolds in any dimension under the assumption that M has finite topological type, meaning that the Betti numbers of M are finite. For $\dim M = 4$, this classification is extended to general hypertoric M in [24]. Here we wish to provide the full classification of hypertoric manifolds in arbitrary dimension, without any restriction on the topology. First let us recall some of the four-dimensional story.

2.2 Dimension Four

Let M be a four-dimensional hyperKähler manifold with an effective tri-Hamiltonian S^1-action of period 2π. Let X be the corresponding vector field on M. Note that the only special orbits for the action are fixed points: if $g \in S^1$ stabilises the point x and $X_x \neq 0$, then g fixes $T_xM = \mathrm{Span}\{X_x, IX_x, JX_x, KX_x\}$ and hence a neighbourhood of x, so by analyticity $g = e$.

The hyperKähler moment map

$$\mu = (\mu_I, \mu_J, \mu_K)\colon M \to \mathbb{R}^3$$

is a local diffeomorphism away from the fixed point set M^X. Locally on $M' = M \setminus M^X$, the hyperKähler metric may be written as

$$g = \frac{1}{V}\beta_0^2 + V(\alpha_I^2 + \alpha_J^2 + \alpha_K^2),$$

where $\alpha_A = X \lrcorner \omega_A = d\mu_A$, $V = 1/g(X,X)$ and $\beta_0 = \alpha_0/\|X\| = g(X,\cdot)V^{1/2}$. The hyperKähler condition is now equivalent to the monopole equation $d\beta_0 = -*_3 dV$, which implies that locally V is a harmonic function on $\mathbb{R}^3$.

Theorem 2 ([24]) *Let M be a complete connected hyperKähler manifold of dimension 4 with a tri-Hamiltonian circle action of period 2π. Then the hyperKähler moment map $\mu\colon M \to \mathbb{R}^3$ is surjective with connected fibres and induces a homeomorphism $\overline{\mu}\colon M/S^1 \to \mathbb{R}^3$. The metric on M is specified by any harmonic function $V\colon \mathbb{R}^3 \setminus Z \to (0,\infty)$ of the form*

$$V(p) = c + \frac{1}{2}\sum_{q\in Z}\frac{1}{\|p-q\|} \tag{3}$$

where $c \geqslant 0$ is constant and $Z \subset \mathbb{R}^3$ is finite or countably infinite. □

The set $Z = \mu(M^X)$ is the image of the fixed-point set M^X. The metrics with $c = 0$ and Z finite are the Gibbons-Hawking metrics [11]; $c > 0$ and Z finite gives the older multi-Taub-NUT metrics [16]. For $Z \neq \varnothing$, the hyperKähler manifold M is simply-connected, with $b_2(M) = |Z| - 1$; for $c > 0$, $Z = \varnothing$, we have $M = S^1 \times \mathbb{R}^3$ with μ projection to the second factor. For Z infinite and $c = 0$, the hyperKähler metrics are of type A_∞ as constructed by Anderson et al. [3] and Goto [13], concentrating on the case $Z = \{(n^2, 0, 0)n \in \mathbb{N}_{>0}\}$, and written down for general Z in Hattori [15]. These are the examples of infinite topological type.

To understand and extend Hattori's general formulation of these structures, we need to study when (3) gives a finite sum on an open subset of $\mathbb{R}^3$. By Harnack's Principle (see e.g. [4]) if (3) is finite at one point of $\mathbb{R}^3$, then it is finite on all of $\mathbb{R}^3 \setminus Z$. This may be seen in a elementary way via the following result.

Lemma 3 *Suppose $(q_n)_{n\in\mathbb{N}}$ is a sequence of points in $\mathbb{R}^3$ is given. Then the series $S_1 = \sum_{n\in\mathbb{N}} \|p - q_n\|^{-1}$ converges at some $p \in \mathbb{R}^3 \setminus \{q_n \mid n \in \mathbb{N}\}$ if and only if the series $S_2 = \sum_{n\in\mathbb{N}} (1 + \|q_n\|)^{-1}$ converges.*

Proof First note that if there is a compact subset C of $\mathbb{R}^3$ containing infinitely many points of the sequence (q_n), then neither sum converges: there is some subsequence $(q_i)_{i\in I}$ that converges to a $q \in \mathbb{R}^3$, and so infinitely many terms are greater than some strictly positive lower bound.

Now putting $c = 1 + \|p\|$, we have $\|p - q\| \leqslant \|p\| + \|q\| \leqslant c(1 + \|q\|)$. It follows that convergence of S_1 implies converges of S_2.

For the converse, we consider $q \in \mathbb{R}^3 \setminus \overline{B}(0; R)$ for $R = 1 + 2\|p\|$ and have

$$\begin{aligned} \|p - q\| &\geqslant \|q\| - \|p\| = \tfrac{1}{2}\|q\| + (\tfrac{1}{2}\|q\| - \|p\|) \\ &> \tfrac{1}{2}(\|q\| + 1). \end{aligned} \tag{4}$$

If S_2 converges, then $\{n \in \mathbb{N} \mid q_n \in \overline{B}(0; R)\}$ is finite, so the inequality (4) implies convergence of S_1. □

Finally let us remark that scaling the hyperKähler metric g by a constant C scales V as a function on M by C^{-1}. However the hyperKähler moment map μ also scales by C, so the induced function $V(p) = V(\mu(x))$ on $\mathbb{R}^3$ has the same form, with a new constant term c/C and the points q replaced by q/C. On the other hand scaling the vector field X by a constant, so that the action it generates is no longer of period 2π, scales V on M and μ by different weights. In particular, such a change alters the factors $1/2$ in (3).

2.3 Construction of Hypertoric Manifolds

Bielawski and Dancer [6] provided a general construction of hypertoric manifolds in all dimensions with finite topological type. Goto [13] gave a particular construction of examples in arbitrary dimension of infinite topological type. Let us now build on Hattori's four-dimensional description [15], to combine these two constructions.

Let $\mathbb{L}$ be a finite or countably infinite set. Choose $\Lambda = (\Lambda_k)_{k\in\mathbb{L}} \in \mathbb{H}^{\mathbb{L}}$ and define $\lambda = (\lambda_k)_{k\in\mathbb{L}}$ by $\lambda_k = -\frac{1}{2}\overline{\Lambda}_k i\Lambda_k \in \operatorname{Im}\mathbb{H}$. For each $k \in \mathbb{L}$, let $u_k \in \mathbb{R}^n$ be a non-zero vector and put $\hat{\lambda}_k = \lambda_k/\|u_k\|$. Suppose

$$\sum_{k\in\mathbb{L}}(1+|\hat{\lambda}_k|)^{-1} < \infty. \tag{5}$$

Consider the Hilbert manifold $M_\Lambda = \Lambda + \mathbb{L}^2(\mathbb{H})$, where

$$\mathbb{L}^2(\mathbb{H}) = \Big\{ v \in \mathbb{H}^{\mathbb{L}} \;\Big|\; \sum_{k\in\mathbb{L}}|v_k|^2 < \infty \Big\}.$$

Let T_λ be the Hilbert group

$$T_\lambda = \Big\{ g \in T^{\mathbb{L}} = (S^1)^{\mathbb{L}} \;\Big|\; \sum_{k\in\mathbb{L}}(1+|\lambda_k|)\,|1-g_k|^2 < \infty \Big\}.$$

If $\|u_k\|$ is bounded away from 0, then $g \in T_\lambda$ implies g_k is arbitrarily close to 1 except for a finite number of $k \in \mathbb{L}$. As $|1-\exp(it)|^2 = 2-2\cos(t) \leqslant t^2$ for all $t \in \mathbb{R}$ and $|1-\exp(it)|^2 \geqslant 2t^2/\pi^2$ on $(-\pi/2, \pi/2)$, we see that the Lie algebra of T_λ is

$$\mathfrak{t}_\lambda = \Big\{ t \in \mathbb{R}^{\mathbb{L}} \;\Big|\; \|t\|^2_{\lambda,\mathfrak{t}} = \sum_{k\in\mathbb{L}}(1+|\lambda_k|)\,|t_k|^2 < \infty \Big\}.$$

Now consider the linear map $\beta\colon \mathfrak{t}_\lambda \to \mathbb{R}^n$ given by $\beta(e_k) = u_k$, where $e_i = (\delta_i^k)_{k\in\mathbb{L}}$, where $\delta_i^k \in \{0,1\}$ is Kronecker's delta. Supposing β is continuous then we define $\mathfrak{n}_\beta = \ker\beta \subset \mathfrak{t}_\lambda$. If $u_k \in \mathbb{Z}^n \subset \mathbb{R}^n$ for each $k \in \mathbb{L}$, we may define a Hilbert subgroup N_β of T_λ by

$$N_\beta = \ker(\exp\circ\beta\circ\exp^{-1}\colon T_\lambda \to T^n).$$

This gives exact sequences

$$0 \longrightarrow \mathfrak{n}_\beta \xrightarrow{\ \iota\ } \mathfrak{t}_\lambda \xrightarrow{\ \beta\ } \mathbb{R}^n \longrightarrow 0, \tag{6}$$

$$0 \longrightarrow N_\beta \longrightarrow T_\lambda \longrightarrow T^n \longrightarrow 0. \tag{7}$$

Our aim now is to construct hypertoric manifolds of dimension $4n$ as hyperKähler quotients of M_Λ by N_β.

Remark 4 The construction of Hattori [15] corresponds to $n = 1$ and $u_k = 1 \in \mathbb{R}$ for each k. For general dimension $4n$, Goto's construction [13] corresponds to $\mathbb{L} = (\mathbb{Z} \setminus \{0\}) \amalg \{1, \dots, n\}$,

$$\Lambda_k = \begin{cases} k\mathbf{i}, & \text{for } k \in \mathbb{Z}_{>0}, \\ k\mathbf{k}, & \text{for } k \in \mathbb{Z}_{<0}, \\ 0, & \text{for } k \in \{1, \dots, n\}, \end{cases} \qquad \text{with} \quad u_k = \begin{cases} \mathbf{e}_1, & \text{for } k \in \mathbb{Z} \setminus \{0\}, \\ \sum_{i=1}^{n} \mathbf{e}_i, & \text{for } k = 1 \in \{1, \dots, n\}, \\ -\mathbf{e}_r, & \text{for } k = r \in \{2, \dots, n\}. \end{cases}$$

Thus Goto's construction is for one concrete choice of $(\lambda_k)_{k\in\mathbb{L}}$ and only one of the u_k's is repeatedly infinitely many times.

Returning to the general situation, note that the integrality of u_k implies $\|u_k\| \geqslant 1$, so the convergence condition (5) implies

$$\sum_{k\in\mathbb{L}} (1 + |\lambda_k|)^{-1} < \infty. \tag{8}$$

The group T_λ acts on M_Λ via $gx = (g_k x_k)_{k\in\mathbb{L}}$: indeed for $g \in T_\lambda$ and $x = \Lambda + v \in M_\Lambda$, we have $gx = g\Lambda + gv = \Lambda - (1 - g)\Lambda + gv$, but $gv \in \mathbb{L}^2(\mathbb{H})$ and $\|(1 - g)\Lambda\|^2 = \sum_{k\in\mathbb{L}} \frac{1}{2}|\lambda_k||1 - g_k|^2 \leqslant \frac{1}{2}\sum_{k\in\mathbb{L}}(1 + |\lambda_k|)|1 - g_k|^2$, which is finite by the definition of T_λ, so $(1 - g)\Lambda \in \mathbb{L}^2(\mathbb{H})$ too. The action preserves the flat hyperKähler structure with $\mathbb{L}^2$-metric and complex structures obtained by regarding $\mathbb{L}^2(\mathbb{H})$ as a right $\mathbb{H}$-module. Identifying $\mathbb{R}^3$ with $\operatorname{Im}\mathbb{H} = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$, a corresponding hyperKähler moment map is given by $\langle \mu_\Lambda(x), t\rangle = \frac{1}{2}\sum_{k\in\mathbb{L}}(\overline{x}_k \mathbf{i} t_k x_k - \overline{\Lambda}_k \mathbf{i} t_k \Lambda_k)$. The terms $-\frac{1}{2}\overline{\Lambda}_k \mathbf{i} t_k \Lambda_k$ ensure that the sum in μ_Λ converges, but otherwise are arbitrary linear terms in t_k with values in $\operatorname{Im}\mathbb{H}$. With our definition of λ_k, we have

$$\mu_\Lambda(x) = \sum_{k\in\mathbb{L}} \bigl(\lambda_k + \tfrac{1}{2}\overline{x}_k \mathbf{i} x_k\bigr) e_k^*.$$

The hyperKähler moment map for the subgroup N_β is then $\mu_\beta = \iota^*\mu_\Lambda\colon M_\Lambda \to \operatorname{Im}\mathbb{H} \otimes \mathfrak{n}_\beta^*$. We define

$$M = M(\beta, \lambda) = M_\Lambda /\!/\!/ N_\beta = \mu_\beta^{-1}(0)/N_\beta.$$

Since (6) is exact, we have dually $\ker\iota^* = \operatorname{im}\beta^*$ and hence the following characterisation of $\mu_\beta^{-1}(0)$.

Lemma 5 *A point $x \in M_\Lambda$ lies in the zero set of the hyperKähler moment map μ_β for N_β if and only if there is an $a \in \operatorname{Im}\mathbb{H} \otimes (\mathbb{R}^n)^*$ with*

$$a(u_k) = \lambda_k + \tfrac{1}{2}\overline{x}_k \mathbf{i} x_k \tag{9}$$

for each $k \in \mathbb{L}$, where $u_k = \beta(e_k)$. □

In Eq. (9), note that $x_k = 0$ if and only if $a(u_k) = \lambda_k$. Indeed, the map $\mathbb{H} \to \operatorname{Im}\mathbb{H}$, $v \mapsto \overline{v}\mathbf{i}v$, is surjective with $|\overline{v}\mathbf{i}v| = |v|^2$. As in [6], we define affine subspaces $H_k \subset \operatorname{Im}\mathbb{H} \otimes (\mathbb{R}^n)^*$ of real codimension 3 by

$$H_k = H(u_k, \lambda_k) = \{a \in \operatorname{Im}\mathbb{H} \otimes (\mathbb{R}^n)^* \mid a(u_k) = \lambda_k\} \tag{10}$$

which we call *flats*. Note that $T^n = T_\lambda / N_\beta$ acts on M and that if M is smooth this action preserves the induced hyperKähler structure and has moment map $\phi\colon M \to \operatorname{Im}\mathbb{H} \otimes (\mathbb{R}^n)^*$ induced by μ_Λ: indeed $\operatorname{Lie} T^n = \mathfrak{t}_\lambda / \mathfrak{n}_\beta$ implies $(\operatorname{Lie} T^n)^* = (\mathfrak{n}_\beta)^0$, the annihilator of n_β in $\mathfrak{t}_\lambda^*$, so on $\mu_\beta^{-1}(0)$ the map μ_Λ takes values in $(\mathfrak{n}_\beta)^0 = (\operatorname{Lie} T^n)^*$ and descends to M as ϕ. It follows, as in [6], that ϕ induces a homeomorphism $M/T^n \to \operatorname{Im}\mathbb{H} \otimes (\mathbb{R}^n)^*$ and that for $p \in M$, the stabiliser $\operatorname{stab}_{T^n}(p)$ is the subtorus with Lie algebra spanned by the u_k such that $\phi(p) \in H_k$.

Theorem 6 *Suppose $u_k = \beta(e_k) \in \mathbb{Z}^n$, $k \in \mathbb{L}$, are primitive and span $\mathbb{R}^n$. Let $\lambda_k \in \operatorname{Im}\mathbb{H}$, $k \in \mathbb{L}$, be given such that the convergence condition* (5) *holds and the flats $H_k = H(u_k, \lambda_k)$, $k \in \mathbb{L}$, are distinct. Then the hyperKähler quotient $M = M(\beta, \lambda)$ is smooth if*

(a) *any set of $n + 1$ flats H_k has empty intersection, and*
(b) *whenever n distinct flats $H_{k(1)}, \dots, H_{k(n)}$ have non-empty intersection the corresponding vectors $u_{k(1)}, \dots, u_{k(n)}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.*

We break the proof into several steps.

Proposition 7 *Suppose $u_k \in \mathbb{Z}^n$, $k \in \mathbb{L}$, are primitive, span $\mathbb{R}^n$ and satisfy condition* (b) *of Theorem 6. Then $\mathcal{U} = \{u_k \mid k \in \mathbb{L}\}$ is finite.*

Proof Note that if $u_{k(1)}, \dots, u_{k(n)}$ are linearly independent then $\bigcap_{j=1}^n H_{k(j)}$ is a single point. Thus (b) implies that $\mathcal{U}$ contains a $\mathbb{Z}$-basis $v_1, \dots, v_n$ for $\mathbb{Z}^n$. Then the matrix A with columns $v_1, \dots, v_n$ is invertible with inverse in $M_n(\mathbb{Z})$, so A lies in $\mathrm{GL}(n, \mathbb{Z}) = \{B \in M_n(\mathbb{Z}) \mid \det B = \pm 1\}$. Multiplying with A^{-1} we may thus assume for the purpose of this proof that $\mathcal{U}$ contains the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_n$.

Suppose $u = (u_1, \dots, u_n) \in \mathcal{U}$ is different from $\mathbf{e}_i$, for all $i = 1, \dots, n$. For $j \in \{1, \dots, n\}$, consider the matrix A_j with columns $u, \mathbf{e}_1, \dots, \widehat{\mathbf{e}_j}, \dots, \mathbf{e}_n$, so $\mathbf{e}_j$ is omitted. We have $\det A_j = \pm u_j$. If $\det A_j$ is non-zero, then its columns are linearly independent and the discussion above gives $\det A_j = \pm 1$. It follows that $u_j \in \{-1, 0, 1\}$ for each $j \in \{1, \dots, n\}$. In particular, there are only finitely many such u's. □

It follows that under condition (b), the set $\{\|u_k\| \mid k \in \mathbb{L}\}$ is bounded, so (8) and (5) are equivalent. Now the map $\beta\colon \mathfrak{t}_\lambda \to \mathbb{R}^n$ has Riesz representation $\beta(t) = \langle t, \gamma\rangle_{\lambda,\mathfrak{t}} = \sum_{k\in\mathbb{L}} (1 + |\lambda_k|) t_k \gamma_k$ given by $\gamma_k = u_k/(1 + |\lambda_k|)$. As $\|\gamma\|^2_{\lambda,\mathfrak{t}} = \sum_{k\in\mathbb{L}} (1 + |\lambda_k|)^{-1} \|u_k\|^2$, boundedness of $\|u_k\|$ and (8) show that γ lies in $\mathfrak{t}_\lambda$. Thus β is continuous and it follows that N_β is a Hilbert subgroup of T_λ of codimension n.

Lemma 8 *If conditions* (a) *and* (b) *hold then the group N_β acts freely on $\mu_\beta^{-1}(0)$.*

Proof Given a subset $\mathbb{K} \subset \mathbb{L}$, we define the subgroup $T_{\mathbb{K}}$ subgroup of T_λ by

$$T_{\mathbb{K}} = \{(g_k)_{k\in\mathbb{L}} \in T_\lambda \mid g_\ell = 1 \ \forall \ell \notin \mathbb{K}\},$$

so that the Lie algebra of $T_{\mathbb{K}}$ is spanned by $\{e_k \mid k \in \mathbb{K}\}$. For $x \in M_\Lambda$, the stabiliser $\mathrm{stab}_{T_\lambda}(x)$ is $T_{\mathbb{K}}$ where $\mathbb{K} = \{k \mid x_k = 0\}$.

Now consider $x \in \mu_\beta^{-1}(0)$. Equation (9) implies that $x_k = 0$ if and only if $\phi(\pi(x)) \in H_k$, where $\pi\colon \mu_\beta^{-1}(0) \to M$ is the quotient map. Thus $\mathrm{stab}_{T_\lambda}(x) = T_{\mathbb{K}(x)}$, where $\mathbb{K}(x) = \{k|\phi(\pi(x)) \in H_k\}$. Condition (a) implies that $\mathbb{K}(x)$ contains at most n elements. The stabiliser of x under N_β consists of those elements in $\mathrm{stab}_{T_\lambda}(x)$ that lie in the kernel of the map $T_\lambda \to T^n$ induced by β. But $\beta(e_k) = u_k$ and implies that $(g_k)_{k\in\mathbb{L}} \in T_{\mathbb{K}(x)}$ maps to $h = \prod_{k\in\mathbb{K}(x)} g_k \exp(u_k) = \prod_{k\in\mathbb{K}(x)} \exp(i\theta_k u_k) \in T^n$, where $g_k = e^{i\theta_k}$.

Condition (b) implies that the u_k, $k \in \mathbb{K}(x)$, are part of a $\mathbb{Z}$-basis for $\mathbb{Z}^n$, so we may change basis via an element of $\mathrm{GL}(n, \mathbb{Z})$ so that the u_k become the first r basis elements. Then h becomes $\mathrm{diag}(e^{i\theta(1)}, \dots, e^{i\theta(r)}, 1, \dots, 1)$, where $\theta(j)$ is a corresponding relabelling of the θ_k's. It follows that $\theta_k \in 2\pi\mathbb{Z}$ and so $g_k = 1$ for each $k \in \mathbb{K}(x)$. Thus $\mathrm{stab}_{N_\beta}(x)$ is trivial, as claimed. □

As in [13], for $x \in M_\Lambda$, let $X_x\colon \mathfrak{t}_\lambda \to T_x M_\Lambda$ be the map sending an element to the corresponding tangent vector at x generated by the action:

$$X_x(t) = \frac{d}{ds}(\exp(st)x)\bigg|_{s=0} = \big(\mathbf{i}t_k x_k\big)_{k\in\mathbb{L}}.$$

Lemma 9 *For $x \in \mu_\beta^{-1}(0)$, the map X_x induces a linear homeomorphism from $\mathfrak{n}_\beta$ to the tangent space $T_x(N_\beta \cdot x)$ of the N_β-orbit through x.*

Proof For general $x\in M_\Lambda$, we have $\|X_x(t)\|^2 = \|(t_k x_k)_{k\in\mathbb{L}}\|^2 = \sum_{k\in\mathbb{L}} |t_k|^2 |\Lambda_k + v_k|^2$. Except for finitely many $k \in \mathbb{L}$, we have $|\Lambda_k| \geqslant 2|v_k|$, so for these k, we have $|\Lambda_k + v_k|^2 \leqslant |3\Lambda_k/2|^2 = 9|\lambda_k|/8$. It follows that there is a constant C_x, independent of t, such that $\|X_x(t)\| \leqslant C_x \|t\|_{\lambda,\mathfrak{t}}$. Thus $X_x\colon \mathfrak{t}_\lambda \to T_x M_\Lambda$ is continuous.

Now let $\mathbb{K}_1 = \mathbb{L} \setminus \{k \in \mathbb{L} \mid |\Lambda_k| \geqslant 1 \geqslant 2|v_k|\}$, which is a finite set by (8) and the condition that $v \in \mathbb{L}^2(\mathbb{H})$. For $k \notin \mathbb{K}_1$, we have $|\lambda_k| \geqslant 1/2$, so $|x_k|^2 = |\Lambda_k + v_k|^2 \geqslant |\Lambda_k/2|^2 = |\lambda_k|/8 \geqslant (1 + |\lambda_k|)/32$. It follows that for $k \notin \mathbb{K}(x) = \{k \mid x_k = 0\}$, there is a constant $c_x > 0$ such that $|t_k x_k|^2 \geqslant c_x(1 + |\lambda_k|)|t_k|^2$.

For $x \in \mu_\beta^{-1}(0)$, the set $\mathbb{K}(x)$ coincides with the previous definition $\mathbb{K}(x) = \{k \mid \phi(\pi(x)) \in H_k\}$ and so contains at most n elements. Let $V_x = \mathrm{Span}\{e_k \mid k \in \mathbb{K}(x)\} \leqslant \mathfrak{t}_\lambda$ and write $\mathrm{pr}^\perp\colon \mathfrak{t}_\lambda \to V_x^\perp$ for the orthogonal projection away from V_x. Then β is injective on V_x, so $\mathrm{pr}^\perp$ is a continuous linear bijection $\mathrm{pr}_\beta\colon \mathfrak{n}_\beta \to \mathrm{pr}^\perp(\mathfrak{n}_\beta)$. The image is the orthogonal complement to $V_x \oplus \beta^\dagger(\beta(V_x)^\perp)$, where $\beta(V_x)^\perp$ is the orthogonal complement in $\mathbb{R}^n$. As β is surjective, its adjoint $\beta^\dagger$ is injective, so $\mathrm{pr}^\perp(\mathfrak{n}_\beta)$ is of finite codimension and thus a Hilbert subspace of $\mathfrak{t}_\lambda$. By the Open Mapping Theorem, we conclude that pr_β^{-1} is continuous, and we note that its norm is non-zero.

Now for $x \in \mu_\beta^{-1}(0)$ and $t \in \mathfrak{n}_\beta$, we have

$$\|X_x(t)\|^2 = \|(t_k x_k)_{k\in\mathbb{L}}\|^2 \geqslant c_x \sum_{k\notin\mathbb{K}(x)} (1+|\lambda_k|)|t_k|^2 = c_x\|\mathrm{pr}_\beta(t)\|^2_{\lambda,\mathfrak{t}}$$
$$\geqslant \frac{c_x}{\|\mathrm{pr}_\beta^{-1}\|^2}\|t\|^2_{\lambda,\mathfrak{t}},$$

showing that X_x has continuous inverse on $T_x(N_\beta \cdot x)$. □

It now follows, as in [13], that for $x \in \mu_\beta^{-1}(0)$, the differential $d\mu_\beta\colon T_x M_\Lambda \to \operatorname{Im}\mathbb{H}\otimes\mathfrak{n}_\beta^*$ is split, with right inverse the $\mathbb{R}$-linear map given by $\mathbf{a}\otimes\delta = AX_x(t)$, where $\delta = \langle t,\cdot\rangle_{\lambda,\mathfrak{t}}$ and $A = a_1 I + a_2 J + a_3 K$ for $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. This implies that $\mu_\beta^{-1}(0)$ is a smooth Hilbert submanifold of M_Λ. On $\mu_\beta^{-1}(0)$, Goto's construction [13] of slices S_x goes through unchanged: one considers the map $F_x\colon \mu_\beta^{-1}(0) \to \mathfrak{n}_\beta^*$ given by $F_x(x+w)(t) = \langle w, X_x(t)\rangle$ and puts $S_x = F_x^{-1}(0)\cap U$, for a sufficiently small neighbourhood U of x. Thus $M(\beta,\lambda) = \mu_\beta^{-1}(0)/N_\beta$ is a smooth manifold.

Fix a point $q \in \operatorname{Im}\mathbb{H}\otimes(\mathbb{R}^n)^* \setminus \bigcup_{k\in\mathbb{L}} H_k$. For $x \in \mu_\beta^{-1}(0)$ with $\phi\pi(x) = q$, we have that $x_k \neq 0$ for all $k\in\mathbb{L}$. Thus T_λ acts freely on x_k. As $e_k \in \mathfrak{t}_\lambda$ for each $k\in\mathbb{L}$, it follows that $e_k \in F = T_x(T_\lambda\cdot x)$, and that $T_X M_\Lambda = F\oplus IF\oplus JF\oplus KF$. As β is surjective, we conclude that $F/T_x(N_\beta\cdot x)$ is of dimension n and that $M(\beta,\lambda) = \mu_\beta^{-1}(0)/N_\beta$ is of dimension $4n$.

We observe that the quotient is Hausdorff as follows. Suppose $g^{(i)}x \in \mu_\beta^{-1}(0)$, $g^{(i)} \in N_\beta$, is a sequence of points converging to $y \in \mu_\beta^{-1}(0)$. Lemma 9 gives a $c_x > 0$ such that $|x_k|^2 \geqslant c_x(1+|\lambda_k|)$ for each $x_k \notin \mathbb{K}(x) = \{k\in\mathbb{L} \mid x_k = 0\}$. Thus $\sum_{k\in\mathbb{L}}(1+|\lambda_k|)|g_k^{(i)} - g_k^{(j)}|^2 \leqslant c_x^{-1}\|g^{(i)}x - g^{(j)}x\|^2 + \sum_{k\in\mathbb{K}(x)}(1+|\lambda_k|)|g_k^{(i)} - g_k^{(j)}|^2$, for all i,j. Taking a subsequence of the $g^{(i)}$ so that g_k^i converges in S^1 for all $k\in\mathbb{K}(x)$, it follows that this subsequence is Cauchy in N_β and has a limit $g\in N_\beta$ with $gx = y$, as required.

The standard considerations of the hyperKähler quotient construction show that $M(\beta,\lambda)$ inherits a smooth hyperKähler structure, completing the proof of Theorem 6.

Just as in Hattori [15], one may use the T_λ action to show that different choices of $(\Lambda_k)_{k\in\mathbb{L}}$ yielding the same $(\lambda_k)_{k\in\mathbb{L}}$ result in hyperKähler structures that are isometric via a tri-holomorphic map.

2.4 *Classification of Complete Hypertoric Manifolds*

Now suppose that M is an arbitrary complete connected hypertoric manifold of dimension $4n$ and write $G = T^n$. Bielawski [5, §4] shows that locally M has much of the structure of the hypertoric manifolds constructed above.

Indeed for each $p \in M$, we may find a G-invariant neighbourhood of the form $U = G \times_H W$, where $H = \operatorname{stab}_G(p)$ and $W = T_p(G \cdot p)^\perp \subset T_x M$. Now H acts trivially on $W_1 = (\operatorname{Im}\mathbb{H})T_p(G \cdot p)$, and effectively as an Abelian subgroup of $\operatorname{Sp}(r)$ on the orthogonal complement $W_2 = W \cap W_1^\perp \cong \mathbb{H}^r$. Counting dimensions, it follows that H acts as T^r on W_2, and hence H is connected. The image of the singular orbits in U is a union of distinct flats $H_k = H(u_k, \lambda_k)$ as in (10), where u_k may be chosen to lie in $\mathbb{Z}^n \subset \mathbb{R}^n = \mathfrak{g}$ and be primitive vectors. The collection $\{u_k \mid \mu(p) \in H_k\}$ is then part of a $\mathbb{Z}$-basis for $\mathbb{R}^n$ spanning the Lie algebra of H. Furthermore, examining the structure of μ on such a neighbourhood U, Bielawski shows that μ induces a local homeomorphism $M/G \to \operatorname{Im}\mathbb{H} \otimes \mathfrak{g}^* \cong \mathbb{R}^{3n}$. In particular, the hyperKähler moment map $\mu\colon M \to \operatorname{Im}\mathbb{H} \otimes \mathfrak{g}^*$ is an open map.

Let $\{H_k \mid k \in \mathbb{L}\}$ be the collection of all flats that arise in this way. The index set $\mathbb{L}$ is finite or countably infinite, since M is second countable.

Lemma 10 *Suppose $\alpha \in (\mathbb{Z}^n)^* \subset (\mathbb{R}^n)^* = \mathfrak{g}^*$ is non-zero. Let T_α be the subtorus of $G = T^n$ whose Lie algebra is spanned by $\ker\alpha = \{u \in \mathfrak{g} \mid \alpha(u) = 0\}$. For $a \in \operatorname{Im}\mathbb{H} \otimes \mathfrak{g}^*$, write $[a]_\alpha = a + \operatorname{Im}\mathbb{H} \otimes \mathbb{R}\alpha$ for the equivalence class of a in $\operatorname{Im}\mathbb{H} \otimes (\mathfrak{g}^*/\mathbb{R}\alpha)$. Then except for countably many choices of $[a]_\alpha$, the group T_α acts freely on $\mu^{-1}([a]_\alpha)$.*

Note that $(\ker\alpha)^* = \mathfrak{g}^*/\mathbb{R}\alpha$.

Proof Consider the intersection $[a]_\alpha \cap H_k$. A general point of $[a]_\alpha$ is $a + q \otimes \alpha$, $q \in \operatorname{Im}\mathbb{H}$, which lies in $H_k = H(u_k, \lambda_k)$ only if $a(u_k) + q\alpha(u_k) = \lambda_k$. If $\alpha(u_k) \neq 0$ this equation has a unique solution for q; if $\alpha(u_k) = 0$ then there is a solution only if $a(u_k) = \lambda_k$ and then $[a]_\alpha \subset H_k$. Thus choosing $a(u_k) \neq \lambda_k$ for each $k \in \mathbb{L}$, ensures that $[a]_\alpha \cap H_k$ is empty for every k for which $u_k \in \ker\alpha$. It follows that T_α acts almost freely on $\mu^{-1}([a]_\alpha)$, but as each stabiliser of the T^n-action is connected, we find that T_α acts freely. □

Corollary 11 *For α, T_α and a as in Lemma 10, the hyperKähler quotient $M(a, \alpha) = \mu^{-1}([a]_\alpha)/T_\alpha$ is a complete hyperKähler manifold of dimension four. Furthermore, $M(a, \alpha)$ carries an effective tri-Hamiltonian circle action.*

Proof As T_α is compact and acts freely on $\mu^{-1}([a]_\alpha)$, it follows that $M(a, \alpha)$ is hyperKähler [17]. Completeness of M implies completeness of the level set $\mu^{-1}([a]_\alpha)$ and hence of the hyperKähler quotient.

As μ is T^n-invariant, the level set $\mu^{-1}([a]_\alpha)$ is preserved by T^n and we get an action of the circle T^n/T_α on the quotient. Identifying $[a]_\alpha$ with $\operatorname{Im}\mathbb{H}$ via $a + q \otimes \alpha \mapsto q$, the restriction of μ to $\mu^{-1}([a]_\alpha)$ descends to a hyperKähler moment map for this action. □

From the four-dimensional classification Theorem 2, we find have that the moment map of $M(a, \alpha)$ surjects on to $\operatorname{Im}\mathbb{H}$. Interpreting this in terms of the moment map μ of M, we have that $[a]_\alpha$ lies in the image of μ. But on M the moment map μ is an open map. And, as $\bigcup\{[a]_\alpha \mid a(u_k) \neq \lambda_k \forall k : u_k \in \ker\alpha\}$ is dense in $\operatorname{Im}\mathbb{H} \otimes \mathbb{R}^n$, we conclude that μ is a surjection.

Furthermore, the metric on $M(a, \alpha)$ is given by a potential of the form (3) up to an overall positive scale. The elements of Z are just the intersection points of $[a]_\alpha \cap H_k$, for $u_k \notin \ker \alpha$. These are the points $a_k = a + q_k \otimes \alpha$ with $q_k = (\lambda_k - a(u_k))/\alpha(u_k)$. Now for $p \in \mu^{-1}([a]_\alpha)$, writing $\mu(p) = a + q \otimes \alpha$, we have

$$\mu(p) - a_k = (q - q_k) \otimes \alpha = \frac{q\alpha(u_k) - (\lambda_k - a(u_k))}{\alpha(u_k)} \otimes \alpha$$
$$= (\langle \mu(p), u_k \rangle - \lambda_k) \otimes \frac{\alpha}{\alpha(u_k)}.$$

Thus the potential for $M(a, \alpha)$ is proportional to

$$V_\alpha(p) = c + \frac{1}{2} \sum_{k \in \mathbb{L}} \frac{1}{\| \langle \mu(p), u_k \rangle - \lambda_k \|} \frac{|\alpha(u_k)|}{\|\alpha\|},$$

where we may include the terms with $u_k \in \ker \alpha$, since they contribute zero, and $\|\alpha\|$ is the norm of α with respect to the standard inner product from the identification $\mathfrak{g} = \mathbb{R}^n$.

Using this inner product we may identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then the function

$$r_k(b) = \|b(u_k) - \lambda_k\| / \|u_k\| = \|b(\hat{u}_k) - \hat{\lambda}_k\|,$$

where $\hat{u}_k = u_k/\|u_k\|$ and $\hat{\lambda}_k = \lambda_k/\|u_k\|$, corresponds to the distance of b from the flat H_k. We may thus write

$$V_\alpha(p) = c + \frac{1}{2} \sum_{k \in \mathbb{L}} \frac{|\hat{\alpha}(\hat{u}_k)|}{r_k(\mu(p))}, \tag{11}$$

for $\hat{\alpha} = \alpha/\|\alpha\|$.

Now choose a so that $a(u_k) \neq \lambda_k$ for all $k \in \mathbb{L}$. Then for $p \in \mu^{-1}(a)$ we have that V_α in (11) is finite for each non-zero integral $\alpha \in \mathfrak{g}^*$. As each unit vector $\hat{u} \in \mathfrak{g} \cong \mathbb{R}^n$ has $\langle \hat{u}, \mathbf{e}_i \rangle \geqslant 1/\sqrt{n}$ for some $i \in \{1, \dots, n\}$, we conclude that $\sum_{k \in \mathbb{L}} 1/r_k(\mu(p))$ converges. In particular the distance of $\mu(p)$ to H_k is bounded below by a uniform constant. It follows that there is an open neighbourhood U of $\mu(p)$ in $\mu(M) \subset \operatorname{Im} \mathbb{H} \otimes \mathfrak{g}^*$ for which $a(u_k) \neq \lambda_k$ for all $a \in U$.

Let M_U be a connected component of $\mu^{-1}(U)$. Then T^n acts freely on M_U and the hyperKähler structure on M_U is uniquely determined via a polyharmonic function F on $U \subset \operatorname{Im} \mathbb{H} \otimes \mathfrak{g}^*$ as follows. The hyperKähler metric is of the form

$$g = \sum_{i,j=1}^{n} (V^{-1})_{ij} \beta_0^i \beta_0^j + V_{ij} (\alpha_I^i \alpha_I^j + \alpha_J^i \alpha_J^k + \alpha_K^i \alpha_K^j),$$

where for $X_1, \dots, X_n$ a basis for $\mathbb{R}^n = \mathfrak{g}$, we have $\alpha^i_A = X_i \lrcorner \omega_A$, $(V^{-1})_{ij} = (g(X_i, X_j))$ and $\beta^1_0, \dots, \beta^n_0$ are the $C^\infty(M_U)$-linear combinations of $g(X_1, \cdot), \dots, g(X_n, \cdot)$ such that $\beta^i_0(X_j) = \delta^i_j$. By a result of Pedersen and Poon [22] and the Legendre transform of Lindström and Roček [17, 20] the functions V_{ij} on $U \subset \operatorname{Im}\mathbb{H} \otimes \mathfrak{g}^*$ are *polyharmonic* meaning that they are harmonic on each affine subspace $a + \operatorname{Im}\mathbb{H} \otimes \mathbb{R}\alpha$, $\alpha \in \mathfrak{g}^* \setminus \{0\}$. Furthermore this matrix of functions is given by a single polyharmonic function $F\colon U \to \mathbb{R}$ via $V_{ij} = F_{x_i x_j}$, where we choose a unit vector $\mathbf{e} \in \operatorname{Im}\mathbb{H}$ and $(x_1, \dots, x_n)$ are standard coordinates on $\mathbb{R}^n = \mathbb{R}\mathbf{e} \otimes \mathfrak{g}^* \subset \operatorname{Im}\mathbb{H} \otimes \mathfrak{g}^*$. We write

$$s_k(b) = \langle \mathbf{e}, b(\hat{u}_k) - \hat{\lambda}_k \rangle.$$

As $\mathbf{e}$ acts on $\mathbf{e}^\perp \subset \operatorname{Im}\mathbb{H}$ as a complex structure, we may choose corresponding standard complex coordinates $(z_1, \dots, z_n)$ on $\mathbf{e}^\perp \otimes \mathfrak{g}^*$. A potential V of the form (3) is then $V = F_{xx}$ with

$$F(x, z) = \frac{1}{4} c(2x^2 - |z|^2) + \frac{1}{2} \sum_{k \in \mathbb{L}} (s_k \log(s_k + r_k) - r_k).$$

As in Bielawski [5], we now deduce that for M_U the function F has the form

$$F = \sum_{k \in \mathbb{L}} a_k (s_k \log(s_k + r_k) - r_k) + \sum_{i,j=1}^{n} c_{ij} (4 x_i x_j - z_i \bar{z}_j - z_j \bar{z}_i)$$

for some real constants a_k, c_{ij}. As Bielawski explains the c_{ij} terms are from a *Taub-NUT deformation* of a metric determined by the first sum, that is there is a hypertoric manifold M_2 and $M_U = M_2 \times (S^1 \times \mathbb{R}^3)^m /\!/\!/ T^m$ with T^m acting effectively on the product of S^1-factors, trivially on the $\mathbb{R}^3$-factors and as a subgroup of T^n on M_2. By analyticity, the hyperKähler metric on M_U determines the hyperKähler metric on M.

Using Bielawski's techniques and the computations of [6], one may now conclude that M_2 comes from the construction of the previous section. In particular, we note that the a_k's are bounded and convergence of $V_\alpha(p)$ in (11) for each non-zero integral α corresponds to the condition (5). To see this first we remark that convergence of the $V_\alpha(p)$'s corresponds to convergence of $R(b) = \sum_{k \in \mathbb{L}} 1/r_k(b)$ for $b = \mu(p)$: this follows from $|\hat{\alpha}(\hat{u}_k)| \leqslant 1$ and $\langle \hat{u}_k, \mathbf{e}_i \rangle \geqslant 1/\sqrt{n}$ for some i. Now $r_k(b) \leqslant |b(u_k)| + |\hat{\lambda}_k| \leqslant (1 + \|b\|)(1 + |\hat{\lambda}_k|)$ gives that convergence of $R(b)$ implies (5). Conversely, note that (5) implies that $|\hat{\lambda}_k| < 1 + 2\|b\|$ for only finitely many $k \in \mathbb{L}$. But for $|\hat{\lambda}_k| \geqslant 1 + 2\|b\|$, we have $r_k(b) \geqslant (|\hat{\lambda}_k| + 1)/2$, as in (4), so we get convergence of $R(b)$.

We have thus proved the following result.

Theorem 12 *Let M be a connected hypertoric manifold of dimension $4n$. Then M is a product $M = M_2 \times (S^1 \times \mathbb{R}^3)^m$ with M_2 a hypertoric manifold of the type constructed in Sect. 2.3, i.e., the hyperKähler quotient of a flat Hilbert hyperKähler* *manifold by an Abelian Hilbert Lie group. The hyperKähler metric in M is either the product hyperKähler metric or a Taub-NUT deformation of this metric.* □

From the proof and the construction of the previous section, we have the following properties of the hyperKähler moment map of in this situation.

Corollary 13 *If M is a connected complete hyperKähler manifold of dimension $4n$ with an effective tri-Hamiltonian action of T^n, then the hyperKähler moment map $\mu\colon M \to \operatorname{Im}\mathbb{H} \otimes (\mathbb{R}^n)^* \cong \mathbb{R}^{3n}$ is surjective with connected fibres.* □

3 Cuts and Modifications

Symplectic cutting was introduced by Lerman in 1995 [19]. The construction starts with a symplectic manifold with Hamiltonian circle action, and produces a new manifold of the same type, but with different topology. Explicitly, given M with circle action and associated moment map μ, we form the symplectic quotient at level ϵ,

$$M^{\epsilon}_{\mathrm{cut}} = (M \times \mathbb{C})/\!/_{\epsilon} S^1$$

where the S^1 is the antidiagonal of the product action obtained from the given action on M and the standard rotation on $\mathbb{C}$. Note that the moment map for the action on $\mathbb{C}$ is $\phi\colon z \mapsto |z|^2$.

The new space M_{cut} is of the same dimension as M and inherits a circle action from the diagonal action on $M \times \mathbb{C}$. Moreover, as

$$M_{\mathrm{cut}} = \{(m, z) \mid \mu(m) - |z|^2 = \epsilon\}/S^1$$

we see that the points $\{m \mid \mu(m) < \epsilon\}$ are removed. Moreover, because $\phi\colon z \mapsto |z|^2$ is a trivial circle fibration over $(0, \infty)$ with the circle fibre collapsing to a point at the origin, we see that the region $\{m \mid \mu(m) > \epsilon\}$ remains unchanged, while the hypersurface $\mu^{-1}(\epsilon)$ is collapsed by a circle action.

We can generalise this to the case of torus actions, by replacing $\mathbb{C}$ by a toric variety associated to a polytope Δ. The region $\mu^{-1}(\mathbb{R}^n \setminus (\Delta + \epsilon))$ will be removed, the preimage of the ϵ-translate of the interior of Δ is unchanged, while collapsing by tori takes place on the preimage of lower-dimensional faces of the translated polytope.

For general geometries, we want to mimic this construction by looking at the appropriate quotient of $M \times N$ by an Abelian group G, where N is a space whose reduction by G is a point. The topological change in M will be controlled by the geometry of the moment map for the G action on N.

The simplest example is that of a hyperKähler manifold with circle action. We now explain the hyperKähler analogue of a cut in this situation, which we call a *modification* [7]. The natural choice of N is now the quaternions $\mathbb{H}$. Several new features now emerge, because the hyperKähler moment map $\mu\colon \mathbb{H} \to \mathbb{R}^3$ has very different properties from that for $\mathbb{C}$. In particular, it is *surjective*, and is a *non-trivial* circle fibration over $\mathbb{R}^3 \setminus \{0\}$; in fact ϕ is given by the Hopf map on each sphere. This means that in forming the hyperKähler quotient $M_{\mathrm{mod}} = (M \times \mathbb{H})///S^1$ we do *not* discard any points in M (hence the use of the terminology modification rather than cut!). We still collapse the locus $\mu^{-1}(\epsilon)$ by a circle action, because ϕ is injective over the origin. The complements $M \setminus \mu^{-1}(\epsilon)$ and $M_{\mathrm{mod}} \setminus (\mu^{-1}(\epsilon)/S^1)$ can no longer be identified, because of the non-triviality of the Hopf fibration. Instead, the topology has been given a 'twist'. An example of this is if we start with $M = \mathbb{H}$. Now iterating the above construction generates the Gibbons-Hawking A_k multi-instanton spaces, where the spheres at large distance are replaced by lens spaces $S^3/\mathbb{Z}_{k+1}$.

We can make this more precise by observing that the space

$$M_1 = \{(m, q) \in M \times \mathbb{H} \mid \mu(m) - \phi(q) = \epsilon\}$$

projects onto both M and M_{mod}. The first map is just projection onto the first factor (onto as ϕ is surjective), while the second is just the quotient map $M_1 \to M_{\mathrm{mod}}$. Note that the first map is not quite a fibration: it has S^1 fibres generically but over $\mu^{-1}(\epsilon)$ the fibres collapse to a point.

On the open sets where both maps are fibrations, note that $M_1 \to M_{\mathrm{mod}}$ is a Riemannian submersion, but that the projection to first factor is not. As shown in [24], the metric $\tilde{g}$ induced on M by M_1 has the form

$$\tilde{g} = g + V(\mu) g_{\mathbb{H}}, \tag{12}$$

where $g_{\mathbb{H}} = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2$, with $\alpha_0 = g(X, \cdot)$, etc., and $V(\mu) = 1/2\|\mu - \epsilon\|$ is the potential of the flat hyperKähler metric on $\mathbb{H}$.

One may now generalise the hyperKähler modification, by replacing $\mathbb{H}$ by any hypertoric manifold N of dimension 4. The modification changes the metric as in (12) with $V(\mu)$ now the potential function of N, so one of the functions (3). In [24] it is proved that metric changes of the form (12) with V now an arbitrary smooth invariant function on M, so called 'elementary deformations', only lead to new hyperKähler metrics when V is $\pm V(\mu)$ for some hypertoric N^4. The case of negative V corresponds precisely to inverting a modification via a positive V.

For general torus actions one can take N to be a hypertoric manifold and we get a similar picture to that above.

4 Non-Abelian Moment Maps

One can also consider cutting constructions for non-Abelian group actions. Now, because the diagonal and anti-diagonal actions no longer commute, one considers the product of a K-manifold M with a space N with $K \times K$ action and then reduces by the antidiagonal action formed from the action on M and (say) the left action on N.

In the symplectic case, following Weitsman [25], with $K = \mathrm{U}(n)$ one can take $N = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ with $K \times K$ action $A \mapsto UAV^{-1}$. The moment map for the right action is $\mu: A \mapsto iA^*A$, with image Δ the set of non-negative Hermitian matrices. We have a picture that is quite reminiscent of the Abelian case, essentially because the fibres of ϕ are orbits of the *left* action. The right moment map ϕ gives a trivial fibration with fibres $\mathrm{U}(n)$ over the interior of the image, while over the lower-dimensional faces of Δ (corresponding to non-negative Hermitian matrices that are not strictly positive), the fibres are $\mathrm{U}(n)/\mathrm{U}(n-k)$ where k is the number of positive eigenvalues. This gives a nice non-Abelian generalisation of the toric cuts described above. To form the cut space we remove the complement of $\mu^{-1}(\Delta + \epsilon)$ and perform collapsing by the appropriate unitary groups on the preimages of the lower-dimensional faces of $\Delta + \epsilon$.

In the hyperKähler case life becomes more complicated. An obvious choice of N is $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)^*$, with action

$$(U, V): (A, B) \mapsto (UAV^{-1}, VBU^{-1}).$$

The hyperKähler moment map for the right $\mathrm{U}(n)$-action is now

$$\mu: (A, B) \mapsto (\frac{\mathbf{i}}{2}(A^*A - BB^*), BA) \in \mathrm{Im}\,\mathbb{H} \otimes \mathfrak{u}(n)^* \cong \mathfrak{u}(n) \oplus \mathfrak{gl}(n, \mathbb{C}).$$

In contrast to the symplectic case, or the Abelian hyperKähler case, the fibres of μ are no longer group orbits in general; in particular the left $U(n)_L$ action need not be transitive and indeed the quotient of a fibre by this action may have positive dimension. The result is that when we perform the non-Abelian hyperKähler modification, blowing up of certain loci occurs.

This is closely related to the phenomenon that the fibres of hyperKähler moment maps over *non-central* elements may have larger than expected dimension, even on the locus where the group action is free. This is because the kernel of the differential of a moment map μ is the orthogonal of $I\mathcal{G} + J\mathcal{G} + K\mathcal{G}$, where $\mathcal{G}$, as in Sect. 2.1, denotes the tangent space to the orbits of the group action. For the fibre over a central element this sum is direct (because $\mathcal{G}$ is tangent to the fibre so is orthogonal to the sum, and hence the three summands are mutually orthogonal), but over non-central elements this is no longer necessarily true. The dimension of the fibre is now no longer determined by the dimension of $\mathcal{G}$, and hence not determined by the dimension of the stabiliser. Note that in the Kähler situation the kernel of $d\mu$ is just the orthogonal of $I\mathcal{G}$, so here the dimension of the fibre is completely controlled by the dimension of the stabiliser, even in the non-Abelian case.

Another example of the unexpected behaviour of hyperKähler moment maps over non-central elements is the phenomenon of disconnected fibres. This is in contrast to symplectic moment maps, whose level sets enjoy many connectivity properties, see [23] and the references therein. As a simple hyperKähler example we may consider the $SU(2)$ action on $\mathbb{H}^2 = \mathbb{C}^2 \times (\mathbb{C}^2)^* = T^*\mathbb{C}^2$

$$A\colon (z, w) \mapsto (Az, wA^{-1}) \tag{13}$$

with hyperKähler moment map

$$\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})\colon (z, w) \mapsto \Big(\frac{\mathbf{i}}{2}(zz^\dagger - w^\dagger w)_0, (zw)_0\Big), \tag{14}$$

where $\cdot_0$ denotes trace-free part, and $\cdot^\dagger$ the conjugate transpose. (This calculation arose in discussions with S. Tolman.)

We are interested in finding the fibre of μ over $(\alpha, \beta) \in \mathfrak{su}(2) \oplus \mathfrak{sl}(2, \mathbb{C}) \cong \operatorname{Im}\mathbb{H} \otimes \mathfrak{su}(2)^*$. Using the $SU(2)$-equivariance, we may take

$$\beta = \begin{pmatrix} \lambda & \mu \\ 0 & -\lambda \end{pmatrix}.$$

We find that the fibre is:

1. empty or a disjoint pair of circles if λ, μ are both non-zero;
2. empty, a circle or a disjoint pair of circles if λ is zero and μ non-zero;
3. empty or a disjoint pair of circles if μ is zero and λ non-zero;
4. a disjoint pair of circles or a point if $\lambda = \mu = 0$ (the point fibre occurs exactly over the origin $\alpha = \beta = 0$).

Now let us turn to the case of SU(3). This acts on $T^*\mathbb{C}^3 = \mathbb{C}^3 \oplus (\mathbb{C}^3)^*$ via (13) and has the same formula (14) for the hyperKähler moment map $\mu\colon T^*\mathbb{C}^3 \to \operatorname{Im}\mathbb{H} \otimes \mathfrak{su}(3)^* \cong \mathfrak{su}(3) \oplus \mathfrak{sl}(3, \mathbb{C})$. In particular $\mu_{\mathbb{C}}(z, w)$ has off-diagonal (i, j)-entries $z_i w_j$ whilst the diagonal entries are of the form $2z_i w_i - \sum_{k \neq i} z_k w_k$. Similarly, for $j > i$, $(\mu_{\mathbb{R}}(z, w))_{ij} = \frac{\mathbf{i}}{2}(z_i \overline{z_j} - \overline{w_i} w_j)$ and

$$(\mu_{\mathbb{R}}(z, w))_{ii} = \frac{\mathbf{i}}{6}\Big\{2(|z_i|^2 - |w_i|^2) - \sum_{k \neq i}(|z_k|^2 - |w_k|^2)\Big\}.$$

We consider the fibre $\mu^{-1}(\alpha, \beta)$, for $\alpha \in \mathfrak{su}(3)$, $\beta \in \mathfrak{sl}(3, \mathbb{C})$. Note that $\dim_{\mathbb{C}} \mathfrak{sl}(3, \mathbb{C}) = 8$ is strictly greater than $\dim_{\mathbb{C}} T^*\mathbb{C}^3 = 6$ so there are now restrictions on β to lie in the image of $\mu_{\mathbb{C}}$. In particular, we see that μ is not surjective.

For more detail, note that the map μ is SU(3)-equivariant and we may use the action to put β in to the canonical upper triangular form

$$\beta = \begin{pmatrix} \lambda_1 & \xi_1 & \zeta \\ 0 & \lambda_2 & \xi_2 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 0$. This gives the three constraints

$$z_2 w_1 = 0, \quad z_3 w_1 = 0, \quad z_3 w_2 = 0. \tag{15}$$

Case 1

Both $\xi_i \neq 0$: we have $z_1 w_2 \neq 0 \neq z_2 w_3$ implying that z_1, z_2, w_2, w_3 are non-zero and so $\zeta \neq 0$. Equation (15) gives $w_1 = 0 = z_3$. This implies

$$\lambda_1 = \lambda_3 = -\tfrac{1}{2}\lambda_2 = -\tfrac{1}{3}z_2 w_2.$$

Now z_1 determines the remaining variables via

$$w_2 = \frac{\xi_1}{z_1}, \quad w_3 = \frac{\zeta}{z_1}, \quad z_2 = \frac{\xi_2}{w_3} = \frac{\xi_2}{\zeta} z_1$$

giving the relation

$$3\lambda_2 \xi_2 = 2\xi_1 \zeta,$$

so $\lambda_2 \neq 0$. Constraints on z_1 come from $\mu_{\mathbb{R}}$ and using the above relations they are seen to only involve linear combinations of $x = |z_1|^2$ and $1/x$. Thus there are at most 2 values for $|z_1|$. However, closer inspection reveals that the entries above the diagonal in $\mu_{\mathbb{R}}$ are

$$\begin{pmatrix} * & p|z_1|^2 & 0 \\ * & * & q/|z_1|^2 \\ * & * & * \end{pmatrix},$$

with $p = \overline{\xi_2/\zeta}$, $q = \overline{\zeta}\xi_1$. Thus $\alpha_{13} = 0$, $4\alpha_{12}\alpha_{23} = -pq = -\overline{\xi_2}\xi_1$ and there is at most one solution for $|z_1|$, which together with β specifies the diagonal entries of α. Thus the fibre is either a circle or empty.

Case 2

$\xi_1 \neq 0, \xi_2 = 0$: we have $z_1 \neq 0 \neq w_2$ and thus $z_3 = 0$. Now $z_2 w_1 = 0 = z_2 w_3$ divides into cases.

(a) $z_2 \neq 0, w_1 = 0 = w_3$: gives $\lambda_2 = \frac{2}{3}z_2 w_2 = -2\lambda_1 = -2\lambda_3 \neq 0$. $z_2 = 3\lambda_2/2w_2, z_1 = \xi_1/w_2$. In α, the $(1,2)$-entry determines $z_1\overline{z}_2 \neq 0$ which specifies $|w_2|$ uniquely. So the fibre is either a circle or empty.
(b) $w_1 \neq 0, z_2 = 0$: gives $\lambda_1 = \frac{2}{3}z_1 w_1 = -2\lambda_2 = -2\lambda_3 \neq 0$, $\zeta = z_1 w_3$. So $w_1 = 3\lambda/2z_1$, $w_2 = \xi_1/z_1$, $w_3 = \zeta/z_1$. The $(1,2)$ entry of α then specifies $|z_1|$ uniquely and the fibre is a single circle.
(c) $w_1 = 0 = z_2$: gives $\lambda_i = 0$ for all i and $\zeta = z_1 w_3$, so $w_2 = \xi_1/z_1$, $w_3 = \zeta/z_1$. For $\zeta \neq 0$, $|z_1|$ is determined by α_{23} and the fibre is a circle or empty. For $\zeta = 0$, the only non-zero entries are the diagonal ones, with a the first entry -2 times

the other two which are equal. These give a single quadratic equation for $|z_1|$, so the fibre is either 2 circles, 1 circle or empty. For example, the first diagonal entry in $\mu_{\mathbb{R}}$ is proportional to $2|z_1|^2 + |\xi_1|^2/|z_1|^2$, which attains any sufficiently large positive value at two different values of $|z_1|$. In particular, the fibre of μ can be disconnected.

Case 3

$\xi_1 = 0 = \xi_2 = \zeta$: there are three types of case:

(a) two z's non-zero: $z_1 \neq 0 \neq z_2$ implies $w \equiv 0$ and $\beta = 0$. The off-diagonal entries of α determine z_2 and z_3 in terms of z_1. If $\alpha_{23} \neq 0$, then this determines $|z_1|$ and the fibre is empty or a circle. Otherwise the diagonal entries lead to a quadratic constraint in $|z_1|$. The fibres are thus 2 circles, one circle or empty.

(b) one z and one w non-zero: then these must have the same index, say $z_1 \neq 0 \neq w_1$. So $w_2 = 0 = w_3 = z_2 = z_3$. β is diagonal with two repeated eigenvalues, $w_1 = -3\lambda_1/2z_1$. α is necessarily diagonal, the diagonal entries give a quadratic constraint on $|z_1|$. The fibres are 2 circles, one circle or empty.

(c) one z or w non-zero: gives $\beta = 0$, α diagonal with the sign of the entries in $i\alpha$ determined by whether it is z or w that is non-zero. The fibre is either a circle, a point or empty.

5 Implosion

Implosion arose as an abelianisation construction in symplectic geometry [14]. Given a Hamiltonian K-manifold M, one forms a new Hamiltonian space M_{impl} with an action of the maximal torus T of K, such that the symplectic reductions agree

$$M/\!/_{\lambda}K = M_{\mathrm{impl}}/\!/_{\lambda}T$$

for λ in the closed positive Weyl chamber. In most cases the implosion is a singular stratified space, even in M is smooth.

The key example is the implosion $(T^*K)_{\mathrm{impl}}$ of T^*K by (say) the right K action. Because T^*K has a $K \times K$ action, the implosion has a $K \times T$ action, and in fact we may implode a general Hamiltonian K-manifold by forming the reduction of $M \times (T^*K)_{\mathrm{impl}}$ by the diagonal K action. In this sense $(T^*K)_{\mathrm{impl}}$ is a universal example for imploding K-manifolds. Concretely, $(T^*K)_{\mathrm{impl}}$ is obtained from the product of K with the closed positive Weyl chamber $\bar{\mathfrak{t}}^*$ by stratifying by the face of the Weyl chamber (i.e. by the centraliser C of points in $\bar{\mathfrak{t}}^*$) and then collapsing by the commutator of C.

There is also a more algebraic description of the universal implosion $(T^*K)_{\mathrm{impl}}$, as the non-reductive Geometric Invariant Theory (GIT) quotient $K_{\mathbb{C}}/\!/N$, where N is the maximal unipotent subgroup. Note that in general non-reductive quotients need not exist as varieties, due to the possible failure of finite generation for the ring of

N-invariants. In the above case (and in the hyperKähler and holomorphic symplectic situation described below) it is a non-trivial result that we do have finite generation, so the quotient does exist as an affine variety.

In a series of papers [8–10] the authors and Kirwan described an analogue of implosion in hyperKähler geometry. There is a hyperKähler metric (due to Kronheimer [18]) with $K \times K$ action on the cotangent bundle $T^*K_{\mathbb{C}}$, and the idea is that the analogue of the universal symplectic implosion should be the complex-symplectic quotient (in the GIT sense) of $T^*K_{\mathbb{C}}$ by N. Explicitly, this quotient is $(K_{\mathbb{C}} \times \mathfrak{n}^\circ)//N$, which it is often convenient to identify with $(K_{\mathbb{C}} \times \mathfrak{b})//N$, where $\mathfrak{b}$ is the Borel subalgebra.

In the case of $K = SU(n)$ it was shown in [8] that $(K_{\mathbb{C}} \times \mathfrak{n}^\circ)//N$ arises, via a quiver construction, as a hyperKähler quotient with residual $K \times T$ action. Moreover the hyperKähler quotients by T may be identified with the Kostant varieties, that is the subvarieties of $\mathfrak{k}_{\mathbb{C}}$ obtained by fixing the values of a generating set of invariant polynomials. For example, reducing at zero gives the nilpotent variety. For general semi-simple K, results of Ginzburg and Riche [12] give the existence of the complex-symplectic quotient $(K_{\mathbb{C}} \times \mathfrak{n}^\circ)//N$ as an affine variety, and the complex-symplectic quotients by the $T_{\mathbb{C}}$ action again give the Kostant varieties.

There is a link here with some intriguing work by Moore and Tachikawa [21]. They propose a category HS whose objects are complex semi-simple groups, and where elements of $\mathrm{Mor}(G_1, G_2)$ are complex-symplectic manifolds with $G_1 \times G_2$ action (together with a commuting circle action that acts on the complex-symplectic form with weight -2). We compose morphisms $X \in \mathrm{Mor}(G_1, G_2)$ and $Y \in \mathrm{Mor}(G_2, G_3)$ by taking the complex-symplectic reduction of $X \times Y$ by the diagonal G_2 action. The Kronheimer space T^*G, where $G = K_{\mathbb{C}}$, gives a canonical element of $\mathrm{Mor}(G, G)$—in fact this functions as the identity in $\mathrm{Mor}(G, G)$. The implosion now may be viewed as giving an element of $\mathrm{Mor}(G, T_{\mathbb{C}})$, where $G = K_{\mathbb{C}}$ and $T_{\mathbb{C}}$ is the complex maximal torus in G.

Acknowledgements Andrew Swann partially supported by the Danish Council for Independent Research, Natural Sciences. We thank Sue Tolman for discussions about the disconnected fibres of hyperKähler moment maps.

References

1. D.V. Alekseevskiĭ, B.N. Kimel'fel'd, Structure of homogeneous Riemannian spaces with zero Ricci curvature. Funktsional. Anal. i Prilozhen. **9**(2), 5–11 (1975). English translation [2]
2. D.V. Alekseevskiĭ, B.N. Kimel'fel'd, Structure of homogeneous Riemannian spaces with zero Ricci curvature. Funct. Anal. Appl. **9**(2), 97–102 (1975)
3. M.T. Anderson, P.B. Kronheimer, C. LeBrun, Complete Ricci-flat Kähler manifolds of infinite topological type. Commun. Math. Phys. **125**(4), 637–642 (1989)
4. S. Axler, P. Bourdon, W. Ramey, Harmonic function theory, in *Graduate Texts in Mathematics*, vol. 137, 2nd edn. (Springer, New York, 2001)
5. R. Bielawski, Complete hyper-Kähler $4n$-manifolds with a local tri-Hamiltonian $\mathbb{R}^n$-action. Math. Ann. **314**(3), 505–528 (1999)

6. R. Bielawski, A.S. Dancer, The geometry and topology of toric hyperkähler manifolds. Commun. Anal. Geom. **8**(4), 727–760 (2000)
7. A.S. Dancer, A.F. Swann, Modifying hyperkähler manifolds with circle symmetry. Asian J. Math. **10**(4), 815–826 (2006)
8. A.S. Dancer, F. Kirwan, A.F. Swann, Implosion for hyperkähler manifolds. Compos. Math. **149**, 1592–1630 (2013)
9. A.S. Dancer, F. Kirwan, A.F. Swann, Implosions and hypertoric geometry. J. Ramanujan Math. Soc. **28A**, 81–122 (2013). Special issue for Professor Seshadri's 80th birthday
10. A. Dancer, F. Kirwan, A.F. Swann, Twistor spaces for hyperkähler implosions. J. Differ. Geom. **97**(1), 37–77 (2014)
11. G.W. Gibbons, S.W. Hawking, Gravitational multi-instantons. Phys. Lett. **B78**, 430–432 (1978)
12. V. Ginzburg, S. Riche, Differential operators on G/U and the affine Grassmannian. J. Inst. Math. Jussieu **14**(3), 493–575 (2015)
13. R. Goto, On hyper-Kähler manifolds of type A_∞. Geom. Funct. Anal. **4**, 424–454 (1994)
14. V. Guillemin, L. Jeffrey, R. Sjamaar, Symplectic implosion. Transform. Groups **7**(2), 155–184 (2002)
15. K. Hattori, The volume growth of hyper-Kähler manifolds of type A_∞. J. Geom. Anal. **21**(4), 920–949 (2011)
16. S.W. Hawking, Gravitational instantons. Phys. Lett. A **60**(2), 81–83 (1977)
17. N.J. Hitchin, A. Karlhede, U. Lindström, M. Roček, HyperKähler metrics and supersymmetry. Commun. Math. Phys. **108**, 535–589 (1987)
18. P.B. Kronheimer, A hyperKähler structure on the cotangent bundle of a complex Lie group (1986, preprint). arXiv:math.DG/0409253
19. E. Lerman, Symplectic cuts. Math. Res. Lett. **2**(3), 247–258 (1995)
20. U. Lindsträm, M. Roček, Scalar tensor duality and $N = 1, 2$ nonlinear σ-models. Nuclear Phys. B **222**(2), 285–308 (1983)
21. G.W. Moore, Y. Tachikawa, On 2d TQFTs whose values are holomorphic symplectic varieties, in *Proceedings of Symposium on Pure Mathematics*. String-Math 2011, vol. 85 (American Mathematical Society, Providence, RI, 2012), pp. 191–207
22. H. Pedersen, Y.S. Poon, Hyper-Kähler metrics and a generalization of the Bogomolny equations. Commun. Math. Phys. **117**, 569–580 (1988)
23. R. Sjamaar, Convexity properties of the moment mapping re-examined. Adv. Math. **138**(1), 46–91 (1998)
24. A.F. Swann, Twists versus modifications. Adv. Math. **303**, 611–637 (2016)
25. J. Weitsman, Non-abelian symplectic cuts and the geometric quantization of noncompact manifolds. Lett. Math. Phys. **56**(1), 31–40 (2001). EuroConférence Moshé Flato 2000, Part I (Dijon)

Harmonic Almost Hermitian Structures

Johann Davidov

Abstract This is a survey of old and new results on the problem when a compatible almost complex structure on a Riemannian manifold is a harmonic section or a harmonic map from the manifold into its twistor space. In this context, special attention is paid to the Atiyah-Hitchin-Singer and Eells-Salamon almost complex structures on the twistor space of an oriented Riemannian four-manifold.

Keywords Almost complex structures • Harmonic maps • Twistor spaces

2010 Mathematics Subject Classification Primary 53C43, Secondary 58E20, 53C28

1 Introduction

Recall that an almost complex structure on a Riemannian manifold (N, h) is called almost Hermitian (or, compatible) if it is h-orthogonal. If a Riemannian manifold admits an almost Hermitian structure, it possesses many such structures (cf. Sect. 3). Thus, it is natural to look for "reasonable" criteria that distinguish some of these structures. A natural way to obtain such criteria is to consider the almost Hermitian structures on (N, h) as sections of its twistor bundle $\pi : \mathcal{T} \to N$ whose fibre at a point $p \in N$ consists of all h-orthogonal complex structures $J_p : T_pN \to T_pN$ on the tangent space of N at p. The fibre of the bundle $\mathcal{T}$ is the compact Hermitian symmetric space $O(2m)/U(m)$, $2m = dim\, N$, and its standard metric $-\frac{1}{2} Trace\, J_1 \circ J_2$ is Kähler-Einstein. The twistor space $\mathcal{T}$ admits a natural Riemannian metric h_1 such that the projection map $\pi : (\mathcal{T}, h_1) \to (N, h)$ is a Riemannian submersion with totally geodesic fibres. This metric is compatible with the natural almost complex structures on $\mathcal{T}$, which have been introduced by Atiyah-Hitchin-Singer [2] and Eells-Salamon [16] in the case $dim\, N = 4$.

J. Davidov (✉)
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G.Bonchev Str. Bl.8, 1113 Sofia, Bulgaria
e-mail: jtd@math.bas.bg

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_6

If N is oriented, the twistor space $\mathcal{T}$ has two connected components often called positive and negative twistor spaces of (N, h); their sections are almost Hermitian structures yielding the orientation and, respectively, the opposite orientation of N.

Calabi and Gluck [6] have proposed to single out those almost Hermitian structures J on (N, h), whose image $J(N)$ in $\mathcal{T}$ is of minimal volume with respect to the metric h_1. They have proved that the standard almost Hermitian structure on the 6-sphere S^6, defined by means of the Cayley numbers, can be characterized by that property.

Motivated by harmonic map theory, Wood [34, 35] has suggested to consider as "optimal" those almost-Hermitian structures $J : (N, h) \to (\mathcal{T}, h_1)$, which are critical points of the energy functional under variations through sections of $\mathcal{T}$. In general, these critical points are not harmonic maps, but, by analogy, in [35] they are referred to as "harmonic almost complex structures". They are also called "harmonic sections" [34], a term, which is more appropriate in the context of this article.

Forgetting the bundle structure of $\mathcal{T}$, we can consider almost Hermitian structures that are critical points of the energy functional under variations through all maps $N \to \mathcal{T}$. These structures are genuine harmonic maps from (N, h) into $(\mathcal{T}, h_1)$ and we refer to [15] for basic facts about such maps.

The main goal of this paper is to survey results about the harmonicity (in both senses) of the Atiyah-Hitchin-Singer and Eells-Salamon almost Hermitian structures on the twistor space of an oriented four-dimensional Riemannian manifold, as well as almost Hermitian structures on such a manifold.

In Sect. 2 we recall some basic facts about the twistor spaces of even-dimensional Riemannian manifolds. Special attention is paid to the twistor spaces of oriented four-dimensional manifolds. In Sects. 3 and 4 we discuss the energy functional on sections of a twistor space, i.e. almost Hermitian structures on the base Riemannian manifold. We state the Euler-Lagrange equation for such a structure to be a critical point of the energy functional (a harmonic section) obtained by Wood [34, 35]. Several examples of non-Kähler almost Hermitian structures, which are harmonic sections are given. Kähler structures are absolute minima of the energy functional. Bor et al. [4] have given sufficient conditions for an almost Hermitian structure to be a minimizer of the energy functional. Their result (in fact, part of it) is presented in Sect. 4 and is used to supply examples of non-Kähler minimizers based on works by LeBrun[26] and Kim [23]. Section 4 ends with a lemma from [10], which rephrases the Euler-Lagrange equation for an almost Hermitian structure (h, J) on a manifold N in terms of the fundamental 2-form of (h, J) and the curvature of (N, h). This lemma has been used in [10] to show that the Atiyah-Hitchin-Singer almost Hermitian structure $\mathcal{J}_1$ on the negative twistor space $\mathcal{Z}$ of an oriented Riemannian 4-manifold (M, g) is a harmonic section if and only if the base manifold (M, g) is self-dual, while the Eells-Salamon structure $\mathcal{J}_2$ is a harmonic section if and only (M, g) is self-dual and of constant scalar curvature. The main part of the proof of this result (slightly different from the proof in [10]) is presented in Sect. 5. In this context, it is natural to ask when $\mathcal{J}_1$ and $\mathcal{J}_2$ are harmonic maps into the twistor space of $\mathcal{Z}$. Recall that a map between Riemannian manifolds is harmonic if and only if the trace of its second fundamental form vanishes. Section 6 contains a computation of the second fundamental form of a map from a Riemannian manifold (N,h) into its twistor space $(\mathcal{T},h_1)$. The corresponding formula obtained in this section is used in Sect. 7 to give an answer to the question above: $\mathcal{J}_1$ or $\mathcal{J}_2$ is a harmonic map if and only if (M,g) is either self-dual and Einstein, or locally is the product of an open interval on $\mathbb{R}$ and a 3-dimensional Riemannian manifold of constant curvature. A sketch of the proof, involving the main theorem of Sect. 5 and several technical lemmas, is given in Sect. 7 following [11]. In Sect. 8 we give geometric conditions on a four-dimensional almost Hermitian manifold (M,g,J) under which the almost complex structure J is a harmonic map of (M,g) into the positive twistor space $(\mathcal{Z}_+,h_1)$, M being considered with the orientation induced by J. We also find conditions for minimality of the submanifold $J(M)$ of the twistor space $\mathcal{Z}_+$. As is well-known, in dimension four, there are three basic classes in the Gray-Hervella classification [18] of almost Hermitian structures: Hermitian, almost Kähler (symplectic) and Kähler structures. As for a manifold of an arbitrary dimension, if (g,J) is Kähler, the map $J:(M,g)\to(\mathcal{Z}_+,h_1)$ is a totally geodesic isometric imbedding. In the case of a Hermitian structure, we express the conditions for harmonicity or minimality of J in terms of the Lee form, the Ricci and star-Ricci tensors of (M,g,J), while for an almost Kähler structure the conditions are in terms of the Ricci, star-Ricci and Nijenhuis tensors. Several examples illustrating these results are discussed at the end of Sect. 8, among them a Hermitian structure that is a harmonic section of the twistor bundle $\mathcal{Z}_+$ and a minimal isometric imbedding in it, but not a harmonic map.

2 The Twistor Space of an Even-Dimensional Riemannian Manifold

Denote by $F(\mathbb{R}^{2m})$ the set of complex structures on $\mathbb{R}^{2m}$ compatible with its standard metric g. This set has the structure of an imbedded submanifold of the vector space $so(2m)$ of skew-symmetric endomorphisms of $\mathbb{R}^{2m}$. The tangent space of $F(\mathbb{R}^{2m})$ at a point J consists of all skew-symmetric endomorphisms of $\mathbb{R}^{2m}$ anti-commuting with J. Then we can define an almost complex structure on the manifold $F(\mathbb{R}^{2m})$ setting

$$\mathcal{J}Q=JQ \quad \text{for} \quad Q\in T_JF(\mathbb{R}^{2m}).$$

This almost complex structure is compatible with the standard metric

$$G(A,B)=-\frac{1}{2m}Trace\,AB$$

of $so(2m)$, where the factor $1/2m$ is chosen so that every $J\in F(\mathbb{R}^{2m})$ should have unit norm. In fact, the almost Hermitian structure $(G,\mathcal{J})$ is Kähler-Einstein.

The group $O(2m)$ acts on $F(\mathbb{R}^{2m})$ by conjugation and the isotropy subgroup at the standard complex structure J_0 of $\mathbb{R}^{2m} \cong \mathbb{C}^m$ is $U(m)$. Therefore $F(\mathbb{R}^{2m})$ can be identified with the homogeneous space $O(2m)/U(m)$.

Note also that the manifold $F(\mathbb{R}^{2m})$ has two connected components: if we fix an orientation on $\mathbb{R}^{2m}$, these components consists of all complex structures on $\mathbb{R}^{2m}$ compatible with the metric g and inducing $\pm$ the orientation of $\mathbb{R}^{2m}$; each of them has the homogeneous representation $SO(2m)/U(m)$.

The twistor space of an even-dimensional Riemannian manifold (N, h), $dim\, N = 2m$, is the bundle $\pi : \mathcal{T} \to N$, whose fibre at every point $p \in N$ is the space of compatible complex structures on the Euclidean vector space (T_pN, h_p). This is the associated bundle

$$\mathcal{T} = O(N) \times_{O(2m)} F(\mathbb{R}^{2m})$$

where $O(N)$ is the principal bundle of orthonormal frames on N. Since the bundle $\pi : \mathcal{T} \to N$ is associated to $O(N)$, the Levi-Civita connection on (N, h) gives rise to a splitting $\mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of the manifold $\mathcal{T}$ into vertical and horizontal subbundles. Using this splitting, we can define a natural 1-parameter family of Riemannian metrics h_t, $t > 0$, as follows. For every $J \in \mathcal{T}$, the horizontal subspace $\mathcal{H}_J$ of $T_J\mathcal{T}$ is isomorphic to the tangent space $T_{\pi(J)}N$ via the differential π_{*J} and the metric h_t on $\mathcal{H}_J$ is the lift of the metric h on $T_{\pi(J)}N$, $h_t|\mathcal{H}_J = \pi^*h$. The vertical subspace $\mathcal{V}_J$ of $T_J\mathcal{T}$ is the tangent space at J to the fibre through J of the bundle $\mathcal{T}$ and $h_t|\mathcal{V}_J$ is defined as t times the metric G of this fibre. Finally, the horizontal space $\mathcal{H}_J$ and the vertical space $\mathcal{V}_J$ are declared to be orthogonal. Then, by the Vilms theorem [33], the projection $\pi : (\mathcal{T}, h_t) \to (N, h)$ is a Riemannian submersion with totally geodesic fibres (this can also be proved directly).

It is often convenient to consider $\mathcal{T}$ as a submanifold of the bundle

$$\pi : A(TN) = O(N) \times_{O(2m)} so(2m) \to N$$

of skew-symmetric endomorphisms of TN. The inclusion of $\mathcal{T}$ into $A(TN)$ is fibre-preserving and for every $J \in \mathcal{T}$ the horizontal subspace $\mathcal{H}_J$ of $T_J\mathcal{T}$ coincides with the horizontal subspace of $T_JA(TN)$ with respect to the connection induced by the Levi-Civita connection of (N, h) since the inclusion of $F(\mathbb{R}^{2m})$ into $so(2m)$ is $O(2m)$-equivariant; the vertical subspace $\mathcal{V}_J$ of $T_J\mathcal{T}$ is the subspace of the fibre $A(T_{\pi(J)}N)$ of $A(TN)$ through J consisting of the skew-symmetric endomorphisms of $T_{\pi(J)}N$ anti-commuting with J.

If the manifold N is oriented, its twistor space has two connected components, the spaces of compatible complex structures on tangent spaces of N yielding the given, or the opposite orientation of N. These are often called the positive, respectively, the negative twistor space.

2.1 The Twistor Space of an Oriented Four-Dimensional Riemannian Manifold

In dimension four, each of the two connected components $F(\mathbb{R}^4)$ can be identified with the unit sphere S^2. It is often convenient to describe this identification in terms of the space $\Lambda^2\mathbb{R}^4$. The metric g of $\mathbb{R}^4$ induces a metric on $\Lambda^2\mathbb{R}^4$ given by

$$g(x_1 \wedge x_2, x_3 \wedge x_4) = \frac{1}{2}[g(x_1, x_3)g(x_2, x_4) - g(x_1, x_4)g(x_2, x_3)], \tag{1}$$

the factor $1/2$ being chosen in consistency with [9, 10]. Consider the isomorphism $so(4) \cong \Lambda^2\mathbb{R}^4$ sending $\varphi \in so(4)$ to the 2-vector $\varphi^\wedge$ for which

$$2g(\varphi^\wedge, x \wedge y) = g(\varphi x, y), \quad x, y \in \mathbb{R}^4.$$

This isomorphism is an isometry with respect to the metric G on $so(4)$ and the metric g on $\Lambda^2\mathbb{R}^4$. Given $a \in \Lambda^2\mathbb{R}^4$, the skew-symmetric endomorphism of $\mathbb{R}^4$ corresponding to a under the inverse isomorphism will be denoted by K_a.

Fix an orientation on $\mathbb{R}^4$ and denote by $F_\pm(\mathbb{R}^4)$ the set of complex structures on $\mathbb{R}^4$ compatible with the metric g and inducing $\pm$ the orientation of $\mathbb{R}^4$.

The Hodge star operator defines an endomorphism $*$ of $\Lambda^2\mathbb{R}^4$ with $*^2 = Id$. Hence we have the orthogonal decomposition

$$\Lambda^2\mathbb{R}^4 = \Lambda^2_-\mathbb{R}^4 \oplus \Lambda^2_+\mathbb{R}^4,$$

where $\Lambda^2_\pm\mathbb{R}^4$ are the subspaces of $\Lambda^2\mathbb{R}^4$ corresponding to the (± 1)-eigenvalues of the operator $*$. Let (e_1, e_2, e_3, e_4) be an oriented orthonormal basis of $\mathbb{R}^4$. Set

$$s_1^\pm = e_1 \wedge e_2 \pm e_3 \wedge e_4, \quad s_2^\pm = e_1 \wedge e_3 \pm e_4 \wedge e_2, \quad s_3^\pm = e_1 \wedge e_4 \pm e_2 \wedge e_3. \tag{2}$$

Then $(s_1^\pm, s_2^\pm, s_3^\pm)$ is an orthonormal basis of $\Lambda^2_\pm\mathbb{R}^4$.

It is easy to see that the isomorphism $\varphi \to \varphi^\wedge$ identifies $F_\pm(\mathbb{R}^4)$ with the unit sphere $S(\Lambda^2_\pm\mathbb{R}^4)$ of the Euclidean vector space $(\Lambda^2_\pm\mathbb{R}^n, g)$. Under this isomorphism, if $J \in F_\pm(\mathbb{R}^4)$, the tangent space $T_J F(\mathbb{R}^4) = T_J F_\pm(\mathbb{R}^4)$ is identified with the orthogonal complement $(\mathbb{R}J)^\perp$ of the space $\mathbb{R}J$ in $\Lambda^2_\pm\mathbb{R}^4$.

