BINARY CONSTANT-LENGTH SUBSTITUTIONS AND MAHLER MEASURES OF BORWEIN POLYNOMIALS

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In memory of Jonathan Michael Borwein (1951–2016)

ABSTRACT. We show that the Mahler measure of every Borwein polynomial—a polynomial with coefficients in $\{-1,0,1\}$ having non-zero constant term—can be expressed as a maximal Lyapunov exponent of a matrix cocycle that arises in the spectral theory of binary constant-length substitutions. In this way, Lehmer's problem for height-one polynomials having minimal Mahler measure becomes equivalent to a natural question from the spectral theory of binary constant-length substitutions. This supports another connection between Mahler measures and dynamics, beyond the well-known appearance of Mahler measures as entropies in algebraic dynamics.

1. Introduction

Let p be a polynomial with complex coefficients. The *logarithmic Mahler measure* of p is defined to be the logarithm of the geometric mean of p over the unit circle; that is,

$$\mathfrak{m}(p) := \int_0^1 \log |p(e^{2\pi i t})| dt.$$

It is well known and easily shown using Jensen's formula [38, Prop. 16.1] that the logarithmic Mahler measure satisfies

$$\mathfrak{m}(p) \,=\, \log |a_s| \,+ \sum_{j=1}^s \log \bigl(\max\{|\alpha_j|,1\} \bigr),$$

where $p(z) = a_s \prod_{j=1}^s (z - \alpha_j)$; see [28] for background. Here, we will only consider integer polynomials. An old result of Kronecker [30] implies that, if p is monic, then $\mathfrak{m}(p) = 0$ if and only if p is a product of cyclotomic polynomials and a monomial. In this way, \mathfrak{m} is a measure of the distance of an integer polynomial to the unit circle, and of a monic integer polynomial to a product of cyclotomic polynomials and a monomial.

One of the most interesting and long-standing problems in this area concerns finding polynomials with small logarithmic Mahler measures. Lehmer found the polynomial

(1)
$$\ell_{\rm L}(z) = 1 + z - z^3 - z^4 - z^5 - z^6 - z^7 + z^9 + z^{10},$$

which is irreducible, and has precisely one real root outside the unit circle, which is a Salem number. The polynomial $\ell_{\rm L}$ is the polynomial with the smallest known positive logarithmic

Mahler measure,

$$\mathfrak{m}(\ell_{L}) \, \approx \, \log(1.176281).$$

Lehmer's problem. Does there exist a constant c > 0 such that any irreducible non-cyclotomic polynomial p with integer coefficients satisfies $\mathfrak{m}(p) \geqslant c$?

There are some special classes of polynomials for which Lehmer's problem has long been answered in the affirmative. In particular, there is a very interesting gap result for non-reciprocal polynomials due to Smyth [39]; see also Breusch [24]. A polynomial p is reciprocal provided $p(z) = z^{\deg(p)}p(1/z)$; that is, a polynomial is reciprocal if its sequence of coefficients is palindromic. Smyth proved that, for non-reciprocal polynomials, one either has $\mathfrak{m}(q) = 0$ or $\mathfrak{m}(q) \ge \log(\lambda_p)$, where λ_p is the smallest Pisot number, which is the real root of $z^3 - z - 1$, also known as the plastic number; compare [8, Ex. 2.17]. Specialising this class a bit more, Borwein, Hare and Mossinghoff [19, Cor. 1.2] showed that all non-reciprocal polynomials q with odd integer coefficients satisfy the bound

$$\mathfrak{m}(q) \geqslant \mathfrak{m}(z^2 - z - 1) = \log(\tau),$$

where $\tau = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio, a well-known Pisot number. The golden ratio is characterised by the property that it is the smallest limit point of Pisot numbers. See Smyth [41] for a general survey, Boyd [22] for work on reciprocal polynomials, and Boyd and Mossinghoff [23] for a statistical study of limit points.

