

# Integral Self-Affine Tiles in $\mathbb{R}^n$

## Part II: Lattice Tilings

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**ABSTRACT.** Let  $A$  be an expanding  $n \times n$  integer matrix with  $|\det(A)| = m$ . A standard digit set  $\mathcal{D}$  for  $A$  is any complete set of coset representatives for  $\mathbb{Z}^n / A(\mathbb{Z}^n)$ . Associated to a given  $\mathcal{D}$  is a set  $T(A, \mathcal{D})$ , which is the attractor of an affine iterated function system, satisfying  $T = \bigcup_{d \in \mathcal{D}} (T + d)$ . It is known that  $T(A, \mathcal{D})$  tiles  $\mathbb{R}^n$  by some subset of  $\mathbb{Z}^n$ . This paper proves that every standard digit set  $\mathcal{D}$  gives a set  $T(A, \mathcal{D})$  that tiles  $\mathbb{R}^n$  with a lattice tiling.

### 1. Introduction

Suppose that  $A$  is an  $n \times n$  real matrix, which is *expanding*, that is, all its eigenvalues  $\lambda_i$  have  $|\lambda_i| > 1$ , and that  $|\det(A)| = m$  is an integer. Associated with any finite set  $\mathcal{D} \subset \mathbb{R}^n$ , with  $|\mathcal{D}| = m$ , there is then a unique compact set  $T = T(A, \mathcal{D})$  that satisfies the set-valued functional equation

$$A(T) = \bigcup_{d \in \mathcal{D}} (T + d), \quad (1.1)$$

which is given explicitly by

$$T(A, \mathcal{D}) := \left\{ \sum_{k=1}^{\infty} A^{-k} d_k : \text{all } d_k \in \mathcal{D} \right\}. \quad (1.2)$$

We call the vectors  $d \in \mathcal{D}$  *digits*, based on the view that (1.2) gives a multidimensional generalization of a radix expansion for the members of  $T$ . The set  $T(A, \mathcal{D})$  is called a *self-affine tile* if it has positive Lebesgue measure. For most pairs  $(A, \mathcal{D})$  the set  $T(A, \mathcal{D})$  has Lebesgue measure zero, and only special pairs  $(A, \mathcal{D})$  yield self-affine tiles.

The term *self-affine tile* refers to a geometric interpretation of the functional equation (1.1): It says that the affinely dilated set  $A(T)$  is perfectly tiled by the  $m$  translates  $T + \mathcal{D}$  of  $T$ , and that the overlaps  $(T + d) \cap (T + d')$  have measure zero for distinct  $d, d' \in \mathcal{D}$ . Moreover, using the functional equation, it then can easily be shown that  $T$  tiles  $\mathbb{R}^n$  by translation. Many examples of such tiles have fractal boundaries (see Falconer [9, §8.3]).

A *lattice self-affine tile* is a self-affine tile  $T = T(A, \mathcal{D})$  produced by a pair  $(A, \mathcal{D})$  such that the difference set  $\Delta(\mathcal{D}) = \mathcal{D} - \mathcal{D}$  is contained in a lattice  $\Lambda$  that is  $A$ -invariant in the sense that

$$A(\Lambda) \subseteq \Lambda. \quad (1.3)$$

Such self-affine tiles always give a tiling of  $\mathbb{R}^n$  by a set of translations  $\mathcal{S}$  contained in  $\Lambda$ . An

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*integral self-affine tile*<sup>1</sup> is a special kind of lattice self-affine tile where  $T = T(A, \mathcal{D})$  for an integer matrix  $A \in M_n(\mathbb{Z})$  and an integer digit set  $\mathcal{D} \subseteq \mathbb{Z}^n$ ; in this case we can take  $\Lambda = \mathbb{Z}^n$ . The study of lattice self-affine tiles can always be reduced to the special case of integral self-affine tiles by an affine transformation (see Lemma 2.1 in §2).

This paper continues a study of integral self-affine tiles, and it studies the following question: Which integral self-affine tiles can tile  $\mathbb{R}^n$  with a lattice tiling?

One motivation for studying the structure of tilings concerns the construction of orthonormal wavelet bases in  $\mathbb{R}^n$ . Gröchenig and Madych [12, Theorem 1] have shown that the characteristic function  $\chi_T(x)$  of an integral self-affine tile  $T$  is a scaling function of a multiresolution analysis that produces an orthonormal wavelet basis of  $L^2(\mathbb{R}^n)$  if and only if  $T$  tiles  $\mathbb{R}^n$  with the lattice  $\mathbb{Z}^n$ . This is equivalent to the Lebesgue measure  $\mu(T(A, \mathcal{D}))$  equals 1.

In studying lattice tilings for integral self-affine tiles, without loss of generality we may restrict consideration to a special subclass of  $(A, \mathcal{D})$ , which we call *primitive*. Associate to any integral pair  $(A, \mathcal{D})$  is the  $A$ -invariant sublattice  $\mathbb{Z}[A, \mathcal{D}]$  of  $\mathbb{Z}^n$  that contains the difference set  $\mathcal{D} - \mathcal{D}$ . When  $0 \in \mathcal{D}$ , this is

$$\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}[\mathcal{D}, A(\mathcal{D}), \dots, A^{n-1}(\mathcal{D})].$$

A pair  $(A, \mathcal{D})$  is *primitive* if  $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n$ , and then we call  $\mathcal{D}$  a *primitive digit set* for  $A$ . In Part I [18], we observed that if  $T = T(A, \mathcal{D})$  is an integral self-affine tile, then there is another integral self-affine tile  $\tilde{T} = T(\tilde{A}, \tilde{\mathcal{D}})$  with  $(\tilde{A}, \tilde{\mathcal{D}})$  primitive and  $0 \in \tilde{\mathcal{D}}$ , such that

$$T = B(\tilde{T}) + v \tag{1.4}$$

for some<sup>2</sup>  $B \in M_n(\mathbb{Z})$  with  $|\det(B)| \neq 0$  and some  $v \in \mathbb{Z}^n$ . This shows that  $T$  has a lattice tiling of  $\mathbb{R}^n$  if and only if  $\tilde{T}$  does. Consequently, it suffices to study primitive digit sets.

In Part I [18], we introduced a distinction between *standard digit sets* and *nonstandard digit sets*. A primitive digit set is called *standard* if it forms a complete residue system (mod  $A$ ), that is, a complete set of co-set representatives of the group  $\mathbb{Z}^n/A(\mathbb{Z}^n)$ ; otherwise it is *nonstandard*. (The extension of this definition to imprimitive digit sets is given in Part I [18].) All standard digit sets give self-affine tiles, that is, the measure  $\mu(T(A, \mathcal{D})) > 0$ . However, most nonstandard digit sets have  $\mu(T(A, \mathcal{D})) = 0$ . Part I [18] showed that if  $|\det(A)| = p$  is prime and  $p\mathbb{Z}^n \not\subseteq A^2\mathbb{Z}$ , then all nonstandard digit sets have  $\mu(T(A, \mathcal{D})) = 0$ . However, when  $|\det(A)| \neq p$  there exist nonstandard digit sets with  $\mu(T(A, \mathcal{D})) > 0$ . Part I [18] also proved that the measure condition  $\mu(T(A, \mathcal{D})) = 1$  necessary to obtain a multiresolution analysis giving a wavelet basis can never hold for nonstandard digit sets.

The distinction between standard and nonstandard digit sets is important for tiling questions. This paper considers only standard digit sets and proves the following theorem.

**Theorem 1.1.**

*Every integral self-affine tile  $T$  coming from a standard digit set gives a lattice tiling of  $\mathbb{R}^n$  with some lattice  $\Gamma \subseteq \mathbb{Z}^n$ .*

This result was conjectured by Gröchenig and Haas [11], who proved that it is true in the one-dimensional case. The hypothesis of a standard digit set cannot be removed from this conjecture, because there are integral self-affine tiles  $T$  coming from nonstandard digit sets that have no lattice tilings, for example,  $A = [4]$  and  $\mathcal{D} = \{0, 1, 8, 9\}$  has  $T = [0, 1] \cup [2, 3]$ .

To indicate why establishing Theorem 1.1 is a nontrivial problem in higher dimensions, we observe that iterating the functional equation (1.1) does not necessarily find lattice tilings. The

<sup>1</sup>Strictly speaking the integrality property is associated with the pair  $(A, \mathcal{D})$ . For any self-affine tile there are infinitely many choices of  $(A, \mathcal{D})$  with  $T = T(A, \mathcal{D})$ . Some of these pairs might be nonintegral.

<sup>2</sup>The columns of  $B$  then form a basis of the lattice  $\mathbb{Z}[A, \mathcal{D}]$ .

functional equation (1.1) can be used to directly produce *self-replicating tilings* of  $\mathbb{R}^n$ , which are translation tilings of  $\mathbb{R}^n$  consistent with (1.1) in the sense that for each tile  $T + v$  in the tiling, the inflated tile  $A(T + v)$  is a finite union of tiles in the tiling. (The concept of self-replicating tiling is due to Kenyon [16].) However, the primitive pair  $(A, \mathcal{D})$ , with

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathcal{D} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}, \quad (1.5)$$

has a standard digit set  $\mathcal{D}$ , and the tile  $T(A, \mathcal{D})$  has the property that *all* self-replicating tilings using  $T(A, \mathcal{D})$  are nonperiodic tilings and hence are not lattice tilings (see Lagarias and Wang [18, Example 2.3]). Nevertheless, this particular tile does have a lattice tiling,<sup>3</sup> using the lattice

$$3\mathbb{Z} \oplus \mathbb{Z} = \left\{ \begin{bmatrix} 3a \\ b \end{bmatrix} : a, b \in \mathbb{Z} \right\}.$$

To place these results in a more general context, we remark that it remains an open question whether every tile  $T$  that tiles  $\mathbb{R}^n$  by translation has a periodic tiling. (A *tile* is a compact set of positive measure, which is the closure of its interior, and has a boundary of measure zero.) Venkov [29] proved that every convex set  $T$  that tiles  $\mathbb{R}^n$  by translation has a lattice tiling, and his result was also found by McMullen [22]. Nonconvex tiles need not have any lattice tilings; for example, on  $\mathbb{R}$ , take  $T = [0, 1] \cup [2, 3]$ .

