CONTEMPORARY MATHEMATICS

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Groups and Model Theory

In Honor of Rüdiger Göbel's 70th Birthday
May 30–June 3, 2011
Conference center "Die Wolfsburg",
Mülheim an der Ruhr, Germany

Lutz Strüngmann
Manfred Droste
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Photograph of Rüdiger Göbel courtesy of Heidi Göbel. Photograph of conference participants courtesy of Lutz Strüngmann.

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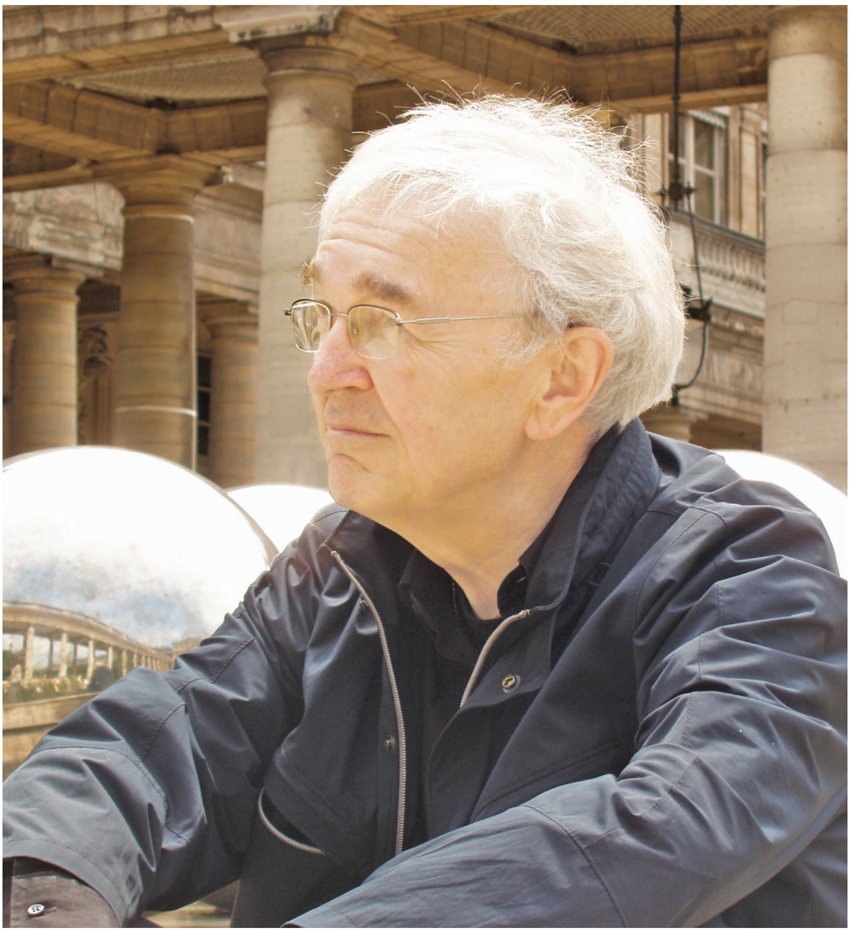
This volume is dedicated to Professor Rüdiger Göbel
on the occasion of his 70^{th} birthday



Rüdiger in the yard of the Grand Palais, Paris 2008

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About the conference

The international conference on Groups and Model Theory was held in honor of Rüdiger Göbel on the occasion of his 70th birthday, at the Catholic academy 'Die Wolfsburg,' Mülheim an der Ruhr (Germany), in June 2011. More than eighty scientists from Australia, the Czech Republic, China, England, Germany, Ireland, Israel, Italy, Russia, Scotland, Spain, Turkey, and the United States of America participated and enjoyed mathematics in the beautiful location.

Group theory and model theory were chosen as the topics of the conference, reflecting Rüdiger's mathematical interests. Little more than three decades have passed since algebra and, in particular group theory, received new impetus from the applications of advanced methods of logic to algebraic questions. In particular, model theory has strongly influenced group theory, both commutative and non-commutative. This led to striking new developments in group theory as a field of application and had an interesting feedback to model theory. Since then this interplay has fruitfully been revisited by algebraists and model theorists alike. It is being studied extensively by a broad community and will surely have a strong and promising future. Rüdiger is one of the pioneers in this interplay between algebra and model theory.

The conference focused on the interdisciplinary relation between these two fields of research. The invited speakers - Bazzoni, Fuchs, Glass, Göbel, Macpherson, Salce, Shelah, Thomas, Trlifaj and Truss - are leading experts in their fields of research. The organizers were especially delighted to have Saharon Shelah as a speaker and his participation was a great asset to the conference. Moreover, several young scientists and students attended the conference and took the opportunity to present their results and to meet leading scientists for discussions on the most recent developments in their fields of interest.

The articles in this volume deal with abelian groups, modules over commutative rings, permutation groups, automorphism groups of homogeneous structures like graphs, relational structures, geometries, topological spaces or groups, consequences of model theoretic properties like stability or categoricity, subgroups of small index, the lattice of normal subgroups and simplicity, the automorphism tower problem, algebraic entropy as well as random constructions. All the articles were refereed according to strict standards.

The conference was supported by generous grants from the German Research Foundation, the Universities of Duisburg-Essen and Münster, the University of Applied Sciences Mannheim; technical support was provided by Enigma Software. We gratefully acknowledge all their support.

The conference was organised by Manfred Droste, Lutz Strüngmann, Katrin Tent, and Martin Ziegler. In particular, we would like to thank Katrin Leistner for the tremendous administrative work she has done and our students Ruth Heselhaus and Malte Heitbrede for their help in organizing a successful conference. Special thanks are due to Heidi Göbel for organizing daily activities for the attending spouses of the participants. .

Manfred Droste, László Fuchs, Lutz Strüngmann, and Katrin Tent

A few words about Rüdiger

Rüdiger Göbel was born on 27th December, 1940 in Fürstenwalde, and spent his early youth in the former German Democratic Republic. His family moved to West Germany, where he studied physics and mathematics at Johann W. Goethe University in Frankfurt am Main. Having found his passion for mathematics and theoretical physics, Rüdiger was heavily influenced and inspired by Reinhold Baer who guided him to his PhD which he received in 1967. They had a close relationship, but with the special flavor of the 'old school.' As Rüdiger told us, even long after he became full professor, they still greeted each other as "Guten Morgen, Herr Professor Baer" — "Guten Morgen, Herr Göbel."

After assistant professorships in Würzburg (Germany) and Austin (Texas) working in relativity theory and general physics, he obtained his Habilitation with a thesis on General Relativity Theory and Group Theory in 1974. In the same year he got a full professorship at the University of Essen (which became later the University of Duisburg-Essen) where he stayed until his recent retirement. Being a very active person, Rüdiger travelled a lot, lecturing at a large number of universities, and giving talks at various conferences. He also held positions as visiting professor in Dortmund, London, Las Cruces, Waco, Jerusalem, and Middletown. He was the main organizer of a number of conferences, co-founder of Forum Mathematicum, and an editor of various proceedings.

It is almost impossible to describe all of Rüdiger's work and contributions to mathematics as well as to physics and computer science; this page has certainly not enough space to do so. We just want to point out that his prolific research activity can be seen from his over 200 papers in prominent international journals (5 of them are still in progress). His research text with Jan Trlifaj has become a classic in the field. In his publications, he solved several open problems in group theory, initiated new theories, and his ideas stimulated a large amount of new research.

Shortly after Rüdiger moved to Essen, he got interested in Abelian Group Theory and, in particular, in applications of set-theoretic and model-theoretic methods to Abelian groups. Soon he became a leading expert in this area and started his fruitful collaboration with Saharon Shelah. Several projects sponsored by the German-Israeli Foundation enabled Rüdiger and Saharon to attack some of the hardest problems in Abelian Group Theory and general Module Theory over commutative rings. Their joint forces led to extraordinary results. His collaboration with more than fifty coauthors produced important results in several other fields as well. He is a hard and careful worker, very knowledgeable and quick, besides being friendly and polite; it is a real pleasure to work with him. To his coauthors and to many researchers world-wide Rüdiger is not just a colleague, but also a close, highly-respected friend. He developed deep friendships with numerous colleagues and the hospitality of the Göbel home to friends, colleagues, visitors, even to students is legendary.

No picture of Rüdiger would be complete without pointing out his exemplary relationship to his students. His lectures had a particular style with partially ordered proofs rather than linearly ordered ones, but always with a lot of enthusiasm and passion in his explanations (we may claim so, since two of the editors were his students). From a student's point of view, he was sometimes very challenging (e.g. when proving the equivalence of Zorn's lemma and the axiom of choice in a first year linear algebra course), but this was his special way of showing the students the beauty of mathematics, in particular, of algebra, and of motivating and even of inspiring them. Rüdiger has always been available for students in a friendly and encouraging manner. He is popular as a research professor as he respects the students, knows how to help them, and is very generous in sharing his ideas. Over the years he attracted many students, and guided them through their diploma theses. He had more than twenty Ph.D. students, six of whom are now full professors.

When he was asked: "What part of your research are you most proud of?", Rüdiger reached back to his roots and answered: "A sentence in a publication by Stephen Hawking thanking me for a seminar that I gave to students at Cambridge University." Seemingly this inspired Hawking (who attended the talk) to write one of his essential papers on Zeeman's conjecture. He was proud that his attempt to make the students understand something on relativity theory inspired a person like Stephen Hawking.

Last, but not least, we should mention that such a successful career would not have been possible without his wife Heidi, whom he married in 1969. Her loving and caring support during the four decades of their marriage and the warmth she showed to the countless visitors to the Göbel family home, played a major role in supporting Rüdiger's extraordinary accomplishments. Rüdiger has always been very close to his daughter, Ines, who has followed her father's footsteps in also studying mathematics. This closeness can be seen from the fact that Rüdiger persuaded his co-authors to adopt the notion of Ines(sential) homomorphisms in Abelian group theory. In recent years Ines has introduced Rüdiger to a new non-academic interest: sailing. No doubt he will bring the same passion to this that he has shown for mathematics!

It is a great pleasure to see that Rüdiger keeps working with a youthful spirit and a great deal of enthusiasm. On the occasion of his 70th birthday all his friends, coauthors and students join us in wishing him good health to continue his activities for many more years to come.

Lieber Rüdiger, wir wünschen Dir alles Gute für die Zukunft!

Katrin, László, Lutz, and Manfred

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Functorial properties of Hom and Ext

Ulrich Albrecht, Simion Breaz, and Phill Schultz

Dedicated to Rüdiger Göbel on the occasion of his 70th birthday

Abstract. The goal of this paper is to present a comprehensive survey of recent results addressing the question for which Abelian groups G the functors $\mathrm{Hom}(G,-), \mathrm{Hom}(-,G), \mathrm{Ext}(G,-)$ or $\mathrm{Ext}(-,G)$ preserve or invert direct products or sums.

1. Introduction

Every right R-module A induces three families of functors, namely the covariant functors $A\otimes_R -$ and $\mathrm{Hom}_R(A,-)$ together with their induced functors $\mathrm{Tor}_n^R(A,-)$ and $\mathrm{Ext}_R^n(A,-)$, and the contravariant functor $\mathrm{Hom}_R(-,A)$ with its derived functors $\mathrm{Ext}_R^n(-,A)$. Being additive, each of these functors commutes with finite direct sums. Although their behavior with respect to infinite sums and products has given rise to many interesting classes of modules, a comprehensive survey of this important topic has so far been absent from the literature. It is the goal of this paper to present such a survey with a special emphasis on the functors Hom and Ext in the context of Abelian groups.

Although we are mostly interested in the functors Hom and Ext, we begin our discussion in a more general setting by considering an additive functor $\mathcal{F} = \mathcal{F}_A$ which is induced by a bifunctor. While any reference to A is usually suppressed, we nevertheless want to mention that the maps constructed in the following are natural with respect to A. For a class $\mathcal{C}$ of R-modules, choose modules $\{N_i\}_{i\in I}$ from $\mathcal{C}$ and consider the modules $\mathcal{F}(\oplus_I N_i)$, $\oplus_I \mathcal{F}(N_i)$, $\mathcal{F}(\Pi_I N_i)$, and $\Pi_I\mathcal{F}(N_i)$. The universal properties of direct sums and products induce natural homomorphisms between these objects. In Section 2, we define whether $\mathcal{F}$ preserves or inverts direct sums (products) with respect to the class $\mathcal{C}$ by specifying which of these maps are isomorphisms (Proposition 2.1 and Proposition 2.4). If no reference to $\mathcal{C}$ is given, then $\mathcal{C} = \mathcal{M}_R$, the class of all right R–modules. Since $\mathcal{F} = \mathcal{F}_A$, the class $\mathcal{C} = \{A\}$ merits special attention. In this case, we say that $\mathcal{F}$ *preserves (inverts) self–sums (self–products).*

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It is well-known that $\mathrm{Hom}_R(A,-)$ and $\mathrm{Ext}^n_R(A,-)$ preserve direct products since the induced maps $\mathrm{Ext}^n_R(A,\Pi_I N_i) \to \Pi_I \mathrm{Ext}^n_R(A,N_i)$ are natural isomorphisms for all $n<\omega$ [**CE56**]. Similarly, the contravariant functors $\mathrm{Hom}_R(-,A)$ and $\mathrm{Ext}^n_R(-,A)$ invert direct sums. However, if a functor $\mathcal{F}$ already has one of these two properties, then the requirement that it has an additional preservation or inversion property for families from a class $\mathcal{C}$ often yields $\mathcal{F}(\mathcal{C})=0$. Nevertheless, the cases when $\mathcal{F}$ does not operate trivially on $\mathcal{C}$ give rise to some important classes $\mathcal{C}$ of modules, for instance the *slender* modules, i.e. those modules A for which $\mathrm{Hom}_R(-,A)$ inverts products, and the *(self–)small* modules, i.e., those for which $\mathrm{Hom}_R(A,-)$ preserves sums (of copies of A). Hence, $\mathrm{Hom}_R(A,-)$ is a covariant functor which preserves direct sums and direct products whenever A is small. While every finitely generated module is small, there are rings for which one can find small modules which are not finitely generated [**EGT97**]. Although this does not occur in the case of Abelian groups (Corollary 4.2), the mixed self-small groups are one of the most widely studied classes of Abelian groups. Section 4 presents further properties of these groups continuing the investigation of self-small groups started in [**AM75**], [**AGW95**] and [**ABW09**]. Finally, the ideas of Proposition 2.1 are not limited to Hom and Ext functors. Using the universal properties of tensor products it is easy to see that $A\otimes_R -$ preserves direct sums. However, the functor $A\otimes_R -$ preserves direct products if and only if the right R-module A is finitely presented [**Go76**].

In the remaining part of the paper, we apply these general results to the special case where $\mathcal{M}_R$ is the category $\mathcal{A}$ of Abelian groups and $\mathcal{F}$ is one of the covariant functors $\mathrm{Hom}(A,-)$ or $\mathrm{Ext}(A,-)$ or one of the contravariant functors $\mathrm{Hom}(-,A)$ or $\mathrm{Ext}(-,A)$. The main results of Section 3 give necessary and sufficient conditions on Abelian groups A for $\mathrm{Hom}(A,-)$ or $\mathrm{Ext}(A,-)$ to invert sums and for $\mathrm{Hom}(-,A)$ or $\mathrm{Ext}(-,A)$ to preserve products from some classes $\mathcal{C}$. In particular, $\mathrm{Hom}(-,A)$ inverts all products from $\mathcal{A}$ if and only if A is strongly slender. However, there are cases in which $\mathrm{Hom}(-,A)$ inverts products from a class $\mathcal{C}$ without A being slender. The investigation of the corresponding properties for $\mathrm{Ext}(-,A)$ leads to the new class $\mathcal{C}$*–coextendible* groups whose study is similar to the study of cotorsion theories and their generalizations [**Sa79**], [**Str02**], [**StrW01**]. Section 5 investigates $\mathcal{C}$*-extendible* groups, which are the groups A for which $\mathrm{Ext}(A,-)$ preserves direct sums. We show that the structure of extendible and self-extendible groups is trivial in the sense that a group is extendible if and only if it is a direct sum of a finitely generated group and a free group. We also prove a similar structure theorem for self-extendible groups (Theorem 5.3).

To conclude this introduction, we want to mention another interesting line of inquiry, which we cannot consider within the framework of this paper: for which modules (Abelian groups) are the natural maps discussed in Section 2 either monomorphisms or epimorphisms? Obtaining answers to these questions would be particularly interesting for the Ext-functors. For tensor products, the former question yields the interesting class of Mittag-Leffler modules, while the latter leads to the class of finitely generated modules.

2. Functors Acting on Sums and Products

Let $\mathcal{C}$ be a non–empty class of R-modules; and consider a non-empty family $\mathcal{N} = \{N_i\}_I$ of modules from $\mathcal{C}$. The symbol f_i denotes the canonical embedding of N_i into the i^{th}-coordinate of $\oplus_{j\in I} N_j$, while f'_i indicates the corresponding embedding into $\Pi_{j\in I} N_j$. Similarly, the symbols $g_i : \oplus_{j\in I} N_j \to N_i$ and $g'_i : \Pi_{j\in I} N_j \to N_i$ denote the canonical projections onto the i^{th}-coordinate. Finally, $\iota_{\mathcal{N}} : \oplus_{j\in I} N_j \to \Pi_{j\in I} N_j$ is the canonical embedding.

We begin our discussion with the case that $\mathcal{F}$ is a covariant functor defined on $\mathcal{M}_R$. Let $\phi_i : \mathcal{F}(N_i) \to \oplus_{j\in I}\mathcal{F}(N_j)$ and $\phi'_i : \mathcal{F}(N_i) \to \prod_{j\in I} \mathcal{F}(N_j)$ denote the canonical injections, while $\gamma_i : \oplus_{j\in I}\mathcal{F}(N_j) \to \mathcal{F}(N_i)$ and $\gamma'_i : \Pi_{j\in I}\mathcal{F}(N_j) \to \mathcal{F}(N_i)$ are the canonical projections. Using the universal properties of sums and products, we obtain natural homomorphisms

$$\Phi_{\mathcal{N}} :\ \oplus_I \mathcal{F}(N_i) \to \mathcal{F}(\oplus_I N_i) \text{ and } \Phi'_{\mathcal{N}} : \oplus_{i\in I}\mathcal{F}(N_i) \to \mathcal{F}(\Pi_I N_i)$$

induced by $\mathcal{F}(f_i) : \mathcal{F}(N_i) \to \mathcal{F}(\oplus_{j\in I} N_j)$ and $\mathcal{F}(f'_i) : \mathcal{F}(N_i) \to \mathcal{F}(\Pi_{j\in I} N_j)$. Similarly, the maps $\mathcal{F}(g'_i) : \mathcal{F}(\Pi_{j\in I} N_j) \to \mathcal{F}(N_i)$ and $\mathcal{F}(g_i) : \mathcal{F}(\oplus_{j\in I} N_j) \to \mathcal{F}(N_i)$ induce natural homomorphisms

$$\Gamma'_{\mathcal{N}} : \mathcal{F}(\prod_I N_i) \to \prod_{i\in I} \mathcal{F}(N_i) \text{ and } \Gamma_{\mathcal{N}} : \mathcal{F}(\oplus_I N_i) \to \prod_I \mathcal{F}(N_i).$$

Proposition 2.1. *These maps fit into the diagram*

$$\begin{tikzcd}
\mathcal{F}(N_i) \arrow[r, "\phi_i"] \arrow[d, equal] & \oplus\mathcal{F}(N_i) \arrow[r, equal] \arrow[d, "\Phi_{\mathcal{N}}"'] & \oplus\mathcal{F}(N_i) \arrow[r, "\gamma_i"] \arrow[d, "\Phi'_{\mathcal{N}}"'] & \mathcal{F}(N_i) \arrow[d, equal] \\
\mathcal{F}(N_i) \arrow[r, "\mathcal{F}(f_i)"] \arrow[d, equal] & \mathcal{F}(\oplus N_i) \arrow[r, "\mathcal{F}(\iota_{\mathcal{N}})"] \arrow[d, "\Gamma_{\mathcal{N}}"'] & \mathcal{F}(\Pi N_i) \arrow[r, "\mathcal{F}(g'_i)"] \arrow[d, "\Gamma'_{\mathcal{N}}"'] & \mathcal{F}(N_i) \arrow[d, equal] \\
\mathcal{F}(N_i) \arrow[r, "\phi'_i"] & \Pi\mathcal{F}(N_i) \arrow[r, equal] & \Pi\mathcal{F}(N_i) \arrow[r, "\gamma'_i"] & \mathcal{F}(N_i)
\end{tikzcd}$$

where all sums and products are taken over I. Moreover, the following hold:

a) *The compositions of the maps in the top and bottom rows yield the respective identity maps; and the left hand and right hand squares commute.*

b) i) $\Gamma'_{\mathcal{N}}\mathcal{F}(\iota_{\mathcal{N}}) = \Gamma_{\mathcal{N}}$ *and* $\mathcal{F}(\iota_{\mathcal{N}})\Phi_{\mathcal{N}} = \Phi'_{\mathcal{N}}$.
ii) $\Gamma_{\mathcal{N}}\Phi_{\mathcal{N}} = \Gamma'_{\mathcal{N}}\Phi'_{\mathcal{N}} = \iota_{\mathcal{F}(\mathcal{N})}$.

In particular, the whole diagram commutes.

Proof. a) is immediate from the definitions. For b) i), observe that

$$\gamma'_i\Gamma'_{\mathcal{N}}\mathcal{F}(\iota_{\mathcal{N}}) = \mathcal{F}(g'_i)\mathcal{F}(\iota_{\mathcal{N}}) = \mathcal{F}(g'_i\iota_{\mathcal{N}}) = \mathcal{F}(g_i) = \gamma'_i\Gamma_{\mathcal{N}}$$

for all $i \in I$. By the universal property of direct products, $\Gamma'_{\mathcal{N}}\mathcal{F}(\iota_{\mathcal{N}}) = \Gamma_{\mathcal{N}}$.

In the same way, $\mathcal{F}(\iota_{\mathcal{N}})\Phi_{\mathcal{N}}\varphi_i = \mathcal{F}(\iota_{\mathcal{N}})\mathcal{F}(f_i) = \mathcal{F}(\iota_{\mathcal{N}} f_i) = \mathcal{F}(f'_i) = \Phi'_{\mathcal{N}}\phi_i$ for all i. The universal property of direct sums yields $\mathcal{F}(\iota_{\mathcal{N}})\Phi_{\mathcal{N}} = \Phi'_{\mathcal{N}}$.

ii) is established similarly. □

Using this notation, we say that the functor $\mathcal{F}$

a) *preserves the product of* $\mathcal{N}$ if $\Gamma'_{\mathcal{N}}$ is an isomorphism, and *preserves products from* $\mathcal{C}$ if it preserves products of all families $\mathcal{N}$ from a class $\mathcal{C}$,

b) *inverts the sum of* $\mathcal{N}$ if $\Gamma_{\mathcal{N}}$ is an isomorphism, and *inverts sums from* $\mathcal{C}$ if it inverts sums of all families $\mathcal{N}$ from $\mathcal{C}$,
c) *preserves the sum of* $\mathcal{N}$ if $\Phi_{\mathcal{N}}$ is an isomorphism, and *preserves sums from* $\mathcal{C}$ if it preserves sums of all families $\mathcal{N}$ from $\mathcal{C}$, and finally
d) *inverts the product of* $\mathcal{N}$ if $\Phi'_{\mathcal{N}}$ is an isomorphism, and *inverts products from* $\mathcal{C}$ if it inverts products of all families $\mathcal{N}$ from $\mathcal{C}$.

The question arises whether a given functor $\mathcal{F}$ can have more than one of these properties for an infinite family $\mathcal{N}$. We will address it in a series of results, and show that there are only a few non-trivial cases where this can occur.

PROPOSITION 2.2. *The following are equivalent for a covariant functor* $\mathcal{F}$ *which preserves the product of a non–empty family* $\mathcal{N}$ *of modules:*

a) $\mathcal{F}$ *inverts the product of* $\mathcal{N}$.
b) $\mathcal{F}(N_i) = 0$ *for all but finitely many* $N_i \in \mathcal{N}$.

PROOF. If $\mathcal{F}$ inverts the product of $\mathcal{N}$, then $\iota_{\mathcal{F}(\mathcal{N})}$ has to be an isomorphism by the previous lemma. However, this is only possible if $\mathcal{F}(N_i) = 0$ for almost all i.
The converse is obvious. □

For example, $\mathrm{Hom}_R(A, -)$ and $\mathrm{Ext}^n_R(A, -)$ satisfy the hypothesis of Proposition 2.2.

In spite of this result, there may exist an isomorphism $\mathcal{F}(\prod_{i\in I} N_i) \cong \oplus_{i\in I}\mathcal{F}(N_i)$ for an infinite family $\mathcal{N} = (N_i)_{i\in I}$ such that $\mathcal{F}(N_i) \neq 0$ for all i. However, this isomorphism cannot be natural. For instance, we can consider $\mathcal{F} = \mathrm{Hom}(\mathbb{Z}, -)$, and I a countable set, $N_i = \mathbb{Q}$ for all $i \in I$ except for a fixed index i_0 and $N_{i_0} = \mathbb{Q}^\omega$. Then $\mathcal{F}(\prod_{i\in I} N_i) \cong \prod_{i\in I} N_i = Q^\omega \times \mathbb{Q}^\omega \cong \mathbb{Q}^{(2^\omega)} \cong \mathbb{Q}^\omega \oplus \mathbb{Q}^{(\omega)} = \oplus_{i\in I} N_i \cong \oplus_{i\in I}\mathcal{F}(N_i)$.

COROLLARY 2.3. a) *A product preserving covariant functor* $\mathcal{F}$ *inverts products from* $\mathcal{C}$ *if and only if* $\mathcal{F}(N) = 0$ *for all* $N \in \mathcal{C}$.
b) *Let* $\mathcal{F}$ *be a product preserving, covariant functor and* $\mathcal{C}$ *a maximal class such that* $\mathcal{F}$ *inverts products from* $\mathcal{C}$.
 i) *If* $\mathcal{F}$ *is left exact, then* $\mathcal{C} = \mathrm{Ker}(\mathcal{F})$ *and is closed under products and submodules.*
 ii) *If* $\mathcal{F}$ *is right exact, then* $\mathcal{C} = \mathrm{Ker}(\mathcal{F})$ *and is closed under products and epimorphic images.* □

We now turn to the case that $\mathcal{F}$ is a contravariant functor defined on $\mathcal{M}_R$ and $\mathcal{N} = \{N_i : i \in I\}$ is a non–empty family from $\mathcal{M}_R$. The families $\{\mathcal{F}(f_i)\}_I$ and $\{\mathcal{F}(f'_i)\}_I$ of maps induce natural homomorphisms

$$\Delta_{\mathcal{N}} : \mathcal{F}(\oplus_{i\in I} N_i) \to \Pi_I \mathcal{F}(N_i) \text{ and } \Delta'_{\mathcal{N}} : \mathcal{F}(\Pi_{i\in I} N_i) \to \Pi_I \mathcal{F}(N_i)$$

Similarly, the families $\{\mathcal{F}(g_i)\}_I$ and $\{\mathcal{F}(g'_i)\}_I$ induce natural homomorphisms

$$\Psi_{\mathcal{N}} : \oplus_I \mathcal{F}(N_i) \to \mathcal{F}(\oplus_{i\in I} N_i) \text{ and } \Psi'_{\mathcal{N}} : \oplus_I \mathcal{F}(N_i) \to \mathcal{F}(\Pi_I N_i).$$

As in the covariant case, we obtain

PROPOSITION 2.4. *These maps induce the diagram*

$$\begin{array}{ccccccc}
\mathcal{F}(N_i) & \xrightarrow{\phi_i} & \oplus\mathcal{F}(N_i) & = & \oplus\mathcal{F}(N_i) & \xrightarrow{\gamma_i} & \mathcal{F}(N_i) \\
\| & & \downarrow{\Psi'_{\mathcal{N}}} & & \downarrow{\Psi_{\mathcal{N}}} & & \| \\
\mathcal{F}(N_i) & \xrightarrow{\mathcal{F}(f'_i)} & \mathcal{F}(\Pi N_i) & \xrightarrow{\mathcal{F}(\iota_{\mathcal{N}})} & \mathcal{F}(\oplus N_i) & \xrightarrow{\mathcal{F}(f_i)} & \mathcal{F}(N_i) \\
\| & & \downarrow{\Delta'_{\mathcal{N}}} & & \downarrow{\Delta_{\mathcal{N}}} & & \| \\
\mathcal{F}(N_i) & \xrightarrow{\phi'_i} & \Pi\mathcal{F}(N_i) & = & \Pi\mathcal{F}(N_i) & \xrightarrow{\gamma'_i} & \mathcal{F}(N_i)
\end{array}$$

where the sums and products are taken over I. Moreover the following hold:

a) *The compositions of the maps in top and bottom rows yield the identity maps; and the left and right hand squares are commutative.*

b) i) $\Delta'_{\mathcal{N}} = \Delta_{\mathcal{N}}\mathcal{F}(\iota_{\mathcal{N}})$ *and* $\mathcal{F}(\iota_{\mathcal{N}})\Psi'_{\mathcal{N}} = \Psi_{\mathcal{N}}$.

ii) $\Delta_{\mathcal{N}}\Psi_{\mathcal{N}} = \Delta'_{\mathcal{N}}\Psi'_{\mathcal{N}} = \iota_{\mathcal{F}(\mathcal{N})}$.

In particular, the whole diagram commutes.

The proof is similar to the covariant case, and will therefore be omitted. □

As in the covariant case, we say that $\mathcal{F}$ *preserves (inverts) products (sums)* using the vertical maps in the diagram, e.g., $\mathcal{F}$ inverts the sum of $\mathcal{N}$ if $\Delta_{\mathcal{N}}$ is an isomorphism and inverts sums from $\mathcal{C}$ if $\Delta_{\mathcal{N}}$ is an isomorphism for all families $\mathcal{N}$ from $\mathcal{C}$. As in Proposition 2.2, we obtain:

PROPOSITION 2.5. *Let $\mathcal{F}$ be a contravariant functor which inverts the sum of a family $\mathcal{N} = \{N_i : i \in I\}$. Then $\mathcal{F}$ preserves the sum of $\mathcal{N}$ if and only if $\mathcal{F}(N) = 0$ for all but finitely many $N \in \mathcal{N}$.* □

For example, $\mathrm{Hom}_R(-, A)$ and $\mathrm{Ext}^n_R(-, A)$ satisfy the hypothesis of Proposition 2.5.

COROLLARY 2.6. a) *A sum inverting contravariant functor $\mathcal{F}$ preserves sums from $\mathcal{C}$ if and only if $\mathcal{F}(H) = 0$ for all $H \in \mathcal{C}$.*

b) *Let $\mathcal{F}$ be a sum inverting, contravariant functor and $\mathcal{C}$ a maximal class such that $\mathcal{F}$ preserves sums from $\mathcal{C}$.*

i) *If $\mathcal{F}$ is left exact, then $\mathcal{C} = \mathrm{Ker}(\mathcal{F})$ and is closed under direct sums and epimorphic images.*

ii) *If $\mathcal{F}$ is right exact, then $\mathcal{C} = \mathrm{Ker}(\mathcal{F})$ and is closed under direct sums and submodules.* □

Therefore it remains to consider only the following properties for the functor $\mathcal{F}$:

a) $\mathcal{F}$ is a product preserving covariant functor which preserves or inverts direct sums.

b) $\mathcal{F}$ is a sum inverting contravariant functor which preserves or inverts direct products.

3. Hom and Ext

For any Abelian group A, the symbol $T(A)$ means the torsion subgroup of A and $\overline{A} = A/T(A)$. Given a prime p, $T_p(A)$ denotes the p-component of A and $A[p]$

its p-socle. Finally, $A^{(I)} = \oplus_I A$ and $A^I = \Pi_I A$ for all index-sets I. To simplify our notation, sum and product always refer to direct sum and direct product. Except where explicitly stated, we adopt the notation of [**F70**, F73]. In particular, J_p is the group of p-adic integers and $\mathbb{Z}_p$ the localization of $\mathbb{Z}$ at the prime p. We are particular interested in the case that $\mathcal{C}$ is one of the following classes of groups: $\mathcal{A}$, all Abelian groups; $\mathcal{TF}$, the torsion-free groups; $\mathcal{T}$, the torsion groups; $\mathcal{D}$, the divisible groups; $\mathcal{R}$, the reduced groups; and $\mathcal{A}[p^\infty]$, the p-groups.

We now discuss the properties introduced in the last section for the special case that $\mathcal{F}$ is induced by either Hom or Ext. Our first results investigate when the covariant functors $\mathrm{Hom}(A, -)$ and $\mathrm{Ext}(A, -)$ invert sums.

Theorem 3.1. *The following are equivalent for Abelian groups A and G:*

a) $\mathrm{Hom}(A, -)$ *inverts sums of copies of G.*
b) $\mathrm{Hom}(A, G) = 0$.

Proof. *a)* $\Rightarrow$ *b)*: Consider the exact sequence $G^{(\omega)} \overset{\iota}{\rightarrowtail} G^\omega \twoheadrightarrow G^\omega/G^{(\omega)}$. Using the first diagram of Section 2, we obtain that $\mathrm{Hom}(A, \iota)$ is an isomorphism. Consequently, $\phi(A) \subseteq G^{(\omega)}$ for all homomorphisms $\phi : A \to G^\omega$. For a non-zero map $\alpha \in \mathrm{Hom}(A, G)$, define $\beta : A \to G^\omega$ by $\beta(x) = (\alpha_n(x))_{n<\omega}$, where $\alpha_n = \alpha$ for all $n < \omega$. Since $\beta(A) \nsubseteq G^{(\omega)}$, we obtain a contradiction. Hence $\mathrm{Hom}(A, G) = 0$.

b) $\Rightarrow$ *a)* is obvious since every non-zero element from $H^{(\omega)}$ can be mapped via a canonical projection into a non-zero element of H. □

Corollary 3.2. *Let A be an Abelian group.*

a) *The following are equivalent:*
   i) $\mathrm{Hom}(A, -)$ *inverts sums.*
   ii) $\mathrm{Hom}(A, -)$ *inverts sums from $\mathcal{T}$.*
   iii) $\mathrm{Hom}(A, -)$ *inverts sums from $\{\mathbb{Z}(p^\infty)\}$.*
   iv) $\mathrm{Hom}(A, -)$ *inverts self-sums.*
   v) $A = 0$.
b) $\mathrm{Hom}(A, -)$ *inverts sums from $\mathcal{R}$ if and only if A is divisible.*
c) $\mathrm{Hom}(A, -)$ *inverts sums from $\mathcal{TF}$ if and only if A is torsion.* □

We now turn to the situation where the covariant functor $\mathrm{Ext}(A, -)$ inverts direct sums.

Theorem 3.3. [**S11**] *Let A be an Abelian group.*

a) *The following are equivalent for a non-zero bounded p-group B:*
   i) $\mathrm{Ext}(A, -)$ *inverts sums from $\{B\}$.*
   ii) $T_p(A) = 0$.
   iii) $\mathrm{Ext}(A, B) = 0$.
b) *The following are equivalent:*
   i) $\mathrm{Ext}(A, -)$ *inverts sums from $\mathcal{A}[p^\infty]$.*
   ii) $A \otimes \mathbb{Z}_p$ *is free over $\mathbb{Z}_p$.*
   iii) $\mathrm{Ext}(A, \mathcal{A}[p^\infty]) = 0$.
c) *The following are equivalent:*
   i) $\mathrm{Ext}(A, -)$ *inverts sums from $\mathcal{A}$.*
   ii) $\mathrm{Ext}(A, -)$ *inverts sums from $\mathcal{TF}$.*
   iii) $\mathrm{Ext}(A, \mathcal{TF}) = 0$.
   iv) *A is free.* □

The dual question when $\mathrm{Hom}(-, A)$ or $\mathrm{Ext}(-, A)$ preserve products is answered in the following theorems.

THEOREM 3.4. [**Br11(1)**, Theorem 2] *The following are equivalent for an Abelian group A:*

a) $\mathrm{Hom}(-, A)$ *preserves direct products.*
b) $\mathrm{Hom}(-, A)$ *preserves self-products.*
c) $A = 0$. □

It is interesting that the property that $\mathrm{Ext}(-, A)$ preserves products produces similar restrictions:

THEOREM 3.5. [**GP78**, Theorem 3.3, Corollary 5.6] *Let A be an Abelian group.*

a) $\mathrm{Ext}(-, A)$ *preserves products if and only if A is divisible.*
b) *The following are equivalent:*
   i) *A is cotorsion, i.e.,* $\mathrm{Ext}(\mathcal{TF}, A) = \mathrm{Ext}(\mathbb{Q}, A) = 0$.
   ii) $\mathrm{Ext}(-, A)$ *preserves products from $\mathcal{TF}$.*
   iii) $\mathrm{Ext}(-, A)$ *preserves products from $\{\mathbb{Q}\}$.* □

The final results of this section address the question when the functors $\mathrm{Hom}(-, A)$ and $\mathrm{Ext}(-, A)$ invert products. The question for which groups A the functor $\mathrm{Hom}(-, A)$ inverts products from a class $\mathcal{C}$ is closely related to the discussion of (strongly) slender groups (see [**GGK09**]). Since our definitions do not impose any immediate restrictions on the cardinals involved, any group A with this property is strongly slender. However, it is consistent with ZFC that there do not exist any non-zero strongly slender groups [**GGK09**]. On the other hand, all slender groups are strongly slender if there do not exist any measurable cardinals (see [**GT06**, Section 1.4]). To avoid any set-theoretic problems, *we assume that there exist no measurable cardinals.* Then,

THEOREM 3.6. [**F73**, Corollary 94.5 and Ex. 6] *Let A be an Abelian group. The functor* $\mathrm{Hom}(-, A)$ *inverts products from $\mathcal{A}$ if and only if A is slender.* □

Moreover, if $\mathrm{Hom}(-, A)$ inverts products from the class $\{H\}$ (regardless of the underlying set-theoretic assumptions), then the diagram constructed in Proposition 2.4 for contravariant functors yields that $\mathrm{Hom}(\iota, A)$ is a monomorphism where $\iota : H^{(\omega)} \to H^{\omega}$ is the canonical embedding. Therefore $\mathrm{Hom}(H^{\omega}/H^{(\omega)}, A) = 0$. Consequently, the study of the property that $\mathrm{Hom}(-, A)$ inverts products is closely related to the investigation of stout groups [**Go75**]. Furthermore,

THEOREM 3.7. [**GGK09**] *There exists a cotorsion-free Abelian group A which is not slender, but* $\mathrm{Hom}(-, A)$ *inverts self-products.* □

To enter further into the discussion of slender and self-slender groups would be beyond the framework of this paper. We refer to [**GT06**, Section 1.5] for a complete survey on slender Abelian groups. We just make one concluding remark:

THEOREM 3.8. *Let A be an Abelian group.*

a) $\mathrm{Hom}(-, A)$ *inverts products of cotorsion groups if and only if A is cotorsion-free.*
b) $\mathrm{Hom}(-, A)$ *inverts products of p-groups if and only if $T_p(A) = 0$.* □

Surprisingly, the situation is simpler for the functor $\mathrm{Ext}(-, A)$. We say that a group A is *$\mathcal{C}$-coextendible* if $\mathrm{Ext}(-, A)$ inverts products from $\mathcal{C}$.

THEOREM 3.9. [**GP78**] *Let A be an Abelian group:*

a) *A is divisible if and only if it is $\mathcal{A}$-coextendible.*
b) *The following are equivalent for A:*
   i) *A is $\mathcal{TF}$-coextendible.*
   ii) *A is $\{\mathbb{Q}\}$-coextendible.*
   iii) *A is cotorsion.* □

For the torsion case, we have in addition:

THEOREM 3.10. [**S11**, Theorem 2.10] *The following are equivalent for an Abelian group A and a prime p:*

a) *A is $\{\mathbb{Z}(p)\}$-coextendible.*
b) *A is $\{B\}$-coextendible for all non-zero bounded p-groups B.*
c) *A is p-divisible.* □

THEOREM 3.11. [**S11**, Theorem 2.11] *The following are equivalent for an Abelian group A and a prime p:*

a) *A is $\mathcal{A}[p^\infty]$-coextendible.*
b) *A is $\{\mathbb{Z}(p^\infty)\}$-coextendible.*
c) *A is $\{B\}$-coextendible for all unbounded p-groups B.*
d) *A is cotorsion and p-divisible.* □

4. $\mathcal{C}$-small groups

We want to remind the reader that A is *$\mathcal{C}$-small* if $\mathrm{Hom}(A,-)$ preserves sums from $\mathcal{C}$. The main characterization of self-small modules [**AM75**, Proposition 1.1] was extended in [**GMN94**, Proposition 2.1] to Grothendieck categories. Since we are mostly concerned with Abelian groups in this paper, we state it only in this context. Consider an Abelian group A and a class $\mathcal{C}$ of groups. For $X \subseteq A$, let $V_{\mathcal{C}}(X) = \{f \in \mathrm{Hom}(A,\mathcal{C}) \mid f(X) = 0\}$.

THEOREM 4.1. *The following are equivalent for an Abelian group A and a non–empty class $\mathcal{C}$:*

a) *A is $\mathcal{C}$-small.*
b) *For every (strictly) ascending chain of proper subgroups*

$$U_0 \subset U_1 \subset \ldots \subset U_n \subset \ldots$$

of A, there exists $n < \omega$ such that $V_{\mathcal{C}}(U_n) = \{0\}$ or $A \neq \cup_{n<\omega} U_n$. □

COROLLARY 4.2. *The following are equivalent for an Abelian group A:*

a) *A is small.*
b) *A is $\mathcal{T}$-small.*
c) *A is $\{\mathbb{Q}/\mathbb{Z}\}$-small.*
d) *A is not the union of a strictly increasing countable chain of subgroups.*
e) *A is finitely generated.*

PROOF. The implications $a) \Rightarrow b) \Rightarrow c)$ and $d) \Rightarrow e) \Rightarrow a)$ are obvious.

For $c) \Rightarrow d)$, it is enough to observe that $\mathrm{Hom}(N,\mathbb{Q}/\mathbb{Z}) \neq 0$ for every non-zero subgroup N of A. Since every non-zero homomorphism $N \to \mathbb{Q}/\mathbb{Z}$ can be extended to a homomorphism $A \to \mathbb{Q}/\mathbb{Z}$, we obtain $V_{\mathbb{Q}/\mathbb{Z}}(N) \neq 0$ for all $0 \neq N \subseteq A$. □

In the same way, we can establish

COROLLARY 4.3. *The following are equivalent for an Abelian group A:*

a) *A is $\mathcal{TF}$-small.*
b) *A is $\{\mathbb{Q}\}$-small.*
c) *$\overline{A}$ is $\{\mathbb{Q}\}$-small.*
d) *$\overline{A}$ cannot be a union of a strictly increasing countable chain of pure subgroups.*
e) *A has finite torsion-free rank.* □

To discuss smallness with respect to bounded groups, we need the following lemma.

LEMMA 4.4. *Let G be an Abelian group and p a prime. If G/p^kG is infinite for some $0 < k < \omega$, then G/p^nG is infinite for all $0 < n < \omega$, and the groups G/p^nG have the same cardinality for all such n.*

PROOF. Since $p^nG \subseteq p^mG$ whenever $n > m > 0$, there is an exact sequence $p^mG/p^nG \rightarrowtail G/p^nG \twoheadrightarrow G/p^mG$. Therefore, $\kappa_n = |G/p^nG|$ for $n < \omega$ is an increasing chain of cardinals with $\kappa_0 = 0$. In particular, κ_n is infinite for almost all n. Suppose there exists $0 < n < \omega$ such that κ_n is infinite, but $\kappa_{n+1} > \kappa_n$. Then $\kappa_{2n} > \kappa_n$ too. Consider the canonical epimorphism $G/p^{2n}G \to G/p^nG$ defined by $g + p^{2n}G \mapsto g + p^nG$ which has $p^nG/p^{2n}G$ as its kernel. Then, $\kappa_{2n} = |p^nG/p^{2n}G| + \kappa_n$. Comparing cardinalities shows $\kappa_{2n} = |p^nG/p^{2n}G|$. But multiplication by p^n induces an epimorphism $\beta : G/p^nG \to p^nG/p^{2n}G$, $g + p^nG \mapsto p^ng + p^{2n}G$. Hence $\kappa_n \geq |p^nG/p^{2n}G| = \kappa_{2n} > \kappa_n$, a contradiction.

On the other hand, suppose that $\kappa_n < \infty$ for some $n > 0$. Then, choose n in such a way that κ_{n+1} is infinite. Since $n > 0$, we have $2n \geq n + 1$, so that $\kappa_{2n} > \kappa_n$. Arguing as in the last paragraph, we obtain $\kappa_{2n} = |p^nG/p^{2n}G| + \kappa_n$, so $p^nG/p^{2n}G$ is infinite. But using the epimorphism β we obtain that $p^nG/p^{2n}G$ is finite, a contradiction. Therefore the cardinals κ_n are all infinite (and equal to κ_1). □

PROPOSITION 4.5. *The following are equivalent for an Abelian group A and a prime p:*

a) *A is $\{\mathbb{Z}(p)\}$-small.*
b) *If B is a bounded p-group, then A is $\{B\}$-small.*
c) *A/pA is finite.*

PROOF. The equivalence $a) \Leftrightarrow c)$ and the implication $b) \Rightarrow a)$ are obvious.

For $c) \Rightarrow a)$, note that if A/pA is finite, then by Lemma 4.4, A/p^nA is finite for every n. So A is $\{\mathbb{Z}(p)\}$-small. □

Although $\mathbb{Z}(p^\infty)$ is clearly not $\{\mathbb{Z}(p^\infty)\}$-small, we have:

PROPOSITION 4.6. *The following are equivalent for an Abelian group A and a prime p:*

a) *A is $\{\mathbb{Z}(p^\infty)\}$-small.*
b) *If B is a p-group, then A is $\{B\}$-small.*
c) *If F is a full free subgroup of A, then $(A/F)_p$ is finite.*

Under these conditions A has finite torsion-free rank.

PROOF. To see that *a)* $\Leftrightarrow$ *b)* $\Leftrightarrow$ *c)*, it is enough to observe that A is $\{\mathbb{Z}(p^\infty)\}$-small if and only if every torsion epimorphic image of A has the same property, and every kernel corresponding to such image contains a full free subgroup.

For the last statement, suppose that A has infinite torsion–free rank and let F be a full free subgroup of A. Then $F/pF \le A/pF$ so $(A/pF)_p$ is not finite, contradicting c). $\square$

Recall that $\mathcal{T} \cap \mathcal{R}$ is the class of reduced torsion groups. The symbol $\mathbb{P}$ denotes the collection of prime numbers.

COROLLARY 4.7. *The following are equivalent for an Abelian group A:*

a) *A is $\mathcal{T} \cap \mathcal{R}$-small.*
b) *A is $\{\mathbb{Z}(p) \mid p \in \mathbb{P}\}$-small;*
c) *$A = D \oplus H$ where D is a divisible group and H is a reduced $\mathcal{T} \cap \mathcal{R}$-small group.* $\square$

COROLLARY 4.8. *The following are equivalent for a reduced Abelian group A:*

a) *A is $\mathcal{T} \cap \mathcal{R}$-small.*
b) *For every full (free) subgroup F of A, the reduced part of A/F is finite.*
c) *There is a full free subgroup F of A such that the reduced part of A/F is finite.* $\square$

We obtain an important characterization for self-small torsion groups.

PROPOSITION 4.9. [**AM75**, Proposition 3.1] *The following are equivalent for a torsion group T:*

a) *T is small.*
b) *T is $\mathcal{T}$-small.*
c) *T is self-small.*
d) *T is finite.* $\square$

Furthermore, we obtain the following result generalizing [**AM75**, Corollary 1.4]:

PROPOSITION 4.10. *Let A and G be Abelian groups. If* $\mathrm{Hom}(A, G)$ *is countable, then A is $\{G\}$-small.*

PROOF. If A is not $\{G\}$-small, then there exists a chain $U_0 \subset \cdots \subset U_n \subset \cdots$ of proper subgroups of A such that $\cup_{n>0} U_n = A$ and $V_G(U_n) \neq 0$ for all $n > 0$. Then $(V_G(U_n))_{n>0}$ is a descending chain of subgroups in $\mathrm{Hom}(A, G)$. Moreover, we may assume that $V_G(U_n) \neq V_G(U_k)$ whenever $n \neq k$. For every $n > 0$, we fix a morphism $f_n \in V_H(U_n) \setminus V_H(U_{n+1})$. Observe that, for every $x \in A$, there exists $n(x) > 0$ such that $f_n(x) = 0$ for all $n \ge n(x)$. For every $L \subseteq \mathbb{N}$, we can construct a homomorphism $g_L : A \to G$, defined by $g_L = \sum_{n \in L} f_n$. Note that g_L is actually a finite sum because there exists $n(x) > 0$ such that $f_n(x) = 0$ for all $n \ge n(x)$. It is easy to see that for all subsets $L \neq K$ of $\mathbb{N}$, we have $g_L \neq g_K$. Thus, $\mathrm{Hom}(A, G)$ is not countable. $\square$

LEMMA 4.11. *If a countable Abelian group A is $\{G\}$-small then* $\mathrm{Hom}(A, G)$ *is finitely G-cogenerated, i.e., embedded in a finite product of copies of G.*

PROOF. Let $A = \{a_1, ..., a_n, ...\}$ be a countable, $\{G\}$-small Abelian group. For every $n < \omega$, consider the subgroup $A_n = \langle a_1, ..., a_n \rangle$. If A is infinite, then

$(A_n)_{n<\omega}$ is an ascending chain of proper subgroups such that $\cup_{n<\omega} A_n = A$. By Theorem 4.1, there exists $m < \omega$ such that $V_G(A_m) = 0$. Consequently, the map $\alpha : \mathrm{Hom}(A, G) \to G^m$ defined by $\alpha(f) = (f(a_1), ..., f(a_m))$ is one-to-one. □

COROLLARY 4.12. *Let A and G be countable Abelian groups. Then, A is $\{G\}$-small if and only if $\mathrm{Hom}(A, G)$ is countable.* □

If A has finite torsion-free rank, then we can improve some of the previous results. The groups described by the following result are called *quotient divisible*.

COROLLARY 4.13. *([**AW04**], [**FW98**]) The following are equivalent for a reduced Abelian group A of finite torsion-free rank:*

a) *A is $\mathcal{T} \cap \mathcal{R}$-small.*
b) *For every full (free) subgroup F of A, the reduced part of A/F is finite.*
c) *There is a full free subgroup F of A such that the reduced part of A/F is finite.*
d) *There is a finitely generated subgroup F of A such that the reduced part of A/F is finite.*
e) *A is a pure subgroup of a product $\Pi_p M_p$ such that, for every prime p, M_p is a finitely generated p-adic module and A satisfies the 'projection condition': there is a full free subgroup F of A such that, for almost all p, the image of the canonical projection $\pi_p(F)$ generates M_p as a p-adic module.* □

For self-small groups of finite torsion-free rank, we want to mention the following results from [**AM75**, Theorem 3.6], [**AGW95**, Theorem 2.4] and [**ABW09**, Section 3]. Recall that an Abelian group A is *torsion-reduced* if $T(A)$ is reduced. Moreover, the *support* of a mixed group A is $S(A) = \{p \in \mathbb{P} \mid T_p(A) \neq 0\}$.

THEOREM 4.14. *Let A and G be torsion-reduced groups of finite torsion-free rank. The following are equivalent:*

a) *A is $\{G\}$-small.*
b) *If $p \in S(G)$, then $T_p(A)$ is finite, and for every full free subgroup F of A, A/F is p-divisible for almost all $p \in S(G)$.*
c) *If $p \in S(G)$, then $T_p(A)$ is finite, and $\mathrm{Hom}(A, T(G))$ is a torsion group.*

PROOF. *a) ⇒ b)*: Let $p \in S(G)$. Since A is $\{G\}$-small, every epimorphic image of A is {K}-small for all $K \leq G$. Hence A/pA is $\{G[p]\}$-small so by Proposition 4.5, A/pA is finite. Hence $T_p(A)/pT_p(A)$ is finite. Since $T(A)$ is reduced, $T_p(A)$ is finite.

If F is a full free subgroup of A, then the torsion group $A/F = \oplus_p (A/F)_p$ is $\{G\}$-small. Let $P = \{p : p \in S(G)$ and $(A/F)_p$ is not divisible$\}$. Then there are non–zero homomorphisms of $(A/F)_p$ into $T_p(G)$ which induce a homomorphism $f : A \to \oplus_{p \in P} T_p(G)$. Since A is $\{G\}$–small, $f(A)$ is finite, so U is finite, i.e., A/F is p–divisible for almost all $p \in S(G)$. Note that if F_1 and F_2 are two full free subgroups of A such that A/F_1 is p–divisible for almost all $p \in S(G)$, then using the canonical exact sequence $(F_1 + F_2)/F_2 \rightarrowtail G/F_2 \twoheadrightarrow G/(F_1 + F_2)$, we we conclude that F_2 has the same property.

b) ⇒ c): It remains to show that $\mathrm{Hom}(A, T(G))$ is torsion. Let $f : A \to T(G)$ be a homomorphism. If F is a full free subgroup of A, then we may assume that $f(F) = 0$. Hence, f induces a homomorphism $\overline{f} : A/F \to T(A)$.

Write $A/F = \oplus_p (A/F)_p$. If $p \notin P$, then $\overline{f}\,((A/F)_p) = 0$. Hence $f(A) = \overline{f}\,(\oplus\{(A/F)_{p\in P})$ is finite.

c) $\Rightarrow$ *b)* is obvious.

b) $\Rightarrow$ *a)*: If A is not $\{G\}$-small, then there exists a strictly ascending chain $U_0 \subset U_1 \subset \dots$ of subgroups of A such that $\cup_n U_n = A$ and $V_G(U_n) \neq 0$ for all n. As before, $A/F = (\oplus_{p \notin S(G)} (A/F)_p) \oplus D \oplus B$, where D is divisible and B is finite. Let $B = \{x_1 + F, \dots, x_k + F\}$. There is a positive integer m such that $F \cup \{x_1, \dots, x_k\} \subseteq U_m$. Since $V_G(U_m) \neq 0$, there exists a non-zero homomorphism $f : A \to G$ such that $f(U_m) = 0$. Then $f(F + \langle x_1, \dots, x_k\rangle) = 0$. Hence f induces a non-zero homomorphism $A/(F + \langle x_1, \dots, x_k\rangle) \to G$. But, $A/(F + \langle x_1, \dots, x_k\rangle)$ is a p-divisible torsion group for all $p \in S(G)$; and $T(G)$ is reduced; which leads to a contradiction. ☐

For self-small groups, the previous results can be summarized in the following way ([**ABW09**]):

COROLLARY 4.15. *The following are equivalent for an Abelian group A of finite torsion-free rank:*

(a) *A is self-small;*
(b) *Each $T_p(A)$ is finite, and there exists a full free subgroup $F \leq A$ such that A/F is p-divisible for almost all p with $T_p(A) \neq 0$;*
(c) *Each $T_p(A)$ is finite, and* $\mathrm{Hom}(A, T(A))$ *is torsion;*
(d) *$A = B \oplus H$ such that B is a finite group and H is an extension of an Abelian group X by an Abelian group Y with the following properties:*
    (i) *X is a finite rank torsion-free group that is $S(A)$-divisible,*
    (ii) *Y is torsion-free or Y is an $S(A)$-pure subgroup of a direct product of (finitely generated) p-adic modules $\prod_{p\in S(A)} M(p)$ such that Y satisfies the following projection condition: there is a full free subgroup of finite rank $F \leq Y$ such that, for every $p \in S(A)$, the natural projection $\pi'_p(F)$ of F into $M(p)$ generates $M(p)$ as a $\widehat{\mathbb{Z}}_p$- module.*

PROOF. The equivalences *a)* $\Leftrightarrow$ *b)* $\Leftrightarrow$ *c)* are consequences of Theorem 4.14. Note that we need not require that A is torsion-reduced since $\mathbb{Z}(p^\infty)$ is not self-small, so every self-small group is torsion-reduced.

The characterisation d) comes from [**ABW09**, Section 3]. ☐

5. $\mathcal{C}$-extendible groups

In this final section, we consider the case that $\mathrm{Ext}(A, -)$ preserves sums. We call an Abelian group A *$\mathcal{C}$-extendible* if $\mathrm{Ext}(A, -)$ preserves sums from $\mathcal{C}$, and *self-extendible* if it preserves self-sums. The $\mathcal{C}$–extendible groups were described by two of the authors in [**S11**] and [**BS11**]. The following is a summary of these results. We begin the discussion with some well-known examples:

a) Every finitely generated Abelian group is $\mathcal{A}$-extendible [**GT06**, Lemma 3.1.6].
b) If B is a bounded p-group, then $\mathbb{Z}(p^\infty)$ is $\{B\}$-extendible, but not $\mathcal{A}[p^\infty]$-extendible ([**BS11**, Example 6.1 (2)]).

The next theorem determines the $\mathcal{C}$-extendible groups for certain classes $\mathcal{C}$ of torsion groups. In all cases, the property holds trivially.

THEOREM 5.1. [**S11**, Theorem 5.3 and 5.4]

a) *Let B be a non-zero bounded p-group. An Abelian group A is $\{B\}$-extendible if and only if the p-component of A has finite rank.*
b) *The following are equivalent for an Abelian group A:*
   i) *A is $\mathcal{A}[p^\infty]$-extendible.*
   ii) *A is $\mathcal{C}_p$-extendible, where $\mathcal{C}_p$ is the class of cyclic p-groups.*
   iii) *$A = B \oplus K$ where B is a finite p-group and K is an Abelian group with trivial p-component such that $K \otimes \mathbb{Z}_p$ is free over $\mathbb{Z}_p$.* □

Self-extendible groups can be characterized in the following way. Recall that the *nucleus of a torsion-free group A* is the *subring* of $\mathbb{Q}$ generated by $\{\frac{1}{p} \mid A = pA\}$.

THEOREM 5.2. [**ABS11**] *An Abelian group A is self-extendible if and only if $A = D \oplus B \oplus H$ where H is a torsion-free group which is free over its nucleus, D is a divisible torsion group, and B is finite subject to the conditions:*

i) *If $D_p \neq 0$, then $H = pH$ is p-divisible.*
ii) *If $B_p \neq 0$, then the p-rank of D is finite.* □

Returning to the case of extendible groups, we obtain that they are exactly the direct sums of a finite group and a free group. The equivalence of a), b) and d) in the following theorem was shown in [**ABS11**] without using the naturalness of the isomorphism. For the implication *c)* $\Rightarrow$ *d)*, we will use the hypothesis that the natural homomorphism is an isomorphism. The proof uses similar techniques as those used in [**ABS11**], and we include it for the reader's convenience.

THEOREM 5.3. *The following are equivalent for an Abelian group A:*

a) *A is $\mathcal{A}$-extendible.*
b) *A is $\mathcal{TF}$-extendible.*
c) *A is $\mathbb{Z}$-extendible.*
d) *$A = B \oplus H$ where B is a finite group and H is free.*

PROOF. *c)* $\Rightarrow$ *d)*: Since $\mathrm{Ext}(A, \mathbb{Z}^{(\omega)}) \cong \mathrm{Ext}(A, \mathbb{Z})^{(\omega)}$, we obtain that $\mathrm{Ext}(A, \mathbb{Z})^{(\omega)}$ is cotorsion. By [**BaSt09**, Proposition 1.8], $\mathrm{Ext}(A, \mathbb{Z})$ is the direct sum of a bounded group and a divisible group. Since $\mathrm{Ext}(\mathbb{Z}(p^\infty), \mathbb{Z}) \cong J_p$ for all primes p, ([**F70**, Corollary 52.4]), it follows that $T(A)$ is reduced. If A has a direct summand isomorphic to $\mathbb{Z}(p^k)$ for some $k > 0$, then $\mathrm{Ext}(A, \mathbb{Z})$ has the same property. Hence $T(A)$ is bounded. Thus $\mathrm{Ext}(A, \mathbb{Z}^{(\kappa)}) \cong \mathrm{Ext}(T(A), \mathbb{Z}^{(\kappa)}) \oplus \mathrm{Ext}(A/T(A), \mathbb{Z}^{(\kappa)})$ for all cardinals κ and we can use [**S11**, Lemma 5.2] to obtain that $T(A)$ is finite. Therefore it is enough to prove that $\mathbb{Z}$-extendible torsion-free groups are free.

Let H be a $\mathbb{Z}$-extendible torsion-free group. We start with the exact sequence $\mathbb{Z}^{(\omega)} \overset{\varphi}{\rightarrowtail} \mathbb{Z}^\omega \twoheadrightarrow \mathrm{Coker}(\varphi)$. By [**F70**, Corollary 2.2], we know that $\mathrm{Coker}(\varphi)$ is cotorsion. Then, $\mathrm{Ext}(H, \varphi)$ is an epimorphism. Using Lemma 2.1, we obtain that the natural homomorphism $\mathrm{Ext}(H, \mathbb{Z})^{(\omega)} \to \mathrm{Ext}(H, \mathbb{Z})^\omega$ is an epimorphism. This is possible only if $\mathrm{Ext}(H, \mathbb{Z}) = 0$. Since H is $\mathbb{Z}$-extendible, $\mathrm{Ext}(H, F) = 0$ for all free groups. Hence H is free. □

REMARK 5.4. The equivalence of a) and d) was established in [**B11(2)**] for right modules over right hereditary rings using different arguments.

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Rigid abelian groups and the probabilistic method

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Dedicated to Rüdiger Göbel on the occasion of his 70th birthday

Abstract. The construction of torsion-free abelian groups with prescribed endomorphism rings starting with Corner's seminal work (see Corner, 1963) is a well-studied subject in the theory of abelian groups. Usually these construction work by adding elements from a (topological) completion in order to get rid of (kill) unwanted homomorphisms. The critical part is to actually prove that every unwanted homomorphism can be killed by adding a suitable element. We will demonstrate that some of those constructions can be significantly simplified by choosing the elements at random. As a result, the endomorphism ring will be almost surely prescribed, i.e., with probability one.

1. Introduction

The probabilistic method, pioneered by Erdős (see [**8, 9**]) is one of the most powerful tools in combinatorics, theoretical computer science, and other branches of mathematics to show the existence of mathematical objects with prescribed properties. It is a non-constructive method which infers the existence of a mathematical object by showing that *the probability of its existence* is non-zero. Since its early days it has lead to a wide range of striking and unexpected results (cf., e.g., [**10, 16, 17**]); for an extensive overview as well as a very nice introduction the interested reader is referred to [**1**]. We will use the probabilistic method in order to show the existence of abelian groups with prescribed endomorphism rings. By doing so, we obtain the probabilistic counterparts of well-known constructions. While the statements of the probabilistic counterparts are more general in some sense, as they assert that almost any choice of, say, elements from the completion suffice, the proofs simplify. Another application of the probabilistic method in abelian group theory, constructing groups with prescribed Ulm sequences, was presented in [**6**].

The structure of the paper is as follows. We start with a brief introduction to the probabilistic method and recall a few concepts from probability theory in Section 2. We will then apply the method to construct infinite abelian groups with prescribed endomorphism rings. For each construction, we will first recall the deterministic construction and provide a sketch of its proof, then we provide the necessary probabilistic tools and specify the distributions from which the elements

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or substructures are drawn, and finally we present the proof of the probabilistic variant of the construction. In the first part, in Section 3, we consider the classical Corner construction (see [**4**] or [**5**]). We first show that a uniform, random choice of countably many p-adic integers forms an algebraically independent set with probability one (Lemma 3.2) and later we generalize this construction to $2^{\aleph_0}$ elements (Lemma 3.3). We then provide a probabilistic version of Corner's construction (Theorem 3.5). In this case the actual distribution chosen for the random elements does matter and we provide an example where using a nearly uniform distribution results in a free group (Theorem 3.7). We then proceed with the Zassenhaus construction (see [**19**]) in Section 4 showing that every ring with a finite-rank free additive group can be realized as the endomorphism ring of a torsion-free abelian group. While the proof of the deterministic version (Theorem 4.1) is rather nontrivial and slightly technical, the proof of the probabilistic version follows more naturally (Theorem 4.4) relying on an old result by Frobenius and Chebotarëv (Lemma 4.2).

In the following, let $\widehat{B}$ denote the p-adic completion of B. Let $J_p := \widehat{\mathbb{Z}}$ denote the ring of p-adic integers. The p-adic completion is mainly of interest when B is naturally a submodule of $\widehat{B}$, which happens exactly when B is *p-reduced*, i.e., satisfying $\bigcap_{n=0}^{\infty} p^n B = 0$. Further let X_{p*} denote the *p-purification* of a submodule X of a p-torsion-free module for some prime p, i.e., X_{p*} consists of all x/p^k from the ambient module with $x \in X$ and $k \in \mathbb{N}$. We omit the ambient module from the notation as it will be clear from the context. All other notation is standard as to be found in [**7, 15, 14**], and [**12, 13**]. Recall that an event happens *almost surely* if the probability of the event is 1. For convenience we define $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$.

2. The probabilistic method: A brief introduction

We will now present a brief introduction to the probabilistic method and recall the necessary notions and concepts from probability theory. For a more complete introduction we refer the interested reader to [**1**]. As mentioned above, the probabilistic method establishes the existence of structures with desired properties by picking the structure randomly and showing that it has the desired properties with a positive probability. Before we continue with an example to illustrate the method, we recall a few notions and concepts from probability theory.

Recall that probability theory works with a collection of events, which form a so-called σ-algebra: it consists of some subsets of a big set closed under countable union and complements, and therefore also countable intersections. There is a *probability measure* $\mathbb{P}[.]$ assigning to each event a number in $[0,1]$, the *probability* of the event. The probability measure has to satisfy various properties, from which we mention only $\mathbb{P}\left[\bigcup_{i<\omega} A_i\right] \leq \sum_{i<\omega} \mathbb{P}[A_i]$ for any countable family of events $A_1, A_2, \dots$. A collection $\{A_i : i \in J\}$ of events is *independent*, if $\mathbb{P}\left[\bigcap_{i\in I} A_i\right] = \prod_{i\in I} \mathbb{P}[A_i]$ for any finite $I \subseteq J$. The following well-known lemma will be crucial:

LEMMA 2.1 (Borel-Cantelli Lemma). *Let $A_1, A_2, \dots \subseteq \mathcal{F}$ be a sequence of events. Further let $\limsup_{i\to\infty} A_i$ denote the set of outcomes that occur infinitely often. The following hold:*

(1) If $\sum_{i<\omega} \mathbb{P}[A_i] < \infty$ then $\mathbb{P}[\limsup_{i\to\infty} A_i] = 0$.

(2) *If* $A_1, A_2, \dots$ *are independent and* $\sum_{i<\omega} \mathbb{P}[A_i] = \infty$*, then* $\mathbb{P}[\limsup_{i\to\infty} A_i] = 1$*. In other words, infinitely many events occur with probability* 1*.*

A *distribution* of a random variable is the minimal σ-algebra of events meaningful for the variable together with the probability measure on it. For a discrete random variable X, i.e., one taking only countably many values, its *expected value* is $\mathbb{E}[X] = \sum_i X_i \, \mathbb{P}[X = X_i]$, where the $X_i \in \mathbb{R}$ form the range of X. Occasionally, we will use expected values of more general variables, but for intuition, it is mostly sufficient to think of the expected value in its discrete form. We will later use Fubini's theorem which allows for iterated computation of expected values:

LEMMA 2.2 (Fubini's Theorem). *Let f be a non-negative function which is measurable (in the respective space) and let X, Y be independent random variables and*

$$\mathbb{E}_{X,Y}[f(X,Y)] < \omega.$$

Then

$$\mathbb{E}_{X,Y}[f(X,Y)] = \mathbb{E}_X[\mathbb{E}_Y[f(X,Y)]].$$

Here expected values are taken in the total distribution of the variables in the subscript, and the expected value is a function of the other random variables.

We will now illustrate the probabilistic method by computing the order of $\mathrm{GL}(n,q)$, the group of invertible $n \times n$ matrices over the field with q elements. This is merely a reformulation of a counting argument in the framework of probability theory, just as many early examples.

PROPOSITION 2.3. *Let $n \in \mathbb{N}$ be a natural number and $\mathbb{F}_q$ be the finite field with q elements. Then the number of $n \times n$ invertible matrices over $\mathbb{F}_q$ is*

$$|\mathrm{GL}(n,q)| = \prod_{k\in[n]} (q^n - q^{k-1}).$$

PROOF. Let A be a random matrix over $\mathbb{F}_q$ chosen with uniform distribution. Clearly, A is invertible if and only if its columns $a_1, \dots, a_n$ are linearly independent. Observe that the probability of A being invertible can be rephrased by breaking it up into probabilities of linear independence of smaller subsets:

$$\begin{aligned}\mathbb{P}[A \text{ invertible}] &= \prod_{k\in[n]} \mathbb{P}[a_1,\dots,a_k \text{ independent} | a_1,\dots,a_{k-1} \text{ independent}] \\ &= \prod_{k\in[n]} (1 - \mathbb{P}[a_k \in \langle a_1,\dots,a_{k-1}\rangle | a_1,\dots,a_{k-1} \text{ independent}])\end{aligned}$$

Provided that $a_1, \dots, a_{k-1}$ are linearly independent, they span a $(k-1)$-dimensional subspace, so the probability that a_k is in this subspace is

$$\mathbb{P}[a_k \in \langle a_1,\dots,a_{k-1}\rangle | a_1,\dots,a_{k-1} \text{ independent}] = \frac{q^{k-1}}{q^n},$$

as the columns are independent random variables. We therefore obtain

$$\mathbb{P}[A \text{ invertible}] = \prod_{k\in[n]} \left(1 - \frac{q^{k-1}}{q^n}\right).$$

On the other hand we have $\mathbb{P}[A \text{ invertible}] = \frac{\ell}{q^{n^2}}$, where ℓ is the number of invertible matrices and q^{n^2} is the total number of $n \times n$ matrices over $\mathbb{F}_q$. We therefore obtain

$$\ell = q^{n^2} \cdot \prod_{k \in [n]} \left(1 - \frac{q^{k-1}}{q^n}\right) = \prod_{k \in [n]} q^n \left(1 - \frac{q^{k-1}}{q^n}\right) = \prod_{k \in [n]} (q^n - q^{k-1}).$$

□

In the following we operate under the same paradigm. However, it is not the abelian groups *per se* that are drawn from random distributions. We will use the concept in a slightly different fashion: we will pick crucial elements of the constructions, such as elements from the completion, at random. Obviously, we have to specify *how* we actually pick these elements, i.e., we have to provide the distribution. The distributions that we will use are very natural and since we are concerned about existence only, we can basically pick any (well-defined) distribution that suits our needs.

Another fact that is worthwhile to be mentioned is the structure of our results. We do not just provide mere *existence statements*, but we will show that the endomorphism properties hold *almost surely*, i.e., every random choice is satisfactory with probability 1. Actually, this is expected in view of Kolmogorov's zero-one law.

3. Groups via p-adic numbers

Our starting point is the following well-known construction of Corner (see [**4**] or [**13**, Theorem 110.1]). For simplicity, we restrict to p-reduced rings R.

Theorem 3.1. *For every countable p-reduced torsion-free ring R, there is a torsion-free left abelian group of countably infinite rank with endomorphism ring R.*

Sketch of proof. Let $\xi_n \in J_p$ with $n < \omega$ be quadratically independent p-adic integers and further let B be a free R-module of countably infinite rank. We define

$$G := \langle B, Rb\xi_b : b \in B \setminus \{0\} \rangle_{p*} \subseteq \widehat{B}.$$

Then $\operatorname{End} G = R$. For details, see [**4**], or for a slightly different construction [**13**, Theorem 110.1], or the proof of Theorem 3.5 below. □

Note that the construction in Theorem 3.1 carries over to uncountable modules up to size $2^{\aleph_0}$. In order to establish the probabilistic version, we will choose continuum many random p-adic integers, which will be almost surely algebraically independent. To be more precise, for each chosen p-adic integer X, the statement $X \equiv a \pmod{p^n}$ should be an event for all non-negative integer n and integer a.

First we present the easier, countable case: countably many, randomly and independently chosen p-adic integers are almost surely algebraically independent. One can satisfy the conditions of the lemma by choosing the uniform distribution (i.e., the Haar probability measure) for every p-adic integer.

Lemma 3.2. *Let $\mathcal{M} = \{\xi_n \mid n < \omega\} \subseteq J_p$ be a set of countably many, randomly and independently chosen p-adic integers such that $\mathbb{P}[\xi_n = \lambda] = 0$ for every $n < \omega$ and $\lambda \in J_p$. Then $\mathcal{M}$ is almost surely algebraically independent.*

PROOF. We show that every finite subset $S \subseteq \mathcal{M}$ is almost surely algebraically independent. The proof is by induction on the cardinality n of S. For $n = 0$ the statement holds trivially as $S = \emptyset$. Therefore let $n \geq 1$ and let $S = \{\xi_1, \ldots, \xi_n\}$ be a finite subset of $\mathcal{M}$. Note that there are only countably many non-zero polynomials f with integer coefficients in n variables. Thus it suffices to show that $f(\xi_1, \ldots, \xi_n) \neq 0$ almost surely for every such f. By assumption ξ_n is independent of $\xi_1, \ldots, \xi_{n-1}$. We can therefore apply Lemma 2.2 to compute the probability $\mathbb{P}[f(\xi_1, \ldots, \xi_n) \neq 0]$ by iterating expected values:

$$\mathbb{P}[f(\xi_1, \ldots, \xi_n) \neq 0] = \mathbb{E}_{\xi_1, \ldots, \xi_{n-1}}[\mathbb{P}[f(\lambda_1, \ldots, \lambda_{n-1}, \xi_n) \neq 0 \mid \xi_i = \lambda_i, i \in [n-1]]].$$

By induction, we conclude that $f(\xi_1, \ldots, \xi_{n-1}, x_n)$ is almost surely a non-zero polynomial in x_n. Therefore it has only finitely many roots and together with the assumption $\mathbb{P}[\xi_n = \lambda] = 0$ for every $n < \omega$ and $\lambda \in J_p$, we infer that ξ_n is none of these roots almost surely. It follows that

$$\mathbb{E}_{\xi_1, \ldots, \xi_{n-1}}[\mathbb{P}[f(\lambda_1, \ldots, \lambda_{n-1}, \xi_n) \neq 0 \mid \xi_i = \lambda_i, i \in [n-1]]] = 1$$

which completes the proof. $\square$

Note that quadratic independence instead of algebraic independence can be easily shown without the use of Fubini's Theorem.

A slightly more involved construction allows us to choose even continuum many random p-adic numbers, which are almost surely algebraically independent. We hasten to emphasize a peculiarity of the statement: it states that almost always none of *uncountably* many events occur. Usually probability theory cannot provide an answer in such cases as it only asserts that the union of *countably* many probability-0 events has again probability 0. However here we can use that J_p is compact, hence we can *approximate* the events via the topology. To ensure this, we construct the numbers as infinite branches of a tree and we show that it suffices to confine ourselves to sufficiently long *finite* initial segments. By doing so we reduce the uncountable case to a countable one. The construction is similar to the one in [**5**]. Let length(s) denote the length of a sequence s.

LEMMA 3.3. *Let p be an integer. We construct $2^{\aleph_0}$ random p-adic numbers as follows. We choose randomly and independently non-negative integers $a_s \in \{0, 1, \ldots, p^{2^{n+1}-2^n} - 1\}$ with uniform distribution for every finite 0-1 sequence s where $n =$ length(s). In particular, $a_{\langle\rangle} \in \{0, 1, \ldots, p-1\}$ for the empty sequence $\langle\rangle$. For every 0-1 infinite sequence f, we define the p-adic number*

$$\xi_f := \sum_{n=0}^{\infty} p^{2^n - 1} a_{f \restriction n},$$

where $f \restriction n$ is the initial segment of f consisting of n elements. Then the ξ_f are almost surely algebraically independent.

PROOF. To handle the ξ_f more easily we define

$$b_s := \sum_{j=0}^{n} p^{2^j - 1} a_{s \restriction j}$$

for every finite 0-1 sequence s where $n :=$ length(s). Then we have

$$\xi_f \equiv b_{f \restriction n} \pmod{p^{2^{n+1}-1}}.$$

First note that every b_s is uniformly distributed on the set of integers $\{0, 1, \dots, p^{2^{n+1}-1}\}$, i.e., on the mod $p^{2^{n+1}-1}$ classes of J_p with $n \coloneqq \operatorname{length}(s)$. This implies, in particular,

$$\mathbb{P}\left[b_s \equiv c \pmod{p^{2^{n+1}-1}} \middle| b_{s\restriction j} \equiv d \pmod{p^{2^{j+1}-1}}\right] = \begin{cases} \frac{1}{p^{2^{n+1}-2^{j+1}}}, & c \equiv d \pmod{p^{2^{j+1}-1}}, \\ 0, & \text{otherwise.} \end{cases}$$

For every positive integers k and n, every non-zero polynomial g with integer coefficients in k variables, and every *pairwise distinct* finite 0-1 sequences $s_1, \dots, s_k$ of length n, we show that there is almost never an extension f_i of the s_i with $g(\xi_{f_1}, \dots, \xi_{f_k}) = 0$. This will prove the lemma, as these are altogether countably many events, whose union is therefore the probability-0 event that the ξ_f are dependent.

We use induction on k. The statement for $k = 0$ is obvious. For $k > 0$, we prove the claim by showing that the probability of the event is at most ε for all positive $\varepsilon > 0$. Let μ denote the Haar probability measure of the compact additive group J_p^{k-1}. We say that a subset $A \subseteq J_p^{k-1}$ is *admissible* if the event that g has a solution $\xi_{f_1}, \dots, \xi_{f_k}$ for some infinite 0-1 sequences f_i extending the s_i with $(\xi_{f_1}, \dots, \xi_{f_{k-1}}) \in A$ has probability at most $\varepsilon\mu(A)$. We will prove the claim by partitioning J_p^{k-1} into countably many admissible subsets.

For this, write g in the form:

$$g(x_1, \dots, x_k) = g_m(x_1, \dots, x_{k-1})x_k^m + \dots + g_0(x_1, \dots, x_{k-1}),$$

where $g_m \neq 0$. We choose one of the partitions to be the solution set of g_m, which is admissible (actually has probability 0) by the induction hypothesis on k. The other partitions will be basic open sets, i.e., mod p^N-classes. As there are only countably many mod p^N-classes and every family of such classes contains a pairwise disjoint subfamily with the same union, it is enough to prove that every $(\eta_1, \dots, \eta_{k-1}) \in J_p^{k-1}$ with $g_m(\eta_1, \dots, \eta_{k-1}) \neq 0$ is contained in an admissible mod p^N-class for some N. Actually, we show that the mod p^N-class A of $(\eta_1, \dots, \eta_{k-1}) \in J_p^{k-1}$ is admissible for N large enough, because even the event that there are extensions f_i of the s_i with $\xi_{f_i} \equiv \eta_i \pmod{p^N}$ and $\xi_{f_1}, \dots, \xi_{f_k}$ is a solution of g mod p^N, i.e., $g(\eta_1, \dots, \eta_{k-1}, \xi_{f_k}) \equiv 0 \pmod{p^N}$ has probability at most $\varepsilon\mu(A)$.

Let $f \succ s$ denote that the sequence f is an extension of s. We consider the probability modulo the values of the b_{s_i}, as this makes the conditions on the ξ_{f_i} independent:

(3.1)
$$\begin{aligned} &\mathbb{P}\left[\exists f_i \succ s_i : \xi_{f_i} \equiv \eta_i \pmod{p^N}, g(\eta_1, \dots, \eta_{k-1}, \xi_{f_k}) \equiv 0 \pmod{p^N} \middle| b_{s_1}, \dots, b_{s_k}\right] \\ &\quad = \prod_{i \in [k-1]} \mathbb{P}\left[\exists f_i \succ s_i : \xi_{f_i} \equiv \eta_i \pmod{p^N} \middle| b_{s_1}, \dots, b_{s_k}\right] \\ &\quad\quad \cdot \mathbb{P}\left[\exists f_k \succ s_k : g(\eta_1, \dots, \eta_{k-1}, \xi_{f_k}) \equiv 0 \pmod{p^N} \middle| b_{s_1}, \dots, b_{s_k}\right]. \end{aligned}$$

Let us fix a positive integer $r \coloneqq \lceil \log_2(N+1) - 1 \rceil = O(\log N)$ so that $\xi_f \equiv b_{f\restriction r} \pmod{p^N}$ for every infinite 0-1 sequence f. For every $i \in [k]$, there are 2^{r-n} extensions of s_i into a 0-1 sequence of length r where n is the length of the s_i.

Therefore the probability that there exists an extension which is equivalent to η_i mod p^N is at most

$$\mathbb{P}\left[\exists f_i \succ s_i : \xi_{f_i} \equiv \eta_i \pmod{p^N} \middle| b_{s_1}, \dots, b_{s_k}\right] \le \frac{2^{r-n}}{p^{N-2^{n+1}}}. \tag{3.2}$$

For $i = k$, by a similar argument,
(3.3)

$$\mathbb{P}\left[\exists f_k \succ s_k : g(\eta_1, \dots, \eta_{k-1}, \xi_{f_k}) \equiv 0 \pmod{p^N} \middle| b_{s_1}, \dots, b_{s_k}\right] \le \frac{2^{r-n} R(N)}{p^{N-2^{n+1}}},$$

where $R(N)$ is the number of roots of $g(\eta_1, \dots, \eta_{k-1}, x)$ in x mod p^N.

To estimate $R(N)$, let us consider the factorization over J_p

$$g(\eta_1, \dots, \eta_{k-1}, x) = h(x) \prod_{i \in [l]} (x - \lambda_i) \tag{3.4}$$

for some p-adic integers λ_i. The polynomial h has no roots among the p-adic integers. So there is a highest p-power p^M which can divide $h(x)$ for any p-adic number x.

Let us estimate the number of roots of (3.4) modulo p^N. For every root x, the product is divisible by p^N. As $h(x)$ is divisible by at most p^M, there must be an i for which $x - \lambda_i$ is divisible by $p^{\lceil (N-M)/l \rceil}$. So every root is contained in the mod $p^{\lceil (N-M)/l \rceil}$-class of some λ_i, and hence

$$R(N) \le l p^{N - \lceil (N-M)/l \rceil}. \tag{3.5}$$

By combining (3.1), (3.2), (3.3) and (3.5), we finally obtain

$$\begin{aligned}
&\mathbb{P}\left[\exists f_i \succ s_i : \xi_{f_i} \equiv \eta_i \pmod{p^N}, g(\eta_1, \dots, \eta_{k-1}, \xi_{f_k}) \equiv 0 \pmod{p^N} \middle| b_{s_1}, \dots, b_{s_k}\right] \\
&\qquad \le \left(\frac{2^{r-n}}{p^{N-2^{n+1}}}\right)^{k-1} \cdot \frac{2^{r-n} l p^{N - \lceil (N-M)/l \rceil}}{p^{N-2^{n+1}}} \\
&\qquad = \frac{l\left(2^{r-n} p^{2^{n+1}}\right)^k}{p^{\lceil (N-M)/l \rceil}} \cdot \underbrace{\frac{1}{p^{N(k-1)}}}_{\mu(A)} = O\left(\frac{N^k}{p^{N/l}}\right) \cdot \mu(A).
\end{aligned}$$

Hence A is indeed admissible for large N. □

For a countable module B, we will randomly and independently choose elements $a_n = \sum_{b \in I_n} b \xi_{n,b} \in J_p B$ for all $n < \omega$. To this end, we select the support I_n and the coefficients $\xi_{n,b}$ according to the following distribution.

Distribution 3.4. For a countable set B and for $n < \omega$, let I_n be independent, identical distributed random variables taking values in the non-empty finite subsets of B. Every non-empty finite subset should be contained in I_n (for a fixed n) with positive probability.

Furthermore, for all n and $b \in I_n$ and $\alpha < 2^{\aleph_0}$ let the $\xi^\alpha_{n,b}$ be random p-adic numbers chosen as in Lemma 3.3. Note that the $\xi^\alpha_{n,b}$ are almost surely algebraically independent for all $n < \omega, b \in I_n, \alpha < 2^{\aleph_0}$.

We can prove the following probabilistic variant of Theorem 3.1.

THEOREM 3.5. *Let R be a countable p-reduced, torsion-free ring. Let B be an at most countably generated, non-zero, free R-module. Furthermore, let I_n be random finite subsets of B and $\xi^{\alpha}_{n,b}$ for $b \in I_n$ and $\alpha < 2^{\aleph_0}$ be random p-adic numbers with Distribution* 3.4, *and define*

$$a^{\alpha}_{n} := \sum_{b \in I_n} b\xi^{\alpha}_{n,b} \in J_p B. \tag{3.6}$$

Then the groups

$$G^{A} := \langle B, Ra^{\alpha}_{n} : n < \omega, \alpha \in A \rangle_{p*} \subseteq \widehat{B}.$$

for $\emptyset \neq A \subseteq 2^{\aleph_0}$ have endomorphism ring $\operatorname{End} G^{A} = R$ *and form a fully rigid system, i.e.,*

$$\operatorname{Hom}(G^{A}, G^{D}) = \begin{cases} R, & A \subseteq D \\ 0, & A \nsubseteq D \end{cases}$$

almost surely.

PROOF. By 3.3 the family $\{\xi^{\alpha}_{n,b} \mid n < \omega, b \in I_n, \alpha < 2^{\aleph_0}\}$ is almost surely algebraically independent. Moreover, every finite $F \subseteq B$ is almost surely contained in some (actually infinitely many) I_n with $n < \omega$. We will show that these two properties guarantee that $\operatorname{Hom}(G^{A}, G^{D})$ is R or 0 almost surely, as claimed, i.e., all homomorphisms are multiplications by ring elements.

Let φ be a homomorphism from G^{A} to G^{D} and let $\alpha \in A \subseteq 2^{\aleph_0}$ be arbitrary but fixed for the moment. Obviously, $b\varphi, a^{\alpha}_{n}\varphi \in G^{D}$ for $b \in B$ so there are $d_b, c_n, \in \mathbb{Z}[1/p]B$ and $t_{m,b}, r_{m,n} \in R[1/p]$ together with $\beta_{m,b}, \delta_{m,n} \in D$ such that

$$\begin{aligned} b\varphi = d_b + \sum_{m} t_{m,b} a^{\beta_{m,b}}_{m} &= d_b + \sum_{m,f\colon f \in I_m} t_{m,b} f \xi^{\beta_{m,b}}_{m,f}, \\ a^{\alpha}_{n}\varphi &= c_n + \sum_{m,f\colon f \in I_m} r_{m,n} f \xi^{\delta_{m,n}}_{m,f}. \end{aligned} \tag{3.7}$$

On the other hand, by continuity, we also obtain from (3.6) and (3.7)

$$a^{\alpha}_{n}\varphi = \sum_{b \in I_n} d_b \xi^{\alpha}_{n,b} + \sum_{\substack{m,f\colon f \in I_m, \\ b \in I_n}} t_{m,b} f \xi^{\beta_{m,b}}_{m,f} \xi^{\alpha}_{n,b}.$$

By combining the two expressions for $a^{\alpha}_{n}\varphi$ we therefore obtain

$$\sum_{b \in I_n} d_b \xi^{\alpha}_{n,b} + \sum_{\substack{m,f\colon f \in I_m, \\ b \in I_n}} t_{m,b} f \xi^{\beta_{m,b}}_{m,f} \xi^{\alpha}_{n,b} = c_n + \sum_{m,f\colon f \in I_m} r_{m,n} f \xi^{\delta_{m,n}}_{m,f}.$$

Using the algebraic independence of the $\xi^{\gamma}_{n,b}$, we compare coefficients and obtain among others

$$\begin{aligned} t_{m,b} f &= 0, && (f \in I_m) \\ d_b &= r_{n,n} b && (b \in I_n) \\ d_b &= 0 && (\alpha \notin D). \end{aligned} \tag{3.8}$$

We have used that for every $b \in B$ there is an n with $b \in I_n$. For example, to obtain the first equation, we choose $n \neq m$ with $b \in I_n$ and compare the coefficients of $\xi^{\beta_{m,b}}_{m,f} \xi^{\alpha}_{n,b}$. We conclude that if $\alpha \notin D$ then $b\varphi = 0$ for all $b \in B$ and hence $\varphi = 0$. This is enough for the case $A \nsubseteq D$. If $\alpha \in D$ then $b\varphi = d_b = r_{n,n} b$ for all n and $b \in I_n$. We now show that essentially all the $r_{n,n}$ are equal, i.e., $b\varphi = rb$ for some

$r \in R[1/p]$. As B is free, there is an element b' with zero annihilator, e.g., a basis element. As a consequence, all the $r_{n,n}$ are equal for which $b' \in I_n$. Let r be the common value of these $r_{n,n}$, choose $b \in B$ arbitrary and pick n with $b', b \in I_n$, which exists by hypothesis.

So $r = r_{n,n}$, and using (3.8) we obtain $b\varphi = r_{n,n}b = rb$ as claimed. We therefore conclude that the homomorphism φ is multiplication by an $r \in R[1/p]$.

As B is free, $R[1/p] \cap \operatorname{End} B = R$, and it follows that $r \in R$ and thus φ is a multiplication with the ring element r. This finishes the case $A \subseteq D$ and hence the proof. □

By an analogous but simpler argument, we also obtain a probabilistic version of Corner's construction of finite-rank groups as a corollary (see [**4**, Theorem B] or [**14**, Corollary 12.1.3]).

THEOREM 3.6. *Let A be a p-reduced, p-torsion-free ring of finite rank n. Let $w = \sum_{i=1}^{n} \zeta_i e_i$ be a random element of the p-adic completion of A, where the e_i form a maximal linearly independent subset of A and the ζ_i are independent random p-adic numbers with $\mathbb{P}[\zeta_i = \lambda] = 0$ for all i and p-adic integer λ. Then*

$$G := \langle A, wA \rangle_{p*}$$

is of rank $2n$ *and* $\operatorname{End}(G) \cong A$ *almost surely.*

In the usual way Theorem 3.5 and Corollary 3.6 can be generalized to $\mathbb{S}$-reduced, $\mathbb{S}$-torsion free algebras A of finite rank over some $\mathbb{S}$-ring R whose completion $\widehat{R}$ has sufficiently high transcendence degree; we confined ourselves to the simplified case purely for expository reasons and the generalization is left to the interested reader.

Note that the elements $a_n^\alpha \in \widehat{B}$ that we chose at random in Theorem 3.5 were contained in the submodule J_pB. It would be natural to expect that a nearly uniform choice of random elements from the completion $\widehat{B}$ should already suffice. However, it fails: the constructed group is actually almost surely free. This shows in a nice way that the actual distribution does matter which is somewhat counterintuitive. It seems that especially the implicit assumption of finite support in Distribution 3.4 is advantageous.

THEOREM 3.7. *Let B be a free abelian group of countably infinite rank and let*

$$G := \langle B, a_n \rangle_{p*} \subseteq \widehat{B}$$

where the a_n with $n < \omega$ are independent, random elements chosen with a nearly uniform distribution from the completion $\widehat{B}$, *i.e., for some $\alpha > 1$, all $n < \omega$, and $x \in B/p^nB$ we have $\mathbb{P}[a_m + p^nB = x] \leq p^{-n^\alpha}$. Then G is almost surely free.*

PROOF. First we claim that the random elements a_m are almost never contained in the J_p-module generated by any fixed $b_1, \ldots, b_k \in \widehat{B}$, i.e.,

$$\mathbb{P}\left[a_m \in \langle b_1, \ldots, b_k \rangle_{J_p}\right] = 0. \tag{3.9}$$

The event is the intersection of the descending sequence of events that a_m is contained in the subgroup generated by the b_i in the factor group $\widehat{B}/p^n\widehat{B}$. We estimate the probability of the latter events:

$$\mathbb{P}[a_m \in \langle b_1, \ldots, b_k \rangle + p^nB] \leq |\langle b_1, \ldots, b_k \rangle \bmod p^n| \cdot \frac{1}{p^{n^\alpha}} \leq \frac{p^{nk}}{p^{n^\alpha}}.$$

This tends to zero as n goes to infinity, proving the claim.

Next we show that the family of all the e_n and a_m is almost surely linearly independent over J_p. If the family is linearly dependent then

$$\sum_{i=0}^{k} \eta_i e_i + \sum_{j=0}^{l} \mu_j a_j = 0$$

for some p-adic integers η_i and μ_j, where not all of those are zero. The e_i form a basis of B, so they remain linearly independent over J_p, hence there must be a non-zero μ_j. Since J_p is a discrete valuation domain, there is a μ_m dividing all the μ_j. It follows that μ_m divides $\sum_{i=0}^{k} \eta_i e_i$. Because $J_p B = \bigoplus_{n=0}^{\infty} J_p e_i$ is pure in $\widehat{B}$, the number μ_m must divide all of the η_i. All in all, we obtain

$$a_m = -\sum_{i=0}^{k} \mu_m^{-1} \eta_i e_i - \sum_{j=0}^{l} \mu_m^{-1} \mu_j a_j$$

and therefore

$$a_m \in \langle e_0, \dots, e_k, a_0, \dots, a_{m-1}, a_{m+1}, \dots, a_l \rangle_{J_p}. \tag{3.10}$$

Since the a_j are independent random variables, the event (3.10) has probability zero by (3.9) for fixed m, k, and l. Varying m, k, and l, there are only countably many such events, so almost surely none of them occurs, and hence the family of all the e_n and a_m are almost surely linearly independent over J_p.

Finally, we show that the linear independence ensures that G is free. Recall that G is countable, and hence we can apply Pontryagin's criterion (see [**12**, Theorem 19.1] or [**7**, Theorem 2.3]): a countable torsion-free abelian group is free if and only if every finite-rank subgroup is free. Therefore it suffices to show that the purifications $\langle e_0, \dots, e_k, a_0, \dots, a_m \rangle_*$ are actually free groups. Recall that $\widehat{B}$ as a J_p-module has the property that every pure finite-rank submodule is a free module. Therefore $\langle e_0, \dots, e_k, a_0, \dots, a_m \rangle_{J_p,*}$ is free, and in particular for some $k > 0$, we have $p^k \langle e_0, \dots, e_k, a_0, \dots, a_m \rangle_{J_p,*} \subseteq \langle e_0, \dots, e_k, a_0, \dots, a_m \rangle_{J_p}$. It follows that every element of $p^k \langle e_0, \dots, e_k, a_0, \dots, a_m \rangle_*$ is a linear combination of the $e_0, \dots, e_k, a_0, \dots, a_m$ with coefficients in J_p. On the other hand, the coefficients are also in $\mathbb{Z}[1/p]$, since $p^k \langle e_0, \dots, e_k, a_0, \dots, a_m \rangle_*$ is a subgroup of G. All in all, using linear independence, the coefficients are in $J_p \cap \mathbb{Z}[1/p] = \mathbb{Z}$, so $p^k \langle e_0, \dots, e_k, a_0, \dots, a_m \rangle_*$ is contained in $\langle e_0, \dots, e_k, a_0, \dots, a_m \rangle$. Therefore $\langle e_0, \dots, e_k, a_0, \dots, a_m \rangle_*$ must be free. ☐

4. Small-rank groups

In this section we provide a probabilistic counterpart for Zassenhaus's construction (see [**19**] or [**14**, Theorem 12.1.6]).

THEOREM 4.1. *Let A be a ring with a finite-rank free additive group. Then there is a torsion-free abelian group M of the same rank with endomorphism ring A.*

SKETCH OF PROOF. For every pair of non-zero elements a_i, e_i of A, choose an integer c_i and a prime p_i such that $c_i - a_i$ is invertible in $\mathbb{Q}A$, and p_i divides the order of $(c_i - a_i)^{-1} e_i$ in $\mathbb{Q}A/A$. Make the choices such that the primes p_i are pairwise distinct. We choose positive integers r_i, d_i such that $p_i^{r_i} d_i (c_i - a_i)^{-1} \in A$ and d_i is relative prime to p_i. Now the abelian group

$$M := \langle A, p_i^{-r_i}(c_i - a_i)A : i < \omega \rangle \subseteq \mathbb{Q}A$$

has endomorphism ring A acting on it by multiplication on the right.

To see this, first we note that for every $m \in \mathbb{Q}A$ with $p_i^{-r_i}(c_i - a_i)m \in M$, the order of m in $\mathbb{Q}A/A$ is not divisible by p_i. Indeed, there are $a, b_j \in A$ such that

$$p_i^{-r_i}(c_i - a_i)m = a + \sum_j p_j^{-r_j}(c_j - a_j)b_j.$$

Multiplying by $\tilde{a} := p_i^{r_i} d_i (c_i - a_i)^{-1} \in A$ on the left

$$d_i m = d_i b_i + \tilde{a}a + \sum_{j \neq i} p_j^{-r_j} \tilde{a}(c_j - a_j)b_j.$$

The order of the right-hand side is clearly not divisible by p_i. As d_i is not divisible by p_i, it also follows that p_i does not divide the order of m.

We will now prove that the only $\phi \in \operatorname{End} M$ mapping 1 to 0 is the zero map. Suppose for contradiction that there is a non-zero ϕ with $1\phi = 0$. Thus there exists $a \in A$ with $a\phi \neq 0$. By multiplying a with a large positive integer if necessary, we can assume $a\phi \in A$. In fact there is an i with $a_i = a$ and $-e_i = a\phi$.

Since

$$p_i^{-r_i} e_i = p_i^{-r_i}(c_i - a_i)\phi \in M,$$

the order of $(c_i - a_i)^{-1} e_i$ in $\mathbb{Q}A/A$ is not divisible by p_i contradicting one of our assumptions.

Next we establish that $\mathbb{Q}A \cap \operatorname{End} M = A$, where every $a \in \mathbb{Q}A$ is identified with multiplication by a on the right. So let $m \in \mathbb{Q}A \cap \operatorname{End} M$. Then $m = 1m \in M$, so the order of m in $\mathbb{Q}A/A$ can have only the p_i as prime divisors. On the other hand, since $p_i^{-r_i}(c_i - a_i)m \in M$, the order of m in $\mathbb{Q}A/A$ is not divisible by p_i. Hence the order of m must be 1, i.e., $m \in A$.

It remains to show that $\operatorname{End} M = A$. Let $\phi \in \operatorname{End} M$. There is a positive integer n with $1 \cdot n\phi \in A$. Now $n\phi - 1 \cdot n\phi$ is an endomorphism of M mapping 1 to 0, hence it is zero. Thus $\phi = 1\phi \in \mathbb{Q}A \cap \operatorname{End} M = A$. □

For the probabilistic version of this construction, we shall use a consequence of a theorem of Frobenius (see [**11**]) or Chebotarëv's density theorem (see [**3, 18**]) which is a generalization of Frobenius's theorem.

Lemma 4.2. *For every non-constant (univariate) polynomial $f \in \mathbb{Z}[x]$ the sum of the reciprocals of primes p for which f has a root modulo p diverges, i.e.,*

$$\sum_{\substack{p \text{ prime} \\ f \text{ has root mod } p}} \frac{1}{p} = \infty.$$

We first specify the distribution according to which we choose the random elements:

Distribution 4.3. Let A be a ring with a finite-rank, free additive group. For every prime p we choose uniformly and independently a non-zero element $a_p \in A$ and an integer $c_p \in [p, 2p-1]$. Whenever $c_p - a_p$ is invertible in $\mathbb{Q}A$, we choose positive integers r_p and d_p arbitrarily with $p^{r_p} d_p (c_p - a_p)^{-1}$ in A and d_p relative prime to p.

We are ready to prove the probabilistic variant of Theorem 4.1.

THEOREM 4.4. *Let A be a ring with a finite-rank free additive group and let M be the torsion-free abelian group*

$$M := \left\langle A, p^{-r_p}(c_p - a_p)A : p \text{ prime and } \exists (c_p - a_p)^{-1} \right\rangle$$

with c_p, a_p, r_p chosen via Distribution 4.3. Then $\operatorname{End} M = A$ almost surely, where A acts on M by multiplication on the right.

PROOF. With the argumentation in the sketch of proof of Theorem 4.1, it suffices to prove that for every pair of non-zero elements e and a of A there is a prime p such that $a_p = a$, the difference $c_p - a_p$ is invertible in $\mathbb{Q}A$, and p divides the order of $(c_p - a_p)^{-1}e$ in $\mathbb{Q}A/A$.

There are only finitely many c for which $c - a$ is non-invertible, namely, the roots of the characteristic polynomial of (left) multiplication by a. Furthermore $(c - a)^{-1}e$ is a rational function of c of degree at most -1, i.e., the coordinates are rational functions in (any) basis of $\mathbb{Q}A$. Now p divides the order of $(c_p - a_p)^{-1}e$ in $\mathbb{Q}A/A$ if and only if there is a coordinate where p occurs with negative exponent, e.g., p divides the denominator but not the numerator. We simplify the coordinates to make the numerator and denominator relative prime polynomials, so there are only finitely many primes such that at any place c, at most these among all primes divide both the numerator and denominator of a coordinate. Note that for non-zero coordinates, the denominator is still non-constant, as the coordinate has negative degree.

All in all, there is a non-constant polynomial f (the denominator of a coordinate) with integer coefficients for which all but finitely many primes p and any root c of f modulo p, the order of $(c - a)^{-1}e$ in $\mathbb{Q}A/A$ is divisible by p. For every p where f has a root modulo p, we choose c_p a root with probability at least $1/p$. Since these events are independent and the sum of their probabilities is infinite by 4.2, infinitely many of these events occur almost surely. Again by independence, $a_p = a$ almost surely for infinitely many of these p, finishing the proof. □

5. Concluding remarks and open questions

So far the proposed method only works for constructions up to continuum in size. This is due to the lack of a strong probability theory beyond $2^{\aleph_0}$. However we believe that this method is likely to be generalized to such cases as well; a step in this direction is Lemma 3.3. A potential route to carry over the probabilistic tools might be to work with a countable model of set theory; however this is speculation. In particular, the following questions remain open, where the last one is probably the most intricate one:

(1) Generalize to Butler (locally free): There is a well-known generalization of Zassenhaus's Theorem 4.1 to the locally free case by Butler (see [**2**]). It turns out that our randomized construction does not easily generalize to this case.

Can the construction be generalized to the locally free case?

(2) Use randomness in a more involved way: So far randomness has been used either to construct algebraically independent elements or to ensure that all elements of a countable set have been chosen. However randomness has not been directly employed in the construction itself.

Can we use randomization in the constructions itself in order to obtain simplified or even stronger constructions?

(3) Generalize beyond $2^{\aleph_0}$: Probability theory is defined on σ-algebras and countability plays a central role in the arguments.

Are there probabilistic constructions of objects larger than $2^{\aleph_0}$?

(4) It is independent of ZFC whether for $\aleph_0 < \lambda < 2^{\aleph_0}$, the union of λ events of probability 0 from the continuous uniform distribution (e.g., of a random real number from $[0,1]$) has again probability zero.

*Is there a randomized construction for a (natural) statement in abelian group theory, so that the actual probability for the theorem to hold is **independent** of ZFC?*

For example, is there a realization theorem for some (family of) ring A so that $\mathrm{End}(G) = A$ with probability 1 in one universe and probability 0 in another?

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On projection-invariant subgroups of Abelian p-groups

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To Rüdiger Göbel on his 70th birthday

Abstract. A subgroup P of an Abelian p-group G is said to be *projection-invariant in* G, if $P\pi \leq P$ for all idempotent endomorphisms π in $\mathrm{End}(G)$. Clearly fully invariant subgroups are projection invariant, but the converse is not true in general. Hausen and Megibben have, however, shown that in many familiar situations these two concepts coincide. In a different direction, the authors have previously introduced the notions of socle-regular and strongly socle-regular groups by focussing on the socles of fully invariant and characteristic subgroups of p-groups. In the present work the authors examine the socles of projection-invariant subgroups of Abelian p-groups.

1. Introduction

Recall that a subgroup P of an Abelian p-group G is said to be *projection-invariant in* G, if $P\pi \leq P$ for all idempotent endomorphisms π in $\mathrm{End}(G)$. Clearly fully invariant subgroups are projection invariant, but the converse is not true in general. It is an easy exercise to show that P is projection-invariant in G if, and only if, $P\pi = P \cap G\pi$ for every projection $\pi \in \mathrm{End}(G)$. Like fully invariant subgroups, projection-invariant subgroups distribute across direct sums i.e., if $G = A \oplus B$ and P is projection-invariant, then $P = (A \cap P) \oplus (B \cap P)$. In fact in some situations the notions coincide: Hausen [**9**] and Megibben [**14**] have established that for separable p-groups, and for transitive, fully transitive groups G satisfying a certain technical condition (*) on Ulm invariants, all projection-invariant subgroups are in fact fully invariant. For the readers' convenience we list this condition (*): If $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots b_m$ are two disjoint finite sequences of ordinals such that the Ulm-Kaplansky invariants $f_G(\alpha_i) \neq 0$ for each positive integer i, then there is a direct decomposition $G = A \oplus B$ where $f_A(\alpha_i) = 1$ for $i = 1, \cdots, n$ and $f_A(\beta_j) = 0$ for $j = 1, \cdots, m$.

In a different direction, the authors have recently investigated the socles of fully invariant and characteristic subgroups of Abelian p-groups. This led to the notions of socle-regular and strongly socle-regular groups, see [**3, 4**]. Recall the definitions: a group G is said to be *socle-regular* (*strongly socle-regular*) if for all fully invariant (characteristic) subgroups F of G, there exists an ordinal α (depending on F) such that $F[p] = (p^\alpha G)[p]$. It is self-evident that strongly socle-regular groups are themselves socle-regular, whereas the converse is not valid (see [**4**]). Motivated by these concepts we make the following definition: a p-group G is said to be

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projectively socle-regular if, for each projection-invariant subgroup P of G, there is an ordinal α (depending on P) such that $P[p] = (p^\alpha G)[p]$.

It is immediate that projectively socle-regular groups are socle-regular.

Throughout our discussion all groups will be additively written Abelian groups; our notation and terminology are standard and may be found in the texts [**7, 12**].

In [**3**] the following *ad hoc* terminology was introduced; it has been useful in [**3, 4, 5**] and we find it convenient to use it again here:

Suppose that H is an arbitrary subgroup of the group G. Set $\alpha = \min\{h_G(y) : y \in H[p]\}$ and write $\alpha = \min_G(H[p])$; clearly $H[p] \leq (p^\alpha G)[p]$. If there is no possibility of confusion we omit the subscript and just write $\min(H[p])$.

In the light of the results of Hausen [**9**] and Megibben [**14**], our first result is not too surprising.

PROPOSITION 1.1. *If P is a projection-invariant subgroup of the p-group G and $\min_G(P[p]) = n$, a positive integer, then $P[p] = (p^n G)[p]$. Consequently, if G is separable, then G is projectively socle-regular.*

PROOF. Suppose that P is a projection-invariant subgroup of G and $\min(P[p]) = n$, a finite integer. Then there is an element $x \in P[p]$ such that $h_G(x) = n$ and so $x = p^n y$ where y is the generator of a direct summand of G; $G = \langle y\rangle \oplus G_1$ say. Suppose that $z \in (p^n G)[p] \setminus (p^{n+1}G)[p]$, so that $z = p^n w$ for some w of height zero; thus $G = \langle w\rangle \oplus G_2$. Note that $\langle w\rangle \cong \mathbb{Z}(p^{n+1}) \cong \langle y\rangle$. Then we have that $w = ry + g_1$ for some integer r and some $g_1 \in G_1$. Now define $\phi : \langle y\rangle \oplus G_1 \to \langle y\rangle \oplus G_1$ by $y\phi = g_1,\ G_1\phi = 0$. It follows easily, or see the proof of Lemma 5 in [**9**], that ϕ is the difference of two idempotent endomorphisms of G. Now define $\psi : G \to G$ by $g\psi = r(g\pi) + g\phi$, where π is the projection map given by $y\pi = y,\ G_1\pi = 0$. Note that ψ is a sum of integer multiples of idempotents and $y\psi = ry + g_1 = w$. Since $x\psi = p^n(y\psi) = p^n w = z$ and $x \in P$, which is a projection-invariant subgroup of G, we conclude that $z \in P[p]$. Hence $(p^n G)[p] \setminus (p^{n+1}G)[p] \subseteq P[p]$. However, if $u \in (p^{n+1}G)[p]$, then $z + u \in (p^n G)[p]$, and so, by the argument above, $z + u \in P[p]$. Thus, we have that $(p^n G)[p] \leq P[p]$ and since the reverse inclusion holds by virtue of $\min(P[p]) = n$, we have equality. The final deduction is immediate. $\square$

COROLLARY 1.2. *If G is a p-group such that $p^\omega G \cong \mathbb{Z}(p)$, then G is projectively socle-regular.*

PROOF. Suppose P is a projection-invariant subgroup of G. If $P[p] \not\leq p^\omega G$, then $\min_G(P[p])$ is finite and we therefore apply Proposition 1.1 above to obtain that $P[p] = (p^n G)[p]$ for some integer n. So, we may assume that $P[p] \leq p^\omega G$. But since the latter group is cyclic of prime order p, it follows at once that either $P = P[p] = 0$, whence $P[p] = (p^{\omega+1}G)[p]$, or $P[p] = p^\omega G = (p^\omega G)[p]$ as required. $\square$

The property of a p-group being projectively socle-regular is inherited by subgroups of the form $p^\alpha G$.

PROPOSITION 1.3. *If G is a projectively socle-regular p-group, then so also is $p^\alpha G$ for all ordinals α.*

PROOF. Let $H = p^\alpha G$ and suppose that P is a projection-invariant subgroup of H. Let π be an arbitrary idempotent in $\mathrm{End}(G)$. Then $\pi^* = \pi \restriction H$ is an idempotent endomorphism of H. Thus $P\pi = P\pi^* \leq P$ since P is a projection-invariant subgroup of H. Consequently P is a projection-invariant subgroup of G

and so there is an ordinal β such that $P[p] = (p^\beta G)[p]$. Since P is contained in $p^\alpha G$, we conclude that $\beta \geq \alpha$; say $\beta = \alpha + \gamma$. But then $P[p] = (p^{\alpha+\gamma}G)[p] = (p^\gamma H)[p]$, showing that H is also a projectively socle-regular group. $\square$

Projectively socle-regular groups of length $< \omega + \omega$ are easy to construct:

Proposition 1.4. *If H is an arbitrary finite p-group, then there exists a projectively socle-regular group G with $p^\omega G = H$.*

Proof. It is convenient to make use of Corner's realization theorem [**2**, Theorem 6.1]. Let $A = \mathrm{End}(H)$, so that A is certainly countable. Then applying Corner's result we find a group G such that $p^\omega G = H$, $\mathrm{End}(G) \restriction H = A$ and $\mathrm{Aut}(G) \restriction H$ acts as the units of A. Moreover, there is a semigroup homomorphism, * say, from the multiplicative semigroup $A^\times \to \mathrm{End}(G)^\times$ such that $\phi^* \restriction H = \phi$ for every $\phi \in A$. Note that the group G is transitive and fully transitive since finite groups have this property; it follows from [**3**, Theorem 0.3] and [**4**, Theorem 2.4] that G is strongly socle-regular and hence socle-regular.

We claim that G is also projectively socle-regular. Let P be a projection-invariant subgroup of G; if $\min(P[p])$ is finite, we are finished by Proposition 1.1. So suppose that $\min(P[p])$ is infinite, so that $P[p] \leq p^\omega G$. Note that $P[p]$ is also a projection-invariant subgroup of G. Observe firstly that $P[p]$ is actually a projection-invariant subgroup of $p^\omega G = H$: for if $\pi \in A$ is an idempotent, there is a mapping $\pi^* \in \mathrm{End}(G)$ such that $\pi^* \restriction H = \pi$; since * is a semigroup homomorphism, the mapping π^* is also an idempotent and we have $(P[p])\pi = (P[p])\pi^* \leq P[p]$. Since H is a finite group, it is certainly projectively socle-regular so that $P[p] = (p^n H)[p]$ for some integer n. But then $P[p] = (p^{\omega+n}G)[p]$ and so G is projectively socle-regular, as required. $\square$

As observed above, projectively socle-regular groups are socle-regular. However, when $p \neq 2$, we can say a little more.

Proposition 1.5. *If $p \neq 2$, then a projectively socle-regular p-group is strongly socle-regular.*

Proof. Let C be an arbitrary characteristic subgroup of the p-group G, where $p \neq 2$. If π is an idempotent in $\mathrm{End}(G)$, then $(2\pi - 1)^2 = 1$, so that $2\pi - 1$ is an involutary automorphism of G. Thus $C(2\pi - 1) \leq C$ implying that $2(C\pi) \leq C$. Since, by assumption, $p \neq 2$, this implies that $C\pi \leq C$. Hence C is a projection-invariant subgroup of G. If G is a projectively socle-regular group, then $C[p] = (p^\alpha G)[p]$ for some ordinal α. Since C was an arbitrary characteristic subgroup of G, we have that G is strongly socle-regular, as required. $\square$

Our next example shows that the class of socle-regular groups is strictly larger than the class of projectively socle-regular groups. First we derive a technical lemma.

Lemma 1.6. *If G is a p-group in which $p^\omega G$ is homogeneous, then if G is fully transitive but not transitive, every subgroup of $p^\omega G$ is a projection-invariant subgroup of G.*

Proof. Let P be an arbitrary subgroup of $p^\omega G$ and let $\pi \in \mathrm{End}(G)$ be an arbitrary idempotent endomorphism. Then $G = G_1 \oplus G_2$, where $G_1 = G\pi$, $G_2 =$

$G(1-\pi)$. Now if $p^\omega G_i > 0$ for $i = 1, 2$, it follows from Proposition 2.2 in [**2**] that G is transitive - contrary to hypothesis. So either $p^\omega G_1 = 0$ or $p^\omega G_2 = 0$.

It is well known, and easy to show, that if ν is an idempotent endomorphism of G, then $(p^\omega G)\nu = p^\omega(G\nu)$. Thus, either $(p^\omega G)\pi = 0$ or $(p^\omega G)(1-\pi) = 0$. Hence, either $P\pi = 0$ or $P(1-\pi) = 0$; in either case $P\pi \leq P$ and so P is a projection-invariant subgroup of G. □

Proposition 1.7. *For each prime p, there exists a socle-regular p-group which is not projectively socle-regular.*

Proof. Let G be a non-transitive fully transitive p-group with $p^\omega G \cong \bigoplus_{\aleph_0} \mathbb{Z}(p)$ - for example the groups of length $\omega + 1$ constructed by Corner in [**2**]. Since G is fully transitive, it is socle-regular by [**3**, Theorem 0.3]. However, G is certainly not projectively socle-regular, since it follows from Lemma 1.6 above that any non-zero proper subgroup of $p^\omega G$ cannot have a socle of the form $(p^\alpha G)[p]$ for any ordinal α. □

Remark 1.8. Corner's example of a non-transitive fully transitive p-group G used in Proposition 1.7 above, provides an easy example of a group which does not satisfy the condition $(*)$ introduced by Megibben. If it did satisfy $(*)$, then, since $f_G(\omega) = \aleph_0$, there would be a direct decomposition $G = G_1 \oplus G_2$ in which $f_{G_1}(\omega) = 1$ and so $f_{G_2}(\omega) = \aleph_0$. Thus, G would decompose as $G_1 \oplus G_2$ with both $p^\omega G_1, p^\omega G_2 > 0$; as pointed out in Lemma 1.6, no such decomposition exists for G.

We have seen in Proposition 1.3 that the projective socle-regularity of a group G is inherited by subgroups of the form $p^\alpha G$. We now wish to investigate the converse situation. Before doing so, we establish an elementary result of independent interest regarding the lifting of idempotents; we remark that we shall need much deeper results than this to handle the situation for subgroups of the form $p^\alpha G$, when $\alpha \geq \omega$.

Proposition 1.9. *If G is a p-group which has no non-trivial p^n-bounded pure subgroups and $\phi \in \mathrm{End}(G)$ is such that $\phi \restriction p^nG$ is an idempotent endomorphism of p^nG, then there is an idempotent endomorphism θ of G such that $\theta \restriction p^nG = \phi \restriction p^nG$.*

Proof. Assume, for the moment that we have shown that if G has no non-trivial p^n-bounded pure subgroups, then an endomorphism ψ of G such that $p^n\psi = 0$ satisfies the relation $\psi^{n+1} = 0$.

Let $I = \{\varphi \in \mathrm{End}(G) | \varphi \restriction p^nG = 0\}$; it is easily checked that I is a 2-sided ideal of $\mathrm{End}(G)$. Moreover, any $\varphi \in I$ satisfies the relationship $p^n\varphi = 0$, and so, by the observation above, $\varphi^{n+1} = 0$; in particular I is a nil ideal of $\mathrm{End}(G)$. Since $\phi \restriction p^nG$ is an idempotent, it is immediate that $\phi + I$ is an idempotent of $\mathrm{End}(G)/I$. It now follows from standard ring theory - see e.g. [**1**, Proposition 27.1] - that idempotents lift modulo I. Thus there is an idempotent endomorphism θ of G such that $\theta + I = \phi + I$ and so $\theta \restriction p^nG = \phi \restriction p^nG$. It remains only to verify the claim in the first paragraph. This follows from the next lemma. □

The next result is true in a wider context than p-groups; the second author learned it from an unpublished manuscript of Tony Corner and the proof below is taken from that source.

Lemma 1.10. *If G is a p-group which has no non-trivial p^k-bounded pure subgroups and ψ is an endomorphism of G such that $p^k\psi = 0$, then $\psi^{k+1} = 0$.*

Proof. If $k = 0$ there is nothing to prove; so we suppose that $k \geq 1$. Note first that if $x \in G$ and the exponent of x, $E(x) = 1$, then $h_G(x) \geq k$. For if $h_G(x) = l < k$, then $x = p^l y$ for some $y \in G$, and it is clear that $\langle y\rangle$ is a pure subgroup of order p^{l+1}, a factor of p^k, contrary to our hypothesis.

Let $\mathcal{P}(n)$ denote the proposition: $x \in G, E(x) = n \leq k \Rightarrow x\psi^n = 0$. We prove $\mathcal{P}(n)$ by induction on n. Since $\mathcal{P}(0)$ is trivial, we may suppose that $1 \leq n \leq k$ and that $\mathcal{P}(r)$ is true for $r < n$. If $x \in G$ and $E(x) = n$, then $E(p^{n-1}x) = 1$, so $h_G(p^{n-1}x) \geq k$ and therefore $p^{n-1}x = p^k z$ for some $z \in G$. So $p^{n-1}(x\psi) = z(p^k\psi) = 0$, whence $E(x\psi) \leq n-1$ and so $(x\psi)\psi^{n-1} = 0$ i.e. $x\psi^n = 0$.

Since for each $x \in G$ we have $p^k(x\psi) = 0$, so that $E(x\psi) \leq k$, therefore it follows that $x\psi^{k+1} = (x\psi)\psi^k = 0$. Thus $\psi^{k+1} = 0$. □

We now have all the ingredients to prove:

Theorem 1.11. *If G is a p-group and $\pi \in \mathrm{End}(p^nG)$ is an idempotent, then there is an idempotent $\theta \in \mathrm{End}(G)$ such that $\theta \restriction p^nG = \pi$.*

Proof. Let $G = (B_1 \oplus \cdots \oplus B_n) \oplus X$ be the Baer decomposition of G; thus $(B_1 \oplus \cdots \oplus B_n)$ is a p^n-bounded summand of a basic subgroup B of G. It follows by a well-known theorem of Szele [**7**, Theorem 33.2], that X has no non-trivial p^n-bounded pure subgroup (or equivalently no non-trivial p^n-bounded summand). Note that $p^nG = p^nX$, so that $\pi \in \mathrm{End}(p^nX)$ and $\pi^2 = \pi$.

Since every endomorphism of p^nX lifts to an endomorphism of X - this follows by a simplification of the proof of [**7**, Proposition 113.3] - there is an endomorphism ϕ of X such that $\phi \restriction p^nX = \pi$. Now apply Proposition 1.9 to obtain an idempotent endomorphism θ_1 of X such that $\theta_1 \restriction p^nX = \phi \restriction p^nX = \pi$. Define $\theta : G \to G$ by $\theta = 0 \oplus \theta_1$; clearly θ is an idempotent endomorphism of G and $\theta \restriction p^nG = \pi$. □

We can now prove the desired partial converse to Proposition 1.3.

Theorem 1.12. *If n is a non-negative integer and p^nG is a projectively socle-regular p-group, then G is a projectively socle-regular p-group.*

Proof. Let P be a projection-invariant subgroup of G. If $\min(P[p])$ is finite, k say, then $P[p] = (p^kG)[p]$ by Proposition 1.1. If $\min(P[p]) \geq \omega$, then $P[p] \leq p^\omega G \leq p^nG$. We claim that $P[p]$ is a projection-invariant subgroup of p^nG: if π is any idempotent endomorphism of p^nG, then by Theorem 1.11, there is an idempotent endomorphism θ of G such that $\theta \restriction p^nG = \pi$. Hence, $(P[p])\pi = (P[p])\theta \leq P[p]$ since $P\theta \leq P$. As p^nG is projectively socle-regular, we have that $P[p] = (p^\alpha(p^nG))[p]$ for some ordinal α. Thus $P[p] = (p^\gamma G)[p]$ where $\gamma = n + \alpha$ and G is projectively socle-regular. □

We have already seen in Proposition 1.7 that there is a socle-regular p-group which is not projectively socle-regular. We now exhibit a strongly socle-regular 2-group which is not projectively socle-regular.

Proposition 1.13. *There is a strongly socle-regular 2-group which is not projectively socle-regular.*

Proof. Consider any of the groups C constructed by Corner in [**2**] which were transitive but not fully transitive; these groups had the property that $2^\omega C =$

$\langle a\rangle \oplus \langle b\rangle$ where $o(a) = 2, o(b) = 8$ and $\mathrm{End}(C) \restriction 2^\omega C = \Phi$, where Φ is the subring generated by the automorphisms of $\langle a\rangle \oplus \langle b\rangle$.

It is shown in [**8**, Example 3.16] that the elements of Φ can be described by two families $\{\theta_{i\lambda}\}$ and $\{\phi_{j\mu}\}$ with the parameters $1 \leq i, j \leq 4$ and $\lambda \in \{\pm 1, \pm 3\},\ \mu \in \{0, \pm 1, 2\}$. A straightforward check using the definitions given in Example 3.16 of [**8**], reveals that the only idempotent endomorphisms in Φ are 0 and 1.

Let $P = \langle a\rangle$; we claim that P is a projection-invariant subgroup of C. For if $\pi \in \mathrm{End}(C)$ is an idempotent, then $\pi \restriction P$ is an idempotent in Φ and so is either 0 or 1. In either case $P\pi \leq P$ and so P is a projection-invariant subgroup of C. However direct calculation shows that $P[2] = P$ is not any of the subgroups $(2^\omega C)[2], (2^{\omega+1}C)[2], (2^{\omega+2}C)[2], (2^{\omega+3}C)[2] = 0$. Since $a \in P[2]$ has height ω in C, $P[2] \neq (p^n C)[2]$ for any finite n. Thus C is not projectively socle-regular but it is strongly socle-regular since it is transitive - see [**4**, Theorem 2.4]. □

The class of projectively socle-regular groups is, however, large. Recall that Megibben [**14**] has shown that a transitive fully transitive group G, which satisfies the technical condition $(^*)$ on its Ulm invariants, has the property that every projection-invariant subgroup P of G is fully invariant; in particular he noted that the class of totally projective groups satisfies the property $(*)$. Consequently we have:

Proposition 1.14. *Transitive, fully transitive p-groups satisfying the condition $(^*)$ are projectively socle-regular. In particular, totally projective p-groups and C_λ-groups of length λ, where λ has cofinality ω, are projectively socle-regular.*

Proof. If P is a projection-invariant subgroup of G, then it follows from Megibben's theorem [**14**] noted above, that P is fully invariant in G. In view of [**3**, Theorem 0.3], G is socle-regular. Thus $P[p] = (p^\alpha G)[p]$ for some ordinal α, whence G is projectively socle-regular as required. Since a totally projective group is both transitive and fully transitive by Theorems 1.2 of [**10**] and satisfies the condition $(*)$, it is projectively socle-regular.

To show the projective socle-regularity of C_λ-groups of length λ, where λ has cofinality ω - see e.g., [**16**, Chapter 5], [**13**] or [**17**] for further details of C_λ-groups - note that, as observed by Megibben [**14**, p. 179], such a C_λ-group satisfies the technical condition $(*)$ - this is essentially because a C_λ-group of length λ, where λ has cofinality ω, has a λ-basic subgroup whose decompositions lift to the whole group (see [**16**, Lemma 30.2] and the λ-basic subgroup satisfies $(*)$ since it is totally projective. Hence, if we can show that G is both transitive and fully transitive, the desired result will follow from the argument above.

Now such a C_λ-group G of length λ, is λ-separable when λ has cofinality ω - see [**16**, Corollary 31.3]. Hence if $x, y \in G$ with $U_G(x) = U_G(y)$ (resp. $U_G(x) \leq U_G(y)$), there is a direct summand H of G, say $G = H \oplus K$, such that H is totally projective and $\langle x, y\rangle \leq H$. Since H is a direct summand of G, we have $U_H(x) = U_G(x)$ and $U_H(y) = U_G(y)$. However, a totally projective group is both transitive and fully transitive, so there exists an automorphism θ (resp. an endomorphism ϕ) of H with $x\theta = y$ (resp. $x\phi = y$). Since θ, ϕ extend to maps $\theta \oplus 1_K, \phi \oplus 1_K$ which are, respectively, an automorphism and an endomorphism of G, we have that G is both transitive and fully transitive, as required. □

We return now to consideration of the converse of Proposition 1.3; recall that in Theorem 1.12 we showed that the property of being projectively socle-regular lifts from the subgroup p^nG to the whole group G. To extend this result to the ordinal ω and beyond, it seems inevitable that we need some condition relating to total projectivity.

THEOREM 1.15. *(i) If $G/p^\omega G$ is a direct sum of cyclic groups and $p^\omega G$ is projectively socle-regular, then G is projectively socle-regular;*

(ii) If α is an ordinal strictly less than ω^2 such that $p^\alpha G$ is projectively socle-regular and $G/p^\alpha G$ is totally projective, then G is projectively socle-regular;

(iii) If $G/p^\beta G$ is totally projective and $p^\beta G$ is separable, then G is projectively socle-regular.

PROOF. (i) Arguing as in the proof of Theorem 1.12, it suffices to consider an arbitrary projection-invariant subgroup P of G, where $P \leq p^\omega G$. We claim that $P[p]$ is a projection-invariant subgroup of $p^\omega G$. Assuming this, it follows immediately that $P[p] = (p^\alpha(p^\omega G))[p] = (p^{\omega+\alpha}G)[p]$ for some ordinal α and so G is projectively socle-regular. To complete the proof of (i) it remains only to substantiate the claim. To this aim, supposing π is an arbitrary idempotent endomorphism of $p^\omega G$, it then follows from [**11**, Theorem 11] that there is an idempotent endomorphism η of G such that $\eta \restriction p^\omega G = \pi$. But then $P[p]\pi = P[p]\eta \leq P[p]$ since P is a projection-invariant subgroup of G.

The proof of (ii) is by transfinite induction; note that when α is finite or equal to ω, the result follows from Theorem 1.12 and (i) above. Since the argument follows exactly as in the proof of Proposition 1.6 (v) of [**4**], we simply refer the reader there.

For the final part (iii) we note firstly that if $l(p^\beta G) = n < \omega$, then $p^\beta G$ is a direct sum of cyclic groups and so G itself is totally projective by a well-known theorem of Nunke [**15**]. The result then follows from Proposition 1.14 above. Suppose then that $l(p^\beta G) = \omega$. Observe that G is then a C_λ-group of length $\lambda = \beta + \omega$ and λ has cofinality ω. To see this note that if $\sigma \leq \beta$, then $p^\sigma(G/p^\beta G) = p^\sigma G/p^\beta G$ and so $G/p^\sigma G \cong (G/p^\beta G)/p^\sigma(G/p^\beta G)$ is totally projective by the previously quoted result of Nunke. If $\beta < \sigma < \lambda$, then σ has the form $\beta + n$ and so if $X = G/p^{\beta+n}G$, $p^\beta X = p^\beta G/p^{\beta+n}G$ implying that $X/p^\beta X \cong G/p^\beta G$ is totally projective. Since $p^\beta X = p^\beta G/p^{\beta+n}G$, we have that $p^\beta X$ is a direct sum of cyclic groups and so X is again totally projective. It follows then G is a C_λ-group and clearly it has length λ. The result then follows from Proposition 1.14 above. $\square$

The next assertion demonstrates that certain subgroups inherit projective socle-regularity.

PROPOSITION 1.16. *If G is a projectively socle-regular p-group and P is a projection-invariant subgroup of G with the same first Ulm subgroup, then P is projectively socle-regular.*

PROOF. Suppose K is an arbitrary projection-invariant subgroup of P. Since the projection-invariant property is obviously transitive, it follows that K is a projection-invariant subgroup of G. Therefore there is an ordinal α such that $K[p] = (p^\alpha G)[p]$. If $\alpha \geq \omega$, it follows at once that $K[p] = (p^\alpha P)[p]$ and we are done. If now α is a finite ordinal number, say t, then $K[p] = (p^t G)[p] \geq (p^t P)[p]$ and so it is easy to check that $\min_P(K[p])$ is finite. Furthermore, Proposition 1.1 applies to infer that $K[p] = (p^s P)[p]$ for some natural s, as required. $\square$

Let L denote a large subgroup of a p-group G. Using standard group-theoretic facts about L (see, e.g., [**16**]), a direct consequence is the following:

COROLLARY 1.17. *If G is projectively socle-regular p-group and L is a large subgroup of G, then L is projectively socle-regular.*

REMARK 1.18. It is worthwhile noticing that the direct sum of two projectively socle-regular groups need not be projectively socle-regular. In fact, consider the example based on an idea of Megibben that has been used in [**3**] and [**4**]: let A and B be p-groups with $p^\omega A \cong p^\omega B \cong \mathbb{Z}(p)$ such that $A/p^\omega A$ is a direct sum of cyclic groups and $B/p^\omega B$ is torsion-complete. We have shown in [**3**, Theorem 1.6] that their direct sum $G = A \oplus B$ is not socle-regular and hence it is clearly not projectively socle-regular. However, utilizing Corollary 1.2, we observe that both A and B are projectively socle-regular but their direct sum is not.

However, as observed in Proposition 1.1, separable p-groups are always projectively socle-regular and so one would expect that the addition of a separable summand to a projectively socle-regular group would result in a projectively socle-regular group; this is, indeed, the case: if G is projectively socle-regular and H is separable, then $A = G \oplus H$ is projectively socle-regular. The proof follows exactly as the proof of the corresponding statement for strongly socle-regular groups - see [**4**, Proposition 3.2]. The converse situation i.e., if G is a summand, with separable complement, of a projectively socle-regular group A, whether G is necessarily projectively socle-regular is not clear and one encounters similar difficulties to those experienced for strongly socle-regular groups - see the discussion and results following the proof of [**4**, Proposition 3.2].

2. Direct Powers

There is another source of projectively socle-regular groups which can be easily exhibited. In fact groups G of the form $G = H^{(\kappa)}$, where H is any p-group and κ is a cardinal, have the property that every projection-invariant subgroup of G is fully invariant in G. We begin with a result which is presumably well known but we could not find an explicit reference to it.

PROPOSITION 2.1. *Let R be an arbitrary ring, then every $\Delta \in M_n(R)$, the ring of (finite) $n \times n$ matrices over R $(n > 1)$, can be expressed as a finite sum $\Delta = \Delta_1 + \cdots + \Delta_k$, where each Δ_i is either idempotent or a product of two idempotents.*

PROOF. We consider first the case where $n = 2$. Let $\Delta = \left(\begin{smallmatrix} r & s \\ u & v \end{smallmatrix}\right)$ and set $\Delta_1 = \left(\begin{smallmatrix} 0 & s \\ 0 & 1 \end{smallmatrix}\right)$, $\Delta_2 = \left(\begin{smallmatrix} 0 & 0 \\ u & 1 \end{smallmatrix}\right)$, $\Delta_3 = \left(\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\Delta_4 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & v-2 \end{smallmatrix}\right)$; clearly $\Delta = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$. A straightforward check shows that Δ_1, Δ_2 are both idempotent.

However, $\Delta_3 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ r-1 & 0 \end{smallmatrix}\right) = XY$ say, and it is easy to check that both X, Y are idempotent.

Finally $\Delta_4 = \left(\begin{smallmatrix} 0 & 0 \\ v-3 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) = ZW$ say, and again it is straightforward to check that both Z, W are idempotents.

So the result is true for $n = 2$. Proceeding by induction, assume the result holds for $n = k$ and consider $\Delta = \left(\begin{smallmatrix} A & b \\ c & d \end{smallmatrix}\right)$, a $(k+1) \times (k+1)$ matrix, where A is $k \times k$, $d \in R$, b is a $k \times 1$ column vector and c is a $1 \times k$ row vector. Let O denote the $k \times k$ zero matrix.

Then $\Delta = \left(\begin{smallmatrix} O & b \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} O & 0 \\ c & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} O & 0 \\ 0 & d-2 \end{smallmatrix}\right)$.

The first two matrices in this sum are easily seen to be idempotent and an identical argument to that used in the 2×2 case, shows that the final matrix is a product of two idempotents.

By induction the matrix A may be expressed as $A = A_1 + \cdots + A_t$ where each A_i is either idempotent or a product of two idempotents. However, if X is an idempotent $k \times k$ matrix, then $\left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is an idempotent $(k+1)\times(k+1)$ matrix, while if X, Y are idempotent then $\left(\begin{smallmatrix} XY & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} Y & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a product of two $(k+1)\times(k+1)$ idempotent matrices. Thus the remaining matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ can be expressed as a sum of idempotents and products of two idempotents. Therefore the matrix Δ has the same property and the proof is completed by induction. $\square$

PROPOSITION 2.2. *If G has the form $G = H^{(\kappa)}$, for some group H and some cardinal κ, then every projection-invariant subgroup of G is fully invariant in G.*

PROOF. If κ is finite, then every endomorphism of G can be expressed as a $\kappa \times \kappa$ matrix Δ over the ring $R = \mathrm{End}(H)$. Thus if P is a projection-invariant subgroup of G, then $P\Delta = P(\Delta_1 + \cdots + \Delta_t)$ where each Δ_i is either idempotent or a product of idempotents. It is immediate that $P\Delta \leq P$ and hence P is fully invariant in G. If κ is infinite, then $G = A \oplus A$, where $A \cong H^{(\kappa)}$ and so every endomorphism of G is a 2×2 matrix over the ring $S = \mathrm{End}(A)$. Again every endomorphism of G is a sum of products of idempotent matrices over S by Proposition 2.1 and so is a sum of products of idempotent endomorphisms of G. The result follows as before. $\square$

It is now easy to exhibit many projectively socle-regular groups.

THEOREM 2.3. *If G is socle-regular, then $G^{(\kappa)}$ is projectively socle-regular for all $\kappa > 1$. In particular, if G is projectively socle-regular, then so also is $G^{(\kappa)}$ for all κ.*

PROOF. Suppose that P is a projection-invariant subgroup of $G^{(\kappa)}$ for some $\kappa > 1$, then, by Proposition 2.2, P is a fully invariant subgroup of $G^{(\kappa)}$. However, it follows from [**3**, Theorem 1.4] that $G^{(\kappa)}$ is socle-regular and so $P[p] = (p^\alpha G^{(\kappa)})[p]$ for some α. Hence $G^{(\kappa)}$ is projectively socle-regular. The final comment is immediate. $\square$

The next surprising statement illustrates that in some cases socle-regularity, strong socle-regularity and projective socle-regularity do coincide.

THEOREM 2.4. *Suppose $\kappa > 1$. Then the following four points are equivalent:*
(i) G is socle-regular;
(ii) $G^{(\kappa)}$ is socle-regular;
(iii) $G^{(\kappa)}$ is strongly socle-regular;
(iv) $G^{(\kappa)}$ is projectively socle-regular.

PROOF. The equivalences $(i) \iff (ii) \iff (iii)$ follow from [**4**, Corollary 3.7] and [**5**, Corollary 3.2]. Moreover, the implication $(i) \Rightarrow (iv)$ follows from Theorem 2.3, whereas the implication $(iv) \Rightarrow (ii)$ is trivial. $\square$

However, the class of projectively socle-regular groups is not closed under taking summands.

PROPOSITION 2.5. *There exists a projectively socle-regular group H having a direct summand G which is not projectively socle-regular.*

Proof. Let G be a socle-regular group which is not projectively socle-regular-for example, the group in Proposition 1.7. Then if $H = G \oplus G$, it follows from Theorem 2.3 that H is projectively socle-regular, but clearly its direct summand G is not projectively socle-regular. □

We close the paper with some open questions:

Problem 1. Are Krylov transitive p-groups satisfying condition $(^*)$ projectively socle-regular groups?

Problem 2. If G is a socle-regular p-group with finite $p^\omega G$, does it follow that G is projectively socle-regular?

Problem 3. Is it true that projectively socle-regular 2-groups are strongly socle-regular?

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Looking for indecomposable right bounded complexes

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Dedicated to Professor Rüdiger Göbel on the occasion of his 70th birthday.

Abstract. We investigate indecomposable right bounded complexes of projective modules orthogonal to rather special partial tilting complexes.

Introduction

Some "proper" non classical partial tilting modules T have the following property: even though their projective resolution $\dot{T}$ is not a tilting complex, for every non zero module M, there is a morphism from $\dot{T}$ to a shift of the projective resolution of M which is not homotopic to zero.