Lemma 1 *The orientation on $\Lambda^2_\pm\mathbb{R}^4$ determined by the basis $s_1^\pm, s_2^\pm, s_3^\pm$ defined by means of an oriented orthonormal basis $\{e_1, \dots, e_4\}$ of $\mathbb{R}^4$ does not depend on the choice of $\{e_1, \dots, e_4\}$.*

Proof Let $\{s_i' = s_i'^+\}$ and $\{s_i = s_i^+\}$ be the bases of $\Lambda^2_+\mathbb{R}^4$ defined by means of two oriented orthonormal bases $\{e_1', \dots, e_4'\}$ and $\{e_1, \dots, e_4\}$ of $\mathbb{R}^4$. Denote by $A \in SO(4)$ the transition matrix between these bases. Thanks to L. van Elfrikhof (1897), it is well-known that every matrix A in $SO(4)$ can be represented as the product $A = A_1A_2$ of two $SO(4)$-matrices of the following types

$$A_1 = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}, \quad A_2 = \begin{pmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{pmatrix}, \tag{3}$$

where $a, \dots, d, p, \dots, s$ are real numbers with $a^2 + b^2 + c^2 + d^2 = 1$, $p^2 + q^2 + r^2 + s^2 = 1$ (isoclinic representation). For an endomorphism L of $\mathbb{R}^4$, denote by Λ_L the induced endomorphism on $\Lambda^2\mathbb{R}^4$ defined by $\Lambda_L(X \wedge Y) = L(X) \wedge L(Y)$. Denote again by A the isomorphism of $\mathbb{R}^4$ with matrix A with respect to the basis $e_1, \dots, e_4$. Then $s_i' = \Lambda_A(s_i) = \Lambda_{A_1} \circ \Lambda_{A_2}(s_i)$, $i = 1, 2, 3$. One easily computes that $\Lambda_{A_2}(s_i) = s_i$, and that $\Lambda_{A_1}(s_i)$ is a basis of $\Lambda^2_+\mathbb{R}^4$ whose transition matrix is

$$\begin{pmatrix} a^2 + b^2 - (c^2 + d^2) & -2(ad - bc) & 2(ac + bd) \\ 2(ad + bc) & a^2 + c^2 - (b^2 + d^2) & -2(ab - cd) \\ -2(ac - bd) & 2(ab + cd) & a^2 + d^2 - (b^2 + c^2) \end{pmatrix}.$$

The determinant of the latter matrix is $(a^2 + b^2 + c^2 + d^2)^3 = 1$. This proves the statement for $\Lambda^2_+\mathbb{R}^4$. Changing the orientation of $\mathbb{R}^4$ interchanges the roles of $\Lambda^2_+\mathbb{R}^4$ and $\Lambda^2_-\mathbb{R}^4$. Therefore, the statement holds for $\Lambda^2_-\mathbb{R}^4$ as well.

The orientation of $\Lambda^2_\pm$ induced by a basis $\{s_1^\pm, s_2^\pm, s_3^\pm\}$ will be called "canonical".

Remark The map assigning the coset of the matrix above in $SO(3)/SO(2) = S^2$ to the unit quaternion $q = a + ib + jc + kd$ is the Hopf map $S^3 \to S^2$.

Consider the 3-dimensional Euclidean space $(\Lambda^2_\pm\mathbb{R}^4, g)$ with its canonical orientation and denote by $\times$ the usual vector-cross product on it. Then, if $a, b \in \Lambda^2_\pm\mathbb{R}^4$, the isomorphism $\Lambda^2\mathbb{R}^4 \cong so(4)$ sends $a \times b$ to $\pm\frac{1}{2}[K_a, K_b]$. Thus, if $J \in F_\pm(\mathbb{R}^4)$ and $Q \in T_JF(\mathbb{R}^4) = T_JF_\pm(\mathbb{R}^4)$, we have

$$(\mathcal{J}Q)^\wedge = \pm(J^\wedge \times Q^\wedge). \tag{4}$$

Now let (M, g) be an oriented Riemannian manifold of dimension four.

According to the considerations above, the twistor space of such a manifold has two connected components, which can be identified with the unit sphere subbundles $\mathcal{Z}_\pm$ of the bundles $\Lambda^2_\pm TM \to M$, the eigensubbundles of the bundle $\pi : \Lambda^2 TM \to M$ corresponding to the eigenvalues ± 1 of the Hodge star operator. These are the positive and the negative twistor spaces of (M, g). If $\sigma \in \mathcal{Z}_\pm$, then K_σ is a complex structure on the vector space $T_{\pi(\sigma)}M$ compatible with the metric g and $\pm$ the orientation of M. Note that, since $g(K_\sigma X, Y) = 2g(\sigma, X \wedge Y)$, the 2-vector 2σ is dual to the fundamental 2-form of (g, K_σ).

The Levi-Civita connection ∇ of M preserves the bundles $\Lambda^2_\pm TM$, so it induces a metric connection on each of them denoted again by ∇. The horizontal distribution of $\Lambda^2_\pm TM$ with respect to ∇ is tangent to the twistor space $\mathcal{Z}_\pm$.

The manifold $\mathcal{Z}_\pm$ admits two almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ compatible with each metric h_t introduced, respectively, by Atiyah-Hitchin-Singer [2], and Eells-Salamon [16]. On a vertical space $\mathcal{V}_J$, $\mathcal{J}_1$ is defined to be the complex structure $\mathcal{J}_J$ of the fibre through J, while $\mathcal{J}_2$ is defined as the conjugate complex structure, i.e. $\mathcal{J}_2|\mathcal{V}_J = -\mathcal{J}_J$. On a horizontal space $\mathcal{H}_J$, $\mathcal{J}_1$ and $\mathcal{J}_2$ are both defined to be the lift to $\mathcal{H}_J$ of the endomorphism J of $T_{\pi(J)}M$. Thus, if $\sigma \in \mathcal{Z}_\pm$

$$\mathcal{J}_k|\mathcal{H}_\sigma = (\pi_*|\mathcal{H}_\sigma)^{-1} \circ K_\sigma \circ \pi_*|\mathcal{H}_\sigma.$$

$$\mathcal{J}_k V = \pm(-1)^{k+1}\sigma \times V \quad \text{for} \quad V \in \mathcal{V}_\sigma, \quad k = 1, 2.$$

Let R be the curvature tensor of the Levi-Civita connection of (M, g); we adopt the following definition for the curvature tensor R: $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. Then the curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of $\Lambda^2 TM$ defined by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X, Y)Z, T), \quad X, Y, Z, T \in TM.$$

The curvature tensor of the connection on the bundle $\Lambda^2 TM$ induced by the Levi-Civita connection on TM will also be denoted by R.

The following easily verified formulas are useful in various computations on $\mathcal{Z}_\pm$.

$$g(R(a)b, c) = \pm g(\mathcal{R}(a), b \times c)) \tag{5}$$

for $a \in \Lambda^2 T_pM$, $b, c \in \Lambda^2_\pm T_pM$,

$$K_b \circ K_c = -g(b, c)Id \pm K_{b\times c}, \quad b, c \in \Lambda^2_\pm T_pM. \tag{6}$$

$$g(\sigma \times V, X \wedge K_\sigma Y) = g(\sigma \times V, K_\sigma X \wedge Y) = \pm g(V, X \wedge Y) \tag{7}$$

for $\sigma \in \mathcal{Z}_\pm$, $V \in \mathcal{V}_\sigma$, $X, Y \in T_{\pi(\sigma)}M$.

Denote by $\mathcal{B} : \Lambda^2 TM \to \Lambda^2 TM$ the endomorphism corresponding to the traceless Ricci tensor. If s denotes the scalar curvature of (M, g) and $\rho : TM \to TM$ is the Ricci operator, $g(\rho(X), Y) = Ricci(X, Y)$, we have

$$\mathcal{B}(X \wedge Y) = \rho(X) \wedge Y + X \wedge \rho(Y) - \frac{s}{2} X \wedge Y.$$

Note that $\mathcal{B}$ sends $\Lambda^2_\pm TM$ into $\Lambda^2_\mp TM$. Let $\mathcal{W} : \Lambda^2 TM \to \Lambda^2 TM$ be the endomorphism corresponding to the Weyl conformal tensor. Denote the restriction of $\mathcal{W}$ to $\Lambda^2_\pm TM$ by $\mathcal{W}_\pm$, so $\mathcal{W}_\pm$ sends $\Lambda^2_\pm TM$ to $\Lambda^2_\pm TM$ and vanishes on $\Lambda^2_\mp TM$.

It is well known that the curvature operator decomposes as ([30], see e.g. [3, Chapter 1 H])

$$\mathcal{R} = \frac{s}{6} Id + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_- \tag{8}$$

Note that this differ by a factor $1/2$ from [3] because of the factor $1/2$ in our definition of the induced metric on $\Lambda^2 TM$.

The Riemannian manifold (M, g) is Einstein exactly when $\mathcal{B} = 0$. It is called self-dual (anti-self-dual), if $\mathcal{W}_- = 0$ (resp. $\mathcal{W}_+ = 0$).

It is a well-known result by Atiyah-Hitchin-Singer [2] that the almost complex structure $\mathcal{J}_1$ on $\mathcal{Z}_-$ (resp. $\mathcal{Z}_+$) is integrable (i.e. comes from a complex structure) if and only if (M, g) is self-dual (resp. anti-self-dual). On the other hand the almost complex structure $\mathcal{J}_2$ is never integrable by a result of Eells-Salamon [16], but nevertheless it plays a useful role in harmonic map theory.

3 The Standard Variation with Compact Support of an Almost Hermitian Structure Through Sections of the Twistor Space

Now suppose that (N, h) is a Riemannian manifold, which admits an almost Hermitian structure J, i.e. a section of the bundle $\pi : \mathfrak{T} \to N$. Take a section V with compact support K of the bundle $J^*\mathcal{V} \to N$, the pull-back under J of the vertical bundle $\mathcal{V} \to \mathfrak{T}$. There exists an $\varepsilon > 0$ such that, for every point I of the compact set $J(K)$, the exponential map exp_I is a diffeomorphism of the ε-ball in $T_I\mathfrak{T}$. The function $||V||_{h_1}$ is bounded on N, so there exists a number $\varepsilon' > 0$ such that $||sV(p)||_{h_1} < \varepsilon$ for every $p \in N$ and $s \in (-\varepsilon', \varepsilon')$. Set $J_s(p) = exp_{J(p)}[sV(p)]$ for $p \in N$ and $s \in (-\varepsilon', \varepsilon')$. For every fixed $p \in N$, the curve $s \to exp_{J(p)}[sV(p)]$ is a geodesic with initial velocity vector $V(p)$ which is tangent to the fibre $\mathfrak{T}_p$ of $\mathfrak{T}$ through $J(p)$. Since this fibre is a totally geodesic submanifold, the whole curve lies in it. Hence J_s is a section of $\mathfrak{T}$, i.e. an almost Hermitian structure on (N, h), such that $J_s = J$ on $N \setminus K$.

In particular, this shows that if (N, h) admits a compatible almost complex structure J, then it possesses many such structures.

4 The Energy Functional on Sections of the Twistor Space

If D is a relatively compact open subset of a Riemannian manifold (N, h), the energy functional assigns to every compatible almost complex structure J on (N, h), considered as a map $J : (N, h) \to (\mathfrak{T}, h_t)$, the integral

$$E_\Omega(J) = \int_D ||J_*||^2_{h,h_t} vol$$

where the norm is taken with respect to h and h_t.

A compatible almost complex structure J is said to be a *harmonic section* ("a harmonic almost complex structure" in the terminology of [35]), if for every D it is a critical point of the energy functional under variations of J through sections of the twistor space of (N, h).

We have $J_*X = \nabla_X J + X^h$ for every $X \in TN$ where X^h is the horizontal lift of X (and $\nabla_X J$ is the vertical part of J_*X). Therefore $|J_*||^2_{h,h_t} = t||\nabla J||^2_h + (dim\, N)vol(D)$. It follows that the critical points of the energy functional E_Ω coincide with the critical points of the vertical energy functional

$$J \to \int_D ||\nabla J||^2_h vol$$

and do not depend on the particular choice of the parameter t. Another obvious consequence is that the Kähler structures provide the absolute minimum of the energy functional.

The Euler-Lagrange equation for the critical points of the energy functional under variations through sections of the twistor bundle has been obtained by C. Wood.

Theorem 1 ([34, 35]) *A compatible almost complex structure J is a harmonic section if and only if*

$$[J, \nabla^* \nabla J] = 0,$$

where ∇^* is the formal adjoint operator of ∇.

Remark Suppose that N is oriented and J is an almost Hermitian structure on (N, h) yielding the orientation of N, so it is a section of the positive twistor bundle $\mathcal{T}_+$. Every variation of J with compact support consisting of sections of the total twistor space $\mathcal{T}$ contains a subvariation consisting of sections of $\mathcal{T}_+$. Thus J is a critical point of the energy functional under variations with compact support through sections of the total twistor space $\mathcal{T}$, if and only if it is a critical point under variations through sections of $\mathcal{T}_+$.

Examples of non-Kähler almost Hermitian structures, which are harmonic sections.

1. ([34]) The standard nearly Kähler structure on S^6.
2. ([34]) The Calabi-Eckmann complex structure on $S^{2p+1} \times S^{2q+1}$ with the product metric.
3. ([34]) The Abbena-Thurston [1, 31] almost Kähler structure on (the real Heisenberg group $\times S^1$)/(discrete subgroup).
4. The complex structure of the Iwasawa manifold = (the complex Heisenberg group)/(discrete subgroup).

Bor et al. [4] have given sufficient conditions for an almost Hermitian complex structure to minimize the energy functional among sections of the twistor bundle.

Theorem 2 ([4]) *Let (N, h) be a compact Riemannian manifold and let J be a compatible almost complex structure on it. Suppose that*

(1) $\dim N = 4$, the manifold (N, h) is anti-self-dual and the almost Hermitian structure (h, J) is Hermitian, or almost Kähler,
or
(2) $\dim N \geq 6$, (N, h) is conformally flat and the almost Hermitian structure (h, J) is of Gray-Hervella [18] type $W_1 \oplus W_4$.

Then the almost complex structure J is an energy minimizer.

Examples of non-Kähler minimizers of the energy functional.

1. ([4]) LeBrun [26] has constructed anti-self-dual Hermitian structures on the blow-ups $(S^3 \times S^1)\sharp\, n\overline{\mathbb{CP}^2}$ of the Hopf surface $S^3 \times S^1$. Blow-ups do not affect the first Betti number, so any blow up of the Hopf surface has Betti number one, and hence it does not admit a Kähler metric.
2. Kim [23] has shown the existence of anti-self-dual strictly almost Kähler structures on $\mathbb{CP}^2\sharp\, n\overline{\mathbb{CP}^2}$, $n \geq 11$, $(S^2\times\Sigma)\sharp\, n\overline{\mathbb{CP}^2}$, genus $\Sigma \geq 2$, $(S^2\times T^2)\sharp\, n\overline{\mathbb{CP}^2}$, $n \geq 6$, where Σ is a Riemann surface and T^2 is a 2-torus.
3. ([4]) The standard Hermitian structure on the Hopf manifold $S^{2p+1} \times S^1$ is conformally flat and locally conformally Kähler, and hence of Grey-Hervella class W_4.

Let (N, h, J) be an almost Hermitian manifold and $\Omega(X, Y) = h(JX, Y)$ its fundamental 2-form. Then the Euler-Lagrange equation $[J, \nabla^*\nabla J] = 0$ is equivalent to the identity

$$(\nabla^*\nabla\,\Omega)(X, Y) = (\nabla^*\nabla\,\Omega)(JX, JY), \quad X, Y \in TN. \tag{9}$$

Note that for the rough Laplacian $\nabla^*\nabla$ we have $\nabla^*\nabla\,\Omega = -Trace\,\nabla^2\Omega$.

The following simple observation is useful in many cases. Let $\widehat{\Omega}$ be the section of $\Lambda^2 TN$ corresponding to the 2-form Ω under the isomorphism $\Lambda^2 TN \cong \Lambda^2 T^*N$ determined by the metric g on $\Lambda^2 TN$ defined by means of the metric h on TN via (1). Thus, $h(\widehat{\Omega}, X \wedge Y) = \Omega(X, Y)$, and if $E_1, \ldots, E_m, JE_1, \ldots, JE_m$ is an orthonormal frame of TN,

$$\widehat{\Omega} = 2\sum_{k=1}^{m} E_k \wedge JE_k.$$

Denote by $\mathcal{R}(\Omega)$ the 2-form corresponding to $\mathcal{R}(\widehat{\Omega})$. Then we have

$$\mathcal{R}(\Omega)(X, Y) = h(\mathcal{R}(\widehat{\Omega}), X \wedge Y).$$

Lemma 2 ([10]) *A compatible almost complex structure J on a Riemannian manifold (N, h) is a harmonic section if and only if*

$$\Delta\Omega(X,Y) - \Delta\Omega(JX,JY) = \mathcal{R}(\Omega)(X,Y) - \mathcal{R}(\Omega)(JX,JY), \quad X,Y \in TN, \tag{10}$$

where Δ is the Laplace-de Rham operator of (N, h).

Proof By the Weitzenböck formula

$$\begin{aligned}\Delta\Omega(X,Y) - (\nabla^*\nabla\,\Omega)(X,Y)\\ = Trace\{Z \to (R(Z,Y)\Omega)(Z,X) - (R(Z,X)\Omega)(Z,Y)\},\end{aligned}$$

$X, Y \in TN$ (see, for example, [15]). We have

$$\begin{aligned}(R(Z,Y)\Omega)(Z,X) &= -\Omega(R(Z,Y)Z,X) - \Omega(Z,R(Z,Y)X)\\ &= h(R(Z,Y)Z,JX) + h(R(Z,Y)X,JZ).\end{aligned}$$

Hence

$$\begin{aligned}\Delta\Omega(X,Y) - (\nabla^*\nabla\,\Omega)(X,Y)\\ = Ricci(Y,JX) - Ricci(X,JY)\\ +Trace\,\{Z \to h(R(Z,Y)X,JZ) - h(R(Z,X)Y,JZ)\}\end{aligned}$$

By the algebraic Bianchi identity

$$\begin{aligned}h(R(Z,Y)X,JZ) - h(R(Z,X)Y,JZ) &= h(R(X,Y)Z,JZ)\\ &= h(\mathcal{R}(Z\wedge JZ), X\wedge Y).\end{aligned}$$

We have

$$\begin{aligned}Trace\,\{Z \to h(\mathcal{R}(Z\wedge JZ), X\wedge Y)\} &= 2\sum_{k=1}^{m} h(\mathcal{R}(E_k\wedge JE_k), X\wedge Y)\\ &= h(\mathcal{R}(\widehat{\Omega}), X\wedge Y) = \mathcal{R}(\Omega)(X,Y).\end{aligned}$$

Thus

$$\Delta\Omega(X,Y) - (\nabla^*\nabla\,\Omega)(X,Y) = Ricci(Y,JX) - Ricci(X,JY) + \mathcal{R}(\Omega)(X,Y),$$

and the result follows from (9).

5 The Atiyah-Hitchin-Singer and Eells-Salamon Almost Complex Structures as Harmonic Sections

Lemma 2 has been used to prove the following statement.

Theorem 3 ([10]) *Let (M,g) be an oriented Riemannian 4-manifold and let $(\mathcal{Z},h_t)$ be its negative twistor space. Then:*

(i) *The Atiyah-Hitchin-Singer almost-complex structure $\mathcal{J}_1$ on $(\mathcal{Z},h_t)$ is a harmonic section if and only if (M,g) is a self-dual manifold.*
(ii) *The Eells-Salamon almost-complex structure $\mathcal{J}_2$ on $(\mathcal{Z},h_t)$ is a harmonic section if and only if (M,g) is a self-dual manifold with constant scalar curvature.*

In order to apply Lemma 2, one needs to compute the Laplacian of the fundamental 2-form $\Omega_{k,t}(A,B) = h_t(\mathcal{J}_k A,B), k = 1,2$, of the almost-Hermitian structure $(h_t,\mathcal{J}_k)$ on $\mathcal{Z}$. A computation, involving coordinate-free formulas for the differential and codifferential of $\Omega_{k,t}$ [28], gives the following expression for the Laplacian of $\Omega_{k,t}$ in terms of the base manifold (M,g).

Lemma 3 ([10]) *Let V be a vertical vector of $\mathcal{Z}$ at a point σ and $X,Y\in T_{\pi(\sigma)}M$. Then*

$$\Delta\Omega_{k,t}(X^h,Y^h)_\sigma = g(\frac{4\sigma}{t}+2(-1)^k\mathcal{R}(\sigma),X\wedge Y)+tg(R(X\wedge Y)\sigma,R(\sigma)\sigma) \tag{11}$$

and

$$\Delta\Omega_{k,t}(V,X^h)_\sigma = (-1)^{k+1}tg(\delta\mathcal{R}(X),V)-tg((\nabla_X\mathcal{R})(\sigma),\sigma\times V). \tag{12}$$

Denote by $\mathcal{R}_{\mathcal{Z}}$ the curvature tensor of the manifold $(\mathcal{Z},h_t)$. To compute the curvature terms $\mathcal{R}_{\mathcal{Z}}(\Omega_{k,t})$ in (10) one can use the following coordinate-free formula for the curvature of the twistor space.

Proposition 1 ([9]) *Let $\mathcal{Z}$ be the negative twistor space of an oriented Riemannian 4-manifold (M,g) with curvature tensor R. Let $E,F\in T_\sigma\mathcal{Z}$ and $X=\pi_*E$, $Y=\pi_*F$, $V=\mathcal{V}E$, $W=\mathcal{V}F$ where $\mathcal{V}$ means "the vertical part". Then*

$$\begin{gathered} h_t(R_{\mathcal{Z}}(E,F)E,F) = g(R(X,Y)X,Y) \\ -tg((\nabla_X\mathcal{R})(X\wedge Y),\sigma\times W)+tg((\nabla_Y\mathcal{R})(X\wedge Y),\sigma\times V) \\ -3tg(\mathcal{R}(\sigma),X\wedge Y)g(\sigma\times V,W) \\ -t^2g(R(\sigma\times V)X,R(\sigma\times W)Y)+\frac{t^2}{4}||R(\sigma\times W)X+R(\sigma\times V)Y||^2 \\ -\frac{3t}{4}||R(X,Y)\sigma||^2+t(||V||^2||W||^2-g(V,W)^2), \end{gathered}$$

where the norm of the vertical vectors is taken with respect to the metric g on Λ^2_-TM.

Using this formula, the well-known expression of the Levi-Civita curvature tensor by means of sectional curvatures (cf. e.g. [19, § 3.6, p. 93, formula (15)]), and the differential Bianchi identity one gets the following.

Corollary 1 *Let $\sigma \in \mathcal{Z}$, $X, Y, Z, T \in T_{\pi(\sigma)}M$, and $U, V, W, W' \in \mathcal{V}_\sigma$. Then*

$$\begin{aligned}
h_t(R_{\mathcal{Z}}(X^h, Y^h)Z^h, T^h)_\sigma &= g(R(X,Y)Z,T) \\
-\frac{3t}{12}[2g(R(X,Y)\sigma, R(Z,T)\sigma) &- g(R(X,T)\sigma, R(Y,Z)\sigma) \\
&+ g(R(X,Z)\sigma, R(Y,T)\sigma)]. \\
h_t(R_{\mathcal{Z}}(X^h, Y^h)Z^h, U)_\sigma &= -\frac{t}{2}g(\nabla_Z \mathcal{R}(X \wedge Y), \sigma \times U). \\
h_t(R_{\mathcal{Z}}(X^h, U)Y^h, V)_\sigma &= \frac{t^2}{4}g(R(\sigma \times V)X, R(\sigma \times U)Y) \\
&+ \frac{t}{2}g(\mathcal{R}(\sigma), X \wedge Y)g(\sigma \times V, U). \\
h_t(R_{\mathcal{Z}}(X^h, Y^h)U, V)_\sigma &= \frac{t^2}{4}[g(R(\sigma \times V)X, R(\sigma \times U)Y) \\
&\qquad - g(R(\sigma \times U)X, R(\sigma \times V)Y)] \\
&+ tg(\mathcal{R}(\sigma), X \wedge Y)g(\sigma \times V, U)
\end{aligned}$$

$$h_t(R_{\mathcal{Z}}(X^h, U)V, W) = 0, \quad h_t(R_{\mathcal{Z}}(U,V)W,W') = g(U,W)g(V,W') - g(U,W')g(V,W).$$

This implies

Lemma 4 ([10]) *Let V, W be vertical vectors of $\mathcal{Z}$ at a point σ and $X, Y \in T_{\pi(\sigma)}M$. Then*

$$\begin{aligned}
\mathcal{R}_{\mathcal{Z}}(\Omega_{k,t})(X^h, Y^h)_\sigma &= 2[1 + (-1)^{k+1}]g(\mathcal{R}(\sigma), X \wedge Y) - tg(R(X \wedge Y)\sigma, R(\sigma)\sigma) \\
&- \frac{t}{2} Trace\{Z \to g(R(X \wedge Z)\sigma, R(Y \wedge K_\sigma Z)\sigma)\} \\
&- \frac{t}{2}(-1)^k Trace\{\mathcal{V}_\sigma \ni \tau \to g(R(\tau)X, R(\sigma \times \tau)Y)\},
\end{aligned} \tag{13}$$

where the latter trace is taken with respect to the metric g on $\mathcal{V}_\sigma$,

$$\mathcal{R}_{\mathcal{Z}}(\Omega_{k,t})(V, X^h)_\sigma = tg((\nabla_X \mathcal{R})(\sigma), \sigma \times V) \tag{14}$$

and

$$\begin{aligned}
\mathcal{R}_{\mathcal{Z}}(\Omega_{k,t})(V, W)_\sigma &= 2[(-1)^{k+1} + tg(\mathcal{R}(\sigma), \sigma)]g(V, \sigma \times W) \\
&+ \frac{t^2}{2} Trace\{Z \to g(R(\sigma \times V)K_\sigma Z, R(\sigma \times W)Z)\}.
\end{aligned}$$

Proof of Theorem 3 According to Lemmas 2–4, the almost complex structure $\mathcal{J}_k$ is a harmonic section if and only if the following two conditions are satisfied:

$$4g(\mathcal{R}(\sigma), X \wedge Y - K_\sigma X \wedge K_\sigma Y) =$$
$$tTrace\{Z \to g(R(X \wedge Z)\sigma, R(Y \wedge K_\sigma Z)\sigma) - g(R(K_\sigma X \wedge Z)\sigma, R(K_\sigma Y \wedge K_\sigma Z)\sigma)\}$$
$$+t(-1)^k Trace\{\mathcal{V}_\sigma \ni \tau \to g(R(\tau)X, R(\sigma \times \tau)Y) - g(R(\tau)K_\sigma X, R(\sigma \times \tau)K_\sigma Y)\} \tag{15}$$

and

$$g(\delta\mathcal{R}(K_\sigma X), \sigma \times V) = (-1)^k g(\delta\mathcal{R}(X), V) \tag{16}$$

for every $\sigma \in \mathcal{Z}$, $V \in \mathcal{V}_\sigma$ and $X, Y \in T_{\pi(\sigma)}M$.

We shall show that condition (15) is equivalent to (M, g) being a self-dual manifold. Note first that (15) holds for every $X, Y \in T_{\pi(\sigma)}M$ if and only if it holds for every $X, Y \in T_{\pi(\sigma)}M$ with $||X|| = ||Y|| = 1$ and $X \perp Y, K_\sigma Y$. For every such X, Y there is a unique $\tau \in \mathcal{V}_\sigma$, $||\tau|| = 1$, such that $Y = K_\tau X$, namely $\tau = X \wedge Y - K_\sigma X \wedge K_\sigma Y$; conversely, if $\tau \in \mathcal{V}_\sigma$, $||\tau|| = 1$ and $Y = K_\tau X$, then $X \perp Y, K_\sigma Y$ in view of (6). Thus, (15) holds if and only if it holds for every $X \in T_{\pi(\sigma)}M$ and $Y = K_\tau X$ with $||X|| = 1$, $\tau \in \mathcal{V}_\sigma$, $||\tau|| = 1$. Given such X and τ, the vectors $E_1 = X$, $E_2 = K_\sigma X$, $E_3 = K_\tau X$, $E_4 = K_{\sigma\times\tau} X$ constitute an oriented orthonormal basis of $T_{\pi(\sigma}M$ such that $s_1^- = \sigma$, $s_2^- = \tau$, $s_3^- = \sigma \times \tau$, where s_1^-, s_2^-, s_3^- are defined by means of $\{E_1, \dots, E_4\}$ via (2). Using the bases $\{E_1, \dots, E_4\}$ of $T_{\pi(\sigma)}M$ and $\tau, \sigma \times \tau$ of $\mathcal{V}_\sigma$ to compute the traces in the right-hand side of (15), we see that identity (15) is equivalent to

$$4g(\mathcal{R}(\sigma), \tau) = tg(R(\sigma)\sigma, R(\tau)\sigma)$$
$$+t(-1)^k g(R(\tau)\sigma, R(\sigma \times \tau)(\sigma \times \tau)) - t(-1)^k g(R(\tau)(\sigma \times \tau), R(\sigma \times \tau)\sigma)$$

for every $\sigma, \tau \in \mathcal{Z}$, $\pi(\sigma) = \pi(\tau)$, $\sigma \perp \tau$. Using (5) we easily see also that the latter identity is equivalent to

$$4g(\mathcal{R}(\sigma), \tau) = tg(\mathcal{R}(\sigma), \sigma \times \tau)g(\mathcal{R}(\tau), \sigma \times \tau) + tg(\mathcal{R}(\sigma), \tau)g(\mathcal{R}(\tau), \tau)$$
$$+t(-1)^{k+1} g(\mathcal{R}(\tau), \sigma \times \tau)g(\mathcal{R}(\sigma \times \tau), \tau) - t(-1)^{k+1} g(\mathcal{R}(\tau), \sigma)g(\mathcal{R}(\sigma \times \tau), \sigma \times \tau). \tag{17}$$

Writing this identity with (σ, τ) replaced by (τ, σ) and comparing the obtained identity with (17) we get

$$g(\mathcal{R}(\sigma), \tau)[g(\mathcal{R}(\sigma), \sigma) - g(\mathcal{R}(\tau), \tau)] = 0. \tag{18}$$

Replacing the pair (σ, τ) by $\left(\frac{3\sigma + 4\tau}{5}, \frac{4\sigma - 3\tau}{5}\right)$ in (18) and using again this identity, we obtain

$$[g(\mathcal{R}(\sigma), \sigma) - g(\mathcal{R}(\tau), \tau)]^2 = 4[g(\mathcal{R}(\sigma), \tau)]^2,$$

which, together with (18), gives

$$g(\mathcal{R}(\sigma), \sigma) = g(\mathcal{R}(\tau), \tau), \quad g(\mathcal{R}(\sigma), \tau) = 0.$$

Thus

$$g(\mathcal{W}_-(\sigma), \sigma) = g(\mathcal{W}_-(\tau), \tau) \text{ and } g(\mathcal{W}_-(\sigma), \tau) = 0$$

Since $Trace\, \mathcal{W}_- = 0$, this implies $\mathcal{W}_- = 0$.

Conversely, if $\mathcal{W}_- = 0$ we have $\mathcal{R}(\sigma) = \frac{s}{6}\sigma + \mathcal{B}(\sigma)$ where $\mathcal{B}(\sigma) \in \Lambda^2_+ TM$, so it is obvious that identity (17) is satisfied.

To analyze condition (16) we recall that $\delta\mathcal{R} = 2\delta\mathcal{B}\,(= -dRicci)$ (cf. e.g. [3]), so it follows from (8) that

$$\delta\mathcal{R}(X) = -\frac{1}{3}\operatorname{grad} s \wedge X + 2\delta\mathcal{W}(X),\ X \in TM.$$

Suppose $\mathcal{W}_- = 0$. Since $\delta\mathcal{W}_+(X) \in \Lambda^2_+ TM$, we have

$$g(\delta\mathcal{R}(X), V) = \frac{1}{3}g(X \wedge \operatorname{grad} s, V)$$

for any $V \in \Lambda^2_- TM$. The latter formula and (7) imply that condition (16) is equivalent (for self-dual manifolds) to the identity

$$g(V, X \wedge \operatorname{grad} s) = (-1)^{k+1} g(V, X \wedge \operatorname{grad} s).$$

Clearly, this identity is satisfied if $k = 1$; for $k = 2$ it holds if and only if the scalar curvature s is constant.

6 The Second Fundamental Form of an Almost Hermitian Structure as a Map into the Twistor Space

Let J be a compatible almost complex structure on a Riemannian manifold (N, h). Then we have a map $J : (N, h) \to (\mathcal{T}, h_t)$ between Riemannian manifolds. Let $J^*T\mathcal{T} \to N$ be the pull-back of the bundle $T\mathcal{T} \to \mathcal{T}$ under the map $J : N \to \mathcal{T}$. We can consider the differential $J_* : TN \to T\mathfrak{T}$ as a section of the bundle $Hom(TN, J^*T\mathfrak{T}) \to N$. Denote by $\widetilde{D}$ the connection on $J^*T\mathfrak{T}$ induced by the Levi-Civita connection D on $T\mathfrak{T}$. The Levi Civita connection ∇ on TN and the connection $\widetilde{D}$ on $J^*T\mathfrak{T}$ induce a connection $\widetilde{\nabla}$ on the bundle $Hom(TN, J^*T\mathfrak{T})$. Recall that the second fundamental form of the map J is, by definition, $\widetilde{\nabla}J_*$. The map $J : (N,h) \to (\mathfrak{T}, h_t)$ is harmonic if and only if

$$Trace\widetilde{\nabla}J_* = 0.$$

Recall also that the map $J : (N,h) \to (\mathfrak{T}, h_t)$ is totally geodesic exactly when $\widetilde{\nabla}J_* = 0$.

Proposition 2 ([11, 13]) *For every $X, Y \in T_pN$,*

$$\widetilde{\nabla}J_*(X,Y) = \frac{1}{2}\mathcal{V}(\nabla^2_{XY}J + \nabla^2_{YX}J)$$
$$-\frac{2t}{n}[(R((J \circ \nabla_X J)^\wedge)Y)^h_{J(p)} + (R((J \circ \nabla_Y J)^\wedge)X)^h_{J(p)}],$$

where $\mathcal{V}$ means "the vertical component", $n = dim\,N$, and $\nabla^2_{XY}J = \nabla_X\nabla_Y J - \nabla_{\nabla_X Y}J$ is the second covariant derivative of J.

The computation of the second fundamental form is based on several lemmas. First, we note that identity (5) can be generalized as follows.

Lemma 5 ([7]) *For every $a, b \in A(T_pN)$ and $X, Y \in T_pN$, we have*

$$G(R(X,Y)a,b) = \frac{2}{n}h(R([a,b]^\wedge)X,Y). \tag{19}$$

Proof Let $E_1, \dots, E_n$ be an orthonormal basis of T_pN. Then

$$[a,b]^\wedge = \frac{1}{2}\sum_{i,j=1}^{n} h([a,b]E_i, E_j)E_i \wedge E_j.$$

Therefore

$$\begin{aligned} h(R([a,b]^\wedge)X,Y) &= \frac{1}{2}\sum_{i,j=1}^{n} h(R(X,Y)E_i,E_j)[h(abE_i,E_j) + h(aE_i,bE_j)] \\ &= -\frac{1}{2}\sum_{i=1}^{n} h(a(R(X,Y)E_i),bE_i) + \frac{1}{2}\sum_{k=1}^{n} h(R(X,Y)aE_k,bE_k) \\ &= \frac{n}{2}G(R(X,Y)a,b). \end{aligned}$$

Lemma 5 implies

$$h_t(R(X,Y)J,V) = \frac{2t}{n}h(R([J,V]^\wedge)X,Y) = \frac{4t}{n}h(R((J\circ V)^\wedge)X,Y). \tag{20}$$

Lemma 6 ([7, 9]) *If X, Y are vector fields on N, and V is a vertical vector field on $\mathfrak{T}$, then*

$$(D_{X^h}Y^h)_I = (\nabla_X Y)^h_I + \frac{1}{2}R_p(X\wedge Y)I \tag{21}$$

$$(D_V X^h)_I = \mathcal{H}(D_{X^h}V)_I = -\frac{2t}{n}(R_p((I\circ V_I)^\wedge)X)^h_I\,, \tag{22}$$

where $I \in \mathfrak{T}$, $p = \pi(I)$, $n = \dim N$, and $\mathcal{H}$ means "the horizontal component".

Proof Identity (21) follows from the Koszul formula for the Levi-Civita connection and the identity $[X^h, Y^h]_I = [X,Y]^h_I + R(X,Y)I$.

Let W be a vertical vector field on $\mathfrak{T}$. Then

$$h_t(D_V X^h, W) = -h_t(X^h, D_V W) = 0,$$

since the fibres are totally geodesic submanifolds, so $D_V W$ is a vertical vector field. Therefore, $D_V X^h$ is a horizontal vector field. Moreover, $[V, X^h]$ is a vertical vector field, hence $D_V X^h = \mathcal{H}D_{X^h}V$. Thus

$$h_t(D_V X^h, Y^h) = h_t(D_{X^h}V, Y^h) = -h_t(V, D_{X^h}Y^h).$$

Now (22) follows from (21) and (20).

Any (local) section a of the bundle $A(TN)$ determines a (local) vertical vector field $\widetilde{a}$ on $\mathfrak{T}$ defined by

$$\widetilde{a}_I = \frac{1}{2}(a(p) + I\circ a(p)\circ I), \quad p = \pi(I).$$

The next lemma is "folklore".

Lemma 7 *If $I \in \mathfrak{T}$ and X is a vector field on a neighbourhood of the point $p = \pi(I)$, then*

$$[X^h, \widetilde{a}]_I = (\widetilde{\nabla_X a})_I.$$

Let $I \in \mathfrak{T}$ and let $U, V \in \mathcal{V}_I$. Take section a and b of $A(TN)$ such that $a(p) = U$, $b(p) = V$ for $p = \pi(I)$. Let $\widetilde{a}$ and $\widetilde{b}$ be the vertical vector fields determined by the sections a and b. Taking into account the fact that the fibre of $\mathfrak{T}$ through the point I is a totally geodesic submanifold, one easily gets by means of the Koszul formula that

$$(D_{\widetilde{a}}\widetilde{b})_I = \frac{1}{4}[UVI + IVU + I(UVI + IVU)I] = 0. \tag{23}$$

Lemma 8 *For every $p \in N$, there exists an h_t-orthonormal frame of vertical vector fields $\{V_\alpha : \alpha = 1, \ldots, m^2 - m\}$, $m = \frac{1}{2}dim\, N$, in a neighbourhood of the point $J(p)$ such that*

(1) $(D_{V_\alpha}V_\beta)_{J(p)} = 0$, $\alpha, \beta = 1, \ldots, m^2 - m$.
(2) *If X is a vector field near the point p, then* $[X^h, V_\alpha]_{J(p)} = 0$.
(3) $\nabla_{X_p}(V_\alpha \circ J) \perp \mathcal{V}_{J(p)}$.

Proof Let $E_1, \ldots, E_n$ be an orthonormal frame of TN in a neighbourhood of p such that $J(E_{2k-1})_p = (E_{2k})_p$, $k = 1, \ldots, m$, and $\nabla E_l|_p = 0$, $l = 1, \ldots, n$. Define sections S_{ij}, $1 \le i, j \le n$, of $A(TN)$ by the formula

$$S_{ij}E_l = \sqrt{\frac{n}{2}}(\delta_{il}E_j - \delta_{lj}E_i), \quad l = 1, \ldots, n.$$

Then S_{ij}, $i < j$, form an orthonormal frame of $A(TN)$ with respect to the metric $G(a,b) = -\dfrac{1}{n}Trace\,(a \circ b)\,;a, b \in A(TN)$. Set

$$A_{r,s} = \tfrac{1}{\sqrt{2}}(S_{2r-1,2s-1} - S_{2r,2s}), \quad B_{r,s} = \tfrac{1}{\sqrt{2}}(S_{2r-1,2s} + S_{2r,2s-1}),$$
$$r = 1, \ldots, m-1,\ s = r+1, \ldots, m.$$

Then $\{(A_{r,s})_p, (B_{r,s})_p\}$ is a G-orthonormal basis of the vertical space $\mathcal{V}_{J(p)}$. Note also that $\nabla A_{r,s}|_p = \nabla B_{r,s}|_p = 0$. Let $\widetilde{A}_{r,s}$ and $\widetilde{B}_{r,s}$ be the vertical vector fields on $\mathcal{T}$ determined by the sections $A_{r,s}$ and $B_{r,s}$ of $A(TN)$. These vector fields constitute a frame of the vertical bundle $\mathcal{V}$ in a neighbourhood of the point $J(p)$.

Considering $\widetilde{A}_{r,s} \circ J$ as a section of $A(TN)$, we have

$$\begin{aligned}\nabla_{X_p}(\widetilde{A}_{r,s} \circ J) &= \tfrac{1}{2}\{(\nabla_{X_p}J) \circ (A_{r,s})_p \circ J_p + J_p \circ (A_{r,s}) \circ (\nabla_{X_p}J)\}\\ &= \tfrac{1}{2}\{-\nabla_{X_p} \circ J_p \circ (A_{r,s})_p + J_p \circ (A_{r,s}) \circ (\nabla_{X_p}J)\}\\ &= \tfrac{1}{2}[(B_{r,s})_p, \nabla_{X_p}J].\end{aligned}$$

For every $I \in \mathcal{T}$, we have the orthogonal decomposition

$$A(T_{\pi(I)}N) = \mathcal{V}_I \oplus \{S \in A(T_{\pi(I)}N) :\ IS - SI = 0\}. \tag{24}$$

The endomorphisms $(B_{r,s})_p$ and $\nabla_{X_p}J$ of T_pN belong to $\mathcal{V}_{J(p)}$, so they anti-commute with $J(p)$, hence their commutator commutes with $J(p)$. Therefore the commutator $[(B_{r,s})_p, \nabla_{X_p}J]$ is G-orthogonal to the vertical space at J. Thus

$$\nabla_{X_p}(\widetilde{A}_{r,s} \circ J) \perp \mathcal{V}_{J(p)},$$

and, similarly, $\nabla_{X_p}(\widetilde{B}_{r,s} \circ J) \perp \mathcal{V}_{J(p)}$.

It is convenient to denote the elements of the frame $\{\widetilde{A}_{r,s}, \widetilde{B}_{r,s}\}$ by $\{\widetilde{V}_1, \ldots, \widetilde{V}_{m^2-m}\}$. In this way we have a frame of vertical vector fields near the point $J(p)$ with property (3) of the lemma. Properties (1) and (2) are also satisfied by this frame according to (23) and Lemma 7, respectively. In particular,

$$(\widetilde{V}_\gamma)_{J(p)}(h_t(\widetilde{V}_\alpha, \widetilde{V}_\beta)) = 0, \quad \alpha, \beta, \gamma = 1, \ldots, m^2 - m.$$

Note also that, in view of (22),

$$\mathcal{V}(D_{X^h}\widetilde{V}_\alpha)_{J(p)} = [X^h, \widetilde{V}_\alpha]_{J(p)} = 0,$$

hence

$$X^h_{J(p)}(h_t(\widetilde{V}_\alpha, \widetilde{V}_\beta)) = 0.$$

Now it is clear that the h_t-orthonormal frame $\{V_1, \ldots, V_{m^2-m}\}$ obtained from $\{\widetilde{V}_1, \ldots, \widetilde{V}_{m^2-m}\}$ by the Gram-Schmidt process has the properties stated in the lemma.

Proof of Proposition 2 Extend the tangent vectors X and Y to vector fields in a neighbourhood of the point p. Let $V_1, \ldots, V_{m^2-m}$ be an h_t-orthonormal frame of vertical vector fields with properties (1)–(3) stated in Lemma 8.

We have

$$J_* \circ Y = Y^h \circ J + \nabla_Y J = Y^h \circ J + \sum_{\alpha=1}^{m^2-m} h_t(\nabla_Y J, V_\alpha \circ J)(V_\alpha \circ J),$$

hence

$$\begin{aligned}\widetilde{D}_X(J_* \circ Y) = (D_{J_*X}Y^h) \circ J &+ \sum_{\alpha=1}^{m^2-m} h_t(\nabla_Y J, V_\alpha)(D_{J_*X}V_\alpha) \circ J \\ &+ t \sum_{\alpha=1}^{m^2-m} G(\nabla_X \nabla_Y J, V_\alpha \circ J)(V_\alpha \circ J).\end{aligned}$$

This, in view of Lemma 6, implies

$$
\begin{aligned}
\widetilde{D}_{X_p}(J_*\circ Y) = (\nabla_X Y)^h_{J(p)} + \frac{1}{2}R(X\wedge Y)J(p) - \frac{2t}{n}(R((J\circ\nabla_X J)^\wedge)Y)^h_{J(p)}\\
+t\sum_{\alpha=1}^{m^2-m} G(\nabla_{X_p}\nabla_Y J, V_\alpha\circ J)_p V_\alpha(J(p))\\
-\frac{2t}{n}(R((J\circ\nabla_Y J)^\wedge)X)^h_{J(p)}\\
= (\nabla_{X_p}Y)^h_{J(p)} + \frac{1}{2}\mathcal{V}(\nabla_{X_p}\nabla_Y J + \nabla_{Y_p}\nabla_X J) + \frac{1}{2}\nabla_{[X,Y]_p}J\\
-\frac{2t}{n}[R((J\circ\nabla_X J)^\wedge)Y)^h_{J(p)} + (R((J\circ\nabla_Y J)^\wedge)X)^h_{J(p)}].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\widetilde{\nabla}J_*(X,Y) = \widetilde{D}_{X_p}(J_*\circ Y) - (\nabla_X Y)^h_\sigma - \nabla_{\nabla_{X_p}Y}J\\
= \frac{1}{2}\mathcal{V}(\nabla_{X_p}\nabla_Y J - \nabla_{\nabla_{X_p}Y}J + \nabla_{Y_p}\nabla_X J - \nabla_{\nabla_{Y_p}X}J)\\
-\frac{2t}{n}[R((J\circ\nabla_X J)^\wedge)Y)^h_{J(p)} + (R((J\circ\nabla_Y J)^\wedge)X)^h_{J(p)}].
\end{aligned}
$$

Proposition 2 implies immediately the following.

Corollary 2 *If (N,h,J) is Kähler, the map $J:(N,h)\to(\mathcal{T},h_t)$ is a totally geodesic isometric imbedding.*

Remark In view of the decomposition (24), the Euler-Lagrange equation $[J,\nabla^*\nabla J]=0$ is equivalent to the condition that the vertical part of $\nabla^*\nabla J = -Trace\,\nabla^2 J$ vanishes. Thus, by Proposition 2, J is a harmonic section if and only if

$$
\mathcal{V}\,Trace\,\widetilde{\nabla}J_* = 0.
$$

This fact, Proposition 2 and Theorem 3 imply

Corollary 3

(i) $\mathcal{V}\,Trace\,\widetilde{\nabla}\mathcal{J}_{1*}=0$ *if and only if (M,g) is self-dual.*
(ii) $\mathcal{V}\,Trace\,\widetilde{\nabla}\mathcal{J}_{2*}=0$ *if and only if (M,g) is self-dual and with constant scalar curvature.*

7 The Atiyah-Hitchin-Singer and Eells-Salamon Almost Complex Structures as Harmonic Maps

The main result in this section is the following.

Theorem 4 *Each of the Atiyah-Hitchin-Singer and Eells-Salamon almost complex structures on the negative twistor space $\mathcal{Z}$ of an oriented Riemannian four-manifold (M, g) determines a harmonic map if and only if (M, g) is either self-dual and Einstein, or is locally the product of an open interval in $\mathbb{R}$ and a 3-dimensional Riemannian manifold of constant curvature.*

Remarks

1. Every manifold that is locally the product of an open interval in $\mathbb{R}$ and a 3-dimensional Riemannian manifold of constant curvature c is locally conformally flat with constant scalar curvature $6c$. It is not Einstein unless $c = 0$, i.e. Ricci flat.
2. According to Theorems 3 and 4, the conditions under which $\mathcal{J}_1$ or $\mathcal{J}_2$ is a harmonic section or a harmonic map do not depend on the parameter t of the metric h_t. Taking certain special values of t, we can obtain metrics h_t with nice properties (cf., for example, [9, 12, 28]).

The proof is based on several technical lemmas.

Note first that the almost complex structure $\mathcal{J}_k$, $k = 1$ or 2, is a harmonic map if and only if $\mathcal{V}\,Trace\,\widetilde{\nabla}\mathcal{J}_{k*} = 0$ and $\mathcal{H}\,Trace\,\widetilde{\nabla}\mathcal{J}_{k*} = 0$. According to Proposition 2, $\mathcal{H}\,Trace\,\widetilde{\nabla}\mathcal{J}_{k*} = 0$, $k = 1, 2$, if and only if for every $\sigma \in \mathcal{Z}$ and every $F \in T_\sigma\mathcal{Z}$

$$Trace_{h_t}\,\{T_\sigma\mathcal{Z} \ni A \to h_t(R_{\mathcal{Z}}((\mathcal{J}_k \circ D_A\mathcal{J}_k)^\wedge)A), F)\} = 0.$$

Set for brevity

$$Tr_k(F) = Trace_{h_t}\,\{T_\sigma\mathcal{Z} \ni A \to h_t(R_{\mathcal{Z}}((\mathcal{J}_k \circ D_A\mathcal{J}_k)^\wedge)A), F)\}.$$

Let $\Omega_{k,t}(A, B) = h_t(\mathcal{J}_k A, B)$ be the fundamental 2-form of the almost Hermitian manifold $(\mathcal{Z}, h_t, \mathcal{J}_k)$, $k = 1, 2$. Then, for $A, B, C \in T_\sigma\mathcal{Z}$,

$$h_t((\mathcal{J}_k \circ D_A\mathcal{J}_k)^\wedge, B \wedge C) = -\frac{1}{2}h_t((D_A\mathcal{J}_k)(B), \mathcal{J}_k C) = -\frac{1}{2}(D_A\Omega_{k,t})(B, \mathcal{J}_k C).$$

Lemma 9 ([28]) *Let $\sigma \in \mathcal{Z}$ and $X, Y \in T_{\pi(\sigma)}M$, $V \in \mathcal{V}_\sigma$. Then*

$$(D_{X^h_\sigma}\Omega_{k,t})(Y^h_\sigma, V) = \frac{t}{2}[(-1)^k g(\mathcal{R}(V), X \wedge Y) - g(\mathcal{R}(\sigma \times V), X \wedge K_\sigma Y)],$$

$$(D_V\Omega_{k,t})(X^h_\sigma, Y^h_\sigma) = \frac{t}{2}g(\mathcal{R}(\sigma \times V), X \wedge K_\sigma Y + K_\sigma X \wedge Y) + 2g(V, X \wedge Y).$$

Moreover, $(D_A\Omega_{k,t})(B, C) = 0$ when A, B, C are three horizontal vectors at σ or at least two of them are vertical.

Corollary 4 *Let $\sigma \in \mathcal{Z}$, $X \in T_{\pi(\sigma)}M$, $U \in \mathcal{V}_\sigma$. If $E_1, \dots, E_4$ is an orthonormal basis of $T_{\pi(\sigma)}M$ and V_1, V_2 is an h_t-orthonormal basis of $\mathcal{V}_\sigma$,*

$$\begin{aligned}(\mathcal{J}_k \circ D_{X^h_\sigma}\mathcal{J}_k)^\wedge = -\frac{1}{2}\sum_{i=1}^{4}\sum_{l=1}^{2}[g(\mathcal{R}(\sigma \times V_l), X \wedge E_i)\\ +(-1)^k g(\mathcal{R}(V_l), X \wedge K_\sigma E_i)](E_i^h)_\sigma \wedge V_l,\\ (\mathcal{J}_k \circ D_U\mathcal{J}_k)^\wedge = \sum_{1\le i<j\le 4}[\frac{t}{2}g(\mathcal{R}(\sigma \times U), E_i \wedge E_j - K_\sigma E_i \wedge K_\sigma E_j)\\ -2g(U, E_i \wedge K_\sigma E_j)](E_i^h)_\sigma \wedge (E_j^h)_\sigma.\end{aligned}$$

By Corollary 3, if the vertical part of *Trace* $\widetilde{\nabla}\mathcal{J}_{k*}$ vanishes, then the manifold (M, g) is self dual. In the case when the base manifold is self-dual, simple, but long computations involving Corollary 4, the algebraic Bianchi identity, Corollary 1, and formula (7), give the next two lemmas, which play an essential role in the proof of Theorem 4.

Lemma 10 *Suppose that (M, g) is self-dual. Then, if $U \in \mathcal{V}_\sigma$,*

$$Tr_k(U) = \frac{t}{4}g(\mathcal{B}(U), \mathcal{B}(\sigma)),\ k = 1, 2.$$

Lemma 11 *Suppose that (M, g) is self-dual. Then, if $X \in T_pM$, $p = \pi(\sigma)$,*

$$\begin{aligned}Tr_k(X^h_\sigma) = [1 + (-1)^k]\frac{s(p)}{144}X(s) + \frac{1}{12}(\frac{ts(p)}{6} - 2)X(s)\\ +Trace_{h_t}\{\mathcal{V}_\sigma \ni V \to [\frac{t}{8}g((\nabla_X\mathcal{B})(V), \mathcal{B}(V))\\ +(-1)^{k+1}\frac{ts(p)}{24}g(\delta\mathcal{B}(K_VX), V)]\}.\end{aligned}$$

Sketch of the Proof of Theorem 4 Suppose that $\mathcal{J}_1$ or $\mathcal{J}_2$ is a harmonic map. By Corollary 3, (M, g) is self-dual or self-dual with constant scalar curvature. Moreover, $Tr_k(U) = 0$ for every vertical vector U, and $Tr_k(X^h) = 0$ for every horizontal vector X^h, $k = 1$ or $k = 2$. Note that in both cases the first term in the expression for $Tr_k(X^h)$ given in Lemma 11 vanishes.

Lemma 10 implies that $||\mathcal{B}(\cdot)||^2 = const$ on every fibre $\mathcal{Z}_p$ of the twistor space. One can show that this holds if and only if, at every point $p \in M$, at least three eigenvalues of the Ricci operator ρ coincide. Then the next step in the proof is to demonstrate that the condition $Tr_k(X^h_\sigma) = 0$ for every $\sigma \in \mathcal{Z}$, $X \in T_{\pi(\sigma)}M$, is equivalent to the pair of identities

$$g(\delta\mathcal{B}(X), \sigma) = 0 \tag{25}$$

and

$$\left(\frac{ts(p)}{144}+\frac{1}{6}\right)X(s)-\frac{t}{24}X(||\rho||^2)=0. \tag{26}$$

It is not hard to see that the identity (25) is equivalent to

$$Trace\,\{E\to g((\nabla_E\rho)(K_\sigma E),X)\}=0.$$

Let $r(X,Y)$ be the Ricci tensor and set

$$dr(X,Y,Z)=(\nabla_Y r)(Z,X)-(\nabla_Z r)(Y,X).$$

Thus

$$dr(X,Y,Z)=g((\nabla_Y\rho)(Z),X)-g((\nabla_Z\rho)(Y),X).$$

Take an oriented orthonormal basis $(E_1,\dots,E_4)$ such that $E_2=K_\sigma E_1$ and $E_4=-K_\sigma E_3$. Then

$$dr(X,E_1,E_2)-dr(X,E_3,E_4)=\sum_{m=1}^{4}g((\nabla_{E_m}\rho)(K_\sigma E_m),X).$$

Denote by W_- the 4-tensor corresponding to the operator $\mathcal{W}_-$,

$$W_-(X,Y,Z,T)=g(\mathcal{W}_-(X\wedge Y),Z\wedge T).$$

By the differential Bianchi identity we have

$$dr(X,E_1,E_2)-dr(X,E_3,E_4)=-2[\delta W_-(X,E_1,E_2)-\delta W_-(X,E_3,E_4)]. \tag{27}$$

Since (M,g) is self-dual, we see from the latter identity that identity (25) is always satisfied. Identity (27) shows also that

$$dr(X,\sigma)=0,\quad \sigma\in\mathcal{Z},\quad X\in T_{\pi(\sigma)}M. \tag{28}$$

Let $\lambda_1(p)\le\lambda_2(p)\le\lambda_3(p)\le\lambda_4(p)$ be the eigenvalues of the symmetric operator $\rho_p : T_pM\to T_pM$ in ascending order. It is well-known that the functions $\lambda_1,\dots,,\lambda_4$ are continuous (see, e.g. [22, Chapter Two, §5.7] or [29, Chapter I, §3]). We have seen that, at every point of M, at least three eigenvalues of the operator ρ coincide. The set U of points at which exactly three eigenvalues coincide is open by the continuity of $\lambda_1,\dots,\lambda_4$. For every $p\in U$, denote the simple eigenvalue of ρ_p by $\lambda(p)$ and the triple eigenvalue by $\mu(p)$, so the spectrum of ρ is (λ,μ,μ,μ) with $\lambda(p)\ne\mu(p)$ for every $p\in U$. As is well-known, the implicit function theorem implies that the function λ is smooth. Then the function $\mu = \frac{1}{3}(s - \lambda)$ is also smooth. It is also well-known that, in a neighbourhood of every point p of U, there is a (smooth) unit vector field E which is an eigenvector of ρ corresponding to λ (for a proof see [25, Chapter 9, Theorem 7]). Let α be the dual 1-form to E, $\alpha(X) = g(E, X)$. Then

$$r(X, Y) = (\lambda - \mu)\alpha(X)\alpha(Y) - \mu g(X, Y)$$

in a neighbourhood of p. Using this representation of the Ricci tensor, identity $\delta r = -\frac{1}{2}ds$, and (28) one can prove that the scalar curvature s is locally constant on U. Then identity (26) implies that $||\rho||^2$ is locally constant. Thus, in a neighbourhood of every point $p \in U$, we have $\lambda + 3\mu = a$ and $\lambda^2 + 3\mu^2 = b^2$, where a and b are some constants. It follows that $\mu = 12^{-1}(3a \pm \sqrt{12b^2 - 3a^2})$. Note that $12b^2 - 3a^2 \neq 0$, since otherwise we would have $\mu = \frac{1}{4}a$, hence $\lambda = a - 3\mu = \frac{1}{4}a = \mu$, a contradiction. Since μ is continuous, we see that μ is constant, hence λ is also constant. Then one can show that the 1-form α is parallel. It follows that the restriction of the Ricci tensor to U is parallel.

In the interior of the closed set $M \setminus U$ the eigenvalues of the Ricci tensor coincide, hence the metric g is Einstein on this open set. Therefore, the scalar curvature s is locally constant on $Int\,(M \setminus U)$ and the Ricci tensor is parallel on it. Thus, the Ricci tensor is parallel on the open set $U \cup Int\,(M \setminus U) = M \setminus bU$, where bU stands for the boundary of U. Since $M \setminus bU$ is dense in M, it follows that the Ricci tensor is parallel on M. This implies that the eigenvalues $\lambda_1 \leq \ldots \leq \lambda_4$ of the Ricci tensor are constant. Thus, either M is Einstein, or exactly three of the eigenvalues coincide. In the second case the simple eigenvalue $\lambda = 0$ by [12, Lemma 1]. Therefore M is locally the product of an interval in $\mathbb{R}$ and a 3-dimensional manifold of constant curvature.

Conversely, suppose that (M, g) is self-dual and Einstein, or locally is the product of an interval and a manifold of constant curvature. Then at least three of the eigenvalues of the Ricci tensor coincide which, as we have noted, implies that $||\mathcal{B}(\cdot)||^2 = const$ on every fibre of $\mathcal{Z}$. It follows that $g(\mathcal{B}(\sigma), \mathcal{B}(\tau)) = 0$ for every $\sigma, \tau \in \mathcal{Z}$ with $g(\sigma, \tau) = 0$. Therefore, $T_k(U) = 0$ for every vertical vector U, $k = 1, 2$, by Lemma 10. Moreover, $T_k(X^h) = 0$ by Lemma 11, since the scalar curvature is constant and $\nabla\mathcal{B} = 0$.

8 Almost Hermitian Structures on 4-Manifolds That are Harmonic Maps

Let (M, g) be a Riemannian 4-manifold and J a compatible almost complex structure on it. Henceforth in this section, we shall consider M with the orientation induced by J, and the *positive* twistor space $\mathcal{Z}_+$ will be denoted by $\mathcal{Z}$.

Denote the Ricci tensor of (M,g) by ρ and let ρ^* be the $*$-Ricci tensor of the almost Hermitian manifold (M,g,J). Recall that the latter is defined by

$$\rho^*(X,Y) = trace\{Z \to R(JZ,X)JY\}.$$

Denote by N the Nijenhuis tensor of J

$$N(Y,Z) = -[Y,Z] + [JY,JZ] - J[Y,JZ] - J[JY,Z].$$

We shall also consider this skew-symmetric tensor as a map from $\Lambda^2 TM$ into TM. It is well-known (and easy to check) that

$$2g((\nabla_X J)(Y),Z) = d\Omega(X,Y,Z) - d\Omega(X,JY,JZ) + g(N(Y,Z),JX), \tag{29}$$

for all $X,Y,Z \in TM$.

8.1 The Case of Integrable J

Suppose that the almost complex structure J is integrable. Denote by B the vector field on M dual to the Lee form $\theta = -\delta\Omega \circ J$ with respect to the metric g. Then (29) and the identity $d\Omega = \theta \wedge \Omega$ imply the following well-known formula

$$2(\nabla_X J)(Y) = g(JX,Y)B - g(B,Y)JX + g(X,Y)JB - g(JB,Y)X. \tag{30}$$

It follows that, considering J as a section of the vector bundle $\Lambda^2_+ TM$,

$$\nabla_X J = \frac{1}{2}(JX \wedge B + X \wedge JB).$$

Using this formula and Proposition 2, one can prove the following.

Theorem 5 ([13]) *Suppose that the almost complex structure J is integrable. Then the map $J : (M,g) \to (\mathcal{Z},h_t)$ is harmonic if and only $d\theta$ is a $(1,1)$-form and $\rho(X,B) = \rho^*(X,B)$ for every $X \in TM$.*

Corollary 5 *The map $J : (M,g) \to (\mathcal{Z},h_t)$ defined by an integrable almost Hermitian structure J on (M,g) is a harmonic section if and only if the 2-form $d\theta$ is of type $(1,1)$.*

Remark The 2-form $d\theta$ of a Hermitian surface (M,g,J) is of type $(1,1)$ if and only if the $\star$-Ricci tensor ρ^* is symmetric.

The map $J : M \to \mathcal{Z}$ is an imbedding and one can ask when this imbedding is minimal, i.e. when $J(M)$ is a minimal submanifold of $(\mathcal{Z},h_t)$. Note that J is minimal exactly when it is a harmonic map from M endowed with the metric J^*h_t into $(\mathcal{Z},h_t)$.

If Π is the second fundamental form of the submanifold $J(M)$, then $\Pi(J_*X, J_*Y)$ is the normal component of $\widetilde{\nabla}J_*(X,Y)$. In particular $J(M)$ is a minimal submanifold if and only if the normal component of $Trace\,\widetilde{\nabla}J_*$ vanishes.

Theorem 6 ([13]) *Suppose that the almost complex structure J is integrable. Then the map $J : M \to (\mathcal{Z}, h_t)$ is a minimal isometric imbedding if and only if $d\theta$ is a $(1,1)$ form and $\rho(X,B) = \rho^*(X,B)$ for every $X \perp \{B, JB\}$.*

8.2 The Case of Symplectic J

Suppose that (M, g, J) is almost Kähler (symplectic).

Denote by Λ^2_0TM the subbundle of Λ^2_+TM orthogonal to J (thus $\Lambda^2_0T_pM = \mathcal{V}_{J(p)}$). Under this notation we have the following.

Theorem 7 ([13]) *Let (M, g, J) be an almost Kähler 4-manifold. Then the map $J : (M, g) \to (\mathcal{Z}, h_t)$ is harmonic if and only if the $*$-Ricci tensor ρ^* is symmetric and*

$$Trace\,\{\Lambda^2_0TM \ni \tau \to R(\tau)(N(\tau))\} = 0.$$

The proof makes use of the Weitzenböck formula.

Theorem 8 ([13]) *Let (M, g, J) be an almost Kähler four-manifold. Then the map $J : M \to (\mathcal{Z}, h_t)$ is a minimal isometric imbedding, if and only if the $\star$-Ricci tensor ρ^* is symmetric, and for every $p \in M$*

$$Trace\,\{\Lambda^2_0T_pM \ni \tau \to R_p(\tau)(N(\tau))\} \in \mathcal{N}_p = span\{N(X,Y) : X, Y \in T_pM\}.$$

8.3 Examples [13]

Primary Kodaira Surfaces Every primary Kodaira surface M can be obtained in the following way [24, p. 787]. Let $\varphi_k(z,w)$ be the affine transformations of $\mathbb{C}^2$ given by

$$\varphi_k(z,w) = (z + a_k, w + \overline{a}_k z + b_k),$$

where $a_k, b_k, k = 1, 2, 3, 4$, are complex numbers such that

$$a_1 = a_2 = 0, \quad Im(a_3\overline{a}_4) = mb_1 \neq 0, \quad b_2 \neq 0$$

for some integer $m > 0$. They generate a group G of transformations acting freely and properly discontinuously on $\mathbb{C}^2$, and M is the quotient space $\mathbb{C}^2/G$.

It is well-known that M can also be described as the quotient of $\mathbb{C}^2$ endowed with a group structure by a discrete subgroup Γ. The multiplication on $\mathbb{C}^2$ is defined by

$$(a,b).(z,w) = (z+a, w+\bar{a}z+b), \quad (a,b),(z,w) \in \mathbb{C}^2,$$

and Γ is the subgroup generated by (a_k, b_k), $k = 1,\dots,4$ (see, for example, [5]).

A frame of Γ-left-invariant vector fields on $\mathbb{C}^2 \cong \mathbb{R}^4$ is given by

$$A_1 = \frac{\partial}{\partial x} - x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}, \quad A_2 = \frac{\partial}{\partial y} - y\frac{\partial}{\partial u} - x\frac{\partial}{\partial v}, \quad A_3 = \frac{\partial}{\partial u}, \quad A_4 = \frac{\partial}{\partial v},$$

where $x+iy = z$, $u+iv = w$. Let g be the left-invariant Riemannian metric on $M \cong \mathbb{C}^2/\Gamma$ obtained from the metric on $\mathbb{C}^2$ for which the frame $A_1,\dots,A_4$ is orthonormal.

It is a result by Hasegawa [20] that every complex structure on M is induced by a left-invariant complex structure on $\mathbb{C}^2$. It is not hard to see [8, 27] that a left-invariant almost complex structure J on $\mathbb{C}^2$ compatible with the metric g is integrable if and only if it is given by

$$JA_1 = \varepsilon_1 A_2, \quad JA_3 = \varepsilon_2 A_4, \quad \varepsilon_1, \varepsilon_2 = \pm 1.$$

It is easy to check that, by Theorem 5, the map $J : (M,g) \to (\mathcal{Z}, h_t)$ is harmonic.

It is also easy to give an explicit description of the twistor space $(\mathcal{Z}, h_t)$ [8], since $\Lambda^2_+ M$ admits a global orthonormal frame defined by

$$s_1 = \varepsilon_1 A_1 \wedge A_2 + \varepsilon_2 A_3 \wedge A_4, \quad s_2 = A_1 \wedge A_3 + \varepsilon_1\varepsilon_2 A_4 \wedge A_2,$$
$$s_3 = \varepsilon_2 A_1 \wedge A_4 + \varepsilon_1 A_2 \wedge A_3.$$

Then we have a natural diffeomorphism $F : \mathcal{Z} \cong M \times S^2$ defined by $\sum_{k=1}^{3} x_k s_k(p)) \to (p, x_1, x_2, x_3)$ under which J becomes the section $p \to (p,1,0,0)$. Denote the pushforward of the metric h_t under F again by h_t. For $x = (x_1,x_2,x_3) \in S^2$, set

$$u_1(x) = \varepsilon_1\varepsilon_2(-x_3, 0, x_1), \quad u_2(x) = \varepsilon_2(x_2, -x_1, 0), \quad u_3(x) = 0, \quad u_4(x) = \varepsilon_1(0, x_3, -x_2).$$

The differential F_* sends the horizontal lifts A_i^h $i = 1,\dots,4$, at a point $\sigma = \sum_{k=1}^{3} x_k s_k(p) \in \mathcal{Z}$ to the vectors $A_i + u_i$ of $TM \oplus TS^2$. Then, if $X, Y \in T_pM$ and $P, Q \in T_xS^2$,

$$h_t(X+P, Y+Q) = g(X,Y)$$
$$+t < P - \textstyle\sum_{i=1}^{4} g(X,A_i)u_i(x), Q - \sum_{j=1}^{4} g(Y,A_j)u_j(x) >$$

where $< .,. >$ is the standard metric of $\mathbb{R}^3$.

Now suppose again that J is a left-invariant almost complex structure on $\mathbb{C}^2$ compatible with the metric g. Then the almost Hermitian structure (g, J) is almost Kähler (symplectic) if and only if J is given by [8, 27]

$$JA_1 = -\varepsilon_1 \sin\varphi A_3 + \varepsilon_1\varepsilon_2 \cos\varphi A_4, \quad JA_2 = -\cos\varphi A_3 - \varepsilon_2 \sin\varphi A_4,$$
$$JA_3 = \varepsilon_1 \sin\varphi A_1 + \cos\varphi A_2, \quad JA_4 = -\varepsilon_1\varepsilon_2 \cos\varphi A_1 + \varepsilon_2 \sin\varphi A_2,$$
$$\varepsilon_1, \varepsilon_2 = \pm 1, \ \varphi \in [0, 2\pi).$$

Suppose that J is determined by these identities and set

$$E_1 = A_1, \quad E_2 = -\varepsilon_1 \sin\varphi A_3 + \varepsilon_1\varepsilon_2 \cos\varphi A_4,$$
$$E_3 = \cos\varphi A_3 + \varepsilon_2 \sin\varphi A_4, \quad E_4 = A_2.$$

Then $E_1, \dots, E_4$ is an orthonormal frame of TM for which $JE_1 = E_2$ and $JE_3 = E_4$. Define an orthonormal frame $s_l = s_l^+$, $l = 1, 2, 3$, of $\Lambda^2_+ TM$ by means of $E_1, \dots, E_4$ via (2). Computing $\rho^*(E_i, E_j)$ one can see that the $*$-Ricci tensor is symmetric. Also, computing the curvature and the Nijenhuis tensor, we have

$$Trace\,\{\Lambda^2_0 TM \ni \tau \to R(\tau)(N(\tau))\} = R(s_2)(N(s_2)) + R(s_3)(N(s_3)) = 0.$$

Thus, by Theorem 8, J defines a harmonic map.

As in the preceding case, it is easy to find an explicit description of the twistor space $\mathcal{Z}$ of M and the metric h_t [8]. The frame $\{s_1, s_2, s_3\}$ gives rise to an obvious diffeomorphism $F : \mathcal{Z} \cong M \times S^2$ under which J becomes the map $p \to (p, 1, 0, 0)$. The differential F_* of this diffeomorphism sends the horizontal lifts E_i^h, $i = 1, \dots, 4$, to $E_i + u_i$ where

$$u_1(x) = (x_3\varepsilon_1\varepsilon_2 \cos\varphi, x_3\varepsilon_2 \sin\varphi, -x_1\varepsilon_1\varepsilon_2 \cos\varphi - x_2\varepsilon_2 \sin\varphi),$$
$$u_2(x) = (x_2\varepsilon_1\varepsilon_2 \cos\varphi, -x_1\varepsilon_1\varepsilon_2 \cos\varphi, 0), \quad u_3(x) = (x_2\varepsilon_2 \sin\varphi, -x_1\varepsilon_2 \sin\varphi, 0)$$
$$u_4(x) = (-x_3\varepsilon_2 \sin\varphi, x_3\varepsilon_1\varepsilon_2 \cos\varphi, x_1\varepsilon_2 \sin\varphi - x_2\varepsilon_1\varepsilon_2 \cos\varphi).$$

for $x = (x_1, x_2, x_3) \in S^2$. Then, if $X, Y \in T_pM$ and $P, Q \in T_xS^2$,

$$\begin{aligned} h_t(X + P, Y + Q) = g(X, Y) \\ +t < P - \textstyle\sum_{i=1}^4 g(X, E_i)u_i(x), Q - \sum_{j=1}^4 g(Y, E_j)u_j(x) > . \end{aligned} \tag{31}$$

8.4 Four-Dimensional Lie Groups

By a result of Fino [17] for every left-invariant almost Kähler structure (g, J) with J-invariant Ricci tensor on a 4-dimensional Lie group M there exists an orthonormal frame of left-invariant vector fields $E_1, \dots, E_4$ such that

$$JE_1 = E_2, \quad JE_3 = E_4$$

and

$$[E_1, E_2] = 0, \quad [E_1, E_3] = sE_1 + \frac{s^2}{t}E_2, \quad [E_1, E_4] = \frac{s^2 - t^2}{2t}E_1 - sE_2$$
$$[E_2, E_3] = -tE_1 - sE_2, \quad [E_2, E_4] = -sE_1 - \frac{s^2 - t^2}{2t}E_2, \quad [E_3, E_4] = -\frac{s^2 + t^2}{t}E_3$$

where s and $t \neq 0$ are real numbers. Using this table one can compute the $*$-Ricci and Nijenhuis tensors. The computation shows that J defines a harmonic map by virtue of Theorem 8.

8.5 Inoue Surfaces of Type S^0

Let us recall the construction of these surfaces [21]. Take a matrix $A \in SL(3, \mathbb{Z})$ with a real eigenvalue $\alpha > 1$ and two complex eigenvalues β and $\overline{\beta}$, $\beta \neq \overline{\beta}$. Choose eigenvectors $(a_1, a_2, a_3) \in \mathbb{R}^3$ and $(b_1, b_2, b_3) \in \mathbb{C}^3$ of A corresponding to α and β, respectively. Then the vectors (a_1, a_2, a_3), (b_1, b_2, b_3), $(\overline{b_1}, \overline{b_2}, \overline{b_3})$ are $\mathbb{C}$-linearly independent. Denote the upper-half plane in $\mathbb{C}$ by $\mathbf{H}$ and let Γ be the group of holomorphic automorphisms of $\mathbf{H} \times \mathbb{C}$ generated by

$$g_o : (w, z) \to (\alpha w, \beta z), \quad g_i : (w, z) \to (w + a_i, z + b_i), \ i = 1, 2, 3.$$

The group Γ acts on $\mathbf{H} \times \mathbb{C}$ freely and properly discontinuously. Then $M = (\mathbf{H} \times \mathbb{C})/\Gamma$ is a complex surface known as the Inoue surface of type S^0. It has been shown by Tricerri [32] that every such a surface admits a locally conformal Kähler metric g (cf. also [14]) obtained from the Γ-invariant Hermitian metric

$$g = \frac{1}{v^2}(du \otimes du + dv \otimes dv) + v(dx \otimes dx + dy \otimes dy), \quad u + iv \in \mathbf{H}, \quad x + iy \in \mathbb{C}.$$

on $\mathbf{H} \times \mathbb{C}$.

By Corollary 5, $J : (M, g) \to (\mathcal{Z}, h_t)$ is a harmonic section. It is also a minimal isometric imbedding by Theorem 6. However, J is not a harmonic map according to Theorem 5.

Acknowledgements This paper is an expanded version on the author's talk at the INdAM workshop in Rome, November, 16–20, 2015, on the occasion of the sixtieth birthday of Simon Salamon. The author would like to express his gratitude to the organizers and Simon for the invitation to take part in the workshop and for the wonderful and stimulating environment surrounding it. Special thanks are also due to the referee whose remarks helped to improve the final version of the article.

The author is partially supported by the National Science Fund, Ministry of Education and Science of Bulgaria under contract DFNI-I 02/14.

References

1. E. Abbena, An example of an almost Kähler manifold which is not Kählerian. Boll. Un. Mat. Ital. A(6) **3**, 383–392 (1984)
2. M.F. Atiyah, N.J. Hitchin, I.M. Singer, Self-duality in four-dimensional Riemannian geometry. Proc. R. Soc. Lond. Ser. A **362**, 425–461 (1978)
3. A. Besse, *Einstein Manifolds*. Classics in Mathematics (Springer, Berlin, 2008)
4. G. Bor, L. Hernández-Lamoneda, M. Salvai, Orthogonal almost-complex structures of minimal energy. Geom. Dedicata **127**, 75–85 (2007)
5. C. Borcea, Moduli for Kodaira surfaces. Compos. Math. **52**, 373–380 (1984)
6. E. Calabi, H. Gluck, What are the best almost-complex structures on the 6-sphere? Proc. Symp. Pure Math. **54**, part 2, 99–106 (1993)
7. J. Davidov, Einstein condition and twistor spaces of compatible partially complex structures. Differ. Geom. Appl. **22**, 159–179 (2005)
8. J. Davidov, Twistorial construction of minimal hypersurfaces. Int. J. Geom. Methods Modern Phys. **11**(6), 1459964 (2014)
9. J. Davidov, O. Mushkarov, On the Riemannian curvature of a twistor space. Acta Math. Hungarica **58**, 319–332 (1991)
10. J. Davidov, O. Mushkarov, Harmonic almost-complex structures on twistor spaces. Israel J. Math. **131**, 319–332 (2002)
11. J. Davidov, O. Mushkarov, Harmonicity of the Atiyah-Hitchin-Singer and Eells-Salamon almost complex structures. Ann. Mat. Pura Appl. https://doi.org/10.1007/s10231-017-0675-y; arXiv:1611.06496v2 [math.DG]
12. J. Davidov, G. Grantcharov, O. Mushkarov, Twistorial examples of $*$-Einstein manifolds. Ann. Glob. Anal. Geom. **20**, 103–115 (2001)
13. J. Davidov, A. Ul Haq, O. Mushkarov, Almost complex structures that are harmonic maps. J. Geom. Phys. https://doi.org/10.1016/j.geomphys.2017.09.010; arXiv:1504.01610v3 [math.DG]
14. S. Dragomir, L. Ornea, *Locally Conformal Kähler Geometry*. Progress in Mathematics, vol. 155 (Birkhäuser, Boston, 1998)
15. J. Eells, L. Lemaire, *Selected Topics in Harmonic Maps*. CBMS Regional Conference Series in Mathematics, vol. 50 (AMS, Providence, RI, 1983)
16. J. Eells, S. Salamon, Twistorial constructions of harmonic maps of surfaces into four-manifolds. Ann. Scuola Norm. Sup. Pisa, Ser. IV, **12**, 589–640 (1985)
17. A. Fino, Almost Kähler 4-dimensional Lie groups with J-invariant Ricci tensor. Differ. Geom. Appl. **23**, 26–37 (2005)
18. A. Gray, L.M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants. Ann. Mat. Pure Appl. **123**, 288–294 (1980)
19. D. Gromol, W. Klingenberg, W. Meyer, *Riemannsche Geometrie in Grossen*. Lecture Notes in Mathematics, vol. 55 (Springer, Berlin, 1968)
20. K. Hasegawa, Complex and Kähler structures on compact solvmanifolds. J. Symplectic Geom. **3**, 749–767 (2005)

21. M. Inoue, On surfaces of class VII_0. Invent. Math. **24**, 269–310 (1974)
22. T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1980)
23. I. Kim, Almost Kähler anti-self-dual metrics, Ph.D. thesis, Stony Brook University, May 2014, www.math.stonybrook.edu/alumni/2014-Inyong-Kim.pdf; see also arXiv:1511.07656v1 [math.DG] 24 Nov 2015
24. K. Kodaira, On the structure of compact complex analytic surfaces I. Am. J. Math. **86**, 751–798 (1964)
25. P. Lax, *Linear Algebra and Its Applications* (Wiley, Hoboken, NJ, 2007)
26. C. LeBrun, Anti-self-dual hermitian metrics on blow-up Hopf surfaces. Math. Ann. **289**, 383–392 (1991)
27. O. Muškarov, Two remarks on Thurston's example, in *Complex Analysis and Applications '85 (Varna, 1985)* (Publishing House of Bulgarian Academy of Sciences, Sofia, 1986), pp. 461–468
28. O. Muškarov, Structures presque hermitienes sur espaces twistoriels et leur types. C. R. Acad. Sci. Paris Sér. I Math. **305**, 307–309 (1987)
29. F. Rellich, *Perturbation Theory of Eigenvalue Problems*. Notes on Mathematics and Its Applications (Gordon and Breach Science Publishers, New York, 1969)
30. I.M. Singer, J.A. Thorpe, The curvature of 4-dimensional Einstein spaces, in *Global Analysis. Papers in Honor of K. Kodaira* (University of Tokyo Press/Princeton University Press, Princeton, 1969), pp. 355–365
31. W.P. Thurston, Some simple examples of symplectic manifolds. Proc. Am. Math. Soc. **55**, 467–468 (1976)
32. F. Tricerri, Some example of locally conformal Kähler manifolds. Rend. Sem. Mat. Univ. Torino **40**, 81–92 (1982)
33. J. Vilms, Totally geodesic maps. J. Differ. Geom. **4**, 73–79 (1970)
34. C.M. Wood, Instability of the nearly-Kähler six-sphere. J. Reine Angew. Math. **439**, 205–212 (1993)
35. C.M. Wood, Harmonic almost-complex structures. Compos. Math. **99**, 183–212 (1995)

Killing 2-Forms in Dimension 4

Paul Gauduchon and Andrei Moroianu

Dedicated to Simon Salamon on the occasion of his 60th birthday

Abstract A Killing p-form on a Riemannian manifold (M, g) is a p-form whose covariant derivative is totally antisymmetric. If M is a connected, oriented, 4-dimensional manifold admitting a non-parallel Killing 2-form ψ, we show that there exists a dense open subset of M on which one of the following three exclusive situations holds: either ψ is everywhere degenerate and g is locally conformal to a product metric, or g gives rise to an ambikähler structure of Calabi type, or, generically, g gives rise to an ambitoric structure of hyperbolic type, in particular depends locally on two functions of one variable. Compact examples of either types are provided.

Keywords Ambikähler structures • Ambitoric structures • Hamiltonian forms • Killing forms

1 Introduction

On any n-dimensional Riemannian manifold (M, g), an exterior p-form ψ is called *conformal Killing* [13] if its covariant derivative $\nabla\psi$ is of the form

$$\nabla_X \psi = \alpha \wedge X^\flat + X \lrcorner \beta, \tag{1}$$

P. Gauduchon (✉)
CMLS, École Polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau, France
e-mail: paul.gauduchon@polytechnique.edu

A. Moroianu
Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France
e-mail: andrei.moroianu@math.cnrs.fr

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_7

for some $(p-1)$-form α and some $(p+1)$-form β, which are then given by

$$\alpha = \frac{(-1)^p}{n-p+1}\,\delta\psi, \qquad \beta = \frac{1}{p+1}\,d\psi. \tag{2}$$

Conformal Killing forms have the following conformal invariance property: if ψ is a conformal Killing p-form with respect to the metric g, then, for any positive function f, $\tilde{\psi} := f^{p+1}\,\psi$ is conformal Killing with respect to the conformal metric $\tilde{g} := f^2\,g$. In other words, if L denotes the real line bundle $|\Lambda^n TM|^{\frac{1}{n}}$ and $\ell, \tilde{\ell}$ denote the sections of L determined by $g, \tilde{g}$, then, for *any* Weyl connection D relative to the conformal class $[g]$, the section $\boldsymbol{\psi} := \psi \otimes \ell^{p+1} = \tilde{\psi} \otimes \tilde{\ell}^{p+1}$ of $\Lambda^p T^*M \otimes L^{p+1}$ satisfies

$$D_X \boldsymbol{\psi} = \boldsymbol{\alpha} \wedge X + X \lrcorner \boldsymbol{\beta}, \tag{3}$$

for some section $\boldsymbol{\alpha}$ of $\Lambda^{p-1}T^*M \otimes L^{p-1}$ and some section $\boldsymbol{\beta}$ of $\Lambda^{p+1}T^*M \otimes L^{p+1}$ (depending on D), cf. e.g. [4, Appendix B].

The p-form ψ is called *Killing*, resp. $*$-*Killing*, with respect to g, if ψ satisfies (1) *and* $\alpha = 0$, resp. $\beta = 0$. In particular, Killing forms are co-closed, $*$-Killing forms are closed, and, if M is oriented and $*$ denotes the induced Hodge star operator, ψ is Killing if and only if $*\psi$ is $*$-Killing.

Although the terminology comes from the fact that Killing 1-forms are just metric duals of Killing vector fields, and thus encode infinitesimal symmetries of the metric, no geometric interpretation of Killing p-forms exists in general in terms of symmetries when $p \geq 2$, except in the case of Killing 2-forms in dimension 4, which is special for various reasons, the most important being the self-duality phenomenon.

