Extensions of Abelian Automata Groups

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Mealy Automata

A **Mealy Automaton** $\mathcal{A}$ is a finite state machine which encodes a family of continuous functions from Cantor Space to itself. For us, these continuous functions will always be homeomorphisms, and thus we may associate to a machine $\mathcal{A}$ a subgroup $\mathcal{G}(\mathcal{A})$ of the automorphisms of Cantor Space. These are known as **Automata Groups** in the literature.

Classifying all groups generated by even 3-state machines is still and open problem, so we will focus attention on those which generate abelain groups.

We can define two operations from $\mathcal{G}(\mathcal{A})$ to itself called **Residuation**, where the 0-residual of f is defined to be the unique function $\partial_0 f$ so that

$$f(0s) = f(0)(\partial_0 f)(s).$$

The 1-residual $\partial_1 f$ is defined analogously.

A function f is called **even** if it copies its first input bit, that is $f(as) = a\partial_a f(s)$, and is called **odd** otherwise.

It is a theorem of Sutner that, in the abelian case, f is even if and only if $\partial_0 f = \partial_1 f$.

Past Results

In their paper "Automorphisms of the binary tree: State-closed subgroups and dynamics of 1/2-endomorphisms", Nerkashevych and Sidki show that abelian automata groups are isomorphic to integer lattices, and moreover, there is a "1/2-integral" matrix $\mathbf{A}_\mathcal{A}$ of irreducible character so that residuation lifts to an affine map. Succinctly, for some φ:

$$\varphi : \mathcal{G}(\mathcal{A}) \cong \mathbb{Z}^m$$

$$\varphi(\partial_0 f) = \begin{cases} \mathbf{A}\varphi(f) & f \text{ even} \\ \mathbf{A}(\varphi(f) - \overline{e}) & f \text{ odd} \end{cases}$$

$$\varphi(\partial_1 f) = \begin{cases} \mathbf{A}\varphi(f) & f \text{ even} \\ \mathbf{A}(\varphi(f) + \overline{e}) & f \text{ odd} \end{cases}$$

Moreover, φ can be chosen so that the first component of $\varphi(f)$ is even iff f is even. Under this additional constraint, $\overline{e}$ must be odd, and we can put $\mathbf{A}$ into rational canonical form ($a_i \in \mathbb{Z}$):

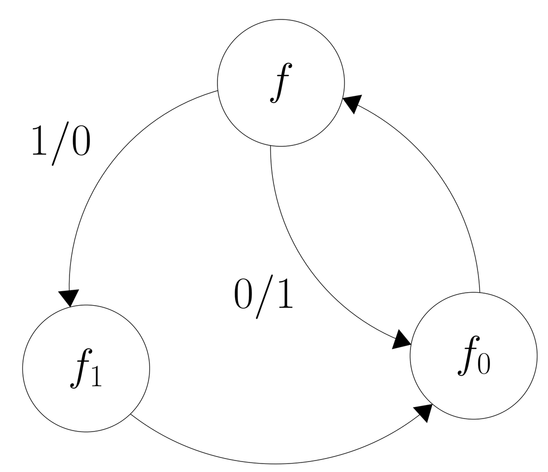
$$\begin{pmatrix} \frac{a_1}{2} & 1 & 0 & \dots & 0 \\ \frac{a_2}{2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n-1}}{2} & 0 & 0 & \dots & 1 \\ \frac{a_n}{2} & 0 & 0 & \dots & 0 \end{pmatrix}$$

$\chi_\mathbf{A}$is $\mathbb{Q}$-irreducible

The Question

By choosing a 1/2-integral matrix $\mathbf{A}$ and an odd residuation vector $\overline{e}$, we can view $\mathbb{Z}^m$ as a Mealy Automaton with countably many states. It is then natural to ask when we can find a particular machine $\mathcal{A}$ as a sub-automaton. It doesn't take much effort to show that the matrix $\mathbf{A}$ must agree with $\mathbf{A}_\mathcal{A}$, but there is no immediate pattern for which choices of $\overline{e}$ admit a particular $\mathcal{A}$ as a subautomaton.

An Important Example: $\mathcal{A}_2^3$

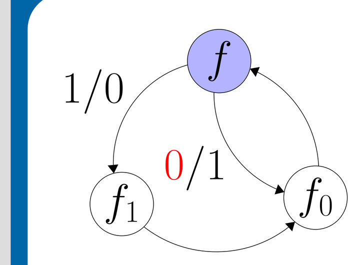


$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$

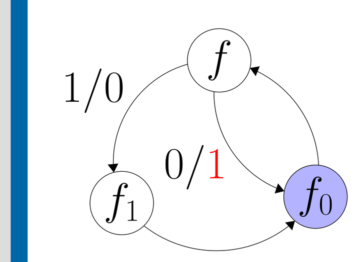
$\partial_0 f = f_0$
$\partial_1 f = f_1$
$\partial_0 f_0 = \partial_1 f_0 = f$
$\partial_0 f_1 = \partial_1 f_1 = f_0$

(Unlabeled edges correspond to both 0/0 and 1/1 edges)

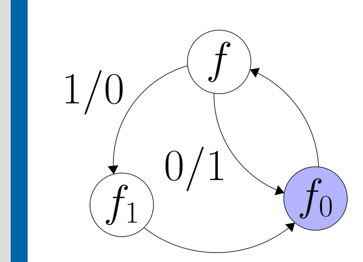
Computing $f(0110\dots)$



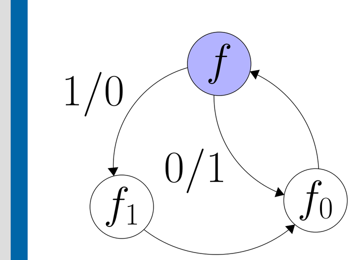
$f(0110\dots)$



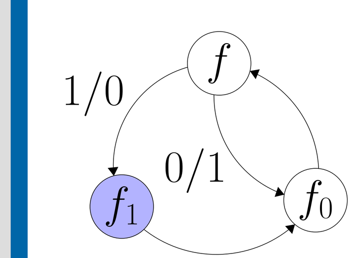
$1f_0(110\dots)$



$1f_0(110\dots)$



$11f(10\dots)$



$110f_1(0\dots)$

An Embedding

For $\overline{e} = (-3, -2)$:

$f = (1, 0)$
$f_0 = (0, 1)$
$f_1 = (-2, -2)$