In this note we investigate indecomposable complexes $\dot{C}$, not left bounded, such that every morphism from $\dot{T}$ to any shift of $\dot{C}$ is homotopic to zero. In Section 1 we fix the notation and the conventions used in the sequel. In Section 2 we describe three completely different situations, where there is an indecomposable not left bounded complex "orthogonal" with all its shifts to a partial tilting complex and with indecomposable non–zero components.

1. Preliminaries

Let R be a ring. We denote by $R-\mathrm{Mod}$ the category of all left R–modules. If $M \in R-\mathrm{Mod}$, then we write $\operatorname{Add} M$ for the class of all modules isomorphic to direct summands of direct sums of copies of M. Next, for every cardinal λ, we write $M^{(\lambda)}$ for the direct sum of λ copies of M. Finally, we write $M^{\perp_\infty}$ for the following class

$$M^{\perp_\infty} = \left\{X \in R-\mathrm{Mod} \,|\, \mathrm{Ext}^i_R(M, X) = 0 \text{ for all } i \geq 1\right\}.$$

The symbol $\mathrm{pdim}(M)$ denotes the projective dimension of M.

We shall say that an R–module T is a *partial n–tilting module* if $\mathrm{pdim}(T) \leq n$ and $\mathrm{Ext}^i_R(T, T^{(\lambda)}) = 0$ for every $i \geq 1$ and every cardinal λ. Given a partial n–tilting module T, we shall say that T is an *n–tilting module* if there is a long exact sequence of the form

$$0 \longrightarrow R \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \ldots \longrightarrow T_n \longrightarrow 0,$$

where $T_i \in \operatorname{Add} T$ for every $i = 0, 1, \ldots, n$.

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From now on, we shall say, for brevity, that a partial n–tilting module T is a *large partial n–tilting module* if

$$\operatorname{Ker}\operatorname{Hom}(T,-)\cap T^{\perp_\infty}=0.$$

Maintaining the terminology introduced above, we recall some properties of partial n–tilting modules. First of all, for every $n \geq 1$, every n–tilting module T is a large partial n–tilting module [**B**, page 371]. Secondly, a finitely presented module T is a 1-tilting module if and only if T is a large partial 1–tilting module [**C**, Theorem 1] (also see [**CbF**, Theorem 3.2.1 and Section 3.1]). Finally, for every $n \geq 2$, there exist non faithful decomposable large partial n–tilting modules of projective dimension n and Loewy length 2 [**D1**, Example 4].

Given an algebra with n simple modules and a multiplicity free partial tilting module T, then T is an *almost complete tilting* module ([**HU**] and [**R2**, page 413]) if T has exactly $n-1$ indecomposable direct summands.

Given a ring R, we write $K(R)$ for the category of complexes over R with morphisms modulo homotopy. Let $\dot{T} \in K(R)$ be a bounded complex of finitely generated projective R–modules. Assume the following conditions hold:

(1) $\operatorname{Hom}_{K(R)}(\dot{T},\dot{T}[i])=0$ for every $i \neq 0$,

(2) For every non–zero right bounded complex $\dot{X} \in K(R)$ of projective R–modules there exists some $i \in \mathbb{Z}$, such that $\operatorname{Hom}_{K(R)}(\dot{T},\dot{X}[i]) \neq 0$.

Then $\dot{T}$ is a *tilting complex*, in the sense of Rickard [**Rk**] as observed by Miyachi [**Mi**, condition (iii)', page 184]. In other words, the global, but functorial, condition (2) can replace a global non functorial condition on triangulated categories, which says the following:

- $\operatorname{add}\dot{T}$, the additive category of direct summands of finite direct sums of copies of $\dot{T}$, generates (as a triangulated category) the category of all bounded complexes of finitely generated projective R–modules.

Consequently, given a noetherian ring R, every tilting complex $\dot{T}$ satisfies the following condition ([**Sc-ZI**, condition (2) in the Definition of page 190]).

- If P is an indecomposable projective module and $\dot{P}$ is the stalk complex $0 \longrightarrow P \longrightarrow 0$, with P in degree 0, then $\dot{P}$ belongs to $\operatorname{add}\dot{T}$.

Finally, let $\dot{T} \in K(R)$ be a bounded complex of finitely generated projective R–modules. Then, following the definition of [**Sc-ZI**] for complexes over noetherian rings, we shall say that $\dot{T}$ is a *partial tilting complex* if $\operatorname{Hom}_{K(R)}(\dot{T},\dot{T}[i])=0$ for every $i \neq 0$.

Throughout the paper, given a module M, the symbol $\dot{M}$ denotes a right bounded complex of projective modules of the form $\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$, where P_0 is in degree 0 and $\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is a fixed projective resolution of M.

Moreover, we often say, for short, that a complex $\dot{W}$ is *orthogonal with all its shifts* (or just *orthogonal*) to a complex $\dot{V}$ if every morphism from $\dot{V}$ to any shift of $\dot{W}$ is homotopic to zero.

Next, K always denotes an algebraically closed field, and we always identify modules with their isomorphism classes (resp. complexes with their homotopy classes). Moreover, if Λ is a K–algebra given by a quiver and relations, according to [**R1**], then we often replace indecomposable finite dimensional modules by some obvious pictures, describing their composition factors. Over a representation–finite algebra given by a quiver, we often denote by x the simple module $S(x)$ corresponding to the vertex x.

When dealing with a complex $\dot{C}$, we often write d, instead of d_i, for the usual morphism $C_i \longrightarrow C_{i-1}$.

We end with the conventions used to describe morphisms between indecomposable projective modules P and Q (defined over the K–algebra Λ) with the following useful combinatorial property: the K–dimension of the vector space $V = \{f \in \mathrm{Hom}_\Lambda(P,Q) | f(P) \neq Q\}$ is at most one. First of all, the symbol $P \longrightarrow Q$ will denote a fixed generator v of V, and the symbol $P \xrightarrow{a} Q$ will denote the morphism av for all $a \in K$. Secondly, we shall use Greek or Latin letters $\alpha, \beta, \ldots, a, b, \ldots$ to denote arbitrary morphisms.

For unexplained terminology, we refer to [**AF**] and [**AuReS**].

2. Examples

We begin with two indecomposable left unbounded complexes of the form

$$\cdots \to P \to P \to P \to Q \to X \to 0,$$

where the modules P and Q are indecomposable modules, while the module X is either indecomposable or equal to 0.

Example 1 (Left cancellation of one component and left addition of infinitely many components). *There are a non faithful large partial 3–tilting module T, defined over a finite dimensional algebra A, and an indecomposable right bounded complex $\dot{C} \in K(A)$ with the following properties:*

(i) $\dot{C}$ is not left bounded, every non zero component of $\dot{C}$ is indecomposable and $\mathrm{Hom}_{K(A)}(\dot{T}, \dot{C}[i]) = 0$ for every $i \in \mathbb{Z}$.
(ii) The direct sum of the homology modules of $\dot{C}$ is a semisimple non homogeneous module of infinite dimension over K.
(iii) T is an almost complete tilting module.

Construction. Let A be the K–algebra given by the quiver

$$\begin{tikzcd} 3 \arrow[dr, "c"] & & \\ & 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd}$$

with relations $ac = 0$ and $ba = 0$, and let T denote the injective module $\begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix} \oplus 3$.

Then $\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$ and 1 are the indecomposable modules belonging to $\mathrm{Ker}\,\mathrm{Hom}_A(T, -)$, and we clearly have

$$\mathrm{Ext}^3_A\left(3\ ,\ \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right) \simeq \mathrm{Ext}^2_A\left(1\ ,\ \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right) \simeq \mathrm{Ext}^1_A\left(2\ ,\ \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right) \neq 0$$

and $\mathrm{Ext}^1_A(\ 3\ ,\ 1\ ) \neq 0$. Consequently T is a non faithful large partial 3–tilting module. Let $\dot{C}$ denote the indecomposable complex

$$\dots \longrightarrow \substack{2\\1\\2} \longrightarrow \substack{2\\1\\2} \longrightarrow \substack{1\\2} \longrightarrow \substack{3\\1} \longrightarrow 0\ .$$

Then $\dot{C}$ satisfies (ii) and (iii), and 2 is not a composition factor of $H(\dot{C})$. Hence we deduce from [**D2**, Lemma 1] that

(1) $\quad \mathrm{Hom}_{K(A)}\left(\dot{\substack{2\\1\\2}}\,,\dot{C}[i]\right) = 0$ for every i.

Let $\dot{\alpha}$ be the morphism described by

(2)
$$\begin{tikzcd}
0 \arrow[r] & \substack{1\\2} \arrow[r] \arrow[d, "x"'] & \substack{2\\1\\2} \arrow[r] \arrow[d, "\phi"'] & \substack{1\\2} \arrow[r] \arrow[d, "y"'] & \substack{3\\1} \arrow[r] \arrow[d, "0"'] & 0 \\
\dots \arrow[r] & \substack{2\\1\\2} \arrow[r] & \substack{2\\1\\2} \arrow[r] & \substack{2\\1\\2} \arrow[r] & P \arrow[r] & \dots
\end{tikzcd}$$

where $P = \substack{1\\2} \oplus \substack{2\\1\\2}$. Next, let $\dot{\beta}$ be the morphism described by:

(3)
$$\begin{tikzcd}
0 \arrow[r] & \substack{1\\2} \arrow[r] \arrow[d, "x'"'] & \substack{2\\1\\2} \arrow[r] \arrow[d, "\phi'"'] & \substack{1\\2} \arrow[r] \arrow[d, "y'"'] & \substack{3\\1} \arrow[r] \arrow[d, "z'"'] & 0 \\
\dots \arrow[r] & \substack{2\\1\\2} \arrow[r] & \substack{2\\1\\2} \arrow[r] & \substack{1\\2} \arrow[r] & \substack{3\\1} \arrow[r] & 0\ .
\end{tikzcd}$$

Then ϕ and ϕ' are not automorphisms, otherwise the first squares in (2) and (3) do not commute. Therefore, we have $z' = 0$. It is now easy to show that $\dot{\alpha}$ and $\dot{\beta}$ are homotopic to zero.

Next, let $\dot{\gamma}$ be the morphism described by

(4)
$$\begin{tikzcd}
0 \arrow[r] & \substack{1\\2} \arrow[r] \arrow[d, "x"'] & \substack{2\\1\\2} \arrow[r] \arrow[d, "y"'] & \substack{1\\2} \arrow[r] \arrow[d, "z"'] & \substack{3\\1} \arrow[r] & 0 \\
\dots \arrow[r] & \substack{2\\1\\2} \arrow[r] & \substack{1\\2} \arrow[r] & \substack{3\\1} \arrow[r] & 0\ . &
\end{tikzcd}$$

Also in this case, proceeding from left to right, we see that $\dot\gamma$ is homotopic to zero. On the other hand, a picture of the form

$$\text{(5)}\qquad \begin{tikzcd} & 0 \arrow[r] & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[r] \arrow[d, "x"'] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "0"'] & \dots \\ & \dots \arrow[r] & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[r] & 0 \end{tikzcd}$$

describes a morphism only if $x = 0$. Finally, any morphism of the form

$$\text{(6)}\qquad \begin{tikzcd} & 0 \arrow[r] & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[r] \arrow[d, "x"'] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \dots \\ \dots \arrow[r] & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[r] & 0 & \end{tikzcd}$$

is clearly homotopic to zero. Putting (2), ...,(5) and (6) together, we conclude that

(7) $\quad \mathrm{Hom}_{K(A)}(\dot 3, \dot C[i]) = 0$ for every $i \in \mathbb{Z}$.

Hence (i) follows from (1) and (7).

One can prove that, up to shift, $\dot C$ is the unique indecomposable right bounded complex satisfying (i). □

In the next example we replace the algebra of Example 1 by its opposite algebra.

Example 2 (Left cancellation of two non–zero components and left addition of infinitely many components). *There are a faithful large partial 3–tilting module T, defined over a finite dimensional K–algebra B, and an indecomposable right bounded complex $\dot C \in K(B)$ with the following properties:*

(i) $\dot C$ is not left bounded, every non–zero component of $\dot C$ is indecomposable and $\mathrm{Hom}_{K(B)}(\dot T, \dot C[i]) = 0$. for every $i \in \mathbb{Z}$.
(ii) $\dot C$ has infinitely many non–zero homology modules (all indecomposable) and one of them is not simple.
(iii) T is an almost complete tilting module.

Construction. Let B denote the K–algebra given by the quiver

$$\begin{tikzcd} & \overset{1}{\bullet} \arrow[r, bend left, "a"] \arrow[dl, "c"'] & \overset{2}{\bullet} \arrow[l, bend left, "b"] \\ \underset{3}{\bullet} & & \end{tikzcd}$$

with relations $ba = 0$ and $cb = 0$, and let T denote the injective module $\begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$.

Then 3 is the unique indecomposable module belonging to $\mathrm{Ker}\,\mathrm{Hom}_B(T, -)$ and

$\operatorname{Ext}^3_B\left(\begin{smallmatrix}1\\3\end{smallmatrix},3\right) \simeq \operatorname{Ext}^2_B(2,3) \simeq \operatorname{Ext}^1_B\left(\begin{smallmatrix}1\\2\end{smallmatrix},3\right) \neq 0$. Thus T is a faithful large partial 3–tilting module. Now let $\dot{C}$ denote the indecomposable complex

$$\cdots \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\,3\end{smallmatrix} \longrightarrow 0 \ .$$

Then (ii) and (iii) holds, and 2 is not a composition factor of $H(\dot{C})$. By [**D2**, Lemma 1], this implies that

$$(1) \qquad \operatorname{Hom}_{K(B)}\left(\overset{\cdot}{\begin{smallmatrix}2\\1\\2\end{smallmatrix}},\dot{C}[i]\right) = 0 \text{ for every } i.$$

Let $\dot{\alpha}$ be a morphism described by

$$(2) \qquad \begin{tikzcd} 0 \arrow[r] & 3 \arrow[r]\arrow[d,"0"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r]\arrow[d,"x"'] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r]\arrow[d,"\phi"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r]\arrow[d,"y"'] & 0 \\ \cdots \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \cdots \end{tikzcd}$$

and let $\dot{\beta}$ be a morphism described by

$$(3) \qquad \begin{tikzcd} 0 \arrow[r] & 3 \arrow[r]\arrow[d,"0"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r]\arrow[d,"x'"'] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r]\arrow[d,"\phi'"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r]\arrow[d,"y'"'] & 0 \\ \cdots \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] & 0 \ . \end{tikzcd}$$

Then ϕ and ϕ' are not automorphisms, otherwise the second squares in (2) and (3) would not commute. It is now easy, proceeding from left to right, to show that $\dot{\alpha}$ and $\dot{\beta}$ are homotopic to zero. We also note that all the morphisms of the form

$$(4) \qquad \begin{tikzcd} 0 \arrow[r] & 3 \arrow[r]\arrow[d,"0"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r]\arrow[d,"x"'] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r]\arrow[d,"y"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] & 0 \\ \cdots \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] & 0 & \end{tikzcd}$$

and

$$
\begin{tikzcd}
0 \arrow[r] & 3 \arrow[r] \arrow[d, "x"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] & \dots \\
\dots \arrow[r] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] & 0 &
\end{tikzcd}
\tag{5}
$$

are homotopic to zero. Finally, a picture of the form

$$
\begin{tikzcd}
0 \arrow[r] & 3 \arrow[r] \arrow[d, "0"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] \arrow[d, "x"'] & \dots \\
\dots \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] & 0
\end{tikzcd}
\tag{6}
$$

describes a morphism only if $x = 0$. Consequently, we deduce from (2),...,(6) that

$$
\mathrm{Hom}_{K(B)}\left(\overset{\cdot}{\begin{smallmatrix}1\\3\end{smallmatrix}}, \dot{C}[i] \right) = 0 \text{ for every } i. \tag{7}
$$

This remark and (1) show that (i) holds. $\square$

The next example shows that the complex $\dot{C}$ of Example 2 is somehow a "limit" of bounded complexes, all orthogonal to $\dot{T}$ together with all their shifts.

EXAMPLE 3 (Left cancellation; left cancellation and central addition). *For any $n \geq 3$ we can find B and T as in the hypotheses of Example 2, and an indecomposable bounded complex $\dot{D} \in K(B)$ with the following properties:*

(i) $\dot{D}$ has exactly n components different from zero (all indecomposable) and $\mathrm{Hom}_{K(B)}\left(\dot{T}, \dot{D}[i]\right) = 0$ for every $i \in \mathbb{Z}$.

(ii) $\dot{D}$ has $n-1$ non–zero homology modules (all indecomposable) and one of them is not simple.

CONSTRUCTION. Fix any $n \geq 3$, and let $\dot{D}$ be the indecomposable complex

$$
0 \longrightarrow \begin{smallmatrix}1\\2\,3\end{smallmatrix} \longrightarrow \underbrace{\begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \dots \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix}}_{n-2} \longrightarrow \begin{smallmatrix}1\\2\,3\end{smallmatrix} \longrightarrow 0 \ .
$$

Then (ii) holds, and we deduce from [**D2**, Lemma 1] that

(1) $\quad \mathrm{Hom}_{K(B)}\left(\begin{smallmatrix}\dot{2}\\1\\2\end{smallmatrix}, \dot{D}[i]\right) = 0$ for every i.

Let $\dot{\alpha}$ be the morphism described by

(2)
$$\begin{tikzcd}
0 \arrow[r] & 3 \arrow[r] \arrow[d, "x"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] \arrow[d, "y"'] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "\phi"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] \arrow[d, "z"'] & 0 \\
0 \arrow[r] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & P \arrow[r] & Q \arrow[r] & \cdots
\end{tikzcd}$$

where the second line contains an indecomposable complex, and we have either $P = Q = \begin{smallmatrix}2\\1\\2\end{smallmatrix}$, or $P = \begin{smallmatrix}2\\1\\2\end{smallmatrix}, Q = \begin{smallmatrix}1\\2\,3\end{smallmatrix}$, or $P = \begin{smallmatrix}1\\2\,3\end{smallmatrix}$, $Q = 0$. Then ϕ cannot be injective and we always have $z = 0$. Proceeding from left to right, we conclude that $\dot{\alpha}$ is homotopic to zero. Next, let $\dot{\beta}$ be the morphism described by

(3)
$$\begin{tikzcd}
0 \arrow[r] & 3 \arrow[r] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] \arrow[d, "x"'] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "\phi"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] \arrow[d, "y"'] & 0 \\
& 0 \arrow[r] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & P \arrow[r] & \cdots
\end{tikzcd}$$

where $P = \begin{smallmatrix}1\\2\,3\end{smallmatrix}$ if $n = 3$, and $P = \begin{smallmatrix}2\\1\\2\end{smallmatrix}$ if $n > 3$. Then $x = 0$, ϕ is not an automorphism and $y = 0$. In this case, proceeding either from left to right or conversely, we conclude that $\dot{\beta}$ is homotopic to zero. Finally, let $\dot{\gamma}$ be the morphism described by

(4)
$$\begin{tikzcd}
0 \arrow[r] & 3 \arrow[r] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "x"'] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] \arrow[d, "y"'] & 0 \\
& & 0 \arrow[r] & \begin{smallmatrix}1\\2\,3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \arrow[r] & \cdots
\end{tikzcd}$$
.

Then x (resp. y) factors on the right (resp. left) through an endomorphism of $\begin{smallmatrix}1\\2\,3\end{smallmatrix}$. Therefore $\dot{\gamma}$ is homotopic to zero.

On the other hand, for any morphism of the form

$$(5)\qquad \begin{tikzcd} \cdots \arrow[r] & \begin{smallmatrix} 2\\ 1\\ 2 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1\\ 2\ 3 \end{smallmatrix} \arrow[r] \arrow[d, "x"'] & 0 & \\ & 0 \arrow[r] & \begin{smallmatrix} 1\\ 2\ 3 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 2\\ 1\\ 2 \end{smallmatrix} \arrow[r] & \cdots \end{tikzcd}$$

we obviously have $x = 0$. Combining the above remarks with (2),. . . ,(5) and (6) in Example 2, we get

$$(6)\qquad \mathrm{Hom}_{K(B)}\left(\dot{\begin{smallmatrix} 1\\ 3 \end{smallmatrix}}, \dot{D}[i] \right) = 0 \text{ for every } i.$$

Hence (i) follows from (1) and (6). □

One can prove that, up to shift, the right bounded complex $\dot{C}$ and the bounded complexes $\dot{D}$, constructed in Examples 2 and 3 respectively, are the unique indecomposable right bounded complexes satisfying condition (i) of the same examples.

The next example describes a completely different situation.

Example 4 (Lego–type construction). *There are a K–algebra R, a large partial 5–tilting module T, and complexes $\dot{C}(m)$ with $1 \le m \le \aleph_0$ with the following properties:*

- *(i) $T = P \oplus S$ with P indecomposable projective–injective, S simple injective, but T is not an almost complete tilting module.*
- *(ii) Every non–zero component of $\dot{C}(m)$ is an indecomposable projective module, and exactly m components of $\dot{C}(m)$ are equal to P.*
- *(iii) $\dot{C}(m)$ is orthogonal to $\dot{T}$, but no proper subcomplex of $\dot{C}(m)$ is orthogonal to $\dot{T}$.*
- *(iv) Every component of $\dot{C}(m)$ is injective if and only if $m \ge 2$.*
- *(v) Up to shift, there exist $2^{\aleph_0}$ indecomposable right bounded complexes orthogonal to $\dot{T}$ and with indecomposable non–zero components.*

Construction. Let R be the K–algebra given by the quiver

$$\begin{tikzcd} \underset{4}{\bullet} \arrow[dr, "d"] & & & \\ & \underset{1}{\bullet} \arrow[r, "a"] & \overset{2}{\bullet} \arrow[r, "b"] & \overset{3}{\bullet} \arrow[ll, bend left, "c"] \end{tikzcd}$$

with relations $cba = 0$ and $ad = 0$. Let T denote the module $\begin{smallmatrix} 2\\ 3\\ 1\\ 2 \end{smallmatrix} \oplus\ 4$. Then T is injective, $p\dim T = 5$ and the indecomposable modules belonging to $\mathrm{Ker}\,\mathrm{Hom}_R(T, -)$ are 1 , 3 , $\begin{smallmatrix} 3\\ 1 \end{smallmatrix}$, $\begin{smallmatrix} 4\\ 1 \end{smallmatrix}$ and $\begin{smallmatrix} 3\ 4\\ 1 \end{smallmatrix}$. Since

$$\mathrm{Ext}^5_R(4, -) \simeq \mathrm{Ext}^4_R(1, -) \simeq \mathrm{Ext}^3_R(\begin{smallmatrix} 2\\ 3 \end{smallmatrix}, -) \simeq \mathrm{Ext}^2_R(\begin{smallmatrix} 1\\ 2 \end{smallmatrix}, -) \simeq \mathrm{Ext}^1_R(3, -),$$

it immediately follows that T is large. Hence (i) holds.

Next, let $\dot{X}$ be the indecomposable complex

$$(1)\qquad 0 \longrightarrow \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \longrightarrow 0\ .$$

Finally, let $\dot{Y}$ be the indecomposable complex

$$(2)\qquad 0 \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\1\end{smallmatrix} \longrightarrow 0\ .$$

We first note that $\dot{4}$ is of the form

$$(3)\qquad 0 \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\1\end{smallmatrix} \longrightarrow 0\ .$$

Consequently, we obtain $\dot{Y}$ from $\dot{4}$ after left cancellation of two components. On the other hand $\dot{T}$ is a partial tilting complex. Therefore any morphism from $\dot{4}$ to a shift of $\dot{Y}$ of the form

$$(4)\qquad \begin{tikzcd} 0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "a"'] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "b"'] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "c"'] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \cdots \\ \cdots \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0 & \end{tikzcd}$$

is homotopic to zero.

Dually, let $\dot{\alpha}$ be a morphism from $\dot{4}$ to a shift of $\dot{Y}$ of the form

$$(5)\qquad \begin{tikzcd} \cdots \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r, "d"] \arrow[d, "\phi"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "\psi"] & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[r] \arrow[d, "0"] & 0\ . \\ & 0 \arrow[r] & L \arrow[r, "d'"] & M \arrow[r] & N \arrow[r] & \cdots \end{tikzcd}$$

If $L = M = 0$ (resp. $L = 0$, and $M = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$), then we have $\phi = \psi = 0$. If $L = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ and $M = \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$, then there is a morphism h such that $\phi = h \circ d$ and $\psi = d' \circ h$. Hence, $\dot{\alpha}$

is always homotopic to zero. Let now $\dot\beta$ be a morphism of the form

(6)
$$\begin{tikzcd}
0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "a"] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "b"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "c"] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "0"] & \dots \\
0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[r] & 0
\end{tikzcd}$$

Proceeding from right to left, we obtain $c = b = a = 0$. Therefore $\dot\beta$ is homotopic to zero. Next, let $\dot\gamma$ be a morphism of the form

(7)
$$\begin{tikzcd}
0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "f"'] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "g"'] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "h"'] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "l"'] & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[r] & 0 . \\
 & 0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[r] & 0 &
\end{tikzcd}$$

In this case, proceeding from left to right, the existence of suitable endomorphisms of $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$, $\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$ and $\begin{smallmatrix}4\\1\end{smallmatrix}$ shows that $\dot\gamma$ is homotopic to zero. Finally, let $\dot\delta$ be a morphism of the form

(8)
$$\begin{tikzcd}
\dots \arrow[r] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r, "d"] \arrow[d, "m"] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "\sigma"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "p"] & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[r] \arrow[d, "q"] & 0 . \\
 & 0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r, "d'"] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[r] & 0
\end{tikzcd}$$

Proceeding from left to right, we have $m = 0$, $\sigma^2 = 0$ and $p = q = 0$. Hence there is a morphism r such that $r \circ d = m$ and $d' \circ r = \sigma$. Therefore $\dot\delta$ is homotopic to zero. This observation completes the proof that

(9) $\quad \mathrm{Hom}_{K(R)}(\dot 4, \dot Y[i]) = 0$ for every $i \in \mathbb{Z}$.

Now, let $\dot{\alpha}$ be a morphism from $\dot{4}$ to a shift of $\dot{X}$ of the form

$$\text{(10)}\qquad \begin{tikzcd} \cdots \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r, "d"] \arrow[d, "f"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "g"] & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[r] \arrow[d, "0"] & 0 \; . \\ & 0 \arrow[r] & L \arrow[r, "d'"] & M \arrow[r] & N \arrow[r] & \cdots \end{tikzcd}$$

If $L = M = 0$ (resp. $L = 0$, $M = \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$), then we have $f = g = 0$. If $L = \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ and $M = \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$, then there is a morphism s such that $f = s \circ d$ and $g = d' \circ s$. Consequently, $\dot{\alpha}$ is homotopic to zero. Let $\dot{\beta}$ be a morphism from $\dot{4}$ to a shift of $\dot{X}$ of the form

$$\text{(11)}\qquad \begin{tikzcd} 0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "r"] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "\sigma"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "t"] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \cdots . \\ \cdots \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

If $M = N = 0$ and $L = \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ (resp. $L = M = \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$ and $N = \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$), then we have $\sigma = t = 0$. If $L = \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$, $M = \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ and $N = 0$, then $t = 0$ and $\sigma^2 = 0$. Proceeding from left to right, we conclude that $\dot{\beta}$ is homotopic to zero. Now let $\dot{\gamma}$ be a morphism of the form

$$\text{(12)}\qquad \begin{tikzcd} 0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "p"] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "q"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "v"] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "w"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \cdots \\ \cdots \arrow[r] & I \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] & 0 \end{tikzcd}$$

with I indecomposable projective–injective. Proceeding from left to right, we see that $\dot\gamma$ is homotopic to zero. Let $\dot\delta$ be a morphism of the form

(13)
$$\begin{tikzcd}
\cdots \arrow[r] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r, "d"] \arrow[d, "a"] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "\tau"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "b"] & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[r] \arrow[d, "0"] & 0 \\
 & 0 \arrow[r] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r, "d'"] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & I \arrow[r] & \cdots
\end{tikzcd}$$

with I indecomposable projective–injective. In this case we have $a = 0$, $\tau^2 = 0$ and $b = 0$. Hence there is a morphism t such that $t \circ d = 0 = a$ and $d' \circ t = \tau$. Thus $\dot\delta$ is homotopic to zero. Finally, let $\dot\epsilon$ denote a morphism of the form

(14)
$$\begin{tikzcd}
0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "\phi"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "p"] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "\psi"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "q"] & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[r] \arrow[d, "0"] & 0 \\
 & 0 \arrow[r] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & I \arrow[r] & L \arrow[r] & \cdots
\end{tikzcd}$$

with I and L projective–injective and I indecomposable. We first note that $\phi^2 = 0$, $\psi^2 = 0$ and $q = 0$. Proceeding from left to right, we conclude that $\dot\epsilon$ is homotopic to zero. This remark completes the proof that

(15) $\quad \mathrm{Hom}_{K(R)}(\dot 4, \dot X[i]) = 0$ for every $i \in \mathbb{Z}$.

Putting (9), (15) and [**D2**, Lemma1] together, we conclude that $\dot Y$ and $\dot X$ satisfy (ii) and (iii) with $m = 1$ and $m = 2$ respectively. Now fix some $m \geq 3$, and let $\dot Z$ denote the indecomposable complex

$$0 \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{m} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to 0$$

Let $\dot\alpha$ be a morphism of the form

(16)
$$\begin{tikzcd}
0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "f"] & \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "g"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "h"] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] \arrow[d, "l"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] \arrow[d, "s"] & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[r] \arrow[d, "0"] & 0 \\
\cdots \arrow[r] & U \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \arrow[r] & W \arrow[r] & I \arrow[r] & \cdots
\end{tikzcd}$$

where U, W and I (resp U and W) are projective–injective (resp. indecomposable) modules. Then f and g factor through a morphism from $\begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ to U. On the other hand, we clearly have $l^2 = 0$ and $s = 0$. Hence, proceeding from left to right, we see that $\dot{\alpha}$ is homotopic to zero. Since we know from (15) that $\dot{X}$ is orthogonal to $\dot{T}$, we deduce from (12), (13), (14), (16) and [**D2**, Lemma 1] that the complex $\dot{Z}$ satisfies (ii) and (iii) for the natural number $m \geq 3$. Let now $\dot{U}$ denote the indecomposable complex

$$\cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to 0.$$

Then any morphism form $\dot{4}$ to a shift of $\dot{U}$ induces a morphism from $\dot{4}$ to a bounded complex of the form

$$0 \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{n} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to 0. \qquad \text{for some } n \geq 2.$$

This remark and [**D2**, Lemma 1] guarantee that $\dot{U}$ satisfies (ii) and (iii) for $m = \aleph_0$.

Since (iv) clearly holds, it remains to prove (v). To see this, we first note that there are $2^{\aleph_0}$ sequences $\sigma = (m(i))_{i \geq 1}$ of integers $m(i) \geq 2$.

Next, for any σ as above and any n-uple $s = (m(1), \dots, m(n))$ where n and the $m(i)$'s are natural numbers ≥ 2, we denote by $\square$ the indecomposable complexes

$$\cdots \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{m(2)} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \xrightarrow{\delta} \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{m(1)} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \quad \text{and}$$

$$0 \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{m(n)} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \underbrace{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}_{m(1)} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$$

respectively. Keeping the above notation, let $\square \to 0$ denote one of the indecomposable complexes $\dot{C}(\sigma)$ or $\dot{C}(s)$. Next, let $\dot{C}(\sigma, 1)$ or $\dot{C}(s, 1)$ denote the indecomposable complex

$$\square \to \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to \begin{smallmatrix}4\\1\end{smallmatrix} \to 0.$$

Finally, let $\blacksquare$ denote an indecomposable bounded complex such that $0 \to \blacksquare$ is a bounded complex, say $\dot{B}$, of the form $\dot{C}(s)$ or $\dot{C}(s, 1)$ for some s. Then we denote by $\dot{B}(\aleph_0)$ the indecomposable complex $\cdots \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to \blacksquare$. Let $\dot{C}$ be one of the complexes just defined, and let $\dot{\alpha}$ be a morphism from $\dot{4}$ to a shift of $\dot{C}$. Then $\dot{\alpha}$ induces a morphism, say $\dot{\beta}$, from $\dot{4}$ to a bounded complex, say $\dot{D}$, as above, obtained by "glueing together" finitely many bounded complexes, say $\dot{C}(n(1)), \dots, \dot{C}(n(r))$, as indicated by the following picture

$$\begin{array}{cccc} \cdots\cdots & \cdots\cdots & & \overset{\delta}{\curvearrowright} \\ \hline \multicolumn{1}{|c|}{\dot{C}(n(r))} & \multicolumn{1}{c|}{\ldots\ldots} & \multicolumn{1}{c|}{\dot{C}(n(2))} & \multicolumn{1}{c|}{\dot{C}(n(1))} \\ \hline \end{array}\ .$$

We also note that the connecting maps δ are morphisms $\begin{smallmatrix}3\\1\\2\\3\end{smallmatrix} \to P$ with simple image.

On the other hand, the canonical inclusion $\begin{smallmatrix}1\\2\\3\end{smallmatrix} \hookrightarrow \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ is the unique arrow in $\dot{4}$ ending in $\begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$. Moreover, if $f : \begin{smallmatrix}1\\2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}$ and $d' : \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\\1\\2\\3\end{smallmatrix}$ are morphisms, then we have $d' \circ f = 0$. Consequently, finitely many applications of [**D3**, Lemma 5] guarantee that $\dot{D}$ is orthogonal to $\dot{4}$. Hence $\dot{\beta}$ is homotopic to zero, and so $\dot{\alpha}$ has the same property. Therefore $\dot{C}$ is orthogonal to $\dot{4}$. Moreover, the definition of $\dot{C}$ and [**D2**, Lemma 1] imply that $\dot{C}$ is orthogonal to $\dot{\begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}}$. This observation completes the proof of (v). $\square$

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Butler's theorem revisited

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Dedicated to Rüdiger Göbel on the occasion of his 70th birthday

Abstract. Call c.d.-group a completely decomposable Abelian group of finite rank. Butler's Theorem proves that the class of pure subgroups of c.d. groups equals the class of torsionfree quotients of c.d.-groups, called B(n)-groups. We give here a constructive proof of one of the implications, and complete the construction for the other, in order to provide a workable algorithm. The developement of the proof shows that, contrary to common perception, the linear side of a Butler group is predominant over its poset structure.

Introduction

In this paper *group = torsionfree Abelian group of finite rank.* A B(n)-group W is the sum of a finite number r of pure rank one subgroups, $W =< w_1 >_*$ $+...+ <w_r>_*$, subject to $n \leq r$ relations; equivalently, it is a *torsionfree quotient* $W = Y/K_Y$ of a completely decomposable (= *c.d.*) group $Y =<w_1>_* \oplus ... \oplus <w_r>_*$, over a rank n pure subgroup K_Y, collecting the linear relations among the w_j's. Thus a B(n)-group is determined by two choices: an order-theoretical one (the isomorphism types of the $<w_j>_*$) and a linear one (the relations in K_Y).

Throughout, as is usual in this subject, we use as a basic equivalence *quasi-isomorphism* (=isomorphism up to finite index, [**F II**]) instead of isomorphism; we write "*isomorphic, direct decomposition ...*" instead of "quasi-isomorphic, quasi-direct decomposition ..." .

In a fundamental paper of 1967 [**Bu**], M. Butler proved that this class of groups coincides with the class of *pure subgroups* of c.d. groups, spurring a vaste amount of research concerning in particular $B(1)$-groups (for the beginnings see [**A**], [**AV**]).

Butler's proof that "a B(n)-group W is the pure subgroup of some c.d. group X" is done by finite induction, and is far from yielding a viable construction; even the later version by Arnold and Vinsonhaler given in [**A**, Lemma 3.2.3.b] is not constructive. We give here an alternative proof of this implication, which yields a construction for X (a pure-container of W), while - in a sense - minimizing it. As is in our chords, the construction is simple, and can be performed by hand in the smaller ranks. Also for the proof of the other implication we provide the explicit step towards an algorithm (Section 9).

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Besides the practical advantage, our proof of the first implication yields a surprising insight in the nature of Butler groups: it shows the linear side to be dominant, while the contribution of the poset side is taken care of here by the decomposition of base types into products of Primes, forming our original tool, a table called *tent*.

E.g. here is the tent of the group W of Example 1.4 where $r = 4$:

$$\begin{array}{lllllllll}
t(w_1) & = & q_{34} & \cdot & q_{23} & \cdot & q_{234} & \cdot & \cdot & \cdot \\
t(w_2) & = & q_{34} & \cdot & \cdot & q_{14} & \cdot & q_{134} & \cdot & \cdot \\
t(w_3) & = & \cdot & q_{12} & \cdot & q_{14} & \cdot & \cdot & q_{124} & \cdot \\
t(w_4) & = & \cdot & q_{12} & q_{23} & \cdot & \cdot & \cdot & \cdot & q_{123}
\end{array}$$

Here the Primes q_F ($F \subseteq I$) are identified (and named) after the base types they don't divide.

Primes mark the sup-irreducible types of the group (see e.g. [**DVM 4**], [**DVM 11**], and a recap in Section 8). After the decomposition of types as products of Primes the problem can be treated 'locally at Primes', where only the linear side intervenes. The main step consists in recognizing that Primes themselves are linear objects (Setion 2). Here is a summary of the procedure.

For $W = Y/K_Y$, set $Y \otimes \mathbb{Q} = V$, $K_Y \otimes \mathbb{Q} = K$ (whence $K_Y = K \cap Y$); the vector space K is called the *creel* of W. The n independent linear relations $\sum_{j=1}^{r} \alpha_{l,j} w_j = 0$ determining W as a quotient of Y yield a base of the subspace K of V, with matrix $\mathbf{A} = (\alpha_{l,j})$ indexed in $L \times J$. The crucial step of the proof (Lemma 2.2) shows that the Primes of W are identified by 'strongly linearly dependent' sets F of columns of the matrix $\mathbf{A}$ of K. These subsets of J represent all Primes allowed for a B(n)-group W of creel K; when all of them are present in its tent, such a group is called W_{tot}. The tent of any B(n)-group W with the same creel K as W_{tot} is obtained by 'cancelling' Primes from the tent of W_{tot}: thus its poset side reduces to the choice of a subset of the set of all Primes allowed by K.

We will show that a pure-container X of W_{tot} yields a pure-container of any B(n)-group W with the same creel K: we get it from X by retaining in its tent only the Primes of W. X is then called a *total* pure-container (*t.p.-container*). Our goal is to build for every K a minimal t.p.-container $X(K)$.

Purity turns out to be the result of p-purity for each Prime p; we start from the criterion given in [**DVM 15**] to determine when a B(n)-group W contained in a c.d. group X is a p-pure subgroup of X (Section 3). In Sections 4, 5 we examine the $B(1)$, $B(2)$ cases; for the first, we will see that one pure-container will do the trick for all $B(1)$-groups of a given rank. The algorithm we build includes linearly a B(n)-group W in a c.d. group X via a matrix $\mathbf{\Omega}$; the minimal t.p.-container $X(K)$ will be given by building a *t.p.-inclusion matrix* $\mathbf{\Omega}(K)$ for each W_{tot} (Section 6). $\mathbf{\Omega}(K)$ is built by columns; each minimal-dependent set of columns of $\mathbf{A}$ determines a homogeneous system of equations whose unique nonzero solution becomes a column of $\mathbf{\Omega}$. This construction includes W_{tot} in an X_{tot} of minimal rank, and ensures directly p-purity for certain Primes. The number m of minimal-dependent sets is the rank of X_{tot}; even the Primes in its tent are computed from the dependent sets (Lemma 6.3). We then determine all Primes of W_{tot} in order to cancel from X_{tot} all other Primes, thus getting $X(K)$.

A tent is a finite object, allowing us to treat big classes of groups with simple combinatorial means; at the end of the process we will need to step down, from tent-abstraction to quasi-isomorphism-concreteness, in order to complete the construction (Section 7) to reach the result abstractly proved in [**A**]. In the meantime, we will still say "*the* B(n)-group with tent...". Finally (Section 9) we complete the construction for the other implication.

As is usual in our work, the treatment of the subject is complemented by many developed examples.

For a last comment: we show that a subspace K of dimension n of a vector space V of dimension r determines uniquely two groups: a B(n)-group W_{tot} of rank $n-r$, indecomposable or trivially decomposable, and its m.t.p.-container, a c.d. group $X(K)$ of rank m. What is the nature of the number m? Does it occur elsewhere in linear algebra with any similar relevance?

1. Notation and basic settings

Lower case greek letters, with the exception of σ, τ, ν and ξ (separately defined), denote rational numbers. In the following we will define four groups: X, W, Y and K, for each of which there will be a dedicated index set:

$I = \{1, ..., m\}$ *for X, and for the generic group G in the definitions;*

$J = \{1, ..., r\}$ *for W and Y;* $\quad L = \{1, ..., n\}$ *for K.*

We will keep basic notation and tools introduced in our previous papers (e.g. [**DVM 4**], [**DVM 11**], [**DVM 12**]). In particular, $\mathbb{T}(\wedge, \vee)$ is the lattice of all types, with the added maximum ∞ for the type of the 0 subgroup. If $G = <g_1>_* + ... + <g_m>_*$ is a B(n)-group and $g \in G$, any expression of $g = \sum\{\gamma_i g_i \mid i \in I\}$ as a linear combination of the *base elements* g_i will be called a *representative* of g; its support $\{i \in I \mid \gamma_i \neq 0\}$ is *a support of* g, and its complement in I is *a zero-block of* g. When G is $B(0)$ (as will be the case for X and Y) hence the representative of g is unique, we will write for the above $supp_G(g)$ resp. $Z_G(g)$.

If $g \in G$, $t_G(g)$ denotes the type in G of the pure subgroup $<g>_*$; $typeset(G) = \{t_G(g) \mid g \in G\}$ is a finite sub-$\wedge$-semilattice of $\mathbb{T}$, hence (having ∞ as a maximum) a lattice. The types $t_G(g_i)$ of the base elements are called *base types.*

For $E \subseteq I$ set $\quad \tau(E) = \wedge\{t_G(g_i) \mid i \in E\}$

$\qquad\qquad\qquad\quad G_E = <g_i \mid i \in E>_*$, $G_\emptyset = 0$.

By including typeset(G) as a sub-$\wedge$-semilattice in ($\mathbb{N}$, g.c.d., l.c.m.) we view each type as a product of Primes (capitalized to distinguish them from the natural primes acting on G as a $\mathbb{Z}$-module); each Prime in the product stands for a $\vee$-irreducible type below it (this is explained in more detail in Section 8). We say *the Prime p divides the type σ*, but also $p \leq \sigma$, and if $\sigma = t_G(g)$ we say *p is a Prime of g*, or *divides g*. Expressing the base types of G as products of Primes we get a finite table called a *tent* (see Example 1.4) where each Prime p (viewed as a column in the tent) has a support *supp*(p) and a zero-block $Z(p)$. When we specialize the group G into X, W, or Y, considering the bases fixed, we will write e.g. $Z_X(p)$, $supp_Y(p')$ ... If $Z(p) = \emptyset$ resp. $= I$, p is the *full* resp. *empty Prime*, and will in general be omitted.

Extending the classical definition to a Prime p of the tent we define the pure fully invariant subgroup of G

$$G(p) = \{g \in G \mid t_G(g) \geq p\}.$$

LEMMA 1.1. *The subset E of I contains the zero-block $Z(p)$ if and only if p divides $\tau(I \backslash E)$.*

The Prime p divides g in G if and only if one of the following holds:

- *some representative g' of g is a linear combination of base elements g_i all divisible by p: $g' \in <g_i \mid i \in supp(p)>_*$, i.e. $p \mid \tau(supp(g'))$;*
- *some zero-block of g contains $Z(p)$; equivalently, there is a representative g' of g with $Z_G(g') \supseteq Z(p)$.* □

In our basic notation, $W = <w_1>_* + ... + <w_r>_*$ is a B(n)-group; setting
($\star$) $<y_j>_* \cong <w_j>_*$ for all $j \in J$ and
$Y = <y_1>_* \oplus ... \oplus <y_r>_*$.

Condition ($\star$) ensures that the tent of Y is the same as the tent of W, and poses conditions on the tent of Y that go under the term 'regularity'.

W is isomorphic to a quotient Y/K_Y where K_Y is a pure subgroup of Y of rank n, and we set

$$\begin{aligned} w_j &= y_j + K_Y \quad \text{for all } j \in J \\ K_Y &= <a_1, a_2, ..., a_n>_* . \end{aligned}$$

Since the elements of K_Y represent relations in W, when this does not impair understanding we will use the *creel* of W: $K = K_Y \otimes \mathbb{Q}$ instead of $K_Y = K \cap Y$. The elements of K

$$\alpha_l = \alpha_{l,1} y_1 + ... + \alpha_{l,r} y_r, \quad l \in L$$

are called *basic relations* of W, and are independent; we have for K the matrix

$$\mathbf{A} = \begin{bmatrix} \alpha_{1,1} & ... & \alpha_{1,j} & ... & \alpha_{1,r} \\ ... & ... & ... & ... & ... \\ \alpha_{l,1} & ... & \alpha_{l,j} & ... & \alpha_{l,r} \\ ... & ... & ... & ... & ... \\ \alpha_{n,1} & ... & \alpha_{n,j} & ... & \alpha_{n,r} \end{bmatrix}, \text{ indexed in } L \times J.$$

For $j \in J$ we denote by c_j the j'th column vector of $\mathbf{A}$.

As shown in [**DVM 15**, 1.1 and 1.2] w.l.o.g. we may suppose
$<w_j>_* \cap \sum\{<w_{j'}>_* \mid j' \neq j\} \neq 0$ for all $j \in J$; equivalently,
($\star\star$) $K \nsubseteq Y_F$ for all $F \subseteq J$,
hence *when $n > 0$ we may assume* among the basic relations of W *the diagonal relation* $w_J (= w_1 + ... + w_r) = 0$: the first row of $\mathbf{A}$ is a row of 1s. As a consequence, if $n > 0$ *no Prime in the tent of W can have just one hole*: if p divides all w_j for $j \neq j'$, then it divides $w_{j'} = -\sum\{w_j \mid j \neq j'\}$.

Moreover, to avoid trivial complications, we assume that
($\star\star\star$) *no relation of K has ≤ 2 terms*
(otherwise it is easy to realize W as a $B(n-1)$-group without that relation).

When we interpret the basic relations in $\mathbb{Q}^m$ we will denote the variables as ξ_j, and by the *basic system* $\mathcal{S}$ we intend the linear system with matrix $\mathbf{A}$

$$\mathcal{S}: \quad \begin{cases} \alpha_{1,1}\xi_1 + ... + \alpha_{1,r}\xi_r = 0 \\ ------------- \\ \alpha_{l,1}\xi_1 + ... + \alpha_{l,r}\xi_r = 0 \\ ------------- \\ \alpha_{n,1}\xi_1 + ... + \alpha_{n,r}\xi_r = 0 \end{cases}, \quad \text{i.e.} \quad \xi_1 c_1 + ... + \xi_r c_r = 0.$$

For $n > 1$, a relevant role is played by the *basic partition* $\mathcal{A} = \{A_1, ..., A_k\}$ of J into equal-coefficient blocks A_s, called *sections*: $j, j' \in J$ are in the same section if and only if $\alpha_{l,j} = \alpha_{l,j'}$ for all $l \in L$; base elements w_j of W indexed in the same section are not distinguished by the relations; in the matrix $\mathbf{A}$, columns are equal if and only if they are indexed in the same section.

We introduce now the *key notation* for our investigation.

DEFINITION 1.2. Denoting by q the generic Prime of Y (and of W, when we view it as a quotient of Y), for $Z_Y(q) = F \subseteq J$ we set

$$q = q_F, \qquad q_{\{j\}} = q_j;$$

the elements of Y divided by q_F form the subgroup $Y(q_F) = Y_{J\setminus F} = \oplus\{<y_j>_* \mid j \notin F\}$; the elements of W divided by q_F form the subgroup $W(q_F) = W_{J\setminus F}$.

A *total c.d. group* Y_{tot} is a c.d. group in whose tent there is a Prime q_F for each $F \subseteq J$. □

Our choice ($\star$) of a c.d. Y where $t_Y(y_j) = t_W(w_j)$ for all $j \in J$ can be realized by starting with a Y_{tot} of the same rank r, and eliminating the Primes that are not in the tent of W. Some of these Primes are in fact not allowed by the creel K itself: e.g., if - say - $y_1 + y_2 + y_3 \in K$, hence $w_1 + w_2 + w_3 = 0$, and q divides w_1 and w_2 in W, then q divides w_3; there is no Prime in the tent of W with support $J\setminus F = \{1, 2\}$, hence $q_{\{1,2\}}$ must be eliminated from the tent of Y_{tot} to obtain *any* Y of creel K. We then call $Y(K)$ the c.d. group obtained from Y_{tot} by cancelling all Primes not allowed by K, and set $Y(K)/(Y(K)\cap K) = W_{tot}$; this is the B($n$)-group with creel K whose tent has all allowed Primes; the tent of any other W with creel K is obtained from it by cutting more Primes.

For a subgroup $W =<w_1>_* +...+ <w_r>_*$ of a c.d. group X we have the following notation:

$$X = R_1x_1 \oplus ... \oplus R_mx_m, \text{ with } \mathbb{Z} \leq R_i \leq \mathbb{Q} \text{ for all } i \in I;$$
$$w_j = \sum\{\omega_{j,i}x_i \mid i \in I\} \text{ for all } j \in J.$$

We get thus the *inclusion matrix* $\mathbf{\Omega}$ of W in X:

$$\mathbf{\Omega} = \begin{bmatrix} \omega_{1,1} & ... & \omega_{1,j} & ... & \omega_{1,m} \\ ... & ... & ... & ... & ... \\ \omega_{j,1} & ... & \omega_{j,i} & ... & \omega_{j,m} \\ ... & ... & ... & ... & ... \\ \omega_{r,1} & ... & \omega_{r,i} & ... & \omega_{r,m} \end{bmatrix}, \text{ indexed in } J \times I.$$

The relations of W: $\alpha_{l1}w_1+...+\alpha_{l,r}w_r = 0$, besides involving (in W) the base elements w_j, involve also (in $\mathbb{Q}$) their coefficients in each column of $\mathbf{\Omega}$: $\sum\{\alpha_{l,j}\omega_{j,i} \mid j \in J\} = 0$ for all $l \in L$.

LEMMA 1.3. $\mathbf{A\Omega} = 0$, *the zero-matrix indexed in* $L \times I$. *Since* $rkK = rk\mathbf{A} = n$, *we have* $rkW = rk\mathbf{\Omega} = n - r$. □

We recall:

- the type in X of an element $w = \gamma_1x_1 + ... + \gamma_mx_m$ is the infimum of the types of those base elements x_i of X which occur in w with coefficient $\gamma_i \neq 0$:

$$t_X(w) = \wedge\{t_X(x_i) \mid i \in \text{supp}_X(w)\} = \tau(\text{supp}_X(w)) = \tau(I\setminus Z_X(w));$$

- the type in W of an element $w = \sum\{\beta_jw_j \mid j \in J\} = y + K_Y$ is the supremum of the types in Y of its representatives $y + a$ ($a \in K$):

$$t_W(w) = \vee\{t_Y(y + a) \mid a \in K\}.$$

Introducing the Prime notation in X we set:

Definition 1.2 (cont.d). If for a Prime p of X we have $Z_X(p) = E \subseteq I$, we set

$$p = p_E, \qquad p_{\{i\}} = p_i,$$

hence the set of elements of X divided by p_E is the subgroup $X(p_E) = X_{I\backslash E} = \oplus\{R_i x_i \mid i \notin E\}$. Thus "$p_E$ *divides* $x \in X$" means "$\tau(I\backslash E) \leq t_X(x)$". □

When $W \leq X$, for $E \subseteq I$ *let* $F(E)$ *index the holes that the Prime* p_E *of* X *has in the tent of* W: p_E will be called $q_{F(E)}$ in the tent of W. If $|F(E)| = r$ or 0, p_E is the empty or the full Prime of W_{tot}, hence is omitted. We have $X(p_E) = X_{I\backslash E}$, $Y(p_E) = Y(q_{F(E)}) = Y_{J\backslash F(E)}$. Note that $J\backslash F(E) = \{j \in J \mid w_j \in Y(q_{F(E)})\} = \{j \in J \mid p_E = q_{F(E)} \text{ divides } w_j\} = \{j \in J \mid \omega_{j,i} = 0 \text{ for all } i \in E\}$ indexes the *rows* of $\mathbf{\Omega}$ whose zero-block is contained in E; $F(E) = \{j \in J \mid \omega_{j,i} \neq 0 \text{ for all } i \in E\}$ indexes the rows of $\mathbf{\Omega}$ whose support contains E. In particular, $F(\{i\})$ *is the support of the i-th column of* $\mathbf{\Omega}$.

It may be useful to observe that, since in our notation p_i divides w_j exactly when the coefficient $\omega_{j,i}$ of x_i in w_j is 0, that is when x_i is missing, the part of the tent of W concerning the Primes p_i is the "photographic negative" of the matrix $\mathbf{\Omega}$ (see Example 4.2).

Example 1.4. Let $X_{tot} = R_1 x_1 \oplus ... \oplus R_4 x_4$, the total c.d. group of rank 4, with $t_i = t_X(x_i)$; its tent in our notation is

$$\begin{array}{llcccccccccccccc}
t_1 & = & \cdot & p_2 & p_3 & p_4 & \cdot & p_{34} & p_{24} & \cdot & p_{23} & \cdot & p_{234} & \cdot & \cdot & \cdot \\
t_2 & = & p_1 & \cdot & p_3 & p_4 & \cdot & p_{34} & \cdot & p_{13} & \cdot & p_{14} & \cdot & p_{134} & \cdot & \cdot \\
t_3 & = & p_1 & p_2 & \cdot & p_4 & p_{12} & \cdot & p_{24} & \cdot & \cdot & p_{14} & \cdot & \cdot & p_{124} & \cdot \\
t_4 & = & p_1 & p_2 & p_3 & \cdot & p_{12} & \cdot & \cdot & p_{13} & p_{23} & \cdot & \cdot & \cdot & \cdot & p_{123}
\end{array}$$

(Observe trivially that the inclusion matrix of X in itself is the identity matrix, mirrored by the Primes p_1, p_2, p_3, p_4 in the tent).

The element - say - $x_2 + x_4$ has type $t_2 \wedge t_4 = p_1 p_3 p_{13}$. Since the types of the elements of X are infima of base-types, the Primes p_i and $p_{i'}$ divide an element w of X *in* X if and only if $p_{ii'}$ divides w in X; this then holds also for the base elements w_j in a B(n)-group W, if all $<w_j>_*$ are pure also in X. What may happen for a generic w in W is shown by $W = <w_1>_* + ... + <w_4>_* \leq X_{tot}$ with all $<w_j>_*$ pure in X and matrix

$$\mathbf{\Omega} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, \qquad \text{that is} \qquad \begin{cases} w_1 & = & x_2 + x_4, \\ w_2 & = & -x_2 + x_3, \\ w_3 & = & x_1 - x_3, \\ w_4 & = & -x_1 - x_4. \end{cases}$$

W is a $B(1)$-group, since the only relation among the base elements is the diagonal one: $w_J = w_1 + ... + w_4 = 0$. As seen, the Primes p_{ijk} become empty Primes in W, hence will be omitted. The Primes p_E of X can be written as q_F in the tent of W, e.g. $F(\{1\}) = \{3, 4\}$, thus in the tent of W we write q_{34} instead of p_1, and analogously q_{234} instead of p_{13}:

$$\begin{array}{llcccccccclcccccccc}
t(w_1) & = & p_1 & \cdot & p_3 & \cdot & p_{13} & \cdot & \cdot & \cdot & = & q_{34} & \cdot & q_{23} & \cdot & q_{234} & \cdot & \cdot & \cdot \\
t(w_2) & = & p_1 & \cdot & \cdot & p_4 & \cdot & p_{14} & \cdot & \cdot & = & q_{34} & \cdot & \cdot & q_{14} & \cdot & q_{134} & \cdot & \cdot \\
t(w_3) & = & \cdot & p_2 & \cdot & p_4 & \cdot & \cdot & p_{24} & \cdot & = & \cdot & q_{12} & \cdot & q_{14} & \cdot & \cdot & q_{124} & \cdot \\
t(w_4) & = & \cdot & p_2 & p_3 & \cdot & \cdot & \cdot & \cdot & p_{23} & = & \cdot & q_{12} & q_{23} & \cdot & \cdot & \cdot & \cdot & q_{123}
\end{array}$$

Again, the first four columns of the tent reproduce the matrix $\mathbf{\Omega}$ "in the negative", empty versus filled.

The Primes missing from the tent of W - those that do not divide any element of W *in* W - are of two kinds: Primes like p_{123}, which do not divide any w even in X, and hence are ininfluencial to purity; and Primes like p_{12}, which divides $w_1 + w_2 = x_3 + x_4$ in X, but not in W, since it does not belong to the tent of W: W is not p_{12}-pure (hence not pure) in X_{tot}. □

2. Regularity

For a matrix $\mathbf{M}$ indexed in $D \times H$ and $H' \subseteq H$, denote by $\mathbf{M}[H']$ the submatrix consisting of the *columns* of $\mathbf{M}$ indexed in H'.

Now 2.1 and 2.2 are the crucial steps for the subsequent development.

Definition 2.1. A subset F of J is K-*regular* (*regular*, if K is given) if it is the zero-block of a Prime of W_{tot}. This means that the Prime q_F *divides* w_j *in* W *(if and) only if it divides* y_j *in* Y*, that is if and only if* $j \notin F$. □

Thus determining the K-regular subsets of J means *determining the Primes of* W_{tot}, that is its tent.

In this section we will show that regularity - an order-theoretical feature - amounts to a strong linear dependence for the columns of $\mathbf{A}[F]$; it implies linear dependence, and coincides with minimal dependence when it is minimal itself.

Given two subsets A, B of a set U, say A *pierces* B if $|A \cap B| = 1$. Here is the connection between Primes and creel, showing the linear nature of Primes:

Lemma 2.2. *A subset F of J is not regular (that is, q_F is not an allowed Prime) if and only if F pierces the support in Y of an element of K. In particular, by $(\star\star\star)$ J is regular.*

Proof. Let $a = \sum\{\beta_j y_j \mid j \in \mathrm{supp}_Y(a)\} \in K$; if $F \cap \mathrm{supp}_Y(a) = \{j_0\}$ then $\mathrm{supp}(a) \backslash \{j_0\} \subseteq J \backslash F$. Since q_F divides all w_j for $j \in J \backslash F$, and $\beta_{j_0} w_{j_0} = -\sum\{\beta_j w_j \mid j \in \mathrm{supp}_Y(a) \backslash \{j_0\}\}$, q_F must divide w_{j_0}, a contradiction.

Conversely, if F is not regular, there is a Prime q with $F = Z_Y(q)$ and a $j_0 \in F$ such that (q does not divide y_{j_0}, but) q divides w_{j_0}, that is, for some $a = \sum\{\beta_j y_j \mid j \in \mathrm{supp}_Y(a)\} \in K$, q divides $\rho y_{j_0} + a$ in Y for some $\rho \neq 0$. Thus $q \mid y_j$ for all $j \in \mathrm{supp}_Y(a) \backslash \{j_0\}$, and $\beta_{j_0} = \rho \neq 0$, hence $j_0 \in \mathrm{supp}_Y(a)$, thus $F \cap \mathrm{supp}_Y(a) = \{j_0\}$. □

Corollary 2.3. *With the convention $(\star\star\star)$, if W_{tot} is represented as a $B(n')$-group then: $n' \geq n = rk(K)$; its representation as a $B(n)$-group is unique; moreover, if $n \geq 1$ W_{tot} is either indecomposable or trivially decomposable (degenerate).*

Proof. A Prime $q_{J \backslash \{j\}}$ of W is called a *locking Prime*, because the base element w_j is bound to belong to every base of W in any representation. By the above result, a locking Prime is irregular if and only if there is an element $a \in K$ with $|\mathrm{supp}_Y(a)| \leq 2$, excluded by $(\star\star\star)$. Thus all locking Primes are present; then either W_{tot} is indecomposable or $\mathbf{A}$ is equivalent to a block-diagonal matrix ([**DVM 10**, Prop. 2.1]). □

Let $j_0 \in F \subseteq J$; say j_0 is *irregular in* F if $\{j_0\} = F \cap supp_Y(a)$ for some $a \in K$; j_0 is *regular in* F otherwise; thus, F is regular if and only if all its elements are.

The next proposition shows in particular (from iv) that if j_0 is irregular in F then, in order to obtain from q_F a Prime of W, we must 'fill its hole' in j_0. The proof is elementary linear algebra, we give a possible sketch.

PROPOSITION 2.4. *For $j_0 \in F \subseteq J$ the following are equivalent:*

(i) *j_0 is irregular;*
(ii) *$rk\mathbf{A}[F\backslash\{j_0\}] = rk\mathbf{A}[F] - 1$;*
(iii) *c_{j_0} is independent from $\{c_j \mid j \in F\backslash\{j_0\}\}$;*
(iv) *whenever $(\zeta_1, ..., \zeta_r)$ is a solution of $\mathcal{S}(F)$ then $\zeta_{j_0} = 0$.*

Proof. (i)⇔(ii) Let $\{j_0\} = F \cap \text{supp}_Y(a)$ for some $a \in K$. We have $a = \sum\{\beta_j y_j \mid j \in J\} = \sum_{l=1,...,n} \lambda_l a_l$, hence $\beta_j = \sum_{l=1,...,n} \lambda_l \alpha_{l,j}$ for all $j \in J$. j_0 is irregular if and only if the linear system consisting of the equations

$$\sum_{l=1,...,n} \lambda_l \alpha_{l,j} = 0 \quad (j \in F\backslash\{j_0\}) \quad \text{and} \quad \sum_{l=1,...,n} \lambda_l \alpha_{l,j_0} = \rho$$

has a solution. The transposed of the incomplete matrix of this system is $\mathbf{A}[F]$; compatibility is equivalent to requiring that if we replace any row with the row $(0, ..., 0, \rho, 0, ..., 0)$ with ρ as the j_0'th entry, $\text{rk}\mathbf{A}[F] = d$ does not increase; it would increase if and only if there was a $d \times d$ submatrix of rank d in $\mathbf{A}[F]$ without the column c_{j_0}. Thus all sets of d independent columns must contain c_{j_0}, and this is exactly (ii).

(ii)⇔(iii) is clear; for (iii)⇔(iv), recall that $(\zeta_1, ..., \zeta_r)$ is a solution of $\mathcal{S}(F)$ if and only if $\zeta_1 c_1 + ... + \zeta_r c_r = 0$. □

Say $F \subseteq J$ is *minimal dependent* if F indexes a linearly dependent set of columns such that any proper subset is independent.

LEMMA 2.5. *The following are equivalent for $F \subseteq J$:*

(a) *F is minimal dependent;*
(b) *$rk(\mathbf{A}[F]) = |F| - 1$ and the solution of $\mathcal{S}(F)$ has support F;*
(c) *F is minimal regular.*

Proof.

(a)⇔(b): if F is dependent, $\text{rk}(\mathbf{A}[F]) \leq |F| - 1$; if $<$ then there would be a smaller dependent set $F' \subset F$. If the solution had $\zeta_j = 0$ for some $j \in F$, then again there would be a smaller dependent set.

(b)⇔(c): from Proposition 2.4 (iv) F is regular; if it was not minimal, then a minimal regular $F' \subset F$ would yield a solution of $\mathcal{S}(F')$ (hence of $\mathcal{S}(F)$) with support F', a contradiction.

(c)⇔(a): regular implies dependent; if F was not minimal dependent then $F \supset F'$ minimal dependent which by (a)⇒(c) would be minimal regular, a contradiction.

□

Define $F \subseteq J$ *minimal* if it satisfies any of the requirements of Lemma 2.5.

COROLLARY 2.6. *A union of regular subsets of J is regular; $F \subseteq J$ is regular if and only if it is a union of minimal F_i's.*

Proof. If F is a union of regular subsets of J then it cannot pierce a support of an element of K, otherwise one of its subsets would as well; in particular, a union of minimal F_i's is regular.

Let then F be regular; by Proposition 2.4 (iv) for any $j_0 \in F$ there is a linear combination $\sum\{\zeta_j c_j \mid j \in F'\}$ with $j_0 \in F' \subseteq F$ and $\zeta_j \neq 0$ for all $j \in F'$. Let $F'' \subseteq F'$ be minimal dependent; either $j_0 \in F''$, or a $j'' \in F''$ depends on $F''\backslash\{j''\}$, hence $F'\backslash\{j''\}$ is dependent and contains j_0; finite induction concludes the argument. $\square$

In particular, J the union of all minimal F_i's.

3. Previous results

We report here the fundamental results of [**DVM 15**] in which we determined the conditions for W to be pure in X.

THEOREM 3.1. [**DVM 15**, 3.1] *With the above notation,* $W = <w_1>_* + ... + < w_r>_*$*, with the* $<w_j>_*$ *pure in* X*, is* p_E*-pure in* X *if and only if*

$$rk(\mathbf{A}[F(E)]) = |F(E)| - rk(\mathbf{\Omega}[E]).$$

A straightforward necessary condition for purity is

COROLLARY 3.2. [**DVM 15**, 3.2, 3.3, 3.5] *If the* $B(n)$*-group* W *is a pure subgroup of* X*, then for all* $E \subseteq I$ *we have*

$$|F(E)| \leq n + |E|.$$

In particular for each $i \in I$ *and* $E = \{i\}$*, that is for the columns of* $\mathbf{\Omega}$*, the order* $|F(\{i\})|$ *of the support must be* $\leq n+1$*, and the entries must satisfy the basic system* $\mathcal{S}$*.*

For $B(1)$*-groups the condition* $|F(E)| \leq 1 + |E|$ *is also sufficient.* $\square$

Since purity is p-purity for each Prime p, if the B(n)-group W_{tot} is a pure subgroup of the c.d. group X, then any B(n)-group W with the same creel is pure in the c.d. group X' obtained from X by cancelling from its tent the Primes that do not occur in the tent of W.

When W_{tot} is pure in X we call X a *total pure-container* (=*t.p.-container*) of the B(n)-groups with creel K.

In the following, we will determine for every W_{tot} a minimal t.p.-container $X(K)$, (= *m.t.p.-container*); it will be obtained from a c.d. X_{tot} of minimal rank, cancelling suitable Primes. [The notation similar to the previous $Y(K)$ should not be confusing, since K is not a subgroup of X].

We will build the inclusion of W_{tot} in its pure-container by way of an inclusion matrix $\mathbf{\Omega}$; everything will be done *up to permutations of base types and of Primes.*

4. The $B(1)$ case

For $n = 1 = \mathrm{rk}(K)$, in our setting there is only one K for each rank, generated by the diagonal relation, hence W_{tot} is uniquely determined by its rank $r-1$. We will thus obtain one m.t.p.-container per rank, and we will call it $X(1)$.

As an immediate consequence of Corollary 3.2, the condition $|F(E)| \leq 1+|E|$ to ensure purity of W in X requires for $|E| = 1$ that *the columns of* $\mathbf{\Omega}$ *have exactly two nonzero entries, opposite to each other due to the diagonal relation*; equivalently, the Primes p_i of X when occurring in the tent of W must have there exactly two holes.

Note that $\mathbf{\Omega}$ will have r rows, and we need to build a sufficient number m of columns to include W_{tot} in the c.d. group X_{tot} of rank m; we will obtain the m.t.p.-pure-container $X(1)$ by reducing the tent of X_{tot} to the Primes coming from

W_{tot}, while ensuring purity. Then we can purely include any $B(1)$-group W of rank $r-1$ in the c.d. group X obtained from $X(1)$ by cancelling all Primes not in W.

This procedure requires an inversion of the function $F(E)$: given a Prime q_F of W_{tot}, find an E such that $F = F(E)$ (i.e. such that q_F is the Prime p_E of X), and moreover such that W_{tot} is p_E-pure in X. We will start with the Primes of W_{tot} with minimal zero-block, i.e. $|F| = 2$; once we settle these, the rest will follow.

As we saw, the purity condition for $B(1)$ requires that if $|F| = 2$, $\mathrm{rk}(\mathbf{\Omega}[E]) = 1$; we can obtain this for $|E| = 1$, that is by a single column of $\mathbf{\Omega}$ with support F, and entries opposite to each other (diag. rel.). (Any other choice for $|E|$ would have to consider more such columns all proportional to each other, thus unnecessarily increasing the rank of X). Clearly, we need one such column for each $F \subseteq J$ with $|F| = 2$; the minimal number of columns of $\mathbf{\Omega}$ is then $\binom{r}{2}$. Here is the state of $\mathbf{\Omega}$ for $r = 5$, when we have settled all such F's:

$$\mathbf{\Omega} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

This inclusion guarantees purity for all Primes $q_{jj'}$ of W_{tot} ($j \neq j' \in J$) in the X_{tot} of rank $m = \binom{r}{2}$. We must now settle the other Primes q_F of W_{tot}.

Say in our example $F = \{1,2,3,4\}$; for any $E \subseteq \{1,2,3,5,6,8\}$ we have that the union $F(E)$ of supports of the columns indexed in E is contained in F. But if we choose - say - $E = \{1,8\}$, we have $\mathrm{rk}(W[E]) = 2 \neq 4-1$, hence W_{tot} is not $p_{\{1,8\}}$-pure. To get $\mathrm{rk}(\mathbf{\Omega}[E]) = 3$ we need $|E| \geq 3$: say, $E(F) = \{1,2,3\}$. In fact, if among all counterimages E of F we choose the biggest one, setting $E(F) = \{1,2,3,5,6,8\}$, then W_{tot} is $q_{\{1,2,3,4\}} = p_{\{1,2,3,5,6,8\}}$-pure in X_{tot}. Note that if we had chosen $E(F) = \{1,2,3\}$ the rank would have been right, but the Prime $p_{\{1,2,3\}}$ has less holes in the tent of X, hence defines a 'bigger' X.

In general, let $\mathbf{\Omega}$ be an $r \times \binom{r}{2}$ - matrix whose columns have for supports the subsets of cardinality 2 of J, with opposite entries; X_{tot} the total c.d. group of rank $\binom{r}{2}$. Let $|F| = f < r$; then the column supports contained in $\mathbf{F}$ are the $\binom{f}{2}$ subsets of F of cardinality 2. Let E be the set of indices of those columns, and set $E = E(F)$. It is not difficult to see (e.g., by re-indexing, the first $|F|$ columns will look just like the first 4 in the example) that $\mathrm{rk}(\mathbf{\Omega}[E]) = f-1$, hence W_{tot} is $q_F = p_E(F)$-pure in X_{tot}. We now need only to *delete from the tent of X_{tot} the Primes p_E with $E \neq E(F)$ for all $F \subseteq J$*, obtaining the c.d. group $X(1)$ (see Example 4.2).

PROPOSITION 4.1. *In the above notation, $X(1)$ is the m.t.p.-container of the $B(1)$-groups of rank r.*

Proof. $X(1)$ is a pure-container of W_{tot}; minimality of rank is insured by the initial settling of the cases $|F| = 2$; minimality of types by the choice of Primes $p_{E(F)}$ with the maximum number of holes. □

What if the W we started with was not a W_{tot}? We still include W_{tot} in $X(1)$; then cancel from $X(1)$ all Primes that do not belong to the tent of W (see Section 7); we get $X(W)$, a c.d group still of rank $\binom{r}{2}$, but with a narrower tent, in which W

is pure. Here, though, minimality is not assured: e.g., if W has only one Prime it is itself completely decomposable, of rank $r-1<\binom{r}{2}$.

EXAMPLE 4.2. **(1)** Let $r=4$ and set $t(w_j)=u_j$ for $j=1,2,3,4$; the tent of the $B(1)$-group W_{tot} has all Primes except those with only one hole:

$$\begin{array}{lcllllllllll}
u_1 & = & \cdot & q_{34} & q_{24} & \cdot & q_{23} & \cdot & q_{234} & \cdot & \cdot & \cdot \\
u_2 & = & \cdot & q_{34} & \cdot & q_{13} & \cdot & q_{14} & \cdot & q_{134} & \cdot & \cdot \\
u_3 & = & q_{12} & \cdot & q_{24} & \cdot & \cdot & q_{14} & \cdot & \cdot & q_{124} & \cdot \\
u_4 & = & q_{12} & \cdot & \cdot & q_{13} & q_{23} & \cdot & \cdot & \cdot & \cdot & q_{123}
\end{array}$$

The matrices are

$$\mathbf{A}=[1,1,1,1], \qquad \mathbf{\Omega}=\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}.$$

W_{tot} will be a pure subgroup of a completely decomposable group of rank $\binom{4}{2}=6$, obtained from the rank 6 X_{tot} (we omit the tent, too big) by keeping only the Primes $p_{E(F)}=q_F$ for $F\subseteq J=\{1,2,3,4\}$.

For $|F|=2$: $E\{1,2\}=\{1\}$ (the only column supported in $F=\{1,2\}$ is the first); analoglusly, $E\{1,3\}=\{2\}$, ..., $E\{3,4\}=\{6\}$: in the tent of X_{tot} we keep all Primes $p_1,...,p_6$.

For $|F|=3$: $E\{1,2,3\}=\{1,2,4\}$ (the columns supported in $F=\{1,2,3\}$ are the first, second and fourth); $E\{1,2,4\}=\{1,3,5\}$; $E\{1,3,4\}=\{2,3,6\}$; $E\{2,3,4\}=\{4,5,6\}$: in the tent of X_{tot} we keep Primes p_{124}, p_{135}, p_{236}, p_{456}.

That is all, since $|F|=4$ indexes the empty Prime. Setting $t_i=t(x_i)$ for $i=1,...,6$, the tent of $X(1)$ is

$$\begin{array}{lcllllllllll}
t_1 & = & \cdot & p_2 & p_3 & p_4 & p_5 & p_6 & \cdot & \cdot & p_{236} & p_{456} \\
t_2 & = & p_1 & \cdot & p_3 & p_4 & p_5 & p_6 & \cdot & p_{135} & \cdot & p_{456} \\
t_3 & = & p_1 & p_2 & \cdot & p_4 & p_5 & p_6 & p_{124} & \cdot & \cdot & p_{456} \\
t_4 & = & p_1 & p_2 & p_3 & \cdot & p_5 & p_6 & \cdot & p_{135} & p_{236} & \cdot \\
t_5 & = & p_1 & p_2 & p_3 & p_4 & \cdot & p_6 & p_{124} & \cdot & p_{236} & \cdot \\
t_6 & = & p_1 & p_2 & p_3 & p_4 & p_5 & \cdot & p_{124} & p_{135} & \cdot & \cdot
\end{array}$$

(2) If we had started with the rk3 indecomposable $B(1)$-group W' with tent

$$\begin{array}{lcllll}
t(w'_1) & = & \cdot & q_{34} & q_{24} & \cdot \\
t(w'_2) & = & \cdot & q_{34} & \cdot & q_{13} \\
t(w'_3) & = & q_{12} & \cdot & q_{24} & \cdot \\
t(w'_4) & = & q_{12} & \cdot & \cdot & q_{13}
\end{array}$$

the relative $X(W')$ would keep only the Primes q_F with $F=\{1,2\}$, $\{3,4\}$, $\{2,4\}$, $\{1,3\}$, corresponding to $E=\{1\}$, $\{6\}$, $\{5\}$, $\{2\}$, hence

$$\begin{array}{lcllll}
t(x'_1) & = & \cdot & p_2 & p_5 & p_6 \\
t(x'_2) & = & p_1 & \cdot & p_5 & p_6 \\
t(x'_3) & = & p_1 & p_2 & p_5 & p_6 \\
t(x'_4) & = & p_1 & p_2 & p_5 & p_6 \\
t(x'_5) & = & p_1 & p_2 & \cdot & p_6 \\
t(x'_6) & = & p_1 & p_2 & p_5 & \cdot
\end{array}$$

(3) We realize now that even in case (1) we are not done yet: given the 'concrete' base types u_j of W, we must convert the tent types of X into 'concrete' types. Define E_j to be the support of the j'th row of $\mathbf{\Omega}$: then we have $w_j=\sum\{w_{j,i}x_i \mid i\in E_j\}$

and

$$u_j = \tau(E_j):$$

thus our unknown types t_i must satisfy

$$\begin{aligned} u_1 &= \tau(\{1,2,3\}) = t_1 \wedge t_2 \wedge t_3; \\ u_2 &= t_1 \wedge t_4 \wedge t_5; \\ u_3 &= t_2 \wedge t_4 \wedge t_6; \\ u_4 &= t_3 \wedge t_5 \wedge t_6. \end{aligned}$$

Therefore - say - $t_1 \geq u_1, u_2$; ...; $t_6 \geq u_3, u_4$; let us then try and *define* $t_1 = u_1 \vee u_2$; $t_2 = u_1 \vee u_3$; $t_3 = u_1 \vee u_4$; $t_4 = u_2 \vee u_3$; $t_5 = u_2 \vee u_4$; $t_6 = u_3 \vee u_4$; and check whether in the c.d. X with these 'concrete' base types the pure subgroup defined by $\mathbf{\Omega}$ has the 'concrete' base types of W (hence is (quasi-) isomorphic to W). We have e.g. for the subgroup W' of X, image of W by $\mathbf{\Omega}$, $u'_1 = t_1 \wedge t_2 \wedge t_3 = (u_1 \vee u_2) \wedge (u_1 \vee u_3) \wedge (u_1 \vee u_4) = u_1 \wedge (u_2 \vee u_3 \vee u_4) \leq u_1$; but $t_1 \wedge t_2 \wedge t_3 \geq u_1$, hence $u'_1 = u_1$; the rest works as well. $\square$

Part (3) of the example, i.e. the construction of the 'concrete' X from the 'concrete' W, will be treated in general in Section 7.

5. The $B(2)$ case

We sketch this as an introduction to the general case. Here we have $\mathbf{A} = \begin{bmatrix} 1 & 1 & ... & 1 \\ \alpha_{2,1} & \alpha_{2,2} & ... & \alpha_{2,r} \end{bmatrix}$, where a role is played by the basic partition $\mathcal{A} = \{A_1, ..., A_k\}$ of J into sections: in particular, $\mathrm{rk}(\mathbf{A}[F]) = 1$ if and only if F is contained in a section of $\mathcal{A}$.

Note that in the tent of a $B(2)$-group there are no Primes with only one hole, while Primes with only two holes are allowed if and only if the two holes are contained in a section: this by Lemma 2.2, because if $a \in K$, $a = \sum\{(\rho - \alpha_{2,j})y_j \mid j \in J\}$, hence $\mathrm{supp}_Y(a)$ is either I or the complement of a section.

Let us start by examining the Primes q_F of W_{tot} such that F is contained in a section: the requirement for purity is $1 = |F| - \mathrm{rk}(\mathbf{\Omega}[E(F)])$. If $|F| = 2$, as in the $B(1)$ case we can satisfy the relation by taking for $E(F)$ a singleton, introducing in $\mathbf{\Omega}$ a column with entries all zero except on F, where they are opposite to each other. After we do this for all subsets of cardinality 2 in a section, all other subsets of the section obtain their $E(F)$ as in the $B(1)$-case. We thus build sets of columns indexed in $E(A_s)$ $(s = 1, ..., k)$ with $|E(A_s)| = \binom{|A_s|}{2}$, which constitute the first part of an inclusion matrix $\mathbf{\Omega}$ for which W_{tot} is q_F-pure for all subsets F of J that are contained in a same section.

To find out what happens in the other cases, let us consider the particular case in which $\mathcal{A} = \{A_1, ..., A_k\}$ is minimal, that is $|A_s| = 1$ for all $s = 1, ..., k$. Here $|F| = 2$ is forbidden by regularity, and in any case as soon as $|F| > 1$ we have $\mathrm{rk}(\mathbf{A}[F]) = 2$, hence the condition is $2 = |F| - \mathrm{rk}(\mathbf{\Omega}[E(F)])$. This means that for the minimal $|F| = 3$, we may choose $E(F)$ a singleton, with entries all zero except for the three entries on F, where the two relations of W must hold. Here is an example:

EXAMPLE 5.1. Let $W_{tot} = \langle w_1 \rangle_* + ... + \langle w_4 \rangle_*$, with $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & -1 & 0 \end{bmatrix}$; here $\mathcal{A} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$. The tent of W_{tot} is

$$\begin{array}{lcllll} t(w_1) & = & q_{234} & \cdot & \cdot & \cdot \\ t(w_2) & = & \cdot & q_{134} & \cdot & \cdot \\ t(w_3) & = & \cdot & \cdot & q_{124} & \cdot \\ t(w_4) & = & \cdot & \cdot & \cdot & q_{123}. \end{array}$$

To obtain $F = \{1,2,3\}$, we build the column of $\mathbf{\Omega}$ by solving the basic system $\mathcal{S}$: $\begin{cases} \xi_1+\xi_2+\xi_3+\xi_4=0, \\ 3\xi_1+2\xi_2-\xi_3=0, \end{cases}$ adding $\xi_4 = 0$; we get e.g. $(-3,4,-1,0)$. For $F = \{1,2,4\}$, i.e. adding $\xi_3 = 0$, we get $(2,-3,0,1)$, etc; that is $\mathbf{\Omega} = \begin{bmatrix} -3 & 2 & 1 & 0 \\ 4 & -3 & 0 & 1 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & -4 & -3 \end{bmatrix}$, which in this simple case settles all Primes and yields the m.t.p.-container (of rank 4) for this W_{tot}. Here $q_{234} = p_4$; $q_{124} = p_3$; $q_{134} = p_2$; $q_{123} = p_1$.

Let us now increase the rank and blow up the partition to $\mathcal{A} = \{\{1\},\{2\},\{3\},\{4,5\}\}$; then $W_{tot} = <w_1>_* + ... + <w_5>_*$, with

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & -1 & 0 & 0 \end{bmatrix}, \qquad \text{i.e.} \qquad \mathcal{S} : \begin{cases} \xi_1+...+\xi_5 & = & 0, \\ 3\xi_1+2\xi_2-\xi_3 & = & 0. \end{cases}$$

For $|F| = 2$, we have only $F = \{4,5\} = A_4$, with $|E(A_4)| = \binom{2}{2} = 1$, yielding the column $\begin{bmatrix} ... & 0 & ... \\ ... & 0 & ... \\ ... & 0 & ... \\ ... & 1 & ... \\ ... & -1 & ... \end{bmatrix}$ of $\mathbf{\Omega}$; for $|F| = 3$, by regularity F cannot contain $\{4,5\}$, thus besides $\{1,2,3\}$ all previous F's double: $\{1,2,4\}$, $\{1,2,5\}$; $\{1,3,4\}$, $\{1,3,5\}$; $\{2,3,4\}$, $\{2,3,5\}$. Since ξ_4 and ξ_5 have the same coefficient (the corresponding columns of $\mathbf{A}$ are equal), the solutions - say - for $\xi_1 = \xi_5 = 0$ are shifted from those for $\xi_1 = \xi_4 = 0$. After settling all minimal F's we get

$$\mathbf{\Omega} = \left[\begin{array}{c|c:cc:cc:cc} 0 & -3 & 2 & 2 & 1 & 1 & 0 & 0 \\ \hdashline 0 & 4 & -3 & -3 & 0 & 0 & 1 & 1 \\ \hdashline 0 & -1 & 0 & 0 & 3 & 3 & 2 & 2 \\ \hdashline 1 & 0 & 1 & 0 & -4 & 0 & -3 & 0 \\ -1 & 0 & 0 & 1 & 0 & -4 & 0 & -3 \end{array}\right].$$

Theorem 6.4 will show that $\mathbf{\Omega} = \mathbf{\Omega}_{tot}$, the m.t.p.-inclusion for the given W_{tot}. $\square$

Observation 5.2. In actual practice, non-regular Primes exclude themselves directly from the computation: for instance, if we try the procedure on $F = \{3,4,5\}$, i.e. add $\xi_1 = \xi_2 = 0$, we get $\xi_3 = 0$, $\xi_4 = -\xi_5$, falling back into the previous $F = \{4,5\}$: the singleton hole fills up automatically. If we try with $F = \{1,2\}$, i.e. add $\xi_3 = \xi_4 = \xi_5 = 0$, we get $\xi_1 = \xi_2 = 0$, which yields the zero column of $\mathbf{\Omega}$; this corresponds to the full Prime $q_\emptyset$ of W, which we do not consider (and in any case W is $q_\emptyset$-pure anywhere).

6. The general case

Given W_{tot} and $F \subseteq J$, consider the F's such that $\mathrm{rk}(\mathbf{A}[F]) = |F| - 1$: then $\mathcal{S}(F)$ has a nonzero solution (viewed as a column vector, its support is contained in F; it equals F if F is minimal) that is unique up to a common factor. Recall that $F \subseteq J$ is minimal if it satisfies the requirements of Lemma 2.5.

Definition 6.1. List the minimal subsets of J: $F_1, ..., F_m$; the $r \times m$ matrix $\mathbf{\Omega}$ has as its i'th column (for each $i = 1, ..., m$) the nonzero solution of $\mathcal{S}(F_i)$. $\square$

View $\mathbf{\Omega}$ - for the moment - as a linear transformation of $W_{tot} \otimes \mathbb{Q}$ into $X_{tot} \otimes \mathbb{Q}$. $F_i \subseteq J$ is then the support of the i'th column of $\mathbf{\Omega}$; analogously, let $E_j \subseteq I$ be the *support of the j'th row* for all $j \in J$. We prove that $\mathbf{\Omega}$ is injective.

Proposition 6.2. *$rk(\mathbf{\Omega}) = r - n$.*
Proof. Each column of $\mathbf{\Omega}$ satisfies the basic system $\mathcal{S}$, posing n independent conditions on the entries; thus the r row-vectors of $\mathbf{\Omega}$ satisfy the same n conditions, and $\mathrm{rk}(\mathbf{\Omega}) = r - n$. To prove equality, we must find $r - n$ independent columns in $\mathbf{\Omega}$. Since $\mathbf{A}$ has rank n, let F_0 index a set of n independent columns of $\mathbf{A}$. If j indexes one of the remaining $r - n$ columns - w.l.o.g. $j = 1, ..., r - n$ - $\mathcal{S}[F_0 \cup \{j\}]$ has a unique nonzero solution, thus $F_0 \cup \{j\} \supseteq F_{i(j)}$ for some $i(j) = 1, ..., m$, and $F_{i(j)} \supseteq \{j\}$, because the j'th entry of the solution cannot be zero, while all entries indexed out of $F_0 \cup \{j\}$ are zero. Let w.l.o.g. $i(j) = j$ for all $j = 1, ..., r - n$; then the relative part of $\mathbf{\Omega}$ looks like this:

$$\mathbf{\Omega} = \begin{bmatrix} \omega_{1,1} \neq 0 & 0 & ... & 0 & ... \\ 0 & \omega_{2,2} \neq 0 & ... & 0 & ... \\ 0 & 0 & ... & 0 & ... \\ 0 & 0 & ... & \omega_{r-n,r-n} \neq 0 & ... \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \end{bmatrix}.$$

Hence the said $r - n$ solutions (columns of $\mathbf{\Omega}$) are independent, q.e.d. $\square$

We do not yet know whether $\mathbf{\Omega}$, viewed as a function restricted to W_{tot}, sends W_{tot} with all its Primes in our X_{tot} of rank m. What we can say is that, by construction, the subgroup $W' = <w'_1>_* + ... + <w'_r>_*$ of X_{tot}, defined by $\mathbf{\Omega}$ as $w'_j = \sum\{\omega_{j,i} x_i \mid i \in I\}$ for all $j \in J$, is $p_i = q_{F_i}$-pure for all $i = 1, ..., m$. Let us now deal with the other Primes q_F of W: we need to show that for each such Prime there is a Prime $p_{E(F)}$ of X_{tot} with $p_{E(F)} = q_F$ of W': then W' is W_{tot}, and we must choose $E(F)$ so that W' is $p_{E(F)} = q_F$-pure.

For $F \subseteq J$, set $E(F) = \emptyset$ if the only solution of $\mathcal{S}(F)$ is the zero solution, i.e. $\mathrm{rk}(\mathbf{A}[F]) = |F|$; otherwise, set

$$E(F) = \{i \mid F_i \subseteq F\}.$$

By Corollary 2.6, the subsets F of J satisfying $F = \cup\{F_i \mid F_i \subseteq F\}$ describe all regular subsets of J.

Lemma 6.3. *E is an injective function transforming the Primes of W_{tot} into the Primes of W'.*
Proof. For $F \subseteq J$ set $\overline{F} = \cup\{F_i \mid F_i \subseteq F\}$; by definition $E(F) = E(F')$ if and only if $\overline{F} = \overline{F'}$; q_F is a Prime of W_{tot} if and only if $F = \overline{F}$. $\square$

THEOREM 6.4. *The matrix* $\mathbf{\Omega}$ *built as above is a m.t.p. inclusion of* W_{tot} *in the c.d group* $X(K)$ *of rank* m *obtained from* X_{tot} *by cancelling all Primes that are not in the tent of* W_{tot}.

Proof. We need to show **(a)** that for all Primes of W_{tot}, that is for all q_F with F regular, q_F divides w_j in W if and only if $p_{E(F)}$ divides $w'_j = \sum\{\omega_{j,i}x_i \mid i \in I\}$ in X_{tot}; and moreover **(b)** that W' is $q_F = p_{E(F)}$-pure in X_{tot}.

W.l.o.g. set $F = \{1, ..., f\} \subseteq J$, $\emptyset \neq E(F) = \{1, ..., e\} \subseteq I$. We have in the tents of W resp. X:

$$
\begin{array}{c|lcl}
 & t(w_1) & = & \cdot \quad ... \\
F & \multicolumn{3}{c}{\dots\dots\dots\dots} \\
 & t(w_f) & = & \cdot \quad ... \\
\hline
 & t(w_{f+1}) & = & q_F \quad ... \\
J\backslash F & \multicolumn{3}{c}{\dots\dots\dots\dots} \\
 & t(w_r) & = & q_F \quad ...
\end{array}
\qquad
\begin{array}{c|lcl}
 & t(x_1) & = & \cdot \quad ... \\
E(F) & \multicolumn{3}{c}{\dots\dots\dots\dots} \\
 & t(x_e) & = & \cdot \quad ... \\
\hline
 & t(x_{e+1}) & = & p_{E(F)} \, ... \\
I\backslash E(F) & \multicolumn{3}{c}{\dots\dots\dots\dots} \\
 & t(x_m) & = & p_{E(F)} \, ...
\end{array}
$$

(a) If F is regular and q_F divides w_j then $j \in J\backslash F$. Since $E(F)$ indexes all columns whose support is contained in F, the entries of the j'th row indexed in $E(F)$ are all zero. Thus $w'_j = \sum\{\omega_{j,i}x_i \mid i \in I\backslash E(F)\}$; then $p_{E(F)}$ divides w'_j.

$$
\mathbf{\Omega} = \begin{array}{c}
 \\ \\ F \\ \\ --- \\ \\ J\backslash F \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccc} & E(F) & \end{array} \quad \begin{array}{ccc} & I\backslash E(F) & \end{array} \\
\left[\begin{array}{ccc:ccc}
\omega_{1,1} & ... & \omega_{1,e} & \omega_{1,e+1} & ... & \omega_{1,m} \\
... & ... & ... & ... & ... & ... \\
\omega_{f,1} & ... & \omega_{f,e} & \omega_{f,e+1} & ... & \omega_{f,m} \\
\hdashline
0 & ... & 0 & \omega_{f+1,e+1} & ... & \omega_{f+1,m} \\
... & ... & ... & ... & ... & ... \\
0 & ... & 0 & \omega_{r,e+1} & ... & \omega_{r,m}
\end{array}\right]
\end{array}
$$

Conversely, if $p_{E(F)}$ divides $w'_{j'}$ then all entries of the j'-th row indexed in $E(F)$ are zero; but then $j' \notin F$, because F is the union of supports of the columns indexed in $E(F)$, thus any index $j \in F$ supports at least one nonzero entry under $E(F)$. Hence q_F divides $w'_{j'}$.

(b) For q_F-purity, by Theorem 3.1 we must show that when F is regular we have $\mathrm{rk}(\mathbf{A}[F]) = |F| - \mathrm{rk}(\mathbf{\Omega}[E(F)])$, which is the dimension of the space of solutions of $\mathcal{S}(F)$. Observe that for $F = J$ we have $\mathbf{\Omega} = \mathbf{\Omega}[E(J)]$, $\mathbf{A} = \mathbf{A}[J]$, $r = |J|$, $n = \mathrm{rk}(\mathbf{A}[J])$; Proposition 6.2 proves our statement for $F = J$. It is easy to see that the same proof applies to any regular F. $\square$

EXAMPLE 6.5. Let $J = \{1, ..., 6\}$ and consider the rank 3 $B(3)$-group $W_{tot} = <w_1>_* + ... + <w_6>_*$, with creel $K = <a_1 = w_J, a_2 = w_2 + w_3 + w_4, a_3 = w_3 - w_4 - w_{\{5,6\}}>_*$, hence basic system

$$
\mathcal{S} : \left\{ \begin{array}{l} \xi_1 = 0 \\ \xi_2 + \xi_3 + \xi_4 = 0 \\ \xi_3 - \xi_4 - \xi_{\{5,6\}} = 0 \end{array} \right. \qquad \text{and} \qquad \mathbf{A} = \left[\begin{array}{c|c|c|c|cc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array} \right].
$$

The subsets F of J (indexing columns of $\mathbf{A}$) with $\mathrm{rk}(\mathbf{A}[F]) = |F| - 1$ are:

- $\{5, 6\}$;
- $\{2, 3, 4\}$; and $\{1, 5, 6\}$, $\{2, 5, 6\}$, $\{3, 5, 6\}$, $\{4, 5, 6\}$;
- all subsets of 4 elements.

Of these, the minimal ones are

- $F_1 = \{5, 6\}$, with solution $(0, 0, 0, 0, 1, -1)^T$ (transposed);
- $F_2 = \{2, 3, 4\}$, with solution $(0, -2, 1, 1, 0, 0)^T$; while - say - $F = \{2, 5, 6\}$ adding equations $\xi_1 = \xi_3 = \xi_4 = 0$ transforms the second equation into $\xi_2 = 0$, hence yields the same solution as $\{5, 6\}$; analogously, $\{2, 5, 6\}$, $\{3, 5, 6\}$, $\{4, 5, 6\}$ are not minimal;
- all subsets of 4 elements not yielding the same solution as the previous ones.

Note that if we had known all minimal F's of the tent of W_{tot} via other means (e.g. via [**DVM 11**]) we would have only needed to compute the solutions of the systems $\mathcal{S}(F)$, eliminating repeated solutions and attributing each solution to the minimum F for which it occurs.

Using Lemma 2.2 consider the element $a = a_1 + a_2 + a_3 = \xi_1 + 2\xi_2 + 3\xi_3 + \xi_4$ of K; the above $F = \{2, 5, 6\}$ intersects $\mathrm{supp}_Y(a) = \{1, 2, 3, 4\}$ in the singleton $\{2\}$; this means that the equations $\xi_j = 0$ indexed in $J \backslash F = \{1, 3, 4\}$ transform the condition $a = 0$ into $\xi_2 = 0$, obtaining the same system as for $F' = \{5, 6\} = F \backslash \{2\}$. Thus the non-minimal F's are those that intersect the supports of K in a singleton, and must be deprived of that singleton to become minimal (regularization). The final computation yields: $F_3 = \{1, 2, 3, 5\}$; $F_4 = \{1, 2, 3, 6\}$; $F_5 = \{1, 2, 4, 5\}$; $F_6 = \{1, 2, 4, 6\}$; $F_7 = \{1, 3, 4, 5\}$; $F_8 = \{1, 3, 4, 6\}$; with inclusion matrix

$$\mathbf{\Omega} = \left[\begin{array}{c|c|cc|cc|cc} 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & -2 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & -2 \end{array}\right].$$

Thus we have W_{tot} as a subgroup of the X_{tot} of rank 8:

$$\begin{aligned} w_1 &= x_3 + x_4 + x_5 + x_6 + 2x_7 + 2x_8, \\ w_2 &= -2x_2 + x_3 + x_4 - x_5 - x_6, \\ w_3 &= x_2 - x_3 - x_4 - x_7 - x_8, \\ w_4 &= x_2 + x_5 + x_6 + x_7 + x_8, \\ w_5 &= x_1 - x_3 - x_5 - x_7, \\ w_6 &= -x_1 - x_4 - x_6 - x_8. \end{aligned}$$

The above F_j yield the beginning of the tent of W_{tot}, those Primes q_F that become p_i's in X; the remaining F's (all locking Primes, since $|F| = 5$) are $F_1 \cup F_2 = \{2, 3, 4, 5, 6\}$ with $E(\{2, 3, 4, 5, 6\}) = \{1, 2\}$, yielding $q_{23456} = p_{12}$; $F_1 \cup F_3 \cup F_4$ yielding $q_{12356} = p_{134}$, $F_1 \cup F_5 \cup F_6$ yielding $q_{12456} = p_{156}$, $F_1 \cup F_7 \cup F_8$ yielding $q_{13456} = p_{178}$, and analogously $q_{12345} = p_{2357}$, $q_{12346} = p_{2468}$.

Setting $t(w_j) = u_j$ we have thus the tent of W_{tot}:

$$\begin{array}{lcccccccccccccc} u_1 = & q_{56} & q_{234} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_{23456} & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_2 = & q_{56} & \cdot & \cdot & \cdot & \cdot & \cdot & q_{1345} & q_{1346} & \cdot & \cdot & \cdot & q_{13456} & \cdot & \cdot \\ u_3 = & q_{56} & \cdot & \cdot & \cdot & q_{1245} & q_{1246} & \cdot & \cdot & \cdot & \cdot & q_{12456} & \cdot & \cdot & \cdot \\ u_4 = & q_{56} & \cdot & q_{1235} & q_{1236} & \cdot & \cdot & \cdot & \cdot & \cdot & q_{12356} & \cdot & \cdot & \cdot & \cdot \\ u_5 = & \cdot & q_{234} & \cdot & q_{1236} & \cdot & q_{1246} & \cdot & q_{1346} & \cdot & \cdot & \cdot & \cdot & \cdot & q_{12346} \\ u_6 = & \cdot & q_{234} & q_{1235} & \cdot & q_{1245} & \cdot & q_{1345} & \cdot & \cdot & \cdot & \cdot & \cdot & q_{12345} & \cdot \end{array}$$

Here is the tent of $X(K)$, c.d. pure container of W_{tot} via $\mathbf{\Omega}$, where $t(x_i) = t_i$, and the Primes $q_F = p_{E(F)}$ follow the above ordering:

t_1	$=$	$\cdot$	p_2	p_3	p_4	p_5	p_6	p_7	p_8	$\cdot$	$\cdot$	$\cdot$	$\cdot$	p_{2357}	p_{2468}
t_2	$=$	p_1	$\cdot$	p_3	p_4	p_5	p_6	p_7	p_8	$\cdot$	p_{134}	p_{156}	p_{178}	$\cdot$	$\cdot$
t_3	$=$	p_1	p_2	$\cdot$	p_4	p_5	p_6	p_7	p_8	p_{12}	$\cdot$	p_{156}	p_{178}	$\cdot$	p_{2468}
t_4	$=$	p_1	p_2	p_3	$\cdot$	p_5	p_6	p_7	p_8	p_{12}	$\cdot$	p_{156}	p_{178}	p_{2357}	$\cdot$
t_5	$=$	p_1	p_2	p_3	p_4	$\cdot$	p_6	p_7	p_8	p_{12}	p_{134}	$\cdot$	p_{178}	$\cdot$	p_{2468}
t_6	$=$	p_1	p_2	p_3	p_4	p_5	$\cdot$	p_7	p_8	p_{12}	p_{134}	$\cdot$	p_{178}	p_{2357}	$\cdot$
t_7	$=$	p_1	p_2	p_3	p_4	p_5	p_6	$\cdot$	p_8	p_{12}	p_{134}	p_{156}	$\cdot$	$\cdot$	p_{2468}
t_8	$=$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	$\cdot$	p_{12}	p_{134}	p_{156}	$\cdot$	p_{2357}	$\cdot$

7. The concrete solution

If we start with a $B(1)$-group W, our algorithm builds the base types t_i of $X(K)$ abstractly, via the Primes. Not surprisingly, the concrete result is as in [**A**], with the relevant advantage that here we know exactly how to build the supports E_j of the infima, where E_j denotes the support of the j'th row of $\mathbf{\Omega}$ ($i \in E_j$ means $j \in F_i$).

We start with a 'concrete' $B(1)$-group W; from its tent, we build the tent of $X(K)$ via a matrix $\mathbf{\Omega}$ which has the property that, whenever we have **(a)** a 'concrete' c.d. group X' with the tent of $X(K)$, and **(b)** a subgroup W' of X' defined by the matrix $\mathbf{\Omega}$, then W' has the tent of W and is pure in X'. What we must prove is that among these W' there is our W; in other words, we must build suitable 'concrete' base types t_i for X from the given 'concrete' base types u_j of W, and show that the subgroup W' of X defined by $\mathbf{\Omega}$ is isomorphic to W. In the 'concrete' (= quasi-isomorphism) world, this can be done by proving equality of base types.

PROPOSITION 7.1. *Set* $t_i = \bigwedge_{i \in E_j} u_j$ *for each* $i \in I$. *Then the c.d. group* X *with base* $\{t_1, ..., t_m\}$ *contains (an isomorphic copy of)* W *as a pure subgroup.*

Proof. For all $j \in J$ set $u_{j,i} = u_j$ if $i \in E_j$; $u_{j,i} = \min$ (the minimum type of W) otherwise, and, for each $i \in I$, the 'concrete' type

$$t_i = \bigvee_{i \in E_j} u_j = \bigvee_{j \in J} u_{j,i};$$

let X be the 'concrete' c.d. group with base types $\{t_i \mid i \in I\}$; then we have for the subgroup W' defined in X by $\mathbf{\Omega}$:

$$u'_j = \bigwedge_{i \in E_j} t_i \qquad \text{for all} \quad j \in J.$$

If $i \in E_j$, for all $j \in J$ we have $t_i \geq u_j$, hence $u'_j \geq u_j$; we claim equality.
Say w.l.o.g. $j = 1$; by distributivity in $\mathbb{T}$

$$u'_1 = \bigwedge_{i \in E_j} t_i = \bigwedge_{i \in E_j} \left(\bigvee_{j \in J} u_{j,i} \right) = \bigvee_{j \in J} \left(\bigwedge_{i \in E_j} u_{j,i} \right);$$

observe that among the $u_{j,i}$ for $i \in E_1$ there is u_1, hence the last term is $\leq u_1$, as desired. $\square$

The descent into the 'concrete' world must be the last operation, performed only after all unwanted Primes are cancelled: the reason is explained in the next Section.

8. Cancelling Primes

This simple tent-theoretical operation has no simple group-theoretical equivalent. Recall from [**DVM 4**] that in a finite lattice of types we call *Prime type* a $\vee$-irreducible type, that is a type σ that has only one immediate predecessor σ'. We then denote σ

- by a Prime p: $\sigma = p$, if σ' is the minimum of the typeset: $\sigma = p\sigma'$;
- by adding a symbol p (a 'Prime') to σ' if σ' is not the minimum.

Since each element of a lattice is the supremum of the $\vee$-irreducible elements below it, each type will be denoted as a (squarefree) products of Primes (and will behave accordingly). If the lattice is the typeset of a B(n)-group, and we build its tent, we may well consider the tent (and the lattice) we obtain by cancelling a Prime. But is there an equivalent group-theoretical operation?

Example 8.1. Consider the following subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$, given by their characteristics [**F II**]:

$$\chi(\mathbb{Q}_2) = (\infty, 0, 0, 0, ...); \qquad \chi(\mathbb{Q}_{2,3}) = (\infty, \infty, 0, 0, 0, ...);$$

$$\chi(R_2) = (2, 2, 2, ...), \qquad \chi(R_1) = (1, 1, 1, ...). \qquad \text{Let}$$

$$X = \mathbb{Q}_2 x_1 \oplus \mathbb{Q}_{2,3} x_2 \oplus \mathbb{Z} x_3 \leq \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$$

$$Y = R_1 y_1 \oplus R_2 y_2 \oplus \mathbb{Z} y_3 \leq \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}.$$

Since for both X and Y the typeset consists of three $\vee$-irreducible types, for both groups the tent is of the form

$$\begin{array}{lcll} t_1 & = & p_1 & \cdot \\ t_2 & = & p_1 & p_2 \\ t_3 & = & \cdot & \cdot \end{array}$$

For X, we can give somehow an algebraic meaning to 'cancelling p_2', because, letting $\chi(\mathbb{Q}_3) = (0, \infty, 0, 0, 0, ...)$, we have $X \oplus \mathbb{Q}_3 = \mathbb{Q}_{2,3} x_1 \oplus \mathbb{Q}_{2,3} x_2 \oplus \mathbb{Q}_3 x_3$, which (eliminating the full Prime) has tent

$$\begin{array}{lcl} t_1 & = & p_1 \\ t_2 & = & p_1 \\ t_3 & = & \cdot \end{array}$$

This is not possible for Y, since there is no type in $\mathbb{T}$ analog to $\mathbb{Q}_3$: the existence of a group-theoretical tensoring operation equivalent to cancelling a Prime depends on the way the lattice of the tent is included in $\mathbb{T}$. Moreover, no other group-theoretical operation on Y we can think of reduces the tent as above.

Thus, if we didn't have tents, the agile instruction «cancel all Primes of X that are not Primes of W» that we need to build the m.t.p.-container X of W would have to be worded «consider each $\vee$-irreducible type σ in the typeset of X that is not in the typeset of W, and reduce it to its immediate predecessor in W; moreover, do the same operation to all the types that are $\geq \sigma$», an instruction that is not simple to express, let alone enact.

This is just another example of how using tents considerably simplifies one's life with B(n)-groups.

9. Realizing the converse of Butler's Theorem

The other implication of Butler's Theorem claims that if G is a pure subgroup of a c.d. $X = <x_1>_* \oplus ... \oplus <x_m>_*$, then G is the epimorphic image of a suitable c.d. $Y = <y_1>_* \oplus ... \oplus <y_r>_*$. The proof (e.g. in [**A**]) shows that this is indeed the case if the $<y_j>_*$ are isomorphic to the rank 1 pure subgroups $<g_S>_*$ of G generated by the *elements* g_S *with minimal support* $S \subseteq I$ *in* X; their types - the required types for the $<y_j>_*$ - are then the

$$t(<y_j>_*) = \wedge\{t(x_i) \mid i \in S\}.$$

In order to have an algorithm, we need to know how to build the sets S.

Let $G = <g_1, ..., g_s>_*$ with $s = \text{rk}G$; write for $k = 1, ..., s$:

$g_k = \sum\{\gamma_{i,k} x_i \mid i \in I\}$; then $\mathbf{\Gamma} = \begin{bmatrix} \gamma_{1,1} & \cdots & \gamma_{1,s} \\ ... & ... & ... \\ \gamma_{m,1} & \cdots & \gamma_{m,s} \end{bmatrix}$ has rank s.

For $g = \sum\{\beta_k g_k \mid k = 1, ..., s\} = \sum\{\beta_k \gamma_{i,k} x_i \mid k = 1, ..., s; i \in I\}$ to have minimal support in X, that is maximal zero-block, we must look for maximal homogeneous subsystems of the system

$$\begin{cases} \gamma_{1,1}\beta_1 + ... + \gamma_{1,s}\beta_s &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ \gamma_{m,1}\beta_1 + ... + \gamma_{m,s}\beta_s &= 0 \end{cases}$$

with all-nonzero solution.

PROPOSITION 9.1. *The minimal supports* S *for the elements of* G *index the maximal row-submatrices of* $\mathbf{\Gamma}$ *of rank* $s - 1$. □

The proof is left to the reader, and completes the algorithm.

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The role of the Jacobson radical in isomorphism theorems

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Dedicated to Rüdiger Göbel on his 70th birthday

Abstract. The Baer-Kaplansky theorem states that two torsion groups are isomorphic if and only if their endomorphism rings are isomorphic rings. Let R be a discrete valuation domain. If R is complete in its p-adic topology, then it is well known that torsion-free R-modules and several classes of mixed R-modules including mixed modules of torsion-free rank one with a totally projective torsion submodule and Warfield modules of finite torsion-free rank also satisfy a Baer-Kaplansky type isomorphism theorem. The purpose of this paper is to show that in many of these classes an isomorphism between only the Jacobson radicals of the endomorphism rings of two modules is sufficient to imply that the modules are isomorphic. Furthermore, if the torsion submodule of a mixed module contains an unbounded basic submodule, the Jacobson radical of its endomorphism ring determines its torsion submodule.

1. Introduction

For a discrete valuation domain R, the Baer-Kaplansky Theorem [**B43, K69**] states that two torsion R-modules are isomorphic if and only if their endomorphism rings are isomorphic R-algebras. Wolfson [**W62**] showed that the class of torsion-free modules over a complete discrete valuation domain satisfies a Baer-Kaplansky type isomorphism theorem. Warren May [**M89, M90, M95**], May and Toubassi [**MT83**] and Files [**Fi95**] showed that several classes of mixed modules over a (complete) discrete valuation domain also satisfy an isomorphism theorem.

The ideal of the endomorphism ring generated by primitive idempotents is sufficient to prove the Baer-Kaplansky theorem for torsion R-modules. The Jacobson radical of an endomorphism ring contains no nonzero idempotents. Expanding the Baer-Kaplansky Theorem in a different direction, consider the question of whether there exist classes of R-modules for which an isomorphism between the Jacobson radicals of the endomorphism rings of two modules in that class implies the modules are isomorphic.

In this vein, Hausen, Praeger and Schultz [**HPS94**] showed that, for the class of p-primary groups which are unbounded modulo their divisible subgroups, an isomorphism between the Jacobson radicals of the endomorphism rings of two groups implies the groups are isomorphic. This result extends naturally to torsion R-modules which are unbounded modulo their divisible submodules. Flagg [**Fl08**]

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showed that torsion-free modules over a complete discrete valuation domain which are not divisible satisfy an isomorphism theorem.

In the case of mixed modules, the first step toward an isomorphism theorem is to show that the torsion submodules of all modules in the class of interest are determined by the Jacobson radical. The author [**Fl09**] began this step by proving the following theorem.

THEOREM 1.1. *Let R be a complete discrete valuation domain and let M and N be R-modules. Assume that M has a torsion submodule T which contains an unbounded basic submodule and that M/T is divisible. If there exists an R-algebra isomorphism between the Jacobson radicals of the endomorphism rings of M and N, then there exists an isomorphism $\phi : T \to T(N)$ which induces the Jacobson radical isomorphism on the torsion.*

The first purpose of this paper is to improve Theorem 1.1 by removing the requirements that R be complete in its p-adic topology and the R-modules be divisible modulo their torsion submodules.

The second purpose of this paper is to use the improved version of Theorem 1.1 to prove that there are two classes of mixed modules which satisfy a Jacobson radical isomorphism theorem.

THEOREM 1.2. *Let R be a complete discrete valuation domain, and let M and N be mixed R-modules of torsion-free rank one. Assume M has a totally projective torsion submodule which is unbounded. Let J_M and J_N be the Jacobson radicals of the respective endomorphism rings. If there exists an R-algebra isomorphism $\Phi : J_M \to J_N$, then there exists an isomorphism $\phi : M \to N$ that induces Φ.*

THEOREM 1.3. *Let R be a complete discrete valuation domain. Let M and N be reduced Warfield modules of finite torsion-free rank. Assume that M has an unbounded torsion submodule. If there exists an R-algebra isomorphism $\Phi : J_M \to J_N$, then $M \cong N$.*

2. Preliminaries

Throughout this paper, R will denote a discrete valuation domain with prime p and quotient field Q. All modules will be left R-modules. For an R-module M, there is a natural embedding of the ring R into the endomorphism ring $E(M)$ given by $r \mapsto r \cdot 1_M$. This embedding provides a natural R-algebra structure to $E(M)$ and its ideals. An R-algebra isomorphism between two R-algebras E_1 and E_2 is simply a ring isomorphism that is the identity on R. Endomorphisms will be written as acting from the right. For an R-module M (ideal I of the ring $E(M)$), the notation $\mathbf{t}M$ ($\mathbf{t}I$) will denote the submodule (ideal) of all elements of finite additive order. Furthermore, for a ring W, $\gamma \in W$ and a set $L \subseteq W$, let $\gamma \cdot L = \{\gamma\lambda : \lambda \in L\}$ and define $L \cdot \gamma$ similarly. Any unspecified notation will be that of Fuchs [**F1, F2**].

The following results summarize the properties of the Jacobson radical of the endomorphism ring of an R-module that are needed. The proofs are the same as those given in Flagg [**Fl09**].

First, the Jacobson radical is bounded by two other radicals of the endomorphism ring. Let M be a mixed R-module with endomorphism ring E. The first important radical is the nilradical $N(E)$, which is sum of all nil ideals of E. The second is called the Pierce radical [**P63**]. It is defined as

$$P(E) = \{\phi \in E : p^n M[p]\phi \subseteq p^{n+1}M \text{ for all integers } n \geq 0\}.$$

LEMMA 2.1. *Let M be an R-module with torsion submodule T, let $E = E(M)$, and let $J = J(E(M))$. Then $N(E) \subseteq J \subseteq P(E)$ and $\mathbf{t}N(E) = \mathbf{t}J = \mathbf{t}P(E)$.*

Second, for a module M, define the set $H_M = \{\beta \in \mathbf{t}J(E(M)) : \mathbf{t}J \cdot \beta = 0\}$. The set H_M is a two-sided ideal of $\mathbf{t}J(E(M))$. The following results explain the crucial properties of the ideal H_M when the torsion submodule contains an unbounded basic submodule.

PROPOSITION 1. *Let M be an R-module with torsion submodule T. Assume T contains an unbounded basic submodule. Let $J = J(E(M))$, $H_M = \{\alpha \in \mathbf{t}J : \mathbf{t}J \cdot \alpha = 0\}$, and let $\alpha \in \mathbf{t}J$. Then $\alpha \in H_M$ if and only if $T\alpha = 0$.*

PROPOSITION 2. *Let M be a mixed R-module with torsion submodule T. Assume T has an unbounded basic submodule. Then M/T is divisible if and only if $H_M = \{\alpha \in \mathbf{t}J(E(M)) : \mathbf{t}J \cdot \alpha = 0\} = 0$.*

Finally, define the quotient ring (without identity) $K = \mathbf{t}J/H_M$. For every $\alpha \in \mathbf{t}J$, let $\overline{\alpha} = \alpha + H_M \in K$.

3. Determining the Torsion

Suppose a torsion module T has a decomposition $T = \langle a \rangle \oplus \langle b \rangle \oplus T'$ for some $T' \leq T$ and $a, b \in T$ such that $o(a) < o(b)$. Define a map $\eta \in E(T)$ by $b\eta = a$ and $(\langle a \rangle \oplus T')\eta = 0$. The basic technique employed by Hausen, Praeger and Schultz [**HPS94**] is to recognize η from its ring-theoretic properties in the Jacobson radical. Their method applies to mixed R-modules if the quotient ring K replaces $\mathbf{t}J$. The technical results will be stated in the appropriate form using K, but the proofs will not be repeated.

LEMMA 3.1. [**HPS94**, Lemma 2.3] *Let M be an R-module whose torsion submodule, T, contains an unbounded basic submodule. Let u be a positive integer and let $\eta \in \mathbf{t}J$. Then $\overline{\eta} \cdot K[p^u] = 0$ if and only if $T\eta \subseteq p^u T$.*

LEMMA 3.2. [**HPS94**, Proposition 2.4] *Let M be an R-module whose torsion submodule, T, contains an unbounded basic submodule. Let u be a positive integer and let $\overline{\eta} \in K[p^u]$. Then $p^{u-1}\overline{\eta} \cdot K[p^u] \neq 0$ if and only if T has a decomposition*

$$T = \langle y \rangle \oplus Y = \langle y\eta \rangle \oplus X$$

such that $o(y\eta) = p^u$.

DEFINITION 3.3. A left K-module L is strongly uniform if, under addition, L is a torsion group and, whenever λ and κ are two nonzero elements in L, then $K\lambda \cap K\kappa$ contains an element τ such that the order of τ equals the minimum of the orders of λ and κ.

PROPOSITION 3. [**HPS94**, Propostion 2.5] *Let M be an R-module whose torsion submodule, T, contains an unbounded basic submodule. Let u be a positive integer and let $\overline{\eta} \in K[p^u]$ such that $p^{u-1}\overline{\eta} \cdot K[p^u] \neq 0$. Then $T\eta$ is cyclic if and only if $K\overline{\eta}$ is strongly uniform.*

PROPOSITION 4. [**HPS94**, Propostion 2.6] *Let M be an R-module with torsion submodule T. Then T has an unbounded basic submodule if and only if there exists an increasing sequence of integers $1 < n_1 < n_2 < \ldots$ and elements $\eta_1, \eta_2, \ldots \in \mathbf{t}J$ such that, for each $i \geq 1$, the following hold:*

1. $o(\overline{\eta_i}) = p^{n_i}$
2. $p^{n_i-1}\overline{\eta_i} \cdot K[p^{n_i}] \neq 0$
3. $K\overline{\eta_i}$ *is a strongly uniform left* K*-module*
4. the function $f_i : \overline{\eta_i}K \to \overline{\eta_{i+1}}K$ *defined by* $\overline{\eta_i\kappa} \mapsto \overline{\eta_{i+1}\eta_i\kappa}$ *for all* $\kappa \in \mathbf{t}J$ *is a monomorphism.*

Furthermore, if there exists an increasing sequence $1 < n_1 < n_2 < \ldots$ *of natural numbers and* $\overline{\eta_i} \in K$ *satisfying conditions (1)-(4), then there exist elements* $d_1, d_2, \ldots$ *in* T *such that, for each integer* $i \geq 1$, $o(d_i) = p^{n_i}$, $d_{i+1}\eta_i = d_i$, $T\eta_i = \langle d_i \rangle$ *and* $\langle d_i \rangle$ *is a direct summand of* M*.*

These technical lemmas provide the necessary machinery to prove the following theorem.

THEOREM 3.4. *Let* R *be a discrete valuation domain. Let* M *and* N *be left* R*-modules with torsion submodules* T_M *and* T_N*, respectively. If* T_M *has an unbounded basic submodule and there exists an* R*-algebra isomorphism* $\Phi : J_M \to J_N$*, then there exists an isomorphism* $\phi : T_M \to T_N$ *which induces* Φ *on the torsion.*

PROOF. For a module M, let T_M denote its torsion submodule, $J_M = J(E(M))$, $H_M = \{\alpha \in \mathbf{t}J_M : \mathbf{t}J_M \cdot \alpha = 0\}$, and $K_M = \mathbf{t}J_M/H_M$. Furthermore, for $\alpha \in J_M$, let $\alpha^* = \alpha\Phi \in J_N$.

Since $\mathbf{t}J_M\Phi = \mathbf{t}J_N$ and $H_M\Phi = H_N$, the isomorphism Φ induces an isomorphism from K_M to K_N. Hence, for $\overline{\alpha} \in K_M$, let $\overline{\alpha}^* = \alpha^* + H_N \in K_N$.

The module T_M has an unbounded basic submodule, so there exists a sequence of natural numbers $1 < n_1 < n_2 < \ldots$ and $\overline{\eta_i} \in K_M$ satisfying (1) through (4) of Proposition 4. Since these properties are preserved under ring isomorphisms, it follows that, for all i, $\overline{\eta_i}^* \in K_N$ have analogous properties $(1^*) - (4^*)$. Hence, T_N also has an unbounded basic submodule. Applying Proposition 4 again, for every $i \geq 1$, there exist $d_i \in T_M$ and $d_i^* \in T_N$ which generate direct summands of order p^{n_i}. For every $i \geq 1$ the elements d_i and d_i^* have the further properties that $d_{i+1}\eta_i = d_i$, $d_{i+1}^*\eta^* = d_i^*$, $T_M\eta_i = \langle d_i \rangle$, and $T_N\eta_i^* = \langle d_i^* \rangle$.

Define a map $\phi : T_M \to T_N$ as follows. For $x \in T_M$, choose a natural number k such that $o(x) < p^{n_k}$. Since $\langle d_k \rangle$ is a direct summand of T_M, it is also a direct summand of M. Therefore, there exists an endomorphism $\varepsilon \in E(M)$ such that $d_i\varepsilon = x$ and the complement of $\langle d_i \rangle$ is mapped to 0. Since $\varepsilon \in \mathbf{t}P(E(M))$, $\varepsilon \in \mathbf{t}J_M$ by Lemma 2.1. Define $x\phi = d_k^*\varepsilon^*$. Showing that ϕ is well-defined and an isomorphism that induces Φ on the torsion follows the standard argument and will be left to the reader. □

Hausen and Johnson [**HJ95**] showed that, in the class of bounded plus divisible p-groups, an isomorphism between the Jacobson radicals of the endomorphism rings of two groups is not generally induced by a isomorphism between the groups. Hence, the requirement that T_M have an unbounded basic submodule is necessary for Theorem 3.4 to be the strong form of an isomorphism theorem in which the Jacobson radical isomorphism is induced on the torsion submodule.

4. Rank One Mixed Modules

May and Toubassi [**MT83**] showed that the class of mixed R-modules of torsion-free rank one with a simply presented torsion submodule satisfies an isomorphism theorem. In light of Theorem 3.4, the class of R-modules will be restricted to those whose torsion submodules contain an unbounded basic submodule. .

For simplicity, the term "rank one module" will be used in this section to mean an R-module of torsion free rank one with a torsion submodule which contains an unbounded basic submodule. For a rank one module M, $M/\mathbf{t}M$ is either isomorphic to R or Q. If $M/\mathbf{t}M \cong R$, then it splits and $M \cong R \oplus \mathbf{t}M$. If $M/\mathbf{t}M \cong Q$, then M may or may not split. An isomorphism theorem will be proved separately in each of the three cases.

4.1. Case One: $M/\mathbf{t}M$ is Isomorphic to R. If the Jacobson radical of the endomorphism rings of M and that of another rank one module N are isomorphic, then Theorem 3.4 and Proposition 2 already imply that the modules are isomorphic. However, this isomorphism may not be induced. Therefore, more information is needed to define an isomorphism that is induced.

Proposition 5. *Let M be a rank one R-module, let $T = T(M)$, $J = J(E(M))$ and let $\varepsilon \in J$. Assume $M/T \cong R$. Then, there exists an idempotent $\pi \in E(M)$ with $M\pi \cong R$ such that $\varepsilon = p\pi$ if and only if ε satisfies the following properties: (1) $\varepsilon^2 = p\varepsilon$, (2) $p^n\varepsilon \neq 0$ for all $n \geq 0$, (3) $\mathbf{t}J \cdot \varepsilon = 0$, and (4) $\varepsilon \cdot J[p] = 0$.*

Proof. First, let $\pi \in E$ with $\pi^2 = \pi$ and $M\pi = \langle w \rangle$ for some $w \in M$ such that $\langle w \rangle \cong R$. Let $\omega = p\pi \in J$. The facts that $M = Rw \oplus T$ and $T\omega = 0$ imply that ω has properties $(1) - (4)$. Conversely, assume $\varepsilon \in J$ satisfies properties $(1) - (4)$. Then property (3) and Proposition 1 show that $T\varepsilon = 0$. Therefore, $T \leq \ker \varepsilon$. Property (2) further implies $T = \ker \varepsilon$. Since $M/T \cong R$, choose $x \in M$ such that $M = Rx \oplus T$ and $Rx \cong R$. Let $\rho : M \to Rx$ be the cannonical projection. Decompose $J = \rho J \rho + \rho J(1-\rho) + (1-\rho)J(1-\rho)$. Then $\rho J \rho = J(E(Rx)) = pR * 1_{Rx} = pR(\rho)$ and $\rho J(1-\rho) = H_M$. Since $T\varepsilon = 0$, $(1-\rho)\varepsilon = 0$. Therefore, there exists $r \in R$ and $\lambda \in H_M$ such that $\varepsilon = (pr)\,\rho + \lambda$. Thus, $x\varepsilon = prx + x\lambda$. By property (1), $\varepsilon^2 = p\varepsilon$, and this implies $r = 1$. So, $x\varepsilon = px + x\lambda$. The claim is that $x\lambda \in pM$. Otherwise, $x\lambda = t \notin pT$ and there exists a map $\gamma \in J[p]$ such that $x\lambda\gamma \neq 0$. This contradicts the fact that ε satisfies property (4). Since $x\lambda \in pT$, there exists an element $s \in T$ and a map $\sigma \in H_M$ such that $x\sigma = s$ and $x\lambda = ps$.

Then $x\varepsilon = p(x+s)$. This means $(x+s)\varepsilon = p(x+s)$. M has a decomposition $M = R(x+s) \oplus T$ and $\varepsilon = p\alpha$ for the projection $\alpha : M \to R(x+s)$ along T. ☐

Notice that properties $(1) - (4)$ of Proposition 5 are preserved under isomorphism. Also for a rank one module, given a primitive idempotent $\pi \in E(M)$ with $M\pi = Rw \cong R$ and $\ker \pi = T$, the ideal $H_M = \{\alpha \in \mathbf{t}J(E(M)) : \mathbf{t}J \cdot \alpha = 0\} = \pi J(1-\pi)$. Furthermore, $H_M \cong T$ as an R-module. This connection is the key to the following theorem:

Theorem 4.1. *Let M and N be rank one modules. Assume M/T_M is isomorphic to the ring R. If there exists an isomorphism $\Phi : J(E(M)) \to J(E(N))$, then there exists an isomorphism $\phi : M \to N$ that induces Φ.*

Proof. For all maps $\alpha \in J_M$, let $\alpha^* = \alpha\Phi \in J_N$. By Lemma 2, $H_M \neq 0$. Since $H_M\Phi = H_N$, $H_N \neq 0$. Hence, $N/T_N \cong R$. Choose $\varepsilon \in J_M$ such that ε satisfies properties $(1) - (4)$ of Proposition 5. Then $\varepsilon^* = \varepsilon\Phi \in J_N$ also satisfies properties $(1) - (4)$ of Proposition 5. There exists a decomposition $M = Ra \oplus T_M$ with $Ra \cong R$ and an idempotent $\pi \in E(M)$ such that π is the projection $M \to Ra$ along T_M and $\varepsilon = p\pi$. There exists a similar decomposition $N = Ra^* \oplus T_N$ with $\varepsilon^* = p\pi^*$ and $\pi^* : N \to Ra^*$ the cannonical projection. Define a map $\phi : M \to N$ as follows: Given $x \in M$, there exists $r \in R$ and $t \in T_M$

such that $x = ra + t$. Furthermore, there exists a unique map $\lambda_t \in H_M$ such that $a\lambda_t = t$. Define $x\phi = ra^* + a^*\lambda_t^*$. The map ϕ is the desired isomorphism by the standard argument. $\square$

4.2. Cases Two and Three: $M/\mathrm{t}M$ is Divisible. For a rank one module M, if $M/T_M \cong Q$, M may or may not be split. Unfortunately, since $J(E(Q)) = 0$, determining splitting is nontrivial. Therefore, the full machinery of the nonsplit case is needed immediately. This requires that in both the split and nonsplit case rank one modules will be restricted to those with totally projective torsion submodules (and totally projective torsion modules are assumed to be reduced). Furthermore, the ring R must be complete in the p-adic topology.

Let M be a nonsplit mixed module of torsion-free rank one with torsion submodule T. Let

$$I = I(M) = \{\gamma \in E(M) : M\gamma \subseteq T\}.$$

It is a straightforward exercise to prove that I is a two-sided ideal of E. May and Toubassi [**MT83**] prove the following description of the endomorphism ring of a nonsplit R-module of torsion-free rank one.

PROPOSITION 6. *Let M be a nonsplit mixed R-module of torsion-free rank one with torsion submodule T. Then $E(M) = R \cdot 1_M + I(M)$.*

This form of the endomorphism ring provides the following lower bound for the Jacobson radical of a nonsplit mixed R-module of torsion-free rank one:

PROPOSITION 7. *Let M be a nonsplit mixed R-module of torsion-free rank one. Then, $pE(M) \subseteq J(E(M))$.*

PROOF. Let $E = E(M)$. The set pE is a two-sided ideal of E. To show that pE is quasi-regular, it is sufficient to show that $\gamma \in pE$ implies $(1+\gamma) \in Aut(M)$. Let $\gamma \in pE$, and by Proposition 6, there exist $r \in R$ and $\beta \in I$ such that

$$\gamma = pr + p\beta.$$

Then $1+\gamma = (1+pr)+p\beta$. Since $\gamma|_T \in pE(T)$ and $pE\,(T) \leq J(E(T))$, it follows that $\gamma|_T \in J(E(T))$. Thus, the map $1+\gamma$ is bijective on the torsion. To see that $1+\gamma$ is injective on all of M, suppose there exists $0 \neq x \in \ker(1+\gamma)$. Since $1+\gamma$ is injective on the torsion, x must be of infinite order. Then $x(1+\gamma) = (1+pr)x + xp\beta = 0$, and note that $xp\beta \in T$. The element x is of infinite order and $(1+pr)x \in T$, creating the needed contradiction. Let $x \in M$ have infinite order. Since $1+\gamma$ is bijective on the torsion, $x \in M(1+\gamma)$ implies $1+\gamma$ is surjective. The element $1+pr$ is a unit in R, so there exists $(1+pr)^{-1} \in R$. Then,

$$(1+pr)^{-1}x(1+\gamma) = (1+pr)^{-1}(1+pr)x + (1+pr)^{-1}xp\beta = x + (1+pr)^{-1}xp\beta.$$

By the fact that $1+\gamma$ is surjective on T, there exists $w \in T$ such that $w(1+\gamma) = (1+pr)^{-1}xp\beta$. Then, $[(1+pr)^{-1}x - w](1+\gamma) = x$. $\square$

The isomorphism question for the nonsplit case is best addressed in the setting of the cotorsion hull of the common torsion submodule. Theorem 3.4 allows the techniques of May [**M90**] to apply to the Jacobson radical version.

Let A be a reduced R-module. Recall that the cotorsion hull of A, denoted A^*, is the module

$$A^* = Ext^1_R(Q/R, A).$$

LEMMA 4.2. *Let M be a reduced R-module with torsion submodule T and assume M/T is divisible. Then,*

(1). There exists an embedding $\iota : M \to T^*$, such that $M\iota = \overline{M}$ is a pure submodule of T^*, $T\iota = \mathbf{t}\,(T^*)$, and $T^*/\overline{M}$ is torsion-free and divisible.

(2). There exists an embedding $\mu : E(M) \to E(T^*)$ such that $\overline{E} = E(M)\mu = \{\beta \in E(T^*) : \overline{M}\beta \subseteq \overline{M}\}$,

(3). The image of the Jacobson radical under the embedding μ will be denoted $\overline{J} = J(E(M))\mu$.

Notice that the set $\overline{J}$ is simply the image of the Jacobson radical under the embedding. A connection to $J(E(T^*))$ is not required.

LEMMA 4.3. *Let M be a reduced module with torsion submodule T which contains an unbounded basic submodule. Assume that M/T is divisible. Let N be a reduced module such that there exists an R-algebra isomorphism $\Phi : J_M \to J_N$. Then, N/tN is divisible and there exists an embedding $\iota_N : N \to T^*$ such that $\overline{N} = N\iota_N$ is a pure submodule of T^* and $T^*/\overline{N}$ is torsion-free and divisible.*

PROOF. By Theorem 3.4, there exists an isomorphism $\phi : T \to \mathbf{t}N$ such that $\alpha\Phi$ and $\phi\alpha\phi^{-1}$ agree for all $\alpha \in J_M$. Also, $H_N = H_M\Phi = 0$ and Lemma 2 implies $N/\mathbf{t}N$ is divisible. Therefore, $N^* \cong (\mathbf{t}N)^* \cong T^*$. Using the isomorphism ϕ^{-1} and Lemma 4.2, N can be embedded into T^*. Denote the image of this embedding $\overline{N}$, and Lemma 4.2 implies $\overline{N}$ has the required properties. □

PROPOSITION 8. *Let M and N be modules satisfying the hypotheses of Lemma 4.3. Let $\overline{J}_M$ and $\overline{J}_N$ be the images of J_M and J_N under the respective embeddings $\chi_M : E_M \to E(T^*)$ and $\chi_N : E_N \to E(T^*)$. Then, $\overline{J}_M = \overline{J}_N$.*

PROOF. By Theorem 3.4, there exists an isomorphism $\phi : T \to \mathbf{t}N$ and the isomorphism ϕ^{-1} is used to embed N into T^*. Thus, for $t \in \overline{T} = \mathbf{t}(T^*)$ and $\overline{\gamma} \in \overline{J}_M$ with $\overline{\gamma^*} = \overline{\gamma\Phi} \in \overline{J}_N$, $t\overline{\gamma} = t\overline{\gamma^*}$. Since $T^*/\overline{T}$ is divisible, homomorphisms in $E\,(T^*)$ are uniquely determined by their actions on the torsion, $\overline{T}$. Therefore, $\overline{\gamma} = \overline{\gamma^*}$, proving the claim. □

Assume that M is a mixed module with a totally projective torsion submodule T which is unbounded and assume that M is nonsplit. Then embed M into T^* by Lemma 4.2. Let $\overline{M}$ be the image of M under this embedding. Then, May [**M90**] defines the submodule $C(\overline{M})$ as follows: Consider the family of submodules $\mathcal{C}$ of $\overline{M}$ such that, $A \in \mathcal{C}$ implies that for every $\gamma^* \in E(T^*)$, $A\gamma^* \subseteq \overline{M}$. The family $\mathcal{C}$ is closed under taking arbitrary sums. Thus, $\mathcal{C}$ has a maximal element. Denote this maximal submodule by $C(\overline{M})$. Alternately, $C(\overline{M})$ may be described as the maximal $E(T^*)$-module contained in $\overline{M}$. Note that since $\overline{\mathbf{t}M} = \mathbf{t}T^*$, $\overline{\mathbf{t}M} \subseteq C(\overline{M})$.

Theorem 7 of May [**M90**] implies the following description of $C(\overline{M})$ for rank one modules (and Warfield modules).

LEMMA 4.4. *Let M be reduced mixed R-module of finite torsion-free rank with torsion submodule T. Assume M/T is divisible. Let $\overline{M}$ and $\overline{T}$ be the embedded image of M and T in T^*, respectively. If T is totally projective or M is a Warfield module, then, $C(\overline{M}) = \overline{T}$.*

Note that if $x \in \overline{M}$ has infinite order, then Lemma 4.4 implies that there exists $\gamma^* \in E(T^*)$ such that $x\gamma^* \notin \overline{M}$.

Lemma 4.4 provides the following criterion for determining splitting.

THEOREM 4.5. *Let M be a reduced rank one R-module and let $T_M = \mathbf{t}M$. Assume that $M/T_M \cong Q$. Then, $J(E(M)) \cong J(E(T_M))$ if and only if M is split.*

PROOF. First, if M is split, then $J(E(M)) \cong J(E(T_M))$.

Conversely, assume that M is nonsplit. Let T be a torsion module such that $T \cong T_M$. By way of contradiction, assume $J(E(M)) \cong J(E(T))$. Let $\Phi : J_M \to J_T$ be this R-algebra isomorphism. Theorem 3.4 constructs an isomorphism $\phi : T_M \to T$ that induces Φ on T_M. The modules M and T are both reduced modules and M/T_M is divisible, so M and T may be embedded into T^*.

Embed T into T^* and note that $\overline{J}_T = J(E(T^*))$.

Embed M into T^* using the isomorphism ϕ. Let $\overline{M}$ be the image of M under this embedding. Let χ_M be the induced embedding of E_M into $E(T^*)$, with $\overline{E}_M = E_M\chi_M$. Define $\overline{J}_M = J_M\chi_M$. Then, Proposition 8 implies $\overline{J}_M = \overline{J}_T = J(E(T^*))$.

Lemma 4.4 implies that $C(\overline{M}) = \overline{T}$. Therefore, there exists $\gamma^* \in E(T^*)$ and $x \in \overline{M}$ of infinite order such that

$$x\gamma^* \notin \overline{M}.$$

Since $pE(T) \subseteq J(E(T))$ for a torsion module T , $p\gamma^* \in J(E(T^*))$. Since $\overline{J}_M = J(E(T^*))$, $p\gamma^* \in \overline{J}_M$. The fact that $\overline{J}_M \subset \overline{E}_M$ implies $\overline{M}p\gamma^* \subseteq \overline{M}$. So,

$$xp\gamma^* \in \overline{M}.$$

Let $w = x\gamma^*$. Then, $xp\gamma^* = pw$. The element w is of infinite order since $\overline{T} \subseteq \overline{M}$ and $w \notin \overline{M}$. The submodule $\overline{M}$ has the property that $T^*/\overline{M}$ is torsion-free and divisible. However,

$$0 \neq w + \overline{M} \in T^*/\overline{M} \text{ and } p(w + \overline{M}) = pw + \overline{M} = 0.$$

This creates the needed contradiction. Therefore, if M is nonsplit, then $J(E(M)) \ncong J(E(T_M))$. □

Theorem 4.5 provides the mechanism for proving Theorem 1.2 in the split case.

THEOREM 4.6. *Let M and N be rank one R-modules and suppose the torsion submodule of M is totally projective. Assume that $M/\mathbf{t}M \cong Q$ and M is split. If there exists an R-algebra isomorphism $\Phi : J_M \to J_N$, then there exists an isomorphism $\phi : M \to N$ that induces Φ.*

PROOF. Since $M \cong Q \oplus \mathbf{t}M$, $J(E(M)) \cong J(E(\mathbf{t}M))$. By Theorem 3.4, $\mathbf{t}M \cong \mathbf{t}N$, and hence $J(E(N)) \cong J(E(\mathbf{t}N))$, which implies N is split also. The isomorphism $\phi : \mathbf{t}M \to \mathbf{t}N$ constructed in Theorem 3.4 induces Φ on the torsion. Since $J(E(Q)) = 0$ and $Hom(Q, \mathbf{t}M) = 0$ for a reduced torsion module $\mathbf{t}M$, the isomorphism ϕ is sufficient in this case. □

The following theorem finishes the proof of Theorem 1.2 in the nonsplit case.