On any oriented four-dimensional manifold (M, g), the Hodge star operator $*$, acting on 2-forms, is an involution and, therefore, induces the well known orthogonal decomposition

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M, \tag{4}$$

where $\Lambda^2 M$ stands for the vector bundle of (real) 2-forms on M and $\Lambda^\pm M$ for the eigen-subbundle for the eigenvalue ± 1 of $*$. Accordingly, any 2-form ψ splits as

$$\psi = \psi_+ + \psi_-, \tag{5}$$

where ψ_+, resp. ψ_-, is the *self-dual*, resp. the *anti-self-dual* part of ψ, defined by $\psi_\pm = \frac{1}{2}(\psi \pm *\psi)$. Since $*$ acting on 2-forms is conformally invariant, a 2-form ψ is conformal Killing if and only if ψ_+ and ψ_- are separately conformal Killing, meaning that

$$\nabla_X \psi_+ = (\alpha_+ \wedge X^\flat)_+, \qquad \nabla_X \psi_- = (\alpha_- \wedge X^\flat)_- \tag{6}$$

for some real 1-forms α_+, α_-, and ψ is Killing, resp. $*$-Killing, if, in addition,

$$\alpha_+ = -\alpha_-, \quad \text{resp.} \quad \alpha_+ = \alpha_-. \tag{7}$$

Throughout this paper, (M, g) will denote a connected, oriented, 4-dimensional Riemannian manifold and $\psi = \psi_+ + \psi_-$ a non-trivial $*$-Killing 2-form on M (the choice of the $*$-Killing ψ, instead of the Killing 2-form $*\psi$ is of pure convenience). We also discard the non-interesting case when ψ is parallel.

On the open set, M_0^+, resp. M_0^-, where ψ_+, resp. ψ_-, is non-zero, the associated skew-symmetric operators Ψ_+, Ψ_-, are of the form $\Psi_+ = f_+ J_+$, resp. $\Psi_- = f_- J_-$, where J_+, resp. J_-, is an almost complex structure inducing the chosen, resp. the opposite, orientation of M, and f_+, resp. f_-, is a positive function. It is then easily checked, cf. Sect. 2 below, that the first, resp. the second, condition in (6) is equivalent to the condition that the pair $(g_+ := f_+^{-2} g, J_+)$, resp. the pair $(g_- := f_-^{-2} g, J_-)$, is *Kähler*. On the open set $M_0 = M_0^+ \cap M_0^-$, which is actually dense in M, cf. Lemma 2.1 below, we thus get *two* Kähler structures, whose metrics belong to the same conformal class and whose complex structures induce opposite orientations (in particular, commute), hence an *ambikähler structure*, as defined in [4]. This actually holds if ψ is simply conformal Killing and had been observed in the twistorial setting by Pontecorvo in [12], cf. also [4, Appendix B2]. The additional coupling condition (7), which, on M_0, reads $J_+ df_+ = J_- df_-$, cf. Sect. 2, then has strong consequences, that we now explain.

A first main observation, cf. Proposition 3.3, is that the open subset, M_S, where ψ is of maximal rank, hence a symplectic 2-form, is either empty or dense in M.

The case when M_S is empty is the case when ψ is *decomposable*, i.e. $\psi \wedge \psi = 0$ everywhere; equivalently, $|\psi_+| = |\psi_-|$ everywhere; on M_0, we then have $f_+ = f_-$, hence $g_+ = g_- =: g_K$, and (M_0, g_K) is locally a product of two (real) Kähler surfaces $(\Sigma, g_\Sigma, \omega_\Sigma)$ and $(\tilde{\Sigma}, g_{\tilde{\Sigma}}, \omega_{\tilde{\Sigma}})$, with $f_+ = f_-$ constant on $\tilde{\Sigma}$, cf. Sect. 6. In this case, no non-trivial Killing vector field shows up in general, but a number of compact examples involving Killing vector fields are provided, coming from [9].

The case when M_S is dense is first handled in Proposition 2.4, where we show that the vector field $K_1 := -\frac{1}{2}\alpha^\sharp$ is then Killing with respect to g—the chosen normalization is for further convenience—and that each eigenvalue of the Ricci tensor, Ric, of g is of multiplicity at least 2; moreover, on the (dense) open set $M_1 = M_S \cap M_0$, K_1 is Killing with respect to g_+, g_- and Hamiltonian with respect to the Kähler forms $\omega_+ := g_+(J_+\cdot, \cdot)$ and $\omega_- := g_-(J_-\cdot, \cdot)$; also, Ric is both J_+- and J_--invariant, cf. Proposition 2.4 below. On M_1, the ambikähler structure (g_+, J_+, ω_+), (g_-, J_-, ω_-) is then of the type described in Proposition 11 (iii) of [4].

In Sect. 3, we set the stage for a *separation of variables* by introducing new functions x, y, defined by $x = \frac{1}{2}(f_+ + f_-)$ and $y = \frac{1}{2}(f_+ - f_-)$, which, up to a factor 2, are the “eigenvalues” of ψ, and whose gradients are easily shown to be orthogonal. In Proposition 3.1, we show that $|dx|^2 = A(x)$ and $|dy|^2 = B(y)$, for some positive functions A and B of one variable. In terms of the new functions x, y, the dual 1-form of K_1 with respect to g is simply $J_+ dx + J_+ dy$. Furthermore, in

Proposition 3.2 a second Killing vector field, K_2, shows up, whose dual 1-form is $y^2 J_+ dx + x^2 J_+ dy$ and which turns out to coincide, up to a constant factor, with the Killing vector field constructed by W. Jelonek in [8, Lemma B], cf. also the proof of Proposition 11 in [4], namely the image of K_1 by the *Killing symmetric endomorphism* $S = \Psi_+ \circ \Psi_- + \frac{(f_+^2 + f_-^2)}{2} \mathrm{I}$, cf. Remark 3.1.

In Proposition 3.3, we then show that either K_2 is a (positive) constant multiple of K_1, and we end up with an ambikähler structure *of Calabi type*, according to Definition 5.1 taken from [1], or K_1, K_2 are independent on a dense open subset of M, determining an *ambitoric structure*, as defined in [3, 4].

The Calabi case is considered in Sect. 5, where it is shown that, conversely, any ambikähler structure of Calabi type gives rise, up to scaling, to a 1-parameter family of pairs $(g^{(k)}, \psi^{(k)})$, where $g^{(k)}$ is a Riemannian metric in the conformal class and $\psi^{(k)}$ a $*$-Killing 2-form with respect to $g^{(k)}$, cf. Theorem 5.1 and Remark 5.1. The example of *Hirzebruch-like* ruled surfaces is described in Sect. 8.

The ambitoric case is the case when dx and dy are independent on a dense open subset of M. In Sect. 4, we show that x, y can be locally completed into a full system of coordinates by the addition of two "angular coordinates", s, t, in such a way that $K_1 = \frac{\partial}{\partial s}$ and $K_2 = \frac{\partial}{\partial t}$ and giving rise to a general Ansatz, described in Theorem 4.1. As an Ansatz for the underlying ambikähler structure, this turns out to be the same as the ambitoric Ansatz of Proposition 13 in [4] for the "quadratic" polynomial $q(z) = 2z$, hence in the *hyperbolic* normal form of [4, Section 5.4], when the functions x, y are identified with the *adapted coordinates* x, y in [4].

The main observation at this point is that, while the adapted coordinates in [4] are obtained via a quadratic transformation, cf. [4, Section 4.3], the functions x, y are here naturally attached to the $*$-Killing 2-form ψ which determines the ambitoric structure. This is quite reminiscent of the *orthotoric* situation, described in [1] in dimension 4 and in [2] in all dimensions, where the separation of variables—and the corresponding Ansatz—are similarly obtained via the "eigenvalues" of a *Hamiltonian 2-form*, which share the same properties as the "eigenvalues" x, y of the $*$-Killing 2-form ψ.

In spite of this, the $*$-Killing 2-forms considered in this paper are *not* Hamiltonian 2-forms in general—for a general discussion about Killing or $*$-Killing 2-forms versus Hamiltonian 2-forms, cf. [10], in particular Theorem 4.5 and Proposition 4.8, and, also, [2, Appendix A]—but, in many respects, at least in dimension 4, the role played by Hamiltonian 2-forms in the orthotoric case is played by $*$-Killing 2-forms in the (hyperbolic) ambitoric case.

The three situations described above, namely the decomposable, the Calabi ambikähler and the ambitoric case, cf. Proposition 3.3, are nicely illustrated in the example of the round 4-sphere described in Sect. 7, on which every $*$-Killing form can be written as the restriction of a constant 2-form $\mathsf{a} \in \mathfrak{so}(5) \simeq \Lambda^2\mathbb{R}^5$, which is also the 2-form associated to the covariant derivative of the Killing vector field induced by a. If a has rank 2, the same holds for its restriction on a dense open subset of the sphere, so this corresponds to the decomposable case. Otherwise, a can be expressed as $\lambda\, e_1 \wedge e_2 + \mu\, e_3 \wedge e_4$—cf. Sect. 7 for the notation—with λ, μ both positive, and, depending on whether λ and μ are equal or not, we obtain on a dense subset of the sphere an ambikähler structure of Calabi type or a hyperbolic ambitoric structure respectively. By using the hyperbolic ambitoric Ansatz of Sect. 4, it is eventually shown that the resulting $*$-Killing 2-forms are actually $*$-Killing with respect to infinitely many non-isometric Riemannian metrics on S^4, cf. Remark 7.2.

2 Killing 2-Forms and Ambikähler Structures

In what follows, (M, g) denotes a connected, oriented, 4-dimensional Riemannian manifold admitting a non-parallel Killing 2-form φ, and $\psi := *\varphi$ denotes the corresponding $*$-Killing 2-form; we then have

$$\nabla_X \psi = \alpha \wedge X^\flat, \tag{8}$$

for some real, non-zero, 1-form α, where ∇ denotes the Levi-Civita connection of g and $X^\flat$ the dual 1-form of X with respect to g, cf. [13]. By anti-symmetrizing and by contracting (8), it is easily checked that ψ is closed and that

$$\delta\psi = 3\alpha, \tag{9}$$

where δ denotes the codifferential with respect to g. Denote by $\psi_+ = \frac{1}{2}(\psi + *\psi)$, resp. $\psi_- = \frac{1}{2}(\psi - *\psi)$, the self-dual, resp. the anti-self-dual, part of ψ, where $*$ is the Hodge operator induced by the metric g and the chosen orientation. Then, (8) is equivalent to the following two conditions

$$\begin{aligned}\nabla_X \psi_+ &= \left(\alpha \wedge X^\flat\right)_+ = \frac{1}{2}\alpha \wedge X^\flat + \frac{1}{2} X \lrcorner * \alpha,\\ \nabla_X \psi_- &= \left(\alpha \wedge X^\flat\right)_- = \frac{1}{2}\alpha \wedge X^\flat - \frac{1}{2} X \lrcorner * \alpha.\end{aligned} \tag{10}$$

Here, we used the general identity:

$$* (X^\flat \wedge \phi) = (-1)^p X \lrcorner * \phi, \tag{11}$$

for any vector field X and any p-form ϕ on any oriented Riemannian manifold. In particular, ψ_+ and ψ_- are *conformally Killing*, cf. [13]. The datum of a (non-parallel) $*$-Killing 2-form ψ on (M, g) is then equivalent to the datum of a pair (ψ_+, ψ_-) consisting of a self-dual 2-form ψ_+ and an anti-self-dual 2-form ψ_-, both conformally Killing and linked together by

$$d\psi_+ + d\psi_- = 0, \tag{12}$$

or, equivalently, by

$$\delta\psi_+ = \delta\psi_-. \tag{13}$$

We denote by Ψ, Ψ_+, Ψ_- the anti-symmetric endomorphisms of TM associated to ψ, ψ_+, ψ_- respectively via the metric g, so that $g(\Psi(X),Y) = \psi(X,Y)$, $g(\Psi_+(X),Y) = \psi_+(X,Y)$, $g(\Psi_-(X),Y) = \psi_-(X,Y)$. On the open set, M_0, of M where Ψ_+ and Ψ_- have no zero, denote by J_+, J_- the corresponding almost complex structures:

$$J_+ := \frac{\Psi_+}{f_+}, \qquad J_- := \frac{\Psi_-}{f_-}, \tag{14}$$

where the positive functions f_+,f_- are defined by

$$f_+ := \frac{|\Psi_+|}{\sqrt{2}}, \qquad f_- := \frac{|\Psi_-|}{\sqrt{2}} \tag{15}$$

(here, the norms $|\Psi_+|$, $|\Psi_-|$, are relative to the conformally invariant inner product defined on the space of anti-symmetric endomorphisms of TM by $(A,B) := -\frac{1}{2}\mathrm{tr}(A \circ B)$); the open set M_0 is then defined by the condition

$$f_+ > 0, \qquad f_- > 0. \tag{16}$$

Notice that J_+ and J_- induce opposite orientations, hence commute to each other, so that the endomorphism

$$\tau := -J_+J_- = -J_-J_+, \tag{17}$$

is an involution of the tangent bundle of M_0.

From (8), we get

$$\nabla_X\Psi = \alpha \wedge X, \tag{18}$$

with the following general convention: for any 1-form α and any vector field X, $\alpha \wedge X$ denotes the anti-symmetric endomorphism of TM defined by $(\alpha \wedge X)(Y) = \alpha(Y)X - g(X,Y)\alpha^\sharp$, where $\alpha^\sharp$ is the dual vector field to α relative to g (notice that the latter expression is actually independent of g in the conformal class $[g]$ of g). Equivalently:

$$\nabla_X\Psi_+ = (\alpha \wedge X)_+, \qquad \nabla_X\Psi_- = (\alpha \wedge X)_-. \tag{19}$$

We infer $(\nabla_X \Psi_+, \Psi_+) = \frac{1}{2}(d|\Psi_+|^2)(X) = (\Psi_+, \alpha \wedge X) = \big(\Psi_+(\alpha)\big)(X)$, hence $\Psi_+(\alpha) = \frac{1}{2} d|\Psi_+|^2$. Similarly, $\Psi_-(\alpha) = \frac{1}{2} d|\Psi_-|^2$. By using (14), we then get

$$\begin{aligned} \alpha &= -2\Psi_+ \left(\frac{d|\Psi_+|}{|\Psi_+|}\right) = -2J_+ df_+ \\ &= -2\Psi_- \left(\frac{d|\Psi_-|}{|\Psi_-|}\right) = -2J_- df_-. \end{aligned} \tag{20}$$

In particular,

$$J_+ df_+ = J_- df_-. \tag{21}$$

Remark 2.1 For any $*$-Killing 2-form ψ as above, denote by $\Phi = \Psi_+ - \Psi_-$ the skew-symmetric endomorphism associated to the Killing 2-form $\varphi = *\psi$ and by S the symmetric endomorphism defined by

$$S = -\frac{1}{2}\,\Phi \circ \Phi = \Psi_+ \circ \Psi_- + \frac{1}{2}(f_+^2 + f_-^2)\,\mathrm{I} = \frac{1}{2}\,\Psi \circ \Psi + (f_+^2 + f_-^2)\,\mathrm{I}, \tag{22}$$

where I denotes the identity of TM. Then, S is *Killing* with respect to g, meaning that the symmetric part of ∇S is zero or, equivalently, that $g((\nabla_X S)X, X) = 0$ for any vector field X, cf. [11], [4, Appendix B]. This readily follows from the fact that $\nabla_X \Phi(X) = X \lrcorner * (\alpha \wedge X) = 0$, so that $g(\nabla_X S(X), X) = -2g(\nabla_X \Phi(X), \Phi(X)) = 0$, for any vector field X.

Lemma 2.1 *The open subset M_0 defined by (16) is dense in M.*

Proof Denote by $M_0^{\pm}$ the open set where $f_\pm \neq 0$, so that $M_0 = M_0^+ \cap M_0^-$. It is sufficient to show that each $M_0^{\pm}$ is dense. If not, $f_\pm = 0$ on some non-empty open set, V, of M, so that $\psi_\pm = 0$ on V, hence is identically zero, since $\psi_\pm$ is conformally Killing, cf. [13]; this, in turn, implies that α, hence also $\nabla\psi$, is identically zero, in contradiction to the hypothesis that ψ is non-parallel. □

In view of the next proposition, we recall the following definition, taken from [4]:

Definition 2.1 ([4]) An *ambikähler structure* on an oriented 4-manifold M consists of a pair of Kähler structures, $\big(g_+, J_+, \omega_+ = g_+(J_+\cdot, \cdot)\big)$ and $\big(g_-, J_-, \omega_- = g_-(J_-\cdot, \cdot)\big)$, where the Riemannian metrics g_+, g_- belong to the same conformal class, i.e. $g_- = f^2\, g_+$, for some positive function f, and the complex structure J_+, resp. the complex structure J_-, induces the chosen orientation, resp. the opposite orientation; equivalently, the Kähler forms ω_+ and ω_- are self-dual and anti-self-dual respectively.

We then have:

Proposition 2.1 *Let (M, g) be a connected, oriented, 4-dimensional Riemannian manifold, equipped with a non-parallel $*$-Killing 2-form $\psi = \psi_+ + \psi_-$ as above. Then, on the dense open subset, M_0, of M defined by (16), the pair (g, ψ) gives* *rise to an ambikähler structure* (g_+, J_+, ω_+), (g_-, J_-, ω_-), *with* $g_\pm = f_\pm^{-2}\, g$ *and* $J_\pm = f_\pm^{-1}\, \Psi_\pm$, *by setting* $f_\pm = |\Psi_\pm|/\sqrt{2}$. *In particular, this ambikähler structure is equipped with two non-constant positive functions* f_+, f_-, *satisfying the two conditions*

$$f = \frac{f_+}{f_-}, \tag{23}$$

where $g_- = f^2\, g_+$, *and*

$$\tau(df_+) = df_-. \tag{24}$$

Conversely, any ambikähler structure (g_+, J_+, ω_+), $(g_- = f^2\, g_+, J_-, \omega_-)$ *equipped with two non-constant positive functions* f_+, f_- *satisfying* (23)–(24) *arises from a unique pair* (g, ψ), *where* g *is the Riemannian metric in the conformal class* $[g_+] = [g_-]$ *defined by*

$$g = f_+^2\, g_+ = f_-^2\, g_-, \tag{25}$$

and ψ *is the* $*$*-Killing* 2*-form relative to* g *defined by*

$$\psi = f_+^3\, \omega_+ + f_-^3\, \omega_-. \tag{26}$$

Proof Before starting the proof, we recall the following general facts. (i) For any two Riemannian metrics, g and $\tilde{g} = \varphi^{-2}\, g$, in a same conformal class, and for any anti-symmetric endomorphism, A, of the tangent bundle with respect to the conformal class $[g] = [\tilde{g}]$, the covariant derivatives $\nabla^{\tilde{g}} A$ and $\nabla^g A$ are related by

$$\nabla^{\tilde{g}}_X A = \nabla^g_X A + \left[A, \frac{d\varphi}{\varphi} \wedge X\right] = A\left(\frac{d\varphi}{\varphi}\right) \wedge X + \frac{d\varphi}{\varphi} \wedge A(X), \tag{27}$$

by setting $A\left(\frac{d\varphi}{\varphi}\right) = -\frac{d\varphi}{\varphi} \circ A$. (ii) For any 1-form β and any vector field X, we have

$$\begin{aligned}(\beta \wedge X)_+ &= \frac{1}{2}\beta \wedge X - \frac{1}{2} J_+\beta \wedge J_+X - \frac{1}{2}\beta(J_+X)\, J_+\\ &= \frac{1}{2}\beta \wedge X + \frac{1}{2} J_-\beta \wedge J_-X + \frac{1}{2}\beta(J_-X)\, J_-,\end{aligned} \tag{28}$$

and

$$\begin{aligned}(\beta \wedge X)_- &= \frac{1}{2}\beta \wedge X - \frac{1}{2} J_-\beta \wedge J_-X - \frac{1}{2}\beta(J_-X)\, J_-\\ &= \frac{1}{2}\beta \wedge X + \frac{1}{2} J_+\beta \wedge J_+X + \frac{1}{2}\beta(J_+X)\, J_+,\end{aligned} \tag{29}$$

for *any* orthogonal (almost) complex structures J_+ and $J-$ inducing the chosen and the opposite orientation respectively.

From (14), (19), (20) and (28), we thus infer

$$\begin{aligned}\nabla_X J_+ &= -2\left(J_+\left(\frac{df_+}{|f_+|}\right)\wedge X\right)_+ - \frac{df_+}{f_+}(X)\,J_+\\ &= -J_+\left(\frac{df_+}{f_+}\right)\wedge X - \frac{df_+}{f_+}\wedge J_+X + \frac{df_+}{f_+}(X)\,J_+ - \frac{df_+}{f_+}(X)\,J_+\\ &= -J_+\left(\frac{df_+}{f_+}\right)\wedge X - \frac{df_+}{f_+}\wedge J_+X = \left[\frac{df_+}{f_+}\wedge X, J_+\right]\end{aligned} \tag{30}$$

which, by using (27), is equivalent to

$$\nabla^{g_+} J_+ = 0, \tag{31}$$

where ∇^{g_+} denotes the Levi-Civita connection of the conformal metric $g_+ = f_+^{-2}\,g$, meaning that the pair (g_+, J_+) is *Kähler*. Similarly, we have

$$\nabla_X J_- = \left[\frac{df_-}{f_-}\wedge X, J_-\right] \tag{32}$$

or, equivalently:

$$\nabla^{g_-} J_- = 0, \tag{33}$$

where ∇^{g_-} denotes the Levi-Civita connection of the conformal metric $g_- = f_-^{-2}\,g$, meaning that the pair (g_-, J_-) is Kähler as well. We thus get on M_0 an *ambikähler structure* in the sense of Definition 2.1. Moreover, because of (21), f_+ and f_- evidently satisfy (23)–(24).

For the converse, define g by

$$g = f_+^2\,g_+ = f_-^2\,g_- \tag{34}$$

and denote by ∇ the Levi-Civita connection of g. By defining $\Psi_+ = f_+\,J_+$, $\Psi_- = f_-\,J_-$ and $\Psi = \Psi_+ + \Psi_-$, we get

$$\begin{aligned}\nabla_X\Psi_+ &= \nabla_X(f_+\,J_+)\\ &= \nabla_X^{g_+}(f_+\,J_+) + \left[\frac{df_+}{f_+}\wedge X, f_+\,J_+\right]\\ &= df_+(X)\,J_+ - J_+df_+\wedge X - df_+\wedge J_+X\\ &= -2(J_+df_+\wedge X)_+.\end{aligned} \tag{35}$$

Similarly,

$$\nabla_X \Psi_- = -2\,(J_- df_- \wedge X)_-. \tag{36}$$

By using (21), we obtain

$$\nabla_X \Psi = \alpha \wedge X, \tag{37}$$

with $\alpha := -2\,J_+ df_+ = -2\,J_- df_-$, meaning that the associated 2-form $\psi(X,Y) := g(\Psi(X),Y)$, is $*$-Killing. Finally $\psi = f_+\, g(J_+\cdot,\cdot) + f_-\, g(J_-\cdot,\cdot) = f_+^3\,\omega_+ + f_-^3\,\omega_-$. □

Remark 2.2 The fact that the pair $(g_+ = f_+^{-2}\, g, J_+)$, resp. the pair $(g_- = f_-^{-2}\, g, J_-)$, is Kähler only depends on, in fact is equivalent to, $\Psi_+ = f_+\, J_+$, resp. $\Psi_- = f_-\, J_-$, being conformal Killing, i.e. ψ being conformally Killing. This was observed in a twistorial setting by Pontecorvo in [12], cf. also Appendix B2 in [4].

We now explain under which circumstances an ambikähler structure satisfies the conditions (23)–(24).

Proposition 2.2 *Let M be an oriented 4-manifold equipped with an ambikähler structure (g_+, J_+, ω_+), $(g_- = f^2\, g_+, J_-, \omega_-)$. Assume moreover that f is not constant. Then, on the open set where $f \neq 1$, there exist non-constant positive functions f_+, f_- satisfying (23)–(24) of* Proposition 2.1 *if and only if the 1-form*

$$\kappa := \frac{\tau(df)}{1-f^2} \tag{38}$$

is exact.

Proof For any ambikähler structure (g_+, J_+, ω_+), $(g_- = f^2\, g_+, J_-, \omega_-)$ and any positive functions f_+, f_- satisfying (23)–(24), we have

$$\begin{aligned}(1-f^2)\,\frac{df_+}{f_+} &= \frac{df}{f} + \tau(df),\\ (1-f^2)\,\frac{df_-}{f_-} &= f\,df + \tau(df).\end{aligned} \tag{39}$$

On the open set where $f \neq 1$, this can be rewritten as

$$\begin{aligned}\frac{df_+}{f_+} &= \frac{df}{f(1-f^2)} + \frac{\tau(df)}{(1-f^2)},\\ \frac{df_-}{f_-} &= \frac{fdf}{(1-f^2)} + \frac{\tau(df)}{(1-f^2)};\end{aligned} \tag{40}$$

in particular, κ is exact on this open set. Conversely, if κ is exact, but not identically zero, then $\kappa = \frac{d\varphi}{\varphi}$, for some, non-constant, positive function, φ, and we then *define*

f_+, f_- by $\frac{df_+}{f_+} = \frac{d\varphi}{\varphi} + \frac{df}{f(1-f^2)}$ and $\frac{df_-}{f_-} = \frac{d\varphi}{\varphi} + \frac{f\,df}{(1-f^2)}$, hence by $f_+ := \frac{f\varphi}{|1-f^2|^{\frac{1}{2}}}$ and $f_- := \frac{\varphi}{|1-f^2|^{\frac{1}{2}}}$, which clearly satisfy (23)–(24). □

Remark 2.3 It follows from (39) that if $f = k$, where k is a constant different from 1, then f_+ and f_- are constant and the corresponding $*$-Killing 2-form ψ is then parallel. More generally, the existence of a pair (g, ψ) inducing an ambikähler structure depends on the chosen relative scaling of the Kähler metrics. More precisely, if the ambikähler structure (g_+, J_+, ω_+), $(g_- = f^2 g_+, J_-, \omega_-)$ arises from a $*$-Killing 2-form in the conformal class, in the sense of Proposition 2.1, then for any positive constant $k \neq 1$, the ambikähler structure (g_+, J_+, ω_+), $(\tilde{g}_- = k^2 g_-, J_-, k^2\omega_-)$ *does not* arise from a $*$-Killing 2-form, unless $\tau(df) = \pm df$. This is because the 1-forms $\frac{\tau(df)}{(1-f^2)}$ and $\frac{\tau(df)}{(1-k^2f^2)}$ would then be both closed, implying that $\tau(df) = \phi\, df$ for some function ϕ; since $|\tau(df)| = |df|$, we would then have $\phi = \pm 1$.

The 1-form κ in Proposition 2.2 is clearly exact on the open set where $f \neq 1$ whenever $\tau(df) = df$ or $\tau(df) = -df$, and it readily follows from (40) that f_+, f_- are then given by

$$f_+ = \frac{cf}{|1-f|}, \quad f_- = \frac{c}{|1-f|} = \pm c + f_+, \tag{41}$$

if $\tau(df) = df$, or by

$$f_+ = \frac{cf}{1+f}, \quad f_- = \frac{c}{1+f} = c - f_+, \tag{42}$$

if $\tau(df) = -df$, for some positive constant c. If

$$TM_0 = T^+ \oplus T^-, \tag{43}$$

denotes the orthogonal splitting determined by τ, where τ is the identity on T^+ and minus the identity on T^-—equivalently, J_+, J_- coincide on T^+ and are opposite on T^-—then $\tau(df) = \pm df$ if and only if $df_{|T^\mp} = 0$ and we also have:

Proposition 2.3 *The distribution $T^\pm$ is involutive if and only if $\tau(df) = \pm df$.*

Proof For a general ambikähler structure (g_+, J_+, ω_+) and $(g_- = f^2 g_+, J_-, \omega_-)$, with $g_- = f^2 g_+$, we have

$$\frac{df(Z)}{f}\,\omega_+(X,Y) = -\omega_+([X,Y],Z), \quad \frac{df(Z)}{f}\,\omega_-(X,Y) = \omega_-([X,Y],Z), \tag{44}$$

for any X, Y in T^+ and any Z in T^-, and

$$\frac{df(Z)}{f}\,\omega_+(X,Y) = \omega_+([X,Y],Z), \quad \frac{df(Z)}{f}\,\omega_-(X,Y) = -\omega_-([X,Y],Z), \tag{45}$$

for any X, Y in T^- and any Z in T^+. This can be shown as follows. Suppose that X, Y are in T^+ and Z is in T^-. Then, since the Kähler form $\omega_+(\cdot,\cdot) = g_+(J_+\cdot,\cdot)$ and $\omega_-(\cdot,\cdot) = g(J_-\cdot,\cdot)$ are closed and T^+, T^- are ω_+- and ω_--orthogonal, we have

$$Z \cdot \omega_+(X, Y) = \omega_+([X, Y], Z) + \omega_+([Y, Z], X) + \omega_+([Z, X], Y), \tag{46}$$

and

$$Z \cdot \omega_-(X, Y) = \omega_-([X, Y], Z) + \omega_-([Y, Z], X) + \omega_-([Z, X], Y), \tag{47}$$

which can be rewritten as

$$Z \cdot \big(f^2\omega_+(X, Y)\big) = -f^2\, \omega_+([X, Y], Z) + f^2\, \omega_+([Y, Z], X) + f^2\, \omega_+([Z, X], Y), \tag{48}$$

or else:

$$\begin{aligned} 2\frac{df(Z)}{f}\, \omega_+(X, Y) + Z \cdot \omega_+(X, Y) = \\ - \omega_+([X, Y], Z) + \omega_+([Y, Z], X) + \omega_+([Z, X], Y). \end{aligned} \tag{49}$$

Comparing (46) and (49), we readily deduce the first identity in (44); the other three identities are checked similarly. Proposition 2.3 then readily follows from (44)–(45). □

In the following statement, M_0 stills denotes the (dense) open subset of M defined by (16); we also denote by M_S the open subset of M defined by

$$f_+ \neq f_-, \tag{50}$$

on which ψ is a symplectic 2-form, and by M_1 the intersection $M_1 := M_0 \cap M_S$.

Proposition 2.4 *Let (M, g) be an oriented Riemannian 4-dimensional manifold admitting a non-parallel $*$-Killing 2-form ψ. Denote by $(g_+ = f_+^2\, g, J_+, \omega_+)$, $(g_- = f_-^2\, g, J_-, \omega_-)$ the induced ambikähler structure on M_0 as explained above. Then, on the open set M_1, the Ricci endomorphism,* Ric*, of g is J_+- and J_--invariant, hence of the form*

$$\mathrm{Ric} = a\, \mathrm{I} + b\, \tau, \tag{51}$$

for some functions a, b, where I *denotes the identity of TM_1 and τ is defined by (17). Moreover, the vector field*

$$K_1 := J_+\mathrm{grad}_g f_+ = J_-\mathrm{grad}_g f_- = -\frac{1}{2}\alpha^\sharp \tag{52}$$

is Killing with respect to g and preserves the whole ambikähler structure.

Proof Let R be the curvature tensor of g, defined by

$$\mathrm{R}_{X,Y}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z, \tag{53}$$

for any vector field X, Y, Z. We denote by Scal its scalar curvature, by Ric_0 the trace-free part of Ric, by W the Weyl tensor of g, and by W^+ and W^- its self-dual and anti-self-dual part respectively. As in the previous section, Ψ denotes the skew-symmetric endomorphism of TM determined by ψ, Ψ_+ its self-dual part, Ψ_- its anti-self-dual part, with $\Psi_+ = f_+ J_+$ and $\Psi_- = f_- J_-$ on M_0. Since $g = f_+^2\, g_+ = f_-^2\, g_-$, where g_+ and g_- are Kähler with respect to J_+ and J_- respectively, W^+ and W^- are both degenerate and $\mathrm{W}^+(\Psi_+) = \lambda_+ \Psi_+$, $\mathrm{W}^-(\Psi_-) = \lambda_- \Psi_-$, for some functions λ_+, λ_-. For any vector fields X, Y on M, the usual decomposition of the curvature tensor reads:

$$\begin{aligned}\mathrm{R}_{X,Y}\Psi &= [\mathrm{R}(X \wedge Y), \Psi] \\ &= \frac{\mathrm{Scal}}{12}\,[X^\flat \wedge Y, \Psi] + \frac{1}{2}[\{\mathrm{Ric}_0, X^\flat \wedge Y\}, \Psi] \\ &\quad + [\mathrm{W}^+(X \wedge Y), \Psi_+] + [\mathrm{W}^-(X \wedge Y), \Psi_-],\end{aligned} \tag{54}$$

by setting $\{\mathrm{Ric}_0, X^\flat \wedge Y\} := \mathrm{Ric}_0 \circ (X^\flat \wedge Y) + (X^\flat \wedge Y) \circ \mathrm{Ric}_0 = \mathrm{Ric}_0(X) \wedge Y + X \wedge \mathrm{Ric}_0(Y)$, cf. e.g. [5, Chapter 1, Section G]. On M_0 we then have:

$$\frac{\mathrm{Scal}}{12}[X \wedge Y, \Psi] = -\frac{\mathrm{Scal}}{12}\,\big(\Psi(X) \wedge Y + X \wedge \Psi(Y)\big), \tag{55}$$

$$\begin{aligned}\frac{1}{2}[\{\mathrm{Ric}_0, X \wedge Y\}, \Psi] = -\frac{1}{2}\,\Big(&\Psi\big(\mathrm{Ric}_0(X)\big) \wedge Y + \mathrm{Ric}_0(X) \wedge \Psi(Y) \\ &+ \Psi(X) \wedge \mathrm{Ric}_0(Y) + X \wedge \Psi\big(\mathrm{Ric}_0(Y)\big)\Big),\end{aligned} \tag{56}$$

and

$$\begin{aligned}\mathrm{W}^+_{X,Y}\Psi_+ &= \frac{\lambda_+}{2}\,\big(\Psi_+(X) \wedge Y + X \wedge \Psi_+(Y)\big), \\ \mathrm{W}^-_{X,Y}\Psi_- &= \frac{\lambda_-}{2}\big(\Psi_-(X) \wedge Y + X \wedge \Psi_-(Y)\big).\end{aligned} \tag{57}$$

We thus get

$$\begin{aligned}\sum_{i=1}^{4} e_i \lrcorner \mathrm{R}_{e_i,Y}\Psi = &\left(\lambda_+ - \frac{\mathrm{Scal}}{6}\right)\,\Psi_+(Y) + \left(\lambda_- - \frac{\mathrm{Scal}}{6}\right)\,\Psi_-(Y) \\ &+ \frac{1}{2}[\mathrm{Ric}_0, \Psi](Y).\end{aligned} \tag{58}$$

Similarly,

$$\sum_{i=1}^{4} e_i \lrcorner \mathrm{R}_{e_i,Y} \Psi_+ = \left(\lambda_+ - \frac{\mathrm{Scal}}{6}\right) \Psi_+(Y) + \frac{1}{2}[\mathrm{Ric}_0, \Psi_+](Y) \tag{59}$$

and

$$\sum_{i=1}^{4} e_i \lrcorner \mathrm{R}_{e_i,Y} \Psi_- = \left(\lambda_- - \frac{\mathrm{Scal}}{6}\right) \Psi_-(Y) + \frac{1}{2}[\mathrm{Ric}_0, \Psi_-](Y). \tag{60}$$

On the other hand, from (18), we get

$$\mathrm{R}_{X,Y} \Psi = \nabla_Y \alpha \wedge X - \nabla_X \alpha \wedge Y, \tag{61}$$

hence

$$\sum_{i=1}^{4} e_i \lrcorner \mathrm{R}_{e_i,Y} \Psi = -2 \nabla_Y \alpha, \tag{62}$$

whereas, from (19), we obtain

$$\mathrm{R}_{X,Y} \Psi_+ = (\nabla_Y \alpha \wedge X - \nabla_X \alpha \wedge Y)_+, \quad \mathrm{R}_{X,Y} \Psi_- = (\nabla_Y \alpha \wedge X - \nabla_X \alpha \wedge Y)_-, \tag{63}$$

hence

$$\sum_{i=1}^{4} e_i \lrcorner \mathrm{R}_{e_i,Y} \Psi_+ = -Y \lrcorner (\nabla \alpha)^s - Y \lrcorner (d\alpha)_+, \tag{64}$$

where $(\nabla\alpha)^s$ denotes the symmetric part of $\nabla\alpha$. Indeed, we have

$$\begin{aligned}
\sum_{i=1}^{4} e_i \lrcorner \big(\nabla_Y \alpha \wedge e_i - \nabla_{e_i} \alpha \wedge Y\big)_+ &= \frac{1}{2} \sum_{i=1}^{4} e_i \lrcorner (\nabla_Y \alpha \wedge e_i) - \frac{1}{2} \sum_{i=1}^{4} e_i \lrcorner (\nabla_{e_i} \alpha \wedge Y) \\
&\quad + \frac{1}{2} \sum_{i=1}^{4} e_i \lrcorner * (\nabla_Y \alpha \wedge e_i) - \frac{1}{2} \sum_{i=1}^{4} e_i \lrcorner * (\nabla_{e_i} \alpha \wedge Y) \\
&= -\nabla_Y \alpha - \frac{1}{2} \sum_{i=1}^{4} e_i \lrcorner * (\nabla_{e_i} \alpha \wedge Y) \\
&= -\nabla_Y \alpha - \frac{1}{2} Y \lrcorner * d\alpha = -Y \lrcorner (\nabla \alpha)^s - Y \lrcorner (d\alpha)_+,
\end{aligned} \tag{65}$$

as $\delta\alpha = 0$ and $e_i \lrcorner * (\nabla_Y \alpha \wedge e_i)$ is clearly equal to zero thanks to the general identity (11). We obtain similarly:

$$\sum_{i=1}^{4} e_i \lrcorner \mathrm{R}_{e_i,Y} \Psi_- = -Y \lrcorner (\nabla\alpha)^s - Y \lrcorner (d\alpha)_-. \tag{66}$$

From the above, we infer

$$\begin{aligned} (d\alpha)_+ &= \left(\frac{\mathrm{Scal}}{6} - \lambda_+\right) \psi_+, \quad (d\alpha)_- = \left(\frac{\mathrm{Scal}}{6} - \lambda_-\right) \psi_-, \\ (\nabla\alpha)^s &= -\frac{1}{2}[\mathrm{Ric}_0, \Psi_+] = -\frac{1}{2}[\mathrm{Ric}_0, \Psi_-]. \end{aligned} \tag{67}$$

It follows that

$$[\mathrm{Ric}, \Psi_+] = [\mathrm{Ric}, \Psi_-], \tag{68}$$

and that the vector field $\alpha^{\sharp_g}$ is Killing with respect to g if and only if $[\mathrm{Ric}, \Psi_+] = [\mathrm{Ric}, \Psi_-] = 0$. We now show that (68) actually *implies* $[\mathrm{Ric}, \Psi_+] = [\mathrm{Ric}, \Psi_-] = 0$ at each point where $f_+ \neq f_-$. Indeed, in terms of the decomposition (4), Ric, J_+, J_- can be written in the following matricial form

$$\mathrm{Ric} = \begin{pmatrix} P & Q \\ Q^* & R \end{pmatrix}, \quad J_+ = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad J_- = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \tag{69}$$

where J denotes the restriction of J_+ on T^+ and on T^-, so that:

$$[\mathrm{Ric}_0, J_+] = \begin{pmatrix} [P,J] & [Q,J] \\ [Q^*,J] & [R,J] \end{pmatrix}, \quad [\mathrm{Ric}_0, J_-] = \begin{pmatrix} [P,J] & -\{Q,J\} \\ \{Q^*,J\} & -[R,J]. \end{pmatrix} \tag{70}$$

Then (68) can be expanded as

$$\begin{aligned} &(f_+ - f_-)[P,J] = 0, \\ &(f_+ + f_-)\, QJ = (f_+ - f_-)\, JQ, \\ &(f_+ + f_-)[R,J] = 0. \end{aligned} \tag{71}$$

Since $f_+ > 0$ and $f_- > 0$ on M_0, from (71) we readily infer $[R,J] = 0$ and $Q = 0$, meaning that

$$\mathrm{Ric} = \begin{pmatrix} P & 0 \\ 0 & R \end{pmatrix}. \tag{72}$$

Moreover, on the open subset $M_1 = M_0 \cap M_S$, where $f_+ - f_- \neq 0$, we also infer from (71) that $[P,J] = 0$, hence that $[\mathrm{Ric}, J_+] = [\mathrm{Ric}, J_-] = 0$. By (67), $(\nabla\alpha)^s = 0$, meaning that the vector field $K_1 := -\frac{1}{2}\alpha^\sharp = J_+ \mathrm{grad}_g f_+$ is Killing with respect to g. Notice that

$$\begin{aligned} K_1 &= J_+ \mathrm{grad}_g f_+ = J_- \mathrm{grad}_g f_- \\ &= -J_+ \mathrm{grad}_{g_+} \frac{1}{f_+} = -J_- \mathrm{grad}_{g_-} \frac{1}{f_-}. \end{aligned} \tag{73}$$

In particular, K_1 is also Killing with respect to g_+ and g_- and is (real) holomorphic with respect to J_+ and J_-. □

3 Separation of Variables

In this section we restrict our attention to the open subset $M_1 := M_0 \cap M_S$, defined by the conditions (16) and (50). Recall that since $\psi \wedge \psi = \psi_+ \wedge \psi_+ + \psi_- \wedge \psi_- = 2(f_+ - f_-)\, v_g$, where v_g denotes the volume form of g relative to the chosen orientation, M_S is the open subset of M where ψ is non-degenerate, hence a symplectic 2-form. According to Proposition 2.4, on M_1 the Ricci tensor Ric is of the form (51), for some functions a, b and the vector field $\alpha^\sharp$ is Killing; we then infer from (67) that $\nabla \alpha^\sharp$ can be written as:

$$\nabla \alpha^\sharp = h_+ J_+ + h_- J_-, \tag{74}$$

with

$$h_+ := \frac{1}{2} f_+ \left(\frac{\mathrm{Scal}}{6} - \lambda_+ \right), \quad h_- := \frac{1}{2} f_- \left(\frac{\mathrm{Scal}}{6} - \lambda_- \right). \tag{75}$$

We then introduce the functions x, y defined by

$$\begin{aligned} x &:= \frac{f_+ + f_-}{2}, \qquad y := \frac{f_+ - f_-}{2}, \\ f_+ &= x + y, \qquad\quad f_- = x - y. \end{aligned} \tag{76}$$

Notice that $(2x, 2y)$, resp. $(2x, -2y)$, are the eigenvalues of the Hermitian operator $-J_+ \circ \Psi = f_+ \, \mathrm{I} + f_- \, \tau$, resp. $-J_- \circ \Psi = f_+ \, \tau + f_- \, \mathrm{I}$, relative to the eigen-subbundle T^+ and T^- respectively. From (16) and (50) we deduce that x, y are subject to the conditions

$$x > |y| > 0, \tag{77}$$

whereas, from (21), we infer

$$\tau(dx) = dx, \qquad \tau(dy) = -dy. \tag{78}$$

In particular, dx, $J_+dx = J_-dx$, dy and $J_+dy = -J_-dy$ are *pairwise orthogonal* and

$$|dx|^2 + |dy|^2 = |df_+|^2 = |df_-|^2, \quad |dx|^2 - |dy|^2 = (df_+, df_-). \tag{79}$$

We then have:

Proposition 3.1 *On each connected component of the open subset of M_1 where $dx \neq 0$ and $dy \neq 0$, the square norm of dx, dy and the Laplacians of x, y relative to g are given by*

$$\begin{aligned} |dx|^2 &= \frac{A(x)}{(x^2-y^2)}, \quad |dy|^2 = \frac{B(y)}{(x^2-y^2)}, \\ \Delta x &= -\frac{A'(x)}{(x^2-y^2)}, \quad \Delta y = -\frac{B'(y)}{(x^2-y^2)}, \end{aligned} \tag{80}$$

where A, B are functions of one variable.

Proof By using (30) and (32) and setting $g_\tau(X,Y) := g(\tau(X), Y)$, we infer from (20) and (74) that

$$\begin{aligned} \nabla df_+ &= \left(-\frac{1}{2}h_+ + \frac{|df_+|^2}{f_+}\right) g - \frac{1}{2}\, h_-\, g_\tau \\ &\quad - \frac{1}{f_+}\big(df_+ \otimes df_+ + J_+df_+ \otimes J_+df_+\big), \\ \nabla df_- &= \left(-\frac{1}{2}h_- + \frac{|df_-|^2}{f_-}\right) g - \frac{1}{2}\, h_+\, g_\tau \\ &\quad - \frac{1}{f_-}\big(df_- \otimes df_- + J_-df_- \otimes J_-df_-\big). \end{aligned} \tag{81}$$

In terms of the functions x, y, this can be rewritten as

$$\begin{aligned} \nabla dx &= \big(\frac{x}{(x^2-y^2)}(|dx|^2+|dy|^2) - \frac{1}{4}(h_+ + h_-)\big)\, g - \frac{1}{4}(h_+ + h_-)\, g_\tau \\ &\quad - \frac{x}{(x^2-y^2)}\,(dx\otimes dx + dy \otimes dy) + \frac{y}{(x^2-y^2)}\,(dx \otimes dy + dy \otimes dx) \\ &\quad - \frac{x}{(x^2-y^2)}\, J_+(dx+dy) \otimes J_+(dx+dy), \\ \nabla dy &= -\big(\frac{y}{(x^2-y^2)}(|dx|^2+|dy|^2) + \frac{1}{4}(h_+ - h_-)\big)\, g + \frac{1}{4}(h_+ - h_-)\, g_\tau \\ &\quad + \frac{y}{(x^2-y^2)}\,(dx\otimes dx + dy \otimes dy) - \frac{x}{(x^2-y^2)}\,(dx \otimes dy + dy \otimes dx) \\ &\quad + \frac{y}{(x^2-y^2)}\,\big(J_+(dx+dy) \otimes J_+(dx+dy)\big). \end{aligned} \tag{82}$$

In particular:

$$\begin{aligned}\Delta x &= (h_+ + h_-) - \frac{2x}{(x^2 - y^2)}\,(|dx|^2 + |dy|^2),\\ \Delta y &= (h_+ - h_-) + \frac{2y}{(x^2 - y^2)}\,(|dx|^2 + |dy|^2).\end{aligned} \tag{83}$$

To simplify the notation, we temporarily put

$$F := |dx|^2, \qquad G := |dy|^2. \tag{84}$$

By contracting ∇dx by dx and ∇dy by dy in (82), and taking (83) into account, we obtain:

$$\begin{aligned}dF &= -\left(\Delta x + \frac{2x\,F}{(x^2 - y^2)}\right) dx + \frac{2y\,F}{(x^2 - y^2)}\,dy,\\ dG &= -\frac{2x\,G}{(x^2 - y^2)}\,dx - \left(\Delta y - \frac{2y\,G}{(x^2 - y^2)}\right) dy.\end{aligned} \tag{85}$$

From (85), we get

$$\begin{aligned}d\big((x^2 - y^2)\,F\big) &= -\big((x^2 - y^2)\,\Delta x\big)\,dx,\\ d\big((x^2 - y^2)\,G\big) &= -\big((x^2 - y^2)\,\Delta y\big)\,dy.\end{aligned} \tag{86}$$

It follows that $(x^2 - y^2)\,F = A(x)$, for some (smooth) function A of one variable and that $A'(x) = -(x^2 - y^2)\,\Delta x$; likewise, $(x^2 - y^2)\,G = B(y)$ and $B'(y) = -(x^2 - y^2)\,\Delta y$. $\square$

A simple computation using (83) shows that in terms of A, B, the functions h_+, h_- appearing in (74) and their derivatives dh_+, dh_- have the following expressions:

$$\begin{aligned}h_+ &= -\frac{A'(x) + B'(y)}{2(x^2 - y^2)} + \frac{(x - y)(A(x) + B(y))}{(x^2 - y^2)^2},\\ h_- &= -\frac{A'(x) - B'(y)}{2(x^2 - y^2)} + \frac{(x + y)(A(x) + B(y))}{(x^2 - y^2)^2},\end{aligned} \tag{87}$$

$$\begin{aligned}dh_+ &= -\frac{A''(x)dx + B''(y)dy}{2(x^2 - y^2)}\\ &\quad + \frac{A'(x)\big((2x - y)\,dx - y\,dy\big) + B'(y)\big(x\,dx + (x - 2y)\,dy\big)}{(x^2 - y^2)^2}\\ &\quad - \frac{\big(A(x) + B(y)\big)(x - y)\big((3x - y)\,dx + (x - 3y)\,dy\big)}{(x^2 - y^2)^3},\end{aligned} \tag{88}$$

and

$$
\begin{aligned}
dh_- = &-\frac{A''(x)dx - B''(y)dy}{2(x^2-y^2)} \\
&+ \frac{A'(x)\big((2x+y)\,dx - y\,dy\big) + B'(y)\big(-x\,dx + (x+2y)\,dy\big)}{(x^2-y^2)^2} \\
&- \frac{\big(A(x)+B(y)\big)(x+y)\big((3x+y)\,dx - (x+3y)\,dy\big)}{(x^2-y^2)^3}.
\end{aligned} \tag{89}
$$

In particular:

$$
J_+ dh_+ - J_- dh_- = \left(\frac{h_+}{f_+} - \frac{h_-}{f_-}\right). \tag{90}
$$

Proposition 3.2 *The vector fields*

$$
\begin{aligned}
K_1 &:= J_+ \mathrm{grad}_g (x+y) = J_- \mathrm{grad}_g (x-y) \\
&= J_+ \mathrm{grad}_{g_+} \left(\frac{-1}{x+y}\right) = J_- \mathrm{grad}_{g_-} \left(\frac{-1}{x-y}\right)
\end{aligned} \tag{91}
$$

(which is equal to the vector field $K_1 = -\frac{1}{2}\alpha^\sharp$ *appearing in* Proposition 2.4*), and*

$$
\begin{aligned}
K_2 &:= y^2 J_+ \mathrm{grad}_g x + x^2 J_+ \mathrm{grad}_g y = y^2 J_- \mathrm{grad}_g x - x^2 J_- \mathrm{grad}_g y \\
&= J_+ \mathrm{grad}_{g_+} \left(\frac{xy}{x+y}\right) = J_- \mathrm{grad}_{g_-} \left(\frac{-xy}{x-y}\right)
\end{aligned} \tag{92}
$$

are Killing with respect to g, g_+, g_- *and Hamiltonian with respect to* ω_+ *and* ω_-*. The momenta,* μ_1^+*,* μ_2^+ *of* K_1, K_2 *with respect to* ω_+*, and the momenta,* μ_1^-*,* μ_2^-*, of* K_1, K_2 *with respect to* ω_-*, are given by*

$$
\begin{aligned}
\mu_1^+ &= \frac{-1}{x+y}, \qquad \mu_2^+ = \frac{xy}{x+y}, \\
\mu_1^- &= \frac{-1}{x-y}, \qquad \mu_2^- = \frac{-xy}{x-y},
\end{aligned} \tag{93}
$$

and Poisson commute with respect to ω_+ *and* ω_-*, meaning that* $\omega_\pm(K_1, K_2) = 0$*, so that* $[K_1, K_2] = 0$ *as well. In particular, on the open set* M_1*, the ambikähler structure* (g_+, J_+, ω_+)*,* (g_-, J_-, ω_-) *is ambitoric in the sense of* [4, Definition 3].

Proof In terms of A, B, (82) can be rewritten as

$$\begin{aligned}
\nabla dx &= \frac{1}{4(x^2-y^2)^2}\Big(2x\big(A(x)+B(y)\big)+(x^2-y^2)\,A'(x)\Big)\,g\\
&\quad-\frac{1}{4(x^2-y^2)^2}\Big(2x\big(A(x)+B(y)\big)-(x^2-y^2)\,A'(x)\Big)\,g_\tau\\
&\quad-\frac{x}{(x^2-y^2)}\,(dx\otimes dx+dy\otimes dy)+\frac{y}{(x^2-y^2)}\,(dx\otimes dy+dy\otimes dx)\\
&\quad-\frac{x}{(x^2-y^2)}\,J_+(dx+dy)\otimes J_+(dx+dy),\\
\nabla dy &= \frac{1}{4(x^2-y^2)^2}\Big(-2y\big(A(x)+B(y)\big)+(x^2-y^2)\,B'(y)\Big)\,g\\
&\quad-\frac{1}{4(x^2-y^2)^2}\Big(2y\big(A(x)+B(y)\big)+(x^2-y^2)\,B'(y)\Big)\,g_\tau\\
&\quad+\frac{y}{(x^2-y^2)}\,(dx\otimes dx+dy\otimes dy)-\frac{x}{(x^2-y^2)}\,(dx\otimes dy+dy\otimes dx)\\
&\quad+\frac{y}{(x^2-y^2)}\,J_+(dx+dy)\otimes J_+(dx+dy).
\end{aligned}\tag{94}$$

By taking (30)–(32) into account, we infer

$$\begin{aligned}
\nabla(J_+dx) &= \frac{1}{2(x^2-y^2)}\left(\frac{(2y-x)\,A(x)+x\,B(y)}{(x^2-y^2)}+\frac{A'(x)}{2}\right)g(J_+\cdot,\cdot)\\
&\quad-\frac{1}{2(x^2-y^2)}\left(\frac{xA(x)+xB(y)}{(x^2-y^2)}-\frac{A'(x)}{2}\right)g(J_-\cdot,\cdot)\\
&\quad-\frac{y\,dx\wedge J_+dx+x\,dy\wedge J_+dy}{(x^2-y^2)}\\
&\quad+\frac{x\,(dx\otimes J_+dy+J_+dy\otimes dx)+y\,(dy\otimes J_+dx+J_+dx\otimes dy)}{(x^2-y^2)}
\end{aligned}\tag{95}$$

and

$$\begin{aligned}
\nabla(J_+dy) &= \frac{1}{2(x^2-y^2)}\left(\frac{(-y\,A(x)+(y-2x)\,B(y)}{(x^2-y^2)}+\frac{B'(y)}{2}\right)g(J_+\cdot,\cdot)\\
&\quad-\frac{1}{2(x^2-y^2)}\left(\frac{y\,A(x)+y\,B(y)}{(x^2-y^2)}+\frac{B'(y)}{2}\right)g(J_-\cdot,\cdot)\\
&\quad+\frac{y\,dx\wedge J_+dx+x\,dy\wedge J_+dy}{(x^2-y^2)}\\
&\quad-\frac{x\,(dx\otimes J_+dy+J_+dy\otimes dx)+y\,(dy\otimes J_+dx+J_+dx\otimes dy)}{(x^2-y^2)}.
\end{aligned}\tag{96}$$

In particular, the symmetric parts of $\nabla(J_+dx)$ and $\nabla(J_+dy)$ are opposite and given by

$$\begin{aligned}\big(\nabla(J_+dx)\big)^s = -\big(\nabla(J_+dy)\big)^s =& \frac{x\,(dx\otimes J_+dy + J_+dy\otimes dx)}{(x^2-y^2)}\\ &+\frac{y\,(dy\otimes J_+dx + J_+dx\otimes dy)}{(x^2-y^2)}.\end{aligned} \tag{97}$$

The symmetric parts of $\nabla(J_+dx + J_+dy)$ and of $\nabla(y^2J_+dx + x^2J_+dy) = y^2\nabla(J_+dx) + x^2\nabla(J_+dy) + 2dy\otimes J_+dx + 2xdx\otimes J_+dy$ then clearly vanish, meaning that K_1 and K_2 are Killing with respect to g. In view of the expressions of K_1, K_2 as symplectic gradients in (91)–(92), K_1 and K_2 are Hamiltonian with respect to ω_+ and ω_-, their momenta are those given by (93) and their Poisson bracket with respect to $\omega_\pm$ is equal to $\omega_\pm(K_1, K_2)$, which is zero, since dx lives in the dual of T^+ and dy in the dual of T^-. This, in turn, implies that K_1 and K_2 commute. □

Remark 3.1 As already observed, the Killing vector field K_1 appearing in Proposition 3.2 is the restriction to M_1 of the smooth vector field, also denoted by K_1, appearing in Proposition 2.4, which is defined on the whole manifold M by

$$K_1 = -\frac{1}{2}\alpha^\sharp = -\frac{1}{6}\delta\Psi. \tag{98}$$

Similarly, it is easily checked that K_2 is the restriction to M_1 of the smooth vector field, still denoted by K_2, defined on M by

$$\begin{aligned}K_2 &= -\frac{1}{8}\delta\big((f_+^2 - f_-^2)\,(\Psi_+ - \Psi_-)\big)\\ &= \frac{1}{8}\big(\Psi_+ - \Psi_-\big)\,\big(\mathrm{grad}_g(f_+^2 - f_-^2)\big)\end{aligned} \tag{99}$$

(recall that the Killing 2-form $\varphi = \psi_+ - \psi_- = *\psi$ is co-closed). It is also easily checked that K_2 and K_1 are related by

$$K_2 = \frac{1}{2}\,S(K_1), \tag{100}$$

where, we recall, S denotes the *Killing* symmetric endomorphism defined by (22) in Remark 2.1; this is because, on the dense open subset M_0, S can be rewritten as

$$S = -(x^2-y^2)\,\tau + (x^2+y^2)\,\mathrm{I}, \tag{101}$$

whereas $K_1^\flat = J_+(dx+dy)$, so that $S(K_1^\flat) = 2y^2\,J_+dx + 2x^2\,J_+dy = 2K_2^\flat$; we thus get (100) on M_0, hence on M. In view of (100), the fact that K_2 is Killing can then be alternatively deduced from [8, Lemma B], cf. also the proof of [4, Proposition 11 (iii)].

In view of the above, we eventually get the following rough classification:

Proposition 3.3 *For any connected, oriented,* 4*-dimensional Riemannian manifold* (M, g) *admitting a non-parallel* $*$*-Killing* 2*-form* ψ*, the open subset* M_S *defined by* (50) *is either empty or dense and we have one of the following three mutually exclusive cases:*

(1) M_S *is dense; the vector fields* K_1, K_2 *are Killing and linearly independent on a dense open set of* M*, or*
(2) M_S *is dense; the vector fields* K_1, K_2 *are Killing and* $K_2 = c\,K_1$*, for some non-zero real number* c*, or*
(3) M_S *is empty, i.e.* ψ *is decomposable everywhere; then,* K_2 *is identically zero, whereas* K_1 *is non-identically zero and is not a Killing vector field in general.*

Proof Being Killing on $M_0 \cap M_S$ and zero on any open set where $f_+ = f_-$, K_2 is Killing everywhere on M. We next observe that, for any x in M_S, $K_2(\mathsf{x}) = 0$ if and only if $K_1(\mathsf{x}) = 0$, as readily follows from (100) and from the fact that S is invertible if and only if x belongs to M_S, as the eigenvalues of S are equal to $\frac{(f_+ + f_-)^2}{2}$ and $\frac{(f_+ - f_-)^2}{2}$.

Suppose now that M_S is not dense in M, i.e. that $M \setminus M_S$ contains some non-empty open subset V; then, K_2 vanishes on V, hence vanishes identically on M, as K_2 is Killing; from (99), we then infer $0 = \Psi(K_2) = \frac{1}{8}(f_+^2 - f_-^2)\mathrm{grad}_g(f_+^2 - f_-^2)$, which implies that the (smooth) function $(f_+^2 - f_-^2)^2$ is *constant* on M, hence identically zero, meaning that M_S is empty. If M_S is empty, then $f_+ = f_-$ everywhere (equivalently, $\psi \wedge \psi$ is identically zero); it follows that K_2 is identically zero, whereas K_1, which is not identically zero since ψ is not parallel, is not Killing in general, cf. Sect. 6.

If M_S is dense, then K_1 and K_2 are both Killing vector fields on M, hence either linearly independent on some dense open subset of M or dependent everywhere and, by the above discussion, K_2 is then a constant, non-zero multiple of K_1. □

In the next sections we consider in turn the three cases listed in Proposition 3.3.

4 The Ambitoric Ansatz

In this section, we assume that M_S is dense and that K_1, K_2 are linearly independent on some dense open set $\mathcal{U}$. In the remainder of this section, we focus our attention on $\mathcal{U}$, i.e. we assume that dx and dy are linearly independent everywhere—equivalently, $\tau(df) \neq \pm df$ everywhere—so that $\{dx, J_+dx = J_-dx, dy, J_+dy = -J_-dy\}$ form a direct orthogonal coframe. By Proposition 3.1, the metric g and the Kähler forms

ω_+, ω_- can then be written as

$$\begin{aligned} g &= (x^2 - y^2)\left(\frac{dx \otimes dx}{A(x)} + \frac{dy \otimes dy}{B(y)}\right) \\ &+ (x^2 - y^2)\left(\frac{J_+dx \otimes J_+dx}{A(x)} + \frac{J_+dy \otimes J_+dy}{B(y)}\right), \end{aligned} \tag{102}$$

$$\begin{aligned} \omega_+ &= \frac{(x-y)}{(x+y)}\left(\frac{dx \wedge J_+dx}{A(x)} + \frac{dy \wedge J_+dy}{B(y)}\right), \\ \omega_- &= \frac{(x+y)}{(x-y)}\left(\frac{dx \wedge J_+dx}{A(x)} - \frac{dy \wedge J_+dy}{B(y)}\right), \end{aligned} \tag{103}$$

and we also have:

Proposition 4.1 *The functions* Scal $= 4a$ *and* b *appearing in the expression* (51) *of the Ricci tensor of* g *are given by:*

$$\mathrm{Scal} = -\frac{A''(x) + B''(y)}{(x^2 - y^2)}, \tag{104}$$

and

$$b = -\frac{A''(x) - B''(y)}{4(x^2 - y^2)} + \frac{xA'(x) + yB'(y)}{(x^2 - y^2)^2} - \frac{A(x) + B(y)}{(x^2 - y^2)^2}. \tag{105}$$

Proof Since $\alpha^\sharp$ is Killing, the Bochner formula reads:

$$\mathrm{Ric}(\alpha^\sharp) = \delta\nabla\alpha^\sharp \tag{106}$$

whereas, by (51),

$$\mathrm{Ric}(\alpha^\sharp) = a\,\alpha^\sharp + b\,\tau(\alpha^\sharp). \tag{107}$$

By using

$$\alpha = f_+\,\delta J_+ = f_-\,\delta J_-, \tag{108}$$

which easily follows from (30)–(32), we infer from (74) that

$$\delta\nabla\alpha^\sharp = \frac{h_+}{f_+}\,\alpha + \frac{h_-}{f_-}\,\alpha - J_+dh_+ - J_-dh_-. \tag{109}$$

By putting together (106), (109) and (90), we get

$$a\,\alpha + b\,\tau(\alpha) = 2\left(\frac{h_+}{f_+}\,\alpha - J_+dh_+\right) = 2\left(\frac{h_-}{f_-}\,\alpha - J_-dh_-\right), \tag{110}$$

hence

$$
\begin{aligned}
dh_+ &= \left(a - 2\frac{h_+}{f_+} + b\right) dx + \left(a - 2\frac{h_+}{f_+} - b\right) dy, \\
dh_- &= \left(a - 2\frac{h_-}{f_-} + b\right) dx + \left(-a + 2\frac{h_-}{f_-} + b\right) dy.
\end{aligned} \tag{111}
$$

We thus get

$$
\begin{aligned}
a &= \frac{1}{2}\left(\frac{\partial h_+}{\partial x} + \frac{\partial h_+}{\partial y}\right) + \frac{2h_+}{x+y} = \frac{1}{2}\left(\frac{\partial h_-}{\partial x} - \frac{\partial h_-}{\partial y}\right) + \frac{2h_-}{x+y} \\
b &= \frac{1}{2}\left(\frac{\partial h_+}{\partial x} - \frac{\partial h_+}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial h_-}{\partial x} + \frac{\partial h_-}{\partial y}\right).
\end{aligned} \tag{112}
$$

By using (87), we obtain (104) and (105). ◻

Recall that a function φ is called J_+-*pluriharmonic* if $d(J_+ d\varphi) = 0$ and J_--*pluriharmonic* if $d(J_- d\varphi) = 0$.

Proposition 4.2

(i) *The space of real J_+-pluriharmonic functions, modulo additive constants, of the form $\varphi^+ = \varphi^+(x, y)$ is generated by φ_1^+, φ_2^+ defined by:*

$$
\varphi_1^+(x, y) = \int^x \frac{\zeta^2 d\zeta}{A(\zeta)} - \int^y \frac{\zeta^2 d\zeta}{B(\zeta)}, \quad \varphi_2^+(x, y) = \int^x \frac{d\zeta}{A(\zeta)} - \int^y \frac{d\zeta}{B(\zeta)}, \tag{113}
$$

where $\int^x$, resp. $\int^y$, stands for any primitive of the variable x, resp. y.

(ii) *The space of real J_--pluriharmonic functions, modulo additive constants, of the form $\varphi^- = \varphi^-(x, y)$ is generated by φ_1^-, φ_2^- defined by:*

$$
\varphi_1^-(x, y) = \int^x \frac{\zeta^2 d\zeta}{A(\zeta)} + \int^y \frac{\zeta^2 d\zeta}{B(\zeta)}, \quad \varphi_2^-(x, y) = \int^x \frac{d\zeta}{A(\zeta)} + \int^y \frac{d\zeta}{B(\zeta)}. \tag{114}
$$

Proof From (95)–(96), we readily infer the following expression of $d(J_\pm dx)$ and $d(J_\pm dy)$:

$$
\begin{aligned}
d(J_+ dx) = d(J_- dx) = &\left(\frac{A'(x)}{A(x)} - \frac{2x}{x^2 - y^2}\right) dx \wedge J_+ dx \\
&+ \frac{2y\,A(x)}{(x^2 - y^2)\,B(y)}\, dy \wedge J_+ dy,
\end{aligned} \tag{115}
$$

and

$$d(J_+dy) = -d(J_-dy) = -\frac{2x\,B(y)}{(x^2-y^2)\,A(x)}\,dx\wedge J_+dx + \left(\frac{B'(y)}{B(y)} + \frac{2y}{x^2-x^2}\right)dy\wedge J_+dy. \tag{116}$$

Let $\varphi = \varphi(x,y)$ be any function of x,y and denote by $\varphi_x, \varphi_y, \varphi_{xx}, etc.\dots$ its derivative with respect to x, y. Then

$$\begin{aligned} d(J_+d\varphi) &= \varphi_x\,d(J_+dx) + \varphi_y\,d(J_+dy) \\ &+ \varphi_{xx}\,dx\wedge J_+dx + \varphi_{yy}\,dy\wedge J_+dy \\ &+ \varphi_{xy}\,(dx\wedge J_+dy + dy\wedge J_+dx). \end{aligned} \tag{117}$$

By (115)–(116), φ is J_+-pluriharmonic if and only if $\varphi_{xy} = 0$—meaning that φ is of the form $\varphi(x,y) = C(x) + D(y)$—and C, D satisfy

$$\begin{aligned} C''(x) + \left(\frac{A'(x)}{A(x)} - \frac{2x}{x^2-y^2}\right)C'(x) - \frac{2x\,B(y)\,D'(y)}{(x^2-y^2)\,A(x)} &= 0, \\ D''(y) + \left(\frac{B'(y)}{B(y)} + \frac{2y}{x^2-y^2}\right)D'(y) + \frac{2y\,A(x)\,C'(x)}{(x^2-y^2)\,B(y)} &= 0 \end{aligned} \tag{118}$$

or, equivalently, by multiplying the first equation by $A(x)$ and the second by $B(y)$, which are both positive, and by setting $F(x) := A(x)C'(x)$, $G(y) := B(y)D'(y)$:

$$F'(x) - \frac{2x\big(F(x)+G(y)\big)}{x^2-y^2} = 0, \quad G'(y) + \frac{2y\big(F(x)+G(y)\big)}{x^2-y^2} = 0. \tag{119}$$

It is easily checked that the general solution of this system is given by:

$$F(x) = k_1\,x^2 + k_2, \qquad G(y) = -k_1\,y^2 - k_2, \tag{120}$$

for real constants k_1, k_2. We thus get Part (i) of Proposition 4.2. Part (ii) is obtained similarly. □

In view of Proposition 4.2, we (locally) define s and t, up to additive constants, by

$$J_+d\varphi_1^+ = J_-d\varphi_1^- = ds, \qquad J_+d\varphi_2^+ = J_-d\varphi_2^- = -dt. \tag{121}$$

Equivalently:

$$ds = \frac{x^2\,J_+dx}{A(x)} - \frac{y^2\,J_+dy}{B(y)}, \qquad dt = -\frac{J_+dx}{A(x)} + \frac{J_+dy}{B(y)}. \tag{122}$$

Notice that $ds \wedge dt = \frac{(x^2-y^2)}{A(x)B(y)} J_+dx \wedge J_+dy$; it then follows from Proposition 3.1 that

$$dx \wedge dy \wedge ds \wedge dt = \frac{v_g}{(x^2-y^2)}, \tag{123}$$

where v_g denotes the volume form of g with respect to the orientation induced by J_+, showing that dx, dy, ds, dt form a direct coframe. In view of (102), (103), (122), on the open set where x, y, s, t form a coordinate system, the metrics g, g_+, g_-, the complex structures J_+, J_-, the involution τ and the Kähler forms ω_+, ω_- have the following expressions:

$$\begin{aligned} g &= (x^2-y^2)\left(\frac{dx \otimes dx}{A(x)} + \frac{dy \otimes dy}{B(y)}\right) \\ &\quad + \frac{A(x)}{(x^2-y^2)}(ds + y^2\,dt) \otimes (ds + y^2\,dt) \\ &\quad + \frac{B(y)}{(x^2-y^2)}(ds + x^2\,dt) \otimes (ds + x^2\,dt) \\ &= (x+y)^2\,g_+ = (x-y)^2\,g_- \end{aligned} \tag{124}$$

$$\begin{aligned} J_+dx &= J_-dx = \frac{A(x)}{(x^2-y^2)}(ds + y^2\,dt) \\ J_+dy &= -J_-dy = \frac{B(y)}{(x^2-y^2)}(ds + x^2\,dt) \\ J_+dt &= \frac{dx}{A(x)} - \frac{dy}{B(y)}, \quad J_-dt = \frac{dx}{A(x)} + \frac{dy}{B(y)} \\ J_+ds &= -\frac{x^2\,dx}{A(x)} + \frac{y^2\,dy}{B(y)}, \quad J_-ds = -\frac{x^2\,dx}{A(x)} - \frac{y^2\,dy}{B(y)} \end{aligned} \tag{125}$$

$$\begin{aligned} \tau(dx) &= dx, \qquad \tau(dy) = -dy \\ \tau(ds) &= \frac{(x^2+y^2)}{(x^2-y^2)}\,ds + \frac{2x^2y^2}{(x^2-y^2)}\,dt \\ \tau(dt) &= \frac{-2}{(x^2-y^2)}\,ds - \frac{(x^2+y^2)}{(x^2-y^2)}\,dt, \end{aligned} \tag{126}$$

$$\omega_+ = \frac{dx \wedge (ds + y^2\,dt) + dy \wedge (ds + x^2\,dt)}{(x+y)^2}$$
$$\omega_- = \frac{dx \wedge (ds + y^2\,dt) - dy \wedge (ds + x^2\,dt)}{(x-y)^2}; \tag{127}$$

while it follows from (26) that the $*$-Killing 2-form ψ is given by

$$\psi = 2x\,dx \wedge (ds + y^2\,dt) + 2y\,dy \wedge (ds + x^2\,dt). \tag{128}$$

Notice that, in view of (124), the (local) vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ are Killing with respect to g and respectively coincide with the Killing vector fields K_1 and K_2 appearing in Proposition 3.2 on their domain of definition.

It turns out that the expressions of $(g_+ = (x+y)^{-2}\,g, J_+, \omega_+)$ and $(g_- = (x-y)^{-2}\,g, J_-, \omega_-)$ just obtained coincide with the *ambitoric Ansatz* described in [4, Theorem 3], in the case where the quadratic polynomial is $q(z) = 2z$, which is the *normal form* of the ambitoric Ansatz in the *hyperbolic* case considered in [4, Paragraph 5.4].

The discussion in this section can then be summarized as follows:

Theorem 4.1 *Let (M, g) be a connected, oriented, 4-dimensional manifold admitting a non-parallel, $*$-Killing 2-form $\psi = \psi_+ + \psi_-$ and assume that the open set, M_S, where $|\psi_+| \neq |\psi_-|$ is dense, cf.* Proposition 3.3. *On the open subset, $\mathcal{U}$, of M_S where ψ_+ and ψ_- have no zero and $d|\psi_+| \wedge d|\psi_-| \neq 0$, the pair (g, ψ) gives rise to an ambitoric structure of hyperbolic type, in the sense of* [4]*, relative to the conformal class of g, which, on any simply-connected open subset of $\mathcal{U}$, is described by* (124)–(125)–(127)*, where the Hermitian structures $(g_+ = (x+y)^{-2}\,g, J_+, \omega_+)$ and $(g_- = (x-y)^{-2}g, J_-, \omega_-)$ are Kähler, while ψ is described by* (128).

Conversely, on the open set, $\mathcal{U}$, of $\mathbb{R}^4$, of coordinates x, y, s, t, with $x > |y| > 0$, the two almost Hermitian structures $(g_+ = (x+y)^{-2}\,g, J_+, \omega_+)$, $(g_- = (x-y)^{-2}g, J_-, \omega_-)$ defined by (124)–(125)–(127)*, with $A(x) > 0$ and $B(y) > 0$, are Kähler and, together with the Killing vector fields $K_1 = \frac{\partial}{\partial s}$ and $K_2 = \frac{\partial}{\partial t}$, constitute an ambitoric structure of hyperbolic type, while the 2-form ψ defined by* (128) *is $*$-Killing with respect to g.*

Proof The first part follows from the preceding discussion. For the converse, we first observe that the 2-forms ω_+ and ω_- defined by (127) are clearly closed and not degenerate. To test the integrability of the almost complex structures J_+ and J_- defined by (125), we consider the complex 1-forms:

$$\beta_+ = dx + i\,J_+ dx = dx + i\frac{A(x)}{(x^2 - y^2)}\,(ds + y^2\,dt),$$
$$\gamma_+ = dy + i\,J_+ dy = dy + i\frac{B(y)}{(x^2 - y^2)}\,(ds + x^2\,dt), \tag{129}$$

which generate the space of $(1,0)$-forms with respect to J_+. We then have:

$$\begin{aligned}
d\beta_+ &= i\,\frac{\big((x^2-y^2)\,A'(x)+x\,A(x)\big)}{(x^2-y^2)}\,dx\wedge(ds+y^2\,dt)\\
&\quad + i\,\frac{2y\,A(x)}{(x^2-y^2)}\,dy\wedge(ds+x^2\,dt)\\
&= \frac{\big(A'(x)-2x\,A(x)\big)}{A(x)}\,dx\wedge\beta_+ + \frac{2y\,A(x)}{B(y)}\,dy\wedge\gamma_+,\\
d\gamma_+ &= i\,\frac{\big((x^2-y^2)\,B'(y)+2y\,B(y)\big)}{(x^2-y^2)}\,dy\wedge(ds+x^2\,dt)\\
&\quad - i\,\frac{2x\,B(y)}{(x^2-y^2)}\,dx\wedge(ds+y^2\,dt)\\
&= \frac{\big(B'(y)+2y\,B(y)\big)}{B(y)}\,dy\wedge\gamma_+ - \frac{2x\,B(y)}{A(x)}\,dx\wedge\beta_+,
\end{aligned} \tag{130}$$

which shows that J_+ is integrable. For J_-, we likewise consider the complex 1-forms:

$$\begin{aligned}
\beta_- &= dx + i\,J_-dx = \beta_+ = dx + i\frac{A(x)}{(x^2-y^2)}\,(ds+y^2\,dt),\\
\gamma_- &= dy + i\,J_-dy = dy - i\frac{B(y)}{(x^2-y^2)}\,(ds+x^2\,dt),
\end{aligned} \tag{131}$$

which generate the space of $(1,0)$-forms with respect to J_+. We then get

$$\begin{aligned}
d\beta_- &= d\beta_+\\
&= \frac{\big(A'(x)-2x\,A(x)\big)}{A(x)}\,dx\wedge\beta_- - \frac{2y\,A(x)}{B(y)}\,dy\wedge\gamma_-,\\
d\gamma_- &= -d\gamma_+\\
&= \frac{\big(B'(y)+2y\,B(y)\big)}{B(y)}\,dy\wedge\gamma_- + \frac{2x\,B(y)}{A(x)}\,dx\wedge\beta_-,
\end{aligned} \tag{132}$$

which, again, shows that J_- is integrable. It follows that the almost Hermitian structures $(g_+ = (x+y)^{-2}\,g, J_+, \omega_+)$ and $(g_- = (x-y)^{-2}\,g, J_-, \omega_-)$ are both *Kähler* and thus determine an *ambikähler structure* on $\mathcal{U}$. Moreover, the vector fields

$\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ are clearly Killing with respect to g, g_+, g_-, and satisfy:

$$\frac{\partial}{\partial s} \lrcorner \omega_+ = -\frac{dx+dy}{(x+y)^2} = d\left(\frac{1}{x+y}\right), \quad \frac{\partial}{\partial s} \lrcorner \omega_- = \frac{-dx+dy}{(x-y)^2} = d\left(\frac{1}{x-y}\right),$$
$$\frac{\partial}{\partial t} \lrcorner \omega_+ = -\frac{y^2\,dx+x^2\,dy}{(x+y)^2} = -d\left(\frac{xy}{x+y}\right), \quad \frac{\partial}{\partial t} \lrcorner \omega_- = -\frac{y^2\,dx-x^2\,dy}{(x-y)^2} = d\left(\frac{xy}{x-y}\right), \tag{133}$$

meaning that they are both Hamiltonian with respect to ω_+ and ω_-, with momenta given by (93) in Proposition 3.2. This implies that $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ preserve the two Kähler structures (g_+, J_+, ω_+) and (g_-, J_-, ω_-) and actually coincide with the vector field K_1 and K_2 respectively defined in a more general context in Proposition 3.2. We thus end up with an *ambitoric structure*, as defined in [4]. According to Theorem 3 in [4], it is an *ambitoric structure of hyperbolic type*, with "quadratic polynomial" $q(z) = 2z$. To check that the 2-form ψ defined by (128)—which is evidently closed—is $*$-Killing with respect to g, denote by f_+, f_- the positive functions on $\mathcal{U}$ defined by $f_+ = x + y, f_- = x - y$, so that $g_+ = f_+^{-2}\, g$, $g_- = f_-^{-2}\, g$ and $\psi = f_+^3\, \omega_+ + f_-^3\, \omega_-$; it then follows from (126) that $\tau(df_+) = df_-$, hence that ψ is $*$-Killing by Proposition 2.1. □

5 Ambikähler Structures of Calabi Type

The second case listed in Proposition 3.3, which is considered in this section, can be made more explicit via the following proposition:

Proposition 5.1 *Let (M, g) be a connected, oriented, Riemannian 4-manifold admitting a non-parallel $*$-Killing 2-form $\psi = \psi_+ + \psi_-$. In view of* Proposition 3.3, *assume that the open set M_S—where ψ is non-degenerate—is dense in M and that the Killing vector fields K_1, K_2 defined by (98)–(99) are related by $K_2 = c\,K_1$, for some non-zero real number c. Then, c is positive and one of the following three cases occurs:*

(1) $f_+(\mathsf{x}) + f_-(\mathsf{x}) = 2\sqrt{c}$, *for any* x *in* M, *or*
(2) $f_+(\mathsf{x}) - f_-(\mathsf{x}) = 2\sqrt{c}$, *for any* x *in* M, *or*
(3) $f_-(\mathsf{x}) - f_+(\mathsf{x}) = 2\sqrt{c}$, *for any* x *in* M,

with the usual notation: $f_+ = |\psi_+|/\sqrt{2}$ and $f_- = |\psi_-|/\sqrt{2}$.

Proof First recall that $(\Psi_+ + \Psi_-) \circ (\Psi_+ - \Psi_-) = -(f_+^2 - f_-^2)\,\mathrm{I}$. From (99) and $K_1 = J_+ \mathrm{grad}_g f_+ = J_- \mathrm{grad}_g f_-$, we then infer

$$\Psi(K_1) = -\frac{1}{2}\mathrm{grad}_g\left(f_+^2 + f_-^2\right),$$
$$\Psi(K_2) = -\frac{1}{16}\mathrm{grad}_g\left(\left(f_+^2 - f_-^2\right)^2\right). \tag{134}$$

On M_S, where Ψ is invertible, the identity $K_2 = c\,K_1$ then reads:

$$(f_+^2 - f_-^2)d(f_+^2 - f_-^2) = 4c\left(df_+^2 + df_-^2\right), \tag{135}$$

or, else:

$$(f_+^2 - f_-^2 - 4\,c)\,df_+^2 = (f_+^2 - f_-^2 + 4\,c)\,df_-^2. \tag{136}$$

Since $|df_+| = |df_-|$ on M_0, on $M_1 = M_0 \cap M_S$ we then get:

$$f_+^2(f_+^2 - f_-^2 - 4\,c)^2 - f_-^2(f_+^2 - f_-^2 + 4\,c)^2 = 0. \tag{137}$$

Since M_1 is dense this identity actually holds on the whole manifold M. It can be rewritten as

$$(f_+^2 - f_-^2)\big((f_+ + f_-)^2 - 4\,c\big)\big((f_+ - f_-)^2 - 4\,c\big) = 0; \tag{138}$$

this forces c to be positive—if not, $f_+^2 - f_-^2$ would be identically zero—and we eventually get the identity:

$$(f_+^2 - f_-^2)(f_+ + f_- + 2\sqrt{c})(f_+ + f_- - 2\sqrt{c})(f_+ - f_- - 2\sqrt{c})(f_+ - f_- + 2\sqrt{c}) = 0. \tag{139}$$

Denote by $\tilde{M}$ the open subset of M obtained by removing the zero locus $K_1^{-1}(0)$ of K_1 from M (notice that $\tilde{M}$ is a connected, dense open subset of M, as $K_1^{-1}(0)$ is a disjoint union of totally geodesic submanifolds of codimension a least 2). It readily follows from (139) that $\tilde{M}$ is the union of the following four *closed* subsets $\tilde{F}_0 := F_0 \cap \tilde{M}$, $\tilde{F}_+ := F_+ \cap \tilde{M}$, $\tilde{F}_- := F_- \cap \tilde{M}$ and $\tilde{F}_S := F_S \cap \tilde{M}$ of $\tilde{M}$, where F_0, F_+, F_-, F_S denote the four closed subsets of M defined by:

$$\begin{aligned}
F_0 &:= \{\mathsf{x} \in M \,|\, f_+(\mathsf{x}) + f_-(\mathsf{x}) = 2\sqrt{c}\},\\
F_+ &:= \{\mathsf{x} \in M \,|\, f_+(\mathsf{x}) - f_-(\mathsf{x}) = 2\sqrt{c}\},\\
F_- &:= \{\mathsf{x} \in M \,|\, f_-(\mathsf{x}) - f_+(\mathsf{x}) = 2\sqrt{c}\},\\
F_S &:= \{\mathsf{x} \in M \,|\, f_+(\mathsf{x}) - f_-(\mathsf{x}) = 0\}.
\end{aligned} \tag{140}$$

We now show that if the interior, V, of $\tilde{F}_0$ is non-empty then $\tilde{F}_0 = \tilde{M}$ (and thus $F_0 = M$ by density); this amounts to showing that the boundary $B := \bar{V} \setminus V$ of V in $\tilde{M}$ is empty. If not, let x be any element of B; then, x belongs to $\tilde{F}_0$, as $\tilde{F}_0$ is closed, and it also belongs to $\tilde{F}_+$ or $\tilde{F}_-$: otherwise, there would exist an open neighbourhood of x disjoint from $\tilde{F}_+ \cup \tilde{F}_-$, hence contained in $\tilde{F}_0 \cup \tilde{F}_S$; as $\tilde{F}_S$ has no interior, this neighbourhood would be contained in contained in $\tilde{F}_0$, which contradicts the fact that x sits on the boundary of V. Without loss, we may thus assume that x belongs to $\tilde{F}_+$, so that $f_+(\mathsf{x}) = 2\sqrt{c}$ and $f_-(\mathsf{x}) = 0$; since $K_1(\mathsf{x}) \neq 0$—by the very definition of $\tilde{M}$—f_+ is regular at x, implying that the locus of $f_+ = 2\sqrt{c}$ is a smooth hypersurface, S, of $\tilde{M}$ near x; moreover, since $\tilde{F}_+$ and $\tilde{F}_-$ are disjoint, $f_- = 0$ on S, meaning that $\Psi_- = 0$ on S; for any X in $T_{\mathsf{x}}S$ we then have $\nabla_X \Psi_- = 0$. On the other hand, $\nabla_X \Psi = (\alpha(\mathsf{x}) \wedge X)_-$, for any X in $T_{\mathsf{x}}M$, cf. (19), and we can then choose X in $T_{\mathsf{x}}S$ in such a way that $(\alpha(\mathsf{x}) \wedge X)_-$ be non-zero, hence $\nabla_X \psi_- \neq 0$, contradicting the previous assertion. We similarly show that $M = F_+$ or $M = F_-$ whenever the interior of $\tilde{F}_+$ or of $\tilde{F}_-$ is non-empty. □

A direct consequence of Proposition 5.1 is that on the (dense) open subset M_0, the associated ambikähler structure $(g_+ = f_+^{-2}\, g, J_+ = f_+^{-1}\, \Psi_+, \omega_+)$, $(g_- = f_-^{-2}\, g = f^2\, g_+, J_- = f_-^{-1}\, \Psi_-, \omega_-)$, with $f = f_+/f_-$, satisfies

$$\tau(df) = -df \tag{141}$$

in the first case listed in Proposition 5.1, and

$$\tau(df) = df \tag{142}$$

in the remaining two cases. The ambikähler structure is then *of Calabi type*, according to the following definition, taken from [1]:

Definition 5.1 An ambikähler structure (g_+, J_+, ω_+), (g_-, J_-, ω_-), with $g_+ = f^{-2}\, g_-$, is said to be *of Calabi type* if $df \neq 0$ everywhere, and if there exists a non-vanishing vector field K, Killing with respect to g_+ and g_- and Hamiltonian with respect to ω_+ and ω_-, which satisfies

$$\tau(K) = \pm K, \tag{143}$$

with $\tau = -J_+ J_- = -J_- J_+$.