The contents of this article are as follows. Section 2 describes a Fourier-analytic tiling criterion taken from Gröchenig and Haas [11], which implies that a lattice tiling always exists when a certain transfer operator has no nonconstant eigenfunction of eigenvalue 1. In §3 we suppose that a nonconstant eigenfunction exists, and we introduce a notion of *special eigenfunction*  $f(x)$ . A key to our approach is a result showing that the zero set  $Z_f$  of a special eigenfunction, when projected onto the torus  $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ , is invariant under the linear map  $A^T$  (Lemma 3.2). The general idea of obtaining information from zero sets of special eigenfunctions is due to Conze and Raugi [6]. In §4 we introduce the notion of a *stretched tile*, which is a tile whose smallest  $A$ -invariant lattice generated by the differenced digit set  $\mathcal{D} - \mathcal{D}$  is  $\mathbb{Z}^n$ , but that has  $\mu(T(A, \mathcal{D})) > 1$ . Stretched tiles  $T(A, \mathcal{D})$  essentially correspond to the case where special eigenfunctions exist for  $(A, \mathcal{D})$ . We use a recent result of Cerveau, Conze, and Raugi [4], together with Lemma 3.2, to prove that the zero sets of special eigenfunctions of stretched tiles contain translates of an  $A^T$ -invariant vector space of dimension greater than or equal to one (Theorem 4.1). In §5 we explicitly construct a class of stretched tiles whose digit sets have a quasi-product form (Theorem 5.1). In §6 we use Theorem 4.1 to prove a structure theorem for those  $(A, \mathcal{D})$  giving stretched tiles, which shows that all of them essentially arise from the construction of §4 (Theorem 6.1). Section 6 uses this structure theorem to prove that all stretched tiles  $T(A, \mathcal{D})$  give lattice tilings by some sublattice of  $\mathbb{Z}^n$ . The lattice tiling property of self-affine tiles is also discussed in a recent preprint of Conze, Hervé, and Raugi [7].

## 2. Fourier-Analytic Tiling Criterion

It is known that  $T(A, \mathcal{D})$  tiles  $\mathbb{R}^n$  by translation with some tiling set  $\Gamma$  satisfying

$$\Gamma \subseteq \mathbb{Z}[A, \mathcal{D}] \quad (2.1)$$

<sup>3</sup>Kenyon [16] states a result (Theorem 12) that would imply the truth of the lattice tiling conjecture and, furthermore that asserts there always is a lattice tiling with an  $A$ -invariant lattice. However, this result is false. The tile  $(A, \mathcal{D})$  in (1.5) is a counterexample to Kenyon's result, as is shown in Lagarias and Wang [18, §4].

(see Gröchenig and Haas [11] or Lagarias and Wang [17]). If  $\Gamma = \mathbb{Z}[A, \mathcal{D}]$ , then  $T(A, \mathcal{D})$  tiles  $\mathbb{R}^n$  with a lattice tiling, and this occurs if and only if the Lebesgue measure  $\mu(T(A, \mathcal{D}))$  of the tile is

$$\mu(T(A, \mathcal{D})) = [\mathbb{Z}^n : \mathbb{Z}[A, \mathcal{D}]] = \det(\mathbb{Z}[A, \mathcal{D}]). \quad (2.2)$$

Vince [30] and Gröchenig and Haas [11] give criteria for the equality  $\Gamma = \mathbb{Z}[A, \mathcal{D}]$ . We follow the latter (see Lemma 2.3 herein).

As a preliminary fact, we recall that the study of general lattice self-affine tiles can be reduced to the study of integral self-affine tiles that are primitive.

**Lemma 2.1.**

Let  $T = T(A, \mathcal{D})$  be an integral self-affine tile in  $\mathbb{R}^n$ . Then there is an invertible affine transformation  $L(x) = Bx + v$  such that  $L(T) = \tilde{T}$ , where  $\tilde{T} = T(\tilde{A}, \tilde{\mathcal{D}})$  is an integral self-affine tile with  $0 \in \tilde{\mathcal{D}}$  and  $(\tilde{A}, \tilde{\mathcal{D}})$  primitive, that is,  $\tilde{A} \in M_n(\mathbb{Z})$ ,  $\tilde{\mathcal{D}} \subset \mathbb{Z}^n$  and  $\mathbb{Z}[\tilde{A}, \tilde{\mathcal{D}}] = \mathbb{Z}^n$ . Furthermore,  $\tilde{A}$  is similar to  $A$  over  $\mathbb{Q}$ .

**Proof.** This is Lemma 2.1 of Lagarias and Wang [18].  $\square$

For a digit set  $\mathcal{D}$  we define the *digit function*  $g_{\mathcal{D}} : \mathbb{R}^n \rightarrow \mathbb{C}$  by

$$g_{\mathcal{D}}(x) := \frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} \exp(2\pi i \langle d, x \rangle). \quad (2.3)$$

We also define the *correlation function*  $u_{\mathcal{D}} : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$u_{\mathcal{D}}(x) := |g_{\mathcal{D}}(x)|^2 = \frac{1}{|\mathcal{D}|^2} \sum_{d, d' \in \mathcal{D}} \exp(2\pi i \langle d - d', x \rangle). \quad (2.4)$$

In the rest of this section, we always assume that  $\mathcal{D}$  is a complete residue system (mod  $A$ ). We also assume that  $\mathcal{D}^T$  is some complete residue system (mod  $A^T$ ).

**Lemma 2.2.**

For all  $x \in \mathbb{R}^n$ , we have

$$\sum_{l \in \mathcal{D}^T} u_{\mathcal{D}}((A^T)^{-1}(x + l)) \equiv 1. \quad (2.5)$$

**Proof.** See Gröchenig and Haas [11, Lemma 5.1].  $\square$

Now we define a linear operator  $\hat{C}_{A, \mathcal{D}}$  on the space  $\Omega(\mathbb{R}^n)$  of exponential polynomials, where  $\Omega(\mathbb{R}^n)$  consists of all

$$f(x) = \sum_{m \in \mathbb{Z}^n} a_m \exp(2\pi i \langle m, x \rangle), \quad a_m \in \mathbb{R}, \quad (2.6)$$

with only finitely many  $a_m \neq 0$ . Define the *transfer operator*  $\hat{C}_{A, \mathcal{D}} : \Omega(\mathbb{R}^n) \rightarrow \Omega(\mathbb{R}^n)$  by

$$\hat{C}_{A, \mathcal{D}} f(x) := \sum_{l \in \mathcal{D}^T} u_{\mathcal{D}}((A^T)^{-1}(x + l)) f((A^T)^{-1}(x + l)). \quad (2.7)$$

It is easy to check that  $\hat{C}_{A, \mathcal{D}}$  is a linear operator that maps  $\Omega(\mathbb{R}^n)$  into itself, by expanding the terms  $u_{\mathcal{D}}(\cdot)$  using (2.4), and  $\hat{C}_{A, \mathcal{D}}$  is independent of the choice of  $\mathcal{D}^T$ .

We are concerned with the action of  $\hat{C}_{A, \mathcal{D}}$  on the space  $\Omega^+(\mathbb{R}^n)$  of real cosine polynomials

$$f(x) = \sum_{m \in \mathbb{Z}^n} a_m \cos(2\pi \langle m, x \rangle), \quad a_m \in \mathbb{R}, \quad (2.8)$$

with only finitely many  $a_m \neq 0$ . This is exactly the set of functions  $f(x)$  in  $\Omega(\mathbb{R}^n)$  left fixed by the involution  $Jf(x) = f(-x)$ . It is easy to check that  $\hat{C}_{A,D}$  commutes with the involution  $J$ , and hence it has  $\Omega^+(\mathbb{R}^n)$  as an invariant subspace.

Lemma 2.2 shows that the constant functions are eigenfunctions of  $\hat{C}_{A,D}$  with eigenvalue one.

**Lemma 2.3 ( $\mathbb{Z}^n$ -Tiling Criterion).**

$T(A, D)$  tiles  $\mathbb{R}^n$  with a  $\mathbb{Z}^n$ -tiling if and only if the only solutions  $f(x) \in \Omega^+(\mathbb{R}^n)$  of

$$\hat{C}_{A,D}f(x) = f(x) \quad (2.9)$$

are constant functions.

**Proof.** This is proposition 5.3 of Gröchenig and Haas [11].  $\square$

### 3. Zero Set of Eigenfunctions

Throughout this section,  $A$  denotes an expanding matrix in  $M_n(\mathbb{Z})$ , and  $D$  denotes a complete residue system (mod  $A$ ).

If  $\mu(T(A, D)) = 1$ , then  $T(A, D)$  lattice tiles  $\mathbb{R}^n$  with the lattice  $\mathbb{Z}^n$ , so we need only study the case in which  $\mu(T(A, D)) > 1$ . Our basic approach to finding a lattice tiling is to study the structure of the zero set of a specially chosen nonconstant eigenfunction  $f(x)$ . This approach was used by Gröchenig and Haas in the one-dimensional case; they attribute the idea to Conze and Raugi [6].

**Lemma 3.1.**

Suppose that there exists a nonconstant  $\tilde{f}(x) \in \Omega^+(\mathbb{R}^n)$  satisfying

$$\hat{C}_{A,D}\tilde{f}(x) = \tilde{f}(x). \quad (3.1)$$

Then there exists such an eigenfunction  $f(x)$  satisfying

$$f(x) \geq 0 \quad \text{and} \quad f(0) > 0, \quad (3.2)$$

which has a nonempty (real) zero set  $Z_f = \{x \in \mathbb{R}^n : f(x) = 0\}$ .

**Proof.** Suppose that  $\tilde{f}(x) \in \Omega^+(\mathbb{R}^n)$  is nonconstant and satisfies (3.1). Define

$$f_1(x) = \tilde{f}(x) - \min_{y \in \mathbb{R}^n} \tilde{f}(y), \quad f_2(x) = \max_{y \in \mathbb{R}^n} \tilde{f}(y) - \tilde{f}(x).$$

Clearly, both  $f_i(x) \geq 0$  for all  $x \in \mathbb{R}^n$ , and

$$f_2(x) + f_1(x) = \max_{y \in \mathbb{R}^n} \tilde{f}(y) - \min_{y \in \mathbb{R}^n} \tilde{f}(y) > 0.$$

Now we define  $f(x)$  to be any one of the  $f_1(x)$ ,  $f_2(x)$  that satisfies  $f_i(0) > 0$ . The zero set  $Z_f$  is nonempty by construction, and our choice of  $f(x)$  guarantees that  $f(x) \geq 0$  and  $f(0) > 0$ , proving the lemma.  $\square$

**Remark.** We call an  $f(x)$  having the properties of Lemma 3.1 a *special eigenfunction* of  $(A, D)$ . The property  $f(0) > 0$  and the periodicity of  $f(x)$  (mod  $\mathbb{Z}^n$ ) guarantees that  $\mathbb{Z}^n \cap Z_f = \emptyset$ , a fact that will be important later.

