SUPPLEMENTAL NOTES ON \SET THEORY AND TOPOLOGY" MATH 445, FALL 2009

Steven Hurder
University of Illinois at Chicago

SUPPLEMENTAL NOTES ON “SET THEORY AND TOPOLOGY” MATH 445, FALL 2009

STEVEN HURDER

Contents

version date: November 20, 2009.

1. Introduction

These notes are not a course on Point-Set Topology. The text for the course, *Set Theory and Metric Spaces*, by Irving Kaplansky, covers the basics of Set Theory and Point-Set Topology.

That being said, the textbook devotes approximately half of its development to Set Theory, and then approaches Point-Set Topology from the "metric sequential topology" point of view. At some point, it is necessary to introduce the alternative, more formal approach to topology, in terms of open sets and axioms. This is relegated to an Appendix in Kaplansky, while it is the basic approach taken, for example, by the textbook *Topology*, (2^{nd} Edition) by James Munkres, among others.

These notes are thus a collection of handouts which discuss points related to the text by Kaplansky, or develop additional topics that this book omits. A good reference for the sections of these notes concerning point-set topology are Chapters 2, 3 and 4 of the textbook by Munkres.

In addition to these notes, various entries from wikipedia also add interesting highlights and background information, especially historical comments, on subjects related to these notes. The wikipedia links are listed in Appendix C, although it is probably much easier to just go to the course web page

`http://www.math.uic.edu/∼hurder/math445`

and follow the links there. Some additional handouts are also linked from the course web page.

One note: in the following, when some property is said to be a "topological property", this means that if we have two topological spaces X and Y and a homeomorphism $h\colon X \to Y$ between them, then X has the property, if and only if, Y also has the property.

2. Some Choice Axioms

We summarize the various forms of the Axiom of Choice.

DEFINITION 2.1. *Let A be a set. Then $\mathcal{P}_*(A)$ is the set of all* non-empty *subsets of A.*

Axiom of Choice [AC]: Let A be a set. Then there exists a "choice function"

$$f\colon \mathcal{P}_*(A) \to A$$

such that for every non-empty subset $B \in \mathcal{P}(A)$, we have $f(B) \in B$.

DEFINITION 2.2. *Let L be a partially ordered set. An* upper bound *for a chain $S \subset L$ is an element $u \in L$ such that $x \le u$ for all $x \in S$.*

Zorn's Lemma [ZL]: Let L be a partially ordered set such that every chain $S \subset L$ has an upper bound. Then L contains a maximal element. That is, there exists $u_* \in L$ such that if $y \in L$ satisfies $u_* \le x$, then $u_* = x$.

DEFINITION 2.3. *An ordered set (a chain) A is* well-ordered *if every subset $S \subset A$ has a least element (or bottom element.) For example the set of integers $\mathbb{N}$ with the natural order is well-ordered, while the set of all integers $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{Z}$ is not.*

Well-Ordering [WO]: Every set A has an ordering which is a well-ordering.

THEOREM 2.4. *Each of the above three "principles" are equivalent. That is*

$$AC \iff ZL \iff WO$$

Chapter 3.3 of the textbook gives the proofs that $AC \Longrightarrow ZL \Longrightarrow WO \Longrightarrow AC$.

$ZL \Longrightarrow WO$ is a standard application of "Zornification". See Theorem 22 on page 52.

$WO \implies AC$ is the simplest. Given a set A, assume it has been well-ordered. Given a subset $B \subset A$ define $f(B) = b$, where $b \in B$ is bottom element in the subset B. Obviously, $f(B) \in B$.

$AC \Longrightarrow ZL$ is the most difficult to prove. See pages 59–64 of the textbook.

3. Tricks of the "Point-Set" Trade

EXAMPLE 3.1. *The Hilbert Hotel*

A set X is infinite if and only if it has a countably infinite subset, $A \subset X$. This works whether X is countable infinite, or uncountably infinite. Of course, if X is uncountably infinite, then $X - A$ is again uncountable, so we can choose yet another countable subset $B \subset X - A$, which is a subset $B \subset X$ that is disjoint from A. For an uncountable set X, we can repeat this trick over and over, countably may times if needed.

Since the countable set A can be counted, we might as well assume that it has the form of a sequence, $A = \{a_1, a_2, a_3, \ldots\}$ with all a_i distinct. Given any other countable set B, there are many bijections $\Phi\colon A \cup B \cong A$. For example, if $B = \{b_1, b_2, \ldots, b_n\}$ is finite, then we can define Φ by setting $\Phi(a_i) = a_{i+n} \in A$, and $\Phi(b_j) = a_j$. This is a bijection.

If $B = \{b_1, b_2, b_3, \ldots\}$ then there are lots of ways to map two copies of the natural numbers into themselves. For example, we can send one copy to the even integers, the other copy to the odd integers. Then the map Φ looks like $\Phi(a_i) = a_{2i}$ and $\Phi(b_j) = a_{2j-1}$. This is a bijection.

Application: X infinite, $B \subset X$ finite, then there is a bijection $X \cong X - B$.

Application: X uncountably infinite, $B \subset X$ countable, then there is a bijection $X \cong X - B$.

EXAMPLE 3.2. *Uncountably many of anything*

To show some set X is countable, it suffices to show X is a subset of some countable set. But how to show that some collection of something is uncountable? Show there is some countably infinite set A, and find a one-to-one (or finite-to-one) map from the power set $\mathcal{P}(A)$ to X.

It helps sometimes to use the bijection $\mathcal{P}(A) \cong \{F\colon A \to \{0,1\}\}$ where to a subset $B \subset A$ we associate its characteristic function χ_B which is the map on A assigning value 1 to elements of A in B, and value 0 to elements of A not in B.

Application: The cardinality of the interval $[0,1]$ is the same as that of the power set $\mathcal{P}(\mathbb{N})$. To each function $F\colon \mathbb{N} \to \{0,1\}$ associate the sequence $\{a_n = F(n)\}$, then define $\Phi\colon \mathcal{P}(\mathbb{N}) \to [0,1]$ by $\Phi(\{a_n\}) = \sum_{n=1}^{\infty} a_n \cdot 2^{-n}$ which exists by the completeness of the real numbers. The map Φ is onto, but not one-to-one. Still, the preimage of any real $x \in [0,1]$ is at most two sequences.

EXAMPLE 3.3. *Open is open, closed is closed*

LEMMA 3.4. *Let $x \in M$ and $\epsilon > 0$. Then the open ball $B(x, \epsilon)$ is an open set.*

Proof. Let $y \in B(x, \epsilon)$ then $R = D(x, y) < \epsilon$, so $\delta = \epsilon - R > 0$. We claim $B(y, \delta) \subset B(x, \epsilon)$. Let $z \in B(y, \delta)$. Then $D(y, z) < \delta$, so $D(x, z) \le D(x, y) + D(y, z) < R + (\epsilon - R) = \epsilon$, so $z \in B(x, \epsilon)$. □

LEMMA 3.5. *Let $x \in M$ and $\epsilon > 0$. Then the closed ball $\{y \in M \mid D(x, y) \le \epsilon\}$ is a closed set.*

Proof 1. Let $y \in M - \{y \in M \mid D(x, y) \le \epsilon\}$ so $R = D(x, y) > \epsilon$. Set $\delta = R - \epsilon$. We claim $B(y, \delta) \subset M - \{y \in M \mid D(x, y) \le \epsilon\}$. Let $z \in B(y, \delta)$ then $D(y, z) < \delta$, so

$$R = D(x, y) \le D(x, z) + D(z, y) < D(x, z) + \delta \implies D(x, z) > R - \delta = \epsilon$$

Thus, $B(y, \delta) \subset M - \{y \in M \mid D(x, y) \le \epsilon\}$ □

Proof 2. Let $\{a_n \mid n = 1, 2, \ldots\} \subset \{y \in M \mid D(x, y) \le \epsilon\}$ converge, $a_n \to u$. Then $D(x, a_n) \le \epsilon$ for all n implies $D(x, u) \le \epsilon$ by the triangle inequality. Hence $\{y \in M \mid D(x, y) \le \epsilon\}$ contains all of its limit points, so is closed. □

EXAMPLE 3.6. *Near the Limit*

Suppose that M is a metric space with metric $D(x,y)$, and $A \subset M$ is some given set. A point $u \in A$ is a *limit point of* A if for every open neighborhood $U \subset M$ with $u \in U$, the intersection $U \cap A$ contains at least one point besides u. This implies that $u \in \overline{A}$. In fact, every $u \in \overline{A} - A$ must be a limit point of A.

LEMMA 3.7 (Limit Lemma)**.** *Let u be a limit point of A. Then there exists a sequence of distinct points, $\{a_1, a_2, a_3, \ldots\} \subset A$ such that $a_n \to u$, and the distances $\epsilon_n = D(u, a_n)$ are monotonically decreasing to zero. That is,*

$$\epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots > \epsilon_n \to 0$$

We can require any estimate on the ϵ_n – for example, $\epsilon_n \leq 2^{-n}$ is sometimes nice to assume.

Proof. This is an inductive construction. Start with $n = 1$ and set $\epsilon_1 = 1/2$. Then the open ball $B(u, \epsilon_1) \cap A$ contains some point besides u. Pick any such point $a_1 \in B(u, \epsilon_1) \cap (A - \{u\})$.

Now assume that $\{a_1, a_2, \ldots, a_n\} \subset A - \{u\}$ have been chosen with $D(u, a_i) < \epsilon_i$ for $1 \leq i \leq n$, and $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n$. Then choose $\epsilon_{n+1} < \min\{\epsilon_n, 2^{-(n+1)}\}$, with $\epsilon_{n+1} > 0$ of course. Then the open ball $B(u, \epsilon_{n+1}) \cap A$ contains some point besides u. Pick any point $a_{n+1} \in B(u, \epsilon_{n+1}) \cap (A - \{u\})$. ☐

Application: Suppose that $u \in M$ is a limit point. Then M has an uncountable number of distinct open subsets. In fact, every open neighborhood U of u contains such an uncountable collection of distinct open subsets!)

EXAMPLE 3.8. *Open shadows of closed sets*

Let $F \subset M$ be a closed subset of a metric space. Given any $\epsilon > 0$ define the ϵ-penumbra of F by

$$Pen(F, \epsilon) = \{x \in M \mid \exists\; y \in F \;,\; D(x,y) < \epsilon\}$$

This is the set of all points in M that are within distance ϵ of some point of F.

LEMMA 3.9. *$Pen(F, \epsilon)$ is an open set; in fact $Pen(F, \epsilon) = \bigcup_{y \in F} B(y, \epsilon)$.*

Proof. Writing $Pen(F, \epsilon)$ as a union of open balls is just a way of rewriting the definition. For the definition states that $x \in Pen(F, \epsilon)$ if and only if there is some $y \in F$ so that $D(x,y) < \epsilon$, and so $x \in B(y, \epsilon)$. Now, this implies that $Pen(F, \epsilon)$ is an open set. ☐

Application: For any closed set F and $\epsilon > 0$, the complement $M - Pen(F, \epsilon)$ is a closed subset of M such that for all $x \in Pen(F, \epsilon)$ and $y \in M - Pen(F, \epsilon)$ we have $D(x,y) \geq \epsilon$.

Caution: For the above definition works for any subset X of M, but sometimes the result is not useful. For example, for $X = \mathbb{Q}$ the rational numbers in the real line $\mathbb{R}$, for every $\epsilon > 0$ the set $Pen(\mathbb{Q}, \epsilon) = \mathbb{R}$ is everything, so its complement is empty.

4. Another Definition of a Topology

DEFINITION 4.1. *A* topology $\mathcal{T}$ *on a set* X *to be a collection of subsets,* $\mathcal{T} \subset \mathcal{P}(X)$.

The subset $\mathcal{T}$ *must satisfy three Axioms:*

(1) $\emptyset,\ X \in \mathcal{T}$
(2) *If* $\{U_\alpha \mid \alpha \in \mathcal{A}\} \subset \mathcal{T}$ *for* any *index set* $\mathcal{A}$*, then* $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha \ \in \mathcal{T}$
(3) *If* $\{U_1, \ldots, U_n\} \subset \mathcal{T}$ *for any finite* n*, then* $V = \bigcap_{i=1}^{n} U_i \ \in \mathcal{T}$

Here is another definition of a topology, which is in terms of an operation on subsets, which should be thought of as taking the *closures* of subsets. It is the approach taken, for example, in the seminal book by Kuratowksi [3].

DEFINITION 4.2. *A* topology $\mathcal{T}$ *on a set* X *is an operation* $A \mapsto \overline{A}$ *on subsets* $A \in \mathcal{P}(X)$.

This operation must satisfy four Axioms: for all $A, B \in \mathcal{P}(X)$,

(1) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
(2) $A \subset \overline{A}$
(3) $\overline{\emptyset} = \emptyset$
(4) $\overline{\overline{A}} = \overline{A}$

PROPOSITION 4.3. *Let* X *be a space equipped with a topology, in the sense of Definition 4.2. Define a subset* $\mathcal{T} \subset \mathcal{P}(X)$ *by the rule*

$$U \in \mathcal{T} \iff A = X - U \quad \text{satisfies } \overline{A} = A$$

Show that $\mathcal{T}$ *is a topology on* X*, in the sense of Definition 4.1.*

Proof. Exercise. □

PROPOSITION 4.4. *Let* X *be a space equipped with a topology* $\mathcal{T}$*, in the sense of Definition 4.1. For* $A \subset X$*, let* A' *denote the set of limit points of* A*. Then define*

$$\overline{A} = A \cup A'$$

Show that this operation satisfies the four Axioms of Definition 4.2.

Proof. Exercise. □

It follows from these two definitions, that the two notions of a topology are equivalent.

5. Definitions of Compactness

Let X be a topological space, and $\mathcal{T}$ a topology on X. We introduce three notions of "compactness" for X and its subsets. The first definition requires the concept of an open cover for a subset.

DEFINITION 5.1 (Open Cover)**.** *A collection of open subsets, $\mathcal{U} = \{U_\beta \mid \beta \in \mathcal{B}\}$, is said to be an* open cover *for $A \subset X$, if for every $x \in A$ there is some index $\beta \in \mathcal{B}$ with $x \in U_\beta$.*

That is, $A \subset \bigcup_{\beta \in \mathcal{B}} U_\beta$. The cardinality of the index set $\mathcal{B}$ can be anything, from finite to uncountable.

DEFINITION 5.2 (Compact)**.** *We say that a subset $A \subset X$ is* compact *if for every open cover $\mathcal{U} = \{U_\beta \mid \beta \in \mathcal{B}\}$ of A, there is a finite subset $\mathcal{B}_0 = \{\beta_1, \ldots, \beta_n\} \subset \mathcal{B}$ so that $A \subset U_{\beta_1} \cup \cdots \cup U_{\beta_n}$.*

The collection $\mathcal{U}_0 = \{U_\beta \mid \beta \in \mathcal{B}_0\}$ is called a *finite subcover* of $\mathcal{U}$. Thus,

$$A \text{ is } \textit{Compact} \iff \text{every open cover of } A \text{ admits a } \textit{finite subcover}$$

Every finite set is compact. The condition that A is compact is a sort-of converse, that from the point of view of open neighborhoods, "A is finite". However, the number of sets required for the finite subcover is not well-defined: two different covers of A may require completely different number of open sets in a finite subcover.

In the above definition, if we take $X = A$, then we just say that X is a *compact topological space.*

The contrapositive of the open cover condition defining compact sets is the following condition:

DEFINITION 5.3 (FIP = Finite Intersection Property)**.** *Let X be a topological space. Then X satisfies the* FIP *if for every collection of closed subsets $\{F_\beta \mid \beta \in \mathcal{B}\}$ of X such that for all finite subsets $\{\beta_1, \ldots, \beta_n\} \subset \mathcal{B}$ we have $F_{\beta_1} \cap \cdots \cap F_{\beta_n} \neq \emptyset$, then $\bigcap_{\beta \in \mathcal{B}} F_\beta \neq \emptyset$.*

PROPOSITION 5.4. *X is compact if and only if it satisfies the FIP.*

Proof. It a formal consequence of de Morgan Laws relating intersections, unions & complements. $\square$

There is a special case of the *FIP* which is often used, called the:

DEFINITION 5.5 (DCC = Descending Chain Condition)**.** *Let X be a topological space. Then X satisfies the DCC if for every descending chain of closed subsets $F_1 \supset F_2 \supset F_3 \supset \cdots$ such that $F_n \neq \emptyset$ for all $n \geq 1$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

The *DCC* condition is just a special case of the *FIP*, so $FIP \implies DCC$. However, the converse need not hold for a general topological space. The FIP condition allows for arbitrary index sets $\mathcal{B}$, while the DCC only allows for countable chains.

The next two definitions relate compactness to the existence of limit points for infinite subsets. The difference between the two notions is somewhat subtle.

DEFINITION 5.6 (Limit Point Compact)**.** *X is* limit point compact *if every infinite subset $A \subset X$ has a limit point in X. That is, there exists $x_* \in X$ such that for every open set $U \subset X$ with $x_* \in U$ then $U \cap (A - \{x_*\}) \neq \emptyset$. This is also called the* Bolzano-Weierstrass Property*.*

DEFINITION 5.7 (Sequentially Compact). *A space X is* sequentially compact *if every infinite sequence $A = \{x_1, x_2, \ldots, x_n, \ldots\} \subset X$ has a convergent subsequence. That is, there exists increasing integers $1 \leq n_1 < n_2 < \ldots$ so that the sequence $a_i = x_{n_i} \to x_* \in X$.*

For limit point compact, the set A can be either countably or uncountably infinite, while A is always countably infinite for sequentially compact. Note that in the case of limit point compact, there is no assertion that one can find a convergent subsequence; typically, to produce such a subsequence, one uses a metric on the space X.

PROPOSITION 5.8 (C $\Longrightarrow$ LPC). *If X is compact, then X is limit point compact.*

Proof. Let $A \subset X$ be a subset without a limit point. Then every point of A is isolated, and A is a closed subset of X. Then $U_0 = X - A$ is an open set. For each $a \in A$ pick an open set $U_a \subset X$ for which $U_a \cap A = \{a\}$. Then $\mathcal{U} = \{U_0\} \cup \{U_a \mid a \in A\}$ is an open cover of X.

As X is compact, there exists a finite subset $\{a_1, \ldots, a_n\} \subset A$ so that $\{U_0, U_{a_1}, \ldots, U_{a_n}\}$ is a cover of X. This implies that $A \subset U_{a_1} \cup \cdots \cup U_{a_n}$ hence $A = \{a_1, \ldots, a_n\}$.

Thus, if A is infinite, it cannot consists of isolated points, so there must exist a limit point in X. ☐

COROLLARY 5.9. *$A \subset X$ be a subset without a limit point. If A is compact, then A is finite.*

Finally, we mention a condition that is useful when studying spaces which are complete, but not necessarily compact:

DEFINITION 5.10. *A topological space X is locally compact if for each $x \in X$, there exists an open set $x \in U \subset X$ such that the closure $\overline{U}$ is compact.*

For example, if $X \subset \mathbb{R}^n$ is closed, then X is locally compact. For given $x \in X$ let $U = B(x, 1) \cap X$ be the open unit ball in X about X. Then the closure $\overline{U} = \overline{B(x, 1) \cap X} = \overline{B(x, 1)} \cap X$ is a closed and bounded subset of $\mathbb{R}^n$, hence by the Heine-Borel Theorem 12.5 it is compact.

6. Lebesgue Numbers and Compactness

Let $\{X, D\}$ be a metric space, and $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ be an open cover of X.

DEFINITION 6.1. *A Lebesgue number for $\mathcal{U}$ is a constant $\delta > 0$ such that for all $x \in X$, there is some $\alpha_x \in \mathcal{A}$ so that $B(x, \delta) \subset U_{\alpha_x}$.*

In words, at any position $x \in X$, you can pick some element of the cover, the U_{α_x} , so that this open set contains a δ-neighborhood of the point. What this says is that the cover $\mathcal{U}$ is "uniform in how much room you have", which is $\delta > 0$.

This is an important application of compactness; we will give two proofs, the first one direct, and the second proof using sequential compactness. This second proof is used in the proof of Proposition 7.3 that sequential compactness implies compactness for a metric space.

THEOREM 6.2. *Let $\{X, D\}$ be a compact metric space. Then every open covering $\mathcal{U}$ of X admits a Lebesgue number.*

Proof. Let $\mathcal{U}$ be a given open cover of X. Then there exists a finite subcollection of open sets $\{U_1, \ldots, U_n\} \subset \mathcal{U}$ so that $X = U_1 \cup \cdots \cup U_n$. The Lebesgue number δ will be defined with respect to this finite subcollection.

Define closed sets $C_i = X - U_i$ for each $1 \leq i \leq n$. Use these to define a function, for $x \in X$,

$$f(x) = \frac{1}{n} \cdot \sum_{i=1}^{n} D(x, C_i)$$

By the homework, each function $x \mapsto D(x, C_i)$ is continuous, hence their average $f(x)$ is continuous.

For each $y \in X$ there is some open set with $y \in U_i$, so there is $\epsilon_y > 0$ with $B(y, \epsilon_y) \subset U_i$. This implies that $D(y, C_i) \geq \epsilon_y$ and hence $f(y) \geq D(y, C_i)/n > 0$.

The function $f \colon X \to \mathbb{R}$ is continuous and X is compact, so f has a minimum on X, which must be positive. Let $\delta > 0$ be a number such that $f(x) \geq \delta$ for every $x \in X$. This is our Lebesgue number.