By replacing the pair (J_+, J_-) by the pair $(J_+, -J_-)$ if needed, we can assume, without loss of generality, that $\tau(K) = -K$. In the following proposition, we recall some general facts concerning this class of ambikähler structures, cf. e.g. [1, Section 3]:

Proposition 5.2 *For any ambikähler structure of Calabi type, with $\tau(K) = -K$:*

(i) *The Killing vector field K is an eigenvector of the Ricci tensor,* Ric^{g_+}*, of g_+ and of the Ricci tensor,* Ric^{g_-}*, of g_-; in particular,* Ric^{g_+} *and* Ric^{g_-} *are both J_+- and J_--invariant;*

(ii) *the Killing vector field K is a constant multiple of* $J_- \mathrm{grad}_{g_-} f = J_+ \mathrm{grad}_{g_+} \frac{1}{f}$.

Proof By hypothesis, $K = J_+ \mathrm{grad}_{g_+} z_+ = J_- \mathrm{grad}_{g_-} z_-$, for some real functions z_+ and z_-. Since $J_- K = -J_+ K$, we infer $\mathrm{grad}_{g_+} z_+ = -\mathrm{grad}_{g_-} z_-$, hence

$$dz_+ = -f^{-2}\, dz_-. \tag{144}$$

Since $df \neq 0$ everywhere, this, in turn, implies that

$$z_+ = F(f), \qquad z_- = G(f) \tag{145}$$

for some real (smooth) functions F, G defined on $\mathbb{R}^{>0}$ up to an additive constant and satisfying:

$$G'(x) = -x^2 F'(x). \tag{146}$$

Moreover,

$$\tau(df) = -df. \tag{147}$$

Since K has no zero and satisfies $\tau(K) = -K$, we have

$$J_+ = \frac{K^\flat \wedge J_+K}{|K|^2} + *\frac{K^\flat \wedge J_+K}{|K|^2}, \quad J_- = -\frac{K^\flat \wedge J_+K}{|K|^2} + *\frac{K^\flat \wedge J_+K}{|K|^2}, \tag{148}$$

so that

$$J_+ - J_- = \frac{2\,K^\flat \wedge J_+K}{|K|^2}, \tag{149}$$

In (148)–(149), the dual 1-form $K^\flat$ and the square norm $|K|^2$ are relative to any metric in $[g_+] = [g_-]$. For definiteness however, we agree that they are both relative to g_+. Since $g^+ = f^{-2}\,g_-$, we have:

$$\nabla^{g+}_X J_- = J_-\frac{df}{f} \wedge X + \frac{df}{f} \wedge J_-X. \tag{150}$$

By using (24), we then infer from (149):

$$\begin{aligned} \nabla^{g+}_X (J_+ - J_-) = -\nabla^{g+}_X J_- &= J_+\frac{df}{f} \wedge X - \frac{df}{f} \wedge J_-X \\ &= \frac{2\,\nabla^{g+}_X K^\flat \wedge J_+K + 2\,K^\flat \wedge J_+\nabla^{g+}_X K}{|K|^2} \\ &- \frac{X \cdot |K|^2}{|K|^2}\,(J_+ + J_-). \end{aligned} \tag{151}$$

By contracting with K, and by using $K^\flat = F' J_+ df$ and $J_+ \nabla^{g_+}_X K = \nabla_{J_+ X} K$ (as K is $J_\pm$-holomorphic), we obtain

$$\begin{aligned}\nabla^{g_+}_X K = &-\frac{|K|^2}{2f F'} J_+ X + \frac{1}{2f F'} \left(K^\flat \wedge J_+ K\right)(X)\\ &+ \frac{1}{2}\frac{d|K|^2}{|K|^2}(X)\, K + \frac{1}{2}\frac{J_+ d|K|^2}{|K|^2}(X)\, J_+ K.\end{aligned} \tag{152}$$

Since K is Killing with respect to g_+, $\nabla^{g_+} K$ is anti-symmetric; in view of (152), this forces $|K|^2$ to be of the form

$$|K|^2 = H(f), \tag{153}$$

for some (smooth) function H from $\mathbb{R}^{>0}$ to $\mathbb{R}^{>0}$, hence

$$\frac{d|K|^2}{|K|^2} = \frac{H'(f)}{H(f)}\, df = -\frac{H'(f)}{H(f) F'(f)}\, J_+ K^\flat. \tag{154}$$

By substituting (154) in (152), we eventually get the following expression of $\nabla^{g_+} K$:

$$\nabla^{g_+} K = \Phi_+(f)\, J_+ - \Phi_-(f)\, J_-, \tag{155}$$

with

$$\Phi_+ = \frac{1}{4}\left(\frac{H'(f)}{F'(f)} - \frac{H(f)}{f\, F'(f)}\right), \quad \Phi_- = \frac{1}{4}\left(\frac{H'(f)}{F'(f)} + \frac{H(f)}{f\, F'(f)}\right). \tag{156}$$

Since K is Killing with respect to g_+, it follows from the Bochner formula that

$$\mathrm{Ric}^{g_+}(K) = \delta \nabla^{g_+} K, \tag{157}$$

whereas, from (155) we get

$$\begin{aligned}(\nabla^{g_+})^2_{X,Y} K = &\ \Phi'_+\, df(X)\, J_+(Y) - \Phi'_-\, df(X)\, J_-(Y)\\ &- \Phi_- \left(\nabla^{g_+}_X J_-\right)(Y),\end{aligned} \tag{158}$$

and, from $\nabla^{g_+}_X J_- = [J_-, \frac{df}{f} \wedge X]$:

$$\delta J_- = -\left(\sum_{i=1}^{4} \nabla^{g_+}_{e_i} J_-\right)(e_i) = -2 J_+ \frac{df}{f} = -\frac{2}{f F'(f)}\, K^\flat. \tag{159}$$

By putting together (155), (157)–(159), we get

$$\mathrm{Ric}^{g_+}(K) = \mu\, K, \tag{160}$$

with

$$\mu = -\frac{\big(f\,\Phi'_+(f) + f\,\Phi'_-(f) - 2\,\Phi_-(f)\big)}{f\,F'(f)}. \tag{161}$$

Since the metric g_+ is Kähler with respect to J_+, in particular is J_+-invariant, (160) implies that the two eigenspaces of Ric^{g_+} are the space $\{K, J_+K\}$ generated by K and J_+K (where $J_- = J_+$) and its orthogonal complement, $\{K, J_+K\}^\perp$ (where $J_- = -J_+$). It follows that Ric^{g_+} is both J_+- *and* J_--invariant. This establishes the part (i) of the proposition (it is similarly shown that Ric^{g_-} is J_+- and J_--invariant).

Before proving part (ii), we first recall the general transformation rules of the curvature under a conformal change of the metric. If g and $\tilde{g} = \phi^{-2}\,g$ are two Riemannian metrics in a same conformal class $[g]$ in any n-dimensional Riemannian manifold $(M, g), n > 2$, then the scalar curvature, $\mathrm{Scal}^{\tilde{g}}$, and the trace-free part, $\mathrm{Ric}_0^{\tilde{g}}$, of $\tilde{g}$ are related to the scalar curvature, Scal^g, and the trace-free part, Ric_0^g, of g by

$$\mathrm{Scal}^{\tilde{g}} = \phi^2\,\big(\mathrm{Scal}^g - 2(n-1)\,\phi\,\Delta_g\phi - n(n-1)\,|d\phi|_g^2\big), \tag{162}$$

and

$$\mathrm{Ric}_0^{\tilde{g}} = \mathrm{Ric}^g - (n-2)\,\frac{(\nabla^g d\phi)_0}{\phi}, \tag{163}$$

where $(\nabla^g d\phi)_0$ is the trace-free part of the Hessian $\nabla^g d\phi$ of ϕ with respect of g, cf. e.g. [5, Chapter 1, Section J]. Applying (163) to the conformal pair $(g_-, g_+ = f^{-2}\,g_-)$, we get

$$\mathrm{Ric}_0^{g_+} = \mathrm{Ric}_0^{g_-} - \frac{2\,(\nabla^{g_-} df)_0}{f}. \tag{164}$$

Since Ric^{g_+} and Ric^{g_+} are both J_+- and J_--invariant, it follows that $(\nabla^{g_-} df)_0$ is J_--invariant, as well as $\nabla^{g_-} df$, since all metrics in $[g_+] = [g_-]$ are J_+- and J_--invariant. This means that the vector field $\mathrm{grad}_{g\_} f$ is J_--holomorphic, hence that $J_-\mathrm{grad}_{g\_} f$ is Hamiltonian with respect to ω_-, hence Killing with respect to g_-; since $J_-\mathrm{grad}_{g\_} f = \frac{1}{G'(f)}\,K$, we conclude that $G'(f)$ is constant, hence, by using (146), that $F(f)$ and $G(f)$ are of the form

$$F(f) = \frac{a}{f} + b, \qquad G(f) = af + c, \tag{165}$$

for a non-zero real constant a and arbitrary real constants b, c. This, together with (24), establishes part (ii) of the proposition. $\square$

Theorem 5.1 *Let* (M, g) *be a connected, oriented* 4*-manifold admitting a non-parallel* $*$*-Killing* 2*-form* $\psi = \psi_+ + \psi_-$*, satisfying the hypothesis of* Proposition 5.1*, corresponding to* Case (2) *of* Proposition 3.3*. Then, on the dense open set* $M_0 \setminus K_1^{-1}(0)$ *the associated ambikähler structure is of Calabi type, with respect to the Killing vector field* $K = K_1$*, with* $\tau(K) = -K$ *in the first case of* Proposition 5.1 *and* $\tau(K) = K$ *in the two remaining cases.*

Conversely, let (g_+, J_+, ω_+)*,* $(g_- = f^2\, g_+, J_-, \omega_-)$ *be any ambikähler structure of Calabi type with non-vanishing Killing vector field* K*, defined on some oriented* 4*-dimensional manifold* M*. If* $\tau(K) = -K$*, there exist, up to scaling, a unique metric* g *in the conformal class* $[g_+] = [g_-]$ *and a unique non-parallel* $*$*-Killing* 2*-form* ψ *with respect to* g*, inducing the given ambikähler structure. If* $\tau(K) = K$*, such a pair* (g, ψ) *exists and is unique outside the locus* $\{f = 1\}$*.*

Proof The first part of the proposition has already been discussed in the preceding part of this section. Conversely, let (g_+, J_+, ω_+), $(g_- = f^2\, g_+, J_-, \omega_-)$ be an ambikähler structure of Calabi type, with respect to some non-vanishing Killing vector field K, with $\tau(K) = -K$ or $\tau(K) = K$. Then, according to Proposition 5.2, K can be chosen equal to

$$K = J_+ \mathrm{grad}_{g_-} f = J_+ \mathrm{grad}_{g_+} \frac{1}{f}, \tag{166}$$

if $\tau(K) = -K$, or

$$K = J_+ \mathrm{grad}_{g_-} f = -J_+ \mathrm{grad}_{g_+} \frac{1}{f}, \tag{167}$$

if $\tau(K) = K$. According to Proposition 2.2 and (42), if $\tau(K) = -K$, hence $\tau(df) = -df$, the ambikähler structure is then induced by the metric g, in the conformal class $[g_+] = [g_-]$, defined by $g = f_+^{-2}\, g_+ = f_-^{-2}\, g_-$, with

$$f_+ = \frac{cf}{1+f}, \quad f_- = \frac{c}{1+f} = c - f_+, \tag{168}$$

for some positive constant c, and the $*$-Killing 2-form ψ defined by

$$\psi = \frac{f^3}{(1+f)^3}\, \omega_+ + \frac{1}{(1+f)^3}\, \omega_-. \tag{169}$$

If $\tau(K) = K$, hence $\tau(df) = df$, it similarly follows from Proposition 2.2 and (41) that the ambikähler structure is induced by the metric $g = f_+^2\, g_+ = f_-^2\, g_-$, with

$$f_+ = \frac{cf}{1-f}, \quad f_- = \frac{c}{1-f} = c + f^+, \tag{170}$$

for some constant c, positive if $f < 1$, negative if $f > 1$, and the $*$-Killing 2-form

$$\psi = \frac{f^3}{(1-f)^3}\,\omega_+ + \frac{1}{(1-f)^3}\,\omega_-, \tag{171}$$

but the pair (g, ψ) is only defined outside the locus $\{f = 1\}$. □

Remark 5.1 Any ambikähler structure (g_+, J_+, ω_+), (g_-, J_-, ω_-) generates, up to global scaling, a 1-parameter family of ambikähler structures, parametrized by a non-zero real number k, obtained by, say, fixing the first Kähler structure (g_+, J_+, ω_+) and substituting $(g_-^{(k)} = k^{-2}\, g_- = f_k^2\, g_+, J_-^{(k)} = \epsilon(k)\, J_-, \omega_-^{(k)} = \epsilon(k)\, k^{-2}\, \omega_-)$ to the second one, with $\epsilon(k) = \frac{k}{|k|}$ and $f_k = \frac{f}{|k|}$. Assume that the ambikähler structure (g_+, J_+, ω_+), (g_-, J_-, ω_-) is of Calabi type, with $\tau(df) = -df$. For any k in $\mathbb{R} \setminus \{0\}$, we then have $\tau^{(k)}(df_k) = -\epsilon(k)\, df_k$, by setting $\tau^{(k)} = -J_+ J_-^{(k)} = -J_-^{(k)} J_+ = \epsilon(k)\, \tau$, whereas, from (40) we infer:

$$f_+^{(k)} = \frac{f}{|k+f|}, \quad f_-^{(k)} = \frac{|k|}{|k+f|}, \tag{172}$$

up to global scaling; the ambikähler structure (g_+, J_+, ω_+), $(g_-^{(k)}, J_-^{(k)}, \omega_-^{(k)})$ is then induced by the pair $(g^{(k)}, \psi^{(k)})$, where $g^{(k)}$ is defined in the conformal class by

$$g^{(k)} = \frac{f^2}{(k+f)^2}\, g_+ = \frac{(1+f)^2}{(k+f)^2}\, g, \tag{173}$$

and $\psi^{(k)}$ is the $*$-Killing 2-form with respect to $g^{(k)}$ defined by

$$\psi^{(k)} = \frac{f^3}{|k+f|^3}\,\omega_+ + \frac{k}{|k+f|^3}\,\omega_-, \tag{174}$$

both defined outside the locus $\{f + k = 0\}$.

Remark 5.2 As observed in [1, Section 3.1], any ambikähler structure of Calabi type (g_+, J_+, ω_+), $(g_- = f^2\, g_+, J_-, \omega_-)$, with $\tau(df) = df$, admits a *Hamiltonian 2-form*, ϕ^+, with respect to the Kähler structure (g_+, J_+, ω_+) and a Hamiltonian 2-form, ϕ^-, with respect to the (g_-, J_-, ω_-), given by

$$\phi^+ = f^{-1}\,\omega_+ + f^{-3}\,\omega_-, \qquad \phi^- = f^3\,\omega_+ + f\,\omega_-. \tag{175}$$

6 The Decomposable Case

Assume now that (M, g, ψ) is as in Case (3) in Proposition 3.3, that is, that the $*$-Killing 2-form $\psi = \psi_+ + \psi_-$ is degenerate (or decomposable). This latter condition holds if and only if $\psi \wedge \psi = 0$, if and only if $|\psi_+| = |\psi_-|$, i.e. $f_+ = f_- =: \varphi$, or $f = 1$, meaning that $g_+ = g_- =: g_K$, whereas $g = \varphi^2 g_K$. Denote by ∇^K the Levi-Civita connection of g_K. Then from (31)–(33) we get $\nabla^K J_+ = \nabla^K J_- = \nabla^K \tau = 0$, which implies that (M, g_K) is locally a Kähler product of two Kähler curves of the form $M = (\Sigma, g_\Sigma, J_\Sigma, \omega_\Sigma) \times (\tilde{\Sigma}, g_{\tilde{\Sigma}}, J_{\tilde{\Sigma}}, \omega_{\tilde{\Sigma}})$, with

$$\begin{aligned} g_K &= g_\Sigma + g_{\tilde{\Sigma}}, \\ J_+ &= J_\Sigma + J_{\tilde{\Sigma}}, \quad J_- = J_\Sigma - J_{\tilde{\Sigma}}, \\ \omega_+ &= \omega_\Sigma + \omega_{\tilde{\Sigma}}, \quad \omega_- = \omega_\Sigma - \omega_{\tilde{\Sigma}}. \end{aligned} \tag{176}$$

Moreover, from (21) we readily infer $\tau(d\varphi) = d\varphi$, meaning that φ is the pull-back to M of a function defined on Σ. Conversely, for any Kähler product $M = (\Sigma, g_\Sigma, J_\Sigma, \omega_\Sigma) \times (\tilde{\Sigma}, g_{\tilde{\Sigma}}, J_{\tilde{\Sigma}}, \omega_{\tilde{\Sigma}})$ as above and for *any* positive function φ defined on Σ, regarded as a function defined on M, the metric $g := \varphi^2 (g_\Sigma + g_{\tilde{\Sigma}})$ admits a $*$-Killing 2-form ψ, given by

$$\psi = \varphi^3 \omega_\Sigma, \tag{177}$$

whose corresponding Killing 2-form $*\psi$ is given by

$$* \psi = \varphi^3 \omega_{\tilde{\Sigma}}. \tag{178}$$

Note that by (9) $\alpha = \frac{1}{3}\delta^g \psi = \frac{1}{\varphi^2} *_\Sigma d\varphi$, so $K_1 = -\frac{1}{2}\alpha^\sharp$ is not a Killing vector field in general.

The above considerations completely describe the local structure of 4-manifolds with decomposable $*$-Killing 2-forms. They also provide compact examples, simply by taking Σ and $\tilde{\Sigma}$ to be compact Riemann surfaces. We will show, however, that there are compact 4-manifolds with decomposable $*$-Killing 2-forms which are not products of Riemann surfaces (in fact not even of Kähler type). They arise as special cases (for $n = 4$) of the classification, in [9], of compact Riemannian manifolds (M^n, g) carrying a Killing vector fields with conformal Killing covariant derivative.

It turns out that if ψ is a non-trivial $*$-Killing 2-form which can be written as $\psi = d\xi^\flat$ for some Killing vector field ξ on M, then either ψ has rank 2 on M, or M is Sasakian or has positive constant sectional curvature (Proposition 4.1 and Theorem 5.1 in [9]). For $n = 4$, the Sasakian situation does not occur, and the case when M has constant sectional curvature will be treated in detail in the next section. The remaining case—when ψ is decomposable—is the one which we are interested in, and is described by cases 3. and 4. in Theorem 8.9 in [9]. We obtain the following two classes of examples:

1. (M, g) is a warped mapping torus

$$M = (\mathbb{R} \times N)/_{(t,x)\sim(t+1,\varphi(x))}, \qquad g = \lambda^2 d\theta^2 + g_N,$$

   where (N, g_N) is a compact 3-dimensional Riemannian manifold carrying a function λ, such that $d\lambda^\sharp$ is a conformal vector field, φ is an isometry of N preserving λ, $\xi = \frac{\partial}{\partial\theta}$ and $\psi = d\xi^\flat = 2\lambda d\lambda \wedge d\theta$. One can take for instance $(N, g_N) = \mathbb{S}^3$ and λ a first spherical harmonic. Further examples of manifolds N with this property are given in Section 7 in [9].
2. (M, g) is a Riemannian join $\mathbb{S}^2 *_{\gamma,\lambda} \mathbb{S}^1$, defined as the smooth extension to S^4 of the metric $g = ds^2 + \gamma^2(s)g_{\mathbb{S}^2} + \lambda^2(s)d\theta^2$ on $(0, l) \times S^2 \times S^1$, where $l > 0$ is a positive real number, $\gamma : (0, l) \to \mathbb{R}^+$ is a smooth function satisfying the boundary conditions

$$\gamma(t) = t(1 + t^2 a(t^2)) \quad \text{and} \quad \gamma(l - t) = \frac{1}{c} + t^2 b(t^2), \quad \forall\ |t| < \epsilon,$$

   for some smooth functions a and b defined on some interval $(-\epsilon, \epsilon)$, $\lambda(s) := \int_s^l \gamma(t)dt$, $\xi = \frac{\partial}{\partial\theta}$ and $\psi = 2\lambda(s)\lambda'(s)ds \wedge d\theta$.

In particular, we obtain infinite-dimensional families of metrics on $S^3 \times S^1$ and on S^4 carrying decomposable $*$-Killing 2-forms.

7 Example: The Sphere $\mathbb{S}^4$ and Its Deformations

We denote by $\mathbb{S}^4 := (S^4, g)$ the 4-dimensional sphere, embedded in the standard way in the Euclidean space $\mathbb{R}^5$, equipped with the standard induced Riemannian metric, g, of constant sectional curvature 1, namely the restriction to $\mathbb{S}^4$ of the standard inner product $(\cdot, \cdot)$ of $\mathbb{R}^5$. We first recall the following well-known facts, cf. e.g. [13]. Let $\psi = \psi_+ + \psi_-$ be any $*$-Killing 2-form with respect to g, so that $\nabla_X \Psi = \alpha \wedge X$, cf. (8). Since g is Einstein, the vector field $\alpha^\sharp$ is Killing and it follows from (74)–(75) that $\nabla\alpha = \psi$. Conversely, for any Killing vector field Z on $\mathbb{S}^4$, it readily follows from the general *Kostant formula*

$$\nabla_X(\nabla Z) = \mathrm{R}_{Z,X}, \tag{179}$$

that, in the current case, $\nabla_X(\nabla Z) = Z \wedge X$, so that the 2-form $\psi := \nabla Z^\flat$ is $*$-Killing with respect to g. The map $Z \mapsto \nabla Z^\flat$ is then an isomorphism from the space of Killing vector fields on $\mathbb{S}^4$ to the space of $*$-Killing 2-forms.

It is also well-known that there is a natural $1 - 1$-correspondence between the Lie algebra $\mathfrak{so}(5)$ of anti-symmetric endomorphisms of $\mathbb{R}^5$ and the space of Killing vector fields on $\mathbb{S}^4$: for any a in $\mathfrak{so}(5)$, the corresponding Killing vector field, Z_{a}, is defined by

$$Z_{\mathsf{a}}(u) = \mathsf{a}(u), \tag{180}$$

for any u in $\mathbb{S}^4$, where $\mathsf{a}(u)$ is viewed as an element of the tangent space $T_u\mathbb{S}^4$, via the natural identification $T_u\mathbb{S}^4 = u^\perp$.

By combining the above two isomorphisms, we eventually obtained a natural identification of $\mathfrak{so}(5)$ with the space of $*$-Killing 2-forms on $\mathbb{S}^4$ and it is easy to check that, for any a in $\mathfrak{so}(5)$, the corresponding $*$-Killing 2-form, ψ_{a}, is given by

$$\psi_{\mathsf{a}}(X, Y) = (\mathsf{a}(X), Y), \tag{181}$$

for any u in $\mathbb{S}^4$ and any X, Y in $T_u\mathbb{S}^4 = u^\perp$; alternatively, the corresponding endomorphism Ψ_{a} is given by

$$\Psi_{\mathsf{a}}(X) = \mathsf{a}(X) - (\mathsf{a}(X), u)\, u, \tag{182}$$

for any X in $T_u\mathbb{S}^4 = u^\perp$.

Since, for any u in $\mathbb{S}^4$, the volume form of $\mathbb{S}^4$ is the restriction to $T_u\mathbb{S}^4$ of the 4-form $u \lrcorner \mathrm{v}_0$, where v_0 stands for the standard volume form of $\mathbb{R}^5$, namely $\mathrm{v}_0 = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4$, for any direct frame of $\mathbb{R}^5$ (here identified with a coframe via the standard metric), we easily check that, for any a in $\mathfrak{so}(5)$, the corresponding Killing 2-form $*\psi_{\mathsf{a}}$ has the following expression

$$(*\psi_{\mathsf{a}})(X, Y) = (u \lrcorner *_5 \mathsf{a})(X, Y) = *_5(u \wedge \mathsf{a})(X, Y), \tag{183}$$

for any u in $\mathbb{S}^4$ and any X, Y in $T_u\mathbb{S}^4 = u^\perp$; here, $*_5$ denotes the Hodge operator on $\mathbb{R}^5$ and we keep identifying vector and covectors via the Euclidean inner product.

From (182), we easily infer

$$|\Psi_{\mathsf{a}}|^2 = |\mathsf{a}|^2 - 2|\mathsf{a}(u)|^2, \tag{184}$$

at any u in $\mathbb{S}^4$, where the norm is the usual Euclidean norm of endomorphisms, whereas the Pfaffian of ψ_{a} is given by:

$$\mathrm{pf}(\psi_{\mathsf{a}}) := \frac{\psi_{\mathsf{a}} \wedge \psi_{\mathsf{a}}}{2\,\mathrm{v}_g} = \frac{(\psi_{\mathsf{a}}, *\psi_{\mathsf{a}})}{2} = \frac{u \wedge \mathsf{a} \wedge \mathsf{a}}{2\,\mathrm{v}_0}. \tag{185}$$

On the other hand, when f_+, f_- are defined by (15), we have

$$|\Psi_{\mathsf{a}}|^2 = 4(f_+^2 + f_-^2), \tag{186}$$

and

$$\mathrm{pf}(\psi_{\mathsf{a}}) = f_+^2 - f_-^2. \tag{187}$$

For any a in $\mathfrak{so}(5)$, we may choose a direct orthonormal basis e_0, e_1, e_2, e_3, e_4 of $\mathbb{R}^5$, with respect to which a has the following form

$$\mathsf{a} = \lambda\, e_1 \wedge e_2 + \mu\, e_3 \wedge e_4, \tag{188}$$

for some real numbers λ, μ, with $0 \leq \lambda \leq \mu$. Then,

$$\begin{aligned} |\mathsf{a}|^2 &= 2(\lambda^2 + \mu^2),\\ \mathsf{a}(u) &= \lambda(u_1 e_2 - u_2 e_1) + \mu(u_3 e_4 - u_3 e_3),\\ |\mathsf{a}(u)|^2 &= \lambda^2(u_1^2 + u_2^2) + \mu^2(u_3^2 + u_4^2),\\ u \wedge \mathsf{a} \wedge \mathsf{a} &= 2\,\lambda\,\mu\, u_0\, e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4, \end{aligned} \tag{189}$$

for any $u = \sum_{i=0}^{4} u_i\, e_i$ in $\mathbb{S}^4$. We thus get

$$\begin{aligned} f_+^2 + f_-^2 &= \frac{1}{2}\left(\lambda^2 + \mu^2 - \lambda^2(u_1^2 + u_2^2) - \mu^2(u_3^2 + u_4^2)\right),\\ f_+^2 - f_-^2 &= \lambda\mu\, u_0, \end{aligned} \tag{190}$$

hence

$$\begin{aligned} f_+(u) &= \frac{1}{2}\left((\lambda + \mu\, u_0)^2 + (\mu^2 - \lambda^2)\,(u_1^2 + u_2^2)\right)^{\frac{1}{2}}\\ &= \frac{1}{2}\left((\mu + \lambda\, u_0)^2 + (\lambda^2 - \mu^2)\,(u_3^2 + u_4^2)\right)^{\frac{1}{2}},\\ f_-(u) &= \frac{1}{2}\left((\lambda - \mu\, u_0)^2 + (\mu^2 - \lambda^2)\,(u_1^2 + u_2^2)\right)^{\frac{1}{2}}\\ &= \frac{1}{2}\left((\mu - \lambda\, u_0)^2 + (\lambda^2 - \mu^2)\,(u_3^2 + u_4^2)\right)^{\frac{1}{2}}. \end{aligned} \tag{191}$$

From (190)–(191), we easily obtain the following three cases, corresponding, in the same order, to the three cases listed in Proposition 3.3:

Case 1: a is of rank 4—i.e. λ and μ are both non-zero—and $\lambda < \mu$. Then:

(i) $f_+(u) = f_-(u)$ if and only if u belongs to the equatorial sphere $\mathbb{S}^3$ defined by $u_0 = 0$;

(ii) $f_+(u) = 0$ if and only u belongs to the circle $\mathrm{C}_+ = \{u_0 = -\frac{\lambda}{\mu}, u_1 = u_2 = 0\}$, and we then have $f_-(u) = \frac{\lambda}{2}$;

(iii) $f_-(u) = 0$ if and only if u belongs to the circle $\mathrm{C}_- = \{u_0 = \frac{\lambda}{\mu}, u_1 = u_2 = 0\}$, and we then have $f_+(u) = \frac{\lambda}{2}$;

(iv) the 2-form $df_+^2 \wedge df_-^2$ is non-zero outside the 2-spheres $\mathrm{S}_+^2 = \{u_1 = u_2 = 0\}$ and $\mathrm{S}_-^2 = \{u_3 = u_4 = 0\}$; this is because

$$\begin{aligned} df_+^2 \wedge df_-^2 &= \frac{\lambda\mu(\lambda^2 - \mu^2)}{2} du_0 \wedge (u_1\, du_1 + u_2\, du_2) \\ &= \frac{\lambda\mu(\mu^2 - \lambda^2)}{2} du_0 \wedge (u_3\, du_3 + u_4\, du_4), \end{aligned} \tag{192}$$

which readily follows from (190).

Case 2: $\mathfrak{a}$ is of rank 4 and $\lambda = \mu$. Then

$$f_+(u) = \frac{\lambda}{2}(1 + u_0), \quad f_-(u) = \frac{\lambda}{2}(1 - u_0); \tag{193}$$

in particular,

$$f_+ + f_- = \lambda; \tag{194}$$

moreover, $f_+(u) = 0$ if and only if $u = -e_0$ and $f_-(u) = 0$ if and only if $u = e_0$.

Case 3: $\mathfrak{a}$ is of rank 2, i.e. $\lambda = 0$. Then, $f_+ - f_-$ is identically zero and $f_+(u) = f_-(u)$ vanishes if and only if u belongs to the circle $\mathrm{C}_0 = \{u_0 = u_1 = u_2 = 0\}$.

Remark 7.1 Consider the functions $x = \frac{f_+ + f_-}{2}, y = \frac{f_+ - f_-}{2}$ defined in Sect. 3, as well as the functions of one variable, A and B, appearing in Proposition 3.1. If $\mathfrak{a}$ is of rank 4, with $0 < \lambda < \mu$, corresponding to Case 1 in the above list, we easily infer from (190) that

$$\begin{aligned} u_0 &= \frac{4xy}{\lambda\mu}, \\ u_1^2 + u_2^2 &= \frac{(\lambda^2 - 4x^2)(\lambda^2 - 4y^2)}{\lambda^2(\lambda^2 - \mu^2)}, \\ u_3^2 + u_4^2 &= \frac{(\mu^2 - 4x^2)(\mu^2 - 4y^2)}{\mu^2(\mu^2 - \lambda^2)}. \end{aligned} \tag{195}$$

Since $x \geq |y|$, the above identities imply that the image of (x, y) in $\mathbb{R}^2$ is the rectangle $R := \left[\frac{\lambda}{2}, \frac{\mu}{2}\right] \times \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]$. A simple calculation then shows that A and B are given by

$$A(z) = -B(z) = -\left(z^2 - \frac{\lambda^2}{4}\right)\left(z^2 - \frac{\mu^2}{4}\right). \tag{196}$$

Notice that $A(x)$ and $B(y)$ are positive in the interior of R, corresponding to the open set of $\mathbb{S}^4$ where dx, dy are linearly independent, and vanish on its boundary. Also notice that the above expressions of A, B fit with the identities (104)–(105), with $\mathrm{Scal} = 12$ and $b = 0$.

Remark 7.2 By using the ambitoric Ansatz in Theorem 4.1, the above situation can easily be deformed in Case 1, where a is of rank 4, with $0 < \lambda < \mu$, and the 2-form ψ_{a} defined by (181) is $*$-Killing with respect to the round metric (We warmly thank Vestislav Apostolov for this suggestion.) On the open set $\mathcal{U} = \mathbb{S}^4 \setminus \left(S^2_+ \cup S^2_-\right)$, where $f_+ \neq 0, f_- \neq 0$ and $df_+ \wedge df_- \neq 0$, the round metric of $\mathbb{S}^4$ takes the form (124), where A and B are given by (196), $x \in \left(\frac{\lambda}{2}, \frac{\mu}{2}\right)$, $y \in \left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ are determined by (195) and ds, dt are explicit exact 1-forms determined by the last two equations of (125). It can actually be shown that outside the 2-spheres S^2_+ and S^2_-, ds and dt are given by:

$$\begin{aligned} ds &= \frac{2}{\mu^2 - \lambda^2} \left(\lambda \, \frac{u_1 du_2 - u_2 du_1}{u_1^2 + u_2^2} - \mu \, \frac{u_3 du_4 - u_4 du_3}{u_3^2 + u_4^2}\right) \\ &= \frac{2}{\mu^2 - \lambda^2} \, d\left(\lambda \, \arctan \frac{u_2}{u_1} - \mu \, \arctan \frac{u_4}{u_3}\right), \\ dt &= \frac{8}{\mu^2 - \lambda^2} \left(-\frac{1}{\lambda} \, \frac{u_1 du_2 - u_2 du_1}{u_1^2 + u_2^2} + \frac{1}{\mu} \, \frac{u_3 du_4 - u_4 du_3}{u_3^2 + u_4^2}\right) \\ &= \frac{8}{\mu^2 - \lambda^2} \, d\left(-\frac{1}{\lambda} \, \arctan \frac{u_2}{u_1} + \frac{1}{\mu} \, \arctan \frac{u_4}{u_3}\right). \end{aligned} \tag{197}$$

Moreover, ψ_{a} is given by (128) with respect to these coordinates.

Consider now a small perturbation $\tilde{A}, \tilde{B}$ of the functions A and B such that $\tilde{A}(x) = A(x)$ near $x = \frac{\lambda}{2}$ and $x = \frac{\mu}{2}$ and $\tilde{B}(y) = B(y)$ near $y = \pm\frac{\lambda}{2}$. If the perturbation is small enough, the expression analogue to (124)

$$\begin{aligned} \tilde{g} := (x^2 - y^2) &\left(\frac{dx \otimes dx}{\tilde{A}(x)} + \frac{dy \otimes dy}{\tilde{B}(y)}\right) \\ &+ \frac{\tilde{A}(x)}{(x^2 - y^2)} \, (ds + y^2 \, dt) \otimes (ds + \, y^2 \, dt) \\ &+ \frac{\tilde{B}(y)}{(x^2 - y^2)} \, (ds + x^2 \, dt) \otimes (ds + x^2 \, dt) \end{aligned} \tag{198}$$

is still positive definite so defines a Riemannian metric on $\mathcal{U}$, which coincides with the canonical metric on an open neighbourhood of $\mathbb{S}^4 \setminus \mathcal{U} = S^2_+ \cup S^2_-$, and thus has a smooth extension to $\mathbb{S}^4$ which we still call $\tilde{g}$. Since the expression (128) of the $*$-Killing form in the Ansatz of Sect. 4 does not depend on A and B, the 2-form ψ_{a} is still $*$-Killing with respect to the new metric $\tilde{g}$. We thus get an infinite-dimensional family (depending on two functions of one variable) of Riemannian metrics on S^4 which all carry *the same* non-parallel $*$-Killing form.

8 Example: Complex Ruled Surfaces

In general, a (geometric) *complex ruled surface* is a compact, connected, complex manifold of the form $M = \mathbb{P}(E)$, where E denotes a rank 2 holomorphic vector bundle over some (compact, connected) Riemann surface, Σ, and $\mathbb{P}(E)$ is then the corresponding projective line bundle, i.e. the holomorphic bundle over Σ, whose fiber at each point y of Σ is the complex projective line $\mathbb{P}(E_y)$, where E_y denotes the fiber of E at y. A complex ruled surface is said to be *of genus* $\mathbf{g}$ if Σ is of genus $\mathbf{g}$.

In this section, we restrict our attention to complex ruled surfaces $\mathbb{P}(E)$ as above, when $E = L \oplus \mathbb{C}$ is the Whitney sum of some holomorphic line bundle, L, over Σ and of the trivial complex line bundle $\Sigma \times \mathbb{C}$, here simply denoted $\mathbb{C}$: M is then the *compactification* of the total space of L obtained by adding the *point at infinity* $[L_y] := \mathbb{P}(L_y \oplus \{0\})$ to each fiber of M over y. The union of the points at infinity is a divisor of M, denoted by Σ_∞, whereas the (image of) the zero section of L, viewed as a divisor of M, is denoted Σ_0; both Σ_0 and Σ_∞ are identified with Σ by the natural projection, π, from M to Σ. The open set $M \setminus (\Sigma_0 \cup \Sigma_\infty)$, denoted M^0, is naturally identified with $L \setminus \Sigma_0$. We moreover assume that the degree, $\mathrm{d}(L)$, of L is *negative* and we set: $\mathrm{d}(L) = -k$, where k is a positive integer.

Complex ruled surfaces of this form will be called *Hirzebruch-like ruled surfaces*. When $\mathbf{g} = 0$, these are exactly those complex ruled surfaces introduced by F. Hirzebruch in [7]. When $\mathbf{g} \geq 2$, they were named *pseudo-Hirzebruch* in [14].

In general, the Kähler cone of a complex ruled surface $\mathbb{P}(E)$ was described by A. Fujiki in [6]. In the special case considered in this section, when $M = \mathbb{P}(L \oplus \mathbb{C})$ is a Hirzebruch-like ruled surface, if $[\Sigma_0]$, $[\Sigma_\infty]$ and $[F]$ denote the Poincaré duals of the (homology class of) Σ_0, Σ_∞ and of any fiber F of π in $\mathrm{H}^2(M, \mathbb{Z})$, the latter is freely generated by $[\Sigma_0]$ and $[F]$ or by $[\Sigma_\infty]$ and $[F]$, with $[\Sigma_0] = [\Sigma_\infty] - k\,[F]$, and the Kähler cone is the set of those elements, Ω_{a_0, a_∞}, of $\mathrm{H}(M, \mathbb{R})$ which are of the form $\Omega_{a_0, a_\infty} = 2\pi \left(- a_0\,[\Sigma_0] + a_\infty\,[\Sigma_\infty]\right)$, for any two real numbers a_0, a_∞ such that $0 < a_0 < a_\infty$.

We assume that Σ comes equipped with a Kähler metric $(g_\Sigma, \omega_\Sigma)$ polarized by L, in the sense that L is endowed with a Hermitian (fiberwise) inner product, h, in such a way that the curvature, R^∇, of the associated Chern connection, ∇, is related to the Kähler form ω_Σ by $\mathrm{R}^\nabla = i\,\omega$; in particular, $[\omega_\Sigma] = 2\pi\, c_1(L^*)$, where $[\omega_\Sigma]$ denotes the de Rham class of ω_Σ, L^* the dual line bundle to L and $c_1(L^*)$ the (de Rham) Chern class of L^*. The natural action of $\mathbb{C}^*$ extends to a holomorphic $\mathbb{C}^*$-action on M, trivial on Σ_0 and Σ_∞; we denote by K the generator of the restriction of this action on $S^1 \subset \mathbb{C}^*$. On $M^0 = L \setminus \Sigma_0$, we denote by t the function defined by

$$t = \log r, \tag{199}$$

where r stands for the distance to the origin in each fiber of L determined by h; on M^0, we then have

$$dd^c t = \pi^* \omega_\Sigma, \qquad d^c t(K) = 1 \tag{200}$$

(beware: the function t defined by (199) has nothing to do with the local coordinate t appearing in Sect. 4). Any (smooth) function $F = F(t)$ of t will be regarded as function defined on M^0, which is evidently K-invariant; moreover:

1. $F = F(t)$ smoothly extends to Σ_0 if and only if $F(t) = \Phi_+(e^{2t})$ near $t = -\infty$, for some smooth function Φ_+ defined on some neighbourhood of 0 in $\mathbb{R}^{\geq 0}$, and
2. $F = F(t)$ smoothly extends to Σ_∞ if and only if $F(t) = \Phi_-(e^{-2t})$ near $t = \infty$, for some smooth function Φ_- defined on some neighbourhood of 0 in $\mathbb{R}^{\geq 0}$, cf. e.g. [14], [1, Section 3.3].

For any (smooth) real function $\varphi = \varphi(t)$, denote by ω_φ the real, J-invariant 2-form defined on M^0 by

$$\omega_\varphi = \varphi\, dd^c t + \varphi'\, dt \wedge d^c t, \tag{201}$$

where φ' denotes the derivative of φ with respect to t. Then, ω_φ is a Kähler form on M^0, with respect to the natural complex structure $J = J_+$, of M, if and only if φ is positive and increasing as a function of t; moreover, ω_ϕ extends to a smooth Kähler form on M, in the Kähler class Ω_{a_0, a_∞}, if and only if φ satisfies the above asymptotic conditions (1)–(2), with $\Phi_+(0) = a_0 > 0$, $\Phi'_+(0) > 0$, $\Phi_-(0) = a_\infty > 0$, $\Phi'_-(0) < 0$. Kähler forms of this form on M, as well as the corresponding Kähler metrics

$$g_\varphi = \varphi\, \pi^* g_\Sigma + \varphi'\, (dt \otimes dt + d^c t \otimes d^c t), \tag{202}$$

are called *admissible*.

Denote by J_- the complex structure, first defined on the total space of L by keeping J on the horizontal distribution determined by the Chern connection and by substituting $-J$ on the fibers, then smoothly extended to M. The new complex structure induces the opposite orientation, hence commutes with $J_+ = J$.

Any admissible Kähler form ω_φ is both J_+- and J_--invariant, as well as the associated 2-form $\tilde{\omega}_\varphi$ defined by

$$\tilde{\omega}_\varphi := \frac{1}{\varphi}\, dd^c t - \frac{\varphi'}{\varphi^2}\, dt \wedge d^c t, \tag{203}$$

which is moreover Kähler with respect to J_-, with metric

$$\tilde{g}_\varphi = \frac{1}{\varphi^2}\, g_\varphi. \tag{204}$$

We thus obtain an ambikähler structure of Calabi-type, as defined in Sect. 5, with $f = \frac{1}{\varphi}$ and $\tau(K) = -K$. According to Theorem 5.1 and Remark 5.1, for any k in

$\mathbb{R} \setminus \{0\}$, the metric $g^{(k)}$ defined, outside the locus $\{1 + k\varphi = 0\}$, by

$$g_\varphi^{(k)} = \frac{1}{(1 + k\varphi)^2}\, g_\varphi, \tag{205}$$

there admits a non-parallel $*$-Killing 2-form $\psi_\varphi^{(k)}$, namely

$$\begin{aligned}\psi_\varphi^{(k)} &= \frac{1}{(1 + k\varphi)^3}\,\omega_\phi + \frac{k\varphi^3}{(1 + k\varphi)^3}\,\tilde{\omega}_\varphi \\ &= \frac{\varphi}{(1 + k\varphi)^2}\, dd^c t + \frac{(1 - k\varphi)\varphi'}{(1 + k\varphi)^3}\, dt \wedge d^c t.\end{aligned} \tag{206}$$

Notice that the pair $(g_\varphi^{(k)}, \psi_\varphi^{(k)})$ smoothly extends to M for any $k \in \mathbb{R} \setminus [-\frac{1}{a_0}, -\frac{1}{a_\infty}]$, including $k = 0$ for which we simply get the Kähler pair (g_φ, ω_+).

Acknowledgements We warmly thank Vestislav Apostolov and David Calderbank for their interest in this work and for many useful suggestions. We also thank the anonymous referee, whose valuable observations allowed us to correct a mistake and significantly improve a part of the paper. This work was partially supported by the Procope Project No. 32977YJ.

References

1. V. Apostolov, D.M.J. Calderbank, P. Gauduchon, The geometry of weakly self-dual Kähler surfaces. Compos. Math. **135**, 279–322 (2003)
2. V. Apostolov, D.M.J. Calderbank, P. Gauduchon, Hamiltonian 2-forms in Kähler geometry I: general theory. J. Differ. Geom. **73**, 359–412 (2006)
3. V. Apostolov, D.M.J. Calderbank, P. Gauduchon, Ambitoric geometry II: extremal toric surfaces and Einstein 4-orbifolds. Ann. Sci. Éc. Norm. Supér. (4) **48**(5), 1075–1112 (2015)
4. V. Apostolov, D.M.J. Calderbank, P. Gauduchon, Ambitoric geometry I: Einstein metrics and extremal ambikähler structures. J. Reine Angew. Math. **721**, 109–147 (2016)
5. A.L. Besse, *Einstein Manifolds*. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 10 (Springer, Berlin, 1987)
6. A. Fujiki, Remarks on extremal Kähler metrics on ruled manifolds. Nagoya Math. J. **126**, 89–101 (1992)
7. F. Hirzebruch, Über eine Klasse von einfachzusammenhängenden komplexen Mannigfaltigkeiten. Math. Ann. **124**, 77–86 (1951)
8. W. Jelonek, Bi-Hermitian gray surfaces II. Differ. Geom. Appl. **27**, 64–74 (2009)
9. A. Moroianu, Killing vector fields with twistor derivative. J. Differ. Geom. **77**, 149–167 (2007)
10. A. Moroianu, U. Semmelmann, Twistor forms on Kähler manifolds. Ann. Sci. Norm. Sup. Pisa **2**(4), 823–845 (2003)
11. R. Penrose, M. Walker, On quadratic first integrals of the geodesic equations for type {22} spacetimes. Commun. Math. Phys. **18**, 265–274 (1970)
12. M. Pontecorvo, On twistor spaces of ant-self-dual Hermitian surfaces. Trans. Am. Math. Soc. **331**, 653–661 (1992)
13. U. Semmelmann, Conformal Killing forms on Riemannian manifolds. Math. Z. **245**, 503–527 (2003)
14. C. Tønnesen–Friedman, Extremal metrics on minimal ruled surfaces. J. Reine Angew. Math. **502**, 175–197 (1998)

Twistors, Hyper-Kähler Manifolds, and Complex Moduli

Claude LeBrun

For my good friend and admired colleague Simon Salamon, on the occasion of his sixtieth birthday.

Abstract A theorem of Kuranishi (Ann Math 75(2):536–577, 1962) tells us that the moduli space of complex structures on any smooth compact manifold is always *locally* a finite-dimensional space. *Globally*, however, this is simply not true; we display examples in which the moduli space contains a sequence of regions for which the local dimension tends to infinity. These examples naturally arise from the twistor theory of hyper-Kähler manifolds.

Keywords Complex structure • Hyper-Kähler • Kodaira-Spencer map • Moduli • Twistor space

If Y is a smooth compact manifold, the moduli space $\mathfrak{M}(Y)$ of complex structures on Y is defined to be the quotient of the set of all smooth integrable almost-complex structure J on Y, equipped with the topology it inherits from the space of almost-complex structures, modulo the action of the group of self-diffeomorphisms of Y. When we focus only on complex structures near some given J_0, an elaboration of Kodaira-Spencer theory [3] due to Kuranishi [4] shows that the moduli space is locally finite dimensional. Indeed, if Θ denotes the sheaf of holomorphic vector fields on (Y, J_0), Kuranishi shows that there is a family of complex structures parameterized by an analytic subvariety of the unit ball in $H^1(Y, \Theta)$ which, up to biholomorphism, sweeps out every complex structure near J_0. This subvariety of $H^1(Y, \Theta)$ is defined by equations taking values in $H^2(Y, \Theta)$, and one must then also divide by the group of complex automorphisms of (Y, J), which is a Lie group with Lie algebra $H^0(Y, \Theta)$. But, in any case, near a given complex structure, this says

C. LeBrun (✉)
Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, USA
e-mail: claude@math.stonybrook.edu

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_8

that the moduli space is a finite-dimensional object, with dimension bounded above by $h^1(Y, \Theta)$.

What we will observe here, however, is that this local finite-dimensionality can completely break down in the large:

Theorem A *Let X^{4k} be a smooth simply connected compact manifold that admits a hyper-Kähler metric. Then the moduli space $\mathfrak{M}$ of complex structures on $S^2 \times X$ is infinite dimensional, in the following sense: for every $N \in \mathbb{Z}^+$, there are holomorphic embeddings $D^N \hookrightarrow \mathfrak{M}$ of the N-complex-dimensional unit polydisk $D^N := D \times \cdots \times D \subset \mathbb{C}^N$ into the moduli space.*

In fact, for every natural number N, we will construct proper holomorphic submersions $\mathfrak{Y} \to D^N$ with fibers diffeomorphic to $X \times S^2$ such that no two fibers are biholomorphically equivalent. Focusing on this concrete assertion should help avoid confusing the phenomenon under study with other possible structural pathologies of the moduli space $\mathfrak{M}$.

Before proceeding further, it might help to clarify how our construction differs from various off-the-shelf examples where Kodaira-Spencer theory produces mirages of moduli that should not be mistaken for the real thing. Consider the Hirzebruch surfaces $F_\ell = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(\ell)) \to \mathbb{CP}_1$. These are all diffeomorphic to $S^2 \times S^2$ or $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$, depending on whether ℓ is even or odd. For $\ell > 0$, $h^1(F_\ell, \Theta_\ell) = (\ell - 1) \to \infty$ and $h^2(F_\ell, \Theta_\ell) = 0$, so it might appear that the dimension of the moduli space is growing without bound. However, when these infinitesimal deformations are realized by a versal family, most of the fibers always turn out to be mutually biholomorphic, because $h^0(F_\ell, \Theta_\ell) = (\ell + 5) \to \infty$, too, and a cancellation arises from the action of the automorphisms of the central fiber on the versal deformation. In fact, the F_ℓ represent *all* the complex structures on $S^2 \times S^2$ and $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$; thus, while the corresponding moduli spaces are highly non-Hausdorff, they are in fact just 0-dimensional. Similar phenomena also arise from projectivizations of higher-rank vector bundles over $\mathbb{CP}_1$; even though it is easy to construct examples with $h^1(\Theta) \to \infty$ in this context, the piece of the moduli space one constructs in this way is once again non-Hausdorff and 0-dimensional.

Let us now recall that a smooth compact Riemannian manifold (X^{4k}, g) is said to be *hyper-Kähler* if its holonomy is a subgroup of **Sp**(k). One then says that a hyper-Kähler manifold is *irreducible* if its holonomy is exactly **Sp**(k). This in particular implies [1] that X is simply connected. Conversely, any simply connected compact hyper-Kähler manifold is a Cartesian product of irreducible ones, since its deRham decomposition [2] cannot involve any flat factors. In order to prove Theorem A, one therefore might as well assume that (X, g) is irreducible, since any hyper-Kähler manifold admits complex structures, and $S^2 \times (X \times \tilde{X}) = (S^2 \times X) \times \tilde{X}$. Note that examples of irreducible hyper-Kähler $(4k)$-manifolds are in fact known [1, 6] for every $k \geq 1$. When $k = 1$, the unique choice for X is $K3$. For $k \geq 2$, the smooth manifold X is no longer uniquely determined by k, but the the Hilbert scheme of k points on a $K3$ surface always provides one simple and elegant example.

The construction we will use to prove Theorem A crucially involves the use of twistor spaces [2, 7]. Recall that the standard representation of **Sp**(k) on $\mathbb{R}^{4k} = \mathbb{H}^k$ commutes with every almost-complex structure arising from a quaternionic scalar in

$S^2 \subset \Im m\, \mathbb{H}$, and that every hyper-Kähler manifold is therefore Kähler with respect to a 2-sphere's worth of parallel almost-complex structures. Concretely, if we let J_1, J_2, and J_3 denote the complex structures corresponding to the quaternions **i**, **j**, and **k**, then the integrable complex structures in question are those given by $aJ_1 + bJ_2 + cJ_3$ for any $(a, b, c) \in \mathbb{R}^3$ with $a^2 + b^2 + c^2 = 1$. We can then assemble these to form an integrable almost-complex structure on $X \times S^2$ by using the round metric and standard orientation on S^2 to make it into a $\mathbb{CP}_1$, and then giving the X the integrable complex structure $aJ_1 + bJ_2 + cJ_3$ determined by $(a, b, c) \in S^2$. For each $x \in X$, the stereographic coordinate $\zeta = (b + ic)/(a + 1)$ on $\{x\} \times S^2$ is thus a compatible complex coordinate system on the so-called *real twistor line* $\mathbb{CP}_1 \subset Z$ near the point $(1, 0, 0)$ representing $J_1|_x$. We will make considerable use of the fact that the factor projection $X \times S^2 \to S^2$ now becomes a holomorphic submersion $\varpi : Z \to \mathbb{CP}_1$ with respect to the twistor complex structure, so that ϖ can therefore be thought of as a family of complex structures on X.

Lemma 1 *Let (X^{4k}, g), $k \geq 1$, be a hyper-Kähler manifold, and let Z be its twistor space. Consider the holomorphic submersion $\varpi : Z \to \mathbb{CP}_1$ as a family of compact complex manifolds, and set $X_\zeta := \varpi^{-1}(\zeta)$ for any $\zeta \in \mathbb{CP}_1$. Then the Kodaira-Spencer map $T^{1,0}_{\zeta_0}\mathbb{CP}_1 \to H^1(X_{\zeta_0}, \mathcal{O}(T^{1,0}X_\zeta))$ is non-zero at every $\zeta_0 \in \mathbb{CP}_1$.*

Proof Since we can always change our basis for the parallel complex structures on (X, g) by the action of $\mathbf{SO(3)}$, we may assume that the value ζ_0 of $\zeta \in \mathbb{CP}_1$ at which we wish to check the claim represents the complex structure on X we have temporarily chosen to call J_1. Observe that the 2-forms $\omega_\alpha = g(J_\alpha \cdot, \cdot)$, $\alpha = 1, 2, 3$, are all parallel. Moreover, notice that, with respect to J_1, the 2-form ω_1 is just the Kähler form of g, while $\omega_2 + i\omega_3$ is a non-degenerate holomorphic $(2, 0)$-form.

By abuse of notation, we will now also use ζ to denote a local complex coordinate on $\mathbb{CP}_1$, with $\zeta = 0$ representing the complex structure J_1 of interest. Now recall that the Kodaira-Spencer map sends $d/d\zeta$ to an element of $H^1(X, \mathcal{O}_{J_1}(T^{1,0}_{J_1}X))$ that literally encodes the derivative of the complex structure J_ζ with respect to ζ. Indeed, since we already have chosen a differentiable trivialization of our family, this element is represented in Dolbeault cohomology by the $(0, 1)$-form φ with values in $T^{1,0}$ given by

$$\varphi(v) := \left[\frac{d}{d\zeta} J_\zeta(v^{0,1}) \right]^{1,0} \Bigg|_{\zeta=0}$$

where the decomposition $T_{\mathbb{C}}X = T^{1,0} \oplus T^{0,1}$ used here is understood to be the one determined by J_1. Now taking ζ to specifically be the stereographic coordinate $\zeta = \xi + i\eta$, where $\xi = b/(1 + a)$ and $\eta = c/(1 + a)$, we then have

$$\frac{d}{d\xi} J_\zeta \Big|_{\zeta=0} = J_2 \quad \text{and} \quad \frac{d}{d\eta} J_\zeta \Big|_{\zeta=0} = J_3,$$

and hence

$$\frac{d}{d\zeta}J_\zeta\bigg|_{\zeta=0} = \frac{1}{2}(J_2 - iJ_3).$$

Since $T^{0,1}$ is the $(-i)$-eigenspace of J_1, we therefore have

$$\begin{aligned}\varphi(v) &= \frac{1}{2}\left[(J_2 - iJ_3)v^{0,1}\right]^{1,0}\\ &= \frac{1}{2}\left[(J_2 + iJ_2J_1)v^{0,1}\right]^{1,0}\\ &= \left[J_2(v^{0,1})\right]^{1,0}\\ &= J_2(v^{0,1})\end{aligned}$$

where the last step uses the fact that J_2 anti-commutes with J_1, and therefore interchanges the $(\pm i)$-eigenspaces $T^{1,0}$ and $T^{0,1}$ of J_1.

On the other hand, since $\omega_2 + i\omega_3$ is a non-degenerate holomorphic 2-form on (X, J_1), contraction with this form induces a holomorphic isomorphism $T^{1,0} \cong \Lambda^{1,0}$, and hence an isomorphism $H^1(X, \mathcal{O}(T^{1,0})) \cong H^1(X, \Omega^1)$. In Dolbeault terms, the Kodaira-Spencer class $[\varphi]$ is thus mapped by this isomorphism to the element of $H^{1,1}_{\bar\partial_{J_1}}(X) = H^1(X, \Omega^1)$ represented by the contraction $\varphi\lrcorner(\omega_2 + i\omega_3)$. Since

$$\begin{aligned}[\varphi(v^{0,1})]\lrcorner(\omega_2 + i\omega_3) &= g([J_2 + iJ_3]\varphi(v^{0,1}), \cdot)\\ &= g([J_2 + iJ_1J_2]J_2(v^{0,1}), \cdot)\\ &= g(-[I + iJ_1]v^{0,1}, \cdot)\\ &= -2i\,\omega_1(v^{0,1}, \cdot)\\ &= 2i\,\omega_1(\cdot, v^{0,1}),\end{aligned}$$

the Kodaira-Spencer class is therefore mapped to $2i[\omega_1] \in H^{1,1}_{\bar\partial_{J_1}}(X)$. However, since $[\omega_1]^{2k}$ pairs with fundamental cycle $[X]$ to yield $(2k)!$ times the total volume of (X, g), $2i\,[\omega_1]$ is certainly non-zero in deRham cohomology, and is therefore non-zero in Dolbeault cohomology, too. The Kodaira-Spencer map of such a twistor family is thus everywhere non-zero, as claimed. ∎

We next define many new complex structures on $X \times S^2$ by generalizing a construction [5] originally introduced in the $k = 1$ case to solve a different problem. Let $f : \mathbb{CP}_1 \to \mathbb{CP}_1$ be a holomorphic map of arbitrary degree ℓ. We then define a holomorphic family $f^*\varpi$ over $\mathbb{CP}_1$ by pulling ϖ back via f:

$$\begin{array}{ccc} f^*Z & \xrightarrow{\hat f} & Z \\ {\scriptstyle f^*\varpi}\downarrow & & {\scriptstyle \varpi}\downarrow \\ \mathbb{CP}_1 & \xrightarrow{f} & \mathbb{CP}_1. \end{array}$$

In other words, if $\Gamma \subset \mathbb{CP}_1 \times \mathbb{CP}_1$ is the graph of f, then f^*Z is the inverse image of Γ under $Z \times \mathbb{CP}_1 \stackrel{\varpi \times 1}{\longrightarrow} \mathbb{CP}_1 \times \mathbb{CP}_1$. Since ϖ is differentiably trivial, so is $\hat{\varpi} := f^*\varpi$, and $\hat{Z} := f^*Z$ may therefore be viewed as $X \times S^2$ equipped with some new complex structure J_f.

Lemma 2 *Let $\hat{Z} = f^*Z$ be the complex $(2k+1)$-manifold associated with a holomorphic map $f : \mathbb{CP}_1 \to \mathbb{CP}_1$ of degree ℓ, and let $\hat{\varpi} = f^*\varpi$ be the associated holomorphic submersion $\hat{\varpi} = f^*\varpi$. Then the canonical line bundle $K_{\hat{Z}}$ is isomorphic to $\hat{\varpi}^*\mathcal{O}(-2k\ell - 2)$ as a holomorphic line bundle.*

Proof The twistor space of any hyper-Kähler manifold (X^{4k}, g) satisfies $K_Z = \varpi^*\mathcal{O}(-2k-2)$. On the other hand, the branch locus B of $\hat{f} : \hat{Z} \to Z$ is the inverse image via $\hat{\varpi}$ of $2\ell - 2$ points in $\mathbb{CP}_1$. Thus

$$K_{\hat{Z}} = [B] \otimes \hat{f}^* K_Z \cong \hat{\varpi}^*[\mathcal{O}(2\ell - 2) \otimes \mathcal{O}(\ell(-2k-2))] = \hat{\varpi}^*\mathcal{O}(-2k\ell - 2),$$

as claimed. ∎

This now provides one cornerstone of our argument:

Proposition 1 *If $\hat{Z} = f^*Z$ is the complex $(2k+1)$-manifold arising from a simply connected hyper-Kähler manifold (X^{4k}, g) and a holomorphic map $f : \mathbb{CP}_1 \to \mathbb{CP}_1$ of degree ℓ, then there is a unique holomorphic line bundle $K^{-1/(2k\ell+2)}$ whose $(2+2k\ell)$th tensor power is isomorphic to the anti-canonical line bundle. Moreover, $h^0(Z, \mathcal{O}(K^{-1/(2k\ell+2)})) = 2$, and the pencil of sections of this line bundle exactly reproduces the holomorphic map $\hat{\varpi} : \hat{Z} \to \mathbb{CP}_1$. Thus the holomorphic submersion $\hat{\varpi}$ is an intrinsic property of the compact complex manifold $\hat{Z} = (X \times S^2, J_f)$, and is uniquely determined, up to Möbius transformation, by the complex structure J_f.*

Proof Because $\hat{Z} \approx X \times S^2$ is simply connected, $H^1(\hat{Z}, \mathbb{Z}_{2k\ell+2}) = 0$, and the long exact sequence induced by the short exact sequence of sheaves

$$0 \to \mathbb{Z}_{2k\ell+2} \to \mathcal{O}^\times \to \mathcal{O}^\times \to 0$$

therefore guarantees that there can be at most one holomorphic line bundle $K^{-1/(2k\ell+2)}$ whose $(2+2k\ell)$th tensor power is the anti-canonical line bundle K^*. Since Lemma 2 guarantees that $\hat{\varpi}^*\mathcal{O}(1)$ is one candidate for this root of K^*, it is therefore the unique such root. On the other hand, since $\hat{\varpi}^*\mathcal{O}(1)$ is trivial on the compact fibers of $\hat{\varpi}$, any holomorphic section of this line bundle on $\hat{Z}$ is fiber-wise constant, and is therefore the pull-back of a section of $\mathcal{O}(1)$ on $\mathbb{CP}_1$. Thus $h^0(Z, \mathcal{O}(K^{-1/(2k\ell+2)})) = h^0(\mathbb{CP}_1, \mathcal{O}(1)) = 2$, and the pencil of sections of $K^{-1/(2k\ell+2)}$ thus exactly reproduces $\hat{\varpi} : \hat{Z} \to \mathbb{CP}_1$. ∎

Here, the role of the Möbius transformations is of course unavoidable. After all, preceding f by a Möbius transformation will certainly result in a biholomorphic manifold!

Since $\hat{\varpi}$ is intrinsically determined by the complex structure of $\hat{Z}$, its complex structure also completely determines those elements of $\mathbb{CP}_1$ at which the Kodaira-Spencer map of the family $\hat{\varpi} : \hat{Z} \to \mathbb{CP}_1$ vanishes; this is the same as asking for fibers for which there is a transverse holomorphic foliation of the first formal neighborhood. Similarly, one can ask whether there are elements of $\mathbb{CP}_1$ at which the Kodaira-Spencer map vanishes to order m; this is the same as asking for fibers for which there is a transverse holomorphic foliation of the $(m+1)^{\mathrm{st}}$ formal neighborhood.

Proposition 2 *The critical points of $f : \mathbb{CP}_1 \to \mathbb{CP}_1$, along with their multiplicities, can be reconstructed from the submersion $f^*\varpi : f^*Z \to \mathbb{CP}_1$.*

Proof The Kodaira-Spencer map is functorial, and transforms with respect to pull-backs like a bundle-valued 1-form. Since the Kodaira-Spencer map of ϖ is everywhere non-zero by Lemma 1, the points at which the Kodaira-Spencer map of $\hat{\varpi} = f^*\varpi$ vanishes to order m are exactly those points at which the derivative of $f : \mathbb{CP}_1 \to \mathbb{CP}_1$ has a critical point of order m. ∎

Taken together, Propositions 1 and 2 thus imply the following:

Theorem B *Modulo Möbius transformations, the configuration of critical points of $f : \mathbb{CP}_1 \to \mathbb{CP}_1$, along with their multiplicities, is an intrinsic invariant of the compact complex manifold $\hat{Z} = f^*Z$.*

By displaying suitable families of holomorphic maps $\mathbb{CP}_1 \to \mathbb{CP}_1$, we will now use Theorem B prove Theorem A. Indeed, for any $(a_1, \dots, a_N) \in \mathbb{C}^N$ with $|a_j - 2j| < 1$, let $P_{a_1,\dots,a_N}(\zeta)$ be the polynomial of degree $N+6$ in the complex variable ζ defined by

$$P_{a_1,\dots,a_N}(\zeta) = \int_0^{\zeta} t^2(t-1)^3(t-a_1)\cdots(t-a_N)dt,$$

and let $f_{a_1,\dots,a_N} : \mathbb{CP}_1 \to \mathbb{CP}_1$ be the self-map of $\mathbb{CP}_1 = \mathbb{C} \cup \{\infty\}$ obtained by extending $P_{a_1,\dots,a_N} : \mathbb{C} \to \mathbb{C}$ via $\infty \mapsto \infty$; in other words,

$$f_{a_1,\dots,a_N}([\zeta_1, \zeta_2]) = [P_{a_1,\dots,a_N}(\zeta_1, \zeta_2), \zeta_2^{N+6}],$$

where $P_{a_1,\dots,a_N}(\zeta_1, \zeta_2)$ is the homogeneous polynomial formally defined by

$$P_{a_1,\dots,a_N}(\zeta_1, \zeta_2) = \zeta_2^{N+6} P_{a_1,\dots,a_N}(\tfrac{\zeta_1}{\zeta_2}).$$

Since the constraints we have imposed on our auxiliary parameters force the complex numbers $0, 1, a_1, \dots, a_N$ to all be distinct, the critical points of $f_{a_1,\dots,a_N} : \mathbb{CP}_1 \to \mathbb{CP}_1$ are just the $a_1, \dots, a_N$, each with multiplicity 1, along with 0, 1, and ∞, which are individually distinguishable by their respective multiplicities of 2, 3, and $N+5$. Since any Möbius transformation that fixes 0, 1, and ∞ must be the identity, Theorem B implies that different values of the parameters $(a_1, \dots, a_N)$, subject the constraints $|a_j - 2j| < 1$, will always result in non-biholomorphic complex manifolds

$\hat{Z}_{a_1,\ldots,a_N} := f^*_{a_1,\ldots,a_N}Z$. Thus, pulling back $\varpi : Z \to \mathbb{CP}_1$ via the holomorphic map

$$\Phi : D^N \times \mathbb{CP}_1 \longrightarrow \mathbb{CP}_1$$
$$(u_1, \ldots, u_n, [\zeta_1, \zeta_2]) \longmapsto f_{u_1+2,\ldots,u_N+2N}([\zeta_1, \zeta_2])$$

now produces a family $\Phi^*\varpi : \Phi^*Z \to D^N$ of mutually non-biholomorphic complex manifolds over the unit polydisk $D^N \subset \mathbb{C}^N$. Since these manifolds are all diffeomorphic to $X \times S^2$, and since this works for any positive integer N, Theorem A is therefore an immediate consequence.

Of course, the above proof is set in the world of general compact complex manifolds, and so has little to say about conditions prevailing in the tidier realm of, say, complex algebraic varieties. In fact, one should probably expect the examples described in this article to never be of Kähler type, since there are results in this direction [5] when $k = 1$. It would certainly be interesting to see this definitively established for general k.

On the other hand, the feature of the $k = 1$ case highlighted in [5] readily generalizes to higher dimensions; namely, the Chern numbers of the complex structures J_f change as we vary the degree of f. Indeed, notice the tangent bundle of $X \times S^2$ is stably isomorphic to the pull-back of the tangent bundle of X, and that TX has some non-trivial Pontrjagin numbers; for example, if we assume for simplicity that X is irreducible, we then have $\hat{A}(X) = k + 1$. Since the fibers of $f^*\varpi$ are Poincaré dual to $c_1(f^*Z)/(2k\ell + 2)$, we have $(\mathbf{c_1}\hat{\mathbf{A}})(f^*Z) = 2(k\ell + 1)(k + 1)$, and a certain combination of the Chern numbers of $f^*(Z)$ therefore grows linearly in $\ell = \deg f$. Consequently, as $N \to \infty$, the families of complex structures we have constructed skip through infinitely many connected components of the moduli space $\mathfrak{M}(X \times S^2)$. Is this necessary for a complex moduli space to fail to be finite-dimensional?

Finally, notice that the dimension of each exhibited component of the moduli space $\mathfrak{M}(X \times S^2)$ is higher than what might be inferred from our construction. Indeed, we have only made use of a single hyper-Kähler metric g on X, whereas these in practice always come in large families. Hyper-Kähler twistor spaces also carry a tautological anti-holomorphic involution, whereas their generic small deformations generally will not. In short, these moduli spaces are still largely *terra incognita*. Perhaps some interested reader will take up the challenge, and tell us much more about them!

Acknowledgements This paper is dedicated to my friend and sometime collaborator Simon Salamon, who first introduced me to hyper-Kähler manifolds and quaternionic geometry when we were both graduate students at Oxford. I would also like to thank my colleague Dennis Sullivan for drawing my attention to the finite-dimensionality problem for moduli spaces.

Claude LeBrun was supported in part by NSF grant DMS-1510094.

References

1. A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differ. Geom. **18**, 755–782 (1983)
2. A.L. Besse, *Einstein Manifolds*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 10 (Springer, Berlin, 1987)
3. K. Kodaira, D.C. Spencer, On deformations of complex analytic structures. I, II. Ann. Math. **67**(2), 328–466 (1958)
4. M. Kuranishi, On the locally complete families of complex analytic structures. Ann. Math. **75**(2), 536–577 (1962)
5. C. LeBrun, Topology versus Chern numbers for complex 3-folds. Pac. J. Math. **191**, 123–131 (1999)
6. K.G. O'Grady, A new six-dimensional irreducible symplectic variety. J. Algebraic Geom. **12**, 435–505 (2003)
7. S. Salamon, Quaternionic Kähler manifolds. Invent. Math. **67**, 143–171 (1982)

Explicit Global Symplectic Coordinates on Kähler Manifolds

Andrea Loi and Fabio Zuddas

Abstract In this survey paper we provide several explicit constructions and examples of global symplectic coordinates on Kähler manifolds found in the last decade by the authors and their collaborators. In particular, we treat the cases of complete Reinhardt domains, LeBrun's Taub-Nut Kähler form, gradient Kähler-Ricci solitons, Calabi's inhomogeneous Kähler–Einstein form on tubular domains, Hermitian symmetric spaces of noncompact type.

Keywords Gromov width • Gromov-Witten invariants • Kähler manifolds • Symplectic maps

2000 Mathematics Subject Classification: 53D05; 53C55; 53D05; 53D45

1 Introduction and Organization of the Paper

Let (M, ω) be a $2n$-dimensional symplectic manifold. By the celebrated Darboux theorem for every point $p \in M$ there exists on open neighborhood U of p and a diffeomorphism $\Phi : U \to \mathbb{R}^{2n}$ such that $\Phi^*(\omega_0) = \omega$ where $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$ is the standard symplectic form on $\mathbb{R}^{2n}$. In other words one can say that the open set (U, ω) can be equipped with *local symplectic coordinates*. An interesting question is to understand how large the set U can be taken and, in particular, when the case $U = M$ occurs, namely when (M, ω) admits *global symplectic coordinates*. The interest for this kind of questions comes, for example, after Gromov's discovery [10] of the existence of exotic symplectic structures on $\mathbb{R}^{2n}$ (see also [1] for an explicit construction of a 4-dimensional symplectic manifold diffeomorphic to $\mathbb{R}^4$ which cannot be symplectically embedded in $(\mathbb{R}^4, \omega_0)$). The only known (for the best of the authors' knowledge) and general result in the Kähler case is given by the following global version of Darboux theorem.

A. Loi (✉) • F. Zuddas
Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale 72, Cagliari, Italy
e-mail: loi@unica.it; fabio.zuddas@unica.it

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_9

Theorem A (McDuff [23]) *Let (M, ω) be a simply-connected and complete complex n-dimensional Kähler manifold of non-positive sectional curvature. Then there exists a diffeomorphism $\Psi : M \to \mathbb{R}^{2n}$ such that $\Psi^*(\omega_0) = \omega$.*

Later on E. Ciriza discovers a Riemannian feature of the map Ψ expressed by the following result.

Theorem B (Ciriza [4]) *Let (M, ω) be as in the previous theorem and let $\Psi : M \to \mathbb{R}^{2n}$ be McDuff's symplectomorphism. Then the image $\Psi(T)$ of any complete complex and totally geodesic submanifold T of M passing through the point p with $\Psi(p) = 0$, is a complex linear subspace of $\mathbb{C}^n$.*

Notice that the global symplectic coordinates obtained by McDuff's map Ψ are not explicit. Therefore it is natural and interesting to find explicit symplectic coordinates on a given symplectic (contractible) manifold (M, ω), possibly satisfying Ciriza's property.

The aim of this survey paper is to collect various explicit examples of symplectic coordinates, obtained in the last decade by the authors of the present paper jointly with F. Cuccu, A.J. Di Scala, R. Mossa and M. Zedda, on the following Kähler manifolds: complete Reinhardt domains; LeBrun's Taub-Nut Kähler form; gradient Kähler-Ricci solitons; Calabi's inhomogeneous Kähler–Einstein form on tubular domains; Hermitian symmetric spaces of noncompact type. In the case of Hermitian symmetric spaces of noncompact type we have tried to avoid technical results (such as Hermitian Jordan triple systems or Gromov–Witten invariants) as much as possible in order to make the paper more readable also by non-experts in this field. The interested reader will find details and other results in the bibliography.

We point out that an explicit description of symplectic coordinates could be useful for a deep understanding of the symplectic geometry of the Kähler manifold involved. For example in the case of Hermitian symmetric spaces of noncompact type, whose symplectic structure is standard by McDuff's theorem, the knowledge of explicit symplectic coordinates allowed the authors of the present paper jointly with R. Mossa to compute the Gromov width of all Hermitian symmetric spaces of compact type.

The paper is organized as follows. In the next section we treat the case of rotation invariant complex domains. The main tool is Lemma 2.1 which provides necessary and sufficient conditions for such domains to be globally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. This lemma is used in the following three subsections to provide an explicit description of global symplectic coordinates for the complete Reinhardt domains, LeBrun's Taub-NUT form and gradient Kähler-Ricci solitons. In Sect. 3 we consider the case of a Kähler form (not rotation invariant) introduced by Calabi as the first example of Kähler–Einstein nonhomogeneous metric. In Sect. 4 we treat the case of Hermitian symmetric spaces. After briefly recalling the basic facts on Hermitian symmetric spaces we state the main result (Theorem 4.1) on the symplectic geometry of such domains which provides explicit symplectic coordinates on every bounded domain and on a dense subset of its compact dual. The proof of this theorem is quite technical and for this reason it is not given here.

Alternatively, in Sect. 4.1 we provide a proof of Theorem 4.1 in the special case of the first Cartan domain and its compact dual, namely the complex Grassmannian. Finally, in Sect. 4.2, after defining an important class of symplectic invariant called symplectic capacities, we define the Gromov width of a symplectic manifold and we give an idea of how the knowledge of explicit symplectic coordinates on a bounded symmetric domain and on a dense subset of its compact dual can be used to compute the Gromov width of all Hermitian symmetric spaces of compact and noncompact type (see Example 4.7 for an explicit computation of the Gromov width in the case of the first Cartan domain and the complex Grassmannian and Theorems 4.8 and 4.9 for the general case).

2 The Rotation Invariant Case

In [18] the authors of the present paper proved the following result on the existence of a symplectomorphism between a rotation invariant Kähler manifold of complex dimension n and $(\mathbb{R}^{2n}, \omega_0)$. For the reader's convenience, we summarize here that result and its proof.

Lemma 2.1 *Let* $\omega_\Phi = \frac{i}{2}\partial\bar\partial\Phi$ *be a rotation invariant Kähler form on* $\mathbb{C}^n$ *i.e. the Kähler potential only depends on* $|z_j|^2$, $j = 1, \dots, n$.[1] *If*

$$\frac{\partial\Phi}{\partial|z_k|^2} \geq 0, \quad k = 1, \dots, n, \tag{1}$$

then the map:

$$\Psi : (M, \omega_\Phi) \to (\mathbb{C}^n, \omega_0) = (\mathbb{R}^{2n}, \omega_0), \quad z = (z_1, \dots, z_n) \mapsto (\psi_1(z)z_1, \dots, \psi_n(z)z_n),$$

where

$$\psi_j = \sqrt{\frac{\partial\Phi}{\partial|z_j|^2}}, \quad j = 1, \dots, n,$$

is a symplectic immersion. If in addition:

$$\lim_{z\to+\infty} \sum_{j=1}^{n} \frac{\partial\Phi}{\partial|z_j|^2}|z_j|^2 = +\infty, \tag{2}$$

then Ψ *is a global symplectomorphism.*

[1] Notice that the rotation invariant condition on the potential Φ is more general then the radial one which requires Φ depending only on $|z_1|^2 + \cdots + |z_n|^2$.

Proof Assume condition (1) holds true. Let us prove first that $\Psi^*\omega_0 = \omega$. We have:

$$\begin{aligned}\Psi^*\omega_0 =& \frac{i}{2}\sum_{j=1}^{n} d\Psi_j \wedge d\bar{\Psi}_j\\ =& \frac{i}{2}\sum_{j=1}^{n}\left(\frac{\partial\Psi_j}{\partial z_j}dz_j + \frac{\partial\Psi_j}{\partial\bar{z}_j}d\bar{z}_j\right)\wedge\left(\frac{\partial\bar{\Psi}_j}{\partial z_j}dz_j + \frac{\partial\bar{\Psi}_j}{\partial\bar{z}_j}d\bar{z}_j\right)\\ =& \frac{i}{2}\sum_{j,k=1}^{n}\left(\left|\frac{\partial\Psi_j}{\partial z_j}\right|^2 - \left|\frac{\partial\bar{\Psi}_j}{\partial z_j}\right|^2\right)dz_j\wedge d\bar{z}_j\end{aligned}$$

Since

$$\frac{\partial\Psi_j}{\partial z_j} = \frac{\partial\psi_j}{\partial z_j}z_j + \Psi_j, \qquad \frac{\partial\Psi_j}{\partial\bar{z}_j} = \frac{\partial\psi_j}{\partial\bar{z}_j}z_j,$$

and

$$\frac{\partial\psi_j}{\partial z_j} = \frac{1}{2}\psi_j^{-1}\left(\frac{\partial^2\Phi}{\partial|z_j|^4}\right)\bar{z}_j,$$

it follows:

$$\begin{aligned}\Psi^*\omega_0 =& \frac{i}{2}\sum_{j=1}^{n}\left(\left|\frac{\partial\psi_j}{\partial z_j}z_j + \psi_j\right|^2 - \left|\frac{\partial\psi_j}{\partial z_j}\right|^2|z_j|^2\right)dz_j\wedge d\bar{z}_j\\ =& \frac{i}{2}\sum_{j=1}^{n}\left(\frac{\partial\psi_j}{\partial z_j}\psi_j z_j + \frac{\partial\psi_j}{\partial\bar{z}_j}\psi_j\bar{z}_j + \psi_j^2\right)dz_j\wedge d\bar{z}_j\\ =& \frac{i}{2}\sum_{j=1}^{n}\left(\left(\frac{\partial^2\Phi}{\partial|z_j|^4}\right)|z_j|^2 + \left(\frac{\partial\Phi}{\partial|z_j|^2}\right)\right)dz_j\wedge d\bar{z}_j\\ =& \frac{i}{2}\sum_{j=1}^{n}\frac{\partial^2\Phi}{\partial z_j\partial\bar{z}_j}dz_j\wedge d\bar{z}_j.\end{aligned}$$

Observe now that since ω and ω_0 are non-degenerate, it follows by the inverse function theorem that Ψ is a local diffeomorphism. If in addition condition (2) holds true, then Ψ is a proper map and hence a global diffeomorphism. □

Example 2.2 As a simple application of Lemma 2.1 we obtain the very well-known fact that the complex hyperbolic space $(\mathbb{C}H^n, \omega_{hyp})$, namely the unit ball $B^{2n}(1) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | \sum_{j=1}^n |z_j|^2 < 1\}$ in $\mathbb{C}^n$ endowed with the hyperbolic form $\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log(1 - \sum_{j=1}^n |z_j|^2)$ is globally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. An explicit global symplectomorphism $\Psi : B^{2n}(1) \to \mathbb{R}^{2n}$ is given by:

$$(z_1, \dots, z_n) \mapsto \left(\frac{z_1}{\sqrt{1 - \sum_{i=1}^{n} |z_i|^2}}, \dots, \frac{z_n}{\sqrt{1 - \sum_{i=1}^{n} |z_i|^2}} \right). \tag{3}$$

2.1 Complete Reinhardt Domains

The material of this subsection is taken from [5] and [18]. Let $x_0 \in \mathbb{R}^+ \cup \{+\infty\}$ and let $F : [0, x_0) \to (0, +\infty)$ be a non-increasing smooth function. Consider the domain

$$D_F = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 < x_0,\ |z_2|^2 < F(|z_1|^2)\}$$

endowed with the 2-form

$$\omega_F = \frac{i}{2} \partial\bar{\partial}\Phi,\ \Phi = \log \frac{1}{F(|z_1|^2) - |z_2|^2}.$$

If the function $A(x) = -\frac{xF'(x)}{F(x)}$ satisfies $A'(x) > 0$ for every $x \in [0, x_0)$, then ω_F is a Kähler form on D_F and (D_F, ω_F) is called the *complete Reinhardt domain* associated with F.