**Lemma 3.2.**

Let  $f(x)$  be a special eigenfunction of  $(A, D)$ . Then

$$Z_f \subseteq A^T(Z_f) + \mathbb{Z}^n. \quad (3.3)$$

**Proof.** If  $x \in Z_f$ , then  $\hat{C}_{A, \mathcal{D}} f(x) = f(x) = 0$ . Let  $\mathcal{D}^T$  be a complete residue system  $(\text{mod } A^T)$ . Then, by definition,

$$\sum_{l \in \mathcal{D}^T} u_{\mathcal{D}}((A^T)^{-1}(x+l)) f((A^T)^{-1}(x+l)) = 0. \quad (3.4)$$

Since  $f(z) \geq 0$  everywhere, every term on the right-hand side must be zero. Now Lemma 2.2 implies that some  $z^* = (A^T)^{-1}(x+l)$  give

$$u_{\mathcal{D}}(z^*) > 0,$$

and hence  $f(z^*) = 0$ . Now,  $A^T(z^*) = x+l$ , and therefore  $x \in A^T(Z_f) + \mathbb{Z}^n$ . Thus  $Z_f \subseteq A^T(Z_f) + \mathbb{Z}^n$ .  $\square$

Now, for each  $l \in \mathbb{Z}^n$  define the map  $\tau_l : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$\tau_l(x) = (A)^{-1}(x+l). \quad (3.5)$$

**Definition 3.1.** Let  $\mathcal{D}^T$  be a complete residue system  $(\text{mod } A^T)$ . We call a subset  $Y$  of  $\mathbb{R}^n$   $\tau$ -invariant with respect to  $\mathcal{D}^T$  if for any  $x \in Y$ ,

$$l \in \mathcal{D}^T \quad \text{and} \quad u_{\mathcal{D}}(\tau_l(x)) > 0 \implies \tau_l(x) \in Y. \quad (3.6)$$

$Y$  is *minimal* if it does not contain a proper subset that is also  $\tau$ -invariant with respect to  $\mathcal{D}^T$ .  $\square$

We shall simply call such a set  $Y$   $\tau$ -invariant when there is no ambiguity. If  $Y$  is *periodic*, that is, if  $Y = Y + \mathbb{Z}^n$ , then  $Y$  is  $\tau$ -invariant with respect to some  $\mathcal{D}^T$  implies that  $Y$  is  $\tau$ -invariant with respect to all complete residue systems  $(\text{mod } A^T)$ . The zero set  $Z_f$  of a special eigenfunction  $f$  is always  $\tau$ -invariant as a result of (3.4). Let  $\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$  be the  $n$ -dimensional torus. We call  $\bar{Y} \subseteq \mathbb{T}^n$   $\tau$ -invariant if  $\pi_n^{-1}(\bar{Y}) \subseteq \mathbb{R}^n$  is  $\tau$ -invariant, where  $\pi_n : \mathbb{R}^n \rightarrow \mathbb{T}^n$  is the canonical covering map.

Gröchenig and Haas [11] settled the one-dimensional case of Theorem 1.1 by showing that  $\mu(T(A, \mathcal{D})) > 1$  can never occur for a primitive pair  $(A, \mathcal{D})$  when  $n = 1$ . The essential part of their proof is contained in the following lemma.

**Lemma 3.3.**

*If there exists a uniformly discrete nonempty invariant subset  $Y_f$  with  $Y_f = Y_f + \mathbb{Z}^n$ , then we have  $\mathbb{Z}(A, \mathcal{D}) \subset \Gamma$ , where  $\Gamma$  is a proper  $A$ -invariant lattice, and  $\mathcal{D}$  is not primitive.*

**Proof.** We argue by contradiction. Suppose  $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n$ . Let  $Y_f \subseteq Z_f$  be a uniformly discrete nonempty  $\tau$ -invariant set with  $Y_f = Y_f + \mathbb{Z}^n$ , and set  $\bar{Y}_f := \pi_n(Y_f)$ . Suppose that  $\mathcal{D}^T$  is a complete residue system  $(\text{mod } A^T)$ . By Lemma 2.2, for each  $y \in Y_f$  there exists at least one  $l \in \mathcal{D}^T$  such that  $u_{\mathcal{D}}(\tau_l(y)) > 0$ , and therefore  $\tau_l(y) \in Y_f$ . Because  $A^T(\tau_l(y)) \equiv y \pmod{\mathbb{Z}^n}$ , we therefore have  $A^T(Y_f) + \mathbb{Z}^n \supseteq Y_f$ . Hence  $A_*^T(\bar{Y}_f) \supseteq \bar{Y}_f$ , where  $A_*^T : \mathbb{T}^n \rightarrow \mathbb{T}^n$  is the induced map from  $A^T$ . This implies  $A_*^T(\bar{Y}_f) = \bar{Y}_f$  because  $\bar{Y}_f$  is finite, and therefore  $A_*^T$  acts as a permutation on  $\bar{Y}_f$ . Thus for any  $y \in Y_f$  we have  $(A^T)^k(y) \equiv y \pmod{\mathbb{Z}^n}$  for some finite  $k$ . Solving this equation shows that  $y$  is rational, that is,  $y \in \mathbb{Q}^n$ .

We show that for each  $y \in Y_f$  there exists exactly one  $l \in \mathcal{D}^T$  such that  $\tau_l(y) \in Y_f$ . Suppose there were distinct  $l_1, l_2 \in \mathcal{D}^T$  such that  $\tau_{l_1}(y), \tau_{l_2}(y) \in Y_f$ . Then we have  $A^T(\tau_{l_1}(y)) \equiv A^T(\tau_{l_2}(y)) \equiv y \pmod{\mathbb{Z}^n}$ . However,  $\tau_{l_1}(y) \not\equiv \tau_{l_2}(y) \pmod{\mathbb{Z}^n}$ . This contradicts the fact that  $A_*^T$  is a permutation on  $\bar{Y}_f$ .

So now for any  $y \in Y_f$  there exists an  $l^* \in \mathcal{D}^T$  such that  $u_{\mathcal{D}}(\tau_{l^*}(y)) > 0$  and  $u_{\mathcal{D}}(\tau_l(y)) = 0$  for all  $l \in \mathcal{D}^T$  and  $l \neq l^*$ ; hence  $u_{\mathcal{D}}(\tau_{l^*}(y)) = 1$ . Since  $y \in Y_f$  is arbitrarily chosen, we have

$$u_{\mathcal{D}}(y) = |g_{\mathcal{D}}(y)|^2 = 1, \quad \text{all } y \in Y_f. \quad (3.7)$$

Using the definition (2.3) of  $g_{\mathcal{D}}(x)$ , and that  $0 \in \mathcal{D}$ , (3.7) holds if and only if

$$\langle d, y \rangle \equiv 0 \pmod{1}, \quad \text{all } d \in \mathcal{D} \text{ and all } y \in Y_f.$$

We use this fact to define a new lattice

$$\Gamma = \{w : w \in \mathbb{Z}^n \text{ and } \langle w, y \rangle \in \mathbb{Z} \text{ for all } y \in Y_f\}. \quad (3.8)$$

Because  $Y_f$  lies in finitely many  $\mathbb{Z}^n$ -equivalence classes,  $\Gamma$  is a full rank sublattice of  $\mathbb{Z}^n$ . Also  $Y_f \cap \mathbb{Z}^n = \emptyset$ , because  $Z_f \cap \mathbb{Z}^n = \emptyset$ ; hence we have  $\Gamma \neq \mathbb{Z}^n$ . Next we show that

$$A(\Gamma) \subseteq \Gamma. \quad (3.9)$$

To see this, given  $w \in \Gamma$  and  $y \in Y_f$ , there is a  $y_1 \in Y_f$  such that  $A^T(y) = y_1 + l$  for some  $l \in \mathbb{Z}^n$ , and

$$\langle Aw, y \rangle = \langle w, A^T y \rangle = \langle w, y_1 + l \rangle = \langle w, y_1 \rangle + \langle w, l \rangle \in \mathbb{Z},$$

since  $\langle w, y \rangle \in \mathbb{Z}$  by definition of  $\Gamma$ , and  $\langle w, l \rangle \in \mathbb{Z}$  since both  $w, l \in \mathbb{Z}^n$ .

Now (3.7) implies that  $\mathcal{D} \subseteq \Gamma$ , and hence  $\Delta(\mathcal{D}) = \mathcal{D} - \mathcal{D} \subseteq \Gamma$ . Therefore, (3.9) implies that  $\mathbb{Z}[A, \mathcal{D}] \subseteq \Gamma$ . However,  $\Gamma$  is a proper subset of  $\mathbb{Z}^n$ , contradicting  $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n$ .  $\square$

Now we can settle the one-dimensional case, where  $A = [\pm m]$  with  $m \geq 2$ , and a standard digit set  $\mathcal{D} = \{d_1, \dots, d_m\} \subseteq \mathbb{Z}$  is just a complete residue system (mod  $m$ ). The primitivity condition  $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}$  is equivalent to

$$\gcd(d - d' : d, d' \in \mathcal{D}) = 1. \quad (3.10)$$

**Theorem 3.4 (Gröchenig and Haas).**

Suppose that  $A = [\pm m]$  and  $\mathcal{D}$  is a complete residue system (mod  $m$ ). Set  $d = \gcd(d' - d'' : d', d'' \in \mathcal{D})$ . Then  $T(A, \mathcal{D})$  tiles  $\mathbb{R}$  by the lattice  $d\mathbb{Z}$ , and  $\mu(T(A, \mathcal{D})) = d$ .

**Proof.** Using Lemma 2.1, we reduce to the case in which that  $d = 1$ . Now Lemma 3.3 applies to show that  $\mu(T(A, \mathcal{D})) = 1$ , because the real zero set  $Z_f$  of any nonconstant trigonometric polynomial must be discrete.  $\square$

When the dimension  $n \geq 2$  the case  $\mu(T(A, \mathcal{D})) > 1$  can occur, as in example (1.5) of §1.

#### 4. Stretched Tiles and Hyperplane Zeros of Special Eigenfunctions

To prove that all standard digit sets give tiles having lattice tilings, it suffices to study the case of  $(A, \mathcal{D})$  such that  $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n$  and  $\mu(T(A, \mathcal{D})) > 1$ .

**Definition 4.1.** We call  $T(A, \mathcal{D})$  a *stretched tile* if

$$\mu(T(A, \mathcal{D})) > [\mathbb{Z}^n : \mathbb{Z}[A, \mathcal{D}]]. \quad \square$$

Lemmas 2.3 and 3.1 combine to show that a stretched tile has a special eigenfunction. The proof of Theorem 1.1 rests on a special property of the real zero set of a special eigenfunction  $Z_f$  of a stretched tile, which is that it contains translates of certain linear subspaces of  $\mathbb{R}^n$ , stated as Theorem 4.1 below.

It appears that the global structure of the set  $Z_f$  is a union of translates of rational subspaces of  $\mathbb{R}^n$  of various dimensions. A *rational subspace*  $V$  of  $\mathbb{R}^n$  is a linear space having a basis consisting of rational vectors  $v \in \mathbb{Q}^n$ . This would be a special case of the following conjecture.