THEOREM 4.7. *Let M be a rank one module with an unbounded totally projective torsion submodule, let $T_M = \mathbf{t}M$. Assume M is nonsplit. Let N be a module of torsion-free rank one with torsion submodule T_N. If there exists an R-algebra isomorphism $\Phi : J_M \to J_N$, then there exists an isomorphism $\varphi : M \to N$ that induces Φ.*

PROOF. Theorem 3.4 implies the existence of an isomorphism $\phi : T_M \to T_N$ that induces Φ on the torsion. This isomorphism shows that T_N is reduced and totally projective. Since M/T_M is divisible, $H_M = \{\alpha \in \mathbf{t}J_M : \mathbf{t}J_M \cdot \alpha = 0\} = 0$. Then, $H_N = H_M\Phi = 0$ showing that N/T_N is divisible.

By Theorem 3.4, there exists an isomorphism $\phi : T_M \to T_N$ that induces Φ on the torsion. Since M is nonsplit, Theorem 4.5 shows that $J_M \ncong J(E(T_M))$. Since $T_M \cong T_N$, this implies $J_N \ncong J(E(T_N))$. Therefore, Theorem 4.5 shows N is nonsplit.

Now, M and N may both be embedded into the cotorsion hull of T_M. To keep the notation consistent, let $T = T_M$. Then, embed M and N into T^*. Let $\overline{M}$ and $\overline{N}$ be the images of M and N under the respective embeddings. Let $\overline{T}$ be the embedded image of T, and recall that $\overline{T}$ is the torsion submodule of T^*. Then, $\overline{M}$ and $\overline{N}$ have the following properties:

(1) $\overline{M}$ and $\overline{N}$ are pure submodules of T^* containing T, and

(2) $T^*/\overline{M}$ and $T^*/\overline{N}$ are torsion-free and divisible.

Embed E_M and E_N into $E(T^*)$ and let $\overline{E}_M$ and $\overline{E}_N$ be the images under the respective embeddings. Then, Lemma 4.2 shows

$$\overline{E}_M = \{\varepsilon^* \in E(T^*) : \overline{M}\varepsilon^* \subseteq \overline{M}\},$$
$$\overline{E}_N = \{\varepsilon^* \in E(T^*) : \overline{N}\varepsilon^* \subseteq \overline{N}\}.$$

Let $\overline{J}_M$ and $\overline{J}_N$ be the images of J_M and J_N under the respective embeddings. Then, Proposition 8 shows

$$\overline{J}_M = \overline{J}_N.$$

Finally, define $C(\overline{M})$ and $C(\overline{N})$ as the maximal $E(T^*)$-modules contained in $\overline{M}$ and $\overline{N}$, respectively. Then, Lemma 4.4 implies $C(\overline{M}) = \overline{T}$ and $C(\overline{N}) = \overline{T}$.

With the setting described, the next step is to show $\overline{E}_M = \overline{E}_N$. Since the argument is symmetric, it will be sufficient to show $\overline{E}_M \subseteq \overline{E}_N$. Let $\gamma \in E_M$ and let $\gamma^* \in \overline{E}_M$ be its unique extension under the embedding. Suppose $\gamma^* \notin \overline{E}_N$ and show that this creates a contradiction. Since $\gamma^* \notin \overline{E}_N$, by Lemma 4.3, there exists $y \in \overline{N}$ such that

$$y\gamma^* \notin \overline{N}.$$

Since $\gamma \in E_M$, Proposition 7 implies

$$p\gamma \in J_M \text{ and } p\gamma^* \in \overline{J}_M.$$

Proposition 8 implies $p\gamma^* \in \overline{J}_N$. Thus, $p\gamma^* \in \overline{E}_N$. Therefore,

$$yp\gamma^* \in \overline{N}.$$

Let $w = y\gamma^*$, and then it follows $yp\gamma^* = pw$. Then $w + \overline{N}$ is a nonzero element of $T^*/\overline{N}$ that has order p. This creates a contradiction with the fact that $T^*/\overline{N}$ is torsion-free. Thus $\gamma^* \in \overline{E}_N$, which shows $\overline{E}_M \subseteq \overline{E}_N$. This concludes the proof that $\overline{E}_M = \overline{E}_N$.

Since $\overline{E}_M = \overline{E}_N$, this implies $E_M \cong E_N$. Then, Theorem 1 and Corollary B of May [**M90**] states $M \cong N$, and this isomorphism induces the isomorphism $E_M \cong E_N$. The isomorphism $\Phi : J_M \to J_N$ reduces to the identity map in $E(T^*)$, by Proposition 8. So, inducing the equality $\overline{E}_M = \overline{E}_N$ implies that the isomorphism $M \cong N$ also induces Φ. □

The proof of Theorem 1.2 follows directly from Theorems 4.1, 4.6 and 4.7. Note that the requirement that the modules be of torsion-free rank one is crucial to the proof of Theorems 4.1 and 4.6. In particular, the Jacobson radical cannot distinguish $Q \oplus T$ from $Q \oplus Q \oplus T$ if T is reduced. Hence, to extend a Jacobson radical isomorphism theorem to modules of torsion-free rank greater than one, the requirement that the module be reduced is necessary.

5. Warfield Modules

Recall that a Warfield module is a direct summand of a simply presented module. Reduced Warfield modules of finite torsion-free rank are included in the list of classes of modules satisfying an isomorphism theorem in [**M90**]. Therefore, many of the techniques of the last section will carry over to reduced Warfield modules. However, the proof of the split case, Theorem 4.1, relies on the simple structure of a rank one torsion-free module over a discrete valuation domain. Since higher rank modules come in many more possible forms, the techniques have to be modified. The price paid is the fact that Theorem 1.3 does not imply that the isomorphism is induced.

The class of Warfield modules will be restricted to reduced Warfield modules of finite torsion-free rank over a complete discrete valuation domain with unbounded torsion submodules. To streamline the terminology, this class will simply be called Warfield modules. The rest of this section is the machinery necessary to prove the Jacobson radical isomorphism theorem for this class, Theorem 1.3.

If M is a Warfield module with torsion submodule T, then M/T is a torsion-free R-module of finite rank. It is well known that a finite rank torsion-free R-module is the direct sum of a free module of finite rank and copies of the quotient field, Q. This decomposition implies the following decomposition of M.

LEMMA 5.1. *Let M be a Warfield module (as defined above). Then there exists a free module F_M and a reduced, nonsplit-mixed Warfield module M' with torsion submodule equal to $\mathbf{t}M$ and $M'/\mathbf{t}M$ divisible such that $M = F_M \oplus M'$.*

Since M' in Lemma 5.1 is fully invariant, the Jacobson radical of the endomorphism ring of M has the form

$$J(E(M)) = J(E(F_M)) + Hom(F_M, M') + J(E(M'))$$

There are several ideals of the Jacobson radical that reflect this decomposition of the Jacobson radical. Let L be the right annihilator of $\mathbf{t}J$ in J,

$$L_M = \{\alpha \in J : \mathbf{t}J \cdot \alpha = 0\}$$

Note that by Lemma 1, $\alpha \in L_M$ if and only if $\mathbf{t}M\alpha = 0$. Let A_M be the right annihilator of L in L:

$$A_M = \{\lambda \in L_M : \lambda \cdot L_M = 0\}$$

Define the quotient rings $W = L_M/A_M$ and $S = J_M/L_M$.

PROPOSITION 9. *Let M be a Warfield module with decomposition $M = F_M \oplus M'$ for some finite rank free module F_M and nonsplit reduced mixed Warfield module M' with $M'/\mathbf{t}M$ divisible. Then, $S = J_M/L_M$ is isomorphic to $J(E(M'))$ and $W = L_M/A_M$ is isomorphic to $J\left(E\left(F_M\right)\right)$.*

PROOF. Let $\alpha \in J = J(E(M))$. Then $\alpha \in L_M$ if and only if $\mathbf{t}M\alpha = 0$. Furthermore, since M' is reduced and $M'/\mathbf{t}M$ is divisible, if $\mathbf{t}M\alpha = 0$, then $M'\alpha = 0$. Hence, $L_M = J(E(F_M)) + Hom(F_M, M')$ and the fact that $S \cong J(E(M'))$ follows. For any map $\gamma \in Hom\,(F_M, M')$, $\gamma \in L$ but $M\gamma \subseteq M'$. Hence, $L_M \cdot \gamma = 0$. Therefore $Hom\,(F_M, M') \subseteq A_M$. Conversely, if $\lambda \in A_M$, $\lambda \in L_M$ so there exists (with the appropriate inclusion and projection maps understood) $\phi \in J(E(F_M))$

and $\eta \in HomF_M, M')$ such that $\lambda = \phi + \eta$. The module F_M is free of finite rank, so $J(E(F_M)) = M_n(pR)$ for some integer n. If $\phi \neq 0$, there exists $\sigma \in J(E(F_M))$ such that $\sigma\lambda \neq 0$ creating a contradiction with $\lambda \in A_M$. Hence, $A_M = Hom(F_M, M')$. Therefore the quotient ring $W \cong J(E(F_M))$. □

Recall that a module is Warfield exactly if it has a nice decomposition basis (see Section 2 of [**GO**] for the definitions and references). Furthermore, Göbel and Opdenhövel [**GO**] showed that a nice decomposition basis can be chosen with an invariance property.

Proposition 10. *Let M be a reduced Warfield module with finite torsion-free rank, and let γ be an endomorphism of M. Then there exists a nice decomposition basis Y such that $M/\langle Y\rangle$ is simply presented and $\langle Y\rangle\gamma \subseteq \langle Y\rangle$.*

This invariant nice decomposition basis is the key to the following bound for the Jacobson radical:

Proposition 11. *Let M be a reduced Warfield module of finite torsion-free rank with an unbounded torsion submodule. Then $pE(M) \subseteq J(E(M))$.*

Proof. Decompose $M = F_M \oplus M'$ as in Lemma 5.1. Then $pE(F_M) \subseteq J(E(F_M))$, so to prove the proposition it is sufficient to prove that $pE(M')) \subseteq J(E(M'))$. Therefore, assume that M is reduced and that M/tM is divisible. Let $\beta \in E(M)$ and choose a nice decomposition basis $Y = \{y_i\}_{1\leq i\leq n}$ of M such that $\langle Y\rangle$ is β-invariant. Let $\varepsilon = 1 + p\beta$. To show that $pE(M)$ is contained in the Jacobson radical, it is sufficient to show that ε is an automorphism of M. Let $D = \langle Y\rangle = \sum_{i=1}^{n}\langle y_i\rangle$. Since D is a free module of finite rank, $E(D) \cong M_n(R)$ and $J(E(D)) \cong M_n(pR)$. The map $p\beta|_D \in J(E(D))$, so ε is an automorphism of D. For a torsion module T, it is well-known that $pE(T) \subseteq J(E(T))$, so $\varepsilon|_T$ is also an automorphism. With the facts that $\varepsilon|_T$ and $\varepsilon|_D$ are isomorphisms, it is straightforward to show $\varepsilon \in Aut(M)$. □

Theorem 5.2. *Let M and N be reduced Warfield modules of finite torsion-free rank. Assume $\mathbf{t}M$ is unbounded. Then there exist finite rank free modules F_M and F_N and reduced Warfield modules M' and N' with $M'/\mathbf{t}M$ and $N'/\mathbf{t}N$ divisible such that $M = F_M \oplus M'$ and $N = F_N \oplus N'$. If there exists an R-algebra isomorphism $\Phi : J_M \to J_N$, then $F_M \cong F_N$ and $M' \cong N'$.*

Proof. Theorem 3.4 shows that $\mathbf{t}M \cong \mathbf{t}N$, hence N also has an unbounded torsion submodule. Ideals H, L and A are all invariant under isomorphism. Therefore, the isomorphism Φ implies $W_M \cong W_N$ and $S_M \cong S_N$. Proposition 9 shows $J(E(F_M)) \cong J(E(F_N))$ and Theorem 3.3 of Flagg [**Fl08**] proves $F_M \cong F_N$. Proposition 9 implies $J(E(M')) \cong J(E(N'))$. By Proposition 11, $pE(M') \subseteq J(E(M'))$ and $pE(N') \subseteq J(E(N'))$. Let $T = \mathbf{t}M = \mathbf{t}M'$. Then M' and N' may be embedding into T^* by Lemma 4.2 and Lemma 4.3. Lemma 4.4 shows $C\left(\overline{M'}\right) = C\left(\overline{N'}\right) = \mathbf{t}T^*$. The same procedure as the proof of Theorem 4.7 may be used to show $E(M') \cong E(N')$. Theorem 1 and Corollary B(3) of May [**M90**] implies $M' \cong N'$. □

Theorem 1.3 is a direct consequence of Theorem 5.2. Since the theorems used for each "piece" of Theorem 5.2 include that the isomorphism is induced, one may ask why Theorem 1.3 does not. The issue is that $H_M \subseteq L_M$ and $H_M \subseteq A_M$. The isomorphisms in Theorem 5.2 use the quotient rings W and S, hence they give no information about maps in H_M. Therefore, whether an isomorphism between the modules that induces the Jacobson radical isomorphism exists is an open question.

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Dimension in topological structures: Topological closure and local property

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Abstract. Let K be a first-order structure with a dimension function, satisfying some natural conditions. Let A be a definable set. If every point in A has a definable neighborhood in A with dimension less than p, then A has dimension less than p.

1. Introduction

Let $\mathbb{K}$ be a first-order topological structure without isolated points; topological structure means that we have a topology on $\mathbb{K}$ which has a basis consisting of a $\emptyset$-definable family of subsets of $\mathbb{K}$ (see [**Pil87**]). By "definable", unless otherwise specified, we will mean "definable in $\mathbb{K}$ using parameters". Let dim be a **dimension** on $\mathbb{K}$, that is a function mapping each nonempty definable set to some natural number and satisfying some axioms.

We show that, under certain extra assumptions, the dimension satisfies two important properties: for every definable set A,

(1) $\dim(\overline{A}) = \dim(A)$ (where $\overline{A}$ is the topological closure of A);
(2) If, for every $a \in A$, there exists U definable neighborhood of a, such that $\dim(A \cap U) \leq p$, then $\dim(A) \leq p$.

It turns out that (1) is easy, whereas (2) really seems to need some work.

We now list the assumptions on the dimension. We set $\dim(\emptyset) = -\infty$. Let A, A' definable subsets of $\mathbb{K}^n$ and B be a definable subset of $\mathbb{K}^{n+1}$. If $a \in \mathbb{K}^n$, we write $B_a = \{b \in \mathbb{K} \mid (a, b) \in B\}$ for the fiber. Moreover, we write $\Pi^m_p : \mathbb{K}^m \to \mathbb{K}^p$ for the projection onto the first p coordinates.

(Dim 1) $\dim(\mathbb{K}) = 1$, $\dim(\{a\}) = 0$ for every $a \in \mathbb{K}$.
(Dim 2) $\dim(A \cup A') = \max(\dim(A), \dim(A'))$.
(Dim 3) For every permutation σ of coordinates, $\dim(A^\sigma) = \dim(A)$.
(Dim 4) Let $B(0) \coloneqq \{a \in \mathbb{K}^n : \dim(B_a) = 0\}$. Then, $B(0)$ is definable, with the same parameters as B.
(Dim 5) Let i be either 0 or 1. Let $C \coloneqq \Pi^{n+1}_n(B)$. Assume that, for every $a \in C$, $\dim(B_a) = i$. Then, $\dim(B) = \dim(C) + i$.

The following axiom relates the dimension to the topology.

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(TD 1) If A is open (in $\mathbb{K}^n$), then $\dim(A) = n$.

We will also need the following condition.

(Group) $\mathbb{K}$ expands an Abelian group $\langle K, +, 0\rangle$; moreover, this group is a Hausdorff topological group.

Axioms (Dim 1–5) are almost the same as the axioms for dimension in [**vdD89**]. The only difference is that in the analogue of Axiom (Dim 3), van den Dries does not require that one can use the parameters of definition of B to define $B(0)$ (but see [**For11a**, §4] and [**vdD89**, Proposition 1.7]).

In topological structures, a natural way to define dimension is the following:

(TD 2) For $A \subset \mathbb{K}^n$, let $\dim(A)$ be the largest integer p, such that, after a permutation of coordinates, $\Pi^n_p(A)$ has nonempty interior in $\mathbb{K}^p$.

Notice that this definition implies Axioms (TD 1), (Dim 3), (Dim 4), and, since $\mathbb{K}$ has no isolated points, (Dim 1). Thus in that case, the only interesting axioms are (Dim 2) and (Dim 5). However, in Theorem 3.1 below, we do not need dimension to be defined that way, so let us simply treat (TD 2) as another axiom.

See §4 for some examples of structures satisfying various subsets of the axioms.

An easy but important remark is the following.

Remark 1.1. Let $A \subseteq \mathbb{K}^n$ be definable.

(1) Assume that Axiom (TD 2) holds. Then, $\dim(A) = n$ iff A has nonempty interior (in $\mathbb{K}^n$).
(2) Assume that Axiom (Dim 2) holds. If $A \subseteq B$, then $\dim(A) \leq \dim(B)$.

2. Closure

Proposition 2.1. *Assume that Axioms (TD 2) and (Dim 2) hold. Then, for every definable set $A \subseteq \mathbb{K}^n$, $\dim(\overline{A}) = \dim(A)$.*

Proof. Let $p \coloneqq \dim(\overline{A})$.

Case 1: $p = n$. Define $\partial A \coloneqq \overline{A} \setminus A$. Notice that $\overline{A} = A \cup \partial A$ and that ∂A has empty interior, and therefore $\dim(\partial A) < n$. If, for a contradiction, $\dim(A) < n$, then, by (Dim 2), $\dim(\overline{A}) < n$, absurd.

Case 2: p any. After a permutation of coordinates, w.l.o.g. $\Pi^n_p(\overline{A})$ has dimension p. Let $B \coloneqq \Pi^n_p(A)$. If, for a contradiction, $\dim(A) < p$, then $\dim(B) < p$. Notice that $\Pi^n_p(\overline{A}) \subseteq \overline{B}$. Thus, by Case 1, $\dim(\overline{B}) = \dim(B) < p$, absurd. □

3. Local property

Theorem 3.1. *Assume that Axioms (TD 1), (Dim 1–5), and (Group) hold. Let $A \subseteq \mathbb{K}^n$ be definable. Assume that, for every $a \in A$, there exists a definable neighborhood U of a, such that $\dim(A \cap U) \leq p$. Then, $\dim(A) \leq p$.*

The proof is quite more difficult than the proof of Proposition 2.1. It might be surprising that (Group) is needed; in Section 4, we will see a counter-example without (Group).

For all this section, we assume that Axioms (TD 1), (Dim 1–5), and (Group) hold.

We make heavy use of the results in [**For11a**, §1–4]. We first need the following definitions.

Definition 3.2. Let $a \in \mathbb{K}^n$ and $C \subseteq \mathbb{K}$. The **rank** of a over C, denoted by $\mathrm{rk}(a/C)$, is the minimum p such that there exists a set $X \subseteq \mathbb{K}^n$ which is definable with parameters from C, with $a \in X$ and $\dim(X) = p$.

Notice that if $a \in \mathbb{K}$, then $\mathrm{rk}(a/C) \leq 1$.

Definition 3.3. Let A, B, C be subsets of $\mathbb{K}$. We define the preindependence relation $A \mathop{\smile\!\!\!\!\!\mid}_B C$ iff, for every finite subset $\bar{a} \subseteq A$, $\mathrm{rk}(\bar{a}/B) = \mathrm{rk}(\bar{a}/BC)$.

Lemma 3.4. *$\mathop{\smile\!\!\!\!\!\mid}$ is symmetric: that is, $A \mathop{\smile\!\!\!\!\!\mid}_B C$ iff $C \mathop{\smile\!\!\!\!\!\mid}_B A$.*

Proof. The following proof is a standard exercise (see [**For11a**, Theorem 4.3 and Lemma 3.6]). Let us assume that $A \mathop{\smile\!\!\!\!\!\mid}_B C$; we have to prove that $C \mathop{\smile\!\!\!\!\!\mid}_B A$; thus, we have to show that, for every finite $C' \subseteq C$, $\mathrm{rk}(C'/BA) = \mathrm{rk}(C'/B)$. Since C' is finite, there exists a finite set $A' \subseteq A$, such that $\mathrm{rk}(C'/BA) = \mathrm{rk}(C'/BA')$. By assumption, we have $\mathrm{rk}(A'/BC) = \mathrm{rk}(A'/B)$. Since $\mathrm{rk}(A'/BC) \leq \mathrm{rk}(A'/BC') \leq \mathrm{rk}(A'/B)$, we also have $\mathrm{rk}(A'/BC') = \mathrm{rk}(A'/B)$. Now we can use the additivity of rank:

$$\mathrm{rk}(A'/BC') + \mathrm{rk}(C'/B) = \mathrm{rk}(A'C'/B) = \mathrm{rk}(C'/BA') + \mathrm{rk}(A'/B).$$

Since all the summands in the above equality are finite, $\mathrm{rk}(A'/BC') = \mathrm{rk}(A'/B)$ implies $\mathrm{rk}(C'/B) = \mathrm{rk}(C'/BA') = \mathrm{rk}(C'/BA)$. □

Let κ be some big cardinal. From now on, we assume that $\mathbb{K}$ is a monster model (since it suffices to prove the theorem in some $\mathbb{K}'$ elementary extension of $\mathbb{K}$), that is a κ-saturated and κ-homogeneous structure. By "small", we will mean of cardinality less than κ. Moreover, we assume that A is definable without parameters (otherwise, we add the parameters of definition of A to the language).

The following fact follows easily from the definitions.

Fact 3.5 ([**For11a**, Lemma 3.69]). *Let $A \subseteq \mathbb{K}^n$ be definable without parameters, and let $\big(U_t : t \in T\big)$ be a family of subsets of $\mathbb{K}^n$, such that each U_t is definable with parameters t. Let $p \leq n$, and assume that, for every $a \in A$, there exists $t \in T$, such that $a \in U_t$, $a \mathop{\smile\!\!\!\!\!\mid} t$, and $\dim(A \cap U_t) \leq p$. Then, $\dim(A) \leq p$.*

We want to apply the above result. Let A be as in Theorem 3.1 and let $a \in A$. By the theorem's assumption, there exists a definable set V which is a neighborhood of a and such that $\dim(A \cap V) \leq p$. If we could find a definable set $V' \subseteq V$ and some parameter t such that V' is definable over t, $a \in V'$, and $a \mathop{\smile\!\!\!\!\!\mid} t$, we could then apply Fact 3.5 to conclude. The remainder of the proof is finding such V'.

Since $\mathbb{K}$ is a first-order topological structure, there exists a definable family $\mathcal{B} := \big(B_t : t \in I\big)$ of subsets of $\mathbb{K}$, definable without parameters, such that $\mathcal{B}$ is a basis for the topology of $\mathbb{K}$. Given $b \in I^n$, we denote by B_b the set $B_{b_1} \times \dots B_{b_n}$. Given $c \in \mathbb{K}^n$ and $E, E' \subseteq \mathbb{K}^n$, we denote by $E + c$ the translation of E by c and by $E + E' := \{e + e' : e \in E, e' \in E'\}$.

Notice that $\big(B_b : b \in I^n\big)$ is a basis for the topology of $\mathbb{K}^n$.

Lemma 3.6 (Cf. [**For11a**, Lemma 9.14]). *Let $d \in \mathbb{K}^n$, V be a definable neighborhood of d, and C be a small subset of $\mathbb{K}$. Then, there exists $b \in I^n$ such that $b \mathop{\smile\!\!\!\!\!\mid}_d C$ and $d \in B_b \subseteq V$.*

Proof. Let $J := \{b \in I^n : d \in B_b\}$. Let $\leq$ be the quasi-ordering on J given by reverse inclusion: $b \leq b'$ iff $B_b \supseteq B_{b'}$. Fix $e \in J$ such that $B_e \subseteq V$. Since $\langle J, \leq\rangle$

is a directed quasi-order definable over d, [**For11a**, Lemma 3.68] implies that there exists $b \in J$ such that $b \mathop{\smile\!\!\!\!\!\!\mid}_d C$ and $b \geq e$. ☐

Notice that if in the above lemma we were able to reach the conclusion that $e \mathop{\smile\!\!\!\!\!\!\mid} D$ instead of $e \mathop{\smile\!\!\!\!\!\!\mid}_d D$, we would be done.

Lemma 3.7 (Cf. [**For11a**, Lemma 9.18]). *Let $a \in \mathbb{K}^n$, V be a definable neighborhood of a, and C be a small subset of $\mathbb{K}$. Then, there exist $e \in \mathbb{K}^n$ and $b \in I^n$ such that $a \in B_b + e \subseteq V$ and $eb \mathop{\smile\!\!\!\!\!\!\mid} C$.*

Proof. Let $V_0 := V - a$; it is a definable neighborhood of 0. Since $\mathbb{K}^n$ is a topological group, there exists V_1 definable open neighborhood of 0, such that $V_1 = -V_1$ and $V_1 + V_1 \subseteq V_0$. By Lemma 3.6, applied to $d = 0$, there exists $b \in I^n$ such that $b \mathop{\smile\!\!\!\!\!\!\mid}_0 C$ and $0 \in B_b \subseteq V_1$; since 0 is in the definable closure of the empty set, we have $b \mathop{\smile\!\!\!\!\!\!\mid} C$. Let $W := a - B_b$. Notice that W is an open subset of $\mathbb{K}^n$; thus, $\dim(W) = n$, and, by monstrosity of $\mathbb{K}$, there exists $e \in W$ such that $\mathrm{rk}(e/Cab) = n$ [**For11a**, Remark 3.10], and therefore $e \mathop{\smile\!\!\!\!\!\!\mid} Cab$. We have to show that b and e satisfy the conclusion.

Claim 1. $a \in B_b + e$

It follows immediately form $e \in a - B_b$.

Claim 2. $B_b + e \subseteq V$.

Let $f \in B_b + e$: we need to prove that $f \in V$. We know that $e \in W = a - B_b$, and therefore $e - a \in -B_b \subseteq V_1$. Since $f - e \in B_b \subseteq V_1$, we have $f - a \in V_1 + V_1 \subseteq V - a$, and therefore $f \in V$.

Claim 3. $eb \mathop{\smile\!\!\!\!\!\!\mid} C$.

We have that $e \mathop{\smile\!\!\!\!\!\!\mid} Cab$ implies $e \mathop{\smile\!\!\!\!\!\!\mid}_b Cab$, which implies $eb \mathop{\smile\!\!\!\!\!\!\mid}_b Ca$, and hence $eb \mathop{\smile\!\!\!\!\!\!\mid}_b C$. Together with $b \mathop{\smile\!\!\!\!\!\!\mid} C$, by transitivity the above implies $eb \mathop{\smile\!\!\!\!\!\!\mid} C$. ☐

Finally, we are in a position to apply Fact 3.5. Let V an open neighborhood of a, such that $\dim(V \cap A) \leq p$. Consider the definable family $\big(B_b + e : b \in I^n, e \in \mathbb{K}^n\big)$. Each set $B_b + e$ is definable with parameters be. By Lemma 3.7, applied with $C := \{a\}$, there exist $e \in \mathbb{K}^n$ and $b \in I^n$ such that $a \in B_b + e \subseteq V$ and $a \mathop{\smile\!\!\!\!\!\!\mid} eb$. Since $B_b + e \subseteq V$, we have $\dim((B_b + e) \cap A) \leq p$. Thus, the assumptions of Fact 3.5 are satisfied (for the family of sets of the form $U_{b,e} := B_b + e$), and the theorem is proved. ☐

4. Some examples

Here is another axiom one could have required:

(Finite) A definable set of dimension 0 is finite.

In the following examples, if we do not specify the dimension, we mean the one defined by (TD 2).

- Axioms (TD 1–2), (Dim 1–5), (Group), and (Finite) hold in arbitrary o-minimal structures expanding an ordered group.
- Axioms (Dim 1–5) hold for the dimension in b-minimal structures $\mathbb{K}$ in the sense of [**CL07**] (by Theorems 4.2 and 4.3 in loc. sit.). Let us say that $\mathbb{K}$ is a "topological b-minimal structure" if it is b-minimal and it has a definable topology such that a set has non-empty interior if and

only if it contains a ball. A topological b-minimal structure additionally satisfies (TD 2) (and hence also (TD 1)). In fact, all standard examples of b-minimal structures are topological b-minimal structures, but in general, the family of balls does not even determine the topology; see the last example below.

Any o-minimal structure is a topological b-minimal structure. Other standard examples are Henselian valued fields of characteristic 0; these fields also satisfy (Group) and (Finite). By [**CL11**], this is still true if one adds an analytic structure in the sense of [**CL11**] to the language. By [**Yin10**], another example of a topological b-minimal structure is an algebraically closed valued field of residue characteristic 0 with a section added to the language, either from the residue field to the valued field or from the "leading term structure" RV to the valued field. In this example, we also have (Group), but (Finite) is not satisfied (since the image of the section is 0-dimensional).

- A d-minimal expansion of the real field satisfies (TD 1–2), (Dim 1–5), and (Group), but not (Finite) (unless it is o-minimal): see [**For10a**]. An example of a d-minimal expansion is the reals with a predicate for the powers of 2. (In fact, this example is also b-minimal. Question: Does d-minimality imply b-minimality?)
- Let $\mathbb{B}$ be an o-minimal structure expanding a field, and $\mathbb{A}$ a topologically dense proper elementary substructure. Then, $\langle \mathbb{B}, \mathbb{A} \rangle$ has a unique dimension function satisfying (TD 1), (Dim 1–5) and (Group), but neither (TD 2) nor (Finite); moreover, the conclusion of Proposition 2.1 does not hold in $\langle \mathbb{B}, \mathbb{A} \rangle$: see [**For10b**].
- Let $\mathbb{K}$ be an expansion of the real field by a closed set $E \subset \mathbb{R}$, such that $\langle \mathbb{K}, E \rangle$ does not define the natural number. Let dim be the dimension defined using Axiom (TD 2). Then, dim satisfies Axioms (Dim 2) and (Group), and, in general, will not satisfy (Finite). Moreover, dim coincides with the Hausdorff dimension. It is an open question if dim satisfies Axiom (Dim 5): see [**For11b**].
- The Sorgenfrey plane shows that Axiom (Group) is necessary in Theorem 3.1. More precisely, let $\mathbb{K} := \langle \mathbb{R}, +, < \rangle$, or any o-minimal expansion of $\langle \mathbb{R}, +, < \rangle$. Let dim be the usual o-minimal dimension on $\mathbb{K}$. We endow $\mathbb{K}$ with, instead of the usual order topology, the right half-open interval topology: it is the topology generated by the basis of all half-open intervals $[a, b)$, where a and b are real numbers. Notice that the half-open interval topology is finer than the order topology; however, a subset of $\mathbb{R}^n$ has nonempty interior in the (product of the) half-open interval topology iff it has nonempty interior in the Euclidean topology. Thus, $\mathbb{K}$ satisfies Axioms (TD 1–2), (Dim 1–5), and (Finite), and moreover $\mathbb{K}$ is Hausdorff. However, $\mathbb{K}$ does not satisfy Axiom (Group), because the function minus is not continuous. Let $A := \{\langle x, y \rangle \in \mathbb{K}^2 : x = -y\}$ be the graph of minus: the set A as dimension 1, but, with the induced topology, it is a discrete set: for every point $a = \langle c, -c \rangle \in A$, the set $U_a := [c, c+1) \times [-c, -c+1)$ is an open neighborhood of a, such that $U_a \cap A = \{a\}$, and thus $\dim(U_a \cap A) = 0$; thus, the conclusion of Theorem 3.1 does not hold in $\mathbb{K}$.

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On kernel modules of cotorsion pairs

László Fuchs

Dedicated to Rüdiger Göbel on his 70th birthday

ABSTRACT. We study kernel modules for cotorsion pairs $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ over commutative rings R, i.e. R-modules in $\mathcal{A} \cap \mathcal{B}$. Several results by Guil–Herzog will be generalized. If the class $\mathcal{A}$ is closed under direct limits, then the endomorphism rings of kernel modules are Enochs-cotorsion (Theorem 2.2) and the indecomposable kernel modules have local endomorphism rings (Corollary 2.4). More can be said if, in addition, the class of kernel modules is closed under direct sums. In this case, each kernel module is a direct sum of indecomposable kernel modules. In Theorem 3.3 we show that, over almost perfect domains R, these indecomposable kernel modules are, besides Q, precisely the localizations of Q/R at the maximal ideals of R (where Q denotes the field of quotients of R). We also describe the construction of weak-injective envelopes over almost perfect domains, and show that in this case they commute with direct sums (Theorem 4.2).

1. Introduction

In what follows, R will denote a commutative ring with $1 \neq 0$, and Q will stand for the field of quotients of R in case R is a domain ($Q \neq R$ is always assumed).

Throughout, let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ denote a cotorsion pair of R-modules. The R-modules M that belong to both $\mathcal{A}$ and $\mathcal{B}$ are called the *kernel modules* of $\mathfrak{C}$. We are interested in learning more about these modules, in particular, when the class of kernel modules is closed under arbitrary direct sums (note that $\mathcal{A}$ is always closed under direct sums). We generalize several results by Guil Asensio and Herzog [**12**] from the special Enochs-cotorsion pair to a class of cotorsion pairs. In particular, we prove that if $\mathcal{A}$ is closed under direct limits, then the endomorphism ring of a kernel module is always Enochs-cotorsion both as an R-module and as a ring over itself (Theorem 2.2) and the indecomposable kernel modules have local endomorphism rings (Corollary 2.4).

In case $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a perfect cotorsion pair of R-modules, every R-module has an $\mathcal{A}$-cover as well as a $\mathcal{B}$-envelope; these are unique up to equivalence (see Göbel–Trlifaj [**10**]). We will call $\mathfrak{C}$ a Σ-*cotorsion pair* if the class $\mathcal{B}$ is also closed under direct sums. Σ-cotorsion pairs are e.g.: ($\mathcal{M}od$-$R, \mathcal{I}$) (where $\mathcal{I}$ is the class of injectives) if R is noetherian, and $(\mathcal{P}_1, \mathcal{D})$ (the classes of modules of projective

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dimension ≤ 1 and divisible modules) if R is a domain. If the class of kernel modules is closed under direct sums, then each kernel module is a direct sum of indecomposable kernel modules (Theorem 2.7). We mention that in [**8**] it was shown that if λ is an infinite cardinal such that the ring R has the property that each R-module in $\mathcal{B}$ is the direct sum of modules of cardinalities $\leq \lambda$, then $\mathfrak{C}$ is a Σ-cotorsion pair.

The results are applied to almost perfect domains; these were introduced by Bazzoni–Salce [**4**] as domains R such that R/I is a perfect ring (in the sense of Bass) for every non-zero ideal I of R. Over such domains, weak-injective modules M will be studied. These modules were defined by Lee [**14**] over arbitrary domains as modules satisfying $\mathrm{Ext}^1_R(A, M) = 0$ for all modules A of weak dimension ≤ 1. Over almost perfect domains, the cotorsion pair $(\mathcal{F}_1, \mathcal{WI})$ is perfect ($\mathcal{F}_1$ is the class of R-modules of weak-dimension ≤ 1, and $\mathcal{WI}$ is the class of weak-injective modules), i.e. $\mathcal{F}_1$-covers and $\mathcal{WI}$-envelopes exist. A main result (Theorem 3.3) describes precisely the weak-injective modules of weak dimension ≤ 1: they admit unique decompositions into the direct sums of modules each of which is isomorphic to Q or to a localization of Q/R at a maximal ideal.

We also study the method how weak-injective envelopes can be obtained over almost perfect domains. In particular, we prove that in this case weak-injective envelopes commute with arbitrary direct sums. Our results will show that weak-injective modules over almost perfect domains behave in several respects similarly to injective modules over Dedekind domains.

For unexplained terminology we refer to Göbel–Trlifaj [**10**] for cotorsion pairs, and Fuchs–Salce [**9**] for modules over domains. Cf. also Enochs–Jenda [**5**].

2. Modules in the kernel

We now concentrate on modules M that belong to the kernel of a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$, i.e. $M \in \mathcal{A} \cap \mathcal{B}$. In general, there are many kernel modules for a perfect cotorsion pair. For instance, the following modules are always kernel modules: the $\mathcal{B}$-envelopes of $\mathcal{A}$-modules as well as the $\mathcal{A}$-covers of $\mathcal{B}$-modules (since both classes $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions). The class of kernel modules is readily seen to be closed under taking finite direct sums, summands and extensions. Clearly, the class of kernel modules in a Σ-cotorsion pair is also closed under direct sums. We will call $\mathfrak{C}$ a *Σ'-cotorsion pair* if we only assume that the class of kernel modules is closed under (arbitrarily large) infinite direct sum formations.

In order to obtain information about the endomorphism rings of kernel modules, we are going to make use of an isomorphism proved in Fuchs–Lee [**6**].

Lemma 2.1 ([**6**, Lemma 2.3]). *Assume that R, S are (non-necessarily commutative) rings and ${}_SA_R, B_R, C_S$ are modules (over R or S as indicated) satisfying*

$$\mathrm{Ext}^1_R(A, B) = 0 \quad \textit{and} \quad \mathrm{Tor}^S_1(C, A) = 0.$$

Then there is an isomorphism as abelian groups

$$\mathrm{Ext}^1_S(C, \mathrm{Hom}_R(A, B)) \cong \mathrm{Ext}^1_R(C \otimes_S A, B). \tag{2.1}$$

This is an R-isomorphism in case the ring $R = S$ is commutative. □

The first claim in the next theorem generalizes a result in Fuchs–Lee [**6**].

Theorem 2.2. *Assume $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair of R-modules and the class $\mathcal{A}$ is closed under direct limits.*

(i) *If* $M \in \mathcal{A}$ *and* $N \in \mathcal{B}$*, then* $\operatorname{Hom}_R(M,N)$ *is an Enochs-cotorsion R-module.*
(ii) *If* $M \in \mathcal{A} \cap \mathcal{B}$*, then* $S = \operatorname{End}_R M$ *is a self-Enochs-cotorsion ring.*

PROOF. The proof for (i) is similar to the one for (ii) by replacing S by R, so we only verify (ii). If $\mathcal{A}$ is closed under direct limits, then for every flat module C_S and for every ${}_SM_R \in \mathcal{A}$, also $C \otimes_S M \in \mathcal{A}$, since the last module is the direct limit of R-modules $P \otimes_S M$ for projective modules P_S, and clearly, $P \otimes_S M \in \mathcal{A}$. For flat modules C_S, the quoted isomorphism for $A = B = M$ is applicable, and it yields

$$\operatorname{Ext}^1_S(C, \operatorname{End}_R M) \cong \operatorname{Ext}^1_R(C \otimes_S M, M) = 0,$$

proving the claim. □

Observe that if in the preceding theorem the condition that $\mathcal{A}$ is closed under direct limits is dropped, then the result is false. Counterexample: the perfect cotorsion pair $(\mathcal{P}, \mathcal{M}od\text{-}R)$ where $\mathcal{P}$ denotes the class of projective modules.

We now apply the preceding result to kernel modules of the perfect cotorsion pairs $(\mathcal{F}, \mathcal{EC})$, $(\mathcal{F}_1, \mathcal{WI})$, and $(\mathcal{TF}, \mathcal{WC})$. Here $M \in \mathcal{TF}$ (*torsion-free*) means that $\operatorname{Tor}_1^R(M, A) = 0$ for all A of w. d. ≤ 1, and N is in $\mathcal{WC}$ (*Warfield-cotorsion*) if $\operatorname{Ext}^1_R(M, N) = 0$ for all torsion-free M. As the classes $\mathcal{F}$, $\mathcal{F}_1$, and $\mathcal{TF}$ are closed under direct limits, from Theorem 2.2 we obtain:

COROLLARY 2.3. (i) (Guil–Herzog [**12**, p.18]) *The endomorphism ring of a flat Enochs-cotorsion R-module is a cotorsion ring.*
(ii) *The endomorphism rings of weak-injective R-modules of weak dimension* ≤ 1 *are cotorsion rings.*
(iii) *The endomorphism ring of a torsion-free Warfield-cotorsion R-module is cotorsion.* □

Note that (iii) is a weak result if R is a domain, because in this case a torsion-free Warfield-cotorsion module has an algebraically compact endomorphism ring.

Observe that if $\mathcal{A}$ is closed under direct limits, then $\mathcal{F} \subseteq \mathcal{A}$, thus the modules in $\mathcal{B}$ are all Enochs-cotorsion. As is shown by (2.3)(i), the flat modules in $\mathcal{B}$ have cotorsion endomorphism rings.

We refer to [**11**, Corollary 7] to argue that an indecomposable ring S that is cotorsion over itself is local. Hence:

COROLLARY 2.4. *Assume the hypotheses of Theorem* 2.2. *If* $M \in \mathcal{A} \cap \mathcal{B}$ *is, in addition, indecomposable, then its endomorphism ring* $\operatorname{End} M$ *is local. Hence such an* M *enjoys the Exchange Property.* □

For more information about the endomorphism ring we quote the following interesting result.

PROPOSITION 2.5 (Guil–Herzog [**11**, Corollary 4, Theorem 6]). *If R is a cotorsion ring, then*
(i) *idempotents lift modulo ideals of R; and*
(ii) *the factor ring* R/J *is von Neumann regular, where* J *denotes the Jacobson radical of* R*.* □

Since the weak-injective envelope of a projective module is a kernel module, from Lemma 2.2 we can derive at once:

Corollary 2.6. *The endomorphism ring of the weak-injective envelope of a projective R-module is a cotorsion ring.* □

We now recall a definition and a result needed in the proof of the next theorem. A *local summand* $N = \bigoplus_{i \in I} N_i$ of a module M is a submodule such that $\bigoplus_{i \in J} N_i$ is a summand of M for each finite subset J of the index set I. By Mohamed–Müller [**15**, Theorem 2.17], if every local summand of M is a direct summand of M, then M is the direct sum of indecomposable modules.

Theorem 2.7. *Let M be a kernel module for a Σ'-cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$, and assume that $\mathcal{A}$ is closed under direct limits. Then*

$$M = \bigoplus_{i \in I} B_i \tag{2.2}$$

where each B_i is an indecomposable kernel module. The B_i have local endomorphism rings, and the decomposition 2.2 *into indecomposable summands is unique up to isomorphism.*

Proof. (This argument is borrowed from [**12**, Proposition 7].) In view of the remark above, we show that every local summand of M is a summand. Let $B = \bigoplus_{j \in J} C_j$ be a local summand of M. Clearly, the C_j are kernel modules, and so is B, since $\mathfrak{C}$ is Σ'-cotorsion. As the class $\mathcal{A}$ is closed under direct limits, we have $M/B \in \mathcal{A}$ as the direct limit of the factor modules $M/(\bigoplus_{j \in K} C_j) \in \mathcal{A}$ for finite subsets K of J. Hence $B \in \mathcal{B}$ is a summand of M, as desired.

The second claim is a consequence of Lemma 2.4, since direct decompositions into summands with local endomorphism rings are always unique (Azumaya [**2**]). □

It is perhaps worth while observing that in the special case when R is noetherian and $\mathfrak{C} = (\mathcal{M}od\text{-}R, \mathcal{I})$, the hypotheses of Theorem 2.7 are satisfied. It implies the well-known fact that the injective modules over noetherian domains are direct sums of indecomposable injectives with local endomorphism rings.

3. Applications to almost perfect domains

Almost perfect domains were introduced by Bazzoni–Salce [**4**] as domains with the property that every proper homomorphic image is a perfect ring in the sense of H. Bass. They can be characterized in several ways, for instance, as domains over which modules of weak dimension ≤ 1 have projective dimension ≤ 1. (For more on almost perfect domains, see Bazzoni [**3**].) We are interested in weak-injective modules over almost perfect domains. Over such domains, weak-injectivity is equivalent to divisibility (see Lee [**14**]), and since divisibility is preserved under arbitrary direct sums, in this case the cotorsion pair $(\mathcal{F}_1, \mathcal{WI}) = (\mathcal{P}_1, \mathcal{D})$ is perfect and Σ-cotorsion.

As the kernel modules are of p. d. ≤ 1 in this case, by a theorem in Lee [**13**], they decompose into the direct sums of countably generated divisible modules. But from Theorem 2.7 we derive a much stronger result:

Corollary 3.1. *Let R be an almost perfect domain. Every weak-injective R-module of* w. d. ≤ 1 *is the direct sum of indecomposable modules, each of which has a local cotorsion ring as endomorphism ring, so enjoys the Exchange Property.*

Thus such a decomposition is unique up to isomorphism. □

Almost perfect domains have Krull dimension 1 (cf. [**4**]), thus all non-zero prime ideals are maximal. Let P be a maximal ideal of R. Then the results of Abuhlail–Jarrar [**1**] imply that either P is projective or $\mathrm{p.d.}\, P = \infty$ (if $\mathrm{p.d.}\, P$, then $\mathrm{p.d.}\, R/P + 1$, but no module over an almost perfect domain has finite $\mathrm{p.d.} > 1$). It is clear that the weak-injective envelope $W(R/P)$ of the simple module R/P has $\mathrm{w.d.} \leq 1$ exactly if P is projective. As almost perfect domains are h-local (cf. [**4**]), every torsion module decomposes into the direct sum of its P-components, and hence it follows that $W(R/P)$ is a P-module, i.e. it coincides with its own P-component.

The next result is reminiscent of Matlis' theorem on indecomposable injective modules over noetherian rings which states that their isomorphy classes are in a bijective correspondence with the prime ideals of the ring. Such a correspondence exists for indecomposable kernel modules over almost perfect domains, as is shown by the next theorem.

THEOREM 3.2. *Suppose R is an almost perfect domain. Then the isomorphy classes of indecomposable torsion kernel modules for the cotorsion pair $(\mathcal{F}_1, \mathcal{WI}) = (\mathcal{P}_1, \mathcal{D})$ are in a bijective correspondence with the simple R-modules S. The correspondence is given by $S \leftrightarrow W(S)/S$.*

Hence the isomorphy classes of indecomposable kernel modules correspond bijectively to the prime ideals P: the module Q corresponds to $P = 0$, and the modules $W(R/P)/(R/P)$ correspond to the maximal ideals P.

PROOF. Let $S = \langle x \rangle$ be a simple R-module, thus $S = R/P$ with a maximal ideal P of R. If the quotient $D = W(S)/S$ is directly decomposable, say, $D = D_1 \oplus D_2$, then one of the summands, say D_1 contains an element d such that $0 \neq rd \in S$ for some $r \in P$ (recall that torsion modules $\neq 0$ have essential socles, see [**4**]). We may assume that $rd = x$. Consider the submodule M of $W(S)$ that is an extension of S by D_1. To show that M is divisible, it suffices to prove that x is divisible by any $0 \neq t \in R$. As D_1 is divisible, there is an $a \in M$ such that $ta \equiv d \bmod S$. Since $rS = 0$, $t(ra) = x$ establishes the divisibility of M. By [**10**] it follows that M is a special weak-injective pre-envelope of S. Therefore, $W(S)$ is a summand of M, whence $M = W(S)$ follows. Thus $W(S)/S$ is an indecomposable kernel module.

By [**4**], the quotient $W(S)/S$ must be a Loewy module, where evidently all the simple modules are $\cong S$. Consequently, non-isomorphic simple modules S define non-isomorphic kernel modules $W(S)/S$.

In order to show that every indecomposable kernel module is either isomorphic to Q or of the form $W(S)/S$, let D be an indecomposable kernel module. D is divisible of p.d. 1. If it is torsion-free, then clearly it must be isomorphic to Q. If torsion, then it is equal to its own P-component for some maximal ideal P of R. We show that then $D \cong W(S)/S$ for $S = R/P$. If $r \neq 0$ is an element of P, then we have a non-splitting exact sequence $0 \to D[r] \to D \xrightarrow{r} D \to 0$. Here $D[r] \neq 0$ is a bounded Loewy module, so there exists a surjective map $\phi : D[r] \to S$ which we use to form by push-out construction the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & D[r] \arrow[r] \arrow[d, "\phi"] & D \arrow[r, "r"] \arrow[d] & D \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & S \arrow[r] & H \arrow[r] & D \arrow[r] & 0. \end{tikzcd}$$

Since there is no surjection $D \to S$, H is a non-splitting extension of S by D, The first paragraph of this proof applies to show that H is divisible, and hence a special pre-envelope of S. It must be indecomposable, so it is an envelope. Consequently, $H \cong W(S)$, and $D \cong W(S)/S$, in fact. □

Before formulating the main structure theorem on kernel modules, we want to identify the modules $W(S)/S$ for simple modules $S = R/P$. We know that they are divisible of projective dimension 1. Such modules are the P-components $(Q/R)_P$ of the module Q/R for maximal ideals P of R. They are Loewy modules built of simple modules $\cong S$. By Corollary 3.1, the P-components of Q/R are direct sums of indecomposable modules, and Theorem 3.2 shows that these indecomposable divisible modules ought to be isomorphic. But all submodules of Q/R are fully invariant in Q/R, and hence also in every summand of Q/R in which they are contained. Consequently, the only possibility is that the P-components $(Q/R)_P$ are indecomposable, i.e. we have

$$W(S)/S \cong (Q/R)_P.$$

Note that the P-component of Q/R can also be obtained as the quotient Q/R_P.

We now conclude with the structure theorem on kernel modules showing that they can be characterized by cardinal invariants.

Theorem 3.3. *If R is an almost perfect domain, then every kernel module for the cotorsion pair $(\mathcal{F}_1, \mathcal{WI}) = (\mathcal{P}_1, \mathcal{D})$ is a direct sum of indecomposable kernel modules. Such a direct decomposition is unique up to isomorphism.*

The indecomposable kernel modules are Q and the P-components of Q/R for the maximal ideals P of R. □

It is worth while pointing out that the weak-injective envelope of a simple module R/P (with maximal ideal P) is exactly the P-component $(Q/P)_P$. This follows at once from the exact sequence

$$0 = \mathrm{Tor}_1^R(R, Q/R) \to \mathrm{Tor}_1^R(R/P, Q/R) \cong R/P \to$$

$$\to P \otimes_R Q/R \cong Q/P \to R \otimes_R Q/R \cong Q/R \to R/P \otimes_R Q/R = 0$$

obtained from the exact sequence $0 \to P \to R \to R/P \to 0$ if tensored with the divisible module Q/R.

Observe that the kernel modules in the cotorsion pair $(\mathcal{F}_1, \mathcal{WI})$ generate the cotorsion pair. To see this, let $A \in \mathcal{F}_1$, and consider the exact sequence $0 \to A \to W(A) \to W(A)/A \to 0$. As $\mathrm{p.d.}\, W(A)/A \leq 1$, for any module N we have the induced exact sequence

$$\mathrm{Ext}_R^1(W(A), N) \to \mathrm{Ext}_R^1(A, N) \to \mathrm{Ext}_R^2(W(A)/A, N) = 0$$

whence it is evident that $\mathrm{Ext}_R^1(A, N) = 0$ whenever $\mathrm{Ext}_R^1(W(A), N)$ vanishes where $W(A)$ is a kernel module.

Hence it follows at once that the summands of Q/R are 1-tilting modules whenever R is an almost perfect domain: $Q/R = \bigoplus_P (Q/R)_P$.

4. Weak-injective envelopes

By making use of the preceding results, we are able to describe more precisely how to form weak-injective envelopes of modules over an almost perfect domain. Recall that over such a domain, the class of weak-injective modules coincides with the class of divisible modules. Actually, the process of forming weak-injective hulls is similar to the one that embeds an abelian group in its divisible hull and to the one that yields the injective hull of modules over Dedekind domains.

We start with semisimple modules. Because of the h-locality of almost perfect domains, it suffices to consider the case when the modules are annihilated by a maximal ideal P.

LEMMA 4.1. *Let R be an almost perfect domain and $M = \bigoplus_{i\in I} S_i$ with simple R-modules $S_i \cong R/P$ for a fixed maximal ideal P. Then*

$$W(M) = \bigoplus_{i\in I} W(S_i).$$

PROOF. We distinguish two cases according as P has p.d. 0 or ∞. If P is a projective ideal, then the $W(S_i)$ are injective R-modules, also injective over the localization R_P, which is a Dedekind domain. In this case the claim is evident. So suppose that $\operatorname{p.d.} P = \infty$, in which case also $\operatorname{p.d.} W(S_i) = \infty$. That the indicated direct sum is a special pre-envelope of M is clear, thus we can write $\bigoplus_{i\in I} W(S_i) = W(M) \oplus N$ for some N. Passing mod M, we conclude that N is a kernel module, thus a direct sum of indecomposable kernel modules. These kernel modules have the Exchange Property (Corollary 3.1). But the $W(S_i)$ are indecomposable, so some of them must be an indecomposable kernel module, an obvious contradiction unless $N = 0$. $\square$

Passing to the general case, we perform the construction separately for each maximal ideal P.

Let $M \neq 0$ be an arbitrary R-module. If M is not P-divisible, then $M/PM \neq 0$ (see e.g. [**1**, Proposition 3.6]), thus M/PM is a non-zero direct sum of simple modules $S_i \cong R/P$. Since $\operatorname{p.d.} W(S_i)/S_i = 1$, the bottom exact sequence exists (though not uniquely up to equivalence) making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & PM & \longrightarrow & M & \longrightarrow & M/PM = \bigoplus S_i & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & PM & \longrightarrow & D & \longrightarrow & \bigoplus W(S_i) & \longrightarrow & 0
\end{array}$$

It is easily seen that D is P-divisible. If we perform this embedding process for every maximal ideal, then the arising module W will be divisible and W/M will have projective dimension ≤ 1. These two properties suffice to guarantee that W will be a weak-injective pre-envelope of M. In order to show that W is actually an envelope, we argue similarly as before. If $W = W(M) \oplus N$, then factoring out PM, the P-component of the left hand side will give the weak-injective envelope of the semisimple module M/PM. This must be isomorphic to the P-component of the direct sum $W(M)/PM \oplus N$, and Lemma 4.1 shows that N must have trivial P-component. Consequently, $N = 0$.

From the construction just described it is clear that

THEOREM 4.2. *Over an almost perfect domain, the formation of weak-injective envelopes commutes with direct sums.* $\square$

5. Flat covers

Using the relation between weak-injective envelopes and flat covers over arbitrary domains (see e.g. [**7**]), we can describe how to obtain the flat cover of a module M over an almost perfect domain once a weak-injective envelope has been found.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & H \arrow[r, equal] \arrow[d] & H \arrow[d] & & \\
0 \arrow[r] & F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & W(M) \arrow[r] \arrow[d] & D \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Given the R-module M, let the bottom exact sequence represent the weak-injective envelope of M. Then form the middle column representing the Matlis equivalence between the h-divisible module $W(M)$ and the Matlis-cotorsion module H (which is even Enochs-cotorsion, as R is almost perfect). The kernel F of the map $E \to D$ will be the flat cover of M. (As E is a direct sum of copies of Q and D is a summand of a direct sum of copies of Q/R, it is pretty obvious that F is a summand of an extension of a free module by a torsion-free divisible module, i.e. it is strongly flat.)

In the special case when $M = S \cong R/P$ is a simple R-module, we have $D \cong (Q/R)_P$ in the last diagram, as is shown above. It follows that $E \cong Q$, and hence $F \cong R_P$ is the flat cover of S.

From this construction and from Theorem 4.2 we conclude:

COROLLARY 5.1. *If R is an almost perfect domain, then flat covers of R-modules commute with direct sums.* $\square$

This corollary can also be proved by using the Matlis category equivalence, taking advantage of the coincidence of the Matlis- and Enochs-cotorsion theories.

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Absolutely rigid fields and Shelah's absolutely rigid trees

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Abstract. Given a field K and a monoid M with right cancelation we will construct an extension field F such that the endomorphism monoid of F is $\operatorname{End} F = M \ltimes \operatorname{Frob}(F)$, where $\operatorname{Frob}(F)$ are the Frobenius endomorphisms of F. Our main goal is to show that the equality, in particular $\operatorname{Aut} F = G$ for groups G *holds absolutely.* Thus the realization of M by F should hold in any generic extension of the universe, e.g., if we change the model of set theory by forcing arguments. The construction must be very robust, so, for example, applications of stationary sets (as in Dugas and Göbel, 1987 and 1994) are not permitted. From Herden-Shelah (2009) it follows that our claim can only hold if the cardinality of F is less than the first ω-Erdős cardinal. Like measurable cardinals, this is a huge cardinal. The formula for endomorphism monoids requires a mild restriction on K (which is also necessary for Theorem 2.4), while for automorphism groups K can be arbitrary. We will proceed as follows: An old paper (from 1982) by Shelah provides the existence of absolutely rigid trees. They can be encoded into P_4-vector spaces (which are vector spaces V with 4 distinguished subspaces) over the prime field P of K (thus $P \in \{\mathbb{Q}, Z_p\}$) such that the monoid P-algebra $A = P(M)$ can be represented absolutely as the endomorphism algebra $A = \operatorname{End}_P(V, V_1, \ldots, V_4)$. Such a P_4-vector space again will be encoded into a field F which is close to a function field and is as desired. The existence of P_4-vector spaces follows by combining earlier results Göbel and Shelah (2007), Fuchs and Göbel (2008), and Göbel (2011). We also extend arguments and results from two other earlier, non-absolute constructions, Dugas and Góbel (1994 and 1997).

1. Introduction

In particular we want to show the existence of absolutely rigid fields. With the exception of an early paper by Shelah [**27**] from 1982 on trees, the topic of absoluteness entered algebra only recently (1998) mainly through Eklof-Shelah [**9**] where they study the existence of absolutely indecomposable abelian groups. Since then, absolute properties of various algebraic objects were considered, among them modules with prescribed endomorphism ring [**15, 11**], endomorphism monoids of

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graphs [**4**], E-rings [**14**] and E-modules [**13**]. In all cases it turns out that there is a (sometimes precise) cardinal barrier $\kappa(\omega)$ for these problems, which is the first ω-Erdős cardinal (see Definition 3.9). The cardinal $\kappa(\omega)$ is huge, much larger than the first inaccessible, but small enough to be consistent with the axiom of constructibility. However, it may not exist at all in any universe of set theory.

The notion of *absoluteness* is old and well established (see [**23**] or [**1, 20**]). A property of a model M (e.g., of a field) holds absolutely if it is preserved in any generic extension of the universe. In particular, the considered property of M must survive the Lévy collapse or other changes of the universe by forcing, e.g., the former implying that $\aleph_1$-free abelian groups become free. Due to this strain on M, absolute properties of huge models M will not come up. The non-existence of absolutely rigid models of cardinality $\geq \kappa(\omega)$, in particular of fields, groups etc., follows by Herden-Shelah [**19**], see also [**18**].

All absolute constructions mentioned above are based on Shelah's [**27**] absolutely rigid trees, this is also the case for our study of absolutely rigid fields. Hence, in the second part of the paper (which is Section 3), we will elaborate on the crucial steps from [**27**] reducing the arguments to their essentials. In the first part (Section 2) we then will apply the results from Section 3 to derive the existence of absolutely rigid fields.

Since we are interested in a more general case of endomorphisms of fields, we will consider extension fields F of a given field K and fix a monoid M. Since endomorphisms of F are injective, we also have to assume that M satisfies right cancelation. (Note that maps will be acting on the right.) Thus we also must take care of Frobenius homomorphisms and put $\mathrm{Frob}(F) = 1$, if K has characteristic $\chi(K) = 0$, and if $\chi(K) = p > 0$, then $\mathrm{Frob}(F)$ is generated by all multiples of $\exp(p) : F \longrightarrow F$ $(x \mapsto x^p)$, which is a central submonoid of $\mathrm{End}\, F$ isomorphic to $\mathbb{N}$. In order to show that there is an extension field F of K such that $\mathrm{End}\, F = M \ltimes \mathrm{Frob}(F)$ holds absolutely, we must require, by the above, that $|\, F\,| < \kappa(\omega)$. In the case of endomorphism monoids we also must be aware of two easy results in case of algebraically closed fields.

Observation 1.1. *Let K be an algebraically closed field with infinite transcendence basis B over the prime field P.*

- *Let $F = K(Y)$ be an extension field with generators Y such that $|\, Y\,| < |\, B\,|$. Then $|\,\mathrm{Aut}\, F\,| = 2^{|\, F\,|}$ and if $|\, Y\,| \leq |\, B\,|$, then trivially $|\,\mathrm{End}\, F\,| = 2^{|\, F\,|}$.*
- *There is no endo-rigid extension F of K such that $\mathrm{End}\, F = \mathrm{Frob}(F)$ holds absolutely.*

Proof. The first part follows from [**8**, p. 3784, Theorem 4.1], and we thank Saharon Shelah for pointing us to the second part. As in the proof showing that there are no absolutely $\aleph_1$-free, indecomposable abelian groups, we consider an extension $K \subseteq F$ and change the universe by applying a Levi collapse such that F is countable. If $\widehat{F}$ denotes the algebraic closure of F, then there are $2^{\aleph_0}$ distinct isomorphisms $\alpha : \widehat{F} \longrightarrow \widehat{F}$. Thus there are $2^{\aleph_0}$ monomorphisms $\alpha \restriction F$, and F cannot be absolutely endo-rigid. $\square$

Hence absolutely endo-rigid extensions of algebraically closed fields do not exist, and in the non-absolute case they must have a larger size. If $|\, K\,| < \lambda$, then endo-rigid fields F of size $\lambda^{\aleph_0}$ exist, as shown in [**6**]. Thus we need a restriction on

K. Hence (as for the non-absolute proofs in [**7**]) we require that K satisfies the condition $\pi(K) \geq 7$, (see Definition 2.3), i.e., K permits at least 7 primes with no primitive roots of unity. These restrictions apply to our results for endomorphisms (cf.,Theorem 2.4).

By the action of M on F it follows that $\mathrm{Frob}(F) \cap M = \{\mathrm{id}_F\}$ and only the trivial Frobenius endomorphism is surjective. Thus our absolute construction (cf., Theorem 2.4) has an immediate

Corollary 1.2. *If K is any field and M a group, both of size $< \kappa(\omega)$, then for each infinite cardinal $\lambda < \kappa(\omega)$ with $\max\{|\,K\,|, |\,M\,|\} \leq \lambda < \kappa(\omega)$, there is a field F of size λ such that $\mathrm{Aut}\, F = M$ holds absolutely.*

However, in the case of automorphism groups Observation 1.1 does not confine us, and we can remove the restriction on K (see Theorem 2.5). This needs some extra work; the arguments in Dugas-Göbel [**8**] carry over without changes and we only discuss the needed changes in Section 2.4.

The strategy of our construction of absolute fields with prescribed endomorphism monoids is based on R_κ-modules which are modules over a commutative ring $R \neq 0$ with a family of κ distinguished submodules. (We can restrict κ to 4 or ω, the first infinite cardinal.) Using [**27**] (see also Section 3,) we will first obtain a family of absolute, rigid R_ω-modules as shown already in [**15**]. Luckily, by a strengthening of a result in [**11**], with the help of [**12**], ω can be replaced 4. If P is the prime field of K (so $P \in \{Z_p, \mathbb{Q}\}$,) let $A = P[M]$ be the monoid P-algebra. We find P_4-modules $\mathbf{V} = (V, V_1, \ldots, V_4)$ with P-endomorphism ring A. In the next step we consider a suitable function field $K(X)$ and mark four distinguished P-subspaces with the help of four distinct primes (given by $\pi(K) \geq 7$). In order to recognize them by field-endomorphisms we adjoin countable sequences of q-th roots (with $q \neq \chi(K)$). The remaining 3 primes are used to show that every endomorphism of F induces an endomorphism on the P_4-vector space $\mathbf{V} = (V, V_1, \cdots, V_4)$ now embedded into F. Finally we can adopt and extend arguments from [**7**] to show the absolute result for $\mathrm{End}\, F$ indicated above.

There are several earlier non-absolute result related to Theorem 2.4 and Theorem 2.5. Fried and Kollár prescribe groups as automorphism groups of fields F of characteristic $\chi(F) \neq 2$ and Prőhle [**26**] added the case $\chi(F) = 2$ (here $K \in \{\mathbb{Q}, Z_p\}$ is just a prime field). A pioneering paper constructing fields is due to Prőhle [**25**], however his extension fields of K are quite large; they have strongly inaccessible cardinality. We will use a special case of his Lemma 3 (see Corollary 2.11) describing function fields which are extended by infinite q-root chains. This was also used to prove (non-absolute) realization theorems for automorphism groups of extension fields over a fixed field K in Dugas-Göbel [**5, 6, 7, 8**], which are basic for our present considerations. In [**5**] Jensen's $\diamondsuit$-principle, an additional axiom of set theory, which holds in Gödel's universe, was applied and in [**6**] Black Box combinatorics (so working in ordinary ZFC-set theory) was employed. These constructions imply a cardinal jump $|\,K\,| < |\,F\,|$ of the extension field, due to the applied combinatorics, but permit endo-rigid extensions of algebraically closed fields (so the $\pi(K)$-condition is not needed) for $\mathrm{End}\, F$. Prőhle's technique [**25**, Lemma 3, p. 19 - 20] is also used for the non-absolute construction in [**7, 8**], and the idea to get rid of the restrictions on K for $\mathrm{Aut}\, F$ is adapted from [**8**]. But clearly Black Box constructions and those used in [**7, 8**] based on the Shelah-elevator (see [**16**, p. 355]) are by no means absolute.

There is also another working paper by Kaplan, Shelah [**21**] on a related topic: the construction of fields with prescribed automorphism group (over $\mathbb{Q}$ and Z_p, respectively) without choice. The authors here apply a strengthening of Prőhle [**25**, Lemma 3, p. 19 - 20] adjoining q-root chains to function fields; a detailed proof of this lemma is elaborated in [**21**, Section 4]. In fact, inspecting the proof in [**21**], it follows from the arguments in Section 2.4 or in [**8**, pp. 3780 - 3783], that we can extend the main result in [**21**], and get as a

Corollary 1.3. *If K is any field and G is any group, then* without the axiom of choice *there is an equipotent extension field F with* $\operatorname{Aut} F = G$.

(The case in [**21**] is $K \in \{\mathbb{Q}, Z_p\}$.)

2. Absolutely rigid fields

We want to investigate the following two problems in an absolute fashion.

Problem 2.1 (Endo-rigid problem). *Given a field K and a monoid M, does there exist an extension field F of K such that* $\operatorname{End} F \cong M$ *modulo Frobenius endomorphisms holds absolutely?*

If this is possible, then we will say that F realizes M over K absolutely and we speak about the *endo-rigid* problem. The second problem that we consider is the following:

Problem 2.2 (Auto-rigid problem). *Given a field K and a group G, does there exist an extension field F of K such that* $\operatorname{Aut} F \cong G$ *holds absolutely?*

If this is possible, then F realizes G over K absolutely and we call this the *auto-rigid* problem. Often an answer to the first problem also gives an answer to the second problem, but in general these questions must be separated. For answering the endo-rigid problem, we will need an algebraic condition on K from [**7**]. For a field K, let $\chi(K)$ denote its characteristic.

Definition 2.3. *Let $\pi(K)$ be the set of all prime integers $q \neq \chi(K)$ such that K contains no primitive q-th root of unity.*

Thus clearly $|\,\pi(K)\,| \leq \aleph_0$. In particular $\pi(\mathbb{C}) = \emptyset$ and $|\,\pi(\mathbb{Q})\,| = |\,\pi(\mathbb{R})\,| = \aleph_0$. Our main result on endomorphism monoids does not apply if $|\,\pi(K)\,| < 7$, in particular we do not deal with $K = \mathbb{C}$, the complex numbers or any other algebraically closed field as such a construction cannot exist (see Observation 1.1). However there are non-absolute constructions in this case [**6**]. The condition on the roots of unity is used to apply the machinery of absolute trees as well as the results on R_4-modules.

With $\operatorname{Frob}(K)$ denoting the Frobenius endomorphisms of a field K, we will prove the following theorem.

Theorem 2.4. *Let M be a monoid with right cancelation, K a field with $|\,\pi(K)\,| \geq 7$ and λ an infinite cardinal with* $\max\{|\,K\,|, |\,M\,|\} \leq \lambda < \kappa(\omega)$. *Then there is a field extension F with the following absolute properties.*

(1) $\operatorname{End} F \cong M \ltimes \operatorname{Frob}(F)$.
(2) *K is point-wise fixed under the action of M*
(3) $|\,F\,| = \lambda$.

In Theorem 2.4 we have $\mathrm{Frob}(F) = \mathrm{id}_F$ if $\chi(F) = 0$ and otherwise, if $\chi(F) = p$, then $\mathrm{Frob}(F)$ is the family of all endomorphisms taking powers of p. However in the latter case the Frobenius endomorphisms are invertible only if they are the identity on F as pointed out in Observation 2.12. Hence $\mathrm{Aut}\, F \cong M^*$, where M^* is the group of units of the monoid M. Thus we have an immediate corollary from Theorem 2.4 for the automorphism case.

However, a few more arguments, using a countable iteration of the arguments from Section 2.3 (as in [**8**]) show that the assumption on K can be avoided in that case. The arguments follow directly from [**8**] and we will only sketch the major steps in Section 2.4. We obtain Theorem 2.5 which holds in particular for algebraically closed fields, e.g., for the field $\mathbb{C}$ of complex numbers.

THEOREM 2.5. *If G is a group, K is any field and λ an infinite cardinal such that* $\max\{|K|, |M|\} \leq \lambda < \kappa(\omega)$, *then there exists a field extension F of K of cardinality λ such that* $\mathrm{Aut}\, F = G$ *holds absolutely.*

The non-absolute version of Theorem 2.5 has been proven in Dugas-Göbel [**8**]. In [**8**] however the Shelah elevator (cf., [**16**]) has been applied. Thus the construction heavily relies on the existence of stationary sets, and it is clear that this earlier result is not absolute. We can find a generic extension of the universe, in which the elevator fails and the equation $\mathrm{Aut}\, F \cong G$ in the initial universe is no longer valid in some generic extension of the universe.

2.1. The link between trees and module theory. We will need the main result from Fuchs-Göbel [**11**] which provides the link between absolutely rigid trees (see Theorem 3.14), and absolutely indecomposable R_κ-modules (and more). The essential idea to rest on absolutely rigid trees comes from Göbel-Shelah [**15**] on R_ω-modules and utilizes a reduction argument from Göbel [**12**] replacing ω by 6, see also [**16**]. In fact, we can even replace 6 by 4 (see [**11**]). This additional reduction to the lower bound 4 will appear in [**17**]. It is known from representation theory of algebras that 4 cannot be replaced by 3; cf., e.g., [**16**].

First recall the notion of an R_κ-module over a commutative ring R with $1 \neq 0$ for some fixed cardinal κ. As indicated, we are only interested in $\kappa = \omega$ or $\kappa = 4$.

DEFINITION 2.6. *Let $R_\kappa - \mathbf{Mod}$ denote the category of R–modules M over a commutative ring R with κ distinguished submodules $\mathbf{M} = (M, M^\alpha \mid \alpha < \kappa)$. If $\kappa = n < \omega$, we will also write $\mathbf{M} = (M, M_1, \ldots, M_n)$ and call $\mathbf{M}$ an R_κ–module.*

Recall that the homomorphisms between two R_κ–modules $\mathbf{M}, \mathbf{M}'$ must respect their distinguished submodules M^k, M'^k. We define

$$\begin{aligned}\mathbf{Hom}_R(\mathbf{M}, \mathbf{M}') &= \mathrm{Hom}_R\big(M, M'; M^k, M'^k \mid k < \kappa\big)\\ &= \big\{\sigma \in \mathrm{Hom}_R(M, M') \mid M^k\sigma \subseteq M'^k \text{ for all } k < \kappa\big\}.\end{aligned}$$

If X is any R–module, then $\mathbf{M} \otimes X = (M \otimes X, M^k \otimes X \mid k < \kappa)$ is again an R_κ–module, defined by the tensor product over R. It holds the following theorem.

THEOREM 2.7. *Let $\lambda < \kappa(\omega)$ be any infinite cardinal and let A any faithful algebra over a commutative ring $R \neq 0$ such that A has fewer than λ generators. Then there exists a family of right A_4–modules*

$$\mathbf{M}_U = (M_U, M_U^1, M_U^2, M_U^3, M_U^4) \qquad (U \subseteq \lambda),$$

where $M, M_U^j, M/M_U^j$ *are free* A*–modules of rank* λ *for all* $1 \leq j \leq 4$ *such that*

$$\mathbf{Hom}_R(\mathbf{M}_U, \mathbf{M}_V) = \begin{cases} A, & \text{if} \quad U \subseteq V \\ 0, & \text{if} \quad U \not\subseteq V \end{cases}$$

holds in any generic extension of the universe, i.e., it is absolute.

We want to apply Theorem 2.7 immediately. Let K be a λ-generated field with prime field P, and let M be a monoid with right cancelation. Consider the monoid P-algebra $A = P[M]$ (defined as a group ring) and suppose that $|\,A\,| < \kappa(\omega)$. With these assumptions we obtain

Corollary 2.8. *Let* $M, K, P,\ A = P[M]$ *be as above and let* $\lambda < \kappa(\omega)$ *be any cardinal such that* A *has fewer than* λ *generators (over* P*). Then there exists an* A*-module with four distinguished* A*-submodules* $\mathbf{V} = (V, V_1, V_2, V_3, V_4)$ *with* V, V_i *and* V/V_i *free* A*-modules of rank* λ *for each* $1 \leq i \leq 4$ *and* $\mathbf{End}_P\mathbf{V} = P[M]$ *holds absolutely.*

2.2. The tools from field theory for applications of R_κ-modules. Besides the results on trees presented in Section 3, we will also need results on field extensions derived in [**6, 7, 8, 25, 26**]. We will summarize the necessary facts in this section. We will also need a special case of a result on extensions of function fields which is due to Fried-Kollár [**10**] for characteristic 0 and Prőhle [**25**] for positive characteristic. In order to formulate these auxiliary tools, we will need certain pure algebraic extensions of a function field over K.

If $K \subseteq F$ is a field extension, then F is *purely transcendental* over K if any element $b \in F$, which is algebraic over K, belongs to K. We now transfer Baer's notion of types from abelian groups (e.g., for torsion-free rank-1 groups) to fields (see also [**8**]). Let $\mathbb{P}_K$ be the set of primes different from $\chi(K)$. Then the notion of types becomes

Definition 2.9. *Let* $a \in K$*. Then the* type of a in K *is defined as*

$$\tau(a) = \tau_K(a) = \{p \in \mathbb{P}_K \mid \forall n \in \mathbb{N} : \exists x \in K \text{ such that } x^{p^n} = a\}.$$

Clearly $\tau(a) \subseteq \tau(a\alpha)$ for all $a \in K$ and $\alpha \in \operatorname{End} K$, and equality holds if $\alpha \in \operatorname{Aut} K$. If K is algebraically closed, then $\tau_K(a) = \mathbb{P}_K$ and $\tau_{\mathbb{Q}}(a) = \emptyset$ for all $a \in \mathbb{Q} \setminus \{0, 1, -1\}$. If $t \in K$ and $q \in \mathbb{P}_K$, then we want to inductively choose elements ${}_nt \in \widehat{K}$ from the algebraic closure of K such that ${}_0t = t$ and ${}_{n+1}t^q = {}_nt$ for all $n < \omega$. We will write $\langle t\rangle_q = \{{}_nt \mid n < \omega\}$ and call this family of elements a *q-root chain* (of t). If $\langle t\rangle_q \subseteq K$, then $q \in \tau_K(t)$. If $K(t)$ is the function field with $t = {}_0t$ transcendental over K, then $K' = K(\langle t\rangle_q)$ is an algebraic extension of $K(t)$, $\tau_{K'}(t) = \{q\}$ and $K' = \bigcup_{n\geq 0} K({}_nt)$ is the union of function fields $K({}_nt)$. We say that K' is $K(t)$ extended by a q-root-chain $\langle t\rangle_q$. More generally we will say that elements $f_n \in F$ $(n \geq 0)$ of an extension field F of K constitute a q-root chain of f if $f = f_0 \in F\backslash K$ and if there are $0 \neq k_n \in K$ such that $(f_{n+1}k_{n+1})^q = f_n$ for all $n \geq 0$.

Such root chains are important to control automorphisms of fields, as observed first by Fried-Kóllar [**10**] and exploited by Prőhle [**25, 26**] and in Dugas-Göbel [**7, 8**]. We now refer to [**10**, p. 296, Lemma 4.3] and [**25**, p. 338, Third Lemma (3)].

LEMMA 2.10. *Let K be a field and z transcendental over K. Let p, q be two distinct primes in $\mathbb{P}_K$ and put $F = K(\langle z\rangle_q \cup \{a_i \mid i \in I\})$ with $\langle z\rangle_q = \{{}_nz \mid n < \omega\}$, ${}_0z = z$, $a_i \in \widehat{K}$ $a_i^p = b_i \in K[z]$ (the polynomial ring in z over K), and each $b_i(z)$ being a polynomial of degree one with $b_i(0) \neq 0$ for all $i \in I$. Then the following holds for F.*

(1) *If $a \in F$ such that $a^p \in K[{}_nz]$, then there is a finite subset $W \subseteq I$ such that*
$$a^p = c({}_nz)^p \prod_{w\in W} b_w^{e_w} \text{ with } 0 < e_w < p \text{ and } c({}_nz) \in K[{}_nz].$$
(2) *F is a purely transcendental extension of K.*
(3) *If $a \in F$ and $p \in \tau_F(a)$, then $a \in K$.*
(4) *$\tau_F({}_nz) = \{q\}$ for all $n < \omega$.*
(5) *If $q \in \tau_F(a)$ for $a \in F$, then $a = c \cdot ({}_\ell z)^e$ for some $e \in \mathbb{Z}$ and $\ell < \omega$, where $c \in K$ satisfies $q \in \tau_K(c)$ and $\ell = 0$ or q does not divide $e \in \mathbb{Z}$.*
(6) *If $a \in K$, then $\tau_K(a) = \tau_F(a)$.*

A new, more detailed proof of an extended version of Lemma 2.10 (based on Prőhle [**26**]) is given in Kaplan-Shelah [**21**, Section 4]; note however that here we will use only little from [**26**]. Recall the well known fact that if K is a field and ε a primitive n-th root of unity, then $[K(\varepsilon) : K] | \varphi(n)$, where $\varphi(n)$ is the Euler function at n.

2.3. The construction of absolutely endo-rigid fields with prescribed endomorphism monoid. For establishing Theorem 2.4 we must find an extension field F of the given field K with $\pi(K) \geq 7$ so that $\operatorname{End} F = M \ltimes \operatorname{Frob}(F)$ holds for the given monoid M with right cancelation. The proof follows the lines of [**7**], but we must carefully replace $\operatorname{Aut} F = M$ by $\operatorname{End} F = M \ltimes \operatorname{Frob}(F)$ and we will use Corollary 2.8 to ensure absoluteness.

The construction of an extension field F of K. By assumption $|\,\pi(K)\,| \geq 7$ we can choose seven primes from $\mathbb{P}_K$

$$p = q_{-1} > q_0 > q_1 \cdots > q_5 \tag{1}$$

such that K contains no primitive q_i-th root of unity for $-1 \leq i \leq 5$. Moreover, let $\{a_i \mid i < \lambda\}$ be a set of generators of K as extension field over the prime field P and let $A = P[M]$ be the P-monoid algebra over the given monoid M. By Corollary 2.8 we have a free A-module V with four distinguished free A-submodules V_i such that $\mathbf{V} = (V, V_1, \dots, V_4)$ satisfies

$$\mathbf{End}_P(V, V_1, \dots, V_4) = A$$

and properties stated in the corollary. Hence we can choose A-bases X and X_i of V and V_i ($i \leq 4$), respectively. Then $XM = \{xg \mid x \in X, g \in M\}$ and $X_iM = \{xg \mid x \in X_i, g \in M\}$ are P-bases of V and V_i, respectively. Next we want to construct an ascending chain of extension fields

$$K \subseteq K_{-1} \subseteq K_0 \subseteq K_1 \subseteq \cdots \subseteq K_5 = F. \tag{2}$$

Let $K := P(a_i \mid i < \lambda)$. For each $a_i \in K$ with $i < \lambda$ we inductively choose a root chain $\langle a_i\rangle_p$ by $a_i = {}_0x_i$, $({}_{j+1}x_i)^p = {}_jx_i$ for all $j < \omega$ from the algebraic closure of K.

Let $K_{-1} = K(\langle a_i\rangle_p \mid i < \lambda)$ be the extension field of K in the algebraic closure of K obtained by adding all root chains $\langle a_i\rangle_p$ with $i < \lambda$. Thus $p \in \tau_{K_{-1}}(a_i)$ for all $i < \lambda$. The field K_{-1} can be expressed as the union $K_{-1} = \bigcup_{\alpha<\lambda} K_\alpha$ of an ascending, continuous chain of field extensions K_α such that $[K_{\alpha+1} : K_\alpha] = p$ (apply Abel's Theorem, see [**22**, p. 221]). By the choice (1) the field K_{-1} contains no primitive q_i-th root of unity for $-1 \le i \le 5$ and if $\varphi \in \operatorname{End} K_{-1}$ with $\varphi \restriction K = \mathrm{id}_K$, then $\varphi = \mathrm{id}_{K_{-1}}$. Let $\operatorname{End}_K H$ denote the endomorphisms φ of K with $\varphi \restriction H = \mathrm{id}_H$. With this notation we have $\operatorname{End}_K K_{-1} = \{\mathrm{id}_{K_{-1}}\}$.

Let $XM\dot\cup\{t_1, t_2, \dots, t_5\}$ be algebraically independent over K_{-1} and pick further root chains from the algebraic closure of the function field $K_{-1}(XM\cup\{t_1, t_2, \dots, t_5\})$.

(1) Choose $\langle t_i\rangle_{q_i}$ by ${}_jt_i$, such that ${}_0t_i = t_i$ and $({}_{j+1}t_i)^{q_i} = {}_jt_i$ for $j < \omega$ and $1 \le i \le 5$.

(2) If $x \in X, g \in M$, then choose $\langle xg\rangle_{q_0}$ by ${}_j(xg)$ with ${}_0(xg) = xg$ and $({}_{j+1}(xg))^{q_0} = {}_j(xg)$ for all $j < \omega$.

and define $K_0 = K_{-1}(\langle xg\rangle_{q_0} \mid xg \in XM)$ by adding (ii). It follows that $q_0 \in \tau_{K_0}(xg)$ for all $xg \in XM$. Clearly K_0 is the union of function fields over K_{-1} and thus K_0 contains no new primitive q-th roots of unity (other than those already in K_{-1}). Now we must take care of the A_4-module $\mathbf{V}$ and define inductively $L_j = K_{j-1}(\langle t_j\rangle_{q_j})$ for $1 \le j \le 4$ and

$$K_j = L_j(\sqrt[p]{t_j - 1},\ \sqrt[p]{t_j - xg} \mid\ xg \in X_jM) \tag{3}$$

an extension of L_j contained in the algebraic closure of L_j. Hence $p_j \in \tau_{K_j}(t_j)$. In the final step of the construction let $X \longrightarrow \{a_i \mid i < \lambda\}$ $(x \mapsto a_x)$ be any bijection between X and $\{a_i \mid i < \lambda\}$ and set

$$F = K_5 = L_5(\sqrt[p]{t_4 - 1},\ \sqrt[p]{t_5 - xg - a_x} \mid\ x \in X,\ g \in M).$$

This finishes the construction of F.

We can apply Prőhle's Lemma 2.10 to F and get the immediate

COROLLARY 2.11. *If $f \in F$ such that $q_i \in \tau_F(f)$, then $f \in K_i$ and f looks like an obvious element with q_i in its type:*

(1) *If $i \ge 1$, then $f = c \cdot ({}_\ell t_i^e)$ and $c \in K_{-1}$ such that $q_i \in \tau_{K_{-1}}(c)$ with $\ell < \omega$ and $\ell = 0$ or q_i does not divide $e \in \mathbb{Z}$.*

(2) *If $i = 0$, then $f = c \cdot \prod_{x\in Y, g\in M'} ({}_{\ell_{xg}}(xg))^{e_{xg}}$ with $\ell_{xg} < \omega$, Y, M' finite subsets of X, M, respectively, $q_0 \in \tau_{K_{-1}}(c)$, and $e_{xg} \in \mathbb{Z}$ such that $\ell_{xg} = 0$ or q_0 does not divide e_{xg}.*

(3) *If $a \in K_i$ and $q \in \tau_F(a)$, then $q = q_i$ or $a \in K_{-1}$ for $i = 1, \dots, 5$.*

If F is any algebraically closed field of characteristic $\chi(F) = r \ne 0$, then clearly $\operatorname{Frob}(F) = \{\exp(r)^z \mid z \in \mathbb{Z}\}$, and if $F = K(x)$ is a function field, then $\operatorname{Frob}(F) = \{\exp(r)^n \mid n < \omega\}$ because $\exp(r)^n$ is not surjective. For example $x \in F$ is not in the image of $\exp(r)^n$. Otherwise, $x = (\frac{f(x)}{g(x)})^{r^n}$ for some reduced $\frac{f(x)}{g(x)} \in F$, i.e., $(g, f) = 1$. Hence, $xg(x)^{r^n} = f(x)^{r^n}$ which leads to a contradiction as x cannot occur as a common factor of $f(x)$ and $g(x)$. The constructed field F above behaves essentially like a function field. We have the immediate

OBSERVATION 2.12. *If the constructed field F has characteristic $\chi(F) = r \ne 0$, then $\varphi \in \operatorname{Frob}(F)$ is invertible if and only if $\varphi = \mathrm{id}_F$.*

PROOF. Let $t_1 \in F$ be as constructed. In particular we have $t_1 \notin K_{-1}$. If now $\mathrm{id} \neq \varphi \in \mathrm{Frob}(F)$ is invertible, then $r \in \tau_F(t_1)$. Together with Corollary 2.11.(iii) it follows that $r = q_1$ which is a contradiction as $r \neq q_1$ by construction. □

Proof of Theorem 2.4.

PROOF. Recall that $F = K_5$ contains no primitive q_i-th root of unity for $-1 \leq i \leq 5$ and F is the union of a chain of intermediate fields $L^{(\alpha)}$ such that $L^{(\alpha+1)}$ is either a function field over L_α or $[L^{(\alpha+1)} : L^{(\alpha)}] = p \geq q_i$ for $-1 \leq i \leq 5$. This implies that $\mathrm{End}_{L_j}\ K_j = \{\mathrm{id}_{K_j}\}$ for $0 \leq j \leq 5$. We obtain an embedding $M \subseteq \mathrm{End}_{K_{-1}}\ K_0$ by defining ${}_j(xg)h = {}_j(xgh)$ for $xg \in XM$, $j < \omega$, $h \in M$. Moreover $M \subseteq \mathrm{End}\, K_1$ by letting M act homogeneously w.r.t. $(+, \cdot)$ on K_1, i.e., we canonically define

$$(4)\quad t_1 h = t_1 \text{ and } (\sqrt[p]{t_1 - 1})h = \sqrt[p]{t_1 - 1} \text{ and } (\sqrt[p]{(t_1 - xg)})h = \sqrt[p]{t_1 - xgh}$$

for all $xg \in XM,\ h \in M$. The cases $i = 2, \ldots, 5$ are defined similarly.

Since $F = K_5$ contains no primitive q_i-th root of unity, $-1 \leq i \leq 5$, any $\alpha \in \mathrm{End}\, F$ satisfying $\alpha \restriction L = \mathrm{id}_L$ for $L = K(XM \cup \{t_1, \ldots, t_5\})$ is also the identity on F, i.e., $\mathrm{End}_L\ K = \{\mathrm{id}_K\}$. Setting ${}_jt_i h = {}_jt_i$ for $1 \leq i \leq 5$, $j < \omega$ and $h \in M$, it follows by the construction of F that $M \subseteq \mathrm{End}_{K_{-1}} F$.

Let $\chi(K) = r$ be the characteristic of F. Then $\mathrm{Frob}(F) = \langle \exp(r) \rangle = \{\exp(r)^z \mid z < \omega\}$ is generated by the Frobenius endomorphism $\exp(r) : F \longrightarrow F$ $(f \mapsto f^r)$ if $r > 0$ and $\exp(0) = \mathrm{id}_F$. In the following we concentrate on the more complicated case $\chi(K) > 0$; the other case follows with obvious simplifications. By the action of M on F (e.g., on some xg) it is clear that $M \cap \mathrm{Frob}(F) = \{\mathrm{id}_F\}$ and $\mathrm{Frob}(F)$ is also central in $\mathrm{End}\, F$, hence $M \ltimes \mathrm{Frob}(F) \subseteq \mathrm{End}\, F$ is a split extension. It remains to show that $M \ltimes \mathrm{Frob}(F) = \mathrm{End}\, F$, hence if $\alpha \in \mathrm{End}\, F$, then we aim for showing that $\alpha \in M \ltimes \mathrm{Frob}(F)$ (with the help of Corollary 2.11).

Note that

$$(5)\qquad p \in \tau_{K_{-1}}(a_i) \text{ for all } \ i < \lambda, \text{ and if } p \in \tau_F(f), \text{ then } f \in K_{-1}.$$

Hence $a_i\alpha \in K_{-1}$ and $K\alpha \subseteq K_{-1}$ follows. Moreover, K_{-1} is generated over K by elements of type containing p and therefore $K_{-1}\alpha \subseteq K_{-1}$. A similar argument, using q_0, shows that $K_0\alpha \subseteq K_0$. This way we can also argue with the partial inverse α^{-1} of α. Let $\alpha^{-1} : \mathrm{Im}(\alpha) \to F$, then for all $s \in K_{-1} \cap \mathrm{Im}(\alpha)$ it follows $s\alpha^{-1} \in K_{-1}$.

Since $q_1 \in \tau_F(t_1)$ and all elements of F of type containing q_1 are in K_1 we have $t_1\alpha \in K_1$ and by Corollary 2.11(i) also $t_1\alpha = s\ {}_et_1^f$, $s \in K_0$ has type containing q_1, and $e = 0$ or q_1 does not divide $f \in \mathbb{Z}$. Since $s \in K_0$ has type containing q_1, also $s \in K_{-1}$. We distinguish three cases.

Case $f > 0$:

Using Lemma 2.10 we obtain

$$(t_1 - 1)\alpha = s\ {}_et_1^f - 1 = c({}_et_1)^p \cdot ({}_et_1^{q_1^e} - 1)^{m_1} \cdot \prod_{ug \in S} ({}_et_1^{q_1^e} - ug)^{m_{ug}}\ ,$$

an equation involving polynomials in $K_0[{}_et_1]$. Since roots of $s\ {}_et_1^f - 1 \in K_{-1}[{}_et_1]$ are algebraic over K_{-1} and ug is transcendental over K_{-1} we infer $S = \emptyset$ and

$$s\ {}_et_1^f - 1 = c({}_et_1)^p({}_et_1^{q_1^e} - 1)^{m_1}\ .$$

Suppose $m_1 > 0$. Then 1 is a zero on the right hand side, $s = 1$ follows and ${}_e t_1^f - 1 = c({}_e t_1)^p ({}_e t_1^{q_1^e} - 1)^{m_1}$. Each q_1^e-th root of unity is an f-th root of unity. Thus q_1^e divides f and $e = 0$. Our equation now turns into $t_1^f - 1 = c(t_1)^p (t_1 - 1)^{m_1}$.

Now suppose that $r \nmid f$. If $f > 1$, pick a primitive f-th root ε of unity. Such a root exists because $r \nmid f$. Then $c(\varepsilon)^p = 0$ and ε has multiplicity a multiple of p in $t_1^f - 1$. Since $\chi(K) \neq p,\ f = 1$ follows and $t_1\alpha = t_1$.

Next, let $f = k \cdot r^n$ where $(k, r) = 1$. Therefore $t_1\alpha = {}_e t_1^{k \cdot r^n} = ({}_e t_1^k)^{r^n}$. We obtain

$$(t_1\alpha)^{1/r^n} = t_1(\alpha \exp(r)^{1/n}).$$

We define $\tilde{\alpha} := \alpha \exp(r)^{1/n}$ and have $t_1\tilde{\alpha} = {}_e t_1^k$ so that

$$(t_1 - 1)\tilde{\alpha} = {}_e t_1^k - 1$$

and we can proceed as before as $r \nmid k$. It follows that $t_1\tilde{\alpha} = t_1$ and so $t_1\alpha = t_1 \exp(r)^n$.

Now let $m_1 = 0$.

Comparing multiplicities of roots of the left and right hand side again leads to the contradiction $p = \chi(K)$ and so the case $m_1 = 0$ cannot occur. This completes the case $f > 0$ and we obtain that $t_1\alpha = t_1 \exp(r)^{n_1}$ for some $n_1 < \omega$, possibly $n_1 = 0$.

<u>Case $f < 0$:</u>

First let $r \nmid f$. Pick $m \in \mathbb{N}$ with $0 \leq mp + f < p$. Then $(s\ {}_e t_1^f - 1) {}_e t_1^{mp} = s\ {}_e t_1^{f+mp} -_e t_1^{mp}$ is a polynomial in ${}_e t_1$ and a p-th power in K_1 by (3). Thus, by Lemma 2.10,

$$s\ {}_e t_1^{f+mp} -_e t_1^{mp} = c({}_e t_1)^p ({}_e t_1^{q_1^e} - 1)^{m_1} \prod_{ug \in S} ({}_e t_1^{q_1^e} - ug)^{m_{ug}}\ .$$

If $f + mp > 0$, then 0 is a root with multiplicity a multiple of p of the right hand side. Thus $f + mp = 0$ and

$$s -_e t_1^{mp} = c({}_e t_1)^p ({}_e t_1^{q_1^e} - 1)^{m_1} \prod_{ug \in S} ({}_e t_1^{q_1^e} - ug)^{m_{ug}}.$$

Since $s \in K_{-1},\ S = \emptyset$ follows and

$$s -_e t_1^{mp} = c({}_e t_1)^p ({}_e t_1^{q_1^e} - 1)^{m_1}.$$

Suppose $m_1 > 0$. Then $s = 1$ and $1 -_e t_1^{mp} = c({}_e t_1)^p ({}_e t_1^{q_1^e} - 1)^{m_1}$.

If $e > 0$, then q_1^e divides $mp = f$, a contradiction. Thus $e = 0$ and $1 - t_1^{mp} = c(t_1)^p (t_1 - 1)^{m_1}$. Let ε be a primitive mp-th root of unity. Then ε has multiplicity a multiple of p. Again we obtain the contradiction $p = \chi(K)$. Now the case $r \mid f$ follows similarly replacing α with $\tilde{\alpha}$ as in the case $f > 0$.

<u>Case $f = 0$:</u>

The assumption $f = 0$ becomes $t_1\alpha = s \in K_0$, $q_1 \in \tau_F(t_1\alpha)$ and $t_1\alpha \in K_{-1}$ by Corollary 2.11 (iii). Now, abusing notation, let $\alpha^{-1} : \mathrm{Im}(\alpha) \to K_{-1}$ be the (partial) inverse of α defined on $K_{-1}\alpha$. We obtain $t_1 = s\alpha^{-1}$ and with the argumentation from above we conclude $t_1 \in K_{-1}$ which is impossible. Therefore the case $f = 0$ cannot occur.

Now consider $(t_1 - xg)\alpha = t_1 \exp(r)^n - xg\alpha,\ xg \in X_1M$ for $n_1 = n$. Since $xg \in K_0$ and $q_0 \in \tau_{K_0}(xg)$ by construction of K_0, the same holds for $(xg)\alpha$, thus

$xg\alpha \in K_0$ and also $(xg)\alpha \notin K_{-1}$ with an argument similar to the one used in case $f = 0$. Note that $t_1 - xg$ has a p-th root in K_0 and so does $(t_1 - xg)\alpha \in K_0[t_1]$. By Lemma 2.10(i) it follows

$$(t_1 - xg)\alpha = t_1 \exp(r)^n - (xg)\alpha = c(t_1)^p (t_1 - 1)^{m_1} \prod_{yh \in M} (t_1 - yh)^{m_{yh}}.$$

From $(xg)\alpha \notin K_{-1}$ follows $m_1 = 0$, $M = \{yh\}$ is a singleton, $m_{yh} = r^n$ and $c(t_1)$ is a constant. Thus $t_1 \exp(r)^n - (xg)\alpha = c(t_1 - yh)^{r^n}$, $c = 1$, and $yh \in X_1 M$ follows. We get $(X_1 M)\alpha \subseteq (X_1 M) \exp(r)^n$.

We repeat the last step four times and obtain in the same fashion that $t_i \alpha = t_i \exp(r)^{n_i}$ for $1 \leq i \leq 4$ and $(X_i M)\alpha \subseteq (X_i M) \exp(r)^{n_i}$ for $1 \leq i \leq 4$. Using $(t_1 + t_i)\alpha = t_1\alpha + t_i\alpha$ we also see that $n_i = n$ does not depend on i, so $t_i\alpha = t_i \exp(r)^n$ for $1 \leq i \leq 4$.

Working with K_5, we use the above argument again, i.e., apply Lemma 2.10 and $m_{xg}\alpha \in (XM)\exp(r)^n$ to show

$$(t_5 - xg - a_x)\alpha = t_5\alpha - (xg + a_x)\alpha = t_5\alpha - (yh \exp(r)^n + a_y)$$

for some $yh \in XM$. Thus $(xg + a_x)\alpha = (xg)\alpha + a_x\alpha = yh \exp(r)^n + a_y$ and $a_y, a_x\alpha \in K_{-1}$. Thus $(xg)\alpha$ is in $K_{-1}[yh \exp(r)^n]$ and $q_0 \in \tau_{K_0}((xg)\alpha)$. The characterization of elements with q_0 in the type implies that $(xg)\alpha \in K_{-1}[yh \exp(r)^n]$ has no constant term. We infer $(xg)\alpha = yh \exp(r)^n$ and $(XM)\alpha \subseteq (XM)\exp(r)^n$. Hence α induces an endomorphism $\hat{\alpha} \in \operatorname{End} \mathbf{V} = P[M]$ that *permutes up to the shift* r^n the elements of XM. This shows that $\hat{\alpha} = g_0 \exp(r)^n \in M \ltimes \operatorname{Frob}(F)$. $\square$

2.4. Sketch of the Proof of Theorem 2.5. The following lemma is the tool for realizing *automorphism groups* of fields. Its proof is given in [**8**], but we must replace the application of [**8**, Theorem 2.2 on p. 3781] by Corollary 2.8 and also use Lemma 2.11. The arguments are similar to the proof of Theorem 2.4. Thus details are left to the reader.

LEMMA 2.13. *Let K be a field and $G \subseteq \operatorname{Aut} K$ with $\Sigma \subseteq \mathbb{P}_K$ and $\max\{|K|, |G|\} \leq \lambda < \kappa(\omega)$ with the following properties.*

(a) *Let $L \subseteq K \subseteq \widehat{L}$ such that L is generated by elements with types in $\{\mathbb{P}_K\} \cup \Sigma$.*
(b) *There are seven primes $p = q_{-1} > q_0 > \cdots > q_5 \in \mathbb{P}_K \setminus \bigcup \Sigma$.*

Then there is an extension field F of K such that the following holds.

(1) *F is purely transcendental over K.*
(2) $\operatorname{Aut} F \restriction K = G$
(3) *There is $K \subseteq L_1 \subseteq F \subseteq \widehat{L}_1$ such that L_1 is generated over K by elements of type $\{q_0\}, \{q_1\}, \{q_2\}, \{q_3\}, \{q_4\}, \{q_5\}$.*
(4) *If $b \in F$ and $\tau_F(b) \in \{\mathbb{P}_K\} \cup \Sigma$, then $b \in K$.*
(5) *If $a \in K$, then $\tau_F(a) = \tau_K(a)$.*
(6) $|F| = \lambda$.

PROOF. The lemma follows by simplification of the proof of Theorem 2.4. See also Dugas-Göbel [**8**, pp. 3781 - 3783] for a similar version for the non-absolute case. Here is a sketch of the proof.

Let F be constructed as before (see Section 2.3). Recall that this construction is absolute. Moreover $G \subseteq \operatorname{Aut} K_1$ by letting M act homogeneously w.r.t. $(+, \cdot)$ on K_1, i.e., we canonically define

$$t_1 h = t_1 \text{ and } (\sqrt[p]{t_1 - 1})h = \sqrt[p]{t_1 - 1} \text{ and } (\sqrt[p]{(t_1 - xg)})h = \sqrt[p]{t_1 - xgh}$$

for all $xg \in XG$, $h \in G$ (and similarly defined for $i = 2, \ldots, 5$). Therefore $G \subseteq \operatorname{Aut} F$ and we have to show that $G = \operatorname{Aut} F$, so let $\alpha \in \operatorname{Aut} F$. With an argumentation similar to the one in Theorem 2.4, however dropping the Frobenius part of the map, it follows that $X_i\alpha \subseteq X_i$ for $1 \leq i \leq 4$. Using K_5 we then show that $xg\alpha \in K_{-1}[yh]$ and $q_0 \in \tau_{K_0}((xg)\alpha)$. The latter implies that $(xg)\alpha$ read in $K_{-1}[yh]$ has no constant term and so $xg\alpha = yh$ follows. We obtain an induced automorphism $\tilde{\alpha} \in \operatorname{Aut} \mathbf{V} = P[G]$ (see (2)) which permutes the elements in XG and so $\alpha \in G$ follows. Using Lemma 2.10 we verify the remaining properties. $\square$

2.4.1. *Proof of Theorem* 2.5. For convenience of the reader we recall the following short, but essential proof given in [**8**, p. 3783 - 3784].

Proof. Let $\mathbb{P}_K = \{p, q\} \cup \{q_n \mid n < \omega\}$. We may assume that $K = \widehat{K}$ is algebraically closed (because K is either finite or $|\widehat{K}| = |K|$). Let $\{x_g \mid g \in G\}$ be algebraically independent over K and choose q-root chains $\langle x_g \rangle_q = \{{}_\ell x_g \mid \ell < \omega\}$ as before with $({}_{\ell+1}x_g)^q = {}_\ell x_g$ and $x_g = {}_0 x_g$. Let $K' = K(\langle x_g \rangle_q \mid g \in G)$. Thus $h \in G$ acts naturally on K' by ${}_\ell x_g h = {}_\ell x_{gh}$ and $G \subseteq \operatorname{Aut} K'$: The field K' is generated by elements of type q and $\{\mathbb{P}_K\}$. By Lemma 2.13 there is an extension field K_0 of K' which is purely transcendental over K' and K' is an algebraic extension of the subfield K which is generated over K by elements of type $\{\mathbb{P}_K\}$ or $\{q_i\}$ (for $-1 \leq i \leq 5$). Moreover, $\operatorname{Aut} K_0 \restriction K' = G$. By repeated application of Lemma 2.13 we find an ascending chain $\{K_n \mid n < \omega\}$ of fields K_n and $F = \bigcup_{n<\omega} K_n$ is the desired field.

Now we claim $\operatorname{Aut} F = G$.

Clearly $G \subseteq \operatorname{Aut} F$ by the above. Conversely, let $\alpha \in \operatorname{Aut} F$. The field F is purely transcendental over K_n for each n and K_n is algebraic over a subfield L_n which is generated by elements of type in $\{\mathbb{P}_F\} \cup \{q_{6n+i} \mid 0 \leq i \leq 5\}$ and all elements of this type are in K_n. Thus $K_n\alpha = K_n$ for all $n < \omega$ and $\alpha \restriction K_{n-1} = g_n \in G$ for all $n < \omega$ by inductive application of Lemma 2.13. Comparing g_0 and g_n we choose any x_g and see that $x_{gg_0} = x_g g_0 = x_g \alpha = x_g g_n = x_{gg_n}$, thus $gg_0 = gg_n$ and $g_0 = g_n$ follows. Hence $\alpha = g_0 \in G$ and the claim follows.

The last claim also holds absolutely, because the construction is based on Corollary 2.7 which is an absolute statement, and the inductive steps are surely absolute constructions. The remaining condition $|\, F \,| = \lambda$ follows by the construction of F. $\square$

3. Absolutely rigid trees

Before we begin the construction of trees, we must clarify that all steps of the construction are carried out absolutely.

We will see that the proof of Shelah's central Lemma 3.11 only uses permitted steps as described in Burgess [**1**, pp. 408 - 412]. The colouring function F (below) exists by the assumption that $\alpha < \kappa(\omega)$. Moreover, in the remaining arguments we only use induction arguments or add 2 or 4 colours to the trees by another inductive process. Hence the proof is carried out absolutely as shown in Silver [**28**]. Formally we carry out the following construction in a (ground) model $\mathfrak{M}$ of set theory. In $\mathfrak{M}$ we will construct a rigid family $\mathcal{T}$ of trees. To show that $\mathcal{T}$ is absolutely rigid, we will consider any extension $\mathfrak{N}$ of the ground model $\mathfrak{M}$. The only places referring to the $\mathfrak{M}$-$\mathfrak{N}$-absoluteness proof are Lemma 3.7 and Theorem 3.13. Otherwise we can argue naively without any special reference to $\mathfrak{M}$, $\mathfrak{N}$.

3.1. Quasi-orders which are not narrow. A set $(\Omega, \leq)$ with a binary relation $\leq$ is a *quasi-order* (for short *qo*) if the relation is reflexive and transitive. Recall that $(\Omega, \leq)$ is a poset if $\leq$ is also anti-symmetric. In order to turn a quasi-order into a better quasi-order (for short a *bqo*), we will need the notion of a barrier.

Let λ be a cardinal and $A \subseteq \lambda$ a non-empty subset of ordinals, then

$$^{\omega>}A = \{\eta : n \longrightarrow A \mid n < \omega\}$$

is the tree $T_A = {}^{\omega>}A$ of all finite sequences (or *finite branches*) in A. Since $n = \{0, \dots n-1\}$, we can view $\eta = \eta(0)^\wedge \dots {}^\wedge\eta(n-1)$ as a sequence of length $\lg(\eta) = n$ with $\eta(i) \in A$ for all $i < n$. For η we further define $\eta^- := \eta(1)^\wedge \dots {}^\wedge\eta(n-1)$. As usual we have

$$\eta \trianglelefteq \eta' \iff \eta \subseteq \eta' \text{ as relation.}$$

The bottom element of the tree T_A is the empty sequence $\perp = \langle\ \rangle$ and $\eta = \eta' \restriction n \trianglelefteq \eta'$ with $\lg(\eta) = n$ is an initial segment of η' (if $n \leq \lg(\eta')$). The maximal linear subsets of T_A, the *infinite branches* of T_A, are denoted by $\mathrm{Br}(T_A)$. If $v : \omega \longrightarrow A$, then $\{v \restriction n \mid n < \omega\}$ is an infinite branch, and we can identify $\mathrm{Br}(T_A) = {}^\omega A = \{\eta : \omega \longrightarrow A\}$ and $\overline{T}_A = {}^{\omega\geq}A = T_A \cup \mathrm{Br}(T_A)$ is the tree of all branches of length $\leq \omega$. Let

$$\overline{T}^s_A = \{\eta \in {}^{\omega\geq}A \mid \eta(i) \neq \eta(i+1) \text{ for all } i < \lg(\eta)\}$$

be the *strict subtree* of $\overline{T}_A$ of all *strict branches*. In particular let $T^s_A = \overline{T}^s_A \cap T_A$ be the *strict subtree* of T_A. Similarly, let $T^\uparrow_A \subseteq T^s_A$ and $T^\downarrow_A \subseteq T^s_A$ denote the finite, strictly increasing (resp. decreasing) sequences with entries in A; often these sets are also denoted by ${}^{\omega>\uparrow}A$ or ${}^{\omega>\downarrow}A$ respectively. In the same way let $\mathrm{Br}(T^\uparrow_A) = \overline{T^\uparrow_A} \setminus T^\uparrow_A$ and observe that $\mathrm{Br}(T^\downarrow_A) = \emptyset$ by well-ordering.

If $\eta \in T_A$, then $[\eta] = \{\eta(i) \mid i < \lg(\eta)\} \subseteq A \subseteq \lambda$ is the *support* of η. This notion extends to subsets $B \subseteq T_A$: we have $[B] = \bigcup_{\eta\in B}[\eta] \subseteq A$. We are ready to define a barrier of T_A. If $B \subseteq T^s_A$, then clearly $B \subseteq T^s_{[B]}$, and we may assume that $A = [B]$.

Definition 3.1. *A subset $B \subseteq T^s_A$ is a* barrier of T_A *(with $A = [B]$) if the following holds.*

(1) *A and B are infinite.*
(2) *If $\eta \in \mathrm{Br}(T^s_A)$, then $\eta \restriction n \in B$ for some $n < \omega$.*
(3) *If $\eta, \nu \in B$ and $\eta \trianglelefteq \nu$, then $\eta = \nu$.*
(4) *If $\eta, \nu \in B$ and $\eta \trianglelefteq \nu^-$, then $\eta = \nu^-$.*

Note that $\perp \notin B$ by (i) and (iii). And B is called a *κ-barrier* if $[B] = \kappa$ for some cardinal $\kappa \leq \lambda$.

Next we show that T_A always has a (trivial) barrier.

Observation 3.2. *If A is infinite, then T_A has a barrier.*

PROOF. Define $B = \{\langle a\rangle \mid a \in A\}$. Clearly, B is infinite and for every $\eta \in \mathrm{Br}(T_A)$ we have $\eta \restriction 1 \in B$. Moreover, for all $\eta, \nu \in B$ we have that $\eta \trianglelefteq \nu$ implies $\eta = \nu$ and the case $\eta \trianglelefteq \nu^-$ cannot occur as $\nu^- = \perp$. □

Note that A cannot be finite, because in this case there is an infinite branch which must also belong to B, in contradiction with the Definition 3.1(i), (ii), and (iii) of a barrier. A simple direct example exhibiting this fact is the following

EXAMPLE 3.3. *Let* $A = \{0,1\}$. *Then* $T_A^s = S_0 \dot\cup S_1$ *with* $S_0 = \{\langle 0,1,0,1,\dots\rangle \restriction n \mid n < \omega\}$ *and* $S_1 = \{\langle 1,0,1,0,\dots\rangle \restriction n \mid n < \omega\}$. *Now suppose that B is a barrier of T_A. Since S_0 and S_1 are linearly ordered by $\trianglelefteq$, from Definition* 3.1 *(iii) follows* $|B \cap S_i| = 1$ *for* $i < 2$. *Therefore* $|\,B\,| = 2$ *which contradicts Definition* 3.1 *(i).*

Any barrier of a tree gives rise to a norm function, which we will need for our absolute arguments.

DEFINITION 3.4. *Let B be a κ-barrier of the tree T_A (as above). We define inductively a continuous ascending chain* $\bigcup_{\alpha<\kappa^+} D_\alpha = T_A^s$, *and define a norm*

$$\|\,\eta\,\| = \alpha \iff \eta \in D_{\alpha+1} \setminus D_\alpha.$$

Let $D_0 = \{\eta \in T_A^s \mid \exists \nu \in B \text{ such that } \nu \trianglelefteq \eta\}$ *and put* $\|\,\eta\,\| = 0$ *for all* $\eta \in D_0$. *If* $\alpha < \kappa^+$ *and* $D_\alpha \neq T_A^s$ *is defined, then let*

$$D_{\alpha+1} = \{\eta \in T_A^s \mid \eta^\wedge\langle a\rangle \in D_\alpha \text{ for all } a \in A \setminus \{\eta(\lg(\eta)-1)\}\}.$$

Recall that $\perp \notin B$, hence $\perp \notin D_0$ we get $0 < \|\perp\| = \sup\{\|\,b\,\| \mid b \in B\} < \kappa^+$ because $|\,B\,| = \kappa$. By the next lemma it follows $D_\alpha \neq D_{\alpha+1}$ for all $D_\alpha \neq T_A^s$, hence $\bigcup_{\alpha<\beta} D_\alpha = T_A^s$ for $\beta = \|\perp\| + 1 < \kappa^+$ as required.

LEMMA 3.5. *If $D_\alpha \neq T_A^s$ (in the last definition), then there is $\eta \in T_A^s \setminus D_\alpha$ with $\lg(\eta) = n$ such that $\|\,\eta^\wedge\langle a\rangle\,\| < \alpha$ for all $a \in A \setminus \{\eta(n-1)\}$, and the norm-function in Definition* 3.4 *is well-defined on all of T_A^s.*

PROOF. We may assume that $\alpha \geq 0$ and $D_\alpha \neq T_A^s$ and claim that

(6) $\exists \eta \in T_A^s \setminus D_\alpha$ such that $\|\,\eta^\wedge\langle a\rangle\,\| < \alpha$ for all $a \in A \setminus \{\eta(n-1)\}$ if $\lg(\eta) = n$.

Choose $\nu \in T_A^s \setminus D_\alpha$. If (6) does not hold, then there is $a_1 \in A$ with $a_1 \neq \nu(n-1)$ if $\lg(\nu) = n$ and $\eta_1 = \nu^\wedge\langle a_1\rangle \in T_A^s \setminus D_\alpha$. We continue inductively and get

$$\eta_n = \eta_{n-1}{}^\wedge\langle a_n\rangle \in T_A^s \setminus D_\alpha \text{ for all } n < \omega.$$

Thus we find an infinite branch

$$\eta' = (\nu^\wedge\langle a_1\rangle^\wedge\langle a_2\rangle \dots) \in \mathrm{Br}(T_A)$$

which is strict. By Definition 3.1(ii) there is $m < \omega$ such that $\eta' \restriction m \in B \subseteq D_0$. If $m \leq n$, then $\eta' \restriction m \trianglelefteq \nu$ and $\nu \in D_0$, a contradiction. If $m > n$, then also $\eta' \restriction m = \nu^\wedge\langle a_1\rangle^\wedge \dots \langle a_{m-1}\rangle \in D_0$ but $\eta' \restriction m \notin D_\alpha$ is also a contradiction. Hence (6) follows and the lemma holds. □

A subset A of a qo $(\Omega, \leq)$ is a weak anti-chain if for any $a, b \in A$ with $a \leq b$ follows $a = b$. (Recall that any anti-chain is a weak anti-chain.) We come to the basic

DEFINITION 3.6. *Let $(\Omega, \leq)$ be a qo, and λ be an infinite cardinal.*

(1) *The qo $(\Omega, \leq)$ is λ-narrow, if any weak anti-chain A of Ω satisfies $|A| < \lambda$.*
(2) *If B is a barrier of $A \subseteq \Omega$ (so $A = [B]$ and $B \subseteq T^s_A$), then $(\Omega, \leq)$ is a B-bqo (a B-*better quasi-order*), if for any enumeration $\{q_\eta \in \Omega \mid \eta \in B\}$ of a subset of Ω there are $\eta, \nu \in B$ such that $\eta^- \trianglelefteq \nu$, $\eta(0) \neq \nu(0)$ and $q_\eta \leq q_\nu$ (in $(\Omega, \leq)$).*

For the rest of this paper let $\omega^* = (\omega, =)$ be a quasi-ordered set which is the anti-chain on ω, hence for all $m, n < \omega$ we have $m \leq n$ if and only if $m = n$. It is the smallest quasi-ordered set which is not $\aleph_0$-narrow. Following Shelah [**27**], but for clarity only concentrating on this special case ω^*, we want to get by induction a continuous, ascending chain of quasi-ordered sets $\mathcal{P}_\alpha(\omega^*)$ for any ordinal α.

We consider the first step. If $(\Omega, \leq)$ is any quasi-ordered set, and A, B are subsets of $(\Omega, \leq)$, then as usual we write $A \leq B$ if $a \leq b$ for all $a \in A$ and $b \in B$ and say that A *is bounded by* B. If $\mathcal{P}(\Omega)$ is the power set of Ω, then we define a (new) order on $\mathcal{P}(\Omega)$ and let $A \leq B$ for $A, B \in \mathcal{P}(\Omega)$ if A is *weakly bounded* by B, i.e., there is a map $h : A \longrightarrow B$ such that $a \leq h(a)$ for all $a \in A$. Hence $(\mathcal{P}(\Omega), \leq)$ is quasi-ordered.

Let $\mathcal{P}_0(\omega^*) = \omega^*$. By continuity we only have to perform the inductive step: If $\mathcal{P}_\alpha(\omega^*)$ is defined, then let $\mathcal{P}_{\alpha+1}(\omega^*) = \mathcal{P}_\alpha(\omega^*) \dot\cup \mathcal{P}(\mathcal{P}_\alpha(\omega^*))$ as sets. If $A \in \mathcal{P}_\alpha(\omega^*)$, then let $[A]_{\omega^*} \subseteq \omega^*$ be the natural support of A over ω^*, i.e., all elements of ω^* in the transitive closure of A.

Given any α we must define the order on $\mathcal{P}_\alpha(\omega^*)$. Since $\mathcal{P}_0(\omega^*) = (\omega^*, =)$, we may assume $\alpha > 0$. If $A, B \in \mathcal{P}_\alpha(\omega^*)$ and α is a limit, then $A, B \in \mathcal{P}_\beta(\omega^*)$ for some $\beta < \alpha$, hence it remains to consider the case with A, B for $\alpha = \beta + 1$. Since $(\mathcal{P}_\beta(\omega^*), \leq)$ is given by induction, we may assume that not both A and B belong to $\mathcal{P}_\beta(\omega^*)$, respectively. Let $A \leq B$ in $\mathcal{P}_\beta(\omega^*) \dot\cup \mathcal{P}(\mathcal{P}_\beta(\omega^*))$ if one of the following three cases hold.

- A is weakly bounded by B in $\mathcal{P}_\beta(\omega^*)$.
- $B \in \omega^*$ and $[A]_{\omega^*} = \{B\}$.
- There is $A' \in B$ and $A \leq A'$.

Thus the following lemma is a particular case of [**27**, Claim 1.7].

LEMMA 3.7. *If $A, B \in \mathcal{P}_\alpha(\omega^*)$ for some ordinal α, then $A \leq B$ if and only if one of the following conditions hold.*

(1) *$A, B \notin \omega^*$ and for all $a \in A$ there is $b \in B$ such that $a \leq b$,*
(2) *$B \in \omega^*, A \notin \omega^*$ and $[A]_{\omega^*} = \{B\}$.*
(3) *$A \in \omega^*, B \notin \omega^*$ and $A \in [B]_{\omega^*}$.*
(4) *$A, B \in \omega^*$ and $A = B$.*

PROOF. The proof is by induction on α. Observe that the four cases are mutually exclusive and completely exhaustive. For $\alpha = 0$ there is nothing to prove as (iv) coincides with the original definition. Therefore consider $\alpha > 0$ and $A, B \in \mathcal{P}_\alpha(\omega^*)$. If α is a limit ordinal, then there exists $\beta < \alpha$ such that $A, B \in \mathcal{P}_\beta(\omega^*)$ and the claim follows by induction. Therefore assume that $\alpha = \beta + 1$ and let $A \leq B$. If $A, B \in \mathcal{P}_\beta(\omega^*)$ (which includes $A, B \in \omega^*$) the proof follows by induction. It remains to consider the case that not both A and B are elements of $\mathcal{P}_\beta(\omega^*)$.

By the definition of the quasi-order $\leq$ on $\mathcal{P}_\alpha(\omega^*)$ three cases can occur. In the first case A is weakly bounded by B in $\mathcal{P}_\beta(\omega^*)$, i.e., there exists a function

$h : A \to B$ such that for all $a \in A$ we have $a \leq h(a)$ which implies (i). Conversely, if (i) holds, we can define $h(a) := b$. In the second case, we have $B \in \omega^*$ and $[A]_{\omega^*} \leq B$ which is (ii). This brings us to the last case. Let $A' \in B$ such that $A \leq A'$. By well-foundedness this case can only occur finitely many times and we can apply the arguments from above with B being replaced by A'.

We finally verify that (iii) implies that $A \leq B$. Let $A \in \omega^*$ and $B \notin \omega^*$ such that $A \in [B]_{\omega^*}$. If $A \in B$ it follows $A \leq B$ by definition. If $A \notin B$ there exists $A_1 \in B$ with $A \in [A_1]_{\omega^*}$ and $A \leq B$ if $A \leq A_1$ by definition of the ordering. We repeat this argument and obtain a chain of the form

$$A \in A_m \in A_{m-1} \in \ldots \in A_1 \in B,$$

and therefore $A \leq B$ holds. □

REMARK 3.8. *The definition of $\leq$ in Lemma 3.7 is given in the ground model $\mathfrak{M}$. However using formally an induction on α it is clear that for $A, B \in \mathcal{P}_\alpha(\omega^*)$*

$$A \leq B \text{ holds in } \mathfrak{M} \iff A \leq B \text{ holds in } \mathfrak{N}.$$

See [**1, 23**] for examples.

DEFINITION 3.9. *The first* Erdős cardinal $\kappa(\omega)$ *is the smallest uncountable cardinal κ such that for every function f from the finite subsets of κ to 2 there exist an infinite subset $X \subset \kappa$ and a function $g : \omega \to 2$ such that $f(Y) = g(|Y|)$ holds for all finite subsets Y of X.*

The cardinal $\kappa(\omega)$ is known to be strongly inaccessible, hence one of the huge cardinals; see Jech [**20**, p. 303]. The above definition for $\kappa = \kappa(\omega)$ is often written as $\kappa \longrightarrow (\omega)_2^{<\omega}$. It is well-known that this condition can be replaced by the (formally) stronger statement that $\kappa \longrightarrow (\omega)_\chi^{<\omega}$ holds, where in Definition 3.9 the number 2 is replaced by any cardinal $\chi < \kappa(\omega)$, see [**3**, p. 239, Corollary 2.2] or [**20**]. By a result due to Silver [**28**, p. 96, 98] this equivalent to the next condition (7), also abbreviated as $\kappa \overset{w}{\to} (\omega)_\chi^{<\omega}$.

(7) $\forall F : T_\kappa^s \longrightarrow \chi \; \exists \eta \in {}^{\omega\uparrow}\kappa$ such that $F(\eta \restriction n) = F(\eta^- \restriction n)$ holds $\forall n < \omega$.

Thus the following Corollary 3.10 is obvious.

F-CONDITION 3.10. *If $\kappa < \kappa(\omega)$, then*

$$\exists F : T_\kappa^s \longrightarrow \chi \; \forall \eta \in \mathrm{Br}(T_\kappa^\uparrow) \exists n < \omega \text{ such that } F(\eta \restriction n) \neq F(\eta^- \restriction n).$$

We will now prove Shelah's [**27**] central lemma about $\mathcal{P}_\alpha(\omega^*)$ establishing that $\mathcal{P}_\alpha(\omega^*)$ is not κ-narrow for a suitable α, i.e., $\mathcal{P}_\alpha(\omega^*)$ does have a large weak anti-chain. The reader who wants to get motivated by the existence of absolutely rigid trees should first read Section 3.2 for its application and then return to the proof of

LEMMA 3.11. *If $\kappa < \kappa(\omega)$, then there is an ordinal α such that $\mathcal{P}_\alpha(\omega^*)$ is not κ-narrow.*

PROOF. By the F-Condition 3.10 there is a function $F' : T_\kappa^s \longrightarrow \omega$ such that for all strictly increasing sequences $\eta = (\eta(0) < \eta(1) < \cdots < \eta(l) < \ldots) \in \mathrm{Br}(T_\kappa^\uparrow)$ there exists $n < \omega$ with $F'(\eta \restriction n) \neq F'(\eta^- \restriction n)$. We may (w.l.o.g.) replace F' by a mapping

$$F : T_\kappa^s \longrightarrow \omega \times \omega \times 2 \; (\eta \mapsto (F'(\eta), \mathrm{lg}(\eta), o(\eta)),$$

where $o(\eta) = 1$ if $\eta(\lg(\eta) - 2) < \eta(\lg(\eta) - 1)$ and $o(\eta) = 0$ otherwise, describes partially the ordering on η. By assumption on F for any $\eta \in \mathrm{Br}(T_\kappa^\uparrow)$ there exists $n < \omega$ such that

$$F(\eta \restriction n) \neq F(\eta^- \restriction n). \tag{8}$$

This will be used to recover the strictly increasing sequences, which comprise a subtree $T_\kappa^\uparrow$ of T_κ^s. Inductively we define an increasing sequence of subsets $C^n \subseteq T_\kappa^s$ $(n < \omega)$ and put $C := \bigcup_{n<\omega} C^n$. Let $C^0 := \{\perp\} \subseteq C^1 := \{\perp, \langle\alpha\rangle \mid \alpha \in \kappa\}$ and define

$$C_{n+1} := \{\eta \in T_\kappa^s \mid \lg(\eta) = n+1, \eta \restriction n, \eta^- \in C^n \text{ and } F(\eta \restriction n) = F(\eta^-)\},$$

and put $C^{n+1} = C^n \dot\cup C_{n+1}$.

We claim that

$$C \subseteq T_\kappa^s \text{ is a subtree contained in } T_\kappa^\uparrow \cup T_\kappa^\downarrow \text{ and } C \text{ is closed under } (\eta \mapsto \eta^-). \tag{9}$$

Let $\eta \in C$, $\nu \in T_\kappa^s$ and $\nu \trianglelefteq \eta$. Then $\eta \in C^n$ for $n = \lg(\eta)$ and $\eta \restriction (n-1) \in C^{n-1}$ by definition of C. By induction it follows that $\nu \in C^m$ for some $m \leq n$. Hence C is a tree and similarly, we obtain that C is closed under $^-$. It remains to show that $C \subseteq T_\kappa^\uparrow \cup T_\kappa^\downarrow$. Suppose there is $\eta \in C$ which is neither strictly increasing nor strictly decreasing. Since $\eta \in T_\kappa^s$, there is $n < \lg(\eta)$ such that (w.l.o.g.) we have $\eta(n-1) < \eta(n)$ but $\eta(n+1) < \eta(n)$. Since C is a tree, it follows that $\eta' := \eta \restriction (n+2) \in C^{n+1}$ and application of F gives $F(\eta' \restriction (n+1)) = F((\eta')^-)$. In particular $o(\eta' \restriction (n+1)) = o((\eta')^-)$, hence $\eta(n-1) < \eta(n)$ implies the contradiction $\eta(n) < \eta(n+1)$. Thus η is either strictly increasing or strictly decreasing.

Next we show

$$\text{If } \eta \in \mathrm{Br}(T_\kappa^s) \text{ is a strict sequence, then } \{\eta \restriction n \mid n < \omega\} \nsubseteq C. \tag{10}$$

Suppose for contradiction that $\{\eta \restriction n \mid n < \omega\} \subseteq C$ for some strict $\eta \in \mathrm{Br}(T_\kappa^s)$, then η is strictly increasing or decreasing by (9). The second case is impossible because κ is well-ordered. Hence $\eta \in \mathrm{Br}(T_\kappa^\uparrow)$ and by definition of F, we get the equality $F(\eta \restriction n) = F(\eta^- \restriction n)$ which contradicts (8) so that (10) holds.

Now we use C to define a barrier, i.e.,

$$B := \{\eta \in T_\kappa^s \mid \eta \restriction (\lg(\eta) - 1) \in C \text{ and } \eta \notin C\} \text{ is a } \kappa\text{-barrier of } T_\kappa^s. \tag{11}$$

If $\eta \triangleleft \nu$ for $\eta \in T_\kappa^s$, $\nu \in B$ then $\nu \restriction (\lg(\nu) - 1) \in C$ by (11) and $\eta \trianglelefteq \nu \restriction (\lg(\nu) - 1)$ which implies $\eta \in C$. Thus $\eta \notin B$ by definition of B and (iii) of the Definition 3.1 holds. Now let $\eta \in B$. Hence $\eta \restriction (\lg(\eta) - 1), (\eta \restriction (\lg(\eta) - 1))^- \in C$. If $\nu \triangleleft \eta^-$, then $\nu \trianglelefteq \eta \restriction (\lg(\eta) - 1)$ and hence $\nu \in C$, thus $\nu \notin B$. Finally, if $\eta \in \mathrm{Br}(T_\kappa^s)$ then by (10) it follows $\eta \restriction k \notin C$ for some $k < \omega$, and if k is minimal, then $\eta \restriction k \in B$. For any $\alpha < \kappa$ we can now choose a sequence $\eta \in \mathrm{Br}(T_\kappa^s)$ with $\eta(0) = \alpha$. It follows that $\eta \restriction k \in B$ for some $k < \omega$, hence $\alpha \in \mathrm{Dom}(B)$ so that $[B] = \kappa$ follows. We conclude that B is a κ-barrier.

In the next step we show that

$$\omega^* \text{ is not a } B\text{-bqo}, \tag{12}$$

by providing a subset of ω^* that is not B-bqo. Recall that ω^* is not $\aleph_0$-narrow, in fact ω^* is a countable anti-chain. Also recall that $\lg(\eta) \in \omega^*$ can be obtained from F by projection onto the second component. For $\eta \in B$ we let $q_\eta = \lg(\eta) - 1$ and $\{q_\eta \mid \eta \in B\}$ represents an enumeration of a subset of ω^*. Suppose that (12) does not hold, so there are $\eta^- \trianglelefteq \nu$ in B with $\eta(0) \neq \nu(0)$ and $q_\eta = q_\nu$. It follows that $\lg(\eta) = \lg(\nu) =: n+1$. From $\eta, \nu \in B$ we obtain $\eta \restriction (\lg(\eta)-1), \nu \restriction (\lg(\nu)-1) \in C^n$. Note that $\eta(n) = \eta(\lg(\eta)-1) = \nu(\lg(\nu)-2) = \nu(n-1)$. Therefore, we have $\eta \restriction (\lg(\eta)-1) \in C^n$ and $\eta^- \in C^n$ and by definition of C^{n+1} we conclude $\eta = \eta \restriction (\lg(\eta)-1)^\wedge \nu(\lg(\nu)-1) \in C^{n+1} \subseteq C$, which contradicts $\eta \in B$, so (12) holds.

Finally we show

$$\text{If } \alpha = \|\perp\|, \text{ then } \mathcal{P}_\alpha(\omega^*) \text{ is not } \kappa\text{-narrow.} \tag{13}$$

Let $H = \{\eta \restriction k \mid \eta \in B, k \leq \lg(\eta)\}$. By induction on $\|\eta\|$ $(\eta \in H)$ we want to construct an enumeration $\{u_\eta \mid \eta \in H\} \subseteq \mathcal{P}_\alpha(\omega^*)$.

If $\|\eta\| = 0$, then $\eta \in B$. In this case we choose $u_\eta = q_\eta = \lg(\eta) - 1 \in \omega^* = \mathcal{P}_0(\omega^*)$ as above.

If $\|\eta\| > 0$, then let $u_\eta := \{u_{\eta^\wedge\langle i\rangle} : \eta^\wedge\langle i\rangle \in H\}$, which is well defined, as $\|\eta^\wedge\langle i\rangle\| < \|\eta\|$, hence $u_{\eta^\wedge\langle i\rangle}$ is already defined and $u_{\eta^\wedge\langle i\rangle} \in \mathcal{P}_{\|\eta^\wedge\langle i\rangle\|}(\omega^*) \subseteq \mathcal{P}_{\|\eta\|}(\omega^*)$. It follows that $u_\eta \subseteq \mathcal{P}_{\|\eta\|}(\omega^*)$.

Observe that $\{i \mid \langle i\rangle \in H\} = \kappa$ as B is a barrier for κ and hence contains an initial segment for all strict sequences $\nu \in \mathrm{Br}(T^s_\kappa)$. For $i < \kappa$ it follows that $u_{\langle i\rangle} \in \mathcal{P}_{\|\langle i\rangle\|}(\omega^*) \subseteq \mathcal{P}_{\|\perp\|}(\omega^*)$.

We claim that $A = \{u_{\langle i\rangle} \mid \langle i\rangle \in H\}$ is a weak anti-chain, thus

$$\forall\, i, j < \kappa, i \neq j \text{ follows } u_{\langle i\rangle} \not\leq u_{\langle j\rangle},$$

which establishes (13).

Assume for contradiction that there are $i, j < \kappa$ with $i \neq j$ and $u_{\langle i\rangle} \leq u_{\langle j\rangle}$. We will show that there exist $\eta_m, \nu_n \in B$ with $\eta_m^- \trianglelefteq \nu_n$ and $\eta_m(0) \neq \nu_m(0)$ such that $u_{\eta_m} \leq u_{\nu_n}$ with contradicts the fact that ω^* is not B-bqo. The construction is in two steps. First we inductively construct η_m and ν_n until $u_{\eta_m} < \omega^*$. We then extend ν_m to ν_n with $n \geq m$ so that $u_{\nu_n} \in \omega^*$ as well.

We define by induction on l the ordinals and $j(l)$ such that:

(1) $j(0) = j$;
(2) $\eta_l = \langle i, j, j(1), \ldots, j(l)\rangle \in H$, $\nu_l = \langle j, j(1), \ldots, j(l)\rangle \in H$;
(3) $u_{\eta_l} \leq u_{\nu_l}$.

We may assume that $l > 0$ and suppose that $j, j(1), \ldots, j(l)$ are defined and $\|\eta_l\| > 0$. Note that $\eta_{l+1} = \eta_l^\wedge\langle j(l)\rangle$. We have $\eta_l^- \vartriangleleft \nu_l$ and $\eta_l(0) = i \neq j = \nu_l(0)$ and hence $\eta_{l+1} = \langle i\rangle^\wedge\nu_l \in T^s_\kappa$. By definition of H and $\|\eta_l\| > 0$ we find $\eta^0 \in B$ such that $\eta_{l+1} \trianglelefteq \eta^0$. This implies that $\eta_{l+1} \in H$. By definition of u_η we obtain that $u_{\eta_{l+1}} \in u_{\eta_l}$. Now, (iv) together with Lemma 3.7 implies that either for some $u \in u_{\nu_l}$ we have that $u_{\eta_{l+1}} \leq u$ or $u_{\nu_l} \in \omega^*$.

In the first case, for some ordinal $j(l+1)$ we deduce $\nu_{l+1} := \nu_l^\wedge\langle j(l+1)\rangle \in H$ such that $u_{\nu_{l+1}} = u$ by definition of u_ν. The induction step is finished in this case.

In the second case, $u_{\nu_l} \in \omega^*$ implies $\|\nu_l\| = 0$. As just seen, for $\eta \in H$ there is $\eta^0 \in H$ such that $\|\eta^0\| = 0$ and $\eta \trianglelefteq \eta^0$. Choose such an $\eta^0_{l+1} \in H$ for η_{l+1}. Hence $\nu_l = \eta^-_{l+1} \trianglelefteq (\eta^0_{l+1})^-$. Recall that by definition of H for $\eta \in H$ and $\|\eta\| = 0$ it follows that $\eta \in B$. Hence $\nu_l \in B$ and $\eta^0_{l+1} \in B$ and as B is a κ-barrier we obtain $\nu_l = (\eta^0_{l+1})^-$ and so $\eta_{l+1} = \eta^0_{l+1}$. Clearly, $\eta^-_{l+1} \trianglelefteq \nu_l$ and $\eta_{l+1}(0) \neq \nu_l(0)$ and, by Lemma 3.7, we obtain that $u_{\eta_{l+1}} \leq u_{\nu_l}$ as $u_{\nu_l} \in \omega^*$ and $u_{\eta_{l+1}} \in u_{\eta_l}$. Together with $\eta_{l+1}, \nu_l \in B$, this contradicts the fact that ω^* is not a B-bqo by (12) and the second case cannot occur.

The constructed sequences η_l are finite: as $\eta_l \lhd \eta_{l+1}$ it follows that $\|\eta_{l+1}\| < \|\eta_l\|$ and so, eventually (by the well-ordering) we reach the case that $\|\eta_m\| = 0$ for some $m < \omega$. Hence, $u_{\eta_m} \in \omega^*$.

This finishes the construction of η_m but we have to ensure that ν_l leads to a final contradiction. For this we extend ν_l until we have $u_{\nu_n} \in \omega^*$ for some $n \geq l$. Hence by induction on $l \geq m$ with $\|\nu_l\| > 0$, we define $\nu_l \lhd \nu_{l+1} \in C$ such that $u_{\eta_m} \leq u_{\nu_{l+1}}$ which is possible by Lemma 3.7 as $u_{\eta_m} \leq u_{\nu_l}$ and $u_{\eta_m} \in \omega^*$. Eventually, for some $n < \omega$ we obtain $\|\nu_n\| = 0$ which finishes the construction. As before, this implies $u_{\nu_n} \in \omega^*$. Hence $u_{\eta_m} \leq u_{\nu_n}$ and $\eta_m^- \unlhd \nu_n$ with $\eta_m(0) \neq \nu_n(0)$, again contradicting the fact that ω^* is not B-bqo. This shows (13) and establishes the lemma. $\square$

3.2. Rigid families of trees. We want to construct an absolutely rigid family $\mathcal{T} = \{T_i \mid i < \lambda\}$ of trees T_i with $|T_i| = \lambda < \kappa(\omega)$ for tree embeddings. Again, the formal construction takes place in the fixed ground model $\mathfrak{M}$ of set theory.

This family will be enlarged to $|\mathcal{T}| = 2^\lambda$ and strengthened to be rigid with respect to homomorphisms in the last section. First we apply Lemma 3.11 and get a quasi-order $\mathcal{P}_\alpha(\omega^*)$ for $\alpha = \|\perp\|$ which is not λ-narrow. Let

$$A = \{t_i \in \mathcal{P}_\alpha(\omega^*) \mid i < \lambda\}$$

be the weak anti-chain given by the lemma. Recall that $\omega^* = \omega$ as a set. Let $[t_i]$ be the transitive closure (under $\in$) of t_i and choose disjoint subsets $S_\omega, S \subseteq \mathbb{N}$ where $S_\omega = \{3n+2 \mid n < \omega\}$ and $S = \{3n+1 \mid n < \omega\}$. Also decompose $S = \dot{\bigcup}_{i<\omega} S_i$ into infinite subsets S_i. Now we are ready to build the tree

$$T_i \subseteq {}^{\omega>}\{[t_i]\dot{\cup}\{\omega, \omega+1, \omega+2, \omega+3\}$$

on the root $\perp = t_i$, where $\{\omega, \omega+1, \omega+2, \omega+3\}$ are four fresh elements. We construct its branches by induction on $n < \omega$. Suppose that $n \geq 0$, and the branches of length $< n+1$ are constructed. Given any branch

$$\eta : n \longrightarrow \{[t_i]\dot{\cup}\{\omega, \omega+1, \omega+2, \omega+3\}$$

of length n, then ν is a successor of η of length $\lg(\nu) = n+1$ if $\nu(n)$ satisfies one of the following conditions.