Let $x_0 \in X$, then

$$f(x_0) = \frac{1}{n} \cdot \sum_{i=1}^{n} D(x_0, C_i) \ \geq \ \delta$$

The average of any sum is bounded between the maximum and the minimum of the terms. So, there must exists some term $D(x_0, C_{i_0}) \geq \delta$. But this means that $B(x_0, \delta) \subset C_{i_0}$ as was to be shown. □

Exercise: The above theorem does not have a converse: Every open covering of the metric space $\mathbb{N}$ with the metric induced from $\mathbb{R}$ admits a Lebesgue number, but $\mathbb{N}$ is not compact.

THEOREM 6.3. *Let $\{X, D\}$ be a sequentially compact metric space. Then every open covering $\mathcal{U}$ of X admits a Lebesgue number.*

Proof. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ be a given open cover of X. Assume that there does not exist a Lebesgue number for $\mathcal{U}$; we show this leads to a contradiction.

For each $x \in X$ there exists some α_x for which $x \in U_{\alpha_x}$. As U_{α_x} is an open set, there exists some $\epsilon_x > 0$ so that $B(x, \epsilon_x) \subset U_{\alpha_x}$. Thus we can define a function

$$\delta(x) = \sup\{\epsilon \mid \ \exists \ \alpha \in \mathcal{A}, \ B(x, \epsilon) \subset U_\alpha\} > 0$$

If $\delta(x) \geq \delta > 0$ for all $x \in X$, then clearly δ is a Lebesgue number for $\mathcal{U}$. By assumption, no Lebesgue number exists, so there must exists a sequence of points $\{x_n\} \subset X$ such that $\delta(x_n) < 1/n$.

Since X is sequentially compact, there exists a convergent subsequence $a_i = x_{n_i} \to x_* \in X$. Let $U_{\alpha_*} \in \mathcal{U}$ be an element of the open cover containing x_*. Then there exists $\epsilon_* > 0$ such that $B(x_*, \epsilon_*) \subset U_{\alpha_*}$.

Let $i \gg 0$ be large enough so that $1/n_i < \epsilon_*/2$ and $D(x_{n_i}, x_*) < \epsilon_*/2$. Then by the triangle inequality,

$$B(x_{n_i}, \epsilon_*/2) \subset B(x_*, \epsilon_*) \subset U_{\alpha_*}$$

and thus $\delta(x_{n_i}) \geq \epsilon_*/2$. This contradicts the choice of i so that $\delta(x_{n_i}) < 1/n_i < \epsilon_*/2$. □

Remark: These two proofs of the existence of a Lebesgue number are similar. Also, both involve a step which is not "effective":

- For X compact, one first passes to a finite subcover. How is this choice made?
- For X sequentially compact, one passes to a convergent subsequence. How is this choice made?

The point is that if you wanted to write a computer program to calculate the Lebesgue number of a covering of X, you would not know how to tell it to stop.

7. Compact Metric Spaces

We show next that if $\{X, D\}$ is metric space, then all three definitions of compactness are equivalent:

(1) X *Compact* $\Longrightarrow X$ *Limit Point Compact* $\Longrightarrow X$ *Sequentially Compact* $\Longrightarrow X$ *Compact*

PROPOSITION 7.1 (LPC $\Longrightarrow$ SC)**.** *Let X be a limit point compact metric space; then every countably infinite sequence $A = \{x_1, x_2, \ldots, x_n, \ldots\} \subset X$ has a convergent subsequence.*

Proof. Let $A = \{x_1, x_2, \ldots, x_n, \ldots\} \subset X$. If A is a finite set, then for some value $a \in A$, by the *Pigeon Hole Principle*, there is some subset $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ so that $x_{n_i} = a$ for all $i = 1, 2, \ldots$. Clearly, $x_{n_i} \to a$.

If A is an infinite set, then by *LPC* there exists a limit point $x_* \in X$ for the set A. We use this to extract a convergent subsequence from A. For each $i = 0, 1, 2, 3, \ldots$ choose

$$a_{i+1} \in (A - \{x_*, x_1, x_2, \ldots, x_{n_i}\}) \cap B(x_*, 1/n)$$

which exists as x_* is a limit point. Then $\{a_i\} \subset A$ and $a_i \to x_*$. $\square$

COROLLARY 7.2. *Let X be a compact metric space. Then X is complete.*

Proof. Let $\{x_n\} \subset X$ be a Cauchy sequence in X. By the above, the set A has a convergent subsequence, and since $\{x_n\}$ is Cauchy, the limit of the sequence and the subsequence must agree. Hence, $x_n \to x_* \in X$. $\square$

Finally, we prove that a sequentially compact metric space is compact. This proof is more involved than the above arguments, reflecting the deeper nature of this assertion: sequential compactness is about countable sets, while compactness concerns arbitrary open covers.

PROPOSITION 7.3 (SC $\Longrightarrow$ C)**.** *Let X be a sequentially compact metric space; then for every open cover $\mathcal{U} = \{U_\beta \mid \beta \in \mathcal{B}\}$ of X, there is a finite subset $\mathcal{B}_0 = \{\beta_1, \ldots, \beta_n\} \subset \mathcal{B}$ so that $X = U_{\beta_1} \cup \cdots \cup U_{\beta_n}$.*

Proof. There are two steps in the proof. First, we show that given any $\epsilon > 0$ there exists a finite covering of X by open ϵ-balls. We argue by contradiction: assume there exists some $\epsilon > 0$ such that X cannot be covered by finitely many open ϵ-balls. We use this to construct by induction an infinite sequence without a limit point. Let $x_1 \in X$ be any point. Then the ball $B(x_1, \epsilon)$ cannot be all of X, as otherwise we would have a finite covering. So we can choose $x_2 \in X - B(x_1, \epsilon)$. Now proceed inductively: given $\{x_1, \ldots, x_n\} \subset X$, choose

$$x_{n+1} \in X - (B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon))$$

This is possible, since otherwise the collection $\{B(x_1, \epsilon), \ldots, B(x_n, \epsilon)\}$ is a finite covering of X. Note that by choice, we have $D(x_{n+1}, x_i) \geq \epsilon$ for all $1 \leq i \leq n$. In this way we obtain an infinite sequence $\{x_1, x_2, \ldots\} \subset X$ which has no converging subsequence, as $D(x_i, x_j) \geq \epsilon > 0$ for all $i \neq j$. But this contradicts the assumption that X is sequentially compact.

Now, let $\mathcal{U} = \{U_\beta \mid \beta \in \mathcal{B}\}$ be an open covering of X. Since X is sequentially compact, by Theorem 6.3 the cover $\mathcal{U}$ has a Lebesgue number $\delta > 0$. Set $\epsilon = \delta/3$, then choose a finite covering of X by open ϵ-balls, say $\{B(x_1, \epsilon), \ldots, B(x_n, \epsilon)\}$.

For each $1 \leq i \leq n$ the set $B(x_i, \epsilon)$ has diameter less than $2\epsilon < \delta$, hence there is some $\beta_i \in \mathcal{B}$ so that $B(x_i, \epsilon) \subset U_{\beta_i}$. Then

$$X = B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon) \subset U_{\beta_1} \cup \cdots \cup U_{\beta_n} \subset X$$

which shows that $\{U_{\beta_1}, \ldots, U_{\beta_n}\}$ is a finite subcover of $\mathcal{U}$. $\square$

8. Separation Axioms

Compact sets in metric spaces have many very strong properties. It turns out that many of these properties hold for more general topological spaces which satisfy an additional hypothesis, called the *Hausdorff Axiom*, which is one of many "separation axioms". Before defining the Hausdorff Axiom, we consider an example.

DEFINITION 8.1 (Zariski Topology)**.** *Let X be any set. Let $\mathcal{T}_z$ be the topology where the only closed sets are finite unions of points. Thus, $U \subset X$ is open if and only if $U = X - F$ for some finite subset $F \subset X$. This is called the* Zariski topology*, after the example from algebraic geometry.*

In some books, this is called the *finite complement topology*, as in Munkres on page 77.

The Zariski topology is actually quite natural for the study of sets which arise in number theory. Here are two of the strange properties it possesses:

LEMMA 8.2. *Let X be an infinite set with the Zariski topology $\mathcal{T}_z$. Then for any two open sets $U, V \subset X$ the intersection $U \cap V \neq \emptyset$.*

Proof. By definition, there exists finite subsets $F_U, F_V \subset X$ so that $U = X - F_U$ and $V = X - F_V$. Then $U \cap V = X - (F_U \cup F_V)$. Since $F_U \cup F_V$ is again finite, it is not all of X, hence $U \cap V \neq \emptyset$. □

PROPOSITION 8.3. *Let X have the Zariski topology $\mathcal{T}_z$. Then every subset of X if compact.*

Proof. Let $A \subset X$ and $\mathcal{U} = \{U_\beta \mid \beta \in \mathcal{B}\}$ be an open cover of A. Pick any $\beta_0 \in \mathcal{B}$ and let $U_0 = U_{\beta_0}$ be an open set in the cover of A. Then there exists some finite subset $F_0 \subset X$ such that $U_0 = X - F_0$. If $A \cap F_0 = \emptyset$, then $A \subset U_0$ and we are done.

Otherwise, enumerate $F_0 = \{x_1, x_2, \ldots, x_n\}$. Since $\mathcal{U}$ is a cover for A, for each $x_i \in A$ we can choose an open set $U_{\beta_i} \in \mathcal{U}$ with $x_i \in U_{\beta_i}$. Then the collection of open sets $\mathcal{B}' = \{U_0\} \cup \{U_{\beta_i} \mid x_i \in A\}$ is a finite subcover for A. □

This result shows that the compact condition can be meaningless if the open sets for the topology are "too large". For the Zariski topology, the problem is that it often fails to be Hausdorff.

DEFINITION 8.4 (Hausdorff Axiom)**.** *A topological space X is* Hausdorff *(or T_2) if for every pair of distinct points, $x, y \in X$, there exists open sets $U, V \subset X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Notice that for a set X, the Zariski topology $\mathcal{T}_z$ on X is Hausdorff if and only if X is a finite set.

On the other hand, we have:

LEMMA 8.5. *Let X be a metric space with the metric topology $\mathcal{T}$. Then X is Hausdorff.*

Proof. Let $x \neq y \in X$ then $\epsilon = D(x, y) > 0$. Take $U = B(x, \epsilon/2)$ and $V = B(y, \epsilon/2)$. □

We mention one other of the separation axioms:

DEFINITION 8.6 (T_1)**.** *A topological space X is or T_1 if for every pair of distinct points, $x, y \in X$, there exists open sets $U, V \subset X$ with $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.*

For a T_1 topology, *points are closed*: $U = X - \{x\}$ is open, hence $\{x\}$ is closed.

LEMMA 8.7. *Let X be a set with the Zariski topology $\mathcal{T}_z$. Then X is T_1.*

Proof. Let $x \neq y \in X$ then set $U = X - \{y\}$ and $V = X - \{x\}$. □

9. Properties of compact spaces

We now gather some of the useful properties of compact sets, and their proofs. The first three results are about closed subsets of compact sets.

PROPOSITION 9.1. *A closed subset $A \subset X$ of a compact topological space X is compact.*

Proof. Since A is closed, the complement $U_0 = X - A$ is open.

Now let $\mathcal{U} = \{U_\beta \mid \beta \in \mathcal{B}\}$ be an open cover of A. Then $\mathcal{U}' = \mathcal{U} \cup \{U_0\}$ is an open cover of X. Since X is compact, there is a finite subcover $\mathcal{U}_0'$. We may as well assume that $U_0 \in \mathcal{U}_0'$ as adding this set we still have a finite subcover. Then $X = U_0 \cup U_{\beta_1} \cup \cdots \cup U_{\beta_n}$. Since $A \cap U_0 = \emptyset$ we have $A \subset U_{\beta_1} \cup \cdots \cup U_{\beta_n}$. Thus $\mathcal{U}_0 = \{U_{\beta_1}, \ldots, U_{\beta_n}\}$ is a finite subcover of $\mathcal{U}$ for A. $\square$

PROPOSITION 9.2. *Let X be* compact Hausdorff*. If $A \subset X$ is compact, then A is closed.*

Proof. We must show that $X - A$ is an open set. It will suffice to show that for each $x \in X - A$ there exists an open set $V \subset X - A$ with $x \in V$; then $X - A$ is a union of open sets, hence is open.

Let $x \in X - A$ then we define an open cover for A. Since X is Hausdorff, for each $y \in A$ there exists open sets $U_y, V_y \subset X$ so that $y \in U_y$ and $x \in V_y$ such that $U_y \cap V_y = \emptyset$. Then the collection $\mathcal{U} = \{U_y \mid y \in A\}$ is an open cover for A.

Since A is compact, there is a finite subcover $\mathcal{U}_0 = \{U_{\beta_1}, \ldots, U_{\beta_n}\}$ of A; so $A \subset U = U_{\beta_1} \cup \cdots \cup U_{\beta_n}$.

Set $V = V_{\beta_1} \cap \cdots \cap V_{\beta_n}$. Then V is open, as it is a finite intersection of open sets. We also have $V \cap (U_{\beta_1} \cup \cdots \cup U_{\beta_n}) = \emptyset$, as $V \cap V_{\beta_i} = \emptyset$ for each $1 \leq i \leq n$. Thus, $V \subset X - A$. $\square$

One of the most useful property of "compactness" is that it is preserved by a continuous map:

PROPOSITION 9.3. *Let X, Y be topological spaces, and $f : X \to Y$ a continuous map. If $A \subset X$ is a compact set, then $f(A) \subset Y$ is a compact set.*

Proof. Let $\mathcal{V} = \{V_\beta \mid \beta \in \mathcal{B}\}$ be an open cover for the image $f(A)$. Let $U_\beta = f^{-1}(V_\beta) \subset X$ for each $\beta \in \mathcal{B}$. Since f is continuous, each set U_β is open. Then the collection $\{U_\beta \mid \beta \in \mathcal{B}\}$ is an open cover for A. As A is compact, there exists a finite subcover $\{U_{\beta_1}, \ldots, U_{\beta_n}\}$ of A. Thus, the collection $\{V_{\beta_1}, \ldots, V_{\beta_n}\}$ is a finite subcover of $\mathcal{V}$ for $f(A)$. $\square$

We now give two definitions, which are related to continuity, but are independent.

DEFINITION 9.4. *$f : X \to Y$ is* closed *if for every closed set $A \subset X$, the image $f(A)$ is closed.*

DEFINITION 9.5. *$f : X \to Y$ is* open *if for every open set $U \subset X$, the image $f(U)$ is open.*

Here is an important application which uses both the Hausdorff and compactness properties. (This problem has been asked in the Topology section for perhaps one-third of all Masters Exams.)

PROPOSITION 9.6. *Let X, Y be topological spaces, and $f : X \to Y$ a continuous map. If X is compact and Y is Hausdorff, then for every closed subset $A \subset X$, the image $f(A)$ is closed.*

Proof. This is a homework problem:-) $\square$

Here is a celebrated application of Proposition 9.6.

THEOREM 9.7. *Let X, Y be topological spaces, and $f : X \to Y$ a continuous bijective map. If X is compact and Y is Hausdorff, then f is a homeomorphism. That is, f^{-1} is continuous.*

10. Uniform continuity

Let $\{X, D_X\}$ and $\{Y, D_Y\}$ be metric spaces. Recall that a function $f\colon X \to Y$ is continuous in the metric sense if the $\epsilon - \delta$ condition holds:

$$\forall\ \epsilon > 0,\ \forall\ x \in X,\ \exists\ \delta_{\epsilon,x} > 0 \ \text{ such that } \ D_X(x,y) < \delta_{\epsilon,x} \implies D_Y(f(x), f(y)) < \epsilon$$

The order of the logical statements allows for $\delta_{\epsilon,x}$ to change with differing choices of x. The classic example is for $f\colon (0,\infty) \to (0,\infty)$ defined by $f(x) = 1/x$. For this example, the constant $\delta_{\epsilon,x}$ is approximately $x^2 \cdot \epsilon$, which tends to 0 as $x \to 0$. (To check this, just use that $f'(x) = -1/x^2$.)

A function $f\colon X \to Y$ is *uniformly continuous* if it satisfies

$$\forall\ \epsilon > 0,\ \exists\ \delta_\epsilon > 0 \ \text{ such that } \ \forall\ x \in X,\ D_X(x,y) < \delta_\epsilon \implies D_Y(f(x), f(y)) < \epsilon \tag{2}$$

In this case, there is a choice of δ_ϵ which works for all $x \in X$.

Here is another "classic" result of point-set topology:

THEOREM 10.1. *Let X and Y be metric spaces, and $f\colon X \to Y$ a continuous function. If X is compact, then f is uniformly continuous.*

Proof. Let $\epsilon > 0$ be given. For each $x \in X$, choose $\delta_x > 0$ such that

$$D_X(x,y) < \delta_x \implies D_Y(f(x), f(y)) < \epsilon/2$$

Then observe that if $y, z \in B(x, \delta_x)$, then by the Triangle Inequality, we have

$$D_Y(f(y), f(z)) \le D_Y(f(y), f(x)) + D_Y(f(x), f(z)) < \epsilon/2 + \epsilon/2 = \epsilon \tag{3}$$

The collection of open balls $\mathcal{U} = \{B(x, \delta_x) \mid x \in X\}$ forms an open cover of X. As X is compact, the open cover $\mathcal{U}$ has a Lebesgue number $\delta > 0$. That is, for any $\xi \in X$ there exists some $x \in X$ such that $B(\xi, \delta) \subset B(x, \delta_x)$. Then for any $\zeta \in B(\xi, \delta)$ we have both $\xi, \zeta \in B(x, \delta_x)$ hence by the calculation (3) we have that $D_Y(f(y), f(z)) < \epsilon$. Thus, $\delta_\epsilon = \delta$ is the uniform constant required so that (2) is satisfied. □

11. Compact subsets of $\mathbb{R}^n$

Compact subsets of metric spaces, and especially Euclidean space, have many special properties, some of them intuitive, some not. We start with a basic fact:

PROPOSITION 11.1. *Let M be a metric space, and $X \subset M$ a compact subset. Then X is closed and bounded.*

Proof. The metric space topology is always Hausdorff, so X is closed by Proposition 9.2.

We show that X must be bounded. Pick $x_0 \in M$ and consider the increasing sequence of open balls $\mathcal{U} = \{B(x_0, \ell) \mid \ell = 1, 2, \ldots\}$. The collection $\mathcal{U}$ is an open cover of all of M, since for any point $y \in M$ choose $\ell > D(x_0, y)$ then $y \in B(x_0, \ell)$. It follows that $\mathcal{U}$ is also an open cover for X. As X is assumed to be compact, there must be a finite subcover, $\{B(x_0, \ell_1), \ldots, B(x_0, \ell_k)\}$. Let $\ell = \max\{\ell_1, \ldots, \ell_k\}$. Then $X \subset B(x_0, \ell)$. □

COROLLARY 11.2. *Let $X \subset \mathbb{R}^n$ be a compact subset. Then X is closed and bounded.* □

Here is an application of this, which is one of the basic theorems of Analysis I.

THEOREM 11.3. *Let X be a compact topological space and $f\colon X \to \mathbb{R}$ a continuous function. Then there exists points $a, b \in X$ such that $f(a) \le f(x) \le f(b)$ for all $x \in X$. That is, f achieves its maximum and minimum on X at points of X.*

Proof. The image $f(X) \subset \mathbb{R}$ is compact by Proposition 9.3. Then $f(A)$ is closed and bounded by Proposition 11.1. Let $\beta = lub(f(A))$. We claim there exists $b \in X$ such that $f(b) = \beta$.

For each integer $n > 0$, by the definition of the least upper bound, there exists $z_n \in f(X)$ such that $\beta - 1/n < z_n \leq \beta$. Choose $x_n \in X$ such that $f(x_n) = z_n$.

If the collection of points $A = \{x_n\}$ is finite, then $\beta - 1/n < f(x_n) \leq \beta$ implies that for some N we have $f(x_n) = \beta$ for all $n \geq N$. Then take $b = x_N$.

If the set A is infinite, then as X is compact, it is sequentially compact, hence the sequence $\{x_n\} \subset X$ has a convergent subsequence $\{x_{n_i}\}$ with $x_{n_i} \to b \in X$. Since f is continuous,

$$f(b) = \lim_{i \to \infty} f(x_{n_i}) = \lim_{i \to \infty} z_n = \beta$$

To show there also exists a minimum point $a \in X$, repeat the above argument for $-f(x)$. □

12. Heine-Borel Theorems

Next, we ask which subsets of Euclidean space are compact. They turn out to have a simple characterization:

THEOREM 12.1 (Heine-Borel)**.** *A subset $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Note that we are assuming $\mathbb{R}^n$ has the metric topology. Since the metric topology on $\mathbb{R}^n$ is Hausdorff, we have shown already that a compact subset must be closed and bounded.

The converse requires proof, that closed and bounded implies compact. The Heine-Borel Theorem is so important, that we give two proofs of it. The first is the one given in beginning Analysis courses, such as Math 313–414. Here is the basic argument, which follows from the existence of a least upper bound for bounded subsets of $\mathbb{R}$.

THEOREM 12.2. *Let $a \leq b \in \mathbb{R}$, then $[a, b] \subset \mathbb{R}$ is compact.*

Proof. Let $\mathcal{U} = \{U_\beta \mid \beta \in \mathcal{B}\}$ be an open cover of the set $[a, b]$. That is, each set $U_\beta \subset \mathbb{R}$ is open, and $[a, b] \subset \bigcup_{\beta \in \mathcal{B}} U_\beta$. Let $a \leq c \leq b$ be the least upper bound of the numbers $a \leq c' \leq b$ such that for $[a, c']$, the restriction of $\mathcal{U}$ admits a finite subcover.