**Hyperplane Zeros Conjecture.**

Let  $h : \mathbb{R}^n \rightarrow \mathbb{C}$  be a real-analytic function that is periodic (mod  $\mathbb{Z}^n$ ). Suppose that there is an expanding integer matrix  $A$  such that

$$Z_h \subseteq A(Z_h) + \mathbb{Z}^n.$$

Then

$$Z_h = \bigcup_{i=1}^m (x_i + V_i) + \mathbb{Z}^n, \quad (4.1)$$

in which each  $x_i \in \mathbb{R}^n$  and each  $V_i$  is a rational subspace of  $\mathbb{R}^n$ . (The  $V_i$  need not all have the same dimension.)

We derive a weak result in the direction of this conjecture for a special eigenfunction of a stretched tile, which will suffice to prove our main result.

**Theorem 4.1.**

Let  $A \in M_n(\mathbb{Z})$  be expanding and  $\mathcal{D}$  be a primitive complete residue system (mod  $A$ ). Let  $\mathcal{D}^T$  be a complete residue system (mod  $A^T$ ). Suppose that  $\mu(T(A, \mathcal{D})) > 1$ , and let  $f(x)$  be a special eigenfunction for  $(A, \mathcal{D})$ . Then the real zero set  $Z_f$  contains a finite number of translates  $\{y_i + W : 0 \leq i \leq k-1\}$  of an  $A^T$ -invariant proper rational subspace  $W$  of  $\mathbb{R}^n$  such that

- i.  $A^T y_{i+1} \equiv y_i \pmod{\mathbb{Z}^n}$  for all  $0 \leq i \leq k-1$ , where  $y_k := y_0$ .
- ii. For every  $x \in y_i + W$ , we have

$$\sum_{\substack{l \in \mathcal{D}^T \\ \tau_l(x) \in y_{i+1} + W + \mathbb{Z}^n}} u_{\mathcal{D}}(\tau_l(x)) = 1. \quad (4.2)$$

The main ingredient in the proof of this theorem is a result of Cerveau, Conze, and Raugi [4]. First, we prove the following lemma.

**Lemma 4.2.**

Let  $V$  be a subspace of  $\mathbb{R}^n$ . Then  $\pi_n(V)$  is closed in  $\mathbb{T}^n$  if and only if  $V$  is a rational subspace of  $\mathbb{R}^n$ .

**Proof.** First we show that if  $V$  is a rational subspace of  $\mathbb{R}^n$ , then  $\pi_n(V)$  is closed in  $\mathbb{T}^n$ . Let  $w_1, w_2, \dots, w_r \in \mathbb{Z}^n$  form a basis of  $V$ . Suppose that  $z^* \in \mathbb{T}^n$  is in the closure of  $\pi_n(V)$ . Then we may find a sequence  $\{x_j\}$  in  $V$  such that  $\lim_{j \rightarrow \infty} \pi_n(x_j) = z^*$ . Write

$$x_j = \sum_{k=1}^r b_{j,k} w_k.$$

Since all  $w_k \in \mathbb{Z}^n$ , we may choose all  $b_{j,k} \in [0, 1)$ . Therefore, we can find a subsequence  $\{j_m\}$  of  $\{j\}$  such that

$$\lim_{m \rightarrow \infty} b_{j_m,k} = b_k^*, \quad \text{all } 1 \leq k \leq r.$$

Let  $x^* = \sum_{k=1}^r b_k^* w_k$ . Clearly,  $\pi_n(x^*) = z^*$ . Hence  $z^* \in \pi_n(W)$ . Therefore,  $\pi_n(V)$  is closed in  $\mathbb{T}^n$ .

Next we prove the following hypothesis: If  $v \in \mathbb{R}^n$ , then the closure of  $\pi_n(\mathbb{R}v)$  in  $\mathbb{T}^n$  is a rational subspace. To see this, let  $v = [\beta_1, \dots, \beta_n]^T$ . Without loss of generality we assume that



$\beta_1, \dots, \beta_r$  are linearly independent over  $\mathbb{Q}$ , while  $\beta_k = \sum_{j=1}^r a_{k,j} \beta_j$  with  $a_{k,j} \in \mathbb{Q}$  for all  $1 \leq k \leq n$ . The set

$$\left\{ m \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \pmod{\mathbb{Z}^r} : m \in \mathbb{Z} \right\}$$

is dense in  $\mathbb{T}^r$  (Cassels [3, Theorem I, p. 64]). Now let  $V_0 = \{Ax : x \in \mathbb{R}^r\}$ , where  $A = [a_{k,j}]$ . Then  $V_0$  is a rational subspace of  $\mathbb{R}^n$ , and  $\pi_n(V_0)$  is contained in the closure of  $\pi_n(\mathbb{R}v)$ . However,  $\pi_n(V_0)$  is closed and  $V_0 \supseteq \mathbb{R}v$ . Hence the closure of  $\pi_n(\mathbb{R}v)$  is  $\pi_n(V_0)$ , proving the hypothesis.

Finally, let  $v_1, \dots, v_r$  be a basis of  $V$ . Suppose that  $\bar{W}_j$  is the closure of  $\pi_n(\mathbb{R}v_j)$  in  $\mathbb{T}^n$ . Then the closure of  $\pi_n(V)$  contains  $\bar{W}_1 + \dots + \bar{W}_r$ . However,  $\bar{W}_1 + \dots + \bar{W}_r$  is closed in  $\mathbb{T}^n$  because it is a rational subspace, and it contains  $\pi_n(V)$ . Hence the closure of  $\pi_n(V)$  is  $\bar{W}_1 + \dots + \bar{W}_r$ , proving the lemma.  $\square$

#### Corollary 4.3.

Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  be continuous and periodic (mod  $\mathbb{Z}^n$ ). Suppose that  $V$  is a subspace of  $\mathbb{R}^n$  such that  $v_0 + V \subseteq Z_f$  where  $v_0 \in \mathbb{R}^n$ . Then  $v_0 + W \subseteq Z_f$  where  $W$  is the smallest rational subspace of  $\mathbb{R}^n$  containing  $V$ .

**Proof.** First, let  $\{V_\alpha\}$  be a set of rational subspaces of  $\mathbb{R}^n$ . Then  $\pi_n(\bigcap_\alpha V_\alpha) = \bigcap_\alpha \pi_n(V_\alpha)$  is closed in  $\mathbb{T}^n$ ; therefore,  $\bigcap_\alpha V_\alpha$  must be a rational subspace of  $\mathbb{R}^n$ . This implies that the minimal rational subspace  $W$  containing  $V$  exists. Since  $f(x)$  is periodic (mod  $\mathbb{Z}^n$ ), we may view it as a continuous function defined on  $\mathbb{T}^n$ . Now,  $\pi_n(v_0) + \pi_n(W)$  is the closure of  $\pi_n(v_0) + \pi_n(V)$  in  $\mathbb{T}^n$ . Hence  $\pi_n(v_0) + \pi_n(W)$  is in the zero set of  $f : \mathbb{T}^n \rightarrow \mathbb{C}$ . Thus  $v_0 + W \subseteq Z_f$ .  $\square$

**Proof of Theorem 4.1.** We construct a nonempty minimal compact  $\tau$ -invariant set  $Y$  with respect to  $\mathcal{D}^T$  in  $Z_f$  as follows, where  $f(x)$  is a special eigenfunction of  $(A, \mathcal{D})$ . Take any point  $x_0 \in Z_f$  and set  $X_0 = \{x_0\}$  and recursively define the finite sets  $\{X_j : j \geq 0\}$  by letting  $X_j$  consist of all points  $x_j$  such that  $x_j = \tau_l(x_{j-1})$  with  $x_{j-1} \in X_{j-1}$  and  $l \in \mathcal{D}^T$  such that  $u_{\mathcal{D}}(x_j) > 0$ . Then the  $\tau$ -invariance of  $Z_f$  with respect to  $\mathcal{D}^T$  gives  $X_j \subseteq Z_f$  for all  $j \geq 0$ . The set  $\bigcup_{j=0}^\infty X_j$  lies in a bounded region in  $\mathbb{R}^n$  because the mappings  $\tau_l$  are uniformly contracting with respect to a suitable norm in  $\mathbb{R}^n$  (see Lagarias and Wang [18, §3] or Cerveau, Conze, and Raugi [4]). Now let  $Y_0$  be the set of all cluster points of sequences  $\{x_j^* : x_j^* \in X_j\}$ . Then  $Y_0$  is a compact set, and we show that  $Y_0$  is  $\tau$ -invariant with respect to  $\mathcal{D}^T$ . If  $y \in Y_0$  and  $u_{\mathcal{D}}(\tau_l(y)) > 0$  where  $l \in \mathcal{D}^T$ , take a subsequence  $x_{j_k} \in X_{j_k}$  that converges to  $y$ , so that  $\tau_l(x_{j_k}) \rightarrow \tau_l(y)$ . Now  $u_{\mathcal{D}}(\tau_l(x_{j_k})) > 0$  for  $k$  sufficiently large, and hence  $\tau_l(x_{j_k}) \in X_{j_k+1}$ ; therefore, we may construct a sequence having  $\tau_l(y)$  as a cluster point, proving  $\tau_l(y) \in Y_0$ . The existence of a nonempty minimal compact  $\tau$ -invariant set  $Y$  with respect to  $\mathcal{D}^T$  contained in  $Y_0$  follows by a Zorn's lemma argument.

It follows from Theorem 2.8 of Cerveau, Conze, and Raugi [4] that there exists an  $A^T$ -invariant subspace  $V$  and  $\{y_i \in Y : 0 \leq i \leq k-1\}$  satisfying property i such that

$$Y \subseteq \bigcup_{i=0}^{k-1} (y_i + V) \subseteq Z_f, \quad (4.3)$$

with the property that the set  $\bigcup_{i=0}^{k-1} (y_i + V)$  is  $\tau$ -invariant with respect to  $\mathcal{D}^T$ . Now let  $W$  be the smallest rational subspace of  $\mathbb{R}^n$  containing  $V$ . Since  $A^T(W)$  is also a rational subspace containing  $V$  and has the same dimension as  $W$ , we must have  $A^T(W) = W$ . By Corollary 3,

$$Y \subseteq \bigcup_{i=0}^{k-1} (y_i + W) \subseteq Z_f. \quad (4.4)$$

Moreover, since  $\pi_n(\bigcup_{i=0}^{k-1}(y_i + W))$  is the closure of  $\pi_n(\bigcup_{i=0}^{k-1}(y_i + V))$  in  $\mathbb{T}^n$ , we conclude that  $\bigcup_{i=0}^{k-1}(y_i + W)$  is  $\tau$ -invariant with respect to  $\mathcal{D}^T$ .