- If $\eta(n-1) \notin \omega^*$ and $(n-1) \in S_\omega$, then $\nu(n) \in \eta(n-1)$.
- If $\eta(n-1) \notin \omega^*$ and $(n-1) \notin S_\omega$, then $\nu(n) = \eta(n-1)$
- If $\eta(n-1) \in \omega^*$ and $n-1 \notin S_{\eta(n-1)}$, then $\nu(n) = \eta(n-1)$.
- If $\eta(n-1) \in \omega^*$ and $n-1 \in S_{\eta(n-1)}$, then $\nu(n) \in \{\omega, \omega+1\}$.
- If $\eta(n-1) \in \{\omega, \omega+1\}$, then $\nu(n) \in \{\omega+2, \omega+3\}$.

This completes the construction of the family $\mathcal{T}$ of trees of size λ for any $\lambda < \kappa(\omega)$. Since $[t_i]$ is well-founded (by $\in$), their branches are finite and their heights are ω. Each maximal branch ends with a node in $\{\omega+2, \omega+3\}$. The final segment of the branches is illustrated in Figure 1.

A homomorphism between two trees is a map that preserves levels and initial segments, and an injective tree-homomorphism is also called a *tree-embedding.* Note that a homomorphism does not need to be injective or preserve $\not\trianglelefteq$.

If $T \in \mathcal{T}$ as above and $\eta \in T$, then we say that $\eta\restriction m$ is a branch point for $m < \lg(\eta)$ if there is $b \in \{[t_i]\dot{\cup}\{\omega, \omega+1, \omega+2, \omega+3\}$ such that $\eta\restriction(m+1) \neq (\eta\restriction m)^\wedge b$ in T. And we also say that $\eta\restriction m$ is a *twin branch point at m* (a TBP at m) if $\eta\restriction m$

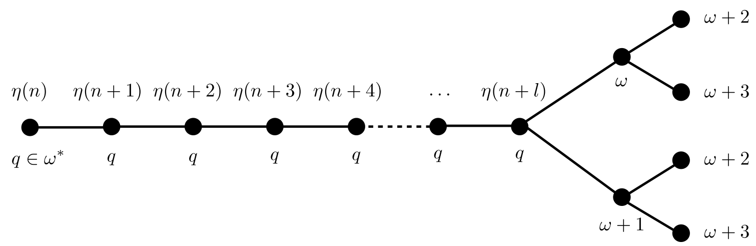

FIGURE 1. final-segment of a branch (here: $n+l \in S_q$)

and $\eta\restriction(m+1)$ are branch points. Inspecting the construction of the branches of T we observe,

(14) if $\eta \in T$ and η has a TBP at m, then $m \geq \lg(\eta) - 2$.

(15) Conversely, if η has a TBP at m, then $\lg(\eta) = m+2$ and η is maximal.

Typical TBPs can be seen from Figure 1. For the next, and final step of the proof we want to point out a general, and remarkable property of the trees (from $\mathcal{T}$) constructed at the beginning of this Section 3.2. Every branch of the trees in $\mathcal{T}$ is an initial segment of a maximal branch (described in Figure 1). By the shape of these maximal branches they are recognized by height and TBP. Let us summarize the immediate consequence. Any ascending sequence of branches of the trees $T \in \mathcal{T}$ terminates after finitely many steps; we say that T is a *noetherian* tree. And we also have the following

OBSERVATION 3.12. *Let T_i, T_j be as above and $\varphi : T_i \to T_j$ be a tree embedding, then φ maps maximal branches to maximal branches of the same length.*

Note that the trees used in connection with the Black Boxes like $T = {}^{\omega>}\kappa$ do not have any maximal branches at all, cf., [**2**].

THEOREM 3.13. *If $\lambda < \kappa(\omega)$ is an infinite cardinal, then the family of trees $\mathcal{T} = \{T_i \mid i < \lambda\}$ of size λ constructed above has the property that any tree-embedding $T_i \hookrightarrow T_j$ (with $i, j < \lambda$) in some generic extension of the universe forces $i = j$.*

We say that $\mathcal{T}$ is *absolutely rigid for embeddings.*

PROOF. Let T_i, T_j be as above with the roots t_i and t_j, respectively and let $\varphi : T_i \hookrightarrow T_j$ be the given tree-embedding, possibly in some extension $\mathfrak{N}$ of the ground model $\mathfrak{M}$ of set theory. We claim that

(16) $$\varphi\restriction\omega^* = \mathrm{id}_{\omega^*}\,.$$

Let $\eta = \eta(0)^\wedge \dots {}^\wedge\eta(n)$ be a maximal branch of length $n+1$ in T_i. By construction it follows that $\eta(n) \in \{\omega+2, \omega+3\}$ and $\eta(n-1) \in \{\omega, \omega+1\}$. Hence $n-2 \in S_{\eta(n-2)}$. Recall that $\bigcup_{i\leq\omega} S_i$ is a decomposition and thus there is a unique $m \in \omega^*$ such that $\eta(n-2) = m$. Hence the height $n+1$ of η corresponds uniquely to an element $m \in \omega^*$. Now we are going to use that φ preserved height and initial segments of η, thus Observation 3.12 and (14) hold. Hence $\varphi\restriction\omega^* = \mathrm{id}_{\omega^*}$ follows as claimed.

Finally we must show that $t_i \leq t_j$. Since t_i, t_j are members of a weak anti-chain of $\mathcal{P}_\alpha(\omega^*)$, it then follows $t_i = t_j$ and hence $i = j$ as claimed in the theorem. In order to show the final claim $t_i \leq t_j$, we apply recursively Lemma 3.7. Hence the claim is equivalent to say that for every sequence η satisfying

(17)
$$\eta = \langle \eta(0), \ldots, \eta(\ell)\rangle \wedge \eta(k+1) \in \eta(k) \in \mathcal{P}_\alpha(\omega^*)\ \forall k < \ell \text{ and } \eta(0) = t_i, \eta(\ell) \in \omega^*$$

there is a similar sequence ν satisfying

$$\nu = \langle \nu(0), \ldots, \nu(m)\rangle \text{ with } \nu(k+1) \in \eta(k)\ \forall k < m, \nu(0) = t_j, \nu(m) \in \omega^* \tag{18}$$

with

$$\eta(\ell) = \nu(m). \tag{19}$$

We want to convert the given sequence η from (17) into a maximal branch η^* of T_i and obtain $\eta^*\varphi \in T_j$. Then we will find a sequence ν such that the corresponding tree-element satisfies $\nu^* = \eta^*\varphi$. By the properties of φ we will show that (18) and (19) hold, thus $t_i \leq t_j$ and $i = j$ follows.

In the first step we now rewrite η as a maximal branch $\eta^* \in T_i$. By inspection of the branch-construction there is a unique maximal branch η^* essentially grown by duplicating the nodes of η and adding the top of a typical maximal branch:

$$\eta^* = \eta(0)^\wedge\eta(0)^\wedge\eta(0)^\wedge\eta(1)^\wedge\eta(1)^\wedge\eta(1)^\wedge\ldots{}^\wedge\eta(\ell)^\wedge\eta(\ell)^\wedge\eta(\ell)^\wedge\omega^\wedge(\omega+2) \in T_i.$$

Conversely we can reconstruct η uniquely from η^* by removing the copies of nodes adjoint to η^*. Now we apply φ and its properties discussed above (see also Observation 3.12). The maximal branch $\eta^*\varphi \in T_j$ has the same length $3\ell + 5$ as η^*, its first initial segment is t_j and the nodes of the two branches at $\lg(\eta^*) - 3$ satisfy $\eta^*(\lg(\eta^*)-3), \eta^*\varphi(\lg(\eta^*\varphi)-3) = \eta^*\varphi(\lg(\eta^*)-3) \in \omega^*$. By (16) now follows $\eta^*(\lg(\eta^*) - 3) = \eta^*\varphi(\lg(\eta^*) - 3) \in \omega^*$. Now we shrink $\eta^*\varphi$ as described for η above and get a sequence ν of length $\lg(\nu) = \lg(\eta) = \ell + 1$ such that (18) and (19) hold for $m = \ell$ and $\nu^* = \eta^*\varphi$. Hence $t_i \leq t_j$ in $\mathfrak{N}$, and by Remark 3.8 this also holds in $\mathfrak{M}$ as desired, which completes the proof of the theorem. $\square$

3.3. Absolute trees with 4 colours. If we want to replace the tree-embeddings of Theorem 3.13 by tree-homomorphisms, then we must add more structure, in this case colour to the trees. In fact we have to ensure Observation 3.12 now by an appropriate colouring.

For applications to algebra it is often desirable to use only as few colours as possible. We will show that four colours suffice, and we will eventually reduce it to two colours. Trivially, we can get the same implication when using more colour. For the present application it will suffice to have an infinite, but countable set of colours.

First, let $C = \{1^*, 2^*, 3^*, 4^*\}$ be a four element set, we say a set with four colours. The tree T is *coloured* by C, if with the tree T we have a colouring map $c : T \longrightarrow C$, which assigns a colour to each branch. To emphasize, that T is coloured by c, we often write (T, c). In the following a tree will always come with four colours and $\mathrm{Hom}_c(T_1, T_2)$ now denotes the colour-preserving tree-homomorphisms between trees (T_1, c_1) and (T_2, c_2). If φ is a tree-homomorphism, then φ is also *colour preserving*, if $c_2(\eta\varphi) = c_1(\eta)$ for all $\eta \in T_1$.

Let $\mathcal{T} = \{T_i \mid i < \lambda\}$ be the family of trees given by Theorem 3.13 for $\lambda < \kappa(\omega)$. We want to replace any T_i by a coloured tree (T_i, c_i) with $c_i : T_i \longrightarrow C$. For $T \in \mathcal{T}$ we define the colouring $c : T \longrightarrow C$ as follows. If $\eta = \eta(0)^\wedge\eta(1)^\wedge \dots {}^\wedge\eta(m) \in T$, then we colour η only by looking at the head of the branch η and let

$$c(\eta) = \begin{cases} 1^* & \textit{if } \eta(m) = \omega \textit{ or } [\eta(m-1) = \omega \textit{ and } \eta(m) = \omega + 2] \\ 2^* & \textit{if } \eta(m) = \omega + 1 \textit{ or } [\eta(m-1) = \omega \textit{ and } \eta(m) = \omega + 3] \\ 3^* & \textit{if } \eta(m-1) = \omega + 1 \textit{ and } \eta(m) = \omega + 2 \\ 4^* & \textit{otherwise} \end{cases} .$$

THEOREM 3.14. *If $\lambda < \kappa(\omega)$ and $\mathcal{T} = \{T_i \mid i < \lambda\}$ is the family of trees given by Theorem 3.13, then we add the above four colours $c_i : T_i \longrightarrow C$ such that in any generic extension of the universe the following holds for $i, j < \lambda$.*

$$\mathrm{Hom}_c((T_i, c_i), (T_j, c_j)) \neq \emptyset \Longrightarrow i = j.$$

PROOF. We will derive from the existence of a colour-preserving tree-homomorphism $\varphi : (T_i, c_i) \to (T_j, c_j)$ that $i = j$.

Any branch ν of T_i is the initial segment of a maximal branch $\eta \in T_i$. By (14) and (15) the branch η ends with a TBP at $m := \lg(\eta) - 2$. By the colouring of the two pairs of branches which describe the TBP and the colouring just defined, these pairs of branches carry *different colours* $(1^*, 2^*), (1^*, 4^*)$ and $(2^*, 4^*)$, respectively; see Figure 1 and the definition of $c(\eta)$. Since φ is a colour-preserving tree-homomorphism, the same holds for $\eta\varphi \in T_j$ and $\eta\varphi$ has a TBP at $\lg(\eta\varphi) - 2$. Hence $\eta\varphi$ is also maximal by (14) and (15). We derived the following claim.

(20) φ maps maximal branches of T_i to maximal branches of T_j the same length,

which is parallel to Observation 3.12, but φ need not be an injection. Now we can argue as in the proof of Theorem 3.13 and get $t_i = t_j$. □

3.4. Reducing colours to 2 and increasing $\mathcal{T}$ to 2^λ. Let $A = \{t_i \in \mathcal{P}_\alpha(\omega^*) \mid i < \lambda\}$ be the weak anti-chain given by Lemma 13 and used already in the proof of Theorem 3.13 and Theorem 3.14. Choose a family $\mathcal{F} \subseteq \mathcal{P}(A)$ such that $|\mathcal{F}| = 2^\lambda$ and for any distinct pair $X, X' \in \mathcal{F}$ we have $X \nsubseteq X'$ and $X' \nsubseteq X$. For each $X \in \mathcal{F}$ we construct a tree T_X as follows. Let T_X be defined as

$$T_X := \{\langle \perp_X, s\rangle \mid s \in T_i, t_i \in X\}.$$

Note that T_X is the union of trees T_i with $t_i \in X$ joint by a common bottom $\perp_X$.

OBSERVATION 3.15. *Let $\varphi : T_X \to T_Y$ with $X, Y \in \mathcal{F}$ a tree-homomorphism. Then $T_i\varphi \subseteq T_j$ for $t_i \in X$ and $t_j \in Y$.*

PROOF. Let $\langle \perp_X, s\rangle \in T_X$. Then $s \in T_i$ for some $t_i \in X$ and $\langle \perp_X, s\rangle \varphi = \langle \perp_Y, s'\rangle$. As φ is preserves levels and initial segments we obtain that $s' \in T_j$ for some $t_j \in Y$. Therefore $\langle \perp_X, T_i\rangle \varphi \subseteq \langle \perp_Y, T_j\rangle$ for $t_i \in X$ and $t_j \in Y$ as φ preserves initial segments and all other sequences $\tilde{s} \in T_i$ share a common bottom $\perp_i$ with s. □

We obtain the following corollaries

COROLLARY 3.16. *Let $\mathcal{T}^* = \{T_X \mid X \in \mathcal{F}\}$ the family of trees of size 2^λ with $\mathcal{F}$ from above. Then*

(1) *if there exists a tree-embedding* $\varphi : T \hookrightarrow T'$ *this implies* $T = T'$ *for* $T, T' \in \mathcal{T}^*$,
(2) $\mathrm{Hom}_c(T, T') \neq \emptyset$ *implies* $T = T'$ *for* $T, T' \in \mathcal{T}^*$.

PROOF. Suppose $\varphi : T_X \hookrightarrow T_Y$ is a tree-embedding with $X, Y \in \mathcal{F}$. Then $T_i\varphi \subseteq T_j$ for all $t_i \in X$ and $t_j \in Y$ by Observation 3.15. Now by Theorem 3.13 we obtain that $i = j$ whenever $T_i\varphi \subseteq T_j$ for $t_i \in X$ and $t_j \in Y$. Therefore $X \subseteq Y$ and as $X, Y \in \mathcal{F}$ we obtain $X = Y$.

The second claim follows similarly with Theorem 3.14. □

Finally we reduce the number of colours to 2. For this we simply change the colouring function to

$$c(\eta) = \begin{cases} 1^* & \textit{if } \eta(m) = \omega \textit{ or } [\eta(m-1) = \omega \textit{ and } \eta(m) = \omega + 2] \\ 2^* & \textit{if } \eta(m) = \omega + 1 \textit{ or } [\eta(m-1) = \omega \textit{ and } \eta(m) = \omega + 3] \\ 2^* & \textit{if } \eta(m-1) = \omega + 1 \textit{ and } \eta(m) = \omega + 2 \\ 1^* & \textit{otherwise} \end{cases}.$$

COROLLARY 3.17. *If* $\lambda < \kappa(\omega)$ *and* $\mathcal{T} = \{T_i \mid i < \lambda\}$ *is the family of trees given by Theorem* 3.13, *and we adjoin the colouring* $c_i : T_i \longrightarrow C$ *given above, then in any generic extension of the universe for* $i, j < \lambda$ *it holds*

$$\mathrm{Hom}_c((T_i, c_i), (T_j, c_j)) \neq \emptyset \Longrightarrow i = j.$$

PROOF. Consider a pair of branches which define a TBP and the colouring given above. Then these branches carry different colours $(1^*, 2^*), (1^*, 1^*)$ and $(2^*, 1^*)$. As before, we derive that TBPs are mapped to TBPs and with an analog argument as in Theorem 3.14 we conclude $\mathrm{Hom}_c((T_i, c_i), (T_j, c_j)) \neq \emptyset$ for $i, j < \lambda$ implies $i = j$. □

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A note on Hopfian and co-Hopfian abelian groups

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Abstract. The notions of Hopfian and co-Hopfian groups have been of interest for some time. In this present work we exploit some unpublished ideas of Corner to answer questions relating to such groups. In particular we extend an answer given by Corner to a problem of Beaumont and Pierce and show how the properties may be lifted from subgroups to the whole group in certain situations.

1. Introduction

The classes of groups which are today called Hopfian and co-Hopfian groups were first studied by Baer [**B**], under the names Q-group and S-group. In modern terminology we say that a group G is **Hopfian** if every surjection $G \to G$ is an automorphism; it is said to be **co-Hopfian** if every injection $G \to G$ is an automorphism. Finite groups are, of course, the prototypes for both Hopfian and co-Hopfian groups. The existence of infinite co-Hopfian p-groups was first established by Crawley [**Cr**]. Hopfian and co-Hopfian groups have arisen recently in the study of algebraic entropy and its dual, adjoint entropy – see e.g. [**DGSZ, GG**]. Despite the seeming simplicity of their definitions, Hopfian and co-Hopfian groups are notoriously difficult to handle, for example, it is still not known whether the direct sum of two co-Hopfian groups which are not torsion-free, is co-Hopfian.

Our motivation for this work arose from some unpublished work of the late A.L.S. Corner, which we have adapted and extended to use in the context of Hopfian and co-Hopfian groups. In the first section we quickly review some standard results and consider the question of when subgroups inherit the Hopfian or co-Hopfian properties. We show, under a suitable simple condition, that the properties 'lift' from certain subgroups to the whole group; our argument is based on a result which may be of independent interest and utilizes arguments reminiscent of those used by Pierce [**P**] in his seminal work on homomorphism groups. In the final section we utilize an idea from an unpublished paper of A.L.S. Corner which answered a conjecture of Beaumont and Pierce [**BP**], to exhibit, without assuming (CH), mixed Hopfian and co-Hopfian groups with torsion subgroups of arbitrary cardinality $\lambda \leq 2^{\aleph_0}$. Moreover, the groups are the extension of a non-Hopfian (non-co-Hopfian) group by a non-Hopfian (non-co-Hopfian) group.

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The word group shall normally mean an additively written Abelian group; the books [**F**] shall serve as a reference to ideas needed in Abelian group theory. We shall denote the set of primes by the symbol $\mathbb{P}$.

2. Lifting Hopficity (co-Hopficity) from Subgroups

We begin by recording some well-known facts of Hopfian and co-Hopfian groups.

Definition 2.1. A group G is said to be **Hopfian** if every surjection $G \to G$ is an automorphism; it is said to be **co-Hopfian** if every injection $G \to G$ is an automorphism.

It is easy to show that the Hopfian property for G is equivalent to G having no proper isomorphic factor group, while co-Hopficity is equivalent to having no proper isomorphic subgroup. The groups $\mathbb{Z}$ and $\mathbb{Z}(p^\infty)$ show that the notions are independent of each other.

Proposition 2.2. *The following record some well-known and easily established properties about Hopficity and co-Hopficity:*

(1) *A torsion-free group is co-Hopfian if and only if it is divisible of finite rank; thus it is isomorphic to a finite dimensional* $\mathbb{Q}-$*space.*
(2) *A torsion-free group of finite rank is Hopfian;*
(3) *Finitely generated groups are Hopfian and dually, finitely co-generated groups are co-Hopfian;*
(4) *A group* G *with endomorphism ring* $\mathrm{End}(G) \cong \mathbb{Z}$ *is Hopfian; thus arbitrarily large Hopfian groups exist;*
(5) *Reduced Hopfian (co-Hopfian) p-groups are semi-standard and so have cardinality at most* $2^{\aleph_0}$*;*
(6) *A reduced countable Hopfian (co-Hopfian) p-group is finite.*

The classes of Hopfian and co-Hopfian groups exhibit some weak closure properties which are well known:

Proposition 2.3. *Let* $0 \to H \to G \to K \to 0$ *be an exact sequence.*

(1) *If* H, K *are both Hopfian and if* H *is left invariant by each surjection* $\phi : G \to G$*, then* G *is Hopfian. In particular, extensions of Hopfian torsion groups by torsion-free Hopfian groups are again Hopfian;*
(2) *If* H, K *are both co-Hopfian and if* H *is left invariant by each injection* $\psi : G \to G$*, then* G *is co-Hopfian. In particular, extensions of co-Hopfian torsion groups by torsion-free co-Hopfian groups are again co-Hopfian.*

Proof. The proofs of the two statements are essentially dual, so we present only the proof of the Hopfian property. Let $\phi : G \to G$ be a surjection, then by assumption, $H\phi \leq H$ and so we get an induced map $\bar{\phi} : G/H \to G/H$ giving the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r] \arrow[d, "\phi\restriction H"] & G \arrow[r, "\beta"] \arrow[d, "\phi"] & K \arrow[r] \arrow[d, "\bar{\phi}"] & 0 \\
0 \arrow[r] & H \arrow[r] & G \arrow[r, "\beta"] & K \arrow[r] & 0
\end{tikzcd}$$

Since ϕ is onto, $\bar{\phi}$ is onto and so K, being Hopfian, gives that $\bar{\phi}$ is an automorphism. If we show that $\phi \restriction H : H \to H$ is onto, then as H is Hopfian, $\phi \restriction H$ will also be an automorphism and the result will follow by an appeal to the "Five Lemma". However the fact that $\phi \restriction H$ is onto follows immediately from the commutativity of the first square of the diagram above. $\square$

The following example, which provided the first examples of unbounded Hopfian and co-Hopfian p-groups, will be useful.

EXAMPLE 2.4. If B is a standard basic p-group and G is a pure subgroup of the torsion-completion $\bar{B}$ of B with $\mathrm{End}(G) = J_p \cdot 1_G \oplus E_s(G)$, where $E_s(G)$ is the ideal of small endomorphisms, then G is both Hopfian and co-Hopfian.

PROOF. The details of this result are contained in Section 16 of Pierce's fundamental work [**P**]. The critical part of his argument is that a group with this type of endomorphism ring does exist; other proofs using variations of a realization theorem due to Corner [**C**] are possible. So assume such a group exists and suppose that $\psi = r + \theta$, where $r \in J_p$ and $\theta \in E_s(G)$, is a monic (epic) endomorphism of G, then using Lemma 16.1 in [**P**], we conclude that r must be a p-adic unit. Moreover, Lemma 16.3 of [**P**] shows that if $B = \bigoplus_{n=1}^{\infty} B_n$, where each B_n is a direct sum of cyclic groups of order p^n, then there are decompositions $G = B_1 \oplus \cdots \oplus B_m \oplus H_m = B_1 \oplus \cdots \oplus B_m \oplus H_m\psi$, where $H_m = \langle p^m G, B_{m+1}, \ldots \rangle$. Now, if ψ is monic then $G\psi/H_m\psi \cong G/H_m \cong G/H_m\psi$ and so, since G/H_m is finite, we have $G = G\psi$ and ψ is an automorphism. If ψ is epic, then $G/(H_m + \mathrm{Ker}\psi) \cong G/H_m\psi \cong G/H_m$ and again finiteness yields that $H_m + \mathrm{Ker}\psi = H_m$. Since r is a unit, it follows that H_m and $\mathrm{Ker}\psi$ are disjoint, whence $\mathrm{Ker}\psi = 0$ and ψ is again an automorphism. $\square$

If G is a Hopfian (co-Hopfian) group then it is easy to see that subgroups of G do not necessarily inherit this property: for example, if G is an unbounded group which is both Hopfian and co-Hopfian as in Example 2.4, then a basic subgroup of G is an unbounded direct sum of cyclic groups and hence is neither Hopfian nor co-Hopfian. However, we do have:

PROPOSITION 2.5. *If G is Hopfian (co-Hopfian), then, for each natural number n, the subgroup nG is Hopfian (co-Hopfian).*

PROOF. If $\phi : nG \to nG$ is epic (monic), then it follows from the proof of Proposition 113.3 in [**F**], that there exists an epic (monic) $\psi : G \to G$ such that $\psi \restriction nG = \phi$. Since G is Hopfian (co-Hopfian), ψ must be an automorphism and hence its restriction to nG is also an automorphism, i.e. ϕ is an automorphism. $\square$

The converse of Proposition 2.5 is not true in general: consider the p-group G constructed by Pierce in [**P**, Theorem 16.4] which is both Hopfian and co-Hopfian - see Example 2.4 above. It follows from the last proposition that pG is also both Hopfian and co-Hopfian. Now set $H = (\bigoplus_{\aleph_0} \mathbb{Z}(p)) \oplus G$; clearly H is neither Hopfian nor co-Hopfian but $pH = pG$ has both properties.

We now show that, under a suitable restriction, a converse to Proposition 2.5 may be obtained. The key result is derived using arguments similar to those used by Pierce in [**P**] and may be of some independent interest.

First we make an *ad hoc* definition first used by Corner in unpublished work: an endomorphism ϵ of the group G is said to be a **q-map** if $q(\epsilon - \alpha) = 0$ for some automorphism α of G.

THEOREM 2.6. *If G is a group which has no q-bounded pure subgroup of infinite rank and qG is Hopfian (co-Hopfian), then G is Hopfian (co-Hopfian).*

Proof. Let ϕ be an epic (monic) endomorphism of G. Then $\phi \restriction qG$ is an epic (monic) endomorphism of qG and hence is an automorphism of qG. It follows from [**F**, Proposition 113.3] that there is an automorphism ψ (say) of G such that $\psi \restriction qG = \phi \restriction qG$. Consequently $q(\psi - \phi) = 0$ and ϕ is a q-map of G. The result now follows immediately from Theorem 2.7 below. $\square$

Theorem 2.7. *Suppose G is a group which has no nonzero q-bounded pure subgroup for some integer $q = p_1^{k_1} p_2^{k_2} \dots p_t^{k_t}$. Then if $\epsilon : G \to G$ is either monic or epic and a q-map, then ϵ is an automorphism of G.*

Proof. We remark at the outset that there is no loss in generality in assuming that $q\epsilon = q1_G$: since ϵ is a q-map, there is an automorphism α with $q\epsilon = q\alpha$, then simply replace ϵ by $\epsilon\alpha^{-1}$ and note that ϵ is epic (monic) if, and only if, $\epsilon\alpha^{-1}$ has the same property. We consider the three possibilities for G, i.e. G is torsion-free, torsion or mixed. If G is torsion-free, then $\epsilon = 1_G$ and hence is an automorphism. Suppose then that G is torsion and let $G_i (1 \leq i \leq t)$ denote the p_i-primary component of G. Then $G = (\bigoplus_{i=1}^t G_i) \oplus G_0$ for some complement G_0 having no p_i-primary component $(1 \leq i \leq t)$. Now each G_i is left invariant by ϵ. Moreover, the assumption that G has no nonzero q-bounded pure subgroup means that the first k_i Ulm invariants of each G_i vanish. Then, it follows from Proposition 2.8 below that $\epsilon \restriction G_i$ is an automorphism of G_i. Clearly ϵ acts as the direct sum of these restrictions and hence is an automorphism.

Finally, suppose G is mixed with torsion subgroup T. From the last paragraph it follows that $\epsilon \restriction T$ is an automorphism of T. Moreover the induced mapping $\bar{\epsilon}$ on G/T is also an automorphism as noted above, since G/T is torsion-free. It follows immediately from the "Five Lemma" that ϵ is an automorphism of G. $\square$

The final step in the proof of Theorem 2.7 is completed by the following more general result:

Proposition 2.8. *Let G be a p-group such that the first r Ulm invariants $f_G(0), f_G(1), f_G(2), \dots, f_G(r-1)$ are finite and let $B = \bigoplus_{i=1}^\infty B_i$, where each B_i is a direct sum of cyclic groups of order p^i, be a basic subgroup of G. If $\epsilon : G \to G$ is monic (respectively epic) such that $p^r\epsilon = p^r 1_G$, then ϵ is an automorphism of G.*

Proof. Let $G = B_1 \oplus \cdots \oplus B_r \oplus H_r$, where $H_r = \{p^r G, B_{r+1}, B_{r+2}, \dots\}$; note that $B_1 \oplus \cdots \oplus B_r$ is then finite.

(i) *Claim that $\epsilon \restriction H_r$ is monic.* Observe firstly that $\epsilon \restriction H_r[p]$ is the identity map since $H_r[p] \leq p^r G$. Hence if $0 \neq x \in H_r \cap \mathrm{Ker}\epsilon$, there is a $k \geq 0$ such that $0 \neq p^k x \in H_r[p]$ and this would lead to the contradiction that $0 = p^k(x\epsilon) = (p^k x)\epsilon = p^k x \neq 0$.

(ii) *Claim that $H_r\epsilon$ is pure in G.* It suffices to check this on the socle, so suppose that $x \in H_r\epsilon \cap G[p] \cap p^n G$ for an arbitrary n. Then $x = y\epsilon$ for some $y \in H_r$. Since ϵ is monic on H_r, then $px = 0$ implies that $py = 0$ and so $y \in H_r[p]$. It follows from (i) above that $x = y\epsilon = y \in H_r[p]$. Thus $x \in H_r \cap p^n G = p^n H_r$ and so $x = p^n z$ for some $z \in H_r$. It now follows that $x = x\epsilon = p^n z\epsilon$ and so $x \in p^n(H_r\epsilon) \cap G[p]$ as required.

(iii) *Claim that $H_r[p] = H_r\epsilon[p]$.* It follows from (i) that the LHS of the above is contained in the RHS. So suppose that $z \in H_r\epsilon[p]$. Then $z = x\epsilon$ for some $x \in H_r$ and $0 = pz = (px)\epsilon$. Since ϵ is monic on H_r, then it is immediate that $x \in H_r[p]$, and then by (i) above, we have $z = x\epsilon = x \in H_r[p]$.

It follows from (ii) and (iii) that H_r and $H_r\epsilon$ are pure subgroups of G having equal socles. Since H_r is a summand, it follows from a result of Irwin and Walker [**IW**, Theorem 16], that $H_r\epsilon$ is also a summand and that it has a common complement to H_r. Thus $G = H_r \oplus X = H_r\epsilon \oplus X$ where $X \cong B_1 \oplus \cdots \oplus B_r$; as observed above, X is finite.

If ϵ is monic, it induces an isomorphism $G/H_r \cong G\epsilon/H_r\epsilon$. Moreover $G/H_r \cong G/H_r\epsilon$, and so both of these groups are then finite. It follows immediately that $G = G\epsilon$ and ϵ is an automorphism, as required. If ϵ is epic, then the mapping $x \mapsto x\epsilon + H_r\epsilon$ has kernel $H_r + \mathrm{Ker}\epsilon$ and so there are isomorphisms $G/H_r + \mathrm{Ker}\epsilon \cong G/H_r\epsilon \cong G/H_r$. Again all the groups are finite and so $H_r + \mathrm{Ker}\epsilon = H_r$. However if $x \in \mathrm{Ker}\epsilon[p]$, then $x \in H_r[p]$ and so from (i) above, $0 = x\epsilon = x$. Thus $\mathrm{Ker}\epsilon = 0$ and ϵ is again an automorphism. $\square$

3. Mixed groups

We now switch our attention to mixed Abelian groups. Our approach in this section is heavily influenced by an unpublished paper of Corner answering a conjecture of Beaumont and Pierce – this is [U16] in [**G**]. We proceed via a series of steps starting with an arbitrary unbounded semi-standard (not necessarily separable) reduced p-group T. Firstly we determine the structure of the quotient A/T, where A is the cotorsion-completion of T and then we construct a subgroup H of the group of p-adic integers J_p with $\mathrm{End}(H) = \mathbb{Z}_p$, the integers localized at the prime p. Finally we use A and H to construct, via a suitable pullback, a mixed group G which will be both Hopfian and co-Hopfian.

(1) So suppose that T is an arbitrary unbounded semi-standard (not necessarily separable) reduced p-group and let $A = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$. Then A is the cotorsion-completion, $T^\bullet$, of T and $A/T \cong \mathrm{Ext}(\mathbb{Q}, T)$ is torsion-free divisible. We claim $A/T \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q}$.

To establish the claim note that if B is a basic subgroup of T, then B is pure and dense in T and so there is an epimorphism $\mathrm{Ext}(\mathbb{Q}, B) \twoheadrightarrow \mathrm{Ext}(\mathbb{Q}, T)$. Moreover, B is, by Szele's Theorem, an epimorphic image of T and so there is an epimorphism $\mathrm{Ext}(\mathbb{Q}, T) \twoheadrightarrow \mathrm{Ext}(\mathbb{Q}, B)$. Since $\mathbb{Q}$ is torsion-free and divisible, both $\mathrm{Ext}(\mathbb{Q}, B)$ and $\mathrm{Ext}(\mathbb{Q}, T)$ are $\mathbb{Q}$-vector spaces and the existence of the above epimorphisms ensures that they are of the same dimension. Hence $\mathrm{Ext}(\mathbb{Q}, B) \cong \mathrm{Ext}(\mathbb{Q}, T)$; furthermore, the algebraic compactness of $\hat{B}$ implies that $\mathrm{Ext}(\mathbb{Q}, B) \cong \mathrm{Hom}(\mathbb{Q}, \hat{B}/B)$. We remark that the exact structure of the quotient $\hat{B}/B$ may be computed but it is not essential for our purposes here: it suffices to note that as it is divisible, it has the form $\hat{B}/B \cong \bigoplus_\lambda \mathbb{Q} \oplus \bigoplus_\kappa \mathbb{Z}(p^\infty)$, where $\max\{\lambda, \kappa\} = 2^{\aleph_0}$. Thus $\mathrm{Hom}(\mathbb{Q}, \hat{B}/B) \cong \mathrm{Hom}(\mathbb{Q}, \bigoplus_\lambda \mathbb{Q}) \oplus \mathrm{Hom}(\mathbb{Q}, \bigoplus_\kappa \mathbb{Z}(p^\infty))$. The first term is easily seen to be isomorphic to $\bigoplus_\lambda \mathbb{Q}$. Now $\mathrm{Hom}(\mathbb{Q}, \bigoplus_\kappa \mathbb{Z}(p^\infty))$ is torsion-free divisible since so is $\mathbb{Q}$, and it has cardinality $\leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ since $\kappa \leq 2^{\aleph_0}$. However, $\mathbb{Z}(p^\infty)$ is an epimorphic image of $\mathbb{Q}$ and so $\mathrm{Hom}(\mathbb{Q}, \bigoplus_\kappa \mathbb{Z}(p^\infty))$ also contains a subgroup isomorphic to $\mathrm{Hom}(\mathbb{Z}(p^\infty), \bigoplus_\kappa \mathbb{Z}(p^\infty))$; this latter is isomorphic to $\widehat{\bigoplus_\kappa J_p}$ and hence has cardinality $\geq 2^{\aleph_0}$. Thus $\mathrm{Hom}(\mathbb{Q}, \bigoplus_\kappa \mathbb{Z}(p^\infty)) \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q}$ and in any event we have that $\mathrm{Hom}(\mathbb{Q}, \hat{B}/B) \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q}$ and so $A/T \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q}$, as claimed.

Note that if $\phi : A \to T$ is any homomorphism, then the image $A\phi$ is both cotorsion and torsion, and hence is bounded [**F**, Corollary 54.4].

(2) Let H be a maximal pure subgroup of J_p containing $\mathbb{Z}_p$, then H has cardinality $2^{\aleph_0}$ and $J_p/H \cong \mathbb{Q}$. Since every endomorphism of H extends to an endomorphism of J_p, it must be multiplication by a p-adic integer. Moreover, this multiplication must induce an endomorphism on the quotient $J_p/H \cong \mathbb{Q}$, and so it must be both a p-adic integer and a rational integer i.e. it is in $\mathbb{Z}_p$. Conversely since J_p is q-divisible for all primes $q \neq p$, any multiplication by an element of $\mathbb{Z}_p$ is an endomorphism of H.

Thus we have a pure subgroup H of the group of p-adic integers J_p such that H contains the subgroup $\mathbb{Z}_p$ of integers localized at p and $\mathrm{End}(H) = \mathbb{Z}_p$. Moreover, as H has rank $2^{\aleph_0}$, we have that $H/\mathbb{Z}_p$ is torsion-free divisible of rank $2^{\aleph_0}$.

(3) We now use the groups A and H constructed above to construct a mixed group G.

The groups A/T and $H/\mathbb{Z}_p$ are isomorphic, fixing such an isomorphism we form the pullback of A and H with kernels T and $\mathbb{Z}_p$. The resulting group G is a subgroup of the direct sum $A \oplus H$ and satisfies

$$G/T \cong H, \ G/\mathbb{Z}_p \cong A.$$

Since G/T is torsion-free, T is the torsion subgroup of G. Note that G/T is *reduced* in this case.

We claim that the group G so constructed is both Hopfian and co-Hopfian.

To see this, suppose that $\epsilon : G \to G$ is any *monomorphism* (respectively *epimorphism*) of G; we shall show that ϵ is an automorphism of G. Now, ϵ induces a mapping $\bar{\epsilon}$ of $G/T \cong H$ and so $\bar{\epsilon}$ is multiplication by a rational n/m where n, m are coprime and $p \nmid m$. Thus, $m\epsilon - n1_G$ induces the zero map on G/T and so $G(m\epsilon - n1_G) \leq T$, in particular $\mathbb{Z}_p(m\epsilon - n1_G) \leq T$. However, every homomorphic image of $\mathbb{Z}_p$ in T is cyclic, and so bounded, so there exists an integer $r \geq 0$ such that $p^r\mathbb{Z}_p(m\epsilon - n1_G) = 0$. Thus, the endomorphism $p^r(m\epsilon - n1_G)$ of G annihilates $\mathbb{Z}_p$ and so passes to the quotient inducing a map: $G/\mathbb{Z}_p \cong A \to T$; as noted above the choice of A means this image is also bounded. So replacing r by a larger integer if necessary, we may suppose that this image is zero i.e. $p^r(m\epsilon - n1_G) = 0$.

We consider firstly the case where ϵ is monic. Since T is unbounded there is a cyclic summand $\langle x \rangle$ of T of order greater than p^r; say $O(x) = r + s$. Thus, $p^r m(p^{s-1}x) \neq 0$ and so $p^r n(p^{s-1}x) = p^r m(p^{s-1}x)\epsilon \neq 0$ as ϵ is monic. Hence, $p \nmid n$ and n/m is a unit in $\mathbb{Z}_p$.

If ϵ is epic, then note that since T is reduced and unbounded, $p^rT > p^{r+1}T$. However, as ϵ is onto and $p \nmid m$, $(p^rT)m\epsilon = p^rT$, but on the other hand $(p^rT)m\epsilon = p^rnT$ and we deduce immediately that $p \nmid n$ and n/m is a unit in $\mathbb{Z}_p$.

Thus, in either case, the endomorphism induced on G/T by ϵ, is in fact an automorphism. To prove that ϵ is an automorphism of G, it is enough, by the Five Lemma, to prove that ϵ induces an automorphism of T. Moreover, as multiplication by n/m effects an automorphism of T, $n/m(\epsilon \restriction T)$ is a monomorphism (respectively an epimorphism) $T \to T$ and it is enough to prove it is an automorphism of T. In other words we may restrict attention to the case $n = m = 1$.

Then $\epsilon : T \to T$ is a monomorphism (respectively an epimorphism) such that $p^r(\epsilon - 1_T) = 0$ for some $r \geq 1$. We show that ϵ is an automorphism of T. (When T is the torsion-completion of a standard basic group, it is possible to give a fairly direct element-wise proof of this fact. To the best of our knowledge, this was first done for monomorphisms by A.L.S. Corner in 1962 in an unpublished paper ([U16] in [**G**]) answering a conjecture of Beaumont and Pierce – see the conjecture before

Example 1, p218 in [**BP**].) This, however, is immediate from Theorem 2.7 above, since we have assumed that T is semi-standard.

Summarizing the above, we have established:

THEOREM 3.1. *If T is an arbitrary semi-standard unbounded reduced p-group, then there is a mixed Abelian group G, which is both Hopfian and co-Hopfian, and which satisfies*

(1) *T is the torsion subgroup of G;*
(2) *the quotient G/T is isomorphic to a pure subgroup H, of cardinality $2^{\aleph_0}$, of the group of p-adic integers J_p (and, in particular, is reduced).*

Our first corollary answers negatively a conjecture of Beaumont and Pierce [**BP**] mentioned above. Note that in their terminology an I-group is precisely a group which is *not* co-Hopfian.

COROLLARY 3.2. *For any infinite cardinal $\lambda \leq 2^{\aleph_0}$, there exists a mixed Abelian group G with torsion subgroup T of cardinality λ, such that G/T is reduced and G is co-Hopfian (and hence not an I-group) and Hopfian.*

PROOF. Let B be a semi-standard p-group so that the quotient $D = \bar{B}/B \cong \bigoplus_{2^{\aleph_0}} \mathbb{Z}(p^\infty)$. Let T be the pre-image in $\bar{B}$ of a subgroup D_1 of D such that $D_1 \cong \bigoplus_\lambda \mathbb{Z}(p^\infty)$. Then T is a semi-standard separable p-group of cardinality λ. So, by Theorem 3.1, there is a co-Hopfian group G with $t(G) = T$ and G/T is a reduced subgroup of J_p. $\square$

The mixed groups we have constructed, in contrast to Proposition 2.3, can have the property that they are the extension of a non-co-Hopfian fully invariant subgroup by a non-co-Hopfian group, and yet are co-Hopfian: simply choose T to be a standard basic group which is clearly not co-Hopfian and observe that a pure subgroup of J_p with endomorphism ring $\mathbb{Z}_p$, is not co-Hopfian since multiplication by p is a monomorphism which is not an automorphism.

We conclude this brief discussion of Hopfian and co-Hopfian groups by raising a number of problems; we believe that solutions to these problems would give a great deal of insight into the structure of such groups. Our first problem comes from the observation that *reduced* p-groups constructed to date seem to possess both properties or neither.

Problem 1 Find a reduced p-group which is Hopfian but not co-Hopfian and *vice versa.*

We have noted above that countable Hopfian (co-Hopfian) p-groups are finite and that there exist Hopfian and co-Hopfian groups of cardinality $2^{\aleph_0}$. Thus we pose:

Problem 2 Assuming the negation of the Continuum Hypothesis, do there exist Hopfian (co-Hopfian) p-groups of cardinality κ for $\aleph_0 < \kappa < 2^{\aleph_0}$?

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Upper cardinal bounds for absolute structures

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Dedicated to Rüdiger Göbel on the occasion of his 70th birthday

Abstract. We introduce $\kappa \xrightarrow{vw} (\omega)^{<\omega}_{\lambda}$ as a new equivalent formulation of the partition principle $\kappa \longrightarrow (\omega)^{<\omega}_{\lambda}$ and use it for deriving a self-contained proof of a fundamental result by Shelah, in *Better quasi-orders for uncountable cardinals*, characterizing the first Erdös cardinal $\kappa(\omega)$ as the sharp upper bound for the cardinality of an absolutely rigid family of coloured trees.

1. Introduction

An important and central concept of set theory is absoluteness, where we call a mathematical object, a notion or property *absolute* if it is preserved under extensions of the underlying universe, or more precisely, if the following holds.

Definition 1.1. *Let* $\mathbf{M}$ *be a transitive model of* ZFC. *Then a formula* ϕ *is called* absolute *if for every transitive model* $\mathbf{M} \subseteq \mathbf{N}$ *of* ZFC *holds*

$$\forall x_1, \dots x_n \in \mathbf{M} : (\phi^{\mathbf{M}}(x_1, \dots x_n) \longleftrightarrow \phi^{\mathbf{N}}(x_1, \dots x_n)).$$

This includes in particular generic extensions of the universe as obtained by the use of forcing.

Finite structures and ordinals are absolute. However, being countable or being a cardinal are non-absolute properties. To see this we apply the Levy collapse $\text{Levy}(\aleph_0, \aleph_1)$ to obtain some generic extension $\mathbf{M}[G]$ of a given countable transitive model $\mathbf{M}$ of ZFC in which ${\aleph_1}^{\mathbf{M}}$ becomes a countable ordinal. Consider now $\aleph_1$-free abelian groups, i.e. groups where every countable subgroup is free. Being $\aleph_1$-free is an absolute property as according to Pontryagin a group is $\aleph_1$-free if and only if every finite rank subgroup is free. Consider next an $\aleph_1$-free abelian group A with the additional property $\operatorname{End} A \cong \mathbb{Z}$. Such a group can easily be constructed using suitable combinatorial techniques. Nevertheless this property is not absolute, because using again the Levy collapse $\text{Levy}(\aleph_0, |\,A|)$ the constructed group A becomes countable in $\mathbf{M}[G]$. Thus in $\mathbf{M}[G]$ by definition A will be free of countable rank with $|\operatorname{End} A| = 2^{\aleph_0}$ contradicting $|\operatorname{End} A| = |\,\mathbb{Z}| = \aleph_0$. The root of this particular

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astounding effect lies in the construction relying on non-absolute notions like stationary sets.
Conversely, absolute objects can be considered set-theoretically as particularly stable, hence they are highly appreciated. One of the first absolute constructions appeared as part of Eklof, Shelah [**4**] dealing with absolutely *rigid families* $\{G_\alpha \mid \alpha < \kappa\}$ of abelian groups, i.e. families for which $\mathrm{Hom}\,(G_\beta, G_\gamma) = \{0\}$ holds for all $\beta, \gamma < \kappa$ with $\beta \neq \gamma$, showing that the *first Erdös cardinal* $\kappa(\omega)$ is a sharp upper bound for the cardinality of such a family. This result has subsequently been corrected and followed by similar results concerning modules with prescribed endomorphism rings [**5, 8**], endomorphism monoids of graphs [**2**], E-rings [**6, 10**] and most recently also rigid fields [**7**], showing in all cases $\kappa(\omega)$ to be the strictly upper bound for an absolute construction. All these results are clever applications of the following celebrated theorem from Shelah [**20**]:

THEOREM 1.2. *Let κ, λ be cardinals and $\{\mathcal{T}_\alpha \mid \alpha < \kappa\}$ be a family of λ-coloured trees.*

(*a*) *For $\kappa \geq \kappa(\omega)$ and $\lambda < \kappa(\omega)$ there exist $\beta, \gamma < \kappa$ with $\beta \neq \gamma$ and* $\mathrm{Hom}\,(\mathcal{T}_\beta, \mathcal{T}_\gamma) \neq \emptyset$.

(*b*) *For $\kappa < \kappa(\omega)$ and $\lambda = 2$ there exists a family $\{\mathcal{T}_\alpha \mid \alpha < 2^\kappa\}$ of λ-coloured subtrees of ${}^{<\omega}\kappa$ such that* $\mathrm{Hom}\,(\mathcal{T}_\beta, \mathcal{T}_\gamma) = \emptyset$ *holds absolutely for all $\beta, \gamma < 2^\kappa$ with $\beta \neq \gamma$, i.e. the family $\{\mathcal{T}_\alpha \mid \alpha < 2^\kappa\}$ is* absolutely rigid.

The provided version of Theorem 1.2(b) was first proven in [**7**, Section 3.4] improving Shelah's original result by reducing the number of colours from $\lambda = \omega$ to $\lambda = 2$ and increasing the family of trees to $\{\mathcal{T}_\alpha \mid \alpha < 2^\kappa\}$.
Our main task in this paper now is a transparent, self-contained proof of this theorem. We will elaborate the set-theoretic foundations for absolute constructions in general. For this we introduce $\kappa \xrightarrow{vw} (\omega)^{<\omega}_\lambda$ in Section 2 as a new equivalent formulation of $\kappa \longrightarrow (\omega)^{<\omega}_\lambda$ which will give a unified treatment of Theorem 1.2. Necessary tools will be results by Silver [**21**], Nash-Williams [**14, 18**] and Shelah [**20**]. In Section 3 we introduce the Erdös cardinal $\kappa(\omega)$ and describe its basic properties. In Section 4 we give an alternative proof of Theorem 1.2(b) different from [**7**, Section 3] demonstrating the particular strength of $\kappa \xrightarrow{vw} (\omega)^{<\omega}_\lambda$. In Section 5 we continue with the far more challenging proof of Theorem 1.2(a) using Nash-Williams theory. Finally, in Section 6 we conclude the paper with an easy application of Theorem 1.2 giving a revisited proof of the following result from [**10**].

THEOREM 1.3. *For every countable transitive model* $\mathbf{M}$ *of* ZFC *and every abelian group $A \in \mathbf{M}$ of cardinality $|A| \geq \kappa(\omega)$ exists a generic extension $\mathbf{M}[G]$, where $|A| = \aleph_0$ and $|\mathrm{End}\, A| = 2^{\aleph_0}$ holds.*

Our notations are standard, see [**3, 9, 11, 12**].

2. Partition Principles

In this section we introduce some basic notions and partition principles and show the equivalence of $\kappa \longrightarrow (\omega)^{<\omega}_\lambda$ and $\kappa \xrightarrow{vw} (\omega)^{<\omega}_\lambda$. We will follow thoughts provided by Silver [**21**, Section 2].
In the following S will always denote a set, n a non-negative integer, and κ, λ will be cardinals.

Definition 2.1.

(a) $[S]^n = \{U \subseteq S \mid |U| = n\}$, $[S]^{<\omega} = \bigcup_{n<\omega} [S]^n$ *and* $[S]^\omega = \{U \subseteq S \mid |U| = \aleph_0\}$,

(b) ${}^n\kappa^\uparrow = \{\nu : n \longrightarrow \kappa \mid \nu(i) < \nu(i+1)$ *for all* $i+1 < n\}$, ${}^{<\omega}\kappa^\uparrow = \bigcup_{n<\omega} {}^n\kappa^\uparrow$ *and*
${}^\omega\kappa^\uparrow = \{\eta : \omega \longrightarrow \kappa \mid \eta(i) < \eta(i+1)$ *for all* $i < \omega\}$,

(c) ${}^n\kappa^{\neq} = \{\nu : n \longrightarrow \kappa \mid \nu(i) \neq \nu(i+1)$ *for all* $i+1 < n\}$, ${}^{<\omega}\kappa^{\neq} = \bigcup_{n<\omega} {}^n\kappa^{\neq}$ *and*
${}^\omega\kappa^{\neq} = \{\eta : \omega \longrightarrow \kappa \mid \eta(i) \neq \eta(i+1)$ *for all* $i < \omega\}$.

To avoid trivial cases we will always assume $\kappa \geq \aleph_0$ when using ${}^n\kappa^\uparrow, {}^{<\omega}\kappa^\uparrow, {}^\omega\kappa^\uparrow$ and $\kappa \geq 2$ when using ${}^n\kappa^{\neq}, {}^{<\omega}\kappa^{\neq}, {}^\omega\kappa^{\neq}$. We are now ready to define the necessary partition properties.

Definition 2.2.

(a) $\kappa \longrightarrow (\omega)^{<\omega}_\lambda$ *holds iff for every function* $F : [\kappa]^{<\omega} \longrightarrow \lambda$ *there exist some* $S \in [\kappa]^\omega$ *and some function* $F' : \omega \longrightarrow \lambda$ *with*

$$F(U) = F'(|U|) \text{ for all } U \in [S]^{<\omega}.$$

(b) $\kappa \xrightarrow{w} (\omega)^{<\omega}_\lambda$ *holds iff for every function* $F : {}^{<\omega}\kappa^\uparrow \longrightarrow \lambda$ *there exists some* $\eta \in {}^\omega\kappa^\uparrow$ *with*

$$F(\eta(0), \eta(1), \ldots, \eta(i)) = F(\eta(1), \eta(2), \ldots, \eta(i+1)) \text{ for all } i < \omega.$$

(c) $\kappa \xrightarrow{vw} (\omega)^{<\omega}_\lambda$ *holds iff for every function* $F : {}^{<\omega}\kappa^{\neq} \longrightarrow \lambda$ *there exists some* $\eta \in {}^\omega\kappa^{\neq}$ *with*

$$F(\eta(0), \eta(1), \ldots, \eta(i)) = F(\eta(1), \eta(2), \ldots, \eta(i+1)) \text{ for all } i < \omega.$$

The partition property $\kappa \xrightarrow{w} (\omega)^{<\omega}_\lambda$ was introduced in Silver [**21**] as a *weak* version of $\kappa \longrightarrow (\omega)^{<\omega}_\lambda$ in the same sense as we introduce $\kappa \xrightarrow{vw} (\omega)^{<\omega}_\lambda$ as a weak version of $\kappa \xrightarrow{w} (\omega)^{<\omega}_\lambda$ and a *very weak* version of $\kappa \longrightarrow (\omega)^{<\omega}_\lambda$ (used to construct trees and branches in Sections 4 and 5). Note also that Definition 2.2(b) is identical with [**7**, F-Condition 3.10].

For Theorem 2.4 we prove the following simple Lemma, which is dual to a result of Rowbottom [**19**] for $\kappa \longrightarrow (\omega)^{<\omega}_\lambda$.

Lemma 2.3. *If* $\kappa \xrightarrow{vw} (\omega)^{<\omega}_2$, *then also* $\kappa \xrightarrow{vw} (\omega)^{<\omega}_{2^{\aleph_0}}$.

Proof. Observe that $|{}^\omega 2| = 2^{\aleph_0}$ and hence let $F : {}^{<\omega}\kappa^{\neq} \longrightarrow {}^\omega 2$ be given. We define a new function $G : {}^{<\omega}\kappa^{\neq} \longrightarrow 2$ as follows: If $\nu \in {}^{<\omega}\kappa^{\neq}$ with $\nu : n \longrightarrow \kappa$ let $n+1 = 2^j(2i+1)$ be the unique representation of $n+1$ by non-negative integers i, j and set

$$G(\nu) = F(\nu \restriction i)(j).$$

With $\kappa \xrightarrow{vw} (\omega)^{<\omega}_2$ there exists some $\eta \in {}^\omega\kappa^{\neq}$ such that

$$\begin{aligned} F(\eta(0), \eta(1), \ldots, \eta(i))(j) &= G(\eta(0), \eta(1), \ldots, \eta(n)) \\ = G(\eta(1), \eta(2), \ldots, \eta(n+1)) &= F(\eta(1), \eta(2), \ldots, \eta(i+1))(j) \end{aligned}$$

whenever $2^j(2i+3) = n+2 < \omega$. Running j through ω for fixed $i < \omega$ gives the desired

$$F(\eta(0), \eta(1), \ldots, \eta(i)) = F(\eta(1), \eta(2), \ldots, \eta(i+1)). \blacksquare$$

We are now ready to get

Theorem 2.4. (ZFC) *The following conditions are equivalent for all cardinals* κ, λ*:*

(α) $\kappa \longrightarrow (\omega)^{<\omega}_{\lambda}$,
(β) $\kappa \xrightarrow{w} (\omega)^{<\omega}_{\lambda}$,
(γ) $\kappa \xrightarrow{vw} (\omega)^{<\omega}_{\lambda}$.

Proof.
$(\alpha) \Longleftrightarrow (\beta)$: We refer to Silver [**21**] for a model theoretic proof of this deep result using Skolem functions and hence AC.
$(\beta) \Longrightarrow (\gamma)$: For a given function $F : {}^{<\omega}\kappa^{\neq} \longrightarrow \lambda$ apply $\kappa \xrightarrow{w} (\omega)^{<\omega}_{\lambda}$ to $F \restriction {}^{<\omega}\kappa^{\uparrow}$ to find a suitable $\eta \in {}^{\omega}\kappa^{\uparrow} \subseteq {}^{\omega}\kappa^{\neq}$.
$(\gamma) \Longrightarrow (\beta)$: Given $F : {}^{<\omega}\kappa^{\uparrow} \longrightarrow \lambda$, we choose any function $G : {}^{<\omega}\kappa^{\neq} \longrightarrow {}^{\omega}2$ such that for all $\nu \in {}^{<\omega}\kappa^{\uparrow}$ with $\nu : n \longrightarrow \kappa$ holds

$$G(\nu)(0) = F(\nu) \tag{1}$$

and

$$G(\nu)(1) = 1 \text{ iff } n \geq 2 \text{ and } \nu(n-2) < \nu(n-1). \tag{2}$$

With Lemma 2.3 we now can apply $\kappa \xrightarrow{vw} (\omega)^{<\omega}_{2^{\aleph_0}}$ to G and find some $\eta \in {}^{\omega}\kappa^{\neq}$ with

$$G(\eta(0), \eta(1), \ldots, \eta(i)) = G(\eta(1), \eta(2), \ldots, \eta(i+1)) \text{ for all } i < \omega.$$

From (1) we have

$$F(\eta(0), \eta(1), \ldots, \eta(i)) = F(\eta(1), \eta(2), \ldots, \eta(i+1)) \text{ for all } i < \omega,$$

while (2) gives either $\eta(0) > \eta(1) > \eta(2) > \ldots$ or $\eta(0) < \eta(1) < \eta(2) < \ldots$, hence $\eta \in {}^{\omega}\kappa^{\uparrow}$ as the ordinals are well-founded. ∎

3. The Erdös Cardinal $\kappa(\omega)$ and Trees

In this section we introduce the Erdös cardinal $\kappa(\omega)$ and discuss some of its basic properties.

Definition 3.1. *The* (first) Erdös cardinal $\kappa(\omega)$ *is defined as the smallest cardinal* κ *such that* $\kappa \longrightarrow (\omega)^{<\omega}_{2}$ *holds.*

The cardinal $\kappa(\omega)$ is considerably large. It is strongly inaccessible, i.e. regular with $2^{\lambda} < \kappa$ for all cardinals $\lambda < \kappa$, and known to be larger than the first weakly compact cardinal. For more details on Erdös cardinals we refer to Drake [**1**, Chapter 7.4, p.217.ff] and Jech [**11**, Chapter 17, p.302.ff]. We note also the following important

Lemma 3.2. *For every cardinal* $\lambda < \kappa(\omega)$ *holds:*

(a) $\kappa(\omega) \longrightarrow (\omega)^{<\omega}_{\lambda}$.
(b) $\kappa(\omega) \xrightarrow{vw} (\omega)^{<\omega}_{\lambda}$.
(c) $\lambda \stackrel{vw}{\not\longrightarrow} (\omega)^{<\omega}_{2}$.

Proof.
(a) : See Drake [**1**, Corollary 8.2.2, p.239] or Jech [**11**, Lemma 17.30, p.302].
(b) : Combine (a) with Theorem 2.4.
(c) : Combine Definition 3.1 with Theorem 2.4. ∎

Next we will need the definitions for trees:
A *tree* T is a non-empty subset of ${}^{<\omega}\kappa$ that is closed under taking initial segments of sequences. In particular $\emptyset \in {}^{<\omega}\kappa$ and $\emptyset \in T$ holds for the empty sequence (the *root* of T). For every $\nu \in T$ we denote by $\lg(t) = \operatorname{Dom}\nu$ the length of the sequence ν, thus $\lg(\emptyset) = 0$. A *homomorphism* $f : T_1 \longrightarrow T_2$ between trees T_1 and T_2 is a mapping $f : T_1 \longrightarrow T_2$ that preserves initial segments and the lengths. For a cardinal λ a λ-*coloured tree* $\mathcal{T}$ is a pair (T, c) consisting of a tree and a colouring function $c : T \longrightarrow \lambda$. A *homomorphism* $f : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ between λ-coloured trees $\mathcal{T}_1$ and $\mathcal{T}_2$ is a mapping $f : T_1 \longrightarrow T_2$ that preserves initial segments, the lengths and the colours. Finally $\operatorname{Hom}(\mathcal{T}_1, \mathcal{T}_2)$ is the set of all homomorphisms $f : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$.
The goal of the next two sections is to provide a new improved and more transparent proof of Theorem 1.2. We will prove Theorem 1.2(a) in Section 5 and Theorem 1.2(b) in Section 4.

4. Constructing Absolutely Rigid Families of Trees

For the absoluteness arguments of this section we will implicitly work with two transitive models $\mathbf{M} \subseteq \mathbf{N}$ of ZFC, where $\mathbf{M}$ will always denote the ground model in which our construction of the absolutely rigid family of λ-coloured trees takes place, while $\mathbf{N}$ denotes the extension where we want to prove the absoluteness. We will often use superscripts to indicate the model in question.
The main target of this section are noetherian trees, see also [**7**]. For this we will identify every element $\eta \in {}^{\omega}\kappa$ with its induced *branch* $\{\eta \restriction n \mid n < \omega\} \subseteq {}^{<\omega}\kappa$.

Definition 4.1. *A tree $T \subseteq {}^{<\omega}\kappa$ is called a* noetherian tree *if it does not contain any induced branch $\eta \in {}^{\omega}\kappa$, and $\mathbb{T}^{\mathrm{n}}$ denotes the class of all coloured noetherian trees.*

For any tree T and any $\nu \in T$ we define its set of immediate successors $S(\nu) = \{\nu' \in T \mid \nu \subseteq \nu', \lg(\nu') = \lg(\nu) + 1\}$. Furthermore, for any $\nu \in T$ we define $T[\nu] = \{\nu' \in T \mid \nu \subseteq \nu' \text{ or } \nu' \subseteq \nu\}$ as the induced subtree of T. We also define for any coloured tree $\mathcal{T} = (T, c)$ and any $\nu \in T$ the induced coloured subtree $\mathcal{T}[\nu] = (T[\nu], c \restriction T[\nu])$. We next introduce from [**20**] depth as another very useful concept for describing noetherian trees.

Definition 4.2. *We call* $\mathrm{dp} : T \longrightarrow \mathrm{ORD}$ *a* depth function *for a given tree T if the following holds.*

(*i*) $\mathrm{dp}(\nu) = 0$ *for* $S(\nu) = \emptyset$,
(*ii*) $\mathrm{dp}(\nu) = \sup\{\mathrm{dp}(\nu') + 1 \mid \nu' \in S(\nu)\}$ *otherwise.*

We now have as a simple but crucial consequence

Lemma 4.3. *The following are equivalent for all trees T:*

(α) *T is a noetherian tree,*
(β) *T admits a depth function* dp.

If the function dp *exists then it is uniquely determined.*

Proof.
$(\alpha) \Longrightarrow (\beta)$: Assume that T does not admit a (uniquely defined) depth function. We then can define inductively some branch $\eta \in {}^{\omega}\kappa$ in T such that there exists no (uniquely defined) depth function on $T[\eta \restriction n]$ for all $n < \omega$. Thus T is not a noetherian tree.

$(\beta) \Longrightarrow (\alpha)$: Assume that T admits a depth function dp and contains the branch η. Then $(\mathrm{dp}(\eta \restriction n))_{n<\omega}$ is a strictly decreasing sequence of ordinals, which is impossible. ■

A crucial point of the following construction is that noetherian trees are absolute.

LEMMA 4.4. *The class* $\mathbb{T}^{\mathrm{n}}$ *is absolute, which means that* $(\mathbb{T}^{\mathrm{n}})^{\mathbf{M}} = (\mathbb{T}^{\mathrm{n}})^{\mathbf{N}} \cap \mathbf{M}$ *holds.*

Proof. The inclusion $\supseteq$ is obvious from Definition 4.1 and ${}^{\omega}\kappa^{\mathbf{M}} \subseteq {}^{\omega}\kappa^{\mathbf{N}}$, while $\subseteq$ follows from Lemma 4.3 as any existing depth function in $\mathbf{M}$ remains a depth function in $\mathbf{N}$. ■

With the help of depth functions the question of existence of homomorphisms between uncoloured trees becomes an easy exercise.

LEMMA 4.5. *For trees* $T_1, T_2 \subseteq {}^{<\omega}\kappa$ *there exists a homomorphism* $f : T_1 \longrightarrow T_2$ *if and only if one of the following conditions holds:*

(*i*) T_2 *is not noetherian* *or*

(*ii*) T_1, T_2 *are noetherian and their depth functions* $\mathrm{dp}_{T_1}, \mathrm{dp}_{T_2}$ *satisfy* $\mathrm{dp}_{T_1}(\emptyset) \leq \mathrm{dp}_{T_2}(\emptyset)$.

Proof. Conditions (*i*) and (*ii*) are necessary for the existence of a homomorphism f. If $T_1 \notin \mathbb{T}^{\mathrm{n}}$ then there is $\eta \in T_1 \cap {}^{\omega}\kappa$. Hence $f(\eta) \in T_2 \cap {}^{\omega}\kappa$ and $T_2 \notin \mathbb{T}^{\mathrm{n}}$, while $T_1, T_2 \in \mathbb{T}^{\mathrm{n}}$ satisfy $\mathrm{dp}_{T_1}(\nu) \leq \mathrm{dp}_{T_2}(f(\nu))$ for all $\nu \in T_1$.
Conditions (*i*) and (*ii*) are also sufficient: For $T_2 \notin \mathbb{T}^{\mathrm{n}}$ and $\eta \in T_2 \cap {}^{\omega}\kappa$ we can define f setting $f(\nu) = \eta \restriction \lg(\nu)$, while for $T_1, T_2 \in \mathbb{T}^{\mathrm{n}}$ with $\mathrm{dp}_{T_1}(\emptyset) \leq \mathrm{dp}_{T_2}(\emptyset)$ we can define $f(\nu)$ recursively over $\lg(\nu)$ preserving initial segments, lengths and $\mathrm{dp}_{T_1}(\nu) \leq \mathrm{dp}_{T_2}(f(\nu))$ for all $\nu \in T_1$. ■

We are ready for the final construction of our absolutely rigid family of trees. For this we define for every sequence $\emptyset \neq \eta \in {}^{<\omega}\kappa \cup {}^{\omega}\kappa$ the sequence $\eta^- \in {}^{<\omega}\kappa \cup {}^{\omega}\kappa$ by omitting the first entry of η, or equivalently $\eta = \eta(0)^\wedge\eta^-$.

Proof of Theorem 1.2*(b).* Suppose $\kappa < \kappa(\omega)^{\mathbf{M}}$. Thus $\kappa \overset{vw}{\not\rightarrow} (\omega)_2^{<\omega}$ follows from Lemma 3.2(c) and we can choose a function $c : {}^{<\omega}\kappa^{\neq} \longrightarrow 2$ in $\mathbf{M}$ such that for no $\eta \in {}^{\omega}\kappa^{\neq}$ holds

$$c(\eta(0), \eta(1), \dots, \eta(i)) = c(\eta(1), \eta(2), \dots, \eta(i+1)) \quad \text{for all} \quad i < \omega. \tag{3}$$

We next define recursively sets $T(n) \subseteq {}^{n}\kappa^{\neq}$ by setting $T(0) = \{\emptyset\}$ and

$$T(n+1) = \{\nu \in {}^{n+1}\kappa^{\neq} \mid \nu \restriction n, \nu^- \in T(n) \text{ and } c(\nu \restriction n) = c(\nu^-)\}. \tag{4}$$

This obviously defines a tree $T = \bigcup_{n<\omega} T(n)$ and $T \in (\mathbb{T}^{\mathrm{n}})^{\mathbf{M}}$ with (3). From this tree we get a family $\{\mathcal{T}_\alpha \mid \alpha < \kappa\}$ of 2-coloured trees in $\mathbf{M}$ setting $\mathcal{T}_\alpha = (T[\langle\alpha\rangle], c \restriction T[\langle\alpha\rangle])$.

We claim that this family $\{\mathcal{T}_\alpha \mid \alpha < \kappa\}$ is absolutely rigid. Assume that in $\mathbf{N}$ there exist $\beta, \gamma < \kappa$ with $\beta \neq \gamma$ and some homomorphism $f : \mathcal{T}_\beta \mapsto \mathcal{T}_\gamma$. We now define recursively some $\eta \in T \cap {}^{\omega}\kappa^{\mathbf{N}}$ setting $\eta(0) = \beta$ and

$$f(\eta(0), \eta(1), \dots, \eta(i)) = f(\eta(1), \eta(2), \dots, \eta(i+1)) \quad \text{for all} \quad i < \omega. \tag{5}$$

Observe here that (5) implies (3) for any fixed i as f is a colour preserving homomorphism; thus in particular $\eta \restriction (i+1) \in \mathcal{T}_\beta$ follows from $\eta \restriction i \in \mathcal{T}_\beta$ and (4). Now the existence of $\eta \in T \cap {}^\omega\kappa^{\mathbf{N}}$ contradicts Lemma 4.4 and $T \in (\mathbb{T}^{\mathrm{n}})^{\mathbf{M}} = (\mathbb{T}^{\mathrm{n}})^{\mathbf{N}} \cap \mathbf{M} \subseteq (\mathbb{T}^{\mathrm{n}})^{\mathbf{N}}$.

Like in [**7**, Section 3.4] we can easily enlarge to a family $\{\mathcal{T}_X \mid X \in \mathcal{F}\}$ with $|\mathcal{F}| = 2^\kappa$. For this just choose some family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ of size 2^κ such that we have $X \nsubseteq X'$ and $X' \nsubseteq X$ for all $X, X' \in \mathcal{F}$ with $X \neq X'$ and set $\mathcal{T}_X = (\bigcup_{\alpha \in X} T[\langle \alpha \rangle], c \restriction \bigcup_{\alpha \in X} T[\langle \alpha \rangle])$. Such a family is given e.g. by

$$\mathcal{F} = \{(S \times \{0\}) \cup ((\kappa \setminus S) \times \{1\}) \mid S \in \mathcal{P}(\kappa)\}$$

when identifying κ with $\kappa \times 2$. ■

Observe that the previous proof also holds for trees of 3 or more colours as well when replacing $\kappa \overset{vw}{\not\longrightarrow} (\omega)_2^{<\omega}$ by $\kappa \overset{w}{\not\longrightarrow} (\omega)_2^{<\omega}$, while 2-coloured trees force the use of the stronger notion $\kappa \overset{vw}{\not\longrightarrow} (\omega)_2^{<\omega}$.

5. Nash-Williams Theory of Coloured Trees

The proof of Theorem 1.2(a) needs a short detour through Nash-Williams theory [**13, 14, 15, 16, 17, 18**]. For this we will introduce the crucial but basic notions of barriers and better quasi-orders.

A set $(\Omega, \leq)$ with a binary relation $\leq$ is a *quasi-order* if the relation $\leq$ is reflexive and transitive. In accordance with [**7**, Definition 3.1] we define

Definition 5.1. *A subset $B \subseteq {}^{<\omega}\kappa^{\neq}$ is called a* κ-barrier *if the following holds.*

(*i*) $\emptyset \notin B$.
(*ii*) *If $\eta \in {}^\omega\kappa^{\neq}$, then $\eta \restriction n \in B$ for some $n \in \omega$.*
(*iii*) *If $\nu_1, \nu_2 \in B$ and $\nu_1 \subseteq \nu_2$, then $\nu_1 = \nu_2$.*
(*iv*) *If $\nu_1, \nu_2 \in B$ and $\nu_1 \subseteq \nu_2^-$, then $\nu_1 = \nu_2^-$.*

For a more extensive discussion of some basic properties of κ-barriers we refer to [**7**, Section 3.1].

Definition 5.2. *A quasi-order $(\Omega, \leq)$ is a* $[\kappa,\lambda]$-bqo (*short for* $[\kappa,\lambda]$-better quasi-order)*, if for any κ-barrier B and any choice of a colouring function $c : B \longrightarrow \lambda$ and elements $q_\nu \in \Omega$ ($\nu \in B$) there exist $\nu_1, \nu_2 \in B$ such that $\nu_1^- \subseteq \nu_2$, $\nu_1(0) \neq \nu_2(0)$, $c(\nu_1) = c(\nu_2)$ and $q_{\nu_1} \leq q_{\nu_2}$.*

Note that the notion of $[\kappa,\lambda]$-bqo is stronger than the notion of B-bqo introduced in [**7**, Definition 3.6(b)] which was defined only with respect to a specific fixed κ-barrier B. In order to show Theorem 1.2(a) we will combine ideas from Shelah [**20**, Section 5] with $\kappa \overset{vw}{\longrightarrow} (\omega)_\lambda^{<\omega}$. The following characterization of $[\kappa,\lambda]$-bqos $(\Omega, \leq)$ from Nash-Williams theory will be crucial. We will use the abbreviation $\Omega^{2,\nleq} = \{(q_1, q_2) \mid q_1, q_2 \in \Omega, q_1 \nleq q_2\}$.

Theorem 5.3. *Let a quasi-order $(\Omega, \leq)$ and some cardinal $\lambda \geq \aleph_0$ be given. Then a* sufficient condition *for $(\Omega, \leq)$ to be a $[\kappa,\lambda]$-bqo is the existence of a rank function* $\mathrm{rk} : \Omega \longrightarrow \mathrm{ORD}$ *and a function $s : \Omega^{2,\nleq} \longrightarrow \Omega$ with the following properties:*

(*i*) *If* $q_1, q_2, q_3 \in \Omega$ *with* $(q_1, q_2), (q_2, q_3) \in \Omega^{2,\not\leq}$ *and* $s(q_1, q_2) \neq q_1, s(q_2, q_3) \neq q_2$*, then* $s(q_1, q_2) \not\leq s(q_2, q_3)$*.*

(*ii*) *If* $t \in \Omega$ *does not have minimal rank, then* $\mathrm{rk}\,(s(t, q)) < \mathrm{rk}\,(t)$ *for all* $(t, q) \in \Omega^{2,\not\leq}$*.*

(*iii*) *If* $\Omega_m = \{t \in \Omega \mid \mathrm{rk}\,(t) \leq \mathrm{rk}\,(q)$ *for all* $q \in \Omega\}$ *is the set of all elements of minimal rank in* Ω*, then the induced quasi-order* $(\Omega_m, \leq)$ *is a* $[\kappa, \lambda]$*-bqo.*

Proof. Assume that $(\Omega, \leq)$ is not a $[\kappa, \lambda]$-bqo, say that there are a κ-barrier B, a colouring function $c : B \longrightarrow \lambda$ and elements $q_\nu \in \Omega$ $(\nu \in B)$ but there are no $\nu_1, \nu_2 \in B$ with $\nu_1^- \subseteq \nu_2$, $\nu_1(0) \neq \nu_2(0)$, $c(\nu_1) = c(\nu_2)$ and $q_{\nu_1} \leq q_{\nu_2}$. We want to show that also $(\Omega_m, \leq)$ is not a $[\kappa, \lambda]$-bqo which contradicts 5.3(*iii*).
In a first step towards this goal we want to define recursively for $n \in \omega$ a κ-barrier B_n, a set $C_n \subseteq B_n$, a colouring function c_n on B_n and elements $q^n_\nu \in \Omega$ $(\nu \in B_n)$ as follows:
For $n = 0$ we define $B_0 = B$, $q^0_\nu = q_\nu$ and $c_0(\nu) = \langle c(\nu), 0\rangle$ for all $\nu \in B_0$ and we set $C_0 = \{\nu \in B_0 \mid q^0_\nu \in \Omega_m\}$. We then proceed recursively

$$(6) \quad B^0_{n+1} = \{\nu_1(0)^\wedge\nu_2 \mid \nu_1 \in B_n \setminus C_n, \nu_2 \in B_n, \nu_1(0) \neq \nu_2(0), \nu_1^- \subseteq \nu_2\},$$

$$B_{n+1} = C_n \,\dot\cup\, B^0_{n+1},$$

$$(7)\ q^{n+1}_\nu = \begin{cases} q^n_\nu & \text{if } \nu \in C_n, \\ s(q^n_{\nu_1}, q^n_{\nu_2}) & \text{if } \nu = \nu_1(0)^\wedge\nu_2 \text{ as in (6) with } (q^n_{\nu_1}, q^n_{\nu_2}) \in \Omega^{2,\not\leq}, \\ q_* & \text{if } \nu = \nu_1(0)^\wedge\nu_2 \text{ as in (6) with } (q^n_{\nu_1}, q^n_{\nu_2}) \notin \Omega^{2,\not\leq}, \end{cases}$$

for some fixed $q_* \in \Omega_m$,

$$(8) \qquad c_{n+1}(\nu) = \begin{cases} c_n(\nu) & \text{if } \nu \in C_n, \\ \langle c_n(\nu_1), c_n(\nu_2), n+1\rangle & \text{if } \nu = \nu_1(0)^\wedge\nu_2 \text{ as in (6)}, \end{cases}$$

and $C_{n+1} = C_n \,\dot\cup\, \{\nu \in B^0_{n+1} \mid q^{n+1}_\nu \in \Omega_m\} \subseteq B_{n+1}$. We first check that
(9)
$B_n, q^{n+1}_\nu, c_{n+1}(\nu)$ *and* C_{n+1} *are well-defined and* B_n *is a* κ*-barrier for* $n \in \omega$*.*

We prove this by induction on n. For $n = 0$ this is obvious. We thus assume $B_n, q^n_\nu, c_n(\nu)$ and C_n are given and well-defined and want to show that also B_{n+1} is a well-defined κ-barrier.
First observe that $B^0_{n+1} \subseteq {}^{<\omega}\kappa^{\neq}$ by definition. Furthermore, any $\nu = \nu_1(0)^\wedge\nu_2 \in B^0_{n+1}$ satisfies $\nu^- = \nu_2$ and $\{\nu \restriction n \mid n \leq \lg(\nu)\} \cap B_n = \{\nu_1\}$ by 5.1(*iii*). Thus, $\nu_1, \nu_2 \in B_n$ are uniquely determined by $\nu = \nu_1(0)^\wedge\nu_2 \in B^0_{n+1}$ and q^{n+1}_ν and $c_{n+1}(\nu)$ are well-defined. Similarly, $C_n \cap B^0_{n+1} = \emptyset$ follows from $\nu_1 \subseteq \nu_1(0)^\wedge\nu_2$, $\nu_1 \in B_n \setminus C_n$ and 5.1(*iii*). Thus, also B_{n+1} and C_{n+1} are well defined with $B_{n+1} = C_n \,\dot\cup\, B^0_{n+1}$.
It remains to show that B_{n+1} is a κ-barrier. Property 5.1(*i*) is obvious. If $\eta \in {}^\omega\kappa^{\neq}$ is given, then there exist $n_1, n_2 \in B_n$ with $\nu_1 = \eta \restriction n_1, \nu_2 = \eta^- \restriction n_2 \in B_n$. Now either $\eta \restriction n_1 \subseteq C_n \subseteq B_{n+1}$ or $\eta \restriction n_1 \subseteq B_n \setminus C_n$ and $\eta \restriction (n_2 + 1) = \nu_1(0)^\wedge\nu_2 \subseteq B^0_{n+1} \subseteq B_{n+1}$. Thus property 5.1(*ii*) always holds. We next prove properties 5.1(*iii*) and (*iv*). For this we need to distinguish the following cases on $\nu_1, \nu_2 \in B_{n+1}$.

- Case 1: $\nu_1, \nu_2 \in C_n \subseteq B_n$.
  In this case both 5.1(*iii*) and (*iv*) are immediate from B_n being a κ-barrier.
- Case 2: $\nu_1 = \eta_1(0)^\wedge\eta_2, \nu_2 = \sigma_1(0)^\wedge\sigma_2 \in B^0_{n+1}$.
  If $\nu_1 \subseteq \nu_2$, then η_1 and σ_1 are comparable as well as η_2 and σ_2. Hence

$\eta_1 = \sigma_1$ and $\eta_2 = \sigma_2$ in B_n and $\nu_1 = \nu_2$ follows proving 5.1(iii).
If $\nu_1 \subseteq \nu_2^- = \sigma_2$, then $\eta_1 \subseteq \sigma_2$. Hence $\eta_1 = \sigma_2$ in B_n and $\nu_1 \subseteq \nu_2^- = \sigma_2 = \eta_1 \subseteq \nu_1$ follows proving 5.1(iv).

- Case 3: $\nu_1 = \eta_1(0)^\wedge\eta_2 \in B^0_{n+1}, \nu_2 \in C_n \subseteq B_n$.
  If $\nu_1 \subseteq \nu_2$, then η_1 and ν_2 are comparable in B_n. Hence $\eta_1 = \nu_2 \in (B_n \setminus C_n) \cap C_n = \emptyset$, a contradiction. If $\nu_1 \subseteq \nu_2^-$, then $\eta_1 \subseteq \nu_2^-$. Hence $\eta_1 = \nu_2^-$ in B_n and $\nu_1 \subseteq \nu_2^- = \eta_1 \subseteq \nu_1$ follows proving 5.1(iv).
- Case 4: $\nu_1 \in C_n \subseteq B_n, \nu_2 = \sigma_1(0)^\wedge\sigma_2 \in B^0_{n+1}$.
  If $\nu_1 \subseteq \nu_2$, then ν_1 and σ_1 are comparable in B_n. Hence $\nu_1 = \sigma_1 \in C_n \cap (B_n \setminus C_n) = \emptyset$, a contradiction. If $\nu_1 \subseteq \nu_2^- = \sigma_2$, then $\nu_1 = \sigma_2 = \nu_2^-$ follows in B_n proving 5.1(iv).

Hence (9) follows. It is clear from the definitions that

$$C_n = \{\nu \in B_n \mid q^n_\nu \in \Omega_m\} \text{ and } |\operatorname{Im} c_n| \leq \lambda. \tag{10}$$

Note in particular that by identifying $\operatorname{Im} c_n$ with λ we easily can derive a proper colouring function $c_n : B_n \longrightarrow \lambda$. We now claim that

(11) *B_n, c_n and the elements q^n_ν ($\nu \in B_n$) witness that $(\Omega, \leq)$ is no $[\kappa, \lambda]$-bqo.*

We prove this by induction on n. For $n = 0$ this is evident. Thus suppose (11) holds for n and we are given $\nu_1, \nu_2 \in B_{n+1}$ with $\nu_1^- \subseteq \nu_2$, $\nu_1(0) \neq \nu_2(0)$ and $c_{n+1}(\nu_1) = c_{n+1}(\nu_2)$. We must show that $q^{n+1}_{\nu_1} \not\leq q^{n+1}_{\nu_2}$. This follows by distinguishing the following cases.

- Case 1: $\nu_1, \nu_2 \in C_n \subseteq B_n$.
  Now $c_n(\nu_1) = c_{n+1}(\nu_1) = c_{n+1}(\nu_2) = c_n(\nu_2)$ and we have $q^{n+1}_{\nu_1} = q^n_{\nu_1} \not\leq q^n_{\nu_2} = q^{n+1}_{\nu_2}$ by induction hypothesis.
- Case 2: $\nu_1 = \eta_1(0)^\wedge\eta_2, \nu_2 = \sigma_1(0)^\wedge\sigma_2 \in B^0_{n+1}$.
  As $\eta_2 = \nu_1^- \subseteq \nu_2$, we have that η_2 and σ_1 are comparable. Hence $\eta_2 = \sigma_1$ in B_n. Furthermore, $\eta_1^- \subseteq \eta_2, \eta_1(0) \neq \eta_2(0)$ and $\sigma_1^- \subseteq \sigma_2, \sigma_1(0) \neq \sigma_2(0)$ with (6), while $c_{n+1}(\nu_1) = c_{n+1}(\nu_2)$ gives $c_n(\eta_1) = c_n(\sigma_1) = c_n(\eta_2)$ and $c_n(\sigma_1) = c_n(\eta_2) = c_n(\sigma_2)$ with (8). Hence $q^n_{\eta_1} \not\leq q^n_{\eta_2}$ and $q^n_{\sigma_1} \not\leq q^n_{\sigma_2}$ using the induction hypothesis and $(q^n_{\eta_1}, q^n_{\eta_2}), (q^n_{\sigma_1}, q^n_{\sigma_2}) \in \Omega^{2,\not\leq}$ follows.
  Also note that $\eta_1, \sigma_1 \notin C_n$ by (6), hence $q^n_{\eta_1}, q^n_{\sigma_1} \notin \Omega_m$ and $s(q^n_{\eta_1}, q^n_{\eta_2}) \neq q^n_{\eta_1}$, $s(q^n_{\sigma_1}, q^n_{\sigma_2}) \neq q^n_{\sigma_1}$ by 5.3(ii). Setting $q_1 = q^n_{\eta_1}, q_2 = q^n_{\eta_2} = q^n_{\sigma_1}, q_3 = q^n_{\sigma_2}$ we have $(q_1, q_2), (q_2, q_3) \in \Omega^{2,\not\leq}$ and $s(q_1, q_2) \neq q_1, s(q_2, q_3) \neq q_2$ and $s(q_1, q_2) \not\leq s(q_2, q_3)$ follows with 5.3(i). In particular $q^{n+1}_{\nu_1} = s(q^n_{\eta_1}, q^n_{\eta_2}) = s(q_1, q_2) \not\leq s(q_2, q_3) = s(q^n_{\sigma_1}, q^n_{\sigma_2}) = q^{n+1}_{\nu_2}$.
- Case 3: $\nu_1 = \eta_1(0)^\wedge\eta_2 \in B^0_{n+1}, \nu_2 \in C_n \subseteq B_n$.
  Due to (8) the last component of $c_{n+1}(\nu_1)$ is $n+1$, while the last component of $c_{n+1}(\nu_2)$ is distinct, contradicting $c_{n+1}(\nu_1) = c_{n+1}(\nu_2)$.
- Case 4: $\nu_1 \in C_n \subseteq B_n, \nu_2 = \sigma_1(0)^\wedge\sigma_2 \in B^0_{n+1}$.
  Due to (8) the last component of $c_{n+1}(\nu_2)$ is $n+1$, while the last component of $c_{n+1}(\nu_1)$ is distinct, again contradicting $c_{n+1}(\nu_1) = c_{n+1}(\nu_2)$.

Note that $(C_i)_{i\in\omega}$ is an increasing sequence of sets. We conclude the proof defining an ultimate κ-barrier B', a colouring function c' on B' and elements $q'_\nu \in \Omega_m$ ($\nu \in B'$) as follows:

$$B' = \bigcup_{i\in\omega} C_i, c'(\nu) = c_n(\nu) \text{ and } q'_\nu = q^n_\nu \in \Omega_m \text{ whenever } \nu \in C_n \setminus \bigcup_{i<n} C_n. \tag{12}$$

We need to check that B' is indeed a κ-barrier. The only interesting condition is 5.1(ii):
Suppose $\eta \in {}^{\omega}\kappa^{\neq}$ does not satisfy 5.1(ii). Then for every $i \in \omega$ there exist some $m(i), n(i) \in \omega$ with $\eta \restriction m(i), \eta^- \restriction n(i) \in B_i$ by (9) and $\eta \restriction m(i) \in B_i \setminus C_i$ must hold because otherwise $\eta \restriction m(i) \in C_i \subseteq B'$ contradicts our assumption on η. With $\eta \restriction m(i) \in B_i \setminus C_i$, $\eta^- \restriction n(i) \in B_i$ follows $\eta \restriction (n(i)+1) \in B^0_{i+1} \subseteq B_{i+1}$ using definition (6), $\eta \restriction m(i+1)$ and $\eta \restriction (n(i)+1)$ are comparable in B_{i+1} and $\eta \restriction (n(i)+1) = \eta \restriction m(i+1) \in B_{i+1} \setminus C_{i+1}$ follows. This is only consistent with (7) if $q^{n+1}_{\eta\restriction m(i+1)} = s(q^n_{\eta\restriction m(i)}, q^n_{\eta\restriction n(i)})$ and $\mathrm{rk}\,(q^{n+1}_{\eta\restriction m(i+1)}) = \mathrm{rk}\,(s(q^n_{\eta\restriction m(i)}, q^n_{\eta\restriction n(i)})) < \mathrm{rk}\,(q^n_{\eta\restriction m(i)})$ follows from 5.3(ii). Thus $(\mathrm{rk}\,(q^n_{\eta\restriction m(i)}))_{i\in\omega}$ is a strictly decreasing sequence of ordinals, a contradiction. This proves that B' is a κ-barrier.
It is now obvious that
(13)
B', c' and the elements $q'_\nu \in \Omega_m$ ($\nu \in B'$) witness that $(\Omega_m, \leq)$ is no $[\kappa,\lambda]$-bqo.

This contradicts condition 5.3(iii), which ends our proof. ∎

We next want to derive Theorem 1.2(a) from Theorem 5.3 for families $\{\mathcal{T}_\alpha \mid \alpha < \kappa\}$ in $\mathbb{T}^{\mathrm{n}}$. This will follow by the next two lemmas. We introduce an equivalence relation $\equiv$ on quasi-orders $(\Omega, \leq)$ defined by

$$q_1 \equiv q_2 \iff (q_1 \leq q_2 \wedge q_2 \leq q_1). \tag{14}$$

Lemma 5.4. *Let $\kappa \geq \kappa(\omega)$, $\aleph_0 \leq \lambda < \kappa(\omega)$ be cardinals and $(\Omega, \leq)$ be a quasi-order with $|\{[q]_\equiv \mid q \in \Omega\}| < \kappa(\omega)$. Then $(\Omega, \leq)$ is a $[\kappa,\lambda]$-bqo.*

Proof. Let a κ-barrier B with a colouring function $c : B \longrightarrow \lambda$ and elements $q_\nu \in \Omega$ ($\nu \in B$) be given. A well-defined function $F : {}^{<\omega}\kappa^{\neq} \longrightarrow (\{[q]_\equiv \mid q \in \Omega\} \times \lambda) \cup \{*\}$ is given by

$$F(\nu) = \begin{cases} ([q_{\nu'}]_\equiv, c(\nu')) & \text{if } B \cap \{\nu \restriction n \mid n \leq \lg(\nu)\} = \{\nu'\}, \\ \quad * & \text{otherwise.} \end{cases} \tag{15}$$

Observe that $\mu = |\operatorname{Im} F| < \kappa(\omega)$ and that $\kappa(\omega) \xrightarrow{vw} (\omega)^{<\omega}_\mu$ and $\kappa \xrightarrow{vw} (\omega)^{<\omega}_\mu$ hold by Lemma 3.2(b). Thus there exists $\eta \in {}^{\omega}\kappa^{\neq}$ with

$$F(\eta(0), \eta(1), \ldots, \eta(i)) = F(\eta(1), \eta(2), \ldots, \eta(i+1)) \text{ for all } i < \omega. \tag{16}$$

Choosing $n \in \omega$ with $\eta \restriction n \in B$ we have now by (15) that $F(\eta \restriction i) = *$ for all $i < n$ and $F(\eta \restriction i) = ([q_{\eta\restriction n}]_\equiv, c(\eta \restriction n))$ for all $i \geq n$. With (16) follows $F(\eta^- \restriction i) = *$ for all $i < n$ and $F(\eta^- \restriction i) = ([q_{\eta\restriction n}]_\equiv, c(\eta \restriction n))$ for all $i \geq n$. We conclude $\eta^- \restriction n \in B$ with $[q_{\eta^-\restriction n}]_\equiv = [q_{\eta\restriction n}]_\equiv$ and $c(\eta^- \restriction n) = c(\eta \restriction n)$, and $q_{\eta\restriction n} \leq q_{\eta^-\restriction n}$ follows. Thus Definition 5.2 holds with $\nu_1 = \eta \restriction n$ and $\nu_2 = \eta^- \restriction n$. ∎

Let $\mathbb{T}^{\mathrm{n}}_\lambda$ be the class of all λ-coloured noetherian trees. In the next lemma we want to investigate the quasi-order $(\mathbb{T}^{\mathrm{n}}_\lambda, \leq)$, where the order $\mathcal{T}_1 \leq \mathcal{T}_2$ is defined by $\operatorname{Hom}(\mathcal{T}_1, \mathcal{T}_2) \neq \emptyset$.

Lemma 5.5.

(a) *The quasi-order $(\mathbb{T}^{\mathrm{n}}_\lambda, \leq)$ is a $[\kappa,\lambda]$-bqo for any $\kappa \geq \kappa(\omega)$ and $\aleph_0 \leq \lambda < \kappa(\omega)$.*

(b) *For any cardinals $\kappa \geq \kappa(\omega)$ and $\lambda < \kappa(\omega)$ and any family $\{\mathcal{T}_\alpha \mid \alpha < \kappa\}$ in $\mathbb{T}^{\mathrm{n}}_\lambda$ there exist $\beta, \gamma < \kappa$ with $\beta \neq \gamma$ and* $\operatorname{Hom}(\mathcal{T}_\beta, \mathcal{T}_\gamma) \neq \emptyset$.

Proof.
(a): Since $\mathcal{T} = (T, c) \in \mathbb{T}^{\mathrm{n}}_{\lambda}$ is noetherian, we have a unique depth function dp of T. For $n(\mathcal{T}) = \min\{n \in \omega \mid |\{\nu \in T \mid \lg(\nu) = n\}| \neq 1\}$ and $\{\nu(\mathcal{T})\} = \{\nu \in T \mid \lg(\nu) = n(\mathcal{T}) - 1\}$ we define a rank function by $\mathrm{rk}(\mathcal{T}) = \mathrm{dp}(\nu(\mathcal{T}))$. Thus in particular $\{\nu \mid \nu \in T, \lg(\nu) = n\} = \{\nu(\mathcal{T}) \restriction n\}$ for all $n < n(\mathcal{T})$, while $|S(\nu(\mathcal{T}))| \neq 1$.
We next define an appropriate function $s : (\mathbb{T}^{\mathrm{n}}_{\lambda})^{2,\not\leq} \longrightarrow \mathbb{T}^{\mathrm{n}}_{\lambda}$. For this let $(\mathcal{T}_1, \mathcal{T}_2) \in (\mathbb{T}^{\mathrm{n}}_{\lambda})^{2,\not\leq}$ be given. If $\mathrm{rk}(\mathcal{T}_1) = 0$ then set $s(\mathcal{T}_1, \mathcal{T}_2) = \mathcal{T}_1$. If $\mathrm{rk}(\mathcal{T}_1) > 0$, then $|S(\nu(\mathcal{T}_1))| > 1$ in $\mathcal{T}_1$ and as $\mathrm{Hom}(\mathcal{T}_1, \mathcal{T}_2) = \emptyset$ it is obviously possible to choose some $\nu' \in S(\nu(\mathcal{T}_1))$ such that for $s(\mathcal{T}_1, \mathcal{T}_2) = \mathcal{T}_1[\nu'] \in \mathbb{T}^{\mathrm{n}}_{\lambda}$ holds $\mathrm{Hom}(s(\mathcal{T}_1, \mathcal{T}_2), \mathcal{T}_2) = \emptyset$ and $(s(\mathcal{T}_1, \mathcal{T}_2), \mathcal{T}_2) \in (\mathbb{T}^{\mathrm{n}}_{\lambda})^{2,\not\leq}$.
We claim now that the functions rk and s satisfy the conditions of Theorem 5.3:
Observe that $(\mathbb{T}^{\mathrm{n}}_{\lambda})_m = \{\mathcal{T} \in \mathbb{T}^{\mathrm{n}}_{\lambda} \mid \mathrm{rk}(\mathcal{T}) = 0\}$ and that $\mathcal{T} = (T, c) \in (\mathbb{T}^{\mathrm{n}}_{\lambda})_m$ if and only if $T = \{\nu(\mathcal{T}) \restriction n \mid n < n(\mathcal{T})\}$. In particular $|\{[\mathcal{T}]_{\equiv} \mid \mathcal{T} \in (\mathbb{T}^{\mathrm{n}}_{\lambda})_m\}| = |{}^{<\omega}\lambda| < \kappa(\omega)$ is obvious and $((\mathbb{T}^{\mathrm{n}}_{\lambda})_m, \leq)$ is a $[\kappa, \lambda]$-bqo with Lemma 5.4. This shows 5.3(iii).
If $(\mathcal{T}_1, \mathcal{T}_2), (\mathcal{T}_2, \mathcal{T}_3) \in (\mathbb{T}^{\mathrm{n}}_{\lambda})^{2,\not\leq}$ with $s(\mathcal{T}_1, \mathcal{T}_2) \neq \mathcal{T}_1$, then $\mathrm{rk}(\mathcal{T}_1) > 0$ and by definition $\mathrm{Hom}(s(\mathcal{T}_1, \mathcal{T}_2), \mathcal{T}_2) = \emptyset$. In particular also $\mathrm{Hom}(s(\mathcal{T}_1, \mathcal{T}_2), s(\mathcal{T}_2, \mathcal{T}_3)) = \emptyset$ as $s(\mathcal{T}_2, \mathcal{T}_3)$ is a subtree of $\mathcal{T}_2$ and 5.3(i) holds.
If $(\mathcal{T}_1, \mathcal{T}_2) \in (\mathbb{T}^{\mathrm{n}}_{\lambda})^{2,\not\leq}$ with $\mathcal{T}_1 \notin (\mathbb{T}^{\mathrm{n}}_{\lambda})_m$, then $\mathrm{rk}(\mathcal{T}_1) > 0$ and by definition $s(\mathcal{T}_1, \mathcal{T}_2) = \mathcal{T}_1[\nu']$ holds. Thus $\mathrm{rk}(s(\mathcal{T}_1, \mathcal{T}_2)) = \mathrm{rk}(\mathcal{T}_1[\nu']) = \mathrm{dp}(\nu') < \mathrm{dp}(\nu(\mathcal{T})) = \mathrm{rk}(\mathcal{T})$ with 4.2(ii) and also 5.3(ii) holds.
Thus we can apply Theorem 5.3 and $(\mathbb{T}^{\mathrm{n}}_{\lambda}, \leq)$ is a $[\kappa, \lambda]$-bqo.
(b): This is an immediate consequence of (a) for the κ-barrier $\{\langle \alpha \rangle \mid \alpha < \kappa\}$. ∎

To prove Theorem 1.2(a) in the case of general trees we will approximate trees with the help of suitable noetherian trees. For this we introduce

Definition 5.6. *Let $\mathcal{T} = (T, c)$ be a λ-coloured tree and α be an ordinal. Then*

$$T^{\mathrm{n}}_{\alpha} = \{(\nu_1, \nu_2) \mid \nu_1 \in T, \nu_2 \in {}^{<\omega}(\alpha + 1), \lg(\nu_2) = \lg(\nu_1) + 1, \\ \alpha = \nu_2(0) > \nu_2(1) > \ldots > \nu_2(\lg(\nu_1))\}$$

with the order relation

$$(\nu_1, \nu_2) \leq (\nu_1', \nu_2') \Longleftrightarrow \nu_1 \leq \nu_1' \wedge \nu_2 \leq \nu_2'$$

defines a new tree. Together with the colouring function $c^{\mathrm{n}}_{\alpha} : T^{\mathrm{n}}_{\alpha} \longrightarrow \lambda$, $c^{\mathrm{n}}_{\alpha}(\nu_1, \nu_2) = c(\nu_1)$ this defines the λ-coloured tree $\mathcal{T}^{\mathrm{n}}_{\alpha} = (T^{\mathrm{n}}_{\alpha}, c^{\mathrm{n}}_{\alpha})$ as α-th approximation *of $\mathcal{T}$.*

We mention as simple but useful

Corollary 5.7. *For every λ-coloured tree $\mathcal{T} = (T, c)$ and ordinal α follows $\mathcal{T}^{\mathrm{n}}_{\alpha} \in \mathbb{T}^{\mathrm{n}}_{\lambda}$ with $\mathrm{dp}(\nu_1, \nu_2) \leq \nu_2(\lg(\nu_1))$ for all $(\nu_1, \nu_2) \in T^{\mathrm{n}}_{\alpha}$.*

Proof. $\mathcal{T}^{\mathrm{n}}_{\alpha}$ is obviously noetherian because every strictly increasing infinite sequence $(\nu^i_1, \nu^i_2)_{i \in \omega}$ in $\mathcal{T}$ produces a strictly decreasing sequence $\bigcup_{i \in \omega} \nu^i_2$ of ordinals, a contradiction. The above inequality follows by transfinite induction over $\mathrm{dp}(\nu_1, \nu_2)$. ∎

We are now all set.

Proof of Theorem 1.2*(b).* Let $\kappa \geq \kappa(\omega)$ and $\lambda < \kappa(\omega)$ be cardinals and consider a family of λ-coloured trees $\{\mathcal{T}_{\alpha} = (T_{\alpha}, c_{\alpha}) \mid \alpha < \kappa\}$. For every regular cardinal μ

we now apply Lemma 5.5(b) to the family $\{(\mathcal{T}_\alpha)^{\mathrm{n}}_\mu \mid \alpha < \kappa\}$ and obtain $\beta_\mu, \gamma_\mu < \kappa$ with $\beta_\mu \neq \gamma_\mu$ and $\mathrm{Hom}\,((\mathcal{T}_{\beta_\mu})^{\mathrm{n}}_\mu, (\mathcal{T}_{\gamma_\mu})^{\mathrm{n}}_\mu) \neq \emptyset$. Thus, there exist $\beta, \gamma < \kappa$ with $\mathrm{Hom}\,((\mathcal{T}_\beta)^{\mathrm{n}}_\mu, (\mathcal{T}_\gamma)^{\mathrm{n}}_\mu) \neq \emptyset$ for arbitrarily large regular cardinals μ, and we may assume $\mu > |\,T_\gamma|$ and fix a homomorphism $f \in \mathrm{Hom}\,((\mathcal{T}_\beta)^{\mathrm{n}}_\mu, (\mathcal{T}_\gamma)^{\mathrm{n}}_\mu)$. We next construct some homomorphism $g \in \mathrm{Hom}\,(\mathcal{T}_\beta, \mathcal{T}_\gamma) \neq \emptyset$ as follows:
Let g map the root of $\mathcal{T}_\beta$ onto the root of $\mathcal{T}_\gamma$ which is possible as $f(\emptyset, \langle\mu\rangle) = (\emptyset, \langle\mu\rangle)$ and in particular $c_\beta(\emptyset) = (c_\beta)^{\mathrm{n}}_\mu(\emptyset, \langle\mu\rangle) = (c_\gamma)^{\mathrm{n}}_\mu(\emptyset, \langle\mu\rangle) = c_\gamma(\emptyset)$ by Definition 5.6. From this we can define $h(\nu)$ recursively over $\lg(\nu)$ such that

$$\sup\{\delta \in \mu \mid \exists \nu_1, \nu_2 \in {}^{<\omega}(\mu+1) : (\nu, \nu_1) \in (\mathcal{T}_\beta)^{\mathrm{n}}_\mu, \nu_1(\lg(\nu)) = \delta \quad \text{and}$$
$$f(\nu \restriction n, \nu_1 \restriction (n+1)) = (g(\nu \restriction n), \nu_2 \restriction (n+1)) \text{ for all } n \leq \lg(\nu)\} = \mu$$

holds for all $\nu \in \mathcal{T}_\beta$. ∎

6. Countable groups with large endomorphism rings

This section is dedicated to proving Theorem 1.3 as an easy application of Theorem 1.2(a) fixing a minor gap in [**10**, Lemma 2.2]. As in [**10**] we first strengthen [**4**, Theorem 4].

LEMMA 6.1. *If* $\mathbf{M} \subseteq \mathbf{N}$ *are two transitive models of* ZFC*, then let* $B \subseteq A$ *be groups* $A, B \in \mathbf{M}$ *of cardinalities* $|\,A| \geq \kappa(\omega)$ *and* $|\,B| < \kappa(\omega)$ *in* $\mathbf{M}$ *and* $|\,A|, |\,B| = \aleph_0$ *in* $\mathbf{N}$*. Then there exists some* $\varphi \in \mathrm{Mon}\,A$ *in* $\mathbf{N}$ *with* $\varphi \restriction B = \mathrm{id}_B$ *and* $\varphi \neq \mathrm{id}_A$*.*

Proof. We first work in $\mathbf{M}$.
Let $s = \langle s_i \mid i < m\rangle$ and $t = \langle t_j \mid j < n\rangle$ be elements in ${}^{<\omega}A$ and let $B_s := \langle B, s_i \mid i < m\rangle$ be the induced subgroup of A. Setting $B_m := B \oplus \bigoplus_{i<m} \mathbb{Z}e_i$ we have a canonical projection $\pi_s : B_m \longrightarrow B_s$ induced by $\pi_s \restriction B = \mathrm{id}_B$ and $\pi_s(e_i) = s_i$ for all $i < m$. Next we define an equivalence relation $\mathcal{E}$ on ${}^{<\omega}A$ setting $s\,\mathcal{E}t$ iff $m = n$ and $\mathrm{Ker}\ \pi_s = \mathrm{Ker}\ \pi_t$. For the induced partition $A/\mathcal{E}$ obviously $|\,A/\mathcal{E}| \leq \aleph_0 \cdot 2^{|B|+\aleph_0} < \kappa(\omega)$ holds because $\kappa(\omega)$ is strongly inaccessible. Note the observation:

We have $s\,\mathcal{E}t$ *if and only if* $n = m$*, and there is an isomorphism*

(17) $\psi : B_s \longrightarrow B_t$ *such that* $\psi \restriction B = \mathrm{id}_B$ *and* $\psi(s_i) = t_i$ *for all* $i < m$*.*

Next choose in $\mathbf{M}$ a list $\langle u_\alpha \mid \alpha < \kappa(\omega)\rangle$ of pairwise distinct elements $u_\alpha \in A$. Let T_α for $\alpha < \kappa(\omega)$ be the tree generated by all finite sequences in ${}^{<\omega}A$ with initial element u_α. Furthermore let $\mathcal{T}_\alpha = (T_\alpha, c_\alpha)$ be the $|\,A/\mathcal{E}|$-coloured tree where the colouring function c_α is defined by $c_\alpha(t) := t/\mathcal{E}$ for every $t \in {}^{<\omega}A$. Using Theorem 1.2(a) there exist $\alpha, \beta < \kappa(\omega)$ with $\alpha \neq \beta$ and $\mathrm{Hom}\,(\mathcal{T}_\alpha, \mathcal{T}_\beta) \neq \emptyset$. Fix some homomorphism $f : \mathcal{T}_\alpha \longrightarrow \mathcal{T}_\beta$. Passing to $\mathbf{N}$, this map f will remain.

In $\mathbf{N}$ the group A is countable and we can enumerate A by $\langle a_i \mid i < \omega\rangle$. As f preserves initial segments and lengths, there exists a sequence $\langle a'_i \mid i < \omega\rangle$ in A with

$$f(\langle u_\alpha, a_0, a_1, \ldots, a_i\rangle) = \langle u_\beta, a'_0, a'_1, \ldots, a'_i\rangle$$

for all $i < \omega$. Since f also preserves colours, we can use (17) to define a monomorphism $\varphi \in \mathrm{Mon}\,A$ setting $\varphi(a_i) := a'_i$ $(i < \omega)$. From (17) it follows that in particular $\varphi \restriction B = \mathrm{id}_B$ and $\varphi(u_\alpha) = u_\beta \neq u_\alpha$, thus $\varphi \neq \mathrm{id}_A$. ∎

We can also find large sets of endomorphisms of A.

LEMMA 6.2. *If A is a countable abelian group such that for every finite set $S \subseteq A$ there exists some $\varphi \in \operatorname{Mon} A$ with $\varphi \restriction S = \operatorname{id}_S$ and $\varphi \neq \operatorname{id}_A$, then* $|\operatorname{End} A| = 2^{\aleph_0}$.

Proof. $|\operatorname{End} A| \leq 2^{\aleph_0}$ is obvious.
Given an element $a \in A$ and a sequence $\langle \psi_i \mid i < \omega \rangle$ in $\operatorname{End} A$ we define for $\eta \in {}^{<\omega}2$

$$a^\eta := \sum_{0 \leq i < \lg(\eta),\, \eta(i)=1} \psi_i(a).$$

Furthermore list A as $\langle a_i \mid i < \omega \rangle$.
Next we specify the sequence $\langle \psi_i \mid i < \omega \rangle$: for $i < \omega$ define recursively a triple (S_i, ψ_i, b_i) consisting of a finite set $S_i \subseteq A$, some $\psi_i \in \operatorname{End} A$ and elements $b_i \in A$. Set $S_0 := \emptyset$. Given S_i choose $\varphi_i \in \operatorname{Mon} A$ such that $\varphi_i \restriction S_i = \operatorname{id}_{S_i}$ while $\varphi_i(b_i) \neq b_i$ for a suitable $b_i \in A$. We set $S_{i+1} := S_i \cup \{a_i, b_i\}$ and $\psi_i := (\varphi_i - \operatorname{id})$ with $\psi_i \restriction S_i = 0$ and $\psi_i(b_i) \neq 0$.
For every $a \in A$, $\eta \in {}^{\omega}2$ we now have the sequence $\langle a^{\eta\restriction n} \mid n < \omega \rangle$ in A. This sequence becomes stationary in A, i.e. for $a = a_i$ we have $a^{\eta\restriction n} = a^{\eta\restriction(i+1)}$ for all $n \geq i+1$. Thus,

$$\psi_\eta(a) := a^{\eta\restriction n} \text{ for large } n$$

defines an endomorphism $\psi_\eta \in \operatorname{End} A$.
For $\eta_0, \eta_1 \in {}^{\omega}2$ with $\eta_0 \neq \eta_1$ choose i minimal such that $\eta_0(i) \neq \eta_1(i)$. Then obviously $\psi_{\eta_0}(b_i) \neq \psi_{\eta_1}(b_i)$ because $\psi_i(b_i) \neq 0$ and $\psi_n(b_i) = 0$ for $n > i$, and $\psi_{\eta_0} \neq \psi_{\eta_1}$ follows. It follows $|\operatorname{End} A| \geq 2^{\aleph_0}$. ∎

Proof of Theorem 1.3. From the universe $\mathbf{M}$ with $|A| \geq \kappa(\omega)$ we get by the Levy collapse $\operatorname{Levy}(\aleph_0, |A|)$ to a generic extension $\mathbf{M}[G]$ where $|A| = \aleph_0$ holds. As the notions of finite and infinite sets are absolute we can derive from Lemma 6.1 that A satisfies Lemma 6.2 in $\mathbf{M}[G]$. ∎

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Undefinability of local Warfield groups in $L_{\infty\omega}$

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Dedicated to Professor Rüdiger Göbel in honor of his 70th birthday

Abstract. Ulm's Theorem presents invariants that classify countable torsion abelian groups up to isomorphism. Barwise and Eklof extended this result to the classification of arbitrary torsion abelian groups up to $L_{\infty\omega}$-equivalence. The problem addressed here is a similar extension to $L_{\infty\omega}$-equivalence of the results of Warfield classifying a certain class of mixed $\mathbb{Z}_p$-modules up to isomorphism. We consider the class of modules with partial decomposition bases, which is closed under $L_{\infty\omega}$-equivalence and includes those studied by Warfield. $L_{\infty\omega}$-invariants have been defined for this class. Here we show that the invariants are expressible in $L_{\infty\omega}$, which completes the classification. However, we find that the class of modules is not definable, nor are decomposition sets, nor is any class of modules that generalizes Warfield modules in any reasonable way.

1. Introduction

Ulm's Theorem defines invariants that classify countable torsion abelian groups up to isomorphism. Generalizations of this theorem have taken two directions. The first extends the class of groups that may be classified up to isomorphism. Warfield [**W1**] developed new invariants that, along with the Ulm invariants, serve to classify a class of $\mathbb{Z}_p$-modules including all countable torsion and all completely decomposable torsion-free modules. The other direction was taken by Barwise and Eklof [**BE**]. They looked at the classification problem in the infinitary language $L_{\infty\omega}$ and classified all torsion abelian groups up to $L_{\infty\omega}$-equivalence using modified Ulm invariants. Since countable groups that are $L_{\infty\omega}$-equivalent are also isomorphic, the result is indeed a generalization of Ulm's Theorem.

This paper seeks to unify these two generalizations of Ulm's Theorem by defining a class of modules that includes those studied by Warfield and that may be classified in $L_{\infty\omega}$. Barwise and Eklof generalized Ulm's Theorem by defining a class of groups and invariants with the following properties:

(i) The class includes the groups classified by Ulm's Theorem.
(ii) The class is closed under $\equiv_\infty$.

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(iii) The invariants agree up to ω with Ulm's invariants for groups covered by Ulm's Theorem.
(iv) The invariants classify members of the class up to $L_{\infty\omega}$-equivalence.
(v) The invariants are expressible in $L_{\infty\omega}$.
(vi) The class is definable in $L_{\infty\omega}$.

We seek a class and invariants that provide analogous results based on Warfield's Theorem. Note that unlike the work of Barwise and Eklof, the resulting classification cannot be a strict generalization of Warfield's Theorem since Warfield modules need not be countable.

The class and invariant were defined by the author in 1980 [**J2**] and items (i) - (iv) were proved. Here we summarize these results and present definitions and theorems only as necessary for this model theoretic part of the problem, items (v) and (vi).

The defining property of the modules studied in this paper is the existence of what we call a partial decomposition basis, a generalization of the concept of decomposition basis that is preserved under $L_{\infty\omega}$-equivalence. This class, however, is not definable in $L_{\infty\omega}$, nor is any class that generalizes the class of Warfield modules in any reasonable way, as we prove.

2. Background

2.1. Algebraic Preliminaries. The goal of the classification theorems is to define numerical invariants that determine a given class of infinite abelian groups. *Warfield groups* are direct summands of simply presented groups, where a group is called *simply presented* if it has a set of generators subject only to defining relations of the form $p^n x = y$ or $p^n x = 0$ for generators $x \neq y$. As in most problems of this sort, the local case was solved first and used to solve the global case. Here we will focus on the local case. The global case is considered in [**J2**] and [**JLLS**]. A group G is called *p-local* if for every prime $q \neq p$ and $x \in G$ there is a unique $y \in G$ such that $x = qy$. Such groups may be considered as $\mathbb{Z}_p$-modules, where p is a prime and $\mathbb{Z}_p$ is the integers localized at p. In the rest of this paper we will consider modules over $\mathbb{Z}_p$ for fixed prime p.

For M a module and α an ordinal, define the submodule $p^\alpha M$ by transfinite induction on α in the obvious way. For $x \in M$, let $|x|$, the *Ulm height* of x, be α if $x \in p^\alpha M \setminus p^{\alpha+1}M$ and ∞ if $x \in p^\alpha M$ for all α. Let the *length* of M, $l(M)$, be the least α such that $p^\alpha M = p^{\alpha+1}M$. We write tM for the torsion submodule of the module M. See Fuchs [**F**] for precise definitions.

If M is a $\mathbb{Z}_p$-module we say $X \subseteq M$ is a *decomposition set* if X is an independent set of elements of infinite order and for all $x_1, \ldots, x_n \in X$ and $a_1, \ldots, a_n \in \mathbb{Z}_p$,

$$|a_1x_1 + \cdots + a_nx_n| = \min_{1\leq i\leq n}\{|a_ix_i|\}.$$

We let $\langle X\rangle$ denote the submodule generated by X. If X is a decomposition set and $M/\langle X\rangle$ is torsion, we say X is a *decomposition basis* for M. Every Warfield module has a decomposition basis [**W1**]. If N is a submodule of M, we write N^0 for $\{x \in M : ax \in N \text{ for some } a \neq 0\}$. Note that each $x \in N^0$ has the same height in N^0 as it does in M.

The following theorem of Richman and Walker shows that we can impose any "reasonable" height structure on a module and then add enough elements to support it. Thus this theorem allows us to construct examples of modules with any desired height structure.

We define a *p-valuation* on a module M to be a function v_p on M satisfying the following properties for any x, $y \in M$, under the convention that $\infty > \alpha$ if α is an ordinal or ∞:

(a) $v_p(x)$ is an ordinal or ∞,
(b) $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$
(c) $v_p(px) > v_p(x)$
(d) $v_p(nx) = v_p(x)$ if $(n, p) = 1$.

It is easy to verify that $|x|$ satisfies these conditions.

THEOREM 2.1. [**RW**, Theorem 1] *Let M be a module and v_p a p-valuation on M. Then there is a module N such that $M \subseteq N$ and for all $x \in M$, $|x|$ in N is equal to $v_p(x)$, N/M is torsion and if $M = \langle x_1, \ldots, x_n\rangle$ for some decomposition set $\{x_1, \ldots, x_n\}$ in N, then N is a Warfield module.*

Ulm [**U**] defined an invariant $u_p(\alpha, G)$ for a group G for each prime p and ordinal α and proved that these invariants classify all countable torsion groups. Barwise and Eklof defined the *modified Ulm invariant* $\hat{u}_p(\alpha, G) = \min\{u_p(\alpha, G), \omega\}$. To classify all torsion groups in $L_{\infty\omega}$ they added the invariant $\hat{u}_p(\infty, G)$. For complete definitions, see [**BE**]. When p is understood, we will write simply $\hat{u}(\alpha, G)$.

Warfield [**W2**] defined his invariants using the decomposition basis. We say a sequence (α_i), $0 \leq i < \omega$, is an *Ulm sequence* if each α_i is either an ordinal or the symbol ∞ and for all i, $\alpha_{i+1} > \alpha_i$, again with the convention that $\infty > \alpha$ if α is ∞ or an ordinal. If x is an element of a module M, $U(x)$, the *Ulm sequence of* x, is the sequence $(|p^i x|)$. We call the Ulm sequences (α_i) and (β_i) *equivalent* if there are positive integers m and n such that $\alpha_{i+n} = \beta_{i+m}$ for all $i \geq 0$. For M a module with a decomposition basis X and e an equivalence class of Ulm sequences, the *Warfield invariant* is defined as

$$w(e, M) = \text{ the cardinality of } \{x \in X : U(x) \in e\}.$$

Warfield proved that this is independent of the choice of X and that these invariants, along with the Ulm invariants, serve to classify Warfield modules.

The author [**J2**] defined a class of modules, those with a partial decomposition basis, and an invariant that classifies them up to partial isomorphism. Let M be a module. We say $\mathcal{C}$ is a *partial decomposition basis* for M if

(i) $\mathcal{C}$ is a nonempty collection of finite subsets of M,
(ii) if $X \in \mathcal{C}$, then X is a decomposition set, and
(iii) if $X \in \mathcal{C}$ and $x \in M$, then there is a $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $x \in \langle Y\rangle^0$.

Note that the finite subsets of a decomposition basis form a partial decomposition basis. Now we define the analogue of the Warfield invariant for a module M with a partial decomposition basis $\mathcal{C}$. Let e be an equivalence class of Ulm sequences and define

$$\hat{w}(e, M) = \text{ the maximum } n \text{ such that for some } X \in \mathcal{C} \text{ and } x_1, ..., x_n \in X,$$
$$U(x_i) \in e \text{ for } 1 \leq i \leq n, \text{ if such a maximum exists, and } \omega \text{ otherwise.}$$

This class and invariant satisfy many of the properties we seek [**J2**]. The Warfield modules are a proper subclass of the class of modules with partial decomposition basis. The invariants are independent of the choice of $\mathcal{C}$. The invariants along with the modified Ulm invariants $\hat{u}$ classify the modules with partial decomposition basis up to $L_{\infty\omega}$-equivalence.

THEOREM 2.2. [**J2**, Theorem 12] *Let M and N be modules with partial decomposition bases. If $\hat{u}(\infty, M) = \hat{u}(\infty, N)$ and for every ordinal α and equivalence class e of Ulm sequences $\hat{u}(\alpha, M) = \hat{u}(\alpha, N)$ and $\hat{w}(e, M) = \hat{w}(e, N)$ then M and N are equivalent in $L_{\infty\omega}$.*

The converse of this theorem will be a corollary of the definability of the invariants. An algebraic proof of the converse is given in [**J2**, Theorem 12].

The following extension lemma is used to prove a more general classification theorem. We state it here because it will be useful for some of the definability proofs in the last section. We say an isomorphism f *preserves heights up to* α for an ordinal α if $\min\{|x|, \alpha\} = \min\{|f(x)|, \alpha\}$ for all $x \in dom(f)$.

LEMMA 2.3. [**J2**, Lemma 12] *Let M and N be $\mathbb{Z}_p$-modules, $f : S \to T$ an isomorphism of submodules of M and N respectively that preserves heights up to α for some ordinal α. Suppose that $x \in M$, $y \in N$, $|x| \geq \alpha$ and $|y| \geq \alpha$. Suppose further that $x + S$ and $y + T$ have the same order and if p^n is the order of $x + S$, then $f(p^n x) = p^n y$. Then $g : \langle S, x\rangle \to \langle T, y\rangle$, defined by setting $g \restriction S = f$ and $g(x) = y$, is an isomorphism preserving heights up to α.*

2.2. The Language $L_{\infty\omega}$. The results of this paper will be considered in light of the language of infinitary logic known as $L_{\infty\omega}$. This is an extension of the familiar language of first order logic to allow infinite conjunctions and disjunctions. Every formula φ of $L_{\infty\omega}$ has a quantifier rank $qr(\varphi)$. We let $L^{\alpha}_{\infty\omega}$ be the set of formulas of $L_{\infty\omega}$ with quantifier rank $\leq \alpha$. For full definitions see [**BE**]. For models of group theory $\mathfrak{A}$ and $\mathfrak{B}$ we may write $\mathfrak{A} \equiv_\infty \mathfrak{B}$ (resp. $\mathfrak{A} \equiv_\alpha \mathfrak{B}$) if they satisfy the same sentences of $L_{\infty\omega}$ ($L^{\alpha}_{\infty\omega}$).

An example of a sentence of $L_{\infty\omega}$ is one that may be abbreviated

$$\forall x \vee \{nx = 0 : n \in \omega, n \neq 0\}$$

where nx is an abbreviation of the expression $x + \cdots + x$ with n terms. It is clear that if G is a group, then it is torsion if and only if it satisfies this sentence. There is no sentence of first order logic with this property, as can be seen by a simple compactness argument. Note that this shows that the class of abelian groups classified by Barwise and Eklof is definable in $L_{\infty\omega}$.

In the process of proving their classification theorem Barwise and Eklof proved the following expressibility lemmas for torsion groups, which are of interest in their own right [**BE**]. They say that height is expressible in $L_{\infty\omega}$ and the torsion-free rank and Ulm invariants $u(\alpha, M)$ are expressible up to ω. The proofs apply just as well to $\mathbb{Z}_p$-modules.

LEMMA 2.4. *Let $\alpha = \omega\delta + n$ be an ordinal. Then there is an existential formula $\theta_\alpha(x)$ whose quantifier rank is δ if $n = 0$, and $\delta + 1$ if $n > 0$, such that for any module M and any $a \in M$, $M \models \theta_\alpha[a]$ if and only if $a \in p^\alpha M$.*

LEMMA 2.5. *Let $\alpha = \omega\delta + n$ be an ordinal, $m < \omega$ and M a $\mathbb{Z}_p$-module. There are sentences $\varphi_{\alpha,m}$, $\psi_{\alpha,m}$ of quantifier rank $\delta + m$, $\delta + m + 1$ respectively such that $M \models \varphi_{\alpha,m}$ ($M \models \psi_{\alpha,m}$) if and only if $rank(p^\alpha M) \geq m$ ($u(\alpha, M) \geq m$).*

The following theorem of Karp gives an algebraic formulation for the model theoretic notion of $\equiv_\alpha$ [**K**]. This will be useful in proving undefinability.

THEOREM 2.6. *Let δ be an ordinal and $\mathfrak{A}$ and $\mathfrak{B}$ models for $L_{\infty\omega}$. Then the following are equivalent:*

(i) *For each ordinal $\alpha \leq \delta$ there is a non-empty set I_α of isomorphisms each between a substructure of $\mathfrak{A}$ and a substructure of $\mathfrak{B}$ such that $I_\beta \subseteq I_\alpha$ whenever $\alpha < \beta$ and for any $\alpha < \delta$, $f \in I_{\alpha+1}$ and $x \in A$ ($y \in B$), there is a $g \in I_\alpha$ such that $f \subseteq g$ and $x \in dom(g)$ ($y \in rng(g)$)*

(ii) $\mathfrak{A} \equiv_\delta \mathfrak{B}$.

In that case, if $\varphi(x_1, \ldots, x_n)$ is a formula of quantifier rank $\alpha \leq \delta$, $f \in I_\alpha$ and $a_1, \ldots, a_n \in dom(f)$, then $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$ if and only if $\mathfrak{B} \models \varphi[f(a_1), \ldots, f(a_n)]$.

This theorem allows us to prove a more general classification theorem that says that the more the Ulm and Warfield invariants agree the more the modules agree. First we define the modified Warfield invariant up to some ordinal.

Let e and e' be equivalence classes of Ulm sequences and α an ordinal. We say e and e' are *α-equivalent*, written $e \sim_\alpha e'$, if there is a $(\beta_i) \in e$ and $(\gamma_i) \in e'$ such that $\min\{\beta_i, \alpha\} = \min\{\gamma_i, \alpha\}$ for all $i \geq 1$. If the module M has a partial decomposition basis $\mathcal{C}$, we define $\hat{w}_\alpha(e, M) =$ the largest n such that for some $X \in \mathcal{C}$ there are $x_1, \ldots, x_n \in X$ such that $[U(x_i)] \sim_\alpha e$ for all $1 \leq i \leq n$, if such an n exists, and ω otherwise. Equivalently [**J1**], $\hat{w}_\alpha(e, M) = \min\{\sum_{e' \sim_\alpha e} \hat{w}(e', M), \omega\}$.

THEOREM 2.7. *Let M and N be modules with partial decomposition bases $\mathcal{C}$ and $\mathcal{C}'$ respectively. Let λ be an ordinal such that $\hat{u}(\alpha, M) = \hat{u}(\alpha, N)$ for all $\alpha < \omega\lambda$. Suppose also that for any $\nu < \lambda$, $\hat{w}_{\omega(\nu+1)}(e, M) = \hat{w}_{\omega(\nu+1)}(e, N)$ for any equivalence class e of Ulm sequences, and if $l(tM) \leq \omega(\nu + 1)$ then $\hat{u}(\infty, M) = \hat{u}(\infty, N)$. Then $M \equiv_\lambda N$.*

The proof is based on a collection $\{I_\alpha\}$ that satisfies (i) of Theorem 2.6. Specifically, for $\alpha \leq \delta$ let I_α be the set of all isomorphisms $f : S \to T$ such that

(i) S and T are finitely generated submodules of M and N respectively,
(ii) f preserves heights up to $\omega\alpha$,
(iii) there are $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$ such that $S \subseteq \langle X\rangle^0$ and $T \subseteq \langle Y\rangle^0$,
(iv) there is a bijection $\varphi : X \to Y$ such that $U(x) = U(\varphi(x))$ up to $\omega\alpha$,
(v) for every $x \in X$ there is an integer n such that $f(p^n x) = p^n\varphi(x)$ and for every $y \in Y$ there is an integer n such that $f^{-1}(p^n y) = p^n\varphi^{-1}(y)$, and
(vi) $\langle X\rangle \subseteq S$ and $\langle Y\rangle \subseteq T$.

This theorem was proved in [**J2**, Theorem 17] for modules over a complete discrete valuation ring with $\hat{w}_{\omega\delta}(e, M) = \hat{w}_{\omega\delta}(e, N)$. The proof holds under the slightly weaker conditions of this theorem. The completeness condition was removed in [**JL2**]. We will prove the converse, for certain λ, in the next section.

3. Definability

Now we will look at the question of definability. We start with the invariant. As we shall see, "decomposition set" is not expressible in $L_{\infty\omega}$, so the invariant must be expressed in some other form.

We have defined $\hat{w}(e, M)$ for modules with partial decomposition bases only, but it is possible to define the Warfield invariant for arbitrary modules. Specifically, Stanton [**St2**] defines an invariant $w_M(u)$ for $u = (\beta_i)$ an Ulm sequence and M a Warfield module. Hunter, Richman and Walker [**HRW**] prove that this agrees with

$w(e, M)$ for $u \in e$ if M has a decomposition basis and is reduced. The definition may be expressed in the following form.

$$w_M(u) = \dim(M(u)/M(u)^*), \text{ where}$$
$$M(u) = \{x \in M : |p^k x| \geq \beta_k \text{ for all } k \geq 0\} \text{ and}$$

$$M(u)^* = \begin{cases} \langle \{x \in M(u) : |p^k x| \neq \beta_k \text{ for infinitely many } k\} \rangle & \text{if } \beta_i \neq \infty \text{ for all } i \\ t(M(u)), \text{ the torsion submodule of } M(u) & \text{if } \beta_i = \infty \text{ for some } i. \end{cases}$$

The dimension is as a vector space over $\mathbb{Z}/p\mathbb{Z}$ in the first case and over $\mathbb{Q}$ in the other. It may be verified that $w_M(u) = w_M(u')$ if $u \sim u'$. Define $\hat{w}_M(u) = \min(w_M(u), \omega)$.

Theorem 3.1. *Let M be a module with a partial decomposition basis, e an equivalence class of Ulm sequences and $u \in e$. Then $\hat{w}_M(u) = \hat{w}(e, M)$.*

The more general global version of this theorem is proved in [**JL1**].

Corollary 3.2. *For M a module, u an Ulm sequence, define $\hat{w}^\alpha_M(u)$ to be $min\{\sum \hat{w}_M(u'), \omega\}$, where the sum is taken over all $u' \sim_\alpha u$. Then $\hat{w}^\alpha_M(u) = \hat{w}_\alpha(e, M)$ for all α, all $u \in e$.*

The theorem and corollary allow us to restate the following results of Göbel, Leistner, Loth and Strüngmann [**GLLS**, Theorem 4.4 and preceding discussion] in terms of $\hat{w}(e, M)$.

Theorem 3.3. [**GLLS**] *If $\beta_i \neq \infty$ for all i then for each $m \in \omega$, there is a sentence $\theta_{\bar{\beta},m}$ such that $M \models \theta_{\bar{\beta},m}$ if and only if $\hat{w}(e, M) \geq m$. Furthermore, if we write $\beta_i = \omega\delta_i + n_i$, where δ_i is an ordinal and $n_i < \omega$, and if $\xi = sup\{\delta_i : i < \omega\}$, then the sentence has quantifier rank $\xi + \omega + m$.*

Note that since $\hat{w}(e, M) = 0$ if M is reduced and e contains some (β_i) with $\beta_i = \infty$, this theorem says that $\hat{w}(e, M)$ is expressible by a sentence in $L_{\infty\omega}$ for all reduced modules.

The next theorem, which was proved for modules with nice decomposition bases, is here extended to the more general case of modules with partial decomposition bases. The proof extends simply by extending the following lemma and combining it with Lemma 2.5.

Lemma 3.4. *Let M be a module with a partial decomposition basis $\mathcal{C}$, α an ordinal and e an equivalence class of Ulm sequences such that for some $\bar{\beta} = (\beta_i) \in e$, $\beta_i > \alpha$ for some i. Then $\hat{w}_\alpha(e, M) \geq m$ if and only if $rank(p^\alpha M) \geq m$.*

Proof. This is proved in [**GLLS**, discussion preceding Lemma 4.4] for modules with decomposition bases. Suppose $\hat{w}_\alpha(e, M) \geq m$. Then there are $X \in \mathcal{C}, x_1, \ldots, x_m \in X$ such that $U(x_i) \sim_\alpha e$ for all i and hence $\hat{w}_\alpha(e, \langle X \rangle^0) \geq m$. Since $\langle X \rangle^0$ has a decomposition basis, $rank(p^\alpha \langle X \rangle^0) \geq m$ and so $rank(p^\alpha M) \geq m$. Conversely, suppose $rank(p^\alpha M) \geq m$. Let $x_1, \ldots, x_m$ be independent representatives and choose $Y \in \mathcal{C}$ such that $x_1, \ldots, x_m \in \langle Y \rangle^0$. Then $rank(p^\alpha \langle Y \rangle^0) \geq m$ and so, since $\langle Y \rangle^0$ has a decomposition basis, $\hat{w}_\alpha(e, \langle Y \rangle^0) \geq m$ and hence $\hat{w}_\alpha(e, M) \geq m$ since $Y \in \mathcal{C}$. □

Theorem 3.5. [**GLLS**] *If $(\beta_i) \in e$, e an equivalence class of Ulm sequences, and $\beta_i > \alpha$ for some α and some i, then for each $m \in \omega$ there is a sentence $\varphi_{\alpha,m}$*

such that for every module M with partial decomposition basis, $M \models \varphi_{\alpha,m}$ if and only if $\hat{w}_\alpha(e,M) \geq m$. If $\alpha = \omega\delta + n$, $\varphi_{\alpha,m}$ has quantifier rank $\delta + m$.

Theorem 3.6. [**GLLS**] *Let M and N be modules. Suppose $M \equiv_\delta N$ for some δ. Then $\hat{u}(\alpha, M) = \hat{u}(\alpha, N)$ for all $\alpha < \omega\delta$ and if $l(t(M)) \leq \omega(\nu+1)$ then $\hat{u}(\infty, M) = \hat{u}(\infty, N)$.*

Theorem 3.7. *If M and N are $\mathbb{Z}_p$-modules with partial decomposition bases and $M \equiv_\infty N$, then $\hat{w}(e,M) = \hat{w}(e,N)$ for every e an equivalence class of Ulm sequences.*

Proof. Let $m < \omega$, e an equivalence class of Ulm sequences, and $\bar{\beta} = (\beta_i) \in e$. Suppose $\beta_i \neq \infty$ for all i. Let $\theta_{\beta,m}$ be as in Theorem 3.3. Then $\hat{w}(e,M) = m$ if and only if $M \models \theta_{\bar{\beta},m} \wedge \neg\theta_{\bar{\beta},m+1}$ if and only if $N \models \theta_{\bar{\beta},m} \wedge \neg\theta_{\bar{\beta},m+1}$ if and only if $\hat{w}(e,N) = m$. Similarly, $\hat{w}(e,M) = \omega$ if and only if $M \models \wedge_{m\in\omega}\theta_{\bar{\beta},m}$, which proves $\hat{w}(e,M) = \hat{w}(e,N)$ in this case as well.

Now suppose $\beta_i = \infty$ for some i. Then $\beta_i > \alpha$ for all α. For each ordinal α, take $\varphi_{\alpha,m}$ as in Theorem 3.5. Then $\hat{w}(e,M) \geq m$ if and only if $M \models \varphi_{\alpha,m}$ for all α if and only if $N \models \varphi_{\alpha,m}$ for all α if and only if $\hat{w}(e,N) \geq m$. $\square$

This theorem, when combined with Theorems 2.2 and 3.6, completes the classification of modules with partial decomposition bases in $L_{\infty\omega}$. Similarly, the following theorem completes the converse of Theorem 2.7 for suitable λ.

Theorem 3.8. *Let M and N be modules with partial decomposition bases $\mathcal{C}$ and $\mathcal{C}'$ respectively. Suppose $\lambda = \omega\gamma$ for some limit ordinal γ and $M \equiv_\lambda N$. Then $\hat{w}_{\omega(\nu+1)}(e,M) = \hat{w}_{\omega(\nu+1)}(e,N)$ for any equivalence class e of Ulm sequences and $\nu < \lambda$.*

Proof. The proof follows that of the converse of a classification theorem of Göbel, Leistner, Loth and Strüngmann [**GLLS**].

We first show for any $\alpha < \omega\lambda$ and $m \geq 0$, "$w_\alpha(e,M) \geq m$" can be written as a sentence of quantifier rank $< \lambda$. The proof breaks down into two cases. First, suppose $\beta_i < \alpha$ for every $i \in \omega$ and $\bar{\beta} \in e$. Then $\hat{w}_\alpha(e,M) = \hat{w}(e,M)$. Write β_i as $\omega\delta_i + n_i$ and let $\xi = sup\{\delta_i\}$. Then "$\hat{w}(e,M) \geq m$" is expressible in a sentence of quantifier rank $\xi + \omega + m$, by Theorem 3.3. Write α as $\omega\delta + n$, and δ as $\omega\delta' + n'$. Then $\delta < \lambda$, so $\delta' < \gamma$. Using the fact that both γ and $\omega\gamma$ are limit ordinals, we get $\xi + \omega + m \leq (\omega\delta' + n') + \omega + m = \omega(\delta'+1) + m < \omega\gamma = \lambda$. Thus the quantifier rank of "$\hat{w}_\alpha(e,M) \geq m$" is less than λ for all m.

Now suppose that for some $\bar{\beta} \in e$ and $i \in \omega, \beta_i > \alpha$. By Theorem 3.5 "$\hat{w}_\alpha(e,M) \geq m$" is expressible by a sentence of quantifier rank $\delta + m < \lambda$.

Thus in either case, "$\hat{w}_\alpha(e,M) \geq m$" is expressible by a sentence $\phi_{\alpha,m}$ of quantifier rank $< \lambda$. As above, it follows that $\hat{w}_\alpha(e,M) = \hat{w}_\alpha(e,N)$ for all $\alpha < \omega\lambda$ and, in particular, since $\nu < \lambda$, a limit ordinal, $\hat{w}_{\omega(\nu+1)}(e,M) = \hat{w}_{\omega(\nu+1)}(e,N)$. $\square$

4. Undefinability

Since decomposition sets are central to this theory, we would like to know whether for each n there is a formula $\psi_n(x_1, \dots, x_n)$ of $L_{\infty\omega}$ that says "$\{x_1, \dots, x_n\}$ is a decomposition set." We will now see that it follows from the preceding theorem that there is no such formula for any $n \geq 2$.

Theorem 4.1. *"X is a decomposition set" is not expressible in $L_{\infty\omega}$.*

Proof. More precisely, we mean that for any $n \geq 2$ there can be no formula $\psi_n(x_1, \dots, x_n)$ such that for any $\mathbb{Z}_p$-module M and $a_1, \dots, a_n \in M$, $M \models \psi_n[a_1, \dots, a_n]$ if and only if $\{a_1, \dots, a_n\}$ is a decomposition set in M. We sketch the proof. The full details are in [**J2**]. Suppose for some n there is such a ψ_n. Let $\delta =$ the quantifier rank of ψ_n.