Note that there exists some $U_{\beta_1} \in \mathcal{U}$ with $a \in U_{\beta_1}$. As U_{β_1} is open, there exists $\epsilon > 0$ so that $(a - \epsilon, a + \epsilon) \subset U_{\beta_1}$. Then for any $a < c < a + \epsilon$ the single open set U_{β_1} is an open cover of $[a, c]$. Thus, $c \geq a + \epsilon$.

Suppose that $c < b$. Choose $U_{\beta_0} \in \mathcal{B}$ with $c \in U_{\beta_0}$. Then there exists $\epsilon > 0$ so that $(c-\epsilon, c+\epsilon) \subset U_{\beta_0}$. Pick $c - \epsilon < c' < c$, and let $\{U_{\beta_1}, \ldots, U_{\beta_n}\}$ be an open cover for $[a, c']$. This exists by the definition of c. Then $\{U_{\beta_0}, U_{\beta_1}, \ldots, U_{\beta_n}\}$ is a finite open subcover for $[a, c'']$ where $c < c'' < c + \epsilon$. This contradicts the maximality of c, so we must have that $c = b$.

Finally, choose $U_{\beta_0} \in \mathcal{B}$ with $b \in U_{\beta_0}$. Then there exists $\epsilon > 0$ so that $(c - \epsilon, c + \epsilon) \subset U_{\beta_0}$. Pick $b - \epsilon < c' < b$, and let $\{U_{\beta_1}, \ldots, U_{\beta_n}\}$ be an open subcover for $[a, c']$. Then $\{U_{\beta_0}, U_{\beta_1}, \ldots, U_{\beta_n}\}$ is a finite open cover for $[a, b]$. □

The following is a standard fact, whose proof we omit:

THEOREM 12.3. *Let $X_1, X_2, \ldots, X_n$ be topological spaces, and let $X = X_1 \times X_2 \times \cdot \times X_n$ be the product space with the product topology. If each space X_i is compact, then X is compact.*

COROLLARY 12.4. *For all $\ell > 0$, the "n-cube"*

$$[-\ell, \ell]^n = [-\ell, \ell] \times \cdots \times [-\ell, \ell] \subset \mathbb{R}^n$$

is compact. $\square$

Now, here is the first proof of the Heine-Borel Theorem for $\mathbb{R}^n$:

THEOREM 12.5. *Let $X \subset \mathbb{R}^n$ be a closed and bounded subset. Then X is compact.*

Proof. As X is bounded, there exists some $\ell > 0$ with $X \subset B(0, \ell) \subset [-\ell, \ell]^n$. By Corollary 12.4 the set $[-\ell, \ell]^n$ is compact, and X is closed, hence by Proposition 9.1, the set X is compact. $\square$

The second proof of the Heine-Borel Theorem is much closer to the method of proof given in class, and introduces a new concept:

DEFINITION 12.6. *A metric space X is* totally bounded*, if for every $\epsilon > 0$, X admits a finite covering by open ϵ-balls. That is, there exists $\{x_1, \ldots, x_n\} \subset X$ such that*

$$X = B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$$

PROPOSITION 12.7. *Let $X \subset \mathbb{R}^n$ be a bounded set. Then X is totally bounded.*

Proof. There exists an integer $\ell > 0$ so that $X \subset B(0, \ell) \subset [-\ell, \ell]^n$ as X is bounded. Choose an integer $m > 0$ so large that $\ell\sqrt{n}/m < \epsilon$. Then any cube of side length ℓ/m has diameter $\ell\sqrt{n}/m$ hence is contained in the ball of radius ϵ centered on any of its vertices.

We subdivide the large cube $[-\ell, \ell]^n$ in $\mathbb{R}^n$ into smaller cubes $c_{\mathcal{I}}$ of side length ℓ/m, where $\mathcal{I} = (i_1, \ldots, i_n)$ is a collection of integers satisfying

$$-\ell \cdot m \leq i_j < \ell \cdot m \ \text{ for } j = 1, 2, \ldots, n$$

The cube corresponding to such $\mathcal{I}$ is given by

$$C_{\mathcal{I}} = \{\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid i_1/m \leq x_1 \leq (i_1 + 1)/m \ , \ \ldots \ , \ i_n/m \leq x_n \leq (i_n + 1)/m\}$$

By the choice of the index sets $\mathcal{I}$, the large cube $[-\ell, \ell]^n$ is the union of all such smaller cubes $C_{\mathcal{I}}$.

For each index set $\mathcal{I}$, define

$$\vec{x}_{\mathcal{I}} = (i_1/m, i_2/m, \ldots, x_n/m) \ \text{ where } \ -\ell \cdot m \leq i_j < \ell \cdot m \ \text{ for } j = 1, 2, \ldots, n$$

The point $\vec{x}_{\mathcal{I}}$ is one of the corners of the cube $C_{\mathcal{I}}$. Then by the choice of m we have $C_{\mathcal{I}} \subset B(\vec{x}_{\mathcal{I}}, \epsilon)$.

It follows that the collection of open balls $\{B(\vec{x}_{\mathcal{I}}, \epsilon)\}$ if a finite cover of $[-\ell, \ell]^n$ by open ϵ-balls, which restrict to an open cover of X. $\square$

In the above proof, notice that the number of ϵ-balls used to cover $[-\ell, \ell]^n$ is $(2\ell m)^n$ which is proportional to $m^n \sim (1/\epsilon)^n$ where n is the dimension of the ambient space $\mathbb{R}^n$. This is a general statement, that the number of open ϵ-balls required in a cover of a totally bounded set is proportional to $(1/\epsilon)^d$ where $0 \leq d \leq \infty$ is a version of the *Hausdorff dimension* of the set X. For $X \subset \mathbb{R}^n$, the above argument shows that the Hausdorff dimension of X is at most n.

Hausdorff dimension of a metric space is a very important concept in many areas of analysis, especially in the study of *fractal sets*. Note that the Hausdorff dimension does not have anything to do with the Hausdorff separation axiom T_2 – Hausdorff did a lot of math! He was one of the principal founders of the modern theory of point-set topology. You can read more about Hausdorff dimension on the wiki page `http://en.wikipedia.org/wiki/Hausdorff_dimension`

Now, we show how to use the totally bounded hypothesis to prove the most general version of the Heine-Borel Theorem.

THEOREM 12.8. *A totally bounded, complete metric space X is sequentially compact.*

Proof. Let $\{x_n\} \subset X$ be an infinite sequence. If the set $A = \{x_n\}$ of values of the sequence is finite, then the sequence has a convergent subsequence. So, we may assume that A is infinite. We will then show that $\{x_n\}$ contains a Cauchy subsequence $\{a_i = x_{n_i} \mid n_1 < n_2 < \ldots\}$, which must then converge to a point $x_* \in X$, as X is complete. The sequence $a_i = x_{n_i}$ will be defined inductively.

Since X is totally bounded, for each $i = 1, 2, 3, \ldots$ we can find a finite covering of X by open ϵ_i-balls, where $\epsilon_i = 1/2^i$. We label these open covers by

$$\mathcal{U}^{(i)} = \{B(z_k^{(i)}, \epsilon_i) \mid 1 \leq k \leq \beta_i\}$$

Here, β_i is just some integer, which probably tends to infinity as $i \to \infty$. The points $z_k^{(i)} \in X$ are the centers of the balls of course, and exist by the assumption that X is totally bounded.

The collection $\mathcal{U}^{(1)}$ is a cover for X, so

$$A = \left(A \cap B(z_1^{(1)}, \epsilon_1)\right) \cup \cdots \cup \left(A \cap B(z_{\beta_1}^{(1)}, \epsilon_1)\right)$$

As the set A is assumed to be infinite, there must exists some index $1 \leq \ell_1 \leq \beta_1$ so that the intersection $A \cap B(z_{\ell_1}^{(1)}, \epsilon_1)$ is also an infinite set. Choose $a_1 = x_{n_1} \in A \cap B(z_{\ell_1}^{(1)}, \epsilon_1)$.

We now proceed inductively. For simplicity of the description, we show how to choose $a_2 = x_{n_2} \in A$.

Define $A_1 = A \cap B(z_{\ell_1}^{(1)}, \epsilon_1)$ which is an infinite set. The collection $\mathcal{U}^{(2)}$ is a cover for X, so

$$A_1 = \left(A_1 \cap B(z_1^{(2)}, \epsilon_2)\right) \cup \cdots \cup \left(A_1 \cap B(z_{\beta_2}^{(2)}, \epsilon_2)\right)$$

Thus, there must exists some index $1 \leq \ell_2 \leq \beta_2$ so that the intersection $A_1 \cap B(z_{\ell_2}^{(2)}, \epsilon_2)$ is also an infinite set. Choose

$$a_2 = x_{n_2} \in (A_1 - \{x_1, x_2, \ldots, x_{n_1}\}) \cap B(z_{\ell_2}^{(2)}, \epsilon_2) \subset B(z_{\ell_1}^{(1)}, \epsilon_1)$$

We continue on in this way, to obtain $\{a_1 = x_{n_1}, \ldots, a_i = x_{n_i}, \ldots\} \subset A$ with $n_1 < n_2 < \ldots$ and

$$\{a_i, a_{i+1}, a_{i+2}, \ldots\} \subset B(z_{\ell_i}^{(i)}, \epsilon_i) \subset \cdots B(z_{\ell_1}^{(1)}, \epsilon_1)$$

Since the ball $B(z_{\ell_i}^{(i)}, \epsilon_i)$ has radius ϵ_i its diameter $2\epsilon_i = 2/2^i \to 0$. Thus, $\{a_i\} \subset A$ is a Cauchy sequence, and we are done. $\square$

COROLLARY 12.9. *A closed and bounded subset X of $\mathbb{R}^n$ is compact.*

Proof. The space $\mathbb{R}^n$ is complete, as $\mathbb{R}$ is complete. The subset X is closed, hence complete. By Proposition 12.7, the set X is totally bounded. Then by Theorem 12.8, the set X is sequentially compact. Then by Proposition 7.3, the set X is compact. $\square$

13. Summary of properties of compact sets

$$\begin{aligned}
A \;\; compact &\iff A \;\text{ has } \textit{Finite Intersection Property} \\
&\implies A \;\text{ has } \textit{Descending Chain Condition}
\end{aligned}$$

$$A \;\; compact \implies \text{f(A) is compact for } f\colon X \to Y \text{ continuous}$$

$$\begin{aligned}
A \;\; compact\ metric &\implies A \;\text{ is closed and bounded} \\
A \;\; compact\ metric &\implies \text{every open cover has a Lebesgue number} \\
A \;\; compact\ metric &\implies f\colon A \to Y \text{ continuous, then f is uniformly continuous}
\end{aligned}$$

$$A \subset \mathbb{R}^n \;\; compact \iff A \;\text{ is closed and bounded}$$

14. Connected sets

We introduce another basic concept of topology. We first give the abstract definition, then derive the *Intermediate Value Theorem*. We also discuss intuitive interpretations of the notion of connectedness, and then discuss local connectedness and some pathological examples.

DEFINITION 14.1. *A topological space X is* connected *if X cannot be written as the disjoint union of non-empty open sets. That is, if $U, V \subset X$ are open with $X = U \cup V$ and $U \cap V = \emptyset$, then either $U = \emptyset$ or $V = \emptyset$.*

If there exists open sets $U, V \subset X$ with $X = U \cup V$ and $U \cap V = \emptyset$, then we say that the pair U, V are *separating* for X.

PROPOSITION 14.2. *Let X be a topological space, and $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of connected subsets of X. Suppose there exists $x \in X$ so that $x \in A_\alpha$ for all $x \in \mathcal{A}$. Then the union $A = \displaystyle\bigcup_{\alpha \in \mathcal{A}} A_\alpha$ is connected.*

Proof. Let $U, V \subset X$ be open sets such that $A \subset U \cup V$ and $A \cap U \cap V = \emptyset$. Suppose that $x \in U$. Then for each set A_α we have that $x \in A_\alpha \cap U$. As A_α is connected, this implies that $A_\alpha \subset U$. As this holds for all $\alpha \in \mathcal{A}$, we have $A \subset U$ and so $A \cap V = \emptyset$. Hence, U, V are not separating. □

Here is a fundamental property of connected sets:

PROPOSITION 14.3. *Let $f\colon X \to Y$ be a continuous map. If $C \subset X$ is connected in the subspace topology of X, then the image $f(C) \subset Y$ is connected in the subspace topology of Y.*

Proof. Suppose that $f(C)$ is not connected. Then there exists open sets $U', V' \subset Y$ with $f(A) \subset U' \cup V'$ and $f(A) \cap U' \cap V' = \emptyset$. Let $U = f^{-1}(U')$ and $V = f^{-1}(V')$ be open sets in X. We have $A \subset U \cup V$, and $A \cap U \cap V = \emptyset$, so A is not connected in the relative topology. □

The case of connected subsets of $\mathbb{R}$ is easy to describe.

LEMMA 14.4. *Let $C \subset \mathbb{R}$ be a connected subset for the relative metric topology. Then for each $a < b \in C$, the segment $[a, b] \subset C$.*

Proof. Let $a < c < b$, and suppose that $c \notin C$. Define open subsets $U = (-\infty, c)$ and $V = (c, \infty)$. Then $A \subset U \cup V$ and $U \cap V = \emptyset$. Thus C is not connected, contrary to assumption. $\square$

It follows that the connected subsets of the real line $\mathbb{R}$ consists of the intervals: for $a < b \in \mathbb{R}$, either $C = [a, b], [a, b), (a, b]$ or (a, b); or the rays $C = (-\infty, b], (-\infty, b), [a, \infty)$ or (a, ∞). The case of the line is thus relatively simple to determine. The situation changes for subsets of more general spaces.

These last two results combine to give a very well-known result:

THEOREM 14.5 (Intermediate Value Theorem)**.** *Let X be connected, and $f: X \to \mathbb{R}$ a continuous function. If $a < b \in f(X)$ then for every $a < c < b$, there exists $\xi \in X$ such that $f(\xi) = c$.*

Proof. The image $f(X) \subset \mathbb{R}$ is connected by Proposition 14.3. Hence by Lemma 14.4, the assumption that $a < b \in f(X)$ implies that $c \in f(X)$. That is, there exists $\xi \in X$ such that $f(\xi) = c$. $\square$

What sort of spaces are connected, besides intervals in $\mathbb{R}$? First note that connected spaces have no isolated points, except in the trivial case:

PROPOSITION 14.6. *Let $x \in X$ be an isolated point in a Hausdorff topological space. Then X is connected if and only if $X = \{x\}$.*

Proof. Let $x \in X$ be an isolated point, then $U = \{x\}$ is open. Also, $\{x\}$ is closed as X is Hausdorff, thus, $V = X - \{x\}$ is open. Note that $U \cup V = X$ and $U \cap V = \emptyset$, so X connected implies $V = \emptyset$. $\square$

For example, if X is a discrete space, then X is connected if and only if X consists of a single point. For metric spaces, connected implies even more:

THEOREM 14.7. *Let X be a connected metric space. Suppose that there exists $x \neq y \in X$, then X is uncountable.*

Proof. Define $f(z) = D(x, u)$ so that $f: X \to \mathbb{R}$ is a continuous function. Note that $f(x) = 0$, and $f(y) = D(x, y) > 0$, so the image of f is not a single point. By Proposition 14.3 the image $f(X)$ is a connected set. Then by Lemma 14.4 the image contains the segment $[0, D(x, y)]$. Since the image of f contains an uncountable set, the set X must also be uncountable. $\square$

Alternately, we can give a slightly shorter (but equivalent) proof using the Intermediate Value Theorem applied to the function $f(z) = D(x, z)$.

We next discuss the relation between connected sets and limit points for sets. Here is the original definition, circa 1910. (See the interesting article *Connected sets and the AMS, 1901–1921*, [18] on the development of the notion of a "connected set".)

DEFINITION 14.8. *Let X be a topological space. Two non-empty sets $A, B \subset X$ are said to be* separating *if $X = A \cup B$ and their closures satisfy $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.*

The space X is limit-point connected *if it cannot be separated. That is, there does not exists a pair of separating sets for X. A subset $Z \subset X$ is* limit-point connected *if Z cannot be separated in the relative topology on Z.*

This condition was historically phrased as saying that A does not contain any limit points of B, and B does not contain any limit points of A.

We show that the limit points notion of "connected" agrees with the modern definition.

PROPOSITION 14.9. *A topological space X is connected if and only if it is limit-point connected.*

Proof. Suppose that $X = U \cup V$ where $U, V \subset X$ are open subsets with $U \cap V = \emptyset$. Then each set U and V are the complements of open sets, hence are also closed. Thus, $\overline{U} = U$ and $\overline{V} = V$. Thus, X is also separated in the sense of Definition 14.8.

Conversely, suppose that there exists non-empty sets $A, B \subset X$ such that $X = A \cup B$ and their closures satisfy $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. The condition $\overline{A} \cap B = \emptyset$ implies that B is the complement of a closed set, hence is open. Similarly, A is also open. Thus, X is not connected. □

Here is a useful lemma, which is just an exercise in the definition of the relative topology.

LEMMA 14.10. *Suppose that X is a topological space, with separating sets $X = A \cup B$. If $C \subset X$ is a connected subset, then either $C \subset A$ or $C \subset B$.*

Proof. We have seen that both A and B must be disjoint open subsets of X, hence their restrictions to C are open in the relative topology. Thus, we have $C = (C \cap A) \cup (C \cap B)$ separates C. As C is connected, one of these two intersection must be empty. □

The following result has been a question on previous Masters Exams:

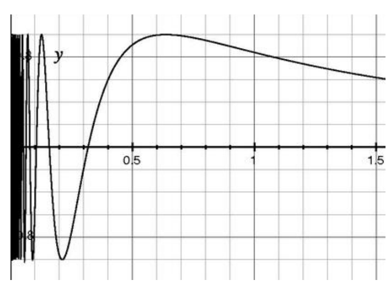
LEMMA 14.11. *Suppose that X is a topological space. If A is a connected subset of X and B is a subset satisfying $A \subset B \subset \overline{A}$, then B is also connected.*

Proof. Suppose that $B = C \cup D$ is a separation of B. Then by Lemma 14.10, we must have either $A \subset C$ or $A \subset D$. Suppose that $A \subset C$, then $\overline{A} \subset \overline{C}$. As $\overline{C} \cap D = \emptyset$, then $\overline{A} \cap D = \emptyset$ and so $B \cap D = \emptyset$. But this contradicts the assumption that $B = C \cup D$ is a separation, which requires that $B \cap D \neq \emptyset$. □

What these results show, is that if a set X is connected, then for any decomposition $X = A \cup B$ into disjoint subsets, they must have common limit points.

For example, the set $X = [0, 2] = [0, 1) \cup [1, 2] \subset \mathbb{R}$ can be written as a disjoint union, but it is not "separated".

The set $X = \{(0, y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin(1/x)) \mid 0 < x < 1\} \subset \mathbb{R}^2$ looks separated, but isn't:



If a topological space X is not connected, we may try to decompose X into connected pieces. The key idea is to introduce an equivalence relation on X, using connected subsets to define the relationship between points.

DEFINITION 14.12. *Let X be a topological space. For points $x, y \in X$, define $x \overset{c}{\sim} y$ if there is a subset $C \subset X$ which is connected in the relative topology such that $x, y \in C$.*

PROPOSITION 14.13. *The relation $x \overset{c}{\sim} y$ is an equivalence relation.*

Proof. First, $x \overset{c}{\sim} x$ as $C = \{x\} \subset X$ is a connected subset. The condition $x \overset{c}{\sim} y \iff y \overset{c}{\sim} x$ is equally trivial to show.

Now, consider the case where $x \overset{c}{\sim} y$ and $y \overset{c}{\sim} z$. Then there exists connected sets $C, D \subset X$ with $x, y \in C$ and $y, z \in D$. Then by Proposition 14.2, the union $C \cup D$ is connected, and hence $x \overset{c}{\sim} z$. $\square$

DEFINITION 14.14. *Let X be a topological space. The equivalence classes of points for the relation $x \overset{c}{\sim} y$ are called the connected components of X. Given $x \in X$, the connected component defined by x is the set $C(x) = \{y \in X \mid y \overset{c}{\sim} x\} \subset X$.*

For example, if $X = [0,1] \cup (3,4]$ then the connected components are $[0,1]$ and $(3,4]$. In general, the situation is much more complicated, as suggested by the following definition.

DEFINITION 14.15. *A topological space X is* totally disconnected *if each connected component of X consists of singleton sets.*

For example, a discrete space is totally disconnected, as each point $x \in X$ is a connected component.

Another simple example is the set $X = \mathbb{R} - \mathbb{Q}$, the set of irrational numbers. This is a totally disconnected set, as every open interval in $\mathbb{R}$ contains a rational, so its intersection with X cannot be connected. This set is not closed in $\mathbb{R}$, obviously.

A totally disconnected closed subset of $\mathbb{R}$ need not be discrete, however:

PROPOSITION 14.16. *The Cantor set $\mathbb{K} \subset [0,1]$ is compact and totally disconnected, and is not discrete.*

Proof. Let $C \subset \mathbb{K}$ be any subset with more than one point, say $x < y \in C$. It is then an exercise to see that there exists some value $x < c < y$ for which $c \notin \mathbb{K}$. Here is a proof "by picture":

Thus, $C \subset (-\infty, c) \cup (c, \infty)$ so C is not connected in the relative topology. As every point of the Cantor set $\mathbb{K}$ is a limit point, $\mathbb{K}$ is not discrete. See Appendix B for details. $\square$

We introduce another topological property of spaces, which distinguishes between the various types of totally disconnected spaces, such as the integers $\mathbb{Z}$ and the Cantor set $\mathbb{K}$.