We now prove property ii. Let  $x \in y_{i-1} + W$ . We show that for any  $l \in \mathcal{D}^T$ ,  $u_{\mathcal{D}}(\tau_l(x)) > 0$  only if  $\tau_l(x) \in y_i + W + \mathbb{Z}^n$ . Suppose this is false, then there exists an  $l^* \in \mathcal{D}^T$  with  $u_{\mathcal{D}}(\tau_{l^*}(x)) > 0$  such that  $\tau_{l^*}(x) \notin y_i + W + \mathbb{Z}^n$ . The  $\tau$ -invariance of  $\bigcup_{i=0}^{k-1}(y_i + W)$  with respect to  $\mathcal{D}^T$  then implies that  $\tau_{l^*}(x) \in y_j + W$  for some  $j$ , where  $y_i + W \neq y_j + W$ . Hence  $x \in A^T(y_j + W + \mathbb{Z}^n) \subseteq y_{j-1} + W + \mathbb{Z}^n$ . However, this could happen only if

$$y_{i-1} + W + \mathbb{Z}^n = y_{j-1} + W + \mathbb{Z}^n.$$

By applying the operator  $(A^T)^{k-1}$  to the equation above, we obtain

$$y_i + W + \mathbb{Z}^n = y_j + W + \mathbb{Z}^n,$$

a contradiction. Property (2) now follows immediately from Lemma 2.2.  $\square$

## 5. Stretched Tiles and Quasi-Product Form Digit Sets

Our object in this section is to present a large class of pairs  $(A, \mathcal{D})$  giving stretched tiles  $T(A, \mathcal{D})$ . In §6 we shall then prove a structure theorem asserting that all pairs  $(A, \mathcal{D})$  with  $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n$  and  $\mu(T(A, \mathcal{D})) > 1$  essentially arise from this class.

Now suppose that  $A$  is an expanding integer matrix having the block-triangular form

$$A = \begin{bmatrix} A_1 & 0 \\ C & A_2 \end{bmatrix}, \quad (5.1)$$

where  $A_1$  and  $A_2$  are  $r \times r$  and  $(n-r) \times (n-r)$ , respectively, with  $1 \leq r \leq n-1$ . We say that a digit set  $\mathcal{D}$  for  $A$  is of *quasi-product form* if it has the form

$$\mathcal{D} := \left\{ \begin{bmatrix} a_i \\ b_i \end{bmatrix} + \begin{bmatrix} 0 \\ Qc_{i,j} \end{bmatrix} : 1 \leq i \leq |\det(A_1)|, 1 \leq j \leq |\det(A_2)| \right\}, \quad (5.2)$$

with the following properties:

1.  $\{a_i\} \subset \mathbb{Z}^r$  is a complete residue system (mod  $A_1$ ), and  $\{b_i\} \subset \mathbb{Z}^{n-r}$ .
2.  $c_{i,j} \in \mathbb{Z}^{n-r}$  for all  $i, j$  and for each  $i$  the set  $\{Qc_{i,j} : 1 \leq j \leq |\det(A_2)|\}$  is a complete residue system (mod  $A_2$ ).
3.  $Q \in M_{n-r}(\mathbb{Z})$  has  $|\det(Q)| \geq 2$  and  $A_2 Q = Q \tilde{A}_2$  for some  $\tilde{A}_2 \in M_{n-r}(\mathbb{Z})$ .

Conditions 1 and 2 imply that  $\mathcal{D}$  is necessarily a standard digit set.

### Theorem 5.1.

Let  $A$  be an expanding integer matrix of block-triangular form

$$A = \begin{bmatrix} A_1 & 0 \\ C & A_2 \end{bmatrix}, \quad (5.3)$$

where  $A_1$  and  $A_2$  are  $r \times r$  and  $(n-r) \times (n-r)$ , respectively. Suppose that  $\mathcal{D}$  is a primitive standard digit set for  $A$  which is of quasi-product form (5.2). Then  $|\det(Q)|$  divides  $\mu(T(A, \mathcal{D}))$ , so that  $\mu(T(A, \mathcal{D})) > 1$ .

**Proof.** Suppose that  $\mathcal{D}$  is of quasi-product form. Consider the block-diagonal matrix

$$\tilde{A} := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

together with the new digit set

$$\tilde{\mathcal{D}} := \left\{ \begin{bmatrix} a_i \\ Qc_{i,j} \end{bmatrix} : 1 \leq i \leq |\det(A_1)|, 1 \leq j \leq |\det(A_2)| \right\}.$$

We pair the digits of  $\mathcal{D}$  and  $\tilde{\mathcal{D}}$  by

$$d = \begin{bmatrix} a_i \\ b_i + Qc_{i,j} \end{bmatrix} \in \mathcal{D} \iff \tilde{d} = \begin{bmatrix} a_i \\ Qc_{i,j} \end{bmatrix} \in \tilde{\mathcal{D}}.$$

There is a simple relationship between  $T(A, \mathcal{D})$  and  $T(\tilde{A}, \tilde{\mathcal{D}})$ , which implies that

$$\mu(T(A, \mathcal{D})) = \mu(T(\tilde{A}, \tilde{\mathcal{D}})). \quad (5.4)$$

Define a map  $\phi : T(A, \mathcal{D}) \rightarrow T(\tilde{A}, \tilde{\mathcal{D}})$  by  $\phi(x) = \tilde{x}$ , where if

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sum_{k=1}^{\infty} A^{-k} d_k, \quad d_k \in \mathcal{D},$$

then

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \sum_{k=1}^{\infty} \tilde{A}^{-k} \tilde{d}_k, \quad \tilde{d}_k \in \tilde{\mathcal{D}},$$

where  $d_k$  and  $\tilde{d}_k$  are paired digits (see (1.2)). Now

$$A^{-k} = \begin{bmatrix} A_1^{-k} & 0 \\ C_k & A_2^{-k} \end{bmatrix}, \quad \tilde{A}^{-k} = \begin{bmatrix} A_1^{-k} & 0 \\ 0 & A_2^{-k} \end{bmatrix},$$

and hence

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - \psi(x_1) \end{bmatrix} \quad (5.5)$$

where

$$\psi(x_1) = \sum_{k=1}^{\infty} (C_k a_{i_k} + A_1^{-k} b_{i_k}).$$

The function  $\psi : T(A_1, \{a_i\}) \rightarrow \mathbb{R}^{n-r}$  is easily checked to be a measurable function, and hence using Fubini's theorem, (5.4) follows from (5.5).

Next we define the expanding matrix

$$\hat{A} = \begin{bmatrix} A_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix},$$

where  $\tilde{A}_2 \in M_{n-r}(\mathbb{Z})$  and  $A_2 Q = Q \tilde{A}_2$ , together with the digit set

$$\hat{\mathcal{D}} := \left\{ \begin{bmatrix} a_i \\ c_{i,j} \end{bmatrix} : 1 \leq i \leq |\det(A_1)|, 1 \leq j \leq |\det(A_2)| \right\}.$$

Set

$$\tilde{Q} := \begin{bmatrix} I_r & 0 \\ 0 & Q \end{bmatrix},$$

and then we have

$$\tilde{A} = \tilde{Q} \hat{A} \tilde{Q}^{-1}, \quad \tilde{\mathcal{D}} = \tilde{Q}(\hat{\mathcal{D}}).$$

Therefore, it follows from Lemma 2.1 that

$$T(\tilde{A}, \tilde{\mathcal{D}}) = \tilde{Q}T(\hat{A}, \hat{\mathcal{D}}), \quad (5.6)$$

and hence

$$\mu(T(\tilde{A}, \tilde{\mathcal{D}})) = |\det(Q)|\mu(T(\hat{A}, \hat{\mathcal{D}})),$$

since  $|\det(\tilde{Q})| = |\det(Q)|$ . Combining this with (5.4) completes the proof.  $\square$

**Remark.** The name *stretched tile* is suggested by (5.6), which shows that in a weak sense the tile  $T(A, \mathcal{D})$  is stretched by the matrix  $\tilde{Q}$  along the  $\mathbb{R}^{n-r}$ -coordinate directions. See Theorem 6.1 for the general case.  $\square$

As an example of Theorem 5.1, consider the pair  $(A, \mathcal{D})$  of (1.5). Let  $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ . Then the digit set  $\mathcal{D}^T = P(\mathcal{D})$  for the matrix  $A^T = P A P^{-1}$  is a standard digit set of quasi-product form with  $Q = [3]$ . Theorem 5.1 asserts that 3 divides  $\mu(T(A^T, \mathcal{D}^T))$ , and therefore 3 divides  $\mu(T(A, \mathcal{D}))$ . In fact,  $\mu(T(A, \mathcal{D})) = 3$ .

## 6. Structure Theorem for Stretched Tiles

We now use Theorem 4.1 to prove a structure theorem concerning stretched tiles, which is the converse of Theorem 5.1.

### Theorem 6.1.

Let  $\mathcal{D}$  be a primitive standard digit set for the expanding matrix  $A \in M_n(\mathbb{Z})$ , and suppose that  $\mu(T(A, \mathcal{D})) > 1$ . Then there exists a matrix  $P \in GL(n, \mathbb{Z})$  such that the following two conditions hold:

1. There is some  $r$  with  $1 \leq r \leq n - 1$  such that

$$P A P^{-1} = \begin{bmatrix} B_1 & 0 \\ C & B_2 \end{bmatrix}, \quad (6.1)$$

where  $B_1, B_2$  are  $r \times r$  and  $(n - r) \times (n - r)$  expanding integer matrices, respectively, and  $C$  is an  $(n - r) \times r$  integer matrix.

2. The digit set  $P(\mathcal{D})$  of  $P A P^{-1}$  is of quasi-product form.

Before proving this result, we derive a corollary. Write  $A_1 \sim_{\mathbb{Z}} A_2$  to mean  $A_1$  is integrally similar to  $A_2$ , that is, there exists some  $Q \in GL(n, \mathbb{Z})$  such that  $A_2 = Q A_1 Q^{-1}$ . We say that  $A$  is (integrally) reducible if

$$A \sim_{\mathbb{Z}} \begin{bmatrix} A_1 & 0 \\ C & A_2 \end{bmatrix}, \quad (6.2)$$

where  $A_1$  and  $A_2$  are nonempty. We call  $A$  *irreducible* if it is not integrally reducible.

### Corollary 6.2.

Suppose that the expanding matrix  $A \in M_n(\mathbb{Z})$  is irreducible. Then for all primitive standard digit sets  $\mathcal{D}$  the tile  $T(A, \mathcal{D})$  lattice tiles  $\mathbb{R}^n$  with lattice  $\mathbb{Z}^n$ .

**Proof.** If  $\mu(T(A, \mathcal{D})) > 1$ , then (6.1) shows that  $A$  is integrally reducible, which contradicts the irreducibility of  $A$ . Thus  $\mu(T(A, \mathcal{D})) = 1$ .  $\square$

A sufficient condition for the irreducibility of  $A$  is that the characteristic polynomial of  $A$  is irreducible over  $\mathbb{Q}$ . Using this criterion, any expanding matrix  $A$  with  $|\det(A)| = p$ , a prime, is irreducible, because if a decomposition (6.2) existed, then  $|\det(A_1)| > 1$  and  $|\det(A_2)| > 1$ .