We have

$$\frac{\partial\Phi}{\partial|z_1|^2} = -\frac{F'(|z_1|^2)}{F(|z_1|^2) - |z_2|^2} > 0, \quad \frac{\partial\Phi}{\partial|z_2|^2} = \frac{1}{F(|z_1|^2) - |z_2|^2} > 0.$$

So, by Lemma 2.1, (D_F, ω_F) admits a symplectic immersion in $(\mathbb{R}^4, \omega_0)$. Moreover, this immersion is a global symplectomorphism if

$$\frac{\partial\Phi}{\partial|z_1|^2}|z_1|^2 + \frac{\partial\Phi}{\partial|z_2|^2}|z_2|^2 = \frac{|z_2|^2 - F'(|z_1|^2)|z_1|^2}{F(|z_1|^2) - |z_2|^2}. \tag{4}$$

tends to infinity on the boundary of D_F.

We now provide two examples of complete Reinhardt domains (D_F, ω_F) where the previous conditions hold true.

Example 2.3 Let F be the real-valued, strictly decreasing smooth function on $[0, 1)$ defined by:

$$F : [0, 1) \to \mathbb{R} : x \mapsto (1 - x)^p,\ p > 0.$$

Its associated complete Reinhardt domain is given by:

$$D_F = \{z \in \mathbb{C}^2 |\ |z_1|^2 + |z_2|^{\frac{2}{p}} < 1\}.$$

Since

$$\frac{xF'}{F} = -\frac{px}{1-x}, \ \left(\frac{xF'}{F}\right)' = -\frac{p}{(1-x)^2} < 0, \ \forall x \in [0,1),$$

we get a well defined Kähler form ω_F on D_F. Moreover one easily verifies that condition (4) is satisfied and hence the map $\Psi : D_F \to \mathbb{R}^4$ given by:

$$(z_1, z_2) \mapsto \left(\left(\frac{p(1-|z_1|^2)^{p-1}}{(1-|z_1|^2)^p - |z_2|^2} \right)^{\frac{1}{2}} z_1, \left(\frac{1}{(1-|z_1|^2)^p - |z_2|^2} \right)^{\frac{1}{2}} z_2 \right)$$

is an explicit global symplectomorphism. Observe that for $p = 1$ our domain is the unitary disk endowed with the hyperbolic metric (cfr. Example 2.2 above).

Example 2.4 Let $F(x) = e^{-x}$ in the interval $[0, +\infty)$. Since $F'(x) = -e^{-x} < 0$, the function F defines a complete Reinhardt domain D_F called the *Spring domain*. Further

$$\frac{xF'}{F} = -x, \ \left(\frac{xF'}{F}\right)' = -1$$

and hence, we get a well defined Kähler form ω_F on D_F. Condition (4) is easily verified and the map $\Psi : D_F \to \mathbb{R}^4$ given by:

$$(z_1, z_2) \mapsto \left(\left(\frac{e^{-|z_1|^2}}{e^{-|z_1|^2} - |z_2|^2} \right)^{\frac{1}{2}} z_1, \left(\frac{1}{e^{-|z_1|^2} - |z_2|^2} \right)^{\frac{1}{2}} z_2 \right)$$

defines global symplectic coordinates on the Spring domain.

2.2 The Taub-NUT Metric

The reader is referred to [18] for the material of this subsection. In [16] LeBrun constructed the following family of Kähler forms on $\mathbb{C}^2$ defined by $\omega_m = \frac{i}{2}\partial\bar{\partial}\Phi_m$, where

$$\Phi_m(u, v) = u^2 + v^2 + m(u^4 + v^4), \ m \geq 0$$

and u and v are implicitly defined by

$$|z_1| = e^{m(u^2 - v^2)}u, \ |z_2| = e^{m(v^2 - u^2)}v.$$

For $m = 0$ one gets the flat metric, while for $m > 0$ each of the metrics of this family represents the first example of complete Ricci flat (non-flat) metric on $\mathbb{C}^2$ having the same volume form of the flat metric ω_0, namely $\omega_m \wedge \omega_m = \omega_0 \wedge \omega_0$. Moreover, for $m > 0$, these metrics are isometric (up to dilation and rescaling) to the Taub-NUT metric.

Now, with the aid of Lemma 2.1, we prove that for every m the Kähler manifold $(\mathbb{C}^2, \omega_m)$ admits global symplectic coordinates. Set $u^2 = U$, $v^2 = V$. Then

$$\frac{\partial \Phi_m}{\partial x_1} = \frac{\partial \Phi_m}{\partial U}\frac{\partial U}{\partial x_1} + \frac{\partial \Phi_m}{\partial V}\frac{\partial V}{\partial x_1},$$

$$\frac{\partial \Phi_m}{\partial x_2} = \frac{\partial \Phi_m}{\partial U}\frac{\partial U}{\partial x_2} + \frac{\partial \Phi_m}{\partial V}\frac{\partial V}{\partial x_2},$$

where $x_j = |z_j|^2, j = 1, 2$. In order to calculate $\frac{\partial U}{\partial x_j}$ and $\frac{\partial V}{\partial x_j}, j = 1, 2$, let us consider the map

$$G : \mathbb{R}^2 \to \mathbb{R}^2,\ (U, V) \mapsto (x_1 = e^{2m(U-V)}U,\ x_2 = e^{2m(V-U)}V)$$

and its Jacobian matrix

$$J_G = \begin{pmatrix} (1 + 2mU)\ e^{2m(U-V)} & -2mU\ e^{2m(U-V)} \\ -2mV\ e^{2m(V-U)} & (1 + 2mV)\ e^{2m(V-U)} \end{pmatrix}.$$

We have $det J_G = 1 + 2m(U + V) \neq 0$, so

$$J_G^{-1} = J_{G^{-1}} = \frac{1}{1 + 2m(U + V)} \begin{pmatrix} (1 + 2mV)e^{2m(V-U)} & 2mUe^{2m(U-V)} \\ 2mVe^{2m(V-U)} & (1 + 2mU)e^{2m(U-V)} \end{pmatrix}.$$

Since $J_{G^{-1}} = \begin{pmatrix} \frac{\partial U}{\partial x_1} & \frac{\partial U}{\partial x_2} \\ \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{pmatrix}$, by a straightforward calculation we get

$$\frac{\partial \Phi_m}{\partial x_1} = (1 + 2mV)e^{2m(V-U)} > 0,\ \frac{\partial \Phi_m}{\partial x_2} = (1 + 2mU)e^{2m(U-V)} > 0,$$

and

$$\lim_{\|x\| \to +\infty} (\frac{\partial \Phi_m}{\partial x_1}x_1 + \frac{\partial \Phi_m}{\partial x_2}x_2) = \lim_{\|x\| \to +\infty} (U + V + 4mUV) = +\infty,$$

namely (1) and (2) above respectively . Hence, by Lemma 2.1, the map

$$\Psi_0 : \mathbb{C}^2 \to \mathbb{C}^2, (z_1, z_2) \mapsto \left((1 + 2mV)^{\frac{1}{2}} e^{m(V-U)} z_1, (1 + 2mU)^{\frac{1}{2}} e^{m(U-V)} z_2\right)$$

is a global symplectomorphism from $(\mathbb{C}^2, \omega_m)$ into $(\mathbb{R}^4, \omega_0)$.

Remark 2.5 Notice that for $m > 0$ we cannot apply McDuff's theorem in the Introduction in order to get the existence of global symplectic coordinates on $(\mathbb{C}^2, \omega_m)$. Indeed, the sectional curvature of $(\mathbb{C}^2, g_m)$ (where g_m is the Kähler metric associated to ω_m) is positive at some point since g_m is Ricci-flat but not flat.

2.3 Gradient Kähler–Ricci Solitons

We now recall what we need about the gradient Kähler–Ricci solitons described by Cao in [3] (to whom we refer for references and further details). Let g_{RS} be the Kähler metric on $\mathbb{C}^n$ generated by the radial Kähler potential $\Phi(z, \bar{z}) = u(t)$, where for all $t \in (-\infty, +\infty)$, u is a smooth function of $t = \log(||z||^2)$ and as $t \to -\infty$ has an expansion:

$$u(t) = a_0 + a_1 e^t + a_2 e^{2t} + \dots, \qquad a_1 = 1. \tag{5}$$

Denote by $\omega_{RS} = \frac{i}{2}\partial\bar{\partial}\Phi$ the Kähler form associated to g_{RS}. If u satisfies the equation:

$$(u')^{n-1} u'' e^{u'} = e^{nt},$$

then the conditions:

$$u'(t) > 0, \qquad u''(t) > 0, \qquad \forall\, t \in (-\infty, +\infty), \tag{6}$$

$$\lim_{t \to +\infty} \frac{u'(t)}{t} = n, \qquad \lim_{t \to +\infty} u''(t) = n \tag{7}$$

are fulfilled and $(\mathbb{C}^n, \omega_{RS})$ is a gradient Kähler–Ricci soliton. The metric g_{RS} is complete and positively curved and for $n = 1$ one recovers the Cigar metric on $\mathbb{C}$ whose associated Kähler form reads:

$$\omega_C = \frac{dz \wedge d\bar{z}}{1 + |z|^2},$$

which was introduced by Hamilton in [11] as first example of Kähler–Ricci soliton on non-compact manifolds. Observe that a Kähler potential for ω_C is given by (see also [24]):

$$\Phi_C = \int_0^{|z|} \frac{\log(1 + s^2)}{s} ds.$$

Furthermore, in this case the Riemannian curvature reads:

$$R = \frac{1}{(1 + |z|^2)^3}. \tag{8}$$

It is interesting to observe that the Kähler metric $\omega_{C,n}$ on $\mathbb{C}^n = \frac{i}{2}\partial\bar{\partial}\Phi_{C,n}$ defined as product of n copies of Cigar metric ω_C, satisfies $\Phi_{C,n} = \Phi_C \oplus \cdots \oplus \Phi_C$ and it is still a complete and positively curved (i.e. with non-negative sectional curvature) gradient Kähler–Ricci soliton, namely it satisfies (5)–(7) above. In particular its Riemannian tensor satisfies $R_{i\bar{j}k\bar{l}} = 0$ whenever one of the indexes is different from the others and by (8) it is easy to see that the nonvanishing components are given by:

$$R_{j\bar{j}j\bar{j}} = \frac{1}{(1+|z_j|^2)^3}. \tag{9}$$

Our results are summarized in the following two theorems.

Theorem 2.6 *A gradient Kähler–Ricci soliton* $(\mathbb{C}^n, \omega_{RS})$ *is globally symplectomorphic to* $(\mathbb{R}^{2n}, \omega_0)$.

Theorem 2.7 *Let* $(\mathbb{C}^n, \omega_{C,n})$ *be the product of n copies of the Cigar soliton. Then there exists a symplectomorphism* $\Psi_{C,n} : (\mathbb{C}^n, \omega_{C,n}) \to (\mathbb{R}^{2n}, \omega_0)$*, with* $\Psi_{C,n}(0) = 0$*, taking complete complex totally geodesic submanifolds through the origin to complex linear subspaces of* $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

The first theorem shows the existence of positively curved complete Kähler manifolds globally symplectomorphic to $\mathbb{R}^{2n}$. In the second one for all positive integers n we provide an example of gradient Kähler–Ricci solitons (the product of n copies of the Cigar soliton) where Ciriza's property (see Theorem B above) holds true.

Proof of Theorem 2.6 Let $\Phi(z, \bar{z}) = u(t)$, where $u(t)$ is given by (5). Then for all $j = 1, \ldots, n$

$$\frac{\partial \Phi}{\partial |z_j|^2} = \frac{\partial \Phi}{\partial ||z||^2} = \frac{u'(\log(||z||^2)}{||z||^2},$$

which is greater than zero for all $||z||^2 \neq 0$ by (6), and evaluated at $||z||^2 = 0$ gives the value 1 by (5). Notice now that by the first of the limit conditions given in (7) it follows that condition (2) in Lemma 2.1 holds true. Therefore by Lemma 2.1 the map:

$$F : (\mathbb{C}^n, g_{RS}) \to (\mathbb{R}^{2n}, g_0), \quad z = (z_1, \ldots, z_n) \mapsto \sqrt{\frac{u'(\log(||z||^2)}{||z||^2}}(z_1, \ldots, z_n),$$

is the desired global symplectomorphism. □

In order to prove Theorem 2.7 we need the following lemma which classifies all totally geodesic submanifolds of $(\mathbb{C}^n, \omega_{C,n})$ through the origin.

Lemma 2.8 *Let S be a totally geodesic complex submanifold (of complex dimension k) of* $(\mathbb{C}^n, \omega_{C,n})$*. Then, up to unitary transformation of* $\mathbb{C}^n$*,* $S = (\mathbb{C}^k, \omega_{C,k})$.

Proof Let us first prove the statement for $n = 2$. For $k = 0, 2$ there is nothing to prove, thus fix $k = 1$. Let

$$f\colon (S, \tilde{\omega}) \hookrightarrow (\mathbb{C}^2, \omega_{C,2}), \quad f(z) = (f_1(z), f_2(z)).$$

be a totally geodesic embedding of a 1-dimensional complex manifold $(S, \tilde{\omega})$ into $(\mathbb{C}^2, \omega_{C,2})$. By $\tilde{\omega} = f^*(\omega_{C,2})$ we get:

$$\tilde{\omega} = \frac{i}{2}\left(\left|\frac{\partial f_1}{\partial z}\right|^2 \frac{1}{1+|f_1(z)|^2} + \left|\frac{\partial f_2}{\partial z}\right|^2 \frac{1}{1+|f_2(z)|^2}\right) dz \wedge d\bar{z}. \tag{10}$$

Let $\tilde{R}$, R_C be the curvature tensor of $(S, \tilde{\omega})$ and $(\mathbb{C}^2, \omega_C)$ respectively. Since $(S, \tilde{\omega})$ is totally geodesic in $(\mathbb{C}^2, \omega_C)$ we have

$$\tilde{R}(X, JX, X, JX) = R_C(X, JX, X, JX)$$

for all the vector fields X on S (see e.g. [15, p. 176]). Taking $X = \partial/\partial z$, we have:

$$\tilde{R}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = -\frac{\partial^2 \tilde{g}}{\partial z \partial \bar{z}} + \tilde{g}^{-1}(z)\left|\frac{\partial \tilde{g}(z)}{\partial z}\right|^2,$$

where $\tilde{g}$ is the Kähler metric associated to $\tilde{\omega}$, i.e.

$$\tilde{g} = \left|\frac{\partial f_1}{\partial z}\right|^2 \frac{1}{1+|f_1(z)|^2} + \left|\frac{\partial f_2}{\partial z}\right|^2 \frac{1}{1+|f_2(z)|^2}.$$

Further, since the vector field $\frac{\partial}{\partial z}$ corresponds through df to $\frac{\partial f_1}{\partial z}\frac{\partial}{\partial z_1} + \frac{\partial f_2}{\partial z}\frac{\partial}{\partial z_2}$, by (9) we get:

$$R_C\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = \left|\frac{\partial f_1}{\partial z}\right|^4 \frac{1}{(1+|f_1(z)|^2)^3} + \left|\frac{\partial f_2}{\partial z}\right|^4 \frac{1}{(1+|f_2(z)|^2)^3}.$$

Since

$$\frac{\partial \tilde{g}}{\partial z} = \sum_{j=1}^{2}\left(\frac{2}{1+|f_j(z)|^2}\overline{\frac{\partial f_j}{\partial z}}\frac{\partial^2 f_j}{\partial z^2} - \left|\frac{\partial f_j}{\partial z}\right|^2 \frac{\bar{f}_j}{(1+|f_j|^2)^2}\frac{\partial f_j}{\partial z}\right),$$

$$\frac{\partial^2 \tilde{g}}{\partial z\, \partial \bar{z}} = \sum_{j=1}^{2}\left[\left|\frac{\partial f_j}{\partial z}\right|^4 \frac{2|f_j|^2}{(1+|f_j(z)|^2)^3} + \left|\frac{\partial^2 f_j}{\partial z^2}\right|^2 \frac{1}{1+|f_j(z)|^2} + \right.$$
$$\left. -\frac{1}{(1+|f_j|^2)^2}\left(\bar{f}_j\left(\frac{\partial f_j}{\partial z}\right)^2 \overline{\frac{\partial^2 f_j}{\partial z^2}} + \left|\frac{\partial f_j}{\partial z}\right|^4 + f_j\left(\overline{\frac{\partial f_j}{\partial z}}\right)^2 \frac{\partial^2 f_j}{\partial z^2}\right)\right]$$

after a long but straightforward computation, we get that $\tilde{R}\left(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}},\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}}\right)-R_C\left(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}},\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}}\right)$ assumes the form:

$$\frac{-|A(f_1,f_2)|^2}{\left(\left|\frac{\partial f_1}{\partial z}\right|^2(1+|f_2|^2)+\left|\frac{\partial f_2}{\partial z}\right|^2(1+|f_1|^2)\right)(1+|f_1|^2)^2(1+|f_2|^2)^2},$$

where

$$\begin{aligned}A(f_1,f_2)=&\left(\frac{\partial^2 f_2}{\partial z^2}\frac{\partial f_1}{\partial z}-\frac{\partial^2 f_1}{\partial z^2}\frac{\partial f_2}{\partial z}\right)(1+|f_1|^2)(1+|f_2|^2)+\\&+\left(\frac{\partial f_1}{\partial z}\right)^2\frac{\partial f_2}{\partial z}\bar{f}_1(1+|f_2|^2)-\left(\frac{\partial f_2}{\partial z}\right)^2\frac{\partial f_1}{\partial z}\bar{f}_2(1+|f_1|^2).\end{aligned}$$

Thus, $\tilde{R}\left(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}},\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}}\right)-R_C\left(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}},\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}}\right)=0$ iff $A(f_1,f_2)=0$, i.e. iff

$$\begin{aligned}&\frac{\partial f_1}{\partial z}(1+|f_2|^2)\left(\frac{\partial^2 f_2}{\partial z^2}(1+|f_1|^2)+\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial z}\bar{f}_1\right)=\\=&\frac{\partial f_2}{\partial z}(1+|f_1|^2)\left(\frac{\partial^2 f_1}{\partial z^2}(1+|f_2|^2)+\frac{\partial f_2}{\partial z}\frac{\partial f_1}{\partial z}\bar{f}_2\right),\end{aligned}\tag{11}$$

which is verified whenever one between $f_1(z)$ and $f_2(z)$ is constant (and thus zero since we assume $f(0,0)=0$), or when $f_1(z)=f_2(z)$. In order to prove that these are the only solutions, write (11) as

$$\frac{\partial f_1}{\partial z}(1+|f_2|^2)\frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial z}(1+|f_1|^2)\right)=\frac{\partial f_2}{\partial z}(1+|f_1|^2)\frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z}(1+|f_2|^2)\right).$$

Assuming f_1,f_2 not constant, it leads to the equation:

$$\left(\frac{\frac{\partial f_1}{\partial z}(1+|f_2|^2)}{\frac{\partial f_2}{\partial z}(1+|f_1|^2)}\right)'\left(\frac{\partial f_2}{\partial z}(1+|f_1|^2)\right)^2=0,$$

which implies that for some complex constant $\lambda\neq 0$,

$$\frac{\partial f_1}{\partial z}(1+|f_2|^2)=\lambda\frac{\partial f_2}{\partial z}(1+|f_1|^2),\tag{12}$$

that is:

$$\frac{\partial \log f_1}{\partial z}\bar{f}_1=\lambda\frac{\partial \log f_2}{\partial z}\bar{f}_2.$$

Comparing the antiholomorphic parts we get $\bar{f}_1 = \alpha\bar{f}_2$, for some complex constant α. Substituting in (12) we get:

$$\alpha(1 + |f_2|^2) = \lambda(1 + |\alpha|^2|f_2|^2).$$

Since $f(0,0) = 0$, from this last equality follows $\alpha = \lambda$ and thus immediately $|\alpha|^2 = 1$. We have proved that a totally geodesic submanifold of $(\mathbb{C}^2, \omega_{C,2})$ is, up to unitary transformation of $\mathbb{C}^2$, $(\mathbb{C}, \omega_C)$ realized either via the map $z \mapsto (f_1, 0)$ (or equivalently $z \mapsto (0, f_1)$) or via $z \mapsto (f_1(z), \alpha f_1(z))$, with $|\alpha|^2 = 1$.

Assume now S to be a k-dimensional complete totally geodesic complex submanifold of $(\mathbb{C}^n, \omega_{C,n})$ and let π_j, $j = 1, \dots, n$, be the projection into the jth $\mathbb{C}$-factor in $\mathbb{C}^n$ and π_{jk} $j, k = 1, \dots, n$, the projection into the space $\mathbb{C}^2$ corresponding to the jth and kth $\mathbb{C}$-factors. Since $\pi_j(S)$, $j = 1, \dots, n$, is totally geodesic into $(\mathbb{C}, \omega_C)$, it is either a point or the whole $\mathbb{C}$. Thus, up to unitary transformation of the ambient space, we can assume S to be of the form:

$$(z_1, \dots, z_k) \mapsto (0, \dots, 0, h_{11}(z_1), \dots, h_{1r}(z_1), \dots, h_{k1}(z_k), \dots, h_{ks}(z_k)). \tag{13}$$

Since also the projections $\pi_{jk}(S)$ have to be totally geodesic into $(\mathbb{C}^2, \omega_{C,2})$, by what we have proven for $n = 2$, we can reduce (13) into the form:

$$(z_1, \dots, z_k) \mapsto (0, \dots, 0, h_1(z_1), \dots, \alpha_r h_1(z_1), \dots, h_k(z_k), \dots, \alpha_s h_k(z_k)),$$

where $|\alpha_t|^2 = 1$ for all t appearing above. Thus, either $S = (\mathbb{C}^k, \omega_{C,k})$ or S is a k dimensional diagonal, which with a suitable unitary transformation can be written again as $(\mathbb{C}^k, \omega_{C,k})$, and we are done. □

Proof of Theorem 2.7 The existence of global symplectic coordinates, namely of a symplectomorphism $\Psi_{C,n} : (\mathbb{C}^n, \omega_{C,n}) \to (\mathbb{R}^{2n}, \omega_0)$ is guaranteed again by Lemma 2.1. In fact for all $j = 1, \dots, n$

$$\begin{aligned}\frac{\partial}{\partial|z_j|^2}\Phi_{C,n} =& 2\frac{\partial}{\partial|z_j|^2}\sum_{j=1}^{n}\int_0^{|z_j|}\frac{\log(1+s^2)}{s}ds\\ =& \frac{1}{|z_j|}\frac{\mathrm{d}}{\mathrm{d}|z_j|}\int_0^{|z_j|}\frac{\log(1+s^2)}{s}ds = \frac{\log(1+|z_j|^2)}{|z_j|^2} > 0.\end{aligned}$$

Moreover, condition (2) in Lemma 2.1 is fulfilled by:

$$\lim_{z\to+\infty}|z_j|^2\sum_{j=1}^{n}\frac{\partial\Phi_{C,n}}{\partial|z_j|^2} = \lim_{z\to+\infty}\sum_{j=1}^{n}\log(1+|z_j|^2) = +\infty.$$

Thus by Lemma 2.1 the map:

$$\Psi_{C,n} \colon (\mathbb{C}^n, \omega_{C,n}) \to (\mathbb{R}^{2n}, \omega_0), \quad z = (z_1, \dots, z_n) \mapsto (\psi_1(z_1)z_1, \dots, \psi_n(z_n)z_n),$$

with

$$\psi_j = \sqrt{\frac{\log(1+|z_j|^2)}{|z_j|^2}},$$

is a global symplectomorphism.

In order to prove the second part of the theorem, let S be a k dimensional totally geodesic complex submanifold of $(\mathbb{C}^n, \omega_{C,n})$ through the origin, which by Lemma 2.8 is given by $(\mathbb{C}^k, \omega_{C,k})$. The image $\Psi_{C,n}(S)$ is of the form:

$$\left(\sqrt{\frac{\log(1+|z_1|^2)}{|z_1|^2}}z_1, \ldots, \sqrt{\frac{\log(1+|z_k|^2)}{|z_k|^2}}z_k, 0, \ldots, 0\right) \simeq \mathbb{C}^k,$$

concluding the proof. □

3 Calabi's Inhomogeneous Kähler–Einstein Metric on Tubular Domains

In this section we construct explicit global symplectic coordinates for the Calabi's inhomogeneous Kähler–Einstein form ω on the complex tubular domains $M = \frac{1}{2}D_a \oplus i\mathbb{R}^n \subset \mathbb{C}^n$, $n \geq 2$, where $D_a \subset \mathbb{R}^n$ is the open ball of $\mathbb{R}^n$ centered at the origin and of radius a. The material of this section in taken from [17].

Let g be the metric on $M \subset \mathbb{C}^n$ whose associated Kähler form is given by:

$$\omega = \frac{i}{2}\partial\bar{\partial} f(z_1 + \bar{z}_1, \ldots, z_n + \bar{z}_n), \tag{14}$$

where $f\colon D_a \to \mathbb{R}$ is a radial function $f(x_1, \ldots, x_n) = Y(r)$, being $r = (\sum_{j=1}^n x_j^2)^{1/2}$ and $x_j = (z_j + \bar{z}_j)/2$, $y_j = (z_j - \bar{z}_j)/2i$, that satisfies the differential equation:

$$(Y'/r)^{n-1}Y'' = e^Y, \tag{15}$$

with initial conditions:

$$Y'(0) = 0,\ Y''(0) = e^{Y(0)/n}. \tag{16}$$

In [2], Calabi proved that the Kähler metric g so defined is smooth, Einstein, complete and not locally homogeneous. This was indeed the first example of such a metric. The reader is also referred to [25] for an alternative and easier proof of the fact that this metric is complete but not locally homogeneous.

The following theorem describes explicit symplectic coordinates for (M, ω).

Theorem 3.1 *For all $n \geq 2$, the Kähler manifold (M, ω) is globally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$ via the map:*

$$\Psi : M \to \mathbb{R}^n \oplus i\mathbb{R}^n \simeq \mathbb{R}^{2n},\ (x, y) \mapsto (\mathrm{grad} f, y)\,, \tag{17}$$

where $f : D_a \to \mathbb{R}, x = (x_1, \dots x_n) \mapsto f(x)$ is a Kähler potential for ω, i.e. $\omega = \frac{i}{2}\partial\bar{\partial} f$, and $\mathrm{grad} f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$.

Notice that in [2, p. 23] Calabi provides an explicit formula for the curvature tensor of (M, g) (he needs this formula to show that the metric g associated to the Kähler form ω is not locally homogeneous). On the other hand it seems a difficult task to compute the sign of the sectional curvature of g using Calabi's formula. Consequently, it is not clear if g satisfies or not the assumptions of McDuff's theorem, namely if its sectional curvature is nonpositive.

Proof of Theorem 3.1 Let us prove first that the map Ψ given by (17) satisfies $\Psi^*\omega_0 = \omega$. In order to simplify the notation we write $\partial f/\partial x_j = f_j$ and $\partial^2 f/\partial x_j \partial x_k = f_{jk}$. The pull-back of ω_0 through Ψ reads:

$$\Psi^*\omega_0 = \sum_{j=1}^{n} df_j \wedge dy_j = \sum_{j,k=1}^{n} f_{jk}\, dx_k \wedge dy_j = \frac{i}{2} \sum_{j,k=1}^{n} f_{jk}\, dz_j \wedge d\bar{z}_k,$$

thus the desired identity follows by:

$$\omega = \frac{i}{2}\partial\bar{\partial} f(z_1 + \bar{z}_1, \dots, z_n + \bar{z}_n) = \frac{i}{2} \sum_{j,k=1}^{n} f_{jk}\, dz_j \wedge d\bar{z}_k.$$

Observe now that since ω and ω_0 are non-degenerate it follows by the inverse function theorem that Ψ is a local diffeomorphism. In order to conclude the proof it is then enough to verify that Ψ is a proper map, from which it follows it is a covering map and hence a global diffeomorphism. In our situation this is equivalent to:

$$\lim_{(x,y) \to \partial M} \Psi(x, y) = \infty$$

or equivalently:

$$\lim_{x \to \partial D_a} ||\, \mathrm{grad} f(x) || = \infty.$$

This readily follows by $f_j(x) = \frac{x_j}{r} Y'(r)$ and the fact that $Y'(r)$ tends to infinity as $r \to a$ (see [2, p. 21]). □

4 The Symplectic Geometry of Hermitian Symmetric Spaces

A *Hermitian symmetric space* is a connected Kähler manifold (N, ω) such that each point $p \in N$ is an isolated fixed point of some holomorphic involutory isometry s_p of N. The component of the identity of the group of holomorphic isometries of N acts transitively on N and hence every Hermitian symmetric space is a homogeneous space. A Hermitian symmetric space N is said to be of *compact* or *noncompact type* if N is compact or noncompact (and non flat). Every Hermitian symmetric space is a direct product $N_0 \times N_- \times N_+$ where all the factors are simply-connected Hermitian symmetric spaces, $N_0 = \mathbb{C}^n$ and N_- and N_+ are spaces of compact and noncompact type, respectively.

Any Hermitian symmetric space of compact or non-compact type is simply connected and is a direct product of irreducible Hermitian symmetric spaces.

An irreducible Hermitian symmetric space of noncompact type (HSSNT in the sequel) is holomorphically isometric to a *bounded symmetric domain* $\Omega \subset \mathbb{C}^n$ centered at the origin $0 \in \mathbb{C}^n$ equipped with the hyperbolic form ω_{hyp} (a multiple of the Bergman form ω_{Berg} (the inclusion $\Omega \subset \mathbb{C}^n$ is often referred as the Harish–Chandra embedding). On the other hand, it is well-known that there exist homogeneous bounded domains (equipped with the Bergman metric) which are not HSSNT (the first example is due to Pyateskii–Shapiro). From now on we identify a HSSNT with its associated bounded symmetric domain. There exists a complete classification of irreducible HSSNT, with four classical series, studied by Cartan, and two exceptional cases.

Let (M, ω_{FS}) be an irreducible Hermitian symmetric space of compact type (HSSCT in the sequel), where ω_{FS} is the *canonical Kähler form*, i.e. the Kähler-Einstein form on M such that

$$\omega_{FS}(A) = \int_A \omega_{FS} = \pi$$

for the generator $A = [\mathbb{C}P^1] \in H_2(M, \mathbb{Z})$. Alternatively one can describe ω_{FS} as follows: there exists a natural number N and a holomorphic embedding

$$BW : M \to \mathbb{C}P^N$$

called the *Borel–Weil embedding*, such that

$$\omega_{FS} = BW^* \Omega_{FS},$$

where Ω_{FS} is the Fubini-Study form on $\mathbb{C}P^N$.

To every bounded symmetric domain $\Omega \subset \mathbb{C}^n$ one can associate an irreducible HSSCT (M, ω) called the *compact dual* of Ω (and viceversa) such that Ω is holomorphically embedded into M. More precisely the Euclidean space $\mathbb{C}^n$ where Ω is embedded can be compactified in order to get M and the inclusion $\mathbb{C}^n \subset M$

is called the *Borel embedding*. Actually one can prove that $\mathbb{C}^n = M \setminus \mathrm{Cut}_0(M)$, where $\mathrm{Cut}_0(M)$ denotes the cut-locus of the origin $0 \in \mathbb{C}^n \subset M$ with respect to the Riemannian metric associated to the Kähler form ω_{FS}.

The following diagram could help the reader to keep in mind the inclusions and the holomorphic embedding described so far:

$$\Omega \overset{Harish-Chandra}{\subset} \mathbb{C}^n = M \setminus \mathrm{Cut}_0(M) \overset{Borel}{\subset} M \overset{BW}{\hookrightarrow} \mathbb{C}P^N.$$

The following theorem summarizes the main results obtained in [6] about the symplectic geometry of Hermitian symmetric spaces.

Theorem 4.1 *Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then there exists a smooth diffeomorphism*

$$\Psi_\Omega : \Omega \to \mathbb{C}^n \tag{18}$$

such that

$$\Psi_\Omega^* \omega_0 = \omega_{hyp} \tag{19}$$

$$\Psi_\Omega^* \omega_{FS} = \omega_0, \tag{20}$$

where ω_{FS} is the restriction of ω_{FS} to $\mathbb{C}^n$ through the Borel embedding. Moreover, the map Ψ_Ω takes any k-dimensional complex and totally geodesic submanifold T of Ω through the origin $0 \in \Omega$ to a complex linear subspace $\mathbb{C}^k \subset \mathbb{C}^n$.

The map (18) then provides global symplectic coordinates on HSSNT and satisfies Ciriza's property described in the introduction. Even if the existence of such coordinates could be deduced by McDuff's theorem the map (18) can be described explicitly in terms of Jordan triple systems (the version given here of Theorem 4.1 differs from that given in [6] to avoid technical tools). Due to the properties (19) and (20) this map was christened in [6] as a *symplectic duality*; its unicity was later considered in [7] (see also [8]). Notice that from the point of view of inducing geometric structures, as in Gromov's programme [10], the importance of the symplectic duality relies on the fact that it is a simultaneous symplectomorphism with respect to different symplectic structures, namely ω_{hyp} and ω_0 on Ω and ω_0 and ω_{FS} on $\mathbb{C}^n$.

4.1 The Basic Example: The First Cartan Domain

Instead of giving the proof of Theorem 4.1 we give here the proof of it in the particular case of the first Cartan domain, namely

$$D_1[k,n] = \{Z \in M_{k,n}(\mathbb{C}) |\, I_k - ZZ^* >> 0\},$$

(the symbol $A >> 0$ means that the matrix A is positive definite) equipped with the hyperbolic form

$$\omega_{hyp} = -\frac{i}{2\pi}\partial\bar{\partial}\log\det(I_k - ZZ^*). \tag{21}$$

Inspired by the map (3) of Example 2.2 we want to show that the smooth map $\Psi_{D_1[k,n]} : D_I[k,n] \to M_{k,n}(\mathbb{C}) = \mathbb{C}^{kn}$ (which agrees with map (3) for $k = 1$) given by:

$$\Psi_{D_1[k,n]}(Z) = (I_k - ZZ^*)^{-\frac{1}{2}}Z, \tag{22}$$

is a symplectic duality, namely is a diffeomorphism satisfying

$$\Psi^*_{D_1[k,n]}\omega_0 = \omega_{hyp}, \tag{23}$$

and

$$\Psi^*_{D_1[k,n]}\omega_{FS} = \omega_0, \tag{24}$$

where ω_0 denotes both the flat Kähler form on $D_1[k,n] \subset M_{k,n}(\mathbb{C})$ and on $\mathbb{C}^{kn}$. We give here a proof due to J. Rawnsley (unpublished). By using the equality

$$XX^*(I_k + XX^*)^{\frac{1}{2}} = (I_k + XX^*)^{\frac{1}{2}}XX^*$$

it is easy to verify that the map

$$\mathbb{C}^{kn} \to D_1[k,n], \;\; X \mapsto (I_k + XX^*)^{-\frac{1}{2}}X \tag{25}$$

is the inverse of $\Psi_{D_1[k,n]}$.

Moreover, we can write

$$\begin{aligned}\omega_{hyp} &= -\frac{i}{2\pi}\partial\bar{\partial}\log\det(I_k - ZZ^*) = \frac{i}{2\pi}d\partial\log\det(I_k - ZZ^*)\\ &= \frac{i}{2\pi}d\partial\,\mathrm{tr}\log(I_k - ZZ^*) = \frac{i}{2\pi}d\,\mathrm{tr}\,\partial\log(I_k - ZZ^*)\\ &= -\frac{i}{2\pi}d\,\mathrm{tr}[Z^*(I_k - ZZ^*)^{-1}dZ],\end{aligned}$$

where we use the decomposition $d = \partial + \bar{\partial}$ and the identity $\log\det A = \mathrm{tr}\log A$. By substituting $X = (I_k - ZZ^*)^{-\frac{1}{2}}Z$ in the last expression one gets:

$$-\frac{i}{2\pi}d\,\mathrm{tr}[Z^*(I_k - ZZ^*)^{-1}dZ] = -\frac{i}{2\pi}d\,\mathrm{tr}(X^*dX) + \frac{i}{2\pi}d\,\mathrm{tr}\{X^*d[(I_k - ZZ^*)^{-\frac{1}{2}}]Z\}.$$

Observe now that $-\frac{i}{2\pi} d\,\mathrm{tr}(X^* dX) = \omega_0$ and the 1-form $\mathrm{tr}[X^* d(I_k - ZZ^*)^{-\frac{1}{2}} Z]$ on $\mathbb{C}^{kn}$ is exactly equal to $d\,\mathrm{tr}(\frac{C^2}{2} - \log C)$, where $C = (I_k - ZZ^*)^{-\frac{1}{2}}$. So (23) is proved.

Let $\mathrm{Grass}(k, n+k)$ denote the complex Grassmannian of complex k-planes in $\mathbb{C}^{n+k}$ (the complex dual of $D_1[k,n]$) endowed with the Fubini–Study form ω_{FS}, namely the Kähler form on $\mathrm{Grass}(k, n+k)$ obtained as the pull-back $P^*\Omega_{FS} = \omega_{FS}$ of the Fubini–Study form Ω_{FS} on $\mathbb{C}\mathrm{P}^N$, $N = \binom{n+k}{k} - 1$, where $P : \mathrm{Grass}(k, n+k) \to \mathbb{C}\mathrm{P}^N$ is the Plücker embedding (a specialization of the Borel–Weil embedding). We identify $\mathbb{C}^{kn}$ as an affine chart in $\mathrm{Grass}(k, n+k)$ (when $k = 1$, $\mathbb{C}^n \subset \mathrm{Grass}(1, n+1) = \mathbb{C}\mathrm{P}^n$ is the natural inclusion where $\mathbb{C}^n$ is identified with the open subset $U_0 \subset \mathbb{C}\mathrm{P}^n$ given in homogeneous coordinates by $U_0 = \{[Z_0, \ldots, Z_{n+1}] \mid Z_0 \neq 0\}$) equipped with the restriction of ω_{FS}.

The proof of (24) follows the same line. Indeed,

$$\begin{aligned} \omega_{FS} &= \frac{i}{2\pi}\partial\bar\partial \log\det(I_n + XX^*) = -\frac{i}{2\pi} d\,\mathrm{tr}\,\partial \log(I_n + XX^*) \\ &= -\frac{i}{2\pi} d\,\mathrm{tr}[X^*(I_n + XX^*)^{-1} dX] \end{aligned}$$

By substituting $Z = (I_n + XX^*)^{-\frac{1}{2}} X$ in the last expression one gets:

$$\begin{aligned} -\frac{i}{2\pi} d\,\mathrm{tr}[X^*(I_n + XX^*)^{-1} dX] &= -\frac{i}{2\pi} d\,\mathrm{tr}(Z^* dZ) + \frac{i}{2\pi} d\,\mathrm{tr}\{Z^* d[(I_n + XX^*)^{-\frac{1}{2}}]X\} \\ &= \omega_0 + \frac{i}{2\pi} d^2\,\mathrm{tr}(\log D - \mathrm{tr}\,\frac{D^2}{2}) = \omega_0, \end{aligned}$$

where $D = (I_n + XX^*)^{-\frac{1}{2}}$.

Remark 4.2 The basic tools to extend the proof of Theorem 4.1 to all classical domains are to combine the previous computation for the first Cartan domain with the fact that every bounded symmetric domain Ω can be complex and totally geodesically embedded into the first Cartan domain $D_1[n,n]$, for n sufficiently large. The case of exceptional domains required more care and the Jordan Algebras and Jordan triple systems tools. The interested reader is referred to [6] and [7] for details.

4.2 Symplectic Capacities and Gromov Width of Hermitian Symmetric Spaces

A map c from the class $\mathcal{C}(2n)$ of all symplectic manifolds of dimension $2n$ to $[0, +\infty]$ is called a *symplectic capacity* if it satisfies the following conditions (see [12] and also [13] for more details):

(**monotonicity**) if there exists a symplectic embedding $(M_1, \omega_1) \to (M_2, \omega_2)$ then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$;

(**conformality**) $c(M, \lambda\omega) = |\lambda| c(M, \omega)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$;
(**nontriviality**) $c(B^{2n}(1), \omega_0) = \pi = c(Z^{2n}(1), \omega_0)$.

Here $B^{2n}(1)$ and $Z^{2n}(1)$ are the open unit ball and the open cylinder in the standard $(\mathbb{R}^{2n}, \omega_0)$, i.e.

$$B^{2n}(r) = \left\{ (x, y) \in \mathbb{R}^{2n} \;\middle|\; \sum_{j=1}^{n} x_j^2 + y_j^2 < r^2 \right\} \tag{26}$$

and

$$Z^{2n}(r) = \left\{ (x, y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2 \right\}. \tag{27}$$

By the previous assumptions it follows that a symplectic capacity is a symplectic invariant. Notice that the previous assumptions do not determine uniquely a symplectic capacity. Indeed one can construct many symplectic capacities (see, e.g. [12]) When $n = 1$, i.e. in the 2-dimensional case, the module of the area

$$c(M, \omega) = |\int_M \omega|$$

is an example of symplectic capacity. If the dimension $n > 1$ then for $(\mathrm{Vol}(M, \omega))^{\frac{1}{n}}$ the nontriviality is not satisfied since the volume of the symplectic cylinder $Z^{2n}(r)$, $n > 1$, is infinite. As we will see in Theorem 4.5 the existence of a symplectic capacity when $n > 1$ is not a trivial fact.

Remark 4.3 Notice that if one takes the cylinder

$$W^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid x_1^2 + x_2^2 < r^2\}, \quad r > 0$$

(instead of the symplectic cylinder $Z^{2n}(r)$) then

$$c(W^{2n}(r), \omega_0) = +\infty,$$

for every symplectic capacity. Indeed the map

$$\varphi : B^{2n}(N) \to W^{2n}(r), (x_1, \dots x_n, y_1, \dots y_n) \mapsto (\frac{r}{N} x_1, \dots, \frac{r}{N} x_n, \frac{N}{r} y_1, \dots, \frac{N}{r} y_n)$$

is a symplectic embedding and then

$$\pi N^2 = c(B^{2n}(N), \omega_0) \le c(W^{2n}(r), \omega_0).$$

The last inequality is true for all N and then one gets $c(W^{2n}(r), \omega_0) = +\infty$.

Example 4.4 In order to get some feeling with symplectic capacities it is useful to describe some examples in $(\mathbb{R}^{2n}, \omega_0)$. First notice that if U is an open subset of $\mathbb{R}^{2n}$ and $\lambda \neq 0$ then

$$c(\lambda U, \omega_0) = \lambda^2 c(U, \omega_0). \tag{28}$$

Indeed the diffeomorphism $\varphi : \lambda U \to U,\ x \mapsto \frac{1}{\lambda}x$ satisfies $\varphi^*(\lambda^2\omega_0) = \lambda^2\varphi^*\omega_0 = \omega_0$ and so φ is a symplectomorphism between $(\lambda U, \omega_0)$ and $(U, \lambda^2\omega_0)$. Then, by the conformality

$$c(\lambda U, \omega_0) = c(U, \lambda^2\omega_0) = \lambda^2 c(U, \omega_0).$$

It follows that

$$c(B^{2n}(r), \omega_0) = c(Z^{2n}(r), \omega_0) = \pi r^2,$$

for all $r > 0$. Indeed by (28) and by the nontriviality one has:

$$c(B^{2n}(r), \omega_0) = c(rB^{2n}(1), \omega_0) = r^2 c(B^{2n}(1), \omega_0) = \pi r^2.$$

With a similar argument, taking the cylinder $Z^{2n}(r)$ instead of $B^{2n}(r)$, one gets $c(Z^{2n}(r), \omega_0) = \pi r^2$.

The existence of a symplectic capacity c provides a proof of the celebrated Gromov nonsqueezing theorem [9] which asserts that *there exists a symplectic embedding* $\varphi : B^{2n}(r) \to Z^{2n}(R)$ *if and only if* $r \leq R$. Indeed if $r \leq R$ then the inclusion $B^{2n}(r) \subset Z^{2n}(R)$ is the desired embedding. Conversely if $\varphi : B^{2n}(r) \to Z^{2n}(R)$ is given then by the monotonicity and nontriviality for c one has:

$$\pi r^2 = c(B^{2n}(r), \omega_0) \leq c(Z^{2n}(R), \omega_0) = \pi R^2$$

and then $r \leq R$.

We refer the reader to [9] for the proof of the nonsqueezing theorem using pseudoholomorphic curves.

The Gromov width of a $2n$-dimensional symplectic manifold (M, ω), introduced in [9], is defined as

$$c_G(M, \omega) = \sup\{\pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M, \omega)\}. \tag{29}$$

By Darboux's theorem $c_G(M, \omega)$ is a positive number or ∞.

The following theorem shows the existence of at least a symplectic capacity, namely the Gromov width.

Theorem 4.5 *The Gromov width is a symplectic capacity. Moreover,*

$$c_G(M, \omega) \leq c(M, \omega),$$

for any symplectic capacity c.

Proof The monotonicity of c_G follows from the fact that if $\varphi : (B^{2n}(r), \omega_0) \to (M, \omega)$ and $\psi : (M, \omega) \to (N, \tau)$ are two symplectic embeddings then $\psi \circ \varphi : (B^{2n}(r), \omega_0) \to (N, \tau)$ is again a symplectic embedding.

To prove the conformality, i.e. $c_G(M, \alpha\omega) = |\alpha| c_G(M, \omega)$, $\alpha \neq 0$, it is enough to show that to any symplectic embedding

$$\varphi : (B^{2n}(r), \omega_0) \to (M, \alpha\omega),$$

one can associate a symplectic embedding

$$\hat{\varphi} : (B^{2n}(\rho), \omega_0) \to (M, \omega),\ \rho = \frac{r}{\sqrt{|\alpha|}},$$

and viceversa. Thus, by the very definition of c_G, one gets $c_G(M, \alpha\omega) = |\alpha| c_G(M, \omega)$. In order to prove this assertion, let $\varphi : (B^{2n}(r), \omega_0) \to (M, \alpha\omega)$ be a symplectic embedding. The diffeomorphism $\psi : B^{2n}(\rho) \to B^{2n}(r)$ given by

$$\psi(x) = \sqrt{|\alpha|} \cdot x,\ x \in B^{2n}(\rho),$$

satisfies

$$\psi^*(\alpha^{-1}\omega_0) = \frac{|\alpha|}{\alpha}\omega_0.$$

If $\alpha > 0$ then the map $\hat{\varphi} = \varphi \circ \psi : (B^{2n}(\rho), \omega_0) \to (M, \omega)$ is the desired symplectic embedding. If $\alpha < 0$ one takes the symplectomorphism

$$\psi_0 : (B^{2n}(\rho), \omega_0) \to (B^{2n}(\rho), \omega_0),\ (u, v) \mapsto (-u, v),\ \ (u, v) \in \mathbb{R}^{2n},$$

and one gets the desired symplectomorphism by taking

$$\hat{\varphi} = \varphi \circ \psi \circ \psi_0 : (B^{2n}(\rho), \omega_0) \to (M, \omega).$$

We now show that c_G satisfies $c_G(B^{2n}(r), \omega_0) = c_G(Z^{2n}(r), \omega_0) = \pi r^2$ (the nontriviality). If $\varphi : B^{2n}(R) \to B^{2n}(r)$ is a symplectic embedding, one gets $R \leq r$ since φ preserves volumes. On the other hand the identity map $B^{2n}(r) \to B^{2n}(r)$ is a symplectic embedding and hence $c_G(B^{2n}(r), \omega_0) = \pi r^2$. By the Gromov nonsqueezing theorem there exists a symplectomorphism $\varphi : B^{2n}(R) \to Z^{2n}(r)$ if and only if $R \leq r$ and hence $c_G(Z^{2n}(r), \omega_0) = \pi r^2$.

Finally, let c be any symplectic capacity and $\varphi : (B^{2n}(r), \omega_0) \to (M, \omega)$ a symplectic embedding. Monotonicity and nontriviality yield

$$\pi r^2 = c(B^{2n}(r), \omega_0) \leq c(M, \omega)$$

and by taking the sup one gets $c_G(M,\omega) \leq c(M,\omega)$. $\square$

Computations and estimates of the Gromov width for various examples can be found in the references given in [21]. In the following examples we compute it for the complex projective space and, more generally, for the complex Grassmannian.

Example 4.6 In order to compute the Gromov width of the n-dimensional complex projective space consider the diffeomorphism $\Psi : B^{2n}(1) \to \mathbb{C}^n$ given by (3) above, namely

$$(z_1,\ldots,z_n) \mapsto \left(\frac{z_1}{\sqrt{1-\sum_{i=1}^n |z_i|^2}},\ldots,\frac{z_n}{\sqrt{1-\sum_{i=1}^n |z_i|^2}}\right).$$

It satisfies $\Psi^*(\omega_{FS}) = \omega_0$, where we are identifying $\mathbb{C}^n$ with the affine chart $U_0 = \{Z_0 \neq 0\} \subset \mathbb{C}\mathrm{P}^n$ and ω_{FS} is the Fubini–Study form on $\mathbb{C}\mathrm{P}^n$. For the monotonicity and nontriviality of the Gromov width one then gets:

$$\pi = c_G(B^{2n}(1),\omega_0) \leq c_G(\mathbb{C}\mathrm{P}^n,\omega_{FS}).$$

From density reasons $c_G(\mathbb{C}\mathrm{P}^n,\omega_{FS}) \leq \pi$. Thus

$$c_G(\mathbb{C}P^n,\omega_{FS}) = \pi.$$

Example 4.7 Let $D_1[k,n] = \{Z \in M_{k,n}(\mathbb{C})|\ I_k - ZZ^* >> 0\}$ be the first Cartan domain as in Sect. 4.1. In matrix notation the unit ball of $\mathbb{C}^{kn}$ reads as

$$B^{2kn}(1) = \{Z \in M_{k,n}(\mathbb{C}) \mid \mathrm{Tr}(ZZ^*) < 1\}$$

and the symplectic cylinder is given by:

$$Z^{2kn}(1) = \{Z \in M_{k,n}(\mathbb{C}) \mid Z_{11} = e^{i\theta}\},$$

where Z_{11} denotes the first entry of the matrix Z. We claim that

$$B^{2kn}(1) \subset D_1[k,n] \subset Z^{2kn}(1).$$

Indeed, if ZZ^* were diagonal then $\mathrm{Tr}(ZZ^*) < 1$ trivially implies $(I - ZZ^*) >> 0$. But, for all $Z \in M_{k,n}(\mathbb{C})$ it always exists $U \in \mathrm{U}(k)$ such that UZZ^*U^* is diagonal, which immediately implies the first inclusion. The second inclusion is obtained as follows. Since $D_1[k,n]$ is convex, for each Z in $\partial D_1[k,n]$ (the boundary of $D_1[k,n]$) one can find a unique hyperplane π_Z of $M_{k,n}(\mathbb{C})$ not intersecting $D_1[k,n]$. Observe now that the point $Z_0 \in M_{k,n}(\mathbb{C})$ defined by the equation $Z_{11} = e^{i\theta}$ and $Z_{jk} = 0$ for all $j,k \neq 1$ is a boundary point for both $B^{2kn}(1)$ and $D_1[k,n]$ and hence $\pi_{Z_0} = T_{Z_0}\partial B^{2kn}(1)$. It follows that $D_1[k,n] \subset Z^{2kn}(1)$.

Using the monotonicity of the Gromov width (and nontriviality) we then get:

$$c_G(D_1[k,n],\omega_0) = \pi.$$

Moreover, since the map (22) satisfies (24) it induces a symplectic embedding between $(B^{2n}(1),\omega_0)$ and $(\mathrm{Grass}(k,n+k),\omega_{FS})$ which implies (again by monotonicity and nontriviality) that $c_G(\mathrm{Grass}(k,n+k),\omega_{FS}) \geq \pi$. Moreover, using Gromov-Witten invariant tools one can prove that $c_G(\mathrm{Grass}(k,n+k),\omega_{FS}) \leq \pi$ (the interested reader is referred either to [14] or to [22]) and hence

$$c_G(\mathrm{Grass}(k,n+k),\omega_{FS}) = \pi.$$

The following two theorems summarize some of the results obtained in [21] (see also [20]). The (ideas of the) proofs given here are extensions of those given in the previous example.

Theorem 4.8 *Let (M,ω_{FS}) be an irreducible HSSCT. Then*

$$c_G(M,\omega_{FS}) = \pi. \tag{30}$$

Idea of the Proof The proof of the upper bound $c_G(M,\omega_{FS}) \leq \pi$ is obtained by the computations of some genus-zero three-points Gromov-Witten invariants for irreducible HSSCT and through nonsqueezing theorem techniques using and extending the ideas in [14] for complex Grassmannians (see [21] for details). The lower bound $c_G(M,\omega_{FS}) \geq \pi$ is obtained by using the symplectic duality. Indeed, by using Jordan triple systems tools one can prove that there exists a symplectic embedding

$$(B^{2n}(1),\omega_0) \hookrightarrow (\Omega,\omega_0),$$

where $\Omega \subset \mathbb{C}^n$ is the bounded symmetric domain noncompact dual of (M,ω_{FS}). By combining this with the symplectic duality map (18) $\Phi_\Omega : (\Omega,\omega_0) \to (M,\omega_{FS})$ one gets a symplectic embedding of $(B^{2n}(1),\omega_0)$ into (M,ω_{FS}) and hence the lower bound $c_G(M,\omega_{FS}) \geq \pi$ thereby follows. Using again Jordan triple system tools one can show $\Omega \subset Z^{2n}(1)$ and hence $c_G(\Omega,\omega_0) = \pi$ follows again from monotonicity. □

Theorem 4.9 *Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then*

$$c_G(\Omega,\omega_0) = \pi. \tag{31}$$

Idea of the Proof Using again Jordan triple system tools one can show $\Omega \subset Z^{2n}(1)$ and hence by the previous theorem and the monotonicity one gets $c_G(\Omega,\omega_0) = \pi$. □

Remark 4.10 The lower bound $c_G(M, \omega_{FS}) \geq \pi$ in Theorem 4.8 could be also obtained by noticing that every HSSCT is an example of homogeneous and simply-connected Kähler manifold with second Betti number equal to 1 (see [20] and [19] for details).

Acknowledgement The first author was supported by PRID 2011/14—University of Cagliari.

References

1. L. Bates, G. Peschke, A remarkable symplectic structure. J. Differ. Geom. **32**, 533–538 (1990)
2. E. Calabi, A construction of nonhomogeneous Einstein metrics, in *Proceedings of Symposia in Pure Mathematics*, vol. 27, Part II (American Mathematical Society, Providence, RI, 1975), pp. 17–24
3. H.-D. Cao, Existence of Gradient Kähler–Ricci Solitons, in *Elliptic and Parabolic Methods in Geometry* (AK Peters, Wellesley, MA, 1996)
4. E. Ciriza, The local structure of a Liouville vector field. Am. J. Math. **115**, 735–747 (1993)
5. F. Cucu, A. Loi, Global symplectic coordinates on complex domains. J. Geom. Phys. **56**, 247–259 (2006)
6. A. Di Scala, A. Loi, Symplectic duality of symmetric spaces. Adv. Math. **217**, 2336–2352 (2008)
7. A. Di Scala, A. Loi, G. Roos, The bisymplectomorphism group of a bounded symmetric domain. Transform. Groups **13**(2), 283–304 (2008)
8. A. Di Scala, A. Loi, F. Zuddas, Symplectic duality between complex domains. Monatsh. Math. **160**, 403–428 (2010)
9. M. Gromov, Pseudoholomorphic curves in symplectic manifolds. Invent. Math. **82**, 307–347 (1985)
10. M. Gromov, *Partial Differential Relations* (Springer, Berlin/Heidelberg, 1986)
11. R.S. Hamilton, The Ricci flow on surfaces, in *Mathematics and General Relativity (Santa Cruz, CA, 1986)*. Contemporary Mathematics, vol. 71 (American Mathematical Society, Providence, RI, 1988), pp. 237–262
12. H. Hofer, E. Zehnder, A new capacity for symplectic manifolds, in *Analysis Et Cetera*, ed. by P. Rabinowitz, E. Zehnder (Academic, New York, 1990), pp. 405–429
13. H. Hofer, E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics* (Birkhäuser, Basel, 1994)
14. Y. Karshon, S. Tolman, The Gromov width of complex Grassmannians. Algebr. Geom. Topol. **5**, 911–922 (2005)
15. S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, vol. 2 (Wiley-Interscience, New York, 1963) (Published 1996 New edition)
16. C. LeBrun, Complete Ricci-flat Kähler metrics on $\mathbb{C}^n$ need not be flat, in *Proceedings of Symposia in Pure Mathematics*, vol. 52, Part 2 (American Mathematical Society, Providence, RI, 1991), pp. 297–304
17. A. Loi, M. Zedda, Calabi's inhomogeneous Einstein manifold is globally symplectomorphic to $\mathbb{R}^{2n}$. Diff. Geom. App. **30**, 145–147 (2012)
18. A. Loi, F. Zuddas, Symplectic maps of complex domains into complex space forms. J. Geom. Phys. **58**, 888–899 (2008)
19. A. Loi, F. Zuddas, On the Gromov width of homogeneous Kähler manifolds. http://arxiv.org/abs/1508.02862
20. A. Loi, R. Mossa, F. Zuddas, Some remarks on the Gromov width of homogeneous Hodge manifolds. Int. J. Geom. Methods Mod. Phys. **11**(2), 1460029 (2014)

21. A. Loi, R. Mossa, F. Zuddas, Symplectic capacities of Hermitian symmetric spaces of compact and non compact type. J. Symplect. Geom. **13**(4), 1049–1073 (2015)
22. G. Lu, Gromov-Witten invariants and pseudo symplectic capacities. Isr. J. Math. **156**, 1–63 (2006)
23. D. McDuff, The symplectic structure of Kähler manifolds of non-positive curvature. J. Diff. Geom. **28**, 467–475 (1988)
24. O. Suzuki, Remarks on continuation problems of Calabi's diastatic functions. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29**(1), 45–49 (1982)
25. J.A. Wolf, On Calabi's inhomogeneous Einstein–Kähler manifolds. Proc. Am. Math. Soc. **63**(2), 287–288 (1977)

Instantons and Special Geometry

Jason D. Lotay and Thomas Bruun Madsen

To Simon Salamon on the occasion of his 60th birthday

Abstract We survey and discuss constructions of instantons on non-compact complete manifolds of special holonomy from the viewpoint of evolution equations and give several explicit examples.

Keywords Gauge theory • Instantons • Special holonomy

1 Introduction

Suppose we have a principal K-bundle $P \to M$ over an oriented Riemannian n-manifold M. Given a connection form $\omega \in \Omega^1(P; \mathfrak{k})$ the associated curvature will be $\Omega \in \Omega^2(M; \mathfrak{k})$, where $\mathfrak{k}$ is the Lie algebra of K. When M comes with a G-structure, $\mathrm{G} \subset \mathrm{SO}(n)$, we can decompose the 2-forms as:

$$\Lambda^2 T^* M \cong \mathfrak{so}(n) \cong \mathfrak{g} \oplus \mathfrak{g}^{\perp},$$

where the fibres of $\mathfrak{g}$ are given by the Lie algebra of G. This splitting gives us a way of distinguishing connections that are particularly adapted to the geometry (cf. [26]).

Definition 1.1 A connection ω on P is called a G-instanton if the 2-form part of its curvature Ω takes values in the subbundle $\mathfrak{g} \subset \Lambda^2 T^* M$.

A natural setting where we have a distinguished G-structure on M is when the metric on M has special holonomy G (and thus the G-structure is torsion-free). There

J.D. Lotay
Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK
e-mail: j.lotay@ucl.ac.uk

T.B. Madsen (✉)
Department of Mathematics, Aarhus University, Ny Munkegade 118, Bldg 1530, 8000 Aarhus, Denmark
e-mail: thomas.madsen@math.au.dk

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_10

are three key dimensions of manifold, and thus three holonomy groups, which will be the focus of this article: $G = SU(2)$, G_2 and Spin(7). In each case we give a reformulation of the criterion of G-instanton from Definition 1.1.

For these groups G we unify the known constructions of G-instantons on non-compact complete manifolds with holonomy G in terms of an evolution procedure. As well as bringing together examples which have occurred in the diverse literature, we analyse the limits of the instantons, including the issue of whether the instantons globally extend. We thus hope to provide insight into future constructions and classifications.

1.1 Dimension 4: G = SU(2)

On a 4-manifold M we can encode the data of an SU(2)-holonomy metric (i.e. a hyperKähler metric in 4 dimensions) in terms of a triple $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of 2-forms satisfying (cf. [19]):

$$\sigma_i \wedge \sigma_j = \tfrac{1}{3}\delta_{ij}\sum_{k=1}^{3}\sigma_k^2 \quad \text{and} \quad d\sigma_i = 0. \tag{1}$$

The triple σ defines a unique metric g such that σ is a triple of self-dual 2-forms and the volume form of g is equal to $\frac{1}{2}\sigma_i^2$ for all i. The metric g has holonomy contained in SU(2).

If we consider $\mathbb{R}^4$ as a representation of SO(4), we have

$$\Lambda^2\mathbb{R}^4 \cong \mathfrak{so}(4) = \Lambda_+ \oplus \Lambda_-,$$

where $\Lambda_\pm \cong \mathfrak{su}(2)_\pm \cong \Sigma^2_\pm$ are the ± 1-eigenspace of the Hodge star operator. Our choice of conventions for SU(2)-structures defined in terms of triples σ corresponds to the choice $\mathfrak{su}(2) = \mathfrak{su}(2)_-$. We thus see that ω is an SU(2)-instanton precisely when its curvature Ω is *anti-self-dual* (ASD):

$$\Omega = -{*}\Omega. \tag{2}$$

These connections have played an important role in the study of the topology of 4-manifolds, so one might hope that G-instantons would encode topological information in the other situations we will discuss.

1.2 Dimension 7: G = G$_2$

On a 7-manifold M a G_2-structure is encoded by a 3-form φ on M whose stabilizer in $GL(7,\mathbb{R})$ at each point is isomorphic to G_2. The form φ defines a metric g and orientation, and thus a Hodge star operator. The torsion-free condition so that g has holonomy contained in G_2 is equivalent to (by Fernandez and Gray [13])

$$d\varphi = 0 \quad \text{and} \quad d{*}\varphi = 0. \tag{3}$$

Let V denote the 7-dimensional irreducible representation of G_2. Then

$$\Lambda^2 V \cong V \oplus \mathfrak{g}_2\,.$$

There are two natural equivariant maps $\Lambda^2 V \to \Lambda^5 V$; one is given by the star operator and the other comes from wedging with φ. It is easy to check that these are both isomorphisms $\Lambda^2 V \cong \Lambda^5 V$, and that they coincide up to a multiple of 2 on V and up to a multiple of -1 on $\mathfrak{g}_2$. It follows that ω is a G_2-instanton precisely when its curvature satisfies

$$\varphi \wedge \Omega = -{*}\Omega\,. \tag{4}$$

Notice the similarity to Eq. (2): this suggests that G_2-instantons are in some sense natural analogues of ASD instantons in 4 dimensions.

For another useful characterization, we notice that the map obtained by wedging 2-forms with the invariant 4-form ${*}\varphi$ gives an equivariant map

$$\Lambda^2 V \to \Lambda^6 V \cong V.$$

It is straightforward to check that this is an isomorphism between copies of V and has kernel $\mathfrak{g}_2 \subset \Lambda^2 V$. So another way of phrasing that ω is a G_2-instanton is that its curvature satisfies the condition

$${*}\varphi \wedge \Omega = 0. \tag{5}$$

Remark 1.2 If a G_2-manifold M is a product $\mathbb{R}^3 \times X$ (or $T^3 \times X$) where X is a hyperKähler 4-manifold with hyperKähler triple σ then, if (x^1, x^2, x^3) are local coordinates on $\mathbb{R}^3$ (or T^3), we have the product G_2-structure

$$\varphi = dx^1 \wedge dx^2 \wedge dx^3 - dx^1 \wedge \sigma_1 - dx^2 \wedge \sigma_2 - dx^3 \wedge \sigma_3.$$

Thus, if ω is a pullback of a connection on X to M, we see that Eq. (4) is equivalent to Eq. (2), since if Ω is ASD on X then $\Omega \wedge \sigma_i = 0$ for all i. Hence, ω is a G_2-instanton if and only if it is an SU(2)-instanton.

1.3 *Dimension 8:* G = Spin(7)

In a similar manner to the G_2 case just discussed, on an 8-manifold M a Spin(7)-structure is equivalent to a 4-form Φ on M whose stabilizer in $\mathrm{GL}(8, \mathbb{R})$ at each point is Spin(7). Again, Φ defines a metric g and orientation, and g has holonomy contained in Spin(7) if and only if (by Fernández [12])

$$d\Phi = 0. \tag{6}$$

Let W denote the 8-dimensional irreducible representation of Spin(7). It is well-known that we have the orthogonal decomposition

$$\Lambda^2 W \cong V \oplus \mathfrak{spin}(7),$$

where V is the 7-dimensional irreducible complement of $\mathfrak{spin}(7)$ inside $\Lambda^2 W \cong \mathfrak{so}(8)$. Consider now the two equivariant maps $\Lambda^2 W \to \Lambda^6 W$:

$$\beta \mapsto *\beta, \quad \beta \mapsto \Phi \wedge \beta.$$

Elementary computations show that these coincide up to a multiple of 3 on V and up to a multiple of -1 on $\mathfrak{spin}(7)$. Hence, a connection ω is a Spin(7)-instanton if and only if its curvature Ω satisfies the following:

$$\Phi \wedge \Omega = -*\Omega. \tag{7}$$

Again, notice the similarity to the ASD condition (2) in 4 dimensions.

Remark 1.3 If we assume that a Spin(7)-manifold M is a product $\mathbb{R} \times N$ (or $S^1 \times N$) where N is a G_2-manifold with torsion-free G_2-structure φ, then if t is a local coordinate on $\mathbb{R}$ (or S^1) we have the product Spin(7)-structure on M:

$$\Phi = \varphi \wedge dt + *_N\varphi.$$

Equations (4), (5) and (7) show that if ω is the pullback of a connection on N to M then ω is a Spin(7)-instanton if and only if it is a G_2-instanton.

Remark 1.4 A related situation that shall occur is when the group $G = SU(3)$, as we shall see in Corollary 3.5. Here, if the SU(3)-structure on a 6-manifold is given by a 2-form σ and 3-form γ then ω is an SU(3)-instanton if and only if:

$$\Omega \wedge \gamma = 0 \quad \text{and} \quad \Omega \wedge \sigma^2 = 0. \tag{8}$$

1.4 Construction Via Evolution

The construction of manifolds with special holonomy, and thus of instantons, is difficult in general, and particularly so in the compact case. This is primarily due to the analytic difficulties involved in solving systems of nonlinear PDE. However, a situation where the problem becomes tractable is where an open dense subset of M

is a product $I \times N$ for an interval $I \subset \mathbb{R}$. A well-known special case of this is when M admits a cohomogeneity one group action.

One can identify G-structures on $I \times N$ with certain natural structures on the hypersurfaces $\{t\} \times N$, or equivalently with an I-family of structures on N. The special holonomy condition on M becomes an evolution equation for the structures on N. Moreover, our bundle P restricts to a principal K-bundle over $I \times N$, which we may always assume is the pullback of a bundle $Q \to N$. Consequently, any connection on P over $I \times N$ can be viewed as a one-parameter family of connections on Q. The next elementary lemma will be instrumental in reformulating the G-instanton condition in terms of an evolution equation for the connections on Q.

Lemma 1.5 *Let ω be a connection on a principal* K*-bundle over $I \times N$. Then ω can be identified with a one-parameter family $I \ni t \to A(t)$ of connections on a principal* K*-bundle over N. In particular, if $F_A = F_A(t)$ is the curvature of $A(t)$ then the curvature* 2*-form Ω of ω can be expressed as*

$$\Omega = dt \wedge A' + F_A. \tag{9}$$

Proof A principal K-bundle $P \to I \times N$ defines a principal bundle $Q \to N$ by composing the bundle projection map with the projection $\pi \colon I \times N \to N$; then the pullback bundle $\pi^* Q$ is isomorphic to P.

In these terms, any connection on P can be written $\alpha dt + A(t)$, where we can regard $A(t)$ as a one-parameter family of connections on Q. The term αdt, however, can be set to zero after performing a t-dependent gauge transformation $\omega \mapsto k^{-1}\omega k + k^{-1}dk$ with k being the unique solution to the ODE $\partial k/\partial t = \alpha k$.

It follows that we can assume $\omega = A(t)$, and from this the expression Eq. (9) immediately follows. □

Hence, when an open dense subset of M is viewed as family of hypersurfaces, the construction of instantons on this manifold with special holonomy reduces to the analysis of ODEs. Whilst this is still challenging, it could allow us to investigate key questions such as the dimension of the moduli space of instantons (including if it is non-empty) and the potential relationship between instantons and calibrated submanifolds.

Remark 1.6 The above approach can be applied to other special geometries. For instance, one can construct instantons on bundles over (open subsets of) the 8-dimensional Wolf spaces $\mathbb{HP}(2)$, $\mathrm{Gr}_2(\mathbb{C}^4)$ and $\mathrm{G}_2 / \mathrm{SO}(4)$; these are all cohomogeneity one spaces with respect to the natural action of SU(3) [15].

In this case, the family of hypersurfaces consists of 7-manifolds with SO(4)-structures in the sense of [8]. By considering suitable connections on bundles over these hypersurfaces, one obtains the type of instantons introduced in [23].

2 The SU(2) Case

2.1 SU(2)-*Structures*

Following [3], a basic construction of metrics with SU(2) holonomy begins with an oriented 3-manifold N equipped with a one-parameter family of oriented coframes

$$I \ni t \mapsto e(t) = (e^1(t), e^2(t), e^3(t)),$$

so that $e^1(t) \wedge e^2(t) \wedge e^3(t) > 0$. These coframes are declared to be orthonormal so that we have a family of induced metrics on N given by $g(t) = e^1(t)^2 + e^2(t)^2 + e^3(t)^2$. From this family of coframes, we can construct a triple σ of 2-forms on the product $M = I \times N$:

$$\sigma_1 = dt \wedge e^1 + e^2 \wedge e^3,\ \sigma_2 = dt \wedge e^2 + e^3 \wedge e^1,\ \sigma_3 = dt \wedge e^3 + e^1 \wedge e^2. \tag{10}$$

These forms are self-dual with respect to $dt^2 + g(t)$ and satisfy the first equations in (1). In terms of data on N the condition $d\sigma = 0$ amounts to

$$d{*_t}e(t) = 0,$$

where $*_t$ is the Hodge star given by $g(t)$ and the orientation $e^1(t) \wedge e^2(t) \wedge e^3(t)$, for each t together with the equations:

$$({*_t}e)' = de. \tag{11}$$

As the condition $d{*e} = 0$ is preserved by Eq. (11), we can in a sense regard Eq. (11) as a way of evolving an initial co-closed coframe on N. This is sometimes called an "SU(2)-flow" though it is not in any sense a parabolic equation, and so does not satisfy the usual analytic properties one would expect of a geometric flow (cf. [3]).

In addition to the flat metric on $\mathbb{R}^4$ there are basically three interesting metrics arising directly from this construction (see, for instance, [16, Proposition 2.7]): the Eguchi-Hanson metric, the Taub-NUT metric and the Atiyah-Hitchin metric. These metrics are complete, have full holonomy SU(2), and are examples of gravitational instantons.

2.2 SU(2)-*Instantons*

If $M = I \times N$ and the bundle P is the pullback of $Q \to N$, then we can express the SU(2)-instanton condition using Lemma 1.5 and Eq. (2) as follows.

Proposition 2.1 *A connection ω on P over $I \times N$ is an* SU(2)*-instanton if and only if the one-parameter family $A(t)$ of connections on $Q \to N$ satisfies:*

$$A'(t) = -*_t F_A(t). \tag{12}$$

As Eq. (12) is in Cauchy form, we immediately deduce:

Corollary 2.2 *Given real-analytic initial data, the* SU(2)*-instanton evolution equation* (12) *admits a unique solution over an open subset of $I \times N$.*

Corollary 2.3 *If ω is asymptotic to a connection on $Q \to N$ at an endpoint of I, then the limiting connection is flat.*

Proof From the form of Ω, we see that it can approach the curvature of a connection on N at an endpoint of I only if $A' \to 0$, and in that case Eq. (12) implies that $F_A \to 0$. □

2.3 Flat $\mathbb{R}^4$

It is easy to see that if $\eta = (\eta^1, \eta^2, \eta^3)$ is the standard left-invariant coframe on $S^3 = \mathrm{SU}(2)$ with $d\eta^1 = 2\eta^2 \wedge \eta^3$ etc. and $\eta^1 \wedge \eta^2 \wedge \eta^3 > 0$, we have a solution $e(t) = t\eta$ to Eq. (11) with corresponding SU(2) triple σ given by $\sigma_1 = tdt \wedge \eta^1 + t^2\eta^2 \wedge \eta^3$ etc. If we take $P = \mathrm{SU}(2) \times \mathbb{R}^4$, we can view the connections $A(t)$ on $Q = \mathrm{SU}(2) \times S^3$ as triples of 1-forms on Q.

The simplest case is when $A(t) = a(t)e(t) = at\eta$, so $F_A(t) = at(1 + at)d\eta$ and thus Eq. (12) is equivalent to

$$(at)' = -\frac{2at(1 + at)}{t} \tag{13}$$

which has solutions

$$a(t) = -\frac{k}{t(t^2 + k)},$$

for $k \in \mathbb{R}$. For non-trivial solutions defined on all of $\mathbb{R}^4$ we take $k > 0$. Then the corresponding SU(2)-instantons have curvature

$$\Omega = \frac{2k}{(t^2 + k)^2}(dt \wedge e^1 - e^{23}, dt \wedge e^2 - e^{31}, dt \wedge e^3 - e^{12}).$$

Taking $k = 1$ gives the basic instanton over $\mathbb{R}^4$. Notice that, indeed, the connection is asymptotic at infinity to a flat connection over S^3 as predicted by Corollary 2.3.

2.4 Eguchi-Hanson

In Sect. 2.3 we have somewhat implicitly used the fact that $\mathbb{R}^4$ is the completion of $\mathbb{R}_+ \times S^3$. In this case, it is evident that the flat hyperKähler structure and (basic) instanton are defined on the whole of $\mathbb{R}^4$ and P, respectively. In less elementary cases, more care needs to be taken if we want to make sure our structures are defined everywhere.

In order to illustrate this, let us look at the Eguchi-Hanson metric as derived in [7]. To this end, we consider a basis $\{\eta^j\}$ for $\mathfrak{so}(3)$ with $d\eta^1 = \eta^{23}$ etc. and $\eta^1 \wedge \eta^2 \wedge \eta^3 > 0$. We then make the ansatz $dt = f^{-1}(r)dr$, $e^1(r) = rf^{-1}(r)\eta^1$, $e^2(r) = f(r)\eta^2$ and $e^3(r) = f(r)\eta^3$. Given this, Eq. (10) is equivalent to the ODE

$$\frac{\partial(f^2)}{\partial r} = rf^{-2},$$

which (up to sign) has the solution $f(r) = (k + r^2)^{1/4}$, where $k \in \mathbb{R}$. To get the Eguchi-Hanson metric on T^*S^2, we should take $k > 0$. Taking $k = 0$ gives the flat metric on $\mathbb{R}^4/\{\pm 1\}$, and $k < 0$ leads to an incomplete metric.

In this Eguchi-Hanson space the principal orbits are SO(3) and the singular orbit is $S^2 = \mathrm{SO}(3)/\,\mathrm{SO}(2)$. To understand the behaviour of the metric near the singular orbit, we consider the vector bundle

$$\mathbb{V} = \mathrm{SO}(3) \times_{\mathrm{SO}(2)} V;$$

here the fibres V correspond to the standard representation of SO(2). We shall write $T = \langle \eta^2, \eta^3 \rangle$ for the $\mathrm{Ad}_{\mathrm{SO}(2)}$-invariant complement $\mathfrak{so}(2)^\perp \subset \mathfrak{so}(3)$. The SO(3)-invariant forms on $\mathbb{V}$ are the elements of $(\Lambda^* T)^{\mathrm{SO}(2)}$ together with "words" whose syllables come about by contracting "letters" (using the inner product an volume form on $\mathbb{R}^2$)

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} da_1 + a_2\eta^1 \\ da_2 - a_1\eta^1 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \eta^2 \\ \eta^3 \end{pmatrix},$$

where a_1, a_2 denote fibre coordinates, and $\boldsymbol{b}$ is the covariant derivative of $\boldsymbol{a}$. For example, the following four 2-forms are SO(3)-invariant:

$$\Sigma(\boldsymbol{b},\boldsymbol{b}) = -b_1b_2, \quad \Sigma(\boldsymbol{c},\boldsymbol{c}) = c_1c_2,$$
$$\boldsymbol{bc} = b_1c_1 + b_2c_2, \quad \Sigma(\boldsymbol{b},\boldsymbol{c}) = b_1c_2 - b_2c_1.$$

The map $\Psi\colon \mathrm{SO}(3) \times \mathbb{R} \to \mathrm{SO}(3) \times V$ given by $\Psi(g, r) = (g, (r, 0))$ induces a map $\mathrm{SO}(3) \times \mathbb{R} \to \mathbb{V}$ that we can use to express the 2-forms σ_j of Eq. (10) in terms of words from the "dictionary". By noting that $\Psi^*(\eta^j) = \eta^j$ for $j = 2, 3$ and

$\Psi^*(b_1) = dr$, $\Psi^*(b_2) = -r\eta^1$, we get that

$$\sigma_1 = f^{-2}\,\Sigma(\boldsymbol{b},\boldsymbol{b}) + f^2\,\Sigma(\boldsymbol{c},\boldsymbol{c}), \quad \sigma_2 = \Sigma(\boldsymbol{b},\boldsymbol{c}), \quad \sigma_3 = \boldsymbol{bc}.$$

For $k > 0$, the fact that these forms extend to the zero section follows immediately by observing that the coefficient functions are smooth even functions of the distance from the zero section of $\mathbb{V}$.

More generally, [6, Theorem 4.1] gives that any SO(3)-invariant 2-form on $\mathbb{V}\setminus S^2$ can be expressed in terms of

$$\begin{aligned} &k_1\boldsymbol{bc} + k_2\Sigma(\boldsymbol{b},\boldsymbol{b}) + k_3\Sigma(\boldsymbol{b},\boldsymbol{c}) + k_4\Sigma(\boldsymbol{c},\boldsymbol{c}) \\ &\quad + k_5\boldsymbol{ab}\,\boldsymbol{ac} + k_6\Sigma(\boldsymbol{a},\boldsymbol{b})\,\boldsymbol{ac}. \end{aligned} \tag{14}$$

Such a 2-form extends smoothly to the zero section if and only if the coefficient functions k_i are smooth even functions of the radial coordinate $r = \sqrt{\boldsymbol{aa}}$; this follows by applying the arguments of [10, Lemma 1.1] and by observing that the basis of Eq. (14) is adapted to the filtration of SO(2)-equivariant homogeneous polynomials $V \supset S^2 \to \Lambda^2(T \oplus V)$.

Let us consider SO(3)-invariant instantons on the natural circle bundle over the Eguchi-Hanson space whose total space is SO(3) $\times\, V$. Away from the zero section we can describe our connection in terms of a potential given by

$$A(r) = p(r)e^1.$$

It follows that

$$A' = f\left(\frac{\partial p}{\partial r} + \frac{f}{r}\frac{\partial(rf^{-1})}{\partial r}p\right)e^1$$

and that the curvature 2-form F_A, on each hypersurface, is given by $F_A = rf^{-3}pe^{23}$. In particular, we have $*_tF_A = rf^{-3}pe^1$.

Altogether Eq. (12) therefore amounts to the following ODE

$$\frac{\partial p}{\partial r} = -\left(\frac{r}{f^4} + \frac{f}{r}\frac{\partial(rf^{-1})}{\partial r}\right)p.$$

If, for concreteness, we take $f(r) = (1+r^2)^{1/4}$, then this equation has the solution

$$p(r) = \frac{c}{r(1+r^2)^{1/4}},$$

for $c \in \mathbb{R}$. Computing the associated curvature 2-form we find

$$\Omega = -\frac{c}{1+r^2}\left(dt \wedge e^1 - e^{23}\right) = -\frac{c}{1+\boldsymbol{aa}}\left(f^{-2}\Sigma(\boldsymbol{b},\boldsymbol{b}) - f^2\Sigma(\boldsymbol{c},\boldsymbol{c})\right),$$

showing that our solution is asymptotically flat.

By generalising our computations slightly, we find that the above instanton is unique in the following sense:

Proposition 2.4 *There is a unique* SO(3)*-invariant* SU(2)*-instanton on the natural circle bundle over the Eguchi-Hanson space whose curvature at the zero section restricts to that of the canonical connection* $SO(3) \to S^2$.

As the distance from the zero section increases, this connection approaches a flat connection on a circle bundle over SO(3).

Remark 2.5 In light of Proposition 2.4, it is tempting to think of the instanton evolution equations as a singular initial value problem, prescribing initial data at the singular orbit as in [10]. This approach, however, still requires knowledge about the explicit solution: when we express our connection (or curvature) in terms of a basis adapted to the filtration of equivariant homogeneous polynomials, we still need to verify that the coefficient functions are smooth even functions of the distance from the zero section.