DEFINITION 14.17. *A topological space X is* locally connected *if for every $x \in X$, there exists an open neighborhood $x \in U \subset X$ such that X is connected.*

For example, the set $\mathbb{Z}$ of integers is locally connected, as each point is open, so is an connected open neighborhood of itself. On the other hand, neither the set of irrationals $\mathbb{R} - \mathbb{Q}$ nor the Cantor set are locally connected.

We can now define an object of considerable study:

DEFINITION 14.18. *A compact, connected metric space X is said to be a* continua.

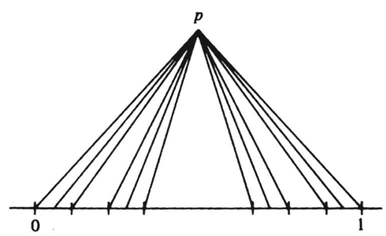
So, the closed intervals $[a, b]$ and the "topologist's sine curve" above are example of continua. The subject of continua in $\mathbb{R}^n$ for $n \geq 2$ has been extensively studied, and includes many curious examples with "pathological" behavior.

The phrase pathological refers to aspects of the construction or properties of the example, and is subjective. However, sometimes there is no doubt that an example is truly bizarre. The book "Counterexamples in Topology" by Steen and Seebach [10] contains a good discussion of such examples. Although, the best source are the books by Kuratowski [3, 4] and Hocking and Young [7].

Here is an example of "pathological" behavior:

DEFINITION 14.19. *Let X be a topological space. A point $x \in X$ is said to be an* explosion point *if X is connected, but the space $Y = X - \{x\}$ is totally disconnected.*

It is hard to believe that an example of such could exist – but there is a famous example called the *Knaster-Kuratowski Fan*, which is approximately pictured below, with details in Appendix B.



Here is another example of a "pathological space", discovered by our own Professor Martin Tangora, while he was an undergraduate student at Cal Tech [8]:

PROPOSITION 14.20. *There exists a topological space X with closed subsets $A, B \subset X$ so that both A and B are totally disconnected, but $X = A \cup B$ is connected.*

15. Path connected and path components

There is another natural notion of "connectedness" for a topological space.

DEFINITION 15.1. *A topological space X is* path connected *if for ever $x, y \in X$ there exists a continuous map, or path, $\sigma_{x,y} \colon [a, b] \to X$ with $\sigma_{x,y}(a) = x$ and $\sigma_{x,y}(b) = y$.*

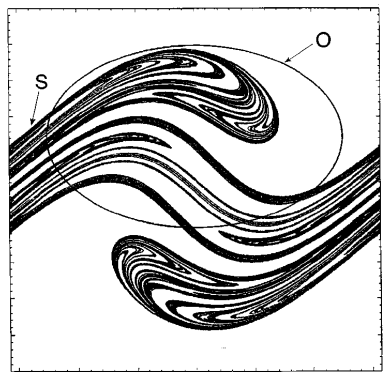
The first remark is that path connected is stronger than connected.

LEMMA 15.2. *If X is path connected, then it is connected.*

Proof. Suppose that $X = A \cup B$ is a separation of X. Let $x \in A$ and $y \in B$. Then we claim there is no path from x to y. Suppose such a path $\sigma_{x,y} \colon [a, b] \to X$ existed, then the image $C = \sigma_{x,y}[a, b] \subset X$ is a connected set. But A, B restrict to a separation for C, which is a contradiction. □

The topologist's sine curve is a connected set, which is not path connected. There is no path between any point on the vertical line segment, to a point on the sine curve itself. The proof of this claim is left as an exercise.

Connected, but not path connected sets occur naturally in the study of dynamical systems. Here is an illustration of such a set, called a *Knaster continuum*, produced by computer modeling of the orbits of a dynamical system. Note this is just a window on the set, which continues outside the part illustrated.



As with connectedness, we can also introduce the notions of path components and locally path connected spaces.

DEFINITION 15.3. *Let X be a topological space. For points $x, y \in X$, define $x \stackrel{p}{\sim} y$ if there is a path $\sigma_{x,y} \colon [a, b] \to X$ with $\sigma_{x,y}(a) = x$ and $\sigma_{x,y}(b) = y$.*

PROPOSITION 15.4. *The relation $x \stackrel{p}{\sim} y$ is an equivalence relation.*

Proof. First, $x \stackrel{p}{\sim} x$ and $x \stackrel{p}{\sim} y \Longleftrightarrow y \stackrel{p}{\sim} x$ is almost immediately obvious.

Now, consider the case where $x \stackrel{p}{\sim} y$ and $y \stackrel{p}{\sim} z$. Then there exists paths $\sigma_{x,y} \colon [a, b] \to X$ with $\sigma_{x,y}(a) = x$ and $\sigma_{x,y}(b) = y$, and $\sigma_{y,z} \colon [c, d] \to X$ with with $\sigma_{y,z}(c) = d$ and $\sigma_{y,z}(d) = z$. Define a new path $\sigma_{x,z} \colon [0, \beta] \to X$ for $\beta = (b - a) + (d - c)$, by

$$\sigma_{x,z}(t) = \begin{cases} \sigma_{x,y}(t + a) \text{ if } 0 \leq t \leq b - a, \\ \sigma_{y,z}(t + c + a - b) \text{ if } b - a \leq t \leq (b - a) + (d - c) \end{cases}$$

Then $\sigma_{x,z}$ is a path with $\sigma_{x,z}(0) = \sigma_{x,y}(a) = x$ and $\sigma_{x,z}(\beta) = \sigma_{y,z}(d) = z$. Hence $x \stackrel{p}{\sim} z$. $\square$

DEFINITION 15.5. *Let X be a topological space. The equivalence classes of points for the relation $x \stackrel{p}{\sim} y$ are called the* path components *of X. Given $x \in X$, the path component defined by x is the set $P(x) = \{y \in X \mid y \stackrel{p}{\sim} x\} \subset X$.*

Finally, we mention one other property of topological spaces, which is very important for the study of covering spaces, and more generally for the study of the algebraic topology of spaces.

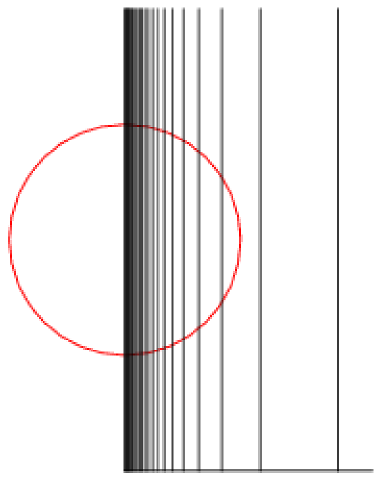
DEFINITION 15.6. *A topological space X is* locally path connected *if for every $x \in X$, there exists an open set $U \subset X$ such that $x \in U$ and U is a path connected topological space.*

A locally path connected space is always locally connected.

For example, any disk in a Euclidean space $\mathbb{R}^n$ is connected and path connected, as is any topological space homeomorphic to a disk. More generally, every open subset $X \subset \mathbb{R}^n$ is locally connected and locally path connected. An *n-manifold* M is a topological space such that every point $x \in M$ has an open neighborhood $x \in U_x \subset M$ which is homeomorphic to an open disk in $\mathbb{R}^n$, hence every manifold is locally connected and locally path connected.

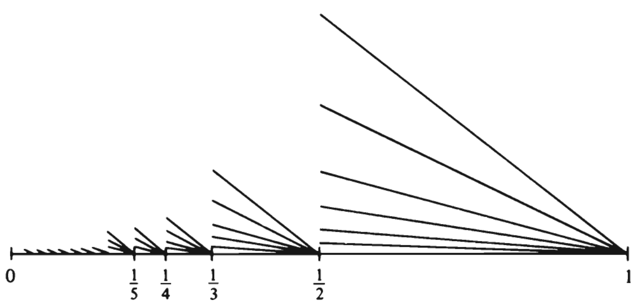
However, for closed subsets $X \subset \mathbb{R}^n$ the situation is much different.

The *Comb Space* $\mathcal{C}$ is a typical example of a closed subspace of $\mathbb{R}^2$ which is path connected, but not locally path connected. Let $K = \{0\} \cup \{1/n \mid n = 1, 2, 3, \ldots\}$. Then $\mathcal{C} = K \times [0,1] \cup [0,1] \times \{0\} \subset \mathbb{R}^2$, which looks like this:



There are many variations on this theme. For example, in the construction of the Knaster-Kuratowski Fan in Appendix 20, if we use the entire line segments $L(c)$ rather than the deleted segments $L('(c)$ and $L''(c)$, we obtain a fan on the Cantor set which is path connected, but not locally path connected at every point except near the point p at the top of the tent. This is because a path from one ray to another must pass through the point p, so an open set which omits this point is not path connected. (Recall that the Cantor set is perfect, so such an open set will intersect an infinite number of rays.)

Here is yet another example. At each "juncture point" $(1/n, 0)$ for $n \geq 1$, the set is not locally connected nor locally path connected:



16. Derived sets

Let X be a topological space. The *derived set* of X is the collection of its limit points,

$$X' = \{x \in X \mid U \cap (X - \{x\}) \neq \emptyset \text{ for all open } U \subset X, x \in U\}$$

Note that if $y \in X - X'$ then y is an isolated point of X, by definition. An isolated point is an open set, so the complement $X - X'$ is the union of open sets, hence X' is a closed subset of X.

The concept of the derived set was first introduced by Georg Cantor in 1872, and he developed set theory in large part to study derived sets on the real line.

Note that X is *discrete* if and only if $X' = \emptyset$.

At the other extreme, a set X is said to be *perfect* if $X' = X$.

If X is not discrete, then we can consider the set of derived points of the derived set, denoted by X''. As above, $X'' \subset X'$ is a closed subset of X', hence X'' is also closed in X.

We now define the n^{th} derived set by induction: if $X^{(n-1)}$ has been defined, then let $X^{(n)}$ denote the derived points of $X^{(n-1)}$. That is, $X^{(n)} = (X^{(n-1)})'$.

This defines $X^{(n)}$ for all integers $n \geq 0$, where we set $X^{(0)} = X$.

Recall that ω is the first infinite ordinal. By definition, ω is the order-isomoprhism class of the set consisting of all the finite ordinals, which is typically represented by the set of positive integers $\mathbb{N}$.

Define the derived set for the ordinal ω by setting

$$X^{(\omega)} = \bigcap_{n=0}^{\infty} X^{(n)}$$

LEMMA 16.1. *Suppose that X is a compact, non-empty topological space. Then either $X^{(\omega)} \neq \emptyset$, or there exists $n > 0$ such that $X^{(n)} = \emptyset$.*

Proof. The sequence

$$X = X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \cdots$$

is a descending chain of closed subsets of a compact space, so by the Descending Chain Condition, either $X^{(n)} = \emptyset$ for some n, or $X^{(\omega)} \neq \emptyset$. □

We can continue this process, using transfinite induction, to define $X^{(\alpha)}$ for any countable ordinal α.

Recall that such α is the order-isomoprhism class of the set consisting of the union of all ordinals β with $\beta < \alpha$. We then have one of two possibilities, either:

- α is a *successor ordinal* if there exists an ordinal $\beta < \alpha$ such that $\alpha = \beta + 1$.
- α is a *limit ordinal* if there does not exist an ordinal $\beta < \alpha$ such that $\alpha = \beta + 1$; then we have

$$\alpha = \bigcup_{\beta < \alpha} [\beta]$$

where $[\beta]$ is the set representing the ordinal class defined by β. Now define $X^{(\alpha)} = (X^{(\beta)})'$ if $\alpha = \beta + 1$; and $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$ if α is a limit ordinal.

We say that X has *finite type* if $X^{(\alpha)} \neq \emptyset$, but $X^{(\alpha+1)} = \emptyset$, for some countable ordinal α. The ordinal α is called the *Cantor-Bendixson rank* of X. Then as above, the Finite Intersection Property for a compact space implies:

LEMMA 16.2. *Suppose that X is a compact, non-empty topological space. If X has finite type, then α is a successor ordinal.* □

Here is an interesting result about compact metric spaces of finite type:

PROPOSITION 16.3. *Let $\{X, D\}$ be a compact metric space. If X has finite type, then X is countable.*

Proof. For each $m > 0$ let $U_m = \{y \in X \mid D(y, X') < 1/m\}$. Note that $X - U_m$ consists of isolated points, and as U_m is open, this set is also closed. Since X is compact, any infinite closed subset must have a limit point, which implies that $X - U_m$ is a finite set.

Since X' is closed, $D(y, X') = 0$ if and only if $y \in X'$, so $X' = \bigcap_{m>0} U_m$. Thus,

$$X - X' = \bigcup_{m=1}^{\infty} (X - U_m)$$

which is a countable union of finite sets, hence is countable.

The above argument holds verbatim for any successor ordinal $\alpha = \beta + 1$, to show that $X^{(\beta)} - X^{(\alpha)}$ is always countable. If α is a limit ordinal, then for any $\beta < \alpha$, the set $X^{\beta} - X^{(\alpha)}$ is a countable union of countable sets, hence countable.

The claim of the Proposition then follows by transfinite induction. □

Recall that Ω is the first uncountable ordinal. By definition, Ω is the order-isomoprhism class of the ordered set consisting of all the countable ordinals, which is itself an uncountable set. The cardinal of Ω is $\aleph_1$ which is the smallest uncountable cardinal.

The ordinal Ω is a limit ordinal, where $\alpha < \Omega$ means simply that α is some countable ordinal. Define the Ω-derived set of X as

$$X^{(\Omega)} = \bigcap_{\alpha<\Omega} X^{(\alpha)} \tag{4}$$

If α is a limit ordinal, then $X^{(\alpha)}$ is defined as an intersection of successor ordinals, so the intersection in (4) can be taken over just the successor ordinals. The set $X^{(\Omega)}$ is called the *perfect kernel* for X, or the *Cantor-Bendixson derivative* of X.

If X does not have finite type, that is $X^{(\alpha)} \neq \emptyset$ for all countable ordinals, then by the Finite Intersection Property, the perfect kernel $X^{(\Omega)}$ is non-empty.

The set $X - X^{(\Omega)}$ is countable, by the same method of proof used to show Proposition 16.3. The points in $X - X^{(\Omega)}$ are said to be "scattered". They include the isolated points in $X - X'$, as well as the limit points of some set $X^{(\alpha)}$ which are not limit points of $X^{(\alpha+1)}$. Thus, the "uncountable" part of a compact metric space must lie in its perfect kernel.

The proof of the following result uses techniques beyond the scope of these notes, but we mention it as it shows the interest in this perfect kernel. The result is proved using an alternate definition of the Cantor-Bendixson derivative which is the point-set topology analogue of the Radon-Nikodyn derivative in Real Analysis.

THEOREM 16.4. *The perfect kernel $X^{(\Omega)}$ is perfect. That is, $(X^{(\Omega)})' = X^{(\Omega)}$.* $\square$

The proof of the following is reminiscent of Cantor's proof via the diagonalization method of showing that the real numbers $\mathbb{R}$ form an uncountable set. Both proofs are due to Cantor.

THEOREM 16.5. *Let X be a compact Hausdorff space. If X is perfect, then X is uncountably infinite.*

Proof. We will prove that if $f\colon \mathbb{N} \to X$ is any map, then there exists some $y \in X$ which is not in the image of f, hence there is no bijection between $\mathbb{N}$ and X, so X must be uncountably infinite.

We use the following property of a perfect Hausdorff space X:

LEMMA 16.6. *Let X be a compact Hausdorff space, and assume that X is perfect. Given any $x \in X$ and non-empty open set $U \subset X$, there exists a non-empty open subset $V \subset U$ with $x \notin \overline{V}$.*

Proof. Let x and U be given. Choose $y \in U$ with $y \neq x$. If $x \notin U$ then we choose any point $y \in U$. If $x \in U$, X perfect implies that x is not isolated, hence there exists some $y \in U$ with $y \neq x$.

Then X Hausdorff implies there exists open sets $W_1, W_2 \subset X$ with $x \in W_1$ and $y \in W_2$ and $W_1 \cap W_2 = \emptyset$. Set $V = W_2 \cap U$. Then $V \subset U$, and $\overline{V} \subset X - W_1$ as W_1 is open, so $X - W_1$ is closed. As $x \in W_1$ this implies $x \notin V$. $\square$

Label $x_n = f(n)$ for $n \in \mathbb{N}$. Choose any open set $U \subset X$ with $x_1 \in U$. Then by Lemma 16.6 there exists a non-empty open set $V_1 \subset U$ so that $x_1 \notin \overline{V_1}$.

We now proceed by induction. Suppose V_n is given with $x_n \notin \overline{V_n}$. Apply Lemma 16.6 for $U = V_n$ and $x = x_{n+1}$. Then there exists non-empty $V_{n+1} \subset V_n$ so that $x_{n+1} \notin \overline{V_{n+1}}$.

Note that by the inductive construction we obtain the descending chain of closed subsets

$$\overline{V_1} \supset \overline{V_2} \supset \cdots \supset \overline{V_n} \supset \cdots$$

so that $x_n \notin \overline{V_n}$ hence $x_n \notin \overline{V_k}$ for all $k \geq n$. As X is compact and each set $\overline{V_n} \neq \emptyset$, there exists $y \in \bigcap_{n=1}^{\infty} \overline{V_n}$. For all $n \geq 1$, the point $y \neq x_n$ as $y \in \overline{V_n}$ and $x \notin \overline{V_n}$. Thus, $y \in X$ is a point not in the image of f. $\square$

Combining Theorems 16.4 and 16.5 yields the following result, which is a special case of a result valid more generally for Polish spaces:

THEOREM 16.7 (Cantor–Bendixson)**.** *A compact metric space X is the disjoint union $X = \mathcal{S} \cup X^{(\Omega)}$ of a countable set S and its perfect kernel $X^{(\Omega)}$, which is uncountable if non-empty.*

In other words, if X is a countably infinite metric space, every point of X need not be isolated, but is a limit of limit points up to some countable ordinal. In a way, this is saying that the topology of X is "tame". In order to obtain "wild" topology, one must consider sets X which are uncountable. This observation set the direction of the study of Point-Set Topology for the next 100 years.

Note, this is all rather remarkable, for when Cantor began his studies of limits of sets, and limits of limits, all the way up to his theory of "infinities", it was considered that there was just "one infinity", which was represented by the integers. In other words, it was assumed that every set was countable, and anyone who argued otherwise was ridiculed. The discovery that the real number line $\mathbb{R}$ is an uncountable set, and the conceptual framework to even discuss when a set was not countable, was a foundation stone of modern mathematics.

17. Separable, second countable, paracompact and Lindelöf spaces

Let X be a topological space, and $\mathcal{T}$ a topology on X. We discuss several axioms to add to those of the topology $\mathcal{T}$ to guarantee that X has a "countable" nature.

Recall that a subset $\mathcal{S} \subset X$ is dense if its closure is all of X.

DEFINITION 17.1. *The space $X, \mathcal{T}$ is* separable *if there exists a countable dense subset $\mathcal{S} \subset X$.*

The model example for this property are the rational numbers $\mathcal{S} = \mathbb{Q}$ and $X = \mathbb{R}$. More generally, the rational points $\mathbb{Q}^n \subset \mathbb{R}^n$ in Euclidean n-space are dense, so $\mathbb{R}^n$ is separable. Other natural examples are provided in sections 23 and 24, where we introduce various metric spaces of functions as the closure of countably dense subspaces.

We introduce three other notions of "countability" for a topology, related to separability, and then discuss their relations. The first of these is a weaker form of compactness:

DEFINITION 17.2 (Lindelöf)**.** *The space $X, \mathcal{T}$ is* Lindelöf *if for every open cover $\mathcal{U}$ of X, there exists a countable subcover $\mathcal{U}_0 \subset \mathcal{U}$ of X.*

As an example, we have the following, whose proof is an exercise:

PROPOSITION 17.3. *Let X be a topological space, and assume there exists a countable collection of compact subsets $K_n \subset X$ such that $X = \bigcup_{n=1}^{\infty} K_n$, then X is Lindelöf.* □

For example, we have

PROPOSITION 17.4. *Let X be a complete metric space, such that each closed and bounded subset is totally bounded. Then X is Lindelöf.*

Proof. Pick $x_0 \in X$ and define $K_n = D(x_0, n) = \{x \in X \mid D(x_0, x) \leq n\}$ for $n \in \mathbb{N}$. Then for each $y \in X$ and $n > D(x_0, y)$ we have $y \in D(x_0, n)$, so the union of the closed balls is all of X. Each closed ball if totally bounded by assumption, so by Theorem 12.8, each K_n is a compact. Hence, by Proposition 17.3 the space X is Lindelöf. □

For example, the hypotheses of Proposition 17.4 are satisfied by any closed subset of $\mathbb{R}^n$.

Recall that a *base* for the topology $\mathcal{T}$ is a collection of subsets, $\mathcal{B} \subset \mathcal{T}$, such that for every $x \in X$ and open $U \subset X$ with $x \in U$, there exists $V \in \mathcal{B}$ with $x \in V \subset U$. Thus, every open set in $\mathcal{T}$ is the union of elements of $\mathcal{B}$. We say that $\mathcal{B}$ generates the topology. For example, in the case of a metric space X, the collection of all open balls $\mathcal{B} = \{B(x, \epsilon) \mid x \in X \ , \ \epsilon > 0\}$ is a base for the topology.

The following concepts refer to special countable bases for the topology.

DEFINITION 17.5 (First Countable)**.** *The space $X, \mathcal{T}$ is* first countable *if for each $x \in X$, there exists a countable base $\mathcal{B}_x$ for the topology at x. That is, $\mathcal{B}_x \subset \mathcal{T}$ is a countable set of open sets, each containing x, and for each open neighborhood $x \in U$ there exists $V \in \mathcal{B}_x$ with $x \in V \subset U$.*

For example, in the case of a metric space X, for each $x \in X$, the collection of all open balls $\mathcal{B}_x = \{B(x, \epsilon) \mid x \in X \ , \ \epsilon \in \mathbb{Q} \ , \epsilon > 0\}$ is a countable base for the topology at x.