Let $M = \sum_{i=1}^n \mathbb{Z}_p = \mathbb{Z}_p(1, \dots, 1) \oplus \sum_{i=1}^{n-1} \mathbb{Z}_p$. We will define a valuation on the summands and hence on M by defining the valuation on a sum as the minimum of the valuations on the summands. Let

$$v_p(m, \dots, m) = \omega\delta + |m| + 2,$$

where $|m|$ represents the ordinary height in $\mathbb{Z}_p$. Let

$$v_p(m_1, \dots, m_{n-1}, 0) = \begin{cases} \omega\delta + \min\{|m_i|\} + 2 & \text{if } |m_i| > 0 \text{ for all } i \leq n-1, \\ \omega\delta + 1 & \text{otherwise.} \end{cases}$$

Embed M into some N as in Theorem 2.1 so that $|x|$ in N is the same as $v_p(x)$ for all $x \in M$. Let $a_i = (0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the i-th position. Then $|a_i| = \omega\delta + 1$ for $1 \leq i \leq n$. But $|a_1 + \cdots + a_n| = |(1, \dots, 1)| = \omega\delta + 2$, so $\{a_1, \dots, a_n\}$ is not a decomposition set. Now let $b_i = pa_i$ for $1 \leq i \leq n$. Then $\{b_1, \dots, b_n\}$ is a decomposition set.

Define $f : \langle a_1, \dots, a_n \rangle \to \langle b_1, \dots, b_n \rangle$ by $f(a_i) = b_i$. Then if we let $S = \langle a_1, \dots, a_n \rangle, T = \langle b_1, \dots, b_n \rangle, X = \{b_1, \dots, b_n\}, Y = \{pb_1, \dots, pb_n\}$ and $\varphi = f \restriction X$, then f is in I_δ as defined in Theorem 2.7 with both modules being N, since all heights are $> \omega\delta$. The hypotheses of Theorem 2.7 are met when the two modules are the same, so $\{I_\delta\}$ satisfies (i) of Theorem 2.6, $f \in I_\delta$ satisfies the last part of that theorem, hence $N \models \psi_n[a_1, \dots, a_n]$ if and only if $N \models \psi_n[b_1, \dots, b_n]$, in other words, $\{a_1, \dots, a_n\}$ is a decomposition set if and only if $\{b_1, \dots, b_n\}$ is, a contradiction. □

This theorem does not rule out the possibility that the class of modules with partial decomposition bases is definable, in other words, that there is a sentence φ of $L_{\infty\omega}$ such that for any module M, $M \models \varphi$ if and only if M has a partial decomposition basis. It merely says that such a sentence, unlike our definition, will not be phrased in terms of decomposition sets. However, the following theorem shows that this is not possible. In fact, no class of modules that generalizes the class of Warfield modules in the sense of the following theorem is definable in $L_{\infty\omega}$. This was proved for groups in [**J2**]. Here we see more generally that no generalization of Warfield groups is definable in $L_{\infty\omega}$ even if we restrict our attention to local groups. In other words, neither Warfield modules, modules with decomposition bases nor modules with partial decomposition bases forms an elementary class in $L_{\infty\omega}$.

Theorem 4.2. *Let C be some class of $\mathbb{Z}_p$-modules that includes all Warfield modules and such that if a module M is in C, then any two elements of M are contained in a submodule with a decomposition basis. Then C is not definable in $L_{\infty\omega}$.*

Proof. Suppose φ is a sentence of $L_{\infty\omega}$ such that if M is a $\mathbb{Z}_p$-module then $M \models \varphi$ if and only if M is in C. Let $\delta = qr(\varphi)$. For each $i \geq 0$, let $\langle x_i \rangle$ be an infinite cyclic group with generator x_i. Define a valuation on this as

$$v_p(p^n a x_i) = \begin{cases} \omega\delta + n & \text{if } n < 3i \\ \infty & \text{if } n \geq 3i \end{cases} \text{ for } (a, p) = 1.$$

Then embed $\langle x_i \rangle$ in a module N_i as in Theorem 2.1. Then N_i is a Warfield module with decomposition basis $\{x_i\}$ such that heights of elements of $\langle x_i \rangle$ agree with the valuation v_p. Every $z \in N_i$ has a multiple in $\langle x_i \rangle$, and since $|p^k x_i| = \infty$ for sufficiently large k, there is a k such that $|p^k z| = \infty$. These properties transfer over to $\sum N_i$: $\sum N_i$ is a Warfield module and for every $z \in \sum N_i$, $|p^k z| = \infty$ for sufficiently large k.

Consider the following two elements of $\prod N_i$:

$$x = (p^i x_i)_i,$$

$$y = (p^{2i} x_i)_i.$$

Now we are ready to define $M \subseteq \prod N_i$. If $z \in \prod N_i$, let z_i denote the ith component of z.

$$M = \{z \in \prod N_i\colon \text{ for some } a, b \in \mathbb{Z}_p, l \geq k \geq 0, z_i = ap^{i-k}x_i + bp^{2i-k}x_i \text{ for all } i \geq l\}.$$

Note that M is a module that includes x, y and $\sum N_i$. Let us examine heights within M of elements of M. We will first prove by transfinite induction on $\alpha \leq \omega\delta$ that $|z| \geq \alpha$ for all $z \in M$ such that $z_i = 0$ for all $i < l$ for some l associated with z. Suppose for all $\beta < \alpha$ and all such $z \in M$, $|z| \geq \beta$. If α is a limit ordinal the result follows immediately from the definition of height. Suppose $\alpha = \beta + 1$. Suppose $z \in M$ with associated a, b, k and l and $z_i = 0$ for $i < l$. Define z' as $z'_i = 0$ if $i < l + 1$, $z'_i = ap^{i-k-1}x_i + bp^{2i-k-1}x_i$ for $i \geq l + 1$. Then $z' \in M$ with associated $a, b, k + 1$ and $l + 1$. Note that $pz'_i = z_i$ for each $i \neq l$. Hence $z = pz' + z_l$. Since $|z_l| \geq |x_l| = \omega\delta \geq \alpha$ and $|z'| \geq \beta$ by the induction hypothesis, we get $|z| \geq \min\{|pz'|, |z_l|\} \geq \beta + 1 = \alpha$. This completes the induction and shows in particular that all such elements of M have height at least $\omega\delta$.

This also shows that if we subtract off components 1 through $l - 1$, we get an element with height $\geq \omega\delta$, so every element of M may be written as the sum of an element of $\sum N_i$ and one of height $\geq \omega\delta$. In fact, each element has a multiple of the form $w + ax + by$ for some $w \in \sum\langle x_i \rangle$. By taking further multiples we may assume $|w| = \infty$. We have shown that

- every element in M may be written in the form $w + z$ where $w \in \sum N_i$ and $|z| \geq \omega\delta$,
- every element in M has a multiple of the form $w+ax+by$ where $w \in \sum\langle x_i \rangle$ and $|w| = \infty$.

We claim that for any $z \in M$, $|z| = \min\{|z_i|\}$. Let $z \in M$ with associated a, b, k and l. We may assume a and b are not both 0 since the claim holds for $z \in \sum N_i$. Suppose $\min\{|z_i|\} \leq \omega\delta + m$ and occurs at z_j. Choose n larger than $max\{j, l+m+1\}$. Define z' as $z'_i = 0$ for $i < n$ and $z'_i = ap^{i-m-k-1}x_i + bp^{2i-m-k-1}x_i$ for $i \geq n$. Then $z = z_1 + \cdots + z_{n-1} + p^{m+1}z'$. By the above claim, $|z'| \geq \omega\delta$, so $|p^{m+1}z'| \geq \omega\delta + m + 1 > \min_{i<n}\{|z_i|\}$. It follows that $|z| = \min_{i<n}\{|z_i|\} = \min\{|z_i|\}$.

Let us examine heights of multiples of x and y. $|p^m x| = \min\{|p^m p^i x_i|\}$. But $|p^{m+i}x_i| \neq \infty$ if and only if $m + i < 3i$, and that height increases with larger i, so this minimum occurs uniquely at the least i such that $m + i < 3i$, which is $i = \lfloor m/2 \rfloor + 1$. Thus $|p^m x| = \omega\delta + m + \lfloor m/2 \rfloor + 1$. Similarly, it can be shown that $|p^m y| = \omega\delta + 3m + 2$ and this occurs uniquely at $i = m + 1$.

We claim that the only torsion elements of M are in $\sum N_i$. Suppose $cz = 0$ for some $z \in M$ and $c \neq 0$. Suppose z has associated a, b, k and l. For $i > l$, $z_i = ap^{i-k}x_i + bp^{2i-k}x_i$. Then for such i we have $0 = cz_i = (a + bp^i)cp^{i-k}x_i$. Since

$cp^{i-k} \neq 0$ and x_i is of infinite order, $a + bp^i = 0$ for all $i > l$. This means that $a = 0$ and $b = 0$, hence $z_i = 0$ for $i > l$ and $z \in \sum N_i$.

We will show that M is not in C. Suppose $y_1, \ldots, y_n \in M$ form a decomposition set such that $x, y \in \langle y_1, \cdots, y_n \rangle^0$. By taking multiples we may assume that each y_j is a linear combination of x, y and an element of $\sum\langle x_i \rangle$, say $y_j = c_j x + d_j y + z_j$, where $z_j \in \sum\langle x_i \rangle$ and $|z_j| = \infty$. Note that from this point on z_j denotes an element of M, not a component of an element.

Now choose an integer $l > |c_j|$ for all $1 \leq j \leq n$ with $c_j \neq 0$. Then for every $m \geq l$, it may be verified that $|p^m c_j x| < |p^m d_j y|$ if $1 \leq j \leq n$ and $c_j \neq 0$. In fact, by multiplying each y_j by p^l, we may assume that for all $1 \leq j \leq n$, $|ac_j x| < |ad_j y|$ for all $a \neq 0$ whenever $c_j \neq 0$.

There must be at least one y_j with c_j and d_j not both 0 since no non-zero multiple of x is in $\sum N_i$. Suppose $c_j = d_j = 0$ for all $j \neq i$ for some i. Let $w = c_i x + d_i y$. Then $ax = bw + z$ and $a'y = b'w + z'$ for some $a, a', b, b' \in \mathbb{Z}_p \setminus \{0\}$ and $z, z' \in \sum N_i$. It follows that the j-th components of $ab'x$ and $a'by$ are equal for all sufficiently large j, a contradiction. Consequently, there are at least two instances of c_j and d_j not both 0. Without loss of generality we may assume that c_1, d_1 are not both 0, nor are c_2, d_2.

First suppose $c_1 \neq 0$ and $c_2 \neq 0$. Then, substituting $y_1 = c_1 x + d_1 y + z_1$ and $y_2 = c_2 x + d_2 y + z_2$, and recalling that $|z_1| = |z_2| = \infty$,

$$|(p-1)c_2 y_1 + c_1 y_2| = |pc_1 c_2 x + (p-1)c_2 d_1 y + c_1 d_2 y|.$$

Since $|c_2 d_1 y| > |c_2 c_1 x|$ and $|c_1 d_2 y| > |c_1 c_2 x|$, all components have height greater than $|c_1 c_2 x|$ and so $|(p-1)c_2 y_1 + c_1 y_2| > |c_1 c_2 x|$. By a similar calculation we find that $|(p-1)c_2 y_1| = |c_1 c_2 x|$ and $|c_1 y_2| = |c_1 c_2 x|$, contradicting the assumption that y_1 and y_2 are in a decomposition set.

Now suppose $c_1 = 0$, $c_2 = 0$ and $d_1, d_2 \neq 0$. By a similar argument we find that

$$|(p-1)d_2 y_1 + d_1 y_2| = |pd_1 d_2 y| > |d_1 d_2 y| = \min\{|(p-1)d_2 y_1|, |d_1 y_2|\},$$

again contradicting our assumption that the y_j's form a decomposition set. Thus there is at most one $c_j = 0$. We already saw that there is at most one nonzero c_j, so we must have, say, $c_1 = 0$ and $c_2 \neq 0$.

Choose $m \geq l$ such that $m \geq |d_1|$ and $2m + 1 \geq |c_2|$. By taking multiples, we may assume that $y_1 = p^m y + z_1$ and $y_2 = p^{2m+1} x + d_2 y + z_2$. Then $|y_1| = |p^m y| = \omega\delta + 3m + 2$, and the unique component of minimum height at $m + 1$ is $p^m p^{2(m+1)} x_{m+1} = p^{3m+2} x_{m+1}$. By our choice of l, $|p^{2m+1} x| < |d_2 y|$, so $|y_2| = |p^{2m+1} x| = \omega\delta + (2m+1) + \lfloor (2m+1)/2 \rfloor + 1 = \omega\delta + 3m + 2$ and this minimal height occurs at $i = \lfloor (2m+1)/2 \rfloor + 1 = m + 1$. So here too the $m+1$ component is the unique component of minimum height. This gives us $\omega\delta + 3m + 2 = \min\{|y_1|, |(p-1)y_2|\} = |y_1 + (p-1)y_2|$, since $\{y_1, y_2\}$ is a decomposition set. This means that the $m + 1$ component of $y_1 + (p-1)y_2 = p^m y + z_1 + (p-1)p^{2m+1} x + (p-1)d_2 y + (p-1)z_2$ must have height $\omega\delta + 3m + 2$. That component is

$$\begin{aligned} &p^m (p^{2(m+1)} x_{m+1}) + z_1 + (p-1)p^{2m+1} p^{m+1} x_{m+1} + (p-1)d_2 y + (p-1)z_2 \\ &= p^{3m+2} x_{m+1} + (p-1)p^{3m+2} x_{m+1} + \text{ terms of height } > \omega\delta + 3m + 2. \end{aligned}$$

But the first two terms sum to $p^{3m+3} x_{m+1}$, of height ∞. Since all other components of $y_1 + (p-1)y_2$ also have height $> \omega\delta + 3m + 2$, we reach a contradiction. Thus $M \notin C$.

Now let $N = \sum N_i \oplus \mathbb{Q} \oplus \mathbb{Q}$. Then N is a Warfield module since each summand is. We will prove that $M \equiv_\delta N$.

For $\nu \leq \delta$, let I_ν be the set of all isomorphisms $f : S \to T$ such that S and T are submodules of M and N respectively and

(i) f preserves heights up to $\omega\nu$.
(ii) $\sum N_i \subseteq S$, $\sum N_i \subseteq T$, $x, y \in S$.
(iii) $f \restriction \sum N_i = id_{\sum N_i}$ and f maps x and y to the unit elements of the first and second copies of $\mathbb{Q}$ respectively.

We may define $S = \sum N_i \oplus \langle x, y\rangle$ and $T = \sum N_i \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$, and then define $f : S \to T$ satisfying (iii). It is easy to see that this is a well-defined isomorphism satisfying (i) and (ii) as well. Thus $I_\nu \neq \emptyset$ for all ν.

Now suppose $f \in I_{\nu+1}$, $z \in M$, $z \notin S$. We wish to extend f to z. We may assume $|z| \geq \omega\delta$ since we may use an element of $\sum N_i \subseteq S$ to subtract off components of smaller height. Since z has a multiple of the form $w + ax + by$, with $w \in \sum N_i \subseteq S$ and $x, y \in S$, for some minimal k, $p^k z \in S$. Let $f(p^k z) = z_0 + z_1 + z_2$, where $z_0 \in \sum N_i$ and z_1 and z_2 are in the first and second copies of $\mathbb{Q}$, respectively. Now $z_1 = p^k z_1'$ and $z_2 = p^k z_2'$ for unique $z_1', z_2' \in \mathbb{Q}$. Also $\omega(\nu + 1) \leq |f(p^k z)| = |z_0 + z_1 + z_2| = |z_0|$, since f preserves heights up to $\omega(\nu + 1) \leq \omega\delta$. Thus we may choose $z_0' \in \sum N_i$ such that $p^k z_0' = z_0$ and $|z_0'| \geq \omega\nu$. Then we have $p^k(z_0' + z_1' + z_2') = f(p^k z)$ and $|z_0' + z_1' + z_2'| \geq \omega\nu$. Furthermore, we claim that the order of $z_0' + z_1' + z_2' + T$ is p^k. Suppose for some $l < k$, $p^l(z_0' + z_1' + z_2') \in T$, say $f(z') = p^l(z_0' + z_1' + z_2')$ for some $z' \in S$. Then $f(p^{k-l} z') = p^k(z_0' + z_1' + z_2') = f(p^k z)$. Since f is injective, $p^{k-l} z' = p^k z$ and $p^{k-l}(z' - p^l z) = 0$. Since all torsion elements of M lie in $\sum N_i \subseteq S$, $z' - p^l z \in S$, so since $z' \in S$, this gives $p^l z \in S$, contradicting the minimality of k. By Lemma 2.3, we may extend f to an isomorphism g such that $g(z) = z_0' + z_1' + z_2'$ and g preserves heights up to $\omega\nu$. Then $g \in I_\nu$, as desired.

Next suppose $f \in I_{\nu+1}$, $z \in N$, say $z = z_0 + z_1 + z_2$, where $z_0 \in \sum N_i$ and z_1 and z_2 are in the first and second copies of $\mathbb{Q}$ respectively. Now $z_0 \in \sum N_i \subseteq T$, so we may assume $z = z_1 + z_2$. Since this has a multiple in $\mathbb{Z}_p \oplus \mathbb{Z}_p \subseteq T$, there is a minimal k such that $p^k z \in T$, say $f(w) = p^k z, w \in S$. Then $|w| \geq \omega(\nu + 1)$ since $|p^k z| = \infty$ and f preserves heights up to $\omega(\nu+1)$. Thus $w = p^k w'$ for some w' with $|w'| \geq \omega\nu$. Also, the order of $w' + S$ is p^k by the same argument as in the previous case, since all torsion elements of N are contained in $\sum N_i \subseteq T$. Then by Lemma 2.3, there is a $g \in I_\nu$ extending f such that $g(w') = z$.

Now we have shown that (i) of Theorem 2.6 is satisfied, so $M \equiv_\delta N$. But then $M \models \varphi$ if and only if $N \models \varphi$, i.e., $M \in C$ if and only if $N \in C$, a contradiction. $\square$

5. Conclusions

We have defined a class of modules, those with partial decomposition bases, and invariants, $\hat{w}(e, M)$ and $\hat{u}(\alpha, M)$ for all α an ordinal or ∞ and equivalence classes of Ulm sequences e, with the following properties, most of which are proved in [**J2**]:

(i) The class includes the modules classified by Warfield's Theorem, and in fact all modules with decomposition bases, since the finite subsets of a decomposition basis form a partial decomposition basis.
(ii) The class is closed under $\equiv_\infty$.
(iii) $\hat{w}$ invariants agree up to ω with Warfield's invariants for Warfield modules by Theorem 3.1.

(iv) The invariants classify members of the class up to $L_{\infty\omega}$-equivalence by Theorem 2.2 and Corollary 3.7.

(v) The invariants are expressible in $L_{\infty\omega}$ for reduced modules by Theorem 3.3.

(vi) The class is not definable in $L_{\infty\omega}$ by Theorem 4.2.

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Abelian groups with partial decomposition bases in $L^{\delta}_{\infty\omega}$, Part I

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Dedicated to Professor Rüdiger Göbel in honor of his 70th birthday

Abstract. We consider the class of abelian groups possessing partial decomposition bases in $L^{\delta}_{\infty\omega}$ for δ an ordinal. This class contains the class of Warfield groups which are direct summands of simply presented groups or, alternatively, are abelian groups possessing a nice decomposition basis with simply presented cokernel. We prove a classification theorem using numerical invariants that are deduced from the classical Ulm-Kaplansky and Warfield invariants. This extends earlier work by Barwise-Eklof, Göbel and the authors.

1. Introduction

One aspect of the model theory of groups is to ask for natural generalizations of classical algebraic results in a model theoretic way. Classifications of certain classes of groups (modules) in terms of numerical invariants are an especially good source for such model theoretic descriptions. It was Szmielew [**Sz**] who first took this point of view and considered abelian groups model-theoretically. Later on Barwise and Eklof [**BE**] took up Szmielew's approach and characterized the equivalence classes of torsion abelian groups with respect to the relation of satisfying the same sentences of some infinitary language L, e.g. $L_{\omega_\alpha\omega}$. This results in a model theoretic version of the classification of totally projective p-groups in terms of their Ulm-Kaplansky invariants (see Ulm [**U**], Hill [**H**] and Walker [**Wal**]). In particular Barwise and Eklof obtained as a corollary the classical theorem by Ulm stating that two countable abelian p-groups are isomorphic if and only if their Ulm-Kaplansky invariants coincide. Their proof used derivatives of the Ulm-Kaplansky invariants and a so-called Karp system of isomorphisms I_α that allows the lifting of partial mappings to global ones. Recall that the usual axioms for abelian groups can be stated in the lower predicate calculus (i.e. in $L_{\omega\omega}$), however it is necessary to allow languages with infinite expressions to characterize torsion groups or simple groups. For instance, the compactness theorem shows that abelian torsion groups are not axiomatizable in $L_{\omega\omega}$.

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Extending the classification of totally projective p-groups algebraically it was Warfield [**War2**] who introduced new invariants, the so-called Warfield invariants, to classify up to isomorphism p-local abelian groups possessing a nice decomposition basis with simply presented cokernel. Motivated by the result of Barwise and Eklof the first author [**J1**], [**J2**] considered Warfield's theorem and proved an analogous result in $L_{\infty\omega}$ for a larger class of R-modules in the case of $R = \mathbb{Z}_p$ and also in the global case of $R = \mathbb{Z}$. Again variants of the classical Warfield invariants were needed. Independently the second, third and fourth authors proved similar results together with Göbel (see [**GLLS**]).

Passing to $L^{\delta}_{\infty\omega}$ the authors were able to prove in the local case analogous results using Karp systems that consist entirely of partial isomorphisms that preserve heights up to ordinals $\leq \omega\delta$. In this paper we will show the global model theoretic classification of modules with partial decomposition bases in $L^{\delta}_{\infty\omega}$.

The rest of the paper is organized as follows: In section 2 we recall the basic definitions and notation concerning p-groups and Warfield modules from module theory. Then in section 3 we review the basics of model theory needed for our paper. Section 4 contains the known classifications from [**BE**], [**GLLS**], [**J1**] and [**J2**] in the language $L_{\infty\omega}$. Section 5 contains a classification in $L^{\delta}_{\infty\omega}$ for $\mathbb{Z}_p$-modules with partial decomposition bases which can be found in [**JL**]. Finally, section 6 contains our main theorem which is the classification theorem in $L^{\delta}_{\infty\omega}$ in the global case for groups with partial decomposition bases.

For notation and terminology on abelian groups and on model theory, the reader may refer to [**F1**], [**F2**], [**L**] and [**R**].

2. Notations and terminology

Throughout this paper, R will denote an arbitrary principal ideal domain, unless stated otherwise, and G will denote an R-module. A module over $\mathbb{Z}$ is referred to as a *group*. The torsion part of G is denoted by tG. For each ordinal α and prime $p \in R$, a submodule $p^{\alpha}G$ is defined as follows: $pG = \{pg : g \in G\}$, $p^{\alpha+1}G = p(p^{\alpha}G)$, and $p^{\alpha}G = \bigcap_{\beta<\alpha} p^{\beta}G$ if α is a limit ordinal. The *p-length* of G is the smallest ordinal τ such that $p^{\tau}G = p^{\tau+1}G$. With every $x \in G$, we associate its *p-height* $|x|_p$, that is, $|x|_p = \alpha$ if $x \in p^{\alpha}G \backslash p^{\alpha+1}G$ and $|x|_p = \infty$ if $x \in p^{\infty}G = \bigcap_{\alpha} p^{\alpha}G$. If S is a subset of G, then $\langle S \rangle$ denotes the submodule of G generated by S, and $\langle S \rangle^0$ denotes the set of all elements $x \in G$ such that $rx \in \langle S \rangle$ for some $0 \neq r \in R$. If x is an element of S, then the p-height of x computed in G coincides with the p-height of x computed in $\langle S \rangle^0$.

For a prime p, $\mathbb{Z}_p$ denotes the ring of integers localized at p. Notice that the natural map $G \to G_p = G \otimes \mathbb{Z}_p$ with $x \mapsto x_p = x \otimes 1$ preserves p-heights (cf. [**F2**, Part 2, Lemma 13]). Suppose S and T are submodules of R-modules G and H, respectively. For an ordinal α, an isomorphism $f : S \to T$ is called *α-height-preserving* if for all $x \in S$ and primes $p \in R$ we have $\min\{|x|_p, \alpha\} = \min\{|f(x)|_p, \alpha\}$ where all p-heights are computed in G and H, respectively. Let $G[p] = \{x \in G : px = 0\}$. Recall that the *Ulm-Kaplansky invariants* of G are defined by

$$u_p(\alpha, G) = \dim(p^{\alpha}G)[p]/(p^{\alpha+1}G)[p]$$

for α an ordinal, and

$$u_p(\infty, G) = \dim(p^{\infty}G)[p].$$

The subscript p may be dropped if it is clear which prime p is considered. Ulm [**U**] proved that these cardinal numbers are isomorphism invariants of countable torsion groups.

We adopt the convention that $\infty > \alpha$ for α an ordinal or ∞. An *Ulm sequence* is an increasing sequence $\overline{\beta} = (\beta_i : i < \omega)$ where each β_i is an ordinal or the symbol ∞. Two Ulm sequences $\overline{\beta} = (\beta_i)$ and $\overline{\gamma} = (\gamma_i)$ are called *equivalent*, and we write $\overline{\beta} \sim \overline{\gamma}$, if there exist $k, l < \omega$ such that $\beta_{i+k} = \gamma_{i+l}$ for all $i < \omega$. The equivalence class of $\overline{\beta}$ is denoted by $[\overline{\beta}]$. The *p-Ulm sequence of $x \in G$* is the sequence $U_p(x) = (|p^i x|_p : i < \omega)$.

For an element x of a group G the *Ulm matrix of* x is the doubly infinite $\mathbf{P} \times \omega$ matrix $U(x)$ having $U_p(x)$ as its p-th row. More generally, an *Ulm matrix* is a matrix $A = [a_{p,i}]_{(p,i)\in \mathbf{P}\times\omega}$ such that each row is an Ulm sequence. If n is a positive integer, we let nA be the Ulm matrix having $a_{p,i+|n|_p}$ as its (p,i) entry where $|n|_p$ is the p-height of n in $\mathbb{Z}$. It follows that $nU(x) = U(nx)$. If $A = [a_{p,i}]$ and $B = [b_{p,i}]$, we write $A \geq B$ in case $a_{p,i} \geq b_{p,i}$ for all (p,i). Two Ulm matrices A and B are called *compatible* if there are positive integers m and n such that $mB \geq A$ and $nA \geq B$. This yields an equivalence relation, and the equivalence classes are called *compatibility classes*. Note that if two Ulm matrices are compatible, their p-th rows coincide for almost all primes p. A subset X of G is called a *decomposition set* if

(1) all elements of X are independent and have infinite order;
(2) for each $x = k_1x_1 + \ldots + k_nx_n$ ($k_1, \ldots, k_n \in R$, $x_1, \ldots, x_n \in X$) and prime $p \in R$, we have $|x|_p = \min\{|k_ix_i|_p\}$.

X is called a *decomposition basis for* G if in addition, $G/\langle X\rangle$ is torsion. A decomposition basis X for G is called *nice* if for every prime p, $\langle X\rangle_p$ is a p-nice subgroup of G_p, that is, if every coset $x + \langle X\rangle_p$ ($x \in G_p$) has an element of maximal p-height. Groups ($\mathbb{Z}_p$-modules, resp.) possessing a nice decomposition basis X with simply presented quotient $G/\langle X\rangle$ are called *Warfield groups* (*Warfield modules*, resp.).

Stanton defined cardinal numbers that, together with the Ulm-Kaplansky invariants, form a complete set of isomorphism invariants for Warfield groups (see Hunter-Richman [**HR**], Stanton [**St**]).

In [**J2**], Jacoby generalized the concept of decomposition basis: If G is an R-module, a system $\mathcal{C}$ is called a *partial decomposition basis for* G if

(1) $\mathcal{C}$ is a nonempty collection of finite subsets of G;
(2) if $X \in \mathcal{C}$, then X is a decomposition set;
(3) if $X \in \mathcal{C}$ and $x \in G$, there is $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $x \in \langle Y\rangle^0$.

It is clear that if X is a decomposition basis for G, then the collection of all finite subsets of X is a partial decomposition basis for G.

3. Model-theoretic preliminaries

Throughout this paper, we will consider the language $L_{\infty\omega}$ which is an extension of an ordinary first order language L that allows infinite conjunctions and disjunctions and has a variable v_α for every ordinal α. In this paper we will take L to be the language of group theory with 0, + and $-$.

We define for each ordinal α a collection $L^\alpha_{\infty\omega}$ of formulas as the smallest collection F of formulas which contains the atomic formulas and is closed under the following logical operations:

(L1) If $\varphi \in F$, then $\neg\varphi \in F$.
(L2) If $\Phi \subseteq F$, then $\bigwedge \Phi, \bigvee \Phi \in F$.
(L3) If $\varphi \in L^{\beta}_{\infty\omega}$ for $\beta < \alpha$ and υ is a variable, then $\exists \upsilon\, \varphi, \forall \upsilon\, \varphi \in F$.

In (L2) $\bigwedge \Phi$ ($\bigvee \Phi$, resp.) denotes the conjunction (disjunction, resp.) of all elements of the set Φ. Then $L_{\infty\omega} = \bigcup_{\alpha} L^{\alpha}_{\infty\omega}$. The *quantifier rank* $qr(\varphi)$ of a formula $\varphi \in L_{\infty\omega}$ is defined to be the least ordinal α such that $\varphi \in L^{\alpha}_{\infty\omega}$.

Models for a language are understood to be sets of elements, which, in combination with the language-specific constants, functions and relations, satisfy the axioms of the language and in which every possible sentence has a distinct truth value. Models are denoted by $\mathfrak{A} = \langle A, ...\rangle$, where A is a set of elements called the universe. If $\varphi \in L_{\infty\omega}$ is a formula with at most n variables, $a_1, ..., a_n \in A$ and $\varphi(a_1, ..., a_n)$ is true, we write $\mathfrak{A} \models \varphi[a_1, ..., a_n]$, and accordingly for a sentence φ which is true, $\mathfrak{A} \models \varphi$.

Let α be an ordinal. Then two models $\mathfrak{A} = \langle A, ...\rangle$ and $\mathfrak{B} = \langle B, ...\rangle$ for $L_{\infty\omega}$ are called $L^{\alpha}_{\infty\omega}$*-equivalent* (resp. $L_{\infty\omega}$-equivalent), and we write $\mathfrak{A} \equiv_{\alpha} \mathfrak{B}$ (resp. $\mathfrak{A} \equiv_{\infty} \mathfrak{B}$), if for all sentences $\varphi \in L^{\alpha}_{\infty\omega}$ (resp. $\varphi \in L_{\infty\omega}$) we have

$$\mathfrak{A} \models \varphi \text{ if and only if } \mathfrak{B} \models \varphi.$$

$L^{\alpha}_{\infty\omega}$-equivalent and $L_{\infty\omega}$-equivalent models can be characterized using partial isomorphisms between them:

THEOREM 3.1 (Karp [**K**]). *Let $\mathfrak{A} = \langle A, \ldots\rangle$ and $\mathfrak{B} = \langle B, \ldots\rangle$ be models for $L_{\infty\omega}$ and δ an ordinal or the symbol ∞. Then the following are equivalent:*

(1) $\mathfrak{A} \equiv_{\delta} \mathfrak{B}$*;*
(2) *For each ordinal $\nu \leq \delta$ there is a non-empty set I_ν of isomorphisms on finitely generated substructures of $\mathfrak{A}$ into $\mathfrak{B}$ such that*
   (a) *if $\nu \leq \mu$, then $I_\mu \subseteq I_\nu$;*
   (b) *if $\nu < \delta$, $f \in I_{\nu+1}$ and $x \in A$ ($y \in B$, resp.), then f extends to a map $f' \in I_\nu$ such that $x \in domain(f')$ ($y \in range(f'), resp.$).*

4. Equivalence in $L_{\infty\omega}$

Ulm [**U**] gave a complete classification of countable p-groups in terms of the Ulm-Kaplansky invariants defined earlier. Warfield [**War2**] extended this theorem to Warfield modules by introducing invariants for an R-module M with decomposition basis X, where R is a discrete valuation ring with prime p. In the definitions that follow e will be an equivalence class of Ulm sequences, c a compatibility class of Ulm matrices and p a prime. Let

$$w(e, M) = |\{x \in X : U_p(x) \in e\}|.$$

Stanton [**St**] extended these invariants to groups and proved that they classify Warfield groups up to isomorphism. If G has a decomposition basis X, the *Warfield invariants* are defined by

$$w(c, p, e, G) = |\{x \in X : U(x) \in c \text{ and } U_p(x) \in e\}|.$$

Ulm's Theorem was extended by Barwise and Eklof [**BE**] who modified the Ulm-Kaplansky invariants: for a p-group G and any ordinal α, define $\hat{u}(\alpha, G) = \min\{u(\alpha, G), \omega\}$ and $\hat{u}(\infty, G) = \min\{u(\infty, G), \omega\}$. They proved that these invariants classify all torsion groups in $L_{\infty\omega}$.

Let M be an R-module with partial decomposition basis $\mathcal{C}$ where R is a discrete valuation ring with prime p. Jacoby [**J2**] defined $\hat{w}(e, M)$ to be the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ such that $U_p(x_i) \in e$ for all $i = 1, \ldots, n$. If no such n exists, put $\hat{w}(e, M) = \omega$. These invariants, along with the invariants of Barwise and Eklof, classify all R-modules with partial decomposition bases up to $L_{\infty\omega}$-equivalence.

Similarly for groups, Jacoby adapted $w(c, p, e, G)$ as follows: Let $\hat{w}(c, p, e, G)$ be the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ such that $U(x_i) \in c$ and $U_p(x_i) \in e$ for all $i = 1, \ldots, n$. If no such n exists, put $\hat{w}(c, p, e, G) = \omega$. The next result, which will be needed, shows that this ordinal is independent of the choice of partial decomposition basis. First we need a lemma.

LEMMA 4.1 (Jacoby [**J2**]). *Let M be a module over a principal ideal domain R which has partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$. Let $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Then there are decomposition sets X' and Y' such that $X \subseteq X', Y \subseteq Y', X'$ and Y' are unions of ascending chains of elements of $\mathcal{C}$ and $\mathcal{D}$ respectively, and $\langle X' \rangle^0 = \langle Y' \rangle^0$.*

PROOF. We will define, by induction on i, sets $X_i \in \mathcal{C}$, $Y_i \in \mathcal{D}$. Let $X_0 = X$ and $Y_0 = Y$. Suppose X_i and Y_i have been chosen. Choose $X_{i+1} \in \mathcal{C}$ such that $X_i \subseteq X_{i+1}$ and $Y_i \subseteq \langle X_{i+1} \rangle^0$, by a finite number of applications of condition (3) of the definition. Then choose similarly $Y_{i+1} \in \mathcal{D}$ such that $Y_i \subseteq Y_{i+1}$ and $X_{i+1} \subseteq \langle Y_{i+1} \rangle^0$.

Let $X' = \bigcup_{i \in \omega} X_i$ and $Y' = \bigcup_{i \in \omega} Y_i$. We claim that $\langle X' \rangle^0 \subseteq \langle Y' \rangle^0$. Let $z \in \langle X' \rangle^0$. Then for some $n \in \mathbb{Z}$, $a \in R$ and $x_1, ..., x_n \in X'$, $az \in \langle x_1, \ldots, x_n \rangle$. Choose $X_i \supseteq \{x_1, ..., x_n\}$. Then $\{x_1, ..., x_n\} \subseteq \langle Y_i \rangle^0$, so $\{bx_1, ..., bx_n\} \subseteq \langle Y_i \rangle$ for some $b \in R$. But then $baz \in \langle Y_i \rangle \subseteq \langle Y' \rangle$, so $z \in \langle Y' \rangle^0$. Similarly, $\langle Y' \rangle^0 \subseteq \langle X' \rangle^0$. $\square$

THEOREM 4.2 (Jacoby [**J2**]). *Suppose G is a group with partial decomposition basis $\mathcal{C}$ and assume c is a compatibility class of Ulm matrices, p is a prime and e is an equivalence class of Ulm sequences such that $\hat{w}(c, p, e, G) \geq n$. If $\mathcal{D}$ is any partial decomposition basis for G and $Y \in \mathcal{D}$, then there exists $\tilde{Y} \in \mathcal{D}$ such that $Y \subseteq \tilde{Y}$ and $\tilde{Y}$ contains elements $y_1, \ldots, y_n$ such that $U(y_i) \in c$ and $U_p(y_i) \in e$ for all $i = 1, \ldots, n$.*

PROOF. Let c, p and e be given and suppose $\hat{w}(c, p, e, G) \geq n$. Then by definition there is a $X \in \mathcal{C}$ containing elements $x_1, \ldots, x_n$ such that $U(x_i) \in c$ and $U_p(x_i) \in e$ for $1 \leq i \leq n$. Let Y be as given and choose X' and Y' as in Lemma 4.1. Then X' and Y' are both decomposition bases for $\langle X' \rangle^0 = \langle Y' \rangle^0$. Stanton [**St**] proved that $w(c, p, e, G)$ is independent of the choice of decomposition basis, so $w(c, p, e, \langle Y' \rangle^0) \geq n$. Thus Y' contains elements $y_1, \ldots, y_n$ such that $U(y_i) \in c$ and $U_p(y_i) \in e$ for $1 \leq i \leq n$. Choose $\tilde{Y} \in \mathcal{D}$ containing Y and $y_1, \ldots, y_n$. $\square$

Since the natural map $G \to G_p$ preserves p-heights, repeated application of this theorem yields

COROLLARY 4.3. *Suppose G is a group with partial decomposition basis, p is a prime and e is an equivalence class of Ulm sequences. Then $\hat{w}(e, G_p) = \min\{\sum_c \hat{w}(c, p, e, G), \omega\}$.*

5. Equivalence in $L^{\delta}_{\infty\omega}$: Local classification

In [**BE**], Barwise and Eklof classified p-groups up to $L^{\delta}_{\infty\omega}$-equivalence:

THEOREM 5.1 (Barwise-Eklof [**BE**]). *Let G and H be p-groups and let δ be an ordinal. Suppose*

(1) *$\hat{u}(\alpha, G) = \hat{u}(\alpha, H)$ for all ordinals $\alpha < \omega\delta$;*
(2) *if $length(G) < \omega\delta$, then $\hat{u}(\infty, G) = \hat{u}(\infty, H)$.*

Then $G \equiv_\delta H$. If δ is a limit ordinal, the converse also holds.

Let α be an ordinal. Two Ulm sequences (β_i) and (γ_i) are said to be *equal up to* α, and we write $(\beta_i) =_\alpha (\gamma_i)$, if

$$\min\{\beta_i, \alpha\} = \min\{\gamma_i, \alpha\}$$

for all $i < \omega$. Two equivalence classes e and e' of Ulm sequences are called *α-equivalent*, and we write $e \sim_\alpha e'$, if there are Ulm sequences $(\beta_i) \in e$ and $(\gamma_i) \in e'$ which are equal up to α. In this case, we will also call any two Ulm sequences $u \in e$ and $u' \in e'$ *α-equivalent* and write $u \sim_\alpha u'$. It is clear that $u \sim_\alpha u'$ if and only if $[u] \sim_\alpha [u']$. If R is a discrete valuation ring, define

$$\hat{w}_\alpha(e, M) = \min\{\sum_{e' \sim_\alpha e} \hat{w}(e', M), \omega\}.$$

The next lemma will be useful and can be easily verified.

LEMMA 5.2. *Let M be a $\mathbb{Z}_p$-module with partial decomposition basis. Then M has a partial decomposition basis $\mathcal{C}$ such that*

(1) *$X \in \mathcal{C}, X' \subseteq X \Rightarrow X' \in \mathcal{C}$;*
(2) *$\{x_1, \ldots, x_n\} \in \mathcal{C}, a_1, \ldots, a_n \in \mathbb{Z}_p \backslash \{0\} \Rightarrow \{a_1 x_1, \ldots, a_n x_n\} \in \mathcal{C}$.*

The following classification of $\mathbb{Z}_p$-modules with partial decomposition bases in $L^\delta_{\infty\omega}$ is the main result of [**JL**] and will be generalized in Theorem 6.10:

THEOREM 5.3 (Jacoby-Loth [**JL**]). *Let M and N be $\mathbb{Z}_p$-modules with partial decomposition bases and let δ be an ordinal. Suppose*

(1) *$\hat{u}(\alpha, M) = \hat{u}(\alpha, N)$ for all $\alpha < \omega\delta$;*
(2) *$\hat{w}_{\omega(\nu+1)}(e, M) = \hat{w}_{\omega(\nu+1)}(e, N)$ for all equivalence classes e of Ulm sequences and ordinals $\nu < \delta$;*
(3) *if $length(tM) < \omega\delta$, then $\hat{u}(\infty, M) = \hat{u}(\infty, N)$.*

Then $M \equiv_\delta N$.

PROOF. A detailed proof can be found in [**JL**]. We would like to give the basic idea of this proof as it uses a construction which is needed for our proof of Theorem 6.10. Suppose M and N are $\mathbb{Z}_p$-modules satisfying conditions (1)-(3) of Theorem 5.3 and let $\mathcal{C}_M$ and $\mathcal{C}_N$ be partial decomposition bases for M and N, respectively, as in Lemma 5.2. Consider the system $\{I_\nu : \nu \leq \delta\}$ where each I_ν is the set of all maps $f : S \to T$ such that there are decomposition sets $X \in \mathcal{C}_M$ and $Y \in \mathcal{C}_N$ with $f(X) = Y$ satisfying the following:

(i) S and T are finitely generated submodules of M and N, respectively;
(ii) f is an $\omega\nu$-height-preserving isomorphism;
(iii) $X \subseteq S \subseteq \langle X \rangle^0$ and $Y \subseteq T \subseteq \langle Y \rangle^0$.

It suffices to show that the system $\{I_\nu : \nu \leq \delta\}$ satisfies condition (2) of Karp's Theorem 3.1. Clearly, each set I_ν is non-empty as it contains the zero function. Let $f : S \to T$ be a map in $I_{\nu+1}$ ($\nu < \delta$) with associated decomposition sets $X \in \mathcal{C}_M$ and $Y \in \mathcal{C}_N$, and let $a \in M$. Then there is $X' = X \cup \{x_1, \ldots, x_m\} \in \mathcal{C}_M$ such that

$a \in \langle X' \rangle^0$. Since M and N have identical modified Warfield invariants (condition (2) of Theorem 5.3), a set $Y' = Y \cup \{y_1, \ldots, y_m\} \in \mathcal{C}_N$ can be constructed such that f extends to a map

$$f' : \langle S, x'_1, \ldots, x'_m \rangle \to \langle T, y'_1, \ldots, y'_m \rangle$$

in $I_{\nu+1}$ for some nonzero multiples x'_i of x_i and y'_i of y_i, where x'_i is mapped onto y'_i $(i = 1, \ldots, m)$. Finally, f' extends to a map $g \in I_\nu$ such that $a \in domain(g)$ due to the imposed conditions on the modified Ulm-Kaplansky invariants (conditions (1) and (3) of Theorem 5.3). By symmetry, condition (2) of Theorem 3.1 is satisfied and $M \equiv_\delta N$ follows. □

Remark 5.4. *It is easy to see that condition (2) in Theorem 5.3 follows from*
(2') $\hat{w}_{\omega\delta}(e, M) = \hat{w}_{\omega\delta}(e, N)$ *for all equivalence classes e of Ulm sequences.*

6. Equivalence in $L^\delta_{\infty\omega}$: Global classification

Let $A = [a_{p,i}]$ and $B = [b_{p,i}]$ be Ulm matrices, and let α be an ordinal. We say that A and B *are equal up to* α and write $A =_\alpha B$ if

$$\min\{a_{p,i}, \alpha\} = \min\{b_{p,i}, \alpha\}$$

for all primes p and $i < \omega$. We then define a relation on compatibility classes of Ulm matrices as follows: $c \sim_\alpha c'$ if and only if there are $A \in c$ and $B \in c'$ such that $A =_\alpha B$. This can be easily verified to be an equivalence relation. If $c \sim_\alpha c'$, then we call any Ulm matrices $C \in c$ and $C' \in c'$ *α-compatible* and write $C \sim_\alpha C'$. Notice that if any two Ulm matrices are α-compatible, their respective p-rows are equal up to α for almost all primes p. Suppose G and H are groups with elements $x \in G$ and $y \in H$ having α-compatible Ulm matrices such that $U_p(x) \sim_\alpha U_p(y)$ for all primes p. Clearly, then there are positive integers k and l such that $U_p(kx) =_\alpha U_p(ly)$ for all primes p.

For any group G with partial decomposition basis, ordinal α, compatibility class c of Ulm matrices, prime p and equivalence class e of Ulm sequences we define

$$\hat{w}_\alpha(c, p, e, G) = \min\{\sum_{c' \sim_\alpha c, e' \sim_\alpha e} \hat{w}(c', p, e', G), \omega\}.$$

Notice that for a finite decomposition set X we have

$$\begin{aligned}\hat{w}_\alpha(c, p, e, \langle X \rangle^0) &= \sum_{c' \sim_\alpha c, e' \sim_\alpha e} |\{x \in X : U(x) \in c' \text{ and } U_p(x) \in e'\}| \\ &= |\{x \in X : [U(x)] \sim_\alpha c \text{ and } [U_p(x)] \sim_\alpha e\}|.\end{aligned}$$

The following fact will be needed and is easily verified:

Lemma 6.1. *Suppose G is a group with partial decomposition basis $\mathcal{C}$. If $X \in \mathcal{C}$, then $\hat{w}_\alpha(c, p, e, G) \geq \hat{w}_\alpha(c, p, e, \langle X \rangle^0)$ for any ordinal α, compatibility class c of Ulm matrices, prime p and equivalence class e of Ulm sequences.*

The next result is the analog to Theorem 4.2 and shows that the cardinals $\hat{w}_\alpha(c, p, e, G)$ are independent of the choice of partial decomposition basis:

Theorem 6.2. *Let G be a group with partial decomposition basis $\mathcal{C}$, α an ordinal and n a positive integer. If $\hat{w}_\alpha(c, p, e, G) \geq n$ and $Y \in \mathcal{C}$, then there exists $Y' \in \mathcal{C}$ such that $Y \subseteq Y'$ and Y' contains elements $y_1, \ldots, y_n$ such that $[U(y_i)] \sim_\alpha c$ and $[U_p(y_i)] \sim_\alpha e$ for all $i = 1, \ldots, n$.*

PROOF. Letting $E = \{(c', e') : \hat{w}(c', p, e', G) \neq 0\}$, we have

$$\hat{w}_\alpha(c, p, e, G) = \min\{\sum_{c' \sim_\alpha c, e' \sim_\alpha e, (c', e') \in E} \hat{w}(c', p, e', G), \omega\}.$$

Let $(c', e') \in E$ and suppose $\hat{w}(c', p, e', G) \geq k$. By Theorem 4.2 there is $Y' \in \mathcal{C}$ such that $Y \subseteq Y'$ and Y' has k elements x with $U(x) \in c' \sim_\alpha c$ and $U_p(x) \in e' \sim_\alpha e$. Repeat this for all elements in E until at least n such elements have been adjoined. □

COROLLARY 6.3. *Let G be a group with partial decomposition basis $\mathcal{C}$, α an ordinal, c a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Then $\hat{w}_\alpha(c, p, e, G)$ is the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ such that $[U(x_i)] \sim_\alpha c$ and $[U_p(x_i)] \sim_\alpha e$ for all $i = 1, \ldots, n$. If no such n exists, $\hat{w}_\alpha(c, p, e, G) = \omega$.*

PROOF. Suppose there is a largest integer n such that there is $X \in \mathcal{C}$ containing n elements x satisfying $[U(x)] \sim_\alpha c$ and $[U_p(x)] \sim_\alpha e$. Then by Theorem 6.2, $\hat{w}_\alpha(c, p, e, G) \leq n$. On the other hand, $\hat{w}_\alpha(c, p, e, G) \geq \hat{w}_\alpha(c, p, e, \langle X \rangle^0) = n$ by Lemma 6.1. If no such n exists, $\hat{w}_\alpha(c, p, e, G) = \omega$ by Lemma 6.1. □

COROLLARY 6.4. *Let G and H be groups with partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$, respectively, α an ordinal, c a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Suppose $\hat{w}_\alpha(c, p, e, G) = \hat{w}_\alpha(c, p, e, H)$, $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and*

$$\hat{w}_\alpha(c, p, e, \langle X \rangle^0) > \hat{w}_\alpha(c, p, e, \langle Y \rangle^0).$$

Then there exists $Y' \in \mathcal{D}$ such that $Y \subseteq Y'$ and there is $y \in Y' \backslash Y$ such that $[U(y)] \sim_\alpha c$ and $[U_p(y)] \sim_\alpha e$.

PROOF. Let $n = \hat{w}_\alpha(c, p, e, \langle Y \rangle^0)$. Then

$$\hat{w}_\alpha(c, p, e, H) = \hat{w}_\alpha(c, p, e, G) \geq \hat{w}_\alpha(c, p, e, \langle X \rangle^0) > \hat{w}_\alpha(c, p, e, \langle Y \rangle^0) = n.$$

By Theorem 6.2, there is $Y' \in \mathcal{D}$ such that $Y \subseteq Y'$ and Y' contains $n+1$ elements y such that $[U(y)] \sim_\alpha c$ and $[U_p(y)] \sim_\alpha e$. Since Y contains only n such elements, one of them is in $Y' \backslash Y$. □

The following lemma will be useful:

LEMMA 6.5 (Stanton [**St**]). *Let G be a group with decomposition basis X, p a prime and $x_1, x_2 \in X$ with α-compatible Ulm matrices. Then there are elements $y_1, y_2 \in \langle X \rangle$ such that $U_p(y_1) =_\alpha U_p(x_2)$, $U_p(y_2) =_\alpha U_p(x_1)$ and $U_q(y_1) =_\alpha U_q(x_1)$, $U_q(y_2) =_\alpha U_q(x_2)$ for all primes $q \neq p$. The set $Y = (X \backslash \{x_1, x_2\}) \cup \{y_1, y_2\}$ is a decomposition basis for G and $\langle X \rangle = \langle Y \rangle$.*

Stanton proved this without the "up to α" conditions. The proof applies just as well in this case. Notice that in the lemma above, the elements x_1 and y_1 have α-compatible Ulm matrices.

LEMMA 6.6 (Jacoby [**J2**]). *Suppose G is a group with partial decomposition basis. Then G has a partial decomposition basis $\mathcal{C}$ for G such that*

(1) *$X \in \mathcal{C}, X' \subseteq X \Rightarrow X' \in \mathcal{C}$;*
(2) *$X \in \mathcal{C}, \langle X \rangle = \langle Y \rangle$, Y finite decomposition set $\Rightarrow Y \in \mathcal{C}$;*
(3) *$\{x_1, \ldots, x_n\} \in \mathcal{C}, a_1, \ldots, a_n \in \mathbb{Z} \backslash \{0\} \Rightarrow \{a_1 x_1, \ldots, a_n x_n\} \in \mathcal{C}$.*

PROOF. The lemma can be easily verified by taking the union of a chain of partial decomposition bases for G that alternately satisfy conditions (1) and (3) and condition (2). $\square$

LEMMA 6.7. *Let G and H be groups with partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$ satisfying conditions (1) and (2) of Lemma 6.6. Suppose α is an ordinal such that $\hat{w}_\alpha(c,p,e,G) = \hat{w}_\alpha(c,p,e,H)$ for every compatibility class c of Ulm matrices, prime p and equivalence class e of Ulm sequences. Assume $X \cup \{x\} \in \mathcal{C}$ and $Y \in \mathcal{D}$ such that*

$$\hat{w}_\alpha(c,p,e,\langle X\rangle^0) = \hat{w}_\alpha(c,p,e,\langle Y\rangle^0)$$

for all c, p and e. Then there exists an element $y \in H$ such that $Y \cup \{y\} \in \mathcal{D}$ and

$$\hat{w}_\alpha(c,p,e,\langle X \cup \{x\}\rangle^0) = \hat{w}_\alpha(c,p,e,\langle Y \cup \{y\}\rangle^0)$$

for all c, p and e. In fact, $U(x) \sim_\alpha U(y)$ and $U_p(x) \sim_\alpha U_p(y)$ for all primes p.

PROOF. Suppose $x \in G\backslash X$ and let c_0 be the compatibility class containing $U(x)$, p_0 a prime and e_0 the equivalence class of Ulm sequences containing $U_{p_0}(x)$. Then

$$\hat{w}_\alpha(c_0,p_0,e_0,\langle X \cup \{x\}\rangle^0) = \hat{w}_\alpha(c_0,p_0,e_0,\langle X\rangle^0) + 1 > \hat{w}_\alpha(c_0,p_0,e_0,\langle Y\rangle^0).$$

By Corollary 6.4 and condition (1) of Lemma 6.6 there is an element $z \in H\backslash Y$ satisfying

$$(*) \quad Y \cup \{z\} \in \mathcal{D},\ U(z) \sim_\alpha c_0 \text{ and } [U_{p_0}(z)] \sim_\alpha e_0.$$

Then $U(x)$ and $U(z)$ are α-compatible and $U_p(x)$ and $U_p(z)$ are α-equivalent for all but finitely many primes p, say, $p_1, \dots, p_n$. We will show by induction on n that z can be replaced by an element $y \in H$ satisfying $(*)$ such that $U_p(x)$ and $U_p(y)$ are α-equivalent for all primes p. For $n = 0$ there is nothing to show, so assume the assertion is true for $n-1$ and let $e_n = [U_{p_n}(x)]$. By our assumption, $[U_{p_n}(z)] \not\sim_\alpha e_n$ and therefore

$$\hat{w}_\alpha(c_0,p_n,e_n,\langle Y \cup \{z\}\rangle^0) = \hat{w}_\alpha(c_0,p_n,e_n,\langle Y\rangle^0) < \hat{w}_\alpha(c_0,p_n,e_n,\langle X \cup \{x\}\rangle^0).$$

By Corollary 6.4, there is $z' \in H$ such that $Y \cup \{z,z'\} \in \mathcal{D}$, $[U(z')] \sim_\alpha c_0$ and $[U_{p_n}(z')] \sim_\alpha e_n$. Now we apply Lemma 6.5 to the group $M = \langle Y \cup \{z,z'\}\rangle^0$ with decomposition basis $Y \cup \{z,z'\}$, elements z, z' and prime p_n. Then there are elements $y, y' \in \langle Y \cup \{z,z'\}\rangle$ such that $Y \cup \{y,y'\}$ is a decomposition basis for M, $\langle Y,y,y'\rangle = \langle Y,z,z'\rangle$ and $U_{p_n}(y) =_\alpha U_{p_n}(z')$, $U_{p_n}(y') =_\alpha U_{p_n}(z)$ and $U_q(y) =_\alpha U_q(z)$, $U_q(y') =_\alpha U_q(z')$ whenever $q \neq p_n$. Notice that condition $(*)$ holds for y as $Y \cup \{y\} \in \mathcal{D}$ by conditions (1) and (2) of Lemma 6.6, y and z have α-compatible Ulm matrices and $U_{p_0}(y) = U_{p_0}(z)$ up to α. To complete the induction, we need to verify that $U_p(x)$ and $U_p(y)$ are α-equivalent for all but $n-1$ primes p. Indeed, $U_q(y) =_\alpha U_q(z) \sim_\alpha U_q(x)$ whenever $q \notin \{p_1, \dots, p_n\}$, and $U_{p_n}(y) =_\alpha U_{p_n}(z') \sim_\alpha U_{p_n}(x)$. This completes the proof. $\square$

LEMMA 6.8 (Jacoby [**J2**]). *Let G be a group with decomposition basis X and S a finitely generated subgroup of G such that $S \cap \langle X\rangle = \langle S \cap X\rangle$. If $y \in X$ $(y \notin S)$, then there is a positive integer n satisfying $|mny + s|_p = \min\{|mny|_p, |s|_p\}$ for all $m \in \mathbb{Z}$, $s \in S$ and primes p.*

Proof. Since S is finitely generated, there is a positive integer k such that $ks \in \langle X \rangle$ for all $s \in S$. Let p be a prime dividing k. Since the natural map $G \to G_p$ sending every $g \in G$ to $g_p = g \otimes 1$ preserves p-heights, the set $X_p = \{x_p : x \in X\}$ is a decomposition basis for G_p and we have $S_p \cap \langle X_p \rangle = \langle S_p \cap X_p \rangle$. Assuming $y_p \in S_p$, there is a positive integer m such that $my \in S$. But then $my \in S \cap \langle X \rangle = \langle S \cap X \rangle$ and therefore $y \in S \cap X$, contradicting $y \notin S$. Thus $y_p \notin S_p$. By the local version of Lemma 6.8 for $\mathbb{Z}_p$-modules (see [**JL**, Lemma 4.4]) there is an $n_p \in \omega$ such that

$$|rp^{n_p} y_p + s_p|_p = \min\{|rp^{n_p} y_p|_p, |s_p|_p\}$$

for all $r \in \mathbb{Z}_p$ and $s \in S$. Now put $n = \prod_{p|k} p^{n_p}$ and let $s \in S$. It is clear that $|mny + s|_p = \min\{|mny|_p, |s|_p\}$ for all integers m whenever p divides k. Suppose p does not divide k. Then $ks \in S \cap \langle X \rangle = \langle S \cap X \rangle$ yields elements $x_1, \ldots, x_n \in S \cap X$ and $a_1, \ldots, a_n \in \mathbb{Z}$ such that $ks = a_1 x_1 + \ldots + a_n x_n$ and hence $s_p = \frac{a_1}{k}(x_1 \otimes 1) + \ldots + \frac{a_n}{k}(x_n \otimes 1)$. Since $\{x_1 \otimes 1, \ldots, x_n \otimes 1, y_p\}$ is a decomposition set, we obtain

$$|mny + s|_p = |mny_p + s_p|_p = \min\{|mny_p|_p, |s_p|_p\} = \min\{|mny|_p, |s|_p\}$$

as required. $\square$

Warfield's local-global lemma will be needed.

Lemma 6.9 (Warfield [**War1**]). *Let A and B be abelian groups, S and T subgroups such that A/S and B/T are torsion, and $f : S \to T$ a homomorphism. Suppose for every prime p, the induced map $f_p : S_p \to T_p$ extends to a homomorphism $g(p) : A_p \to B_p$. Then f extends to a homomorphism $g : A \to B$ such that $g_p = g(p)$ for all primes p. If each map $g(p)$ is injective [bijective], then g is injective [bijective].*

We are now ready to prove the main result of this paper:

Theorem 6.10. *Let G and H be groups with partial decomposition bases and let δ be an ordinal. Suppose*

(1) *$\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ for all primes p and $\alpha < \omega\delta$;*
(2) *$\hat{w}_{\omega(\nu+1)}(c, p, e, G) = \hat{w}_{\omega(\nu+1)}(c, p, e, H)$ for every compatibility class c of Ulm matrices, prime p, equivalence class e of Ulm sequences and $\nu < \delta$;*
(3) *if $length(t(G_p)) < \omega\delta$, then $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$.*

Then $G \equiv_\delta H$.

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be partial decomposition bases for G and H as in Lemma 6.6. For $\nu \leq \delta$ let I_ν be the set of all maps $f : S \to T$ such that there are sets $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ with $f(X) = Y$ satisfying:

(i) S and T are finitely generated subgroups of G and H, respectively;
(ii) f is an $\omega\nu$-height-preserving isomorphism;
(iii) $X \subseteq S \subseteq \langle X \rangle^0$ and $Y \subseteq T \subseteq \langle Y \rangle^0$;
(iv) $U(x)$ and $U(f(x))$ are $\omega\nu$-compatible for every $x \in X$.

To prove that $G \equiv_\delta H$, we will show that the system $\{I_\nu : \nu \leq \delta\}$ satisfies condition (2) of Karp's Theorem 3.1. Suppose $f \in I_{\nu+1}$ where $\nu < \delta$, say $f : S \to T$ with associated $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, and let $x \in G \backslash S$. To find an extension $g \in I_\nu$ of f with $x \in domain(g)$, we will show

(A) If x has a multiple in S, then there is such a map $g \in I_\nu$

and

(B) If $X \cup \{x\} \in \mathcal{C}$, then there is a map $g' \in I_{\nu+1}$ extending f such that $rx \in domain(g')$ for some positive integer r.

Then repeated application of (B) followed by an application of (A) yields an extension $g \in I_\nu$ of f with $x \in domain(g)$. To prove (A), suppose $rx \in S$ for some positive integer r. In order to construct the map g, we will apply Warfield's Lemma 6.9 to the groups $A = \langle S, x\rangle$ and $B = T^0$, so let p be a prime and consider the induced map $f_p : S_p \to T_p$. Since the natural map $G \to G_p$ preserves p-heights, the modules G_p and H_p have induced partial decomposition bases. Let $\alpha < \omega\delta$, $\nu < \delta$ and e an equivalence class of Ulm sequences. Then $\hat{u}(\alpha, G_p) = \hat{u}(\alpha, H_p)$ by [**F2**, Part 2, Lemma 16].

Let C be a set of representatives of the $\omega(\nu+1)$-compatibility classes, one for each class. By Corollary 4.3, $\hat{w}(e, G_p) = \min\{\sum_c \hat{w}(c, p, e, G), \omega\}$. So

$$\begin{aligned}
\hat{w}_{\omega(\nu+1)}(e, G_p) &= \min\{\sum_{e' \sim_{\omega(\nu+1)} e} \hat{w}(e', G_p), \omega\} \\
&= \min\{\sum_{e' \sim_{\omega(\nu+1)} e} \min\{\sum_c \hat{w}(c, p, e', G), \omega\}, \omega\} \\
&= \min\{\sum_{e' \sim_{\omega(\nu+1)} e} \sum_{c \in C} \sum_{c' \sim_{\omega(\nu+1)} c} \hat{w}(c', p, e', G), \omega\} \\
&= \min\{\sum_{c \in C} (\sum_{e' \sim_{\omega(\nu+1)} e} \sum_{c' \sim_{\omega(\nu+1)} c} \hat{w}(c', p, e', G)), \omega\} \\
&= \min\{\sum_{c \in C} \hat{w}_{\omega(\nu+1)}(c, p, e, G), \omega\} \\
&= \min\{\sum_{c \in C} \hat{w}_{\omega(\nu+1)}(c, p, e, H), \omega\} = \hat{w}_{\omega(\nu+1)}(e, H_p).
\end{aligned}$$

Now let $\mathcal{C}_{G_p}$ and $\mathcal{C}_{H_p}$ be the induced partial decomposition bases of G_p and H_p as in Lemma 5.2 and notice that the map f_p with associated sets $\{x \otimes 1 : x \in X\} \in \mathcal{C}_{G_p}$ and $\{y \otimes 1 : y \in Y\} \in \mathcal{C}_{H_p}$ satisfies the conditions (i)-(iii) as stated in the outlined proof of Theorem 5.3. Then by Theorem 5.3, f_p satisfies condition (2)(b) of Karp's Theorem 3.1, hence it can be extended to an $\omega\nu$-height-preserving isomorphism $g(p)$ with $x_p \in domain(g(p))$. By Lemma 6.9 we have a homomorphism $g : A \to B$ where $g(x) = y$ for some $y \in B$ and $g_p = g(p)$ for all p. Each map $g(p) : A_p \to B_p$ is injective and $\omega\nu$-height-preserving, therefore $g : \langle S, x\rangle \to \langle T, y\rangle$ is an $\omega\nu$-height-preserving isomorphism. Then g with associated sets X and Y satisfies conditions (i)-(iv), hence $g \in I_\nu$.

To verify (B), assume that $X \cup \{x\} \in \mathcal{C}$. By condition (iv) we have

$$\hat{w}_{\omega(\nu+1)}(c, p, e, \langle X\rangle^0) = \hat{w}_{\omega(\nu+1)}(c, p, e, \langle Y\rangle^0)$$

for all compatibility classes c of Ulm matrices, primes p and equivalence classes e of Ulm matrices. By Lemma 6.7, there is an element $y \in H$ such that $Y \cup \{y\} \in \mathcal{D}$ and

$$\hat{w}_{\omega(\nu+1)}(c, p, e, \langle X \cup \{x\}\rangle^0) = \hat{w}_{\omega(\nu+1)}(c, p, e, \langle Y \cup \{y\}\rangle^0)$$

for all c, p and e where $U(x) \sim_{\omega(\nu+1)} U(y)$ and $U_p(x) \sim_{\omega(\nu+1)} U_p(y)$ for all primes p. Then there are positive integers k and l such that $U_p(kx) =_{\omega(\nu+1)} U_p(ly)$ for all primes p. Let $x' = kx$ and $y' = ly$. Now proceed as in the proof of the classification in $L_{\infty\omega}$ (see [**J2**, Theorem 14]): Letting $\tilde{X} = X \cup \{x'\}$, we have $S \cap \langle \tilde{X}\rangle = \langle S \cap \tilde{X}\rangle$, so we can apply Lemma 6.8 to the group $\langle S, x'\rangle^0$ with decomposition basis $\tilde{X}$ and the subgroup S, and similarly to $\langle T, y'\rangle^0$, $Y \cup \{y'\}$ and T. Then there is a positive integer n such that

$$|mnx' + s|_p = \min\{|mnx'|_p, |s|_p\} \text{ and } |mny' + t|_p = \min\{|mny'|_p, |t|_p\}$$

for all $m \in \mathbb{Z}$, $s \in S$, $t \in T$ and primes p. Finally, let $S' = \langle S, nx'\rangle$ and $T' = \langle T, ny'\rangle$. Then f extends to the map

$$g' : S' \to T'$$

by sending nx' onto ny'. It is clear that g' is $\omega(\nu+1)$-height-preserving. Let $X' = X \cup \{nx'\}$ and $Y' = Y \cup \{ny'\}$. Then $X' \subseteq S' \subseteq \langle X'\rangle^0$ and $Y' \subseteq T' \subseteq \langle Y'\rangle^0$, therefore g' is a map in $I_{\nu+1}$ with associated sets $X' \in \mathcal{C}$ and $Y' \in \mathcal{D}$ such that $nx' \in domain(g')$.

Consequently, f extends to a map $g \in I_\nu$ with $x \in domain(g)$, as desired. By symmetry, the conditions of Theorem 3.1(2) are satisfied and it follows that $G \equiv_\delta H$. □

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Abelian groups with partial decomposition bases in $L^{\delta}_{\infty\omega}$, Part II

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Dedicated to Professor Rüdiger Göbel in honor of his 70th birthday

Abstract. We consider abelian groups with partial decomposition bases in $L^{\delta}_{\infty\omega}$ for ordinals δ. Jacoby, Leistner, Loth and Strüngmann developed a numerical invariant deduced from the classical global Warfield invariant and proved that if two groups have identical modified Warfield invariants and Ulm-Kaplansky invariants up to $\omega\delta$ for some ordinal δ, then they are equivalent in $L^{\delta}_{\infty\omega}$. Here we prove that the modified Warfield invariant is expressible in $L^{\delta}_{\infty\omega}$ and hence the converse is true for appropriate δ.

1. Introduction

The classical theorem of Ulm [**U**] gives a complete classification of countable abelian p-groups in terms of numerical invariants, the *Ulm-Kaplansky invariants*. For uncountable p-groups this theorem is false; however, Hill [**H**] and Walker [**Wal**] were able to extend it to the class of totally projective p-groups, the largest natural class of abelian p-groups in which the Ulm-Kaplansky invariants distinguish between non-isomorphic groups. It was Szmielew [**Sz**] who first considered abelian groups from a model-theoretic point of view. Barwise and Eklof [**BE**] took up Szmielew's approach and investigated abelian p-groups of arbitrary cardinality in $L^{\delta}_{\infty\omega}$, leading to a generalization of Ulm's theorem and a characterization of $L_{\infty\omega}$-equivalence classes of torsion groups.

Warfield groups are defined to be direct summands of simply presented groups or, alternatively, are abelian groups possessing a nice decomposition basis with simply presented cokernel. By adding new invariants, the *Warfield invariants*, Ulm's theorem was generalized to Warfield groups by Warfield [**War**] in the p-local case and by Hunter, Richman [**HR**] and Stanton [**St**] in the global case. Generalizing Warfield groups, Jacoby [**J1**], [**J2**] defined abelian groups possessing a partial decomposition basis and was able to prove classification theorems in $L_{\infty\omega}$ in the global case and in $L^{\delta}_{\infty\omega}$ for modules over a complete discrete valuation ring. Independently, similar results were obtained by Göbel, Leistner, Loth and Strüngmann [**GLLS**] who considered $\mathbb{Z}_p$-modules with nice decomposition bases. In [**JLLS**], the authors together with Leistner and Strüngmann introduced invariants deduced from the global Warfield invariants and proved a classification theorem for abelian

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groups with partial decomposition bases in $L^{\delta}_{\infty\omega}$ (see Theorem 4.1). In this paper, the converse of this result is proved for appropriate ordinals δ. More specifically, we show that the aforementioned invariants are expressible in $L^{\delta}_{\infty\omega}$ whenever $\delta = \omega\gamma$ and γ is a limit ordinal (Theorem 4.9). Consequently, the converse of Theorem 4.1 holds for those ordinals δ (Corollary 4.10).

For notation and terminology on abelian groups and on model theory, we may refer to the books [**F1**], [**F2**], [**L**] and [**R**].

2. Algebraic background

All groups considered in this paper are abelian. The reader should be familiar with the concepts of the p-height of an element x, written $|x|_p$, the length of a group A, written $l(A)$, the concept of a Warfield group and that of a decomposition set and decomposition basis. The height is used to define the Ulm-Kaplansky invariants of a group A, $u_p(\alpha, A)$ for each ordinal α and prime p. The global Warfield invariants $w(c, p, e, A)$ are defined for each compatibility class of Ulm matrices c, prime p and equivalence class of Ulm sequences e. These cardinal numbers, together with the Ulm-Kaplansky invariants, form a complete set of isomorphism invariants for Warfield groups (see Hunter-Richman [**HR**], Stanton [**St**]).

Barwise and Eklof's [**BE**] modified Ulm-Kaplansky invariant is defined as

$$\hat{u}_p(\alpha, A) = \min\{u_p(\alpha, A), \omega\}.$$

If we add the invariant $\hat{u}_p(\infty, A) = \min\{\dim(p^{\infty}A[p]), \omega\}$, these invariants classify all torsion groups in $L_{\infty\omega}$ (see [**BE**]). Because of this, a subgroup of particular interest is that of all torsion elements of A, which we denote $t(A)$. For a subgroup G of A, let $G^0 = \{a \in A : ra \in G \text{ for some nonzero integer } r\}$.

The class of groups considered in this paper consists of those with a partial decomposition basis [**J2**]. We say $\mathcal{C}$ is a *partial decomposition basis for* A if

(i) $\mathcal{C}$ is a nonempty collection of finite subsets of A,
(ii) if $X \in \mathcal{C}$, then X is a decomposition set, and
(iii) if $X \in \mathcal{C}$ and $x \in A$, then there is a $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $x \in \langle Y \rangle^0$.

Note that if $\mathcal{C}$ is a partial decomposition basis, so is $\{\{a_1x_1, \ldots, a_nx_n\} : a_1, \ldots, a_n \in \mathbb{Z}\backslash\{0\}$ and $x_1, \ldots, x_n \in X$ for some $X \in \mathcal{C}\}$, a fact we will use repeatedly.

We will need the following definitions "up to α" for an ordinal α.

For β and γ ordinals we define $\beta \geq_\alpha \gamma$ to mean $\beta \geq \gamma$ if $\gamma < \alpha$ and $\beta \geq \alpha$ if $\gamma \geq \alpha$. We write $\beta =_\alpha \gamma$ if

$$\min\{\beta, \alpha\} = \min\{\gamma, \alpha\}.$$

Then by $\beta >_\alpha \gamma$ we mean $\beta \geq_\alpha \gamma$ and $\beta \neq_\alpha \gamma$.

If $M = [m_{p,i}]$ and $N = [n_{p,i}]$ are Ulm matrices, we define $M \geq_\alpha N$ as $m_{p,i} \geq_\alpha n_{p,i}$ for all p and i, and other relations are defined similarly. We say $M \sim_\alpha N$ if and only if for some m and n, $mM \geq_\alpha N$ and $nN \geq_\alpha M$. If c is a compatibility class, we write $M \sim_\alpha c$ if $M \sim_\alpha N$ for some $N \in c$. We write $c \sim_\alpha c'$ to mean $M \sim_\alpha N$ for some $M \in c$ and $N \in c'$. Notice that $c \sim_\alpha c'$ if and only if there are $M = [m_{p,i}] \in c$ and $N = [n_{p,i}] \in c'$ such that $m_{p,i} =_\alpha n_{p,i}$ for all p and i (cf. [**JLLS**]).

In the same way, if (α_i) and (β_i) are Ulm sequences, we define $(\alpha_i) \geq_\alpha (\beta_i)$ if and only if for all i $\alpha_i \geq_\alpha \beta_i$. The other relationships are defined similarly. $(\alpha_i) \sim_\alpha (\beta_i)$ if and only if for some m and n, $\alpha_{i+m} =_\alpha \beta_{i+n}$ for all i. If e is an equivalence class of Ulm sequences, we write $(\alpha_i) \sim_\alpha e$ if $(\alpha_i) \sim_\alpha (\beta_i)$ for some $(\beta_i) \in e$ and $e \sim_\alpha e'$ if $(\alpha_i) \sim_\alpha (\beta_i)$ for some $(\alpha_i) \in e$ and $(\beta_i) \in e'$.

If A is a group with a partial decomposition basis $\mathcal{C}$, we define

$$\hat{w}(c, p, e, A) = \sup|\{x \in X : X \in \mathcal{C}, U(x) \in c \text{ and } U_p(x) \in e\}|$$

where c is a compatibility class of Ulm matrices, p a prime and e is an equivalence class of Ulm sequences. Notice that if A has a decomposition basis, then $\hat{w}(c, p, e, A) = \min\{w(c, p, e, A), \omega\}$. Jacoby, Leistner, Loth and Strüngmann [**JLLS**] generalized this definition up to α an ordinal:

$$\hat{w}_\alpha(c, p, e, A) = \min\{\sum_{e' \sim_\alpha e, c' \sim_\alpha c} \hat{w}(c', p, e', A), \omega\}.$$

They proved that this definition is independent of the choice of $\mathcal{C}$ and gave a simpler characterization of the invariant:

Lemma 2.1. *Let A be a group with partial decomposition basis $\mathcal{C}$, α an ordinal, c a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Then $\hat{w}_\alpha(c, p, e, A)$ is the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ such that $U(x_i) \sim_\alpha c$ and $U_p(x_i) \sim_\alpha e$ for all $i = 1, \ldots, n$. If no such n exists, $\hat{w}_\alpha(c, p, e, A) = \omega$.*

3. Model theory background

Throughout this paper, we will consider the language $L_{\infty\omega}$ which is an extension of an ordinary first order language L that allows infinite conjunctions and disjunctions. In this paper we will take L to be the language of group theory with 0, $+$ and $-$. Each formula φ in this language has a quantifier rank $qr(\varphi)$, an ordinal that represents how deeply nested the quantifiers are. The collection of all formulas of quantifier rank less than or equal to α is called $L^\alpha_{\infty\omega}$. See Barwise and Eklof [**BE**] for details. We will write $A \equiv_\infty B$ to represent equivalence in $L_{\infty\omega}$ and $A \equiv_\alpha B$ to represent equivalence in $L^\alpha_{\infty\omega}$.

In the process of proving their classification theorem Barwise and Eklof proved the following expressibility lemma for torsion groups, which is of interest in its own right [**BE**]. It says that p-height is expressible in $L_{\infty\omega}$ and gives the quantifier rank of the formula. The proof applies just as well to arbitrary groups.

Lemma 3.1. *Let $\alpha = \omega\delta + n$ be an ordinal where $n < \omega$. Then there is an existential formula $\theta_\alpha(x)$ whose quantifier rank is*

$$\delta \textit{ if } n = 0,$$
$$\delta + 1 \textit{ if } n > 0$$

such that for any group A and any $a \in A$, $A \models \theta_\alpha[a]$ if and only if $a \in p^\alpha A$.

4. The classification theorem

Let $\mathbb{Z}_p$ denote the ring of integers localized at the prime p. Jacoby, Leistner, Loth and Strüngmann [**JLLS**] proved that the modified Ulm-Kaplansky and Warfield invariants classify all groups with partial decomposition bases in $L^\delta_{\infty\omega}$ in the following sense:

Theorem 4.1. *Let A and B be groups with partial decomposition bases and let δ be an ordinal. Suppose*

(1) $\hat{u}_p(\alpha, A) = \hat{u}_p(\alpha, B)$ *for all primes* p *and* $\alpha < \omega\delta$;

(2) $\hat{w}_{\omega(\nu+1)}(c, p, e, A) = \hat{w}_{\omega(\nu+1)}(c, p, e, B)$ *for every compatibility class* c *of Ulm matrices, prime* p*, equivalence class* e *of Ulm sequences and* $\nu < \delta$;

(3) *if* $l(t(A \otimes \mathbb{Z}_p)) < \omega\delta$*, then* $\hat{u}_p(\infty, A) = \hat{u}_p(\infty, B)$ *for all primes* p.

Then $A \equiv_\delta B$.

In this paper we seek to prove the converse of this theorem, specifically, if $A \equiv_\delta B$ for appropriate δ then $\hat{w}_{\omega(\nu+1)}(c, p, e, A) = \hat{w}_{\omega(\nu+1)}(c, p, e, B)$ for every compatibility class c of Ulm matrices, prime p, equivalence class e of Ulm sequences and $\nu < \delta$.

Notice that $A \equiv_\delta B$ implies (1) and (3) whenever δ is a limit ordinal (cf. [**BE**, Theorem 3.1]). We will need to be able to express the modified Warfield invariant in $L^\delta_{\infty\omega}$. As it turns out, partial decomposition bases are not expressible in $L_{\infty\omega}$ [**J2**], so we turn to a modification of an invariant defined by Stanton [**St**], which we then prove is expressible and equivalent to the definition above. The following are definitions of Stanton [**St**], modified to consider only ordinals up to α. Let α be an ordinal, A a group, $M = [m_{q,i}]$ an Ulm matrix, p a prime and $\mu = M_p$.

$M_\alpha(A) = \{x \in A : U(x) \geq_\alpha M\}$.
$M^*_\alpha(A) = \langle x \in M_\alpha(A) : U(x) \nsim_\alpha M$ or x is torsion$\rangle$.
$\mu_\alpha A = \{x \in A : U_p(x) \geq_\alpha M_p\}$.
$\mu^*_\alpha A = \langle x \in \mu_\alpha A : |p^i x|_p \neq_\alpha m_{p,i}$ for infinitely many i or x is torsion$\rangle$.
$(M; p)^*_\alpha A = (M^*_\alpha(A) + \mu^*_\alpha A) \cap M_\alpha(A)$.

Let c be a compatibility class of Ulm matrices, p a prime, and e an equivalence class of Ulm sequences. We define the modified Stanton invariant as follows:

$$\widehat{ST}_\alpha(c, p, e, A) = \min\{\sup\{rank(M_\alpha(A)/(M; p)^*_\alpha A)\}, \omega\}$$

where the supremum is defined over all $M \in c$ with $M_p \in e$.

Lemma 4.2. *Let* A *be a group,* α *an ordinal,* p *a prime,* $M = [m_{q,i}]$ *an Ulm matrix and* $M' = pM$.

(1) *Suppose* $m_{p,0} < \alpha$. *Then* $x \mapsto px$ *defines a surjective map* $M_\alpha(A) \to M'_\alpha(A)$. *If* $x \in M_\alpha(A)$*, then* $x \in (M; p)^*_\alpha A$ *if and only if* $px \in (M'; p)^*_\alpha A$. *Hence* $x \mapsto px$ *induces an isomorphism*

$$M_\alpha(A)/(M; p)^*_\alpha A \xrightarrow{\sim} M'_\alpha(A)/(M'; p)^*_\alpha A.$$

(2) *If* $m_{p,0} \geq \alpha$ *then* $M_\alpha(A) = M'_\alpha(A)$ *and* $(M; p)^*_\alpha A = (M'; p)^*_\alpha A$.

Proof. Letting $M' = [m'_{q,i}]$ we have for every $i < \omega$, $m'_{p,i} = m_{p,i+1}$ and $m'_{q,i} = m_{q,i}$ if $q \neq p$. Let $\mu' = M'_p$.

For case (1), suppose $m_{p,0} < \alpha$. If $x \in M_\alpha(A)$ then $U(x) \geq_\alpha M$, so for any $i < \omega$ we have $|p^i(px)|_p \geq_\alpha m_{p,i+1} = m'_{p,i}$, hence $px \in M'_\alpha(A)$. If $x' \in M'_\alpha(A)$, then $U(x') \geq_\alpha M'$, and in particular $|x'|_p \geq_\alpha m'_{p,0} = m_{p,1} > m_{p,0}$. Then there is an $x \in A$ such that $|x|_p \geq m_{p,0}$ and $px = x'$. It may be verified that $x \in M_\alpha(A)$, hence $x \mapsto px$ maps $M_\alpha(A)$ onto $M'_\alpha(A)$.

Now let $x \in M_\alpha(A)$. First we assume that $x \in (M; p)^*_\alpha A$, say $x = z_1 + \cdots + z_k$ where $z_1, \ldots, z_k$ are generators of $(M; p)^*_\alpha A$. Consider $z = z_i$ for some i. If z is torsion, so is pz. Suppose z is not torsion. If z is a generator of $M^*_\alpha(A)$, then $z \in M_\alpha(A)$ and $U(pz) \sim U(z) \nsim_\alpha M \sim M'$. Since $pz \in M'_\alpha(A)$ we obtain $pz \in M'^*_\alpha(A)$.

Now suppose z is a generator of $\mu_\alpha^* A$. Then $|p^i pz|_p = |p^{i+1}z|_p \neq_\alpha m_{p,i+1} = m'_{p,i}$ for infinitely many i. In any case, $pz \in M'^*_\alpha(A) + \mu'^*_\alpha A$. Since $z_1 + \ldots + z_k \in M_\alpha(A)$ we have $p(z_1 + \ldots + z_k) \in (M'; p)^*_\alpha A$, hence $px \in (M'; p)^*_\alpha A$.

Conversely, assume that $px \in (M'; p)^*_\alpha A$, say $px = z'_1 + \cdots + z'_k$ for $z'_1, \ldots, z'_k$ generators of $(M'; p)^*_\alpha A$. Let z' be one of the z'_i. Then $U_p(z') \geq_\alpha M'_p$. In particular, $|z'|_p \geq_\alpha m'_{p,0} = m_{p,1} > m_{p,0}$, so there is $z \in A$ such that $pz = z'$ and $|z|_p \geq m_{p,0}$. Also, for $j \geq 1$, $|p^j z|_p = |p^{j-1}z'|_p \geq_\alpha m'_{p,j-1} = m_{p,j}$, so $z \in \mu_\alpha A$. If z' is torsion, so is z. Now suppose z' is not torsion. If z' is a generator of $\mu'^*_\alpha A$ then $|p^i z'|_p \neq_\alpha m'_{p,i}$ for infinitely many i, so $|p^{i+1}z|_p = |p^i z'|_p \neq_\alpha m'_{p,i} = m_{p,i+1}$ for infinitely many i and $z \in \mu^*_\alpha A$ follows. Now suppose z' is a generator of $M'^*_\alpha(A)$. Then there is $z \in M_\alpha(A)$ such that $z' = pz$ and we have $U(z) \sim U(z') \nsim_\alpha M' \sim M$, hence $z \in M^*_\alpha(A)$. In all cases we obtain $z \in M^*_\alpha(A) + \mu^*_\alpha A$. Let $y = z_1 + \ldots + z_k$. Then $y \in M^*_\alpha(A) + \mu^*_\alpha A$ and $py = z'_1 + \ldots + z'_k = px$. Since $U_p(y) \geq_\alpha M_p$ and $|q^i y|_q = |q^i py|_q = |q^i px|_q \geq_\alpha m_{q,i}$ for all $i < \omega$ and $q \neq p$ we have $y \in M_\alpha(A)$, hence $y \in (M; p)^*_\alpha A$. Since $x - y \in t(M_\alpha(A))$ we conclude that $x = y + (x - y) \in (M; p)^*_\alpha A$.

For case (2), suppose $m_{p,0} \geq \alpha$. Then $M =_\alpha M'$ and a straightforward computation yields $M_\alpha(A) = M'_\alpha(A)$, $M^*_\alpha(A) = M'^*_\alpha(A)$ and $\mu^*_\alpha A = \mu'^*_\alpha A$. $\square$

Applying Lemma 4.2 successively allows us to substitute the Ulm matrix M by a p-multiple when calculating the Stanton invariant:

Corollary 4.3. *Let A be a group, α an ordinal, p a prime, M an Ulm matrix and $n < \omega$. Then $M_\alpha(A)/(M; p)^*_\alpha A \cong (p^n M)_\alpha(A)/(p^n M; p)^*_\alpha A$.*

Lemma 4.4. *Let A be a group with partial decomposition basis $\mathcal{C}$, α an ordinal, p a prime, c a compatibility class of Ulm matices and e an equivalence class of Ulm sequences. If $M \in c$ and $M_p \in e$, then $rank(M_\alpha(A)/(M; p)^*_\alpha A) =_\omega \hat{w}_\alpha(c, p, e, A)$.*

Proof. By a suitable choice of $\mathcal{C}$ we may assume that if $X \in \mathcal{C}$ then for any $a_1, \ldots, a_n \in \mathbb{Z}\backslash\{0\}$ and $x_1, \ldots, x_n \in X$, $\{a_1 x_1, \ldots, a_n x_n\} \in \mathcal{C}$. Let $M = [m_{q,i}] \in c$ such that $M_p \in e$. Suppose $\hat{w}_\alpha(c, p, e, A) \geq n$, say $x_1, \ldots, x_n \in X$, $X \in \mathcal{C}$ and $U(x_i) \sim_\alpha c$, $U_p(x_i) \sim_\alpha e$ for $1 \leq i \leq n$ (cf. Lemma 2.1).

Consider x_i for some $1 \leq i \leq n$. Since $U(x_i) \sim_\alpha c$ and $M \in c$, there is an m_i such that $m_i U(x_i) \geq_\alpha M$. Then for $m = \prod_{i=1}^n m_i$ we have $U(mx_i) \geq_\alpha M$. Since $U_p(mx_i) \sim U_p(x_i) \sim_\alpha M_p$, there are k_i and k'_i such that $U_p(p^{k_i} m x_i) =_\alpha p^{k'_i} M_p$. Letting $k' = \sum_{i=1}^n k'_i$ and $l_i = k_i + \sum_{j \neq i} k'_j$ for each i, we have $U_p(p^{l_i} m x_i) =_\alpha p^{k'} M_p$. Note that for $q \neq p$, $U_q(p^{l_i} m x_i) = U_q(mx_i) \geq_\alpha M_q = (p^{k'} M)_q$, so $U(p^{l_i} m x_i) \geq_\alpha p^{k'} M$. Writing x_i instead of $p^{l_i} m x_i$, as the choice of $\mathcal{C}$ allows us to do, and writing M instead of $p^{k'} M$, as the corollary allows us to do, we see that we may assume $x_i \in M_\alpha(A)$ and $U_p(x_i) =_\alpha M_p$ for all i.

We wish to prove that $x_1, \ldots, x_n$ are independent representatives of $M_\alpha(A)$ over $(M; p)^*_\alpha A$ and of either prime or infinite order. Suppose

$$a_1 x_1 + \cdots + a_n x_n = z_1 + \cdots + z_r \in M_\alpha(A)$$

where $Z = \{z_1, \ldots, z_r\}$ is a set of generators of $\mu^*_\alpha A$ and $M^*_\alpha(A)$. Let $Y \in \mathcal{C}$, where $X \subseteq Y$ and $Z \subseteq \langle Y \rangle^0$. There is a multiplier $p^k a$, where a is relatively prime to p, such that $p^k a Z \subseteq \langle Y \rangle$.

Consider $z \in Z$. First suppose z is not torsion. Suppose z is a generator of $M^*_\alpha(A)$. Then $z \in M_\alpha(A)$ and $U(z) \nsim_\alpha M$. Let $p^k a z = c_1 x_1 + \cdots + c_n x_n + y$ for some $y \in \langle Y \setminus \{x_1, \ldots, x_n\} \rangle$. For every $m > 0$ there are q and j such that

$|q^j z|_q > m_{q,j+|m|_q}$ and $m_{q,j+|m|_q} < \alpha$. Take arbitrary m and associated q and j and get $m_{q,j+|m|_q} < |q^j p^k az|_q = \min\{|q^j c_i x_i|_q, |q^j y|_q\} \leq |q^j c_i x_i|_q$ for all i. Then $m_{q,j+|m|_q} < \alpha$ and either $|q^j c_i x_i|_q \geq \alpha$ or else $m_{q,j+|m|_q} < |q^j c_i x_i|_q < \alpha$. This means that $U(c_i x_i) \nsim_\alpha M$. If $c_i \neq 0$, this would contradict $U(c_i x_i) \sim U(x_i) \sim_\alpha M$, so we must have $c_i = 0$ for all i and so $p^k az \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$.

Now let $z \in Z$ be a generator of $\mu^*_\alpha A$. Given any $j_0 < \omega$ we can find j such that $j + k \geq j_0$ and $|p^{j+k} z|_p >_\alpha m_{p,j+k}$ therefore $m_{p,j+k} < \alpha$. We may write $p^k az = b_1 x_1 + \cdots + b_n x_n + y$ for some $y \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. Then for all i, $|p^{j+k} x_i|_p = m_{p,j+k} < |p^j p^k az|_p \leq |p^j b_i x_i|_p$, since the x_i form a decomposition set. From this it follows that $p^{k+1} | b_i$ for all i. Now we may write $p^k az = p^{k+1} x + y$, for some $x \in \langle x_1, \ldots, x_n\rangle$ and $y \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. Finally, if z is torsion, $p^k az = 0$ is also in this form.

In either case it follows that $p^k a(a_1 x_1 + \cdots + a_n x_n)$ may be written in the form $p^{k+1} x + y$ for some $x \in \langle x_1, \ldots, x_n\rangle$ and $y \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. Then for any j, $p^j p^k a(a_1 x_1 + \cdots + a_n x_n) = p^{j+k+1} x + p^j y$. Equating x_i terms, $p^{j+k} a a_i x_i$ is a multiple of $p^{j+k+1} x_i$ for all i.

First suppose $m_{p,j} < \alpha$ for all j and consider $1 \leq i \leq n$. From this we see $|p^{j+k} a a_i x_i|_p > |p^{j+k} x_i|_p = m_{p,j+k}$ for all j and so $a a_i x_i \in \mu^*_\alpha A$. Since a is relatively prime to p, $a_i x_i \in \mu^*_\alpha A$. Also $a_i x_i \in M_\alpha(A)$, so $a_i x_i \in (M;p)^*_\alpha A$ for all i, proving independence. For each i $px_i \in \mu^*_\alpha A \cap M_\alpha(A)$. Also $x_i \notin (M;p)^*_\alpha A$ since otherwise $p^k a x_i$ would be a multiple of $p^{k+1} x_i$ for some $k \geq 0$ and a relatively prime to p, a contradiction. This proves that each $x_i + (M;p)^*_\alpha A$ has order p. It follows that $rank(M_\alpha(A)/(M;p)^*_\alpha A) \geq n$.

Next suppose $m_{p,j} \geq \alpha$ for some j. By taking p-multiples of each x_i and M we may assume $m_{p,j} \geq \alpha$ for all j. Then for any z of $\mu_\alpha A$, $|p^j z|_p \geq \alpha$ and $|p^j z|_p =_\alpha \alpha =_\alpha m_{p,j}$ for all j. It follows that $\mu^*_\alpha A = t(\mu_\alpha A)$. If $z \in Z$ is not torsion, z must be a generator of $M^*_\alpha(A)$ and hence $p^k az \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. If z is torsion, $p^k az = 0 \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. Equating the x_i terms gives $p^k a a_i x_i = 0$, so $a_i x_i$ is torsion and hence in $(M;p)^*_\alpha A$ for all i, again proving independence. We claim the order of each $x_i + (M;p)^*_\alpha A$ is infinite. Suppose $mx_i \in (M;p)^*_\alpha A$ for some $m \neq 0$. Then, as before, $p^k a m x_i = 0$, a contradiction since x_i is in a decomposition set. Again it follows that $rank(M_\alpha(A)/(M;p)^*_\alpha A) \geq n$.

Now suppose $\hat{w}_\alpha(c, p, e, A) = n$ and $\{x_1, \ldots, x_n\}$ is a maximal set. Again we may assume that $x_i \in M_\alpha(A)$ and $U_p(x_i) =_\alpha M_p$ for all i. We wish to show that $\{x_1 + (M;p)^*_\alpha A, \ldots, x_n + (M;p)^*_\alpha A\}$ is a maximal independent set in $M_\alpha(A)/(M;p)^*_\alpha A$. Suppose $x_1, \ldots, x_n, y$ are independent representatives. Choose $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $y \in \langle Y\rangle^0$.

We may write $ay = a_1 x_1 + \cdots + a_n x_n + b_1 y_1 + \cdots + b_m y_m$ where $a > 0$ and $y_1, \ldots, y_m \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. Since $y \in M_\alpha(A)$, for all q and k,

$$m_{q,k} \leq_\alpha |q^k y|_q \leq |q^k ay|_q = \min_{i,j}\{|q^k a_j x_j|_q, |q^k b_i y_i|_q\}$$

and so for all i, $|q^k b_i y_i|_q \geq_\alpha m_{q,k}$ and $b_i y_i \in M_\alpha(A)$. Since $\{x_1, \ldots, x_n, b_i y_i\} \in \mathcal{C}$ for all i by our choice of $\mathcal{C}$, we find that for each i either $U(b_i y_i) \nsim_\alpha c$ or $U_p(b_i y_i) \nsim_\alpha e$, by the maximality of $\{x_1, \ldots, x_n\}$. First suppose $U(b_i y_i) \nsim_\alpha c$. Then $U(b_i y_i) \nsim_\alpha M$ and $b_i y_i \in M^*_\alpha(A)$. Now suppose $U_p(b_i y_i) \nsim_\alpha e$. Suppose for some k_0 $|p^k b_i y_i|_p =_\alpha m_{p,k}$ for all $k \geq k_0$. Then $p^{k_0} U_p(b_i y_i) =_\alpha p^{k_0} M_p$ and $M_p \sim_\alpha U_p(b_i y_i) \nsim_\alpha e \sim M_p$, a contradiction. This leads to $|p^k b_i y_i| \neq_\alpha m_{p,k}$ for infinitely many k and so $b_i y_i \in \mu^*_\alpha A \cap M_\alpha(A)$. In either case $b_i y_i \in (M;p)^*_\alpha A$.

Then $ay + (M;p)^*_\alpha A = a_1x_1 + \cdots + a_nx_n + (M;p)^*_\alpha A$, contradicting independence. This proves that $rank(M_\alpha(A)/(M;p)^*_\alpha A) =_\omega \hat{w}_\alpha(c,p,e,A)$ for any $M \in c$ with $M_p \in e$. □

Corollary 4.5. *Suppose A is a group with a partial decomposition basis. Then $\widehat{ST}_\alpha(c,p,e,A) = \hat{w}_\alpha(c,p,e,A)$ for all α, c, p and e.*

Corollary 4.6. *Suppose A is a group with a partial decomposition basis. Then $\widehat{ST}(c,p,e,A) = \hat{w}(c,p,e,A)$ for all c, p and e.*

Proof. Choose $\alpha = \sup\{p\text{-}length(A) : p \text{ prime}\}$ □

Corollary 4.7. *If A is a group with partial decomposition basis, then*

$$\widehat{ST}_\alpha(c,p,e,A) = \min\{rank(M_\alpha(A)/(M;p)^*_\alpha A), \omega\}$$

for every $M \in c$ with $M_p \in e$.

Lemma 4.8. *For any $\alpha = \omega\delta + n$ where $n < \omega$, "$\widehat{ST}_\alpha(c,p,e,A) \geq m$" is expressible in a formula of quantifier rank $\leq \delta + \omega + m$.*

Proof. We follow the proof of the local case [**GLLS**]. Let $M \in c$ such that $M_p \in e$.

"$x \in M_\alpha(A)$" if and only if for all q, i, either $q^ix \in q^{m_{q,i}}A$ and $m_{q,i} < \alpha$ or $q^ix \in q^\alpha A$. By Lemma 3.1, "$x \in q^\alpha A$" has quantifier rank δ or $\delta+1$, so this formula has quantifier rank $\leq \delta + 1$.

"x is a generator of $M^*_\alpha(A)$" if and only if $x \in M_\alpha(A)$ and either $U(x) \nsim_\alpha M$ or x is torsion. "x is torsion" has quantifier rank 0 and "$x \in M_\alpha(A)$" has quantifier rank $\leq \delta + 1$. For each $x \in M_\alpha(A)$, "$U(x) \nsim_\alpha M$" if and only if $jM \not\geq_\alpha U(x)$ for all j if and only if for every j there are q and i such that $|q^ix| > m_{q,i+|j|_q}$ and $\alpha > m_{q,i+|j|_q}$ if and only if $\bigwedge_j \bigvee_{q,i}(q^ix \in q^{m_{q,i+|j|_q}+1}A \wedge m_{q,i+|j|_q} < \alpha)$. This can be expressed by a formula of quantifier rank $\leq \delta + 1$.

"$x \in \mu_\alpha A$" if and only if for all i either $p^ix \in p^{m_{p,i}}A$ and $m_{p,i} < \alpha$ or $p^ix \in p^\alpha A$, so it can be expressed by a formula of quantifier rank $\leq \delta + 1$.

"x is a generator of $\mu^*_\alpha A$" if and only if $x \in \mu_\alpha A$ and either $|p^ix|_p > m_{p,i}$ for infinitely many i with $m_{p,i} < \alpha$ or x is torsion. This can be expressed by a formula of quantifier rank $\leq \delta + 1$.

"$x \in (M;p)^*_\alpha A$" if and only if $x \in M_\alpha(A)$ and for some $k \in \omega$ $\exists x_1 \ldots \exists x_k$ such that $x = \sum \lambda_jx_j$ for some $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}$ and each x_j either is one of the generators of $\mu^*_\alpha A$, or else is one of the generators of $M^*_\alpha(A)$. It follows that "$x \in (M;p)^*_\alpha A$" can be expressed by a formula of quantifier rank $\leq \delta + \omega$.

"$\widehat{ST}_\alpha(c,p,e,A) \geq m$" if and only if $\exists x_1 \ldots \exists x_m$ ($x_1, \ldots, x_m \in M_\alpha(A)$ and $x_1, \ldots, x_m$ are independent elements of infinite and prime power order modulo $(M;p)^*_\alpha A)$. Since "independent modulo $(M;p)^*_\alpha A$" has quantifier rank $\leq \delta + \omega$, this formula has quantifier rank $\leq \delta + \omega + m$. □

Theorem 4.9. *Suppose A and B are groups with partial decomposition bases. If $A \equiv_\lambda B$ where $\lambda = \omega\gamma$ for some γ a limit ordinal, then for all $\alpha < \omega\lambda$, $\hat{w}_\alpha(c,p,e,A) = \hat{w}_\alpha(c,p,e,B)$ for all c,p,e.*

Proof. Write $\alpha = \omega\delta + n$ and $\delta = \omega\delta' + n'$. By Corollary 4.5, $\hat{w}_\alpha = \widehat{ST}_\alpha$. By Lemma 4.8, $\widehat{ST}_\alpha(c,p,e,A) \geq m$ may be expressed in a formula $\phi_{p,m,\alpha}$ with quantifier rank $\leq \delta + \omega + m$. Note $\delta' < \gamma$, so since γ is a limit ordinal, $\delta' + 3 < \gamma$,

$\lambda = \omega\gamma > \omega(\delta' + 3) > \omega\delta' + n' + \omega + m = \delta + \omega + m$. Since $A \equiv_\lambda B$, A and B satisfy the same formulas of quantifier rank $\leq \lambda$. In particular, $A \models \phi_{p,m,\alpha}$ if and only if $B \models \phi_{p,m,\alpha}$. In other words, $\widehat{ST}_\alpha(c,p,e,A) \geq m$ if and only if $\widehat{ST}_\alpha(c,p,e,B) \geq m$. Then $\hat{w}_\alpha(c,p,e,A) = m$ if and only if $\widehat{ST}_\alpha(c,p,e,A) = m$ if and only if $\widehat{ST}_\alpha(c,p,e,B) = m$ if and only if $\hat{w}_\alpha(c,p,e,B) = m$. □

COROLLARY 4.10. *Let A and B be groups with partial decomposition bases such that $A \equiv_\lambda B$ where $\lambda = \omega\gamma$ and γ is a limit ordinal. Then*

(1) *$\hat{u}_p(\alpha, A) = \hat{u}_p(\alpha, B)$ for all primes p and $\alpha < \omega\lambda$;*
(2) *$\hat{w}_{\omega(\nu+1)}(c,p,e,A) = \hat{w}_{\omega(\nu+1)}(c,p,e,B)$ for every compatibility class c of Ulm matrices, prime p, equivalence class e of Ulm sequences and $\nu < \lambda$;*
(3) *if $l(t(A \otimes \mathbb{Z}_p)) < \omega\lambda$, then $\hat{u}_p(\infty, A) = \hat{u}_p(\infty, B)$.*

PROOF. (1) and (3) follow from [**BE**, Theorem 3.1], and (2) follows from Theorem 4.9. □

COROLLARY 4.11. *Suppose A and B are groups with partial decomposition bases. If $A \equiv_\infty B$ then $\hat{w}(c,p,e,A) = \hat{w}(c,p,e,B)$ for all c, p, e.*

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Automorphism towers and automorphism groups of fields without Choice

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Dedicated to Professor Rüdiger Göbel on his 70th birthday

Abstract. This paper can be viewed as a continuation of the article by Kaplan and Shelah that dealt with the automorphism tower problem without Choice. Here we deal with the inequality $\tau_\kappa^{\text{nlg}} \leq \tau_\kappa$ without Choice and introduce a new proof to a theorem of Fried and Kollár that any group can be represented as an automorphism group of a field. The proof uses a simple construction: working more in graph theory, and less in algebra.

1. Introduction and preliminaries

Background. Although this paper hardly mentions automorphism towers, it is the main motivation for it. So we shall start by giving the story behind them.

Given any centerless group G, $G \cong \mathrm{Inn}(G) \leq \mathrm{Aut}(G)$ so we can embed G into its automorphism group. Also, an easy exercise shows that $\mathrm{Aut}(G)$ is also without center, so we can do this again, and again:

Definition 1.1. For a centerless group G, we define *the automorphism tower* $\langle G^\alpha \mid \alpha \in \mathbf{ord} \rangle$ by

- $G^0 = G$.
- $G^{\alpha+1} = \mathrm{Aut}(G^\alpha)$.
- $G^\delta = \cup \{G^\alpha \mid \alpha < \delta\}$ for δ limit.

Remark 1.2. The union in limit stages can be understood as the direct limit. But we shall think of the tower as an increasing continuous sequence of groups.

The natural question that arises, is whether this process stabilizes, and when. We define

Definition 1.3. For such a group, define $\tau_G = \min \{\alpha \mid G^{\alpha+1} = G^\alpha\}$.

In 1939, Weilandt proved in [**Wie39**] that for finite G, τ_G is finite. What about infinite G? There exist examples of centerless infinite groups such that

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this process does not stop in any finite stage. For example — the infinite dihedral group $D_\infty = \left\langle x, y \,\middle|\, x^2 = y^2 = 1 \right\rangle$ satisfies $\mathrm{Aut}\,(D_\infty) \cong D_\infty$ while the automorphism replacing x with y is not in $\mathrm{Inn}\,(D_\infty)$. The question remained open until the works of Faber [**Fab78**] and Thomas [**Tho85, Tho98**] (who was not aware of Faber's work), that showed $\tau_G < \left(2^{|G|}\right)^+$.

Definition 1.4. For a cardinal κ we define τ_κ as the smallest ordinal such that $\tau_\kappa > \tau_G$ for all centerless groups G of cardinality $\leq \kappa$, or in other words

$$\tau_\kappa = \bigcup \{\tau_G + 1 \,|\, G \text{ is centerless and } |G| \leq \kappa\}.$$

Since $(2^\kappa)^+$ is regular we can immediately conclude $\tau_\kappa < (2^\kappa)^+$.

This paper is concerned with a Choiceless universe, i.e. we do not assume the axiom of Choice. As a consequence, the previous definition is generalized to

Definition 1.5. For a set k, we define $\tau_{|k|}$ to be the smallest ordinal α such that $\alpha > \tau_G$ for all groups G with power $\leq |k|$.

Note that when we write $|X| \leq |Y|$ as in the definition above, we mean that there is an injective function from X to Y. Below we provide a short glossary.

A helpful and close notion is that of *the normalizer tower* $\langle \mathrm{nor}^\alpha_G\,(H) \,|\, \alpha \in \mathbf{ord} \rangle$ of a subgroup H of G in G.

Definition 1.6. Let

- $\mathrm{nor}^0_G\,(H) = H$.
- $\mathrm{nor}^{\alpha+1}_G\,(H) = \mathrm{nor}_G\,(\mathrm{nor}^\alpha_G\,(H))$.
- $\mathrm{nor}^\delta_G\,(H) = \bigcup \{\mathrm{nor}^\alpha_G\,(H) \,|\, \alpha < \delta\}$ for δ limit.

And we let the normalizer length be $\tau^{\mathrm{nlg}}_{G,H} = \min\left\{\alpha \,\middle|\, \mathrm{nor}^{\alpha+1}_G\,(H) = \mathrm{nor}^\alpha_G\,(H)\right\}$ (sometimes we just write $\tau_{G,H}$).

Analogously to τ_κ, we define

Definition 1.7. For a cardinal κ, let $\tau^{\mathrm{nlg}}_\kappa$ be the smallest ordinal such that $\tau^{\mathrm{nlg}}_\kappa > \tau_{\mathrm{Aut}(\mathfrak{A}),H}$, for every structure $\mathfrak{A}$ of cardinality $\leq \kappa$ and $H \leq \mathrm{Aut}\,(\mathfrak{A})$ of cardinality $\leq \kappa$.

In general (i.e. without assuming Choice), for a set k, we define $\tau^{\mathrm{nlg}}_{|k|}$ as the smallest ordinal α, such that for every structure $\mathfrak{A}$ of power $||\mathfrak{A}|| \leq |k|$, $\tau_{\mathrm{Aut}(\mathfrak{A}),H} < \alpha$ for every subgroup $H \leq \mathrm{Aut}\,(\mathfrak{A}) = G$ of power $|H| \leq |k|$. In other words, $\tau^{\mathrm{nlg}}_{|k|} = \sup\{\tau_{G,H} + 1 \,|\, \text{ for such } G, H\}$.

In [**JST99**, Lemma 1.8], Just, Shelah and Thomas proved the following inequality

$$\tau_\kappa \geq \tau^{\mathrm{nlg}}_\kappa.$$

In fact it was essentially already proved by Thomas in [**Tho85**].

In [**KS09**] we dealt with an upper bound of τ_κ without assuming Choice. Here we prove $\tau_\kappa \geq \tau^{\mathrm{nlg}}_\kappa$ without Choice, and also provide a Choiceless variant of $\tau_{|k|} \geq \tau^{\mathrm{nlg}}_{|k|}$.

It is worth mentioning some previous results regarding τ_κ that were proved using this inequality.

In [**Tho85**], Thomas proved that $\tau_\kappa \geq \kappa^+$. It is a easy to conclude from Main Theorem A below that this result still holds without Choice. We will elaborate in the end of this section (See Corollary 2.5).

In [**JST99**] the authors found that for uncountable κ one cannot find an explicit upper bound for τ_κ better than $(2^\kappa)^+$ in ZFC (using set theoretic forcing). In [**She07**], Shelah proved that if κ is strong limit singular of uncountable cofinality then $\tau_\kappa > 2^\kappa$ (using results from PCF theory). In the proofs the authors construct normalizer towers to find lower bound for τ_κ, but we did not check how much Choice was used.

It remains an open question whether or not there exists a countable centerless group G such that $\tau_G \geq \omega_1$.

Description of paper. As mentioned before, we wish to prove $\tau_\kappa \geq \tau_\kappa^{\mathrm{nlg}}$ without Choice. So we started by reading what was done in [**JST99**] (which is also described in detail in [**Tho**]).

The proof contains three parts:

(1) Given some structure, code it in a graph (i.e. find a graph with the same cardinality and automorphism group).
(2) Given a graph code it in a field. Now we have a field K with some subgroup $H \leq \mathrm{Aut}\,(K)$ such that $|K| = |H| = \kappa$.
(3) Use some lemmas from group theory and properties of $PSL\,(2, K)$ to find a centerless group whose automorphism tower coincides with the normalizer tower of H in $\mathrm{Aut}\,(K)$.

Our first intention was to mimic this proof, and to prove some version of $\tau_{|k|} \geq \tau_{|k|}^{\mathrm{nlg}}$ (see definitions 1.7 and 1.1 above). To explain what we did prove, we need some notation:

Definition 1.8. Let X be a set.

(1) $X^{<\omega}$ is the set of all finite sequences of members of X.
(2) $[X]^{<\aleph_0} = \{a \subseteq X \mid |a| < \aleph_0\}$.
(3) $X^{\langle<\omega\rangle} = [X^{<\omega}]^{<\aleph_0}$, i.e. the set of all finite subsets of finite sequences of elements of X.

Our methods cannot tackle $\tau_{|k|} \geq \tau_{|k|}^{\mathrm{nlg}}$ without Choice, since one often needs to code finite sequences. The natural way to overcome this is to replace k with $k^{<\omega}$, so that we get $\tau_{|k^{<\omega}|} \geq \tau_{|k^{<\omega}|}^{\mathrm{nlg}}$. However, we managed to proved a slightly different version:

Main Theorem A. *For any set k, $\tau_{\left|k^{\langle<\omega\rangle}\right|} \geq \tau_{\left|k^{\langle<\omega\rangle}\right|}^{\mathrm{nlg}'}$.*

Where $\tau_{|k|}^{\mathrm{nlg}'}$ is a variant of $\tau_{|k|}^{\mathrm{nlg}}$. See Definition 2.2 below.

With Choice there is no difference, and moreover, we get as a corollary the original inequality for a cardinal κ (see Corollary 2.4 below). It is a matter of taste whether replacing $k^{<\omega}$ and nlg by $k^{\langle<\omega\rangle}$ and nlg$'$ matters. Still, one may ask whether $\tau_{|k^{<\omega}|}^{\mathrm{nlg}} \leq \tau_{|k^{<\omega}|}$ or even $\tau_{|k|} \geq \tau_{|k|}^{\mathrm{nlg}}$ holds without Choice.

Part (1) was easy enough. However, it needs a passage to a structure with countable language. This stage uses Choice. In order to fix this, we just bypassed the problem all together and replaced $\tau_{|k|}^{\mathrm{nlg}}$ by $\tau_{|k|}^{\mathrm{nlg}'}$.

Part (3) was easy as well: An algebraic lemma which obviously did not need Choice (Lemma 4.1); And two lemmas regarding $PSL\,(2, K)$ — Lemma 4.4 and

Lemma 4.8. The latter is a theorem of Van der Waerden and Schreier which described $\mathrm{Aut}\,(PSL\,(2, K))$. There is a simple model theoretic argument that shows that these lemmas do not require Choice (Lemma 4.5).

However, part (2) seemed to be somewhat harder. In [**JST99**], the authors referred to the work of Fried and Kollár [**FK82**]. In [**Tho**], the author gives a less technical proof that the construction in [**FK82**] works. The proof, in both cases, was a little bit complicated, and we were suspicious that Choice was used in it. After some time we realized that it is most likely not used, but by then we already came up with a proof of our own, in which the construction of the field is much simpler, and thought that it is worth presenting. So, for part (2) we prove:

Main Theorem B. *Let $\Gamma = \langle X, E \rangle$ be a connected graph. Then for any choice of characteristic there exists a field K_Γ of that characteristic such that $|K_\Gamma| \leq \left|X^{\langle < \omega \rangle}\right|$ and $\mathrm{Aut}\,(K_\Gamma) \cong \mathrm{Aut}\,(\Gamma)$.*

The proof of Main Theorem B is given in Section 6. Here we will give a brief outline of the construction.

The plan was this: work a little bit on the graph, so that the algebra would be easier. First code the given graph as a graph with the following properties: its edges are colorable with some finite number N of colors, and the subgraphs induced by any particular color is a union of disjoint stars. This is done in Lemma 6.4.

Now the construction of the field is as follows: first let $\langle p_0, p_1, \ldots, p_N \rangle$ be a list of distinct odd primes. Start with $\mathbb{Q}$ (or any prime field), and add the set of vertices X as transcendental elements over it. For each one, add p_0^n roots to it for all $n < \omega$. Now, for each edge, $e = \{s, t\}$, colored with the color $l < N$, adjoin p_{l+1}^n roots for all $n < \omega$ to $(s + t + 1)$. This is it. The reader is invited to compare to [**FK82**].

This construction can be done without Choice.

In the proof we use a generalized form of a lemma by P. Pröhle that appears in [**Prö84**]. In their original paper, Fried and Kollár could construct K_Γ with the restriction that $char\,(K_\Gamma) \neq 2$ and Pröhle removed this restriction. His "third lemma" from [**Prö84**] seemed to be perfect for our situation. However, we needed to generalize it in order to suit our purposes (and prove the generalization). This is Lemma 6.8.

The proof of Lemma 6.8 can be found in full detail in [**KS11**] which is an online copy of the present paper, but with the added proof of this technical lemma. We felt that the details of the proof can be omitted since it is technical and similar to the proof in [**Prö84**].

Acknowledgment. We would like to thank the referee for many useful remarks and to Haran Pilpel for drawing a graph with certain properties in record time.

A note about reading this paper. If the reader is not interested in Choice, but still wants to see the proof of Main Theorem A and Main Theorem B, he should ignore all the computations of cardinalities, since they become trivial. Also, with Choice, the construction of the field is somewhat easier — in our construction, we took the polynomial ring $\mathbb{Q}\,[Y]$ (where Y is a set containing the vertices) and then the quotient by an ideal. Then we had to show the ideal is prime in order to take the field of fractions. But with Choice we can construct the field by adding roots from the algebraic closure. See also Remark 6.14.

A small glossary.

- $|X| \leq |Y|$ means: There is an injective function from X to Y.
- $|X| = |Y|$ means: There is a bijection from X onto Y.
- For a structure $\mathfrak{A}$, $|\mathfrak{A}|$ is its universe and $||\mathfrak{A}||$ is its cardinality.
- $\mathbb{V}$ is the universe and $\mathbb{L}$ is Gödel's constructible universe.

2. A variant of $\tau^{\mathrm{nlg}}_{|k|}$ and some corollaries of Main Theorem A

Definition 2.1. A structure $\mathfrak{A}$ is called rigid if $\mathrm{Aut}(\mathfrak{A}) = 1$, i.e. it has no non-trivial automorphism.

Definition 2.2. For a set k, we define $\tau^{\mathrm{nlg}'}_{|k|}$ as the smallest ordinal α which is greater than $\tau_{\mathrm{Aut}(\mathfrak{A}),H}$ where $\mathfrak{A}, H$ are as in Definition 1.7 and in addition the vocabulary (language) L of $\mathfrak{A}$ satisfies

(1) There is some rigid structure with universe L and a countable vocabulary (for instance, L is well-orderable); and
(2) $|L| \leq \left||\mathfrak{A}|^{\langle<\omega\rangle}\right|$.

Remark 2.3. If κ is a cardinal number (i.e. an $\aleph$), then $\tau^{\mathrm{nlg}}_{\kappa} = \tau^{\mathrm{nlg}'}_{\left|\kappa^{\langle<\omega\rangle}\right|}$ and $\tau_{\kappa} = \tau_{\left|\kappa^{\langle<\omega\rangle}\right|}$. This is true since $\left|\kappa^{\langle<\omega\rangle}\right| = |\kappa|$, and because given any $\mathfrak{A}$ as in the definition, we may assume that $|\mathfrak{A}| \subseteq \kappa$ and that L is $|\mathfrak{A}|^{<\omega} \subseteq \kappa^{<\omega}$ which is well-orderable (see [**KS09**, Observation 2.3]).

Hence, by Main Theorem A

Corollary 2.4. *(ZF) For a cardinal κ, $\tau^{\mathrm{nlg}}_{\kappa} \leq \tau_{\kappa}$.*

The following is another easy conclusion of Main Theorem A

Corollary 2.5. *(ZF) for any cardinal κ, $\tau_{\kappa} \geq \kappa^{+}$. Moreover, letting $\upsilon_{k^{\langle<\omega\rangle}}$ be the smallest nonzero ordinal α such that there is no injective function $f : \alpha \to k^{\langle<\omega\rangle}$, then $\tau_{\left|k^{\langle<\omega\rangle}\right|} \geq \upsilon_{k^{\langle<\omega\rangle}}$* for any set k.

Proof. By [**Tho85**], we know that this result is true with Choice. Moreover, he proves that $\tau^{\mathrm{nlg}}_{\kappa} \geq \kappa^{+}$ (see Lemma in the proof of Theorem 2 there). Let $\alpha < \upsilon_{k^{\langle<\omega\rangle}}$ be some ordinal. We know that $\mathbb{L} \models \tau^{\mathrm{nlg}}_{|\alpha|} \geq |\alpha|^{+} > \alpha$ and that $|\alpha| \leq k^{\langle<\omega\rangle}$.

For a moment we work in $\mathbb{L}$. So there is a group G (the automorphism group of some structure) and a subgroup group $H \leq G$ such that $|H| \leq |\alpha|$ and $\alpha \leq \tau_{G,H}$. We may assume that $|G| \leq |\alpha|$. For one reason, this is the way it is constructed in [**Tho85**]. However, we give a self-contained explanation:

Let L be the language $\{P, Q, <, R\} \cup L_{\mathrm{Groups}}$ where P, Q are predicates, $<, R$ are binary relation symbols and L_{Groups} is the language of groups. Consider the L-structure $\mathfrak{G}$ with universe the disjoint union of G and α where $P^{\mathfrak{G}} = G$, $Q^{\mathfrak{G}} = \alpha$, with the group structure on P, the order on Q and $R^{\mathfrak{G}}(x, \beta)$ holds iff $x \in \mathrm{nor}^{\beta}_{G}(H)$. Let $\mathfrak{G}' \prec \mathfrak{G}$ be an elementary substructure of size $\leq |\alpha|$ such that $H \subseteq P^{\mathfrak{G}'}$, $\alpha \subseteq Q^{\mathfrak{G}'}$ (so $\alpha = Q^{\mathfrak{G}'}$), and let $G' = P^{\mathfrak{G}'}$. As a group G' is a subgroup of G containing H of size $\leq |\alpha|$ and for all $\beta < \alpha$, $\mathrm{nor}^{\beta}_{G'}(H) \neq \mathrm{nor}^{\beta+1}_{G'}(H)$, and in particular $\alpha \leq \tau_{G',H}$.

Now we go back to $\mathbb{V}$, so $|G| \leq |\alpha| \leq \left|k^{\langle<\omega\rangle}\right|$ by assumption. By [**KS09**, Claim 2.8], $\alpha \leq \tau^{\mathbb{L}}_{G,H} = \tau^{\mathbb{V}}_{G,H}$. Let $\mathfrak{A}$ be the structure with universe G and for each $g \in G$ a unary function f_g taking x to $x \cdot g$. Then $\mathrm{Aut}(\mathfrak{A}) \cong G$. So we conclude that $\tau^{\mathrm{nlg}'}_{k^{\langle<\omega\rangle}} \geq \alpha$ (because G is well-orderable as in Remark 2.3 above). By Main Theorem A, $\tau_{\left|k^{\langle<\omega\rangle}\right|} \geq \alpha$. $\square$

3. Coding structures as graphs

The next lemma allows us to present any automorphism group of an (almost) arbitrary structure as the automorphism group of a graph.

Lemma 3.1. *Let $\mathfrak{A}$ be a structure for the vocabulary (=language) L such that*

(1) There is some rigid structure on L with vocabulary L' such that $|L'| \leq \aleph_0$.
(2) $|L| \leq \left||\mathfrak{A}|^{\langle<\omega\rangle}\right|$.

Then there is a structure $\mathfrak{B}$ with vocabulary $L_{\mathfrak{B}}$ such that

- $||\mathfrak{B}|| \leq ||\mathfrak{A}|| + |L|$ (so $\leq \left||\mathfrak{A}|^{\langle<\omega\rangle}\right|$)
- $\mathrm{Aut}(\mathfrak{B}) \cong \mathrm{Aut}(\mathfrak{A})$
- $|L_{\mathfrak{B}}| = \aleph_0$

Proof. We may assume that both L and L' are relational languages.
Define $\mathfrak{B}$ by:

- $|\mathfrak{B}| = |\mathfrak{A}| \times \{0\} \cup L \times \{1\}$.
- The vocabulary is $L_{\mathfrak{B}} = \{R_n \,|\, n \in \omega\} \cup L' \cup \{P\}$ where P is a unary predicate and each R_n is an $n+1$ place relation.

Where:

- $Q^{\mathfrak{B}} = Q^L$ on $L \times \{1\}$ for each $Q \in L'$.
- $R^{\mathfrak{B}}_n = \left\{ ((a_0,0),\ldots,(a_{n-1},0),(R,1)) \,\middle|\, \begin{array}{c} R \in L \text{ is an } n \text{ place relation and} \\ (a_0,\ldots,a_{n-1}) \in R^{\mathfrak{A}} \end{array} \right\}$
- $P^{\mathfrak{B}} = L \times \{1\}$.

It is easy to see that $\mathfrak{B}$ is as desired. $\square$

This is well known:

Theorem 3.2. *Let $\mathfrak{A}$ be a structure for the first order language L which is as in the conditions of* 3.1. *Then there is a connected graph $\Gamma = \langle X_\Gamma, E_\Gamma\rangle$ such that* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathfrak{A})$, *and* $|X_\Gamma| \leq ||\mathfrak{A}||^{<\aleph_0}$.

Proof. For details see e.g. [**Tho**, Lemma 4.2.2] or [**Hod93**, Thereom 5.5.1]. From the construction (which does not use Choice) described there, one can deduce the part regarding the cardinality. The proof uses the fact that we can reduce to structures with countable languages, but this is exactly Lemma 3.1. $\square$

4. Some group theory

Lemma 4.1. *Let S be a simple non-abelian group, and let G be a group such that* $\mathrm{Inn}(S) \leq G \leq \mathrm{Aut}(S)$. *Then the automorphism tower of G is naturally isomorphic to the normalizer tower of G in* $\mathrm{Aut}(S)$.

The proof of this lemma can be found in [**Tho**, Theorem 4.1.4] (and, of course, it does not use Choice).

So we need a simple group. Recall

DEFINITION 4.2. Let K be a field, $n < \omega$, then:

- $GL(n, K)$ is the group of invertible $n \times n$ matrices over K.
- $PGL(n, K) = GL(n, K) / Z(GL(n, K))$ (Here, $Z(GL(n, K))$ is the group $K^{\times} \cdot I$ where I is the identity matrix).
- $SL(n, K) = \{x \in GL(n, K) \mid \det(x) = 1\}$.
- $PSL(n, K) = SL(n, K) / Z(SL(n, K))$ (The denominator is just $Z(GL(n, K)) \cap SL(n, K)$).

FACT 4.3. *$PSL(n, K)$ is a normal subgroup of $PGL(n, K)$.*

LEMMA 4.4. *$PSL(2, K)$ is simple for any field K such that $|K| \geq 3$.*

The proof of this lemma can be found in many books, e.g. [**Rot95**]. It is also true in ZF, by the following Lemma and Claim:

LEMMA 4.5. *Suppose P is a claim, such that $ZFC \vdash P$, and ψ is a first order sentence (in some language) such that $ZF \vdash$'P is true iff ψ does not have a model'. Then $ZF \vdash P$.*

PROOF. If we have a model $\mathbb{V}$ of ZF, such that $\mathbb{V} \models \neg P$, then ψ has a model so cannot prove contradiction (there is no use of Choice here). Hence ψ is consistent in $\mathbb{L} = \mathbb{L}^{\mathbb{V}}$ as well. (If ψ was not consistent in $\mathbb{L}$, then a proof of a contradiction from ψ would exist in $\mathbb{V}$ as well). Hence, by Gödel Completeness Theorem in ZFC, $\mathbb{L} \models \neg P$, but $\mathbb{L} \models ZFC$ — a contradiction. $\square$

CLAIM 4.6. There is a first order sentence ψ such that ψ has a model iff there is a field K, $|K| \geq 3$ such that $PSL(2, K)$ is not simple.

PROOF. Let L be the language of fields with an extra 4-ary relation H, i.e. $L = \{+, \cdot, 0, 1, H\}$. Let the sentence ψ say that the universe is a field K of size ≥ 3 and that $H \subseteq K^4$ is a normal subgroup of $SL(2, K)$ (after some choice of coordinates), and that H contains $Z(SL(2, K))$ and also some element outside $Z(SL(2, K))$. $\square$

We close this section by showing one final algebraic fact holds over ZF. Recall:

DEFINITION 4.7. Given any two groups N and H and a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$, we denote by $N \rtimes_{\varphi} H$ (or simply $N \rtimes H$ if φ is known) the semi-direct product of N and H with respect to φ.

Note that for a field K, there are canonical homomorphisms $\mathrm{Aut}(K) \to \mathrm{Aut}(PSL(2, K))$ and $\mathrm{Aut}(K) \to \mathrm{Aut}(PGL(2, K))$.

FACT 4.8. *(Van der Waerden, Schreier [**vdWS28**]) Let K be a field. Then every automorphism of $PSL(2, K)$ is induced via conjugation by a unique element of $P\Gamma L(2, K) := PGL(2, K) \rtimes \mathrm{Aut}(K)$. Hence $\mathrm{Aut}(PSL(2, K)) \cong P\Gamma L(2, K)$.*

This means that if $\varphi \in \mathrm{Aut}(PSL(2, K))$ then there are unique $\alpha \in \mathrm{Aut}(K)$ and $g \in PGL(2, K)$ such that for every $x \in PSL(2, K)$, $\varphi(x) = g\alpha(x) g^{-1}$.

We again use the model theoretic argument of Lemma 4.5 to give a proof of this fact in ZF:

Claim 4.9.

(1) There is a first order sentence ψ such that ψ has a model iff there is a field K, and an automorphism $\varphi \in \mathrm{Aut}(PSL(2,K))$ such that φ is not in $P\Gamma L(2,K)$. (This implies the existence of (α, g) required by the fact).
(2) There is a first order sentence ψ' such that ψ' has a model iff there is a field K, and some $1 \neq g \in PGL(2,K)$, $\alpha \in \mathrm{Aut}(K)$, such that for every $x \in PSL(2,K)$, $\alpha(x) = gxg^{-1}$. (This implies the uniqueness of (α, g) required by the fact).

Proof. (1): Let K be a field. Recall that $x_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $z_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ generate $SL(2,K)$. Let $g \in PGL(2,K)$, $\sigma \in \mathrm{Aut}(PSL(2,K))$.

Then $\alpha \in \mathrm{Aut}(K)$ satisfies $\sigma(x) = g\alpha(x)g^{-1}$ iff the map $x \mapsto g^{-1}\sigma(x)g$ takes $\bar{x}_t$ to $\bar{x}_{\alpha(t)}$ and $\bar{z}_t$ to $\bar{z}_{\alpha(t)}$. Let L be the language of fields augmented with 4-place function symbols $\{\sigma_i \mid i < 4\}$. ψ says that the universe K is a field, and that σ is an automorphism of $PSL(2,K)$ ($SL(2,K)$ is a definable subset of K^4, as is $Z(SL(2,K))$), such that for all $g \in PGL(2,K)$, the maps $t \mapsto g^{-1}\sigma(\bar{x}_t)g$ and $t \mapsto g^{-1}\sigma(\bar{z}_t)g$ do not induce a well defined automorphism of K.

(2): Let L be the language of fields. ψ' says that the universe K is a field and that there is some nontrivial $g \in PGL(2,K)$ such that the maps $t \mapsto g^{-1}\bar{x}_t g$ and $t \mapsto g^{-1}\bar{z}_t g$ are induced by an automorphism α of K. $\square$

5. Proof of Main Theorem A from Main Theorem B

From Main Theorem B which is proved in the next section, we can now deduce

Main Theorem A. *For any set k, $\tau^{\mathrm{nlg}'}_{|k^{\langle<\omega\rangle}|} \leq \tau_{|k^{\langle<\omega\rangle}|}$.*

Proof. (essentially the same proof as in [**JST99**]). We are given a structure $\mathfrak{A}$, with language L such that on the set L there is a rigid structure with countable vocabulary, and $||\mathfrak{A}|| \leq |k^{\langle<\omega\rangle}|$. By Theorem 3.2 and Main Theorem B we may assume that $\mathfrak{A}$ is an infinite field, K. We are also given a subgroup $H \leq \mathrm{Aut}(K)$, $|H| \leq |k^{\langle<\omega\rangle}|$.

Let $G = PGL(2,K) \rtimes H$. Obviously $|G| \leq |k^{\langle<\omega\rangle}|$.

G is centerless, because by Fact 4.8, the centralizer of $PSL(2,K)$ in $P\Gamma L(2,K)$ is trivial, and $PSL(2,K) \leq G$. So $PSL(2,K) \leq G \leq P\Gamma L(2,K)$. By Lemmas 4.1, 4.4, and 4.8, G^{α} is isomorphic to $\mathrm{nor}^{\alpha}_{P\Gamma L(2,K)}(G)$.

Now, by induction on α, one has $\mathrm{nor}^{\alpha}_{P\Gamma L(2,K)}(G) = PGL(2,K) \rtimes \mathrm{nor}^{\alpha}_{\mathrm{Aut}(K)}(H)$ and we are done. $\square$

6. Coding graphs as fields

In the introduction we mentioned that the following theorem of Fried and Kollár [**FK82**] was used in [**JST99**]:

Theorem 6.1. *(Fried and Kollár) (ZFC) For every connected graph Γ there is a field K such that* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(K)$, *and* $|K| = |\Gamma| + \aleph_0$.

Here we will offer a different proof of the Choiceless version, namely

Main Theorem B. *Let $\Gamma = \langle X, E \rangle$ be a connected graph. Then there exists a field K_Γ of any characteristic such that $|K_\Gamma| \leq |X^{\langle<\omega\rangle}|$ and* $\mathrm{Aut}(K_\Gamma) \cong \mathrm{Aut}(\Gamma)$.

COROLLARY 6.2. *If G is a group and there is some rigid structure with countable vocabulary on it, then there is a field K such that* $\mathrm{Aut}(K) \cong G$*, and* $|K| \leq \left|G^{\langle<\omega\rangle}\right|$.

PROOF. (of corollary) Let $\mathfrak{A}$ be the structure with universe G and for each $g \in G$ a unary function f_g taking x to $x \cdot g$ so that $\mathrm{Aut}(\mathfrak{A}) \cong G$. Now apply 3.2 and Main Theorem B. □

6.1. Coding graphs as colored graphs. We start by working a bit on the graph, to make the algebra easier.

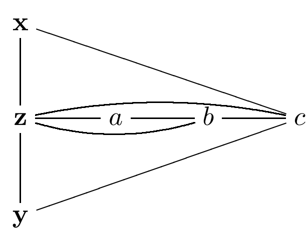
DEFINITION 6.3. A graph $G = \langle X, E\rangle$ is called a star if there is a vertex v (the center) such that $E \subseteq \{\{v, u\} \mid u \in V - \{v\}\}$.

LEMMA 6.4. *There is some number N such that for every connected graph $\Gamma = \langle X_\Gamma, E_\Gamma\rangle$, there is a connected graph $\Gamma^+ = \langle X_{\Gamma^+}, E_{\Gamma^+}\rangle$ with the following properties:*

(1) $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Gamma^+)$.
(2) *There is a coloring $C : E_{\Gamma^+} \to N$ of the edges of Γ^+ in N colors such that for all $l < N$ the l-th colored subgraph is a disjoint union of stars.*
(3) *Every $\varphi \in \mathrm{Aut}(\Gamma^+)$ preserves the coloring.*
(4) $|X_{\Gamma^+}| \leq \left|X_\Gamma^{\langle<\omega\rangle}\right|$*, in fact* $|X_\Gamma| \leq |X_{\Gamma^+}| \leq |X_\Gamma| + 4|E_\Gamma|$.

PROOF. The idea is to replace each edge $\{x, y\}$ by a copy of the graph G described below.

Recall that the valency of a vertex is the number of edges incident to the vertex, and will be denoted by $val(x)$. Let $G = \langle X_G, E_G\rangle$ be the following auxiliary graph:



Note the following properties of G:

- It has only 2 automorphisms: id and σ, where σ switches $\mathbf{x}$ and $\mathbf{y}$, but fixes all other vertices: $\mathbf{z}, b, c$ are characterized by their valency and a is the only vertex with valency 2 which is adjacent to $b, \mathbf{z}$.
- $\mathbf{z}$ is adjacent to all the vertices, its valency is unique and is not divisible by $val(\mathbf{x})$.
- $\mathbf{x}$ and $\mathbf{y}$ are not adjacent.

Description of Γ^+:

The set of vertices is

$$X_{\Gamma^+} = \{(1, x) \mid x \in X_\Gamma\} \cup \{(2, u, w) \mid u \in E_\Gamma, w \in X_G - \{\mathbf{x}, \mathbf{y}\}\}.$$

And the edges are:

- $(2, u, w)$ and $(2, u', w')$ are adjacent iff $u = u'$ and $\{w, w'\} \in E_G$.
- $(1, x)$ and $(2, u, w)$ are adjacent iff $x \in u$ and $\{\mathbf{x}, w\} \in E_G$ (iff $\{\mathbf{y}, w\} \in E_G$).

- That is all.

So, for each edge $\{x, y\} = u \in E_\Gamma$ there is an induced subgraph Γ_u^+ of Γ^+, whose vertices are $\{(1, x), (1, y)\} \cup \{(2, u, w) \mid w \neq \mathbf{x}, \mathbf{y}\}$, and $\Gamma_{\{x,y\}}^+ \cong G$ (by sending $\mathbf{x}$ to $(1, x)$, $\mathbf{y}$ to $(1, y)$ and $w \neq \mathbf{x}, \mathbf{y}$ to $(2, u, w)$).

Let G' be the subgraph of G induced by removing $\mathbf{y}$, let $N = |E_{G'}|$ (so $N = 7$), and denote $E_{G'} = \{e_0, \ldots, e_{N-1}\}$. Let $f : \Gamma^+ \to G'$ be a homomorphism of graphs defined as follows: $f(1, x) = \mathbf{x}$, $f(2, u, w) = w$. The coloring $C : E_{\Gamma^+} \to N$ is defined by $C(e) = i$ iff $f(e) = e_i$.

Let us now show (2). For each $i < N$, let $\Gamma_i^+ = \langle X_i, E_i \rangle$ be the subgraph induced by the color i. If $\mathbf{x} \notin e_i$, then Γ_i^+ is a union of disjoint edges by the definitions (and an edge is a star). If $\mathbf{x} \in e_i$, then Γ_i^+ is a disjoint union of $|X_\Gamma|$ stars, with centers $\{(1, x) \mid x \in X_\Gamma\}$, each having $val_\Gamma(x)$ edges.

For (1), note that $val_{\Gamma^+}(1, x) = val_G(\mathbf{x}) \cdot val_\Gamma(x)$ (or ∞, if $val_\Gamma(x) \geq \aleph_0$), while $val_{\Gamma^+}(2, u, w) = val_G(w)$, hence $val_{\Gamma^+}(2, u, \mathbf{z})$ is not divisible by $val_{\Gamma^+}(1, x)$.

Hence if $\varphi \in \operatorname{Aut}(\Gamma^+)$ then $\varphi(2, u, \mathbf{z}) = (2, u', \mathbf{z})$ for some $u' \in E_\Gamma$. Since $\mathbf{z}$ is adjacent to all the vertices in G, $\Gamma_{\{x,y\}}^+$ consists of all the vertices $(2, u, \mathbf{z})$ is adjacent to and itself. So $\varphi \restriction \Gamma_u^+$ is an isomorphism onto $\Gamma_{u'}^+$. Since $\operatorname{Aut}(G) = \{\mathrm{id}, \sigma\}$, for all $w \neq \mathbf{x}, \mathbf{y}$, $\varphi(2, u, w) = (2, u', w)$. This allows us to define $\psi_\varphi = \psi \in \operatorname{Aut}(\Gamma)$ by $\psi(x) = x'$ where $\varphi(1, x) = (1, x')$. It is now easy to see that $\varphi \mapsto \psi_\varphi$ is an isomorphism from $\operatorname{Aut}(\Gamma^+)$ onto $\operatorname{Aut}(\Gamma)$.

(3) and (4) should be clear. $\square$

6.2. Coding colored graphs as fields. Now we may assume that our graph is as in 6.4, and we start constructing the field.

We use the somewhat nonstandard notation of r as the characteristic of a field, so that $\mathbb{F}_r$ is the prime field with r elements.

Definition 6.5. Let $F \subseteq K$ be a field extension. F is said to be relatively algebraically closed in K if every $x \in K \backslash F$ is transcendental over F.

Definition 6.6. Let p be a prime. An element x in a field F is called *p-high*, if there is a sequence $\langle x_i \mid i < \omega \rangle$ of elements in F, such that $x_0 = x$, and $x_{i+1}^p = x_i$. With Choice this means that x has a p^n-th root for all $n < \omega$.