One of the most studied classes of integer polynomials in relation to Lehmer's problem are the *Borwein polynomials*—polynomials of height 1 (coefficients in $\{-1,0,1\}$) with non-zero constant term; see [27]. Special importance is placed on this class, since for any integer polynomial p with $\mathfrak{m}(p) < \log(2)$ there is an integer polynomial q such that pq has height 1; see Pathiaux [37]. Boyd [21] notes that, in his experience, such a q can be taken to be cyclotomic and of fairly small degree relative to the degree of p; see also Mossinghoff [35]. If Boyd's observation were proved true in general, then to solve Lehmer's problem, it would be enough to consider only Borwein polynomials; unfortunately, this is still unknown.

Before we continue, let us mention that there is a well-known connection between Mahler measures and algebraic dynamics. Here, logarithmic Mahler measures show up as entropies of \mathbb{Z}^d -shifts of algebraic origin [28, 32, 38]. The first appearance of a Mahler measure in such a context actually dates back to a paper by Wannier [43] on the groundstate entropy of the antiferromagnetic Ising model on the triangular lattice; see Remark 6.3 below for details. This general connection between Mahler measures and entropy has initiated many investigations and enhanced our knowledge about Mahler measures significantly; see [32, 38] and the references therein for a detailed account.

In this paper, under some quite natural assumptions, we relate the logarithmic Mahler measure of Borwein polynomials to a Lyapunov exponent from the spectral theory of substitutions; see [26] for an earlier appearance of a connection between Mahler measures and

Lyapunov exponents. A binary constant-length substitution ϱ is defined on $\Sigma_2 := \{0,1\}$ by

$$\varrho: \begin{cases} 0 \mapsto w_0 \\ 1 \mapsto w_1 \end{cases},$$

where w_0 and w_1 are finite words over Σ_2 of equal length $|w_0| = |w_1| = L \ge 2$. Such substitutions are important objects of research in many areas of mathematics, ranging from dynamics and combinatorics (as substitutions) to number theory (this is the class of 2-automatic sequences) and theoretical computer science (under the name of uniform morphisms).

To continue, we say that such a substitution is *primitive* if the substitution matrix M_{ϱ} is primitive, and *aperiodic* if the hull (or shift) defined by it does not contain any element with a non-trivial period. When ϱ is primitive, this is the case if and only if any of the two-sided fixed points of ϱ (or of ϱ^n with a suitable $n \in \mathbb{N}$) with legal core (or seed) is non-periodic. If one of these fixed points is non-periodic, they all are, due to primitivity; see [8, Sec. 4.2] for notions and further details.

Our main result is the following theorem; the relevant concepts concerning Lyapunov exponents are recalled in Section 3.

Theorem 1.1. For any primitive, binary constant-length substitution ϱ , the extremal Lyapunov exponents are explicitly given by

$$\chi_{\min}^B = 0$$
 and $\chi_{\max}^B = \mathfrak{m}(p_\varrho),$

where p_{ϱ} is a Borwein polynomial, easily determined by $\varrho.$

Theorem 1.1 is a statement relating the logarithmic Mahler measure of Borwein polynomials to Lyapunov exponents of binary constant-length substitutions. Depending on which object one is interested in, it can be used in a couple of different ways. As it reads above, if one has a binary substitution, one can easily compute the extremal Lyapunov exponents using the associated Borwein polynomial. Alternatively, if one has a Borwein polynomial, one can determine an associated binary constant-length substitution. This relationship can be exploited to give some general results about extremal Lyapunov exponents for certain binary substitutions. For example, one now has a rather general result considering bijective substitutions, which are the substitutions where the letters in the words w_0 and w_1 are different at each position. Smyth's result [39] implies the following consequence of Theorem 1.1, in conjunction with Lemma 4.4 below.

Corollary 1.2. Suppose that the primitive, binary constant-length substitution ϱ is also bijective, and that w_0 is not a palindrome. Then, $\chi^B_{\max} \geqslant \log(\lambda_p)$, where λ_p is the plastic number.