A topological space which is not first countable is most likely a very nasty, as locally (near any point $x \in X$) the topology is not even defined by a countable set of open sets.

By far the most important concept is the following:

DEFINITION 17.6 (Second Countable)**.** *The space $X, \mathcal{T}$ is* second countable *if there exists a countable base $\mathcal{B}$ for the topology. That is, there exists a countable collection of open sets, $\mathcal{B} \subset \mathcal{T}$, such that for each $x \in X$ and open neighborhood $x \in U$, there exists $V \in \mathcal{B}_x$ with $x \in V \subset U$.*

Clearly, if X is second countable, then it is first countable, as we simply take $\mathcal{B}_x = \mathcal{B}$ for all x.

We next show two results, which illustrate how these four definitions of "countability" are related.

THEOREM 17.7. *A second countable topological space X is separable and Lindelöf.*

Proof. Let $\mathcal{B} = \{B_1, B_2, \ldots\} \subset \mathcal{T}$ be a countable basis for the topology of X.

We first show that X is separable. For each n choose $x_n \in B_n$. Then we claim that the set $\mathcal{S} = \{x_1, x_2, \ldots\}$ is dense in X. That is, for each $y \in X$, either $y \in \mathcal{S}$, or x is a limit point for the set $\mathcal{S}$. Suppose that $y \notin \mathcal{S}$, and let $U \subset X$ be any open set with $x \in U$. Since $\mathcal{B}$ is a basis for the topology, there exists some n such that $x \in B_n \subset U$. Then $x_n \in B_n \subset U$, so $(U - \{x\}) \cap \mathcal{S} \neq \emptyset$. Hence, $x \in \overline{\mathcal{S}}$.

We next show that X is Lindelöf. Let $\mathcal{U}$ be any open covering of X. For each n, pick $U \in \mathcal{U}$ with $B_n \subset U$, if this is possible. Label $V_n = U$. Let $\mathcal{J} \subset \mathbb{N}$ be the collection of integers for which such a choice is possible. Then the collection $\mathcal{V} = \{V_n \mid n \in \mathcal{J}\}$ is a countable subcollection of $\mathcal{U}$.

We claim that $\mathcal{V}$ is an open covering of X. Given $x \in X$, there exists some $U \in \mathcal{U}$ with $x \in U$. Since U is open, there exists some basis element $B_m \in \mathcal{B}$ with $x \in B_m \subset U$. Thus, B_m is contained in some element of $\mathcal{U}$, hence $m \in \mathcal{J}$. The open set $V_m \in \mathcal{V}$ was chosen so that $B_m \subset V_m$ and thus $x \in V_m$. We have then shown that for any $x \in X$, there exists some $V_m \in \mathcal{V}$ with $x \in V_m$, so that $\mathcal{V}$ is an open covering of X. □

The next two results shows that all of these concepts coincide for metric spaces. We start with a pleasant observation, which is Theorem 58 on page 95 of Kaplansky.

PROPOSITION 17.8. *If X is a separable metric space, then X is second countable.*

Proof. Let $\mathcal{S} = \{x_n \mid n \in \mathbb{N}\}$ be a dense subset of X. Let $\mathbb{Q}^+ \subset \mathbb{Q}$ be the collection of all positive rational numbers. This set is again countable. Define the collection of open balls with rational radii, $\mathcal{B} = \{B(x_n, r) \mid x_n \in \mathcal{S} \ \& \ r \in \mathbb{Q}^+\}$. As both sets $\mathcal{S}$ and $\mathbb{Q}^+$ are countable, the collection $\mathcal{B}$ is also countable. We claim that $\mathcal{B}$ is a basis for the topology.

Let $U \subset X$ be an open set. For $y \in V$, there exists $\epsilon_y > 0$ so that $B(y, \epsilon_y) \subset V$. Choose a rational number $0 < r_y < \epsilon_y$ then $B(y, r_y) \subset B(y, \epsilon_y) \subset U$. Thus, $r = r_y/2$ is again rational.

Choose $x_n \in \mathcal{S}$ so that $D(y, x_n) < r$, then $y \in B(x_n, r)$. For any $z \in B(x_n, r)$ we have by the Triangle Inequality that

$$D(y, z) \leq D(y, x_n) + D(x_n, z) < r + r = r_n$$

Thus, $z \in B(y, r_n)$ which implies that $B(x_n, r) \subset B(y, r_n) \subset U$, as was to be shown. □

THEOREM 17.9. *Let X be a metric space. Then the following are equivalent:*

$$X \text{ is separable} \iff X \text{ is second countable} \iff X \text{ is Lindelöf}$$

Note that this is Theorem 59 on page 95 of Kaplansky.

Proof. By Proposition 17.8 above, X separable implies X is second countable.

By Theorem 17.7 above, X second countable implies that X is Lindelöf and separable.

Thus, it suffices to show that X Lindelöf implies that X is separable. For each positive integer n, define the cover $\mathcal{U}_n$ of X by open balls of radius $\epsilon = 1/n$, or $\mathcal{U}_n = \{B(x, 1/n) \mid x \in X\}$.

As X is Lindelöf, there exists a countable subcover $\mathcal{V}_n \subset \mathcal{U}_n$. Enumerate the centers of the open balls in $\mathcal{V}_n$ as $\mathcal{S}_n = \{x_{n,i} \mid i = 1, 2, \ldots\}$. Thus, $x \in \mathcal{S}_n$ if and only if $B(x, 1/n) \in \mathcal{V}_n$. Define

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_n \cup \cdots$$

Given $y \in X$ and $\epsilon > 0$, chose n so that $1/n < \epsilon$. Then $y \in B(x_{n,i}, 1/n)$ for some i, as $\mathcal{V}_n$ is an open cover for X. But this implies that $D(y, x_{n,i}) < 1/n < \epsilon$ so $\mathcal{S}_n \cap B(y, \epsilon) \neq \emptyset$. Thus, $\mathcal{S} \cap B(y, \epsilon) \neq \emptyset$ which shows that $\mathcal{S}$ is dense in X. □

To introduce the last condition, we need a definition.

DEFINITION 17.10 (Refinement)**.** *Let $\mathcal{U}$ be an open covering of X. We say that an open covering $\mathcal{V}$ of X is a* refinement *of $\mathcal{U}$, if for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ with $V \subset U$.*

Clearly, a subcover of $\mathcal{U}$ is a refinement of $\mathcal{U}$, as is any covering of X formed from intersections of sets in $\mathcal{U}$. The notion of a refinement is more general than these two examples, however, as it allows us to take a given covering $\mathcal{U}$ and replace its members with smaller subsets, each contained in some element of $\mathcal{U}$, but which still form a covering $\mathcal{V}$. This is a common technique used when studying properties of manifolds, for example. (See section 18 for definitions.)

A cover $\mathcal{V}$ of X is *locally finite* if for each $x \in X$ there are only finitely many $V \in \mathcal{V}$ with $x \in V$.

DEFINITION 17.11 (Paracompact)**.** *The space $X, \mathcal{T}$ is* paracompact *if for each open covering $\mathcal{U}$ of X, there exists an open cover $\mathcal{V}$ which is a locally finite refinement of $\mathcal{U}$.*

It is obvious that compact space X is paracompact, as it admits finite subcovers, not just locally finite ones. Here is a general criteria for when a space is paracompact, whose proof we omit.

THEOREM 17.12. *A metric space X is always paracompact.* □

18. Topological manifolds

The theme of this course is the study of sets equipped with a topology, especially the "local properties" of X, like when is a point $x \in X$ a limit of some other set. The various topologies introduced, as well as pathological examples of topological spaces, show that these local properties can differ greatly from the intuition developed when thinking about open subsets of the line $\mathbb{R}$ or the plane $\mathbb{R}^2$. This sort of study is often called "general point-set topology".

In contrast, there is another class of examples of topological space which are locally as nice as possible. These are the topological manifolds, which have been very extensively studied for the past 60 years or so, since 1950.

DEFINITION 18.1. *A topological space M is an n-dimensional manifold if for every $x \in M$ there exists an open set $U_x \subset M$ and a homeomorphism $h_x \colon U_x \to B(0,1) \subset \mathbb{R}^n$, where $B(0,1)$ is the open ball about the origin or radius 1.*

The local topological properties of a manifold are as nice as those are for $\mathbb{R}^n$. For example, a topological manifold is always locally path connected, as this is true for the open disk $B(0,1) \subset \mathbb{R}^n$. In other words, they are not *locally* pathological.

On the other hand, global properties of a manifold may have pathologies. For example, the Long Line $\mathcal{L}$ of Example 21 is a connected 1-dimensional manifold, but it is not path connected. Thus, one often imposes additional hypothesis such as that M is connected and separable, or paracompact. Even with these assumptions, a manifold can have very complicated topology "at infinity", and can exhibit many pathologies.

The study of manifolds of dimension 2 is properly the subject of the sequel to this course. Just as an appetizer, section 22 gives some of the classic examples of compact manifolds in dimension 2.

The study of compact manifolds in dimensions $n \geq 3$ and above has been the subject of ongoing research for the past 100 years. Usually, one requires also that the manifold be *differentiable*, and so can be given a *Riemannian metric* which allows using metric techniques and generalized techniques of calculus to study global properties. The literature on these topics is <u>extensive</u>. At UIC, these subjects are developed in the courses Math 446, Math 549, 550, 551, and various Math 569 topics courses.

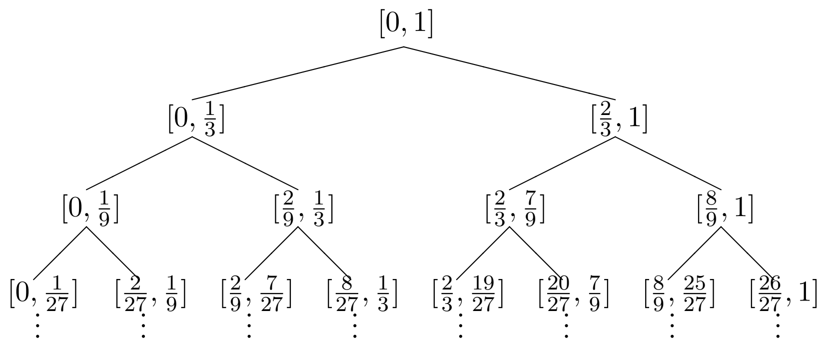
In addition, *dynamical systems* are usually studied as properties of homeomorphisms of a manifold into itself. Part of this research area includes the study of compact invariant subsets of manifolds, which can exhibit a wide variety of pathological behavior.

The conclusion, is that the study of manifolds and the maps between them, have kept very many mathematicians in business for many, many years, and earned researchers the top prizes in mathematics, such as the Fields Medal, the MacArthur Fellowship awards, as well as prestigious awards such as the Sloan Research Fellowships.

The Cantor set $\mathbb{K} \subset [0, 1]$ is a compact set that does not look anything like an interval. It is an important example for many ideas of topology of $\mathbb{R}$.

Actually, there are many "Cantor Sets", all of them homeomorphic. We describe the construction of what is called the "middle thirds" Cantor set, called this for reasons that will be obvious.

This set $\mathbb{K}$ is built by induction. We start with the unit interval $[0, 1]$ and throw out the middle third $(\frac{1}{3}, \frac{2}{3})$. This leaves two closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. We next throw out the middle third of each of those intervals and are left with four closed intervals. We continue this way. We build a tree of intervals that looks like this:



We describe the construction more precisely as follows.

Let S_n be all sequences of zeros and ones of length n. A typical $\sigma \in S_n$ is a finite sequence of zeros and ones of length n, $\sigma = \{i_1, i_2, \dots, i_n\}$. The set $S_0 = \emptyset$; the set S_1 has two elements, $\{0\}$ and $\{1\}$; the set S_2 has four elements, $\{0, 0\}$, $\{1, 0\}$, $\{1, 0\}$, $\{1, 1\}$. In general, S_n has 2^n elements.

For each $\sigma \in S_n$, we define inductively an interval $I_\sigma = I_{i_1,i_2,\dots,i_n}$ of length $\dfrac{1}{3^n}$ as follows. For $n = 0$, $S_0 = \emptyset$ the empty sequence. The corresponding interval is $I_\emptyset = [0, 1]$.

Now, suppose that for $\sigma \in S_n$ we have defined $I_\sigma = [a, b]$, then there are two sequences $\tau \in S_{n+1}$ of length $n + 1$ with initial segment σ, $\tau = \{\sigma, 0\}$ and $\tau = \{\sigma, 1\}$. Then set

$$I_{\sigma,0} = [a, \frac{b-a}{3}] \ \& \ I_{\sigma,1} = [\frac{2(b-a)}{3}, b]$$

Note that if σ is an initial segment of τ, then $I_\tau \subseteq I_\sigma$. That is, if $\sigma = \{i_1, i_2, \dots, i_n\}$ and $\sigma = \{i_1, i_2, \dots, i_n, \dots, i_m\}$, then $I_{i_1,i_2,\dots,i_m} \subseteq I_{i_1,i_2,\dots,i_n}$.

For each $n > 0$, let $\mathbb{K}_n = \displaystyle\bigcup_{\sigma \in S_n} I_\sigma$. The set $\mathbb{K}_n$ is the union of 2^n closed intervals, so is a closed set.

Let U_n be the set of points in $[0, 1]$ that are "thrown out" by stage n: $U_n = [0, 1] - \mathbb{K}_n$. There are 2^n sequences in S_n, including the endpoints of $[0, 1]$, so U_n is the union of $2^n - 1$ open intervals. Moreover, the set $V_n = U_m - U_{n-1}$ is the union of 2^{n-1} disjoint open intervals, each of length 3^{-n}. Note that the 2^{n-1} intervals in the set V_n can be labeled by their order in the interval $[0, 1]$,

$$V_n = V_{n,1} \cup \dots \cup V_{n,2^{n-1}}$$

Let $U = \bigcup_{n>0} U_n$ denote the union of all the open intervals $V_{n,\ell}$, $n > 0$ and $1 \le \ell < 2^n$.

DEFINITION 19.1. $\mathbb{K} = [0,1] - U = \bigcap_{n>0} \mathbb{K}_n$.

Here is the picture of the sets $[0,1]$, and $\mathbb{K}_1$ through $\mathbb{K}_7$:

The infinite intersection $\mathbb{K}$ is called "Cantor's Dust" as it looks like just a cloud of points.

It is easy to see that $0, 1 \in \mathbb{K}$, as these points lie in every $\mathbb{K}_n$. What other numbers are in $\mathbb{K}$? Let S_∞ be the set of all functions from $\mathbb{N}$ to $\{0,1\}$. That is, $\sigma \in S_\infty$ corresponds to an infinite sequence $\{i_1, i_2, \ldots\}$ where each $i_n \in \{0,1\}$. Let $\sigma|n$ be the finite sequence $\{i_1, \ldots, i_n\}$. Then we get an infinite sequence of nested closed intervals,

$$I_{\sigma|0} \supset I_{\sigma|1} \supset I_{\sigma|2} \supset \ldots$$

where each $I_{\sigma|n}$ has length 3^{-n}. By the Descending Chain Condition, Definition 5.5, for compact sets, there is a unique real number $x_\sigma \in [0,1]$ such that $x_\sigma = \bigcap_{n=1}^{\infty} I_{\sigma|n}$. Since $x_\sigma \in I_{\sigma|n}$ for all n, $x_\sigma \in \mathbb{K}_n$ for all n, thus $x_\sigma \in \mathbb{K}$.

PROPOSITION 19.2. *Define $G\colon S_\infty \to \mathbb{K}$ by $G(\sigma) = x_\sigma$. Then G is a bijection.*

Proof. Suppose $\sigma, \tau \in S_\infty$ and $\sigma \ne \tau$. Let n be least such that $\sigma(n) \ne \tau(n)$. Then $I_{\sigma|n} \cap I_{\tau|n} = \emptyset$. Since $x_\sigma \in I_{\sigma|n}$ and $x_\tau \in I_{\tau|n}$, $x_\sigma \ne x_\tau$. Thus G is one to one.

Suppose $x \in \mathbb{K}$. We inductively define $\sigma \in S_\infty$ such that $x = x_\sigma$. Suppose we know $\sigma(1), \ldots, \sigma(n)$ and $x \in I_{\sigma|n}$. Since $x \in \mathbb{K}$, $x \in I_{\sigma|n,i}$ for $i = 0$ or 1. Let $\sigma(n+1) = i$. □

The set S_∞ is isomorphic with the set $2^{\mathbb{N}}$, so is an uncountable set.

The intervals $U_{n,\ell}$ which are deleted from $[0,1]$ to form $\mathbb{K}$ are called the "gaps" in the Cantor set $\mathbb{K}$. These gaps are in the complement of $\mathbb{K}$ in $\mathbb{R}$, and it is an exercise in the definition of $\mathbb{K}$ to see that each gap is an interval (a,b) where $a \in I_\sigma$ and $b \in I_\tau$ for $\sigma, \tau \in S_n$ for some n. These points a, b which are endpoints of gaps are all contained in $\mathbb{K}$. For each n there are only finitely many endpoints of gaps, so the endpoints of gaps form a countable set. These are the "obvious" points in $\mathbb{K}$. For example, when $n = 2$ the endpoints of gaps are the points at the end of the intervals

Since $\mathbb{K}$ is uncountable, there are many more points in $\mathbb{K}$!

However, there is no open, non-empty interval $(a,b) \subset \mathbb{K}$. If $\mathbb{K} \cap (a,b) \ne \emptyset$, then for all $n > 0$, $\mathbb{K}_n \cap (a,b) \ne \emptyset$ and hence there is an interval $I_\sigma \cap (a,b) \ne \emptyset$ for some $\sigma \in S_n$. We can then find some $m > n$ and $\tau \in S_m$ so the subinterval $I_\tau \subset I_\sigma$ with $I_\tau \subset (a,b)$. Then the gaps next to the endpoints of I_τ have non-empty intersection with (a,b). Thus, $(a,b) \ne \emptyset$ implies the intersection $(a,b) \cap (\mathbb{R} - \mathbb{K})$ non-empty.

The Cantor set $\mathbb{K}$ is not connected. Given any gap $U_{n,\ell}$ and point $z \in U_{n,\ell}$ then $\mathbb{K} \subset (-\infty, z) \cup (z, \infty)$. Moreover, since $\mathbb{K}$ contains no intervals, for any open set U if $U \cap \mathbb{K} \neq \emptyset$, then $U \cap \mathbb{K}$ is not connected. The Cantor set is said to "totally disconnected".

There are many variations on the above construction – what matters only is that at each stage, all of the closed intervals get "chopped up", maybe into two equal pieces like the middle third construction, or maybe into a randomly varying collection of closed subintervals. Also, the sets do not have to be in $\mathbb{R}$ – "Cantor sets" exists in every compact connected metric space X. Here is the formal statement:

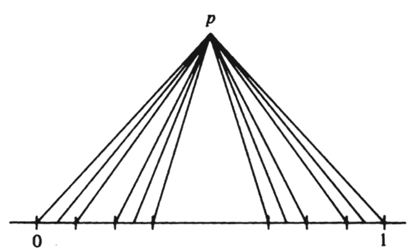
THEOREM 19.3. *Let K be a* compact, perfect, and totally disconnected *topological space. Then there exists a homeomorphism $h\colon K \to \mathbb{K}$, between K and the middle thirds Cantor set $\mathbb{K}$.*

Proof. Exercise! (Hard, but doable. It uses the uniform convergence of sequences of continuous maps on compact metric spaces. The idea is to construct continuous maps $h_n\colon K \to \mathbb{K}$ which are better and better "approximations" to the final map h, so that $h_n \to h$ uniformly.) □

Cantor sets arise very naturally in the study of invariant sets and attractors in dynamical systems, in all dimensions. Yet for all the shapes they may appear as, all of the sets are homeomorphic.

20. Example: Knaster–Kuratowski Fan

This example is also known as the the "Cantor Leaky Tent" or the "Cantor Teepee", for reasons which will be obvious. It is a connected topological space such that the removal of a single point makes it totally disconnected.



Let $\mathbb{K} \subset [0,1]$ be the one-thirds Cantor set constructed in Example 19. Consider $\mathbb{R} \subset \mathbb{R}^2$ embedded as the x-axis, so that the Cantor set lies on the x-axis between the points $(0,0)$ and $(1,0)$.

Let $p = (1/2, 1/2) \in \mathbb{R}^2$.

For each point $c \in \mathbb{K}$ let $L(c) \subset \mathbb{R}^2$ be the line segment between p and the point c in the Cantor set. Some of these lines are pictured above.

Let $L'(c)$ denote the subset of $L(c)$ defined by $L'(c) = \{(x, y \in L(c) \mid y \in \mathbb{Q}\}$.

Let $L''(c)$ denote the subset of $L(c)$ defined by $L''(c) = \{(x, y \in L(c) \mid y \notin \mathbb{Q}\}$.

Let $Q \subset \mathbb{K}$ denote the set of endpoints of the intervals that are deleted in the construction of the Cantor set, together with 0 and 1. Thus, $Q \subset [0,1]$ is a subset of the rational numbers in this interval which can be expressed as $\ell/3^n$ for some positive integer n.

The *Knaster–Kuratowski Fan* $\mathbb{F}$ is defined as the union of all the sets $L'(c)$ for $c \in Q$, together with all sets $L''(c)$ where $c \in \mathbb{K} - Q$. It is given the relative (metric) topology as a subset of $\mathbb{R}^2$.