2.5 Taub-NUT

So far we have only considered left-invariant coframes on SU(2). If we instead view a coframe as a 1-form taking values in the imaginary quaternions $e = ie^1 + je^2 + ke^3$ and suppose that $e = q\epsilon q^{-1}$ where ϵ is left-invariant, then if $\eta = i\eta^1 + j\eta^2 + k\eta^3$ is the standard left-invariant coframe on SU(2), as in Sect. 2.3, we see that

$$q^{-1}(de)q = i(d\epsilon^1 - 2\epsilon^2 \wedge \eta^3 + 2\epsilon^3 \wedge \eta^2) + j(d\epsilon^2 - 2\epsilon^3 \wedge \eta^1 + 2\epsilon^1 \wedge \eta^3) \\ + k(d\epsilon^3 - 2\epsilon^1 \wedge \eta^2 + 2\epsilon^2 \wedge \eta^1).$$

As is well-known (see e.g. [1]), if we take $\epsilon = if_1\eta^1 + jf_2\eta^2 + kf_3\eta^3$ then Eq. (11) is equivalent to $(f_2f_3)' = 2(f_1 - f_2 - f_3)$ etc. Making the ansatz $dt = -\frac{1}{2}(r + m)f^{-1}(r)dr$, $f_1 = 2m(r+m)^{-1}f(r)$ and $f_2 = f_3 = f(r)$ for a constant $m \geqslant 0$ and function f quickly yields the ODE

$$\frac{\partial(f^2)}{\partial r} = 2r.$$

The solution (up to sign) defined for $r \geqslant m$ is $f(r) = (r^2 - m^2)^{\frac{1}{2}}$, which gives the so-called Taub-NUT metric (with mass m) defined on $\mathbb{R}^4$. The unit coframe for each r is given by $q\epsilon q^{-1}$ where

$$\epsilon = i\epsilon^1 + j\epsilon^2 + k\epsilon^3 = 2m\left(\frac{r-m}{r+m}\right)^{\frac{1}{2}} i\eta^1 + (r^2 - m^2)^{\frac{1}{2}}(j\eta^2 + k\eta^3). \tag{15}$$

We can now study instantons on say the trivial SU(2)-bundle over Taub-NUT. The natural family of connections on S^3 to consider is

$$A(r) = q(a(r)i\eta^1 + b(r)j\eta^2 + b(r)k\eta^3)q^{-1}.$$

The curvature is then given by

$$q^{-1}F_A q = 2i(a - 2b + b^2)\eta^2 \wedge \eta^3 + 2ja(b-1)\eta^3 \wedge \eta^1 + 2ka(b-1)\eta^1 \wedge \eta^2.$$

We readily find that the instanton evolution (12) is equivalent to

$$\frac{\partial a}{\partial r} = \frac{2m(a - 2b + b^2)}{r^2 - m^2} \quad \text{and} \quad \frac{\partial b}{\partial r} = \frac{a(r+m)(b-1)}{2m(r-m)}.$$

There is an obvious solution with $b = 1$ and, for a constant c,

$$a = 1 + c\frac{r-m}{r+m}.$$

Using the notation of Eq. (15) we obtain the following.

Proposition 2.6 *The connection on the Taub-NUT space with mass m given by*

$$\omega = q\left(\frac{1}{2m}\left(\frac{r+m}{r-m}\right)^{\frac{1}{2}}\left(1 + c\frac{r-m}{r+m}\right)i\epsilon^1 + \frac{1}{(r^2-m^2)^{\frac{1}{2}}}(j\epsilon^2 + k\epsilon^3)\right)q^{-1}$$

for a constant c is an SU(2)*-instanton. The connection blows up at the "nut" $r = m$ and the curvature is*

$$\begin{aligned}\Omega &= \frac{c}{(r+m)^2}qiq^{-1}\left(\left(\frac{r+m}{r-m}\right)^{\frac{1}{2}} dr \wedge \epsilon^1 + 2\epsilon^2 \wedge \epsilon^3\right)\\ &= -\frac{2c}{(r+m)^2}qiq^{-1}\left(dt \wedge \epsilon^1 - \epsilon^2 \wedge \epsilon^3\right),\end{aligned}$$

which shows that ω is asymptotic to the flat connection on S^3 as $r \to \infty$.

Notice that we just get the flat connection on Taub-NUT when $c = 0$.

Remark 2.7 We can perform the same study for the Atiyah-Hitchin metric [1], where f_1, f_2, f_3 are distinct, obtaining ODEs describing SU(2)-instantons. The analysis of these ODEs is more involved, so we do not pursue this here.

3 The G_2 Case

3.1 G_2-Structures

In this setting we consider an oriented 6-manifold N equipped with a one-parameter family of SU(3)-structures

$$I \ni t \mapsto (\sigma(t), \gamma(t));$$

for each t, the pair $(\sigma(t), \gamma(t)) \in \Omega^2(N) \times \Omega^3(N)$ determines a reduction of the principal frame bundle to an SU(3)-subbundle. Recall that for each t the SU(3)-structure determines $\hat{\gamma}(t) \in \Omega^3(N)$ such that $\gamma + i\hat{\gamma}$ is a nowhere vanishing $(3,0)$-form on N. From this family, we can build a G_2-structure on the product $M = I \times N$ by setting

$$\begin{cases} \varphi = \sigma \wedge dt + \gamma \\ *\varphi = \hat{\gamma} \wedge dt + \frac{1}{2}\sigma^2. \end{cases} \tag{16}$$

The torsion-free condition Eq. (3) then amounts to requiring that each SU(3)-structure $(\sigma(t), \gamma(t))$ is half-flat, meaning

$$d\sigma^2 = 0 = d\gamma, \tag{17}$$

and that the family satisfies "*Hitchin's flow equations*" [20]:

$$\frac{\partial \gamma}{\partial t} = d\sigma, \quad \frac{\partial \sigma^2}{\partial t} = -2d\hat{\gamma}. \tag{18}$$

As the condition Eq. (17) of being half-flat is preserved by Eq. (18), we can regard these equations as a way of evolving an initial half-flat SU(3)-structure on N to construct a metric with holonomy contained in G_2. Again, the "flow" terminology is somewhat specious given the system's lack of parabolicity and the results in [3].

3.2 G_2-Instantons

Given a G_2-manifold $M = I \times N$ as in Eq. (16), we can rewrite the G_2-instanton condition on a connection ω on the pullback of a bundle Q on N as follows:

Lemma 3.1 *In terms of t-dependent data on N, the* G_2*-instanton equation* (4) *is equivalent to the condition*

$$\begin{cases} A' = *_t(F_A \wedge \gamma) \\ \sigma \wedge F_A + *_t F_A = \gamma \wedge *_t(F_A \wedge \gamma). \end{cases} \tag{19}$$

The alternative condition (5) *can be rephrased as*

$$\begin{cases} A' \wedge \sigma^2 = 2F_A \wedge \hat{\gamma} \\ F_A \wedge \sigma^2 = 0. \end{cases} \tag{20}$$

Proof Using Lemma 1.5, the left hand side of Eq. (4) can be written:

$$(\sigma \wedge dt + \gamma) \wedge (dt \wedge A' + F_A) = dt \wedge (\sigma \wedge F_A - \gamma \wedge A') + F_A \wedge \gamma.$$

The right hand side of Eq. (4) reads

$$-{*}\Omega = -{*}(dt \wedge A') - {*}(F_A).$$

This tells us that

$$-{*_t}A' = F_A \wedge \gamma \quad \text{and} \quad -{*_t}(F_A) = \sigma \wedge F_A - \gamma \wedge A'.$$

Clearly, these two expressions are equivalent to Eq. (19). Similarly,

$$\begin{aligned} {*}\varphi \wedge \Omega &= (\hat{\gamma} \wedge dt + \tfrac{1}{2}\sigma^2) \wedge (dt \wedge A' + F_A) \\ &= dt \wedge (\tfrac{1}{2}A' \wedge \sigma^2 - F_A \wedge \hat{\gamma}) + F_A \wedge \tfrac{1}{2}\sigma^2, \end{aligned}$$

from which Eq. (20) follows. □

It is then a question of straightforward computations to obtain:

Proposition 3.7 *The evolution equations for* G_2*-instantons may be phrased as:*

$$\begin{cases} A' = {*_t}(F_A \wedge \gamma) \\ F_A(t_0) \wedge \sigma^2(t_0) = 0, \end{cases} \tag{21}$$

for some initial $t_0 \in I$.

Proof Let W be the 6-dimensional irreducible representation of SU(3). Elementary computations show that the equivariant maps $\Lambda^2 W \to \Lambda^5 W$ given by

$$\beta \mapsto \sigma^2 \wedge {*}(\beta \wedge \gamma) \quad \text{and} \quad \beta \mapsto 2\beta \wedge \hat{\gamma}$$

coincide. It therefore follows that the evolution of A is completely determined by the equation for A' in Eq. (19).

Next, we show that if a 2-form β has $\beta \wedge \sigma^2 = 0$ then it automatically satisfies the constraint of Eq. (19), that is,

$$\sigma \wedge \beta + {*}\beta = \gamma \wedge {*}(\beta \wedge \gamma).$$

This is because the two equivariant maps $\Lambda^2 W \to \Lambda^4 W$ given by

$$\beta \mapsto \sigma \wedge \beta + *\beta \quad \text{and} \quad \beta \mapsto \gamma \wedge *(\beta \wedge \gamma)$$

agree on the two irreducible submodules orthogonal to $\langle \sigma \rangle$ in $\Lambda^2 W$.

In order to prove the proposition, we need to show that the 6-form $F_A \wedge \sigma^2$ is preserved as A evolves. This assertion follows by

$$\begin{aligned}(F_A \wedge \sigma^2)' &= F_A' \wedge \sigma^2 + F_A \wedge (\sigma^2)' \\ &= (dA' + [A, A']) \wedge \sigma^2 + F_A \wedge (-2d\hat{\gamma}) \\ &= 2dF_A \wedge \hat{\gamma} + 2F_A \wedge d\hat{\gamma} + 2[A, F_A] \wedge \hat{\gamma} - 2F_A \wedge d\hat{\gamma} \\ &= 2(dF_A + [A, F_A]) \wedge \hat{\gamma} = 2DF_A \wedge \hat{\gamma} = 0,\end{aligned}$$

where the last equality follows from the Bianchi identity. □

Remark 3.3 The condition $F_A(t_0) \wedge \sigma^2(t_0) = 0$ in Proposition 3.7 is not very restrictive: it simply means that the 2-form part of $F_A(t_0)$ is not allowed to have a component (pointwise) proportional to $\sigma(t_0)$.

Note that the evolution equation for A is in Cauchy form, which means that we immediately have:

Corollary 3.4 *Given real analytical initial data A on N satisfying the condition $F_A \wedge \sigma^2 = 0$, the* G_2*-instanton evolution equations* (21) *have a unique solution over an open set in $I \times N$.*

From the form of Eq. (21), we have the following:

Corollary 3.5 *If a* G_2*-instanton ω is asymptotic to a connection on $Q \to N$ at an endpoint of I, then the limiting connection is an* $\mathrm{SU}(3)$*-instanton.*

Proof As we must have $A' \to 0$, we have

$$F_A \wedge \gamma \to 0 \quad \text{and} \quad F_A \wedge \sigma^2 = 0,$$

which means that A tends to a connection whose curvature has values in $\mathfrak{su}(3) \subset \mathfrak{so}(6) \cong \Lambda^2$, which is an $\mathrm{SU}(3)$-instanton as in Eq. (8). □

3.3 Flat $\mathbb{R}^7$

To construct the flat metric on $\mathbb{R}^7$, which corresponds to a trivial torsion-free G_2-structure, we need a half-flat $\mathrm{SU}(3)$-structure (σ, γ) on S^6: this is provided by the standard nearly Kähler structure on S^6 which satisfies

$$d\sigma = 3\gamma \quad \text{and} \quad d\hat{\gamma} = -2\sigma^2.$$

The evolution equations (18) starting at a nearly Kähler structure always lead to the solution:

$$\varphi = t^2\sigma \wedge dt + t^3\gamma \quad \text{and} \quad *\varphi = t^3\hat{\gamma} \wedge dt + \tfrac{1}{2}t^4\sigma^2$$

for $t > 0$, yielding the cone metric over the nearly Kähler manifold. Using this description of $\mathbb{R}^7$ together with an appropriate ansatz for A, one can use the evolution equation (21) to give a higher dimensional generalisation of the basic instanton on $\mathbb{R}^4$. Indeed, we can reconstruct the instanton on the trivial bundle $\mathrm{G}_2 \times \mathbb{R}^7$ described in [18, 21].

There are other ways to think of $\mathbb{R}^7$ as a family of hypersurfaces. For example, we could think of $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4$ and take hypersurfaces $\mathbb{R}^3 \times S^3$. If $(x^1, x^2, x^3) \in \mathbb{R}^3$ and η^1, η^2, η^3 is our standard left-invariant coframe on S^3, our ansatz for the evolving $\mathrm{SU}(3)$-structures on $\mathbb{R}^3 \times S^3$ would be

$$\begin{aligned}\sigma &= f(t)(dx^1 \wedge \eta^1 + dx^2 \wedge \eta^2 + dx^3 \wedge \eta^3),\\ \gamma &= dx^{123} - f(t)^2(dx^1 \wedge \eta^2 \wedge \eta^3 + dx^2 \wedge \eta^3 \wedge \eta^1 + dx^3 \wedge \eta^1 \wedge \eta^2).\end{aligned}$$

The evolution equations (18) quickly yield a solution $f(t) = t$ for $\varphi = \sigma \wedge dt + \gamma$ to be torsion-free. We see that this is equivalent to allowing the coframe on S^3 to evolve as $e(t) = t\eta$ just as in the flat $\mathbb{R}^4$ case, as expected. If we take evolving connections $A(t) = a(t)t\eta$ on $\mathbb{R}^3 \times S^3$, the G_2-instanton evolution equations give Eq. (13) so that ω is the pullback of the basic instanton on $\mathbb{R}^4$, which is clear in light of Remark 1.2.

Alternatively, we could view $\mathbb{R}^7$ in terms of hypersurfaces $S^2 \times \mathbb{R}^4 \subseteq \mathbb{R}^3 \times \mathbb{R}^4$, and perform the same analysis to get the flat metric by evolving the volume form on S^2 by the obvious scaling. In this case, the pullback of an evolving connection on S^2 to $S^2 \times \mathbb{R}^4$ will define a G_2-instanton if and only if the corresponding connection on $\mathbb{R}^3$ is flat. This situation is equivalent to considering $\mathbb{R}^7 = \Lambda^2_+\mathbb{R}^4$ and having the sphere subbundles as hypersurfaces. We shall see a related construction in the next section which yields non-trivial results.

3.4 Self-dual 2-Forms Over S^4

Following [4, 6], we first describe a space that in a sense is related to that of Eguchi-Hanson, described in Sect. 2.1. In this case, we are considering a manifold with a cohomogeneity one action of $\mathrm{SO}(5)$. The principal stabiliser is $\mathrm{U}(2) \subset \mathrm{SO}(4)$, and the singular stabiliser is the whole subgroup $\mathrm{SO}(4)$.

We shall need a suitable local coframe on $\mathbb{CP}(3) = \mathrm{SO}(5)/\mathrm{U}(2)$. Let us write $\mathfrak{so}(5) = \langle v^1, v^2, v^3, v^4\rangle \oplus \langle \eta^1, \eta^2, \eta^2\rangle \oplus \langle \gamma^1, \gamma^2, \gamma^3\rangle$ in terms of the explicit identification:

$$\begin{pmatrix} 0 & -\frac{1}{2}(\eta^1+\gamma^1) & -\frac{1}{2}(\eta^2+\gamma^2) & -\frac{1}{2}(\eta^3+\gamma^3) & -v^1 \\ \frac{1}{2}(\eta^1+\gamma^1) & 0 & -\frac{1}{2}(\eta^3-\gamma^3) & \frac{1}{2}(\eta^2-\gamma^2) & -v^2 \\ \frac{1}{2}(\eta^2+\gamma^2) & \frac{1}{2}(\eta^3-\gamma^3) & 0 & -\frac{1}{2}(\eta^1-\gamma^1) & -v^3 \\ \frac{1}{2}(\eta^3+\gamma^3) & -\frac{1}{2}(\eta^2-\gamma^2) & \frac{1}{2}(\eta^1-\gamma^1) & 0 & -v^4 \\ v^1 & v^2 & v^3 & v^4 & 0 \end{pmatrix}. \tag{22}$$

Choosing $\mathfrak{u}(2) = \langle \eta^1, \gamma^1, \gamma^2, \gamma^3 \rangle$, we have d acting as:

$$\begin{aligned} d\eta^1 &= -\eta^{23} - (v^{12} + v^{34}), \\ d\eta^2 = -\eta^{31} - (v^{13} + v^{42}), d\eta^3 &= -\eta^{12} - (v^{14} + v^{23}), \\ d(v^{12} + v^{34}) &= -(v^{14} + v^{23})\eta^2 + (v^{13} + v^{42})\eta^3, \\ d(v^{13} + v^{42}) &= (v^{14} + v^{23})\eta^1 - (v^{12} + v^{34})\eta^3, \\ d(v^{14} + v^{23}) &= -(v^{13} + v^{42})\eta^1 + (v^{12} + v^{34})\eta^2. \end{aligned}$$

We now look for an ansatz with $dt = f^{-1}(r)dr$, $e^1(r) = rf^{-1}\eta^2$, $e^2(r) = rf^{-1}\eta^3$ and $e^3(r) = fv^1$, $e^4(r) = -fv^2$, $e^5(r) = fv^4$, $e^6(r) = fv^3$ so that

$$\begin{aligned} \sigma(r) &= (rf^{-1})^2\eta^{23} - f^2(v^{12} + v^{34}), \\ \gamma(r) &= rf((v^{14} + v^{23})\eta^2 - (v^{13} + v^{42})\eta^3), \\ \hat{\gamma}(r) &= rf((v^{13} + v^{42})\eta^2 + (v^{14} + v^{23})\eta^3). \end{aligned}$$

In this case, Eq. (18) reduces to the following ODE

$$\frac{\partial f}{\partial r} = rf^{-3},$$

which has the solution $f(r) = 2^{1/4}(k + r^2)^{1/4}$, for some constant $k \in \mathbb{R}$. To get the Bryant-Salamon metric on $\Lambda^2_+ S^4$ we must take $k > 0$.

As for the Eguchi-Hanson case, we next consider a tubular neighbourhood of the singular orbit. This is modelled on the vector bundle

$$\mathbb{V} = \mathrm{SO}(5) \times_{\mathrm{SO}(4)} V$$

with fibres $V \cong \langle v^{12} + v^{34}, v^{13} + v^{42}, v^{14} + v^{23} \rangle = \Lambda^2_+(T)$, $T \cong \mathfrak{so}(5)/\,\mathfrak{so}(4)$.

In this case, we can form the words

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} da_1 + a_3\eta^2 - a_2\eta^3 \\ da_2 - a_3\eta^1 + a_1\eta^3 \\ da_3 + a_2\eta^1 - a_1\eta^2 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} v^{12} + v^{34} \\ v^{13} + v^{42} \\ v^{14} + v^{23} \end{pmatrix}.$$

The map Ψ: $\mathrm{SO}(5) \times \mathbb{R} \to \mathrm{SO}(5) \times V$, given by $\Psi(g, r) = (g, (r, 0, 0))$, induces a map $\mathbb{CP}(3) \times \mathbb{R} \to \mathbb{V}$ that we can use to express the 3-form φ and 4-form $*\varphi$ in terms of words from the dictionary. For this, we note that $\Psi^*(b_1) = dr$, $\Psi^*(b_2) = r\eta^3$, $\Psi^*(b_3) = -r\eta^2$. It then follows that, up to suitable rescaling of invariant forms (similar to the Eguchi-Hanson case, these are obtained via contractions coming from the inner product and volume form on $\mathbb{R}^3$), we have

$$\varphi = f^{-3}\boldsymbol{bbb} + f\boldsymbol{bc} \quad \text{and} \quad *\varphi = \boldsymbol{bcc} + f^4\boldsymbol{cc}.$$

As before, smoothness of φ and $*\varphi$ follows from the fact that the coefficient functions are smooth even functions of the distance from the zero section.

More generally, we may ask when an SO(5)-invariant 3- or 4-form on $\mathbb{V} \setminus S^4$ extends smoothly to the zero section. By [6, Theorem 4.6] the invariant 3- and 4-forms can be expressed as

$$p_1\boldsymbol{bc} + p_2\boldsymbol{bbb} + p_3\boldsymbol{abc} + p_4\boldsymbol{ab}\,\boldsymbol{ac}$$

and

$$q_1\boldsymbol{cc} + q_2\boldsymbol{bbc} + q_3\boldsymbol{ab}\,\boldsymbol{bc} + q_4\boldsymbol{ab}\,\boldsymbol{abc},$$

respectively. Smoothness then amounts to the functions p_i and q_i being smooth even functions of r; again we are using that the above basis elements are adapted to the filtration of SO(4)-equivariant homogeneous polynomials $V \supset S^2 \to \Lambda^p(T \oplus V)$, $p = 3, 4$.

Remark 3.6 In addition to the above, [4] provides a similar construction of a complete G_2-holonomy metric on $\Lambda^2_-\mathbb{CP}(2)$; whilst for S^4 the construction works with both bundles $\Lambda^2_\pm S^4$, this is not the case for $\mathbb{CP}(2)$ in the sense that one needs to take $\overline{\mathbb{CP}(2)}$ to use Λ^2_+.

Let us consider the $\mathrm{SO}(3)_-$-bundle over S^4, $\mathrm{SO}(5)/\mathrm{SO}(3)_+ \to S^4$; the following G_2-instanton on this space was also considered in [24]. If we regard the canonical connection of this bundle as an $\mathfrak{so}(3)_-$-valued 1 form on $\mathrm{SO}(5) \times_{\mathrm{SO}(3)_+} V$, given by

$$A(t) = \begin{pmatrix} 0 & -\gamma^1 & -\gamma^2 \\ \gamma^1 & 0 & \gamma^3 \\ \gamma^2 & -\gamma^3 & 0, \end{pmatrix},$$

it is SO(4)-invariant and has $A'(t) = 0$. Then, using $d\gamma^1 = \gamma^{23} - (v^{12} - v^{34})$ etc., we find that

$$F_A(t) = \begin{pmatrix} 0 & v^{12} - v^{34} & v^{13} - v^{42} \\ -v^{12} + v^{34} & 0 & -v^{14} + v^{23} \\ -v^{13} + v^{42} & v^{14} - v^{23} & 0 \end{pmatrix}.$$

Since $F_A \wedge \gamma = 0$, we see that $A(t)$ is a static solution the instanton evolution equations (indeed, it is the "lift" of an SU(3)-instanton on $\mathbb{CP}(3)$).

In terms of an r-dependent frame, we can write the associated curvature 2-form as:

$$\Omega(r) = f^{-2} \begin{pmatrix} 0 & -e^{34} + e^{56} & -e^{36} + e^{45} \\ e^{34} - e^{56} & 0 & -e^{35} + e^{46} \\ e^{36} - e^{45} & e^{35} - e^{46} & 0 \end{pmatrix};$$

in particular, we have $\|\Omega(r)\|_{g(r)} \to 0$ as $r \to \infty$.

We should mention that [24] includes another (non static) example of a G_2-instanton on an SU(2)-bundle over $\Lambda^2_+ S^4$; of course this can also be reproduced using our instanton evolution equation (21).

3.5 The Spinor Bundle Over S^3

We want to construct G_2-metrics on cohomogeneity one spaces with principal orbits $S^3 \times S^3 \cong \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ and singular orbit S^3, corresponding to stabiliser Sp(1) with Lie algebra given by the diagonal $\mathfrak{sp}(1) \subset \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_-$, where

$$\mathfrak{sp}(1) = \langle \eta^1_+ + \eta^1_-, \eta^2_+ + \eta^2_-, \eta^3_+ + \eta^3_- \rangle,$$
$$\mathfrak{sp}(1)^\perp = \langle \eta^1_+ - \eta^1_-, \eta^2_+ - \eta^2_-, \eta^3_+ - \eta^3_- \rangle$$

and $d\eta^1_\pm = 2\eta^{23}_\pm = 2\eta^2_\pm \wedge \eta^3_\pm$ etc.

The first known complete G_2-metric arising in this context was constructed by Bryant and Salamon in [4]. To obtain their example, we must take $dt = -\sqrt{2}g\,dr$, $e^{2j-1} = f(\eta^j_+ - \eta^j_-)/\sqrt{2}$ and $e^{2j} = rg(\eta^j_+ + \eta^j_-)/\sqrt{2}$, for $j = 1, 2, 3$. The corresponding SU(3)-structure is then given by

$$\sigma = rfg\left(\eta^1_+\eta^1_- + \eta^2_+\eta^2_- + \eta^3_+\eta^3_-\right),$$

$$\gamma = \frac{f(f^2-3r^2g^2)}{2\sqrt{2}}\left(\eta^{123}_+ - \eta^{123}_-\right) - \frac{f(f^2+r^2g^2)}{4\sqrt{2}} d\left(\eta^1_+\eta^1_- + \eta^2_+\eta^2_- + \eta^3_+\eta^3_-\right),$$

$$\hat{\gamma} = \frac{rg(3f^2-r^2g^2)}{2\sqrt{2}}\left(\eta^{123}_+ + \eta^{123}_-\right) - \frac{rg(f^2+r^2g^2)}{2\sqrt{2}}\Big(\eta^1_+\eta^2_+\eta^3_- + \eta^3_+\eta^1_+\eta^2_- + \eta^1_+\eta^2_-\eta^3_- + \eta^2_+\eta^3_+\eta^1_- + \eta^2_+\eta^3_-\eta^1_- + \eta^3_+\eta^1_-\eta^2_-\Big).$$

From the above, we see that Eq. (18) is equivalent to the ODEs

$$\frac{\partial f}{\partial r} = 2rg^2f^{-1}, \quad \frac{\partial g}{\partial r} = -rg^3f^{-2},$$

with corresponding solution:

$$f(r) = 3^{1/3}(k_1^2r^2+k_2)^{1/3}, \quad g(r) = k_1 3^{-1/6}(k_1^2r^2+k_2)^{-1/6}.$$

Completeness requires that $k_2 > 0$. For concreteness we shall take $k_1 = 1 = k_2$ so that $f = g^{-2}$.

In order to understand the geometry near the singular orbit, we consider the vector bundle

$$\mathbb{V} = (\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-) \times_{\mathrm{Sp}(1)} V,$$

where the fibres $V \cong \mathbb{H}$ correspond to the standard representation of Sp(1) and $T = \frac{\mathfrak{sp}(1)_+\oplus\mathfrak{sp}(1)_-}{\mathfrak{sp}(1)} \cong \langle \eta^j_+ - \eta^j_-\rangle$.

As usual, we have the fibre coordinate letter $\boldsymbol{a}$ and its covariant derivative, the letter $\boldsymbol{b}$. Now, choose the map $\Psi\colon \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- \times \mathbb{R} \to \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- \times V$ given by $\Psi(g,r) = (g,(r,0,0,0))$. From this, we find that $\Psi^*(b_1) = dr$, $\Psi^*(b_2) = r(\eta^1_+ + \eta^1_-)/\sqrt{2}$, $\Psi^*(b_3) = r(\eta^2_+ + \eta^2_-)/\sqrt{2}$ and $\Psi^*(b_4) = r(\eta^3_+ + \eta^3_-)/\sqrt{2}$.

To describe the Bryant-Salamon 3-form in these terms, we need the two elements spanning $\Lambda^3(T\oplus V)^{\mathrm{Sp}(1)} = \Lambda^3(T) \oplus (T\otimes\Lambda^2V)^{\mathrm{Sp}(1)} = \langle \boldsymbol{v}\rangle \oplus \langle\Sigma_1\rangle$. Similarly, for the 4-form, we need the two non-trivial elements of $\Lambda^4(T\oplus V)^{\mathrm{Sp}(1)} = (\Lambda^2(T)\otimes\Lambda^2(V))^{\mathrm{Sp}(1)} \oplus \Lambda^4(V) = \langle\Sigma_2\rangle \oplus \langle \boldsymbol{bbbb}\rangle$. Using these forms, defined up to scaling, we can write:

$$\varphi = (1+\boldsymbol{aa})\boldsymbol{v} + \Sigma_1,$$

$$*\varphi = (1+\boldsymbol{aa})^{-2/3}\boldsymbol{bbbb} + (1+\boldsymbol{aa})^{1/3}\Sigma_2.$$

The coefficient functions are smooth even functions of the distance from the zero section, again implying that these forms extend smoothly to the zero section of $\mathbb{V}$.

In the above model, the principal stabiliser is trivial. As a consequence, the space of invariant p-forms is relatively large, so we shall not write down the general expressions for invariant 3- and 4-forms that extend smoothly to the zero section. It is worthwhile mentioning, however, that the high degree of flexibility leads to (more than) one other complete G_2-holonomy metric [2]; this metric has a different behaviour far away from the zero section: it is similar to the Taub-NUT metric in the sense that it is asymptotically locally conical.

We now address the construction of instantons on an Sp(1)-bundle over the Bryant-Salamon G_2-structure on the spinor bundle over S^3. Instantons on this space were also the subject of [5]. The construction is reminiscent of that of the Eguchi-Hanson space in Sect. 2.4. Motivated by the expression for the canonical connection on the natural Sp(1)-bundle $\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- \to S^3$ over the singular orbit, we consider the one-parameter family of connections specified by the potential

$$A(t) = p(t)\left(ie^2(t) + je^4(t) + ke^6(t)\right),$$

corresponding to a connection on the bundle $\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- \times V \to \mathbb{V}$.

Straightforward computations give that

$$\begin{aligned}
A'(t) &= -\frac{1}{\sqrt{2}g}\left(\frac{\partial p}{\partial r} + pr^{-1}g^{-1}\frac{\partial(rg)}{\partial r}\right)\left(ie^2 + je^4 + ke^6\right),\\
F_A &= i\sqrt{2}p\left(rg^5e^{35} + (\sqrt{2}p + r^{-1}g^{-1})e^{46}\right)\\
&\quad + j\sqrt{2}p\left(rg^5e^{51} + (\sqrt{2}p + r^{-1}g^{-1})e^{62}\right)\\
&\quad + k\sqrt{2}p\left(rg^5e^{13} + (\sqrt{2}p + r^{-1}g^{-1})e^{24}\right),\\
*_t(F_A \wedge \gamma) &= -\sqrt{2}p\left(\sqrt{2}p + r^{-1}g^{-1} - rg^5\right)\left(ie^2 + je^4 + ke^6\right).
\end{aligned}$$

The instanton evolution equation is then the following non-linear ODE of Bernoulli type:

$$\frac{\partial p}{\partial r} = 2\sqrt{2}gp^2 + \left(2r^{-1} - 2rg^6 - r^{-1}g^{-1}\frac{\partial(rg)}{\partial r}\right)p.$$

We find the following non-zero solution

$$p(r) = \frac{2r}{2c(1+r^2)^{1/6} - 3^{5/6}\sqrt{2}(1+r^2)^{5/6}},$$

which is a smooth function of $r \geqslant 0$, so long as we choose our constant $c < 3^{5/6}/\sqrt{2}$. Clearly $\lim_{r\to 0} A(r) = 0$.

For the curvature, we find that

$$\Omega(r) = 2p\left(\sqrt{2}gp + r^{-1} - rg^6\right)\left(idr \wedge e^2 + jdr \wedge e^4 + kdr \wedge e^6\right) + F_A(r)$$

which vanishes at the zero section and satisfies $\|\Omega(r)\|_{g(r)} \to 0$ as $r \to \infty$.

Slightly more generally, as in [5], we could consider the potential

$$A(t) = p(t)\left(ie^1(t) + je^3(t) + ke^5(t)\right) + q(t)\left(ie^2(t) + je^4(t) + ke^6(t)\right).$$

As in op. cit., we find that solutions that extend smoothly to the zero section necessarily have $q \equiv 0$.

Computations for the Brandhuber et al. G_2-metric on $\mathbb{V} \cong S^3 \times \mathbb{H}$ [2] show that there is a globally defined instanton on a circle bundle over this space.

4 The Spin(7) Case

4.1 Spin(7)-*Structures*

From a family of G_2-structures $I \ni\to \varphi(t)$ on a 7-manifold N, we can construct a Spin(7)-structure on the product $M = I \times N$ via

$$\Phi = \varphi \wedge dt + *\varphi. \tag{23}$$

In terms of Eq. (23), the condition (6) for the induced metric to have holonomy contained in Spin(7) amounts to requiring that $d *_t \varphi(t) = 0$ for each t (i.e. that $\varphi(t)$ is co-calibrated) and that the family satisfies the evolution equations [20]:

$$\frac{\partial(*\varphi)}{\partial t} = -d\varphi. \tag{24}$$

This preserves closedness of $*\varphi$ and therefore gives us a way of evolving an initial co-calibrated G_2-structure on N to give our torsion-free Spin(7)-structure. This is again sometimes referred to as "Hitchin's flow equation".

4.2 Spin(7)-*Instantons*

Given a Spin(7)-manifold $M = I \times N$ of the form (23), we can rephrase the instanton condition for a connection ω on the pullback of a bundle Q on N as follows:

Proposition 4.1 *In terms of t-dependent data on N, the* Spin(7)*-instanton condition* (7) *is given by:*

$$A' = *_t(F_A \wedge *_t\varphi). \tag{25}$$

Proof By Eq. (7), we have to compute $\Phi \wedge \Omega = -*\Omega$ in terms of the t-dependent data. We get two equations:

$$A' = *_t(F_A \wedge *_t\varphi) \quad \text{and} \quad *_t\varphi \wedge A' = *_tF_A + \varphi \wedge F_A.$$

If V denotes the irreducible 7-dimensional representation of G_2 then the following two equivariant maps $\Lambda^2 V \to \Lambda^5 V$ coincide:

$$\beta \mapsto *\varphi \wedge *(\beta \wedge *\varphi) \quad \text{and} \quad \beta \mapsto *\beta + \beta \wedge \varphi.$$

The assertion now follows. □

As Eq. (25) is in Cauchy form, we have:

Corollary 4.2 *Given real analytic initial data, Eq.* (25) *has a unique solution over an open subset of $I \times N$.*

From the form of Eq. (25), we see from Eq. (5) that:

Corollary 4.3 *If ω is asymptotic to a connection on $Q \to N$ at an endpoint of I, then the limiting connection is a* G_2*-instanton.*

4.3 Flat $\mathbb{R}^8$

To obtain the standard Spin(7)-structure on $\mathbb{R}^8$ defining the flat metric, we will evolve the standard G_2-structure on S^7 which satisfies $d\varphi = -4 * \varphi$ (a so-called "nearly parallel G_2-structure"). Equation (24) quickly yields that the evolving G_2-structures are $\varphi(t) = t^3\varphi$ so that the Spin(7)-structure is

$$\Phi = t^3\varphi \wedge dt + t^4 * \varphi$$

for $t > 0$. This is a conical solution which always occurs when the initial 7-dimensional hypersurface is endowed with a nearly parallel G_2-structure. Based on this description of $\mathbb{R}^8$ and a suitable ansatz for A, we can use Eq. (25) so as to obtain the "basic" Spin(7)-instanton on the trivial bundle $\mathrm{Spin}(7) \times \mathbb{R}^8$ that appeared in [11, 14, 21].

We could also view $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$ and take as hypersurfaces $S^3 \times \mathbb{R}^4$. If we let η^1, η^2, η^3 be the coframe on S^3 with $d\eta^1 = -2\eta^{23}$ etc. and let τ_1, τ_2, τ_3 be the standard hyperKähler triple on $\mathbb{R}^4$, we may take the ansatz

$$\varphi(t) = f(t)^3\eta^1 \wedge \eta^2 \wedge \eta^3 + f(t)(\tau_1 \wedge \eta^1 + \tau_2 \wedge \eta^2 + \tau_3 \wedge \eta^3)$$

for our G_2-structures on $S^3 \times \mathbb{R}^4$. Equation (24) yields $f(t) = -t$ as a solution, which is equivalent to choosing the evolving coframe $e(t) = -t\eta$ on S^3 as expected. Hence, if we write $\sigma_1 = tdt \wedge \eta^1 - t^2\eta^2 \wedge \eta^3$ etc., so that $(\sigma_1, \sigma_2, \sigma_3)$ is the standard hyperKähler triple on $\mathbb{R}^4$, then

$$\Phi = \tfrac{1}{6}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \sigma_1 \wedge \tau_1 + \sigma_2 \wedge \tau_2 + \sigma_3 \wedge \tau_3 - \tfrac{1}{6}(\tau_1^2 + \tau_2^2 + \tau_3^2)$$

is the solution, which is a re-writing of the standard Spin(7)-form. If we choose our evolving connections $A(t) = -a(t)t(i\eta^1 + j\eta^2 + k\eta^3)$ on $S^3 \times \mathbb{R}^4$, the Spin(7)-instanton evolution equation (25) will yield the pullback of the basic (SD) instanton on $\mathbb{R}^4$, as expected from Remarks 1.2 and 1.3. This situation is equivalent to considering $\mathbb{R}^8$ as the (negative) spinor bundle of $\mathbb{R}^4$ and taking the sphere subbundles as hypersurfaces. We will see a closely related construction in Sect. 4.4 which yields nontrivial results.

4.4 The Spinor Bundle Over S^4

Let us consider the 7-sphere, eventually leading to the Bryant-Salamon metric on the negative spinor bundle over S^4 [4]. Similarly to Eq. (22) we write $S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)_+$ using the explicit identification

$$\begin{pmatrix} 0 & -\gamma^1 & -\gamma^2 & -\gamma^3 & -v^1 & -v^2 & -v^3 & -v^4 \\ \gamma^1 & 0 & -\gamma^3 & \gamma^2 & v^2 & -v^1 & -v^4 & v^3 \\ \gamma^2 & \gamma^3 & 0 & -\gamma^1 & v^3 & v^4 & -v^1 & -v^2 \\ \gamma^3 & -\gamma^2 & \gamma^1 & 0 & v^4 & -v^3 & v^2 & -v^1 \\ v^1 & -v^2 & -v^3 & -v^4 & 0 & -\eta^1 & -\eta^2 & -\eta^3 \\ v^2 & v^1 & -v^4 & v^3 & \eta^1 & 0 & -\eta^3 & \eta^2 \\ v^3 & v^4 & v^1 & -v^2 & \eta^2 & \eta^3 & 0 & -\eta^1 \\ v^4 & -v^3 & v^2 & v^1 & \eta^3 & -\eta^2 & \eta^1 & 0 \end{pmatrix},$$

where $\mathfrak{sp}(1)_+ = \langle \eta^1, \eta^2, \eta^3 \rangle$. The associated structure equations imply that

$$d\gamma^1 = 2(-\gamma^{23} + v^{12} - v^{34}), \quad d\gamma^2 = 2(-\gamma^{31} + v^{13} - v^{42}),$$
$$d\gamma^3 = 2(-\gamma^{12} + v^{14} - v^{23}),$$
$$d(v^{12} - v^{34}) = -2\gamma^2(v^{14} - v^{23}) + 2\gamma^3(v^{13} - v^{42}),$$
$$d(v^{13} - v^{42}) = 2\gamma^1(v^{14} - v^{23}) - 2\gamma^3(v^{12} - v^{34}),$$
$$d(v^{14} - v^{23}) = -2\gamma^1(v^{13} - v^{42}) + 2\gamma^2(v^{12} - v^{34}),$$

and then

$$\begin{aligned} d\gamma^{123} &= 2\gamma^{23}(v^{12}-v^{34}) + 2\gamma^{31}(v^{13}+v^{24}) + 2\gamma^{12}(v^{14}-v^{23}),\\ d(\gamma^1(v^{12}-v^{34}) + \gamma^2(v^{13}+v^{24}) + \gamma^3(v^{14}-v^{23})) &= -12v^{1234}\\ &\quad +2\gamma^{23}(v^{12}-v^{34}) + 2\gamma^{12}(v^{14}-v^{23}) + 2\gamma^{31}(v^{13}-e^{42}). \end{aligned}$$

We now look for an ansatz with $dt = f^{1/2}dr$, $e^j = rf^{1/2}\gamma^j$, $j = 1,2,3$, and $e^4 = g^{1/2}v^1$, $e^5 = g^{1/2}v^2$, $e^6 = g^{1/2}e^3$, $e^7 = g^{1/2}v^4$ so that the associated G_2-structure reads

$$\begin{aligned} \varphi &= r^3f^{3/2}\gamma^{123} + rf^{1/2}g\left(\gamma^1(v^{12}-v^{34}) + \gamma^2(v^{13}-v^{42}) + \gamma^3(v^{14}-v^{23})\right),\\ *\varphi &= g^2v^{1234} - r^2fg\left(\gamma^{23}(v^{12}-v^{34}) + \gamma^{31}(v^{13}-v^{42}) + \gamma^{12}(v^{14}-v^{23})\right). \end{aligned}$$

Using the above computations, we can verify that φ is co-calibrated, i.e. $d{*\varphi} = 0$. The evolution equations (24) are

$$\frac{\partial g}{\partial r} = 6rf, \quad \frac{\partial f}{\partial r} = -4rf^2g^{-1},$$

which then have the solution

$$f(r) = (\tfrac{2}{5})^{2/5}\left(\frac{2r^2-k_1k_2}{k_1}\right)^{-2/5}, \quad g(r) = (\tfrac{5}{2})^{3/5}k_1\left(\frac{2r^2-k_1k_2}{k_1}\right)^{3/5}.$$

For concreteness, let us fix $k_1 = 1$ and $k_2 = -2$, giving

$$f(r) = 5^{-2/5}\left(1+r^2\right)^{-2/5}, \quad g(r) = 5^{3/5}\left(1+r^2\right)^{3/5}.$$

Note that for this choice of integration constants, we have $f(r)^3 = g(r)^{-2}$, and this will lead to a complete holonomy Spin(7)-metric.

Turning to the geometry near the singular orbit, we need to study the vector bundle

$$\mathbb{V} = \mathrm{Sp}(2) \times_{\mathrm{Sp}(1)_+\times\mathrm{Sp}(1)_-} V,$$

where the fibres $V = \mathbb{R}^4 \cong \mathbb{H}$ are the standard representation of $\mathrm{Sp}(1)_-$ (again acting on the right); in our conventions $\mathfrak{sp}(1)_- = \langle\gamma^1,\gamma^2,\gamma^3\rangle$. Obviously, the volume on $T \cong \langle v^1,v^2,v^3,v^4\rangle$ gives us an invariant 4-form $\boldsymbol{v}$ on $\mathbb{V}$, and relevant letters of our dictionary are the fibre coordinates $\boldsymbol{a}$ and its covariant derivative $\boldsymbol{b}$.

Using contraction via the volume form on $\mathbb{R}^4$, we get the invariant 4-form $\boldsymbol{bbbb}$. In addition, note that we have a map $\Sigma_1\colon V\otimes V \to \Sigma^2_-$ and a similar map $\widetilde{\Sigma}_1\colon T\otimes T \to \Sigma^2_-$. This means there is an invariant 4-form $\Sigma_1(\boldsymbol{b},\boldsymbol{b})\widetilde{\Sigma}_1(\boldsymbol{v},\boldsymbol{v})$ corresponding to the contraction $\Sigma^2_- \otimes \Sigma^2_- \to \mathbb{R}$.

As in the previous cases, we use a map $\Psi\colon \mathrm{Sp}(2)\times\mathbb{R}\to \mathrm{Sp}(2)\times V$, now given by $\Psi(g,r)=(g,(r,0,0,0))$, so as to get $\Psi^*(b_0)=dr$, $\Psi^*(b_1)=r\gamma^1$, $\Psi^*(b_2)=r\gamma^3$ and $\Psi^*(b_3)=r\gamma^2$. In invariant terms, we can then express the 4-form Φ by

$$\Phi = (1+\boldsymbol{aa})^{-4/5}\,\boldsymbol{bbbb}$$
$$+(1+\boldsymbol{aa})^{1/5}\Sigma_1(\boldsymbol{b},\boldsymbol{b})\widetilde{\Sigma}_1(\boldsymbol{v},\boldsymbol{v})+(1+\boldsymbol{aa})^{6/5}\,\boldsymbol{v},$$

where, as usual, we have chosen suitable rescalings of the invariant forms. As the coefficient functions are smooth even functions of the distance from the zero section, this form extends to smoothly to the zero section, by the usual arguments.

Let us now turn to the construction of instantons on Sp(1)-bundles over $\mathbb{V}$. Examples of such instantons were also discussed in [5, 22]. On the bundle $\mathrm{Sp}(2)\times_{\mathrm{Sp}(1)_+}V\to\mathbb{V}$, we can consider a connection corresponding to the following one-parameter family of potentials (defined along the principal orbits):

$$A(t)=p(t)\left(ie^1+je^2+ke^3\right).$$

We then have

$$A'(t)=\left(f^{-1/2}\frac{\partial p}{\partial r}+\frac{p}{rf}\frac{\partial(rf^{1/2})}{\partial r}\right)\left(ie^1+je^2+ke^3\right),$$
$$\begin{aligned}F_A =&2ip\left((p-r^{-1}f^{-1/2})e^{23}+rf^2(e^{45}-e^{67})\right)\\&+2jp\left((p-r^{-1}f^{-1/2})e^{31}+rf^2(e^{46}-e^{75})\right)\\&+2kp\left((p-r^{-1}f^{-1/2})e^{12}+rf^2(e^{47}-e^{56})\right).\end{aligned}$$

Straightforward computations show that

$$*_t(F_A\wedge *_t\varphi)=2p\left(p-r^{-1}f^{-1/2}+2rf^2\right)(ie^1+je^2+ke^3).$$

So the instanton evolution equation reads:

$$\frac{\partial p}{\partial r}=2f^{1/2}p^2-p\left(2r^{-1}-4rf^{5/2}+\frac{1}{rf^{1/2}}\frac{\partial(rf^{1/2})}{\partial r}\right)$$

which is a Bernoulli equation.

It is possible to solve explicitly for p; the solution can be expressed in terms of the generalised hypergeometric function $x\mapsto {}_2F_1(1,1;\frac{8}{5};x)$. Explicitly, we have that $p(r)$ is given by

$$\frac{15\left(1+r^2\right)^{6/5}}{3r\left(r^2\left(5c\left(1+r^2\right)^{3/5}+5^{4/5}\right)+5^{4/5}\right)+2\,5^{4/5}r\left(1+r^2\right)\,{}_2F_1\left(1,1;\frac{8}{5};-r^{-2}\right)}.$$

For suitably chosen $c \in \mathbb{R}$, the limiting behaviour of this solution is, in a sense, very similar to that of our SO(3)-invariant instanton on the Eguchi-Hanson space. In particular, we find that

$$\lim_{r\to 0} A(r) = i\gamma^1 + j\gamma^2 + k\gamma^3,$$

corresponding to the canonical connection of the Sp(1)-bundle $\mathrm{Sp}(2)/\mathrm{Sp}(1)_+ \to S^4$. Far away from the zero section, we find that $\|\Omega(r)\|_{g(r)} \to 0$ as $r \to \infty$.

There also an instanton on the bundle $\mathrm{Sp}(2) \times_{\mathrm{Sp}(1)_-} \mathbb{H} \to \mathbb{V}$. To see this, consider the connection form

$$A(t) = \begin{pmatrix} 0 & -\eta^1 & -\eta^2 \\ \eta^1 & 0 & -\eta^3 \\ \eta^2 & \eta^3 & 0 \end{pmatrix},$$

which has $A'(t) = 0$. Straightforward computations show that $F_A \wedge *_t\varphi = 0$, giving that A solves the instanton evolution equations statically.

To find other explicit examples of Spin(7)-instantons, one could consider other known complete Spin(7)-metrics, obtainable via Eq. (24). Some of these arise in the context of cohomogeneity one SU(3)-actions (see, for instance, [17, 25]). Other examples, including an analogue of the Taub-NUT metric, were studied in [9]. As the associated metrics have a more complicated asymptotic behaviour (asymptotically locally conical), we do not expect to get instantons from an ansatz as above. Instead, one should use an approach similar to the one mentioned for the Brandhuber et. al. holonomy G_2-metric [2] in Sect. 3.5.

Acknowledgements JDL was partially supported by EPSRC grant EP/K010980/1. TBM gratefully acknowledges financial support from Villum Fonden.

References

1. M. Atiyah, N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles (M. B. Porter Lectures)* (Princeton University Press, Princeton, NJ, 1988)
2. A. Brandhuber, J. Gomis, S.S. Gubser, S. Gukov, Gauge theory at large N and new G_2 holonomy metrics. Nucl. Phys. B **611**(1–3), 179–204 (2001)
3. R.L. Bryant, Non-embedding and non-extension results in special holonomy, in *The Many Facets of Geometry* (Oxford University Press, Oxford, 2010), pp. 346–367
4. R.L. Bryant, S.M. Salamon, On the construction of some complete metrics with exceptional holonomy. Duke Math. J. **58**(3), 829–850 (1989)
5. A. Clarke, Instantons on the exceptional holonomy manifolds of Bryant and Salamon. J. Geom. Phys. **82**, 84–97 (2014)
6. D. Conti, Special holonomy and hypersurfaces, PhD thesis, Scuola Normale Superiore, Pisa, 2005
7. D. Conti, Invariant forms, associated bundles and Calabi-Yau metrics. J. Geom. Phys. **57**(12), 2483–2508 (2007)

8. D. Conti, T.B. Madsen, Harmonic structures and intrinsic torsion. Transform. Groups **20**(3), 699–723 (2015)
9. M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, New complete noncompact Spin(7) manifolds. Nucl. Phys. B **620**(1–2), 29–54 (2002)
10. J.-H. Eschenburg, M.Y. Wang, The initial value problem for cohomogeneity one Einstein metrics. J. Geom. Anal. **10**(1), 109–137 (2000)
11. D.B. Fairlie, J. Nuyts, Spherically symmetric solutions of gauge theories in eight dimensions. J. Phys. A **17**(14), 2867–2872 (1984)
12. M. Fernández, A classification of Riemannian manifolds with structure group Spin(7). Ann. Mat. Pura Appl. (4) **143**, 101–122 (1986)
13. M. Fernandez, A. Gray, Riemannian manifolds with structure group G_2. Ann. Mat. Pura Appl. (4) **132**, 19–45 (1982)
14. S. Fubini, H. Nicolai, The octonionic instanton. Phys. Lett. B **155**(5–6), 369–372 (1985)
15. A. Gambioli, SU(3)-manifolds of cohomogeneity one. Ann. Glob. Anal. Geom. **34**(1), 77–100 (2008)
16. G.W. Gibbons, P.J. Ruback, The hidden symmetries of multicentre metrics. Commun. Math. Phys. **115**(2), 267–300 (1988)
17. S. Gukov, J. Sparks, M-theory on Spin(7) manifolds. Nucl. Phys. B **625**(1–2), 3–69 (2002)
18. M. Günaydin, H. Nicolai, Seven-dimensional octonionic Yang-Mills instanton and its extension to an heterotic string soliton. Phys. Lett. B **351**(1–3), 169–172 (1995)
19. N.J. Hitchin, The self-duality equations on a Riemann surface. Proc. Lond. Math. Soc. (3) **55**, 59–126 (1987)
20. N.J. Hitchin, Stable forms and special metrics, in *Global Differential Geometry: The Mathematical Legacy of Alfred Gray (Bilbao, 2000)*. Contemporary Mathematics, vol. 288 (American Mathematical Society, Providence, RI, 2001), pp. 70–89
21. T.A. Ivanova, A.D. Popov, (Anti)self-dual gauge fields in dimension $d \geqslant 4$. Teor. Mat. Fiz. **94**(2), 316–342 (1993)
22. H. Kanno, Y. Yasui, Octonionic Yang-Mills instanton on quaternionic line bundle of Spin(7) holonomy. J. Geom. Phys. **34**(3–4), 302–320 (2000)
23. M. Mamone Capria, S.M. Salamon, Yang-Mills fields on quaternionic spaces. Nonlinearity **1**(4), 517–530 (1988)
24. G. Oliveira, Monopoles on the Bryant-Salamon G_2-manifolds. J. Geom. Phys. **86**, 599–632 (2014)
25. F. Reidegeld, Exceptional holonomy and Einstein metrics constructed from Aloff-Wallach spaces. Proc. Lond. Math. Soc. (3) **102**(6), 1127–1160 (2011)
26. R. Reyes Carrión, A generalization of the notion of instanton. Differ. Geom. Appl. **8**(1), 1–20 (1998)

Hermitian Metrics on Compact Complex Manifolds and Their Deformation Limits

Antonio Otal, Luis Ugarte, and Raquel Villacampa

Dedicated to Simon Salamon on the occasion of his 60th birthday

Abstract We consider various types of special Hermitian metrics on compact complex manifolds, namely Kähler, Hermitian-symplectic, SKT, balanced and strongly Gauduchon metrics. We study the strongly Gauduchon metrics in relation to the SKT condition on complex nilmanifolds and to the Hermitian-symplectic condition on certain complex solvmanifolds. We also review the behaviour of the existence properties of these special metrics on compact complex manifolds under holomorphic deformations.

Keywords Complex manifold • Hermitian metric • Holomorphic deformation • Solvmanifold

1 Introduction

It is well known that the existence of a Kähler metric imposes strong topological conditions on the (compact) manifold [15]. Given a compact complex manifold X, several special classes of Hermitian metrics, weaker than the Kähler ones, appear in connection with different geometrical aspects. Let us denote by F the fundamental form associated to a Hermitian metric on X and let n be the complex dimension of X. If F^{n-1} is $\partial\bar{\partial}$-closed, then the Hermitian metric is called *standard* or *Gauduchon*.

A. Otal (✉) • R. Villacampa
Centro Universitario de la Defensa - I.U.M.A., Academia General Militar, Ctra. de Huesca s/n., 50090 Zaragoza, Spain
e-mail: aotal@unizar.es; raquelvg@unizar.es

L. Ugarte
Departamento de Matemáticas - I.U.M.A., Universidad de Zaragoza, Campus Plaza San Francisco, 50009 Zaragoza, Spain
e-mail: ugarte@unizar.es

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_11

By [28] there exists a Gauduchon metric in the conformal class of any Hermitian metric. A particularly interesting class of Gauduchon metrics is the one given by the *balanced* Hermitian metrics, defined by the condition $dF^{n-1} = 0$. Important aspects of these metrics were first investigated by Michelsohn in [39], and many authors have constructed balanced manifolds and studied their properties (see e.g. [1, 2, 9, 23, 26, 27, 47, 55, 58] and the references therein). Popovici has introduced and studied in [43, 44] the class of *strongly Gauduchon* metrics, defined by the condition $\partial F^{n-1} = \bar{\partial}\alpha$, for some complex form α of bidegree $(n, n-2)$ on X (for recent results on the geometry of strongly Gauduchon manifolds, see for instance [13, 48, 63, 64]). By definition it is clear that any Kähler metric is balanced, and any balanced metric is strongly Gauduchon.

On the other hand, a Hermitian metric that satisfies the condition $\partial\bar{\partial}F = 0$ is called *pluriclosed* or *strong Kähler with torsion*. These metrics were first introduced by Bismut in [11] and further studied by many authors (see e.g. [12, 17, 18, 20, 24, 25, 54, 59] and the references therein). Recall that a complex structure J on a symplectic manifold (M, ω) is said to be *tamed* by the symplectic form ω if $\omega(X, JX) > 0$ for any non-zero vector field X on M. The pair (ω, J) is called a *Hermitian-symplectic* structure in [53]. By [17, Proposition 2.1] the existence of a Hermitian-symplectic structure on a complex manifold $X = (M, J)$ is equivalent to the existence of a J-compatible strong Kähler with torsion metric whose fundamental form F satisfies $\partial F = \bar{\partial}\beta$, for some ∂-closed $(2, 0)$-form β on X. No example of non-Kähler compact complex manifold admitting Hermitian-symplectic metric is known (see [36, p. 678] and [53, Question 1.7]).

To sum up, for the Hermitian metrics introduced above, one has the following implications:

$$
\begin{tikzcd}
 & \text{Balanced (B)} \arrow[r, Rightarrow] & \text{Strongly Gauduchon (SG)} \\
\text{Kähler} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & \text{Hermitian-Symplectic (HS)} \arrow[r, Rightarrow] & \text{Strong Kähler with Torsion (SKT)}
\end{tikzcd}
\tag{1}
$$

In complex dimension $n = 2$, an SKT metric is just a Gauduchon metric, and by definition a metric is balanced if and only if it is Kähler. Moreover, if a compact complex surface X admits a Hermitian symplectic (i.e. strongly Gauduchon) metric, then X is Kähler (see for instance [36, 53]).

Hence, we will focus on compact complex manifolds X of complex dimension $n \geq 3$. Recall that it is proved in [3, 38] that an SKT metric F on a compact complex manifold cannot be balanced for $n > 2$ unless it is Kähler (see [21, 31] for an extension of this result to generalized Gauduchon metrics). A recent conjecture by Fino and Vezzoni [22] asserts that if X has an SKT metric and another metric which is balanced, then X is Kähler.

Our aim in Sect. 2 is to show two deep differences of the SG geometry with respect to the balanced geometry: first, SG metrics which are also SKT can exist on a non-Kähler compact complex manifold X; second, there exist non-Kähler SG metrics that additionally satisfy the HS condition (in this case the metrics we provide live on compact complex manifolds admitting a Kähler metric).

For the construction of the examples we consider solvmanifolds, of real dimension 6, endowed with certain classes of invariant complex structures. First, we consider the class of 6-dimensional nilmanifolds. The nilpotent Lie algebras admitting a complex structure are classified by Salamon in [51]. A classification of the complex structures up to isomorphism is obtained in [13], while the SKT, HS, B and SG geometries of these complex nilmanifolds are studied in [13, 17, 24, 56] (see Table 1 for more details). We use those results to classify in Proposition 2.1 the invariant complex structures on 6-dimensional nilmanifolds that admit SG metrics which are also SKT. The nilpotent Lie algebras underlying such nilmanifolds are $\mathfrak{h}_2$, $\mathfrak{h}_4$ and $\mathfrak{h}_5$.

In order to provide examples in complex dimension 3 of non-Kähler SG metrics that additionally satisfy the HS condition, we consider certain complex solvmanifolds. Notice that by [17] such examples cannot exist on nilmanifolds. On the one side, we consider 6-dimensional solvmanifolds admitting an invariant complex structure with holomorphically trivial canonical bundle. They are classified in [26], where a complete study of the Hermitian metrics that they admit is also given (see Table 2 for a summary of the results). On the other hand, we consider 6-dimensional solvmanifolds endowed with a complex structure of splitting type (see Definition 2.2). The study of their complex and Hermitian geometries is carried out in [10] and a summary of the main results is given in Table 3.

In Proposition 2.3 we prove that if X is a 6-dimensional solvmanifold endowed with an invariant complex structure of splitting type, then any invariant HS metric on X is SG. This allows us to conclude that there exist SG metrics that are in addition HS, but which are not Kähler (Corollary 2.4). One of the examples (the one corresponding to $\mathfrak{g}_2^0 \cong \mathfrak{s}_7^1$ in Tables 2 and 3) has holomorphically trivial canonical bundle.

In Sect. 3 we focus on the existence properties of special Hermitian metrics on compact complex manifolds and their behaviour under holomorphic deformations of the complex structure. More concretely, we say that a compact complex manifold X *has the property* $\mathcal{K}$ if X admits a Kähler metric. We define the properties $\mathcal{HS}$, $\mathcal{SKT}$, $\mathcal{B}$ and $\mathcal{SG}$ of compact complex manifolds in a similar way. It is clear that analogous implications to those in (1) are valid for these properties.

We discuss the openness and closedness of the properties $\mathcal{K}$, $\mathcal{HS}$, $\mathcal{SKT}$, $\mathcal{B}$ and $\mathcal{SG}$ of compact complex manifolds of complex dimension $n \geq 3$ (for compact complex surfaces all these properties are both open and closed). We first review the stability of the Kähler, strongly Gauduchon and Hermitian-symplectic properties (proved in [35, 45] and [64], respectively). However, as it is proved in [1] and [20], the complex geometry of the Iwasawa manifold allows to show that the properties $\mathcal{B}$ and $\mathcal{SKT}$ are not open (see Proposition 3.1).

In relation to the closedness of the previous properties, in [29, 30] Hironaka proved that the Kähler property is not closed. In Proposition 3.3 we conclude, as a consequence of Egidi's results on special metrics under modifications [16], that the properties $\mathcal{HS}$ and $\mathcal{SKT}$ are not closed. Alessandrini and Bassanelli proved in [2] that the balanced property is stable under modifications, so the central limit in the Hironaka family is balanced.

We consider other analytic families in order to investigate the closedness of the $\mathcal{B}$ and $\mathcal{SG}$ properties. Concerning the $\mathcal{SG}$ property, we review the construction in [57] of an analytic family $\{X_t = (N \times N, J_t)\}_{t\in\Delta}$, where N is the Heisenberg nilmanifold of real dimension 3, such that X_t has SG metrics for each $t \in \Delta\backslash\{0\}$, but the central fiber X_0 does not admit any SG metric (see Proposition 3.4). For the property $\mathcal{B}$, using the results of [37] about deformations of abelian complex structures on nilmanifolds, in [13] it is given an analytic family $\{X_t\}_{t\in\Delta}$ such that X_t has balanced metrics for each $t \in \Delta\backslash\{0\}$, but the central fiber X_0 does not admit any SG metric (see Proposition 3.7 for details). In conclusion, for compact complex manifolds of complex dimension $n \geq 3$, the properties $\mathcal{K}$, $\mathcal{HS}$, $\mathcal{SKT}$, $\mathcal{B}$ and $\mathcal{SG}$ are not closed (see Theorem 3.8).

2 Special Classes of Strongly Gauduchon Metrics

It is proved in [3, 38] that SKT metrics on compact complex manifolds cannot be balanced for $n > 2$ unless they are Kähler (see [21, 31] for a recent extension of this result to generalized Gauduchon metrics). In this section we show that this result does not extend to SG metrics, that is, there are compact complex manifolds of complex dimension 3 having non-Kähler SG metrics which satisfy in addition the SKT condition. Moreover, we also show the existence of non-Kähler SG metrics which are in addition HS.

For the construction of metrics which are simultaneously SG and SKT we will use the class of nilmanifolds endowed with an invariant complex structure. This class provides an important source of compact complex manifolds with special (non-Kähler) Hermitian metrics. In complex dimension 3, nilmanifolds are classified by Salamon in [51]. The existence of SKT metrics is investigated in [24], and the existence of balanced and SG metrics is carried out, respectively, in [56] and [13]. In [17] it is proved that there are not HS metrics on nilmanifolds, except for the complex tori. In these studies the description of invariant complex structures plays a central role [13, 51], as well as the symmetrization process [19] in order to reduce the existence problem to invariant Hermitian metrics.

In Table 1 we summarize the known existence results for special Hermitian metrics on nilmanifolds of complex dimension 3 endowed with an invariant complex structure J. The Lie algebras in the list are the nilpotent Lie algebras underlying such nilmanifolds. We follow the notation given in the paper [51] to name the Lie algebras as well as for their description. For instance, the notation $\mathfrak{h}_2 = (0, 0, 0, 0, 12, 34)$

Table 1 Special Hermitian metrics on 6-nilmanifolds

[13, 17, 24, 56]	Kähler	HS	SKT	B	SG
$\mathfrak{h}_1 = (0,0,0,0,0,0)$	✓	✓	✓	✓	✓
$\mathfrak{h}_2 = (0,0,0,0,12,34)$	–	–	✓$_{(J)}$	✓$_{(J)}$	✓$_{(J)}$
$\mathfrak{h}_3 = (0,0,0,0,0,12+34)$	–	–	–	✓$_{(J)}$	✓$_{(J)}$
$\mathfrak{h}_4 = (0,0,0,0,12,14+23)$	–	–	✓$_{(J)}$	✓$_{(J)}$	✓$_{(J)}$
$\mathfrak{h}_5 = (0,0,0,0,13+42,14+23)$	–	–	✓$_{(J)}$	✓$_{(J)}$	✓
$\mathfrak{h}_6 = (0,0,0,0,12,13)$	–	–	–	✓	✓
$\mathfrak{h}_7 = (0,0,0,12,13,23)$	–	–	–	–	–
$\mathfrak{h}_8 = (0,0,0,0,0,12)$	–	–	✓	–	–
$\mathfrak{h}_9 = (0,0,0,0,12,14+25)$	–	–	–	–	–
$\mathfrak{h}_{10} = (0,0,0,12,13,14)$	–	–	–	–	–
$\mathfrak{h}_{11} = (0,0,0,12,13,14+23)$	–	–	–	–	–
$\mathfrak{h}_{12} = (0,0,0,12,13,24)$	–	–	–	–	–
$\mathfrak{h}_{13} = (0,0,0,12,13+14,24)$	–	–	–	–	–
$\mathfrak{h}_{14} = (0,0,0,12,14,13+42)$	–	–	–	–	–
$\mathfrak{h}_{15} = (0,0,0,12,13+42,14+23)$	–	–	–	–	–
$\mathfrak{h}_{16} = (0,0,0,12,14,24)$	–	–	–	–	–
$\mathfrak{h}_{19}^{-} = (0,0,0,12,23,14-35)$	–	–	–	✓	✓
$\mathfrak{h}_{26}^{+} = (0,0,12,13,23,14+25)$	–	–	–	–	–

means that there is a basis of 1-forms $\{e^j\}_{j=1}^6$ satisfying $de^1 = de^2 = de^3 = de^4 = 0$, $de^5 = e^1 \wedge e^2$, $de^6 = e^3 \wedge e^4$.

The symbol ✓ means that **for any** invariant J on the nilmanifold, there exist J-Hermitian metrics of the corresponding type. On the other hand, the symbol ✓$_{(J)}$ means that there exist complex structures J admitting J-Hermitian metrics of the corresponding type, but there are also other complex structures for which Hermitian metrics of that type do not exist. In contrast, the symbol "–" means that none of the complex structures admits the corresponding kind of metrics. Here "HS" means Hermitian-symplectic, "B" refers to balanced and "SG" to strongly Gauduchon metrics.

For the description of the complex structures we use a complex basis of (invariant) forms $\{\omega^j\}_{j=1}^3$ of bidegree (1,0) with respect to the complex structure. As it is well known, the integrability is equivalent to $d\omega^j$ to have zero component of bidegree (0,2). In the case of nilmanifolds of complex dimension n, in addition, a result of Salamon [51] asserts that there exists a basis such that each complex 2-form $d\omega^j$ belongs to the ideal $\mathcal{I}(\omega^1, \ldots, \omega^{j-1})$ generated by $\{\omega^1, \ldots, \omega^{j-1}\}$. In particular, the invariant $(n,0)$-form $\Psi = \omega^{1\cdots n} := \omega^1 \wedge \ldots \wedge \omega^n$ is closed, and so the canonical bundle of any nilmanifold endowed with an invariant complex structure is holomorphically trivial.

In the following result we consider the class of nilmanifolds to show that an SG metric satisfying the SKT condition is not necessarily Kähler. Moreover, we classify the nilmanifolds of complex dimension 3 that admit such type of metrics.

Proposition 2.1 *Let $X = (\Gamma\backslash G, J)$ be a (non-toral) nilmanifold endowed with an invariant complex structure,* $\dim_{\mathbb{C}} X = 3$, *and denote by $\mathfrak{g}$ the Lie algebra of G. Then, X has an SG metric which is in addition SKT, if and only if*

- $\mathfrak{g}$ *is isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_4$ or $\mathfrak{h}_5$, and*
- *the complex structure J belongs to one of the following families:*

$$\mathfrak{h}_2: \quad d\omega^1 = d\omega^2 = 0, \;\; d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + (1 + iy)\omega^{2\bar{2}}, \;\; y > 0;$$

$$\mathfrak{h}_4: \quad d\omega^1 = d\omega^2 = 0, \;\; d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + \omega^{2\bar{2}};$$

$$\mathfrak{h}_5: \quad d\omega^1 = d\omega^2 = 0, \;\; d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + (\frac{1}{2} + iy)\omega^{2\bar{2}}, \;\; \frac{\sqrt{3}}{2} > y \geq 0.$$

Furthermore, in such cases all the invariant Hermitian metrics are both SKT and SG.

Proof By the well-known symmetrization process (see [19] for details), we can restrict our attention to invariant metrics. By [24] and [56] (see also [17]), if $\mathfrak{g}$ admits a J-Hermitian metric which is SKT then $\mathfrak{g}^*_{\mathbb{C}}$ has a basis $\{\omega^1, \omega^2, \omega^3\}$ of bidegree $(1, 0)$ with respect to J such that

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\, \omega^{12} + \omega^{1\bar{1}} + B\, \omega^{1\bar{2}} + D\, \omega^{2\bar{2}}, \tag{2}$$

where $B, D \in \mathbb{C}$ and $\rho = 0, 1$ satisfy

$$\rho + |B|^2 = 2\, \mathfrak{Re}\, D. \tag{3}$$

Since this condition only depends on the complex structure, if it is satisfied then any J-Hermitian metric is SKT.

Recall that a complex structure J is called abelian if it satisfies $[JX, JY] = [X, Y]$ for all $X, Y \in \mathfrak{g}$. By [13, Corollary 5.2], if J is abelian, i.e. $\rho = 0$, then an invariant Hermitian metric is SG if and only if it is balanced. So, necessarily the coefficient $\rho = 1$, or in other words, J is a non-abelian nilpotent complex structure. Now, by [13, Proposition 5.3 and Remark 5.4] for a non-abelian nilpotent complex structure, any invariant Hermitian metric is SG. In conclusion, we are led to complex structures given by (2) with $\rho = 1$ and $\mathfrak{Re}\, D = \frac{1+|B|^2}{2}$, as a consequence of the SKT condition (3).

Moreover, by the classification of complex structures given in [13, Table 1], we can take $\mathfrak{Im}\, D \geq 0$ and suppose B to be a real non-negative number λ, i.e. $B = \lambda \in \mathbb{R}^{\geq 0}$. Therefore, instead of (2) and (3), we can focus on the complex structures J given by the simpler equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\, \omega^{1\bar{2}} + D\, \omega^{2\bar{2}},$$
$$\text{where } D = \frac{1 + \lambda^2}{2} + i\, y, \text{ and } \lambda, y \in \mathbb{R}^{\geq 0}.$$

Now the result follows by looking at the precise values of the complex parameters defining complex structures up to isomorphism. We discuss on the possible values of λ (see [13, Table 1] for details):

- If $\lambda = 1$ and $y > 0$, then the Lie algebra is $\mathfrak{h}_2$ and $D = 1 + i\,y$ in the complex structure equations.
- If $\lambda = 1$ and $y = 0$, then the Lie algebra is $\mathfrak{h}_4$ and $D = 1$ in the complex structure equations.
- If $\lambda \neq 1$, then the Lie algebra is $\mathfrak{h}_5$. Moreover, since the real part of D never vanishes, necessarily the coefficient $\lambda = 0$. In addition, the coefficient $D = \frac{1}{2} + i\,y$ must satisfy the condition $4(\mathfrak{Im}\, D)^2 < 1 + 4\,\mathfrak{Re}\, D$, that is, $4y^2 < 3$. In conclusion, $\frac{\sqrt{3}}{2} > y \geq 0$. □

Once we have seen that, in complex dimension 3, there are non-Kähler SG metrics satisfying the SKT condition, the next question that we address is the existence of SG metrics that are also HS, but not Kähler. As we mentioned above, HS metrics do not exist on (non-toral) nilmanifolds by [17], so we are led to consider more general complex spaces. A natural class to explore is the class of solvmanifolds endowed with an invariant complex structure J. Next we will consider two special types: complex solvmanifolds admitting J with holomorphically trivial canonical bundle, and solvmanifolds admitting an invariant complex structure J of splitting type. The former class is studied in [26] and the latter in [10].

The first extension, i.e. solvmanifolds admitting an invariant complex structure J with holomorphically trivial canonical bundle, constitutes a natural generalization of complex nilmanifolds. They have also several applications in Mathematical Physics providing solutions in heterotic string theories [42]. In Table 2 we summarize the existence results obtained in [26] for special Hermitian metrics in complex dimension 3. We follow a similar description as in the previous Table 1 for nilmanifolds. Notice that [26, Proposition 2.10] ensures the existence of a lattice

Table 2 Special Hermitian metrics on 6-solvmanifolds with holomorphically trivial canonical bundle

[26]	Kähler	HS	SKT	B	SG
$\mathfrak{g}_1 = (15, -25, -35, 45, 0, 0)$	–	–	–	✓	✓
$\mathfrak{g}_2^0 = (25, -15, 45, -35, 0, 0)$	✓	✓	✓	✓	✓
$\mathfrak{g}_2^\alpha = (\alpha\cdot 15+25, -15+\alpha\cdot 25, -\alpha\cdot 35+45, -35-\alpha\cdot 45, 0, 0),\ \alpha > 0$	–	–	–	✓	✓
$\mathfrak{g}_3 = (0, -13, 12, 0, -46, -45)$	–	–	–	✓	✓
$\mathfrak{g}_4 = (23, -36, 26, -56, 46, 0)$	–	–	✓	–	–
$\mathfrak{g}_5 = (24+35, 26, 36, -46, -56, 0)$	–	–	–	✓	✓
$\mathfrak{g}_6 = (24+35, -36, 26, -56, 46, 0)$	–	–	–	–	–
$\mathfrak{g}_7 = (24+35, 46, 56, -26, -36, 0)$	–	–	–	✓	✓
$\mathfrak{g}_8 = (16-25, 15+26, -36+45, -35-46, 0, 0)$	–	–	–	✓$_{(J)}$	✓
$\mathfrak{g}_9 = (45, 15+36, 14-26+56, -56, 46, 0)$	–	–	–	–	–

for the connected and simply-connected Lie groups associated to the Lie algebras in the list of Table 2, although for $\mathfrak{g}_2^{\alpha}$ a lattice is found only for a countable number of values of α (note that one cannot expect a lattice to exist for any real $\alpha > 0$ according to [61, Proposition 8.7]).

As one can appreciate from Table 2, the complex geometry of solvmanifolds with holomorphically trivial canonical bundle is more "rigid" than the complex geometry of nilmanifolds. In fact, only solvmanifolds with underlying Lie algebra $\mathfrak{g}_8$ have both complex structures admitting balanced metrics, and complex structures not admitting any balanced metric. This fact plays an important role in the construction of holomorphic deformations with interesting properties in their central limit (see Sect. 3). Notice that the algebra $\mathfrak{g}_8$ corresponds to the holomorphically parallelizable Nakamura (solv)manifold [41].

The second extension consists of the class of complex solvmanifolds with splitting-type complex structures. These structures arise as certain semi-direct products of nilpotent Lie groups by $\mathbb{C}^m$, so they also constitute a natural solvable extension of complex nilmanifolds. Furthermore, some complex cohomological invariants of the manifold can be obtained explicitly, which allows to investigate several aspects of their complex [7, 33] and Hermitian [25, 32] geometries. For instance, in [33] a technique is developed to compute the Dolbeault cohomology groups by means of a certain finite-dimensional subalgebra of the de Rham complex.

Let us recall the precise definition of a solvmanifold $X = G/\Gamma$ endowed with a complex structure of *splitting type*.

Definition 2.2 ([33, Assumption 1.1]) A solvmanifold $X = G/\Gamma$ endowed with an invariant complex structure J is said to be of *splitting type* if G is a semi-direct product $G = \mathbb{C}^m \ltimes_{\varphi} N$ such that:

- N is a connected simply-connected $2k$-dimensional nilpotent Lie group endowed with an N-left-invariant complex structure J_N;
- for any $\mathbf{z} \in \mathbb{C}^m$, it holds that $\varphi(\mathbf{z}) \in Aut(N)$ is a holomorphic automorphism of N with respect to J_N;
- φ induces a semi-simple action on the Lie algebra $\mathfrak{n}$ associated to N;
- G has a lattice Γ (then Γ can be written as $\Gamma = \Gamma_{\mathbb{C}^m} \ltimes_{\varphi} \Gamma_N$ such that $\Gamma_{\mathbb{C}^m}$ and Γ_N are lattices of $\mathbb{C}^m$ and N, respectively, and, for any $\mathbf{z} \in \Gamma_{\mathbb{C}^m}$, it holds $\varphi(\mathbf{z})\,(\Gamma_N) \subseteq \Gamma_N$);
- the inclusion $\wedge^{\bullet,\bullet}(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C})^* \hookrightarrow \wedge^{\bullet,\bullet}(N/\Gamma_N)$ induces the isomorphism in cohomology

$$H_{\bar\partial}^{\bullet,\bullet}\left(\wedge^{\bullet,\bullet}(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C})^*\right) \xrightarrow{\cong} H_{\bar\partial}^{\bullet,\bullet}(N/\Gamma_N)\ .$$

In complex dimension $n = m + k = 3$, a general study is carried out in [10]. By [10, Theorem 1.7], if a solvmanifold $X = G/\Gamma$ admits an invariant complex structure J of splitting type, then the Lie algebra $\mathfrak{g}$ of G is isomorphic to one in the list given in Table 3. For a discussion on the existence of lattices in the connected and simply-connected solvable Lie group G_k corresponding to the Lie algebra $\mathfrak{s}_k$,

Table 3 Special Hermitian metrics on splitting-type complex 6-solvmanifolds

[10]	Kähler	HS	SKT	B	SG
$\mathfrak{s}_1 = (23, 34, -24, 0, 0, 0)$	–	–	✓	–	–
$\mathfrak{s}_2 = (0, -13, 12, 0, 0, 0)$	✓	✓	✓	✓	✓
$\mathfrak{s}_3 = (0, -13, 12, 0, -46, 45)$	✓	✓	✓	✓	✓
$\mathfrak{s}_4 = (15, -25, -35, 45, 0, 0)$	–	–	–	✓	✓
$\mathfrak{s}_5^{\alpha} = (15, 25, -35+\alpha\cdot 45, -\alpha\cdot 35-45, 0, 0)$, $\alpha > 0$	–	–	–	✓	✓
$\mathfrak{s}_6^{\alpha,\beta} = (\alpha\cdot 15+25, -15+\alpha\cdot 25, -\alpha\cdot 35+\beta\cdot 45, -\beta\cdot 35-\alpha\cdot 45, 0, 0)$, $\alpha > 0$, $\beta \in (0, 1)$	–	–	–	✓	✓
$\mathfrak{s}_7^{\alpha} = (25, -15, \alpha\cdot 45, -\alpha\cdot 35, 0, 0)$, $0 < \alpha \leq 1$	✓	✓	✓	✓	✓
$\mathfrak{s}_8^{\alpha} = (\alpha\cdot 15+25, -15+\alpha\cdot 25, -\alpha\cdot 35+45, -35-\alpha\cdot 45, 0, 0)$, $\alpha > 0$	–	–	–	✓	✓
$\mathfrak{s}_9 = (-16, -26, 36-45, 35+46, 0, 0)$	–	–	–	✓	✓
$\mathfrak{s}_{10}^{\alpha,\beta} = (15+\beta\cdot 16-26, 16+25+\beta\cdot 26, -35-\beta\cdot 36-\alpha\cdot 45, \alpha\cdot 35-45-\beta\cdot 46, 0, 0)$, $\alpha \neq 0, \quad \beta \in \mathbb{R}$	–	–	–	✓	✓
$\mathfrak{s}_{11}^{\alpha} = (16-25, 15+26, -36-\alpha\cdot 45, \alpha\cdot 35-46, 0, 0)$, $\alpha \in (0, 1)$	–	–	–	✓	✓
$\mathfrak{s}_{12} = (16-25, 15+26, -36+45, -35-46, 0, 0)$	–	–	–	✓	✓

see [10, Remark 1.17 and Proposition 1.18]. In particular, there is a lattice for G_k $(1 \leq k \leq 4)$, G_5^{α}, $G_6^{\alpha,\beta}$, G_7^{α} and G_8^{α} for a countable family of α's and β's, and also for G_{12}.

Some of the Lie algebras in Table 3 already appeared in Table 2. In fact, $\mathfrak{s}_4 \cong \mathfrak{g}_1$, $\mathfrak{s}_7^1 \cong \mathfrak{g}_2^0$, $\mathfrak{s}_8^{\alpha} \cong \mathfrak{g}_2^{\alpha}$ and $\mathfrak{s}_{12} \cong \mathfrak{g}_8$. This implies that the corresponding solvmanifolds admit complex structures of splitting type with holomorphically trivial canonical bundle. This happens for instance for the Nakamura manifold, since its underlying Lie algebra is $\mathfrak{g}_8 \cong \mathfrak{s}_{12}$.

The existence of special Hermitian metrics for splitting-type complex structures on each Lie algebra $\mathfrak{g}$ is studied in [10] and the results are summarized in Table 3. We follow a similar description as in the previous Table 1.

Next we focus our attention on the cases when HS metrics exist. In Table 2 one has that $\mathfrak{g}_2^0$ is the only algebra admitting HS metrics. But, as we noticed before, $\mathfrak{g}_2^0 \cong \mathfrak{s}_7^1$ and so we can restrict our attention to Table 3 for the study of HS metrics, that is, we consider solvmanifolds with an invariant complex structure J of splitting type. In the following proposition we show the relation of the conditions SG and HS on such solvmanifolds. In the proof one can find a detailed description of the invariant HS metrics which are not Kähler.

Proposition 2.3 *Let $X = (\Gamma\backslash G, J)$ be a 6-dimensional solvmanifold endowed with an invariant complex structure J of splitting type. Then, any invariant HS metric on X is SG.*

Proof From Table 3 it follows that if $\mathfrak{g}$ is the Lie algebra of G, then $\mathfrak{g} \cong \mathfrak{s}_2, \mathfrak{s}_3$ or $\mathfrak{s}_7^{\alpha}$. We need a detailed description of the space of splitting-type complex structures J on these algebras as well as of their spaces of J-Hermitian metrics.

By [10], the existence of an HS structure (J, F) on $\mathfrak{g}$ implies that there is a basis $\{\omega^1, \omega^2, \omega^3\}$ of forms of bidegree (1,0) with respect to J satisfying

$$\begin{cases} d\omega^1 = A\,\omega^{13} - \bar{A}\,\omega^{1\bar{3}}, \\ d\omega^2 = -\varepsilon\,\omega^{23} + \varepsilon\,\omega^{2\bar{3}}, \\ d\omega^3 = 0, \end{cases} \tag{4}$$

where $A \in \mathbb{C}$ and $\varepsilon \in \{0, 1\}$. Now, a generic Hermitian metric F can be written as

$$2F = i\,\omega^{1\bar{1}} + i\,\omega^{2\bar{2}} + i\,t^2\,\omega^{3\bar{3}} + u\,\omega^{1\bar{2}} - \bar{u}\,\omega^{2\bar{1}} + v\,\omega^{2\bar{3}} - \bar{v}\,\omega^{3\bar{2}} + z\,\omega^{1\bar{3}} - \bar{z}\,\omega^{3\bar{1}}, \tag{5}$$

where $t \in \mathbb{R}\backslash\{0\}$ and $u, v, z \in \mathbb{C}$ satisfy the conditions that ensure that F is positive-definite: $1 > |u|^2$, $t^2 > |v|^2$, $t^2 > |z|^2$ and $t^2 + 2\,\mathfrak{Re}\,(i\bar{u}\bar{v}z) > t^2|u|^2 + |v|^2 + |z|^2$.

In what follows, we will denote the complex structure simply by $J = (A, \varepsilon)$ and the J-Hermitian metric by $F = (t^2, u, v, z)$.

We use next the description of HS metrics as obtained in [10, Proposition 2.5]. In that proposition it is proved that a Hermitian structure (J, F) on $\mathfrak{g}$ is HS if and only if it is SKT, and moreover, any HS structure (J, F) on $\mathfrak{g}$ is one of the following:

(HS.i) $(\mathfrak{s}_2, J, F)$, where $J = (1, 0)$ and $F = (t^2, 0, v, z)$;
(HS.ii) $(\mathfrak{s}_3, J, F)$, where $J = (A, 1)$, $\mathfrak{Im}\, A \neq 0$, and $F = (t^2, 0, v, z)$;
(HS.iii) $(\mathfrak{s}_7^{\alpha}, J, F)$, where $J = (A, 1)$, $A \in \mathbb{R} \setminus \{0, -1\}$, and $F = (t^2, 0, v, z)$; here $\alpha = |A|$ or $|1/A|$;
(HS.iv) $(\mathfrak{s}_7^1, J, F)$, where $J = (-1, 1)$ and $F = (t^2, u, v, z)$.

These HS metrics are Kähler if and only if $z = 0$ in the first case (HS.i), and if and only if $v = z = 0$ in the cases (HS.ii), (HS.iii) and (HS.iv). Hence, there are many HS metrics which are not Kähler.