Example 6.7. If $F = \mathbb{Q}$, then for p odd, the only p-high element in F are $1, -1, 0$. If $F = \mathbb{F}_r$ for some prime r, then for every p such that $(p, r-1) = 1$ (i.e. the map $x \mapsto x^p$ is onto), every element in F is p-high.

This next lemma is the technical key. Its proof may use Choice, and this is OK, because we use it for finite Γ (see Remark 6.10 below).

Lemma 6.8. *(taken from* [**Prö84**, The third lemma] *with some adjustments) Let r be a prime number or 0, p a prime number different from r and let $\{p_0, \ldots, p_{n-1}\}$ be a set of pairwise distinct primes, different from p, r. Let F be a field of characteristic r. For $k < n$, let V_k be some set such that $k \neq l \Rightarrow V_k \cap V_l = \emptyset$, and let $V = \bigcup_{k<n} V_k$.*

For each $v \in V$, let $T_v \in F[X]$ be polynomials such that:

- *none of them is constant.*
- *none of them is divisible by X.*
- *they are separable polynomials.*
- *they are pairwise relatively prime (i.e. no nontrivial common divisor).*

Suppose that K is an extension of F generated by the set $\{z_i \mid i < \omega\} \cup \{t_i^v \mid v \in V, i < \omega\}$ from the algebraic closure of $F(z_0)$ where:

- *z_0 is transcendental over F.*
- *$(z_{i+1})^p = z_i$ for all $i < \omega$.*
- *For $v \in V$, $t_0^v = T_v(z_0)$*
- *if $v \in V_k$ then $\left(t_{i+1}^v\right)^{p_k} = t_i^v$.*

Then we have the following properties:

(1) *F is relatively algebraically closed in K.*
(2) *An equivalent definition of K is the following one: Suppose F is the field of fractions of an integral domain S. Then K is the field of fractions of the integral domain R/I (in particular I is prime) where $R = S[Y_i, S_l^v \mid i, l < \omega, v \in V]$ (i.e. the ring generated freely by S and these elements) and $I \leq R$ is the ideal generated by the equations:*
   (a) *$Y_{i+1}^p = Y_i$ for $i < \omega$.*
   (b) *$S_0^v = T_v(Y_0)$ for $v \in V$.*
   (c) *If $v \in V_k$, then $\left(S_{l+1}^v\right)^{p_k} = S_l^v$ for $k < n$, $l < \omega$.*
(3) *Each q-high element of K belongs to F whenever q is a prime different from p and $\langle p_k \mid k < n\rangle$.*
(4) *Each p-high element of K is of the form $c \cdot (z_i)^m$, where c is a p-high element of F, $i < \omega$ and m is an integer.*
(5) *If p' is a prime different from p then z_0 does not have a p' root.*
(6) *If V is finite then $|K| \leq \left|F^{\langle<\omega\rangle}\right|$. Furthermore, the injection witnessing this is definable from the parameters given when constructing K (i.e. the function $v \mapsto T_v$, etc).*

The proof may be found in [**KS11**].

The rest of the section is devoted to proving

Theorem 6.9. *Let $\Gamma = \langle X, E, C\rangle$ be an N-colored graph as in Lemma 6.4. Then there exists a field K_Γ such that $|K_\Gamma| \leq \left|X^{\langle<\omega\rangle}\right|$ and $\mathrm{Aut}(K_\Gamma) \cong \mathrm{Aut}(\Gamma)$. Furthermore, $X \subseteq K_\Gamma$ and $\pi \mapsto \pi \restriction X$ is an isomorphism from $\mathrm{Aut}(K_\Gamma)$ onto $\mathrm{Aut}(\Gamma)$. We can choose K_Γ to be of any characteristic.*

So Main Theorem B immediately follows from this and Lemma 6.4.

<u>The construction of K_Γ</u>: Let L be the field $\mathbb{Q}$ or $\mathbb{F}_r$ for some prime r. Let $\langle p_i \mid i \leq N\rangle$ list odd prime numbers which are different then r, and do not divide $r-1$ (so that in L there are no p_i-roots of unity). Let R be the ring $L[Y_\Gamma]$ where $Y_\Gamma = \left\{x_s^i \mid i < \omega, s \in X\right\} \cup \left\{x_e^i \mid i < \omega, e \in E\right\}$[1] is an algebraically independent set. Let $I_\Gamma \subseteq R$ be the ideal generated by the equations:

- $\left(x_s^{i+1}\right)^{p_0} = x_s^i$ for all $s \in X$ and $i < \omega$.
- If $e = \{s, t\}$ then $x_e^0 = x_s^0 + x_t^0 + 1$ for all $s, t \in X$ and $e \in E$.
- If $C(e) = l$ then $\left(x_e^{i+1}\right)^{p_{l+1}} = x_e^i$ for all $e \in E$.

Now let R_Γ be the ring R/I_Γ.

Remark 6.10.

(1) If Γ, Γ' are N-colored graphs, and $\Gamma \cong \Gamma'$ (and the isomorphism respects the coloring) then $R_\Gamma \cong R_{\Gamma'}$.

[1] The i s are indices not exponents! Later we will use parentheses in order not to confuse a superscript with an exponent.

(2) Hence we may use Choice when proving properties regarding R_Γ (and later K_Γ) when Γ is finite because we may assume $\Gamma \in \mathbb{L}$ (hence also $R_\Gamma \in \mathbb{L}$ etc). In that case we may use Lemma 6.8 even if there is Choice in the proof.

PROPOSITION 6.11. *I_Γ is prime, so we let K_Γ be be the field of fractions of R_Γ.*

The proof uses the following remark (when it makes sense)

REMARK 6.12. If $\Gamma_0 \subseteq \Gamma_1$ are finite where $\Gamma_i = \langle X_i, E_i, C_i\rangle$ for $i < 2$ and $X_1 = X_0 \cup \{t\}$, $t \notin X_0$, then the field extension $K_{\Gamma_0} \subseteq K_{\Gamma_1}$ is as in Lemma 6.8, where

- F is the field K_{Γ_0}; r is its characteristic; p is p_0; $\{p_0, \dots, p_{n-1}\}$ is $\{p_{l+1} \mid l < N\}$; V_k is the set of edges $\{t, s\} \in E$ of color k; for $s \in X_0$ such that $v = \{t, s\} \in E$, T_v is the polynomial $X + x_s^0 + 1$; z_i is x_t^i and for $v = e = \{t, s\}$, t_i^v is x_e^i.

PROOF. (of proposition) We may assume Γ is finite, so the proof is by induction on $|X|$. Suppose that $\Gamma_0 \subseteq \Gamma_1$ where $\Gamma_i = \langle X_i, E_i, C_i\rangle$ for $i < 2$ and that $X_1 = X_0 \cup \{t\}$, $t \notin X_0$. By induction, I_{Γ_0} is prime, so $R = R_{\Gamma_0}$ is an integral domain.

Let $Y_t = \left\{x_t^i \mid i < \omega\right\} \cup \left\{x_e^i \mid i < \omega, t \in e \in E_1\right\}$; $I_t \subseteq R\,[Y_t]$ be the ideal generated by the equations related to t and $\{e \in E_1 \mid t \in e\}$.

By Lemma 6.8, clause (2), I_t is prime.

Consider the canonical projection $\pi : L\,[Y_{\Gamma_1}] \to R\,[Y_t]$ so that $\pi\,(I_{\Gamma_1}) = I_t$ and $\langle I_{\Gamma_0}\rangle = \ker(\pi)$. Hence, π induces an isomorphism $L[Y_{\Gamma_1}]/I_{\Gamma_1} \to R[Y_t]/I_t$ and we are done since the right hand side is an integral domain. $\square$

DEFINITION 6.13. (ZFC) Let F be a field and let p be a natural number. Let S be a set of elements from F. Then $F\,(S, p)$ denotes the field which is obtained by adjoining the elements $\{s\,(l) \mid s \in S, l < \omega\}$ from the algebraic closure of F where:

- $s\,(0) = s$.
- $s\,(l+1)^p = s\,(l)$, $l < \omega$.

REMARK 6.14. Choice is a priori needed in this definition because the construction implicitly assumes the existence of an algebraic closure, and some ordering of S and of the p-roots of the $s\,(l)$s.

DEFINITION 6.15. Let $K_{-1} = L\,(Y)\,(Y, p_0)$, where $Y = \left\{x_t^0 \mid t \in X\right\}$, and $L\,(Y)$ denotes the purely transcendental extension of L, and for $l < N$, $K_l = K_{l-1}\,(E_l, p_{l+1})$, where $E_l = \left\{x_s^0 + x_t^0 + 1 \mid \{s, t\} = e \in E, C\,(e) = l\right\}$.

LEMMA 6.16.

(1) *For Γ finite*[2]*, K_Γ is canonically isomorphic to K_{N-1}.*
(2) *If $\Gamma_0 \subseteq \Gamma_1$ then $K_{\Gamma_0} \subseteq K_{\Gamma_1}$.*

PROOF. (1) follows from Lemma 6.8, (2) by induction on the size of Γ, similarly to the proof of Proposition 6.11. (2) follows from (1) for finite Γ, which is enough. $\square$

From now on, fix some Γ.

[2]The assumption that Γ is finite is only to insure that K_{N-1} is well defined, with Choice this assumption is not needed.

Definition 6.17. For each $Y \subseteq X$, let Γ_Y be the induced subgraph generated by Y (i.e. $\Gamma_Y = \langle Y, E \restriction Y\rangle$) and let $R_Y = R_{\Gamma_Y}$, $K_Y = K_{\Gamma_Y}$.

Some properties of K_Γ:

Lemma 6.18. *For Γ as in Lemma 6.4,*

1. *For any prime p, if $a \in K_Y$ for some $Y \subseteq X$ and is p-high in K_Γ then a is already p-high in K_Y.*
2. *For each $i < \omega$, the set $\left\{x_s^i \mid s \in X\right\}$ is algebraically independent over L.*
3. *If $X_1 \subseteq X_2$ then K_{X_1} is relatively algebraically closed in K_{X_2} (in particular L is r.a.c in K_Γ).*

Proof. (1) and (2) follows from (3). For (3), we may assume X_1, X_2 are finite, and then it is enough to prove it for the case $X_2 = X_1 \cup \{t\}, t \notin X_1$. Now use Remark 6.12, and clause (1) of Lemma 6.8. □

Now we shall define the isomorphism from $\mathrm{Aut}(\Gamma)$ to $\mathrm{Aut}(K_\Gamma)$:

Proposition 6.19. *For Γ as in 6.4, there is a canonical injective homomorphism $\sigma : \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(K_\Gamma)$ defined by $\sigma(\varphi)\left(x_t^i\right) = x_{\varphi(t)}^i$, and $\sigma(\varphi)\left(x_e^i\right) = x_{\varphi(e)}^i$, for $\varphi \in \mathrm{Aut}(\Gamma)$ and all $t \in X, e \in E$.*

Proof. σ is well defined because of clause (3) of Lemma 6.4. σ is obviously a homomorphism. It is injective: If $\sigma(\varphi) = \mathrm{id}$, while $\varphi(s) = t \neq s$, then $x_s^0 = \sigma(\varphi)\left(x_s^0\right) = x_t^0$ — a contradiction to clause (2) of Lemma 6.18. □

Our aim is to prove that σ is onto. We start with:

Claim 6.20. Suppose that $a \in K_\Gamma$ is p-high, then:

1. If $p = p_0$ then a can be written in the form $\varepsilon \cdot \prod\left\{(x_s^{n_s})^{m_s} \mid s \in X_0\right\}$ for some finite $X_0 \subseteq X$, some choice of $m_s \in \mathbb{Z}, n_s < \omega$ for $s \in X_0$ and a p_0-high element $\varepsilon \in L$.
2. If $p = p_{l+1}$ for some $l < N$ then a can be written in the form $\varepsilon \cdot \prod\left\{(x_e^{n_e})^{m_e} \mid e \in E_0\right\}$ for some finite $E_0 \subseteq E$ such that $C \restriction E_0 = l$, some choice of $n_e < \omega, m_e \in \mathbb{Z}$ for $e \in E_0$ and a p_{l+1}-high element $\varepsilon \in L$.

Proof. By Lemma 6.18, clause (1), there is some $X_0 \subseteq X$ such that a is p-high in K_{X_0}. The proof is by induction on $|X_0|$. The base of the induction — $X_0 = \emptyset$ — is clear. For the induction step, we prove that if $X_0 \subseteq X_1$ are finite and $X_1 = X_0 \cup \{t\}$, $t \notin X_0$, and the claim is true for X_0, then every $a \in K_{X_1}$ which is p-high has the desired form.

For clause (1), Remark 6.12 implies that we can use Lemma 6.8, clause (4).

For (2), we shall use the assumption on the coloring.

Case 1. There is no edge $e_0 \ni t$ in Γ_{X_1} such that $C(e_0) = l$. In that case, we use clause (3) of Lemma 6.8, and conclude that $a \in K_{X_0}$.

Case 2. There is an edge $e_0 \ni t$ in Γ_{X_1} with $C(e_0) = l$, but only one such edge. If $e_0 = \{s, t\}, s \in X_0$ then $x_{e_0}^0 = x_s^0 + x_t^0 + 1 \in K_{X_1}$ is transcendental over K_{X_0} (because x_t^0 is). In addition $x_t^0 = x_{e_0}^0 - x_s^0 - 1$ and for all vertices $r \in X_0$ such that $e_r = \{t, r\}$ is an edge (of some other color), $x_{e_r}^0 = x_{e_0}^0 - x_s^0 + x_r^0$. The polynomials $X - x_s^0 - 1$, $X - x_s^0 + x_r^0$ satisfy the conditions of Lemma 6.8, and so, by clause (4), a is of the form $\left(x_{e_0}^i\right)^m \cdot c$ for c which is p_{l+1}-high in K_{Γ_0} and we are done (we do not use the lemma in the same way as in Remark 6.12 — here z_0 is played by $x^0_{e_0}$, but it is the same idea).

Case 3. There is more than one edge $e_0 \ni t$ in Γ_{X_1} with color l. Then t is the center of a star in the subgraph of Γ_1 induced by that color. Assume that $s_1, \dots, s_k \in X_0$ list the vertices such that $C(s_i, t) = l$, $(k \geq 2)$. Let $X^- = X_0 \setminus \{s_1, \dots, s_k\}$, and $X' = X^- \cup \{t\}$. Note that $|X'| < |X_1|$, so by the induction hypothesis, the claim is true for $K_{X'}$. Γ_{X_1} is built from $\Gamma_{X'}$ by adding $s_1, \dots, s_k$ and in each step we are in the previous case (because t was the center of a star), so we are done. ☐

LEMMA 6.21. *For all $s \in X$, x^0_s does not have a p' root for p' a prime different from p_0.*

PROOF. Again, it is enough to prove this finite $X_0 \subseteq X$, and the proof is by induction on $|X_0|$, and follows from clause (5) of Lemma 6.8. ☐

This is the main proposition:

PROPOSITION 6.22. *Assume $\varphi \in \operatorname{Aut}(K_\Gamma)$ and that $\{s_0, t_0\} \in E$ of color l. Then there is an edge $\{s_1, t_1\} \in E$ of the same color such that $\varphi\left(x^0_{s_0}\right) = x^0_{s_1}$ and $\varphi\left(x^0_{t_0}\right) = x^0_{t_1}$.*

PROOF. Let $f_1 = \varphi\left(x^0_{s_0}\right)$, $f_2 = \varphi\left(x^0_{t_0}\right)$, $f = \varphi\left(x^0_{s_0} + x^0_{t_0} + 1\right) = f_1 + f_2 + 1$. From Claim 6.20 it follows that

- $f_1 = \varepsilon_1 \cdot \prod \left\{ \left(x^{i_s}_s\right)^{m_s} \mid s \in X_0 \right\}$, $f_2 = \varepsilon_2 \cdot \prod \left\{ \left(x^{i_t}_t\right)^{m_t} \mid t \in Y_0 \right\}$ and $f = \varepsilon_3 \cdot \prod \left\{ \left(x^{i_e}_e\right)^{m_e} \mid e \in E_0 \right\}$,

where $X_0, Y_0 \subseteq X$ and $E_0 \subseteq E$ are finite nonempty; $i_s < \omega$, $m_s \in \mathbb{Z}$ for $s \in X_0$; $i_t < \omega$, $m_t \in \mathbb{Z}$ for $t \in Y_0$; and E_0 is homogeneous of color l and $i_e < \omega$, $m_e \in \mathbb{Z}$ for $e \in E_0$. Let $p = p_{l+1}$, so f is p-high.

We can assume that unless $i_s = 0$, $p_0 \nmid m_s$ for $s \in X_0 \cup Y_0$, and that unless $i_e = 0$, $p \nmid m_e$ for $e \in E_0$.

Raising the equation $f_1 + f_2 + 1 = f$ by p^k where $k = \max\{i_e \mid e \in E_0\}$, we have an equation of the form

$$\left(\varepsilon_1 \prod \left(x^{i_s}_s\right)^{m_s} + \varepsilon_2 \prod \left(x^{i_t}_t\right)^{m_t} + 1\right)^{p^k} = \varepsilon_3^{p^k} \prod \left(x^0_r + x^0_w + 1\right)^{p^{k - i_{\{r,w\}}} m_{\{r,w\}}}.$$

Let $i = \max\{i_t \mid t \in X_0 \cup Y_0\}$. We can replace $x^{i_t}_t$ by $\left(x^i_t\right)^{p_0^{i - i_t}}$ and the same for $x^{i_s}_s$. Also replace x^0_r by $\left(x^i_r\right)^{p_0^i}$ and the same for x^0_w. For $t \in T := X_0 \cup Y_0 \cup \bigcup E_0$, let $y_t = x^i_t$, then we get

$$\left(\varepsilon_1 \prod (y_s)^{p_0^{i - i_s} m_s} + \varepsilon_2 \prod (y_t)^{p_0^{i - i_t} m_t} + 1\right)^{p^k} = \varepsilon_3^{p^k} \prod \left((y_r)^{p_0^i} + (y_w)^{p_0^i} + 1\right)^{p^{k - i_{\{r,w\}}} m_{\{r,w\}}}.$$

By Lemma 6.18, these elements are algebraically independent so this is an equation in the field of rational functions $L(y_t \mid t \in T)$.

The next step is to see that the exponents (m_t and $m_{\{r,w\}}$) are non-negative. For that we use valuations.

Recall that for any field, F and any irreducible $g \in F[X]$ there is a unique discrete (i.e. with value group $\mathbb{Z}$) valuation on the field of rational functions $F(t)$ defined by $v(g(t)) = 1$, $v \restriction F^{\times} = 0$. In this case, $v \restriction F[t] \geq 0$ and $v(m(t)) > 0$ iff $g|m$ for $m \in F[X]$. This is the g-adic valuation.

Suppose m_{t_0} is negative for some $t \in X_0 \cup Y_0$. Consider the discrete valuation v on the field $L(y_t \,|\, t \in T)$ defined by $v(y_{t_0}) = 1$, $v \restriction L(y_t \,|\, t \neq t_0)^{\times} = 0$. Then on the left hand side we get $v(LHS) < 0$ while on the right hand side, $v(RHS) = 0$ — contradiction.

Suppose $m_{\{r,w\}} < 0$ for some $\{r, w\} \in E_0$. Consider the valuation v on the field $L(y_t \,|\, t \in T)$ defined by $v(g(y_r)) = 1$, $v \restriction L(y_t \,|\, t \neq r)^{\times} = 0$ where g is any irreducible polynomial dividing $X^{p_0^i} + (y_w)^{p_0^i} + 1$. So $v\left((y_r)^{p_0^i} + (y_w)^{p_0^i} + 1\right) > 0$, while g does not divide $\left(X^{p_0^i} + (y_{w'})^{p_0^i} + 1\right)$ for $w \neq w$' (they relatively prime) so $v(RHS) < 0$. On the other hand, since $v(y_r) = 0$, $v(RHS) \geq 0$ — contradiction.

Hence we can consider this equation as one in the polynomial ring $L[y_t \,|\, t \in T]$. Moreover, since these elements are algebraically independent, each one appearing in the left hand side must appear in the right hand side and vice versa, i.e. $T = X_0 \cup Y_0 = \bigcup E_0$.

By examining the free factor, $\varepsilon_3^{p^k} = 1$.

By substituting y_r and y_w with 0 for some r, w , we can show that $E_0 = \{\{r, w\}\}$ (so $k = i_{\{r,w\}}$) and that there are no mixed monomials in the left hand side, i.e. we get an equation of the form

$$\left(\varepsilon_1 (y_r)^{p_0^{i-i_r} m_r} + \varepsilon_2 (y_w)^{p_0^{i-i_w} m_w} + 1\right)^{p^k} = \left((y_r)^{p_0^i} + (y_w)^{p_0^i} + 1\right)^{m_{\{r,w\}}}.$$

Suppose $i = i_r$ and $i \neq 0$, then $p_0 \nmid m_r$, by examining the degree of y_r, we get a contradiction, so $i = 0$ and by choice of i, $i_w = 0$ as well. In the same way we can deduce that $k = 0$. From this it follows that $\varepsilon_1 = \varepsilon_2 = 1$ and $m_r = m_w$. So we have

$$f_1 + f_2 + 1 = \left(x_r^0\right)^{m_r} + \left(x_w^0\right)^{m_w} + 1 = \left(x_r^0 + x_w^0 + 1\right)^{m_{\{r,w\}}} = f.$$

So $\{r, w\}$ is an edge of color l, $m := m_{\{r,w\}} = m_w = m_r$, and $m = 1$ or a power of r (the characteristic).

So finally we have that $\varphi\left(x_{t_0}^0\right)$ is a power of m which is a power of r. This implies that $x_{t_0}^0$ itself has an m-root. But if $m > 1$, this is a contradiction, because $x_{t_0}^0$ has no r-roots by Lemma 6.21.

This concludes the proof of the proposition. □

Corollary 6.23. *The map* $\sigma : \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(K_\Gamma)$ *is a bijection.*

Proof. Recall that all that is left is to show that σ is onto (by Proposition 6.19).

Let $\varphi \in \mathrm{Aut}(K_\Gamma)$. Let $t \in X$ and suppose $\{t, t_0\} \in E$. By Proposition 6.22, $\varphi\left(x_t^0\right) = x_{t'}^0$ for the some $t' \in X$. Since the graph Γ is connected, we can define $\varepsilon \in \mathrm{Aut}(\Gamma)$ by $\varepsilon(t) = t'$ (note that t' does not depend on the choice of t_0). Proposition 6.22 implies that ε is indeed an automorphism.

Since there are no p_i-roots of unity in L for all the primes we chose, it follows then that $\varphi\left(x_t^i\right) = x_{\varepsilon(t)}^i$ and that $\varphi\left(x_e^i\right) = x_{\varepsilon(e)}^i$, and hence $\varphi = \sigma(\varepsilon)$. □

We still have to prove that $|K_\Gamma| \leq \left|X^{\langle<\omega\rangle}\right|$.

LEMMA 6.24. *If $X_i \subseteq X$ $(i = 1, 2)$ are two subsets of the vertices set then* $K_{X_1} \cap K_{X_2} = K_{X_1 \cap X_2}$.

PROOF. We may assume that X_1, X_2 are finite. Assume $x \in K_{X_1} \cap K_{X_2}$ and that $|X_1|$ is minimal with respect to $x \in K_{X_1}$. If $X_1 \subseteq X_2$ then we are done. If not, let $t \in X_1 \backslash X_2$ be some vertex, and let $X' = X_1 \backslash \{t\}$. So $x \notin K_{X'}$, and x is transcendental over $K_{X'}$ while x_t^0 is algebraic over $K_{X'}(x)$. Let $X_2' = X' \cup X_2$, $X_3 = X_2' \cup \{t\}$. We have $x \in K_{X_2} \subseteq K_{X_2'}$, and $x_t^0 \in K_{X_3}$ is transcendental over $K_{X_2'}$. This is a contradiction, because x_t^0 is algebraic over $K_{X'}(x) \subseteq K_{X_2'}$. Hence there is no such t i.e. $X_1 \subseteq X_2$. $\square$

And now it is easy to define an injective map $\Psi : K_\Gamma \to X^{\langle < \omega \rangle}$. Define by induction on n injective functions $\Psi_Y : K_Y \to X^{\langle < \omega \rangle}$ for $|Y| \leq n$ such that $Y_1 \subseteq Y_2$ implies $\Psi_{Y_1} \subseteq \Psi_{Y_2}$. This is enough, since by the lemma above, $\bigcup \{\Psi_Y \mid Y \subseteq X, |Y| < \omega\}$ is an injection from K_Γ to $X^{\langle < \omega \rangle}$.

For the construction of $\Psi_Y : K_Y \to X^{\langle < \omega \rangle}$, the idea is that given $x \in K_Y$ such that $x \notin K_{Y'}$ for any $Y' \subsetneq Y$ we can code x using the set Y and the set of codes that Lemma 6.8, clause (6) gives us for any choice of $Y' \subsetneq Y$ of size $|Y| - 1$.

This (and Lemma 6.8, clause (6)) was the reason we chose $X^{\langle < \omega \rangle}$ and not $X^{< \omega}$: in order to code $x \in K_\Gamma$, we need first to code the minimal set Y such that $x \in K_Y$, and then x can be coded in $|Y|$ different ways, depending on the choice of $|Y'|$ as above. However, there is no well ordering of Y, so we have no way of ordering these codes. For instance, the code of $x_t^0 + x_s^0$ for $s, t \in X$, should be $\{\langle s \rangle, \langle t \rangle, \ldots\}$.

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On subgroups of totally projective primary abelian groups and direct sums of cyclic groups

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Dedicated to Rüdiger Göbel on the occasion of his 70th birthday.

Abstract. For abelian p-groups, a new characterization of C-decomposability is given. It implies, for example, that it is consistent with ZFC that every group whose final rank has countable cofinality is C-decomposable. The class of groups that can be embedded in totally projective groups of length $\omega \cdot 2$ is also considered. It is known that such groups might not be C-decomposable, but it is shown that they possess a related property; that is, they are far from thick.

1. Terminology and introduction

By the term "group" we will mean an abelian p-group, where p is a fixed prime. Our group theoretic terminology and notation will generally follow that found in [**F**]. Recall that the final rank of a group G is given by

$$\text{fin r}(G) = \min_{k<\omega}\{\ r(p^k G)\ \}.$$

A group is Σ-*cyclic* if it is isomorphic to a direct sum of cyclic groups. Of course, the Σ-cyclic groups play a central role in the study of primary abelian groups. The group G is said to be *C-decomposable* if it is isomorphic to $A \oplus B$, where B is a Σ-cyclic group with $\text{fin r}(G) = \text{fin r}(B)$.

If α is an ordinal, a group G is said to be p^α*-projective* if $p^\alpha \text{Ext}(G, X) = \{0\}$ for all groups X. It can be shown that G is p^ω-projective iff it is Σ-cyclic, and if $n < \omega$, then G is $p^{\omega+n}$-projective iff there is a subgroup $Y \subseteq G[p^n]$ such that G/Y is Σ-cyclic. It is readily verified that an arbitrary subgroup of a $p^{\omega+n}$-projective group retains that property.

Interestingly, if G is $p^{\omega+1}$-projective, then it is C-decomposable (see [**G**]). On the other hand, in [**D**], Cutler-Missel constructed an ingenious example of a $p^{\omega+2}$-projective group that does not have any unbounded Σ-cyclic summands.

In [**J**], the concept of C-decomposability was expanded in a way that circumvents this difficulty. A group G of final rank κ is said to be *far from thick* if there is a Σ-cyclic group C and a homomorphism $\phi : G \to C$ such that for all $n < \omega$ the rank of $\phi((p^n G)[p])$ is at least κ. [The terminology stems from the fact that G

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fails to be thick iff there is a Σ-cyclic group C and a homomorphism $\phi : G \to C$ such that for all $n < \omega$, $\phi((p^n G)[p])$ is non-zero.] Clearly, if G is C-decomposable, then the projection onto a Σ-cyclic summand of final rank κ shows that it is far from thick. Though for $n > 1$ not every $p^{\omega+n}$-projective group is C-decomposable, it is true for all $n < \omega$ that every $p^{\omega+n}$-projective group is far from thick ([**J**], Corollary 25).

The class of groups that are far from thick has several useful properties. For example, if $m > 0$, the $p^{\omega+m}$-projective group G is said to be *proper* if it is not $p^{\omega+m-1}$-projective. It is known that a group G is not Σ-cyclic iff it has a subgroup that is a proper $p^{\omega+1}$-projective (see [**B**]). A group G has the *generalized core class property* (or GCCP for short) if for all $m < \omega$, G is either $p^{\omega+m}$-projective or it has a subgroup that is a proper $p^{\omega+m+1}$-projective. It is unknown whether every reduced group has the GCCP, but the class of groups that do is quite extensive. In particular, every group that is far from thick (so, for example, every group that is $p^{\omega+n}$-projective for some $0 < n < \omega$) has the GCCP (see [**J**], Corollary 27).

The purpose of this note is to extend these ideas. We begin by giving a new characterization of C-decomposability in terms of countable ascending sequences of pure subgroups (Theorem 2.1). This fairly straightforward result does have some surprising consequences. For example, we can use it to conclude that any group whose final rank has countable cofinality will necessarily satisfy the GCCP (Corollary 2.6), and in the presence of the generalized continuum hypothesis, it will also be C-decomposable (Corollary 2.5).

Following [**I**], a group G is said to be a *weak p^α-projective* if it has the following property: whenever A is a group with a subgroup K such that $A/K \cong G$, then if B is any other group, then any homomorphism $K \to p^\alpha B$ will extend to a homomorphism $A \to B$. For our purposes, a much more concrete description of the class is as follows: G is a weak p^α-projective iff it can be embedded as a subgroup of a p^α-projective group H. Note that this H will be isomorphic to a summand of $\mathrm{Tor}(H, H_\alpha)$, where H_α denotes the "generalized Prüfer group of length α" (see, for example, Theorem 84 of [**H**]). If $D \cong \bigoplus_\kappa \mathbb{Z}_{p^\infty}$ is a divisible hull of H, then $\mathrm{Tor}(H, H_\alpha)$ will be isomorphic to a subgroup of $\mathrm{Tor}(D, H_\alpha) \cong \bigoplus_\kappa H_\alpha$. Since the latter group is a p^α-bounded totally projective group, it is p^α-projective. In other words, there is no loss of generality in assuming that H is a p^α-bounded totally projective group.

If $n < \omega$, then since a subgroup of a $p^{\omega+n}$-projective group retains this property, the weak $p^{\omega+n}$-projective groups agree with the $p^{\omega+n}$-projective groups. For $\omega+\omega = \omega \cdot 2$, by ([**I**],Corollary 2.17), if G is a group, then G is a weak $p^{\omega\cdot 2}$-projective iff it is an extension of a Σ-cyclic subgroup by a factor that is also Σ-cyclic; i.e., it has a subgroup X such that both X and $Y = G/X$ are Σ-cyclic (we will review this important fact in Proposition 3.1(a)). Nunke showed that there are weak $p^{\omega\cdot 2}$-projective groups that are not p^α-projective for any ordinal α (see [**M**], [**N**]; we also review this fact in Proposition 3.1(b)).

In this paper we show that if G is a weak p^α-projective whose final rank has countable cofinality, then it is C-decomposable (Corollary 2.3(b)). Again, since there are $p^{\omega+2}$-projective groups that are not C-decomposable, this restriction on final ranks cannot be eliminated. On the other hand, we do establish that any group that is a weak $p^{\omega\cdot 2}$-projective is far from thick (Theorem 3.3). It follows that every weak $p^{\omega\cdot 2}$-projective group has the GCCP. It is, perhaps, of interest that this argument extensively uses ideas regarding *large* subgroups that go back to Pierce ([**O**]).

2. A new characterization of C-decomposability

By the *basic final rank* of G we will mean the final rank of one of its basic subgroups; we let bfin r(G) denote the basic final rank of G. So if f_G denotes the Ulm function of G, then

$$\text{bfin r}(G) = \min_{n<\omega}\{ \sum_{n \le i < \omega} f_G(i) \}.$$

It follows easily that if G is C-decomposable, then fin r(G) = bfin r(G).

If G is a group, then let $n < \omega$ be chosen so that $r(p^n G) = \text{fin r}(G)$. There is a decomposition $G = G' \oplus B$, where B is p^n-bounded and $r(G') = \text{fin r}(G)$ (see, for example, [**F**], Theorem 33.2). For the properties considered in this paper, such as C-decomposability, it is easy to check that G satisfies the property iff G' does. In our proofs, therefore, we will usually assume $r(G) = \text{fin r}(G)$ without explicitly stating it. In particular, our groups will always be unbounded. Similarly, any group G is isomorphic to a direct sum $G' \oplus D$ where G' is reduced and D is divisible. Again, replacing G by G' in the proofs in this paper will not change anything materially. Therefore, though we state our theorems in full generality, in their proofs our standing assumptions will be that the groups are reduced, unbounded and that their ranks and final ranks agree.

Suppose P is a pure subgroup of G and $A = G/P$. If B is a basic subgroup of P, then we can extend it to a basic subgroup of G of the form $B \oplus C$. It is easy to verify that the natural epimorphism $G \to A$ maps C to a basic subgroup of A. It follows that $f_G(n) = f_P(n) + f_A(n)$ for all $n < \omega$ and bfin r(G) = bfin r(P) + bfin r(A).

We note in passing that every countable reduced group is C-decomposable. To see this, suppose G is as described. It follows from ([**F**], Lemma 78.1) that the Ulm factor $G' \stackrel{\text{def}}{=} G/p^\omega G$ is Σ-cyclic. Let $G' = A \oplus B$, where A and B are unbounded countable Σ-cyclic groups. By ([**F**], Corollary 76.2) there is a countable group H such that $p^\omega H \cong p^\omega G$ and $H/p^\omega H \cong A$. Note that $H \oplus B$ will have the same Ulm invariants as G, so by ([**F**], Theorem 77.3) they are isomorphic. Therefore, $G \cong H \oplus B$ is necessarily C-decomposable.

THEOREM 2.1. *A group G is C-decomposable iff it is the ascending union of a sequence $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ of pure subgroups such that for all $n < \omega$,*

$$\text{fin r}(G) = \text{bfin r}(G/G_n).$$

Proof: Let $\kappa = r(G) = \text{fin r}(G)$. Suppose first that G is C-decomposable; let $G = K \oplus L$, where L is Σ-cyclic of final rank κ. Clearly, L is isomorphic to $\bigoplus_{m<\omega} L_m$, where each L_m also has final rank κ. If we set $G_n = K \oplus (\bigoplus_{m \le n} L_m)$, then

$$G/G_n = [K \oplus (\bigoplus_{m \le \omega} L_m)]/[K \oplus (\bigoplus_{m \le n} L_m)] \cong \bigoplus_{n<m} L_m$$

clearly has a basic subgroup of final rank κ (namely, the group itself).

Conversely, suppose we are given the G_n, as above. If κ is countable, then G is countable, and from the above discussion, C-decomposable. We may assume, therefore, that κ is uncountable.

Define a sequence of cardinals κ_n as follows: If κ has countable cofinality, then let κ_n be a strictly increasing sequence of cardinals with limit κ; and if κ has uncountable cofinality, let each $\kappa_n = \kappa$.

If $n < \omega$, then

$$\kappa = \text{fin r}(G/G_n) = \sum_{n \le i < \omega} f_{G/G_n}(i) = \sum_{n \le i, j < \omega} f_{G_{j+1}/G_j}(i).$$

It follows that there is a strictly increasing sequence $\{n_k\}_{k<\omega}$ in ω such that for all $k < \omega$,

$$\kappa_k \le f_{G_{n_{k+1}}/G_{n_k}}(m),$$

for some $m \ge n_k$. If we replace the sequence G_n by the subsequence G_{n_k}, we may assume that for all $n < \omega$ we have

$$\kappa_n \le f_{G_{n+1}/G_n}(m),$$

for some $m \ge n$.

We construct two sequences of pure subgroups, B_n and C_n, of G such that for all $n < \omega$ we have:

(a) $B_n \subseteq G_{n+1}$ is isomorphic to $\bigoplus_{\kappa_n} \mathbb{Z}_{p^{m+1}}$ for some $m \ge n$;
(b) The B_ns are linearly independent subgroups of G;
(c) $C_{n-1} \subseteq C_n \subseteq G_{n+1}$;
(d) If $Y_n = B_0 \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n$, then $G_{n+1} = Y_n \oplus C_n$.

Before constructing these objects, we note that they imply the result:

Claim: If $Y = \cup_{n<\omega} Y_n = \bigoplus_{n<\omega} B_n$ and $C = \cup_{n<\omega} C_n$, then Y is a Σ-cyclic group of final rank κ and $G = Y \oplus C$.

To verify our claim, note that the first statement follows directly from (a) and (b). Next, $Y + C = (\cup_n Y_n) + (\cup_n C_n) = \cup_n (Y_n + C_n) = \cup_n G_{n+1} = G$, and $Y \cap C = (\cup_n Y_n) \cap (\cup_n C_n) = \cup_n (Y_n \cap C_n) = \{0\}$. This gives the claim.

We now construct B_n and C_n satisfying (a)-(d). In fact, it will be convenient to start by letting $C_{-1} = G_0$ and $X_{-1} = Y_{-1} = \{0\}$. So assume we have constructed B_ℓ and C_ℓ for all $-1 \le \ell < n$ and we want to construct these subgroups for n.

If $G' \stackrel{\text{def}}{=} G_{n+1}/G_n$, then $f_{G'}(m) \ge \kappa_n$, for some $m \ge n$. It follows that G' has a summand B' which is isomorphic to $\bigoplus_{\kappa_n} \mathbb{Z}_{p^{m+1}}$.

Assume $G' = B' \oplus X'$, and let X be the subgroup of G_{n+1} containing G_n so that $X/G_n = X'$. Since G_n is pure in G_{n+1} and X' is pure in G', it follows that X is pure in G_{n+1}. Therefore, since $G_{n+1}/X \cong B'$ is Σ-cyclic, it follows that $G_{n+1} = B_n \oplus X$, where $B_n \cong B'$ is as in (a). Since $Y_{n-1} \subseteq G_n \subseteq X$, we can also conclude that (b) holds.

Now G_n is a pure subgroups of X containing C_{n-1}, so $G_n/C_{n-1} \cong Y_{n-1}$ is pure in X/C_{n-1}. Since Y_{n-1} is bounded, it follows that G_n/C_{n-1} is a summand of X/C_{n-1}. Let C_n be a subgroup of X containing C_{n-1} so that

$$X/C_{n-1} = (G_n/C_{n-1}) \oplus (C_n/C_{n-1}).$$

We now verify (c) and (d). The first of these is obvious, so consider statement (d). Note that $Y_{n-1} + C_n = Y_{n-1} + C_{n-1} + C_n = G_n + C_n = X$ and $Y_{n-1} \cap C_n = (Y_{n-1} \cap G_n) \cap C_n = Y_{n-1} \cap C_{n-1} = \{0\}$. Therefore, X is the internal direct sum

$Y_{n-1} \oplus C_n$, so that $G_{n+1} = B_n \oplus X = B_n \oplus (Y_{n-1} \oplus C_n) = Y_n \oplus C_n$. This gives (d), and so the Theorem. □

Theorem 2.1 says that to verify that a given group is C-decomposable, we do not have to explicitly produce a Σ-cyclic summand of sufficiently large final rank; we need only exhibit a well-behaved chain of pure subgroups. It does have some surprising consequences, especially for groups whose final ranks are of countable cofinality. The following result states that the C-decomposability of such groups can be completely reduced to a simple question of cardinality.

COROLLARY 2.2. *Suppose G is a group such that* $\operatorname{fin} \operatorname{r}(G)$ *has countable cofinality. Then G is C-decomposable iff* $\operatorname{fin} \operatorname{r}(G) = \operatorname{bfin} \operatorname{r}(G)$.

Proof: We have already observed that if G is C-decomposable, then $\operatorname{fin} \operatorname{r}(G) = \operatorname{bfin} \operatorname{r}(G)$; so assume $r(G) = \operatorname{fin} \operatorname{r}(G) = \operatorname{bfin} \operatorname{r}(G) = \kappa$ has countable cofinality.

Suppose first that $\kappa = \aleph_0$. Since any countable reduced group is C-decomposable, the result immediately follows.

Assume, therefore, that κ is uncountable. Let $\{\kappa_n\}_{n<\omega}$ be a strictly increasing sequence of infinite cardinals with limit κ. We can clearly express G as the ascending union of a sequence of pure subgroups $\{G_n\}_{n<\omega}$ such that for each $n < \omega$, G_n has cardinality κ_n. We therefore have $\kappa = \operatorname{fin} \operatorname{r}(G) = \operatorname{bfin} \operatorname{r}(G) = \operatorname{bfin} \operatorname{r}(G_n) + \operatorname{bfin} \operatorname{r}(G/G_n) \le \kappa_n + \operatorname{bfin} \operatorname{r}(G/G_n)$. So $\operatorname{bfin} \operatorname{r}(G/G_n) = \kappa$, and appealing to Theorem 2.1, we are done. □

Again, observe that the last result does not depend upon the *structure* of G, but only its final rank and Ulm function; it does have several immediate consequences, though. We mention a few of them in our next result. Though we include a proof, they essentially follow directly from Corollary 2.2 together with the fact that the torsion product of any two reduced groups is "fully $*$-ed" (a group is M is fully starred if for every subgroup $P \subseteq M$, the rank of P agrees with the rank of any basic subgroup of P).

COROLLARY 2.3. *Suppose G is a group such that* $\operatorname{fin} \operatorname{r}(G)$ *has countable cofinality and G is isomorphic to a subgroup of a group H.*

(a) If $H = \operatorname{Tor}(X, Y)$ where X and Y are reduced, then G is C-decomposable;

(b) If H is p^α-projective for some ordinal α, then G is C-decomposable;

(c) If H is a reduced totally projective group, then G is C-decomposable.

Proof: We begin with (a); we may clearly assume that G actually is a subgroup of this $H = \operatorname{Tor}(X, Y)$. Let B be a basic subgroup of G; by Corollary 2.2, we need to show $\operatorname{fin} \operatorname{r}(G) = \operatorname{fin} \operatorname{r}(B) \stackrel{\text{def}}{=} \kappa$. Let $X_0 \subseteq X$ and $Y_0 \subseteq Y$ be subgroups of rank at most κ such that $B \subseteq \operatorname{Tor}(X_0, Y_0)$. There is a left-exact sequence

$$0 \to \operatorname{Tor}(X_0, Y_0) \to \operatorname{Tor}(X, Y) \to \operatorname{Tor}(X/X_0, Y) \oplus \operatorname{Tor}(X, Y/Y_0).$$

Since X and Y are reduced, it follows that the right-hand group in this sequence is reduced. Since G/B is divisible and $B \subseteq \operatorname{Tor}(X_0, Y_0)$, $G \subseteq \operatorname{Tor}(X, Y)$ maps to $\{0\}$ in this right-hand group. Therefore, G must be contained in $\operatorname{Tor}(X_0, Y_0)$. This implies that $\operatorname{fin} \operatorname{r}(G) = r(G) \le r(\operatorname{Tor}(X_0, Y_0)) = \kappa$, so that by Corollary 2.2, G must be C-decomposable.

Turning to (b), note that if H is p^α-projective, then it is isomorphic to a summand of $\operatorname{Tor}(H, H_\alpha)$ (where H_α is the "generalized Prüfer group of length α"). So (b) follows from (a).

For (c), if H is a reduced totally projective group and α is its length, then H is p^{α}-projective, so that (c) follows from (b). □

Observe that Corollary 2.3(b) implies that the construction in [**D**] of a $p^{\omega+2}$-projective group that is not C-decomposable necessarily produces a group of final rank of uncountable cofinality. The next result describes precisely, in any model of ZFC, when all reduced groups of a given final rank are C-decomposable.

Theorem 2.4. *Suppose κ is an infinite cardinal. Then every reduced group of final rank κ is C-decomposable if and only if κ has countable cofinality and for every cardinal $\gamma < \kappa$ we have $\gamma^{\aleph_0} < \kappa$.*

Proof: Suppose first that these cardinality restrictions hold. Let G be a reduced group of final rank κ and B be a basic subgroup of G. By ([**F**],Corollary 34.4), we must have $\kappa = r(G) \leq r(B)^{\aleph_0}$. So our cardinality hypotheses implies that $\operatorname{fin\,r}(G) = r(G) = \kappa = r(B) = \operatorname{fin\,r}(B) = \operatorname{bfin\,r}(G)$. An appeal to Corollary 2.2 completes the proof.

Conversely, suppose these cardinality conditions do not hold. Suppose first that we have $\gamma^{\aleph_0} \geq \kappa$ for some $\gamma < \kappa$. Let B be a Σ-cyclic group of rank and final rank γ. If $\overline{B}$ is the torsion completion of B, then $\overline{B}/B$ will be isomorphic to the direct sum of $\gamma^{\aleph_0}$ copies of the infinite cocyclic group $\mathbb{Z}_{p^\infty}$. Let G be a pure subgroup of $\overline{B}$ containing B of rank κ. It follows that G has final rank κ and basic final rank $\gamma < \kappa$. Therefore, G is not C-decomposable.

Suppose next that for all $\gamma < \kappa$ we have $\gamma^{\aleph_0} < \kappa$, but that κ has uncountable cofinality. It follows that

$$\kappa \leq \kappa^{\aleph_0} = \sup\{\gamma^{\aleph_0} : \gamma < \kappa\} \leq \sup\{\mu : \mu < \kappa\} = \kappa.$$

If we let B be a Σ-cyclic group of rank and final rank κ, and G be the torsion completion of B, then G will also have rank and final rank κ. On the other hand, any summand of a torsion complete group that is Σ-cyclic must be bounded, so G will not be C-decomposable. □

Comparing the last result with the generalized continuum hypothesis, we have the following.

Corollary 2.5. *The following statements are equivalent in ZFC:*

(a) Every reduced group G such that $\operatorname{fin\,r}(G)$ *has countable cofinality is C-decomposable;*

(b) For every infinite cardinal $\aleph_\alpha$ (where α is an ordinal), we have

$$\aleph_\alpha{}^{\aleph_0} < \aleph_{\alpha+\omega}.$$

Both of these conditions are consequences of the generalized continuum hypothesis, but they are independent of ZFC.

Proof: The equivalence of (a) and (b) follows by applying Theorem 2.4 to *all* cardinals of countable cofinality. Clearly, the second statement is a consequence of generalized continuum hypothesis. On the other hand, by ([**K**], Theorem 15.18), it is consistent with ZFC that $\aleph_0{}^{\aleph_0} = c$ is greater than $\aleph_\omega$. Therefore, it is consistent with ZFC that (b), and hence (a), fails. □

Another way to view the above calculation is as follows: Let $\kappa_0 = \aleph_0$; if we have defined κ_β for all $\beta < \alpha$, let $\kappa_\alpha = \cup_{\beta<\alpha}(\kappa_\beta{}^{\aleph_0} + \kappa_\beta{}^+)$. It can then be checked that every reduced group of final rank κ is C-decomposable iff $\kappa = \kappa_\lambda$, where λ is either 0 or a limit ordinal of countable cofinality.

We now derive a couple of further consequences of the above. Recall that a group G of final rank κ is said to be far from thick if there is a Σ-cyclic group Y and a homomorphism $\phi : G \to Y$ such that for all $n < \omega$, the rank of $\phi((p^n G)[p])$ is at least κ. We call such a ϕ a *κ-homomorphism.* It is easy to check that if the kernel of ϕ is pure in G, then G is, in fact, C-decomposable. So the difference between a group being C-decomposable and far from thick lies in our ability to find a κ-homomorphism that has such a pure kernel.

Corollary 2.6. *Suppose G is a group and* fin r$(G) = \kappa$ *has countable cofinality.*

(a) G is C-decomposable iff it is far from thick.

(b) G has the GCCP.

Proof: As to (a), if G is C-decomposable, then it is certainly far from thick. On the other hand, if G is far from thick, then by ([**J**], Corollary 19), $G[p]$ is isometric as a valuated vector space to $V \oplus F$, where F is an ω-bounded free valuated vector space of final rank κ. This implies that for all $n < \omega$ we have $\sum_{n \le i < \omega} f_G(i) \geq \sum_{n \le i < \omega} f_V(i) = \kappa$. Therefore, fin r$(G) =$ bfin r$(G) = \kappa$, and we need only appeal to Corollary 2.2.

Regarding (b), if fin r$(G) =$ bfin r(G), then G is C-decomposable, and by ([**C**], Theorem 8), all C-decomposable groups have the GCCP. On the other hand, if fin r$(G) >$ bfin r(G), it follows from ([**C**], Corollary 7) that G has the GCCP. □

3. Weak $p^{\omega \cdot 2}$-Projective Groups

Recall that if α is an ordinal, then a group is a weak p^α-projective if it is a subgroup of a group that is p^α-projective. We include the following well-known description of the weak $p^{\omega \cdot 2}$-projective groups (see [**I**],Corollary 2.17), as well as an elaboration on an observation of Nunke, which shows that this class is at least as complicated as the class of all separable groups, and finally, we recall a well-known result of Dieudonné ([**E**]).

Proposition 3.1. *(a) If G is a group, then G is a weak $p^{\omega \cdot 2}$-projective iff it is an extension of a Σ-cyclic subgroup by a factor that is also Σ-cyclic; i.e., it has a subgroup X such that both X and $A \overset{\text{def}}{=} G/X$ are Σ-cyclic.*

(b) If S is any separable group, then there is a weak $p^{\omega \cdot 2}$-projective group G such that $G/p^\omega G \cong S \oplus A$, where A is Σ-cyclic.

(c) In the notation in (a), G is Σ-cyclic iff X is the ascending union of subgroups $\{X_m\}_{m<\omega}$ such that for each $m < \omega$ there is an $n < \omega$ such that $p^n G \cap X_m = \{0\}$.

Proof: Starting with (a), suppose G is a subgroup of the $p^{\omega \cdot 2}$-bounded totally projective group H. Let $X = G \cap p^\omega H$; since $p^\omega H$ is Σ-cyclic, so is X. On the other hand, $A = G/X$ embeds in $H/p^\omega H$, and since H is totally projective, $H/p^\omega H$ will be Σ-cyclic. It follows that A is also Σ-cyclic.

Conversely, suppose that X is a Σ-cyclic subgroup of G such that $A = G/X$ is also Σ-cyclic. In particular, since A is separable, it follows that X is nice in G. Let L be a totally projective group such that there is an isomorphism $\phi : X \to p^\omega L$; since X is separable, L is clearly $p^{\omega \cdot 2}$-bounded.

For every $n < \omega$, $\phi(X \cap p^{\omega+n} G) = \phi(X \cap p^n(p^\omega G)) \subseteq \phi(p^n X) = p^{\omega+n} L$. It follows from ([**F**], Corollary 81.4) that ϕ extends to a homomorphism $\nu : G \to L$.

If we define $\mu : G \to H \stackrel{\mathrm{def}}{=} L \oplus A$ by $\mu(y) = (\nu(y), y + X)$, then it is easy to check that μ is injective. And since L is a $p^{\omega\cdot 2}$-bounded totally projective group and A is Σ-cyclic, it follows that H is also a $p^{\omega\cdot 2}$-bounded totally projective group. So (a) has been established.

Turning to (b), suppose X is Σ-cyclic, Y is a subgroup of X and $X/Y \cong S$. There is a totally projective group L with $p^\omega L \cong Y$. Let $A = L/p^\omega L$, which is Σ-cyclic, and define G as in the push-out diagram

$$\begin{array}{ccccccccc}
0 & \to & Y & \to & X & \to & S & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & L & \to & G & \to & S & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & A & = & A & & & &
\end{array}$$

Interpreting X as a Σ-cyclic subgroup of G such that $G/X \cong A$ is also Σ-cyclic, it follows that G is a weak $p^{\omega\cdot 2}$-projective. Since $Y = p^\omega L$ and $G/Y \cong S \oplus A$ is separable, it follows that $Y = p^\omega G$, as required.

For (c), we refer the reader to [**E**]. $\square$

Part (b) of the last proposition shows that the class of weak $p^{\omega\cdot 2}$-projective groups, though apparently quite simple and tractable, is in fact at least as complicated as the class of all separable groups. For example, if S is a separable group such that $\mathrm{bfin\ r}(S) < \mathrm{fin\ r}(S)$, then an argument such as that in Corollary 2.3 shows that S is not p^α-projective for any ordinal α. On the other hand, Nunke showed in [**N**] that if G is p^α-projective then so is $G/p^\alpha G$. So in this case, the construction in Proposition 3.1(b) yields a group which is a weak $p^{\omega\cdot 2}$-projective, but not p^α-projective for any α.

We now review some well-known facts relating to large subgroups. If G is a group and $\sigma = (n_0, n_1, n_2, \dots)$ is a strictly increasing sequence of finite ordinals, then $G(\sigma)$ consists of those $x \in G$ such that for all $m < \omega$, $\mathrm{ht}_G(p^m x) \geq n_m$. We call such a σ a *finite height sequence.*

Proposition 3.2. *Suppose G is a group.*

(a) If L is a subgroup of G, then there is a finite height sequence σ such that $G(\sigma) \subseteq L$ iff for every $n < \omega$ there is an $m < \omega$ such that $(p^m G)[p^n] \subseteq L$.

(b) If σ is a finite height sequence, K is a pure subgroup of G and $A = G/K$, then there is a pure exact sequence

$$0 \to K(\sigma) \to G(\sigma) \to A(\sigma) \to 0.$$

(c) If σ is a finite height sequence, then G is Σ-cyclic iff $G(\sigma)$ is Σ-cyclic.

Proof: Since (a) and (b) are particularly well known (see, for example, section 67 of [**F**]), we omit their proofs and concentrate on verifying (c) (also, see [**A**], Theorem 4.3). Certainly, if G is Σ-cyclic, then so is its subgroup $G(\sigma)$. Conversely, if $G(\sigma)$ is Σ-cyclic, then $G(\sigma)$ is the ascending union of p^n-bounded summands B_n. Choose $k < \omega$ such that $(p^k G)[p^{2n}] \subseteq G(\sigma)$. It follows that $p^{k+n}G \cap B_n = (p^{k+n}G)[p^n] \cap B_n = p^n((p^k G)[p^{2n}]) \cap B_n \subseteq p^n(G(\sigma)) \cap B_n = \{0\}$. By ([**F**], Proposition 67.4), $G/G(\sigma)$ is Σ-cyclic, so by Proposition 3.1(c), we can conclude that G is Σ-cyclic. $\square$

We now review an idea from [**L**]: If G and H are groups and κ is a cardinal, then a homomorphism $f : G \to H$ is said to be *κ-bijective* if its kernel and cokernel have cardinality less than κ. If $\kappa = \text{fin r}(G) = \text{fin r}(H)$, it follows from ([**L**], Corollary 2.5) that G is far from thick iff H is far from thick.

THEOREM 3.3. *If G is a weak $p^{\omega\cdot 2}$-projective group, then it is far from thick.*

Proof: Let $\kappa = r(G) = \text{fin r}(G)$. If κ has countable cofinality, then by Corollary 2.3(b), G will be C-decomposable; and since a C-decomposable group is clearly far from thick, the result follows. We may assume, therefore, that κ has uncountable cofinality.

Let H be a $p^{\omega\cdot 2}$-bounded dsc group containing G, and fix a decomposition $H = \bigoplus_{i\in I} K_i$, where each K_i is countable. Let $H' = H/p^\omega H$, and $\phi : H \to H'$ be the natural epimorphism. If $J \subseteq I$, then let $H_J = \bigoplus_{i\in J} K_i \subseteq H$ and $H'_J = [H_J + p^\omega H]/p^\omega H \subseteq H'$.

For each $n < \omega$, let κ_n be the minimum of the ranks of the subgroups

$$\{\phi((p^m G)[p^n]) \subseteq H' : m < \omega\}.$$

Clearly, $\kappa_n \leq \kappa_{n+1}$ for each $n < \omega$. We now divide the argument into two cases.

CASE 1: For some $n < \omega$ we have $\kappa_n = \kappa$.

Choose n minimally so that $\kappa_n = \kappa$. Since $\kappa_0 = 0$, we can conclude that $n > 0$. Choose an $m < \omega$ and a subset $J \subseteq I$ of cardinality κ_{n-1} such that $\phi((p^m G)[p^{n-1}]) \subseteq H'_J$.

We let ν denote the composition $G \to H' \to H'_{I-J}$. Since $\phi((p^{m+k}G)[p^n])$ has rank κ for all $k < \omega$, we can conclude that $\nu((p^{m+k}G)[p^n])$ will also have rank κ. Multiplication by p^{n-1} induces an isomorphism $(p^m G)/(p^m G)[p^{n-1}] \to p^{m+n-1}G$. In addition, since $\nu((p^m G)[p^{n-1}]) = \{0\}$, we can conclude that ν determines a homomorphism $\mu : p^{m+n-1}G \to H'_{I-J}$.

For any $k < \omega$, the isomorphism $(p^m G)/(p^m G)[p^{n-1}] \cong p^{m+n-1}G$ restricts to an isomorphism

$$[(p^{m+k}G)[p^n] + (p^m G)[p^{n-1}]]/(p^m G)[p^{n-1}] \cong (p^k(p^{m+n-1}G))[p].$$

From this we can conclude that $\mu((p^k(p^{m+n-1}G))[p]) = \nu((p^{m+k}G)[p^n])$ has rank κ for all $k < \omega$.

Therefore, $\mu : p^{m+n-1}G \to H'_{I-J}$ is a κ-homomorphism, so $p^{m+n-1}G$ is far from thick. By ([**J**], Corollary 24(b)), this implies that G is also far from thick.

CASE 2: For all $n < \omega$ we have $\kappa_n < \kappa$.

Let γ be the limit of the κ_n. Since we are assuming that κ has uncountable cofinality, we have $\gamma < \kappa$. Therefore, there is a subset $J \subseteq I$ of cardinality γ such that for every $n < \omega$, there is an $m < \omega$ such that $\phi((p^m G)[p^n]) \subseteq H'_J$. By Proposition 3.2(a), this implies that there is a finite height sequence σ such that $\phi(G(\sigma)) \subseteq H'_J$, or equivalently, $G(\sigma) \subseteq H_J \oplus (p^\omega H_{I-J})$.

Note that $G(\sigma)/[H_J \cap G(\sigma)]$ embeds in $p^\omega H_{I-J}$, which is Σ-cyclic; so that $G(\sigma)/[H_J \cap G(\sigma)]$ is also Σ-cyclic. Using a standard “back-and-forth” argument, there is a subgroup K of G satisfying the following:

(1) K is pure in G;
(2) $r(K) \leq \gamma$;
(3) $H_J \cap G(\sigma) \subseteq K$;
(4) $(K \cap G(\sigma))/(H_J \cap G(\sigma))$ is a summand of $G(\sigma)/[H_J \cap G(\sigma)]$.

If we let $A = G/K$, then by Proposition 3.2(b) there is a pure exact sequence

$$0 \to K(\sigma) \to G(\sigma) \to A(\sigma) \to 0.$$

Since $K(\sigma) = K \cap G(\sigma)$, by (4) we can conclude that

$$A(\sigma) \cong G(\sigma)/K(\sigma) \cong [G(\sigma)/(H_J \cap G(\sigma))]/[K(\sigma)/(H_J \cap G(\sigma))]$$

is Σ-cyclic. Therefore, Proposition 3.2(c) implies that A must also be Σ-cyclic. So by (1), $G \cong K \oplus A$. Now, (2) implies that the final rank of A is also κ. So G must be C-decomposable, and in particular, it is far from thick. □

Again, since not all $p^{\omega+2}$-projective groups are C-decomposable, we cannot conclude that all weak $p^{\omega\cdot 2}$-projective groups are C-decomposable, even though they are all far from thick. We now list some direct consequences of Theorem 3.3.

Corollary 3.4. *If G is a weak $p^{\omega\cdot 2}$-projective, then the following hold:*

(a) $G[p]$ is C-decomposable as a valuated vector space (i.e., it is isometric to $V \oplus F$, where F is an ω-bounded free valuated vector space of the same final rank as G).

(b) G has the GCCP.

(c) There is a C-decomposable weak $p^{\omega\cdot 2}$-projective group H such that G and H can be embedded in each other.

Proof: Statements (a) and (b) follow directly from ([**J**], Corollaries 19 and 27). Since a subgroup of a weak $p^{\omega\cdot 2}$-projective is also a weak $p^{\omega\cdot 2}$-projective, (c) follows from ([**J**], Theorem 16). □

We now extend ([**L**], Corollary 2.5) a bit.

Lemma 3.5. *Suppose G and H are groups of final rank κ and $f : G \to H$ is a homomorphism whose kernel and cokernel have final rank less than κ. Then G is far from thick iff H is far from thick.*

Proof: Let K be the kernel of f and C be its cokernel. Suppose $K = X \oplus K'$, where X is bounded and $r(K') = \text{fin r}(K)$, and let $C = Y \oplus C'$ be a similar decomposition for C. Let $G' = G/X$, and let $H' \subseteq H$ be defined by the equation $H'/f(G) = C'$.

Since X and $H/H' \cong Y$ are bounded, it follows from ([**J**], Corollary 24(b)) that G is far from thick iff G' is far from thick, and the same holds for H and H'. Clearly, f induces a homomorphism $f' : G' \to H'$ whose kernel (which is isomorphic to K') and cokernel (which is isomorphic to C') have rank less than κ. So f' is a κ-bijection, and the result follows from ([**L**], Corollary 2.5). □

Corollary 3.6. *Suppose T is a totally projective group, G is a subgroup of T and for every ordinal α we let $G_\alpha = G \cap p^\alpha T$. If there is an ordinal β such that the final ranks of G/G_β and $G_{\beta+\omega\cdot 2}$ are strictly less than the final rank of G, then G is far from thick.*

Proof: Note that $G_\beta/G_{\beta+\omega\cdot 2}$ will be a weak $p^{\omega\cdot 2}$-projective since it embeds in $p^\beta T/p^{\beta+\omega\cdot 2}T$; so it is far from thick. Applying Lemma 3.5 to the natural short exact sequences

$$0 \to G_\beta/G_{\beta+\omega\cdot 2} \to G/G_{\beta+\omega\cdot 2} \to G/G_\beta \to 0$$
$$0 \to G_{\beta+\omega\cdot 2} \to G \to G/G_{\beta+\omega\cdot 2} \to 0$$

shows that first $G/G_{\beta+\omega\cdot 2}$, and then G, is far from thick. □

We close with one particularly simple case of the last result.

COROLLARY 3.7. *If $n < \omega$ and G is a weak $p^{\omega\cdot 2+n}$-projective group, then G is far from thick.*

It appears to the author that it is quite probable that any group that can be embedded in a (reduced) totally projective group is, in fact, far from thick. Extending Theorem 3.3 to ordinals $\alpha \geq \omega \cdot 3$ will require other techniques than those employed above.

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Generic endomorphisms of homogeneous structures

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Dedicated to Rüdiger Göbel on his 70th birthday

Abstract. We define a notion of 'genericity' for endomorphisms, and give basic properties and some examples.

1. Introduction

The notion of a 'generic' automorphism of a homogeneous structure was introduced in [**25**]. This word has been applied to automorphisms in some different, but related ways, for instance also by Lascar in [**16, 17**], and some comparisons between the different definitions is given in [**26**]. The definition is that an automorphism is *generic* if it lies in a comeagre conjugacy class. This was mainly applied to (countable) homogeneous structures, but could in principle be applied more generally.

Here, a structure is said to be *homogeneous* if any isomorphism between finite substructures extends to an automorphism. The classic case of a homogeneous structure is the set of rational numbers under the usual ordering, $(\mathbb{Q}, <)$, and from this example Fraïssé developed his theory of amalgamation classes (see [**5**]). Since then, a wide variety of other homogeneous structures have been studied and classified, for instance [**15**] for graphs, [**3**] for digraphs, [**21**] for partial orders, and [**22**] for coloured partial orders.

We only refer to a small number of homogeneous structures explicitly in this paper, so do not explore these classifications any further. We do however require some knowledge of Fraïssé's methods. If for any relational structure $\mathcal{A}$, we let its *age* be the family of finite structures (in the same language) which can be embedded in $\mathcal{A}$, then Fraïssé's Theorem gives necessary and sufficient conditions for a class $\mathcal{C}$ of finite structures to equal the age of some countable homogeneous structure. These conditions are that the class should be closed under isomorphisms and substructures, should have at most countably many non-isomorphic members, and should have the joint embedding and amalgamation properties JEP and AP. Here JEP says that for any two members of $\mathcal{C}$ there is a member of $\mathcal{C}$ in which they both embed, and AP says that if $f : a \to b$ and $g : a \to c$ are embeddings of members of $\mathcal{C}$, then there are embeddings $h : b \to d$ and $k : c \to d$ of b and c to some member d of $\mathcal{C}$ such that the 'diagram commutes', meaning that $hf = kg$.

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A class of structures fulfilling all these properties is called an *amalgamation class*, and it is furthermore shown that the homogeneous structure resulting is uniquely determined by $\mathcal{C}$. It is called the corresponding *Fraïssé limit*, or *Fraïssé generic*. See [**8**] for more details.

It turns out that for many homogeneous structures, it is possible to show that 'most' automorphisms are of the same type. The natural way to interpret 'of the same type' in a group is to say that they are conjugate, and in view of the fact that the *structure* is approximated by means of finite structures (as a Fraïssé limit), the natural notion of 'most' is in the sense of Baire category, since this also arises from notions of approximating by means of finite objects. A set is viewed as being 'large' in the sense of Baire category if it is comeagre, meaning that it contains a countable intersection of dense open sets, and the non-triviality of this notion is guaranteed by the Baire category theorem, which says that in a complete metric space, any comeagre set is dense.

The goal of this paper is to extend the notion of genericity to other contexts, specifically to endomorphisms. The definition carries over straight away, but we have to explain and make sense of the word 'comeagre'. In the case originally studied, it is well known that the automorphism group of any countable structure carries a natural metric, under which it is complete, and so the definition makes sense.

Let Ω be a countably infinite set. Then we write Ω^Ω for the set of all functions from Ω to itself. This is a monoid under the natural operation of composition. In addition, it is a topological space under a topology arising from a complete metric, so we may apply the notions of Baire category. Specifically, if $\{x_n : n \in \omega\}$ is some fixed enumeration of Ω, then we may define the distance between two members f and g of Ω^Ω to equal $\frac{1}{2^n}$ where n is the least such that $f(x_n) \neq g(x_n)$ (if any, and 0 if $f = g$). This makes Ω^Ω a complete separable metric space whose topology is generated by basic open sets of the form $[p] = \{g \in \Omega^\Omega : p \leq g\}$ for maps p from a finite subset of Ω into Ω (where $p \leq g$ here means that p is a restriction of g).

Now suppose that Ω carries some first order structure. The principal examples we shall discuss in this paper are the random graph and the rationals as an ordered set (though we shall also consider the trivial structure on a countably infinite set, and the countable dense circular order), and these we write as Γ and $(\mathbb{Q}, \leq)$ respectively. Note that in the latter case, we definitely opt for the reflexive relation $\leq$ rather than the strict relation $<$. This is because, when considering *homomorphisms*, they are not equivalent. A homomorphism is required to respect the relations named in the signature, but not necessarily their negations. Thus a homomorphism of Γ must map edges to edges, but is allowed to map a non-edge to an edge, or even to a point, and for $\mathbb{Q}$, $a < b$ are allowed to be mapped to the same point (but then, all points in between must also be mapped to this point).

If we use a script letter $\mathcal{A}$ for a structure, then its domain will be denoted by the corresponding Roman letter A. 'Substructure' will be indicated by $\leq$, and if $a \subseteq A$, then we may abuse notation by also using a for the substructure induced on a. Note that in this paper (unlike [**25**]) we write actions of maps on the left of their arguments.

There are several different monoids which we may wish to consider, and which each potentially gives rise to a different notion of genericity. These are as follows:

the monoid $\mathrm{End}(\mathcal{A})$ of all endomorphisms of the structure,

the monoid $\mathrm{Epi}(\mathcal{A})$ of all epimorphisms (that is, surjective endomorphisms),
the monoid $\mathrm{Mon}(\mathcal{A})$ of all monomorphisms (1–1 endomorphisms),
the monoid $\mathrm{Bi}(\mathcal{A})$ of all bimorphisms (1–1 and surjective endomorphisms),
the monoid $\mathrm{Emb}(\mathcal{A})$ of all embeddings (monomorphisms which are isomorphisms),
the monoid (group) $\mathrm{Aut}(\mathcal{A})$ of automorphisms.

The relations

$$\mathrm{Aut}(\mathcal{A}) \subseteq \mathrm{Emb}(\mathcal{A}) \subseteq \mathrm{Mon}(\mathcal{A}) \subseteq \mathrm{End}(\mathcal{A});$$

$$\mathrm{Aut}(\mathcal{A}) \subseteq \mathrm{Bi}(\mathcal{A}) \subseteq \mathrm{Epi}(\mathcal{A}) \subseteq \mathrm{End}(\mathcal{A});$$

and

$$\mathrm{Bi}(\mathcal{A}) \subseteq \mathrm{Mon}(\mathcal{A})$$

are clear, as well as the facts that

$$\mathrm{Emb}(\mathcal{A}) \cap \mathrm{Epi}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})$$

and

$$\mathrm{Mon}(\mathcal{A}) \cap \mathrm{Epi}(\mathcal{A}) = \mathrm{Bi}(\mathcal{A}).$$

We can illustrate the fact that these are the only relations between the six monoids by considering Γ. Since this graph is universal, it embeds an infinite complete graph on X say. Let a and b be vertices of Γ which are not joined, and let f map Γ onto X, so that a and b are mapped to the same point, but otherwise, f is 1–1 (that is, $f(x) = f(y) \Rightarrow x = y$ or $\{x, y\} = \{a, b\}$). Then f is an endomorphism, but clearly not a monomorphism or an epimorphism. A map which takes Γ 1–1 onto X is a monomorphism which is not an embedding (since it destroys many non-edges) or an epimorphism, hence also not a bimorphism. In [**24**], the group of 'almost' automorphisms of Γ was studied, and it was shown for instance that there is a permutation of the vertices of Γ which maps just one non-edge to an edge, and which preserves all other edges and non-edges, and this lies in $\mathrm{Bi}(\Gamma)$ but not $\mathrm{Aut}(\Gamma)$. In a similar way, the chosen non-edge can be collapsed to a point, so that all other edges and non-edges are preserved, and the map which does this will lie in $\mathrm{Epi}(\Gamma)$ but not $\mathrm{Bi}(\Gamma)$. Finally if a single point a is removed from Γ, then the result is still isomorphic to Γ, and an isomorphism of Γ to $\Gamma - \{a\}$ is an embedding which is not an automorphism.

For $\mathbb{Q}$ however the situation is rather different in that $\mathrm{Mon}(\mathbb{Q}, \leq) = \mathrm{Emb}(\mathbb{Q}, \leq)$, since if f is a monomorphism and $f(x) \leq f(y)$, then as $\leq$ is total, $x \leq y$ or $y < x$, but the latter would imply that $f(y) \leq f(x)$, and hence that $f(y) < f(x)$ since f is 1–1, which is a contradiction. Therefore $x \leq y \Leftrightarrow f(x) \leq f(y)$ so f is an embedding. Similarly $\mathrm{Bi}(\mathbb{Q}, \leq) = \mathrm{Aut}(\mathbb{Q}, \leq)$. We still have examples of endomorphisms which are not monomorphisms, for instance the map which takes all points to 0, and of embeddings which are not automorphisms, for instance any isomorphism from the whole rational line to the set of positive rationals. In the trivial structure Ω, it is immediate that $\mathrm{Mon}(\Omega) = \mathrm{Emb}(\Omega)$ and $\mathrm{Bi}(\Omega) = \mathrm{Aut}(\Omega)$, though once again, the other two monoids are distinct.

In all these cases we can define a corresponding notion of 'generic'. Namely, if M is a monoid which forms a complete metric space and is such that $\mathrm{Aut}(\mathcal{A}) \subseteq M \subseteq \mathrm{End}(\mathcal{A})$, then $g \in M$ is *generic* if $\{h^{-1}gh : h \in \mathrm{Aut}(\mathcal{A})\}$ is a comeagre subset of M. Although this is the 'official' definition, the intuition is that a member of M is generic provided that it satisfies everything which can be demanded of it on the basis of finite pieces of information. This is because the topologies we have in mind are generated by basic open sets of the form 'the set of all maps in M extending a given finite partial map' (which we again write as $[p]$). If the topology was of a different nature, then the intuition would have to be revised. Note that since $M \subseteq \mathrm{End}(\mathcal{A})$, and in each case M is easily seen to be a G_δ subset of $\mathrm{End}(\mathcal{A})$, by the Alexandrov Theorem (see [**13**] page 408) it follows that the induced topology on M arises from a complete metric. (We are grateful to James Mitchell for pointing this out to us.) In the case of $\mathrm{Aut}(\mathcal{A})$ this metric may for instance be given by $d(f,g) = \frac{1}{2^n}$ where n is least such that $f(x_n) \neq g(x_n)$ or $f^{-1}(x_n) \neq g^{-1}(x_n)$ (if any, and $d(f,f) = 0$) under some enumeration of A, but it is harder to give the explicit metric for $\mathrm{Epi}(\mathcal{A})$.

In [**2**] the usual notion of homogeneity (otherwise known as 'ultrahomogeneity') is generalized to the homomorphism situation (and some results about these notions are given in [**1, 10, 19, 20**]). A structure $\mathcal{A}$ is said to be HH-*homomorphism-homogeneous* (abbreviated to just 'HH') if every finite partial homomorphism of $\mathcal{A}$ extends to a homomorphism of $\mathcal{A}$ (and similarly for MM; the usual notion of homogeneity may then be written as II to fit this pattern). Here by a *finite partial homomorphism* of $\mathcal{A}$ is meant a homomorphism p from a finite subset of $\mathcal{A}$ to $\mathcal{A}$, and similarly for finite partial monomorphism or isomorphism. We may write P for the family of finite partial homomorphisms, monomorphisms, or isomorphisms respectively, and $[p]$ stands for the set of all homomorphisms, monomorphisms, or isomorphisms extending p. Since we here consider monoids which were not treated in [**2**], namely Epi, Bi, and Emb, we get further notions of homomorphism-homogeneity, explored in [**18**]. Abbreviating endomorphism, epimorphism, monomorphism, bimorphism, embedding, and automorphism by H, E, M, B, I, and A, we have, apparently at any rate, 18 notions to consider. For instance, $\mathcal{A}$ HE-homomorphism-homogeneous means that every finite partial homomorphism of $\mathcal{A}$ extends to an epimorphism, and IA is another way of expressing usual 'homogeneity'. In fact some of them are equivalent, but we do not go into that here, and instead examine the situation just for generics, where there is quite a lot that we are able to say, at least for some well-known structures.

2. General conditions for the existence of generics

In this section we develop necessary and sufficient conditions for generics to exist in the various monoids, based on the sufficient condition given in [**25**], and the modification which gives a necessary and sufficient condition in [**12, 11**]. First however we make an easy observation on the connection between $\mathrm{Emb}(\mathcal{A})$ and $\mathrm{Aut}(\mathcal{A})$ under the assumption of homogeneity (presumably this does not hold without that assumption).

Theorem 2.1. *If $\mathcal{A}$ is homogeneous (that is,* IA*), then any generic member of* $\mathrm{Emb}(\mathcal{A})$ *is an automorphism (and is generic as an automorphism). Similarly, if $\mathcal{A}$ is* MB *or* HE *then any generic member of* $\mathrm{Mon}(\mathcal{A})$, $\mathrm{End}(\mathcal{A})$ *lies in* $\mathrm{Bi}(\mathcal{A})$, $\mathrm{Epi}(\mathcal{A})$ *respectively (and is generic as a bimorphism or epimorphism).*

Proof: Let $g \in \mathrm{Emb}(\mathcal{A})$ be generic. Then there are countably many dense open sets D_n such that $\bigcap D_n$ is contained in the conjugacy class of g.

For $a \in \mathcal{A}$ let $E_a = \{e \in \mathrm{Emb}(\mathcal{A}) : a \in \mathrm{range}(e)\}$. We show that E_a is a dense open subset of $\mathrm{Emb}(\mathcal{A})$. For openness, let $e \in E_a$. Then for some b, $e(b) = a$. If

$p = \{(b, a)\}$ then $e \in [p] \subseteq E_a$ as required. For density, let p be a finite partial embedding of $\mathcal{A}$. Then p is also a finite partial automorphism, so by homogeneity extends to an automorphism f say. Clearly $f \in E_a$ and so $[p] \cap E_a \neq \emptyset$.

By the Baire category theorem, $\bigcap_{n\in\omega} D_n \cap \bigcap_{a\in\mathcal{A}} E_a \neq \emptyset$. Let f lie in this set. Then $a \in \mathrm{range}(f)$ for every a, so $f \in \mathrm{Aut}(\mathcal{A})$. As $f \in \bigcap D_n$, f is conjugate to g. Hence g is also an automorphism.

Finally we have to see that g is generic *as an automorphism*. Now every member of $\bigcap D_n$ is conjugate to g, and hence so is every member of $\bigcap D_n \cap \mathrm{Aut}(\mathcal{A})$. So it suffices to show that each $D_n \cap \mathrm{Aut}(\mathcal{A})$ is a dense open subset of $\mathrm{Aut}(\mathcal{A})$. First for openness, let $f \in D_n \cap \mathrm{Aut}(\mathcal{A})$. Then as D_n is an open subset of $\mathrm{Emb}(\mathcal{A})$, there is a finite partial embedding p of $\mathcal{A}$ such that $p \leq f$ and every embedding of $\mathcal{A}$ extending p lies in D_n. Hence every automorphism of $\mathcal{A}$ extending p lies in $D_n \cap \mathrm{Aut}(\mathcal{A})$. For density, let p be any finite partial automorphism of $\mathcal{A}$. Then p is also a finite partial embedding, so as D_n is dense, there is an embedding f extending p lying in D_n. Since D_n is open, there is a finite restriction q of f extending p such that every embedding of $\mathcal{A}$ extending q lies in D_n. By homogeneity of $\mathcal{A}$, there is an automorphism of $\mathcal{A}$ extending q, and this is an extension of p lying in $D_n \cap \mathrm{Aut}(\mathcal{A})$. This establishes that $D_n \cap \mathrm{Aut}(\mathcal{A})$ is a dense open subset of $\mathrm{Aut}(\mathcal{A})$ as required.

The proofs for the other two cases are similar and are omitted. □

Now the natural family of finite approximations to an automorphism of $\mathcal{A}$ consists of the family P of pairs of the form (a, p) where a lies in the age of $\mathcal{A}$ and p is a partial automorphism of a. There is a clear sense in which we may write $(a, p) \leq (b, q)$, namely provided a is a substructure of b, and q is an extension of p. (A slightly stronger requirement would be that p equal the restriction of q to a.) It is natural to attempt to apply to these approximations the same notions that arise in Fraïssé's Theorem, namely the joint embedding and amalgamation properties. In any non-trivial cases, the amalgamation property fails when applied to the whole of P, roughly speaking because inside a it is possible to find p such that it can be completed to an automorphism in more than one essentially different way. Therefore the existence of a generic automorphism cannot possibly be equivalent to the truth of the amalgamation property for P. Rather in [**25**], it was shown that P having a cofinal subset closed under conjugacy, and which has the amalgamation property gives a sufficient condition. It was left open whether this, or a modification, would be necessary *and* sufficient, and this was solved, independently, in [**11, 12**]. The solution was that there is a generic automorphism if and only if P satisfies an (apparently) weaker condition, called in [**12**] the *weak amalgamation property* WAP, and in [**11**] the 'almost amalgamation property' (so the extra condition that the class be closed under conjugacy turned out to be unnecessary). The definition is that P is said to satisfy the WAP if any $(a, p) \in P$ has an extension (b, q) such that for any $(c_0, r_0), (c_1, r_1) \geq (b, q)$ there are (d, s) and embeddings of (c_i, r_i) into (d, s) such that the composites of the three maps from (a, p) to (d, s) via (c_0, r_0) or (c_1, r_1) are equal. Under these circumstances we say that (c_0, r_0) and (c_1, r_1) have been *amalgamated over* (a, p). To make what follows easier to read, let us say (slightly inaccurately) that an extension (b, q) of (a, p) in P is *amalgamable* if any two of its extensions can be amalgamated over (a, p).

Since the proofs given in [**11, 12**] didn't depend on the precise definition of the partial order on P (that is, in the above notation, whether q should just be required to be an extension of p, or that p should equal $q \restriction a$), we may regard our use of P in this section, where we specify a finite structure a such that p is a partial automorphism of a, as 'the same' as P given in the last paragraph of section 1, where it just consisted of the map on its own. It is important that a be included in (a, p), as we need to have a structure of the correct similarity type on which p can act; if p is already a finite partial automorphism of $\mathcal{A}$ then we may take a to be $\operatorname{dom} p \cup \operatorname{range} p$. When discussing other monoids, we still use the same letter P to stand for the natural family of finite approximations. Thus for instance, in $\mathrm{Epi}(\mathcal{A})$, P will stand for the family of pairs (a, p) where a is isomorphic to a finite substructure of $\mathcal{A}$, and p is a partial endomorphism of a, partially ordered by letting $(a, p) \leq (b, q)$ if a is a substructure of b, and q is an extension of p.

In fact Kechris and Rosendal give additional information in their paper, and also analyze the situation for automorphisms whose conjugacy class is dense (not necessarily dense open). They show that such elements exist if and only if P has the joint embedding property. So in the remainder of this section we do two jobs; first we give necessary and sufficient conditions for homomorphisms in the various monoids to lie in a dense conjugacy class, and then move on to the similar results for comeagre conjugacy classes (the generic case). In both cases our arguments are simple adaptations of those in [**12, 11, 25**]. Note that in Theorems 2.2 and 2.3 we need to assume that $\mathcal{A}$ is IA, in addition to HE or MB, since at various points in the proof we need to extend partial isomorphisms to automorphisms (which will be conjugacies).

Theorem 2.2. *If $\mathcal{A}$ is* IA *and* HE*, then there is a dense conjugacy class in* $\mathrm{Epi}(\mathcal{A})$ *if and only if P has the joint embedding property. The analogous statement holds for the monoid* $\mathrm{Bi}(\mathcal{A})$ *under the assumption that $\mathcal{A}$ is* IA *and* MB.

Proof: We just prove the result in the first case, since the two cases are just the same (and are both simple adaptations of the method from [**12**]).

First suppose that f lies in a dense conjugacy class, and we show that P satisfies the joint embedding property. Let $(a, p), (b, q) \in P$, where we may assume that a and b are substructures of $\mathcal{A}$. Since $\mathcal{A}$ is HE, p and q may be extended to epimorphisms of $\mathcal{A}$, so by density of the conjugacy class of f, there are automorphisms g, h such that $p \leq g^{-1}fg$ and $q \leq h^{-1}fh$. One checks that (ga, gpg^{-1}), (hb, hqh^{-1}) are isomorphic to (a, p) and (b, q) respectively, and since they are both restrictions of $(\mathcal{A}, f)$, their union is a member of P, giving what is required.

Conversely, assume the joint embedding property for P, and we find an epimorphism with dense conjugacy class. For each finite partial homomorphism p of $\mathcal{A}$, let $D_p = \{f \in \mathrm{Epi}(\mathcal{A}) : (\exists g \in \mathrm{Aut}(\mathcal{A}))gpg^{-1} \leq f\}$. Then we see that D_p is dense open. Openness follows from the fact that as gpg^{-1} is finite, verification that $f \in D_p$ only depends on finitely many values. For density, suppose that q is a given finite partial homomorphism of $\mathcal{A}$, and let $a = \mathrm{dom}(p) \cup \mathrm{range}(p)$ and $b = \mathrm{dom}(q) \cup \mathrm{range}(q)$, so that $(a, p), (b, q) \in P$. Then by the joint embedding property, there are isomorphisms φ, ψ from a, b respectively into c such that $(\varphi a, \varphi p \varphi^{-1})$, $(\psi b, \psi q \psi^{-1}) \leq (c, r) \in P$. By changing φ and ψ if necessary we may suppose that c is a substructure of $\mathcal{A}$, and as $\mathcal{A}$ is HE, r may be extended to $h \in \mathrm{Epi}(\mathcal{A})$. As $\mathcal{A}$ is IA, φ and ψ may be extended to automorphisms φ', ψ'. Let $f = (\psi')^{-1}h\psi' \in \mathrm{Epi}(\mathcal{A})$. Then $f \in D_p$ since $(\psi')^{-1}\varphi' p((\psi')^{-1}\varphi')^{-1} \leq (\psi')^{-1}r\psi' \leq (\psi')^{-1}h\psi' = f$. Also $q \leq (\psi')^{-1}h\psi' = f$. Since every D_p is dense open, the intersection of all the D_p is non-empty, and any member of this intersection has a dense conjugacy class. $\square$

Theorem 2.3. *If $\mathcal{A}$ is* IA *and* HE*, then* $\mathrm{Epi}(\mathcal{A})$ *has a generic member if and only if P has the joint embedding property and the weak amalgamation property, and similarly for* $\mathrm{Bi}(\mathcal{A})$ *assuming that $\mathcal{A}$ is* IA *and* MB*.*

Proof: First suppose that $M = \mathrm{Epi}(\mathcal{A})$ has a generic member. Then M has a dense conjugacy class, so by Theorem 2.2, P has the JEP. Next we show that P has the WAP. Suppose not, for a contradiction, and let (a, p) be chosen to violate the definition of WAP. This means that (a, p) has no amalgamable extension. We assume that $a \le \mathcal{A}$.

Our strategy is to work in the product M^2 and to produce a pair (g, h) of generic epimorphisms which are however not conjugate, which will give a contradiction. Since M has a generic, there is a countable family of dense open subsets D_n of M such that all members of $\bigcap_{n\in\omega} D_n$ are generic. Now working in M^2, we let $D'_n = \{(g, h) \in M^2 : g, h \in D_n\}$ and for each embedding θ of a into $\mathcal{A}$ we let

$$E_\theta = \{(g, h) \in M^2 : (\forall k \in \mathrm{Aut}(\mathcal{A}))(\theta \le k \wedge p \le g \to kgk^{-1} \ne h)\}.$$

Clearly D'_n is dense open and we show that E_θ is too. Openness of E_θ is clear, since membership in E_θ only depends on finitely many points of A, so we concentrate on density. Let (p_0, p_1) be any pair of finite partial endomorphisms of $\mathcal{A}$. We shall find epimorphisms g, h such that $p_0 \le g$, $p_1 \le h$, and $(g, h) \in E_\theta$.

If for some $x \in \mathrm{dom}\, p \cap \mathrm{dom}\, p_0$, $px \ne p_0x$ then we take any epimorphisms $g \ge p_0$ and $h \ge p_1$ (which exist since $\mathcal{A}$ is HE) and vacuously $(g, h) \in E_\theta$ since $p \le g$ is necessarily false. Next if for every extension k of θ to an automorphism of $\mathcal{A}$ and all epimorphisms $g \ge p \cup p_0$ and $h \ge p_1$, $kgk^{-1} \ne h$, then any such choice of g, h will serve. Otherwise, suppose that such k, g, h are chosen so that $kgk^{-1} = h$. Let $q_0 = g \restriction (\mathrm{dom}(p \cup p_0) \cup k^{-1}\mathrm{dom}\, p_1)$ and $q_1 = kq_0k^{-1}$. Then $p \cup p_0 \le q_0 \le g$. Also $p_1 \le h = kgk^{-1}$ and $x \in \mathrm{dom}\, p_1 \Rightarrow k^{-1}x \in \mathrm{dom}\, q_0 \Rightarrow x \in \mathrm{dom}\, q_1$, showing that $p_1 \le q_1 \le h$. Let $b = a \cup \mathrm{dom}\, q_0 \cup \mathrm{range}\, q_0$. Thus $(a, p) \le (b, q_0)$ so by assumption there are (c_0, r_0), $(c_1, r_1) \ge (b, q_0)$ which cannot be amalgamated over (a, p). Since $\mathcal{A}$ is IA and $b \le \mathcal{A}$, we may assume that c_0 and c_1 are substructures of $\mathcal{A}$. As $\mathcal{A}$ is HE there are $g', h' \in M$ extending r_0 and kr_1k^{-1} respectively. Then $p_0 \le q_0 \le r_0$ so $p_0 \le g'$ and $p_1 \le q_1 = kq_0k^{-1} \le kr_1k^{-1} \le h'$. It suffices therefore to see that $(g', h') \in E_\theta$.

If $(g', h') \notin E_\theta$ then $k'g'(k')^{-1} = h'$ for some automorphism k' extending θ. Let $(d, s) = (k'c_0 \cup kc_1, k'r_0(k')^{-1} \cup kr_1k^{-1})$. This lies in P since $k'r_0(k')^{-1} \cup kr_1k^{-1} \le k'g'(k')^{-1} \cup h' = h'$. If we embed (c_0, r_0) in (d, s) by k' and (c_1, r_1) by k, then this gives an amalgamation over (a, p) since k, k' both extend θ, so agree on a.

Since all D'_n and E_θ are dense open, and θ only takes countably many values, there is some (g, h) lying in $\bigcap_{n\in\omega} D'_n \cap \bigcap\{E_\theta : \theta \text{ an embedding } a \to \mathcal{A}\}$. Furthermore, (g, h) may be chosen so that $p \le g$. Since $g, h \in \bigcap_{n\in\omega} D_n$ they are generic, so for some $k \in \mathrm{Aut}(\mathcal{A})$, $kgk^{-1} = h$. Let $\theta = k \restriction a$. This now gives a contradiction to $(g, h) \in E_\theta$.

Conversely, assume that P has the JEP and WAP. We find countably many dense open sets all members of whose intersection are conjugate, giving the desired comeagre conjugacy class.

For each $(a, p) \in P$ such that $a \le \mathcal{A}$ let

$$D(p) = \{g \in M : (\exists f \in \mathrm{Aut}(\mathcal{A})) fpf^{-1} \le g\},$$

$E((a,p)) = \{g \in M : p \le g \to (\exists (b,q) \in P)(p \le q \le g \wedge (b,q)$ is amalgamable over $(a,p))\}$,

$F((a_1,p_1),(a_2,p_2),(a_3,p_3)) = \{g \in M : p_2 \le g \to (\exists f \in \mathrm{Aut}(\mathcal{A}))(f$ fixes a_1 pointwise $\wedge\, fp_3f^{-1} \le g)\}$.

First we show that for any $(a,p) \in P$ where $a \le \mathcal{A}$, $D(p)$ and $E((a,p))$ are dense open, and for any $(a_1,p_1) \le (a_2,p_2) \le (a_3,p_3)$ in P such that (a_2,p_2) is amalgamable over (a_1,p_1) and $a_3 \le \mathcal{A}$, so is $F((a_1,p_1),(a_2,p_2),(a_3,p_3))$. The fact that they are open follows since in each case membership may be determined on a suitable finite subset of A. Density of $D(p)$ follows from the JEP, and of $E((a,p))$ from the WAP.

To see that $F((a_1,p_1),(a_2,p_2),(a_3,p_3)) = F$ is dense provided that (a_2,p_2) is an amalgamable extension of (a_1,p_1) and a_3 is a substructure of $\mathcal{A}$, consider an arbitrary finite partial endomorphism p, and let g' be an epimorphism extending it (by HE). If $p_2 \not\le g'$ then automatically $g' \in F \cap [p]$, so we assume that $p \le g'$. Hence $q = p_2 \cup p$ is a function, and we let $b = a_2 \cup \mathrm{dom}\, q \cup \mathrm{range}\, q$. Then $(a_2,p_2) \le (a_3,p_3),(b,q)$, so as (a_2,p_2) is an amalgamable extension of (a_1,p_1) there are $(c,r) \in P$ and embeddings $\theta_1 : (a_3,p_3) \to (c,r)$ and $\theta_2 : (b,q) \to (c,r)$ which agree on a_1. Thus $\theta_1 p_3 \theta_1^{-1} \le r$. Let f_1, f_2 be automorphisms of $\mathcal{A}$ extending θ_1, θ_2 respectively (by IA). Then as $\mathrm{dom}\,\theta_1 = a_3$, $f_1 p_3 f_1^{-1} = \theta_1 p_3 \theta_1^{-1} \le r$. Also, $f_2^{-1} r f_2$ is a finite partial endomorphism, so can be extended to an epimorphism g. Let $f = f_2^{-1} f_1$. Then f fixes a_1 pointwise and $fp_3f^{-1} = f_2^{-1} f_1 p_3 f_1^{-1} f_2 \le f_2^{-1} r f_2 \le g$. Hence $g \in F \cap [p]$.

Since we have countably many dense open sets, we can consider their intersection, and it is required to show that any two members g_1, g_2 of this intersection are conjugate. Let Q be the family of triples of the form $(\theta,(a,p),(b,q))$ such that $(a,p),(b,q) \in P$ with $a, b \le \mathcal{A}$, $p \le g_1$, $q \le g_2$, and θ is an isomorphism from a to b such that $\theta p \theta^{-1} = q$. We use back-and-forth and choose a sequence of members $(\theta_i,(a_i,p_i),(b_i,q_i))$ of Q such that (a_{i+1},p_{i+1}), (b_{i+1},q_{i+1}) are amalgamable extensions of (a_i,p_i), (b_i,q_i) respectively, $\theta_1 \restriction a_0 \le \theta_2 \restriction a_1 \le \theta_2 \restriction a_1 \le \ldots$, and $\bigcup a_i = \bigcup b_i = A$.

We let $\theta_0 = a_0 = b_0 = \emptyset$. Since $g_1 \in E(\emptyset)$ there is an amalgamable extension (a_1,p_1) of (a_0,p_0). Since $g_2 \in D(p_1)$ there is an automorphism f such that $fp_1f^{-1} \le g_2$. We let $b_1 = fa_1$, $q_1 = fp_1f^{-1}$, and $\theta_1 = f \restriction a_1$.

Now assume that $(\theta_i,(a_i,p_i),(b_i,q_i))$ have been chosen for $i \le n$ where $n \ge 1$ satisfying the above properties. For the back-and-forth we let x be the mth member of A in some fixed enumeration where $2m = n$ if n is even and $2m+1 = n$ if n is odd. If n is even we extend to the $(n+1)$th step so that $x \in \mathrm{dom}\, p_{n+1} \cap \mathrm{range}\, p_{n+1}$ and if n is odd we extend so that $x \in \mathrm{dom}\, q_{n+1} \cap \mathrm{range}\, q_{n+1}$. We just do the former. Since $g_1 \in E((a_n \cup \{x, g_1x, g_1^{-1}x\}, p_n \cup \{(x,g_1x),(g_1^{-1}x,x)\}))$ and $p_n \cup \{(x,g_1x),(g_1^{-1}x,x)\} \le g_1$, there is an amalgamable extension (a_{n+1},p_{n+1}) of $(a_n \cup \{x, g_1x, g_1^{-1}x\}, p_n \cup \{(x,g_1x),(g_1^{-1}x,x)\})$ with $a_{n+1} \le \mathcal{A}$. Let f be an automorphism extending θ_n. Then (a_n,p_n) is an amalgamable extension of (a_{n-1},p_{n-1}), and hence, applying θ_n, (b_n,q_n) is an amalgamable extension of (b_{n-1},q_{n-1}). Therefore $g_2 \in F((b_{n-1},q_{n-1}),(b_n,q_n),(fa_{n+1},fp_{n+1}f^{-1}))$, so there is an automorphism h of $\mathcal{A}$ fixing b_{n-1} pointwise such that $hfp_{n+1}f^{-1}h^{-1} \le g_2$. Let $b_{n+1} = hfa_{n+1}$, $q_{n+1} = hfp_{n+1}f^{-1}h^{-1}$, and $\theta_{n+1} = hf \restriction a_{n+1}$. To see that $\theta_n \restriction a_{n-1} \le \theta_{n+1}$, let

$y \in a_{n-1}$. Then $\theta_{n+1}y = hfy = h\theta_n y = \theta_n y$ since h fixes $b_{n-1} = \theta_n a_{n-1}$ pointwise. This gives the induction step.

It is clear that $\bigcup a_i = \bigcup b_i = A$ since the back-and-forth has specifically ensured that this is true. Since $\theta_1 \restriction a_0 \le \theta_2 \restriction a_1 \le \theta_3 \restriction a_2 \le \ldots$, we can define θ to be the union of all the $\theta_{n+1} \restriction a_n$, and this is an automorphism of $\mathcal{A}$ such that $\theta g_1 \theta^{-1} = g_2$, establishing that g_1 and g_2 are conjugate, as required. □

3. The trivial structure

In this section, we describe the generics in $\mathrm{Epi}(\Omega)$. This is particularly instructive, since it leads on to a similar, but more complicated characterization for Γ. For Ω, Aut and Bi coincide, as do Emb and Mon, so we just have four monoids to consider. But by Theorem 2.1, any generic member of $\mathrm{Emb}(\Omega)$ lies in $\mathrm{Aut}(\Omega)$ (and we know what these are from [**25**]) and any generic member of $\mathrm{End}(\Omega)$ lies in $\mathrm{Epi}(\Omega)$, so it is just this case that we treat. (After writing this, we found that the first part of the following theorem is also given in [**28**], Theorem 3.)

In Ω^Ω, we have the notion of an *orbit* of f, which is defined to be an equivalence class under the relation $x \sim y$ if there are $m, n \in \mathbb{N}$ such that $f^m x = f^n y$. If there is some x in an orbit which is fixed by f^k for some $k > 0$, then we say that it *cycles*, and then the whole orbit consists of $\bigcup_{n\in\mathbb{N}} f^{-n}x$ for any x in its cycle, and we can describe this as consisting as the cycle itself, together with all the elements not in the cycle, which are arranged in 'trees' leading into the points of the cycle. If there is no cycle in the orbit, then it consists of a single tree with outdegree 1 at all vertices. We can view each orbit as a connected directed graph where there is an edge from x to fx for each x, and when describing the structure of the orbits, we refer to this digraph when required without further mention. The particular orbit pattern which captures what happens for generic members of $\mathrm{End}(\Omega)$ is the disjoint union of some finite number k of copies of $\omega^{<\omega}$, where this denotes the set of finite sequences of natural numbers, viewed as a digraph in which $(\sigma^\wedge(i), \sigma)$ is an edge for each $\sigma \in \omega^{<\omega}$ and $i \in \omega$ (where $\sigma^\wedge(i)$ is the sequence obtained from σ by appending i at the end), and the k copies of the empty sequence form a cycle of length k. Formally, this digraph is equal to $k \times \omega^{<\omega}$, where $((i, \sigma), (j, \tau))$ is an edge if and only if $\sigma = \tau =$ the empty sequence and $j = i + 1$ or $j = 0$ and $i = k - 1$, or $i = j$ and for some k, $\sigma = \tau^\wedge(k)$. Let us call this a *generic k-cycle.*

THEOREM 3.1. *(i) An element g of* $\mathrm{End}(\Omega)$ *is generic if and only if all its orbits are generic k-cycles for some finite k, and every k arises infinitely often.*

(ii) An element of $\mathrm{Emb}(\Omega)$ *is generic if and only if it lies in* $\mathrm{Sym}(\Omega)$ *and is generic as a member of* $\mathrm{Sym}(\Omega)$ *(which as shown in* [**25**] *means that it has no infinite cycles, and has infinitely many cycles of each finite length).*

Proof: (i) It is clear that all elements of $\mathrm{End}(\Omega)$ all of whose orbits are generic k-cycles for some finite k, and such that each such k arises infinitely often, are conjugate, since the two corresponding digraphs are isomorphic, and an isomorphism between the digraphs provides the desired conjugacy (and note that such an element also lies in $\mathrm{Epi}(\Omega)$). So it suffices to show that the family of endomorphisms having this orbit pattern is comeagre. As in the corresponding proof in [**25**], we find countably many dense open sets whose intersection is contained in (equals actually) the given set.

For $x \in \Omega$, let D_x be the set of members f of Ω^Ω such that for some $i > j \ge 0$, $f^i x = f^j x$. This is a dense open subset of Ω^Ω. For if $f \in D_x$, witnessed by the fact that $f^i x = f^j x$, then any member h of Ω^Ω which agrees with f on $\{f^k x : 0 \leq k \leq i\}$ also satisfies $h^i x = h^j x$, so lies in D_x, and given any $f \in \Omega^\Omega$ and finite subset A of Ω, we can alter f outside A to ensure that the result lies in D_x, to establish density.

Next consider $E_{k,m} = \{f \in \Omega^\Omega : f$ has at least m orbits of cycle-length $k\}$. As in [**25**], this is dense open, so g lies in it.

Finally let $F_{x,m} = \{f \in \Omega^\Omega : |f^{-1}\{x\}| \geq m\}$, which is also easily seen to be dense open.

We have therefore found countably many dense open sets, and their intersection is precisely equal to the set of all members of $\mathrm{End}(\Omega)$ having the stated orbit pattern.

(ii) This follows at once from Theorem 2.1. □

4. The ordered rationals

To explain what happens in this case, we have to recall some standard material about order-preserving permutations of the rationals, and we also have to give corresponding information about endomorphisms. Now if g is an automorphism of $(\mathbb{Q}, \leq)$ (or indeed any linearly ordered set, usually though assumed to be doubly homogeneous), then we define an *orbital* to be the convex closure of an orbit. Thus it is a subset of $\mathbb{Q}$ of the form $\{y \in \mathbb{Q} : \exists m, n \in \mathbb{Z}(g^m x \leq y \leq g^n x)\}$ for some x, and this set is the orbital containing x. Since g is order-preserving, there are three possibilities for an orbital X: for all $x \in X$, $x < gx$; for all $x \in X$, $gx < x$; and $gx = x$ where $X = \{x\}$. In these cases we say that the orbital has *parity* $+1$, -1, and 0 respectively. Since all orbitals are convex, the family of all orbitals of g receives a natural linear ordering, and together with the assignment of ± 1 and 0 it becomes a '3-coloured' linear order. The family of all orbitals, together with this colouring, is called the *orbital pattern* of g. A standard result on ordered permutation groups says that two automorphisms of $\mathrm{Aut}(\mathbb{Q}, \leq)$ are conjugate if and only if they have isomorphic orbital patterns, see [**6**] Theorem 2.2.5 for example.

For endomorphisms, one can attempt to carry out a similar kind of analysis, though for now we just restrict attention to the way that things work out for generics. We want to describe the behaviour of a generic on a typical convex subset of $\mathbb{Q}$. This will be a little like the non-trivial orbitals in what we have just described, but there will only be one kind in this instance. Let $\mathbb{Q}^{\neq 0} = \mathbb{Q}^{<0} \cup \mathbb{Q}^{>0}$ be the set of non-zero rationals, expressed as the union of the sets of negative and positive rationals, and consider $X = \bigcup_{n>0} \mathbb{Q}^{\neq 0} \times \mathbb{Q}^{n-1} \cup \{0\}$ (where this is of course a disjoint union). By describing this set as a family of finite sequences, we are intending to make it easier to describe a natural order-homomorphism (and of course, we must also say what the ordering is). The map g on X is given by deletion of the final entry, for sequences of length greater than 1, and all sequences of length 1 are mapped to 0 (including 0 itself). The ordering is given as follows:

$$\ldots < \mathbb{Q}^{<0} \times \mathbb{Q}^2 < \mathbb{Q}^{<0} \times \mathbb{Q} < \mathbb{Q}^{<0} < \{0\} < \mathbb{Q}^{>0} < \mathbb{Q}^{>0} \times \mathbb{Q} < \mathbb{Q}^{>0} \times \mathbb{Q}^2 < \ldots$$

and inside each block the ordering is lexicographic. The fact that $x \leq y \Rightarrow g(x) \leq g(y)$ is clear. The effect of the map is to send each non-zero block one step toward the middle, and 0 is fixed. We describe a linear order isomorphic to this X under the action of g as a *typical block*.

THEOREM 4.1. *(i) An element g of* $\mathrm{End}(\mathbb{Q}, \leq)$ *is generic if and only if* $\mathbb{Q}$ *can be written as* $\bigcup_{q \in \mathbb{Q}} X_q$ *where each* X_q *is a typical block under the restriction of* g, *and* $q < r \Rightarrow X_q < X_r$.

(ii) An element of $\mathrm{Emb}(\mathbb{Q}, \leq)$ *is generic if and only if it lies in* $\mathrm{Aut}(\mathbb{Q}, \leq)$ *and is generic as a member of* $\mathrm{Aut}(\Omega)$ *(which as shown in* [**25**] *means that its family of orbitals is a densely* $\{0, \pm 1\}$*-coloured linear order without endpoints, where orbitals are coloured by their parity, in which between any two distinct points, all three colours occur).*

Proof: (i) We first remark that elements of the form described exist. This is because the ordering $\ldots < \mathbb{Q}^{<0} \times \mathbb{Q}^2 < \mathbb{Q}^{<0} \times \mathbb{Q} < \mathbb{Q}^{<0} < \{0\} < \mathbb{Q}^{>0} < \mathbb{Q}^{>0} \times \mathbb{Q} < \mathbb{Q}^{>0} \times \mathbb{Q}^2 < \ldots$ certainly exists and is countable, and hence $\mathbb{Q}$ copies of it is a countable dense linear ordering without endpoints, so isomorphic to $\mathbb{Q}$. Furthermore, they are clearly all conjugate.

The main point therefore is to find countably many dense open subsets of $\mathrm{End}(\mathbb{Q}, \leq)$ such that any member of their intersection has this special form. We remark that epimorphisms g of 'this special form' may be characterized thus: g has fixed points ordered in type $\mathbb{Q}$, and each fixed point lies in a unique typical block. We therefore first find countably many dense open sets such that any member of their intersection has fixed points ordered like $\mathbb{Q}$, and then show that each such point lies in a unique typical block, so that the union of all these typical blocks equals the whole of $\mathbb{Q}$.

First let $D_a = \{g \in \mathrm{End}(\mathbb{Q}, \leq) : (\exists x, y)(x < a < y \wedge gx = x \wedge gy = y)\}$ for each $a \in \mathbb{Q}$, and for each $a < b$ in $\mathbb{Q}$, let $D_{ab} = \{g \in \mathrm{End}(\mathbb{Q}, \leq) : ga = a \wedge gb = b \to (\exists x \in (a, b))gx = x\}$. Now each of these is open, since if $g \in D_a$, there are x and y witnessing the truth of the formula, and any endomorphism agreeing with g on x and y lies in D_a, and if $g \in D_{ab}$ where $a < b$, if $ga \neq a$ or $gb \neq b$ then any endomorphism agreeing with g on a and b lies in D_{ab}, and if $ga = a$ and $gb = b$ then there is x witnessing the truth of the formula, and any endomorphism agreeing with g on a, x and b lies in D_{ab}. Next we verify openness. Let p be a finite partial homomorphism of $(\mathbb{Q}, \leq)$, and let x, y be such that for all points z of $\mathrm{dom}(p) \cup \mathrm{range}(p) \cup \{a\}$, $x < z < y$. Then $p \cup \{(x, x), (y, y)\}$ is an extension of p lying in D_a, giving openness of D_a. Now considering $a < b$, first extend p if necessary so that pa and pb are both defined. If $pa \neq a$ or $pb \neq b$ then already $p \in D_{ab}$. If $pa = a$ and $pb = b$, let y be the greatest point of $[a, b]$ such that for some $m > 0$, $p^m y$ is defined and equals a, and let z be the least point of $[a, b]$ such that for some $m > 0$, $p^m z$ is defined and equals b. Since p is order-preserving, $y < z$ and if $y < x < z$ where x is strictly less than every member of $(\mathrm{dom}(p) \cup \mathrm{range}(p)) \cap (y, z]$ then $p \cup \{(x, x)\}$ is an extension of p lying in D_{ab}.

Any g lying in all the dense open sets so far has fixed points ordered like $\mathbb{Q}$. To define 'block', we first let $O^+(x) = \{y : (\exists n \geq 0)(x \leq y \leq g^n x \vee g^n x \leq y \leq x)\}$, and then a block is an equivalence class under the relation $x \sim y$ if $O^+(x) \cap O^+(y) \neq \emptyset$. The fact that the blocks of a generic element are typical is then established by considering dense open sets of the following forms:

$E_a = \{g \in \mathrm{End}(\mathbb{Q}, \leq) : (\exists x)gx = a\}$, any g lying in all of these sets is surjective;

$F_a = \{g \in \mathrm{End}(\mathbb{Q}, \leq) : (\exists n \geq 0)g^{n+1}a = g^n a\}$, any g lying in all of these sets lies in the block of some fixed point;

$G_a = \{g \in \mathrm{End}(\mathbb{Q}, \leq) : ga = a \to \exists x \exists y(x < a < y \wedge gx = gy = a)\}$, if g lies in all these sets then the inverse image of any fixed point a is an interval I_a such that a is in the interior of I_a;

$H_{ab} = \{g \in \mathrm{End}(\mathbb{Q}, \leq) : gb = a \to \exists x \exists y(x < b < y \wedge gx = gy = a)\}$, saying that I_a is open, and so is the inverse image of every point.

We omit the verification that all these sets are dense open. Clearly any endomorphism lying in all these dense open sets is of the form stated.

(ii) We use Theorem 2.1, together with the characterization of generic members of $\mathrm{Aut}(\mathbb{Q}, \leq)$ given in [**25**]. □

5. The random graph

We now look at the random graph. Here things become more complicated, because, even the *existence* of a single generic requires some labour, and we begin by examining this. They key issue concerns so-called 'extension lemmas'. The one which was established (implicitly) in [**25**] is that if p is a finite partial automorphism of a finite graph, then there there is an automorphism of a finite graph containing the originally given one extending p. This was isolated by Lascar as the key ingredient in showing that there exist 'mutual' generics, required in the proof of the small index property for Γ [**7**], and Hrushovski proved the relevant result, namely that given any finite graph Δ, there is a finite graph Δ' containing Δ such that *every* partial automorphism of Δ can be extended to an automorphism of Δ'.

LEMMA 5.1. *If p is a partial endomorphism of a finite graph Δ, then there is a finite graph Δ' containing Δ, and an endomorphism of Δ' extending p.*

Proof: We may assume that p is not an isomorphism, since in that case we could use the result from [**25**] to extend to an automorphism of some Δ'.

Let D be the orbit digraph of p on Δ, that is the digraph having a directed edge from each member x of $\mathrm{dom}(p)$ to px, so that each vertex of D has outdegree 1 or 0, and write $D = \bigcup_{i<k} D_i$, where the D_is are the connected components of D, that is, the orbits of p on Δ. For each $i < k$, let Δ_i be the induced subgraph of Δ restricted to the (vertices in the) orbit D_i, and let $p_i = p \restriction \Delta_i$. Note that Δ and D have the same sets of vertices, for which we must view points not lying in the domain or range of p as lying in singleton orbits.

Then each D_i is either a directed tree with one sink at the root (including as a degenerate case, the singletons just mentioned), which we call v_i, or a directed cycle with directed trees feeding in. Note that if D_i has a cycle, then p_i is defined on the whole of Δ_i; and if not it is defined everywhere except on the root (sink).

For each $i < k$ for which the orbit D_i cycles, let $\Delta'_i = \Delta_i$ and $q_i = p_i$ (and let $X_i \subseteq \Delta_i$ be the induced subgraph on which p_i cycles).

For each $i < k$ for which the orbit D_i does not cycle, we find $(\Delta'_i, q_i) > (\Delta_i, p_i)$ such that q_i contains a cycle as follows. For each vertex $u \in D_i$, let $f(u) = d(u, v_i)$, the (arc) length of the unique directed path from u to v_i in D_i, and let $n_i = 1 + \max f$, where $\max f$ is the length of the longest directed path to v_i in D_i. We let Δ'_i be obtained from Δ_i by adding a set $X_i = \{x_1, \ldots, x_{n_i}\}$ of new vertices, all pairs of which are adjacent, and where the edges between Δ_i and X_i are given by $u \sim x_j$ if $f(u) + j < n_i$.

Now let q_i be the map given by $q_i u = p_i u$ for $u \in \Delta_i - \{v_i\}$, $q_i v_i = x_1$, $q x_j = x_{j+1}$ for each $j \leq n_i - 1$, and $q_i x_{n_i} = x_1$. Then q_i is an endomorphism of Δ'_i, with a cycle on X_i. To see that it preserves the graph relation, note that it clearly preserves it on each of Δ_i and X_i separately, so we just need to consider $u \in \Delta_i$ and $x_j \in X_i$, and show that if $u \sim x_j$ then $q_i u \sim q_i x_j$. Since $u \sim x_j$, $f(u) + j < n_i$. Hence if $u = v_i$ then $f(u) = 0$, so that $j < n_i$, and $q_i(u) = x_1$, $q_i(x_j) = x_{j+1} \neq x_1$, and these are joined by definition. If however $u \neq v_i$, then

$f(u) > 0$, and $f(q_iu) = f(p_iu) = f(u) - 1$, so that $f(q_iu) + (j+1) < n_i$ and $q_iu \sim x_{j+1} = q_ix_j$ as required.

Finally let $\Delta' = \bigcup_{i<k} \Delta'_i$, where all vertices of Δ'_i are joined by an edge to all vertices of X_j, for $j \neq i$, and for which the orbit on D_j does not cycle, and let $q = \bigcup_{i<k} q_i$. Then q is an endomorphism of $\Delta' \supseteq \Delta$ which extends p. $\square$

This lemma is a key point in establishing the existence of generic endomorphisms in this case, but more is required than for automorphisms, as the orbit structure will be much more complicated. As 'unars' (that is, structures with a single unary function) the orbits will have the same pattern as in Theorem 3.1 (by the same proof), but this time there is a graph structure to consider too. Given a finite approximation to an orbit, which we can assume is closed under the action of the endomorphism (and hence cycles), we can always extend by putting in non-edges further out from the cycle. Of course, any two members of $g^{-1}(x)$ must necessarily be joined by a non-edge (since if they were joined by an edge, they could not be collapsed to a single point). However, we can join other pairs of vertices by non-edges, even if they are farther apart in the unar. It is much harder to join vertices by an edge, and this can only be done if their images are or can be joined by an edge. Generically therefore, we expect there to be many non-edges. Pinning down the possibilities seems to be quite involved, and so we resort to a subsidiary Fraïssé construction to give what we want. For any positive integer k and graph γ on $k = \{0, 1, 2, \ldots, k-1\}$ which is preserved by the cyclic permutation of k ($+1$ mod k), let P_γ be the family of all pairs (X, p) such that X is a graph extending γ, and p is an endomorphism of X extending the cyclic permutation on k.

LEMMA 5.2. *For any k and graph γ on k preserved by the cyclic permutation of k, P_γ is an amalgamation class.*

Proof: Note that in this instance, the 'empty' structure does not lie in P_k, since we must always have k at least (and the graph and action of p on it), but this doesn't matter, and k performs the role of the empty structure. Thus joint embedding will follow from the amalgamation property as usual, by amalgamating over k. The hereditary property also holds, recalling that by definition, substructures have to be closed under the action of p. So we can concentrate on amalgamation. Letting A, B, C be an amalgamation diagram, where there are embeddings of A into each of B and C, and $A, B, C \in P_k$, by replacing by isomorphic copies, we may suppose that A is a substructure of B and C, and furthermore that $A = B \cap C$. The amalgam will have domain $B \cup C$, and there is no choice over the action of the map, since it is already defined on each of B and C separately. All that is necessary is to specify the graph structure, specifically, to say how members of $B - C$ and $C - B$ are joined. The decision is that *no* new edges are added, and this clearly ensures that the map on $B \cup C$ is an endomorphism, as required. (It is possible that we would be able to insert some new edges, but we cannot be sure.) $\square$

Motivated by this lemma, we say that a generic k-cycle of the endomorphism g (as defined in section 3) is a *generic graph orbit* of g if it is Fraïssé-generic corresponding to the family P_k, where the graph structure and action on k are induced by Γ and g respectively. The crucial point is that there are only countably many generic graph orbits 'up to isomorphism'. If there were uncountably many, then this would rule out the existence of generic endomorphisms.

THEOREM 5.3. *(i) There are generic elements of* $\mathrm{End}(\Gamma)$*, and* $g \in \mathrm{End}(\Gamma)$ *is generic if and only if all its orbits are generic graph orbits, and up to isomorphism, each generic graph orbit occurs infinitely often as an orbit of* g*, and for any finite* $B \subseteq \Gamma$ *closed under the action of* g*, and graph of the form* $A \cup B$ *where* A *is a finite graph disjoint from* B*, and endomorphism* p *of* $A \cup B$ *extending* $g \restriction B$*, there is* $A' \subseteq \Gamma - B$ *such that* $A \cup B \cong A' \cup B$ *under a map fixing* B *pointwise, and which carries the action of* p *on* A *to that of* g *on* A'*.*

(ii) There is no generic element of $\mathrm{Mon}(\Gamma)$*.*

(iii) An element of $\mathrm{Emb}(\Gamma)$ *is generic if and only if it lies in* $\mathrm{Aut}(\Gamma)$ *and is generic as a member of* $\mathrm{Aut}(\Gamma)$ *(which is characterized explicitly in* [**25**]*).*

Proof: (i) The fact that any two elements as stated are conjugate is established by back-and-forth. For suppose that g and g' have the form described, and let Q be the family of all finite partial isomorphisms q of Γ taking a subset of Γ closed under g to a subset closed under the action of g', and carrying the action of g to that of g' (meaning that for x in the domain, $qg(x) = g'q(x)$). The rather involved condition in the statement of (i) was specifically designed to make the back and forth steps possible. Notice that the fact that all orbits of g and g' have finite cycles is used here in an essential way, and this ensures that we can close up under their actions and obtain endomorphisms of finite substructures.

As usual our main job is to find suitable dense open sets whose intersection contains precisely endomorphisms of the type described. Some of these are similar to those in Theorem 3.1(i). For $x \in \Gamma$, let D_x be the set of members f of $\mathrm{End}(\Gamma)$ such that for some $i > j \geq 0$, $f^i x = f^j x$. This is a dense open subset of $\mathrm{End}(\Gamma)$. Openness is shown as before, and density follows by appeal to Lemma 5.1. If g lies in all these sets, then all orbits of g cycle.

Next for each graph γ on $k = \{0, 1, 2, \ldots, k-1\}$ preserved by cyclic permutation, consider

$E_{\gamma,m} = \{f \in \mathrm{End}(\Gamma) : f$ has at least m orbits of cycle-length k whose action on the k-cycle is isomorphic to γ by an isomorphism respecting the cycle structure$\}$.

This is easily seen to be dense open, and if g also lies in all these sets, then it has infinitely many orbits corresponding to each γ.

Next, we have to ensure that all the orbits are generic graph orbits. For this, first for each $x \in \Gamma$, k and γ as in the previous paragraph, and $(X, p) \in P_\gamma$, let

$F_{x,(X,p)} = \{f \in \mathrm{End}(\Gamma) : x$ lies in an orbit of f which cycles as a copy of $\gamma \to (X, p)$ embeds in that orbit$\}$.

Clearly $F_{x,(X,p)}$ is open. To see that it is dense, let $f \in \mathrm{End}(\Gamma)$ be arbitrary, and Y be a finite subset of Γ. Then by altering f outside Y, using Lemma 5.1, we may assume that the orbit of x under f cycles. If it does not cycle in a graph of length k isomorphic to γ, then automatically $f \in F_{x,(X,p)}$. Otherwise, as in the proof of Lemma 5.2, we may alter f outside Y preserving the action on $\bigcup_{i \in \omega} f^i x$ to ensure that (X, p) embeds. If g lies in all these sets, then the orbit of x must embed all members of the relevant 'age'.

To ensure homogeneity of the orbit containing x, we may find a dense open set corresponding to each instance of the amalgamation property. Namely, for each amalgamation diagram $\mathcal{D}$ in $age(P_\gamma)$, let a corresponding $G_{\mathcal{D}}$ comprise all those endomorphisms of Γ such that *if* x lies in an orbit which cycles as γ, and an embedding of $\mathcal{D}$ in the orbit of x is given, then they can be amalgamated in the orbit. An argument similar to that for $F_{x,(X,p)}$ shows that $G_{\mathcal{D}}$ is dense open. If f lies in all these dense open sets as well, then its orbits are all generic graph orbits.

Finally let $A \cup B$ be a finite graph such that $A \cap B = \emptyset$, $B \subseteq \Gamma$, and let $p \in \mathrm{End}(A \cup B)$. We let $H_{(A,B,p)}$ be the family of all endomorphisms g of Γ such that either $gB \not\subseteq B$, or p does not extend $g \restriction B$, or there is $A' \subseteq \Gamma - B$ and an isomorphism from $A \cup B$ to $A' \cup B$ fixing B pointwise and carrying the action of p on $A \cup B$ to that of g on $A' \cup B$. Then $H_{(A,B,p)}$ is clearly open (since membership of this set depends on just finitely many points). To see that it is dense, let $g \in \mathrm{End}(\Gamma)$ and finite $X \subseteq \Gamma$ be given, and assume that $A \cap \Gamma = \emptyset$. If $gB \not\subseteq B$, or p does not extend $g \restriction B$ then automatically $g \in H_{(A,B,p)}$, so we now suppose that $gB \subseteq B$ and $g \restriction B \subseteq p$. Turn $A \cup \Gamma$ into a graph extending $A \cup B$ by adding no new edges. By Lemma 5.1 there is a finite graph $B' \supseteq B \cup X \cup gX$ contained in Γ and an endomorphism q of B' extending $g \restriction B \cup X$. As $(A \cup B) \cap B' = B$, p and q agree on B, and we added no new edges in forming $A \cup \Gamma$, $p \cup q$ is an endomorphism of $A \cup B'$. By genericity of Γ there is an isomorphism θ taking $A \cup B'$ into Γ fixing B' pointwise. Let $A' = \theta A$ and let h be an endomorphism of Γ extending $\theta(p \cup q)\theta^{-1} = \theta p \theta^{-1} \cup q$. Then $h \in H_{(A,B,p)}$ and h agrees with g on X, giving density of $H_{(A,B,p)}$.

We have therefore found countably many dense open sets, and their intersection is precisely equal to the set of all members of $\mathrm{End}(\Omega)$ having the stated orbit pattern.

(ii) We use ideas from a proof in [**26**] of a theorem due to Hodkinson that there are no mutually generic pairs in $\mathrm{Aut}(\mathbb{Q}, <)$. The intuition is that we can show that there is a property of any generic member g of $\mathrm{Mon}(\Gamma)$ which has to be able to take $2^{\aleph_0}$ values, but g can only exhibit countably many of these. Officially, the proof is done by a diagonalization.

Let us remark on the cycle structure that any generic member g of $\mathrm{Mon}(\Gamma)$ would have to have. For a start, it is easy to see that g must be a permutation of Γ (that is, onto), since no finite approximation can exclude any particular point of Γ from the range of g. Now as for $\mathrm{Aut}(\Gamma)$, there will have to be infinitely many finite cycles of all possible finite lengths, and with all possible (compatible) graph structures, and so on. The key new remark in this case, is that there must also be *infinite* cycles. For any finite approximation can always be extended to include points a, b, c such that ab is a non-edge and bc is an edge, and a is mapped to b, and b to c. But in any extension of such a partial map, a cannot now lie in a finite cycle, as this would mean that under some iterate of g, the edge bc is mapped to the non-edge ab, which is not allowed (it is allowed to create an edge, but not to destroy one). Hence g has at least one infinite cycle (infinitely many of them, actually).

Now the key idea in our argument is that there are uncountably many 'patterns' which can be exhibited by infinite cycles realized by a generic member of $\mathrm{Mon}(\Gamma)$, but there is only 'room' for countably many, and so this situation is contradictory. We actually do this by an explicit diagonalization. First we explain what is meant by a 'pattern' in this context. Let a lie in an infinite cycle of $g \in \mathrm{Mon}(\Gamma)$. Then for each $r > 0$ there is a least $k \in \mathbb{Z}$ which we write as $k(r)$ such that $g^k a$ is joined by an edge to $g^{r+k} a$. In fact for *any* monomorphism there is such a k provided we allow it to have values in $\pm\infty$, but for a generic, $k(r)$ is necessarily finite. This is because each finite partial monomorphism p has an extension in which for some but not all values of k, $p^k a$ is joined to $p^{r+k} a$ (and as g must preserve edges, if this holds for some value of k, then it holds for all greater values). Here k is therefore a function from the set of positive integers to $\mathbb{Z}$, and so we expect that it could take $2^{\aleph_0}$ values. We refer to these functions as *patterns*, and the ones which arise for g are said to be *exhibited* by g.

Now it is clear that any two generic monomorphisms must exhibit the same patterns (since the conjugating element must preserve everything about them), and using this observation, we are now able to reach a contradiction. Since g is generic, there are dense open subsets D_n of $\mathrm{Mon}(\Gamma)$ such that the conjugacy class of g contains $\bigcap_{n\in\omega} D_n$. In particular this means that all members of $\bigcap_{n\in\omega} D_n$ are generic. Let $\mathcal{F}$ be the set of functions which are exhibited by g, and let $\{(f_n, x_n) : n \in \omega\}$ enumerate all ordered pairs whose first co-ordinate lies in $\mathcal{F}$ and whose second co-ordinate lies in Γ. We now construct another generic h as follows in countably many steps, $h = \bigcup_{n\in\omega} p_n$ where $p_0 \subseteq p_1 \subseteq p_2 \subseteq \ldots$ is an increasing sequence of finite partial monomorphisms. Let p_0 be empty. Assuming that p_{2n} has been defined, let p_{2n+1} be an extension of p_{2n} such that all its extensions lie in D_n. Now assuming that p_{2n+1} has been defined, we choose an extension p_{2n+2} such that either x_n lies in a finite cycle of p_{2n+2}, or for some m, r, there is a partial orbit of p_{2n+2} containing x_n such that for some r and k, $p_{2n+2}^k x_n$ and $p_{2n+2}^{k+r} x_n$ are joined by an edge but $p_{2n+2}^{k-1} x_n$ and $p_{2n+2}^{k-1+r} x_n$ are not (and these are all defined) and $k \neq f_n(r)$. To see that this is possible, note that as p_{2n+1} is finite, the partial cycle containing x_n is finite (possibly empty). If it is already a cycle, then no extension is necessary. If not, then we can extend (taking r to be greater than the length of the partial cycle containing x_n) so that the stated condition holds for some $k \neq f_n(r)$.

Now let h be the union of all the p_n. Extensions of the second kind ensure that h is defined on the whole of Γ, and as $h \in \bigcap_{n\in\omega} D_n$, it is generic. However, by construction, h exhibits a pattern which does not lie in $\mathcal{F}$. (In fact, *none* of the patterns exhibited by h lie in $\mathcal{F}$.) Since, by our remark above, the family $\mathcal{F}$ is the same for any generic, this gives the desired contradiction.

(iii) Once more we appeal to Theorem 2.1. □

We remark that in Theorem 5.3 we have given a non-existence proof for generic monomorphisms very much in the style of the one for pairs in $\mathrm{Aut}(\mathbb{Q}, \leq)$ presented in [**26**] (a result originally proved by Hodkinson using games). Now that we have a necessary and sufficient condition for the existence of generics of all types in Theorem 2.3, we can obtain this result much more simply, avoiding diagonalization. It just suffices to note that WAP fails for P. For consider a finite graph with vertices x_0, x_1, x_2 in which x_1x_2 is an edge but x_0x_1 is not (and whether x_0x_2 is or not doesn't matter). Then the map p taking x_0 to x_1 and x_1 to x_2 is a finite partial monomorphism. If $\mathrm{Mon}(\Gamma)$ had a generic member, then by Theorem 2.3, P would have to fulfil the WAP, and so there would be an amalgamable extension (b, q) of (a, p) in P, where $a = \{x_0, x_1, x_2\}$. Let $\{x_i : m \leq i \leq n\}$ be the orbit of x_0 under q. Then all x_i must be distinct, since q must preserve the edge x_1x_2 so that x_ix_{i+1} is an edge if and only if $1 \leq i < n$. We can now form two extensions of (b, q) of the form (c_j, r_j) where $c_j = \{x_i : m \leq i \leq n - m + 1\}$ for $j = 0, 1$ and x_0, x_{m-n+1} are joined in c_0 but not in c_1, and these cannot be amalgamated over (a, p). (This is essentially the same as the diagonal argument.)

Presumably one can find a similar revised proof for Theorem 2.4 in [**26**], but we have not worked out the details of this.

6. Other structures

The situation regarding generics in the countable dense circular order was remarked on in [**25**], namely, there are no generics, but $\aleph_0$ conjugacy classes of local generics. We here recall what this means, and investigate what happens for the other monoids. The structure was also studied in [**27**]. We also consider 2-transitive trees as described in [**4**]. These are not homogeneous, but in a slightly enlarged language, some of them *are*, so we can attempt to describe generics here too.

The countable dense circular order (C, R) may be axiomatized by a ternary relation, and as for $\mathbb{Q}$ we have to decide whether to use the strict or reflexive relation. Since if we use the strict relation we get nothing new (since any endomorphism is 1–1) we work with the reflexive relation, which we write as R. We may take the domain to comprise all complex numbers of the form $\mathrm{e}^{\mathrm{i}\theta}$ for rational θ, where $R(a, b, c)$ if the anticlockwise arc from a to c round the circle passes through b. In accordance with the remark just made, $R(a, a, c)$ and $R(a, b, b)$ are always regarded as holding. Some results about this structure are given in [**27**], in particular, a characterization of the conjugacy classes of its automorphisms. None of these are generic, and this is because there are countably many pairwise incompatible behaviours, which can be guaranteed by finite partial automorphisms. For instance, a map which just fixes a single point is incompatible with one which interchanges two distinct points. We say that an automorphism is *locally generic* if its conjugacy class is comeagre on some non-empty open set, and then we can see that (C, R) has exactly $\aleph_0$ conjugacy classes of local generics, which are determined by finite cycle-types and 'winding number'. See [**25**].

As for the ordered rationals, Mon = Emb and Bi = Aut. For this we have to show that any monomorphism g preserves $\neg R$. Suppose therefore that $\neg R(a, b, c)$. Then a, b, c are distinct, and as R is 'total', $R(a, c, b)$. As g preserves R, $R(ga, gc, gb)$, and from this it follows that $\neg R(ga, gb, gc)$. By analogy with the earlier results, we only need to try to characterize locally generic epimorphisms.

Now we can consider the same family of finite maps which serve as representatives of the conjugacy classes of local generics in $\mathrm{Aut}(C, R)$, namely p_{mn} for $0 < m < n$ where m and n are coprime, whose domain is a subset $\{x_i : i < n\}$ of C of size n in anticlockwise enumeration, defined by $p_{mn}(x_i) = x_j$ where $j \equiv i+m \bmod n$. These are partial automorphisms, so are also partial endomorphisms, and indeed are still incompatible as endomorphisms, and they constitute a complete list of representatives of locally generic epimorphisms. We can describe what these epimorphisms are by an adaptation of the method from [**25**] using the generic epimorphisms of $(\mathbb{Q}, \leq)$ given in section 4. Instead of taking $\mathbb{Q}$ copies of $\mathbb{Q}$, we take a circular order consisting of C copies of $\mathbb{Q}$. We regard each copy of $\mathbb{Q}$ as a typical block. Given any locally generic *auto*morphism of C we can then determine a corresponding locally generic *epi*morphism of C copies of $\mathbb{Q}$ which permutes the copies of $\mathbb{Q}$ in exactly the same way and which acts on the copies as typical blocks. Since C copies of $\mathbb{Q}$ is isomorphic to C this provides a corresponding locally generic epimorphism of C.

As a final example, we consider trees, otherwise know as 'semilinear orders', which we take to be partially ordered sets $(T, <)$ such that any two elements have a common lower bound, and for any element, the points below it are linearly ordered. (We could also consider $(T, \leq)$, which would be more involved, since 'collapse' along a branch is possible, as well as across different ones; $(T, <)$ is already rich enough to illustrate what we want to show.) An extensive study of the possible structure of trees with appropriate transitivity assumptions was carried out by Droste in [**4**], and he classified those which are 2-transitive, meaning that for any two isomorphic 2-element substructures, there is an automorphism taking the first to the second. In non-trivial cases, the maximal chains are ordered like $\mathbb{Q}$, and they are of two possible types, 'positive' and 'negative', where the ramification points lie in the structure, or do not do so (in which case they lie in its 'completion', so can be specified by lower cuts), and the ramification order is fixed, but may be any integer ≥ 2 or $\aleph_0$. We write T_k^+, T_k^- for the countable 2-transitive tree of ramification order k and of positive or negative type respectively.

Our reason for studying the $T_k^{\pm}$ particularly is that none of them are IA [**4**] (since not all 4-element antichains are in the same orbit of $\mathrm{Aut}(T,<)$), but the ones of negative type are HH, as is shown in [**1**]. This is slightly misleading however, since each $T_k^{\pm}$ is still $\aleph_0$-categorical, and an expansion of it *is* homogeneous, namely the one where we view it as a lower semilattice. In the case of positive type this is easy to describe; the required expansion in the negative case is a little harder, since the meets are ramification points, which do not lie in the original structure, so have to be represented via an interpretation. In all cases we get generic automorphisms, which shows therefore that a structure does not have to be homogeneous in order for it to have generic automorphisms (though a closely related (biinterpretable) structure *is* homogeneous).

We first attempt to give an informal description of what a generic automorphism g of $(T,<)$ would have to be like. For ease we focus on 2-transitive trees of positive type and infinite ramification order. Since the structure is not homogeneous, we must either work with the lower semilattice in mind, or else restrict attention to partial automorphisms which *can be* extended to automorphisms, which we do now without mentioning it explicitly. First note that any finite partial automorphism can be extended to one with a fixed point, and so it follows that there must be some point x_0 fixed by g, and hence $S = \{x \in T : x < x_0\}$ is fixed setwise. Since S is order-isomorphic to $\mathbb{Q}$, the action of g on S will be as described in Theorem 4.1(ii). On an orbital of S of non-zero parity, the cones must be correspondingly permuted, and g must induce an isomorphism between these cones, and this completely specifies the action of g on these up to conjugacy. An orbital of S of parity 0 is a fixed point, and here, g must permute the cones (and similarly those at x_0), and clearly it acts like a generic member of the infinite symmetric group, that is to say, with infinitely many finite cycles of each possible finite length (and no infinite ones). Furthermore, for a cone C which lies in a cycle of length r, g^r fixes C setwise; the powers of g carry the action on C to the other cones in the cycle, and finally, each cone is isomorphic to the whole of T, and so the action of g^r on C is isomorphic to that of g on T. Once we have chosen the action of g^r on one such cone, its action on each other cone in the cycle is carried to it by the relevant power of g.

This description of g is fine as far as it goes. It is however 'recursive', in that in describing what g is, we have also assumed that we know what it is on subtrees (cones), so for a fuller description, one should unravel this. This seems very involved, and instead, we shall at least demonstrate that generic automorphisms exist by verifying that P has a cofinal subset with the AP. Now, unlike for the generic partial order [**14**], it can be checked that any finite partial automorphism can be extended to one in which all points 'spiral'. Here a point *a spirals* under the action of p if for some integers $m < n$, $p^m a$ and $p^n a$ are defined and are comparable ($p^m a < p^n a$ or $p^m a = p^n a$ or $p^m a > p^n a$). Therefore, the family of all finite partial automorphisms such that all points of its domain and range spiral is cofinal. To find our desired cofinal subset having AP we have to refine this somewhat.

If a spirals, we say it has *parity* $+1$ if for some $i < j$, $p^i a$ and $p^j a$ are defined and $p^i a < p^j a$. Similarly for *parity* -1 or *parity* 0 if for some $i < j$, $p^i a > p^j a$, $p^i a = p^j a$ respectively. First we note that no a can have more than one parity. For suppose for instance that $p^i a < p^j a$ and $p^{i'} a \geq p^{j'} a$ where $i < j$ and $i' < j'$. Choose such i, j, i', j' so that $j - i$ and $j' - i'$ are as small as possible. Since p is a partial automorphism, we may suppose that $i = i'$, and this gives $p^{j'} a \leq p^{i'} a = p^i a < p^j a$, so $j \neq j'$. If $j' < j$ then $j - j' < j - i' = j - i$, contrary to minimality of $j - i$, and if $j < j'$ then $j' - j < j' - i = j' - i'$, contrary to minimality of $j' - i'$. Next let the *spiral length* be the smallest value of $j - i$ for which $i < j$ and $p^i a$ and $p^j a$ are comparable. Finally, we take for our cofinal subset of P all those finite partial automorphisms p of $(T, <)$ such that all members of the domain or range of p spiral, and such that if $a < b$ in $\operatorname{dom} p \cup \operatorname{range} p$, if a and b spiral with the same parity and same spiral length, then there is $c \in \operatorname{dom} p \cup \operatorname{range} p$ such that $a < c < b$ which either spirals with different parity from a and b, or with different spiral length. Then as in [**14**], one sees that this is a cofinal subset of P having the amalgamation property. The key step is to show how to extend a finite partial automorphism to one such that all points of its domain spiral (c.f. Lemma 5.1).

We remark that though we were principally thinking here of the case of positive type and infinite ramification order, the same methods apply in negative type and/or finite ramification order. In the latter case, when we come to the description about the cones at a fixed point, we just have to say that for some such, the cones will both be fixed, and for some they will be permuted, and all possible ways of doing so will occur throughout the tree. So the generic behaviour is exhibited *overall*, but not at each individual vertex.