Then $\mathbb{F}$ is a connected set. It is not locally connected at any point of $\mathbb{F}$. Moreover, $\mathbb{F} - \{p\}$ is totally disconnected. The proofs of these claims are left as exercises.

21. Example: The "Long Line" $\mathcal{L}$

The real line $\mathbb{R}$ is the standard example of a topological space which is "nice". However, there is another version of the line, which is not nice at all! It is called the "Long Line", or perhaps should be called the "Very Long Line" as it is connected, locally path connected, but not path connected.

As a set, $\mathcal{L} = \mathbb{R}^2$. However, the topology on $\mathcal{L}$ is defined using the dictionary ordering on the product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. This results in a very strange topology for $\mathcal{L}$.

Given $(a, b) \in \mathbb{R}^2$ and $(c, d) \in \mathbb{R}^2$ we declare $(a, b) < (c, d)$ if either $a < c$, or if $a = c$ then $b < d$.

Give $\mathcal{L}$ the basis of open sets, which for pairs $(a, b) < (c, d)$, is defined by:

$$U[(a, b), (c, d)] = \{(x, y) \mid (a, b) < (x, y) < (c, d)\}$$

where we allow $b = -\infty$ and $d = \infty$. When $a = c$, we see that this basis of open sets includes all the open subsets $U[(a, b), (a, d)] \subset \{a\} \times \mathbb{R}$ for the standard topology on $\mathbb{R}$.

However, if $a < c$ then a basic open set $U[(a, b), (c, d)]$ contains an uncountable number of copies of the real line, as each line $\{x\} \times \mathbb{R} \subset U[(a, b), (a, d)]$ for $a < x < c$. It is impossible to get from the point (a, c) to the point (b, d) via a continuous path $\sigma \colon [0, 1] \to \mathcal{L}$. The image of σ would then be compact, which is it isn't!

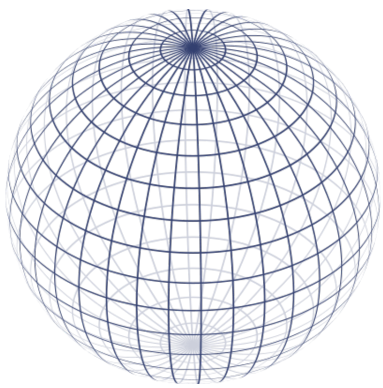
This is the classic example of a topological space which is locally as nice as possible, as every point has an open neighborhood consisting of an intervel in $\mathbb{R}$. But $\mathcal{L}$ is not separable and not second countable. It is however, paracompact.

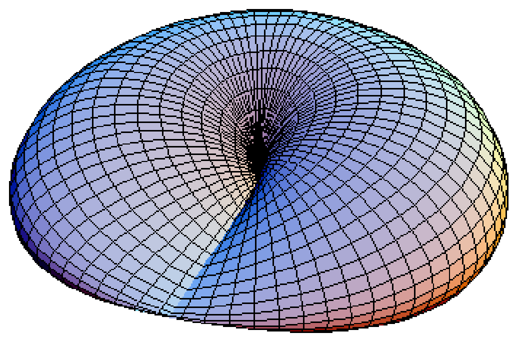
22. Example: 2-dimensional compact manifolds $\mathbb{S}^2$, $\mathbb{P}^2$, $\mathbb{T}^2$, Σ_g

A two-dimensional manifold is called a surface. All surfaces are classified, up to homeomorphism. In the case of the two-dimensional compact manifolds, there are just two invariants needed: the genus, or how many handles the surface has, and whether the surface is oriented or not. Shown below are examples of some of these, as they appear in $\mathbb{R}^3$. Note that if a surface is not orientable, then it cannot be faithfully drawn in $\mathbb{R}^3$, and thus every representation will have crossings.The topology on the surface is the subspace topology induced from the metric topology on $\mathbb{R}$.

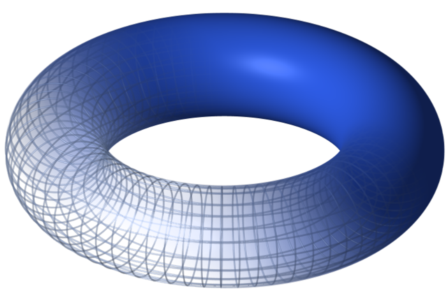
The illustrations below include a hatching by "coordinate lines". These can be thought of as the local patches which are homeomorphic to the open disk (or square) in the plane.

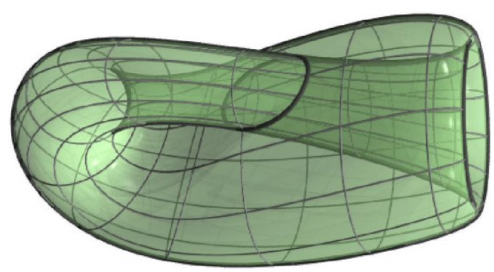
2-Sphere $\mathbb{S}^2$ & *Projective 2-Space* $\mathbb{P}^2$, or *Cross-Cap*



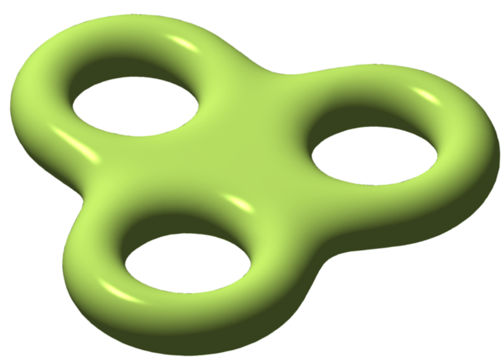


2-Torus $\mathbb{T}^2$ & *Klein Bottle* $\mathbb{P}^2$





Genus 2 *surface* Σ_2 & *genus* 3 *surface* Σ_3



23. Example: Metric spaces of functions

Many interesting examples of topological spaces arise as subsets of some space of functions with metric. We describe a few basic examples of function metric spaces M with metric D.

EXAMPLE 23.1. *Uniform function spaces*

Let X be any set, then $\mathcal{B}(X, \mathbb{R})$ is the collection of all *bounded functions* from X to $\mathbb{R}$. Recall that a function $f: X \to \mathbb{R}$ is *bounded* if there exists some constant C_f such that $|f(x)| \leq C_f$ for all $x \in X$. Bounded does not imply continuous, or place any other restriction on f. For example, given any subset $A \subset X$, the *characteristic function* of A is always bounded,

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ otherwise.} \end{cases}$$

DEFINITION 23.2. *The uniform (or* sup*) metric on $\mathcal{B}(X, \mathbb{R})$ by setting, for $f, g \in \mathcal{B}(X, \mathbb{R})$,*

$$D(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

The "sup norm" on $\mathcal{B}(X, \mathbb{R})$ is defined by setting $\|f\| = \sup_{x \in X} |f(x)|$.

We must show that D is a metric. First, $D(f,f) = 0$, $D(f,g) \geq 0$, and $D(f,g) = D(g,f)$ are all clear from the definitions. Moreover, $D(f,g) = 0$ implies the supremum of $|f(x) - g(x)| \geq 0$ is actually zero, so this must be identically zero. That is, $f = g$. The Triangle Inequality implies that for $f, g, h \in \mathcal{B}(X, \mathbb{R})$ and each $x \in X$, we have $|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$, so

$$\begin{aligned} D(f,h) &= \sup_{x \in X} |f(x) - h(x)| \\ &\leq \sup_{x \in X} \{|f(x) - g(x)| + |g(x) - h(x)|\} \\ &\leq \sup_{x \in X} \{|f(x) - g(x)|\} + \sup_{x \in X} \{|g(x) - h(x)|\} \\ &= D(f,g) + D(g,h) \end{aligned}$$

Let $\{f_n\} \subset \mathcal{B}(X, \mathbb{R})$ be a sequence of functions. Then $f_n \to h \in \mathcal{B}(X, \mathbb{R})$ means that for all $\epsilon > 0$, there exists N_ϵ so that

$$n \geq N_\epsilon \implies \quad h(x) - \epsilon < f_n(x) < h(x) + \epsilon \quad \text{for all } x \in X$$

The sequence of functions f_n are trapped in the *uniform tube* of radius ϵ about the graph of the limit function h.

PROPOSITION 23.3. *The metric space $\mathcal{B}(X, \mathbb{R})$ is complete. That is, given any Cauchy sequence $\{f_n\} \subset \mathcal{B}(X, \mathbb{R})$, there exists $h \in \mathcal{B}(X, \mathbb{R})$ such that $f_n \to h$.*

Proof. Let $\{f_n\}$ be a Cauchy sequence, so that for all $\epsilon > 0$ there exists N_ϵ such that $m, n \geq N_\epsilon$ implies that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in X$. Then for each $x \in X$, the sequence $\{f_n(x)\} \subset \mathbb{R}$ is Cauchy, hence by the completeness of the Reals, it has a limit, which we set to be $h(x) = \lim_{n \to \infty} f_n(x)$.

It is necessary to show that the function h is bounded on X. Let $\epsilon = 1$, and N_1 be the corresponding index so that $m, n \geq N_1$ implies $|f_n(x) - f_{N_1}(x)| < 1$ for all $x \in X$. That is, $f_{N_1}(x) - 1 < f_n(x) < f_{N_1}(x) + 1$ for all $n \geq N_1$ and all $x \in X$. So in the limit, $f_{N_1}(x) - 1 \leq h(x) \leq f_{N_1}(x) + 1$ for each $x \in X$, which shows that $h(x)$ is a bounded function. □

EXAMPLE 23.4. *Bounded continuous functions*

Let X be metric space, then $\mathcal{C}(X,\mathbb{R})$ is the collection of all *bounded continuous functions* from X to $\mathbb{R}$. This is a subset of $\mathcal{B}(X,\mathbb{R})$, and for $f,g \in \mathcal{C}(X,\mathbb{R})$ set

$$D(f,g) \;=\; \sup_{x\in X} \;|f(x)-g(x)|$$

PROPOSITION 23.5. *The metric space $\mathcal{C}(X,\mathbb{R})$ is complete: given a Cauchy sequence $\{f_n\} \subset \mathcal{C}(X,\mathbb{R})$, there exists $h \in \mathcal{C}(X,\mathbb{R})$ such that $f_n \to h$.*

Proof. Let $\{f_n\}$ be a Cauchy sequence, so that for all $\epsilon > 0$ there exists N_ϵ such that $m,n \geq N_\epsilon$. By the proof above, there exists $h \in \mathcal{B}(X,\mathbb{R})$ such that $f_n \to h$ in the uniform norm. We must show that h is a continuous function. This follows from the same argument as used in Math 414, via the "three-epsilon" trick.

Fix $x_0 \in X$, then let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that $D_X(x_0,x) < \delta$ implies that $|h(x_0) - h(x)| < \epsilon$. Let N be sufficiently large such that $D(h,f_N) < \epsilon/3$. That is, $|h(x) - f_N(x)| < \epsilon/3$ for all $x \in X$. By assumption, f_N is a continuous function, so there exists $\delta > 0$ so that $D_X(x_0,x) < \delta$ implies that $|f_N(x_0) - f_N(x)| < \epsilon/3$. Then we have for $D_X(x_0,x) < \delta$,

$$\begin{aligned} |h(x_0) - h_x)| &= |h(x_0) - f_N(x_0) + f_N(x_0) - f_N(x) + f_N(x) - h_x)| \\ &\leq |h(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - h_x)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$ □

The above is a special case of the following:

PROPOSITION 23.6. *Let M be a complete metric space. Then every closed subspace $Z \subset M$ is also a complete metric space.*

Proof. First, note that the metric D_Z on Z is defined by restriction of the metric D_M on M: for $u,v \in Z \subset M$ then $D_Z(u,v) = D_M(u,v)$. In particular, for $u \in Z$ and $\epsilon > 0$, the open metric ball in Z is defined as

$$B_Z(u,\epsilon) \;=\; \{z \in Z \mid D_Z(u,z) < \epsilon\} \;=\; \{z \in Z \mid D_M(u,z) < \epsilon\} \;=\; B_M(u,\epsilon) \cap Z$$

As a consequence, the open sets of Z are just the open sets of M intersected with Z.

Now let $\{u_n\} \subset Z$ be a Cauchy sequence for the metric D_Z. Then $\{u_n\}$ is Cauchy in M, which is complete, hence $u_n \to u_* \in M$. But Z is closed in M so it contains all its limit points. As $\{u_n\} \subset Z$ we conclude $u_* \in Z$ which shows that the Cauchy sequence $\{u_n\}$ converges to some point of Z. Hence, Z is complete. □

Application: Let $A \subset \mathcal{B}(X,\mathbb{R})$ be any subset. Then the closure $\overline{A} \subset \mathcal{B}(X,\mathbb{R})$ is a complete metric space containing A. The space $\overline{A}$ is called the *metric completion* of A.

EXAMPLE 23.7. *L^1-spaces*

Fix an interval $[a,b] \subset \mathbb{R}$. Define the L^1-norm on the space $\mathcal{C}([a,b],\mathbb{R})$ of continuous functions on $[a,b]$, by setting $\|f\|_{(1)} \;=\; \displaystyle\int_a^b \;|f(x)|dx$. Define a metric on $\mathcal{C}([a,b],\mathbb{R})$ by setting

$$D_1(f,g) \;=\; \|f-g\|_{(1)} \;=\; \int_a^b \;|f(x)-g(x)|\;dx$$

Exercise: Check that this is a metric. Why does $\|f\|_{(1)} = 0$ imply $f = 0$?

DEFINITION 24.1. *A* norm *on a vector space* $\mathbb{V}$ *is a linear function* $\|\cdot\|: \mathbb{V} \to \mathbb{R}$ *that satisfies*

(1) $\|\vec{v}\| \geq 0$, *with* $\|\vec{v}\| = 0$ *if and only if* $\vec{v} = 0$ *(non-degenerate)*
(2) $\|\alpha \cdot \vec{v}\| = |\alpha| \cdot \|\vec{v}\|$ *for all* $\alpha \in \mathbb{R}$ *and* $\vec{v} \in \mathbb{V}$ *(scalar linear)*
(3) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ *for all* $\vec{v}, \vec{w} \in \mathbb{V}$ *(Triangle Inequality)*

These are sometimes referred to as "topological vector spaces"

A norm defines a distance function on $\mathbb{V}$ be setting $D_{\mathbb{V}}(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$.

DEFINITION 24.2. *A* Banach Space *is a* complete*, normed vector space* $\mathcal{B}$*.*

Every normed vector space $(\mathbb{V}, \|\cdot\|)$ has a completion $\mathcal{B} = \overline{\mathbb{V}}$ by Theorem 57, page 92 of Kaplansky. The resulting topological vector space is called a "Banach Space". Read more about these space on the wiki links in Appendix C. The completion depends totally on the norm used; in fact, for $\mathbb{V}$ infinite dimensional, it is impossible to classify all of the Banach Spaces obtained via completion. Timothy Gowers, of Cambridge University, UK, received the 1998 Fields Medal for his work on the structure of Banach Spaces associated with number theory.

In this note we will discuss the simplest examples of Banach spaces, the ℓ^p-sequence spaces, obtained by completing the countably infinite-dimensional vector space of sequences for which only a finite number of terms are non-zero. This is denoted by:

$$\ell_f \;=\; \{(a_n) \mid a_n \in \mathbb{R}\ ,\ \exists\, N \text{ such that } a_n = 0 \text{ for } n > N\}$$

EXAMPLE 24.3. *Hilbert Space*

The *Euclidean metric* or ℓ^2- norm, on sequences spaces is given by $\|(a_n)\|_2 \;=\; \{\sum_{n=0}^{\infty} |a_n|^2\}^{1/2}$. The sum is finite, as there are only a finite number of terms a_n which are non-zero. The completion of this example is called ℓ^2-space, and is homeomorphic as topological vector spaces to the space defined by

$$\ell^2 \;\equiv \{(a_n) \mid a_n \in \mathbb{R}\ ,\ \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

The Cauchy–Bunyakovsky–Schwarz Inequality implies that the norm $\|(a_n)\|_2$ is obtained from an inner product on ℓ^2, defined for $(a_n), (b_n) \in \ell^2$ by

$$\langle (a_n), (b_n) \rangle \;=\; \sum_{n=0}^{\infty} a_n \cdot b_n$$

DEFINITION 24.4. *A (real) Hilbert Space is a complete normed vector space* $(\mathcal{H}, \|\cdot\|)$ *with an inner product* $\langle \cdot, \cdot \rangle: \mathcal{H} \times \mathcal{H} \to \mathbb{R}$*, such that* $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ *for all* $\vec{v} \in \mathcal{H}$*.*

One of the standard results of Real Analysis is that this is the "only" example, as contrasted with the case of Banach spaces.

THEOREM 24.5. *Let* $\mathcal{H}$ *be an infinite-dimensional, separable Hilbert Space. Then there is a linear isometric isomorphism* $\Phi: \mathcal{H} \to \ell^2$*.*

EXAMPLE 24.6. *Lebesgue ℓ^p spaces, for $1 \leq p \leq \infty$*

Let $p \geq 1$. Define the ℓ^p- norm on finite sequences spaces by $\|(a_n)\|_p \ = \ \{\sum_{n=0}^{\infty} \ |a_n|^p\}^{1/p}$. The sum is finite as there are only a finite number of terms a_n which are non-zero.

LEMMA 24.7. *For $1 \leq p < \infty$, and all $(a_n), (b_n) \in \ell_f$, $\|(a_n + b_n)\|_p \ \leq \|(a_n)\|_p + \|(b_n)\|_p$.* □

Thus, $\|\cdot\|_p$ defines a norm on ℓ_f, and the completion of ℓ_f with respect to this norm is called ℓ^p-space, for $1 \leq p \leq \infty$. The completion is homeomorphic as topological vector spaces to the space defined by

$$\ell^p \ \equiv \{(a_n) \mid a_n \in \mathbb{R} \ , \ \sum_{n=0}^{\infty} \ |a_n|^p < \infty\}$$

THEOREM 24.8. *For $1 \leq p < \infty$, the space ℓ_p is separable. Moreover, there is a linear isometric isomorphism $\Phi \colon \ell_p \cong \ell_q$ if and only if $\frac{1}{p} = \frac{1}{q} = 1$.*

The case for $p = \infty$ is defined using the limiting norm on ℓ_f, defined by $\|(a_n)\|_\infty \ = \ \lim_{p\to\infty} \ \|(a_n)\|_p$.

LEMMA 24.9. *For $(a_n) \in \ell_f$, $\|(a_n)\|_\infty \ = \ \sup\{|a_0|, |a_1|, |a_2|, \ldots\}$. This is the "sup norm" on sequences.*

PROPOSITION 24.10. *The closure of ℓ_f for the norm $\|\cdot\|_\infty$ is the Banach Space*

$$c_0 \ = \ \{(a_n) \mid a_n \in \mathbb{R} \ , \ \lim_{n\to\infty} \ |a_n| = 0\}$$

Proof. Left as an exercise. □

The space of all bounded sequences is also a Banach Space, defined by:

DEFINITION 24.11. $\ell_\infty \ = \ \{(a_n) \mid a_n \in \mathbb{R} \ , \ \sup\{|a_n|\} < \infty\} \equiv \mathcal{B}(\mathbb{N}, \mathbb{R})$ *with the sup-norm* $\|(a_n)\|_\infty = \sup\{|a_n| \mid n = 1, 2, \ldots\}$.

The surprising fact about this space is the following:

THEOREM 24.12. *The vector space ℓ^∞ has uncountable dimension. That is, it does not admit any vector space basis which is countably infinite.*

Even more is true: the topological space ℓ^∞ is not separable. Given any countable subset $C \subset \ell_\infty$ the closure $\overline{C} \subset \ell_\infty$ is a proper closed subspace.

Proof. Construct an uncountable set of non-empty subsets $\{A_\alpha \subset \mathbb{N} \mid \alpha \in \mathcal{A}\}$ with the property $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta \in \mathcal{A}$. Let $\mathcal{S} = \{f_\alpha = \chi_{A_\alpha} \mid \alpha \in \mathcal{A}\}$ be the set of all characteristic functions of the sets A_α. It is easy to show that all the vectors in $\mathcal{S}$ are linearly independent. [Do it - it is actually easy.] Thus, ℓ_∞ has uncountable dimension.

Observe that $\|f_\alpha\|_\infty = 1$ for all $\alpha \in \mathcal{A}$, and $\|f_\alpha - f_\beta\|_\infty = 1$ if $\alpha \neq \beta$. The set $\mathcal{S}$ is an uncountable, discrete subset of ℓ_∞. Suppose that $C \subset \ell_\infty$ is a dense subset. Then for each $\alpha \in \mathcal{A}$ there must exists at least one element of C in the open ball $B(f_\alpha, 1/2) \subset \ell_\infty$ of radius $1/2$ about f_α. For $\alpha \neq \beta$ we have $B(f_\alpha, 1/2) \cap B(f_\beta, 1/2) = \emptyset$, so the points from C is distinct open balls as these must all be distinct. Thus, C must contain an subset with the same cardinality as $\mathcal{A}$, so is uncountable. □

DEFINITION 25.1. *A metric space $\{M, D\}$ is* complete *if for every Cauchy sequence $\{x_n\} \subset X$, there exists $x_* \in M$ with $x_n \to x_*$.*

That is, suppose $\{x_n\}$ satisfies the Cauchy criterion: for all $\epsilon > 0$ there exists N_ϵ such that $n, m \geq N_\epsilon$ implies that $D(x_n, x_m) < \epsilon$. Then there exists a limit point $x_* \in M$: for all $\epsilon > 0$ there exists N'_ϵ such that $n \geq N'_\epsilon$ implies that $D(x_n, x_*) < \epsilon$.