**Lemma 6.3.**

Let  $\mathcal{D}$  be a primitive standard digit set for  $A$ . Suppose that  $P \in M_n(\mathbb{Z})$  is unimodular. Then  $P(\mathcal{D})$  is a primitive standard digit set for  $PAP^{-1}$ . Furthermore,

$$g_{P(\mathcal{D})}(x) = g_{\mathcal{D}}(P^T x), \quad u_{P(\mathcal{D})}(x) = u_{\mathcal{D}}(P^T x). \quad (6.3)$$

**Proof.** Since  $\mathcal{D}$  is a complete residue system (mod  $A$ ), for distinct  $d_1, d_2 \in \mathcal{D}$ , we have  $d_1 - d_2 \notin A(\mathbb{Z}^n)$ . Hence  $P(d_1 - d_2) \notin PAP^{-1}(\mathbb{Z}^n)$ . Therefore,  $P(\mathcal{D})$  is a complete residue system (mod  $PAP^{-1}$ ). It is primitive because

$$\mathbb{Z}[PAP^{-1}, P(\mathcal{D})] = P\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n.$$

Now for any  $x \in \mathbb{R}^n$ ,

$$g_{P(\mathcal{D})}(x) = \frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} \exp(2\pi i \langle Pd, x \rangle) = \frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} \exp(2\pi i \langle d, P^T x \rangle) = g_{\mathcal{D}}(P^T x).$$

Therefore,  $g_{P(\mathcal{D})}(x) = g_{\mathcal{D}}(P^T x)$ . Similarly,  $u_{P(\mathcal{D})}(x) = u_{\mathcal{D}}(P^T x)$ .  $\square$

**Proof of Theorem 6.1.** Since  $\mu(T(A, \mathcal{D})) > 1$  there exists a special eigenfunction  $f(x)$  for  $(A, \mathcal{D})$  by Lemma 3.1. Now Theorem 4.1 states that there exists a rational subspace  $W$  of  $\mathbb{R}^n$  having  $\dim(W) = r$  with  $1 \leq r \leq n-1$ , and with  $A^T(W) = W$  such that  $Z_f$  contains at least one translate of  $W$ . It is well known that we can choose a unimodular matrix  $P_1 \in GL(n, \mathbb{Z})$  that maps a given rational subspace  $W$  onto the first  $k$ -coordinate axes, that is,

$$P_1(W) = E_r := \left\{ \begin{bmatrix} x_r \\ 0 \end{bmatrix} : x_r \in \mathbb{R}^r \right\}. \quad (6.4)$$

This directly follows from the Hermite normal form decomposition for a rational basis of the vector space  $W$  (Schrijver [25, Theorem 4.1 and Corollary 4.3b]). Now  $W$  is an invariant subspace of  $A^T$ , and hence

$$P_1 A^T P_1^{-1} = \begin{bmatrix} B_1^T & C^T \\ 0 & B_2^T \end{bmatrix} \quad (6.5)$$

for integer matrices  $B_1^T, B_2^T$  and  $C^T$ . Therefore, taking the transpose yields

$$PAP^{-1} = \begin{bmatrix} B_1 & 0 \\ C & B_2 \end{bmatrix},$$

with  $P = (P_1^{-1})^T$ . Both  $B_1$  and  $B_2$  are expanding because  $A$  is expanding. This proves condition 1. To prove condition 2, let

$$B := PAP^{-1} = \begin{bmatrix} B_1 & 0 \\ C & B_2 \end{bmatrix}$$

and  $\mathcal{E} := P(\mathcal{D})$ . Then  $\mathcal{E}$  is a primitive standard digit set for  $B$ . We have  $u_{\mathcal{E}}(x) = u_{\mathcal{D}}(P^T x)$ .

Our object is to analyze the structure of the digit set  $\mathcal{E}$ , to eventually prove that it is of quasi-product form. For any vector  $v \in \mathbb{R}^n$  the notation  $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  always means that  $v_1 \in \mathbb{R}^r$  and  $v_2 \in \mathbb{R}^{n-r}$ .

Let  $\mathcal{E}_1^T, \mathcal{E}_2^T$  be complete residue systems (mod  $B_1^T$ ) and (mod  $B_2^T$ ), respectively. Let

$$\mathcal{E}^T := \mathcal{E}_1^T \oplus \mathcal{E}_2^T = \left\{ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} : b_1 \in \mathcal{E}_1^T, b_2 \in \mathcal{E}_2^T \right\}. \quad (6.6)$$

Then the fact that  $B^T$  is block upper-triangular implies that  $\mathcal{E}^T$  a complete residue system (mod  $B^T$ ). Let  $\mathcal{D}^T = (P^T)^{-1} \mathcal{E}^T$ . Then  $\mathcal{D}^T$  is a complete residue system (mod  $A^T$ ), because  $A^T = P^T B^T (P^T)^{-1}$ .

Now, let  $\{y_j + W : 0 \leq j \leq k-1\}$  satisfy the properties of Theorem 4.1. We have  $A_\star^T \circ \pi_n(y_{j+1}) = \pi_n(y_j)$ , where  $y_k := y_0$ , and

$$\sum_{\substack{l \in \mathcal{D}^T \\ (A^T)^{-1}(x+l) \in y_{j+1} + W + \mathbb{Z}^n}} u_{\mathcal{D}}((A^T)^{-1}(x+l)) \equiv 1, \quad x \in y_j + W. \quad (6.7)$$

Applying the transformations  $v_j = (P^T)^{-1}y_j$  and  $l = (P^T)^{-1}b$  with  $b \in \mathcal{E}^T$  in (6.7), we may rewrite (6.7) as

$$\sum_{\substack{b \in \mathcal{E}^T \\ (B^T)^{-1}(x+b) \in v_{j+1} + E_r + \mathbb{Z}^n}} u_{\mathcal{E}}((B^T)^{-1}(x+b)) \equiv 1, \quad x \in v_j + E_r, \quad (6.8)$$

using (6.4) and Lemma 6.3.

We proceed to simplify the formula (6.8). Choose  $x \in v_j + E_r$  and define

$$\Lambda_j := \{m \in \mathbb{Z}^n : (B^T)^{-1}(x+m) \in v_{j+1} + E_r + \mathbb{Z}^n\}. \quad (6.9)$$

We show that  $\Lambda_j$  is well-defined independent of the choice of  $x \in v_j + E_r$ . More precisely, denote

$$v_j = \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix}, \quad 0 \leq j \leq k,$$

where  $\alpha_k := \alpha_0$  and  $\beta_k := \beta_0$ . Then

$$\Lambda_j = \begin{bmatrix} 0 \\ n_j^* \end{bmatrix} + \Lambda \quad (6.10)$$

in which  $\Lambda$  is the lattice  $\mathbb{Z}^r \oplus B_2^T(\mathbb{Z}^{n-r})$  and

$$n_j^* := B_2^T \beta_{j+1} - \beta_j. \quad (6.11)$$

To prove these facts, let  $x = \begin{bmatrix} x_1 \\ \beta_j \end{bmatrix} \in v_j + E_r$ . Then  $m \in \Lambda_j$  if and only if

$$\begin{bmatrix} B_1^T & C^T \\ 0 & B_2^T \end{bmatrix}^{-1} \left( \begin{bmatrix} x_1 \\ \beta_j \end{bmatrix} + \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \right) = \begin{bmatrix} \alpha_{j+1} \\ \beta_{j+1} \end{bmatrix} + \begin{bmatrix} z \\ k_2 \end{bmatrix}$$

for some  $z \in \mathbb{R}^r$  and  $k_2 \in \mathbb{Z}^{n-r}$ . Therefore, the condition for  $m \in \Lambda_j$  is

$$(B_2^T)^{-1}(\beta_j + m_2) \equiv \beta_{j+1} \pmod{\mathbb{Z}^{n-r}},$$

which is equivalent to

$$\beta_j + m_2 \equiv B_2^T \beta_{j+1} \pmod{B_2^T(\mathbb{Z}^{n-r})},$$

that is,  $m_2 \equiv n_j^* \pmod{B_2^T(\mathbb{Z}^{n-r})}$ , which gives (6.10) and (6.11).

Using these formulas, the identity (6.8) becomes

$$\sum_{b_1 \in \mathcal{E}_1^T} \sum_{\substack{b_2 \in \mathcal{E}_2^T \\ b_2 - n_j^* \in B_2^T(\mathbb{Z}^{n-r})}} u_{\mathcal{E}} \left( (B^T)^{-1} \left( x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \right) \equiv 1 \quad (6.12)$$

for  $x \in v_j + E_r$ . Note that for each  $0 \leq j \leq k-1$  there is exactly one  $b_2 \in \mathcal{E}_2^T$  such that  $b_2 \equiv n_j^* \pmod{B_2^T(\mathbb{Z}^{n-r})}$ ; denote this  $b_2 \in \mathcal{E}_2^T$  by  $b_{2,j}^*$ . Then (6.12) is reduced further to

$$\sum_{b_1 \in \mathcal{E}_1^T} u_{\mathcal{E}} \left( w + (B^T)^{-1} \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \right) \equiv 1, \quad (6.13)$$

where

$$w := (B^T)^{-1} \left( x + \begin{bmatrix} 0 \\ b_{2,j}^* \end{bmatrix} \right) \quad \text{and} \quad x \in v_j + E_r.$$

We now use (6.12) to establish a series of claims.