Next we prove that all these HS metrics satisfy in addition the SG condition, i.e. $\bar\partial F^2 = \partial\gamma$ for some (1,3)-form γ. First, Eq. (4) imply

$$\begin{cases} \partial\omega^{1\bar1\bar2\bar3} = \varepsilon\, \omega^{13\bar1\bar2\bar3}, \\ \partial\omega^{2\bar1\bar2\bar3} = -A\, \omega^{23\bar1\bar2\bar3}, \\ \partial\omega^{3\bar1\bar2\bar3} = 0, \end{cases} \tag{6}$$

and for a generic metric F given by (5), using again (4), we get

$$4F \wedge \bar\partial F = (i\,\bar v - u\,\bar z)\varepsilon\, \omega^{13\bar1\bar2\bar3} + (i\,\bar z + \bar u\, \bar v)\bar A\, \omega^{23\bar1\bar2\bar3}. \tag{7}$$

Now, we study each one of the previous cases:

- In the case (HS.i), i.e. $A = 1$, $\varepsilon = 0$ and $F = (t^2, 0, v, z)$, it follows from (6) and (7) that

$$2\,\bar\partial F^2 = i\,\bar z\, \omega^{23\bar1\bar2\bar3} = \partial(-i\,\bar z\, \omega^{2\bar1\bar2\bar3}).$$

  Thus, any HS metric is SG. The metric F is not Kähler whenever $z \neq 0$. Notice that $z = 0$ if and only if F is balanced, that is, $dF^2 = 0$.
- In the case (HS.ii), i.e. $A \in \mathbb{C}$ with $\mathfrak{Im}\, A \neq 0$, $\varepsilon = 1$ and $F = (t^2, 0, v, z)$, it follows from (6) and (7) that

$$2\,\bar\partial F^2 = i\,\bar v\, \omega^{13\bar1\bar2\bar3} + i\,\bar z \bar A\, \omega^{23\bar1\bar2\bar3} = \partial\left(i\,\bar v\, \omega^{1\bar1\bar2\bar3} - i\,\bar z\, \frac{\bar A}{A}\, \omega^{2\bar1\bar2\bar3}\right).$$

  Hence, any HS metric is SG. The metric is Kähler if and only if $v = z = 0$, and the latter condition is precisely the balanced condition.
- In the case (HS.iii), i.e. $A \in \mathbb{R} \setminus \{0, -1\}$, $\varepsilon = 1$ and $F = (t^2, 0, v, z)$, Eqs. (6) and (7) imply

$$2\,\bar\partial F^2 = i\,\bar v\, \omega^{13\bar1\bar2\bar3} + i\,\bar z A\, \omega^{23\bar1\bar2\bar3} = \partial\left(i\,\bar v\, \omega^{1\bar1\bar2\bar3} - i\,\bar z\, \omega^{2\bar1\bar2\bar3}\right).$$

  Again, any HS metric is SG. The metric F is Kähler if and only if $v = z = 0$, which is equivalent to F be balanced.

- Finally, in the (HS.iv) case, i.e. $A = -1$, $\varepsilon = 1$ and $F = (t^2, u, v, z)$, it follows from (6) and (7) that

$$2\,\bar\partial F^2 = (i\,\bar v - u\,\bar z)\,\omega^{13\bar1\bar2\bar3} - (i\,\bar z + \bar u\,\bar v)\,\omega^{23\bar1\bar2\bar3} = \partial\left((i\,\bar v - u\,\bar z)\,\omega^{1\bar1\bar2\bar3} - (i\,\bar z + \bar u\,\bar v)\,\omega^{2\bar1\bar2\bar3}\right).$$

Hence, any HS metric is SG. The metric is Kähler if and only if $v = z = 0$, and the latter condition is precisely the balanced condition. □

As a consequence of the previous result, we conclude

Corollary 2.4 *In complex dimension* 3*, there exist SG metrics that are in addition HS, but which are not Kähler.*

Remark 2.5 We notice that Remark 4.2 in [26] should be corrected in relation to HS metrics on $\mathfrak{g}_2^0$ with respect to its complex structure with holomorphically trivial canonical bundle, because it is not true that any invariant HS metric is Kähler. Indeed, since $\mathfrak{g}_2^0 \cong \mathfrak{s}_7^1$, such complex structure corresponds to $\mathfrak{s}_7^1$ with the complex structure $J = (1, 1)$, i.e. to the case (HS.iii) with $A = 1$ in the proof of Proposition 2.3. In that case one has that any invariant metric $F = (t^2, 0, v, z)$ is HS, but F is Kähler if and only if $v = z = 0$.

At the level of compact complex manifolds X, a conjecture of Fino and Vezzoni [22] states that if X is non-Kähler, then it is never possible to find on X an SKT metric and also a balanced one. In [23] they study the conjecture for nilmanifolds. Notice that this is in accord to Table 1, since a (real) nilmanifold based on $\mathfrak{h}_2$, $\mathfrak{h}_4$ or $\mathfrak{h}_5$ admits SKT metrics and balanced metrics, but with respect to different complex structures, which is in agreement with the meaning of the symbol $\checkmark_{(J)}$.

The conjecture is proved in [22] for 6-dimensional solvmanifolds having holomorphically trivial canonical bundle, and in [10] for any splitting-type complex structure on a 6-dimensional solvmanifold (see Tables 2 and 3).

On the other hand, Streets and Tian posed in [53, Question 1.7] the problem of finding compact HS manifolds X not admitting Kähler metrics. On (non-toral) nilmanifolds HS metrics never exist [17] and, as we have seen above, for 6-dimensional solvmanifolds endowed with an invariant complex structure having holomorphically trivial canonical bundle or of splitting type, the existence of a HS metric implies the existence of a (possibly different) Kähler one.

3 Deformation Limits of Hermitian Manifolds

In this section we focus on the existence properties of special Hermitian metrics on compact complex manifolds and their behaviour under holomorphic deformations.

We will say that a compact complex manifold X *has the property* $\mathcal{K}$, resp. $\mathcal{HS}$, $\mathcal{SKT}$, $\mathcal{B}$ or $\mathcal{SG}$, if X admits a Kähler metric, resp. an HS, SKT, B or SG metric.

Let Δ be an open disc around the origin in $\mathbb{C}$. Following [45, Definition 1.12], a given property $\mathcal{P}$ of a compact complex manifold is said to be *open* under holomorphic deformations if for every holomorphic family of compact complex manifolds $\{X_t = (M, J_t)\}_{t\in\Delta}$ and for every $t_0 \in \Delta$ the following implication holds:

$X_{t_0} = (M, J_{t_0})$ has the property $\mathcal{P} \Longrightarrow X_t = (M, J_t)$ has the property $\mathcal{P}$ for all $t \in \Delta$ sufficiently close to t_0.

A given property $\mathcal{P}$ of a compact complex manifold is said to be *closed* under holomorphic deformations if for every holomorphic family of compact complex manifolds $\{X_t = (M, J_t)\}_{t\in\Delta}$ and for every $t_0 \in \Delta$ the following implication holds:

$X_t = (M, J_t)$ has the property $\mathcal{P}$ for all $t \in \Delta\backslash\{t_0\} \Longrightarrow X_{t_0} = (M, J_{t_0})$ has the property $\mathcal{P}$.

Next we discuss the openness and closedness of each property $\mathcal{P} \in \{\mathcal{K}, \mathcal{HS}, \mathcal{SKT}, \mathcal{B}, \mathcal{SG}\}$ on compact complex manifolds X of complex dimension n.

In complex dimension $n = 2$, an SKT metric is just a Gauduchon metric (so it always exists by [28]), and by definition a metric is balanced if and only if it is Kähler. Moreover, by Li and Zhang [36] and Streets and Tian [53] (see also [46, Section 3]), if a compact complex surface X admits a Hermitian symplectic or a strongly Gauduchon metric, then X is Kähler. But X is Kähler if and only if its first Betti number $b_1(X)$ is even [40, 52]. Hence, in complex dimension $n = 2$, all the properties $\mathcal{K}$, $\mathcal{HS}$, $\mathcal{SKT}$, $\mathcal{B}$ and $\mathcal{SG}$ are both open and closed.

From now on, we will focus on compact complex manifolds of complex dimension $n \geq 3$ and we first discuss the stability of these properties. A classical result of Kodaira and Spencer [35] asserts that the Kähler property of compact complex manifolds is open under holomorphic deformations.

Concerning the property of existence of balanced Hermitian metrics, Alessandrini and Bassanelli proved in [1] (see also [19]) that it is not deformation open. Indeed, the Iwasawa manifold is balanced, however there are small deformations that do not admit any balanced metric. The Iwasawa manifold is the nilmanifold associated to the Lie algebra $\mathfrak{h}_5$ in Table 1 endowed with its complex-parallelizable structure. The deformations of the Iwasawa manifold were studied in [41]. The space of complex structures on this manifold are studied in [34] (see also [13]). Cohomological aspects of the Iwasawa manifold and of its small deformations are studied in [5] (see also [6]).

In [9] conditions on a compact balanced manifold are found so that any small holomorphic deformation X_t of X_0 still admits a balanced metric (see also [49]). It turns out that any abelian complex structure on the (real nilmanifold underlying the) Iwasawa manifold admits balanced metrics, and moreover, any sufficiently small deformation too. In other words, the property $\mathcal{B}$ is stable in the subclass of abelian complex structures on the Iwasawa manifold.

Concerning the $\mathcal{SKT}$ property, Fino and Tomassini proved in [20, Theorem 2.2] that the Iwasawa manifold has complex structures having SKT metrics and that there are small deformations of them not admitting any SKT metric. Another example is given in Remark 3.6 below. Hence, the $\mathcal{SKT}$ property is not deformation open. Some aspects of the deformation theory of SKT structures are investigated in [12].

In contrast to the SKT and the balanced cases, both the $\mathcal{HS}$ and the $\mathcal{SG}$ properties are open under holomorphic deformations (see [45, Theorem 3.1] for a proof of the latter and, for instance [64, Proposition 2.4], for the former). In the following result we collect the previous discussion, and we will provide a "unified" argument for the openness of the properties $\mathcal{HS}$ and $\mathcal{SG}$.

Proposition 3.1 *For compact complex manifolds of complex dimension $n \geq 3$, we have:*

- *the properties $\mathcal{K}$, $\mathcal{HS}$ and $\mathcal{SG}$ are open;*
- *the properties $\mathcal{SKT}$ and $\mathcal{B}$ are not open.*

Proof As we recalled above, the Kähler property of compact complex manifolds is stable under holomorphic deformations [35].

We also reminded above that different complex structures on the nilmanifold underlying the Iwasawa manifold admit balanced, resp. SKT, metrics, but sufficiently small deformations of them do not admit such metrics, see [1, Theorem 2.2], resp. [20]. Hence, the properties $\mathcal{SKT}$ and $\mathcal{B}$ are not open.

For the openness of the properties $\mathcal{HS}$ and $\mathcal{SG}$, we reproduce the proofs given in [44] and [64]. Let F be an HS or an SG metric on X. By definition, $\partial F^k = \bar{\partial}\alpha$ for some ∂-closed form α of bidegree $(k+1, k-1)$, where $k = 1$ when the metric is HS, and $k = n-1$ when the metric is SG (note that in this case α has bidegree $(n, n-2)$, so it is always ∂-closed). This is equivalent to have a $2k$-form Ω satisfying

(1) Ω is real, i.e. $\Omega = \overline{\Omega}$,
(2) Ω is closed, i.e. $d\Omega = 0$, and
(3) the (k, k)-component $\Omega^{k,k}$ of Ω is positive-definite.

(See [17, Proposition 2.1] for $k = 1$, and [44, Proposition 4.2] for $k = n-1$). In the case $k = n-1$, one uses the observation due to Michelsohn [39] that every positive-definite $(n-1, n-1)$-form has a unique $(n-1)^{st}$ root, i.e. $\Omega^{n-1,n-1} = F^{n-1}$ for a unique positive-definite $(1, 1)$-form F.

Let $\{X_t = (M, J_t)\}_{t\in\Delta}$, Δ containing 0, be a holomorphic family of compact complex manifolds such that $X_0 = (M, J_0)$ has the $\mathcal{HS}$ or the $\mathcal{SG}$ property. Take a form Ω on $X_0 = (M, J_0)$ satisfying (1)–(3), and decompose it as $\Omega = \Omega_t^{k+1,k-1} + \Omega_t^{k,k} + \Omega_t^{k-1,k+1}$ with respect to the complex structure J_t. Notice that the conditions (1) and (2) do not depend on the complex structure. Since $\Omega_0^{k,k}$ is positive-definite by (3), then $\Omega_t^{k,k}$ is also positive-definite for $t \in \Delta$ sufficiently close to $0 \in \Delta$. Thus, Ω satisfies (1)–(3) on $X_t = (M, J_t)$, that is, X_t satisfies the $\mathcal{HS}$ or the $\mathcal{SG}$ property if X_0 does. □

In the rest of this section, we will study the closedness of each property $\mathcal{P} \in \{\mathcal{SKT}, \mathcal{HS}, \mathcal{K}, \mathcal{B}, \mathcal{SG}\}$.

We recall that a *Moišezon manifold* is a compact complex manifold X which is bimeromorphic to a projective manifold, that is, there exists a modification (i.e. a proper holomorphic bimeromorphic map) $f : \tilde{X} \longrightarrow X$ from a projective manifold $\tilde{X}$. Egidi studies in [16] the behaviour of special Hermitian metrics under modifications. In particular, the following result is proved.

Proposition 3.2 ([16, Theorem 7.4]) *Let X be a Moišezon manifold. Then:*

$$X \text{ is SKT} \iff X \text{ is HS} \iff X \text{ is Kähler} \iff X \text{ is projective}.$$

Hironaka constructed in [29, 30] the first example of a Moišezon manifold which is not algebraic given as a modification of $\mathbb{P}_3(\mathbb{C})$. Moreover, Hironaka proved that the Kähler property of compact complex manifolds of complex dimension ≥ 3 is not closed under holomorphic deformations. The idea of this construction is as follows. Let C and D be two conics in $\mathbb{P}_3(\mathbb{C})$ that intersect in two points p and q. Varying D in a family $D(t)$ such that $p \in D(t)$ for all t, but only $D(0) = D$ passes through q, and blowing up appropriately the line pq, C and $D(t)$, a family of compact complex manifolds X_t is obtained. The manifold X_t is projective for every $t \neq 0$, however the central limit X_0 has a positive 1-cycle which is algebraically equivalent to zero, hence X_0 is not projective.

Notice that Popovici proves in [44] that the central limit of an analytic family of projective manifolds is a Moišezon manifold in two important special cases, namely, when the Hodge numbers $h^{0,1}_{\bar\partial}$ of the fibres are locally constant and when the limit fibre is assumed to be an SG manifold.

From Hironaka's family and Proposition 3.2, we conclude:

Proposition 3.3 *For compact complex manifolds of complex dimension $n \geq 3$, the properties $\mathcal{K}$, $\mathcal{HS}$ and $\mathcal{SKT}$ are not closed.*

The central limit in the Hironaka family is balanced. In fact, Alessandrini and Bassanelli proved in [2] that the balanced property is stable under modifications, so in particular any Moišezon manifold is balanced. Therefore, one needs to consider other analytic families in order to investigate the closedness of the $\mathcal{B}$ and $\mathcal{SG}$ properties.

Next we show that the complex geometry of nilmanifolds allows to construct analytic families of compact complex manifolds $\{X_t\}_{t\in\Delta}$, Δ being an open disc around 0 in $\mathbb{C}$, such that X_t is SG, resp. balanced, for every $t \in \Delta\backslash\{0\}$, but X_0 does not admit any SG, resp. balanced, metrics. Such analytic families are based on the complex geometries of $\mathfrak{h}_2$ and $\mathfrak{h}_4$ (see Table 1).

Let us start with $\mathfrak{h}_2$, which is the product of two 3-dimensional Heisenberg algebras. Indeed, recall that the Heisenberg group H is the nilpotent Lie group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}. \tag{8}$$

Since $\{\alpha^1 = dx, \alpha^2 = dy, \alpha^3 = xdy - dz\}$ is a basis of left-invariant 1-forms on H, the structure equations are given by $d\alpha^1 = d\alpha^2 = 0$, $d\alpha^3 = \alpha^{12}$, i.e. the Lie algebra of H is $\mathfrak{h} = (0, 0, 12)$.

Let us consider the lattice Γ given by the matrices in (8) with (x, y, z)-entries lying in $\mathbb{Z}$. Hence, Γ is a lattice of maximal rank in H. From now on, we denote by N the 3-dimensional nilmanifold $N = \Gamma\backslash H$ and we will refer to N as the Heisenberg nilmanifold.

Let us take another copy of N with basis of 1-forms $\{\beta^1, \beta^2, \beta^3\}$ satisfying the equations $d\beta^1 = d\beta^2 = 0$ and $d\beta^3 = \beta^{12}$. We consider the invariant almost-complex structure J_0 on $N \times N$ defined by

$$J_0(\alpha^1) = -\alpha^2, \quad J_0(\beta^1) = -\beta^2, \quad J_0(\alpha^3) = -\beta^3. \tag{9}$$

Notice that the Lie algebra underlying the 6-dimensional product nilmanifold $N \times N$ is isomorphic to $\mathfrak{h} \oplus \mathfrak{h} = (0, 0, 0, 0, 12, 34) = \mathfrak{h}_2$. Indeed, in terms of the basis of 1-forms $\{e^1, e^2, e^3, e^4, e^5, e^6\}$ given by

$$e^1 = \alpha^1, \ e^2 = \alpha^2, \ e^3 = \beta^1, \ e^4 = \beta^2, \ e^5 = \alpha^3, \ e^6 = \beta^3, \tag{10}$$

the structure equations are

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{12}, \quad de^6 = e^{34}. \tag{11}$$

Now, using (9) and (10), the complex 1-forms $\{\omega_0^1, \omega_0^2, \omega_0^3\}$ given by

$$\omega_0^1 = e^1 - iJ_0e^1 = e^1 + i\,e^2, \quad \omega_0^2 = e^3 - iJ_0e^3 = e^3 + i\,e^4,$$
$$\omega_0^3 = 2e^6 - 2iJ_0e^6 = 2e^6 - 2i\,e^5,$$

constitute a basis of invariant forms of bidegree $(1, 0)$ with respect to J_0. By a direct calculation using (11), we get

$$d\omega_0^1 = d\omega_0^2 = 0, \quad d\omega_0^3 = \omega_0^{1\bar{1}} + i\,\omega_0^{2\bar{2}}. \tag{12}$$

These equations immediately imply that J_0 is an abelian complex structure.

The compact complex manifold $X = (N \times N, J_0)$ does not admit any SG metric by [13], i.e. X does not satisfy the $\mathcal{SG}$ property. Next we review a construction given in [57], which shows that the holomorphic deformations of this simple complex structure have very interesting properties in relation to the existence problem of SG metric.

By Console and Fino [14] (see also [50]) the Dolbeault cohomology of the compact complex manifold $(N \times N, J_0)$ can be computed explicitly from the pair $(\mathfrak{h}_2, J_0)$, i.e. $H^{p,q}_{\bar{\partial}}(N \times N, J_0) \cong H^{p,q}_{\bar{\partial}}(\mathfrak{h}_2, J_0)$ for any $0 \leq p, q \leq 3$. In order to perform an appropriate holomorphic deformation of J_0 we first compute the particular Dolbeault cohomology group

$$H^{0,1}_{\bar{\partial}}(N \times N, J_0) \cong H^{0,1}_{\bar{\partial}}(\mathfrak{h}_2, J_0) = \langle [\omega_0^{\bar{1}}], [\omega_0^{\bar{2}}], [\omega_0^{\bar{3}}] \rangle.$$

We consider the small deformation J_t given by

$$t\frac{\partial}{\partial z_2} \otimes \omega_0^{\bar{1}} + it\frac{\partial}{\partial z_1} \otimes \omega_0^{\bar{2}} \in H^{0,1}(X_0, \ T^{1,0}X_0),$$

where X_0 denotes the complex manifold $(N \times N, J_0)$. This deformation is defined for any $t \in \mathbb{C}$, i.e. we can take $\Delta = \mathbb{C}$. The analytic family of compact complex manifolds $(N \times N, J_t)$ has a complex basis $\{\omega_t^1, \omega_t^2, \omega_t^3\}$ of type (1,0) with respect to J_t given by

$$J_t:\ \omega_t^1 = \omega_0^1 + it\,\omega_0^{\bar 2}, \quad \omega_t^2 = \omega_0^2 + t\,\omega_0^{\bar 1}, \quad \omega_t^3 = \omega_0^3. \tag{13}$$

By a direct calculation using (12), we get that the (1,0)-basis $\{\omega_t^1, \omega_t^2, \omega_t^3\}$ given in (13) satisfies

$$d\omega_t^1 = d\omega_t^2 = 0, \quad d\omega_t^3 = \frac{2i\bar t}{1+|t|^4}\omega_t^{12} + \frac{1-i|t|^2}{1+|t|^4}\omega_t^{1\bar 1} + \frac{i-|t|^2}{1+|t|^4}\omega_t^{2\bar 2}, \tag{14}$$

for any $t \in \mathbb{C}$.

The following result shows that the $\mathcal{SG}$ property is not closed.

Proposition 3.4 ([57, Theorem 5.2]) *Let $(N \times N, J_t)_{t\in\Delta}$ be the analytic family given by the product of two copies of the 3-dimensional Heisenberg nilmanifold N endowed with the complex structures J_t given by (13). Then, the complex manifolds $(N \times N, J_t)$ have strongly Gauduchon metrics for each $t \in \Delta\backslash\{0\}$, but the central fiber $(N \times N, J_0)$ does not admit any strongly Gauduchon metric.*

In particular, the $\mathcal{SG}$ property is not closed.

The proof is as follows: let us consider on $(N \times N, J_t)$ the Hermitian metric

$$F_t = \frac{i}{2}(\omega_t^{1\bar 1} + \omega_t^{2\bar 2} + \omega_t^{3\bar 3}). \tag{15}$$

It follows from (14) that $\partial(F_t)^2$ is given by

$$\partial(F_t)^2 = 2F_t \wedge \partial F_t = -\frac{1}{2}\partial(\omega_t^{1\bar 1 2\bar 2} + \omega_t^{1\bar 1 3\bar 3} + \omega_t^{2\bar 2 3\bar 3}) = \frac{1-i}{2}\,\frac{1-|t|^2}{1+|t|^4}\,\omega_t^{123\bar 1\bar 2}.$$

Since $\bar\partial\omega_t^{123\bar 3} = \frac{2it}{1+|t|^4}\,\omega_t^{123\bar 1\bar 2}$, we immediately conclude that the Hermitian metric F_t on $(N \times N, J_t)$ is SG for any $t \neq 0$.

The central limit $(N \times N, J_0)$ is not SG. Indeed, by [13, Corollary 5.2], since J_0 is abelian, an invariant Hermitian metric is SG if and only if it is balanced, but in that case the underlying Lie algebra $\mathfrak{g}$ must be isomorphic to $\mathfrak{h}_3$ or $\mathfrak{h}_5$ by [58]. In other words, abelian complex structures on nilmanifolds with underlying Lie algebra isomorphic to $\mathfrak{h}_2$ or $\mathfrak{h}_4$ do not admit SG metrics.

Remark 3.5 For each $t \in \Delta\backslash\{0\}$, the complex manifold $(N \times N, J_t)$ satisfies a stronger condition in relation to SG metrics. In [48] it is introduced and investigated the sGG manifolds, which are defined as those compact complex manifolds whose *SG cone* coincides with the *Gauduchon cone*. Hence, on an sGG manifold *every Gauduchon metric is SG*. Using the numerical characterizations of the sGG manifolds obtained in [48], which involve the Bott-Chern, Hodge and Betti numbers $h^{0,1}_{\mathrm{BC}}$, $h^{0,1}_{\bar\partial}$ and b_1, one can prove that the compact complex manifold $(N \times N, J_t)$ in Proposition 3.4 is sGG for each $t \in \Delta\backslash\{0\}$ (see [57, Theorem 5.2] for more details). Notice that the central fiber $(N \times N, J_0)$ is not sGG because its SG cone is empty.

Remark 3.6 A direct calculation shows that the metric (15) is SKT if and only $t = 0$. Since by [24] on a 6-dimensional SKT nilmanifold all the invariant Hermitian metrics are SKT, one has that $X_t = (N \times N, J_t)$ is an analytic family such that X_0 is SKT but X_t does not admit any (invariant or not) SKT metric for $t \neq 0$.

One can show that $(N \times N, J_t)$ does not admit balanced metrics (see [57, Remark 5.3] for details), so in order to study the behaviour of the $\mathcal{B}$ property in the central limit we are led to consider other analytic families. The result in Proposition 3.4 is based on the (product) algebra $\mathfrak{h}_2$. The complex geometry of nilmanifolds still allows to construct another interesting analytic families, as we will show next, but now the constructions are based on the complex geometry of a nilmanifold with $\mathfrak{h}_4$ as underlying Lie algebra.

As in the case of $\mathfrak{h}_2$, we will consider an appropriate deformation of the abelian complex structure J_0 of $\mathfrak{h}_4$ (it is proved in [4] that there exists only one abelian structure up to isomorphism). Let M be a nilmanifold with underlying algebra $\mathfrak{h}_4$ and consider on M the abelian complex structure J_0. As we recalled above, the complex manifold (M, J_0) does not admit SG metrics, thus there are not balanced metrics.

Maclaughlin, Pedersen, Poon and Salamon studied in [37] the deformation parameter space of J_0. More concretely, they proved in [37, Example 8] that J_0 has a locally complete family of deformations consisting entirely of invariant complex structures and found that the Kuranishi space has dimension 4. One can find a particular holomorphic deformation for J_0 having balanced metrics. Indeed, for each $t \in \Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, we consider a basis of complex 1-forms $\{\mu_t^1, \mu_t^2, \mu_t^3\}$ of bidegree (1,0) satisfying the equations

$$\begin{cases} d\mu_t^1 = 0, \\ d\mu_t^2 = 0, \\ d\mu_t^3 = \frac{\bar t}{1-|t|^2}\,\mu_t^{12} + \frac{i}{2(1-|t|^2)}\,\mu_t^{1\bar 1} + \frac{1}{2(1-|t|^2)}\,\mu_t^{1\bar 2} + \frac{1}{2(1-|t|^2)}\,\mu_t^{2\bar 1} - \frac{i|t|^2}{2(1-|t|^2)}\,\mu_t^{2\bar 2}, \end{cases} \tag{16}$$

for each $t \in \Delta$. Notice that this basis defines implicitly an invariant complex structure J_t on the nilmanifold M just by declaring that the forms $\mu_t^1, \mu_t^2, \mu_t^3$ are of type (1,0) with respect to J_t. By (16) one can immediately see that the complex structure J_t is abelian if and only if $t = 0$.

Let us consider on the complex nilmanifold (M, J_t) the real 2-form

$$F_t = \frac{i}{2}(\mu_t^{1\bar 1} + |t|^2\,\mu_t^{2\bar 2} + \mu_t^{3\bar 3}).$$

Using (16) it is easy to check that F_t^2 is closed, so it defines a balanced Hermitian for every $t \neq 0$.

In conclusion, one has the following result:

Proposition 3.7 ([13, Theorem 5.9]) *The holomorphic family $(M, J_t)_{t\in\Delta}$ of compact complex manifolds given by (16), where $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, has balanced metrics for each $t \in \Delta\backslash\{0\}$, but (M, J_0) does not admit any SG metric.*

In particular, the $\mathcal{B}$ property is not closed.

We sum up the above conclusions about closedness, i.e. Propositions 3.3, 3.4 and 3.7, in the following result.

Theorem 3.8 *For compact complex manifolds of complex dimension $n \geq 3$, the properties $\mathcal{K}$, $\mathcal{HS}$, $\mathcal{SKT}$, $\mathcal{B}$ and $\mathcal{SG}$ are not closed.*

Recall that an astheno-Kähler metric is a Hermitian metric F satisfying $\partial\bar{\partial}F^{n-2} = 0$. Since in complex dimension $n = 3$, SKT and astheno-Kähler metrics coincide, the previous theorem implies that the astheno-Kähler property is neither open nor closed under holomorphic deformations.

Remark 3.9 Another important property of compact complex manifolds whose behaviour under holomorphic deformations has also been investigated is the $\partial\bar{\partial}$-lemma property. This property is stable under small deformations of the complex structure [60, 62]. Angella and Tomassini give in [8] another proof of this result, based on a characterization of the $\partial\bar{\partial}$-lemma property in terms of the Bott-Chern, Aeppli and Betti numbers of the manifold.

Angella and Kasuya obtain in [7] a holomorphic deformation that shows that the $\partial\bar{\partial}$-lemma property is not closed. The construction in [7] consists in a suitable deformation of the holomorphically parallelizable Nakamura manifold. More recently, based on the rich complex geometry of splitting type of the Nakamura manifold, it is given in [10], for each $k \in \mathbb{Z}$, a compact complex manifold X_k that does not satisfy the $\partial\bar{\partial}$-Lemma, but X_k admits a small holomorphic deformation $\{(X_k)_t\}_{t\in\Delta_k}$, Δ_k being an open disc in $\mathbb{C}$ around 0, such that $(X_k)_t$ is a compact complex $\partial\bar{\partial}$-manifold for any $t \neq 0$. When $k = -1$, one recovers the main result in [7] because it corresponds precisely to the complex-parallelizable structure. The case $k = 0$ corresponds to the abelian complex structure (which is unique up to isomorphism [4]), so the abelian complex structure on the Nakamura manifold (which does not satisfy the $\partial\bar{\partial}$-Lemma) is the central limit of an analytic family of compact complex $\partial\bar{\partial}$-manifolds. It is worthy to note that all the compact complex manifolds $(X_k)_t$ in these analytic families have holomorphically trivial canonical bundle and admit balanced metrics.

Remark 3.10 The Nakamura manifold has $\mathfrak{g}_8 = \mathfrak{s}_{12}$ as underlying solvable Lie algebra (see Tables 2 and 3). By [26, Proposition 3.7] and [10, Proposition 3.1], there exist exactly two (up to isomorphism) complex structures J' and J'' giving rise to complex solvmanifolds with holomorphically trivial canonical bundle, which are not of splitting type. The complex structures J' and J'' do not admit any balanced metric. By using appropriate holomorphic deformations of these complex structures, in [26, Theorem 5.2] it is constructed an analytic family of compact complex manifolds $(X_t)_{t\in\Delta}$ such that X_t satisfies the $\partial\bar{\partial}$-lemma and admits balanced metric for any $t \in \Delta\backslash\{0\}$, but the central limit X_0 neither satisfies the $\partial\bar{\partial}$-lemma nor admits balanced metrics.

Notice that the complex structures J' and J'' have SG metrics (see Table 2), so the central limit X_0 is an SG manifold (this is consistent with a result in [46, Proposition 4.1] about deformation limits of $\partial\bar{\partial}$-manifolds).

Acknowledgements This work has been partially supported by the projects MINECO (Spain) MTM2014-58616-P, and Gobierno de Aragón/Fondo Social Europeo, grupo consolidado E15-Geometría. We are grateful to the referee for helpful comments and suggestions.

References

1. L. Alessandrini, G. Bassanelli, Small deformations of a class of compact non-Kähler manifolds. Proc. Am. Math. Soc. **109**, 1059–1062 (1990)
2. L. Alessandrini, G. Bassanelli, Modifications of compact balanced manifolds. C.R. Acad. Sci. Paris Math. **320**, 1517–1522 (1995)
3. B. Alexandrov, S. Ivanov, Vanishing theorems on Hermitian manifolds. Diff. Geom. Appl. **14**, 251–265 (2001)
4. A. Andrada, M.L. Barberis, I. Dotti, Classification of abelian complex structures on 6-dimensional Lie algebras. J. Lond. Math. Soc. (2) **83**(1), 232–255 (2011); Corrigendum: J. Lond. Math. Soc. (2) **87**(1), 319–320 (2013)
5. D. Angella, The cohomologies of the Iwasawa manifold and of its small deformations. J. Geom. Anal. **23**(3), 1355–1378 (2013)
6. D. Angella, *Cohomological Aspects in Complex Non-Kähler Geometry*. Lecture Notes in Mathematics, vol. 2095 (Springer, Cham, 2014)
7. D. Angella, H. Kasuya, Cohomologies of deformations of solvmanifolds and closedness of some properties, North-West. Eur. J. Math. **3**, 75–105 (2017)
8. D. Angella, A. Tomassini, On the $\partial\bar{\partial}$-Lemma and Bott-Chern cohomology. Invent. Math. **192**, 71–81 (2013)
9. D. Angella, L. Ugarte, On small deformations of balanced manifolds. Diff. Geom. Appl. **54**, Part B, 464–474 (2017)
10. D. Angella, A. Otal, L. Ugarte, R. Villacampa, Complex structures of splitting type. Rev. Mat. Iberoam. **33**(4), 1309–1350 (2017)
11. J.M. Bismut, A local index theorem for non-Kähler manifolds. Math. Ann. **284**, 681–699 (1989)
12. G.R. Cavalcanti, Hodge theory and deformations of SKT manifolds. Preprint (2012). arXiv:1203.0493 V5 [math.DG]
13. M. Ceballos, A. Otal, L. Ugarte, R. Villacampa, Invariant complex structures on 6-nilmanifolds: classification, Frölicher spectral sequence and special Hermitian metrics. J. Geom. Anal. **26**(1), 252–286 (2016)
14. S. Console, A. Fino, Dolbeault cohomology of compact nilmanifolds. Transform. Groups **6**, 111–124 (2001)
15. P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, Real homotopy theory of Kähler manifolds. Invent. Math. **29**, 245–274 (1975)
16. N. Egidi, Special metrics on compact complex manifolds. Diff. Geom. Appl. **14**, 217–234 (2001)

17. N. Enrietti, A. Fino, L. Vezzoni, Tamed symplectic forms and SKT metrics. J. Sympl. Geom. **10**, 203–224 (2012)
18. M. Fernández, A. Fino, L. Ugarte, R. Villacampa, Strong Kähler with torsion structures from almost contact manifolds. Pac. J. Math. **249**, 49–75 (2011)
19. A. Fino, G. Grantcharov, Properties of manifolds with skew-symmetric torsion and special holonomy. Adv. Math. **189**, 439–450 (2004)
20. A. Fino, A. Tomassini, Blow-ups and resolutions of strong Kähler with torsion metrics. Adv. Math. **221**, 914–935 (2009)
21. A. Fino, L. Ugarte, On generalized Gauduchon metrics. Proc. Edinb. Math. Soc. **56**, 733–753 (2013)
22. A. Fino, L. Vezzoni, Special Hermitian metrics on compact solvmanifolds. J. Geom. Phys. **91**, 40–53 (2015)
23. A. Fino, L. Vezzoni, On the existence of balanced and SKT metrics on nilmanifolds. Proc. Am. Math. Soc. **144**, 2455–2459 (2016)
24. A. Fino, M. Parton, S. Salamon, Families of strong KT structures in six dimensions. Comment. Math. Helv. **79**, 317–340 (2004)
25. A. Fino, H. Kasuya, L. Vezzoni, SKT and tamed symplectic structures on solvmanifolds. Tohoku Math. J. **67**, 19–37 (2015)
26. A. Fino, A. Otal, L. Ugarte, Six-dimensional solvmanifolds with holomorphically trivial canonical bundle. Int. Math. Res. Not. IMRN **24**, 13757–13799 (2015)
27. J. Fu, J. Li, S.-T. Yau, Balanced metrics on non-Kähler Calabi-Yau threefolds. J. Differ. Geom. **90**, 81–129 (2012)
28. P. Gauduchon, La 1-forme de torsion d'une variété hermitienne compacte. Math. Ann. **267**, 495–518 (1984)
29. H. Hironaka, On the theory of birational blowing up, Thesis, Harvard, 1960
30. H. Hironaka, An example of a non-Kählerian complex-analytic deformation of Kählerian complex structures. Ann. Math. (2) **75**, 190–208 (1962)
31. S. Ivanov, G. Papadopoulos, Vanishing theorems on $(l|k)$-strong Kähler manifolds with torsion. Adv. Math. **237**, 147–164 (2013)
32. H. Kasuya, Vaisman metrics on solvmanifolds and Oeljeklaus-Toma manifolds. Bull. Lond. Math. Soc. **45**, 15–26 (2013)
33. H. Kasuya, Techniques of computations of Dolbeault cohomology of solvmanifolds. Math. Z. **273**, 437–447 (2013)
34. G. Ketsetzis, S. Salamon, Complex structures on the Iwasawa manifold. Adv. Geom. **4**, 165–179 (2004)
35. K. Kodaira, D.C. Spencer, On deformations of complex analytic structures. III. Stability theorems for complex structures. Ann. Math. **71**(2), 43–76 (1960)
36. T.-J. Li, W. Zhang, Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds. Commun. Anal. Geom. **17**, 651–683 (2009)
37. C. Maclaughlin, H. Pedersen, Y.S. Poon, S. Salamon, Deformation of 2-step nilmanifolds with abelian complex structures. J. Lond. Math. Soc. **73**, 173–193 (2006)
38. K. Matsuo, T. Takahashi, On compact astheno-Kähler manifolds. Colloq. Math. **89**, 213–221 (2001)
39. M.L. Michelsohn, On the existence of special metrics in complex geometry. Acta Math. **149**, 261–295 (1982)
40. Y. Miyaoka, Kähler metrics on elliptic surfaces. Proc. Jpn. Acad. **50**, 533–536 (1974)
41. I. Nakamura, Complex parallelisable manifolds and their small deformations. J. Differ. Geom. **10**, 85–112 (1975)
42. A. Otal, L. Ugarte, R. Villacampa, Invariant solutions to the Strominger system and the heterotic equations of motion. Nuclear Phys. B. **920**, 442–474 (2017)
43. D. Popovici, Stability of strongly Gauduchon manifolds under modifications. J. Geom. Anal. **23**(2), 653–659 (2013)
44. D. Popovici, Deformation limits of projective manifolds: Hodge numbers and strongly Gauduchon metrics. Invent. Math. **194**, 515–534 (2013)

45. D. Popovici, Deformation openness and closedness of various classes of compact complex manifolds; examples. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13**(2), 255–305 (2014)
46. D. Popovici, Limits of projective manifolds under holomorphic deformations. Preprint (2009). arXiv:0910.2032 V2 [math.AG]
47. D. Popovici, Holomorphic deformations of balanced Calabi-Yau $\partial\bar{\partial}$-manifolds. Preprint (2013). arXiv:1304.0331 [math.AG]
48. D. Popovici, L. Ugarte, The sGG class of compact complex manifolds. Preprint (2014). arXiv:1407.5070 [math.DG]
49. S. Rao, X. Wan, Q. Zhao, Power series proofs for local stabilities of Kähler and balanced structures with mild $\partial\bar{\partial}$-lemma. Preprint (2016). arXiv:1609.05637 [math.CV]
50. S. Rollenske, Geometry of nilmanifolds with left-invariant complex structure and deformations in the large. Proc. Lond. Math. Soc. **99**, 425–460 (2009)
51. S. Salamon, Complex structures on nilpotent Lie algebras. J. Pure Appl. Algebra **157**, 311–333 (2001)
52. Y.-T. Siu, Every K3 surface is Kähler. Invent. Math. **73**, 139–150 (1983)
53. J. Streets, G. Tian, A Parabolic flow of pluriclosed metrics. Int. Math. Res. Not. **16**, 3101–3133(2010)
54. A. Swann, Twisting Hermitian and hypercomplex geometries. Duke Math. J. **155**, 403–431 (2010)
55. V. Tosatti, B. Weinkove, Hermitian metrics, $(n-1, n-1)$-forms and Monge-Ampère equations. J. Reine Angew. Math. (to appear 2017)
56. L. Ugarte, Hermitian structures on six dimensional nilmanifolds. Transform. Groups **12**, 175–202 (2007)
57. L. Ugarte, Special Hermitian metrics, complex nilmanifolds and holomorphic deformations. Rev. R. Acad. Cienc. Exactas Fís. Quím. Nat. Zaragoza (2) **69**, 7–36 (2014)
58. L. Ugarte, R. Villacampa, Balanced Hermitian geometry on 6-dimensional nilmanifolds. Forum Math. **27**, 1025–1070 (2015)
59. M. Verbitsky, Rational curves and special metrics on twistor spaces. Geom. Topol. **18**, 897–909 (2014)
60. C. Voisin, *Théorie de Hodge et Géométrie Algébrique Complexe*. Cours Spécialisés, vol. 10 (Société Mathématique de France, Paris, 2002)
61. D. Witte, Superrigidity of lattices in solvable Lie groups. Invent. Math. **122**, 147–193 (1995)
62. C-C. Wu, On the geometry of superstrings with torsion, Ph.D. thesis, Harvard University, Proquest LLC, Ann Arbor, MI, 2006
63. J. Xiao, On strongly Gauduchon metrics of compact complex manifolds. J. Geom. Anal. **25**, 2011–2027 (2015)
64. S. Yang, On blow-ups and resolutions of Hermitian-symplectic and strongly Gauduchon metrics. Arch. Math. (Basel) **104**, 441–450 (2015)

On the Cohomology of Some Exceptional Symmetric Spaces

Paolo Piccinni

Dedicated to Simon Salamon on the occasion of his 60th birthday

Abstract This is a survey on the construction of a canonical or "octonionic Kähler" 8-form, representing one of the generators of the cohomology of the four Cayley-Rosenfeld projective planes. The construction, in terms of the associated even Clifford structures, draws a parallel with that of the quaternion Kähler 4-form. We point out how these notions allow to describe the primitive Betti numbers with respect to different even Clifford structures, on most of the exceptional symmetric spaces of compact type.

Keywords Canonical differential form • Even Clifford structure • Exceptional symmetric space • Primitive cohomology

2010 Mathematics Subject Classification: Primary 53C26, 53C27, 53C35, 53C38

1 Introduction

The exceptional Riemannian symmetric spaces of compact type

$$\mathrm{E\,I},\ \mathrm{E\,II},\ \mathrm{E\,III},\ \mathrm{E\,IV},\ \mathrm{E\,V},\ \mathrm{E\,VI},\ \mathrm{E\,VII},\ \mathrm{E\,VIII},\ \mathrm{E\,IX},\ \mathrm{F\,I},\ \mathrm{F\,II},\ \mathrm{G\,I}$$

are part of the E. Cartan classification.

P. Piccinni (✉)
Dipartimento di Matematica, Sapienza-Università di Roma, Piazzale Aldo Moro 2, I-00185 Roma, Italy
e-mail: piccinni@mat.uniroma1.it

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*, Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_12

Among them, the two Hermitian symmetric spaces

$$\mathrm{E\,III} = \frac{\mathrm{E}_6}{\mathrm{Spin}(10)\cdot \mathrm{U}(1)} \quad \text{and} \quad \mathrm{E\,VII} = \frac{\mathrm{E}_7}{\mathrm{E}_6\cdot \mathrm{U}(1)}$$

are certainly notable. As Fano manifolds, they can be realized as smooth complex projective varieties. As such, E III is also called the *fourth Severi variety*, a complex 16-dimensional projective variety in $\mathbb{C}P^{26}$, characterized as one of the four smooth projective varieties of small critical codimension in their ambient $\mathbb{C}P^N$, and that are unable to fill it through their secant and tangent lines [28]. The projective model of E VII is instead known as the *Freudenthal variety*, a complex 27-dimensional projective variety in $\mathbb{C}P^{55}$, considered for example in the sequel of papers [7].

Next, among the listed symmetric spaces, the five Wolf spaces

$$\mathrm{E\,II} = \frac{\mathrm{E}_6}{\mathrm{SU}(6)\cdot \mathrm{Sp}(1)}, \quad \mathrm{E\,VI} = \frac{\mathrm{E}_7}{\mathrm{Spin}(12)\cdot \mathrm{Sp}(1)}, \quad \mathrm{E\,IX} = \frac{\mathrm{E}_8}{\mathrm{E}_7\cdot \mathrm{Sp}(1)},$$
$$\mathrm{F\,I} = \frac{\mathrm{F}_4}{\mathrm{Sp}(3)\cdot \mathrm{Sp}(1)}, \quad \mathrm{G\,I} = \frac{\mathrm{G}_2}{\mathrm{SO}(4)}$$

give evidence for the long lasting LeBrun-Salamon conjecture [14], being the only known sporadic examples of positive quaternion Kähler manifolds.

Thus, seven of the twelve exceptional Riemannian symmetric spaces of compact type are either Kähler or quaternion Kähler. Accordingly, one of their de Rham cohomology generators is represented by a Kähler 2-form or a quaternion Kähler 4-form, and any further cohomology generators can be looked as primitive in the sense of the Lefschetz decomposition.

The notion of *even Clifford structure*, introduced some years ago by Moroianu and Semmelmann [15], allows not only to deal simultaneously with Kähler and quaternion Kähler manifolds, but also to recognize further interesting geometries fitting into the notion. Among them, and just looking at the exceptional Riemannian symmetric spaces of compact type, there are even Clifford structures, related with octonions, on the following *Cayley-Rosenfeld projective planes*

$$\mathrm{E\,III} = \frac{\mathrm{E}_6}{\mathrm{Spin}(10)\cdot \mathrm{U}(1)}, \qquad \mathrm{E\,VI} = \frac{\mathrm{E}_7}{\mathrm{Spin}(12)\cdot \mathrm{Sp}(1)},$$
$$\mathrm{E\,VIII} = \frac{\mathrm{E}_8}{\mathrm{Spin}^+(16)}, \qquad \mathrm{F\,II} = \frac{\mathrm{F}_4}{\mathrm{Spin}(9)}.$$

An even Clifford structure is defined as the datum, on a Riemannian manifold (M, g), of a real oriented Euclidean vector bundle (E, h), together with an algebra bundle morphism

$$\varphi :\ \mathrm{Cl}^0(E) \to \mathrm{End}(TM)$$

mapping $\Lambda^2 E$ into skew-symmetric endomorphisms. The rank r of E is said to be the *rank of the even Clifford structure*. One easily recognizes that Kähler and quaternion Kähler metrics correspond to a choice of such a vector bundle E with $r = 2, 3$ respectively, and that for the four Cayley-Rosenfeld projective planes E III, E VI, E VIII, F II there is a similar vector bundle E with $r = 10, 12, 16, 9$, cf. [15].

Thus, among the exceptional symmetric spaces of compact type, there are two spaces admitting two distinct even Clifford structures. Namely, the Hermitian symmetric E III has both a rank 2 and a rank 10 even Clifford structure, and the quaternion Kähler E VI has both a rank 3 and a rank 12 even Clifford structure. Moreover, all the even Clifford structures we are here considering on our symmetric spaces are *parallel*, i.e. there is a metric connection ∇^E on (E, h) such that φ is connection preserving:

$$\varphi(\nabla^E_X \sigma) = \nabla^g_X \varphi(\sigma),$$

for every tangent vector $X \in TM$, section σ of $\mathrm{Cl}^0(E)$, and where ∇^g is the Levi Civita connection of the Riemannian metric g. For simplicity, we will call *octonionic Kähler* the parallel even Clifford structure defined by the vector bundles $E^{10}, E^{12}, E^{16}, E^9$ on the Cayley-Rosenfeld projective planes E III, E VI, E VIII, F II.

In conclusion, and with the exceptions of

$$\mathrm{E\,I} = \frac{\mathrm{E}_6}{\mathrm{Sp}(4)}, \quad \mathrm{E\,IV} = \frac{\mathrm{E}_6}{\mathrm{F}_4}, \quad \mathrm{E\,V} = \frac{\mathrm{E}_6}{\mathrm{SU}(8)},$$

nine of the twelve exceptional Riemannian symmetric spaces of compact type admit at least one parallel even Clifford structure.

Aim of the present paper is to describe how, basing on the recent work [19, 21, 22] about Spin(9), Spin(10)·U(1) and further even Clifford structures, one can construct canonical differential 8-forms on the symmetric spaces E III, E VI, E VIII, F II. Their classes are one of the cohomology generators, namely the one corresponding to their octonionic Kähler structure. I will discuss in particular for which of our nine exceptional Riemannian symmetric spaces of compact type the de Rham cohomology is fully canonical, i.e. fully generated by classes represented by canonical forms associated with parallel even Clifford structures.

2 Poincaré Polynomials

The following Table 1 collects some informations on the exceptional symmetric spaces of compact type. For each of them the dimension, the existence of torsion in the integral cohomology, the Kähler or quaternion Kähler or octonionic Kähler (K/qK/oK) property, the Euler characteristic χ, and the Poincaré polynomial (up to mid dimension) are listed. The last column contains the references where the

Table 1 Exceptional symmetric spaces of compact type

	Dim	Torsion	K/qK/oK	χ	Poincaré polynomial $P(t) = \sum_{i=0,\dots} b_i t^i$	Reference
E I	42	Yes		4	$1 + t^8 + t^9 + t^{16} + t^{17} + t^{18} + \dots$	[11]
E II	40	Yes	qK	36	$1 + t^4 + t^6 + 2t^8 + t^{10} + 3t^{12} + 2t^{14} + 3t^{16} + 2t^{18} + 4t^{20} + \dots$	[12]
E III	32	No	K/oK	27	$1 + t^2 + t^4 + t^6 + 2(t^8 + t^{10} + t^{12} + t^{14}) + 3t^{16} + \dots$	[26]
E IV	26	No		0	$1 + t^9 + \dots$	[1]
E V	70	Yes		72	$1 + t^6 + t^8 + t^{10} + t^{12} + 2(t^{14} + t^{16} + t^{18} + t^{20}) +$ $+3(t^{22} + t^{24} + t^{26} + t^{28}) + 4(t^{30} + t^{32}) + 3t^{34} + \dots$	
E VI	64	Yes	qK/oK	63	$1 + t^4 + 2t^8 + 3t^{12} + 4t^{16} + 5t^{20} + 6(t^{24} + t^{28}) + 7t^{32} + \dots$	[16]
E VII	54	No	K	56	$1 + t^2 + t^4 + t^6 + t^8 + 2(t^{10} + t^{12} + t^{14} + t^{16}) +$ $+3(t^{18} + t^{20} + t^{22} + t^{24} + t^{26}) + \dots$	[27]
E VIII	128	Yes	oK	135	$1 + t^8 + t^{12} + 2(t^{16} + t^{20}) + 3(t^{24} + t^{28}) + 5t^{32} +$ $+4t^{36} + 6(t^{40} + t^{44}) + 7(t^{48} + t^{52}) + 8t^{56} + 7t^{60} + 9t^{64} + \dots$	
E IX	112	Yes	qK	120	$1 + t^4 + t^8 + 2(t^{12} + t^{16}) + 3t^{20} + 4(t^{24} + t^{28}) +$ $+5t^{32} + 6(t^{36} + t^{40}) + 7(t^{44} + t^{48} + t^{52}) + 8t^{56} + \dots$	[25]
F I	28	Yes	qK	12	$1 + t^4 + 2(t^8 + t^{12}) + \dots$	[13]
F II	16	No	oK	3	$1 + t^8 + \dots$	[4]
G I	8	Yes	qK	3	$1 + t^4 + \dots$	[4]

above informations are taken from. The Euler characteristics can be confirmed via the theory of elliptic genera, cf. [10].

Most of the de Rham cohomology structures are in the literature, according to the mentioned references. The author was not able to find a reference for the cohomology computations of E V and E VIII. Their de Rham cohomologies can however be obtained through the following Borel presentation, cf. the original Borel article [3], as well as [16].

Let G be a compact connected Lie group, let H be a closed connected subgroup of G of maximal rank, and T a common maximal torus. The de Rham cohomology of G/H can be computed in terms of those of the classifying spaces BG, BH, BT according to

$$H^*(G/H) \overset{\sim}{\longleftarrow} H^*(BH)/\rho^* H^+(BG) \cong H^*(BT)^{W(H)}/(H^+(BT)^{W(G)}),$$

thus as quotient of the ring $H^*(BT)^{W(H)}$ of invariants of the Weyl group $W(H)$. Here notations refer to the fibration

$$G/H \overset{i}{\to} BH \overset{\rho}{\to} BG.$$

Also $H^+ = \oplus_{i>0} H^i$, and $(H^+(BT)^{W(G)})$ is the ideal of $H^*(BT)^{W(H)}$ generated by $H^+(BT)^{W(G)}$.

The two cases of interest for us are

$$\mathrm{E\,V}: \ (G,H) = (\mathrm{E}_7, \mathrm{SU}(8)) \quad \text{and} \quad \mathrm{E\,VIII}: \ (G,H) = (\mathrm{E}_8, \mathrm{Spin}^+(16)).$$

In fact, the rings of invariants of the Weyl groups $W(\mathrm{E}_7)$, $W(\mathrm{E}_8)$ have been computed in [16, 17, 26]. They read:

$$H^*(BT)^{W(\mathrm{E}_7)} \cong \mathbb{R}[\sigma_2, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{18}],$$

$$H^*(BT)^{W(\mathrm{E}_8)} \cong \mathbb{R}[\rho_2, \rho_8, \rho_{12}, \rho_{14}, \rho_{18}, \rho_{20}, \rho_{24}, \rho_{30}],$$

with $\sigma_\beta, \rho_\beta \in H^{2\beta}$.

As mentioned, E V and E VIII are quotients respectively of E_7 by SU(8) and of E_8 by the subgroup $\mathrm{Spin}^+(16)$, a $\mathbb{Z}_2$ quotient of Spin(16) that is not SO(16), see [13]. Since SU(8) has the same rank 7 of E_7 and $\mathrm{Spin}^+(16)$ has the same rank 8 of E_8, we can use Borel presentation.

With Chern, Euler and Pontrjagin classes notations:

$$H^*(BT)^{W(\mathrm{SU}(8))} \cong \mathbb{R}[c_2, c_3, c_4, c_5, c_6, c_7, c_8],$$

$$H^*(BT)^{W(\mathrm{Spin}^+(16))} \cong \mathbb{R}[e, p_1, p_2, p_3, p_4, p_5, p_6, p_7],$$

where $c_\alpha \in H^{2\alpha}, e \in H^{16}, p_\alpha \in H^{4\alpha}$. The cohomologies of E V and E VIII are now easily obtained. By interpreting the σ_β and ρ_β as relations among polynomials in the mentioned Chern, Euler and Pontrjagin classes, the Poincaré polynomials of E V and E VIII included in Table 1 follow from a straightforward computation.

Next, the following Table 2 contains the primitive Poincaré polynomials $\widetilde{P}(t) = \sum_{i=0,\dots} \widetilde{b}_i t^i$ of the nine exceptional Riemannian symmetric spaces that admit an even parallel Clifford structures. Here "primitive" has a different meaning, according to the considered K/qK/oK structure. Thus, for the Hermitian symmetric spaces E III and E VII, they are simply the polynomials with coefficients the primitive Betti numbers

$$\widetilde{b}_i = \dim\,(\ker[L_\omega^{n-i+1} : H^i \to H^{2n-i+2}]),$$

where L_ω is the Lefschetz operator, the multiplication of cohomology classes with that of complex Kähler form ω, and n is the complex dimension.

In the positive quaternion Kähler setting, one has the vanishing of odd Betti numbers and the injectivity of the Lefschetz operator $L_\Omega : H^{2k-4} \to H^{2k}$, $k \le n$, now with Ω the quaternion 4-form and n the quaternionic dimension [23, 24]. A remarkable aspect of the primitive Betti numbers

$$\widetilde{b}_{2k} = \dim(\mathrm{coker}[L_\Omega : H^{2k-4} \to H^{2k}])$$

for positive quaternion Kähler manifolds is their coincidence with the ordinary Betti numbers of the associated Konishi bundle, the 3-Sasakian manifold fibering over it, cf. [9, p. 56]. Indeed, one can check this coincidence on the exceptional Wolf spaces: just compare the quaternion Kähler part of Table 2 with Table III in the quoted paper by Krzysztof Galicki and Simon Salamon.

Finally, on the four Cayley-Rosenfeld planes, one still has the vanishing of odd Betti numbers and the injectivity of the map

$$L_\Phi : H^{2k-8} \to H^{2k},$$

defined by multiplication with the octonionic 8-form Φ, and with $k \le 2n$, n now the octonionic dimension, cf. Sects. 3 and 4.

Note that E III and E VI appear twice in Table 2. The intersection of their primitive cohomology with respect with to the K/ok, respectively qK/oK structure, give rise to the "fully primitive Poincaré polynomials", listed in Table 3 for the nine exceptional symmetric spaces of compact type admitting an even Clifford structure.

Table 2 Primitive Poincaré polynomials $\widetilde{P}(t) = \sum_{i=0,\dots} \widetilde{b}_i t^i$

Hermitian symmetric spaces	*Kähler primitive Poincaré polynomial*
E III	$1 + t^8 + t^{16}$
E VII	$1 + t^{10} + t^{18}$
Wolf spaces	*Quaternion Kähler primitive Poincaré polynomial*
E II	$1 + t^6 + t^8 + t^{12} + t^{14} + t^{20}$
E VI	$1 + t^8 + t^{12} + t^{16} + t^{20} + t^{24} + t^{32}$
E IX	$1 + t^{12} + t^{20} + t^{24} + t^{32} + t^{36} + t^{44} + t^{56}$
F I	$1 + t^8$
G I	1
Cayley-Rosenfeld planes	*Octonionic Kähler primitive Poincaré polynomial*
E III	$1 + t^2 + t^4 + t^6 + t^8 + t^{10} + t^{12} + t^{14} + t^{16}$
E VI	$1 + t^4 + t^8 + 2(t^{12} + t^{16} + t^{20}) + 3(t^{24} + t^{28} + t^{32})$
E VIII	$1 + t^{12} + t^{16} + t^{20} + t^{24} + t^{28} + t^{32} + t^{36} + t^{40} +$ $t^{44} + t^{48} + t^{52} + t^{56} + t^{60} + t^{64}$
F II	1

3 The Octonionic Kähler 8-Form

An even Clifford structure on the Cayley-Rosenfeld projective planes F II, E III, E VI, E VIII is given by a suitable vector sub-bundle of their endomorphism bundle.

To describe these vector sub-bundles look first, as proposed by Friedrich [8], at the following matrices, defining self-dual anti-commuting involutions in $\mathbb{R}^{16}$. It is natural to name them the *octonionic Pauli matrices*:

$$S_0 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \quad S_1 = \left(\begin{array}{c|c} 0 & -R_i \\ \hline R_i & 0 \end{array}\right), \quad S_2 = \left(\begin{array}{c|c} 0 & -R_j \\ \hline R_j & 0 \end{array}\right),$$

$$S_3 = \left(\begin{array}{c|c} 0 & -R_k \\ \hline R_k & 0 \end{array}\right), \quad S_4 = \left(\begin{array}{c|c} 0 & -R_e \\ \hline R_e & 0 \end{array}\right), \quad S_5 = \left(\begin{array}{c|c} 0 & -R_f \\ \hline R_f & 0 \end{array}\right),$$

$$S_6 = \left(\begin{array}{c|c} 0 & -R_g \\ \hline R_g & 0 \end{array}\right), \quad S_7 = \left(\begin{array}{c|c} 0 & -R_h \\ \hline R_h & 0 \end{array}\right), \quad S_8 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right).$$

Here R_u is the right multiplication by the unit basic octonion $u = i, j, k, e, f, g, h$, and matrices S_α act on $\mathbb{O}^2 \cong \mathbb{R}^{16}$. As mentioned, for all $\alpha = 0, \dots, 8$ and $\alpha \neq \beta$, one has

$$S_\alpha^* = S_\alpha, \qquad S_\alpha^2 = \mathrm{Id}, \qquad S_\alpha S_\beta = -S_\beta S_\alpha.$$

Next, on the real vector spaces

$$\mathbb{C}^{16} = \mathbb{C} \otimes \mathbb{R}^{16}, \qquad \mathbb{H}^{16} = \mathbb{H} \otimes \mathbb{R}^{16}, \qquad \mathbb{O}^{16} = \mathbb{O} \otimes \mathbb{R}^{16},$$

besides the endomorphisms S_α ($\alpha = 0, \ldots, 8$), thought now acting on the factor $\mathbb{R}^{16}$, look also at the skew-symmetric endomorphisms:

$$\mathfrak{I} \text{ on } \mathbb{C}^{16}, \qquad \mathfrak{I}, \mathfrak{J}, \mathfrak{K} \text{ on } \mathbb{H}^{16}, \qquad \mathfrak{I}, \mathfrak{J}, \mathfrak{K}, \mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \text{ on } \mathbb{O}^{16},$$

the multiplication on the left factor of the tensor product by the basic units in $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. By enriching the nine S_α with such complex structures, we generate real vector subspaces

$$E^{10} \subset \text{End}\,(\mathbb{C}^{16}), \qquad E^{12} \subset \text{End}\,(\mathbb{H}^{16}), \qquad E^{16} \subset \text{End}\,(\mathbb{O}^{16}).$$

In order to get a Clifford map

$$\varphi : \ \text{Cl}^0(E) \to \text{End}(- \otimes \mathbb{R}^{16}),$$

assume all generators to be anti-commuting with respect to a product formally defined as $\mathfrak{I} \wedge S_\alpha = -S_\alpha \wedge \mathfrak{I}, \ldots$. Of course, the products $S_\alpha \wedge S_\beta$ and $\mathfrak{I} \wedge \mathfrak{J}, \ldots$ are the usual compositions of endomorphisms.

It is convenient to use the notations:

$$S_{-1} = \mathfrak{I},\ S_{-2} = \mathfrak{J},\ S_{-3} = \mathfrak{K},\ S_{-4} = \mathfrak{E},\ S_{-5} = \mathfrak{F},\ S_{-6} = \mathfrak{G},\ S_{-7} = \mathfrak{H},$$

allowing to exhibit the mentioned even Clifford structures as

$$E^9 =< S_0, \ldots, S_8 >, \quad E^{10} =< S_{-1} > \oplus < S_0, \ldots, S_8 >,$$

$$E^{12} =< S_{-1}, S_{-2}, S_{-3} > \oplus < S_0, \ldots, S_8 >,$$

$$E^{16} =< S_{-1}, \ldots, S_{-7} > \oplus < S_0, \ldots, S_8 >,$$

respectively defined on $\mathbb{R}^{16}, \mathbb{C}^{16}, \mathbb{H}^{16}, \mathbb{O}^{16}$. Note that the first summands of E^{10}, E^{12}, E^{16} are the complex, the quaternionic and the octonionic structure in these linear spaces.

On Riemannian manifolds M, like the symmetric spaces F II, E III, E VI, E VIII, the even Clifford structures are defined as vector bundles

$$E^9, \qquad E^{10} = E^1 \oplus E^9, \qquad E^{12} = E^3 \oplus E^9, \qquad E^{16} = E^7 \oplus E^9,$$

with the line bundle $E^1 =< S_{-1} >$ trivial, and E^3, E^7, E^9 locally generated by the complex structures S_α with negative index α, and with the mentioned properties.

To allow a uniform notation, use the lower bound index

$$A = 0, -1, -3, -7,$$

according to whether $M = \mathrm{F\,II}, \mathrm{E\,III}, \mathrm{E\,VI}, \mathrm{E\,VIII}$, so that the mentioned generators of the even Clifford structure can be written as

$$\{S_\alpha\}_{A \le S_\alpha \le 8}.$$

In all four cases one has a matrix of local almost complex structures

$$J = \{J_{\alpha\beta}\}_{A \le \alpha, \beta \le 8},$$

where $J_{\alpha\beta} = S_\alpha \wedge S_\beta$, so that J is skew-symmetric. It is easily recognized that on the model linear spaces $\mathbb{R}^{16}, \mathbb{C}^{16}, \mathbb{H}^{16}, \mathbb{O}^{16}$, the upper diagonal elements $\{J_{\alpha\beta}\}_{A \le \alpha < \beta \le 8}$ are a basis of the Lie algebras

$$\mathfrak{spin}(9) \subset \mathfrak{so}(16), \quad \mathfrak{spin}(10) \subset \mathfrak{su}(16), \quad \mathfrak{spin}(12) \subset \mathfrak{sp}(16), \quad \mathfrak{spin}(16).$$

Next, look at the skew-symmetric matrix of the (local) associated Kähler 2-forms:

$$\psi = \{\psi_{\alpha\beta}\}_{A \le \alpha, \beta \le 8},$$

and note that by the invariance property, the coefficients of its characteristic polynomial

$$\det(tI - \psi) = t^{A+9} + \tau_2(\psi) t^{A+7} + \tau_4(\psi) t^{A+5} + \dots$$

give rise to global differential forms on $\mathrm{F\,II}, \mathrm{E\,III}, \mathrm{E\,VI}, \mathrm{E\,VIII}$.

Definition 3.1 For $A = 0, -1, -3, -7$, that is on the linear spaces $\mathbb{R}^{16}, \mathbb{C}^{16}, \mathbb{H}^{16}, \mathbb{O}^{16}$, and on symmetric spaces $\mathrm{F\,II}, \mathrm{E\,III}, \mathrm{E\,VI}, \mathrm{E\,VIII}$, we call

$$\Phi = \tau_4(\psi)$$

the *octonionic Kähler 8-form.*

4 Cayley-Rosenfeld Planes

The following statement has been proved in [19, 21]; see also [18, 20].

Theorem 4.1

(a) *On the Cayley projective plane* F II*, assuming the lower index* $A = 0$ *in the matrix* ψ*, one has* $\tau_2(\psi) = 0$*. Moreover, the octonionic Kähler 8-form*

$$\Phi_{\mathrm{Spin}(9)} = \tau_4(\psi)$$

is closed and its cohomology class generates the cohomology ring $H^*(\mathrm{F\,II})$.

(b) *On the Hermitian symmetric space* E III *one has (assuming now the lower index* $A = -1$*)* $\tau_2(\psi) = -3\omega^2$*, where* ω *is the complex Kähler 2-form of* E III*. The octonionic Kähler 8-form*

$$\Phi_{\mathrm{Spin}(10)} = \tau_4(\psi)$$

is closed and its class generates the Kähler primitive cohomology ring of E III.

Another algebraic approach to the 8-form $\Phi_{\mathrm{Spin}(9)}$, equivalent to the one outlined in the previous Section, has been proposed by Castrillón Lòpez et al., cf. [5, 6].

Remark 4.2 The above construction of the octonionic Kähler 8-form $\Phi_{\mathrm{Spin}(9)}$ can be seen as parallel with the following way of looking at the quaternion Kähler 4-form Ω on quaternion Hermitian manifolds M^8. For this, consider the following *quaternionic Pauli matrices*:

$$S_0^{\mathbb{H}} = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \quad S_1^{\mathbb{H}} = \left(\begin{array}{c|c} 0 & -R_i^{\mathbb{H}} \\ \hline R_i^{\mathbb{H}} & 0 \end{array}\right), \quad S_2^{\mathbb{H}} = \left(\begin{array}{c|c} 0 & -R_j^{\mathbb{H}} \\ \hline R_j^{\mathbb{H}} & 0 \end{array}\right),$$

$$S_3^{\mathbb{H}} = \left(\begin{array}{c|c} 0 & -R_k^{\mathbb{H}} \\ \hline R_k^{\mathbb{H}} & 0 \end{array}\right), \quad S_4^{\mathbb{H}} = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right),$$

acting on $\mathbb{H}^2 \cong \mathbb{R}^8$, where $R_u^{\mathbb{H}}$ is the right multiplication by the unit basic quaternion $u = i, j, k$. As before, for $\alpha = 0, \ldots, 4$ and $\alpha \neq \beta$:

$$(S_\alpha^{\mathbb{H}})^* = S_\alpha^{\mathbb{H}}, \qquad (S_\alpha^{\mathbb{H}})^2 = \mathrm{Id}, \qquad S_\alpha^{\mathbb{H}} S_\beta^{\mathbb{H}} = -S_\beta^{\mathbb{H}} S_\alpha^{\mathbb{H}}.$$

This can be applied to some Riemannian manifolds M^8, and the symmetric space $\mathrm{G\,I} = \mathrm{G}_2/\mathrm{SO}(4)$ is an example, by defining over it a Euclidean vector bundle E^5 locally spanned by five involutions with such properties, thus giving an even Clifford structure. This is equivalent to a quaternion Hermitian structure on M^8, as one can see by looking at the skew-symmetric matrix

$$J = \{J_{\alpha\beta}\}_{0 \le \alpha, \beta \le 4}$$

of almost complex structures $J_{\alpha\beta} = S^{\mathbb{H}}_\alpha \circ S^{\mathbb{H}}_\beta$, and at its associated matrix $\varphi = (\varphi_{\alpha\beta})$ of Kähler 2-forms. In the characteristic polynomial

$$\det(tI - \varphi) = t^5 + \tau_2(\varphi)t^3 + \tau_4(\varphi)t,$$

the coefficient $\tau_2(\varphi) \in \Lambda^4$ is easily seen to coincide with

$$-2\Omega_L = -2[\omega^2_{L^{\mathbb{H}}_i} + \omega^2_{L^{\mathbb{H}}_j} + \omega^2_{L^{\mathbb{H}}_k}],$$

where Ω_L is the left quaternionic 4-form, cf. [19, p. 329].

Going back to Theorem 4.1, some words of comment on the analogy between the definitions of

$$\Phi_{\mathrm{Spin}(9)} \in \Lambda^8(\mathrm{F\,II}) \quad \text{and} \quad \Phi_{\mathrm{Spin}(10)} \in \Lambda^8(\mathrm{E\,III}).$$

When applied to the Spin(9) $\subset$ SO(16) and Spin(10) $\subset$ SU(16) structures on the linear spaces $\mathbb{R}^{16}$ and $\mathbb{C}^{16}$, both 8-forms can be written in terms of the cartesian coordinates. For example, the computation of $\Phi_{\mathrm{Spin}(9)}$ in $\mathbb{R}^{16}$ gives a sum of 702 non zero monomials in the dx_α $(\alpha = 1, \dots, 16)$ with coefficients $\pm 1, \pm 2, -14$ (cf. [19, pp. 339–343] for the full description).

One sees in particular that $\Phi_{\mathrm{Spin}(9)} \in \Lambda^8\mathbb{R}^{16}$ is, up to a constant, the 8-form defined by integrating the volume of octonionic lines in the octonionic plane. Namely, if ν_l denotes the volume form on the line $l = \{(x, mx)\}$ or $l = \{(0, y)\}$ in $\mathbb{O}^2$, a computation shows that

$$\Phi_{\mathrm{Spin}(9)}(\mathbb{R}^{16}) = c \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl,$$

where $p_l : \mathbb{O}^2 \cong \mathbb{R}^{16} \to l$ is the orthogonal projection, $\mathbb{O}P^1 \cong S^8$ is the octonionic projective line of all the lines $l \subset \mathbb{O}^2$ and the constant c turns out to be $\frac{39916800}{\pi^4}$, cf. [19, pp. 338–339]. The integral in the right hand side of the previous formula is the definition of the octonionic 8-form in $\mathbb{O}^2$ proposed by Berger [2]. Of course, the octonionic lines of $\mathbb{O}^2$ are distinguished 8-planes in $\mathbb{R}^{16}$, so that this 1972 definition anticipates the spirit of calibrations in this context.

Now, in spite of the analogy between our constructions of $\Phi_{\mathrm{Spin}(9)}$ and $\Phi_{\mathrm{Spin}(10)}$, it is clear that a similar approach is not possible for $\Phi_{\mathrm{Spin}(10)} \in \Lambda^8(\mathbb{C}^{16})$, due to the lack of a Hopf fibration to refer to.

The following homological interpretation of $\Phi_{\mathrm{Spin}(10)}$ on E III relates to its projective algebraic geometry, cf. [21].

Proposition 4.3 *Look at* E III *as the closed orbit of the action of* E_6 *in the projectified Jordan algebra of* 3×3 *Hermitian matrices over complex octonions, i.e. as the fourth Severi variety in* $\mathbb{C}P^{26}$ *mentioned in the Introduction. Then the de Rham* *dual of the basis represented in* $H^8(\mathrm{E\,III};\mathbb{Z})$ *by the forms* $(\frac{1}{(2\pi)^4}\Phi_{\mathrm{Spin}(10)}, \frac{1}{(2\pi)^4}\omega^4)$ *is given by the pair of algebraic cycles*

$$\left(\mathbb{C}P^4 + 3(\mathbb{C}P^4)',\ \ \mathbb{C}P^4 + 5(\mathbb{C}P^4)'\right),$$

where $\mathbb{C}P^4, (\mathbb{C}P^4)'$ *are maximal linear subspaces, belonging to the two different families ruling a totally geodesic non-singular quadric* Q_8 *contained in* $\mathrm{E\,III} \subset \mathbb{C}P^{26}$ *as a complex octonionic projective line.*

The described construction of the 8-form Φ has similarly properties on the remaining Cayley-Rosenfeld planes.

Theorem 4.4

(a) On the quaternion Kähler symmetric space E VI*, assuming the lower index* $A = -3$ *for the matrix* ψ*, the second coefficient* $\tau_2(\psi)$ *is proportional to the quaternion Kähler 4-form* Ω*. Also, the octonionic Kähler 8-form*

$$\Phi_{\mathrm{Spin}(12)} = \tau_4(\psi)$$

is closed and its class is one of the two generators of the quaternion Kähler primitive cohomology ring of E VI.

(b) On the symmetric space E VIII *one has again, assuming now* $A = -7$*,* $\tau_2(\psi) = 0$*. The octonionic Kähler 8-form*

$$\Phi_{\mathrm{Spin}(16)} = \tau_4(\psi)$$

is closed and its class is one of the four generators of cohomology ring of E VIII.

The closeness of the octonionic Kähler 8-form $\Phi = \tau_4(\psi)$ on the symmetric spaces F II, E III, E VI, E VIII is recognized by looking at the $\psi_{\alpha\beta}$ $(A \leq \alpha < \beta \leq 9$ and $A = 0, -1, -3, -7)$, as local curvature forms of a metric connection on the vector bundle defining their non-flat even Clifford structure. This allows, in the proof of Proposition 4.3, to relate the class of $\tau_4(\psi)$ with the second Pontrjagin class of this bundle, see [21] for details.

Remark 4.5 Note in Tables 2 and 3 the different behavior of the primitive cohomology in the four Cayley-Rosenfeld projective planes. Concerning F II, E III, there is no "fully primitive cohomology" (and no torsion): the classes of $\Phi_{\mathrm{Spin}(9)}$ in the first case and the classes of the Kähler 2-form ω and of the 8-form $\Phi_{\mathrm{Spin}(10)}$ generate the whole cohomology. On the quaternion Kähler Wolf space there is (besides torsion) a unique primitive generator in H^{12}, whose description does not seem to follow our techniques. Finally E VIII has a much richer primitive cohomology with generators in H^{12}, H^{16}, H^{20}.

Table 3 Fully primitive Poincaré polynomials $\overline{P}(t) = \sum_{i=0,\dots} \overline{b}_i t^i$

Even Clifford symmetric spaces	Fully primitive Poincaré polynomial
E II	$1 + t^6 + t^8 + t^{12} + t^{14} + t^{20}$
E III	1
E VI	$1 + t^{12} + t^{24}$
E VII	$1 + t^{10} + t^{18}$
E VIII	$1 + t^{12} + t^{16} + t^{20} + t^{24} + t^{28} + t^{32} + t^{36} + t^{40} + t^{44} + t^{48} + t^{52} + t^{56} + t^{60} + t^{64}$
E IX	$1 + t^{12} + t^{20} + t^{24} + t^{32} + t^{36} + t^{44} + t^{56}$
F I	$1 + t^8$
F II	1
G I	1

5 Essential Clifford Structures

The discussion in the previous two Sections shows that the even Clifford structures allowing to get the octonionic Kähler form $\Phi = \tau_4(\psi)$ on E III, E VI, E VIII follow a slightly different construction from that of F II. The latter is in fact obtained by what is called a *Clifford system* C_m, i.e. a vector sub-bundle of the endomorphisms bundle locally generated by $(m + 1)$ self-dual anti-commuting involutions S_α. For F II we chose $m + 1 = 9$ and in the linear setting the S_α are the octonionic Pauli matrices. The parallel definition of the quaternionic Pauli matrices, outlined in Remark 4.2, gives a similar approach to $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ structures in dimension 8, using a Clifford system with $m + 1 = 5$.

The vector bundles

$$E^{10} \subset \mathrm{End}\,(T\,\mathrm{E\,III}), \quad E^{12} \subset \mathrm{End}\,(T\,\mathrm{E\,VI}), \quad E^{16} \subset \mathrm{End}\,(T\,\mathrm{E\,VIII}),$$

defining the even Clifford structure on the remaining Cayley-Rosenfeld planes, have both symmetric and skew-symmetric endomorphisms as local generators. This suggests the following:

Definition 5.1 We say that an even Clifford structure is *essential* if it is not given as a Clifford system, i.e. if it cannot be locally generated by self-dual anti-commuting involutions.

We have, cf. [22]:

Theorem 5.2

(i) Any even Clifford structure in dimension 64 and 128 is necessarily essential. In particular such are the parallel even Clifford structures on E VI and E VIII.

(ii) The even Clifford structure on E III is also essential, although in dimension 32 one can have a non-essential even Clifford structure.

The statement *(i)* follows by a dimensional comparison of the representation spaces of pairs of Clifford algebras like $\mathcal{C}l^0_{12}$ and $\mathcal{C}l_{0,12}$, or $\mathcal{C}l^0_{16}$ and $\mathcal{C}l_{0,16}$.

As for statement *(ii)*, the same comparison of dimensions for the representation spaces of $\mathcal{C}l^0_{10}$ and $\mathcal{C}l_{0,10}$ leaves open the possibility of having a Clifford system with $m + 1 = 10$ in the linear space $\mathbb{R}^{32}$. In fact it is not difficult to write down such a Clifford system. However, the structure group of a 32-dimensional manifold carrying such a Clifford system reduces to $\mathrm{Spin}(10) \subset \mathrm{SU}(16)$, and this would be the case of the holonomy group, assuming that such a Clifford system induces the parallel even Clifford structure of E III. Thus, E III would have a trivial canonical bundle, in contradiction with the positive Ricci curvature property of Hermitian symmetric spaces of compact type.

Acknowledgements The author was supported by the GNSAGA group of INdAM and by the research project "Polynomial identities and combinatorial methods in algebraic and geometric structures" of Sapienza Università di Roma.

References

1. S. Araki, Cohomology modulo 2 of the compact exceptional groups E_6 and E_7. J. Math. Osaka City Univ. **12**, 43–65 (1961)
2. M. Berger, Du côté de chez Pu. Ann. Sci. École Norm. Sup. **5**, 1–44 (1972)
3. A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. Ann. Math. **57**, 115–207 (1953)
4. A. Borel, F. Hirzebruch, Characteristic classes of homogeneous spaces, I. Am. J. Math. **80**, 458–538 (1958)
5. M. Castrillón López, P.M. Gadea, I.V. Mykytyuk, The canonical eight-form on manifolds with holonomy group Spin(9). Int. J. Geom. Methods Mod. Phys. **7**, 1159–1183 (2013)
6. M. Castrillón López, P.M. Gadea, I.V. Mykytyuk, On the explicit expression of the canonical 8-form on Riemannian manifolds with Spin(9) holonomy. Abh. Math. Sem. Univ. Hamburg **87**, 17–22 (2017)
7. P.E. Chaput, L. Manivel, N. Perrin, Quantum cohomology of minuscule homogeneous spaces. Transform. Groups **13**, 47–89 (2008). Part II: hidden symmetries. Int. Math. Res. Not. **22**, 29 (2007). Part III: semisimplicity and consequences. Can. J. Math. **62**(6), 1246–1263 (2010)
8. T. Friedrich, Weak Spin(9)-structures on 16-dimensional Riemannian manifolds. Asian J. Math. **5**, 129–160 (2001)
9. K. Galicki, S. Salamon, Betti numbers of 3-sasakian manifolds. Geom. Dedicata **63**, 45–68 (1996)
10. F. Hirzebruch, P. Slodowy, Elliptic genera, involutions, and homogeneous spin manifolds. Geom. Dedicata **35**, 309–343 (1990); Appendix by J.G. Bliss, R.V. Moody, A. Pianzola, Geom. Dedicata **35**, 345–351 (1990)
11. K. Ishitoya, Cohomology of the symmetric space E I. Proc. Jpn. Acad. Ser. A Math. Sci. **53**, 56–60 (1977)
12. K. Ishitoya, Integral cohomology ring of the symmetric space E II. Proc. Jpn. Acad. Ser. A Math. Sci. **53**, 56–60 (1977)
13. K. Ishitoya, H. Toda, On the cohomology of irreducible symmetric spaces of exceptional type. J. Math. Kyoto Univ. **17**, 225–243 (1977)
14. C.R. LeBrun, S.M. Salamon, Strong rigidity of positive quaternion Kähler manifolds. Invent. Math. **118**, 109–132 (1994)

15. A. Moroianu, U. Semmelmann, Clifford structures on Riemannian manifolds. Adv. Math. **228**, 940–967 (2011)
16. M. Nakagawa, The mod. 2 cohomology ring of the symmetric space E VI. J. Math. Kyoto Univ. **41**, 535–557 (2001)
17. M. Nakagawa, The integral cohomology ring of E_8/T. Proc. Jpn. Acad. **86**, 64–68 (2010)
18. L. Ornea, M. Parton, P. Piccinni, V. Vuletescu, Spin(9) geometry of the octonionic hopf fibration. Transform. Groups **18**, 845–864 (2013)
19. M. Parton, P. Piccinni, Spin(9) and almost complex structures on 16-dimensional manifolds. Ann. Glob. Anal. Geom. **41**, 321–345 (2012)
20. M. Parton, P. Piccinni, Spheres with more than 7 vector fields: all the fault of Spin(9). Linear Algebra Appl. **438**, 113–131 (2013)
21. M. Parton, P. Piccinni, The even Clifford structure of the fourth Severi variety. Complex Manifolds **2**, 89–104 (2015). Topical Issue on Complex Geometry and Lie Groups
22. M. Parton, P. Piccinni, V. Vuletescu, Clifford systems in octonionic geometry. Rend. Sem. Mat. Univ. Pol. Torino. **74**, 267–288 (2016). Volume in memory of Sergio Console
23. S.M. Salamon, Index theory and quaternionic Kähler manifolds, in *Conference in Differential Geometry and Its Applications* (Opava, 1992), pp. 387–404. http://www.emis.de/proceedings/5ICDGA/
24. S.M. Salamon, Cohomology of Kähler manifolds with $c_1 = 0$, in *Manifolds and Geometry, Symposia Mathematica* (Pisa, 1993), vol. 36 (Cambridge University Press, Cambridge, 1996), pp. 294–310
25. S.M. Salamon, Index theory and special geometries, in *Luminy Meeting Spin Geometry and Analysis on Manifolds*, Oct 2014. https://nms.kcl.ac.uk/simon.salamon/T/luminy.pdf (see also the conference talks https://nms.kcl.ac.uk/simon.salamon/T/ober.pdf and http://calvino.polito.it/~salamon/T/levico.pdf)
26. H. Toda, T. Watanabe, The integral cohomology ring of F_4/T and E_6/T. J. Math. Kyoto Univ. **14**, 257–286 (1974)
27. T. Watanabe, The integral cohomology ring of the symmetric space E VII. J. Math. Kyoto Univ. **15**, 363–385 (1975)
28. F.L. Zak, Severi varieties. Math. USSR Sbornik **54**, 113–127 (1986)

Manifolds with Exceptional Holonomy

Simon Salamon

Abstract This is a survey of results concerning special and exceptional Riemannian holonomy from a historical and personal perspective.

Keywords Lie group • Ricci-flat • Riemannian metric • Special holonomy

Introduction and Acknowledgement

This article is based on the talk I gave at the *Giornata INdAM* in Bologna on 10 June 2015. The purpose of that talk was both to introduce the subject to non-experts, and bring the audience up to date with some of the more exciting developments in this fast-moving field. The last two years have seen yet more striking discoveries.