Intuitively, completeness means that there are no "holes" in M which can be approached arbitrarily close by a sequence of points in M. The rationals $\mathbb{Q}$ are not complete, while the reals $\mathbb{R}$ are, and the inclusion $\mathbb{Q} \subset \mathbb{R}$ is the basic example of the completion of a metric space.

DEFINITION 25.2. *Let $\{M, D\}$ be a metric space. A map $f \colon M \to M$ is said to be a* contraction *if for all $x \neq y \in M$ we have $D(f(x), f(y)) < D(x, y)$. The map is said to be a* strict contraction *if there exists a constant $0 < c < 1$ such that for all $x, y \in M$ we have $D(f(x), f(y)) \leq c \cdot D(x, y)$.*

The *strength* of the contraction is the upper bound

$$C_f \equiv \sup_{x \neq y} \frac{D(f(x), f(y))}{D(x, y} = 1$$

Notice that the difference between the two notions is that for a strict contraction, the strength $C_f \leq c < 1$, while for a contraction it is possible that the strength $C_f = 1$. For example, the map $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x/3$ is a strict contraction (with $C_f = 1/3$). On the other hand, the map $g \colon [0, 1] \to [0, 1]$ defined by $g(x) = \sin(x)$ is a contraction, as $g'(x) = \cos(x) < 1$ for all $0 < x \leq 1$, but it is not a strict contraction, as for $x = 0$,

$$\lim_{y \to 0} \frac{D(\sin(0), \sin(y))}{D(0, y)} = \frac{|\sin(y) - 0|}{|y - 0|} = \sin'(0) = \cos(0) = 1$$

Contractions and strict contractions arise in many applications - we give two important examples. First we show a basic fact:

LEMMA 25.3. *Let $\{M, D\}$ be a metric space, and $f \colon M \to M$ a contraction. Then f is continuous.*

Proof. We use the ϵ-δ definition to show that f is continuous at every $x \in M$. Let $x \in M$ be given, and $\epsilon > 0$. Set $\delta = \epsilon$. Then for all $y \in M$ with $D(x, y) < \delta$, we have $D(f(x), f(y)) < D(x, y) < \epsilon$. That's it. □

Here is a fundamental result, called the "Contraction Mapping Principle". (This is discussed in section 6.2, page 108, of Kaplansky.)

THEOREM 25.4. *Let $\{M, D\}$ be a non-empty, complete metric space. Let $f \colon M \to M$ be a strict contraction. Then f has a unique fixed-point in M. That is, there exists a* unique *point $x_* \in M$ such that $f(x_*) = x_*$.*

Proof. Let $0 < c < 1$ be the constant in the definition of a strict contraction. First we show existence. Pick any point $x_0 \in M$. Recursively define $x_{n+1} = f(x_n)$, so we have $x_n = f^n(x_0)$.

We claim that the sequence $\{x_n\}$ is Cauchy. The key idea is the observation that for $n > 0$,

$$D(x_n, x_{n+1}) = D(f(x_{n-1}), f(x_n)) \leq c \cdot D(x_{n-1}, x_n) \tag{5}$$

By induction, the equation (5) implies that

$$D(x_n, x_{n+1}) \le c \cdot D(x_{n-1}, x_n) \le c^2 \cdot D(x_{n-2}, x_{n-1}) \le \cdots \le c^n \cdot D(x_0, x_1)$$

Now suppose that we are given $m < n$ so $n = m + k$ for some integer $k > 0$. Then by the Triangle Inequality,

$$\begin{aligned}
D(x_m, x_n) = D(x_m, x_{m+k}) &\le D(x_m, x_{m+1}) + \cdots + D(x_{m+k-1}, x_{m+k}) \\
&\le c^m \cdot D(x_0, x_1) + \cdots + c^{m+k-1} \cdot D(x_0, x_1) \\
&\le \left\{ c^m + \cdots + c^{m+k-1} \right\} \cdot D(x_0, x_1) \\
&\le c^m \left\{ 1 + \cdots + c^{k-1} \right\} \cdot D(x_0, x_1) \\
&= c^m \cdot \frac{1 - c^k}{1 - c} \cdot D(x_0, x_1) \\
&\le \frac{c^m}{1 - c} \cdot D(x_0, x_1)
\end{aligned}$$

Since $0 < c < 1$, given $\epsilon > 0$ we can choose N_ϵ so that $c^m/(1-c) \cdot D(x_0, x_1) < \epsilon$. Then for $n > m \ge N_\epsilon$, we have $D(x_m, x_n) < \epsilon$. Thus $\{x_n\}$ is Cauchy as claimed.

Let $x_* = \lim\limits_{n\to\infty} x_n \in M$, which exists as M is complete. We claim next that $f(x_*) = x_*$

$$f(x_*) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x_*$$

as the function f is continuous. This completes the proof of the existence of a fixed-point.

Uniqueness is very easy to show: Suppose that $x_* \ne y_* \in M$ are both fixed-points for f. Then

$$D(x_*, y_*) = D(f(x_*), f(y_*)) \le c \cdot D(x_*, y_*) < D(x_*, y_*)$$

which is a contradiction. Thus, there can be at most one fixed-point for a contraction. $\square$

There is a standard method to show that a map $f\colon M \to M$ is a contraction, or a strict contraction.

Let $M \subset \mathbb{R}^n$ be a convex subset, and let M has the "induced metric". Recall this means that for all $x, y \in M$, the distance $D(x, y)$ is the Euclidean distance between x and y. Let $\sigma\colon [0,1] \to \mathbb{R}^n$ be a parametrized straight line between x and y, so $\sigma(0) = x$, $\sigma(1) = y$ and the derivative $\vec{v} = \frac{d}{dt}\sigma(t) \in \mathbb{R}^n$ is a constant vector. In other words, $\sigma(t) = x + t \cdot \vec{v}$ where $\vec{v} = y - x$. Then

$$D(x, y) \; = \; \int_0^1 \|\frac{d}{dt}\sigma(t)\| \; dt \; = \; \int_0^1 \|\vec{v}\| \; dt \; = \; \|\vec{v}\|$$

where $\|\vec{v}|$ is the Euclidean norm of the vector $\vec{v}$.

The set M is *convex* if for all $x, y \in M$, the straight line between x and y is contained in M.

Now we make a detour back to multi-variable calculus. Let $f\colon M \to M$ be a continuous map so that the derivative of f exists at every point of M. The derivative of f at a point $u \in M$ is the matrix of partial derivatives, evaluated at u. For example, when $n = 2$ so

$$f(u) = \begin{bmatrix} f_1(u) \\ f_2(u) \end{bmatrix}^{2\times 1}$$

the derivative of f is the 2×2 matrix of partial derivatives

$$Df(u) \; = \; \left[\frac{\partial f}{\partial x}(u)\right]^{2\times 2} \; = \; \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(u) & \frac{\partial f_2}{\partial x_2}(u) \\ \frac{\partial f_2}{\partial x_1}(u) & \frac{\partial f_2}{\partial x_2}(u) \end{bmatrix}$$

The Chain Rule for vector functions then implies, in terms of matrix products, that

$$Df \circ \sigma(t) = Df(\sigma(t)) \cdot D\sigma(t) = \left[\frac{\partial f}{\partial x}(\sigma(t)\right]^{n\times n} \cdot \left[\frac{d\sigma(t)}{dt}\right]^{n\times 1}$$

We need another property of linear algebra, not always covered in Math 320. The *norm* of an $m\times n$ matrix A is defined by, for $\vec{v} \in \mathbb{R}^n$,

$$\|A\| \;=\; \sup_{\vec{v}\neq 0} \frac{\|A\cdot\vec{v}\|}{\|\vec{v}\|}$$

This definition satisfies all of the properties of a norm, including the Triangle Inequality: Suppose that A is an $m \times n$ matrix, and B is an $n \times p$ matrix, then for the $m \times p$ matrix $A \cdot B$ we have

$$\|A \cdot B\| \;\leq\; \|A\| \cdot \|B\|$$

It is a good exercise in matrix properties to check that this is a norm.

The set of $m\times n$ matrices, $\mathrm{Mat}(m, n)$, is a real vector space of dimension mn. The matrix norm then defines a distance function on $\mathrm{Mat}(m, n)$. For $A, B \in \mathrm{Mat}(m, n)$ the distance is $D(A, B) = \|A-B\|$. The vector space is linearly isomorphic to $\mathbb{R}^{mn}$, so this gives another distance function on this space besides the usual Euclidean distance function.

One of the basic facts of Euclidean geometry is that the straight line is the shortest path between any two points. What this means in terms of calculus, is that given $x, y \in \mathbb{R}^n$ and any differentiable path $\tau\colon [0, 1] \to \mathbb{R}$ with $\tau(0) = x$ and $\tau(1) = y$, then

$$D(x, y) \;\leq\; \int_0^1 \left\|\frac{d\tau(t)}{dt}\right\| \, dt$$

We are done with the calculus detour.

PROPOSITION 25.5. *Let $M \subset \mathbb{R}^n$ be a convex subset, and $f\colon M \to M$ a differentiable map. If for every $u \in M$, the matrix norm satisfies $\|Df(u)\| < 1$, then f is a contraction on M.*

If there exists a $0 < c < 1$ such that the stronger condition $\|Df(u)\| \leq c$ holds for every $u \in M$, then f is a strict contraction for the same constant c.

Proof. The proof is an application of the Chain Rule and the Triangle Inequality for matrix norms.

Let $x \neq y \in M$ and let $\sigma(t) = x + t \cdot \vec{v}$ where $\vec{v} = y - x$ be the linear path between then. The assumption that M is convex implies that $\sigma(t) \in M$ for all $0 \leq t \leq 1$. So for all $0 \leq t \leq 1$ the composition $f \circ \sigma(t) = f(\sigma(t))$ is defined. Let $tau = f \circ \sigma\colon [0, 1] \to M$ obtain by the composition. Then $\tau(0) = f(x)$ and $\tau(1) = f(y)$. Combine the previous remarks and observations to obtain

$$\begin{aligned}
D(f(x), f(y)) \;&\leq\; \int_0^1 \left\|\frac{d\tau(t)}{dt}\right\| \, dt \quad \text{[Shortest Path]}\\
&=\; \int_0^1 \|Df(\sigma(t)) \cdot D\sigma(t)\| \; dt \quad \text{[Chain Rule]}\\
&\leq\; \int_0^1 \|Df(\sigma(t))\| \cdot \|D\sigma(t)\| \; dt \quad \text{[Triangle Inequality]}\\
&<\; \int_0^1 \|D\sigma(t)\| \; dt \quad \text{[Assumption } \|Df(u)\| < 1]\\
&=\; D(x, y)
\end{aligned}$$

This completes the proof of the first part of the Proposition. If we are given that $\|Df(u)\| < c$ then in the fourth line above, we can insert the constant c and obtain that f is a strict contraction. □

For example, let $M = (a,b) \subset \mathbb{R}$ and suppose that $f\colon (a,b) \to (a,b)$ is a differentiable function which satisfies $|f'(x)| < 1$ for all $a < x < b$. Then f is a contraction on (a,b).

EXAMPLE 25.6. *Newton's Method*

Suppose that $f\colon (a,b) \to \mathbb{R}$ is given and we know (somehow) that there is some point $a < x_* < b$ with $f(x_*) = 0$. Newton's Method gives a recursive method to finding the approximate value of x_* to whatever accuracy we want – assuming that $f(x)$ has a derivative on the interval $a < x < b$ and satisfies some conditions on this derivative.

Recall the basic strategy: pick any point $a < x_0 < b$ which is "close" to to the sought-after point where $f(x_*) = 0$. In general, it helps to have an idea of what the graph of $y = f(x)$ looks like to make this method work. Set $y_0 = f(x_0)$. If $y_0 = 0$ then we are done, and extremely lucky.

Otherwise, find the tangent line to $y = f(x)$ at $x = x_0$, given by a linear function $h_0(x) = m_0 x + c_0$. Then we solve for the point x_1 such that $h_0(x) = 0$. This is the point where the graph of the tangent line at x_0 intersects the x-axis, and is the first "linear" approximation to x_*. The tangent line is

$$y = h_0(x) = f'(x_0)(x - x_0) + f(x_0) = f'(x_0)x + (f(x_0) - f'(x_0) \cdot x_0)$$

Solve for $h_0(x_1) = 0$ to obtain $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

Obviously, we need $f'(x_0) \neq 0$. If we happen to choose x_0 such that $f'(x_0) = 0$ then we are very unlucky. If this happens, the standard procedure is to chose another candidate for x_0 where this is not true. If we have some idea of where x_* should be, then choose the new x_0 to be about half the distance between the first choice of x_0 and the expected value.

Repeat the process above to obtain the second approximation x_2, and so on. This yields a recursion formula, which given an approximate solution x_n to $f(x) = 0$, the next approximate solution is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We interpret this as an example of the Contraction Mapping Principle. Define the function

$$F(x) = x - \frac{f(x)}{f'(x)}$$

In order for this to be well-defined, we must know that $f'(x) \neq 0$ at all points on some interval $a' < x < b'$ where $a' < x_* < b'$. Assuming that F is well-defined, then $x_n = F^n(x_0)$ defines the sequence of approximate answers. This is Newton's Method, and the amazing thing is that if you write a program to do the recursion, for many functions $y = f(x)$ - especially for polynomials, the sequence does converge to a solution. But not always - that is why computer modeling of this dynamical systems is so interesting.

If we can show that F is a strict contraction for all x in some interval near to the expected solution x_* then the Contraction Mapping Principle implies the sequence converges to the solution of $f(x_*) = 0$. This is the basic idea why the Newton's Method works.

The map $x \mapsto F(x)$ is a contraction if $|F'(x)| < 1$. Let's calculate:

$$F'(x) \; = \; 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} \; = \; \frac{f(x)f''(x)}{f'(x)^2}$$

The condition $|F'(x)| < 1$ becomes $|f(x)f''(x)| < f'(x)^2$.

This looks impossible to verify, and it is, except in very special cases, such as when $f(x) = x^2 - a$. Then $f'(x) = 2x$ and $f'' = 2$. The requirement to be a contraction is that $|(x^2 - a) \cdot 2| < (2x)^2$ or $|x^2 - a| < 2x^2$. If $x^2 > a > 0$, then $|x^2 - a| = x^2 - a$ and so the condition is $x^2 - a < 2x^2$ or $a < x^2$ which is true because that is what we assumed. This means that is choose $x_0 \gg 0$ large enough, then the Newton's Method process $x_n = F^n(x_0)$ always converges to the positive square root of a, as the map F is a contraction.

As arbitrary as the above description seems, it works "in general" which means on an open subset of the line. Where it doesn't work is a starting point into the study of dynamical systems, attracting and repelling points and invariant cycles.

All of the above also applies verbatim if we consider maps $f\colon \mathbb{C} \to \mathbb{C}$ where $f(z)$ is a complex polynomial, such as $f(z) = z^2 - z_0$ for some complex number $z_0 \in \mathbb{C}$. The study of the Newton's Method solutions for these maps has been the source for enormous amount of research in the subject called "complex dynamical systems".

EXAMPLE 25.7. *Existence of solutions to differential equations*

In this example, we apply the Contraction Mapping Principle to the complete space of bounded continuous functions on some closed interval of the line, in order to obtain the proof of the existence of solutions to a differential equation "in small time". Note that this fact is never actually proved in Math 220 - it is just stated. Below is the proof. One nuance of the proof is that as we proceed, the domain of definition must be restricted (or shrunk) in order to obtain a contracting map. Examples show that this is necessary, and not just an artifact of the proof we give.

Suppose that $f(x, y)$ is continuous in x and y and that $\frac{\partial f}{\partial y}$ exists and is continuous on an interval $[x_0 - \delta, x_0 + \delta]$ for some $\delta > 0$. Consider the differential equation $y' = f(x, y)$.

The function $g(x)$ is a solution if $g'(x) = f(x, g(x))$. If g satisfies the initial condition $g(x_0) = y_0$ then, by the Fundamental Theorem of Calculus, it is equivalent that g be continuous and satisfy the integral equation

$$g(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt \tag{6}$$

We will show how to solve all such equations "in theory" using the Contraction Mapping Principle.

For a function g which is continuous on $[x_0 - \delta, x_0 + \delta]$, define a mapping T which takes g to a new function Tg defined by

$$(Tg)(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt$$

The new function Tg is differentiable at every point $x_0 - \delta' \leq x \leq x_0 + \delta'$ by the Fundamental Theorem of Calculus.

Then g is a solution to (6) on the interval $[x_0 - \delta, x_0 + \delta]$ if and only if $Tg = g$. The idea is to use T to construct a sequence of functions $g_n = T^n(g)$ which converge uniformly to a solution to (6) on a smaller domain $[x_0 - \delta', x_0 + \delta']$ for some $0 < \delta' < \delta$. This assumption implies that $[x_0 - \delta', x_0 + \delta'] \subset [x_0 - \delta, x_0 + \delta]$. The proof of convergence uses estimates which depend on bounds on f and $\frac{\partial f}{\partial y}$ to get a contraction estimate, and the shrinking of the domain with the choice of $\delta' < \delta$ is made so that map is a strict contraction in the corresponding metric space of continuous functions.

THEOREM 25.8. *Assume $f(x,y)$ is continuous, and there are constants $B, M > 0$ which are upper bounds:*

$$|f(x,y)| \leq B \quad \& \quad \left|\frac{\partial f}{\partial y}(x,y)\right| \leq M$$

for points (x,y) in the rectangle $[x_0 - \delta, x_0 + \delta] \times [y_0 - \epsilon, y_0 + \epsilon]$. Choose $0 < \delta' < \delta$ such that $\delta' \cdot M \leq c < 1$ and $\epsilon' = \delta' \cdot B < \epsilon$. The new rectangle satisfies

$$[x_0 - \delta', x_0 + \delta'] \times [y_0 - \epsilon', y_0 + \epsilon'] \subset [x_0 - \delta, x_0 + \delta] \times [y_0 - \epsilon, y_0 + \epsilon]$$

Then there is a unique function g defined for x with $|x - x_0| \leq \delta'$ such that $g(x_0) = y_0$ and $g'(x) = f(x, g(x))$.

Proof. Let $\mathcal{F}$ be the set of continuous functions g such that $g(x_0) = y_0$ and, if $|x - x_0| \leq \delta'$, then $|g(x) - y_0| \leq \epsilon'$. Notice that the function $g_0(x) = y_0$ is in $\mathcal{F}$. Also notice that if g lies in $\mathcal{F}$ and $|x - x_0| \leq \delta'$, then by the choice of the constant B, we have

$$|(Tg)(x) - y_0| \leq \int_{x_0}^{x} |f(t, g(t))|\, dt \leq |x - x_0| \cdot B \leq \delta' \cdot B = \epsilon'$$

so Tg lies in $\mathcal{F}$. That is, we have defined a mapping $T\colon \mathcal{F} \to \mathcal{F}$. A fixed-point $g_* \in \mathcal{F}$ for this mapping, $Tg_* = g_*$, is a solution of the differential equation on the interval $x_0 - \delta' \leq x \leq x_0 + \delta'$.

We claim that the sequence g_n defined inductively by

$$g_0(x) = y_0 \text{ and } g_n = Tg_{n-1} = T^n g_0$$

converges uniformly to a fixed-point g_*.

For a function g in $\mathcal{F}$ define the norm $||g|| = \sup\{|g(x)| \mid |x - x_0| \leq \delta'\}$. With this norm, the metric space $\mathcal{F}$ is complete by our Theorem in class. We claim that $T\colon \mathcal{F} \to \mathcal{F}$ is a strict contraction.

Let $y_1, y_2 \in [y_0 - \epsilon', y_0 + \epsilon']$. Then for all $x_0 - \delta' \leq x \leq x_0 + \delta'$, by the Mean Value Theorem,

$$|f(x, y_2) - f(x, y_1)| = |frac\partial f\partial y(x, \eta)(y_2 - y_1)| \leq M \cdot |y_2 - y_1| \tag{7}$$

Hence, if g and h are in $\mathcal{F}$ and $|x - x_0| \leq \delta'$, then

$$\begin{aligned} |(Tg)(x) - (Th)(x)| &\leq \int_{x_0}^{x} |f(t, g(t)) - f(t, h(t))|\, dt \\ &\leq M \int_{x_0}^{x} |g(t) - h(t)|\, dt \\ &\leq M \cdot \delta' \cdot ||g - h|| \end{aligned}$$

Thus, by the choice of δ' so that $M \cdot \delta'$, we have $||Tg - Th|| \leq c \cdot ||g - h||$.

We can now apply the Contraction Mapping Principle. The sequence of differentiable functions $g_n = T^n g_0$ converges uniformly to a limiting continuous function $g_* \in \mathcal{F}$ which satisfies $Tg_* = g_*$.

Now since $g_* = Tg_*$ is a function in the image of the map T, the function g_* is also differentiable.

Finally, since T is a contraction, the function g_* must be the unique solution to the equation $Tg = g$. But this means that g_* is the unique solution to the integral equation (6) hence is the unique solution to the differential equation $g'(x) = f(x, g(x))$. $\square$

There are many other applications of the Contracting Mapping Principles which are used in all areas of Pure & Applied Mathematics, especially in the study of Dynamical Systems, or any topic where one looks for solutions of equations by an iteration method.

Math 445, Fall 2009 Exercise Set #1 Turn in: September 4

1. [#10, page 9] Call a subset B of a set A *cofinite* if the complement of B in A is finite. If B and C are cofinite subsets of A, prove that $B \cap C$ is cofinite.

2. [#2, page 13] Let L be a partially ordered set in which *every* subset has a top and bottom element. Prove that L is a finite chain.

3. [#3, page 13] Let $\mathbb{N}$ be the chain of positive integers, in its usual order. Is $\mathbb{N}$ complete? Is $\mathbb{N}$ complete if $\omega =$ "∞" is placed on top?