**Claim 1.**

Suppose that  $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$  and  $d' = \begin{bmatrix} d'_1 \\ d'_2 \end{bmatrix}$  are two digits in  $\mathcal{E}$  such that  $d_1 - d'_1 \in B_1(\mathbb{Z}')$ . Then  $d_1 = d'_1$  and

$$\langle d_2 - d'_2, \beta_j \rangle \equiv 0 \pmod{1}, \quad 0 \leq j \leq k-1. \quad (6.14)$$

**Proof of Claim 1.** We make use of the *orthogonality relations*<sup>4</sup> on the abelian group  $\mathbb{Z}'/B_1(\mathbb{Z}')$ : For all  $m = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \in \mathbb{Z}^n$ , we have

$$\sum_{b_1 \in \mathcal{E}_1^T} \exp \left( 2\pi i \left\langle \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \begin{bmatrix} (B_1^T)^{-1} b_1 \\ 0 \end{bmatrix} \right\rangle \right) = \begin{cases} |\det(B_1)| & \text{if } m_1 \in B_1(\mathbb{Z}'), \\ 0 & \text{if } m_1 \notin B_1(\mathbb{Z}'). \end{cases}$$

Define

$$\mathcal{F} := \{(d, d') \in \mathcal{E} \times \mathcal{E} : d - d' \in B_1(\mathbb{Z}') \oplus \mathbb{Z}^{n-r}\}.$$

Using the orthogonality relation above and the definition

$$u_{\mathcal{E}}(x) = \frac{1}{|\det(B)|^2} \sum_{d, d' \in \mathcal{E}} \exp(2\pi i \langle d - d', x \rangle),$$

we obtain

$$\begin{aligned} & \sum_{b_1 \in \mathcal{E}_1^T} \sum_{d, d' \in \mathcal{E}} u_{\mathcal{E}} \left( w + \begin{bmatrix} (B_1^T)^{-1} b_1 \\ 0 \end{bmatrix} \right) \\ &= \frac{1}{|\det(B)|^2} \sum_{b_1 \in \mathcal{E}_1^T} \sum_{d, d' \in \mathcal{E}} \exp \left( 2\pi i \left\langle d - d', w + \begin{bmatrix} (B_1^T)^{-1} b_1 \\ 0 \end{bmatrix} \right\rangle \right) \\ &= \frac{1}{|\det(B)|^2} \sum_{d, d' \in \mathcal{E}} \sum_{b_1 \in \mathcal{E}_1^T} \exp \left( 2\pi i \left\langle d - d', w + \begin{bmatrix} (B_1^T)^{-1} b_1 \\ 0 \end{bmatrix} \right\rangle \right) \\ &= \frac{1}{|\det(B_1)| |\det(B_2)|^2} \sum_{(d, d') \in \mathcal{F}} \exp(2\pi i \langle d - d', w \rangle). \end{aligned}$$

Now, the equation above combines with (6.12) to give

$$\sum_{(d, d') \in \mathcal{F}} \exp(2\pi i \langle d - d', w \rangle) \equiv |\det(B_1)| |\det(B_2)|^2. \quad (6.15)$$

Next we show that

$$|\mathcal{F}| = |\det(B_1)| |\det(B_2)|^2, \quad (6.16)$$

which forces all the exponentials on the left side of (6.15) to be 1. To prove (6.16), note that  $\mathcal{E}$  is a complete residue system (mod  $B$ ); hence the set  $\mathcal{F}$  viewed as a subset of  $\mathbb{Z}^n/B(\mathbb{Z}^n) \times \mathbb{Z}^n/B(\mathbb{Z}^n)$  is a subgroup. Its quotient group is isomorphic to  $\mathbb{Z}'/B_1(\mathbb{Z}')$ , which has size  $|\det(B_1)|$ . Therefore (6.16) follows.

<sup>4</sup>The functions  $\chi_{b_1}(m_1) := \exp(2\pi i \langle m_1, (B_1^T)^{-1} b_1 \rangle)$  for  $b_1 \in \mathcal{E}_1^T$  form a complete set of characters on  $\mathbb{Z}'/B_1(\mathbb{Z}')$ .

Now we know that  $\exp(2\pi i \langle d - d', w \rangle) = 1$  for all pairs  $(d, d') \in \mathcal{F}$ , and therefore

$$\langle d - d', w \rangle \equiv 0 \pmod{1}. \quad (6.17)$$

For  $x \in v_j + E_r$ , write  $x = \begin{bmatrix} x_1 \\ \beta_j \end{bmatrix}$ . Then

$$w = (B^T)^{-1} \left( \begin{bmatrix} x_1 \\ \beta_j \end{bmatrix} + \begin{bmatrix} 0 \\ b_{2,j}^* \end{bmatrix} \right) = \begin{bmatrix} z_1 \\ (B_2^T)^{-1}(\beta_j + b_{2,j}^*) \end{bmatrix}$$

for some  $z_1 \in \mathbb{R}^r$ . Note that  $b_{2,j}^* - n_j^* \in B_2^T(\mathbb{Z}^{n-r})$ . By (6.11),

$$(B_2^T)^{-1}(\beta_j + b_{2,j}^*) = (B_2^T)^{-1}(\beta_j + n_j^* + B_2^T m_2) = \beta_{j+1} + m_2$$

for some  $m_2 \in \mathbb{Z}^{n-r}$ . Hence

$$w = \begin{bmatrix} z_1 \\ \beta_{j+1} + m_2 \end{bmatrix}$$

Equation (6.17) now becomes

$$\langle d_1 - d'_1, z_1 \rangle + \langle d_2 - d'_2, \beta_{j+1} \rangle \equiv 1 \pmod{1}.$$

However,  $\langle d_1 - d'_1, z_1 \rangle$  is a continuous function of  $z_1 \in \mathbb{R}^r$ , and as  $x_1$  runs through  $\mathbb{R}^r$  so does  $z_1$ . Hence we must have  $d_1 - d'_1 = 0$ , and

$$\langle d_2 - d'_2, \beta_{j+1} \rangle \equiv 1 \pmod{1},$$

proving Claim 1.  $\square$

**Claim 2.**

There exists a  $B_2$ -invariant proper sublattice  $\Gamma$  of  $\mathbb{Z}^{n-r}$  such that for all  $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$  and  $d' = \begin{bmatrix} d'_1 \\ d'_2 \end{bmatrix}$  in  $\mathcal{E}$ , if  $(d, d') \in \mathcal{F}$  then

$$d_2 - d'_2 \in \Gamma. \quad (6.18)$$

**Proof of Claim 2.** Define the lattice  $\Gamma$  in  $\mathbb{Z}^{n-r}$  by

$$\Gamma := \{m_2 \in \mathbb{Z}^{n-r} : \langle m_2, \beta_j \rangle \equiv 0 \pmod{1}, 0 \leq j \leq k-1\}.$$

Then  $\Gamma$  is a sublattice of  $\mathbb{Z}^{n-r}$ , and it is full rank because all  $\beta_j \in \mathbb{Q}^{n-r}$ . Claim 1 gives

$$\langle d_2 - d'_2, \beta_j \rangle \equiv 0 \pmod{1}, \quad 0 \leq j \leq k-1.$$

Hence  $d_2 - d'_2 \in \Gamma$ .

It remains to check that  $\Gamma$  is a proper sublattice of  $\mathbb{Z}^{n-r}$  and  $B_2(\Gamma) \subseteq \Gamma$ . First, all  $y_j + W$  are contained in  $Z_f$  for some special eigenfunction  $f(x)$  of  $(A, \mathcal{D})$ , and therefore  $(y_j + W) \cap \mathbb{Z}^n = \emptyset$ . Since  $v_j + E_r$  are the images of  $y_j + W$  under a unimodular linear map,  $(v_j + E_r) \cap \mathbb{Z}^n = \emptyset$ . However,  $v_j = \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix}$ . Hence  $\beta_j \notin \mathbb{Z}^{n-r}$ ; therefore,  $\Gamma$  must be a proper sublattice of  $\mathbb{Z}^{n-r}$ . Next we show that  $\Gamma$  is  $B_2$ -invariant. Formula (6.11) states that

$$B_2^T \beta_{j+1} \equiv \beta_j \pmod{\mathbb{Z}^{n-r}},$$

and hence for any  $m_2 \in \Gamma$ ,

$$\begin{aligned} \langle B_2 m_2, \beta_{j+1} \rangle &= \langle m_2, B_2^T \beta_{j+1} \rangle \\ &\equiv \langle m_2, \beta_j \rangle \pmod{1} \\ &\equiv 0 \pmod{1}, \end{aligned}$$

proving  $B_2(\Gamma) \subseteq \Gamma$ .  $\square$



**Claim 3.**

The digit set  $\mathcal{E} = P(\mathcal{D})$  is of quasi-product form.

**Proof of Claim 3.** For a given residue class  $m_1 + B_1(\mathbb{Z}^r)$  we choose a digit  $\begin{bmatrix} a_i \\ b_i \end{bmatrix} \in \mathcal{E}$  with  $a_i \equiv m_1 \pmod{B_1(\mathbb{Z}^r)}$ , which exists since  $\mathcal{E}$  is a complete residue system  $\pmod{B}$ . Consider all other digits  $\begin{bmatrix} a'_i \\ b'_i \end{bmatrix} \in \mathcal{E}$  having

$$a'_i \equiv a_i \pmod{B_1(\mathbb{Z}^r)}.$$

It follows from Claims 1 and 2 that  $a'_i = a_i$  and  $b_i - b'_i \in \Gamma$ . Taking a basis matrix  $Q \in M_{n-r}(\mathbb{Z})$  for  $\Gamma$ , we can write

$$b_i - b'_i = Qc_{i,j}$$

for some  $c_{i,j} \in \mathbb{Z}^{n-r}$ . Since  $\mathcal{E}$  is a complete residue system  $\pmod{B}$ , the set of such  $\begin{bmatrix} a'_i \\ b'_i \end{bmatrix} \in \mathcal{E}$  has cardinality  $|\det(B_2)|$ , and  $\{Qc_{i,j} : 1 \leq j \leq |\det(B_2)|\}$  forms a complete residue system  $\pmod{B_2}$ . Now because  $\Gamma$  is  $B_2$ -invariant, there exists a  $\tilde{B}_2 \in M_{n-r}(\mathbb{Z})$  such that

$$B_2 Q = Q \tilde{B}_2.$$

Finally,  $|\det(Q)| > 1$  because  $\Gamma$  is a proper sublattice of  $\mathbb{Z}^{n-r}$ .  $\square$

Finally, Theorem 6.1 follows from Lemma 6.3 and Claim 3.  $\square$

## 7. Lattice Tilings

We now use Theorem 6.1 to prove Theorem 1.1.

**Proof of Theorem 1.1.** We prove the theorem by induction on the Lebesgue measure  $\mu(T(A, \mathcal{D}))$  of the tile  $T(A, \mathcal{D})$ , where  $\mathcal{D}$  is a standard digit set for  $A \in M_n(\mathbb{Z})$ . This measure is an integer, by Theorem 1.1 of Part I [18]. The base case therefore is  $\mu(T(A, \mathcal{D})) = 1$ , in which case  $T(A, \mathcal{D})$  tiles by  $\mathbb{Z}^n$ .

For the induction step, suppose that it is true for all tiles of measure less than  $k$ , with  $k \geq 2$ , and that  $\mu(T(A, \mathcal{D})) = k$ . First we consider the case in which  $\mathbb{Z}[A, \mathcal{D}] \neq \mathbb{Z}^n$ . The proof of Lemma 2.1 shows that

$$T(A, \mathcal{D}) = Q(T(\tilde{A}, \tilde{\mathcal{D}})) + v, \quad (7.1)$$

where  $\tilde{A} \in M_n(\mathbb{Z})$  is similar to  $A$  over  $\mathbb{Q}$ , and  $Q \in M_n(\mathbb{Z})$  with  $|\det(Q)| \geq 2$ , and hence

$$\mu(T(\tilde{A}, \tilde{\mathcal{D}})) = \frac{\mu(T(A, \mathcal{D}))}{|\det(Q)|} < k.$$

The induction hypothesis applies to  $(\tilde{A}, \tilde{\mathcal{D}})$ , and therefore  $T(\tilde{A}, \tilde{\mathcal{D}})$  tiles with a lattice  $\tilde{\Gamma} \subseteq \mathbb{Z}_n$ , and (7.1) then shows that  $T(A, \mathcal{D})$  tiles  $\mathbb{R}^n$  using the lattice  $\Gamma = Q(\tilde{\Gamma}) \subseteq \mathbb{Z}^n$ .