I would like to express my gratitude to the staff of the Istituto Nazionale di Alta Matematica, both for inviting me on that occasion and for hosting the conference in Rome later that year. The organizational link between the two events is the excuse for including this article in the present volume, and I thank the editors for all the work in putting the volume together. I am aware that this contribution gives only a fleeting presentation of recent topics, and I have omitted some more standard bibliography that can be found in [22, 67, 86].

My own interest in holonomy dates back to my year (1976/77) as masters student in Oxford, when Nigel Hitchin had suggested that I attempt a systematic classification of curvature tensors of metrics enjoying a holonomy group from Berger's list. This was a far-sighted idea at a time in which properties of tensors tended to be computed on a case-by-case basis, and general theory was more obscure. However, having more or less grasped the algebra, I discovered during the course of the year that the problem had already been solved by Dmitry Alekseevsky [4] some time earlier. (It was incidentally an honour to become a close colleague

The author acknowledges support from the Simons Foundation (#488635, Simon Salamon).

S. Salamon (✉)
Department of Mathematics, King's College London, Strand, London WC2, UK
e-mail: simon.salamon@kcl.ac.uk

S.G. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry*,
Springer INdAM Series 23, https://doi.org/10.1007/978-3-319-67519-0_13

of his later in my career.) This led to a back-up plan involving a diversion into more general G structures and lengthy papers of Guillemin–Sternberg on Spencer cohomology. Both topics influenced my later research, for example in my PhD thesis on quaternionic geometry, though holonomy won out thanks also to the influence of Alfred Gray.

In the Riemannian context, "exceptional holonomy" means G_2 or $\mathrm{Spin}(7)$, whilst "special holonomy" might (depending on one's interests) include other groups in Berger's list [14]. The development of both fields has been greatly influenced by ideas arising in theoretical physics. Calabi-Yau spaces with holonomy SU(3) and manifolds with holonomy G_2 play roles in String Theory and M-theory, whilst those with holonomy $\mathrm{Sp}(n)$ and $\mathrm{Sp}(n)\mathrm{Sp}(1)$ arise in supersymmetry and relate to so-called special Kähler manifolds and flat symplectic connections.

Real 4-dimensional manifolds, both compact and non-compact, with holonomy group equal to $\mathrm{Sp}(1) = \mathrm{SU}(2)$ provide the starting point of the Ricci-flat theory. Work in four dimensions remains active; a good example is [15]. My own understanding of G_2 structures grew out of familiarity with self-duality arising from the study of instantons and twistor spaces in the 4-dimensional set-up. Since the talk in Bologna, the subject has received a significant impetus from the setting up of the Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics, and lectures from its workshops informed some of the text.

1 What Is Holonomy?

Holonomy is the subgroup generated by *parallel transport* of tangent vectors along all possible loops on a smooth manifold M. The notion makes sense whenever a *connection* is assigned on the tangent bundle TM to the manifold.

More generally, one can consider the holonomy generated by any connection on a vector bundle because of an important property of connections: they *pull back* under smooth mappings, in particular to the pulled-back bundle over the domain of a smooth curve. Holonomy can be studied both locally and globally, and for manifolds with a non-trivial fundamental group there is an important distinction. In particular, the holonomy of a flat connection is related to the notion of *monodromy*.

In this article, we shall be discussing the local theory of holonomy associated to a Riemannian manifold with metric tensor

$$\sum_{i,j} g_{ij} dx^i dx^j.$$

This tensor gives rise to Christoffel symbols

$$\Gamma_{ij}^k = \tfrac{1}{2} \sum_m g^{km} \Big(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \Big).$$

The symmetry $\Gamma^i_{jk} = \Gamma^i_{kj}$ reflects the fact that this connection has zero torsion. The associated Levi-Civita covariant derivative operator ∇ is defined by setting

$$\nabla_{\partial/\partial x_j} \frac{\partial}{\partial x^k} = \sum_i \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

It is characterized by the fact that (1) it renders the metric constant, and (2) it is torsion-free. In more abstract notation, $\nabla g = 0$ and

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Often, g and ∇ will not be known explicitly; this is the crux of the matter that leads one into more abstract territory. We begin with an elementary example.

1.1 The 2-Sphere

The standard metric g on the unit 2-dimensional sphere S^2 is obtained by restricting the standard dot product on $\mathbb{R}^3$. Adopting longitude $x^1 = u$ and latitude $x^2 = v$ as local coordinates on S^2, one can compute the lengths of the corresponding vectors $\partial/\partial u$ (tangent to parallels, circles with v constant) and $\partial/\partial v$ (tangent to meridians, semicircles with u constant). Indeed,

$$g = (\cos v \, du)^2 + (dv)^2,$$

reflecting the fact that meridians have constant length, whereas parallels shrink to zero as one approaches the poles and $v \to \pm\pi/2$.

One computes the table of Christoffel coefficients

$\Gamma^1_{11} = 0$	$\Gamma^1_{12} = -\tan v$	$\Gamma^1_{22} = 0$
$\Gamma^2_{11} = \frac{1}{2}\sin 2v$	$\Gamma^2_{12} = 0$	$\Gamma^2_{22} = 0$

so as to differentiate the vectors

$$e_1 = (\sec v)\frac{\partial}{\partial u}, \qquad e_2 = \frac{\partial}{\partial v}$$

that form an orthonormal basis of each tangent space. Consider the unit vector

$$X = e_1 \cos\theta + e_2 \sin\theta$$

defined along a parallel with v constant, where $\theta = \theta(u)$; then $\nabla_{\partial/\partial u} X = 0$ if and only if $\theta = -u \sin v$ up to a constant.

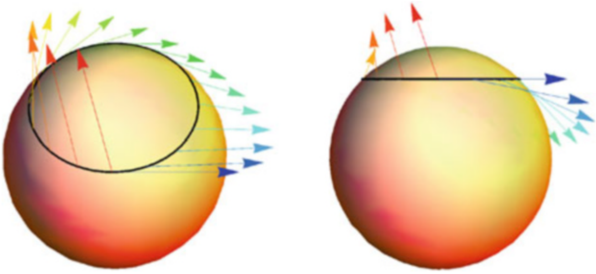


Fig. 1 Vector field at the latitude of Bologna

Like any *great* circle, the equator is a geodesic, which means that its unit tangent vector $e_1 = \partial/\partial u$ is invariant under parallel translation. This is consistent with the fact that θ is constant, giving $\nabla_{e_1} e_1 = 0 = \nabla_{e_1} e_2$. Near the north pole, the surface is like a flat plane, so a parallel vector field points in the same direction and rotates through almost 2π relative to the tangent vector to a small circle. Figure 1 displays a parallel vector field X on the circle $v = \pi/4$, roughly the latitude of Bologna. As one moves around the circle, the length of X is preserved, because the metric tensor is itself constant. When one returns to the starting point, the vector has been rotated through $2\pi \sin v \simeq 255^{\circ}$ relative to the tangent vector to the curve. (I confess that I got the calculation wrong in my lecture. As Gerard Watts reminded me, 2π minus this turning angle is the integral of the geodesic curvature, which by Gauss-Bonnet is proportional to the area enclosed by the curve.) By varying the latitude, one can accomplish a rotation through any angle, and it follows that the holonomy group of g on S^2 equals SO(2).

Now SO(2) can be identified with $U(1)$, and we can equally well say that the holonomy group is $U(1)$, and preserves a complex structure. The latter is defined infinitesimally by the tensor J given by rotation by $\pi/2$ (counter-clockwise relative to our ordering of the basis): $Je_1 = e_2$ and $Je_2 = -e_1$. This leads naturally to the next subsection.

1.2 Complex Projective Space

The compact complex manifold $\mathbb{CP}^n$ is the set of 1-dimensional subspaces in $\mathbb{C}^{n+1}$. It has a natural Riemannian metric g, called the Fubini-Study metric, which arises from the Hermitian product

$$\langle \mathbf{z}, \mathbf{w} \rangle = \langle \mathbf{z} \,|\, \mathbf{w} \rangle = \sum_{i=1}^{n+1} \overline{z}_i w_i.$$

A point $[\mathbf{z}]$ of $\mathbb{CP}^n$ then determines a hyperplane, namely the Hermitian complement of $\mathbf{z}$. The construction gives rise to a distance d (or "metric" in the sense of metric space) which is in fact easier to describe than its infinitesimal counterpart.

The distance $d([\mathbf{w}],[\mathbf{z}])$ between two points represented by unit vectors $\mathbf{w},\mathbf{z}\in\mathbb{C}^{n+1}$ equals $2\arccos\sqrt{\rho}$ where

$$\rho=\frac{\langle\mathbf{w},\mathbf{z}\rangle\,\langle\mathbf{z},\mathbf{w}\rangle}{\langle\mathbf{w},\mathbf{w}\rangle\,\langle\mathbf{z},\mathbf{z}\rangle}=\frac{|\,\langle\mathbf{w},\mathbf{z}\rangle\,|^2}{\|\mathbf{w}\|^2\|\mathbf{z}\|^2}.$$

This quantity is the cross ratio of four points, namely $[\mathbf{w}]$, $[\mathbf{z}]$ and the intersections of the complex projective line $\mathbb{CP}^1$ they generate with the associated hyperplanes. It can also be interpreted as a transition probability of pure quantum states [20].

The Fubini-Study metric g is now obtained as the second-order term in the expansion of $d([\mathbf{z}+t\xi],[\mathbf{w}+t\zeta])$ in powers of t. The holonomy group of g is the subgroup $\mathrm{U}(n)$ of $\mathrm{SO}(2n)$, and requiring that the holonomy be $\mathrm{U}(n)$ or a subgroup thereof is one way of defining a *Kähler metric*. Wigner's theorem states that *the isometry group of* $\mathbb{CP}^n$ *is generated by* $\mathrm{SU}(n+1)/\mathbb{Z}_{n+1}$ *and complex conjugation*. Freed explains how it can be proved as an application of the holonomy idea [52]. The study of configurations of nine points in $\mathbb{CP}^2$ up to such isometries is the subject of the author's joint paper [63], which contains references to a well-known problem in quantum information concerning the existence of a "Symmetric Informationally Complete Positive Operator Valued Measure" for each n.

The reduction in holonomy to $\mathrm{U}(n)$ at each point of a Kähler manifold M determines an *orthogonal* complex structure J (so $J^2=-1$), and the holonomy condition is the assertion that $\nabla J=0$. There is an associated *symplectic form* ω given by

$$\omega(X,Y)=g(X,JY),$$

the closure of ω being a consequence of the condition $\nabla\omega=0$. Both g and ω can be regarded as objects of type $(1,1)$ relative to J, meaning that they will annihilate a pair of complex vectors of the same type, $(1,0)$ or $(0,1)$.

On a Kähler manifold, there is a very intimate relationship between the complex geometry (defined by J and the associated differential operators ∂ and $\overline{\partial}$) and the Riemannian geometry (defined by the pointwise scalar product g). Locally, ω can be expressed as

$$\omega=i\partial\overline{\partial}f$$

for some real-valued function f, a so-called Kähler potential. For $\mathbb{CP}^n$, one may take $f=\log\|\mathbf{z}\|^2$ on $\mathbb{C}^{n+1}\setminus\{0\}$, and note that the resulting 2-form passes to the quotient.

Any algebraic submanifold M of $\mathbb{CP}^N$ has an induced metric g compatible with both the induced complex and symplectic structure, and is of course Kähler.

Example 1.1 Let $a_0, \dots, a_5$ be distinct complex numbers. The smooth intersection

$$\mathscr{K} = \Big\{\sum_{i=0}^{5} z_i^2 = 0\Big\} \cap \Big\{\sum_{i=0}^{5} a_i z_i^2 = 0\Big\} \cap \Big\{\sum_{i=0}^{5} a_i^2 z_i^2 = 0\Big\}$$

of three quadrics in $\mathbb{CP}^5$ is obviously Kähler with holonomy $U(2)$. On the other hand, such a submanifold is a K3 surface: it is simply connected and has $c_1 = 0$. It is well known that such a 4-manifold has $b_2 = 22$ and an associated lattice

$$H^2(\mathscr{K}, \mathbb{Z}) \cong (-\mathrm{E}_8) \oplus (-\mathrm{E}_8) \oplus \mathrm{U} \oplus \mathrm{U} \oplus \mathrm{U}.$$

By Yau's theorem, $\mathscr{K}$ admits a Ricci-flat Kähler metric. Such a metric, whilst not known explicitly, has holonomy group equal to SU(2), making it *hyperkähler*. The same is true of a quartic hypersurface in $\mathbb{CP}^3$, the smooth intersection of a quadric and cubic in $\mathbb{CP}^4$, and many other complete intersections in products of projective spaces [60], though the *triple* intersection above will be used in the sequel, see Sect. 5.1.

1.3 Hermitian Manifolds

A brief digression is called for in order to describe the relationship between the Riemannian structure (g and the Levi-Civita connection ∇) and an orthogonal almost-complex structure (the tensor J) when the holonomy group does *not* reduce.

The Newlander-Nirenberg theorem tells us that (M, J) is a complex manifold (i.e. has charts with holomorphic transition functions compatible with J) if and only if

$$X, Y \in \Gamma(M, T^{1,0}) \quad \Rightarrow \quad [X, Y] \in \Gamma(M, T^{1,0}).$$

It turns out that this is equivalent to the assertion [88]

$$X, Y \in \Gamma(M, T^{1,0}) \quad \Rightarrow \quad \nabla_X Y \in \Gamma(M, T^{1,0}).$$

This is significant because the second assertion is apparently weaker, yet it can be used to fully describe the intrinsic torsion of a $\mathrm{U}(n)$ structure.

A consequence of this is that the Riemann curvature tensor R of a Hermitian manifold of real dimension n is constrained by the condition

$$X, Y, Z \in \Gamma(M, T^{1,0}) \quad \Rightarrow \quad R_{XY} Z \in \Gamma(M, T^{1,0}).$$

This condition only affects the Weyl tensor W, because J will be compatible with any conformally related metric. When $n = 4$ it forces the value of W_+ at each point to lie in a 3-dimensional subspace of what (in the generic case) would be the 5-dimensional space $S^2_0(\Lambda^2_+)$.

At each point, the tensor R belongs to a vector space whose dimension equals

$$\frac{4}{3}n^4 + O(n^3) \quad \text{as} \quad n \to \infty,$$

and the constraint above eliminates a subspace of dimension $\frac{1}{6}n^4 + O(n^3)$. Thus, asymptotically, the existence of a Hermitian structure "knocks out" one-eighth of R, or rather W.

Example 1.2 Suppose that J is a hypothetical complex structure J on S^6, and consider the compatible Riemannian metric $\hat{g}$ defined by

$$\hat{g}(X, Y) = g(X, Y) + g(JX, JY),$$

where g is the standard "round" metric. LeBrun showed that g itself (or, therefore, any conformally flat metric) cannot be compatible with J, so the Weyl tensor $\hat{W}$ must be non-zero. At each point, it belongs to a subspace

$$S^{2,2}_0 \subset \Lambda^3_+ \otimes \Lambda^3_-$$

of dimension 84, and the above constraint can be studied for a fixed orbit type.

2 Timeline

In this section, we shall present a selective list of events in the theory of holonomy groups of linear connections.

2.1 First Act

This can be said to begin with

- 1926: É. Cartan classifies Riemannian symmetric spaces.

  These are homogeneous spaces $M = G/H$ with a transitive group of isometries G and Riemann tensor R satisfying $\nabla R = 0$. This means that the holonomy group preserves R, which is then part of the bracket structure of the Lie algebra $\mathfrak{g}$ of G. If G acts effectively on M then the isotropy subgroup H is the holonomy group. Examples of such symmetric space that are neither Hermitian nor quaternion-Kähler are most real and quaternionic Grassmannians, E_6/F_4, $E_7/SU(8)$ (see Example 3.3(i)) and $E_8/Spin(12)$. Authors wishing a

Fig. 2 Georges de Rham and Marcel Berger. Reproduced with permission

more complete list of inclusions between exceptional groups (and incidentally a bigger class of simple Einstein manifolds) are referred to Wolf's classification of isotropy-irreducible spaces [94].

- 1952: Borel and Lichnerowicz prove that the holonomy group of a Riemannian manifold is a closed subgroup of $O(n)$. De Rham shows that if the holonomy groups acts reducibly then the manifold is locally a product. This feature does not generalize to the pseudo-Riemannian case.
- 1955: Berger lists potential irreducible holonomy groups that do not arise from symmetric spaces. It is important to realize that each is a Lie group H endowed with a *representation* of H on the tangent space $\mathbb{R}^N$ of a hypothetical manifold. All the groups he listed happen to act transitively on the sphere S^{N-1}. Simons gives a direct proof in 1962 that this is the case. The representations of G_2 on $\mathbb{R}^7$ and $\mathrm{Sp}(n)\mathrm{Sp}(1)$ on $\mathbb{R}^{4n}$ are studied (in particular, by Bonan and Kraines respectively) using the exterior differential forms that they stabilize. During the period 1968–1972, the groups Spin(9) and $\mathrm{Sp}(n)U(1)$ are eliminated from Berger's list. The former can only occur as the holonomy group of the Cayley plane and its dual symmetric space. Whilst $\mathrm{Sp}(n)U(1)$ never occurs as a Riemannian holonomy group, it highlights the situation in which one fixes a complex structure on a quaternionic manifold, and such geometries have turned out to be of independent significance [53, 64].

We now know that the remaining holonomy groups from Berger's list can all be realized by non-symmetric spaces, though some questions remain regarding the abundance of compact examples.

Real dim	Holonomy	Geometric type
n	$SO(n)$	Generic
$2n$	$U(n)$	Kähler
$2n$	$SU(n)$	Ricci-flat Kähler
$4n$	$Sp(n)$	Hyperkähler
$4n$	$Sp(n)Sp(1)$	Quaternion-Kähler
7	G_2	Exceptional
8	Spin(7)	Exceptional

2.2 Second Act

Attention focusses on those holonomy groups that produce metrics with zero Ricci tensor. Metrics with exceptional holonomy have this property, but the groups associated to complex structures are more tractable.

- 1977: Yau proves the Calabi conjecture, and thereby the existence of compact manifolds with holonomy equal to $SU(n)$. The theory is formulated on an underlying complex manifold, the problem being that of finding a suitable Kähler-Einstein metric [95]. The case $n = 1$ corresponds to K3 surfaces. Calabi himself had given constructions of explicit metrics with reduced holonomy $SU(n)$ and $Sp(n)$ on total spaces of vector bundles and coined the name "hyperkähler" [29]. Nonetheless, there are claims that there do not exist compact manifolds with holonomy group $Sp(n)$ with $n > 1$ [16], and there is considerable pessimism about the existence of metrics with holonomy G_2 or Spin(7) even locally as computer studies of curvature jets reveal no insight.
- 1983: Beauville discovers two families of compact hyperkähler manifolds (meaning those admitting metrics with holonomy $Sp(n)$) on Hilbert schemes of points [12, 17]. It is now understood that the language of Hilbert schemes provides a general setting for the description of hyperkähler moduli spaces, in particular monopole moduli spaces [7]. The author has been fascinated by the topology underlying the Beauville examples, and open problems concerning the Betti numbers of low-dimensional hyperkähler manifolds. The appearance of logarithms of Poincaré series in [87] suggests a link with analytic torsion.
- 1986: Bryant establishes local existence of metrics with exceptional holonomy G_2 and Spin(7) [23]. In view of the popular expectation, this result came as a surprise to experts. Bryant's proof used the machinery of exterior differential systems (EDS) that (whilst acknowledging this as Cartan's technique) he had developed with increasing skill to hone in on specialized geometrical problems. As such it was not constructive, though his paper incorporates explicit examples of conical metrics with exceptional holonomy based on manifolds of one dimension less that the author had studied from the point of view of "weak holonomy". We joined forces to describe the first complete examples of metrics with holonomy equal to G_2 (on vector bundles over S^3, S^4 and $\mathbb{CP}^2$), and Spin(7) (on the spin bundle over S^4) [26, 55]. As Bobby Acharya recently admitted to

me, the complexity of the differential forms is well disguised in the notation. The approach has been generalized in [61] to emphasize the importance of "positive-definite connections".

- 1993: By now, it is well understood that Ricci-flat holonomy reductions can be defined by parallel spinors, though the relevance of Killing spinors (and so-called nearly-parallel geometries) had emerged more slowly. The full picture is described by Bär [10], who characterizes those metrics whose *cones* have exceptional holonomy. Later Acharya would highlight the importance of *sine cones* in this regard, see Example 4.7.

2.3 Third Act

This again concerns again the analytic construction of metrics with reduced holonomy on compact spaces, but now for the exceptional groups. The problem is harder than the case of $\mathrm{SU}(n)$ because of the absence of any complex structures, or indeed any underlying "integrable" non-metric structure.

- 1996: Joyce publishes a proof of the existence of metrics with holonomy G_2 on compact manifolds, after becoming interested in the subject during a train journey in connection with a conference I co-organized in Cortona in 1994. He applies this to many examples, the first (Example 5.2) being a resolution of an orbifold T^7/Γ where Γ is the abelian group $(\mathbb{Z}_2)^3$. Analogous constructions for *Spin* 7 follow 2 years later. He is able to mass produce new families of examples by resolving Calabi-Yau orbifolds in increasingly sophisticated ways.
- Work continues in parallel on non-Riemannian linear holonomy. Following Bryant's example [24], a complete classification for torsion-free holonomy groups is provided by Merkulov and Schwachhöfer [77, 90]. Significant results in the Lorentzian case are accomplished in [11, 13].
- New complete metrics with exceptional holonomy are discovered [18] and interpreted in terms of evolution equations [62] and intrinsic torsion [38].
- 2003: Kovalev publishes a construction, suggested by Donaldson, of gluing asymptotically cylindrical spaces (obtained from Fano threefolds and admitting metrics with holonomy SU(3)) to obtain new compact G_2 manifolds. This is the "twisted connect(ed) sum". Later he will extend this work to Spin(7). In 2012, Corti, Haskins, Nordström and Pacini introduce new rigour into the procedure, and construct many new compact G_2 manifolds using "semi-Fano' threefolds.
- 2015: Foscolo and Haskins find new compact nearly-Kähler (Einstein) metrics on S^6 and $S^3 \times S^3$, invariant by actions of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of cohomogeneity one. The models were proposed by Podestà and Spiro [81, 82]. The new techniques involve "matching" rather than gluing, with input from the 5-dimensional hypo set-up [39, 50]. Analysis is needed with regard to solutions of the relevant ODE's.

3 Exceptional Holonomy Representations

Let us begin with Spin(8). It has three inequivalent irreducible representations on $\mathbb{R}^8$ that are permuted by an outer (triality) automorphism of order 3. Exactly one of these representations factors through SO(8), and by fixing a vector in the corresponding $\mathbb{R}^8$ one may reduce to SO(7). The subgroup Spin(7) that covers this SO(7) acts irreducibly on the two remaining $\mathbb{R}^8$'s, and this is the relevant holonomy representation (both are equivalent):

$$\mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Aut}(\mathbb{R}^8).$$

In this context, the 7-sphere can of course be viewed as the homogeneous space $\mathrm{Spin}(8)/\mathrm{Spin}(7)$.

The subgroup of Spin(7) fixing a non-zero vector in the irreducible $\mathbb{R}^8$ is the 14-dimensional exceptional Lie group G_2. Moreover,

$$\mathrm{G}_2 \subset \mathrm{SO}(7) \subset \mathrm{Aut}(\mathbb{R}^7),$$

$S^7 \cong \mathrm{Spin}(7)/\mathrm{G}_2$, and

$$\mathrm{Spin}(8)/\mathrm{G}_2 \cong S^7 \times S^7.$$

The subgroup of G_2 fixing a non-zero vector in $\mathbb{R}^7$ is SU(3) and $S^6 \cong \mathrm{G}_2/\mathrm{SU}(3)$. Recall that the possible (non-symmetric, irreducible) holonomy groups all act *transitively* on the unit sphere in T_xM, in accordance with Simons' result [91].

3.1 Curvature Constraint

The Riemann curvature tensor $R = R_{ijkl}$ of an n-dimensional manifold is skew in i, j and k, l but symmetric in $(i, j) \leftrightarrow (k, l)$. Moreover, the first Bianchi identity imposes that R belongs to the kernel of the linear mapping

$$S^2(\Lambda^2(\mathbb{R}^n)^*) \longrightarrow \Lambda^4(\mathbb{R}^n)^*$$

at each point. The Ricci tensor $R_{jl} = \sum_{i,k} g^{ik} R_{ijkl}$ defines an element of $S^2(\mathbb{R}^n)^*$. If the holonomy group is H then $\mathfrak{h} \subseteq \mathfrak{so}(n) \cong \Lambda^2$, and one has

$$R \in S^2(\mathfrak{h}).$$

This is the basis of Alekseevsky's description [4]. One can also say that the Levi-Civita connection is an "instanton" for the holonomy structure [83].

Example 3.1 The group G_2 has two inequivalent representations of dimension 77, one isomorphic to $\Lambda^2(\mathfrak{g}_2)/\mathfrak{g}_2$ ($\mathfrak{so}(\mathfrak{g})/\mathfrak{g}$ is always irreducible if $\mathfrak{g}$ is the Lie algebra of a compact simple Lie group [94]). The other arises from the decomposition

$$S^2(\mathfrak{g}_2) \cong V^{77} \oplus S_0^2(\mathbb{R}^7) \oplus \mathbb{R}.$$

As a corollary, a metric with holonomy G_2 is necessarily Ricci flat and has $R \in V^{77}$. The situation for the other exceptional holonomy group is analogous, a metric with holonomy Spin(7) is Ricci-flat, and R belongs to the subspace

$$W^{168} \subset S^2(\mathfrak{so}(7))$$

generated by Weyl tensors on a 7-manifold. (This particular representation is encoded in a social media identifier SMS168.)

3.2 *Three-forms and four-forms*

It is well known that G_2 is the stabilizer of a 3-form $\varphi \in \Lambda^3(\mathbb{R}^7)^*$, in which $\mathbb{R}^7$ is viewed as the space of imaginary octonians, with multiplication defined by Fig. 3. Forgetting the arrows, this is the matroid characterizing linear independence in the Fano projective plane $\mathbb{P}((\mathbb{Z}/2\mathbb{Z})^3)$, and is one of the “trademarks” of higher mathematics. It was I.M. Gelfand who impressed on me the importance of matroids and combinatorics generally during his visit to Pisa to collect an honorary doctorate in 1981. (At the time, he was equally enthusiastic to hear about my paper with Fran Burstall [28], a conversation that took place in the presence of Soviet security personnel.)

We use the conventions from [86, 89]. In suitable coordinates,

$$\varphi = e^{125} - e^{345} + e^{136} - e^{426} + e^{147} - e^{237} + e^{567}$$

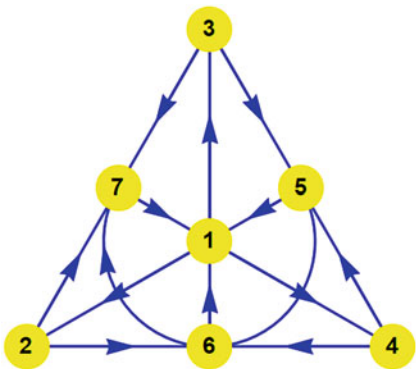


Fig. 3 Cayley multiplication

(see below), and so

$$\mathrm{G}_2 = \{g \in \mathrm{GL}(7,\mathbb{R}) : g \cdot \varphi = \varphi\}.$$

Since $49 - 14 = \binom{7}{3}$, the orbit $\mathrm{GL}(7,\mathbb{R}) \cdot \varphi$ is *open* in $\Lambda^3(\mathbb{R}^7)^*$ or *stable* in the sense of [62].

The 3-form φ determines the metric and the 4-form $*\varphi$. The following fundamental result is due to Fernández and Gray.

Proposition 3.2 *Given a* G_2 *structure defined by a 3-form* φ, *the holonomy of its associated metric is contained in* G_2 *if and only if* $d\varphi = 0$ *and* $d * \varphi = 0$.

Nowadays, this would be seen as an immediate consequence of the decomposition of the space

$$T^* \otimes \mathfrak{h}^\perp \cong \mathbb{R}^7 \otimes \mathbb{R}^7 \cong \mathbb{R}^7 \oplus \mathfrak{g}_2 \oplus \mathbb{R} \oplus S^2_0(\mathbb{R}^7),$$

of intrinsic torsion that contains $\nabla\varphi$, though it was extensive discussions of such theory with Alfred Gray (often carried out in Westbourne Terrace) that kept me enthralled with the problem of exceptional holonomy.

Both φ and $*\varphi$ have open orbits in Λ^3 and Λ^4, and any exterior form of degree 3 or 4 in $\mathbb{R}^7$ can be written a the sum of at most 7 simple (indecomposable) ones. It follows that any 4-form on $\mathbb{R}^8$ can be written as the sum of at most 14 simple forms on $\mathbb{R}^8$. Such an example is the 4-form

$$\Phi = \varphi \wedge e^8 + *\varphi$$

whose stabilizer is Spin(7).

Comparatively little is known about exterior forms of higher degree in higher dimensions, though we highlight two topics next.

Examples 3.3

(i) A description of 4-forms in $\mathbb{R}^8$ is given in [5, 9]. The trick is to use the fact that the isotropy of the symmetric space $\mathrm{E}_7/\mathrm{SU}(8)$ can be identified with $\Lambda^4(\mathbb{R}^8)$.
(ii) A description of the Spin(9)-invariant 8-form on $\mathbb{R}^{16}$ that arises as the isotropy of the rank-one symmetric space $F_4/\mathrm{Spin}(9)$ (the Cayley plane) can be found in [30]. See also [80].

4 Explicit Metrics in Different Dimensions

Looking back over the early history of the subject, leading up to Bryant's discovery of the local existence of metrics with exceptional holonomy, it is perhaps surprising that one can write down such metrics with relative ease.

Many results in mathematics become self-evident with the passage of time, but it is important to understand that results that seem obvious today were far from obvious when they were first studied. There are many instances in the theory of reduced holonomy and special geometrical structures that attest to this phenomenon. It is a privilege to be involved in the development of a subject and witness theorems taking shape. This first happened for me with the theory of self-duality, which I learnt from lectures of I.M. Singer, before the integrability condition for the twistor space was fully understood.

4.1 Resolving Conical Holonomy

In polar coordinates, the Euclidean metric on $\mathbb{R}^7$ can be expressed as

$$g = dr^2 + r^2 h,$$

where h is the standard metric on the 6-sphere. Of course, the holonomy group of g is the identity, but once we realize that $S^6 = \mathrm{G}_2/\mathrm{SU}(3)$ we can bring G_2 into the picture.

A metric h on a 6-manifold M is *nearly-Kähler* iff the cone $dr^2 + r^2 h$ has holonomy contained in G_2 [10]. Such a manifold has an SU(3) structure such that ∇J is anti-symmetric, i.e. $(\nabla_X J)(X) = 0$ for all X. The 6-sphere is nearly-Kähler, and one way of defining such a structure is by means of a 2-form ω and a 3-form ψ such that

$$d\omega = 3\psi, \qquad d * \psi = 2\omega \wedge \omega$$

(see [89]). Then

$$\varphi = r^3 dr \wedge \omega + r^3 \psi = d(\tfrac{1}{3} r^3 \omega)$$

defines a G_2 structure with

$$*\varphi = r^3 dr \wedge * \psi + \tfrac{1}{2} r^4 \omega \wedge \omega = d(\tfrac{1}{4} r^4 \psi),$$

and satisfying the condition in Proposition 3.2. If we replace S^6 with one of the other homogeneous nearly-Kähler spaces

$$N = S^3 \times S^3, \qquad \mathbb{CP}^3, \qquad \mathbb{F} = \mathrm{SU}(3)/T^2,$$

the conical metric on $\mathbb{R}^+ \times N$ has holonomy group *equal* to G_2.

Another construction has its origins in the so-called Eguchi-Hanson space with holonomy SU(2). The singular space $X = \mathbb{C}^2/\pm 1$ admits functions $u = z_1^2$, $v = z_2^2$, $w = z_1 z_2$, embedding it in $\mathbb{C}^3$ as the cone

$$uv = w^2,$$

which provides the local model for the singularities in Fig. 5.

Let Y denote the (total space of the) cotangent bundle $T^*\mathbb{CP}^1$ of the complex projective line, itself isomorphic to $\mathcal{O}(-2) = L \otimes L$. It follows that, with the origin removed, $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_2$ can be identified with the complement

$$\{\alpha \in Y : \alpha \neq 0\}$$

of the zero section. There is a *crepant* resolution

$$\rho\colon Y \to X.$$

The cotangent bundle admits a canonical holomorphic symplectic from $\omega_2 + i\omega_3$ and a complete metric g of the form

$$(r^2+1)^{1/2}(e^1 \otimes e^1 + e^2 \otimes e^2) + (r^2+1)^{-1/2}(e^3 \otimes e^4 + e^4 \otimes e^4),$$

consisting of the sum of a "horizontal" and "vertical" component. The metric g and complex structure also define a closed symplectic form ω_1, so g has holonomy SU(2). The space Y is the simplest example of one that is *asymptotically locally Euclidean* (ALE).

We can now mimic this construction to find a metric with G_2 holonomy. The next result is one from [26].

Theorem 4.1 *There exist complete metrics with holonomy* G_2 *on the rank 3 vector bundles* $\Lambda^2_- T^*M$ *for* $M = S^4$ *or* $\mathbb{CP}^2$.

The metrics arise from the subgroup SO(4) of G_2 relative to which

$$\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3 = \langle e^1, e^2, e^3, e^4 \rangle \oplus \langle e^5, e^6, e^7 \rangle.$$

This enables one to identify $\mathbb{R}^3 \cong \Lambda^2_-(\mathbb{R}^4)$ and define a 3-form

$$\varphi = (e^{12} - e^{34}) \wedge e^5 + (e^{13} - e^{42}) \wedge e^6 + (e^{14} - e^{23}) \wedge e^7 + e^{567}.$$

Ignoring universal constants, the associated metric

$$(r^2+1)^{1/2} \sum_1^4 e^i \otimes e^i + (r^2+1)^{-1/2} \sum_5^7 e^i \otimes e^i$$

has holonomy equal to G_2, and is asymptotic to a conical metric on $\mathbb{R}^+ \times \mathbb{CP}^3$ or on $\mathbb{R}^+ \times \mathbb{F}$, as $r \to \infty$.

Although the resulting metrics are not compact, they have been used as testing grounds for various studies, for example of G_2 instantons. The author is involved in the following two projects.

Examples 4.2 Let X denote the total space of $\Lambda^2_- T^*M$, where M is S^4 or $\mathbb{CP}^2$.

(i) Let Σ denote an embedded real surface in M. The pullback of $\Lambda^2_- T^*S^4$ to Σ splits into the direct sum of a rank 1 and a rank 2 vector bundle (the rank 1 one being essentially the twistor lift of Σ). The rank 2 bundle defines a 4-dimensional submanifold of X that is coassociative (meaning that φ restricts to zero) if and only if Σ is *superminimal* (a minimal surface with a horizontal twistor lift) [21, 68]. In the case $M = S^4$, Kovalev and the author have shown that the coassociative submanifold is hyperkähler if Σ has constant Gaussian curvature. If Σ is a totally geodesic 2-sphere, then we recover the Eguchi-Hanson metric that motivated the G_2 construction above.

(ii) The manifold X can be realized as a quotient of an open set of the quaternionic projective plane $\mathbb{HP}^2$ by the action of a circle subgroup (which one depends on whether M is S^4 or $\mathbb{CP}^2$). Such quotients were studied in [8], and the relevance to G_2 metrics observed in [78]. Gambioli, Nagatomo and the author have shown how to generate a closed G_2 3-form on such a quotient [40, 54] from the $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure on $\mathbb{HP}^2$, and there is an analogous theory starting from Spin(7).

4.2 Examples from the 4-Torus

Let $W = \Gamma\backslash H$ be the Iwasawa manifold, so here

$$H = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C} \right\}$$

is the complex Heisenberg group, and Γ is the discrete subgroup for which $z_i \in \mathbb{Z}$. Then W is the compact total space of a T^2 bundle that fibres (via the map to (z_1, z_2)) over T^4. It has a basis of real 1-forms

$$\begin{gathered} e^1 = d\lambda, \quad e^2 = d\mu, \quad e^3 = d\ell, \quad e^4 = dm, \\ e^5 = dx - \lambda\, d\ell + \mu\, dm, \quad e^6 = dy - \mu\, d\ell - \lambda\, dm, \end{gathered}$$

notation as in [6, Example 1].

Example 4.3 One can define an SU(3) structure that extends to a metric

$$\frac{1}{9}dt^2 + t^{2/3}\sum_{i=1}^{4} e^i \otimes e^i + t^{-2/3}\sum_{i=5}^{6} e^i \otimes e^i.$$

with holonomy G_2 on $W \times (0, \infty)$ [6, 32]. The corresponding almost-complex structure is J_3 in the notation of [1]. This a "half-complete" example, and it can be written down completely explicitly in local coordinates $(\lambda, \mu, \ell, m, x, y, t)$.

We have

$$(g_{ij}) = t^{-2/3} \begin{pmatrix} t^{4/3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{4/3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{4/3}+r^2 & 0 & x & y & 0 \\ 0 & 0 & 0 & t^{4/3}+r^2 & -y & x & 0 \\ 0 & 0 & x & -y & 1 & 0 & 0 \\ 0 & 0 & y & x & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t^{2/3} \end{pmatrix},$$

where $r^2 = \lambda^2 + \mu^2$. One can then check directly that it is Ricci-flat by computer.

This example is generalized by the following result involving a Monge-Ampère evolution equation.

Theorem 4.4 ([6]) *Suppose that a complex surface M has a holomorphic 2-form $\omega_2 + i\omega_3$ and a one-parameter family of Kähler forms $\omega = \omega(t)$ and functions $f = f(t)$ such that*

$$\omega''(t) = 2i\,\partial\overline{\partial} f(t),$$

where $t\,\omega \wedge \omega = f(t)\,\omega_2 \wedge \omega_2$. Then a rank 3 bundle over M admits a Ricci-flat metric g with holonomy in G_2.

If f is constant on M then $\omega''(t) = 0$ and

$$\omega = (p+qt)\omega_0 + (r+st)\omega_1,$$

with $(\omega_1, \omega_2, \omega_3)$ a hyperkähler structure and ω_0 an additional closed 2-form, constructible via the Gibbons-Hawking ansatz. Example 4.3 is such a solution in which M is a torus $\mathbb{C}^2/\mathbb{Z}^4$ and

$$\omega_1 = \tfrac{1}{2} i(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2), \qquad \omega_2 + i\omega_3 = dz_1 \wedge dz_2.$$

Consider now a non-trivial deformation

$$\omega = \omega_1 + i\partial\overline{\partial}\phi$$

where $\phi = \phi(t, z_1)$ with $z_1 = \lambda + i\mu$ Then $\partial\overline{\partial}\phi \wedge \partial\overline{\partial}\phi = 0$ and $f(t) = \frac{1}{2}\phi''(t)$. There are solutions

$$\phi = \tfrac{1}{3} t^3 + A(t)B(\lambda, \mu),$$

with $A'' - ctA = 0$ and $\Delta B + cB = 0$, and the following is copied from [6, Example 3].

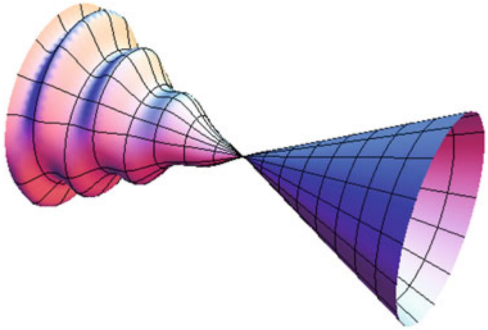


Fig. 4 The separable solution f in Example 4.5

Example 4.5 Consider the Airy function

$$\mathrm{Ai}(t) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{u^3}{3} + tu\right) du,$$

the integral being improper. Take $A(t) = \mathrm{Ai}(t)$ and $B(x) = \sin x$, so

$$f(t,x) = t + \tfrac{1}{2}t\,\mathrm{Ai}(t)\sin x.$$

Then

$$t(f\,d\lambda^2 + fd\mu^2 + d\ell^2 + dm^2) + f^{-1}(dx - \mathrm{Ai}'(t)\cos\lambda\, d\mu)^2 + t^{-2}(dy - \lambda\, d\ell + \mu\, dm)^2 + t^2 f\, dt^2,$$

defined on $T^2 \times \mathbb{R}^4 \times (0, \pm\infty)$, has holonomy G_2. The function f is plotted using cylindrical polar coordinates (with $\lambda \in S^1$) in Fig. 4.

4.3 *Weak Holonomy*

The realization that exceptional holonomy can be constructed from classes of Einstein metrics with "nearly parallel" or "weak holonomy" reductions was understood by Gray [56] before studies of Killing spinors and intrinsic torsion took off. The classification of manifolds with intrinsic torsion invariant by various subgroups has been promoted by the author, see for example [31, 35, 37, 38]. See also [3] and references therein.

The theory in Sect. 4.2 is based to some extent on the theory of hypersurfaces in spaces with special holonomy. Hitchin studied the corresponding evolution equations, and characterized various structures in terms of stable forms [62].

Let us illustrate this in the case of 7 to 8 dimensions. Any hypersurface of $\mathbb{R}^8$ inherits a G_2 structure φ with $d * \varphi = 0$ from a standard Spin(7) invariant 4-form.

In fact, *any* 7-manifold M (with $w_1 = w_2 = 0$) admits such a "co-calibrated" G_2 structure [45]. If this evolves in time t, the 4-form

$$\Phi = dt \wedge \varphi(t) + *\varphi(t),$$

defines a Spin(7) structure on $\mathbb{R} \times M$. It is closed provided

$$\frac{\partial}{\partial t}(*\varphi) = -d\varphi$$

a flow which has short-time existence. Therefore, $(0, \varepsilon) \times M$ will always admit a metric with holonomy contained in Spin(7).

Example 4.6 The homogeneous space $M^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$ admits a G_2 structure with $d\varphi = *\varphi$. Therefore, $\mathbb{R}^+ \times M^7$ has an explicit conical metric with holonomy Spin(7). This was the first example that led the author to realize a metric with exceptional holonomy.

The theory for dimensions 5 to 6 was developed in [39]. A metric k on a 5-manifold is *Einstein-Sasaki* if and only if the cone $dy^2 + y^2 k$ has holonomy contained in SU(3). This is true of $S^2 \times S^3$: its conifold has two Calabi-Yau desingularizations in which the vertex is replaced by S^2 and S^3 respectively.

Example 4.7 The sine cone $d\theta^2 + (\sin\theta)^2 k$ makes

$$\mathbb{E} = (0, \pi) \times S^2 \times S^3$$

nearly-Kähler. To prove this, observe that the conical metric

$$dr^2 + r^2\big(d\theta^2 + (\sin\theta)^2 k\big) = dx^2 + \big(dy^2 + y^2 k\big)$$

has holonomy contained in $\mathrm{SU}(3) \subset G_2$. This is explained in [2], and is the basis of newer constructions [50].

The nearly-Kähler spaces S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ all admit an action by $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of cohomogeneity one with generic orbit $S^2 \times S^3$. Desingularizing the sine cone in two ways and deforming the metrics, one recovers the homogeneous nearly-Kähler spaces

$$\begin{aligned} S^6 :&\quad S^2 \leftarrow \mathbb{E} \rightarrow S^3 \\ S^3 \times S^3 :&\quad S^3 \leftarrow \mathbb{E} \rightarrow S^3 \\ \mathbb{CP}^3 :&\quad S^2 \leftarrow \mathbb{E} \rightarrow S^2. \end{aligned}$$

An analogous construction for SU(3) has been used to find Sp(2)Sp(1) structures on $G_2/\mathrm{SO}(4)$ with a closed 4-form [40].

Foscolo and Haskins show that the first two compactifications of $\mathbb{E}$ above can be modified so as to realize new nearly-Kähler metrics on these spaces:

Theorem 4.8 ([51]) *$S^3 \times S^3$ and S^6 each admit a non-homogeneous nearly-Kähler metric (and compatible almost complex structure), so that their cones have metrics with holonomy* G_2.

This discovery of these metrics has thrown the subject open. Cortés and Vásquez [41] show that only $S^3 \times S^3$ can admit finite quotients that are nearly-Kähler, but that many such quotients do in fact exist.

5 Compact Manifolds with Exceptional Holonomy

Dual to the theory of hypersurfaces is the idea of realizing a metric with special holonomy on a manifold with a circle action. Whilst such a Ricci-flat manifold cannot be compact, the methods are relevant in the quest for new compact examples with exceptional holonomy. When passing to a 7-dimensional G_2 quotient, one has the luxury of starting from either Sp(2)Sp(1) or Spin(7) (see Example 4.2(ii)). The passage from 7 to 6 dimensions is especially fruitful, and a special situation in which the 6-manifold in Kähler was considered in [6].

5.1 Kummer Surface

We begin by investigating, partly as a diversion, a geometrical model defined by a quadratic line complex. The idea was popular in the 1920s, and assigns a K3 surface to each point of a pseudo-Riemannian 4-manifold [74]. The theory below is derived from [57, Chapter 6] and [47].

Given $\alpha \in \Lambda^2(\mathbb{C}^4)$, the condition $\alpha \wedge \alpha = 0$ is equivalent to asserting that α is a decomposable 2-form. It follows that

$$\mathscr{Q} = \{[\alpha] \in \mathbb{P}(\Lambda^2(\mathbb{C}^4)) : \alpha \wedge \alpha = 0\}$$

can be identified with the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^4)$ parametrizing the projective lines in $\mathbb{CP}^3$. It is a smooth hypersurface of $\mathbb{CP}^5$ (the "Klein quadric"), and each point $[x] \in \mathbb{CP}^3$ defines the projective plane $\sigma(x) \subset \mathscr{Q}$ parametrizing lines through $[x]$.

Once we have identified $\mathbb{R}^4$ with its dual, the Riemann curvature tensor R in 4 dimensions also defines a quadric

$$\mathscr{R} = \{[\alpha] \in \mathbb{P}(\Lambda^2(\mathbb{C}^4)) : R(\alpha, \alpha) = 0\}$$

(see Sect. 3.1). The intersection $\mathscr{Q} \cap \mathscr{R}$ is smooth precisely because R satisfies the Bianchi identity, and it distinguishes a set of lines in $\mathbb{CP}^3$. For each $[x] \in \mathbb{CP}^3$, the lines of this "quadratic complex" that pass through $[x]$ is a conic

$$C_x = \{[x \wedge u] \in \sigma(x) : R_{xuxu} = 0\}.$$

The quadratic form $R_{x \cdot x \cdot}$ has rank less than 3 if and only if

$$\sum R_{xapq} R_{bxxr} R_{cdsx}\, \varepsilon^{abcd} \varepsilon^{pqrs} = 0,$$

since the left-hand side reduces to a determinant (ε stands for antisymmetrization that is effectively carried out over 3 letters). This cubic identity in R then determines a quartic surface K in $\mathbb{CP}^3$, whose singular locus D (occurring where C_x is a double line) can be shown to consist of 16 singular points: $\mathscr{K}$ is the Kummer surface associated to R.

In order to distinguish the lines in $\sigma(x)$ generated by C_x for $[x] \in K$, one defines a third quadratic form $P \in S^2(\Lambda^2\mathbb{C}^4)^*$, and associated quadric $\mathscr{P} \subset \mathbb{CP}^5$, by setting

$$P_{xyxy} = \sum R_{xyab} R_{xycd}\, \varepsilon^{abcd} = 8(R_{xy12}R_{xy34} + R_{xy13}R_{xy42} + R_{xy14}R_{xy23}),$$

with $x, y \in \mathbb{C}^4$ and $\{1, 2, 3, 4\}$ indicating a standard a basis of $\mathbb{C}^4$. Similar covariants were considered by the author in an early paper [85] on harmonic spaces, inspired by Tom Willmore. Each point of the intersection

$$\mathscr{K} = \mathscr{P} \cap \mathscr{Q} \cap \mathscr{R}$$

is associated to a line in $\mathbb{CP}^3$ that passes through a point $[x]$ of K, and $[x]$ is unique. This defines a mapping $\mathscr{K} \to K$ that is one-to-one over $K \setminus D$.

Proposition 5.1 *$\mathscr{K}$ is a K3 surface and a resolution of K.*

A projective line in $\mathbb{CP}^5$ lying on $\mathscr{Q} \cap \mathscr{R}$ corresponds to a pencil of lines in a plane $\sigma(x)$ for some unique $[x] \in K$. There are *two* such pencils for each $[x] \in K \setminus D$, but only *one* for $[x] \in D$. The set of such lines is an abelian variety A equipped with an involution ι, and $\mathscr{K}$ can be identified with $A/\langle\iota\rangle$; each singularity is modelled on $\mathbb{C}^2/\pm 1$, just as in Fig. 5. Moreover, A is the Jacobian of a genus 2 curve B consisting of those point in $\mathscr{Q} \cap \mathscr{R}$ whose associated lines in $\mathbb{CP}^3$ intersect a fixed line L.

5.2 Joyce's Construction for G_2

The existence of a hyperkähler metric on $\mathscr{K}$ mentioned in Example 1.1 follows analytically because it is possible to resolve each singularity of K using the Eguchi-Hanson space of Sect. 4.1. This provides a model for producing *compact* 7-manifolds with holonomy G_2.

Joyce's original idea was to write down a finite subgroup Γ acting on $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ preserving the 3-form φ, so T^7/Γ is a G_2 orbifold with amenable singulari-

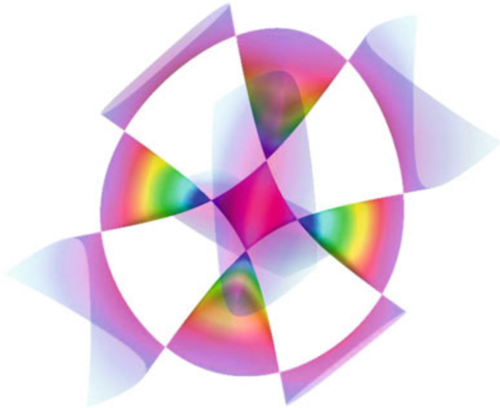


Fig. 5 Real part of a Kummer surface

ties [66]. To achieve the G_2-invariance, one works with inclusions

$$\mathrm{SU}(2)_+ \subset \mathrm{SO}(4) \subset \mathrm{G}_2 \subset \mathrm{F}_4.$$

The presence of F_4 is irrelevant here, but completes the "scrabble" chain and the earlier reference to [94]. Joyce's prototype orbifold is the following.

Example 5.2 Let Γ be the abelian group $(\mathbb{Z}_2)^3$ generated by

$$\begin{aligned}
\alpha(\mathbf{x}) &= (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7,)\\
\beta(\mathbf{x}) &= (x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2}-x_6, -x_7)\\
\gamma(\mathbf{x}) &= (-x_1, x_2, -x_3, x_4, \tfrac{1}{2}-x_5, x_6, \tfrac{1}{2}-x_7)
\end{aligned}$$

While α, β, γ each fix 16 tori T^3, $\beta\gamma, \gamma\alpha, \alpha\beta, \alpha\beta\gamma$ have no fixed points. The singular set of T^7/Γ consists of 12 disjoint 3-tori T^3 each with normal space $\mathbb{C}^2/\pm 1$. Each $\mathbb{Z}_2$ acts within $\mathrm{SU}(2)\times\{e\}$ on $\mathbb{R}^4 \oplus \mathbb{R}^3 = \mathbb{R}^7$ for different $\mathbb{R}^4$'s, so $\Gamma \subset \mathrm{G}_2$.

In the example T^7/Γ, one resolves each singular point by replacing its transverse neighbourhood by an open subset of the ALE space. Each singular T^3 is then surrounded by a "tube" $\mathbb{RP}^3 \times T^3$ outside of which the ALE metric merges with the flat one. In this way, one can define a smooth 7-manifold with a 1-parameter family of *closed* 3-forms φ_t of G_2 type for $t > 0$.

An analytic theorem is needed to deform φ_t into a 3-form with holonomy in G_2. The situation is controlled by the existence of a 3-form ψ_t for which $d{*_t}\varphi_t = d{*_t}\psi_t$, where

$$\|\psi_t\|_{L^2} \lesssim t^4, \quad \|\psi_t\|_{C^0} \lesssim t^3, \quad \|d *_t \varphi_t\|_{L^{14}} \lesssim t^{16/7},$$

whilst the metric determined by φ_t has injectivity radius at least of order t and curvature bounded above by t^{-2}.

Theorem 5.3 ([67]) *In this and similar situations, for sufficiently small t, there exists a* G_2 *3-form $\tilde{\varphi} = \varphi_t + d\eta$ such that $\nabla\varphi = 0$.*

Once one knows that the holonomy lies in G_2, it will *equal* to G_2 if and only if π_1 is finite. Example 5.2 gives rise to a simply-connected manifold admitting a metric with holonomy G_2, and with $b_2 = 12$ and $b_3 = 43$. The latter is the dimension of the moduli space of G_2 metrics up to diffeomorphism.

Joyce further developed this technique in [67]. He was able to resolve more complicated singularities, and realize many diffeomorphism classes of 7-manifolds with holonomy G_2. A partial analysis of relevant finite group actions was carried out in unpublished work by Coílín Nunan.

5.3 *The Twenty-First Century Approach*

We now turn to a discussion of more recent "G_2" activity. The start of this was (by coincidence) marked by the 2002 Fano Conference in Turin, which I was fortunate to attend, though it was also the last occasion many of us had to meet Andrei Tyurin.

At the time, I was more interested in Fano manifolds (of dimension $2n + 1$) generalizing twistor spaces of quaternion-Kähler manifolds. Although I was aware of Kovalev's work, I did not predict the importance that Fano and related threefolds would have for stimulating the construction and classification of compact manifolds with exceptional holonomy.

A *Fano* threefold is a projective variety with ample anti-canonical bundle $\overline{\kappa}$. There are 105 deformation types, of which 17 have $b_2 = 1$ (including of course a smooth quadric, cubic, quartic in $\mathbb{CP}^4$). A table of the 88 Fano threefolds with $b_2 \geqslant 2$ is described by Mori and Mukai in the Fano proceedings [79]; one case had previously been omitted. A Fano threefold with $b_2 \geqslant 2$ that is not the blow up of another along a curve is a conic bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2$, or fibres over a curve. For an original discussion of Fano threefolds case by case, by means of their mirrors, see [36].

A *weak Fano* threefold Y is a generalization (requiring $\overline{\kappa}$ to be nef and $\overline{\kappa}^3 = 2g-2 > 0$) that comes equipped with a resolution

$$\rho \colon Y \to X$$

where X is a (possibly) singular Fano threefold. If ρ is semi-small (meaning that no divisor maps to a point) then Y is said to be *semi-Fano*.

Example 5.4 An important case is that in which X is *nodal*, with a finite number of ordinary double points. Then ρ replaces each such point by a rational curve $\mathbb{CP}^1$ with normal bundle $\nu \cong \mathscr{O}(-1) \oplus \mathscr{O}(-1)$. A general quartic X containing a plane

Fig. 6 The Fano Proceedings [79]

Π in $\mathbb{CP}^4$ has 9 nodes on Π, and two small projective resolutions $Y \to X$ with Y semi-Fano.

Fano and semi-Fano threefolds can be used to construct spaces that are *asymptotically cylindrical Calabi-Yau (ACCY)*. By blowing up a semi-Fano threefold Y along a curve (the base locus of a pencil of anti-canonical divisors), one obtains a smooth threefold Z with a morphism

$$f: Z \to \mathbb{CP}^1$$

such that $f^{-1}(\infty) = \mathscr{K} \in |\overline{\kappa}|$ is a smooth K3 surface that generates $H^2(Z, \mathbb{Z})$. This ensures that the normal bundle of $\mathscr{K}$ in Z is trivial, and topologically

$$V = Z \setminus \mathscr{K} \ \supset \ \mathscr{K} \times S^1 \times \mathbb{R}^+.$$

By construction, Z admits a holomorphic 3-form with a simple pole along $\mathscr{K}$. The following result is in the spirit of work of Tian and Yau [93] on the existence of complete Ricci-flat Kähler metrics on quasi-projective varieties.

Theorem 5.5 ([58, 71]) *$V = Z \setminus \mathscr{K}$ is ACCY: it possesses a Ricci-flat metric with holonomy* SU(3) *that approximates the product metric with holonomy* $\mathrm{SU}(2) \times \{e\}$ *on the cylinder, with all derivatives bounded by $e^{-\lambda t}$ (for some $\lambda > 0$) as $t \to \infty$.*

The *semi* (as opposed to *weak*) Fano condition gives better control over the deformation theory of Z equipped with an anti-canonical divisor, and consequently the hyperkähler structures induced on $\mathscr{K}$. This is important for solving the matching problem described next.

5.4 Twisted Connect Sums

Take two ACCY threefolds $V_\pm$, as constructed above. The trick is to identify the 7-manifolds $V_\pm \times S^1$ by gluing along the neck using an isometry r that performs a "hyperkähler rotation" to mimic a switch of S^1 factors:

$$\begin{array}{ccccccc} \mathscr{K}_+ & \times & S^1 & \times & S^1 & \times & (T, T+1) \\ \downarrow r & & & \swarrow\!\!\!\!\searrow & & & \downarrow - \\ \mathscr{K}_- & \times & S^1 & \times & S^1 & \times & (T, T+1) \end{array}$$

This yields a compact simply-connected 7-manifold M. By the Torelli theorem, the isometry r is determined by the map

$$r^*\colon H^2(\mathscr{K}_-, \mathbb{Z}) \to H^2(\mathscr{K}_+, \mathbb{Z})$$

between the K3 lattices, see Example 1.1.

The theory has been developed by Corti, Haskins, Kovalev, Lee, Nordström and Pacini [42, 43, 72, 73]. The gluing ensures that the underlying G_2 structures on each

$$\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3$$

are identified, and M acquires a *closed* G_2 form φ satisfying

$$\|\nabla^k(d * \varphi)\| = O(e^{-\lambda T}), \quad \forall k.$$

For sufficiently large T, φ can be perturbed (in its cohomology class) as in Joyce's theory, so that $\nabla\tilde{\varphi} = 0$.

The enumeration of examples requires "building-block" threefolds $Z_\pm$ for ACCY's to glue together. There is at least one for almost all Fano or semi-Fano threefolds. Now, $H^2(Z_\pm, \mathbb{Z})$ defines a lattice $N_\pm$ in $\mathrm{Pic}(\mathscr{K}) \subseteq H^2(\mathscr{K}, \mathbb{Z})$ (cf. Example 1.1). The most straightforward case is that in which N_+, N_- are "orthogonal', which is easy to arrange if $\mathrm{rk}N_+ + \mathrm{rk}N_- \leqslant 11$.

There is no problem matching the 17 Fano threefolds with $b_2 = 1$. The $153 = \binom{18}{2}$ choices give at least 82 topologically distinct smooth manifolds M with holonomy G_2, plotted by $b_3(M) = 23 + b$ in Fig. 7. Here b is even and $32 \leqslant b \leqslant 174$ (with almost all values realized) or $b = 216$. More generally,

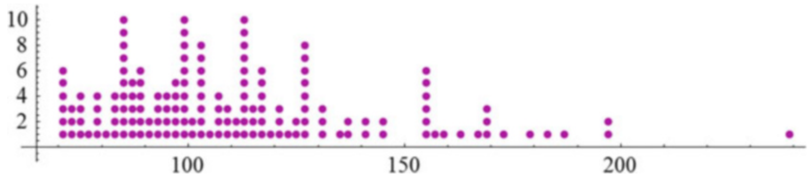


Fig. 7 G_2 manifolds plotted by b_3

5401 of the $5564 = \binom{106}{2}$ pairs taken from all 105 deformation classes can be matched and have $b_2 = 0$. But one finds at most 560 diffeomorphism classes (which are determined by b_3 and the divisibility of p_1). Using one of 75 known ACCY threefolds of "non-symplectic type" defined directly from a K3 surface [72] is one way of generating G_2 manifolds with $b_2 > 0$.

Toric Fano threefolds correspond to 3-dimensional reflexive polytopes. There are 4319 of these, including 18 from smooth Fano threefolds and 82 terminal (with nodal singularities).

Theorem 5.6 ([43]) *There are 526,130 isomorphism classes of smooth toric semi-Fano threefolds (1009 with nodal base), and 435,459 are rigid.*

Of these, 39,584 pairs satisfy $\mathrm{rk}N_+ + \mathrm{rk}N_- \leqslant 11$ but give no new b_3's. Restricting one summand to semi-Fano's of rank 1 and 2, one can generate at least

$$246\,446.(17+36+150) = 50{,}027{,}726$$

types of metrics with holonomy G_2 on 7-manifolds with $b_2 = 0$ and $55 \leqslant b_3 \leqslant 287$. This bound may be exceeded by matching in the weak Fano category and using methods of [19].

The analysis above suggests that a given smooth 7-manifold could admit many different types of metrics with holonomy G_2, which may or may not define G_2 structures of the same homotopy type, and may or may not be deformation equivalent. This leads to some exciting questions that are outlined below.

5.5 Concluding Remarks

Whilst there is an expectation that many of the constructions for G_2 holonomy can be carried over in one dimension higher (and have been [33, 65, 69]), it is true to say that to date the Spin(7) theory is less advanced. There has been less motivation from string theorists to develop the theory.

The defining structures are rather different: a 4-form Φ defining a Spin(7) structure is far from being stable, since the dimension of $\mathrm{GL}(8,\mathbb{R})/\mathrm{Spin}(7)$ is 43. On the other, its closure is enough to guarantee the reduction of holonomy. Any oriented spin 7-manifold admits a G_2 structure, but there is a well-known constraint in the 8-dimensional case involving the Euler characteristic χ and Pontrjagin numbers:

Lemma 5.7 *If a closed 8-manifold has a* Spin(7) *or* Sp(2)Sp(1) *structure then* $8\chi = 7p_1^2 - 4p_2$.

This simple result is another that unites two special structure groups in 8 dimensions, though holonomy reductions impose stronger constraints since $\frac{1}{24}(7p_1^2 - 4p_2)$ can also be expressed as $\sigma - 16\widehat{\mathrm{A}}$ where σ is the signature and $\widehat{\mathrm{A}}$ the A-hat genus. The latter is the index of the Dirac operator when the closed 8-manifold W is spin, and must equal 1 if W admits a metric with holonomy *equal* to Spin(7). In this case,

W is simply-connected and its Betti numbers satisfy

$$b_4^+ - 2b_4^- + b_3 - b_2 = 25.$$

This provides a useful constraint that can be checked when using the techniques of Example 5.2 to resolve orbifolds T^8/Γ with a Spin(7) structure.

Example 5.8 $\mathbb{HP}^2$ and $\mathrm{G}_2/\mathrm{SO}(4)$ both have $\sigma = 1$. Since they carry metrics of positive scalar curvature, they have $\widehat{\mathrm{A}} = 0$ and cannot admit a metric with holonomy Spin(7) or a subgroup thereof. Both admit Spin(7) structures. For example, the positive spin representation of Spin(8) decomposes as

$$\Delta_+ \cong S^2H \oplus \Lambda_0^2 E$$

relative to its subgroup Sp(2)Sp(1), in standard notation. The rank 5 vector bundle with fibre $\Lambda_0^2 E$ admits a family of nowhere-zero sections over $\mathbb{HP}^2$ (cf. [45, Remark 2.6]), each of which defines a reduction to a $\mathbb{Z}_2$ quotient of $\mathrm{Sp}(1)^3$.

There remain fascinating questions concerning compact manifolds with holonomy Spin(7), yet the proposition above has important consequences for G_2. Let M be a closed connected 7-manifold with a spin structure (i.e. $w_1 = 0 = w_2$). Any such manifold admits a G_2 structure (in the non-holonomy sense) and can be represented as the boundary of a manifold W^8 with a Spin(7) structure. This allowed Crowley and Nordström to define an invariant

$$\nu(M) = \chi(W) - 3\sigma(W) \mod 48$$

of the G_2 structure, helping to understand the set $\pi_0(\mathcal{G}_2(M)/\mathrm{Diff}(M))$ of deformation classes of such structures. Here $\mathcal{G}_2(M)$ denotes the "naïve" set of G_2 structures, each of which can be identified with a unit section (up to sign) of the spin bundle, and the set $\pi_0(\mathcal{G}_2(M))$ of its homotopy classes is parametrized by $\mathbb{Z}$.

Example 5.9 ([45]) There is a "parity check" for the invariant, namely $\nu(M) = 1 + b_1 + b_2 + b_3 \bmod 2$. The standard structure on $S^7 = \mathrm{Spin}(7)/G_2$ has $\nu = 1$. The canonical bundle $\kappa = \mathcal{O}(-4)$ over $\mathbb{CP}^3$ has a natural SU(4) structure and the G_2 structure induced on its unit circle bundle (isomorphic to $\mathrm{Spin}(7)/(\mathrm{G}_2 \times \mathbb{Z}_4)$) has $\nu = 7$. The G_2 structure induced on S^7 by the construction [26] over S^4 has $\nu = -1$.

The definition of ν makes it easier to formulate questions raised above:

- How many deformation classes of G_2 structures are there on a given smooth manifold?
- Which (if any) admit a metric with holonomy G_2?
- In each such class, is the moduli space of holonomy G_2 metrics connected?

It follows from the definition of ν (and the fact that reversing the sign of the 3-form reverses the sign of ν) that there are at least 24 deformation types. Concerning the second question, it turns out that manifolds constructed from twisted connected sums have $\nu = 24$, but a refinement of the Eells-Kuiper invariant can distinguish matchings [45].

Thanks to additional results of Crowley and Nordström, the existence is now known of pairs of metrics with holonomy G_2 that:

- exist on homeomorphic non-diffeomorphic manifolds [44],
- are disconnected within the same homotopy class of G_2 structures [46].

The non-diffeomorphic examples are formed by pairwise matchings from a triple of Fano threefolds with $b_2 = 2$ obtained by blowing up $\mathbb{CP}^3$ (or a quadric in $\mathbb{CP}^4$) along a curve. The second result on G_2 moduli is accomplished by lifting ν to an integer invariant defined (using the Atiyah-Patodi-Singer η-invariant for manifolds with boundary) that is locally constant for a family of metrics with *holonomy* G_2. By contrast, any two homotopic G_2 structures with closed 4-form are always connected within that category.

Let us close by mentioning the following topics, the subject of parallel investigations:

- Calibrated submanifolds in manifolds of exceptional holonomy. This means associative threefolds or coassociative fourfolds in G_2 manifolds, and Cayley fourfolds in Spin(7) manifolds. The last two give rise in principle to fibrations by K3 surfaces, and generalize the Strominger-Yau-Zaslow approach to mirror symmetry [92]. The twisted connect sum can produce G_2 manifolds with a finite number of rigid compact associative submanifolds diffeomorphic to $S^1 \times S^2$ [42]. The weakening of the Fano condition is crucial here, since the associatives arise from complex curves C in a semi-Fano manifold that satisfy $\kappa \cdot C = 0$. Moreover, it is possible to glue two K3 fibrations so as to obtain a *co*associative fibration of a manifold with G_2 holonomy over a 3-dimensional sphere [70], a situation that is studied in [48].
- Analogues of Ricci flow for metrics with exceptional holonomy. The Laplacian flow for G_2 structures was introduced by Bryant [25, 27]. When restricted to a closed G_2 3-forms, this takes the form

$$\frac{\partial \varphi}{\partial t} = -d * d * \varphi,$$

  and can be regarded as a gradient flow for the volume functional defined by Hitchin [62]. Lotay and Wei have established convergence to holonomy G_2 starting from a closed 3-form with small torsion on a compact manifold (involving a C^9 estimate), in analogy to Theorem 5.3. Homogeneous cases of the Laplacian flow have been examined in [49, 75].
- A Lie-theoretic account of instantons defined by a G structure was given in [83]. Instantons over manifolds with holonomy G_2, in both the compact and the explicit setting are now being studied [34, 76, 84]. The current volume contains a paper on the last topic, and some recent conjectures appear in [59].

Examples described earlier relating structures in dimensions 5–8 suggest that one should not consider one dimension in isolation. As in the classification of Fano varieties, there are likely to be underlying theories that transcend dimension.

Photographic Credits Figure 2 is an image recorded at a dinner in honour of Henri Cartan at the Tour d'Argent in 1979. I am grateful to Daniel Amiguet, Jean-Pierre Bourguignon and Manuel Ojanguren for their help in its identification, and to Odile Berger for permission to reproduce it. Figure 6 is copied with permission of the Mathematics Department at the University of Turin, which published the Fano volume. Figures 1, 3, 4, 5 and 7 were produced by the author using Mathematica..

References

1. E. Abbena, S. Garbiero, S. Salamon, Almost Hermitian geometry on the Iwasawa manifold. Ann. Scuola Norm. Sup. Pisa **30**(1), 147–170 (2001)
2. B.S. Acharya, F. Denef, C. Hofman, N. Lambert, Freund-Rubin revisited. (2003). arXiv:hep-th/0308046
3. I. Agricola, S.G. Chiossi, T. Friedrich, J. Höll, Spinorial description of $SU(3)$- and G_2-manifolds. J. Geom. Phys. (2014). arXiv:1411.5663
4. D.V. Alekseevsky, Riemannian spaces with unusual holonomy groups. Funk. Anal. Prilozen **2**, 1–10 (1968)
5. L. Antonyan, Classification of four-vectors of an eight-dimensional space. Trudy Sem. Vektor. Tenzor. Anal. **20**, 144–161 (1981)
6. V. Apostolov, S. Salamon, Kähler reduction of metrics with holonomy G_2. Commun. Math. Phys. **246**, 43–61 (2004)
7. M.F. Atiyah, N.J. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*. M.B. Porter Lectures, Rice University (Princeton University Press, Princeton, 1988)
8. M. Atiyah, E. Witten, M-theory dynamics on a manifold of G_2 holonomy. Adv. Theor. Math. Phys. **6**, 1–106 (2002)
9. V. Bangert, M.G. Katz, S. Shnider, S. Weinberger, E_7, Wirtinger inequalities, Cayley 4-form, and homotopy. Duke Math. J. **146**(1), 35–70 (2009)
10. C. Bär, Real killing spinors and holonomy. Commun. Math. Phys. **154**(3), 509–521 (1993)
11. H. Baum, K. Lärz, T. Leistner, On the full holonomy group of special Lorentzian manifolds. Math. Z. **277**(3–4), 797–828 (2014)
12. A. Beauville, Variétés kähleriennes dont la première classe de Chern est nulle. J. Differ. Geom. **18**(4), 755–782 (1983)
13. L. Bérard-Bergery, A. Ikemakhen, On the holonomy of Lorentzian manifolds, in *Differential Geometry: Geometry in Mathematical Physics and Related Topics*. Proceedings of Symposia in Pure Mathematics, vol. 54 (American Mathematical Society, Providence, RI, 1993), pp. 27–40
14. M. Berger, Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes. Bull. Soc. Math. Fr. **83**, 279–330 (1955)
15. O. Biquard, V. Minerbe, A Kummer construction for gravitational instantons. Commun. Math. Phys. **308**(3), 773–794 (2011)
16. F.A. Bogomolov, Hamiltonian Kählerian manifolds. Dokl. Akad. Nauk SSSR **243**(5) (1978)
17. J.P. Bourguignon, Groupes d'holonomie des variétés riemanniennes, in *Astérisque 126* (Soc. Math. France, 1985), pp. 169–180
18. A. Brandhuber, J. Gomis, S.S. Gubser, S. Gukov, Gauge theory at large N and new G_2 holonomy metrics. Nucl. Phys. B **611**(1–3), 179–204 (2001)
19. A.P. Braun, Tops as building blocks for G_2 manifolds. (2016). arXiv:1602.03521
20. D.C. Brody, L.P. Hughston, Geometric quantum mechanics. J. Geom. Phys. **38**, 19–53 (2001)

21. R. Bryant, Conformal and minimal immersions of compact surfaces into the 4-sphere. J. Differ. Geom. **17**(3), 455–473 (1982)
22. R.L. Bryant, Metrics with exceptional holonomy. Ann. Math. (2) **126**(3), 525–576 (1987)
23. R.L. Bryant, A survey of Riemannian metrics with special holonomy groups, in *Proceedings of International Congress of Mathematicians 1986*, Berkeley (1987), pp. 505–514
24. R.L. Bryant, Two exotic holonomies in dimension four, path geometries, and twistor theory, in *Proceedings of Symposia in Pure Mathematics*, vol. 53 (American Mathematical Society, Providence, RI, 1991), pp. 33–88
25. R. Bryant, Some remarks on G_2-structures, in *Proceedings of Gökova Geometry-Topology Conference 2005* (International Press, Providence, RI, 2006), pp. 75–109
26. R.L. Bryant, S.M. Salamon, On the construction of some complete metrics with exceptional holonomy. Duke Math. J. **58**(3), 829–850 (1989)
27. R.L. Bryant, F. Xu, Laplacian flow for closed G_2-structures: short time behavior. (2011). arXiv:1101.2004
28. F.E. Burstall, S.M. Salamon, Tournaments, flags, and harmonic maps. Math. Ann. **277**, 249–266 (1987)
29. E. Calabi, Métriques kählériennes et fibrés holomorphes. Ann. Sci. Ecole Norm. Sup. (4) **12**(2), 269–294 (1979)
30. M. Castrillón López, P.M. Gadea, I.V. Mykytyuk, The canonical eight-form on manifolds with holonomy group Spin(9). Int. J. Geom. Methods Mod. Phys. **07**, 1159 (2010)
31. S.G. Chiossi, Ó. Maciá, $SO(3)$-structures on 8-manifolds. Ann. Glob. Anal. Geom. **43**(1), 1–18 (2013)
32. S. Chiossi, S. Salamon, The intrinsic torsion of $SU(3)$ and G_2 structures, in *Differential Geometry, Valencia 2001* (World Scientific, Singapore, 2002)
33. R. Clancy, New examples of compact manifolds with holonomy Spin(7). Ann. Glob. Anal. Geom. **40**, 203–222 (2011)
34. A. Clarke, Instantons on the exceptional holonomy manifolds of Bryant and Salamon. J. Geom. Phys. **82**, 84–97 (2014)
35. R. Cleyton, A. Swann, Einstein metrics via intrinsic or parallel torsion. Math. Z. **247**, 513–528 (2004)
36. T. Coates, A. Corti, S. Galkin, A. Kasprzyk, Quantum periods for 3-dimensional Fano manifolds. Geom. Topol. **20**, 103–256 (2016)
37. D. Conti, T.B. Madsen, Harmonic structures and intrinsic torsion. Transform. Groups **20**(3), 699–723 (2015)
38. D. Conti, T.B. Madsen, Invariant torsion and G_2-metrics. Complex Manifolds **2**, 140–167 (2015)
39. D. Conti, S. Salamon, Generalized Killing spinors in dimension 5. Trans. Am. Math. Soc. **359**(11), 5319–5343 (2007)
40. D. Conti, T.B. Madsen, S. Salamon, Quaternionic geometry in dimension eight. (2016). arXiv:1610.04833
41. V. Cortés, J.J. Vásquez, Locally homogeneous nearly Kähler manifolds. Ann. Glob. Ann. Geom. **48**(3), 269–294 (2015)
42. A. Corti, M. Haskins, J. Nordström, T. Pacini, Asymptotically cylindrical Calabi-Yau 3-folds from weak Fano 3-folds. Geom. Topol. **17**(4), 1955–2059 (2013)
43. A. Corti, M. Haskins, J. Nordström, T. Pacini, G_2-manifolds and associative submanifolds via semi-Fano 3-folds. Duke Math. J. **164**(10), 1971–2092 (2015)
44. D. Crowley, J. Nordström, Exotic G_2 manifolds. (2014). arXiv:1411.0656
45. D. Crowley, J. Nordström, New invariants of G_2-structures. Geom. Topol. **19**(5), 2949–2992 (2015)
46. D. Crowley, S. Goette, J. Nordström, An analytic invariant of G_2 manifolds. (2015). arXiv:1505.02734
47. I.W. Dolgachev, *Classical Algebraic Geometry, A Modern View* (Cambridge University Press, Cambridge, 2012)

48. S. Donaldson, Adiabatic limits of co-associative Kovalev-Lefschetz fibrations. (2016). arXiv:1603.08391
49. M. Fernández, A. Fino, V. Manero, Laplacian flow of closed G_2-structures inducing nilsolitons. J. Geom. Anal. **26**(3), 1808–1837 (2016)
50. M. Fernández, S. Ivanov, V. Muñoz, L. Ugarte, Nearly hypo structures and compact nearly Kähler 6-manifolds with conical singularities. J. Lond. Math. Soc. (2) **78**(3), 580–604 (2008)
51. L. Foscolo, M. Haskins, New G_2-holonomy cones and exotic nearly Kähler structures on S^6 and $S^3 \times S^3$. Ann. Math. (2) **185**(1), 59–130 (2017)
52. D.S. Freed, On Wigner's theorem, in *Proceedings of the Freedman Fest*. Geom. Topol. Monogr., vol. 18, pp. 83–89. Coventry (2012)
53. A. Fujiki, M. Pontecorvo, Anti-self-dual bihermitian structures on Inoue surfaces. J. Differ. Geom. **85**(1), 15–72 (2010)
54. A. Gambioli, Y. Nagatomo, S. Salamon, Special geometries associated to quaternion-Kähler 8-manifolds. J. Geom. Phys. **91**, 146–162 (2015)
55. G.W. Gibbons, D.N. Page, C.N. Pope, Einstein metrics on S^3, R^3, and R^4 bundles. Commun. Math. Phys. **127**, 529–553 (1990)
56. A. Gray, Weak holonomy groups. Math. Z. **123**, 290–300 (1971)
57. P. Griffiths, J. Harris, *Principles of Algebraic Geometry*. Wiley Classics Library (Wiley-Interscience, New York, 1994)
58. M. Haskins, H.-J. Hein, J. Nordström, Asymptotically cylindrical Calabi-Yau manifolds. J. Differ. Geom. **101**(2), 213–265 (2015)
59. A. Haydys, G_2-instantons and the Seiberg-Witten monopoles. (2017). arXiv:1703.06329
60. A. He, P. Candelas, On the number of complete intersection Calabi-Yau manifolds. Commun. Math. Phys. **135**, 193–199 (1990)
61. Y. Herfray, K. Krasnov, C. Scarinci, Y. Shtanov, A 4D gravity theory and G_2-holonomy manifolds. (2016). arXiv:1602.03428
62. N. Hitchin, Stable forms and special metrics, in *Global Differential Geometry: The Mathematical Legacy of Alfred Gray*. Contemporary Mathematics, vol. 288 (American Mathematical Society, Providence, RI, 2001), pp. 70–89
63. L.P. Hughston, S.M. Salamon, Surveying points in the complex projective plane. Adv. Math. **286**, 1017–1052 (2016)
64. D. Joyce, The hypercomplex quotient and the quaternionic quotient. Math. Ann. **290**(2), 323–340 (1991)
65. D. Joyce, Compact 8-manifolds with holonomy Spin(7). Invent. Math. **123**, 507–552 (1996)
66. D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2. I. J. Differ. Geom. **43**, 291–328 (1996)
67. D.D. Joyce, *Compact Manifolds with Special Holonomy*. Oxford Mathematical Monographs (Oxford University Press, Oxford, 2000)
68. S. Karigiannis, M. Min-Oo, Calibrated sub-bundles in non-compact manifolds of special holonomy. Ann. Global Anal. Geom. **28**, 371–394 (2005)
69. A. Kovalev, Asymptotically cylindrical manifolds with holonomy Spin(7). I. (2013). arXiv:1309.5027
70. A. Kovalev, Coassociative K3 fibrations of compact G_2-manifolds. (2005). arXiv:math/0511150
71. A. Kovalev, Twisted connected sums and special Riemannian holonomy. J. Reine Angew. Math. **565**, 125–160 (2003)
72. A. Kovalev, N.-H. Lee, K3 surfaces with non-symplectic involution and compact irreducible G_2-manifolds. Math. Proc. Camb. Philos. Soc. **151**(2), 193–218 (2011)
73. A. Kovalev, J. Nordström, Asymptotically cylindrical 7-manifolds of holonomy G_2 with applications to compact irreducible G_2-manifolds. Ann. Glob. Anal. Geom. **38**(3), 221–257 (2010)
74. K.W. Lamson, Some differential and algebraic consequences of the Einstein field equations. Trans. Am. Math. Soc. **32**(5), 709–722 (1930)

75. J. Lauret, Laplacian flow of homogeneous G_2-structures and its solitons. Proc. Lond. Math. Soc. **114**(3), 527–560 (2017)
76. J.D. Lotay, G. Oliveira, $SU(2)^2$-invariant G_2-instantons. (2016). arXiv:1608.07789
77. S. Merkulov, L. Schwachhöfer, Classification of irreducible holonomies of torsion-free affine connections. Ann. Math. (2) **150**(1), 77–149 (1999)
78. R. Miyaoka, The Bryant-Salamon G_2-manifolds and hypersurface geometry. (2006). math-ph/0605074
79. S. Mori, S. Mukai, Extremal rays and Fano 3-folds, in *The Fano Conference*. Università di Torino (2004), pp. 37–50
80. M. Parton, P. Piccinni, Spin(9) and almost complex structures on 16-dimensional manifolds. Ann. Glob. Anal. Geom. **41**(3), 321–345 (2012)
81. F. Podestà, A. Spiro, Six-dimensional nearly Kähler manifolds of cohomogeneity one. J. Geom. Phys. **60**(2), 156–164 (2010)
82. F. Podestà, A. Spiro, Six-dimensional nearly Kähler manifolds of cohomogeneity one (II). Commun. Math. Phys. **312**(2), 477–500 (2012)
83. R. Reyes-Carrión, A generalization of the notion of instanton. Differ. Geom. Appl. **8**(1), 1–20 (1998)
84. H. Sá Earp, T. Walpuski, G_2-instantons on twisted connected sums. Geom. Topol. **19**(3), 1263–1285 (2015)
85. S.M. Salamon, Harmonic 4-spaces. Math. Ann. **269**, 169–178 (1984)
86. S. Salamon, *Riemannian Geometry and Holonomy Groups*. Pitman Research Notes Maths, vol. 201 (Longman Scientific and Technical, Harlow, 1989)
87. S.M. Salamon, On the cohomology of Kähler and hyper-Kähler manifolds. Topology **35**(1), 137–155 (1996)
88. S.M. Salamon, Almost Hermitian geometry, in *Invitations to Geometry and Topology*. Oxford Graduate Texts in Mathematics, vol. 7 (Oxford University Press, Oxford, 2002), pp. 233–291
89. S. Salamon, A tour of exceptional geometry. Milan J. Math. **71**, 59–94 (2003)
90. L.J. Schwachhöfer, Riemannian, symplectic and weak holonomy. Ann. Glob. Anal. Geom. **18**, 291–308 (2000)
91. J. Simons, On the transitivity of holonomy systems. Ann. Math. **76**, 213–234 (1962)
92. A. Strominger, S.-T. Yau, E. Zaslow, Mirror symmetry is T-duality. Nucl. Phys. B **479**(1–2), 243–259 (1996)
93. G. Tian, S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature. I. J. Am. Math. Soc. **3**(3), 579–609 (1990)
94. J.A. Wolf, The geometry and structure of isotropy irreducible homogeneous spaces. Acta Math. **120**, 59–148 (1968)
95. S.-T. Yau, Calabi's conjecture and some new results in algebraic geometry. Proc. Natl. Acad. Sci. USA **74**(5), 1798–1799 (1977)