4. [#4, page 13] Let $\mathbb{N}$ be the set of positive integers and define "$m \leq n$" to mean that m divides n. Is $\mathbb{N}$ a lattice? Is it complete? If not, how could we make it complete?

5. [#9, page 13] Let L be a distributive lattice with a top element "1" and a bottom element "0". (Recall this means that $0 \leq a \leq 1$ for all $a \in L$.)

Prove: If an element $a \in L$ has a complement, then the complement a' is unique. (Recall that $b \in L$ is a *complement* for a, if b satisfies $a \cup b = 1$ and $a \cap b = 0$.)

6. [#7, page 17] Given a function $f\colon A \to A$, we write f^n for the function on A obtained by taking the composite of f with itself n times. Suppose that f^n equals the identity function for some n (one then says that f is *periodic*.) Prove that such f is one-to-one and onto.

7. [#8, page 17] As a generalization of periodic functions, we say that $f\colon A \to A$ is *locally periodic* if for every $x \in A$ there exists an integer $n(x) \geq 1$, depending on x, such that $f^{n(x)}(x) = x$. Prove that a locally periodic function is one-to-one and onto.

8. [#14, page 18] Fix a set A. For a subset $S \subset A$, the characteristic function ϕ_S of S is the function from A to the set $\{0, 1\}$ which takes the value 1 on every element of S, and the value 0 on every element of the complement $S' = A - S$. Prove, for subsets $S, T \subset A$:

(1) $\phi_{S \cap T} = \phi_S \cdot \phi_T$ (the product of the two functions)
(2) $\phi_{S'} = 1 - \phi_S$
(3) $\phi_S + \phi_T = \phi_{S \cup T} - \phi_{S \cup T}$

Optional [#11–16, page 14] These are "*" problems, and will likely require some thought to show. But the results are considered interesting problems about chains in partially ordered sets, so worth spending the extra time on. They are not required, as not even sure how hard the solutions might be; but will read and mark your solutions if you turn them in.

Math 445, Fall 2009 | Exercise Set #2 | Turn in: September 11

1. [#1, page 26] Let A be a countable set and suppose there exists a function $f\colon A \to B$ which is surjective. Prove that B is also countable. (Recall that a set is *countable* if there is a bijection with the set of natural numbers $\mathbb{N}$.)

2. [#4, page 26] Show that the set $\mathbb{N}$ of natural numbers can be represented as a union $\mathbb{N} = \cup A_i$ of an infinite number of disjoint *infinite* sets.

3. [#10, page 27] Let A be an infinite set, $B \subset A$ a finite subset, and $C = A - B$ the complement of B in A. Prove there exists a one-to-one correspondence between A and C.

4. [#11*, page 27] Let A be an uncountable set, $B \subset A$ a countable subset, and $C = A - B$ the complement of B in A. Prove there exists a one-to-one correspondence between A and C.

5. [#1, page 31] Prove that the set of positive real numbers has cardinal c. (Recall that c is the cardinal of the real number line $\mathbb{R}$.)

6. [#5, page 31] What is the cardinal number of the set of irrational numbers? Of the set of transcendental real numbers? (Recall that a real number is *transcendental* if it is not algebraic. A real number is *algebraic* if it is the solution of a non-trivial polynomial equation with integer coefficients.)

7. [#2, page 39] Let L be a lattice in which every chain has an upper bound. Prove that L has a unique maximal element; that is, a top element. (You can assume Zorn's Lemma.)

1. [#11, page 70] Let D_1 and D_2 be metrics on a single space M. Which of the following are metrics on M: $D_1 + D_2$, $\max\{D_1, D_2\}$, $\min\{D_1, D_2\}$?

2. [#14, page 71] Let M be a metric space in which the distance function assumes only the values $0, 1, 3$. Define $x \sim y$ to means $D(x, y) \leq 1$. Prove that $\sim$ is an equivalence relation on M. Show also that $\sim$ determines the metric D.

3. [#1, page 74] Let M, D be a metric space. Prove that:

a) For every $x \in M$, the complement $V_x = M - \{x\}$ is open. [Points are closed.]

b) For any set $X \subset M$, then X is the intersection of open sets. [The problem is to find enough open sets. A finite number will not suffice, unless X is itself open.]

2. [#2, page 74] Let $x, y \in M$ be distinct points in a metric space M, D. Prove that there exists disjoint open sets $U, V \subset M$ with $x \in U$ and $y \in V$.

5. [#5, page 74] Let $M = \mathbb{R}$ be the real line, with the metric $D(x, y) = |x - y|$. Prove that there are no isolated points in $\mathbb{R}$. [A point $x \in M$ is *isolated* if there exists an open set U such that $U \cap M = \{x\}$.]

6. [#8, page 74] Let x be a point of as metric space M. Prove that the following two statements are equivalent:

a) x is not isolated.

b) Every neighborhood of x contains an infinite number of points of M.

7. [#9, page 74] Let M be an infinite metric space. Prove that M contains an open set U such that both U and its complement $M - U$ are infinite.

1. [#1, page 78] Let M be a metric space with metric D. Prove that if $\{x_n \mid n = 1, 2, \ldots\} \subset M$ is a sequence which converges to points $x \in M$ and $y \in M$, then $x = y$.

2. [#4, page 78] Given distinct points x and y in a metric space M, prove that there exist open sets U and V such that $x \in U$, $y \in V$, and their closures $\overline{U} \cap \overline{V} = \emptyset$.

3. [#8, page 79] Let M be a metric space with metric D. Prove that the diameter of a set A in M equals the diameter of its closure, $\overline{A}$.

4. [#11*, page 79] Prove that in a metric space, the closure of a countable set has cardinal number at most c. [Recall that c is the cardinal of the continuum $\mathbb{R}$, which equals the cardinal of the power set of the natural numbers, $\mathcal{P}(\mathbb{N})$.]

5. [#12*, page 79] Prove that the following statements are equivalent for a metric space M:

(a) Every subset of M is either open or closed;
(b) At most one point of M is not isolated.

[Hint: Draw an example of a set with exactly one limit point.]

6. [#13*, page 79] Let M be a metric space in which the closure of every open set is open. Prove that M is discrete. That is, show that every point of M is an open set.

7. [#14*, page 79] Prove that in a metric space, every open set is the union of a countable number of closed sets. Deduce from this that every closed set is the intersection of a countable number of open sets.

8. [#16, page 79] Prove that a metric space is discrete if and only if every convergent sequence is ultimately constant.

9. [#17, page 79] Prove that a metric space is discrete if and only if it has no limit points.

10. [#19*, page 79] If a metric space M has only countably many open sets, prove that M is finite.

Math 445, Fall 2009 Exercise Set #5 Turn in: October 9

The following five problems are all related. Each one builds on the previous ones.

1. [#10, page 78] Let $A, B \subset M$ be subsets of a metric space M. Definethe distance between the sets to be

$$D(A, B) = \inf\{D(a, b) \mid a \in A \ , \ b \in B\}$$

a) Suppose that $B = \{x\}$ consists of a single point. Prove that $D(A, B) = 0$ if and only if $x \in \overline{A}$.

b) Give an example in the Euclidean plane of two closed subsets, $A, B \subset \mathbb{R}^2$, such that $A \cap B = \emptyset$ and yet $D(A, B) = 0$. [Hint: the sets A and B cannot be bounded.]

2. [#2, page 82] Let $u \in M$ be a fixed point in a metric space M. The function $f(x) = D(u, x)$ maps M into the real numbers, $f \colon M \to \mathbb{R}$. Prove that f is continuous.

3. [#3, page 82] Let $A \subset M$ be a fixed subset of a metric space M. The function $f(x) = D(A, x)$ maps M into the real numbers, $f \colon M \to \mathbb{R}$. Prove that f is continuous.

4. [#4, page 82] Let $A \subset M$ be a *closed* subset and y a point in a metric space M, with $y \notin A$. Prove that there exists a continuous real-valued function on M which vanishes on A but not at y.

5. Let $A \subset M$ be a subset and y a point in a metric space M. Suppose that $y \notin \overline{A}$. Prove that there exists a continuous real-valued function $f \colon M \to [0, \infty)$ which vanishes on A but not at y.

1. [#1, page 93] Prove that the space $X = \mathcal{B}(\mathbb{N}, \mathbb{R})$ of bounded sequences with the "sup norm" is complete. [This is Example 7, page 119 of Appendix 1 in the text.]

The metric is defined by, for $x = \{x_n\} \in \mathcal{B}(\mathbb{N}, \mathbb{R})$ and $y = \{y_n\} \in \mathcal{B}(\mathbb{N}, \mathbb{R})$

$$D(x, y) \;=\; \sup_{n \in \mathbb{N}} |x_n - y_n|$$

2. [#5, page 93] If every countable closed subset of a metric space M is complete, prove that M is complete.

3. [#6, page 93] Let $\{M, d\}$ be a metric space. If for every $u \in M$ and $\epsilon > 0$, the closed ball $D(u, \epsilon) = \{y \in M \mid d(u, y) \leq \epsilon\}$ is complete, prove that $\{M, d\}$ is complete.

4. [#11, page 93] In the space of real numbers $\mathbb{R}$, give an example of a descending sequence of non-empty closed sets with empty intersection. That is, find $F_1 \supset F_2 \supset \cdots$ where each $F_n \subset \mathbb{R}$ is closed, and $\displaystyle\bigcap_{n=1}^{\infty} F_n \;=\; \emptyset$.

5. [#12, page 93] The following functions are continuous from the real numbers to the real numbers. Which are *uniformly* continuous?

a) $f(x) = x^2$

b) $g(x) = |x|$

c) $h(x) = \dfrac{1}{1 + x^2}$

6. [#13, page 93] Let $\mathbb{R}$ be the real line with the standard metric, $d(x, y) = |x - y|$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function which has derivative $f'(x)$ for every $x \in \mathbb{R}$. Assume there exists $K \geq 0$ such that $|f'(x)| \leq K$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous. [Hint: Mean Value Theorem]

1. Let X be a topological space, and suppose that $A, B \subset X$ are compact subsets. Show that $A \cup B$ is again compact.

2. Let X, Y be topological spaces, $f\colon X \to Y$ a continuous map, and assume that X is *compact* and Y is *Hausdorff*. Show that f is a closed map. That is, if $F \subset X$ is closed, then $f(F) \subset Y$ is closed.

3. [#3, page 103] Let $f\colon X \to Y$ be a continuous one-to-one mapping of a compact metric space X onto a metric space Y. Prove that $f^{-1}\colon Y \to X$ is continuous (and thus f is a homeomorphism.)

4. Show that a topological space X is Hausdorff if and only if the diagonal

$$\Delta(X) = \{(x, x) \mid x \in X\} \subset X \times X$$

is closed for the product topology.

The graph of a function $f\colon X \to Y$ is the set $\mathcal{G}_f = \{(x, f(x)) \mid x \in X\}$.

5. Let X, Y be topological spaces, and suppose that Y is compact Hausdorff.

a) Show that if $f\colon X \to Y$ is continuous, then the graph $\mathcal{G}_f$ of f is closed in $X \times Y$.

b) Show that if the graph $\mathcal{G}_f$ of f is closed, then $f\colon X \to Y$ is continuous.

6. [#5, page 104] Let $A, B \subset X$ be disjoint subsets of a metric space X. Suppose that A is closed, and B is compact. Prove that the distance between A and B is positive. That is, show that

$$\inf \{D(a, b) \mid a \in A,\ b \in B\} > 0$$

7. Let $\{X, d\}$ be a compact metric space. Suppose that $f\colon X \to X$ is a function which satisfies

$$d(f(x), f(y)) \ < \ d(x, y) \text{ for all } x \neq y$$

Prove that f has a unique fixed point.

1. Consider a Hausdorff topological space $T = \{A, \mathcal{T}\}$. Consider the collection of open subsets

$$\mathcal{T}' = \{U \subset A \mid A - U \text{ is compact in } T\} \cup \{\emptyset\}$$

a) Show that $T' = \{A, \mathcal{T}'\}$ is a topological space. (i.e., that $\mathcal{T}'$ satisfies the axioms of a topology.)

b) Show that T' is a compact topological space.

2. Let $X = [0, 2]$ be the closed interval in $\mathbb{R}$, but with a topology $\mathcal{T}$ on X as follows:

$$U \in \mathcal{T} \quad \Longleftrightarrow \quad \text{either } 1 \notin U, \text{ or } (0, 2) \subset U$$

Find the closure of the subset $A = \{\frac{1}{2}\}$ of X.

3. Prove that there is no continuous bijective map $f: \mathbb{S}^1 \to [0, 1]$. [Hint: think connected.]

4. [#1, page 97] Prove that any subspace of a separable metric space is separable.

5. Let X be a locally path connected space. Show that every *open* connected subset of X is path connected.

6. [#3a, page 97] Let X be a separable metric space, and let $h: X \to Y$ be a continuous onto map. Prove that Y is separable.

7. [#3b, page 97] Let X be a complete metric space, and let $h: X \to Y$ be a continuous onto map. Either prove that Y is complete, or give a counter-example.

8. [#4*, page 97] Let M be a metric space. Prove that M is separable if and only if every collection of disjoint open sets of M is countable.

Math 445, Fall 2009 | Midterm Exam | Turn in: October 26

You may use your textbook and classnotes, but no other references.

There are 6 problems. Turn in 5 for grading.

1. Let $\{M, D\}$ be a metric space. Let $A \subset M$ be a subset and suppose $y \in \overline{A}$. Show that every continuous real-valued function $f\colon M \to \mathbb{R}$ which vanishes on A also vanishes at y.

2. Let $\{M, D\}$ be a metric space, and suppose that there exists open sets $U, V \subset M$ such that $M = U \cup V$ and $U \cap V = \emptyset$. Prove that for if $\{x_n\} \subset U$ is a convergent sequence with limit $x_n \to u \in M$, then $u \in U$. *Use only the definitions of an open set and of a convergent sequence in your proof - do not just quote a Theorem.*

3. Let X be an infinite set.

a) Show that there exists two countably-infinite sets $A, B \subset X$ with $A \cap B = \emptyset$.

b) Show that there exists a countable collection $\{A_i \mid i = 1, 2, \ldots\} \subset X$ where each A_i is a countably-infinite set, and $A_i \cap A_j = \emptyset$ for $i \neq j$.

4. Let $\{M, D\}$ be a metric space.

a) Prove that if M is a finite set, then the metric topology on M is discrete.

b) Give an example of a countably-infinite, complete metric space which is discrete.

c) Give an example of a countably-infinite, complete metric space which is not discrete.

5. Let $\{f_n\colon [0,1] \to \mathbb{R} | n = 1, 2, \ldots\}$ be a sequence of continuous real-valued functions. Assume the sequence is monotonically increasing: for each $0 \leq x \leq 1$ we have $f_n(x) \leq f_{n+1}(x)$ for all $n \geq 1$. Suppose that the sequence $\{f_n\}$ converges pointwise to a continuous function $f\colon [0,1] \to \mathbb{R}$. That is, for each $0 \leq x \leq 1$, $f_n(x) \to f(x)$. Prove that the sequence $\{f_n\}$ converges uniformly to f.

6. a) Let $X \subset [0,1] \times [0,1] \subset \mathbb{R}^2$ be a subset of the plane contained in the unit square. Show that if X is infinite, then X has a limit point.

b) Let $X \subset \mathbb{R}^2$ be a subset of the plane. Show that if X is uncountable, then X has a limit point.

1.

2.

3.

4.

5.

6.

7.

8.

Appendix C. Wikipedia & Online References

Topology & Set Theory:

- http://en.wikipedia.org/wiki/Topology [Topology]
- http://en.wikipedia.org/wiki/Point-set_topology [Point-Set Topology]
- http://en.wikipedia.org/wiki/Tychonoff_separation_axiom [Separation Axioms]
- http://en.wikipedia.org/wiki/List_of_general_topology_topics [Topology Glossary]

- http://en.wikipedia.org/wiki/Cantor_set [Cantor Set]
- http://en.wikipedia.org/wiki/Cantor_function [Cantor Function]
- http://en.wikipedia.org/wiki/Hausdorff dimension [Hausdorff Dimension]

- http://en.wikipedia.org/wiki/Set_theory [Set Theory]
- http://en.wikipedia.org/wiki/Descriptive_set_theory [Descriptive Set Theory]
- http://en.wikipedia.org/wiki/Cardinal_number [Cardinal numbers]
- http://en.wikipedia.org/wiki/Ordinal_number [Ordinal numbers]
- http://en.wikipedia.org/wiki/Derived_set_(mathematics) [Derived Set]
- http://en.wikipedia.org/wiki/Howard_ordinal [Howard ordinal]

- http://en.wikipedia.org/wiki/Function_space [Function Spaces]
- http://en.wikipedia.org/wiki/Banach_space [Banach Space]
- http://en.wikipedia.org/wiki/Hilbert_space [Hilbert Spaces]
- http://en.wikipedia.org/wiki/Lp_spaces [L^p-Spaces]

Mathematicians

- http://en.wikipedia.org/wiki/Stefan_Banach [Stefan Banach]
- http://en.wikipedia.org/wiki/Georg_Cantor [Georg Cantor]
- http://en.wikipedia.org/wiki/David_hilbert [David Hilbert]
- http://en.wikipedia.org/wiki/Timothy_Gowers

MacTutor History of Math

- A History of Topology
  http://www-history.mcs.st-and.ac.uk/HistTopics/Topology_in_mathematics.html
- The real numbers: Pythagoras to Stevin
  http://www-history.mcs.st-and.ac.uk/HistTopics/Real_numbers_1.html
- The real numbers: Stevin to Hilbert
  http://www-history.mcs.st-and.ac.uk/HistTopics/Real_numbers_2.html
- The real numbers: Attempts to understand
  http://www-history.mcs.st-and.ac.uk/HistTopics/Real_numbers_3.html

- Stefan Banach
  http://www-history.mcs.st-and.ac.uk/Biographies/Banach.html
- Georg Ferdinand Ludwig Philipp Cantor
  http://www-history.mcs.st-and.ac.uk/Biographies/Cantor.html
- Felix Hausdorff
  http://www-history.mcs.st-and.ac.uk/Biographies/Hausdorff.html
- Kazimierz Kuratowski
  http://www-history.mcs.st-and.ac.uk/Biographies/Kuratowski.html
- Waclaw Sierpinski
  http://www-history.mcs.st-and.ac.uk/Biographies/Sierpinski.html

References

[1] Felix Hausdorff, **Grundzüge der Mengenlehre**, (Foundations of Set Theory), 1914. Reprinted by Chelsea Publishing Company, 1949.

[2] Georg Cantor, **Contributions to the Founding of the Theory of Transfinite Numbers**, Open Court Publishing Company, 1915. Reprinted by Dover Publications, Inc, 1955.

[3] Casimir Kuratowski, **Topology. Vol. I**, Academic Press, New York, 1966 Originally published 1933; new edition, revised and augmented. Translated from the French by J. Jaworowski. Available online at `http://matwbn.icm.edu.pl/kstresc.php?tom=20&wyd=10&jez=`

[4] Casimir Kuratowski, **Topology. Vol. II**, Academic Press, New York, 1968 (Originally published 1933; new edition, revised and augmented. Translated from the French by A. Kirkor.) Available online at `http://matwbn.icm.edu.pl/kstresc.php?tom=21&wyd=10&jez=`

[5] John Kelley, **General Topology**, Graduate Texts in Mathematics, No. 27. Springer-Verlag, New York-Berlin, 1975 (Reprint of original 1955 edition.)

[6] Wacław Sierpiński, **Cardinal and ordinal numbers**, in series Polska Akademia Nauk. Monografie matematyczne, Warszawa [Pa?stwowe Wydawn. Naukowe], 1958.

[7] John Hocking and Gail Young, **Topology**, Addison-Wesley Publishing Co., Inc, 1961. Reprinted by Dover Publications, Inc, 1988.

[8] Martin Tangora, *Connected unions of disconnected sets*, **The American Mathematical Monthly**, Vol. 72:1038–1039, 1965.

[9] James Dugundji, **Topology**, Allyn and Bacon, 1967.

[10] Lynn Steen and J. Arthur Seebach, **Counterexamples in Topology**, Rinehart and Winston, Inc., New York, 1970. Reprinted by Dover Publications, Inc, 1995.

[11] Irving Kaplansky, **Set Theory and Metric Spaces**, 2nd edition, American Mathematical Society, Providence, RI, 2001. (Reprint of original 1971 edition.)

[12] Wilson A. Sutherland, **Introduction to Metric and Topological Spaces**, 2nd edition, Oxford University Press, 2009. (Reprint of original 1975 edition.)

[13] Mary Tiles, **The Philosophy of Set Theory: An Historical Introduction to Cantor's Paradise**, Basil Blackwell, Ltd., Oxford, UK, 1989. Reprinted by Dover Publications, Inc, 2004.

[14] Joseph W. Dauben, **Georg Cantor**, Princeton University Press, Princeton, NJ, 1990.

[15] Stan Wagon, **The Banach-Tarski Paradox**, Cambridge University Press, Cambridge, UK, 1993.

[16] David Foster Wallace, **Everything and More: A Compact History of Infinity**, 2003.

[17] Mohammad Abry, Jan Dijkstra and Jan van Mill, *On one-point connectifications*, **Topology and its Applications**, Vol. 154:725–733, 2007.

[18] David Zitarelli, *Connected sets and the AMS, 1901–1921*, **Notices of the AMS**, Vol. 56:450–458, 2009.

Steven Hurder, Department of Mathematics, University of Illinois at Chicago, 322 SEO (m/c 249), 851 S. Morgan Street, Chicago, IL 60607-7045

E-mail address: `hurder@uic.edu`