Concluding remarks

In conclusion we mention some questions which we have so far been unable to resolve. In the first place, we have examples where two of the three relevant monoids, Aut, Bi, and Epi, are equal (the trivial structure and the ordered rationals), and where there are generics in each, and we also have an example (the random graph) in which all three monoids are distinct but there are generics in just two of the three. Presumably there are examples in which all three monoids are distinct, and they all have generic members, but we have not as yet found any such. We had hoped that trees would provide such an example, and they may still do so. Analysis of the situation for Bi or Epi could however be complicated, partly in view of the argument used to show that $(T, <)$ is not IA. We cannot any more add the operation $\wedge$ to the signature, and this is because any endomorphism which preserves $<$ and $\wedge$ must, as one easily sees, be an embedding. So we attempt to work without $\wedge$. Now restrict attention to $T = T_k^-$ (since as remarked in [**1**] T_k^+ is not even HH or MM), and let $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ be antichains of the two different types, so that $a_1 \wedge a_2$ and $a_3 \wedge a_4$ are incomparable, whereas $b_1 \wedge b_2 > b_1 \wedge b_3 > b_1 \wedge b_4$. We may now map $\{x \in T : x < a_i \text{ for some } i\}$ onto $C = \{x \in T : x < b_i \text{ for all } i\}$, even by a monomorphism (since the chains between $a_1 \wedge a_3$ and the a_i may be 'interleaved' to map to C since incomparability is allowed to be destroyed). The map p sending a_i to b_i may now be extended to a bimorphism of T (at least assuming $k = \aleph_0$) by finding other chains above $a_1 \wedge a_3$ to map to the chains $(b_1 \wedge b_4, b_i)$. Full details of the proof that $(T, <)$ is MB (and HE) are omitted; see [**18**].

In section 5.3 we showed that $\mathrm{Mon}(\Gamma)$ has no generics by a diagonalization argument, and then remarked that this could alternatively be derived by appealing directly to the necessary and sufficient conditions given in Theorem 2.3. Presumably it is also possible to show that there are no mutual generics in $\mathrm{Aut}(\mathbb{Q}, \leq)$ from the same theorem; a proof using diagonalization was given in [**26**].

In practice, where we want to verify the existence of generics, without spelling out explicitly what they are like, as in [**14**] for instance, we have used the existence of a cofinal subset of P having the amalgamation property. This raises the question of whether there are cases in which WAP really must be used; in other words, does the existence of a cofinal subset having AP follow from WAP? We remark in passing that the extra condition required in [**26**] that the cofinal subset of P having AP should also be closed under conjugacy is actually unnecessary, since it is easy to check that *if* P has a cofinal subset with AP, then the family of all those $(a, p) \in P$ such that any two extensions of (a, p) in P can be amalgamated over (a, p) is itself cofinal, with AP, and as it is also definable (since we have defined it), it is also closed under conjugacy.

These questions are rather technical. The question we would really like to know the answer to is whether the generic partial order has two mutual generics. The likelihood is that if this could be established, then one could also show that it has 'ample generics' in the sense of [**12**], and the small index property would follow. But this seems quite difficult. The generic partial order shares some features both of the ordered rationals (as its maximal chains have that order-type) and the trivial structure (its maximal antichains), but the relation between the two is quite complex as shown in [**14**]. Related to this, one would also like to know if there are connections between the small index property and the existence of *single* generics.

Finally, we propose that the body of literature on *automorphism groups* (small index property, simplicity, cofinality) should have some counterpart for the case of the monoids, though exactly what this is is not clear. And, although the monoids we have studied seem the most natural, are there others which one should also take into account for other purposes, and do similar constructions apply to them?

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Special pairs and automorphisms of centreless groups

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Dedicated to Rüdiger Göbel on the occasion of his seventieth birthday.

Abstract. Let G be a group, A be a subset of the domain of G and $\mathcal{L}_A$ be the first-order language of group theory expanded by constant symbols for elements in A. We call the pair $\langle G, A\rangle$ *special* if every element g of G is uniquely determined by the set $\mathsf{qft}_{G,A}(g)$ consisting of all $\mathcal{L}_A$-terms $\mathrm{t}(v)$ with one free variable and $\mathrm{t}^G(g) = 1_G$. The pair $\langle G, A\rangle$ is *strongly special* if $\mathsf{qft}_{G,A}(g) \subseteq \mathsf{qft}_{G,A}(h)$ implies $g = h$ for all $g, h \in G$. Special pairs were introduced by Itay Kaplan and Saharon Shelah to analyze automorphism towers of centreless groups. The purpose of this note is the further analysis of special pairs and their interaction with automorphism groups. This analysis will allow us to prove an absoluteness result for the first three stages of the automorphism tower of countable, centreless groups. Moreover, we develop methods that enable us to construct a variety of examples of such pairs, including special pairs that are not strongly special.

1. Introduction

We let $\mathcal{L}_{GT} = \langle *, {}^{-1}, \mathbb{1}\rangle$ denote the first-order language of group theory. Given a group G and a subset A of the domain of G, we define $\mathcal{L}_A$ to be the first-order language that extends $\mathcal{L}_{GT}$ by introducing a new constant symbol $\dot{g}$ for each element g of A. We regard G as an $\mathcal{L}_A$-model in the obvious way.

We define $\mathcal{T}_A$ to be the set of all $\mathcal{L}_A$-terms $t \equiv t(v)$ with exactly one free variable. If g is an element of the domain of G, then we define

$$\mathsf{qft}_{G,A}(g) = \{t(v) \in \mathcal{T}_A \mid G \models \text{“}t(g) = \mathbb{1}\text{”}\}$$

and call this set the *quantifier-free A-type of g*.

Definition 1.1. Given a group G and a subset A of the domain of G, the pair $\langle G, A\rangle$ is *special* if the function

$$\mathsf{qft}_{G,A} : G \longrightarrow \mathcal{P}(\mathcal{T}_A);\ g \longmapsto \mathsf{qft}_{G,A}(g)$$

is injective.

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Special pairs were introduced by Itay Kaplan and Saharon Shelah in [**KS09**] to analyze automorphism towers of centreless groups. Given a special pair $\langle G, A\rangle$, this notion allows us to measure the complexity of the group G by interpreting it as a set of subsets of $\mathcal{T}_A$. For example, if A is countable, then we can easily identify subsets of $\mathcal{T}_A$ with elements of *Cantor space* ${}^{\omega}2$ (i.e. *reals*) and talk about the complexity of G in terms of *descriptive set theory* (i.e. as *definable sets of reals*). This approach is used in [**KS09**] to find new upper bounds for the heights of automorphism towers.

The aim of this note is to further investigate this notion and the following strengthening of it.

DEFINITION 1.2. Given a group G and a subset A of the domain of G, we call the pair $\langle G, A\rangle$ *strongly special* if $\mathsf{qft}_{G,A}(g) \subseteq \mathsf{qft}_{G,A}(h)$ implies $g = h$ for all $g, h \in G$.

We outline the content of this note. In Section 2, we will introduce automorphism towers and quote the results developed in [**KS09**] that connect special pairs with automorphism towers. We will show that those results also hold for strongly special pairs. In the next section, we introduce concepts and results from the theory of *Polish groups* and derive helpful consequences for special pairs consisting of a Polish group and a countable subset of its domain. These results are applied in Section 4 to prove an absoluteness result for the first three stages of automorphism towers of countable, centreless groups. This result contrasts existing non-absoluteness results for the automorphism towers of certain uncountable groups (see [**Tho98**], [**HT00**] and [**FL**]). Section 5 introduces another way to construct strongly special pairs using groups of autohomeomorphisms of certain Hausdorff spaces. This construction relies on methods and results developed by Robert R. Kallman in [**Kal86**]. In the last section, we will use a result of Manfred Droste, Michèle Giraudet and Rüdiger Göbel to show that there are special pairs that are not strongly special.

Notations. Given a group G, we will also use the letter G to denote the domain of G. We denote applications of the group operation by $g \cdot h$ and we will abbreviate the term $g \cdot h \cdot g^{-1}$ by h^g. If A is a subset of the domain of G, then we let $\langle A\rangle$ denote the subgroup of G generated by A.

If f is a function, A is a subset of the domain of f and B is a subset of the range of f, then $f[A]$ is the pointwise image of A under f and $f^{-1}[B]$ denotes the preimage of B under f. Given functions f and g with $\mathrm{ran}(g) \subseteq \mathrm{dom}(f)$, we use $f \circ g$ to denote the corresponding composition of functions.

We let $\mathrm{Sym}(X)$ denote the *symmetric group* of a set X and $\mathrm{Alt}(X)$ denote the corresponding *alternating group* consisting of all finite even permutations of X. If $a, b \in X$, then $(a\ b)$ denotes the *transposition* of a and b.

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2. Automorphism towers

We start this section by introducing automorphism towers of centreless groups. An extensive account of all aspects of the automorphism tower problem can be found in Simon Thomas' forthcoming monograph [**Tho**].

Let G be a group with trivial centre. For each $g \in G$, the *inner automorphism corresponding to g* is the map

$$\iota_g : G \longrightarrow G;\ h \longmapsto h^g.$$

It is easy to see that the map

$$\iota_G : G \longrightarrow \mathrm{Aut}(G);\ g \longmapsto \iota_g$$

is an embedding that maps G onto the subgroup $\mathrm{Inn}(G)$ of all inner automorphisms of G. An easy computation shows that $\pi \circ \iota_g \circ \pi^{-1} = \iota_{\pi(g)}$ holds for all $g \in G$ and $\pi \in \mathrm{Aut}(G)$. This implies that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ and $\mathrm{C}_{\mathrm{Aut}(G)}(\mathrm{Inn}(G)) = \{\mathrm{id}_G\}$. In particular, $\mathrm{Aut}(G)$ is also a group with trivial centre. By iterating this process, we construct the automorphism tower of G.

Definition 2.1. A sequence $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ of groups is *an automorphism tower of a centreless group G* if the following statements hold.

(1) $G = G_0$.
(2) If $\alpha \in \mathrm{On}$, then G_α is a normal subgroup of $G_{\alpha+1}$ and the induced homomorphism

$$\varphi_\alpha : G_{\alpha+1} \longrightarrow \mathrm{Aut}(G_\alpha);\ g \mapsto \iota_g \restriction G_\alpha$$

is an isomorphism.
(3) If $\alpha \in \mathrm{Lim}$, then $G_\alpha = \bigcup\{G_\beta \mid \beta < \alpha\}$.

In this definition, we replaced $\mathrm{Aut}(G_\alpha)$ by an isomorphic copy $G_{\alpha+1}$ that contains G_α as a normal subgroup. This allows us to take unions instead of direct limits at limit stages. Given $\alpha \in \mathrm{On}$, it is easy to see that the α-th group in an automorphism tower of some centreless group G is uniquely determined up to isomorphisms that induce the identity on G.

We are now ready to state the result from [**KS09**] that establishes a connection between automorphism towers and special pairs.

Theorem 2.2 ([**KS09**, Conclusion 3.10]). *Let $\langle G, A\rangle$ be a special pair with $\mathrm{C}_G(A) = \{1_G\}$ and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. If $\alpha \in \mathrm{On}$, then $\langle G_\alpha, A\rangle$ is a special pair and $\mathrm{C}_{G_\alpha}(A) = \{1_G\}$ holds.*

Corollary 2.3. *If $\langle G, A\rangle$ is a special pair with A infinite and $\mathrm{C}_G(A) = \{1_G\}$, then there is an $\alpha < \left(2^{|A|}\right)^+$ with $G_\alpha = G_{\alpha+1}$.*

Proof. Let $\nu = 2^{|A|}$ and assume, toward a contradiction, that $G_\alpha \neq G_{\alpha+1}$ holds for all $\alpha < \nu^+$. Pick a sequence $\langle g_\alpha \mid \alpha < \nu^+\rangle$ with $g_\alpha \in G_{\alpha+1} \setminus G_\alpha$. Given $\alpha < \nu^+$ and $\beta > \alpha$, it is easy to see that $\mathsf{qft}_{G_{\alpha+1},G}(g_\alpha) = \mathsf{qft}_{G_\beta,G}(g_\alpha)$ holds. By Theorem 2.2, $\langle \mathsf{qft}_{G_{\alpha+1},G}(g_\alpha) \mid \alpha < \nu^+\rangle$ is a sequence of pairwise distinct subsets of $\mathcal{T}_A$. But, $\mathcal{T}_A$ has cardinality $|A|$ and there are only ν-many subsets of $\mathcal{T}_A$, a contradiction. □

The above result allows a short proof of Simon Thomas' *automorphism tower theorem.*

Corollary 2.4 ([**Tho98**, Theorem 1.3]). *If G is an infinite centreless group of cardinality κ, then there is an $\alpha < (2^\kappa)^+$ with $G_\alpha = G_{\alpha+1}$.*

Proof. Since $\dot{g} * v^{-1} \in \mathsf{qft}_{G,G}(g)$ holds for all $g \in G$, it is easy to see that $\langle G, G\rangle$ is a special pair with $\mathrm{C}_G(G) = \mathrm{Z}(G) = \{1_G\}$. □

Note that, in the above situation, $G_\alpha = G_{\alpha+1}$ implies $G_\alpha = G_\beta$ for all $\beta \geq \alpha$. The automorphism tower theorem allows us to make the following definition.

Definition 2.5. If G is a centreless group, then we let $\tau(G)$ denote the least ordinal α such that $G_\alpha = G_{\alpha+1}$ holds whenever $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ is an automorphism tower of G.

In the remainder of this section, we will show that the statement of Theorem 2.2 still holds if we replace *special pair* by *strongly special pair*. We start by generalizing the following characterization of special pairs in terms of local homomorphisms to strongly special pairs.

Lemma 2.6 ([**KS09**, Remark 3.5 (1)]). *If G is a group and A is a subset of the domain of G, then the following statements are equivalent.*

(1) *$\langle G, A\rangle$ is a special pair.*
(2) *If $g \in G$ and $\varphi : \langle A \cup \{g\}\rangle \longrightarrow G$ is a group monomorphism with $\varphi \restriction A = \mathrm{id}_A$, then $\varphi(g) = g$.*

This characterization generalizes to strongly special pairs in the following way.

Lemma 2.7. *If G is a group and A is a subset of the domain of G, then the following statements are equivalent.*

(1) *$\langle G, A\rangle$ is a strongly special pair.*
(2) *If $g \in G$ and $\varphi : \langle A \cup \{g\}\rangle \longrightarrow G$ is a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$, then $\varphi(g) = g$.*

Proof. Let $\langle G, A\rangle$ be a strongly special pair, $g \in G$ and $\varphi : \langle A \cup \{g\}\rangle \longrightarrow G$ be a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$. An easy induction shows that $\mathrm{t}^G(g) \in \langle A \cup \{g\}\rangle$ and $\varphi(\mathrm{t}^G(g)) = \mathrm{t}^G(\varphi(g))$ hold for every term $\mathrm{t}(v) \in \mathcal{T}_A$. In particular, $\mathsf{qft}_{G,A}(g) \subseteq \mathsf{qft}_{G,A}(\varphi(g))$ and we can conclude $g = \varphi(g)$.

Assume that the second statement holds. Let $g_0, g_1 \in G$ with $\mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$. Pick $\mathrm{t}_0, \mathrm{t}_1 \in \mathcal{T}_A$ with $\mathrm{t}_0^G(g_0) = \mathrm{t}_1^G(g_0)$. Then $\mathrm{t}_0 * \mathrm{t}_1^{-1} \in \mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$ and $\mathrm{t}_0^G(g_1) = \mathrm{t}_1^G(g_1)$. Given $h \in \langle A \cup \{g_0\}\rangle$, there is a term $\mathrm{t}(v) \in \mathcal{T}_A$ with $\mathrm{t}^G(g_0) = h$ and, if we define $\varphi(h) = \mathrm{t}^G(g_1)$, then the above computations show that $\varphi(h)$ does not depend on the choice of t. Moreover, these computations directly imply that $\varphi : \langle A \cup \{g_0\}\rangle \longrightarrow G$ is a group homomorphism with $\varphi(g_0) = g_1$ and $\varphi \restriction A = \mathrm{id}_A$. By our assumption, we have $g_0 = g_1$. □

This characterization allows us to prove a version of [**KS09**, Claim 3.8] for strongly special pairs. Note that the proofs of the two statements are almost identical.

Lemma 2.8. *Let $\langle G, A\rangle$ be a strongly special pair and H be a group such that G is a normal subgroup of H and $\mathrm{C}_H(G) = \{1_G\}$. Then $\langle H, A\rangle$ is a strongly special pair.*

PROOF. Let $h \in H$ and $\varphi : \langle A \cup \{h\}\rangle \longrightarrow H$ be a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$. Pick $a \in A$. Then $a^h \in G$, $\varphi(a^h) = a^{\varphi(h)} \in G$ and, if we define $\psi = \varphi \restriction \langle A \cup \{a^h\}\rangle$, then $\psi : \langle A \cup \{a^h\}\rangle \longrightarrow G$ is a group homomorphism with $\psi \restriction A = \mathrm{id}_A$. By our assumption, we have $a^h = \psi(a^h) = a^{\varphi(h)}$. This argument shows $h \cdot \varphi(h^{-1}) \in \mathrm{C}_H(A)$.

Now fix $g \in G$ and define $\xi : \langle A \cup \{g\}\rangle \longrightarrow G$ by $\xi = \iota_{h\cdot\varphi(h^{-1})} \restriction \langle A \cup \{g\}\rangle$. By the above computations, we have $\xi \restriction A = \mathrm{id}_A$ and this means $g = \xi(g) = g^{h\cdot\varphi(h^{-1})}$. We can conclude $h \cdot \varphi(h^{-1}) \in \mathrm{C}_H(G) = \{1_G\}$ and $h = \varphi(h)$. □

We are now ready to prove the promised version of Theorem 2.2 for strongly special pairs. Again, the proofs of both results are almost identical.

THEOREM 2.9. *Let $\langle G, A\rangle$ be a strongly special pair with $\mathrm{C}_G(A) = \{1_G\}$ and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. If $\alpha \in \mathrm{On}$, then $\langle G_\alpha, A\rangle$ is a strongly special pair.*

PROOF. We prove the statement of the theorem by induction.

Assume $\langle G_\alpha, A\rangle$ is a strongly special pair. If $h \in \mathrm{C}_{G_{\alpha+1}}(G_\alpha)$, then $\iota_h \restriction G_\alpha = \mathrm{id}_{G_\alpha}$ and $h = 1_G$. Since G_α is a normal subgroup of $G_{\alpha+1}$, we can apply Lemma 2.8 to see that $\langle G_{\alpha+1}, A\rangle$ is also a strongly special pair.

Let α be a limit ordinal and assume that $\langle G_\beta, A\rangle$ is a strongly special pair for every $\beta < \alpha$. Given $g_0, g_1 \in G_\alpha$ with $\mathsf{qft}_{G_\alpha,A}(g_0) \subseteq \mathsf{qft}_{G_\alpha,A}(g_1)$, there is a $\beta < \alpha$ with $g_0, g_1 \in G_\beta$ and it is easy to see that $\mathsf{qft}_{G_\alpha,A}(g_i) = \mathsf{qft}_{G_\beta,A}(g_i)$. In particular, we have $g_0 = g_1$. □

3. Unique Polish group topologies

We introduce techniques from the theory of *Polish groups* that will be essential for the proof of the absoluteness result for the automorphism towers of countable, centreless groups mentioned in the introduction. Remember that a *topological group* is a pair $\langle G, \tau\rangle$ consisting of a group G and a topology τ on the domain of G such that the map $\left[\langle g, h\rangle \mapsto g \cdot h^{-1}\right]$ is continuous with respect to τ. We call a topological space $\langle X, \tau\rangle$ *Polish* if τ is induced by a complete metric on X and there is a countable subset of X that is dense in τ. Finally, we call a topological group $\langle G, \tau\rangle$ a *Polish group* if the corresponding topological space is Polish. In this case, we call τ a *Polish group topology* on G.

PROPOSITION 3.1. *Let $\langle G, \tau\rangle$ be a topological group such that the corresponding topological space is a Hausdorff space. If $\mathrm{t} \in \mathcal{T}_G$, then the set $\{g \in G \mid \mathrm{t}^G(g) = 1_G\}$ is closed in τ.*

PROOF. An easy induction on shows that the map

$$\xi_{\mathrm{t}} : G^n \longrightarrow G;\ \vec{g} \longmapsto \mathrm{t}^G(\vec{g})$$

is continuous with respect to τ for every $\mathcal{L}_G$-term $\mathrm{t} \equiv \mathrm{t}(v_0, \ldots, v_{n-1})$ with n free variables. Since τ is a Hausdorff space, we can conclude that the set

$$\{g \in G \mid \mathrm{t}^G(g) = 1_G\} = \xi_{\mathrm{t}}^{-1}[\{1_G\}]$$

is closed in τ for every $t \in \mathcal{T}_G$. □

Next, we consider Polish groups whose topology is completely determined by the algebraic structure of the group.

Definition 3.2. Let G be a group. We say that G has a *unique Polish group topology* if there is exactly one topology τ on the domain of G such that $\langle G, \tau\rangle$ is a Polish group.

We state a theorem of George W. Mackey that allows a nice characterization of groups with unique Polish group topologies. Remember that a measurable space $\langle X, \mathcal{S}\rangle$ is a *standard Borel space* if there is a Polish topology τ on X such that $\mathcal{S}$ is equal to the σ-algebra $\mathcal{B}(\tau)$ of all subsets of X that are Borel with respect to τ.

Theorem 3.3 ([**Mac57**, Theorem 3.3]). *Let $\langle X, \mathcal{S}_0\rangle$ and $\langle X, \mathcal{S}_1\rangle$ be standard Borel spaces. If there is a countable point-separating family*[1] *of subsets of X whose members are elements of both $\mathcal{S}_0$ and $\mathcal{S}_1$, then $\mathcal{S}_0 = \mathcal{S}_1$.*

Corollary 3.4. *The following statements are equivalent for a Polish group $\langle G, \tau\rangle$.*

(1) *τ is the unique Polish group topology on G.*
(2) *There is a countable point-separating family of subsets of the domain of G whose members are Borel with respect to any Polish group topology on G.*

Proof. If τ is the unique Polish group topology on G and B is a countable basis of τ, then B satisfies the above properties.

In the other direction, assume that $\mathcal{F}$ is a family of subsets with the above properties and $\bar{\tau}$ is a Polish group topologies on G. If we define $\mathcal{B}(\tau)$ and $\mathcal{B}(\bar{\tau})$ as above, then Theorem 3.3 and our assumptions imply $\mathcal{B}(\tau) = \mathcal{B}(\bar{\tau})$. Since Borel sets have the Baire Property (see [**Kec95**, Proposition 8.22]), the identity map on G is a Baire-measurable group homomorphism with respect to τ and $\bar{\tau}$. By [**BK96**, Theorem 1.2.6], it is continuous and open with respect to τ and $\bar{\tau}$. This shows $\tau = \bar{\tau}$. □

Proposition 3.5. *Let $\langle G, \tau\rangle$ be a Polish group. If there is a countable subset A of the domain of G such that $\langle G, A\rangle$ is a special pair, then τ is the unique Polish group topology on G.*

Proof. If $\mathrm{t} \equiv \mathrm{t}(v)$ is a term in $\mathcal{T}_A$, then we define $T^0_{\mathrm{t}} = \{g \in G \mid \mathrm{t}^G(g) = 1_G\}$ and $T^1_{\mathrm{t}} = \{g \in G \mid \mathrm{t}^G(g) \neq 1_G\}$. Let $\mathcal{F}$ denote the family consisting of all subsets of the domain of G of the form T^0_{t} or T^1_{t} for some $\mathrm{t} \in \mathcal{T}_A$. Then $\mathcal{F}$ is countable and separates points, because $\langle G, A\rangle$ is a special pair. If $\bar{\tau}$ is a Polish group topology on G, then all elements of $\mathcal{F}$ are contained in $\mathcal{B}(\bar{\tau})$ by Proposition 3.1. Corollary 3.4 implies that τ is the unique Polish group topology on G. □

Remark 3.6. The converse of the above implication is not true: Bojana Pejić and Paul Gartside showed that the group $\mathrm{SO}(3, \mathbb{R})$ has a unique Polish group topology (see [**GP08**, Theorem 11]) and there is no countable subset I of $\mathcal{T}_{\mathrm{SO}(3,\mathbb{R})}$ such that the family $\{T^i_{\mathrm{t}} \mid \mathrm{t} \in I,\ i < 2\}$ separates points (see [**GP08**, Lemma 12]).

We close this section by introducing a consequence of the existence of a unique Polish group topology that allows us to deduce the absoluteness result in the next section. This consequence is called *automatic continuity of automorphisms*.

[1] We call a family $\mathcal{F}$ of subsets of X *separating* if for any pair $\langle x, y\rangle$ of distinct elements in X, there is an $F \in \mathcal{F}$ with $x \in F$ and $y \notin F$.

PROPOSITION 3.7. *Let G be a group with a unique Polish group topology. Then every group automorphism of G is continuous with respect to the unique Polish group topology on G.*

PROOF. Let τ be the unique Polish group topology on G and assume, toward a contradiction, that there is an automorphism π of G that is not continuous with respect to τ. Define $\bar{\tau}$ to be the collection of all subsets of G of the form $\pi[U]$, where U is open in τ. It is easy to check that $\bar{\tau}$ is a Polish group topology that is not equal to τ, a contradiction. □

4. An absoluteness result for automorphism towers of countable centreless groups

The aim of this section is to prove the following theorem.

THEOREM 4.1. *Let M be a transitive model of* ZFC[2], *G be a centreless group that is an element of M, $\langle G_\alpha^M \mid \alpha \in \mathrm{On} \cap M\rangle$ be an automorphism tower of G in M and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. If G is countable in M, then there is an embedding $\pi : G_2^M \longrightarrow G_2$ with $\pi \restriction G = \mathrm{id}_G$.*

This theorem directly implies the following absoluteness result for automorphism towers of countable centreless groups.

COROLLARY 4.2. *Let M be a transitive model of* ZFC, *G be a centreless group that is an element of M and $\langle G_\alpha^M \mid \alpha \in \mathrm{On} \cap M\rangle$ be an automorphism tower of G in M. If G is countable in M and $G_1^M \neq G_2^M$, then $\tau(G) > 1$.*

PROOF OF THE COROLLARY FROM THEOREM 4.1. Let $\pi : G_2^M \longrightarrow G_2$ be the embedding given by Theorem 4.1. It suffices to show that $\pi^{-1}[G_1] \subseteq G_1^M$ holds.

Let $h \in G_2^M$ with $\pi(h) \in G_1$. Given $g \in G$, we have $\iota_{\pi(h)}(g) \in G$ and therefore $\pi(\iota_{\pi(h)}(g)) = \iota_{\pi(h)}(g) = \iota_{\pi(h)}(\pi(g)) = \pi(\iota_h(g))$. Since π is an embedding, we can conclude that $\iota_h(g) = \iota_{\pi(h)}(g)$ holds for all $g \in G$ and hence $\iota_h \restriction G = \iota_{\pi(h)} \restriction G \in \mathrm{Aut}(G) \cap M$. By the definition of G_2^M, there is an $\bar{h} \in G_1^M$ with $\iota_{\bar{h}} \restriction G = \iota_h \restriction G$ and this shows $h^{-1} \cdot \bar{h} \in \mathrm{C}_{G_2^M}(G)$. An application of Theorem 2.2 in M yields $h = \bar{h} \in G_1^M$. □

The above result should be compared with the following non-absoluteness result due to Simon Thomas.

THEOREM 4.3 ([**Tho98**, Theorem 2.4]). *There exists a group G with $\tau(G) = 2$ and $1_{\mathbb{P}} \Vdash$ "$\tau(\check{G}) = 1$" for every notion of forcing $\mathbb{P}$ that adds a real.*

We outline how the results of Section 3 can be applied to analyze the first stages of the automorphism tower of a countable, centreless group. If $\mathcal{L}$ is a first-order language and $\mathcal{M}$ is an $\mathcal{L}$-model with domain ω, then $\mathrm{Aut}(\mathcal{M})$ is a subset of Baire space ${}^\omega\omega$ and the corresponding subspace topology induces a Polish group topology on $\mathrm{Aut}(\mathcal{M})$ (see [**Kec95**, Example 9.B 7]). If B is the family of subsets of $\mathrm{Aut}(\mathcal{M})$ of the form $\{\sigma \in \mathrm{Aut}(\mathcal{M}) \mid \pi \restriction X = \sigma \restriction X\}$ for some $\pi \in \mathrm{Aut}(\mathcal{M})$ and a finite subset X of ω, then B forms a countable basis of this group topology.

[2]Note that M can be set-sized or even countable. In addition, we only need to assume that M is a transitive model of a "*suitable*" finite fragment of ZFC which enables us to run all the arguments of this section that take place inside of M.

Let G be a countable group and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. Let B denote the family of all subsets of G_1 of the form $\{h \in G_1 \mid \iota_g \upharpoonright X = \iota_h \upharpoonright X\}$ for some $g \in G_1$ and a finite subset X of G. By the above remarks, B is a countable basis of a Polish group topology on G_1. Moreover, Theorem 2.2 and Proposition 3.5 imply that this is the unique Polish group topology on G_1 and $\iota_\pi \upharpoonright G_1$ is continuous with respect to this topology for every $\pi \in G_2$ by Proposition 3.7.

The following folklore result is the last ingredient in our proof of Theorem 4.1. A proof of this statement can be found in [**BK96**, page 6].

Proposition 4.4. *Let $\langle G, \tau\rangle$ be a Polish group, H be a subgroup of G that is dense in τ and $\varphi : H \longrightarrow G$ be a group homomorphism that is continuous with respect to the subspace topology induced by τ on H and τ. Then there is a unique group homomorphism $\varphi^* : G \longrightarrow G$ that extends φ and is continuous with respect to τ.*

Proof of Theorem 4.1. Let M be a transitive model of ZFC, G be a centreless group with domain ω contained in M, $\langle G^M_\alpha \mid \alpha \in \mathrm{On} \cap M\rangle$ be an automorphism tower of G in M and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. Since every automorphism of G in M is an automorphism of G, we may replace G_1 by an isomorphic copy and assume that G^M_1 is a subgroup of G_1. We fix the following collections of sets.

(1) Let τ denote the unique Polish group topology on G_1.
(2) Let τ^M denote the unique Polish group topology on G^M_1 in M.
(3) Let $\bar{\tau}$ denote the subspace topology induced by τ on G^M_1.

Note that τ^M is contained in $\bar{\tau}$, because every basic open set in τ^M is an element of $\bar{\tau}$.

Remember that a *tree on* ω^n is a set T of n-tuples of finite sequences of natural numbers with the following properties.

(a) If $\langle t_0, \dots, t_{n-1}\rangle \in T$, then $\mathrm{lh}(t_0) = \dots = \mathrm{lh}(t_{n-1})$.
(b) If $\langle t_0, \dots, t_{n-1}\rangle \in T$ and $m < \mathrm{lh}(t_0)$, then $\langle t_0 \upharpoonright m, \dots, t_{n-1} \upharpoonright m\rangle \in T$.

Given a tree T on ω^n and $\vec{x} = \langle x_0, \dots, x_{n-1}\rangle \in ({}^\omega\omega)^n$, we call $\vec{x}$ a *cofinal branch through* T if $\langle x_0 \upharpoonright m, \dots, x_{n-1} \upharpoonright m\rangle \in T$ for every $m < \omega$.

Let $U = \{h \in G_1 \mid \iota_g \upharpoonright X = \iota_h \upharpoonright X\}$ be a nonempty basic open set in τ with $g \in G_1$ and X is a finite subset of ω. Then both X and $\iota_g \upharpoonright X$ are elements of M and there is a tree T on $\omega \times \omega$ in M such that every cofinal branch through T is of the form $\langle x, y\rangle \in {}^\omega\omega \times {}^\omega\omega$ with $x, y \in \mathrm{Aut}(G)$, $y = x^{-1}$ and $\iota_g \upharpoonright X \subseteq x$. It is easy to see that this property is absolute between transitive ZFC-models. Since U is nonempty, there is a cofinal branch through T and, by Mostowski's Absoluteness Theorem (see [**Jec03**, Theorem 25.4]), there is a branch through T that is an element of M. We can conclude $G^M_1 \cap U \neq \emptyset$. This argument shows that G^M_1 is dense in τ.

Fix $h \in G^M_2$. Let U be a basic open set in τ defined by $g \in G_1$ and $X \subset \omega$ as above. The above computations show that we may assume $g \in G^M_1$ and

$$U \cap G^M_1 = \{h \in G^M_1 \mid \iota_g \upharpoonright X = \iota_h \upharpoonright X\}$$

is a basic open set in τ^M. The subset

$$(\iota_h^{-1} \upharpoonright G^M_1)[U] = (\iota_h^{-1} \upharpoonright G^M_1)[G^M_1 \cap U]$$

is an element of τ^M, because $\iota_h \upharpoonright G^M_1$ is continuous with respect to τ^M in M. By the above remarks, the subset is also an element of $\bar{\tau}$. This shows that the map

$\iota_h \restriction G_1^M : G_1^M \longrightarrow G_1$ is a group homomorphism that is continuous with respect to $\bar{\tau}$ and τ. By Proposition 4.4, there is a unique group homomorphism $h^* : G_1 \longrightarrow G_1$ that extends $\iota_h \restriction G_1^M$ and is continuous with respect to τ.

For all $h \in G_2^M$, the map $(h^{-1})^* \circ h^*$ is the identity on the dense subset G_1^M and is therefore the identity on G_1. This shows $h^* \in \mathrm{Aut}(G_1)$ with $(h^*)^{-1} = (h^{-1})^*$. We let $\pi(h)$ denote the unique element of G_2 with $h^* = \iota_{\pi(h)} \restriction G_1$. This means $\iota_{\pi(h)} \restriction G_1^M = \iota_h \restriction G_1^M$ and π is injective. Moreover, if $g \in G_1^M \subseteq G_1$, then $\iota_{\pi(g)} \restriction G = \iota_g \restriction G$ and this shows $g = \pi(g)$.

Given $h_0, h_1 \in G_2^M$, our definitions imply that $\iota_{\pi(h_0 \cdot h_1)}$ is equal to $\iota_{\pi(h_0) \cdot \pi(h_1)}$ on G_1^M and therefore on G_1. This shows $\pi(h_0 \cdot h_1) = \pi(h_0) \cdot \pi(h_1)$ holds for all $h_0, h_1 \in G_2^M$ and π is a group homomorphism. □

5. Groups of autohomeomorphism

In this section, we produce a variety of examples of strongly special pairs using certain group actions on Hausdorff spaces. Given a group G that consists of autohomeomorphisms of a Hausdorff space and satisfies a *locally movability condition*, we will construct a subset A of the domain of G such that $\langle G, A\rangle$ is strongly special pair and the cardinality of A is equal to the cardinality of a basis of the corresponding Hausdorff space.

DEFINITION 5.1. Let G be a group and $\langle X, \tau\rangle$ be a Hausdorff space. We say that G *acts locally mixing on* $\langle X, \tau\rangle$ if the following statements hold.

(1) G is a subgroup of the group $\mathcal{H}(\tau)$ of all autohomeomorphisms of $\langle X, \tau\rangle$.
(2) If U is an element of τ and consists of more than one point, then there is a $g \in G \setminus \{1_G\}$ with $g \restriction (X \setminus U) = \mathrm{id}_{X\setminus U}$.

This condition also appears in the study of topological spaces that can be reconstructed from their autohomeomorphism groups (see [**Rub89**]).

We present some easy examples of autohomeomorphism groups acting locally mixing on the corresponding topological space. Given a topological space $\langle X, \tau\rangle$ and a subset A of X, we let $\bar{A}$ denote the closure of A with respect to τ, δA denote the boundary of A with respect to τ and τ_A denote the corresponding subspace topology on A induced by τ.

PROPOSITION 5.2. *Let $\langle X, \tau\rangle$ be a Hausdorff space. Assume that for every subset U in τ with at least two points, there is a $V \subseteq U$ in τ such that $\bar{V} \subseteq U$ and $\langle \bar{V}, \tau_{\bar{V}}\rangle$ has a nontrivial autohomeomorphisms π with $\pi \restriction \delta V = \mathrm{id}_{\delta V}$. Then $\mathcal{H}(\tau)$ acts locally mixing on $\langle X, \tau\rangle$.*

PROOF. Let U be an element of τ with more than one point. Pick V and π as above and define $\pi^* = \pi \cup \mathrm{id}_{X\setminus\bar{V}}$. We show that π^* is continuous with respect to τ in every $x \in X$.

If $x \in X \setminus \bar{V}$, then this statement is trivial, because $\pi^* \restriction (X \setminus \bar{V}) = \mathrm{id}_{X\setminus\bar{V}}$ and $X \setminus \bar{V}$ is open. Given $x \in \delta V$ and W_1 open in τ with $x = \pi^*(x) \in W_1$, there is $\tilde{W}_0$ in $\tau_{\bar{V}}$ with $x \in \tilde{W}_0$ and $\tilde{W}_0 \subseteq \pi^{-1}[\bar{V} \cap W_1]$. Pick W_0 in τ with $\tilde{W}_0 = \bar{V} \cap W_0$. Then $x \in W_0 \cap W_1$ and $W_0 \cap W_1 \subseteq {\pi^*}^{-1}[W_1]$. Finally, if $x \in V$ and W_1 is open in τ with $\pi^*(x) \in W_1$, then $\pi(x) = \pi^*(x) \in V \cap W_1$ and there is $\tilde{W}_0$ in $\tau_{\bar{V}}$ with $x \in \tilde{W}_0$ and $\tilde{W}_0 \subseteq \pi^{-1}[V \cap W_1]$. Pick W_0 in τ with $\tilde{W}_0 = \bar{V} \cap W_0$. Then $x \in V \cap W_0$ and $V \cap W_0 \subseteq {\pi^*}^{-1}[W_1]$. □

EXAMPLE 5.3. Let $\langle X, \tau\rangle$ be an n-dimensional topological manifold. If U is an element of τ and $x \in U$, then there is a W in τ with $x \in W$ and $\langle W, \tau_W\rangle$ is homeomorphic to an open Euclidean n-ball. The preimage of $U \cap W$ under this homeomorphism is nonempty and therefore contains an open n-ball. This shows that there is a V in τ such that $\bar{V} \subseteq U \cap W \subseteq U$ and there is an homeomorphism of $\langle \bar{V}, \tau_{\bar{V}}\rangle$ and $[-1,1]^n$ that maps δV onto the boundary of $[-1,1]^n$ in $\mathbb{R}^n$. There are nontrivial autohomeomorphisms of $[-1,1]^n$ that map its boundary in $\mathbb{R}^n$ onto itself and, by the above calculations, this shows that $\mathcal{H}(\tau)$ acts locally mixing on $\langle X, \tau\rangle$.

EXAMPLE 5.4. Remember that a partial order $\mathbb{P} = \langle P, <_{\mathbb{P}}\rangle$ is a *tree* if the set $prec(p) = \{q \in P \mid q <_{\mathbb{P}} p\}$ is a well-ordered by $<_{\mathbb{P}}$ for every $p \in P$. Given a tree $\mathbb{T} = \langle T, <_{\mathbb{T}}\rangle$, we call a subset of T a *branch through* $\mathbb{T}$ if it is linearly ordered by $<_{\mathbb{T}}$ and downwards-closed. We let $[\mathbb{T}]$ denote the set of all maximal branches through T. Let $\tau_{\mathbb{T}}$ denote the topology on $[\mathbb{T}]$ generated by basic open sets of the form $U_t = \{b \in [\mathbb{T}] \mid t \in b\}$ with $t \in T$.

Let $\mathbb{T} = \langle T, <_{\mathbb{T}}\rangle$ be a tree with the property that for every $t \in T$ there is an automorphism π of $\mathbb{T}$ with $\pi(t) = t$ and $\pi(s) \neq s$ for some $s \in T$ with $t <_{\mathbb{T}} s$. We show that $\mathcal{H}(\tau_{\mathbb{T}})$ acts locally mixing on $\langle [\mathbb{T}], \tau_{\mathbb{T}}\rangle$. By Proposition 5.2, it suffices to show that the space $\langle U_t, (\tau_{\mathbb{T}})_{U_t}\rangle$ has a nontrivial autohomeomorphism for every $t \in T$, because

$$[\mathbb{T}] \setminus U_t = \bigcup \{U_s \mid s \textit{ and } t \textit{ are incompatible in } \mathbb{T}\}$$

and this shows that U_t is also closed in $\tau_{\mathbb{T}}$. If $t \in T$ and $\pi \in \mathrm{Aut}(\mathbb{T})$ with $\pi(t) = t$ and $\pi(s) \neq s$ for some $s \in T$ with $t <_{\mathbb{T}} s$, then we define $\pi^*(b) = \pi[b]$ for every $b \in U_t$. It is easy to check that $\pi^* : U_t \longrightarrow U_t$ is continuous with respect to $(\tau_{\mathbb{T}})_{U_t}$ and if $s \in b \in U_t$, then $\pi^*(b) \neq b$, because $\pi(s) <_{\mathbb{T}} s$ or $s <_{\mathbb{T}} \pi(s)$ would contradict the well-foundedness of $<_{\mathbb{T}}$ below s.

In particular, if α is an ordinal, X is a set with at least two elements and ${}^{<\alpha}X$ is the tree consisting of functions f with $\mathrm{dom}(f) \in \alpha$ and $\mathrm{ran}(f) \subseteq X$ ordered by inclusion, then $[{}^{<\alpha}X]$ can be identified with the set ${}^{\alpha}X$ of all functions from α to X and the group of autohomeomorphisms of the corresponding topological space acts locally mixing on it.

EXAMPLE 5.5. Let $\mathbb{L} = \langle L, <_{\mathbb{L}}\rangle$ be a linear order without end-points that has a nontrivial automorphism and the property that every nonempty, open interval $(a, b) = \{l \in L \mid a <_{\mathbb{L}} l <_{\mathbb{L}} b\}$ is order-isomorphic to $\mathbb{L}$. If $\tau_{\mathbb{L}}$ denotes the order-topology on $\mathbb{L}$, then Proposition 5.2 directly implies that $\mathrm{Aut}(\mathbb{L})$ acts locally mixing on $\langle L, \tau_{\mathbb{L}}\rangle$. In particular, the group of order-preserving bijections of the rational numbers $\mathbb{Q}$ acts locally mixing on $\mathbb{Q}$ equipped with the order topology.

We use methods and computations from Robert R. Kallman's proof of [**Kal86**, Theorem 1.1] to derive the following result.

THEOREM 5.6. *Let G be a group, $\langle X, \tau\rangle$ be a Hausdorff space and* B *be a basis of τ. If G acts locally mixing on $\langle X, \tau\rangle$ and $\langle X, \tau\rangle$ does not have exactly two isolated points, then there is a subset A of the domain of G of cardinality* $|\mathrm{B}| + \aleph_0$ *such that $\langle G, A\rangle$ is a strongly special pair and* $\mathrm{C}_G(A) = \{1_G\}$.

For the rest of this section, we fix a Hausdorff space $\langle X, \tau\rangle$, a basis B of τ and a group G that acts locally mixing on $\langle X, \tau\rangle$. Given $Y \subseteq X$, we define

$$\mathrm{Sub}_{\mathrm{B}}(Y) = \{U \in \mathrm{B} \mid U \subseteq Y,\ |U| > 1\}.$$

and define $\bar{Y}$ to be the closure of Y with respect to τ. Finally, we fix a sequence $\langle g_U \in G \setminus \{1_G\} \mid U \in \mathrm{Sub}_{\mathrm{B}}(X)\rangle$ such that $g_U \restriction (X \setminus U) = \mathrm{id}_{X\setminus U}$ holds for all $U \in \mathrm{Sub}_{\mathrm{B}}(X)$.

In the following, we adopt the arguments of [**Kal86**, Section 2] to our setting to prove Theorem 5.6.

Lemma 5.7. *Let U be open in τ such that U contains either no points isolated in τ or more than two points isolated in τ. The following statements are equivalent for all $h \in G$.*

(1) $h \restriction \bar{U} = \mathrm{id}_{\bar{U}}$.
(2) $g_{U'}^h = g_{U'}$ *holds for all* $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$.

Proof. Assume $h \restriction \bar{U} = \mathrm{id}_{\bar{U}}$ and fix $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$. Then $h \circ g_{U'} = g_{U'} \circ h$ holds, because we have $g_{U'} \restriction (X \setminus \bar{U}) = \mathrm{id}_{\bar{U}}$.

Now, assume that $g_{U'}^h = g_{U'}$ holds for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$. By the continuity of h, it suffices to show $h \restriction U = \mathrm{id}_U$. Let I_U denote the set of all points in U that are isolated in τ. We start by showing $h \restriction I_U = \mathrm{id}_{I_U}$. If U contains no isolated points, then this is trivial. We may therefore assume $|I_U| > 2$.

Assume, toward a contradiction, that there is an $a \in I_U$ with $h(a) \neq a$. We can find distinct $b_0, b_1 \in I_U$ with $a \notin \{b_0, b_1\}$. Then $\{a, b_i\} \in \mathrm{Sub}_{\mathrm{B}}(U)$ and $g_{\{a,b_i\}} = (a\ b_i)$. Our first assumption yields $(a\ b_i)^h = (a\ b_i)$ and this implies $h[\{a, b_i\}] = \{a, b_i\}$. We can conclude $b_0 = h(a) = b_1$, a contradiction. This shows $h \restriction I_U = \mathrm{id}_{I_U}$.

Assume, toward a contradiction, that there is an $x \in U$ with $h(x) \neq x$. Since x is not isolated in τ and $\langle X, \tau\rangle$ is a Hausdorff space, we can find $V \in \mathrm{Sub}_{\mathrm{B}}(U)$ with $V \cap h[V] = \emptyset$. If $y \in V$ with $g_V(y) \neq y$, then $g_V^h = g_V$, $g_V(h(y)) = h(y)$ and therefore $h(y) = (g_V \circ h)(y) = (h \circ g_V)(y) \neq h(y)$, a contradiction. □

Set $A = \{g_U \mid U \in \mathrm{Sub}_{\mathrm{B}}(X)\}$ and, for all $U, V \in \mathrm{Sub}_{\mathrm{B}}(X)$, we define

$$\mathrm{t}_{U,V}(v) \equiv v * \dot{g}_U * v^{-1} * \dot{g}_V * v * \dot{g}_U^{-1} * v^{-1} * \dot{g}_V^{-1} \in \mathcal{T}_A.$$

Lemma 5.8. *Let U and V be open subsets in τ. Assume that both U and $X \setminus \bar{V}$ contain either no points isolated in τ or more than two points isolated in τ. Then the following statements are equivalent for all $h \in G$.*

(1) $\mathrm{t}^G_{U',V'}(h) = 1_G$ *for all* $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ *and* $V' \in \mathrm{Sub}_{\mathrm{B}}(X \setminus \bar{V})$.
(2) $h[\bar{U}] \subseteq \bar{V}$.

Proof. The first statement is equivalent to $g_{U'}^h \circ g_{V'} = g_{V'} \circ g_{U'}^h$ for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ and $V' \in \mathrm{Sub}_{\mathrm{B}}(X\setminus\bar{V})$. By Lemma 5.7, this is equivalent to $g_{U'}^h \restriction (X\setminus\bar{V}) = \mathrm{id}_{X\setminus\bar{V}}$ for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ and we can reformulate this to

(1)* $(g_{U'} \circ h^{-1}) \restriction (X \setminus \bar{V}) = h^{-1} \restriction (X \setminus \bar{V})$ for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$.

By our assumptions, the set of all points which are moved by some $g_{U'}$ with $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ is dense in U with respect to τ. This shows that (1)* is equivalent to $U \cap h^{-1}[X \setminus \bar{V}] = \emptyset$. This statement holds if and only if $h[U] \subseteq \bar{V}$ and this is equivalent to the second statement of the lemma. □

Proof of Theorem 5.6. We may assume that B is closed under finite unions. By our assumptions, there are not exactly two points in X which are isolated in τ. If there is exactly one point $x_0 \in X$ which is isolated in τ, then it is easy to check that there is a group isomorphic to G that acts locally mixing on $\langle X \setminus \{x_0\}, \tau^*\rangle$, where τ^* is the subspace topology induced by τ. We may therefore assume that there are either no points isolated in τ or more than two.

Pick $g_0, g_1 \in G$ with $\mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$ and assume, toward a contradiction, that $g_0 \neq g_1$ holds. Then $U = \{x \in X \mid g_0(x) \neq g_1(x)\}$ is nonempty and open in τ. Let I_U denote the set of all points in U that are isolated in τ.

First, assume that there is an $x \in U \setminus I_U$. We can find disjoint subsets V_0 and V_1 in B such that $g_i(x) \in V_i$ for $i < 2$ and $X \setminus \bar{V}_0$ contains either no points isolated in τ or more than two. Now we can find $U' \in$ B with $x \in U'$, $g_i[U'] \subseteq V_i$ and U' contains either no points isolated in τ or more than two. This means $g_0[\bar{U}'] \subseteq \bar{V}_0$ and we can apply Lemma 5.8 to conclude $\mathrm{t}_{U'',V'} \in \mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$ for all $U'' \in \mathrm{Sub}_{\mathrm{B}}(U')$ and $V' \in \mathrm{Sub}_{\mathrm{B}}(X \setminus \bar{V}_0)$. Another application of the lemma yields $g_1[\bar{U}'] \subseteq \bar{V}_0$ and this means $g_1(x) \in \bar{V}_0 \subseteq X \setminus V_1$, a contradiction.

This shows $I_U = U \neq \emptyset$. Pick $x \in I_U$. By the above assumptions, we can find distinct $y_0, y_1 \in X$ isolated in τ with $x \notin \{y_0, y_1\}$. For all $i < 2$, we have $\{x, y_i\}, \{g_0(x), g_0(y_i)\} \in$ B, $g_{\{x,y_i\}} = (x\ y_i)$ and

$$g_{\{x,y_i\}}^{g_0} = (g_0(x)\ g_0(y_i)) = g_{\{g_0(x),g_0(y_i)\}}.$$

The above equalities allow us to conclude

$$v * \dot{g}_{\{x,y_i\}} * v^{-1} * \dot{g}_{\{g_0(x),g_0(y_i)\}} \in \mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1).$$

In particular, $g_1[\{x, y_i\}] = \{g_0(x), g_0(y_i)\}$ and this shows $g_1(x) = g_0(y_i)$, because $g_1(x) \neq g_0(x)$. We can conclude $g_0(y_0) = g_1(x) = g_0(y_1)$ and therefore $y_0 = y_1$, a contradiction.

If $h \in \mathrm{C}_G(A)$, then $g_U^h = g_U$ holds for all $U \in \mathrm{Sub}_{\mathrm{B}}(X)$. By our assumptions and the above remark, we can apply Lemma 5.7 to conclude $h = \mathrm{id}_X = 1_G$. □

6. Special pairs that are not strongly special

This section contains the construction of special pairs that are not strongly special using simple groups as building blocks. A theorem of Manfred Droste, Michèle Giraudet and Rüdiger Göbel will allow us to prove the following result.

Theorem 6.1. *If κ is an uncountable regular cardinal, then there is a special pair $\langle G, A\rangle$ such that G has cardinality 2^κ, A has cardinality κ, $\mathrm{C}_G(A) = \{1_G\}$ and $\langle G, A\rangle$ is not strongly special.*

We start with a simple statement about normal subgroups of automorphism groups of centreless groups.

Proposition 6.2. *Let G be a centreless group and N be a normal subgroup of $\mathrm{Aut}(G)$. Then $N \neq \{\mathrm{id}_G\}$ if and only if $\mathrm{Inn}(G) \cap N \neq \{\mathrm{id}_G\}$.*

Proof. Assume $\mathrm{Inn}(G) \cap N = \{\mathrm{id}_G\}$. Given $\pi \in N$, we have

$$\iota_{\pi(g)\cdot g^{-1}} = \pi \circ \iota_g \circ \pi^{-1} \circ \iota_g^{-1} \in \mathrm{Inn}(G) \cap N$$

and therefore $\pi(g) = g$ for all $g \in G$. This shows $N = \{\mathrm{id}_G\}$. □

In the proof of Theorem 6.1, we start by constructing a special pair $\langle G, A\rangle$ with $|G| = |A|$ that is not strongly special. The following proposition will allow us to replace G by a group of higher cardinality.

PROPOSITION 6.3. *Let G and H be groups, A be a subset of the domain of G and $A^* = A \times \{1_H\} \cup \{1_G\} \times H \subseteq G \times H$.*

(1) *If $\langle G, A\rangle$ is a special pair and $\mathrm{Z}(H) = \{1_H\}$, then $\langle G \times H, A^*\rangle$ is a special pair.*
(2) *If $\langle G, A\rangle$ is not a strongly special pair, then $\langle G \times H, A^*\rangle$ is not a strongly special pair.*

PROOF. (1) Assume that $\mathrm{Z}(H) = \{1_H\}$ holds, $\langle g_*, h_*\rangle \in G \times H$ and

$$\varphi : \langle A^* \cup \{\langle g_*, h_*\rangle\}\rangle \longrightarrow G \times H$$

is a monomorphism with $\varphi \restriction A^* = \mathrm{id}_{A^*}$ and $\varphi(\langle g_*, h_*\rangle) \neq \langle g_*, h_*\rangle$. Then $\langle k, 1_H\rangle \in \mathrm{dom}(\varphi)$ for every $k \in \langle A \cup \{g_*\}\rangle$ and $\varphi(\langle g_*, 1_H\rangle) \neq \langle g_*, 1_H\rangle$. Let $p_H : G \times H \longrightarrow H$ denote the canonical projection and define

$$\xi : \langle A \cup \{g_*\}\rangle \longrightarrow H;\ k \longmapsto (p_H \circ \varphi)(\langle k, 1_H\rangle).$$

Given $k \in \langle A \cup \{g_*\}\rangle$ and $h \in H$, we have

$$\begin{aligned} \xi(k) \cdot h &= (p_H \circ \varphi)(\langle k, 1_H\rangle) \cdot (p_H \circ \varphi)(\langle 1_G, h\rangle) = (p_H \circ \varphi)(\langle k, h\rangle) \\ &= (p_H \circ \varphi)(\langle 1_G, h\rangle) \cdot (p_H \circ \varphi)(\langle k, 1_H\rangle) = h \cdot \xi(k) \end{aligned}$$

and this shows $\mathrm{ran}(\xi) \subseteq \mathrm{Z}(H) = \{1_H\}$. We get a function $\bar{\varphi} : \langle A \cup \{g_*\}\rangle \longrightarrow G$ with $\varphi(\langle k, 1_H\rangle) = \langle \bar{\varphi}(k), 1_H\rangle$ for all $k \in \langle A \cup \{g_*\}\rangle$. By our assumptions, $\bar{\varphi}$ is a monomorphism, $\bar{\varphi} \restriction A = \mathrm{id}_A$ and $\bar{\varphi}(g_*) \neq g_*$. This shows that $\langle G, A\rangle$ is not a special pair.

(2) Assume $g_* \in G$ and $\bar{\varphi} : \langle A \cup \{g\}\rangle \longrightarrow G$ is a homomorphism with $\bar{\varphi} \restriction A = \mathrm{id}_A$ and $\bar{\varphi}(g_*) \neq g_*$. If $\langle k, h\rangle \in \langle A^* \cup \{\langle g_*, 1_H\rangle\}\rangle$, then $k \in \langle A \cup \{g_*\}\rangle$ and we can define

$$\varphi : \langle A^* \cup \{\langle g_*, 1_H\rangle\}\rangle \longrightarrow G \times H;\ \langle k, h\rangle \longmapsto \langle \bar{\varphi}(k), h\rangle.$$

Then $\langle G \times H, A^*\rangle$ is not a strongly special pair, because φ is a homomorphism with $\varphi \restriction A^* = \mathrm{id}_{A^*}$ and $\varphi(\langle g_*, 1_H\rangle) \neq \langle g_*, 1_H\rangle$. □

For the remainder of this section, we fix simple non-abelian groups H, S and a homomorphism $\mathfrak{c} : \mathrm{Aut}(S) \longrightarrow \mathrm{Aut}(H)$ with $\mathrm{Inn}(H) \subseteq \mathrm{ran}(\mathfrak{c})$. Define

$$G = H \rtimes_{\mathfrak{c}} \mathrm{Aut}(S)$$

and $A = \{1_H\} \times \mathrm{Aut}(S)$.

LEMMA 6.4. *The following statements are equivalent.*

(1) *There is an isomorphism $\Psi : H \longrightarrow S$ with $\mathfrak{c}(\pi) = \Psi^{-1} \circ \pi \circ \Psi$ for all $\pi \in \mathrm{Aut}(S)$.*
(2) *$\langle G, A\rangle$ is not a special pair.*

PROOF. Assume (1) holds. Define

$$\phi : G \longrightarrow G;\ \langle h, \pi\rangle \longmapsto \langle h^{-1}, \iota_{\Psi(h)} \circ \pi\rangle.$$

Clearly, ϕ is injective and $\phi \upharpoonright A = \mathrm{id}_A$. If $\langle h^{-1}, \iota_{\Psi(h)} \circ \pi\rangle = \langle h, \pi\rangle$ holds with $h \in H$ and $\pi \in \mathrm{Aut}(S)$, then $\iota_{\Psi(h)} = \mathrm{id}_S$ and this means $h = 1_H$. This shows $\phi \neq \mathrm{id}_G$. Given $\langle h_0, \pi_0\rangle, \langle h_1, \pi_1\rangle \in G$, we have

$$\begin{aligned}
\phi(\langle h_0, \pi_0\rangle \cdot \langle h_1, \pi_1\rangle) &= \phi(\langle h_0 \cdot \mathfrak{c}(\pi_0)(h_1), \pi_0 \circ \pi_1\rangle) \\
&= \langle \mathfrak{c}(\pi_0)(h_1^{-1}) \cdot h_0^{-1}, \iota_{\Psi(h_0 \cdot \mathfrak{c}(\pi_0)(h_1))} \circ \pi_0 \circ \pi_1\rangle \\
&= \langle h_0^{-1} \cdot \mathfrak{c}(\pi_0)(h_1^{-1})^{h_0}, \iota_{\Psi(h_0)} \circ \iota_{(\pi_0 \circ \Psi)(h_1)} \circ \pi_0 \circ \pi_1\rangle \\
&= \langle h_0^{-1} \cdot (\iota_{h_0} \circ \mathfrak{c}(\pi_0))(h_1^{-1}), \iota_{\Psi(h_0)} \circ \iota^{\pi_0}_{\Psi(h_1)} \circ \pi_0 \circ \pi_1\rangle \\
&= \langle h_0^{-1} \cdot \mathfrak{c}(\iota_{\Psi(h_0)} \circ \pi_0)(h_1^{-1}), \iota_{\Psi(h_0)} \circ \pi_0 \circ \iota_{\Psi(h_1)} \circ \pi_1\rangle \\
&= \langle h_0^{-1}, \iota_{\Psi(h_0)} \circ \pi_0\rangle \cdot \langle h_1^{-1}, \iota_{\Psi(h_1)} \circ \pi_1\rangle \\
&= \phi(\langle h_0, \pi_0\rangle) \cdot \phi(\langle h_1, \pi_1\rangle),
\end{aligned}$$

because our assumption implies that $\mathfrak{c}(\iota_{\Psi(h)}) = \iota_h$ holds for all $h \in H$. This computation shows that ϕ is a group monomorphism and $\langle G, A\rangle$ is not a special pair by Lemma 2.6.

In the other direction, assume that $\langle G, A\rangle$ is not a special pair. By Lemma 2.6, there is a $g_* = \langle h_*, \pi_*\rangle \in G$ and a monomorphism $\phi : \langle A \cup \{g_*\}\rangle \longrightarrow G$ with $\phi \upharpoonright A = \mathrm{id}_A$ and $\phi(g_*) \neq g_*$. This implies $h_* \neq 1_H$, $\langle h_*, \mathrm{id}_S\rangle \in \mathrm{dom}(\phi)$ and $\phi(\langle h_*, \mathrm{id}_S\rangle) \neq \langle h_*, \mathrm{id}_S\rangle$.

Let $N = \{h \in H \mid \langle h, \mathrm{id}_S\rangle \in \mathrm{dom}(\phi)\}$. If $h \in N$ and $k \in H$, then $\iota_k = \mathfrak{c}(\pi)$ for some $\pi \in \mathrm{Aut}(S)$,

$$\langle h^k, \mathrm{id}_S\rangle = \langle \mathfrak{c}(\pi)(h), \mathrm{id}_S\rangle = \langle 1_H, \pi\rangle \cdot \langle h, \mathrm{id}_S\rangle \cdot \langle 1_H, \pi^{-1}\rangle = \langle h, \mathrm{id}_S\rangle^{\langle 1_H, \pi\rangle} \in \mathrm{dom}(\phi)$$

and $h^k \in N$. This shows that N is a normal subgroup of H and therefore $N = H$, because $1_H \neq h_* \in N$.

Let $p_{\mathrm{Aut}(S)} : G \longrightarrow \mathrm{Aut}(S)$ denote the canonical projection map and define

$$\bar{\Psi} : H \longrightarrow \mathrm{Aut}(S);\ h \longmapsto (p_{\mathrm{Aut}(S)} \circ \phi)(\langle h, \mathrm{id}_S\rangle).$$

Assume, toward a contradiction, that $\ker(\bar{\Psi}) = H$. This assumption gives us a map $\xi : H \longrightarrow H$ with $\phi(\langle h, \mathrm{id}_S\rangle) = \langle \xi(h), \mathrm{id}_S\rangle$ for all $h \in H$. By our assumptions, ξ is a monomorphism. If $h, k \in H$ and $\pi \in \mathrm{Aut}(S)$ with $\mathfrak{c}(\pi) = \iota_k$, then

$$\phi(\langle h^k, \mathrm{id}_S\rangle) = \phi(\langle h, \mathrm{id}_S\rangle^{\langle 1_H, \pi\rangle}) = \phi(\langle h, \mathrm{id}_S\rangle)^{\langle 1_H, \pi\rangle} = \langle \xi(h)^k, \mathrm{id}_S\rangle,$$

and $\xi(h)^k = \xi(h^k) \in \mathrm{ran}(\xi)$. This shows that $\mathrm{ran}(\xi)$ is a normal subgroup of H. Since ϕ is injective and H is nontrivial, we can conclude that $H = \mathrm{ran}(\xi)$ and ξ is a nontrivial automorphism of H. Pick $h \in H$ and $\pi \in \mathrm{Aut}(S)$ with $\mathfrak{c}(\pi) = \iota_h$. If $k \in H$, then

$$\begin{aligned}
\langle k^{\xi(h)}, \pi\rangle &= \langle k^{\xi(h)}, \mathrm{id}_S\rangle \cdot \langle 1_H, \pi\rangle = \phi(\langle \xi^{-1}(k)^h, \mathrm{id}_S\rangle) \cdot \phi(\langle 1_H, \pi\rangle) \\
&= \phi(\langle \mathfrak{c}(\pi)(\xi^{-1}(k)), \pi\rangle) = \phi(\langle 1_H, \pi\rangle) \cdot \phi(\langle \xi^{-1}(k), \mathrm{id}_S\rangle) = \langle 1_H, \pi\rangle \cdot \langle k, \mathrm{id}_S\rangle \\
&= \langle \mathfrak{c}(\pi)(k), \pi\rangle = \langle k^h, \pi\rangle
\end{aligned}$$

and therefore $h^{-1} \cdot \xi(h) \in \mathrm{Z}(H) = \{1_H\}$. This shows $\xi = \mathrm{id}_H$, a contradiction.

By the above computations, $\bar{\Psi} : H \longrightarrow \mathrm{Aut}(S)$ is a monomorphism. If $\pi \in \mathrm{Aut}(S)$ and $h, k \in H$ with $\phi(\langle h, \mathrm{id}_S\rangle) = \langle k, \bar{\Psi}(h)\rangle$, then

$$(\star) \qquad \langle \mathfrak{c}(\pi)(k), \bar{\Psi}(h)^\pi\rangle = \langle k, \bar{\Psi}(h)\rangle^{\langle 1_H, \pi\rangle} = \phi(\langle h, \mathrm{id}_s\rangle^{\langle 1_H, \pi\rangle}) = \phi(\langle \mathfrak{c}(\pi)(h), \mathrm{id}_S\rangle)$$

and therefore $\bar{\Psi}(h)^\pi = \bar{\Psi}(\mathfrak{c}(\pi)(h)) \in \operatorname{ran}(\bar{\Psi})$. This shows that $\operatorname{ran}(\bar{\Psi})$ is a nontrivial normal subgroup of $\operatorname{Aut}(S)$. By Proposition 6.2, we have $\operatorname{Inn}(S) \cap \operatorname{ran}(\bar{\Psi}) \neq \{\mathrm{id}_S\}$ and this implies $\operatorname{Inn}(S) = \operatorname{Inn}(S) \cap \operatorname{ran}(\bar{\Psi}) = \operatorname{ran}(\bar{\Psi})$, because both $\operatorname{Inn}(S)$ and $\operatorname{ran}(\bar{\Psi})$ are simple groups. We have shown that $\bar{\Psi} : H \longrightarrow \operatorname{Inn}(S)$ is an isomorphism.

Define $\Psi : H \longrightarrow S$ to be the isomorphism $\iota_S^{-1} \circ \bar{\Psi}$. Given $\pi \in \operatorname{Aut}(S)$ and $h \in H$, the equalities in $(\star)$ show $\bar{\Psi}(\mathfrak{c}(\pi)(h)) = \bar{\Psi}(h)^\pi$ and this implies

$$\mathfrak{c}(\pi)(h) = \bar{\Psi}^{-1}(\bar{\Psi}(h)^\pi) = \bar{\Psi}^{-1}\left(\iota^\pi_{\Psi(h)}\right) = (\Psi^{-1} \circ \iota_S^{-1})\left(\iota_{(\pi\circ\Psi)(h)}\right) = (\Psi^{-1} \circ \pi \circ \Psi)(h).$$

This equality shows that Ψ is an isomorphism with the desired properties. □

Corollary 6.5. *If $\langle G, A\rangle$ is not a special pair, then $\mathfrak{c}$ is injective.* □

Proposition 6.6. *$\langle G, A\rangle$ is not a strongly special pair.*

Proof. Define

$$\varphi : G \longrightarrow G;\ \langle h, \pi\rangle \longmapsto \langle 1_H, \pi\rangle.$$

Then φ is a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$ and $\varphi(\langle h, \mathrm{id}_S\rangle) \neq \langle h, \mathrm{id}_S\rangle$ for all $h \in H \setminus \{1_H\} \neq \emptyset$. By Lemma 2.7, this implies the statement of the proposition. □

We finish this note by stating the coding result mentioned above and proving Theorem 6.1.

Theorem 6.7 ([**DGG01**, Corollary 4.7]). *Let κ be an uncountable regular cardinal and G be a group of cardinality at most κ. Then there exists a simple group S of cardinality κ such that G is isomorphic to* $\operatorname{Aut}(S)/\operatorname{Inn}(S)$.

Proof of Theorem 6.1. Let κ be a regular uncountable cardinal. It is well-known that the group $\operatorname{Alt}(\kappa)$ is a simple, non-abelian group of cardinality κ. By Theorem 6.7, there is a simple group S of cardinality κ such that there is an isomorphism $\xi : \operatorname{Aut}(S)/\operatorname{Inn}(S) \longrightarrow \operatorname{Alt}(\kappa)$. If we define

$$\mathfrak{c} : \operatorname{Aut}(S) \longrightarrow \operatorname{Aut}(\operatorname{Alt}(\kappa));\ \pi \longmapsto \iota_{\xi(\pi \operatorname{Inn}(S))},$$

then $\mathfrak{c}$ is a non-injective group homomorphism with $\operatorname{Inn}(\operatorname{Alt}(\kappa)) \subseteq \operatorname{ran}(\mathfrak{c})$. We set $\bar{G} = \operatorname{Alt}(\kappa) \rtimes_{\mathfrak{c}} \operatorname{Aut}(S)$ and $\bar{A} = \{\mathrm{id}_\kappa\} \times \operatorname{Aut}(S)$. Since both S and $\operatorname{Aut}(S)/\operatorname{Inn}(S)$ have cardinality κ, $\operatorname{Aut}(S)$ has the same cardinality and $\bar{G}$ is a group of cardinality κ. Corollary 6.5 implies that $\langle \bar{G}, \bar{A}\rangle$ is a special pair and Proposition 6.6 shows that it is not strongly special.

Pick $\langle h, \pi\rangle \in \mathrm{C}_{\bar{G}}(\bar{A})$. Given $\sigma \in \operatorname{Aut}(S)$, we have

$$\langle h, \pi\rangle = \langle h, \pi\rangle^{\langle \mathrm{id}_\kappa, \sigma\rangle} = \langle \mathfrak{c}(\sigma)(h), \pi^\sigma\rangle$$

and this implies $\pi \in \mathrm{Z}(\operatorname{Aut}(S)) = \{\mathrm{id}_S\}$. If $k \in \operatorname{Alt}(\kappa)$ and $\sigma \in \operatorname{Aut}(S)$ with $\mathfrak{c}(\sigma) = \iota_k$, then

$$\langle h, \mathrm{id}_S\rangle = \langle h, \mathrm{id}_S\rangle^{\langle \mathrm{id}_\kappa, \sigma\rangle} = \langle \mathfrak{c}(\sigma)(h), \mathrm{id}_S\rangle = \langle h^k, \mathrm{id}_S\rangle$$

and hence $h \in \mathrm{Z}(\operatorname{Alt}(\kappa)) = \{\mathrm{id}_\kappa\}$.

Define $G = \bar{G} \times \operatorname{Alt}(\kappa)$ and $A = \bar{A} \times \{\mathrm{id}_\kappa\} \cup \{1_{\bar{G}}\} \times \operatorname{Alt}(\kappa)$. By Proposition 6.3, $\langle G, A\rangle$ is a special pair that is not strongly special. Moreover, it is easy to see that both G and A have cardinality κ and $\mathrm{C}_G(A) = \mathrm{C}_{\bar{G}}(\bar{A}) \times \mathrm{Z}(\operatorname{Alt}(\kappa)) = \{\langle 1_{\bar{G}}, \mathrm{id}_\kappa\rangle\}$.

Let $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. Then G_1 has cardinality 2^κ, because the automorphism group of $\operatorname{Alt}(\kappa)$ is isomorphic to the group $\operatorname{Sym}(\kappa)$

of all permutations of κ and every automorphism of $\mathrm{Alt}(\kappa)$ induces a unique automorphism of G. By Theorem 2.2, $\langle G_1, A\rangle$ is a special pair with $\mathrm{C}_{G_1}(A) = \{1_G\}$. Finally, $\langle G_1, A\rangle$ is not a strongly special pair, because otherwise $\langle G, A\rangle$ would be a strongly special pair. □

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Pseudofinite groups with NIP theory and definability in finite simple groups

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For Rüdiger Göbel, in celebration of his seventieth birthday

Abstract. We show that any pseudofinite group with NIP theory and with a finite upper bound on the length of chains of centralisers is soluble-by-finite. In particular, any NIP rosy pseudofinite group is soluble-by-finite. This generalises, and shortens the proof of, an earlier result for stable pseudofinite groups. An example is given of an NIP pseudofinite group which is not soluble-by-finite. However, if $\mathcal{C}$ is a class of finite groups such that all infinite ultraproducts of members of $\mathcal{C}$ have NIP theory, then there is a bound on the index of the soluble radical of any member of $\mathcal{C}$. We also survey some ways in which model theory gives information on families of finite simple groups, particularly concerning products of images of word maps.

1. Introduction

We consider in this paper groups G which are *pseudofinite*, that is, infinite groups which satisfy every first order sentence (in the language L_g of groups) which holds in all finite groups. Equivalently, G is elementarily equivalent to an infinite ultraproduct of finite groups. Or equivalently again, G is an infinite group with the *finite model property*: every sentence in the theory of the group has a finite model. We consider the structure of G, under the assumption that the first order theory $\mathrm{Th}(G)$ of G satisfies various generalisations of model theoretic stability.

It was shown in [**23**] that any stable pseudofinite group G has a definable soluble normal subgroup of finite index. This is not surprising; for by a classification due to Wilson [**34**] (with a slight strengthening due to Ryten – see [**9**, Proposition 2.14]) – any infinite pseudofinite *simple* group is a group of Lie type over a pseudofinite field, and in particular interprets a pseudofinite field [**29**, 5.2.4, 5.3.3, 5.4.3], and so has unstable theory by Duret [**5**]. However, an intricate argument with centralisers was needed in [**23**] to bound the derived length of soluble normal subgroups.

One generalisation of stability is the notion of *simple* theory. Pseudofinite fields (and certain difference fields, that is, fields equipped with a specified automorphism) are simple, in fact supersimple of finite rank, and it follows from Wilson's classification that every simple pseudofinite group is interpretable in such a structure.

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Hence, every simple pseudofinite group has supersimple finite rank theory; this follows from the results of Hrushovski [**12**] and is made explicit in [**9**] (note that measurable structures are supersimple of finite rank – see e.g. [**8**, Corollary 3.7]). A satisfactory structure theory for pseudofinite groups with supersimple finite rank theory – under an additional and probably unnecessary assumption that $\exists^\infty$ is definable in T^{eq} – was initiated in [**9**]. The class of supersimple finite rank structures is sufficiently rich to include a lot of pseudofinite group theory, as indicated by, for example, [**20**, 4.11, 4.12]. Possible applications of the model theory of supersimple theories to finite simple groups are discussed in the final section of the present paper.

Another generalisation of stability of considerable current interest is that of *NIP*, or *dependent* theory. A formula $\phi(\bar{x}, \bar{y})$ has the *independence property* with respect to T if there is $M \models T$ and a set $\{\bar{a}_i : i \in \omega\} \subset M^{l(\bar{x})}$ such that for all $S \subseteq \omega$ there is $\bar{b}_S \in M^{l(\bar{y})}$ such that for all $i \in \omega$, $M \models \phi(\bar{a}_i, \bar{b}_S)$ if and only if $i \in S$. A theory T is *NIP* if no formula has the independence property with respect to T. Any stable theory is simple and NIP, and any theory which is *both* simple and NIP is stable. For groups, by the Baldwin-Saxl Theorem (see [**2**], or [**6**, Fact 0.17]) the NIP condition implies a useful chain condition: if G is an NIP group, then for every formula $\phi(x, \bar{y})$ there is a natural number n_ϕ such that every *finite* intersection of ϕ-definable groups is an intersection of n_ϕ ϕ-definable groups. By Wilson's theorem, there is no simple pseudofinite group with NIP theory, and we expected this, together with the above chain condition, to yield virtual solubility for pseudofinite groups with NIP theory. However, this is false, and in Section 3 below we give a construction of a pseudofinite group G with NIP theory which is not soluble-by-finite.

Our main theorem is the following. We say that a group G has the *centraliser chain condition* if there is a natural number $n = n(G)$ such that there do not exist subsets $F_1, \ldots, F_{n+1} \subset G$ with

$$C_G(F_1) < \ldots < C_G(F_{n+1}).$$

THEOREM 1.1. *Let G be a pseudofinite group with NIP theory, and suppose that G satisfies the centraliser chain condition. Then G has a soluble definable normal subgroup of finite index.*

We obtain some information about finite groups just under an NIP assumption. Let us say that the class $\mathcal{C}$ of finite structures is an *NIP class* if every infinite ultraproduct of members of $\mathcal{C}$ has NIP theory. As a step in the proof of Theorem 1.1 we obtain the following result. Here, and throughout the paper, if G is a finite group we denote by $R(G)$ its *soluble radical*, that is, the unique largest soluble normal subgroup of G.

PROPOSITION 1.2. Let $\mathcal{C}$ be an NIP class of finite groups. Then there is $d = d(\mathcal{C}) \in \mathbb{N}$ such that $|G : R(G)| \leq d$ for every $G \in \mathcal{C}$.

The notion of *rosy* theory is a common generalisation of the notions of *o-minimal* theory and *simple* (and hence also of *stable*) theory. The concept was introduced in [**24**] and developed in [**1**]. We omit the definition of rosiness, but note that by [**6**, Definition 0.3], a theory T is rosy if and only if there is an independence relation $\mathop{\smile\!\!\!\!\!\mid}$ on real *and imaginary* tuples which satisfies the following natural conditions :

(i) $\mathop{\smile\!\!\!\!\!\!\;|}$ is automorphism invariant.

(ii) If $c \in \mathrm{acl}(aB) \setminus \mathrm{acl}(B)$, then $a \not\downarrow_B c$.

(iii) If $a \downarrow_B C$ and $B \cup C \subseteq D$, then there is $a' \in \mathrm{tp}(a/BC)$ with $a' \downarrow_B D$.

(iv) There is λ such that for any a, if $(B_i)_{i<\alpha}$ are sets with $B_i \subset B_j$ whenever $i < j$ and $a \not\downarrow_{B_i} B_j$ for $i < j < \alpha$, then $\alpha < \lambda$.

(v) If $B \subseteq C \subseteq D$, then $a \downarrow_B D$ if and only if $a \downarrow_B C$ and $a \downarrow_C D$.

(vi) $C \downarrow_A B$ if and only if $c \downarrow_A B$ for any finite $c \subseteq C$.

(vii) $a \downarrow_C b$ if and only if $b \downarrow_C a$.

A structure with an infinite descending chain of uniformly definable equivalence relations can never be rosy – see for example the proof of Proposition 1.3 in [**6**]. In particular, a field with a non-trivial definable valuation can never be rosy, and more generally a group with an infinite strictly descending chain of uniformly definable subgroups cannot be rosy. In combination with the consequence mentioned above of the Baldwin-Saxl Theorem this yields the following, for groups.

PROPOSITION 1.3. [**6**, Corollary 1.8] Any group definable in an NIP rosy theory has the centraliser chain condition.

By Theorem 1.1, this yields immediately the following.

COROLLARY 1.4. Let G be a pseudofinite group with NIP rosy theory. Then G has a soluble definable normal subgroup of finite index.

We should not expect here to replace 'soluble' by 'nilpotent', since examples (involving Chapuis, Simonetta, Khelif, and Zilber) are mentioned at the end of [**23**] of stable pseudofinite groups which are not nilpotent-by-finite.

Theorem 1.1 is proved in Section 2. In addition to Proposition 1.3, and the classification of simple pseudofinite groups, we use the following two results.

THEOREM 1.5. [**35**, Wilson] *There is a formula $\psi(x)$ such that for every finite group G, we have $R(G) = \{x \in G : G \models \psi(x)\}$.*

THEOREM 1.6. [**14**, Khukhro] *There is a function $f : \mathbb{N} \to \mathbb{N}$, such that for any $d \in \mathbb{N}$, if G is a finite soluble group with no strictly descending chain of centralisers of length $d+1$, then G has derived length at most $f(d)$.*

The final section of the paper is a discussion of some possible applications of model theory to structural questions on families of finite simple groups of fixed Lie rank. There are three main sources of applications: a generalisation of the Zilber Indecomposability Theorem for groups in supersimple theories; some still-unpublished work of Ryten showing that any family of finite simple groups is an 'asymptotic class', so that cardinalities of definable sets satisfy Lang-Weil-like uniformities; and information on generic types of groups in simple theories. No new results here are given. However the methods give, for example, an alternative approach to some recent advances on word maps, admittedly proving weaker results. For the Suzuki and Ree groups there is heavy dependence on a major result of Hrushovski [**12**].

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2. Proof of Theorem 1.1

Proof of Proposition 1.2. Let $\mathcal{C} = \{G_i : i \in \mathbb{N}\}$ be a class of finite groups such that every non-principal ultraproduct of members of $\mathcal{C}$ has NIP theory. By Theorem 1.5, with $\psi(x)$ the formula given in that theorem, for each $i \in \omega$ we have $R(G_i) = \{x \in G_i : G_i \models \psi(x)\}$. By Łos's Theorem, ψ defines a normal subgroup, denoted by $\psi(G)$, of any ultrapower G of members of $\mathcal{C}$.

Write $\bar{G}_i := G_i/R(G_i)$, and let $S_i := \mathrm{Soc}(\bar{G}_i)$, the direct product of the minimal normal subgroups of $\bar{G}_i$. By the maximality of $R(G_i)$, each minimal normal subgroup of $\bar{G}_i$ is non-abelian and hence each S_i can be written as a direct product of non-abelian simple groups.

Claim 1. There is $t \in \mathbb{N}$ such that each S_i is a direct product of at most t distinct non-abelian simple groups.

Proof of Claim. Otherwise for each $n \in \mathbb{N}$ there are infinitely many groups G_i such that S_i contains at least n non-abelian simple factors. If $T_1 \times \ldots \times T_n$ is such a product, pick $x_j, y_j \in T_j$ with $[x_j, y_j] \neq 1$. For any $w \subset \{1, \ldots n\}$ we find z_w such that $[x_j, z_w] = 1$ if and only if $j \in w$ by putting $z_w = \Pi_{j \notin w} y_j$. It follows by Łos's Theorem that a non-principal ultrafilter can be chosen on $\mathbb{N}$ so that the formula $\chi(y, z)$ of form $yz \neq zy$ witnesses that $\Pi_i \bar{G}_i/\mathcal{U}$ has the independence property. Thus, as $\bar{G}_i$ is uniformly interpretable in G_i, the infinite group $\Pi_{i\in\mathbb{N}} G_i/\mathcal{U}$ does not have NIP theory, a contradiction.

Thus, we may reduce to the case when each S_i is a direct product of exactly c non-abelian simple groups, namely $S_i = T_{i,1} \times \ldots \times T_{i,c}$, where each $T_{i,j}$ is non-abelian simple.

Claim 2. There is $e \in \mathbb{N}$ such that any non-abelian simple subgroup of $\bar{G}_i$ has Lie rank at most e (where we define the Lie rank of the alternating group Alt_n to be n, and that of the sporadic simple groups to be 1).

Proof of Claim. We argue as in the proof of Claim 1. It suffices to note that for any n, a sufficiently large alternating group contains a direct product of n copies of Alt_5. Likewise, non-abelian classical simple groups of large rank contain many commuting copies of $\mathrm{PSL}_2(q)$.

Claim 3. Let $\mathcal{F}$ be a family of finite simple groups of fixed Lie rank e. Then there is $d = d(e) \in \mathbb{N}$ such that if $K \in \mathcal{F}$ and $g, h \in K \setminus \{1\}$ then g is a product of at most d copies of h and h^{-1}.

Proof of Claim. This is well-known. It follows for example from the theorem in [**27**] that any non-principal ultraproduct of members of $\mathcal{F}$ is a group of the same Lie type over a pseudofinite field, and so is simple.

By Claims 2 and 3 we obtain the following: there is $b \in \mathbb{N}$ such that for each i, j and $x_{i,j} \in T_{i,j} \setminus \{1\}$ any element of $T_{i,j}$ is a product of at most b $T_{i,j}$-conjugates of $x_{i,j}$ and $x_{i,j}^{-1}$. As $\bar{G}_i$ normalises S_i, it follows easily that the minimal normal subgroups of $\bar{G}_i$ and finally the $T_{i,j}$ themselves are uniformly definable in the $\bar{G}_i$.

To complete the proof of the proposition, it suffices to show that there is $e \in \mathbb{N}$ such that $|S_i| \leq e$ for all i. For suppose this holds. Then $C_i := C_{\bar{G}_i}(S_i)$ is a normal subgroup of $\bar{G}_i$. Since $C_i \cap S_i = 1$ we have $C_i = 1$. Thus, $\bar{G}_i$ embeds in $\mathrm{Aut}(S_i)$, so has order at most $e!$.

So suppose for a contradiction that there is no finite upper bound on $|S_i|$. Then by the classification of finite simple groups, there is some Lie type Chev (possibly twisted, but with the Lie rank fixed) such that the $R_{i,j}$ include arbitrarily large finite simple groups of type Chev. Relabelling if necessary, we may suppose there is a subsequence $(n_i : i \in \mathbb{N})$ of $\mathbb{N}$ such that each finite simple group $R_{n_i,1}$ has Lie type Chev, and $|R_{n_i,1}| \to \infty$ as $i \to \infty$. We may suppose that $R_{n_i,1}$ is defined in $\bar{G}_{n_i}$ by the formula $\phi(x, \bar{a}_i)$.

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ containing $N = \{n_i : i \in \mathbb{N}\}$, and hence all cofinite subsets of N. Put $G = \Pi_{i\in\mathbb{N}} G_i/\mathcal{U}$, and $\bar{G} := G/H$, where H is the normal subgroup of G defined by ψ. Then there is $\bar{a} \in \bar{G}$ such that $\phi(x, \bar{a})$ defines an infinite ultrapower of groups of type Chev, and hence, by [**27**], a group of Lie type Chev over a pseudofinite field. Such a subgroup has the independence property, by the results of Ryten and Duret mentioned above. It follows that $\bar{G}$, and hence G, does not have NIP theory, a contradiction. □

The following lemma is standard.

LEMMA 2.1. *Let L be a countable language and M be a pseudofinite L-structure. Then there is an infinite class $\mathcal{C}$ of finite structures such that every infinite ultrapower of members of $\mathcal{C}$ is elementarily equivalent to M.*

PROOF. We may suppose that $M = \Pi_{n\in\mathbb{N}} M_i/\mathcal{U}$ where the M_i are finite with $|M_i| \to \infty$ as $i \to \infty$. Let $\{\sigma_i : i \in \mathbb{N}\}$ list $\mathrm{Th}(M)$. Iteratively, we find a sequence $U_0 \supset U_1 \supset \dots$ of members of $\mathcal{U}$ such that for each $i \in \mathbb{N}$, U_i contains the i smallest elements $n_{i1} < \dots < n_{ii}$ of U_{i-1}, and such that for all $i \in \mathbb{N}$ and $j \in U_i$ with $j > n_{ii}$, $M_j \models \sigma_i$. Put $U := \bigcap_{i\in\mathbb{N}} U_i$. Then U is infinite, and by Łos's Theorem, $\mathcal{C} := \{M_i : i \in U\}$ satisfies the lemma. □

Proof of Theorem 1.1. Let G be a pseudofinite group with NIP theory, such that every chain of centralisers has length at most e. Observe that there is a sentence τ_e in the language L_g of groups such that for every group H, we have $H \models \tau_e$ if and only if every chain of centralisers in H has length at most e. By Lemma 2.1 there is a set $\mathcal{C} := \{G_i : i \in \mathbb{N}\}$ and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that (after replacing G by an elementarily equivalent group if necessary) $G = \Pi_{i\in\mathbb{N}} G_i/\mathcal{U}$, and every infinite ultraproduct of members of $\mathcal{C}$ is elementarily equivalent to G. It follows that $\mathcal{C}$ is an NIP class of finite groups, so by Proposition 1.2 there is $d \in \mathbb{N}$ such that $|G_i : R(G_i)| \le d$ for all $i \in \mathbb{N}$. Also $M_i \models \tau_e$ for cofinitely many $i \in \mathbb{N}$. Hence, by Theorem 1.6, $R(G_i)$ has derived length at most $f(e)$ for cofinitely many $i \in \mathbb{N}$. The property that the derived length is at most $f(e)$ is first order expressible by a sentence asserting that a certain word vanishes on a group. Thus, by Łos's Theorem, the normal subgroup $R(G) := \{x \in G : G \models \psi(x)\}$ is soluble of derived length at most $f(e)$, and index at most d in G. □

3. A pseudofinite NIP group which is not soluble-by-finite

We here prove the following theorem.

THEOREM 3.1. *There is a pseudofinite group G with NIP theory which is not soluble-by-finite.*

If M is a structure and $\phi(x_1, \dots, x_m, y_1, \dots, y_n)$ is a formula which does not have the independence property in M, then there is a greatest natural number d such that there are distinct $\bar{a}_1, \dots, \bar{a}_d \in M^m$ such that for each $S \subseteq \{1, \dots d\}$ there is $\bar{b}_S \in M^n$ with, for all $i \in \{1, \dots, d\}$, $M \models \phi(\bar{a}_i, \bar{b}_S) \Leftrightarrow i \in S$. Such d is called the *Vapnik-Cervonenkis dimension*, or *VC-dimension*, of the family of definable sets in the $\bar{x}$-variables determined by ϕ (or just of the formula ϕ). We note the following lemma.

LEMMA 3.2. *Let L, L' be first order languages, and let M be an L-structure with NIP theory. Suppose that $\{M_i : i \in I\}$ is a set of L'-structures which is uniformly interpretable in M (with I an interpretable set of M). Let J be an infinite subset of I and $\mathcal{V}$ a non-principal ultrafilter on J. Then the ultraproduct $N = \Pi_{j \in J} M_j / \mathcal{V}$ has NIP theory.*

PROOF. It suffices to observe that the VC-dimension of any L'-formula $\phi(\bar{x}, \bar{y})$ is uniformly bounded across the class of structures M_i. We leave the details as an exercise. $\square$

Proof of Theorem 3.1. Fix a prime p. It is well-known that the valued field $\mathbb{Q}_p$, and hence its valuation ring $\mathbb{Z}_p$, has NIP theory. Hence, the group $H := \mathrm{SL}_2(\mathbb{Z}_p)$, which is interpretable in $\mathbb{Z}_p$, also has NIP theory. Let $\mathcal{M} := p\mathbb{Z}_p$, the maximal ideal of $\mathbb{Z}_p$. For each $k > 0$ let H_k be the congruence subgroup of H consisting of matrices $\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}$ which lie in H and satisfy $a, b, c, d \in p^k \mathcal{M}$. Then H_k is normal in H, and the quotient $\bar{H}_k := H/H_k$ is finite.

Let $\mathcal{U}$ be a non-principal ultrafilter on ω, and let G be the ultraproduct $\Pi \bar{H}_k / \mathcal{U}$. Then G is a pseudofinite group, and is NIP by the previous lemma, since the groups $\bar{H}_k$ are uniformly interpretable in an NIP theory.

Note that if a group is soluble-by-finite, then so are all its subgroups and quotients. Therefore, in order to show that G is not soluble-by-finite, we first prove the following claim.

Claim 1. The group G has a normal subgroup N such that $G/N \cong \mathrm{SL}_2(\mathbb{Z}_p)$.

Proof of Claim. We view the groups $\bar{H}_k$ and G as structures in the language $L^+ := L_g \cup \{P_i : i < \omega\}$ where the P_i are unary predicates. In $\bar{H}_k$, P_i is interpreted by H_i/H_k for $i \leq k$ and by $1 = H_k/H_k$ for $i > k$. Thus, the P_i are interpreted by a descending chain of normal subgroups of $\bar{H}_k$. The group G has by Łos's Theorem a corresponding strictly descending chain $P_0^G > P_1^G > \ldots$ consisting of normal subgroups of G. Put $N := \bigcap_{i \in \omega} P_i^G$. Compactness together with ω_1-saturation of G (viewed as an L^+-structure) yields that $G/N \cong \mathrm{SL}_2(\mathbb{Z}_p)$.

To complete the proof of Theorem 3.1 we now note:

Claim 2. The group $\mathrm{SL}_2(\mathbb{Z}_p)$ is not soluble-by-finite.

Proof of Claim. This must be well-known: if it were soluble-by-finite, then so would be $\mathrm{SL}_2(\mathbb{Z}) < \mathrm{SL}_2(\mathbb{Z}_p)$ and its quotient $PSL_2(\mathbb{Z})$, which is a free product of a cyclic group of order two and a cyclic group of order three, and clearly not soluble-by-finite (see [**28**], Section 6.2). $\square$

REMARK 3.3. Let G be a pseudofinite NIP group which is not soluble-by-finite. By Lemma 2.1 $G \equiv H$ for some ultraproduct $H = \Pi_{i \in \mathbb{N}} H_i / \mathcal{U}$, such that every infinite ultraproduct of the H_i is elementarily equivalent to G.

The formula $\psi(x)$ defines a non-soluble normal subgroup $\psi(H)$ of finite index in H. By the methods of Section 2, it can be shown that $\psi(H)$ has subgroups $N_1 < N_2$ which are normal in H, such that $\psi(H)/N_2$ is pro-soluble (an inverse limit of soluble groups) but not soluble, and N_1 is the union of a chain of soluble groups but is not soluble. We have not investigated the possible structure of N_2/N_1.

In fact, these conclusions can be shown to hold for *any* infinite NIP group which is a non-principal ultraproduct of distinct finite groups and is not soluble-by-finite.

4. Model theory of finite simple groups

In this section we make some remarks about possible applications of model theory to finite group theory, via pseudofinite groups. As mentioned in the introduction, one generalisation of the notion of *stable* first order theory is that of *simple theory.* This notion was introduced by Shelah in [**32**] and developed in the 1990s in [**15**] and [**16**] and further in other papers. Many ideas first appeared in [**4**] and in early versions of [**11**]. A convenient source, mainly used below, is [**33**]. Simplicity theory is a context for an abstract theory of independence, given by 'non-forking', which is less powerful than the corresponding independence theory in stability theory, but stronger than that in rosy theories. In stable theories, over a suitable base, the first order type of tuples $\bar{a}$ and $\bar{b}$, combined with the knowledge that they are independent, determines the type of $\bar{a}\bar{b}$, but this is false in general in simple theories.

We emphasise the distinction between the group-theoretic notion of *simple group* and the model-theoretic notion of *group definable in a simple theory.* We also stress that our methods below only seem to have applications for families of finite simple groups *of fixed Lie rank.*

Among the simple theories are the *supersimple ones*, for which there are global model-theoretic notions of *rank* or *dimension* for definable sets. We shall only deal with supersimple finite rank theories, in which all the main notions of model-theoretic rank coincide on any definable set (though not on types). Below, we shall refer to SU-rank, described later in more detail.

It can be shown that any family of finite simple groups of fixed Lie rank is uniformly interpretable in a family of finite fields, or (in the case of Suzuki and Ree groups) in a family of finite *difference fields*, that is, fields equipped with an automorphism. In fact, by [**29**, Ch. 5], if parameters are allowed then the groups are uniformly bi-interpretable with the (difference) fields. Thus, the groups $\mathrm{PSL}_3(q)$ are uniformly parameter bi-interpretable with the fields $\mathbb{F}_q$, the Ree and Suzuki groups ${}^2F_4(2^{2k+1})$ and ${}^2B_2(2^{2k+1})$ are uniformly parameter bi-interpretable with the difference fields $(\mathbb{F}_{2^{2k+1}}, x \mapsto x^{2^k})$, and the Ree groups ${}^2G_2(3^{2k+1})$ are uniformly parameter bi-interpretable with the difference fields $(\mathbb{F}_{3^{2k+1}}, x \mapsto x^{3^k})$. Now infinite ultraproducts of finite fields have supersimple SU-rank rank 1 theory – that is, the set defined by the formula $x = x$ has SU-rank 1 – by for example [**3**]. The ultraproducts of the corresponding difference fields also have supersimple SU-rank 1 theory, by the results of Hrushovski [**12**] and of Ryten (see e.g. [**29**, Theorem 3.5.8]). For the difference fields this rests on deep work from the 1990s in [**12**], and Hrushovski was clearly aware then of the supersimplicity of pseudofinite simple groups, and applications similar to some of those below.

We mention three possible lines of application to finite simple groups. Some methods of this kind were used (though not for Ree and Suzuki groups), in the important paper [**13**].

1. *Zilber Indecomposability.* The Irreducibility Theorem for linear algebraic groups was reworked by Zilber for groups of finite Morley rank. Other model-theoretic versions have appeared, but for us the following result of Wagner is convenient. See [**33**, 4.5.6], or, for the guise below, [**10**, Remark 2.5].

THEOREM 4.1 (Indecomposability Theorem). *Let G be a group interpretable in a supersimple finite* SU*-rank theory, and let $\{X_i : i \in I\}$ be a collection of definable subsets of G. Then there exists a definable subgroup H of G such that:*

(i) $H \leq \langle X_i : i \in I\rangle$, and there are $n \in \mathbb{N}$, $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$, and $i_1, \ldots, i_n \in I$, such that $H \leq X_{i_1}^{\epsilon_1} \ldots X_{i_n}^{\epsilon_n}$.

(ii) X_i/H is finite for each $i \in I$.

If the collection of X_i is setwise invariant under some group Σ of definable automorphisms of G, then H may be chosen to be Σ-invariant.

This has the following almost immediate application to finite simple groups. The result below can also be deduced from [**21**, Theorem 1], in combination with Theorem 4.4 below.

THEOREM 4.2. *Let $\mathcal{C}_\tau$ be a family of finite simple groups of fixed Lie type τ, and let $\phi(x, y_1, \ldots, y_m)$ be a formula in the language of groups. Then there is a positive integer $d = d(\phi, \tau)$ with the following property: if $G \in \mathcal{C}_\tau$, $\bar{a} \in G^m$, and $X = \phi(G, \bar{a})$ satisfies $|X| > d$, then G is a product of at most d conjugates of the set $X \cup X^{-1}$.*

PROOF. Suppose that this is false, and let $\mathcal{C}_\tau := \{G_i : i \in \mathbb{N}\}$. Then there is a decreasing sequence of infinite subsets $(I_j : j \in \mathbb{N})$ of $\mathbb{N}$ with infinite intersection I such that for any $d \in \mathbb{N}$, and for all but finitely many $j \in I_d$, G_j is not a product of at most d conjugates of $X_j \cup X_j^{-1}$. Choose a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ which contains the set I. Let $G := \Pi_{j\in\mathbb{N}} G_i/\mathcal{U}$ and $X := \Pi_{j\in\mathbb{N}} G_i/\mathcal{U}$. Then G is a simple pseudofinite group so has supersimple finite SU-rank theory, and X is an infinite definable subset of G such that for each $d \in \mathbb{N}$, G is not a product of at most d conjugates of $X \cup X^{-1}$. By Theorem 4.1 (including the final assertion), G has an infinite definable normal subgroup H which is contained in a product of a bounded number of conjugates of $X \cup X^{-1}$. This is a contradiction, since by simplicity of G, we have $H = G$. □

Other applications of Theorem 4.1 were found in [**20**]. In particular, it was shown in Corollary 4.11 that certain maximal subgroups (those which are not 'subfield subgroups') of finite simple groups are uniformly definable in the groups, and hence, if also unbounded in order, they are 'uniformly maximal' [**20**, Proposition 4.2(ii)].

2. *Asymptotic classes.* The following definition is due to Elwes [**7**], extending the 1-dimensional case of [**22**].

DEFINITION 4.3. A class $\mathcal{C}$ of finite first order structures is, for some positive integer N, an *N-dimensional asymptotic class*, if the following holds.

(i) For every L-formula $\phi(\bar{x}, \bar{y})$ where $l(\bar{x}) = n$ and $l(\bar{y}) = m$, there is a finite set of pairs $D \subseteq (\{0, \ldots, Nn\} \times \mathbb{R}^{>0}) \cup \{(0, 0)\}$ and for each $(d, \mu) \in D$ a collection $\Phi_{(d,\mu)}$ of pairs of the form $(M, \bar{a})$ where $M \in \mathcal{C}$ and $\bar{a} \in M^m$, so that $\{\Phi_{(d,\mu)} : (d, \mu) \in D\}$ is a partition of $\{(M, \bar{a}) : M \in \mathcal{C}, \bar{a} \in M^m\}$, and

$$\big||\phi(M^n, \bar{a})| - \mu|M|^{\frac{d}{N}}\big| = o(|M|^{\frac{d}{N}})$$

as $|M| \to \infty$ and $(M, \bar{a}) \in \Phi_{(d,\mu)}$.

(ii) Each $\Phi_{(d,\mu)}$ is $\emptyset$-definable, that is to say $\{\bar{a} \in M^m : (M, \bar{a}) \in \Phi_{(d,\mu)}\}$ is uniformly $\emptyset$-definable across $\mathcal{C}$.

By the main theorem of [**3**], the class of finite fields is a 1-dimensional asymptotic class, and by Theorem 3.5.8 of [**29**] the classes of difference fields ($\mathbb{F}_{2^{2k+1}}, x \mapsto x^{2^k}$) and ($\mathbb{F}_{3^{2k+1}}), x \mapsto x^{3^k}$) also form 1-dimensional asymptotic classes. The bi-interpretability results of Ryten mentioned above now yield the following.

THEOREM 4.4. [**29**, Ryten] *If $\mathcal{C}$ is a family of finite simple groups of fixed Lie type, then $\mathcal{C}$ is an N-dimensional asymptotic class for some N.*

REMARK 4.5. Let $\mathcal{C} = \{G_i : i \in \mathbb{N}\}$ be an asymptotic class of finite simple groups as above, and let $G^* := \Pi_{i\in\mathbb{N}} G_i/\mathcal{U}$ be an infinite ultraproduct of members of $\mathcal{C}$. Let $\phi(\bar{x}, \bar{y})$ be a formula with $l(\bar{x}) = m$ and $l(\bar{y}) = n$, let $\bar{a} \in (G^*)^n$ with $\bar{a} = (\bar{a}_i)/\mathcal{U}$, and suppose that there is $U \in \mathcal{U}$ such that for all $i \in U$, $\phi(G_i^m, \bar{a}_i)$ has size approximately $\mu|G_i|^d$ (in the sense of asymptotic classes). Then it follows that $\mathrm{SU}(\phi((G^*)^m, \bar{a})) = d.\mathrm{SU}(G^*)$. This can be deduced from [**7**, 5.4], since $\mathcal{C}$ is parameter-bi-interpretable with a 1-dimensional asymptotic class (of fields or difference fields).

3. *Word maps.* Let $w(x_1, \ldots, x_d)$ be a non-trivial group word in $x_1, \ldots, x_d$, that is, a non-identity element of the free group F_d with free basis $\{x_1, \ldots, x_d\}$. Then w defines, in any group G, a map $w : G^d \to G$, the *word map* corresponding to w, with image denoted by $w(G)$. It is shown in [**17**] that there is a function f such that if G is a finite simple group, w is a non-trivial word, and $\epsilon > 0$, then $|w(G)| \geq |G|^{1-\epsilon}$ for sufficiently large G. In fact (and this could also be deduced from the last statement using Theorem 4.4), we have: if $\mathcal{C}$ is a family of finite simple groups of fixed Lie type, and w is a non-trivial word, then there is $\mu > 0$ such that if $G \in \mathcal{C}$ is sufficiently large then $|w(G)| \geq \mu|G|$.

THEOREM 4.6. [**20**] *For any non-trivial words w_1, w_2 there are $N = N(w_1, w_2)$ such that if G is a finite simple group with $|G| \geq N$ then $w_1(G)w_2(G) = G$.*

This result is the culmination of work in several other related papers. For example, it was shown by Shalev [**31**] that if w is a non-trivial word then there is $N = N(w)$ such that if G is a non-abelian finite simple group with $|G| > N$ then $(w(G))^3 = G$; and Theorem 4.6 was already proved for groups of fixed Lie type (other than the Ree and Suzuki groups) in [**18**].

We mention a possible alternative approach, which yields weaker statements than that of Theorem 4.6, but has potential for further applications, since it depends just on the definability of $w(G)$ and its asymptotic size. For one such application, see Theorem 4.11 below. The approach rests on the above-stated result of Larsen from [**17**], and some general model theory of groups in (super)simple theories. An advantage is that Suzuki and Ree groups can be treated simultaneously with other families of finite simple groups with no extra work, though this rests on the major work of Hrushovski in [**12**], in combination with [**29**].

First, for groups definable in simple theories there is a theory of generic types, analogous to that in stable theories, developed by Pillay [**25**] and described in [**33**, Sections 4.3–4.5]. We shall consider a simple theory T, such that in any $M \models T$ there is an $\emptyset$-definable group G. Let $\downarrow$ denote the relation of non-forking (i.e. independence) in simple theories: for subsets A, B, C of M, $A \downarrow_C B$ denotes that A and B are independent over C in the sense of non-forking, that is, for any $\bar{a}$ from A, $\mathrm{tp}(\bar{a}/B \cup C)$ does not fork over C. If A is a set of parameters in $M \models T$, then $S_G(A)$ denotes the set of types over A which contain the formula $x \in G$; that is the set of maximal consistent (with T) sets of formulas in the variable x, with parameters from A, which include the formula $x \in G$. Following [**33**] (see Definition 4.3.2 and also Lemma 4.3.4) a type $p \in S_G(A)$ is *generic* if for any $b \in G$ and a realising p with $a \downarrow_A b$, we have $ba \downarrow A, b$. The group G has a certain subgroup G^o_A (the 'connected component over A'), and a generic type is *principal* if it is realised in G^o_A (where G is interpreted in a sufficiently saturated model of T). Part (i) of the following result was first proved in [**26**, Proposition 2.2], and (ii) is an immediate consequence.

THEOREM 4.7. *Let T be a simple theory over a countable language, $\bar{M}$ an ω_1-saturated model of T with a countable elementary substructure M, and G an $\emptyset$-definable group in $\bar{M}$. Let p_1, p_2, p_3 be three principal generic types of G over M.*

(i) There are $g_1, g_2 \in \bar{M}$ such that $g_i \models p_i$ for $i = 1, 2$, $g_1 \downarrow_M g_2$, and $g_1 g_2 \models p_3$.

(ii) If $r \in S_G(M)$ has realisations in G^o_M then there are $a_i \in G$ with $a_i \models p_i$ (for $i = 1, 2, 3$) such that $a_1 a_2 a_3 \models r$.

PROOF. (i) See for example [**33**, Proposition 4.5.6], though as phrased above one must use ω_1-saturation to find the g_i in $\bar{M}$.

(ii) Choose $a_3, b \in \bar{M}$ such that $a_3 \models p_3$, $b \models r$, and $a_3 \downarrow_M b$, and put $c_3 := ba_3^{-1}$. Let $p_3' := \mathrm{tp}(c_3/M)$. Then p_3' is a generic type of G^* over M. Indeed (repeatedly using 4.3.2 and 4.3.4 of [**33**]), $\mathrm{tp}(a_3^{-1}/M)$ is generic, so as $a_3^{-1} \downarrow_M b$, we find $\mathrm{tp}(a_3^{-1}/Mb)$ is generic, so $\mathrm{tp}(ba_3^{-1}/Gb)$ is generic. As $\mathrm{tp}(a_3^{-1}/M)$ is generic and $a_3^{-1} \downarrow_M b$ we also get $ba_3^{-1} \downarrow M, b$, so $ba_3^{-1} \downarrow_M b$, and this forces that $p_3' = \mathrm{tp}(ba_3^{-1}/M)$ is generic. Also, by the assumptions on p_3 and r, p_3' has realisations in G^o so is principal.

It follows by (i) that there are $a_1, a_2 \in \bar{M}$ such that $a_1 \models p_1$, $a_2 \models p_2$, and $a_1 a_2 = c_3$. Hence $a_1 a_2 a_3 = b \models r$. ☐

We also observe the following, which can be found for example in [**33**]. The SU-rank on types is an ordinal-valued rank defined by transfinite induction: for any type p over A, $\mathrm{SU}(p) \geq \alpha + 1$ if there is $B \supset A$ such that p has a forking extension q over B with $\mathrm{SU}(q) \geq \alpha$, and for limit ordinals δ, $\mathrm{SU}(p) \geq \delta$ if $\mathrm{SU}(p) \geq \beta$ for all ordinals $\beta < \delta$. If X is a set defined by a formula $\phi(x, \bar{a})$ with $\bar{a}$ from A, then $\mathrm{SU}(X)$ is the supremum (which will be the maximum in the finite rank theories considered here) of the $\mathrm{SU}(p)$ for types p over A containing the formula $\phi(x, \bar{a})$.

LEMMA 4.8. *Let G be a group definable in a finite* SU*-rank supersimple theory, and let A be a parameter set. Then*

(i) If $p \in S_G(A)$ then p is generic if and only if $\mathrm{SU}(p) = \mathrm{SU}(G)$.

(ii) If X is an A-definable subset of G, then $\mathrm{SU}(G) = \mathrm{SU}(X)$ if and only if some generic type $p \in S_G(A)$ contains a formula defining X.

PROOF. (i) See [**33**, p. 168].

(ii) Immediate from (i) and the definition of SU-rank for types and formulas. ☐

THEOREM 4.9. *Let $\mathcal{C}_\tau$ be a family of finite simple groups of fixed Lie type τ, and let $w_i(x_1, \ldots, x_{d_i})$ be non-trivial words, for $i = 1, 2, 3$.*

(i) There is $N = N(\tau, w_1, w_2, w_3) \in \mathbb{N}$ such that if $H \in \mathcal{C}_\tau$ with $|H| > N$ then $w_1(H) w_2(H) w_3(H) = H$.

(ii) $|H \setminus w_1(H) w_2(H)| = o(|H|)$ for sufficiently large $H \in \mathcal{C}_\tau$.

(iii) $|w_1(H)w_2(H)|/|H| \to 1$ *as* $|H| \to \infty$*, for* $H \in \mathcal{C}_\tau$*.*

PROOF. (i) Suppose that (i) is false. Then there is an infinite ultraproduct G^* of members of $\mathcal{C}_\tau$ such that $w_1(G^*)w_2(G^*)w_3(G^*)$ is a proper subset of G^*. Also, G^* is ω_1-saturated, and has a countable elementary substructure G. By the result of Larsen [**17**] mentioned above, there is $\mu > 0$ such that if $H \in \mathcal{C}_\tau$ is sufficiently large then $|w_i(H)| \geq \mu|H|$ for $i = 1, 2, 3$. It follows from Remark 4.5 that $SU(w_i(G^*)) = SU(G^*)$ for each i. Hence, by Lemma 4.8, there is for each $i = 1, 2, 3$ a generic type p_i of G^* over G containing the formula $x \in w_i(G^*)$. As all models of $T := \mathrm{Th}(G)$ are simple, a very saturated model of T cannot have a proper subgroup of bounded index, so $G^* = (G^*)^o_M$ and all generic types of G^* are principal.

Let r be any type over G realised in $G^* \setminus w_1(G^*)w_2(G^*)w_3(G^*)$. Then by Theorem 4.7(ii), there are $a_1, a_2, a_3, b \in G^*$ such that $a_i \models p_i$ and $b \models r$ and $a_1a_2a_3 = b$. In particular, $a_i \in w_i(G^*)$, so $b \in w_1(G^*)w_2(G^*)w_3(G^*)$, which is a contradiction.

(ii) Again, suppose this is false. Then by Theorem 4.4 there is $\nu > 0$ and infinitely many groups $H \in \mathcal{C}_\tau$ such that $|H \setminus w_1(H)w_2(H)| > \nu|H|$. Then, by Remark 4.5, we may choose an infinite ultraproduct G^* of members of $\mathcal{C}_\tau$ such that $\mathrm{SU}(G^* \setminus w_1(G^*)w_2(G^*)) = \mathrm{SU}(G^*)$. Again let G be a countable elementary substructure of G^*. By Lemma 4.8 for $i = 1, 2, 3$ there are generic types p_i of G^* over G such that p_1 contains the formula $x \in w_1(G^*)$, p_2 contains the formula $x \in w_2(G^*)$, and p_3 contains the formula $x \in G^* \setminus w_1(G^*)w_2(G^*)$. By ω_1-saturation and Theorem 4.7(i) there are $a_1, a_2 \in G^*$ such that $a_1 \models p_1$, $a_2 \models p_2$, and $a_3 := a_1a_2 \models p_3$. In particular, $a_i \in w_i(G^*)$ for $i = 1, 2$ so $a_3 \in w_1(G^*)w_2(G^*)$, which is a contradiction.

(iii) This is immediate from (ii). $\square$

REMARK 4.10. 1. Part (i) above is of course just a weakening of a special case of Theorem 4.6. Part (iii) was proved in [**30**]. We do not know whether these model-theoretic methods can yield the stronger assertion that if $\mathcal{C}$ is a family of finite simple groups of fixed Lie rank and w_1, w_2 are non-trivial words, then $w_1(G)w_2(G) = G$ for sufficiently large $G \in \mathcal{C}$.

2. It should be possible to strengthen the asymptotic statements in (ii), (iii), by working with tighter error terms in the definition of 'asymptotic class', in the manner of [**3**] rather than with the o-notation. More precisely, Theorem 4.4 should still hold if, in Definition 4.3, the condition

$$\big||\phi(M^n, \bar{a})| - \mu|M|^{\frac{d}{N}}\big| = o(|M|^{\frac{d}{N}})$$

is replaced by, for some constant c,

$$\big||\phi(M^n, \bar{a})| - \mu|M|^{\frac{d}{N}}\big| \leq c|M|^{\frac{d}{N} - \frac{1}{2}}.$$

We have not checked this.

3. If $w(x_1, \ldots, x_d)$ is a non-trivial word, and $\mathcal{C}$ is a class of finite simple groups of fixed Lie type, then w defines the word map $w : G^d \to G$ for $G \in \mathcal{C}$. Theorem 4.4 is applicable in the class $\mathcal{C}$ to the formula $\phi(x_1, \ldots, x_d, y)$ which says $w(x_1, \ldots, x_d) = y$ and hence yields information on the distribution of the solution sets, that is, on the sizes of the fibres.

Finally, we stress that the proof of Theorem 4.9 depends just on the fact that the sets $w_i(H)$ (for $H \in \mathcal{C}_t$) are uniformly definable and have cardinality a positive proportion of H. This gives the possibility of further applications. For example, translates $hw(H)$ of sets $w(H)$ have the same properties. Thus, the same proof yields the following, with an analogue also of Theorem 4.9(ii), (iii). (The definition of the sets $hw(H)$ requires a parameter, but this causes no problems as, in the proof, the countable elementary submodel G of G^* can be assumed to include any required parameters.)

THEOREM 4.11. *Let $\mathcal{C}_\tau$ be a family of finite simple groups of fixed Lie type τ, and let w_1, w_2, w_3 be non-trivial words. Then there is $N = N(w_1, w_2, w_3, \tau)$ such that if $H \in \mathcal{C}_\tau$ and $|H| > N$ and $h_1, h_2 \in H$, then*

$$w_1(H)h_1w_2(H)h_2w_3(H) = H.$$

Added in proof: Since the final version of this paper was submitted, the authors became aware of the paper [N. Nikolov, L. Pyber, *Product decompositions of quasirandom groups and a Jordan type theorem.* J. Eur. Math. Soc. **13** (2011), 1063–1077]. The above results around word maps (including Theorem 4.11) all follow quickly from the main theorem of Nikolov and Pyber. We consider, though, that the model-theoretic approach has independent interest.

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(1,2)-groups for a regulator quotient of exponent p^4

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Dedicated to Rüdiger Göbel on his 70th birthday

1. Introduction

A torsion-free abelian group G is *completely decomposable of finite rank* if G is isomorphic to a finite direct sum of subgroups of $\mathbb{Q}$, the additive group of rational numbers, and *almost completely decomposable* if G contains a completely decomposable subgroup A with G/A a finite group. A subgroup R of an almost completely decomposable group G is a *regulating subgroup* of G if and only if R is completely decomposable and $|G/R|$ is the least integer in the set $\{|G/A| : A \text{ is completely decomposable with } G/A \text{ finite}\}$, [**7**]. The *regulator* $\mathrm{R}(G)$ is the intersection of all regulating subgroups of G and is again a completely decomposable subgroup of finite index in G. The isomorphism types of the regulator $\mathrm{R}(G)$ and the regulator quotient $G/\mathrm{R}(G)$ are *near-isomorphism* invariants of an almost completely decomposable group, cf. [**3**] The well-known and important theorem of Arnold ([**1**, 12.9, p.144], [**8**, Theorem 10.2.5]) states that two near-isomorphic torsion-free groups of finite rank have (up to near-isomorphism of summands) the same decomposition properties.

We refer to the introduction of [**3**] for the general definitions above.

In this paper we consider a special case. Let p be a prime, $(1,2) = (\tau_1, \tau_2 < \tau_3)$ a set of types, partially ordered as indicated with $\tau_i(p) \neq \infty$. An almost completely decomposable group G is called a $(1,2)$*-group* if $G/\mathrm{R}(G)$ is p-primary and $\mathrm{R}(G) = R_1 \oplus R_2 \oplus R_3$ where R_i is completely decomposable and τ_i-homogeneous. Such a group has a unique regulating subgroup that coincides with its regulator R, and, up to near-isomorphism, unique indecomposable decompositions by the theorem of Arnold. Hence, for $(1,2)$-groups, the main question is to determine the near-isomorphism classes of indecomposable $(1,2)$-groups.

It is already proved in [**10**] that there are no indecomposable $(1,2)$-groups G with $\exp\big(G/\mathrm{R}(G)\big) = p$. In this paper we show that

- there are four near-isomorphism classes of indecomposable $(1,2)$-groups G with

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$\exp\big(G/\mathrm{R}(G)\big) \leq\ p^3$, c.f. [**10**]. The regulator quotients are isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$, $\mathbb{Z}/p^3\mathbb{Z}$, $\big(\mathbb{Z}/p^3\mathbb{Z}\big)\oplus(\mathbb{Z}/p\mathbb{Z})$,

- there are eight near-isomorphism classes of indecomposable $(1,2)$-groups G with $\exp\big(G/\mathrm{R}(G)\big) =\ p^4$. The regulator quotients are isomorphic to $\mathbb{Z}/p^4\mathbb{Z}$, $\big(\mathbb{Z}/p^4\mathbb{Z}\big)\oplus(\mathbb{Z}/p\mathbb{Z})$ and $\big(\mathbb{Z}/p^4\mathbb{Z}\big)\oplus\big(\mathbb{Z}/p^2\mathbb{Z}\big)$.

In [**4**] is shown that there are indecomposable $(1,2)$-groups G of arbitrarily large rank with $\exp\big(G/\mathrm{R}(G)\big)\geq p^6$. The decomposability question for $(1,2)$-groups with $\exp\big(G/\mathrm{R}(G)\big)= p^5$ is not resolved. Clearly, similar questions arise for other small typesets such as $(1,3)$, $(2,2)$ etc.

Our method consists in turning the decomposition question into an equivalence problem for integer matrices.

2. Coordinate Matrices

We need some general notations. The rank of an integer matrix modulo p is called its *p-rank*. A square integer matrix Y is *p-invertible* if $\det Y$ is prime to p, equivalently, if there is an integer matrix Z such that $YZ = ZY \equiv I \mod p^k$ for any integer $k>0$. The *type* of a subgroup $S\subset\mathbb{Q}$ is denoted by $\mathrm{tp}(S)$, and $\leq$ is the order relation in the lattice of types.

A diagonal matrix $S=\mathrm{diag}(p^{k_1},\dots,p^{k_r})$ is called a *structure matrix*, and if $k_1\geq\cdots\geq k_r$, then S is called an *ordered structure matrix*.

Let G be a $(1,2)$-group with regulator $R=R_1\oplus R_2\oplus R_3$ where R_i is homogeneous completely decomposable of rank $r_i\geq 1$ and type τ_i. In particular, $n=\operatorname{rank} G = r_1+r_2+r_3$.

The goal of this section is to describe a $(1,2)$-group by means of an integer matrix, the “coordinate matrix”. The coordinate matrix is obtained by means of “bases” of $R=\mathrm{R}(G)$ and G/R. Let $R_1=\bigoplus_{i=1}^{r_1} S_i x_i$, $R_2=\bigoplus_{i=r_1+1}^{r_1+r_2} S_i x_i$, and $R_3=\bigoplus_{i=r_1+r_2+1}^{r_1+r_2+r_3} S_i x_i$, where $\mathbb{Z}\subset S_i\subset\mathbb{Q}$ and $p^{-1}\notin S_i$. The ordered set $(x_1,\dots,x_n)$ is called a *standard p-basis* of R. Note that the purification of $\langle x_i\rangle$ in R is $\langle x_i\rangle_*^R = S_i x_i$ and $\mathrm{tp}(x_i)=\mathrm{tp}(S_i)$.

Definition. Let G be a $(1,2)$-group. A matrix $\delta=[\delta_{i,j}]$ is a *coordinate matrix* of G modulo R if δ is integral, there is a basis $(g_1+R,\dots,g_r+R)$ of G/R and a standard p-basis $(x_1,\dots,x_n)$ of R such that

$$g_i = p^{-k_i}\big(\textstyle\sum_{j=1}^n \delta_{i,j}x_j\big) \quad\text{where}\quad \langle g_i+R\rangle\cong\mathbb{Z}_{p^{k_i}}\,.$$

A standard p-basis of R divides the coordinate matrix in three blocks α, β, γ, of sizes $r\times r_i$, $i=1,2,3$, and we have $\delta=[\alpha|\beta|\gamma]$.

We will call $(g_1,\dots,g_r)$ *a basis of G modulo R*. Since $(g_1+R,\dots,g_r+R)$ is a basis of G/R, an $r\times n$ coordinate matrix has p-rank r.

The matrix S is called the *structure matrix of G modulo R corresponding* to the basis $(g_1+R,\dots,g_r+R)$ of G/R if $p^{k_i}=\mathrm{ord}(g_i+R)$. It is usually convenient and at places crucial that also the generators of G/R of equal orders are grouped together. Let $G/R\cong\bigoplus_{h=1}^f(\mathbb{Z}_{p^{k_h}})^{l_h}$ where $l_h\geq 1$ and $k=k_1>k_2>\cdots>k_f\geq 1$. Then $S=\mathrm{diag}(p^{k_1}I_{l_1},\dots,p^{k_f}I_{l_f})$ is the ordered structure matrix of size $r=\sum_{h=1}^f l_h$. We will use throughout these orderings of the standard p-basis of R and the basis of G/R. We call a coordinate matrix relative to a standard p-basis a *standard coordinate matrix* if the corresponding structure matrix is ordered.

Coordinate matrices exist in abundance and they are uniquely determined by the bases (x_i) and (g_i), cf. [**3**, Lemmata 6 and 7].

Definition. Let $S = \operatorname{diag}(p^{k_1}, \dots, p^{k_r})$.

(1) Two integer matrices M, M' (both of size $r \times n$) are called *S-congruent* if $m_{i,j} \equiv m'_{i,j} \mod p^{k_i}$, for all i, j. If so, we write $M \equiv_S M'$.
(2) A pair (U, U') of integer matrices that are p-invertible is called an *S-pair* if $US = SU'$.

If instead of g_i only the coset $g_i + R$ is fixed, then the coordinate matrix is unique up to S-congruence, cf. [**3**, Lemma 10].

Note that with δ all $\epsilon \equiv_S \delta$ are coordinate matrices of the group G. Since a matrix M is always S-congruent to a matrix M' where $0 \le m'_{i,j} < p^{k_i}$ we replace the entries in a coordinate matrix that are $\equiv_S 0$ by 0 without changing the group.

The significance of S-pairs lies in their connection with automorphisms of finite abelian groups, see ([**6**, Theorem 3.15]).

The ordering of the basis of G/R has effects for S-pairs. Let (U, U') be an S-pair for $S = \operatorname{diag}(p^{k_1} I_{l_1}, \dots, p^{k_f} I_{l_f})$ where $k = k_1 > k_2 > \dots > k_f \ge 1$. The integers l_h define a block structure on U, namely, $U = [U_{h,m}]_{1 \le h,m \le f}$, where the block $U_{h,m}$ is an $l_h \times l_m$ matrix. A p-invertible $r \times r$ block matrix $U = [U_{h,m}]_{1 \le h,m \le f}$ is the first component of an S-pair (U, U') if and only if all entries of the block $U_{h,m}$ are divisible by $p^{k_h - k_m}$ for $h \le m$. In particular, all p-invertible lower block triangular matrices U, i.e., $U_{h,m} = 0$ for all $h < m$, serve as first components of S-pairs (U, U').

The following theorem is a specialization for $(1, 2)$-groups of the general theorem [**3**, Theorem 12]. It gives a 1-1-translation of the classification of $(1, 2)$-groups to some equivalence problem for coordinate matrices.

Theorem 1. *Let G be a $(1, 2)$-group with the standard coordinate matrix δ relative to the standard p-basis $(x_1, \dots, x_n)$, and the (ordered) basis $(g_1, \dots, g_r)$ of G modulo R. Let S be the (ordered) structure matrix.*

(i) *Let ϵ be the standard coordinate matrix of G relative to the standard p-basis $(y_1, \dots, y_n)$ and the (ordered) basis $(h_1, \dots, h_r)$ of G modulo R such that x_i and y_i have equal types for $i = 1, \dots, n$. Then there is an S-pair (U, U') and a matrix*

$$Y = \begin{bmatrix} Y_{1,1} & 0 & 0 \\ 0 & Y_{2,2} & Y_{2,3} \\ 0 & 0 & Y_{3,3} \end{bmatrix} \tag{1}$$

where $Y_{i,j}$ is an $r_i \times r_j$ integer matrix and the diagonal blocks $Y_{i,i}$ are p-invertible such that $\epsilon \equiv_S U\delta Y$.

(ii) *Conversely, suppose that an S-pair (U, U') and a matrix Y of the form (1) are given. Then there is a group H nearly isomorphic to G containing R as regulator, a basis of H modulo R, and a standard p-basis $(y_1, \dots, y_n)$ of R such that H has the structure matrix S, x_i and y_i have equal types for $i = 1, \dots, n$, and $U\delta Y$ is a standard coordinate matrix of H.*

We show next how one can recognize the regulator of a $(1, 2)$-group ([**3**, Lemma 13]).

Lemma 2. (**Regulator Criterion**) *Let G be a $(1, 2)$-group that is the extension of the completely decomposable R by the finite p-group G/R. Then the following statements are equivalent:*

(1) *R is the regulator of G;*
(2) *$R(\tau) = G(\tau)$ for all critical types;*
(3) *if $\delta = [\alpha \mid \beta \mid \gamma]$ is a coordinate matrix of G with r rows, then α and $[\beta \mid \gamma]$ both have p-rank r.*

By Arnold's Theorem two near-isomorphic torsion-free groups of finite rank have (up to near-isomorphism of summands) the same decomposition properties. Hence, given a coordinate matrix we may manipulate the matrix in the ways described in Theorem 1 which means that we obtain coordinate matrices of the same group or of a nearly isomorphic group. If we arrive at a matrix that shows that the group to which it belongs decomposes or not, then the original group is decomposable or not.

3. Some matrix results

We want a reduced form for coordinate matrices and introduce some necessary notation. The term *line* means a row or a column. An integer u is a *p-unit* if $\gcd(p, u) = 1$. If so, for any integer $k > 0$, there is $u' \in \mathbb{Z}$ such that $uu' \equiv 1 \mod p^k$. Often, we simply say "unit" in place of p-unit because there are no other units in use.

Let $A = [a_{i,j}]$ be an integer $r \times n$ matrix and let $S = \operatorname{diag}(p^{k_1}, \ldots, p^{k_r})$. We extend S-congruence to entries of A by defining: $a_{i,j}$ is *S-congruent to a*, denoted as $a_{i,j} \equiv_S a$, if $a_{i,j} \equiv a \mod p^{k_i}$.

A matrix is *decomposed* if it is of the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Here either one of the matrices A, B is allowed to have no rows or no columns, i.e., the decomposed matrices include the special cases

$$[0\ B], \quad \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad [A\ 0], \quad \begin{bmatrix} A \\ 0 \end{bmatrix}.$$

A matrix is *properly decomposed* if the blocks A, B both have rows and columns.

A matrix A is called *decomposable* if there are row and column permutations that transform it to a decomposed form, i.e., there are permutation matrices P, Q such that PAQ is decomposed. Similarly, as above we use the term *properly decomposable.*

A matrix is *S-decomposed* or *S-decomposable* if it is S-congruent to a decomposed or decomposable matrix, respectively. Note that A is S-decomposable if there are permutation matrices P, Q such that PAQ is PSP^{-1}-decomposed.

Let $A = [a_{i,j}]$ be an integer matrix and let S be an ordered structure matrix. Then A is called *S-reduced* if

(1) modulo p the matrix A has at most one entry $\neq 0$ in a line,
(2) if the nonzero entries of $A \mod p$ are at the positions (i_s, j_s), then $a_{i_s,j} \equiv_S 0$ for all $j > j_s$ and $a_{i,j_s} \equiv_S 0$ for all $i > i_s$, and $a_{i_s,j}, a_{i,j_s} \in p\,\mathbb{Z}$ for all $j < j_s$ and all $i < i_s$.

A row or column transformation of a matrix is equivalent to left or right multiplication by a corresponding matrix, respectively. We use both approaches simultaneously. The context clarifies the meaning. Often we use elementary row transformations that add the multiple of a row to a row below itself with a corresponding lower triangular elementary matrix, and elementary column transformations that add a multiple of a column to a column to the right of itself with corresponding upper triangular elementary matrix.

LEMMA 3. ([**3**, Lemma 14]) *Let A be an $r \times n$ integer matrix and S an ordered structure matrix. Then there are two p-invertible matrices U, Y with the following properties.*

(1) *U is a product of lower triangular elementary matrices, where each elementary factor annihilates an entry $\not\equiv_S 0$,*
(2) *Y is a product of upper triangular elementary matrices, where each elementary factor annihilates an entry $\not\equiv_S 0$,*

such that UAY is S-reduced.

In particular, if the i^{th} line of A is $\equiv_S 0$, then the i^{th} line of UAY is $\equiv_S 0$.

The following configuration in a coordinate matrix leads immediately to direct summands of small rank. Let $A = [a_{i,j}]$ be an integer matrix and S a structure matrix. The matrix A has a *cross at* (i_0, j_0) if $a_{i_0,j_0} \not\equiv_S 0$ and $a_{i_0,j} \equiv_S 0$, $a_{i,j_0} \equiv_S 0$ for all $i \neq i_0$ and $j \neq j_0$. We say that the cross is *located* in a subblock of a matrix if the position (i_0, j_0) is in this subblock. Note that a matrix with a cross is S-decomposable.

It is convenient to call an integer $r \times n$ matrix $D = [d_{i,j}]$ *p-diagonal* if all entries $d_{i,j} = 0$ for $i \neq j$ and the diagonal entries are p-powers or 0, i.e., $d_{i,i} = p^{s_i}$ for nonnegative integers s_i, or $d_{i,i} = 0$.

We continue this section with a lemma that is a modification of the elementary divisor theorem for integer matrices.

LEMMA 4. ([**3**, Lemma 15]) *Let l, r, n, k be natural numbers where $1 \leq l \leq r$ and $n \geq 1$. For an $r \times n$ integer matrix H there are p-invertible matrices Y and U where $U = \begin{bmatrix} U_1 & pU_2 \\ U_3 & U_4 \end{bmatrix}$ and U_1 is a (p-invertible) $l \times l$ matrix such that UHY is congruent modulo p^k to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where A, B are p-diagonal with l and $r - l$ rows, respectively.*

If $l = r$, then B has no rows and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \end{bmatrix}$; if A has no columns, then $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 \\ B \end{bmatrix}$; if B has no lines then $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = [A]$.

For an ordered structure matrix S the S-decomposability of an integer matrix is inherited by its S-reduced forms.

COROLLARY 5. ([**3**, Corollary 17]) *Let A be an integer matrix and S an ordered structure matrix. If A is S-decomposable, then the S-reduced forms of A are S-decomposable. More precisely, if for permutation matrices P, Q the matrix PAQ is S-decomposed, and if B is an S-reduced form of A, then PBQ is also S-decomposed, and this S-decomposition is possibly finer than the S-decomposition of PAQ.*

In particular, if the matrix A has a 0-line modulo S or a cross, then an S-reduced form of A has a 0-line modulo S or a cross at the same position, respectively.

4. Direct Decomposition and Coordinate Matrices

We are mainly interested in direct decompositions of our groups. Lemma 6 clarifies how the decomposability of an almost completely decomposable group appears in coordinate matrices.

A group G is *decomposable* if $G = G_1 \oplus G_2$ for some $G_1 \neq 0 \neq G_2$ and *indecomposable* otherwise. A group is *clipped* if it has no completely decomposable direct summands.

LEMMA 6. ([**3**, Lemma 19]) *A clipped, p-reduced, p-local almost completely decomposable group G with regulating regulator R is directly decomposable if and only if it has a properly decomposable coordinate matrix.*

COROLLARY 7. ([**3**, Corollary 20]) *A $(1,2)$-group G with standard coordinate matrix $\delta = [\alpha|\beta|\gamma]$, and corresponding structure matrix S, is decomposable if and only if there is a first component U of an S-pair and a matrix Y of the form* (1) *such that $U\delta Y$ is S-decomposable.*

In the following the left multiplication of a standard coordinate matrix by the first component U of an S-pair is realized by a sequence of row transformations and the right multiplication by a matrix Y of the form (1) is realized by a sequence of column transformations. However, due to the required structure of the matrices U and Y that are allowed as multipliers, only certain special row and column transformations are allowed as follows.

LEMMA 8. ([**3**, Lemma 21 and 24]) *Let the $(1,2)$-group G be given by a standard coordinate matrix $\delta = [\alpha|\beta|\gamma]$ and an ordered structure matrix $S = \operatorname{diag}(p^{k_1}, \dots, p^{k_r})$. Then the following row and column operations on the coordinate matrix lead to a standard coordinate matrix of a group that is nearly isomorphic to G.*

(1) *Any multiple of a row may be added to any row below it.*
(2) *Any multiple of the $p^{k_{i_1}-k_{i_2}}$-fold of row i_2 may be added to a row $i_1 < i_2$.*
(3) *Any line may be multiplied by an integer relatively prime to p.*
(4) *All elementary column operations can be applied to α and γ.*
(5) *Any multiple of a column of β may be added to another column of $[\beta|\gamma]$.*

The transformations in Lemma 8 are called *allowed.* Modulo S-congruence, the column operations (4) allow getting the reduced column-echelon form for α and γ.

If it happens that, while annihilating an entry, other entries that were zero modulo S change to nonzero entries, then those entries are called *fill-ins.*

LEMMA 9. *Let G be a clipped $(1,2)$-group with a standard coordinate matrix $[\alpha|\beta|\gamma]$ and ordered structure matrix $S = \operatorname{diag}(p^{k_1} I_{l_1}, \dots, p^{k_h} I_{l_h})$. Then the following statements hold:*

(1) *There are allowed column transformations that transform α to I_r modulo S without changing $[\beta|\gamma]$.*
(2) *There are allowed row and column transformations that transform β into an S-reduced form $\tilde{\beta}$. Let s be the number of units in $\tilde{\beta}$. Then γ is an $r \times (r-s)$ matrix.*
Conversely, if $\tilde{\beta}$ is a S-reduced form of β then $[I_r|\tilde{\beta}|\tilde{\gamma}]$ is a coordinate matrix of G for an arbitrary $r \times (r-s)$ matrix $\tilde{\gamma}$ if $[\tilde{\beta}|\tilde{\gamma}]$ is of rank r.
(3) *There are allowed row and column transformations that turn the first l_1 rows of β into a p-diagonal matrix.*
(4) *If $k_1 = k_2 + 1$, then the first $l_1 + l_2$ rows of β can be transformed into the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where A, B are p-diagonal matrices (possibly without columns) with l_1 and l_2 rows, respectively.*

(5) *For fixed regulator and fixed regulator quotient a clipped* $(1,2)$*-group is, up to near isomorphism, uniquely determined by* β.

PROOF. (1) Since G is clipped, there is no allowed column transformation that creates a 0-column in α. Moreover, α has p-rank r, thus α is p-invertible. So the reduced column-echelon form of α is $\equiv_S I_r$. This reduced column-echelon form is achieved by column transformations in α that do not change the part $[\beta|\gamma]$ of the coordinate matrix.

(2) By Lemma 3 there are allowed row and column transformations that change even $[\beta|\gamma]$ into an S-reduced form, say $[\tilde{\beta}|\tilde{\gamma}]$. Each unit in $\tilde{\beta}$ creates a 0-row in $\tilde{\gamma}$. Thus the rank of $\tilde{\gamma}$ is $r-s$, i.e., the number of columns is $\geq r-s$. Since γ can be transformed to reduced column echelon form by allowed transformations and since G is clipped the number of columns of $\tilde{\gamma}$ is precisely $r-s$. Only allowed transformations were done so $[I_r|\tilde{\beta}|\tilde{\gamma}]$ is a coordinate matrix of G for an arbitrary $r\times(r-s)$ matrix $\tilde{\gamma}$ if $[\tilde{\beta}|\tilde{\gamma}]$ is of rank r.

(3) In the first l_1 rows of β any row and column transformation is allowed.

(4) Lemma 4.

(5) Suppose that the groups G and G' have standard coordinate matrices $[\alpha|\beta|\gamma]$ and $[\alpha'|\beta|\gamma']$, respectively. By (2) we may assume that γ and γ' have the same 0-rows and have the same reduced column echelon form, i.e., they are equal. Finally, as in (1) we may assume that $\alpha\equiv_S\alpha'\equiv_S I_r$. This means that G and G' are both near-isomorphic to the same group and therefore near-isomorphic to one another.

Certain features of the coordinate matrix of a $(1,2)$-group signal the existence of direct summands of small ranks.

COROLLARY 10. *Let* $\delta=[\alpha|\beta|\gamma]$ *be a standard coordinate matrix of a* $(1,2)$*-group* G*, and let* $S=\operatorname{diag}(p^{k_1}I_{l_1},\dots,p^{k_h}I_{l_h})$ *be the ordered structure matrix. Then the following statements hold:*

(1) *If* δ *contains a* 0*-column, then* G *has a direct summand of rank* 1.
(2) *If* β *contains a* 0*-row, then* G *has a direct summand of rank* 2.
(3) *If* β *contains a cross, then* G *has a summand of rank* 2 *or* 3. *Rank* 3 *happens if and only if the cross entry* $\neq 0$ *is not a unit.*
(4) *If* β *has a unit in the first* l_1 *rows, then* G *has a summand of rank* 2.
(5) *If* $k_1=k_2+1$ *and* β *has a unit in the first* l_1+l_2 *rows, then* G *has a summand of rank* 2.

PROOF. (1) is obvious.

The claims (2) to (4) follow easily using Lemma 9 (1) and (2).

(5) If $k_1=k_2+1$ and β has a unit in a row with index between l_1 and l_1+l_2, then, by Lemma 9 (4), there is a cross located at the first l_1+l_2 rows of β. Since the nonzero entry of this cross is a unit we may extend this cross to a cross of the whole $[\beta|\gamma]$ by allowed row and column transformations. Thus G has a summand of rank 2. □

EXAMPLE 11. Let G be a $(1,2)$-group with regulator quotient of exponent p^4.

(1) Let $R=\mathbb{Z}[q^{-1}]x_1\oplus\mathbb{Z}[q^{-1}]x_2\oplus\mathbb{Z}[r^{-1}]y_1\oplus\mathbb{Z}[(rs)^{-1}]z_1\oplus\mathbb{Z}[(rs)^{-1}]z_2$ be the regulator of G, and $R\subset G=\langle R,g_1,g_2\rangle$, $g_1=p^{-4}(x_1+z_1)$, $g_2=p^{-4}(x_2+z_2)$.