Next suppose that  $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n$ . Since  $\mu(T(A, \mathcal{D})) = k \geq 2$ , the tile  $T(A, \mathcal{D})$  is a stretched tile. Theorem 6.1 shows that there exists a  $P \in GL(n, \mathbb{Z})$  with

$$B := PAP^{-1} = \begin{bmatrix} B_1 & 0 \\ C & B_2 \end{bmatrix}, \quad (7.2)$$

where  $B_1 \in M_r(\mathbb{Z})$  and  $B_2 \in M_{n-r}(\mathbb{Z})$  with  $1 \leq r \leq n-1$ , and  $\mathcal{E} := P(\mathcal{D})$  has the quasi-product form

$$\mathcal{E} = \left\{ \begin{bmatrix} a_i \\ b_i + Qc_{i,j} \end{bmatrix} : 1 \leq i \leq |\det(B_1)|, 1 \leq j \leq |\det(B_2)| \right\}, \quad (7.3)$$

with  $|\det(Q)| \geq 2$ . Since

$$T(A, \mathcal{D}) = P^{-1}T(PAP^{-1}, P(\mathcal{D})) = P^{-1}T(B, \mathcal{E}),$$

we only need to show that  $T(B, \mathcal{E})$  lattice tiles  $\mathbb{R}^n$ .

Consider the new pair  $(B_1, \mathcal{E}_1)$ , where

$$\mathcal{E}_1 = \{a_i : 1 \leq i \leq |\det(B_1)|\}.$$

$\mathcal{E}_1$  is a complete residue system (mod  $B_1$ ). Furthermore,  $\mathbb{Z}[B, \mathcal{E}] = \mathbb{Z}^n$  implies that  $\mathbb{Z}[B_1, \mathcal{E}_1] = \mathbb{Z}^r$ . Hence  $\mathcal{E}_1$  is a primitive standard digit set for  $B_1$ .

We claim that we can always find a factorization (7.2), (7.3) with the additional property of

$$\mu(T(B_1, \mathcal{E}_1)) = 1. \quad (7.4)$$

To see this, assume that  $r$  is the smallest positive integer with which the factorization (7.2), (7.3) exists. If  $r = 1$ , then by Theorem 3.4 we already have  $\mu(T(B_1, \mathcal{E}_1)) = 1$ . Suppose that  $\mu(T(B_1, \mathcal{E}_1)) > 1$ . Then  $r > 1$ , and by Theorem 6.1 there exists a  $P_1 \in GL(r, \mathbb{Z})$  such that

$$P_1 B_1 P_1^{-1} = \begin{bmatrix} \tilde{B}_1 & 0 \\ C_1 & \tilde{B}_2 \end{bmatrix},$$

where  $B_1 \in M_{r_1}(\mathbb{Z})$ ,  $B_2 \in M_{r-r_1}(\mathbb{Z})$  with  $1 \leq r_1 < r$ , and  $P_1(\mathcal{E})$  has the quasi-product form. Now if we let

$$\hat{P} = \begin{bmatrix} P_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} P,$$

then

$$\hat{P} A \hat{P}^{-1} = \begin{bmatrix} \tilde{B}_1 & 0 \\ C' & B'_2 \end{bmatrix}$$

for some integer matrices  $B'_2$  and  $C'$ , with  $\hat{P}(\mathcal{D})$  having the quasi-product form. This is a contradiction because  $r_1 < r$ . Hence we have  $\mu(T(B_1, \mathcal{E}_1)) = 1$ , proving the claim.

Next we associate with the pair  $(B, \mathcal{E})$  a new pair  $(\hat{B}, \hat{\mathcal{E}})$  given by

$$\hat{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad (7.5)$$

$$\hat{\mathcal{E}} = \left\{ \begin{bmatrix} a_i \\ Q c_{ij} \end{bmatrix} : 1 \leq i \leq |\det(B_1)|, 1 \leq j \leq |\det(B_2)| \right\}. \quad (7.6)$$

The proof of Theorem 5.1 already shows that

$$\mu(T(\hat{B}, \hat{\mathcal{E}})) = \mu(T(B, \mathcal{E})) = k,$$

and it also shows the new pair  $(B^\dagger, \mathcal{E}^\dagger)$  given by

$$B^\dagger = \begin{bmatrix} B_1 & 0 \\ 0 & B_2^\dagger \end{bmatrix}, \quad (7.7)$$

$$\mathcal{E}^\dagger = \left\{ \begin{bmatrix} a_i \\ c_{i,j} \end{bmatrix} : 1 \leq i \leq |\det(B_1)|, 1 \leq j \leq |\det(B_2)| \right\}, \quad (7.8)$$

where  $B_2 Q = Q B_2^\dagger$  has

$$T(\hat{B}, \hat{\mathcal{E}}) = \begin{bmatrix} I_r & 0 \\ 0 & Q \end{bmatrix} T(B^\dagger, \mathcal{E}^\dagger). \quad (7.9)$$

Note that

$$\mu(T(B^\dagger, \mathcal{E}^\dagger)) = \frac{\mu(T(\widehat{B}, \widehat{\mathcal{E}}))}{|\det(Q)|} < k,$$

and it is easy to check that  $\mathcal{E}^\dagger$  is a complete residue system (mod  $B^\dagger$ ). Therefore, the induction hypothesis applies to show that  $T(B^\dagger, \mathcal{E}^\dagger)$  lattice tiles  $\mathbb{R}^n$ , and hence  $T(\widehat{B}, \widehat{\mathcal{E}})$  also lattice tiles  $\mathbb{R}^n$  as a result of (7.9).

Assume that  $T(\widehat{B}, \widehat{\mathcal{E}})$  tiles  $\mathbb{R}^n$  with the lattice  $\Gamma$ . We now come to the main point of the proof: We show that  $T(\widehat{B}, \widehat{\mathcal{E}})$  also tiles  $\mathbb{R}^n$  with a (possibly different) lattice  $\Gamma^*$  that is a direct sum  $\mathbb{Z}^r \oplus \Gamma_1$ , where  $\Gamma_1 \subseteq \mathbb{Z}^{n-r}$ . We start by observing that the orthogonal projection of  $T(\widehat{B}, \widehat{\mathcal{E}})$  to its first  $r$ -coordinate plane is  $T(B_1, \mathcal{E}_1)$ . Therefore since  $\Gamma \subseteq \mathbb{Z}^n$ , every tile

$$T(\widehat{B}, \widehat{\mathcal{E}}) + \gamma, \quad \gamma \in \Gamma,$$

in the tiling by  $\Gamma$  orthogonally projects to

$$T(B_1, \mathcal{E}_1) + \gamma_1, \quad \gamma_1 \in \mathbb{Z}^r,$$

where  $\gamma := [\begin{smallmatrix} \gamma_1 \\ \gamma_2 \end{smallmatrix}]$ . These projections are measure-disjoint for different  $\gamma_1$ 's. Thus the tiling  $T(\widehat{B}, \widehat{\mathcal{E}}) + \Gamma$  of  $\mathbb{R}^n$  using  $\Gamma$  naturally divides up into cylinders:

$$U(\gamma_1) := (T(B_1, \mathcal{E}_1) + \gamma_1) \oplus \mathbb{R}^{n-r}. \quad (7.10)$$

Look at the tiling of the particular cylinder  $U(0)$ , which is given by  $T(\widehat{B}, \widehat{\mathcal{E}}) + \Gamma'$ , where

$$\Gamma' = \Gamma \cap (\{0\} \oplus \mathbb{Z}^{n-r}). \quad (7.11)$$

Clearly  $\Gamma'$  is a sublattice of  $\mathbb{Z}^n$ . Write  $\Gamma' = \{0\} \oplus \Gamma_1$ , where  $\Gamma_1 \subseteq \mathbb{Z}^{n-r}$ . Now  $\Gamma_1$  is a sublattice of  $\mathbb{Z}^{n-r}$ , and  $T(\widehat{B}, \widehat{\mathcal{E}})$  tiles  $U(\gamma_1)$  by  $\{\gamma_1\} \oplus \Gamma_1$ . Hence  $T(\widehat{B}, \widehat{\mathcal{E}})$  tiles  $\mathbb{R}^n$  by  $\Gamma^* := \mathbb{Z}^r \oplus \Gamma_1$ .

Next we claim that the tile  $T(B, \mathcal{E})$  also tiles  $\mathbb{R}^n$  using the lattice  $\Gamma^* = \mathbb{Z}^r \oplus \Gamma_1$ . To prove this claim we note that the orthogonal projection of  $T(B, \mathcal{E})$  onto its first  $r$ -coordinate plane is also  $T(B_1, \mathcal{E}_1)$  as a result of the triangular form of  $B$  and the quasi-product form of  $\mathcal{E}$ . Hence  $T(B, \mathcal{E})$  also tiles the cylinder  $U(\gamma_1)$  for each  $\gamma_1 \in \mathbb{Z}^r$ . Thus it suffices to prove that  $T(B, \mathcal{E}) + \Gamma'$  tiles the cylinder  $U(0)$ , where  $\Gamma'$  is defined in (7.11). Recall that the proof of Theorem 5.1 shows that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in T(B, \mathcal{E}) \iff \widehat{x} = \begin{bmatrix} x_1 \\ x_2 - \psi(x_1) \end{bmatrix} \in T(\widehat{B}, \widehat{\mathcal{E}}),$$

where  $\psi : T(B_1, \mathcal{E}_1) \rightarrow \mathbb{R}^{n-r}$  is a certain measurable function, (see (5.5)). This relation shows that translates of  $T(B, \mathcal{E})$  by  $\Gamma'$  inherit the measure-disjointness property from that of translates of  $T(\widehat{B}, \widehat{\mathcal{E}})$  by  $\Gamma'$ . It also yields the covering property for the cylinder  $U(0)$ , since the map  $[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}] \rightarrow [\begin{smallmatrix} x_1 \\ x_2 - \psi(x_1) \end{smallmatrix}]$  maps  $U(0)$  one-to-one onto itself. Thus  $T(B, \mathcal{E})$  tiles  $\mathbb{R}^n$  using the lattice  $\mathbb{Z}^r \oplus \Gamma_1$ . This proves the theorem.  $\square$

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