Then the coordinate matrix

$$\delta = \begin{bmatrix} 1 & 0 & | & 0 & | & 1 & 0 \\ 0 & 1 & | & 0 & | & 0 & 1 \end{bmatrix}$$

has a 0-column, and $\langle y_1 \rangle_*$ is a direct summand of rank 1.

(2) Let $R = \mathbb{Z}[q^{-1}]x_1 \oplus \mathbb{Z}[q^{-1}]x_2 \oplus \mathbb{Z}[r^{-1}]y_1 \oplus \mathbb{Z}[(rs)^{-1}]z_1$ be the regulator of G, and $R \subset G = \langle R, g_1, g_2 \rangle$, $g_1 = p^{-4}(x_1 + z_1)$, $g_2 = p^{-4}(x_2 + y_1)$. Then the coordinate matrix

$$\delta = \begin{bmatrix} 1 & 0 & | & 0 & | & 1 \\ 0 & 1 & | & 1 & | & 0 \end{bmatrix}$$

has a 0-row (and also a unit in the first l_1 rows), and $\langle x_1, z_1 \rangle_*$ (and also $\langle x_2, y_1 \rangle_*$) is a direct summand of rank 2.

(3) Let $R = \mathbb{Z}[q^{-1}]x_1 \oplus \mathbb{Z}[q^{-1}]x_2 \oplus \mathbb{Z}[r^{-1}]y_1 \oplus \mathbb{Z}[r^{-1}]y_2 \oplus \mathbb{Z}[(rs)^{-1}]z_1$ be the regulator of G, and $R \subset G = \langle R, g_1, g_2 \rangle$, $g_1 = p^{-4}(x_1 + y_1)$, $g_2 = p^{-4}(x_2 + py_2 + z_1)$. Then the coordinate matrix

$$\delta = \begin{bmatrix} 1 & 0 & | & 1 & 0 & | & 0 \\ 0 & 1 & | & 0 & p & | & 1 \end{bmatrix}$$

has two crosses, one with cross entry a unit, the other not, and $\langle x_1, y_1 \rangle_*$ and $\langle x_2, y_2, z_1 \rangle_*$ are direct summands of rank 2 and 3, respectively.

(4) Let $R = \mathbb{Z}[q^{-1}]x_1 \oplus \mathbb{Z}[q^{-1}]x_2 \oplus \mathbb{Z}[q^{-1}]x_3 \oplus \mathbb{Z}[r^{-1}]y_1 \oplus \mathbb{Z}[(rs)^{-1}]z_1 \oplus \mathbb{Z}[(rs)^{-1}]z_2$ be the regulator of G, and $R \subset G = \langle R, g_1, g_2 \rangle$, $g_1 = p^{-4}(x_1 + py_1 + z_1)$, $g_2 = p^{-3}(x_2 + y_1)$. Then the coordinate matrix

$$\delta = \begin{bmatrix} 1 & 0 & 0 & | & p & | & 1 & 0 \\ 0 & 1 & 0 & | & 1 & | & 0 & 0 \end{bmatrix}$$

has a unit in the first $l_1 + l_2$ rows and can be transformed to

$$\delta' = \begin{bmatrix} 1 & 0 & 0 & | & 0 & | & 1 & 0 \\ 0 & 1 & 0 & | & 1 & | & 0 & 0 \end{bmatrix}.$$

Hence $\langle x_2, y_1 \rangle_*$ is a direct summand of rank 2.

5. Indecomposable $(1, 2)$-Groups

Proposition 12. *If the part β of a standard coordinate matrix $[\alpha|\beta|\gamma]$ of a $(1, 2)$-group G is S-decomposable, then G is decomposable. Conversely, if G is decomposable without direct summands of rank ≤ 2, then it has some standard coordinate matrix with properly decomposable submatrix β.*

Proof. Let β be S-decomposable. By Lemma 3 there is an S-reduced form $\tilde{\beta} = U\beta Y'$ where U is a lower triangular matrix and Y' is an upper triangular matrix. The two matrices U and $Y = \operatorname{diag}(I_r, Y', I_{r_3})$ are allowed row and column transformations for the coordinate matrix $[\alpha|\beta|\gamma]$, and $U[\alpha|\beta|\gamma]Y = [U\alpha|\tilde{\beta}|U\gamma]$. By Lemma 9 (1) the part $U\alpha$ can be changed to the identity matrix I_r modulo S. By Lemma 9 (2) the part $U\gamma$, can be changed to $\tilde{\gamma}$ modulo S, where $\tilde{\gamma}$ is the identity matrix enlarged by some 0-rows.

These transformations do not affect $\tilde{\beta}$. Now, since β is S-decomposable, also $\tilde{\beta} = U\beta Y'$ is S-decomposable by Corollary 5. Hence the new coordinate matrix is

$[I_r \mid \tilde{\beta} \mid \tilde{\gamma}]$ modulo S. So the S-decomposability of β is inherited and by Corollary 7 the group G is decomposable.

Conversely, let G be decomposable without direct summands of rank ≤ 2. Then, by Lemma 6, our group G has a properly decomposable coordinate matrix. By permutations of the rows and of the columns we get a standard coordinate matrix $[\alpha|\beta|\gamma]$ for G. Clearly, this coordinate matrix is decomposable. Since G has no direct summand of rank ≤ 2 the part β of the coordinate matrix has no 0-line by Corollary 10 (2). But then the proper decomposability of the coordinate matrix $[\alpha|\beta|\gamma]$ implies the proper decomposability of β. □

Lemma 13. *The following four* $(1,2)$*-groups* G *with regulator quotient of exponent* $\leq p^3$ *given by the isomorphism types of their regulator with fixed types, their regulator quotient, and their coordinate matrix* $\delta = [\alpha|\beta|\gamma]$ *are indecomposable and pairwise not near-isomorphic.*

(i) $\delta = [1|p|1]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 3$.
(ii) $\delta = [1|p|1]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3}$ *and* $\operatorname{rank} G = 3$.
(iii) $\delta = [1|p^2|1]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3}$ *and* $\operatorname{rank} G = 3$.
(iv) $\delta = \left[\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\ \middle|\ \begin{matrix}p\\ 1\end{matrix}\ \middle|\ \begin{matrix}1\\ 0\end{matrix}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$ *and* $\operatorname{rank} G = 4$.

Proof. The claims on ranks are clear and the regulator property and the structure of the regulator quotient are easily verified.

First we will show that the indicated groups are indecomposable. By Proposition 12 a group without direct summands of rank ≤ 2 and with coordinate matrix $[\alpha|\beta|\gamma]$ is indecomposable if and only if β is S-indecomposable and, by Theorem 1 this is the case if and only if $U\beta Y_\beta$ is not S-decomposable where U is the first component of an S-pair (in particular p-invertible) and $Y_\beta = \begin{bmatrix} Y_{1,1} & Y_{1,2} \\ 0 & Y_{2,2}\end{bmatrix}$ is the relevant submatrix of a matrix Y as in Equation (1). In our examples the $Y_{i,j}$ are integers and we may assume that $Y_\beta = \begin{bmatrix} 1 & a \\ 0 & 1\end{bmatrix}$ since multiplication by a p-invertible diagonal matrix from the right hand side will not change a decomposition but allows to get entries 1 on the diagonal.

(i),(ii), and (iii) are obvious.

For (iv) we recall that the 2×2 matrix $U = [u_{i,j}]$ is p-invertible, so by the same argument with the diagonal matrix, as above, but multiplying by a diagonal matrix from the left hand side, we may assume that either $u_{1,1} = u_{2,2} = 1$ or $u_{1,2} = u_{2,1} = 1$. We deal with only the first case, the second case is similar. It is obvious that the following matrix has no 0 line and is not decomposable modulo $S = \operatorname{diag}(p^3, p)$.

$$\begin{bmatrix} 1 & p^2 b \\ c & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \equiv \begin{bmatrix} p + p^2 b \\ 1 \end{bmatrix}.$$

Now we will show that the indicated groups are pairwise not near-isomorphic. Recall that the isomorphism types of the regulator and the regulator quotient are invariants. Only the groups G under (ii) and (iii) coincide in their regulator and regulator quotient, but they differ in the minimal p-power p^h such that $p^h G$ is decomposable. So they are not near-isomorphic. □

Smith Normal Form

In the Theorems 14 and 16 we establish Smith Normal Forms for a matrix A, i.e., p-diagonal matrices. By this we mean firstly that this is an allowed transformation, i.e., there are p-invertible matrices U, Y such that UAY is a p-diagonal matrix. We mean secondly that it is possible to reestablish submatrices affected by these transformations. This may require a number of steps. We always want to reestablish all submatrices that were originally either 0 or p^hI, $h \geq 0$. There is a detailed treatment in [**3**, Example 29].

We present a shorter proof of a known result for (1,2)-groups with a regulator quotient of exponent p^3.

Theorem 14. [**10**, Corollary 5.5] *There are precisely the four near isomorphism types in* Lemma 13 *of indecomposable* $(1,2)$*-groups with regulator quotient of exponent* $\leq p^3$.

Proof. If the regulator quotient $\mathbb{Z}_{p^2}^{l_1} \oplus \mathbb{Z}_p^{l_2}$ is of exponent p^2, then by Corollary 10 (4) and (5) all entries of β are in $\mathbb{Z}_p$. To avoid a 0-row we have $l_2 = 0$, and the regulator quotient is homocyclic. But then we may establish the Smith Normal Form pI for β. Hence only the group of type (i) is indecomposable.

Now let us assume that the regulator quotient is isomorphic to $\mathbb{Z}_{p^3}^{l_1} \oplus \mathbb{Z}_{p^2}^{l_2} \oplus \mathbb{Z}_p^{l_3}$, where $l_1 \geq 1$. Further assume that without loss of generality G has no direct summand of rank ≤ 2. By Corollary 10 (1), (2) and (3), this assumption means that the part β of a standard coordinate matrix of G has no 0-lines and crosses located at a unit. Then by Lemma 4, avoiding units in the first $l_1 + l_2$ rows and avoiding 0-rows we may assume that β has the form

$$\left[\begin{array}{cc|cc} p^2I & 0 & 0 & 0 \\ 0 & pI & 0 & 0 \\ \hline 0 & 0 & pI & 0 \\ \hline A_1 & A_2 & A_3 & A_4 \end{array}\right] \quad \begin{array}{c} p^3 \\ p^3 \\ \hline p^2 \\ \hline p \end{array} \; \begin{array}{l} \\ l_1 \\ l_1 + l_2 \end{array}$$

The entries of A_1, A_2, A_3 and A_4 are either units or zero. A unit in A_4 can be used to create a cross located at this unit, a contradiction. Thus, $A_4 = 0$. But since G is clipped, A_4 and the column with A_4 do not exist. Similarly, a unit in A_3 can be used to create a cross located at this unit, a contradiction. Hence $A_3 = 0$. But this leads to a group with a regulator quotient of exponent p^2. So we may assume that the A_3-column and the p^2-row are not present. Hence $l_2 = 0$, and β has the form

$$\left[\begin{array}{cc} p^2I & 0 \\ 0 & pI \\ \hline A_1 & A_2 \end{array}\right] \quad \begin{array}{c} p^3 \\ p^3 \\ \hline p \end{array} \; l_1$$

A unit in A_1 allows to create zeros in its row. If the A_2-column is present, then also the p^2I-row is present and the fill-ins in the A_2-column can be removed using pI below. Then the unit can be used to create a cross located at this unit, a contradiction. Thus, $A_1 = 0$. Since by assumption G has no direct summand of rank ≤ 2 there is no 0-row in A_2. Then the Smith Normal Form of A_2 is $\begin{bmatrix} I & 0 \end{bmatrix}$. Hence we end up with the final form for β:

$$\beta = \left[\begin{array}{ccc} p^2I & 0 & 0 \\ 0 & pI & 0 \\ 0 & 0 & pI \\ \hline 0 & I & 0 \end{array}\right] \quad \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ \hline p \end{array} \; l_1$$

By Proposition 12, the S-decomposability of β determines the decomposability of the group G. Thus, by the final form of β that we obtained above it is easy to read off the groups of types (ii), (iii) and (iv). For instance, the row vector of the first row displays a direct summand with coordinate matrix $[1, p^2, 1]$ of type (iii). $\square$

PROPOSITION 15. *The following eight* $(1,2)$*-groups* G *with regulator quotient of exponent* p^4 *given by the isomorphism types of their regulator with fixed types, their regulator quotient and their coordinate matrix* $\delta = [\alpha|\beta|\gamma]$ *are indecomposable and pairwise not near-isomorphic.*

(i) $\delta = [1|p|1]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^4}$ *and* $\operatorname{rank} G = 3$.

(ii) $\delta = [1|p^2|1]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^4}$ *and* $\operatorname{rank} G = 3$.

(iii) $\delta = [1|p^3|1]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^4}$ *and* $\operatorname{rank} G = 3$.

(iv) $\delta = \left[\begin{array}{cc|c|c} 1 & 0 & p^2 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^4} \oplus \mathbb{Z}_p$ *and* $\operatorname{rank} G = 4$.

(v) $\delta = \left[\begin{array}{cc|c|c} 1 & 0 & p & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^4} \oplus \mathbb{Z}_p$ *and* $\operatorname{rank} G = 4$.

(vi) $\delta = \left[\begin{array}{cc|c|c} 1 & 0 & p & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^4} \oplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 4$.

(vii) $\delta = \left[\begin{array}{cc|c|cc} 1 & 0 & p^2 & 1 & 0 \\ 0 & 1 & p & 0 & 1 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^4} \oplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 5$.

(viii) $\delta = \left[\begin{array}{cc|cc|c} 1 & 0 & p & 0 & 1 \\ 0 & 1 & 1 & p & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^4} \oplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 5$.

PROOF. The claims on ranks are clear and the regulator property and the structure of the regulator quotient are easily verified.

By Proposition 12 a group without direct summands of rank ≤ 2 and with standard coordinate matrix $[\alpha|\beta|\gamma]$ is indecomposable if and only if β is S-indecomposable and, by Theorem 1 this is the case if and only if $U\beta Y_\beta$ is not S-decomposable where U is the first component of an S-pair (in particular p-invertible) and $Y_\beta = \begin{bmatrix} Y_{1,1} & Y_{1,2} \\ 0 & Y_{2,2} \end{bmatrix}$ is the relevant submatrix of a matrix Y as in Equation (1). In our examples the $Y_{i,j}$ are integers and we may assume that $Y_\beta = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ since multiplication by a p-invertible diagonal matrix from the right hand side will not change a decomposition but allows to get entries 1 on the diagonal.

First we will show that the indicated groups are indecomposable.

(i), (ii) and (iii) are obvious.

For (iv) we recall that the 2×2 matrix $U = [u_{i,j}]$ is p-invertible, so by the same argument with the diagonal matrix, as above, but multiplying by a diagonal matrix from the left hand side, we may assume that either $u_{1,1} = u_{2,2} = 1$ or $u_{1,2} = u_{2,1} = 1$. We deal with only the first case, the second case is similar. It is enough to observe that there are no 0-entries modulo $S = \operatorname{diag}(p^4, p)$ in the column

$$\begin{bmatrix} 1 & ap^3 \\ c & 1 \end{bmatrix} \begin{bmatrix} p^2 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} p^2 + ap^3 \\ cp^2 + 1 \end{bmatrix} \equiv \begin{bmatrix} p^2 + ap^3 \\ 1 \end{bmatrix}. \tag{2}$$

Similarly, for (v) it is enough to note that there are no 0-entries modulo $S = \operatorname{diag}(p^4, p)$ in the column

$$\begin{bmatrix} 1 & ap^3 \\ c & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \equiv \begin{bmatrix} p + ap^3 \\ cp + 1 \end{bmatrix} \equiv \begin{bmatrix} p + ap^3 \\ 1 \end{bmatrix}.$$

For (vi) it is enough to verify that modulo p^4 there are no 0-entries modulo $S = \operatorname{diag}(p^4, p^2)$ in the column

$$\begin{bmatrix} 1 & ap^2 \\ c & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \equiv \begin{bmatrix} p + ap^2 \\ cp + 1 \end{bmatrix}. \tag{3}$$

For (vii) it is obvious that the following matrix has no 0-entries modulo $S = \operatorname{diag}(p^4, p^2)$ in the column

$$\begin{bmatrix} 1 & ap^2 \\ c & 1 \end{bmatrix} \begin{bmatrix} p^2 \\ p \end{bmatrix} \equiv \begin{bmatrix} p^2 + ap^3 \\ cp^2 + p \end{bmatrix} \equiv \begin{bmatrix} p^2 + ap^3 \\ p \end{bmatrix}.$$

For (viii) it is enough to state that the following matrix has no 0-line modulo $S = \operatorname{diag}(p^4, p^2)$ and is not S-decomposable.

$$\begin{bmatrix} 1 & ap^2 \\ c & 1 \end{bmatrix} \begin{bmatrix} p & 0 \\ 1 & p \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} p + ap^2 & b(p + ap^2) + ap^3 \\ cp + 1 & b(cp + 1) + p \end{bmatrix}.$$

Since the entries in the first column are both not 0 modulo S, the only possibility for a decomposition is $b \equiv 0$ modulo p^2. But then the entry at position $(2, 2)$ is not 0 modulo p^2.

It remains to show that the eight groups above are pairwise not near-isomorphic. Since the isomorphism types of the regulator and the regulator quotient are near-isomorphism invariants we distinguish the groups first by their rank, then by the isomorphism types of the regulator and the regulator quotient.

The groups G under (i), (ii) and (iii) coincide in the regulator and the regulator quotient, but they differ in the minimal p-power p^h such that $p^h G$ is decomposable. So they are not near-isomorphic.

Among the groups G of rank 4 we have to consider more closely only the cases (iv) and (v), but they differ in the minimal p-power p^h such that $p^h G$ is decomposable. So they are not near-isomorphic.

The groups under (vii) and (viii) of rank 5 have different regulators. So they are not near-isomorphic. □

Theorem 16. *There are precisely the eight near isomorphism types in* Proposition 15 *of indecomposable* $(1, 2)$*-groups with regulator quotient of exponent* p^4*.*

Proof. The regulator quotient is isomorphic to $\mathbb{Z}_{p^4}^{l_1} \oplus \mathbb{Z}_{p^3}^{l_2} \oplus \mathbb{Z}_{p^2}^{l_3} \oplus \mathbb{Z}_{p}^{l_4}$, where $l_1 \geq 1$. It easy to see that every indecomposable $(1, 2)$-group of rank 3 is of type Proposition 15 (i), (ii) or (iii). Therefore we further assume without loss of generality that G has no direct summand of rank ≤ 3. This assumption means that the part β of the coordinate matrix of G has no 0-lines and no crosses, cf. Corollary 10 (1)-(3). We also assume that there is no summand with regulator quotient of exponent p^3.

We apply Lemma 4 to the first $l_1 + l_2$ rows of β. This is an allowed transformation. So we get for the first $l_1 + l_2$ rows of β the form $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ where X, Y are

p-diagonal. By Corollary 10 (5) the matrices X, Y have no units. By ordering the p-powers, by shifting the 0-columns to the right, and since β has no 0-row we get for the first $l_1 + l_2$ rows of β the form as in (4). The $l_3 + l_4$ rows below are denoted by "place holders" as

$$\begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{bmatrix}.$$

With a unit in A_6 we can annihilate in its column and in its row. The same holds for the matrices A_1, A_2, A_3, A_4 and A_5. So this leads to crosses located in the A-row, a contradiction. Here let us be more detailed. With a unit, for example, in A_1 we annihilate all entries in its row in A_1. This is done by a right multiplication with a p-invertible matrix, say Q, that multiplies the whole column of β. In particular, the matrix p^3I changes to p^3Q. But then we multiply this row with Q^{-1} from the left and reestablish p^3I without any further change. After that we annihilate in p^3I and get a 0-row, a contradiction. Altogether, β has the form:

$$\beta = \left[\begin{array}{ccc|ccc} p^3I & 0 & 0 & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & 0 & pI & 0 \\ \hline pA_1 & pA_2 & A_3 & pA_4 & pA_5 & pA_6 \\ \hline C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array}\right] \begin{array}{c} p^4 \\ p^4 \\ p^4 \\ \hline p^3 \\ p^3 \\ \hline p^2 \\ \hline p \end{array} \begin{array}{l} \\ \\ l_1 \\ \\ l_1 + l_2 \\ l_1 + l_2 + l_3 \\ \\ \end{array} \tag{4}$$

The submatrix pA_5 can be annihilated by means of pI in the p^3-row. Moreover, units in C_6, C_4, C_1 lead to crosses, the same way as for the A's, a contradiction. Hence those blocks are 0 and we get:

$$\beta = \left[\begin{array}{ccc|ccc} p^3I & 0 & 0 & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & 0 & pI & 0 \\ \hline pA_1 & pA_2 & A_3 & pA_4 & 0 & pA_6 \\ \hline 0 & C_2 & C_3 & 0 & C_5 & 0 \end{array}\right] \begin{array}{c} p^4 \\ p^4 \\ p^4 \\ \hline p^3 \\ p^3 \\ \hline p^2 \\ \hline p \end{array} \begin{array}{l} \\ \\ l_1 \\ \\ l_1 + l_2 \\ l_1 + l_2 + l_3 \\ \\ \end{array} \tag{5}$$

With a unit in C_2 we annihilate its row in C_5. The fill-ins in the (p^4, C_5)-block and in the (p^2, C_5)-block can be removed by pI in the p^3-row. So, if there is a unit in C_5, then its row in C_2 is 0 and we annihilate in C_3. This creates fill-ins in the C_3-column to the left of pI in the p^3-row. Those fill-ins are in $p\,\mathbb{Z}$, so they can be removed by pI in the p^4-row. This way we get a direct summand with regulator quotient of exponent p^3, a contradiction. Thus $C_5 = 0$ and there is a cross located in pI, again a contradiction. Altogether, the C_5-column and the part of the p^3-row with pI are not present.

The Smith Normal form of C_2 is $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. This splits the p-row in two rows and the C_2-column in two columns. So β has the form

$$\beta = \left[\begin{array}{cccc|cc} p^3I & 0 & 0 & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & p^2I & 0 \\ \hline pA_1 & pA_2 & pA_2' & A_3 & pA_4 & pA_6 \\ \hline 0 & I & 0 & C_3 & 0 & 0 \\ 0 & 0 & 0 & C_3' & 0 & 0 \end{array}\right] \begin{array}{cc} p^4 & \\ p^4 & \\ p^4 & \\ p^4 & \\ \hline p^3 & l_1 \\ \hline p^2 & l_1 + l_2 \\ \hline p & l_1 + l_2 + l_3 \\ p & \end{array} \tag{6}$$

There are two cases depending on whether the A-row is present or not.

(i) Assume that the A-row is not present. Then the row and column with p^3I and the second p^2I in the p^4-row do not exist to avoid a cross. Moreover, the p^3-row and the last two columns of β do not exist to avoid a cross or a 0-column.
The submatrix C_3 can be annihilated by I in the p-row. The fill-ins in the C_3-column above can be removed by pI below in the p^4-row. But then the Smith Normal of C_3' is I since it has no 0-row to avoid a 0-row in β, and no 0-column to avoid a cross. Thus,

$$\beta = \left[\begin{array}{cc} p^2I & 0 \\ 0 & pI \\ \hline I & 0 \\ 0 & I \end{array}\right] \begin{array}{c} p^4 \\ p^4 \\ \hline p \\ p \end{array} \tag{7}$$

Now we read off the indecomposable groups of types (iv) and (v) in Proposition 15.

(ii) Assume that the A-row is present. Then the Smith Normal form of pA_6 is $\begin{bmatrix} pI \\ 0 \end{bmatrix}$ since there is no 0-column. This splits the p^2-row into two. Hence β has the form:

$$\beta = \left[\begin{array}{cccc|cc} p^3I & 0 & 0 & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & p^2I & 0 \\ \hline pA_1 & pA_2 & pA_2' & A_3 & pA_4 & pI \\ pB_1 & pB_2 & pB_2' & B_3 & pB_4 & 0 \\ \hline 0 & I & 0 & C_3 & 0 & 0 \\ 0 & 0 & 0 & C_3' & 0 & 0 \end{array}\right] \begin{array}{cc} p^4 & \\ p^4 & \\ p^4 & \\ p^4 & \\ \hline p^3 & l_1 \\ \hline p^2 & l_1 + l_2 \\ p^2 & l_1 + l_2 + l_3 \\ \hline p & \\ p & \end{array} \tag{8}$$

The matrix (8) includes the case that pA_6 is missing in (6) by omitting the A-row and the last column and renaming B to A.
The submatrices pA_1, pA_2, pA_2' and pA_4 can be annihilated by pI in the same row. This annihilation creates no fill-ins. Moreover, the submatrix pB_2 can be annihilated by means of I in the (p, B_2)-block, and the submatrix C_3 can be annihilated by I in the same row. The fill-ins in the

C_3-column above can be removed by pI below in the p^4-row. Thus,

$$\beta = \left[\begin{array}{cccc|cc} p^3I & 0 & 0 & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & p^2I & 0 \\ \hline 0 & 0 & 0 & A_3 & 0 & pI \\ pB_1 & 0 & pB_2' & B_3 & pB_4 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_3' & 0 & 0 \end{array}\right] \begin{array}{c} p^4 \\ p^4 \\ p^4 \\ p^4 \\ \hline p^3 \\ \hline p^2 \\ p^2 \\ \hline p \\ p \end{array} \begin{array}{l} \\ \\ \\ \\ l_1 \\ l_1+l_2 \\ \\ l_1+l_2+l_3 \\ \\ \end{array} \tag{9}$$

Suppose that there is a p in pB_1. This p can be used to create 0 in its row in pB_1. This changes p^3I to p^3Q with p-invertible Q, as discussed above. But p^3Q can be transformed back to p^3I by row transformations alone. Now we annihilate with this p in p^3I. There are fill-ins in the p^3I-row and the columns of pB_2', B_3, pB_4 that are in $p^2\,\mathbb{Z}$ and in $p^3\,\mathbb{Z}$, just so that they can be removed by p^2I, pI in the p^4-row and by p^2I in the p^3-row, respectively. But then there is a 0-row in β, a contradiction. Hence $pB_1 = 0$ and consequently the row and column with p^3I is not present. So we get:

$$\beta = \left[\begin{array}{ccc|cc} p^2I & 0 & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 \\ \hline 0 & 0 & 0 & p^2I & 0 \\ \hline 0 & 0 & A_3 & 0 & pI \\ 0 & pB_2' & B_3 & pB_4 & 0 \\ \hline I & 0 & 0 & 0 & 0 \\ 0 & 0 & C_3' & 0 & 0 \end{array}\right] \begin{array}{c} p^4 \\ p^4 \\ p^4 \\ \hline p^3 \\ \hline p^2 \\ p^2 \\ \hline p \\ p \end{array} \begin{array}{l} \\ \\ \\ l_1 \\ l_1+l_2 \\ \\ l_1+l_2+l_3 \\ \\ \end{array} \tag{10}$$

Similarly, a p in pB_4 allows to annihilate in its row in pB_4. This changes p^2I to p^2Q with p-invertible Q, as discussed above. But p^2Q can be transformed back to p^2I by row transformations alone. With this p we annihilate in p^2I in the p^3-row. This creates fill-ins in the (p^3, B_3)-block that are in $p\,\mathbb{Z}$, and they can be removed by means of pI in the p^4-row. Further, there are fill-ins in the (p^3, B_2')-block that are in $p^2\,\mathbb{Z}$ and those can be removed by p^2I above. But this causes a 0-row in β, a contradiction. Hence $pB_4 = 0$ and the row and column with p^2I in the p^3-row do not exist. Thus β has the form:

$$\beta = \left[\begin{array}{ccc|c} p^2I & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 \\ 0 & 0 & pI & 0 \\ \hline 0 & 0 & A_3 & pI \\ 0 & pB_2' & B_3 & 0 \\ \hline I & 0 & 0 & 0 \\ 0 & 0 & C_3' & 0 \end{array}\right] \begin{array}{c} p^4 \\ p^4 \\ p^4 \\ \hline p^2 \\ p^2 \\ \hline p \\ p \end{array} \tag{11}$$

The entries of B_3 which are in $p\,\mathbb{Z}$ can be annihilated by pI in the p^4-row. Hence the Smith Normal form of B_3 is $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and it splits the matrix pI in the p^4-row above and we get $\begin{bmatrix} pI & 0 \\ 0 & pI \end{bmatrix}$ instead. Moreover, the columns with matrices A_3 and C_3', and the row with pB_2' split in two columns and rows, respectively, labeled as shown in (12). Then β has the form:

$$\beta = \left[\begin{array}{cccc|c} p^2I & 0 & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & A_3 & A_3' & pI \\ 0 & pB_2' & I & 0 & 0 \\ 0 & pB_2'' & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 & 0 \\ 0 & 0 & C_3' & C_3'' & 0 \end{array}\right] \begin{array}{c} p^4 \\ p^4 \\ p^4 \\ p^4 \\ \hline p^2 \\ p^2 \\ p^2 \\ \hline p \\ p \end{array} \tag{12}$$

By using I in the p^2-row the submatrix pB_2' can be annihilated. The fill-ins in the (p^4, B_2'')-block are in $p^2\,\mathbb{Z}$ and can be annihilated by p^2I in the p^4-row, the fill-ins in the (p^2, B_2'')-block can be removed by pI in the same row and the fill- ins in the (p, B_2'')-block can be disregarded. Then the submatrices A_3 and C_3' can be annihilated by means of I in the p^2-row. This creates no fill-ins. Thus we get

$$\beta = \left[\begin{array}{cccc|c} p^2I & 0 & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & 0 & A_3' & pI \\ 0 & 0 & I & 0 & 0 \\ 0 & pB_2'' & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_3'' & 0 \end{array}\right] \begin{array}{c} p^4 \\ p^4 \\ p^4 \\ p^4 \\ \hline p^2 \\ p^2 \\ p^2 \\ \hline p \\ p \end{array} \tag{13}$$

The Smith Normal form of pB_2'' is pI since it has no 0-line. Moreover, the Smith Normal Form of A_3' is $[I \mid 0]$ since A_3' has no 0-row. This splits the matrix pI in the p^4-row and the submatrix C_3''. Then β has the form:

$$\beta = \left[\begin{array}{ccccc|c} p^2I & 0 & 0 & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 \\ 0 & 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & 0 & I & 0 & pI \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & pI & 0 & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_3'' & C_3''' & 0 \end{array}\right] \begin{array}{c} p^4 \\ p^4 \\ p^4 \\ p^4 \\ p^4 \\ \hline p^2 \\ p^2 \\ p^2 \\ \hline p \\ p \end{array} \tag{14}$$

The submatrix C_3'' can be annihilated by I in the p^2-row above. The fill-ins are in $p\,\mathbb{Z}$ and can be ignored. But then the Smith Normal Form of

C_3''' is I since it has no 0-line to avoid a cross. Thus,

$$\beta = \left[\begin{array}{ccccc|c} p^2I & 0 & 0 & 0 & 0 & 0 \\ 0 & p^2I & 0 & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 \\ 0 & 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & 0 & I & 0 & pI \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & pI & 0 & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{array}\right] \begin{array}{cc} 1 & p^4 \\ 2 & p^4 \\ 3 & p^4 \\ 4 & p^4 \\ 5 & p^4 \\ \hline 6 & p^2 \\ 7 & p^2 \\ 8 & p^2 \\ \hline 9 & p^1 \\ 10 & p^1, \end{array} \tag{15}$$

Now we read off the indecomposable groups. The pairs of rows (1,9), (2,8), (3,7), (4,6), and (5,10) lead to the groups of types (iv), (vii), (vi), (viii), and (v) in Proposition 15, respectively.

□

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On the macro structure of torsion free groups

J. D. Reid

To Rüdiger Göbel on the occasion of his retirement

Abstract. We view torsion free abelian groups of finite rank from a certain rather macro perspective, and give a few applications of this point of view to suggest its usefulness.

1. Introduction

We discuss a way of viewing torsion free groups, restricted here to finite rank, which has applications to various classes of groups that have arisen in the literature. The general idea is very simple and must have occurred to many, but we want to point out how this approach can be used to gain insight into a variety of otherwise perhaps isolated phenomena, notably examples of weird structural elements [**4**], the so-called omega groups [**3**] and the quasi divisible groups of Beaumont and Pierce [**2**].

Our second major purpose is to supply proofs, as promised in [**5**], of a few results announced without proofs there.

In Section two we set up the umbrella environment and make a few comments on it in an attempt to whet the reader's interest, comments as well that will be useful later. Section three discusses the first of the major results, motivation for it, and a full proof. Section four discusses the endomorphism rings of the groups introduced in Section three. We provide detailed information on these rings. Finally in Section five we indicate how our general environment and some of the results give a new (and insightful, we believe) approach to some classical results on quotient divisible groups.

The title of the paper is meant to suggest that our approach here is an explicitly global one, an alternative to localization as in for example the local treatment of q.d. groups by Beaumont and Pierce. This is in a sense dual to localization. The referee, to whom we hereby express our thanks for this and other remarks, has in fact suggested that the groups ${}_pG$ defined in the next section can be viewed as *co-localizations* of the group G. We happily adopt this terminology, which also motivates the choice of position for the subscript, distinguishing thereby the co-localization ${}_pG$ from the localization G_p. A sense of this duality is given by Proposition 1 together with the representation (†) indicated at the end of Section two.

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2. Some Generalities

For a torsion free group G of finite rank, we first choose a fixed full free subgroup X of G. Recall that this just means that the quotient G/X is a torsion group. Write $\overline{G}$ for this quotient and $\overline{G}_p$ for its p-primary component so $\overline{G} = \oplus_p \overline{G}_p$ is the direct sum. Each component $\overline{G}_p$ has the form ${}_pG/X$ for a uniquely determined subgroup ${}_pG$ of G and we then have $G = \sum_p {}_pG$. As indicated, we call the groups ${}_pG$ *co-localizations* of G.

Now consider the set $\Phi = \Phi_X$ of all endomorphisms of G that map the chosen free subgroup X into itself:

$$\Phi = \{\alpha \in End(G) \,|\, \alpha(X) \subseteq X\}.$$

We call these *sited* endomorphisms of G, or *endomorphisms sited at* X if it is necessary to reference the site[1]. We observe the obvious fact that Φ is a subring of the ring of endomorphisms of G and in fact is a full subring since, for any endomorphism β of G, $\beta(X)$ is finitely generated and X is full in G. Note also that $\overline{G}$ is in a natural way a Φ-module since G and X are, and therefore the primary components $\overline{G}_p$ are Φ-modules so that, finally, the co-localizations ${}_pG$ of G are as well. We will sometimes view G and these associated structures in this way.

We note that finiteness of rank of G carries with it the fact that any two full free subgroups X, X' of G are quasi equal so the corresponding rings Φ_X, $\Phi_{X'}$ are also quasi equal. It is easy to see as well that the co-localizations ${}_pG$, which of course depend on the choice of X, are never-the-less uniquely determined up to quasi equality. So we have a certain invariance, but we will usually stick with a fixed choice of X. Here is a first indication of the possible usefulness of all this.

PROPOSITION 1. *If any one of the co-localizations ${}_pG$ is strongly indecomposable then G is strongly indecomposable. The rank of the endomorphism ring of each ${}_pG$ is at least equal to the rank of the endomorphism ring of G, and if any one of the rings $End({}_pG)$ is commutative then $End(G)$ is commutative as well.*

PROOF. The first statement is clear since a quasi decomposition of G corresponds to an endomorphism γ of G such that $\gamma^2 = n\gamma$ for some integer n; and such will exist in the endomorphism ring of G only if a similar map exists in the associated ring Φ. Now Φ acts faithfully in ${}_pG$ and so may be viewed as a subring of the endomorphism ring of ${}_pG$. Therefore $\mathrm{rank}(End(G)) = \mathrm{rank}(\Phi) \leq \mathrm{rank}(End({}_pG))$. Certainly Φ is commutative if $End({}_pG)$ is, and then $End(G)$ is commutative too, since Φ is full in $End(G)$. $\square$

Note however that ${}_pG$ might quasi decompose without G doing so since Φ will in general be smaller than the full ring of endomorphisms of ${}_pG$.

We have the obvious homomorphism σ of the direct sum of the co-localizations ${}_pG$ onto G which we might describe by $\sigma(\oplus g_p) = \Sigma g_p \in G$. Denote by ν the natural map of G onto its quotient $\overline{G} = G/X$ and suppose that $\oplus g_p$ lies in the kernel of σ, $\sigma(\oplus g_p) = \Sigma g_p = 0$. Then certainly $\Sigma \overline{g}_p = 0$ as well. But in the quotient the sum $\oplus_p \overline{G}_p$ is direct so that we must have $\overline{g}_p = 0$ for each p, that is, $g_p \in X$ for all p. We conclude that the kernel of the map σ is just $\oplus_p X$ with X imbedded in each

[1] Obseerve the author's steely resolve in avoiding notation such as "X-sited" here.

${}_pG$ in the obvious way. This gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \oplus_p X & \to & \oplus_p G & \to & \oplus \overline{G}_p & \to & 0 \\ & & \downarrow \sigma & & \downarrow \sigma & & \| & & \\ 0 & \to & X & \to & G & \to & \oplus \overline{G}_p & \to & 0 \end{array}$$

so that we can realize G as the pushout of the diagram

$$\begin{array}{ccc} \oplus_p X & \to & \oplus_p G \\ \downarrow \sigma & & \\ X & & \end{array} .$$

Finally we note a certain representation of the group G. As just indicated, the map σ has kernel consisting of elements $\oplus g_p \in \oplus_p G$ with all g_p lying in X. Choosing a basis $\{x_k\}$ for X and denoting $\operatorname{rank}(X) = n$, let us write

$$g_p = \sum_{k=1}^{n} a_{pk} x_k$$

with the coefficients $a_{pk} \in Z$. Then $\sigma(\oplus g_p) = 0$ means that

$$\begin{aligned} 0 = \sum g_p &= \sum_p \sum_{k=1}^{n} a_{pk} x_k \\ &= \sum_{k=1}^{n} \left(\sum_p a_{pk} \right) x_k \end{aligned}$$

so $\sum_p a_{pk} = 0$ for all k. We let $\mathbb{A}$ denote the set of all $m \times n$ matrices over Z with row sums equal to zero. Here m denotes the number of primes in play, and we allow it perhaps to be infinite. Now we have the map of $\mathbb{A}$ into $\oplus_p G$

$$A \mapsto \oplus g_p$$

where $g_p = \sum_{k=1}^{n} a_{pk} x_k$. It follows that

$$(\dagger) \qquad 0 \to \mathbb{A} \to \oplus_p G \xrightarrow{\sigma} G \to 0$$

is exact.

3. Some special groups

Definition 3.1. The symbol $S(G)$ will denote the set of all subgroups of G. An ω-group is a group G for which

$$|S(G)| < 2^{|G|}.$$

It is easy to see [**3**] that the study of ω-groups reduces to the torsion free case, and clearly a torsion free ω-group must have finite rank. From [**3**] we have

Theorem 3.2. *A torsion free group G is an ω-group if and only if there is an exact sequence*

$$0 \to X \to G \to \prod_{p \in F(G)} Z(p^\infty) \to 0$$

with X free of finite rank and $F(G)$ a finite set of primes.

From our general representation theorem and this result we are led to consider the special case of groups G that fit into an exact sequence as above in which the set $F(G)$ of primes has cardinality 1. That is, groups G such that there is an exact sequence

$$0 \to X \to G \to Z(p^\infty) \to 0$$

with X free of finite rank. These include for example the groups that arise in an interesting (classical) example of Fuchs [**4**]. So we concentrate our attention on these groups, or if one prefers, on $Ext(Z(p^\infty), X)$ for a while. This cohomology group is easy to compute, though we give some details here for the reader's convenience, but for our purposes we need much more information than just its structure. Here is the computation.

Having chosen a fixed basis $\{x_k\}_{k=1}^n$ for X, and fixed generators $z_1, z_2, \ldots$ for $Z(p^\infty)$ with $pz_1 = 0, pz_{k+1} = z_k, k \geqslant 1$, we consider the sequence

$$0 \to X \to X_p \to \Delta_p \to 0$$

where $X_p = \mathbb{Z}\left[\frac{1}{p}\right] \otimes X$, $\Delta_p = X_p/X = Z(p^\infty)^n$ and $\mathbb{Z}\left[\frac{1}{p}\right]$ is the set of rational numbers whose denominators are powers of the fixed prime p. This yields[2]

$$0 \to Hom(Z(p^\infty), \Delta_p) \to Ext(Z(p^\infty), X) \to 0$$

so the sequences

$$0 \to X \to G \to Z(p^\infty) \to 0$$

of interest are parameterized by J_p^n, J_p the p-adic integers, i.e.

$$J_p^n = Ext(Z(p^\infty), X) \cong Hom(Z(p^\infty), \Delta_p).$$

Explicitly, the exact sequence given by $(\sigma_1, \sigma_2, \ldots, \sigma_n) \in J_p^n$ is defined by the pullback

$$(*) \qquad \begin{array}{ccccccccc} 0 & \to & X & \to & G_h & \overset{\nu}{\to} & Z(p^\infty) & \to & 0 \\ & & \| & & \downarrow \lambda & & \downarrow h & & \\ 0 & \to & X & \to & X_p & \overset{\mu}{\to} & \Delta_p & \to & 0 \end{array}$$

where h is the map defined by $h(z_k) = (\sigma_1 z_k, \sigma_2 z_k, \ldots, \sigma_n z_k) \in \Delta_p$. Thus G_h is given by

$$G_h = \{(y,z) \in X_p \times Z(p^\infty) \mid \nu(y) = h(z)\}$$

and $\lambda(y,z) = y$, $\nu(y,z) = z$. Here is our result:

Given h, and the corresponding sequence

$$0 \to X \to G_h \to Z(p^\infty) \to 0,$$

we get

$$0 \to Hom(G_h, \mathbb{Z}) \to Hom(X, \mathbb{Z}) \overset{\partial_h}{\to} Ext(Z(p^\infty), \mathbb{Z}) \cong J_p.$$

We remark that the exact sequence just above and associated maps have the following interpretation. Firstly, $Ext(Z(p^\infty), \mathbb{Z})$, as identified with classes of short exact sequences, is the free rank one J_p module generated by the sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}\left[\frac{1}{p}\right] \to Z(p^\infty) \to 0.$$

[2]We use the fact that $Ext(Z(p^\infty), \mathbb{Z}\left[\frac{1}{p}\right]) = 0$ here. To see this, apply $Hom(Z(p^\infty), -)$ to the sequence $0 \to \mathbb{Z}\left[\frac{1}{p}\right] \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}\left[\frac{1}{p}\right] \to 0$. Note also that $Ext(Z(p^\infty), X) = Ext(Z(p^\infty), \mathbb{Z})^n = J_p^n$.

A p-adic integer α acts on this sequence by assigning to it the pullback determined by

$$\begin{array}{ccccccccc} & & & & & & Z(p^\infty) & & \\ & & & & & & \downarrow \alpha & & \\ 0 & \to & \mathbb{Z} & \to & \mathbb{Z}\left[\frac{1}{p}\right] & \to & Z(p^\infty) & \to & 0 \end{array}$$

Secondly the connecting homomorphism ∂_h assigns to a map $f : X \to \mathbb{Z}$ that uniquely determined p-adic integer (i.e. endomorphism of $Z(p^\infty)$, denoted here again by α) that makes the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \to & X & \to & G_h & \xrightarrow{\nu} & Z(p^\infty) & \to & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow \alpha & & \\ 0 & \to & \mathbb{Z} & \to & \mathbb{Z}\left[\frac{1}{p}\right] & \to & Z(p^\infty) & \to & 0 \end{array}.$$

The mapping g in the middle of this diagram is the composite

$$G_h \xrightarrow{\lambda} \mathbb{Z}\left[\frac{1}{p}\right] \otimes X \xrightarrow{1 \otimes f} \mathbb{Z}\left[\frac{1}{p}\right].$$

THEOREM 3.3. *The group $G = G_h$ is torsion free if and only if at least one of the σ_i is a p-adic unit. The image in the sequence above of the j^{th} projection $\pi_j \in Hom(X, \mathbb{Z})$ under the connecting homomorphism ∂_h is the j^{th} p-adic integer σ_j belonging to h.*

PROOF. We leave the proof of the first statement to the reader. For the second statement, we recall that the image of a map $f \in Hom(X, Z)$ under the connecting homomorphism is given by the pushout, and results in the defining commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \to & G_h & \to & Z(p^\infty) & \to & 0 \\ & & \downarrow f & & \downarrow & & \| & & \\ 0 & \to & \mathbb{Z} & \to & G' & \to & Z(p^\infty) & \to & 0 \end{array}.$$

Now in addition to the diagram (*), we also have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \to & \mathbb{Z}\left[\frac{1}{p}\right] \otimes X & \to & Z(p^\infty)^n & \to & 0 \\ & & \downarrow \pi_j & & \downarrow \pi_j & & \downarrow \overline{\pi}_j & & \\ 0 & \to & \mathbb{Z} & \to & \mathbb{Z}\left[\frac{1}{p}\right] & \to & Z(p^\infty) & \to & 0 \end{array}$$

in which we have made explicit the notation $X_p = \mathbb{Z}\left[\frac{1}{p}\right] \otimes X$ and $\Delta_p = Z(p^\infty)^n$, so the meaning of the vertical maps is clear. We may combine this with (*) to obtain

$$\begin{array}{ccccccccc} 0 & \to & X & \to & G_h & \to & Z(p^\infty) & \to & 0 \\ & & \| & & \downarrow & & \downarrow h & & \\ 0 & \to & X & \to & X_p & \xrightarrow{\nu} & \Delta_p & \to & 0 \\ & & \downarrow \pi_j & & \downarrow \pi_j & & \downarrow \overline{\pi}_j & & \\ 0 & \to & \mathbb{Z} & \to & \mathbb{Z}\left[\frac{1}{p}\right] & \to & Z(p^\infty) & \to & 0 \end{array}$$

and this is commutative. Note that here the composition $\overline{\pi}_j \circ h = \sigma_j$, the j^{th}component of h. Clearly now, $\partial_h(\pi_j) = \sigma_j$ as asserted. $\square$

4. Endomorphisms

We now have a good hold on the groups discussed in the previous section, i.e., those with $G/X = Z(p^\infty)$, just one copy for a single prime. Given h, or the sequence $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, what can we say about the ring $End(G)$ of endomorphisms of the group $G = G_h$?

We will use the following notation:

$$\begin{aligned}
\Phi &= \{\varphi \in End(G) \,|\, \varphi(X) \subseteq X\}.\\
U &= \mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2 \ldots + \mathbb{Z}\sigma_n \subseteq J_p.\\
\Lambda &= \{\lambda \in J_p \,|\, \lambda U \subseteq U\}.
\end{aligned}$$

Here is our second main result:

THEOREM 4.1. *For indecomposable G_h, the ring Φ of sited endomorphisms is isomorphic to the subring Λ of J_p.*

PROOF. Recalling the diagram (*) giving G_h,

$$\begin{array}{ccccccccc}
0 & \to & X & \to & G_h & \to & Z(p^\infty) & \to & 0\\
 & & \| & & \downarrow\lambda & & \downarrow h & & \\
0 & \to & X & \to & X_p & \xrightarrow{\nu} & \Delta_p & \to & 0
\end{array}$$

we note that indecomposability implies that the vertical maps are monic. In particular, λ is an imbedding of G into $X_p = \mathbb{Z}\left[\frac{1}{p}\right] \otimes X$. We have chosen a basis $\{x_k\}$ of X and these elements give a basis of X_p as free module over the ring $\mathbb{Z}\left[\frac{1}{p}\right]$. The map h is given on the generators z_k of $Z(p^\infty)$ by

$$h(z_k) = (\sigma_1 z_k, \sigma_2 z_k, \ldots, \sigma_n z_k).$$

Now each σ_i is a p-adic integer and so we have a representation

$$\sigma_i = \sum_{j=0}^{\infty} s_{ij} p^j$$

where the s_{ij} are residues mod p. We write

$$\sigma_{ik} = \sum_{j=0}^{k-1} s_{ij} p^j$$

for the partial sum whose limit is σ_i and then note that since z_k in $Z(p^\infty)$ has order p^k we have $\sigma_i z_k = \sigma_{ik} z_k$. Therefore the element

$$y_k = \sum_{i=1}^{n} \sigma_{ik} \frac{x_i}{p^k}$$

in X_p has image $h(z_k) = (\sigma_1 z_k, \sigma_2 z_k, \ldots, \sigma_n z_k) = (\sigma_{1k} z_k, \sigma_{2k} z_k, \ldots, \sigma_{nk} z_k)$ in Δ_p. This tells us that each element of G, as imbedded in X_p has the form $c_k y_k + w$ for some integer c_k and element $w \in X$.

Now let α be any endomorphism of X, say

$$\alpha(x_i) = \sum_{j=1}^{n} a_{ji} x_j$$

with the $a_{ji} \in Z$. Of course α extends to an endomorphism of X_p and this maps $\lambda(G)$ into itself, thus corresponds to a sited endomorphism in Φ, if and only if $\alpha(y_k) \in \lambda(G)$ for all k, and it is easy to see that this holds if and only if

$$\alpha(y_k) = c_k y_k + w_k$$

for all k, where $c_k \in \mathbb{Z}$ and $w_k \in X$. Suppose this is the case now. Then from $py_{k+1} \equiv y_k \bmod X$, we get

$$p\alpha(y_{k+1}) \equiv \alpha(y_k) \equiv c_k y_k \bmod X$$

or

$$pc_{k+1} y_{k+1} \equiv c_k y_k \bmod X.$$

Since $py_{k+1} \equiv y_k$ this implies that $c_{k+1} y_k \equiv c_k y_k \bmod X$ and therefore that $c_{k+1} \equiv c_k \bmod p^k$.Thus the sequence $\{c_k\}$determines a p-adic integer, γ say, and it is clear that γ gives the action of the map α on Δ_p.

Finally now,

$$\begin{aligned} \alpha(y_k) &= \alpha\left(\sum_{i=1}^{n} \sigma_{ik} \frac{x_i}{p^k}\right) = \sum_{i=1}^{n} \sigma_{ik} \frac{\alpha(x_i)}{p^k} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ik} a_{ij} \frac{x_j}{p^k} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \sigma_{ik} a_{ij}\right) \frac{x_j}{p^k} \end{aligned}$$

and

$$\begin{aligned} c_k y_k + w_k &= \sum_{j=1}^{n} c_k \sigma_{jk} \frac{x_i}{p^k} + \sum_{j=1}^{n} m_j x_j \\ &= \sum_{j=1}^{n} \left(c_k \sigma_{jk} + p^k m_j\right) \frac{x_j}{p^k}. \end{aligned}$$

We conclude that

$$\sum_{i=1}^{n} \sigma_{ik} a_{ij} \equiv c_k \sigma_{jk} \bmod p^k$$

for all k. This gives

$$\sum_{i=1}^{n} \sigma_i a_{ij} = \gamma \sigma_j$$

and we see that γ lies in the ring Λ as asserted.

The proof of the converse is similar, but easier and we leave it to the reader. □

COROLLARY 1. *Indecomposable groups G_h have commutative endomorphism rings (subrings of algebraic number fields).*

PROOF. The ring Φ is commutative and full in $End(G_h)$ so the latter is commutative as well. It is clearly contained isomorphically in the pure closure of Λ in J_p,and this is an algebraic number field. □

5. Applications

The groups discussed in detail above have many remarkable properties. As mentioned, Fuchs [**4**], see also [**1**], constructed such groups using algebraically independent sets of p-adic integers $\{\sigma_1 = 1, \sigma_2, \ldots, \sigma_n\}$ and showed that any proper pure subgroup of the constructed group is free, but the group itself is indecomposable. Refining the approach in Arnold [**1**] by fully exploiting the structure of $Z(p^\infty)$, it is easy to see [**5**] that in the torsion free case every subgroup, pure or not, of such a group G_h is either free or has finite index. As a consequence these groups are either (strongly) indecomposable or have free summands. It then follows from Theorem 3.3 and the exactness of the sequence

$$0 \to Hom(G_h, \mathbb{Z}) \to Hom(X, \mathbb{Z}) \xrightarrow{\partial_h} Ext(Z(p^\infty), \mathbb{Z})$$

that G_h is (strongly) indecomposable if and only if the associated sequence $\{\sigma_k\}$ of p-adic integers is rationally independent.

We might remark on the classification problem for the groups corresponding to the sequences $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of p-adic integers. Since they are quotient divisible groups, they have been assigned a *q.d. invariant* by Beaumont and Pierce [**2**], that is, a sequence (δ_q) indexed by the primes q, in which δ_q is a vector space over the corresponding field $\widehat{\mathbb{Q}}_q$ of q-adic numbers. Then two quotient divisible groups G and G' with invariants (δ_q) and (δ'_q) are quasi-isomorphic if and only if there is a $\mathbb{Q}$-linear map φ whose q-adic extension maps δ_q onto δ'_q for each q. We refer to [**2**] for details in general, but in our case the situation is quite simple and transparent. The q.d. invariant for the group G corresponding to the sequence $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of p-adic integers has $\delta_q = 0$ for $q \neq p$, while δ_p is the one dimensional space generated by $y = \sigma_1 x_1 + \sigma_2 x_2 + \cdots + \sigma_n x_n$. The criterion amounts to the statement that two such groups G and G' are quasi-isomorphic if and only if there is an $n \times n$ non-singular matrix over $\mathbb{Q}$ which takes the vector $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of parameters for G onto the vector $(\sigma'_1, \sigma'_2, \ldots, \sigma'_n)$ for G'. Thus there are continuum many quasi-isomorphism classes of these groups.

We do not do it here but it seems that the representation results of Section two here allow a global approach to the invariants for q.d. groups dual to the usual local approach. We have one final remark along these lines though, and it concerns a result of Beaumont and Pierce giving a sufficient condition for the indecomposability of a q.d. group in terms of the invariants. This result, Corollary 5.27 in [**2**], has always seemed somewhat "out of the blue" to this author but in terms of our results, it has a very natural and simple form and explanation. Here is our version of the result and a new proof:

Theorem 5.1. *Let G be a quotient divisible group, X a free subgroup modulo which G is divisible, and let ${}_pG$, p a prime, be the groups defined in Section two, the co-localizations of G. Suppose that for some prime p, ${}_pG/X = Z(p^\infty)$ and that the sequence $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ of p-adic integers giving ${}_pG$ is rationally independent. Then G is (strongly)) indecomposable.*

Proof. As we have seen above, the hypothesis on the p-adic integers σ_i implies that the corresponding subgroup ${}_pG$ is (strongly) indecomposable. Then by Proposition 1 we conclude that G is as well. $\square$

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Some results on the algebraic entropy

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Dedicated to Rüdiger Göbel in occasion of his 70th birthday

Abstract. Three new results on the algebraic entropy are presented. The first result removes one of the sufficient conditions for the uniqueness of the algebraic L-entropy, where L is a length function on the category of modules over an arbitrary ring. The second result proves that the finite topology of the endomorphism ring of a left module over an arbitrary ring determines the entropical behaviour of the endomorphisms. The third result shows that the Corner's realization theorem for endomorphism rings of Abelian p-groups, as well as its generalization by Corner-Göbel, do not produce, in a plausible setting, p-groups with global algebraic entropy zero.

1. Introduction

The goal of this paper is to prove three new results on the algebraic entropy. The first one removes one sufficient condition ensuring the uniqueness of the algebraic L-entropy as an extension to the category Mod$R[X]$ of a length function L defined on the category Mod$R$ of R-modules, where R is an arbitrary ring. The Uniqueness Theorem for the L-entropy was one of the main achievements of the recent paper by Vámos, Virili and the author [**SVV**]. The second result shows that the topological endomorphism ring of a left module over an arbitrary ring R, endowed with the finite topology, determines the entropical behaviour of the endomorphisms, in the sense that two modules with isomorphic endomorphism rings, algebraically and topologically, have simultaneously all their endomorphisms with algebraic entropy zero or not; this is not true if the isomorphism between the two endomorphism rings is only algebraic. The third result concerns a problem settled in a recent paper by Göbel and the author [**GS**], which asks to construct two Abelian p-groups G and H such that $End(G)/Small(G) \cong End(H)/Small(H)$, and $ent(G) = 0$, while $ent(H) = \infty$. We show that the Corner's realization theorem for endomorphisms of p-groups cannot provide a positive solution to the problem for a plausible setting of possible examples, differently to the Corner-Göbel realization theorem in the torsion-free case, as proved in [**GS**]. The same Corner-Göbel realization theorem, in the "torsion theory" version, cannot be used to provide p-groups G with $ent(G) = 0$, since the groups produced by that theorem are not semi-standard.

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2. On the Uniqueness of the L-Entropy

Let R be a ring and Mod$R$ the category of the left R-modules. Let $L : \mathrm{Mod}R \to \mathbb{R}^*$ ($\mathbb{R}^* = \mathbb{R}_{\geq 0} \cup \{\infty\}$) be a length function, that is, an additive upper continuous invariant. Recall that the function L is an invariant if $L(M) = L(N)$ for two isomorphic R-modules M and N, and $L(\{0\}) = 0$; L is additive if, given an exact sequence $0 \to A \to B \to C \to 0$ of R-modules, $L(B) = L(A) + L(C)$; L is upper continuous if, for every R-module M, $L(M) = \sup_F L(F)$, where F ranges over the family of the finitely generated submodules of M. The length function L is called discrete if the set of finite values of L is order isomorphic to $\mathbb{N}$ (see [**NR**] or [**SVV**] for more details on these notions). Two typical examples of discrete length functions are the following ones.

EXAMPLE 2.1. *Let $R = \mathbb{Z}$ be the ring of the integers. The invariant $L_1 : Mod\mathbb{Z} \to \mathbb{R}^*$, which assignes to every Abelian group G the logarithm of its cardinality: $L_1(G) = log|G|$ if $|G|$ is finite, and $L_1(G) = \infty$ if $|G|$ is infinite, is additive, by Lagrange Theorem, and it is clearly upper continuous and discrete. Notice that $L_1(G) = 0$ implies that $G = 0$; this property is expressed by saying that the invariant L_1 is faithful.*

Another invariant $L_2 : Mod\mathbb{Z} \to \mathbb{R}^*$ is the rank, that is, for every Abelian group G, $L_2(G) = \mathrm{rk}(G) = \dim_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q})$. It is well known that the rank is additive and upper continuous. Notice that L_2 is not faithful, since $L_2(G) = 0$ for every torsion group G.

In the following we will always consider non-trivial length function, that is, functions $L : \mathrm{Mod}R \to \mathbb{R}^*$ not identically zero, equivalently, such that $L(R) > 0$. Given $M \in \mathrm{Mod}R$ and an endomorphism $\phi : M \to M$, let us denote by $Fin_L(M)$ the family of the submodules of M of finite L-length. We consider the L-entropy $ent_L(\phi)$ of ϕ, defined in [**SZ**] as follows (endomorphisms will be written on the left).

For every $F \in Fin_L(M)$ and for every positive integer n, consider the so-called nth-partial ϕ-trajectory of F: $T_n(\phi, F) = \sum_{0 \leq k \leq n-1} \phi^k F$; this module has also finite L-length. The ϕ-trajectory $T(\phi, F)$ of F is the ϕ-invariant submodule of M generated by F: $T(\phi, F) = \sum_{k \geq 0} \phi^k F$, that is, $T(\phi, F)$ is the union of all the nth-partial trajectories ($n \geq 1$).

Then one can prove that the limit $lim_{n\to\infty} L(T_n(\phi, F))/n$ does exists, and that it is finite; set

$$ent_L(\phi, F) = lim_{n\to\infty} \frac{L(T_n(\phi, F))}{n}$$

and define the L-entropy of ϕ by:

$$ent_L(\phi) = \sup_F \{ent_L(\phi, F),\ F \in Fin_L(M)\}.$$

The L-entropy can be interpreted as a map $h_L : \mathrm{Mod}R[X] \to \mathbb{R}^*$ as follows: every $R[X]$-module can be viewed as a pair (M, ϕ), where M is an R-module and $\phi \in End_R(M)$ is the R-endomorphism which acts as the multiplication by X; conversely, every pair (M, ϕ) (shortly denoted by M_ϕ), with $M \in \mathrm{Mod}R$ and $\phi \in End_R(M)$, can be viewed as an $R[X]$-module under the action of ϕ. Thus one can set $h_L(M_\phi) = ent_L(\phi)$. See [**SVV**] for more details.

In [**SVV**] it is also shown that, if L is discrete, the map ent_L has a nice behaviour on the subcategory $lFin_L[X]$ of $\mathrm{Mod}R[X]$ consisting of the $R[X]$-modules M_ϕ whose cyclic R-submodules have finite L-length. In fact, on this subcategory

ent_L is a length function (this result is the so-called Addition Theorem), while on the whole category $\mathrm{Mod}R[X]$ it is not, in general. Note that $lFin_L[X] = \mathrm{Mod}R[X]$ exactly when $0 < L(R) < \infty$.

The Uniqueness Theorem proved in [**SVV**] states that ent_L is the unique length function on $lFin_L[X]$ once we require that it satisfies two natural conditions relating it to the discrete length function L. In order to state these two conditions, we need to recall the definition of the Bernoulli functor $B : \mathrm{Mod}R \to \mathrm{Mod}R[X]$. For every R-module M, $B(M) = (M^{(\mathbb{N})})_\beta$, where $\beta : M^{(\mathbb{N})} \to M^{(\mathbb{N})}$ is the right shift on the direct sum $M^{(\mathbb{N})}$ and, if $\phi : M \to N$ is an R-homomorphism, then $B(\phi) : B(M) \to B(N)$ is defined by setting: $B(\phi)(x_n)_{n\geq 0} = (\phi(x_n))_{n\geq 0}$ for every $(x_n)_{n\geq 0} \in M^{(\mathbb{N})}$.

It is not difficult to see that the Bernoulli functor B is isomorphic to the tensor product over R by $R[X]$, that is, for each R-module M, $B(M)$ is naturally isomorphic as $R[X]$-module to $R[X] \otimes_R M$; the isomorphism is induced by the assignments

$$(x_n)_{n\geq 0} \mapsto \sum_n X^n \otimes x_n$$

for every $(x_n)_{n\geq 0} \in (M^{(\mathbb{N})})_\beta$; as noted in [**SVV**], this fact has the remarkable consequence that B has as right adjoint the forgetful functor.

The two conditions required in the Uniqueness Theorem on a length function L_X on $lFin_L[X]$ in order that it coincides with the L-entropy ent_L are:

(i) $L_X \cdot B = L$, and

(ii) $L_X(M_\phi) = 0$ for every ϕ, whenever $L(M)$ is finite.

The goal of this Section is to show that the second condition is a consequence of the first one.

PROPOSITION 2.2. *Let R be a ring and $L : \mathrm{Mod}R \to \mathbb{R}^*$ a discrete length function on* $\mathrm{Mod}R$. *Let $L_X : lFin_L[X] \to \mathbb{R}^*$ be a length function satisfying $L_X \cdot B = L$, where $B : \mathrm{Mod}R \to \mathrm{Mod}R[X]$ is the Bernoulli functor. Then, given any endomorphism ϕ of an R-module M such that $L(M)$ is finite, $L_X(M_\phi) = 0$.*

PROOF. Consider the canonical homomorphism of $R[X]$-modules $\Phi : R[X] \otimes_R M \to M_\phi$ induced by the assignments

$$f(X) \otimes_R x \mapsto f(\phi)(x)$$

for every $f(X) \in R[X]$ and $x \in M$. Clearly Φ is surjective and its kernel K contains $\sum_k M_k$, where, for each $k \in \mathbb{N}$, M_k is the R-submodule of $R[X] \otimes_R M$ defined as follows:

$$M_k = \{X^k \otimes \phi(x) - X^{k+1} \otimes x \ : \ x \in M\}.$$

The multiplication by X induces an isomorphism of R-modules from M_k to M_{k+1} for every k. It is immediate to check that $\sum_k M_k = \oplus_k M_k$, so $\oplus_k M_k$ is an $R[X]$-submodule of K. The assignments

$$\sum_{0\leq i\leq k} X^i \otimes x_i \mapsto \sum_{0\leq i\leq k} (X^i \otimes \phi(x_i) - X^{i+1} \otimes x_i)$$

($x_i \in M$) induce an isomorphism of $R[X]$-modules between $R[X] \otimes_R M$ and $\oplus_k M_k$, hence

$$L_X(R[X] \otimes_R M) = L_X(\oplus_k M_k) \leq L_X(K) \leq L_X(R[X] \otimes_R M)$$

therefore $L_X(K) = L_X(R[X] \otimes_R M)$. As we mentioned before, $R[X] \otimes_R M$ is isomorphic as $R[X]$-module to $B(M)$, thus $L_X(R[X] \otimes_R M) = L_X(B(M)) = L(M)$ is finite. From the exact sequence of $R[X]$-modules induced by Φ:

$$0 \to K \to R[X] \otimes_R M \to M_\phi \to 0$$

we obtain $L_X(M_\phi) = L_X(R[X] \otimes_R M) - L_X(K) = 0$, as desired.

□

3. Algebraic vs Topological Isomorphisms between Endomorphism Rings

Let us consider now, for every left module M over an arbitrary ring R, its endomorphism ring $End_R(M)$ as a topological ring endowed by the finte topology $\mathcal{F}(M)$. A basis of neighborhoods of 0 for $\mathcal{F}(M)$ is the family of left ideals $K_F = \{\phi \in End_R(M) | \phi(F) = 0\}$, where F ranges over the set of the finitely generated submodules of M. If $x \in M$, we simply set K_x instead of K_{Rx}. It is well known that $(End_R(M), \mathcal{F}(M))$ is a complete topological ring (see Theorem 107.1 in [**F**]).

The next result shows that the ϕ-trajectories of the cyclic submodules of a module $M \in lFin_L$ can be detected in the topological ring $(End_R(M), \mathcal{F}(M))$.

Lemma 3.1. *Let ϕ be an endomorphism of the R-module $M \in lFin_L$, and let $x \in M$. Then there exists an R-isomorphism*

$$T(\phi, Rx) \cong R[\phi]/(R[\phi] \cap K_x)$$

where $R[\phi]$ is the R-subalgebra of $End_R(M)$ generated by ϕ.

Proof. Every element of $T(\phi, Rx)$ is of the form $f(\phi)(x)$, where $f(X) \in R[X]$. Therefore the map $v_x : R[\phi] \to T(\phi, Rx)$ defined by setting $v_x(f(\phi)) = f(\phi)(x)$ is a surjective homomorphism of R-modules, whose kernel is obviously $R[\phi] \cap K_x$. □

Passing from the trajectories of cyclic submodules to those of finitely generated submodules, we have the following

Lemma 3.2. *Let ϕ be an endomorphism of the R-module $M \in lFin_L$, and $F = \sum_{i \leq k} Rx_i$ a finitely generated submodule of M. Then $R[\phi]/(R[\phi] \cap K_F)$ is isomorphic to a submodule of the direct sum $\oplus_{i \leq k} T(\phi, Rx_i)$.*

Proof. Using the notation of the proof of Lemma 3.1, the diagonal map

$$v : R[\phi] \to \oplus_{i \leq k} T(\phi, Rx_i)$$

of the maps v_{x_i} has kernel $\cap_{i \leq k} K_{x_i} = K_F$, so the conclusion follows. □

We start now to examine the consequences of the existence of an isomorphism between the topological rings $(End_R(M), \mathcal{F}(M))$ and $(End_R(H), \mathcal{F}(H))$, where M and H are two locally L-finite R-modules, on the L-entropy of endomorphisms induced by a discrete length function L on Mod$R$. We need to recall that, if $\phi \in End_R(M)$, then $ent_L(\phi) = 0$ if and only if, for every element $x \in M$, there exists a natural number n such that $L(T(\phi, Rx)) = L(T_n(\phi, Rx))$; this amounts to say that $L(T(\phi, Rx))$ is finite for every $x \in M$.

Proposition 3.3. *Let M and H be two R-modules, $\Phi : (End_R(M), \mathcal{F}(M)) \to (End_R(H), \mathcal{F}(H))$ an isomorphism of topological rings, and L a discrete length function on* Mod$R$*. If $M \in lFin_L$ and $\phi \in End_R(M)$ is an endomorphism of M such that $ent_L(\phi) = 0$, then also $H \in lFin_L$ and $ent_L(\Phi(\phi)) = 0$.*

PROOF. Let $\psi = \Phi(\phi)$ and $x \in H$. Since $\Phi^{-1}(K_x)$ is open in the finite topology of $End_R(M)$, it contains a left ideal of the form K_F, for F a suitable finitely generated submodule of M (note that K_x is an ideal of $End_R(H)$, and K_F is an ideal of $End_R(G)$). Thus $R[\psi]/(R[\psi] \cap K_x) \cong R[\phi]/(R[\phi] \cap \Phi^{-1}(K_x))$, and the last factor is a quotient of $R[\phi]/(R[\phi] \cap K_F)$. By Lemma 3.2, $R[\phi]/(R[\phi] \cap K_F)$ embeds into a finite direct sum of cyclic ϕ-trajectories in M. The hypothesis $ent_L(\phi) = 0$ ensures that the L-length of this direct sum is finite, hence also $L(R[\phi]/(R[\phi] \cap K_F))$ is finite, and consequently $L(T(\psi, Rx)) = L(R[\psi]/(R[\psi] \cap K_x))$ is finite. This shows that $L(Rx)$ is finite, hence $H \in lFin_L$, and also that $ent_L(\psi) = 0$. $\square$

COROLLARY 3.4. *Let* $\Phi : (End_R(M), \mathcal{F}(M)) \to (End_R(H), \mathcal{F}(H))$ *be an isomorphism of topological rings, and let* L *be a discrete length function on* Mod$R$. *Then* $M \in lFin_L$ *if and only if* $H \in lFin_L$.

PROOF. If $M \in lFin_L$, consider $\phi = id_M$, which has L-entropy zero. Apply Proposition 3.3 to deduce that $H \in lFin_L$. For the converse use Φ^{-1}. $\square$

Recall that the module M has global entropy zero, and we write $ent_L(M) = 0$, if $ent_L(\phi) = 0$ for every $\phi \in End(M)$.

COROLLARY 3.5. *Let* $\Phi : (End_R(M), \mathcal{F}(M)) \to (End_R(H), \mathcal{F}(H))$ *be an isomorphism of topological rings, with* $M, H \in lFin_L$, *where* L *is a discrete length function on* Mod$R$. *Then* $ent_L(M) = 0$ *if and only if* $ent_L(H) = 0$.

PROOF. If is an immediate consequence of Proposition 3.3. $\square$

It is natural to ask whether Corollary 3.4 remains valid if we only assume to have an algebraic isomorphism $\Phi : End_R(M) \to End_R(H)$. The next example shows that the answer is in the negative.

EXAMPLE 3.6. *Let* T *be a reduced unbounded torsion group and* C *its cotorsion hull. Then we can identify* T *with the torsion subgroup of* C, *and the quotient* C/T *is non-zero divisible and torsion-free. The exact sequence*

$$0 = Hom(C/T, C) \to End(C) \to Hom(T, C) = End(T) \to 0 \ ,$$

where the second map is the restriction map, shows that the two endomorphism rings $End(T)$ *and* $End(C)$ *are isomorphic. Let* L *be the length function on* Mod$\mathbb{Z}$ *defined by* $L(G) = log|G|$. *Then* $lFin_L$ *consists of the torsion groups, hence* $T \in lFin_L$ *and* $C \notin lFin_L$. *Actually, the isomorphism induced by the restriction map from* $End(C)$ *to* $End(T)$ *is not topological, if we consider the finite topologies; in fact, fixed a torsion-free element* $x \in C$, *for every finite subgroup* F *of* T *there exists an endomorphism* $\phi : C \to C$ *such that* $\phi(F) = 0$ *and* $\phi(x) \neq 0$, *as* F *is contained in a direct summand of* C *contained in* T, *therefore the open ideal* K_x *of* $\mathcal{F}(C)$ *does not contain any open ideal* K_F *of the neighborhoods base of* 0 *of* $\mathcal{F}(T)$. *Notice however that, given a* $\phi \in End(C)$, *denoting* ent_L *simply by ent (the classical algebraic entropy defined in* [**AKM**] *and investigated in* [**W**] *and* [**DGSZ**]*),* $ent(\phi) = ent(\phi_{\restriction T})$, *we still have, as in Proposition* 3.3, *that two endomorphisms of* C *and* T *corresponding in the isomoprphism between* $End(T)$ *and* $End(C)$ *have simultaneously zero algebraic entropy.*

Also Corollary 3.5 does not hold if we only assume to have an algebraic isomorphism Φ between $End_R(M)$ and $End_R(H)$. This fact was investigated in the recent paper by Göbel and the author [**GS**], where Abelian groups are considered and L

is the rank function. In fact, in [**GS**] two torsion-free Abelian groups M and H are exhibited with isomorphic endomorphism rings, such that all the endomorphisms of M have rank-entropy zero, while H has endomorphisms of positive rank-entropy. Note that, when $L = rk$, $lFin_L$ coincides with the whole category Mod$\mathbb{Z}$.

4. Endomorphism rings of p-groups with different algebraic entropy supports

In this Section we explain why the results in papers by Corner [**C**] and Corner-Göbel [**CG**] are of no use to solve a problem posed for Abelian p-groups in the last section of the paper [**GS**] quoted above, which is similar, *mutatis mutandis*, to that solved for torsion-free groups in [**GS**].

By the well known Baer-Kaplansky theorem, two p-groups G and H with isomorphic endomorphism ring are isomorphic (see [**F**, Theorem 108.1]), so one has to modify the problem, solved in [**GS**] for torsion-free groups, as follows. For every reduced p-group G one has that $End(G) = A \oplus Small(G)$ is a split extension of a torsion-free J_p-algebra A, which is the completion of a free J_p-module, by the two-sided ideal $Small(G)$ of the small endomorphisms of G (recall that an endomorphism $\phi : G \to G$ is small if, for every positive integer k there exists a positive integer n such that $\phi(p^n G[p^k]) = 0$, where $G[p^k]$ is the kth-socle of the p-group G).

A careful analysis of the proof of the Baer-Kaplansky theorem reveals that G is determined by $Small(G)$; thus in [**SZ1**] we called the subalgebra A of $End(G)$ the "disengaged" part of $End(G)$. This being the situation, we asked in [**GS**] whether, fixed the complete J_p-algebra A, two reduced p-groups G and H exist such that $End(G) \cong A \oplus Small(G)$, $End(H) \cong A \oplus Small(H)$, and $ent(G) = 0$, $ent(H) = \infty$ (where $ent(X) = sup_{\phi \in End(X)} ent(\phi)$).

An advice for the possibility of the existence of such a pair of groups comes from a celebrated realization theorem by Corner [**C**], which states that, under suitable hypotheses, there exist $2^{2^{\aleph_0}}$ non isomorphic p-groups, with the same basic subgroup and with isomorphic "disengaged" part A. Thus one could hope to find, in this big amount of groups, two p-groups with different entropical behaviour. However, as we will explain in this Section, Corner's realization theorem in [**C**] produces, in a plausible setting of possible examples, only p-groups G with $ent(G) = \infty$.

Theorem 2.1 in [**C**] deals with a torsion-complete p-group $\bar{B}$ having an unbounded basic subgroup B of cardinality $\leq 2^{\aleph_0}$, and constructs pure subgroups G of $\bar{B}$ containing B such that $End(G) \cong A \oplus Small(G)$, where A is a separable closed subring of $End(\bar{B})$ leaving B invariant and satisfying the so-called Crawley condition (C):

(C) if $\alpha \in A \setminus pA$, then $\alpha(p^n \bar{B}[p]) \neq 0$ for all n.

As we want to admit the possibility that $ent(G) = 0$, the basic subgroup B must be semi-standard (see [**DGSZ**, Proposition 4.1]), that is, $B = \oplus_n B_n$ where each B_n is a finite direct sum of groups isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. This assumption strongly simplifies Corner's construction, because, instead of working with the family $\mathbf{D}$ of all unbounded countable direct summands D of B, we can deal with the countable group B alone, that is, $\mathbf{D} = \{B\}$.

On the other hand, as we want to admit also the possibility that $ent(G) = \infty$, one has to start with A which is not integral over J_p, otherwise, by [**DGSZ**, Corollary 5.7], every reduced semi-standard p-group G with $End(G) \cong A \oplus Small(G)$ satisfies $ent(G) = 0$. In particular, A cannot be a free J_p-module of finite rank.

Thus the simplest possibility for the J_p-algebra A is of being the p-adic completion of $J_p[\sigma]$, where σ is a transcendental endomorphism of $\bar{B}$ leaving B invariant. Furthermore, A must satisfy the Crawley's condition mentioned abiove.

Examples of endomorphisms $\sigma : \bar{B} \to \bar{B}$ such that the completion A of $J_p[\sigma]$ satisfies the conditions above can be found in [**DGSZ**, Section 5.3 and Theorem 4.4].

Looking at the construction of G in the Corner's long and technical proof of Theorem 2.1 in [**C**, pages 278-285], another strong simplification in our setting comes from the fact that we are not interested in producing many non-isomorphic p-groups with only small mutual homomorphisms, but we want to construct a pair of p-groups. Therefore, in the proof of Theorem 2.1 at pages 284-285, we can consider a pure subgroup G of $\bar{B}$ such that

$$G[p] \leq B \oplus (\oplus_t A x_t) \leq G$$

where the elements $x_t \in \bar{B}$ are obtained using the technical Lemmas 2.8 and 2.9.

In our simplified situation, with A equal to the completion of $J_p[\sigma]$, it is immediate to see that each subgroup Ax_t is nothing else but the σ-trajectory of the element x_t. In order to obtain $ent(G) = 0$, according with [**DGSZ**, Proposition 2.4], we need that Ax_t is finite for all indices t. Unfortunately, Corner's construction in Lemmas 2.8 and 2.9 produce non-zero elements $x_t \in \bar{B}[p^e]$ such that $\phi(x_t) \neq 0$ for all $\phi \in A \setminus p^e A$. As we can assume that ϕ is a polynomial in σ with coefficients in $\mathbb{Z}_p$, this implies that, for every positive integer n, $\sigma^n(x_t) \notin T_n(\sigma, \mathbb{Z}_p x_t)$, hence the σ-trajectory of x_t is infinite.

We can summarize what we have seen above in the following

Proposition 4.1. *Let B be an unbounded semi-standard direct sum of cyclic p-groups. Let σ be an endomorphism of $\bar{B}$ leaving B invariant, transcendental over J_p. Assume that A, the completion of $J_p[\sigma]$ in the p-adic topology, satisfies the Crawley's condition (C). Then all the p-groups G constructed by means of the Corner's realization theorem in* [**C**] *with $End(G) = A \oplus Small(G)$ satisfy $ent(G) = \infty$.*

One could be tempted to use, as in the torsion-free case, the topological realization theorem for representable (in a technical sense) topological algebras A proved in [**CG**]: fixed a topological J_p-algebra A satisfying suitable hypotheses, a p-group G can be constructed such that $End(G) = A \oplus Ines(G)$, where $Ines(G)$ is the two-sided ideal of the inessential endomorphisms (see [**CG**] for details), and the topology on A coincides with the topology induced by the finite topology on $End(G)$. This construction is howevere useless for our purposes, as the basic subgroup B used to construct G is not semi-standard, hence also in this case $ent(G) = \infty$, by [**DGSZ**, Proposition 4.1].

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Modules and Infinitary Logics

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Dedicated to Rüdiger Göbel for this 70th birthday

Abstract. We deal with abelian groups and R-modules. We consider theories in infinitary logic of the form $\mathbb{L}_{\lambda,\theta}$ of such structures M and prove they have elimination of quantifiers up to positive existential formulas, so ones defining subgroups of some power of M. Hence in the appropriate sense those theories are stable and understood to some extent.

§ 0. Introduction

Much is known on classes of R-modules and first order logic. Szmielew [**Szm49**] prove the decidability of the theory of Abelian groups. Szmielew [**Szm55**] prove an elimination of quantifiers in the theory of Abelian groups up to Boolean combinations of pe (= positive existential) formulas.

Eklof [**Ekl71**] proves the existence of universal homogeneous R-models in λ if $\lambda = \lambda^{<\gamma}$ where γ depending on R only. Fisher improved this to saturated models of elementary classes (see his review of [**Ekl71**]), this implies stability by a general criterion ([**Sh:11**, §0],[**Sh:c**, Ch.III]).

Baur [**Bau76**] proved that for the class of R-modules any first order formula is equivalent to a Boolean combination of positive existential formulas and also prove stability (of $\mathrm{Th}(M)$) for M and R-module.

We like to know for a given ring R how complicated the class of R-modules which are models of a sentence ψ in an infinitary logic.

Question 0.1. Given a ring R, for the class Mod_R of left R-modules:
1) Does it have for the logic $\mathbb{L}_{\lambda,\mu}$ a kind of elimination of quantifiers (say up to some depth).
2) Is it stable? (say no formula $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\infty,\infty}(\tau_R)$ linearly ordering arbitrarily long sequence of tuples in some models of ψ)?
3) Can we define something like non-forking?

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Question 0.2. Do we have a parallel of the main gap, i.e. proving that either every $M \in$ Mod_ψ can be characterized by some suitable cardinal invariants or there are many complicated $M \in$ Mod_ψ?

Here we first show that for any R-module, in $\mathbb{L}_{\lambda,\theta}(\tau_R)$ or better $\mathbb{L}_{\infty,\theta,\gamma}(\tau_R)$ we have a version of eliminating quantifiers up to positive existential formulas however we add parameters. Second, by this we can prove some versions and consequences of stability. More specifically

- after expanding by enough individual constants, every formula in $\mathbb{L}_{\infty,\theta,\gamma}(\tau_R)$ is equivalent to a Boolean combination of positive existential such formulas
- the number of added individual constants is reasonable: $\leq \beth_\gamma(|\tau|^{<\theta})$
- stability, i.e. no long sequences of linearly ordered $(< \theta)$-tuples
- $(\Lambda^{\rm ep}_{\varepsilon,\alpha}, 2)$-indiscernible implies $\Lambda^{\rm ep}_{\varepsilon,\alpha}$-indiscernible
- convergence follows, see Definition 3.4

§ 1. Preliminaries

Notation 1.1. 1) Let θ^- be σ if $\theta = \sigma^+$ and θ if θ is a limit cardinal.

Definition 1.2. 1) A vocabulary τ consists of function symbols (e.g. individual constants) and predicates (= relation symbols), in addition the vocabulary assign generally to each of them its arity = number of places $\mathrm{arity}_\tau(-)$; here it can be an infinite ordinal; an individual constant is a 0-place function.
2) For a vocabulary τ we say M is a τ-structure when M, writing $\tau_M = \tau(M) = \tau$, consisting of:

- (a) $|M|$, the universe of M, a non-empty set of the so-called elements of M, so we may write $a \in M, \bar{a} \in {}^\varepsilon M$ and $A \subseteq M$, etc., instead $a \in |M|, \bar{a} \in {}^\varepsilon(|M|)$ and $A \subseteq |M|$, etc.
- (b) F^M a function from ${}^\varepsilon M$ to M, possibly partial where ε is the ordinal $\mathrm{arity}_\tau(F)$, for F a function symbol from τ
- (c) $P^M \subseteq {}^\varepsilon M$ where ε is the ordinal $\mathrm{arity}_\tau(P)$ for P a predicate from τ.

Definition 1.3. 1) We say τ is a θ-additive (or a θ-Abelian) vocabulary when τ has the two-place function symbols $x + y, x - y$, the individual constant 0 and the other predicates and function symbols has arity $< \theta$.
2) M is a θ-additive structure (or model) when:

- (a) τ_M, the vocabulary of M is a θ-additive vocabulary
- (b) $G_M := (|M|, +^M, -^M, 0^M)$ is an Abelian group
- (c) if $P \in \tau_M$ is an ε-place predicate then P^M is a sub-group of $(G_M)^\varepsilon$
- (e) if $F \in \tau_M \backslash \{+, -, 0\}$ is an ε-place function symbol then F^M is a partial ε-place function from M to M and $\mathrm{graph}(F^M) = \{\bar{a}\char`^\langle F^M(\bar{a})\rangle : \bar{a} \in \mathrm{Dom}(F^M)\}$ is a subgroup of $(G_M)^{\varepsilon+1}$.

Remark 1.4. Fisher [**Fis77**] defines and deals with "Abelian structure" in other directions.

DEFINITION 1.5. 1) We consider an R-module M as a $\tau(R)$-structure, where $\tau_R = \tau(R)$ be the vocabulary of R-modules, i.e. have binary functions $x+y, x-y$, individual constant 0 and unary function symbol F_a, interpreted as multiplication by a from the left for every $a \in R$.
2) If $\bar{x}, \bar{y}$ has length ε <u>then</u> we let $\bar{x}+\bar{y} = \langle x_\zeta + y_\zeta : \zeta < \varepsilon\rangle, \bar{x}-\bar{y} = \langle x_\zeta - y_\zeta : \zeta < \varepsilon\rangle$ and similarly $a\bar{x}$ for $a \in R$, and when we replace $\bar{x}$ and/or $\bar{y}$ by a member of ${}^\varepsilon M$.

OBSERVATION 1.6. *1) For a ring R, an R-module is an $\aleph_0$-additive structure in the vocabulary τ_R.*
2) For a τ-additive model M, for every τ-term $\sigma(\bar{x})$ we have

- (a) $M \models$ "$\sigma(\bar{a} \pm \bar{b}) = \sigma(\bar{a}) \pm \sigma(\bar{b})$" *meaning (when F is partial), if two of the terms are well defined then so is the third and the equality hold*
- (b) $M \models P(\bar{a} \pm \bar{b})$ *when* $M \models P(\bar{a}) \wedge P(\bar{b})$.

§ 2. Eliminating quantifiers

CONTEXT 2.1. 1) R is a fixed ring $\tau = \tau_R$, see 1.5(1) <u>or</u> just τ is an θ-additive vocabulary, see 1.3(1), 1.6(1).
2) $\mathbf{K}$ is the class of R-modules <u>or</u> of τ-additive models.
3) M, N will denote R-modules <u>or</u> are τ-additive models.
4) $\theta = \operatorname{cf}(\theta)$.

DEFINITION 2.2. For $\varepsilon < \theta$ and ordinal α (and τ as in 2.1). We shall define $\Lambda^{\rm pe}_{\alpha,\varepsilon} = \Lambda^{{\rm pe},\theta}_{\alpha,\varepsilon} = \Lambda^{{\rm pe},\theta}_{\alpha,\varepsilon}(\tau)$, a set of formulas $\varphi(\bar{x})$ in $\mathbb{L}_{\infty,\theta}(\tau)$ in fact in $\mathbb{L}_{\infty,\theta,\alpha}(\tau)$ with $\ell g(\bar{x}) = \varepsilon < \theta$, so $\bar{x} = \langle x_\xi : \xi < \varepsilon\rangle$ if not said otherwise, by induction on the ordinal α.

For $\zeta < \theta$ we write $\Lambda^{\rm pe}_{\alpha,\varepsilon,\zeta}$ for the set of $\varphi = \varphi(\bar{x},\bar{y}), \ell g(\bar{x}) = \varepsilon, \ell g(\bar{y}) = \zeta$ (so $\bar{y} = \langle y_\xi : \xi < \zeta\rangle$ if not said otherwise) with $\varphi \in \Lambda^{\rm pe}_{\alpha,\varepsilon+\zeta}$ and $\Lambda^{\rm pe}_\alpha = \cup\{\Lambda^{\rm pe}_{\alpha,\varepsilon} : \varepsilon < \theta\}, \Lambda^{\rm pe}_{\alpha,\varepsilon,<\theta} = \cup\{\Lambda^{\rm pe}_{\alpha,\varepsilon,\zeta} : \zeta < \theta\}$. If $\tau = \tau_R$ we may write $\Lambda^{\rm pe}_{\alpha,\varepsilon}(R)$.

The definition is as follows:

<u>Case 1</u>: $\alpha = 0$

<u>For R-modules</u>:

It is the set of $\varphi = \varphi(\bar{x})$ of the form: $\sum\limits_{\ell<n} a_\ell x_{\zeta(\ell)} = 0$ with $\zeta(\ell) < \ell g(\bar{x})$ or better, $\sum\limits_{\zeta<\varepsilon} a_\zeta x_\zeta = 0$ where $a_\zeta \in R$ is 0_R for all but finitely many ζ's.

<u>For general τ, so here the τ-additive case:</u>

It is the set of $\varphi(\bar{x})$ has the form $P(\bar{\sigma}(\bar{x})), \bar{\sigma}$ a sequence of length $\text{arity}_\tau(P)$ of terms (in the variables $\bar{x}$), P may be equality or any predicate from τ of arity the length of $\bar{\sigma}$.

<u>Case 2</u>: α a limit ordinal

It is $\cup\{\Lambda^{\rm pe}_{\beta,\varepsilon}(R) : \beta < \alpha\}$.

<u>Case 3</u>: $\alpha = \beta + 1$

For some $\zeta < \theta$ and $\Phi \subseteq \Lambda^{\rm pe}_{\beta,\varepsilon+\zeta}$ we have $\psi(\bar{x}) = \exists \bar{y}(\bigwedge\{\varphi(\bar{x}\hat{\ }\bar{y}) : \varphi(\bar{x},\bar{y}) \in \Phi\})$.

Claim 2.3. *1) In 2.2, $\Lambda^{\rm pe}_{\alpha,\varepsilon}$ is $\subseteq$- increasing with α and is of cardinality $\le \beth_\varepsilon(|\tau|+\aleph_0)$ if $\theta=\aleph_0$ and $\beth_\varepsilon(|\tau|^{<\theta})$ in general.*
2) For $M\in\mathbf{K}$ and $\varphi(\bar x)\in\Lambda^{\rm pe}_{\alpha,\varepsilon}(\tau)$, the set $\varphi(\bar M)=\{\bar b\in{}^\varepsilon M: M\models\varphi[\bar b]\}$ is a sub-abelian group of ${}^\varepsilon M$ and the set $\{\bar b\in{}^\varepsilon M: M\models\varphi[\bar b-\bar a]\}$ is affine (= closed under $\bar x-\bar y+\bar z$) for any $\bar a\in{}^\varepsilon M$.

Proof. Easy. $\square$

Theorem 2.4. *For every α for every $M\in\mathbf{K}$ there is a subset $\mathbf{I}=\mathbf{I}_\alpha$ of ${}^{\theta>}M$ of cardinality $\le\kappa_\alpha=\beth_\alpha(|\tau|^{<\theta})$ such that: in M every formula $\psi(\bar x)$ from $\mathbb{L}_{\infty,\theta,\alpha}(\tau)$, so $\ell g(\bar x)<\theta$, is equivalent in M to a Boolean combination of formulas of the form $\varphi(\bar x-\bar a)$ with $\varphi(\bar x)\in\Lambda^{\rm pe}_{\alpha,\ell g(\bar x)}(\tau)$ and $\bar a\in\mathbf{I}\cap{}^{\ell g(\bar x)}M$.*

Before we shall prove

Conclusion 2.5. *For every $M\in\mathbf{K}$, limit ordinal $\alpha,\varepsilon<\theta$ and $\bar a\in{}^\varepsilon M$, for some $i(*),j(*)\le\kappa_\alpha$ and $\varphi_i(\bar x_\varepsilon),\psi_j(\bar x_\varepsilon)\in\Lambda^{\rm pe}_{\alpha,\varepsilon}$ for $i<i(*),j<j(*)$ we have $\{\bar a'\in{}^\varepsilon M:{\rm tp}_{\mathbb{L}^{\rm pe}_{\infty,\theta,\alpha}}(\bar a',\emptyset,M)={\rm tp}_{\mathbb{L}^{\rm pe}_{\infty,\theta,\alpha}}(\bar a,\emptyset,M)\}$ is equal to $\{\bar a'\in{}^\varepsilon M: M\models \bigwedge\limits_{i<i(*)}\varphi_i(\bar a'-\bar a)\wedge\bigwedge\{\neg\psi_j(\bar a'-\bar a''):j<j(*)$ and $\bar a''\in I_\gamma\cap{}^\varepsilon M\}$.*

Definition 2.6. 1) We say $\bar b_1,\bar b_2\in{}^\varepsilon M$ are α-equivalent over $\mathbf{I}\subseteq{}^{\theta>}M$ <u>when</u> $\varphi(\bar x_\varepsilon)\in\Lambda^{\rm pe}_{\alpha,\varepsilon}(R),\bar a\in\mathbf{I}\Rightarrow M\models$ "$\varphi[\bar b_1-a]\equiv\varphi[\bar b_2-\bar a]$".
2) Replacing $\mathbf{I}$ by A means $\mathbf{I}=\cup\{{}^\varepsilon A:\varepsilon<\theta\}$.

We shall use freely

Observation 2.7. *The sequence $\bar b_1,\bar b_2\in{}^\varepsilon M$ are α-equivalent over $\mathbf{I}\subseteq{}^\varepsilon M$ <u>iff</u> for any $\varphi(\bar x)\in\Lambda^{\rm pe}_{\alpha,\varepsilon}$ we have $(a)\vee(b)$ where:*

- *(a) for some $\bar a\in\mathbf{I}\cap{}^\varepsilon M$ we have $M\models\varphi[\bar b_1-\bar a]\wedge\varphi[\bar b_2-\bar a]$*
- *(b) for every $\bar a\in\mathbf{I}\cap{}^\varepsilon M$ we have $M\models\neg\varphi[\bar b_1-\bar a]\wedge\neg\varphi[\bar b_2-\bar a]$*

Proof. Straight. $\square_{2.7}$

Proof. <u>Proof of 2.4</u>
By induction on α we choose $\mathbf{I}_\alpha$ and prove the statement. For $\alpha=0$ choose $\mathbf{I}_\alpha=\{0^M\}$ and for α a limit ordinal this is obvious, use $\cup\{\mathbf{I}_\beta:\beta<\alpha\}$ so assume $\alpha=\beta+1$ and we shall choose $\mathbf{I}_\alpha$.
Choose $\mathbf{I}_\alpha$ such that

$\boxplus_\alpha$
- (a) $\mathbf{I}_\alpha$ is a subset of ${}^{\theta>}M$
- (b) $|\mathbf{I}_\alpha|\le 2^{\kappa_\beta}$ where $\kappa_\beta=\beth_\beta(|\tau|^{<\theta})$
- (c) $\mathbf{I}_\beta\subseteq\mathbf{I}_\alpha$
- (d) If $\varepsilon<\theta$ and $\varphi_i(\bar x)\in\Lambda^{\rm ep}_{\beta,\varepsilon}$ and $\bar a_i\in\mathbf{I}_\beta\cap{}^\varepsilon M$ for $i<i(*)\le\kappa_\beta$ and there is $\bar d\in{}^\varepsilon M$ such that $M\models\varphi_i[\bar d-\bar a_i]$ for $i<i(*)$ <u>then</u> there is such $\bar d\in\mathbf{I}_\alpha$
- (e) Assume $\varepsilon<\theta,\bar x=\bar x_\varepsilon,\psi(\bar x)$ is a conjunction of formulas from $\Lambda^{\rm ep}_{\beta,\varepsilon}$ and $\varphi_i(\bar x)\in\Lambda^{\rm ep}_{\beta,\varepsilon}$ for $i<\kappa_\beta$ and apply 4.1 with $\lambda_\alpha=(2^{\kappa_\beta})^+,\kappa_\beta,\psi({}^\varepsilon M),\psi({}^\varepsilon M)\cap\varphi_i({}^\varepsilon M)$ for $i<\kappa_\beta$ here standing for $\lambda,S,G,G_s(s\in S)$ there; (i.e. the subgroups of $({}^\varepsilon|M|,+^M)$ with universes as above) getting the ideal I on κ_β and

further assume $\kappa_\beta \notin I$

(α) there are $\bar{d}_\iota \in \mathbf{I}_\alpha \cap \varphi_i({}^\varepsilon M)$ for $\iota < \iota(*) \leq 2^{\kappa_\beta}$ such that for every $\bar{a} \in \psi({}^\varepsilon M)$ there is $\iota < \iota(*)$ satisfying $\{i < \kappa : \bar{a} - \bar{d}_\iota \notin \varphi_i({}^\varepsilon M)\} \in I$

(β) for any $u \in I$ there are u_* such that $u \subseteq u_* \in I$ and $\bar{d}_\iota \in \cap\{\varphi_i({}^\varepsilon M) : i \in \kappa_\beta \backslash u_*\} \cap \psi({}^\varepsilon M) \cap \mathbf{I}_\alpha$ for $\iota < (2^{\kappa_\beta})^+$ such that:
$i \in u_* \wedge \iota(1) \leq \iota(2) < (2^{\kappa_\beta})^+ \Rightarrow \bar{d}_{\iota(1)} - \bar{d}_{\iota(2)} \notin \varphi_i({}^\varepsilon M)$

(f) if $\varepsilon < \theta$ and $\bar{d}_1, \bar{d}_2 \in \mathbf{I}_\alpha \cap {}^\varepsilon M$ <u>then</u> $\bar{d}_1 + \bar{d}_2 \in \mathbf{I}_\alpha, \bar{d}_1 - \bar{d}_2 \in \mathbf{I}_\alpha$ and $\xi < \theta \Rightarrow \bar{0}_\xi{}^\frown \bar{d}_1 \in \mathbf{I}_\alpha$.

This is possible for $(e)(\alpha)$ by clause (c) of 4.1 and for $(e)(\beta)$ by clause (d) of 4.1.

To prove the induction statement for α clearly it suffices to prove:

$\boxdot$ assume $\varepsilon, \xi < \theta$; if $\bar{b}_1, \bar{b}_2 \in {}^\varepsilon M$ are α-equivalent over $\mathbf{I}_\alpha$ and $\bar{c}_1 \in {}^\xi M$ <u>then</u> for some $\bar{c}_2 \in {}^\xi M$ the sequences $\bar{b}_1{}^\frown \bar{c}_1, \bar{b}_2{}^\frown \bar{c}_2 \in {}^{(\varepsilon+\xi)}M$ are β-equivalent over $\mathbf{I}_\beta$.

Why $\boxdot$ holds? Let $\bar{x}$ be of length ε and $\bar{y}$ of length ξ. Let $\Phi_1 = \{\varphi(\bar{x}, \bar{y}) \in \Lambda^{\text{ep}}_{\beta,\varepsilon+\xi}$: for some $\bar{a} \in \mathbf{I}_\beta \cap {}^{\varepsilon+\xi}M$ we have $M \models \varphi[\bar{b}_1{}^\frown \bar{c}_1 - \bar{a}]\}$ and for $\varphi(\bar{x}, \bar{y}) \in \Phi_1$ choose $\bar{a}_{\varphi(\bar{x},\bar{y})} \in \mathbf{I}_\beta \cap {}^{\varepsilon+\xi}M$ such that $M \models \varphi[\bar{b}_1{}^\frown \bar{c}_1 - \bar{a}_{\varphi(\bar{x},\bar{y})}]$. Let $\Phi_2 = \{\varphi(\bar{x}, \bar{y}) \in \Lambda^{\text{ep}}_{\beta,\varepsilon+\xi} : \varphi(\bar{x}, \bar{y}) \notin \Phi_1\}$.

So by $\boxplus_\alpha(d)$ there is a $\bar{b}^*{}^\frown \bar{c}^* \in \mathbf{I}_\alpha$ be such that $\ell g(\bar{b}^*) = \ell g(\bar{b}_1), \ell(\bar{c}^*) = \ell g(\bar{c}_1)$ and $\varphi(\bar{x}, \bar{y}) \in \Phi_1 \Rightarrow M \models \varphi[\bar{b}^*{}^\frown \bar{c}^* - \bar{a}_{\varphi(\bar{x},\bar{y})}]$. For transparency note that if $\Phi_2 = \emptyset$ then as the formula $\wedge\{\varphi(\bar{x}, \bar{y}) : \varphi(\bar{x}, \bar{y}) \in \Phi_1\} \in \Lambda^{\text{pe}}_{\alpha,\varepsilon+\xi}$ clearly by the assumption of $\boxdot$ there is $\bar{c}_2 \in {}^\xi M$ such that $\varphi(\bar{x}, \bar{y}) \in \Phi_1 \Rightarrow M \models \varphi(\bar{b}_2{}^\frown \bar{c}_2 - \bar{a}_{\varphi(\bar{x},\bar{y})})$, so $\bar{c}_2$ is as required, hence we are done so without loss of generality $\Phi_2 \neq \emptyset$. Clearly $|\Phi_2| \leq \kappa_\beta$ and let $\Phi'_\ell = \{\varphi(\bar{0}_\varepsilon, \bar{y}) : \varphi(\bar{x}, \bar{y}) \in \Phi_\ell\}$ for $\ell = 1, 2$.

Let $\{\neg\varphi_i(\bar{x}{}^\frown \bar{y} - \bar{a}_i) : i < \kappa_\beta\}$ list possibly with repetitions the set of formulas $\neg\varphi(\bar{x}{}^\frown \bar{y} - \bar{a})$ satisfied by $\bar{c}_1{}^\frown \bar{b}_1$ with $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\text{ep}}_{\beta,\varepsilon,\zeta}, \bar{a} \in \mathbf{I}_\beta$ and let $\varphi'_i(\bar{y}) = \varphi_i(0_\varepsilon, \bar{y})$. Let $\psi'(\bar{y}) = \wedge\{\varphi(\bar{y}) : \varphi(\bar{y}) \in \Phi'_1\}$.

Let the ideal I on κ_β be defined as in 4.1 with $G = \psi'({}^\varepsilon M)$ where $\psi'(\bar{x}_\xi) = \bigwedge\{\varphi(\bar{y}) : \varphi(\bar{y}) \in \Phi'_1\}$ and $G_i = G \cap \varphi'_i({}^\xi M)$ for $i \in S := \kappa_\beta, \lambda = (2^{\kappa_\beta})^+$.

<u>Case 1</u>: $\kappa_\beta \in I$.

So clearly $M \models \varphi[\bar{b}_1 - b^*, \bar{c}_1 - \bar{c}^*]$ for every $\varphi(\bar{x}, \bar{y}) \in \Phi_1$.

Let $\psi_*(\bar{x}, \bar{y}) = \bigwedge\{\varphi(\bar{x}, \bar{y}) : \varphi(\bar{x}, \bar{y}) \in \Phi_1\}$, so clearly it $\in \Lambda^{\text{ep}}_{\alpha,\varepsilon,\zeta}$ and $M \models \psi_*[\bar{b}_1 - \bar{b}^*, \bar{c}_1 - \bar{c}^*]$ hence $M \models (\exists\bar{y})\psi_*[\bar{b}_1 - b^*, \bar{y}]$. But $(\exists\bar{y})\psi(\bar{x}, \bar{y}) \in \Lambda^{\text{ep}}_{\alpha,\varepsilon}$ so by the assumption on $\bar{b}_1, \bar{b}_2$ we have $M \models (\exists\bar{y})\psi_*[\bar{b}_2 - \bar{b}^*, \bar{y}]$ hence for some $\bar{c}'_2$ we have $M \models \psi_*[\bar{b}_2 - \bar{b}^*, \bar{c}'_2]$ and let $\bar{c}''_2 = \bar{c}'_2 + \bar{c}^*$, so $M \models \psi_*[\bar{b}_2 - \bar{b}^*, \bar{c}''_2 - \bar{c}^*]$. As we are in case 1, by $\boxplus_\alpha(e)(\beta)$ there is a sequence $\langle \bar{e}_\iota : \iota < \lambda \rangle$ of members of G, i.e. of $\{\bar{a} \in {}^\xi M : M \models \psi_*(\bar{0}_\varepsilon, \bar{a})\}$ such that $i < \kappa_\beta \wedge (\iota(1) < \iota(2) < \lambda) \Rightarrow \bar{e}_{\iota(2)} - \bar{e}_{\iota(1)} \notin G_i$.

So for every $\iota < \lambda$, the sequence $(\bar{b}_2 - \bar{b}^*)^\frown(\bar{c}''_2 - \bar{c}^* + \bar{e}_\iota)$ belongs to $\psi_*({}^{\varepsilon+\xi}M)$ and for each $i < \kappa_\beta$ the set $\{\iota < \lambda : (\bar{b}_2 - \bar{b}^*)^\frown(\bar{c}''_2 - \bar{c}^* + \bar{e}_\iota)$ belongs to $(\bar{a}_i - \bar{b}^*{}^\frown \bar{c}^*) + G_i\}$ has at most one member. As $\kappa_\beta < \lambda$ for some $\iota < \lambda, (\bar{b}_2{}^\frown \bar{b})^\frown(\bar{c}''_2 - c^* + \bar{e}_\iota) \notin \cup\{\bar{a}_i - \bar{b}^*{}^\frown \bar{c}^* + G_i : i < \kappa_\beta\}$.

So $\bar{c}_2 := \bar{c}''_2 + \bar{e}_\iota$ is as required.

<u>Case 2</u>: $\kappa_\beta \notin I$

So there is a sequence $\langle \bar{d}_\iota : \iota < \iota(*)\rangle$ of members of $\mathbf{I}_\alpha$ as in $\boxplus_\alpha(e)(\alpha)$ for $\xi, G, G_i(i < \kappa_\beta)$ as above, i.e. with $\bar{\psi}'(\bar{y}), \langle \varphi'_i(\bar{y}) : i < \kappa_\beta\rangle$ here stands for $\psi(\bar{x}), \langle \varphi_i(\bar{x}) : i < \kappa_\beta\rangle$ there; so $\iota(*) < (2^{\kappa_\alpha})^+$ and $\iota < \iota(*) \Rightarrow \bar{d}_\iota \in \mathbf{I}_\alpha \cap {}^\varepsilon M$. As clearly $\bar{c}_1 - \bar{c}^* \in G$ necessarily for some $\iota < \iota(*)$ the set $u := \{i < \kappa_\beta : (\bar{c}_1 - \bar{c}^* - \bar{d}_\iota) \notin G_i\}$ belongs to I and, of course, $\bar{b}^* {}^\frown (\bar{c}^* + \bar{d}_\iota) \in \mathbf{I}_\alpha \cap {}^{\varepsilon+\xi} M$ and we have:

$(*)_1$ $M \models \varphi[\bar{b}_1 - \bar{b}^*, \bar{c}_1 - \bar{c}^* - d_\iota]$ for $\varphi \in \Phi_1$

$(*)_2$ if $i \in \kappa_\beta \backslash u$ then $M \models \varphi_i[\bar{b}_1 - \bar{b}^*, \bar{c}_1 - \bar{c}^* - \bar{d}_\iota]$.

As in Case 1 there is $\bar{c}''_2 \in {}^\xi M$ such that

$(*)_3$ $M \models \varphi[\bar{b}_2 - \bar{b}^*, \bar{c}''_2 - \bar{c}^* - \bar{d}_\iota]$ for $\varphi \in \Phi_1$

$(*)_4$ if $i \in \kappa_\beta \backslash u$ then $M \models \varphi_i[\bar{b}_2 - \bar{b}^*, \bar{c}''_2 - \bar{c}^* - \bar{d}_\iota]$.

As $u \in I$ by $\boxplus_\alpha(e)(\beta)$ that is, by 4.1 there are $\bar{\mathbf{e}}, u_*$ such that $\bar{\mathbf{e}}$ is a sequence of the form $\langle \bar{e}_j : j < \kappa^+_\beta\rangle$ and $u \subseteq u_* \in I$ such that:

$(*)_5$ $\bar{e}_j \in G_i$ for $i \in \kappa_\beta \backslash u_*$

$(*)_6$ $e_{j_2} - \bar{e}_{j_1} \notin G_i$ for $j_1 < j_2 < \kappa^+_\beta, i \in u_*$.

So

$(*)_7$ $(\bar{b}_2 - \bar{b}^*){}^\frown(\bar{c}''_2 - \bar{c}^* - \bar{d}_\iota - \bar{e}_j)$ belongs to $\cap\{\varphi({}^{\varepsilon+\xi}M) : \varphi \in \Phi_1\}$

$(*)_8$ if $i \in \kappa_\beta \backslash u_*$ then also $i \in \kappa_\beta \backslash u$ so by $(*)_4 + (*)_5$ the sequence $(\bar{b}_2 - \bar{b}^*){}^\frown(\bar{c}''_2 - \bar{c}^* - \bar{d}_\iota - \bar{e}_j)$ satisfies $\varphi_i(\bar{x}{}^\frown\bar{y} - \bar{a}_i)$ in M hence $\bar{b}_2{}^\frown(\bar{c}''_2 - \bar{e}_j)$ satisfies the formula $\neg\varphi_i(\bar{x}{}^\frown\bar{y} - \bar{a}_i)$ in M.

Lastly, by $(*)_6$

$(*)_9$ for each $i \in u_*$, there is $j_i < \kappa^+_\beta$ such that for every $j \in \kappa^+_\beta \backslash \{j_i\}$ the sequence $(\bar{b}_2{}^\frown\bar{b}^*){}^\frown(\bar{c}''_2 - c^* - e_j)$ satisfies $\neg\varphi_i(\bar{x}{}^\frown\bar{y} - \bar{a}_i)$, so for some $j, (\bar{c}''_2 - \bar{c}^* - \bar{e}_j)$ this holds for every $i \in u_*$.

Putting together $(*)_7 + (*)_8 + (*)_9$ clearly $(\bar{c}''_2 - \bar{c}^* - \bar{d}_\iota - \bar{e}_j)$ is as required in $\boxdot$ so we are done. $\square_{2.4}$

Definition 2.8. Let $\theta = \mathrm{cf}(\theta), \gamma$ an ordinal, $\bar{\lambda} = \langle \lambda_\beta : \beta < \gamma\rangle$.
1) For an R-module M we say $\bar{\mathbf{I}}$ is a (θ, γ)-witness for M <u>when</u> $\bar{\mathbf{I}} = \langle \mathbf{I}_\beta : \beta \leq \gamma\rangle$ and for each $\alpha \leq \gamma, \mathbf{I}_\alpha$ satisfies the conclusion of 2.4.
2) We say $\bar{\mathbf{I}}$ is a $(\bar{\lambda}, \theta, \gamma)$-witness when if in addition $\bar{\lambda} = \langle \lambda_\beta : \beta \leq \gamma\rangle$ and $\beta \leq \gamma \Rightarrow \lambda_\beta > |\mathbf{I}_\beta|$.

§ 3. Stability

Context 3.1. 1)

(a) R a fixed ring, $\tau = \tau_R$ <u>or</u>

(b) τ is a θ-additive vocabulary; $\mathbf{K}$ the class of τ-additive models.

2) $M \in \mathbf{K}$ a fixed R-module.
3) $\theta = \mathrm{cf}(\theta)$ and an ordinal, $\gamma(*)$ limit for simplicity.
4) $\bar{\lambda} = \langle \lambda_\alpha : \alpha \le \gamma(*) \rangle, \lambda_\alpha > \kappa_\alpha := \beth_\alpha(|R| + \theta^-)$.
5) $\bar{\mathbf{I}}^*$ is a $(\bar{\lambda}, \theta, \gamma(*))$-witness, see 2.8.
6) $A_* = \cup\{\bar{a} : \bar{a} \in \mathbf{I}_{\gamma(*)}\}$.
7) $\Lambda_\varepsilon = \Lambda^{\mathrm{pe}}_{\gamma(*),\varepsilon}$ for $\varepsilon < \theta$ and $\Lambda = \cup\{\Lambda_\varepsilon : \varepsilon < \theta\}$.
8) $M_* = M_{A_*} := (M, a)_{a \in A_*}$.

Definition 3.2. Assume $\varepsilon < \theta, \Lambda \subseteq \Lambda^{\mathrm{pe}}_{\theta,\gamma(*)}$ and $A_* \subseteq A \subseteq M \in \mathbf{K}$ and $\bar{a} \in {}^\varepsilon M$.
1) $\mathbf{S}^\varepsilon_\Lambda(A, M) = \{\mathrm{tp}_\Lambda(\bar{a}, A, M) : \bar{a} \in {}^\varepsilon M\}$, see below.
2) For $\bar{a} \in {}^\varepsilon M$ let $\mathrm{tp}_\Lambda(\bar{a}, A, M) = \{\varphi(\bar{x}\char`^\bar{b} - \bar{c}) : \bar{b} \in {}^\xi A$ and $\bar{c} \in {}^{\varepsilon+\xi} M$ and $M \models \varphi[\bar{a}_1\char`^\bar{b} - \bar{c}]$ and $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma,\varepsilon+\xi} \cap \Lambda\}$.

The Stability Theorem 3.3. *Assume* $\Lambda \subseteq \Lambda^{\mathrm{pe}}_{\gamma(*)}$ *and* $A \subseteq M \in \mathbf{K}$.
1) The set $\mathbf{S}^\varepsilon_\Lambda(A, M)$ *has cardinality* $\le ((|A|)^{<\theta})^{|\Lambda|}$.
2) For any $\kappa \ge 4$*, yes! four, there are no* $\bar{a}_\alpha \in {}^\varepsilon M, \bar{b}_\alpha \in {}^\xi M$ *for* $\alpha < \kappa$ *and*[1] $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma(*),\varepsilon,\xi}$ *such that for* $\alpha < \beta < \kappa$ *we have* $M \models$ "$\varphi[\bar{a}_\alpha, \bar{b}_\beta] \wedge \neg\varphi[\bar{a}_\beta, \bar{b}_\alpha]$".
3) If the formula $\varphi(\bar{x}, \bar{y})$ *from* $\mathbb{L}_{\infty,\theta,\gamma(*)}$ *or just is a Boolean combination of such formulas and* $\kappa \ge \beth_{\gamma(*)+2}(|\tau|^{<\theta})^+$ *<u>then</u> there are no* $M \in \mathbf{K}, \bar{a}_\alpha \in {}^\varepsilon M, \bar{b}_\alpha \in {}^\zeta M$ *for* $\alpha < \kappa$ *such that* $M \models \varphi[\bar{a}_\alpha, \bar{b}_\beta] \wedge \neg\varphi[\bar{a}_\beta, \bar{b}_\alpha]$ *whenever* $\alpha < \beta < \kappa$*. Actually* $\kappa \ge \beth_{\gamma(*)+1}(|\tau|^{<\theta})^+$ *suffice.*
4) If $p \in \mathbf{S}^\varepsilon_\Lambda(A, M)$ *and* $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{ep}}_{\gamma(*),\varepsilon,\xi}$ *and* $p \cap \{\varphi(\bar{x}, \bar{b}) : \bar{b} \in {}^\xi A\} \ne \emptyset$ *<u>then</u> for some* $\bar{a}_\varphi \in {}^\varepsilon A$ *and* $\bar{b} \in {}^\xi A$ *we have* $\varphi(\bar{x} - \bar{a}_\varphi, \bar{b}) \vdash p\restriction\{\pm\varphi\}$ *and* $\varphi(\bar{x} - \bar{a}_\varphi, \bar{b}) \in p$.

Proof. 1) Consider the statement

($*$) if $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma(*),\varepsilon,\xi} \cap \Lambda$ and $p_\ell(\bar{x}) = \mathrm{tp}_{\{\varphi(\bar{x},\bar{y})\}}(\bar{a}_\ell, A, M) \in \mathbf{S}^\varepsilon_{\{\varphi(\bar{x},\bar{y})\}}(A, M)$ for $\ell = 1, 2$ and $\bar{b} \in {}^\xi A, \bar{c} \in {}^{\varepsilon+\xi} A$ and $\varphi(\bar{x}\char`^\bar{b} - \bar{c}) \in p_1(\bar{x}) \cap p_2(\bar{x})$ <u>then</u> $p_1(\bar{x}) = p_2(\bar{x})$.

Why ($*$) is true? Assume $\varphi(\bar{x}\char`^\bar{b}' - \bar{c}') \in p_1(\bar{x})$, so $\bar{a}_1\char`^\bar{b}' - \bar{c}' \in \varphi(\bar{M})$. But we are assuming $\varphi(\bar{x}\char`^\bar{b} - \bar{c}) \in p_\ell(\bar{x}) = \mathrm{tp}_{\{\varphi(\bar{x},\bar{y})\}}(\bar{a}_\ell, A, M)$ hence $\bar{a}_\ell\char`^\bar{b} - \bar{c} \in \varphi(M)$ for $\ell = 1, 2$. Together $\bar{a}_2\char`^\bar{b}' - \bar{c}' = (\bar{a}_2\char`^\bar{b} - \bar{c}) - (\bar{a}_1\char`^\bar{b} - \bar{c}) + (\bar{a}_1\char`^\bar{b}' - \bar{c}')$ belongs to $\varphi(M)$, hence $\varphi(x\char`^\bar{b}' - c') \in p_2(x)$. So $\varphi(\bar{x}\char`^\bar{b}' - \bar{c}') \in p_1 \Rightarrow \varphi(\bar{x}\char`^b' - \bar{c}') \in p_2$ and by symmetry we have $\Leftrightarrow$ hence $p_1(\bar{x}) = p_2(\bar{x})$, i.e. we have proved ($*$).

Why ($*$) is sufficient? For every $\xi < \theta, \varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma(*),\varepsilon,\xi} \cap \Lambda$ and $p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)$ choose $(\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}, \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})})$ such that

$\oplus_1$
- $\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} \in {}^\varepsilon A$ and $\bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} \in {}^{\varepsilon+\xi} A$
- if possible $\varphi(\bar{x}\char`^\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} - \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}) \in p(\bar{x})$.

For $p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)$ let $\Phi_{p(\bar{x})} = \{\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma,\varepsilon,\xi}$: in $\oplus_1$ we have "possible"$\}$ and let $q_{p(\bar{x})} = \{\varphi(\bar{x}\char`^\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} - \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}) : \varphi(\bar{x}, \bar{y}) \in \Phi_{p(\bar{x})}\}$.

Now

$\oplus_2$ if $p_1(\bar{x}), p_2(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)$ and $\Phi_{p_1(\bar{x})} = \Phi_{p_2(\bar{x})}$ and $q_{p_1(\bar{x})} = q_{p_2(\bar{x})}$ then $p_1(\bar{x}) = p_2(\bar{x})$.

[1] This holds also for $\neg\varphi(\bar{x}, \bar{y})$ but for κ finite we can invert the order.

[Why? Just think.]

$\oplus_3$ the set $\{(\Phi_{p(\bar{x})}, q_{p(\bar{x})}) : p(\bar{x}) \in \mathbf{S}^{\varepsilon}_{\Lambda}(A, M)\}$ has cardinality $\leq 2^{|\Lambda|} + (|A|^{<\theta})^{|\Lambda|}$.

[Why? Straightforward.]

Clearly we are done.

2) Note that $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\rm pe}_{\gamma,\varepsilon,\xi}$ implies that

$\boxplus$ if $M \models \varphi[\bar{a}, \bar{b}] \wedge \varphi[\bar{a}, \bar{b}'] \wedge \varphi[\bar{a}', \bar{b}]$ then $M \models \varphi[\bar{a}', \bar{b}']$.

[Why? As $\varphi({}^{\varepsilon+\zeta}M)$ is a subgroup of ${}^{\varepsilon+\zeta}M$ and $\bar{a}\hat{\ }\bar{b}, \bar{a}'\hat{\ }\bar{b}, \bar{a}\hat{\ }\bar{b}'$ belongs to it then so does $\bar{a}'\hat{\ }\bar{b}' = \bar{a}'\hat{\ }\bar{b} + (\bar{a}\hat{\ }\bar{b}') - (\bar{a}\hat{\ }\bar{b})$ but the latter is equal to $\bar{a}'\hat{\ }\bar{b}'$.]

So we can choose $\bar{a} = \bar{a}_0, \bar{a}' = \bar{a}_3, \bar{b} = \bar{b}_1, \bar{b}' = \bar{b}_2$ and get a contradiction.

3) Toward contradiction let $\langle \bar{a}_\alpha : \alpha < \kappa\rangle, \bar{a}_\alpha \in {}^{\varepsilon}M$ form a counterexample. By Erdös-Rado theorem $\beth_{\gamma(*)+2}(|\tau|^{<\theta})^+ \rightarrow (4)^2_{\beth_{\gamma(*)+1}(|\tau|^{<\theta})}$. Now for $\alpha < \beta < \kappa$ let $p_{\alpha,\beta} = {\rm tp}_{\Lambda^{\rm pe}_{\gamma(*),\varepsilon,\varepsilon}}(\bar{a}_\alpha\hat{\ }\bar{a}_\beta; \emptyset, M)$ so $\{p_{\alpha,\beta} : \alpha < \beta\}$ has cardinality $\leq \beth_{\gamma(*)+1}(|\tau|^{<\theta})$ hence by the arrow above for some $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ and $p, \ell < m < u \Rightarrow p_{\alpha_\ell,\alpha_n} = p$; we get contradiction by part (2). If κ is just $\geq \beth_{\gamma(*)+1}(|\tau|^{<\theta})^+$, use $\boxplus$ from the proof of part (2) and repeat a proof of the Erdös-Rado theorem.

4) Should be clear. $\square_{3.2}$

Recall ([**Sh:300a**])

Definition 3.4. For $\Phi \subseteq \Lambda$ we say $\mathbf{I} \subseteq {}^{\varepsilon}M$ is (μ, Φ)-convergent <u>when</u> $|\mathbf{I}| \geq \mu$ and for every $\xi < \theta$ and $\varphi(\bar{x}) \in \Phi_{\varepsilon+\xi}$ and $\bar{b} \in {}^{\xi}M, \bar{c} \in {}^{\xi+\varepsilon}M$ for all but $< \mu$ of the $\bar{a} \in I$ the truth value of $\bar{a}\hat{\ }\bar{b} - \bar{c} \in \varphi(M)$ is constant.

Claim 3.5. *1) A sufficient condition for $\mathbf{I} = \{\bar{a}_i : i < \lambda\} \subseteq {}^{\varepsilon}M$ to be (μ, Φ)-convergent is: for some $\varepsilon, \mathbf{I} \subseteq {}^{\varepsilon}M$ and $i < j < \lambda \wedge \varphi(\bar{x}) \in \Phi \cap \Lambda_\varepsilon \Rightarrow \bar{a}_j - \bar{a}_i \in \varphi(M)$.*
2) If $\varepsilon < \theta, \lambda = {\rm cf}(\lambda) > \mu \geq \mu_{\gamma(*)}$ and $(\forall i < \lambda)(|i|^{\mu_{\gamma(*)}} < \lambda)$ and $\bar{a}_i \in {}^{\varepsilon}M$ for $i < \lambda$ with no repetition <u>then</u> for some stationary $S \subseteq \lambda, \{\bar{a}_i : i \in S\}$ is (μ^+, Φ)-convergent.

Remark 3.6. 1) Note that being $(\mu, \mathbf{I})$-convergent is very close to being $(< \omega)$-indiscernible, and sometimes is the reasonable generalization of indiscernible.
2) So 3.5(1) says that 2-indiscernible almost implies $(< \omega)$-indiscernible.
2) Also 3.5(2) says there are $(< \omega)$-indiscernibles.

Proof. Should be clear. $\square_{3.5}$

§ 4. How much does the subgroup exhaust a group

Claim 4.1. *Assume the groups G_s (for $s \in S$) are subgroups of the group G and $\lambda > |S|^+$. There is an ideal I on S (possibly $I = \mathscr{P}(S)$) such that:*

(a) *for every $u \in I$ there is a sequence $\bar{g} = \langle g_\alpha : \alpha < \lambda\rangle$ of members of G such that $s \in u \wedge \alpha < \beta < \lambda \Rightarrow g_\alpha G_s \neq g_\beta G_s$*
(b) *for $u \in \mathscr{P}(S) \backslash I$, clause (a) fails*
(c) *if $S \notin I$, ${\rm cf}(\lambda) > 2^{|S|}$ and $\alpha < \lambda \Rightarrow |\alpha|^{|S|} < \lambda$, e.g. $(\exists \mu)(\lambda = (\mu^{|S|})^+)$ <u>then</u> there is $A \subseteq G$ of cardinality $< \lambda$ such that for every $g \in G$ for some $a \in A$ we have $\{s \in S : gG_s \neq aG_s\} \in I$*
(d) *under the assumptions of clause (c) and in addition λ is regular <u>then</u> for every $u \in I$ for some $\bar{g}$ and v we have*

- $u \subseteq v \in I$
- $\bar{g} = \langle g_\alpha : \alpha < \lambda \rangle$
- *$g_\alpha G_s = g_0 G_s$ moreover $g_\alpha \in G_s$ for $s \in S \backslash v$*
- *if $s \in v, \alpha < \beta < \lambda$ then $g_\alpha G_s \neq g_\beta G_s$.*

(e) *$I \subseteq \mathscr{P}(S), I$ is closed under subsets*

(f) *I is an ideal provided that G is Abelian <u>or</u> just each G_s is a normal subgroup.*

Definition 4.2. For G and $\bar{G} = \langle G_s : s \in S \rangle$ as in 4.1 and $\lambda \geq \aleph_0$ let $I = I_{G,\bar{G},\lambda}$ be as defined in clauses (a),(b) of 4.1, it is an ideal (but may be $\mathscr{P}(S)$).

Proof. Let I be the set of $u \subseteq S$ such that clause (a) holds.
Now

(∗) (α) $I \subseteq \mathscr{P}(\kappa)$

(β) I is $\subseteq$-downward closed, i.e. is closed under subsets.

[Why? Obvious.]

Now for 4.1, we have chosen I such that clauses (a),(b),(e) hold.

Toward proving clause (c) of 4.1 for each $u \in I^+ := \mathscr{P}(S) \backslash I$, let $\bar{g}_u = \langle g_{u,\alpha} : \alpha < \alpha(u) \rangle$ be a maximal sequence of members of G such that $\alpha < \beta < \alpha(u) \wedge s \in u \Rightarrow g_{u,\alpha} G_s \neq g_{u,\beta} G_s$. By the definition of I as $u \notin I$, necessarily $\alpha(u) < \lambda$, and as we are assuming $\mathrm{cf}(\lambda) > 2^{|S|}$, clearly $\alpha(*) = \sup\{\alpha(u) : u \in I^+\} < \lambda$. So $B := \{g_{u,\alpha} : u \in I^+ \text{ and } \alpha < \alpha(u)\}$ is a subset of G of cardinality $< \lambda$. For every $u \in I$ and $h : S \backslash u \to B$ choose $g_h \in G$ such that, if possible, $(\forall s \in S \backslash u)(g_h G_s = h(s) G_s)$, so $A = \{g_h : h \text{ is a function from } S \backslash u \text{ into } B \text{ and } u \in I\}$ is a subset of G of cardinality $\leq |B|^{|S|} < \lambda$, recalling we are assuming now $(\forall \alpha < \lambda)(|\alpha|^{|S|} < \lambda)$.

We shall show that A is as required (in clause (c)), then we are done. Let $g_* \in G$. Let $u = \{s \in S$: for no $w \in I^+$ and $\alpha < \alpha(w)$ do we have $g G_s = g_{u,\alpha} G_s\}$. Now if $u \in I^+$ then $\bar{g}_u = \langle g_{u,\alpha} : \alpha < \alpha(u) \rangle$ is well defined and g_* satisfies the demand on $g_{u,\alpha(u)}$ contradicting the maximality of $\bar{g}_u$. So $u \in I$ and we can find $h : (S \backslash u) \to B$ such that $s \in S \backslash u \Rightarrow g_* G_s = h(s) G_s$. So g_h is well defined and $\in A$ and is as required, so we are done.

For clause (d) let $u \in I$ be given and let $\langle g_\alpha : \alpha < \lambda \rangle$ witness that $u \in I$. For each $\alpha < \beta$ let $u_\alpha = \{s \in S$: there is $\beta < \alpha$ such that $g_\alpha G_s = g_\beta G_s\}$, clearly $u_\alpha \cap u = \emptyset$ and let $h_\alpha : u_\alpha \to \alpha$ be such that $s \in u_\alpha \Rightarrow g_\alpha G_s = g_{h_\alpha(s)}$.

As λ is regular, recalling $(\forall \alpha < \lambda)(|\alpha|^{|S|} < \lambda)$ by the present assumption on λ, for some $h : u_* \to \lambda$, the set $\mathscr{W}$ is a stationary subset of λ where $\mathscr{W} = \{\alpha < \lambda : \mathrm{cf}(\alpha) = |S|^+ \text{ and } h_\alpha = h, u_\alpha = u_*\}$. Clearly $\alpha, \beta \in \mathscr{W} \wedge s \in u_* \Rightarrow g_\alpha G_s = g_{h(s)} G_s = g_\beta G_s$ and $\alpha \neq \beta \in \mathscr{W} \wedge s \in S \backslash u_* \Rightarrow g_\alpha G_s \neq g_\beta G_s$. Letting $\langle \alpha_i : i < \lambda \rangle$ list $\mathscr{W}$ and $g_i = g_{\alpha_i}$ for $\alpha < \lambda$, clearly $v = u_*, \langle g'_i : i < \lambda \rangle$ are as promised in clause (d). Well for the "moreover" and, i.e. use $\langle g_0^{-1} g_{1+i} : i < \lambda \rangle$.

We are left with clause (f).

I is an ideal <u>when</u> the assumption of clause (f) holds. Let $u_1, u_2 \in I$ be disjoint and we shall prove that $u := u_1 \cup u_2 \in I$. Let $\langle g_{\ell,\alpha} : \alpha < \lambda \rangle$ witness $u_\ell \in I$ for $\ell = 1, 2$. We try to choose $g_{3,\varepsilon} \in G$ such that $\zeta < \varepsilon \wedge s \in u \Rightarrow g_{3,\varepsilon} G_s \neq g_{3,\zeta} G_s$; we can add $g_{3,\varepsilon} \in \{g_{1,i} g_{2,j} : i, j < \lambda\}$. Arriving to ε, if for some $i < \lambda \wedge j < \lambda$ we can choose $g_{3,\varepsilon} := g_{1,i} g_{2,j}$ fine.

Otherwise there are $f : \lambda \times \lambda \to \varepsilon$ and $g : \lambda \times \lambda \to u$ such that for $(i, j) \in \lambda \times \lambda$ we have $g_{1,i}g_{2,j}G_{g(i,j)} = g_{3,f(i,j)}G_{g(i,j)}$.

For each $i < \lambda, \zeta < \varepsilon$ and $s \in u \subseteq S$ let $\mathscr{U}^2_{i,\zeta,s} = \{j < \lambda : f(i,j) = \zeta, g(i,j) = s\}$. Now $j \in \mathscr{U}^2_{i,\zeta,s} \Rightarrow g_{1,i}g_{2,j}G_s = g_{3,\zeta}G_s \Rightarrow g_{2,j}G_s = g^{-1}_{1,i}g_{3,\zeta}G_s$ hence if $s \in u_2$ then $j(1) \neq j(2) \in \mathscr{U}^2_{i,\zeta,s} \Rightarrow g_{2,j(1)}G_s = (g^{-1}_{1,i}g_{3,\zeta})G_s = g_{2,j(2)}G_s$ contradiction. Hence $\mathscr{U}^2_{i,\zeta,s}$ has cardinality ≤ 1 when $i < \lambda, \zeta < \varepsilon, s \in u_2$.

For $j < \lambda, \zeta < \varepsilon$ and $s \in u$ let

$$\mathscr{U}^1_{j,\zeta,s} = \{i < \lambda : f(i,j) = \zeta \text{ and } g(i,j) = s\}.$$

If G is abelian, as above we have $\zeta < \varepsilon \wedge j < \lambda \wedge s \in u_1 \Rightarrow |\mathscr{U}^1_{j,\zeta,s}| \leq 1$. If not but still every G_s is a normal subgroup of G then for any $j < \lambda, \zeta < \mu, s \in u_1$ we have $i \in \mathscr{U}^1_{j,\zeta,s} \Rightarrow g_{1,i}g_{2,j}G_s = g_{3,\zeta}G_s \Rightarrow g_{1,i}(G_s g_{2,j}) = g_{1,i}(g_{2,j}G_s) = g_{3,\zeta}G_s \Rightarrow g_{1,i}G_s = g_{3,\zeta}(G_s g^{-1}_{2,j})$ hence $i(1) \neq i(2) \in \mathscr{U}^1_{j,\zeta,s} \Rightarrow g_{1,i(1)}G_s = g_{3,\zeta}(G_s g^{-1}_{2,j}) = g_{1,i(2)}G_s$, a contradiction so again $\mathscr{U}^1_{j,\zeta,s}$ has at most one member.

For $\ell \in \{1, 2\}$ and $i < \lambda$ let $\mathscr{U}^\ell_i = \cup\{\mathscr{U}^\ell_{i,\zeta,s} : \zeta < \varepsilon \text{ and } s \in u_\ell\}$, so as $|u_\ell| \leq |S|$ clearly $|\mathscr{U}^\ell_i| \leq |S|$. As $\lambda > |S|^+$ there are $i, j < \lambda$ such that $i \notin \mathscr{U}^1_j \wedge j \notin \mathscr{U}^2_i$; hence the member $g_{1,i}g_{2,j}$ of G satisfies the demand on $g_{3,\varepsilon}$.

So we can carry the induction on $\varepsilon < \lambda$, so we are done proving clause (f). $\square_{4.1}$

Claim 4.3. *In 4.1 there is a $W \subseteq S$ such that*

(*a*) *there is a sequence $\bar{s} = \langle s_i : i < i(*)\rangle$ listing W satisfying $(\bigcap\limits_{i<j} G_{s_i}, \bigcap G_{s_i})$ is finite for $j < i(*)$ stipulating $\bigcap\limits_{i<0} G_{s_i} = G$*

(*b*) *if $W' \subseteq S$ satisfies (A) then $W' \subseteq W$.*

Proof. Immediate. $\square$

§ 5. Concluding Remark

Example 5.1. An example of additive structure is a ring satisfying $xy = -yx$, i.e. if $(R, +^R)$ is $\oplus\{\mathbb{Z}x_s : s \in I\}, f$ is a function from $I \times I$ into R is such that $f(x, y) = -f(y, x)$ and $f(x, x) = 0$ and we have

$$\Big(\sum_{\ell<\ell(*)} a_\ell x_{s_\ell}\Big)\Big(\sum_{m<n(*)} b_m x_{t_n}\Big) = \Sigma\{a_\ell b_m x_{f(s_\ell,t_m)} : \ell < \ell(*), m < m(*)\}.$$

Remark 5.2. 1) We may use $\tau \supseteq \{+, -, 0, 1\} \cup \{P_i : i < i(*)\}, P_i$ unary and instead modules use τ-models M such that $|M|$ is the disjoint union $\cup\{P^M_i : i < i(*)\}, +^M$ is a partial two-place function, $+^M = \cup\{+^M{\restriction}P^M_i : i < i(*)\}, (P^M_i, +^M)$ an abelian group, all relations and functions commute with $+$ or at least every relation is affine, i.e. let $F_*(x, y, z) = x - y + z$, and demand $G(\ldots, F_*(x_i, y_i, z_i), \ldots)_{i<i(*)} = F_*(G(\bar{x}), G(\bar{y}), G(\bar{z}))$ and $\bar{a}, \bar{b}, \bar{c} \in P^M \Rightarrow F_*(\bar{a}, \bar{b}, \bar{c}) = \langle F_*(a_i, b_i, c_i) : i < \text{arity}(P)\rangle \in P^M$.
2) However, as we use infinitary logics, if M is the disjoint union of Abelian groups $G^M_i := (P^M_i, +^M_i)$ for $i < i(*)$ and we define G_M as the direct sum having predicate for those subgroups then we have bi-interpretability. Concerning having "affine structure" only, we can expand by choosing an element in each to serve as zero.
3) It is natural to extend our logic by cardinality quantifiers saying "the definable subgroup G divided by the definable subgroup H has cardinality $\geq \lambda$".

REMARK 5.3. Concerning 2.4 note
1) Note that instead of an R-module M we can use $(M, c_\alpha)_{\alpha<\kappa}$, i.e. expand M by κ individual constants; the only difference is using $\beth_\alpha(|R|^{<\theta}+\kappa)$ instead $\beth_\alpha(|R|^{<\theta})$.
2) The theorem 5.3 has an arbitrary choice: the $\mathbf{I}_\alpha$, so e.g. not every formula $\varphi(\bar{x}) \in \mathbb{L}_{\infty,\theta,\gamma}$ and $\bar{a} \in \mathbf{I}_\partial$ is $\varphi(\bar{x}, \bar{a}_\gamma)$ equivalent to a formula without parameters. Instead of using extra individual constants, in the proof (see $\boxplus_\alpha$ in the proof of 2.4) for any $\psi(\bar{x}), \psi(\bar{x}) \wedge \varphi_i(\bar{x})$ for $i < i(*) < \kappa_\beta, I, G, G_i(i < i(*))$ and the ideal I on κ_β can expand M by:

(a) $P^M = \{\bar{a} : M \models \psi[\bar{a}] \text{ and } \{i < \kappa_\beta : \bar{a} \notin G_i\} \in I\}$ is a subgroup
(b) predicates for the set $\{\bar{a} + P^M : \bar{a} \in \psi(M)\}$.

So the proof shows that we can in M eliminate quantifier to quantifier-free formulas in this expansion.
3) Also this may give too much information. Still the result gives elimination of quantifiers: not as low as in the first order case.
4) We can now define non-forking and hopefully [**Sh:F1210**] will deal with this.

QUESTION 5.4. 1) Are there arbitrarily large Abelian groups G which are not only indecomposable, but even potentially so, i.e. absolutely, even after any forcing G is indecomposable.
2) Relatives, e.g. no potential non-trivial automorphism.

DISCUSSION 5.5. We know that for the minimal $\lambda, \lambda \to (\omega)^{<\omega}_{\aleph_0}$, up to λ the answer is yes (and more) but if $|G| \geq \lambda$ then potentially it has non-trivial endomorphisms and even non-trivial embedding of G into itself (Eklof-Shelah [**EkSh:678**], Göbel-Shelah [**GbSh:880**]). We can improve this to "for some $a_1 \neq a_2$ from G", potentially there are an embeddings f_1, f_2 of G into itself such that $f_1(a_1) = a_2, f_2(a_2) = a_1$, see [**Sh:F1210**].

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