As it turns out, primitive, binary constant-length substitutions which are also periodic, do not satisfy the assumptions of this corollary; in fact, for such periodic substitutions one has

that $\chi_{\text{max}}^B = 0$. A characterisation and further details regarding these periodic substitutions are given later; see Lemma 4.4 and Theorem 4.5.

Given the above results, in the case of Borwein polynomials, Lehmer's problem can be restated in terms of the Lyapunov exponents for binary constant-length substitutions.

Lehmer's problem (dynamical analogue). Does there exist a constant c > 0 such that, for any primitive, binary constant-length substitution ϱ with $\chi_{\max}^B \neq 0$, we have $\chi_{\max}^B \geqslant c$?

Remark 1.3. Strong versions of both Lehmer's problem and our dynamical analogue would ask whether the constant c can be taken to be $\mathfrak{m}(\ell_L)$, the logarithmic Mahler measure of Lehmer's polynomial from Eq. (1).

Viewing Lehmer's problem in a dynamical setting, as related to constant-length substitutions, has some heuristic benefits. In this area, especially at the interface with number theory, gap results are common and expected. For example, if f(n) denotes the n^{th} letter of a one-sided fixed point of a constant-length substitution, then, for any positive integer $b \geq 2$, the number $\sum_n f(n)b^{-n}$ is either rational or transcendental [1, 3, 14]. Also, this number cannot be a Liouville number, that is, it has finite irrationality exponent [2, 14]. The partial sums $S(N) := \sum_{n \leq N} f(n)$ satisfy even stronger gap properties. If S(N) is unbounded, there is a constant c > 0 such that $|S(N)| \geq c \log(N)$ for infinitely many integers N; see [15, 16]. Viewing Lehmer's problem for Borwein polynomials in this context may, at the least, take away some of the surprise of its conclusion, and provide an additional reason to believe in the conjecture for this class of polynomials.

The remainder of this paper is organised as follows. In Section 2, we give details regarding binary substitutions and their associated Fourier matrices, while we give the relevant definitions on Lyapunov exponents in Section 3. Section 4 contains the proof of Theorem 1.1 as a consequence of a more detailed version. In Section 5, we provide several examples, including those related to Littlewood, Newman, and Borwein polynomials. Finally, in Section 6, we explore extensions to higher dimensions and their relationship to logarithmic Mahler measures of multivariable polynomials.

2. Substitutions and their Fourier matrices

As stated in the Introduction, we are concerned with binary constant-length substitutions ϱ defined on $\Sigma_2 := \{0, 1\}$ by

(2)
$$\varrho: \begin{cases} 0 \mapsto w_0 \\ 1 \mapsto w_1 \end{cases},$$

where w_0 and w_1 are finite words over Σ_2 of equal length $|w_0| = |w_1| = L \geqslant 2$.

¹Those comfortable with the dynamical setting will note that, by working with two prototiles of unit length, the tiling and symbolic pictures of these systems are equivalent (topologically conjugate by a sliding block map).

We denote the m^{th} column of ϱ by

$$\mathcal{C}_m := \begin{bmatrix} (w_0)_m \\ (w_1)_m \end{bmatrix},$$

where $(w_i)_m$ is the m^{th} letter of the word w_i . We follow the convention of indexing the columns starting with 0; compare [8, Ch. 4]. A binary substitution is said to have a *coincidence* at position m, if the column at that specific position is either $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A binary substitution is called *bijective* if there are no coincidences.

For $0 \le i, j \le 1$, let T_{ij} be the set of all positions m where the letter i appears in w_j , and let $T := (T_{ij})_{0 \le i, j \le 1}$ be the resulting 2×2 -matrix. Note that the substitution matrix M_{ϱ} , as usually obtained via Abelianisation, satisfies

$$M_{\varrho} = \left(\operatorname{card}(T_{ij})\right)_{0 \leqslant i,j \leqslant 1}.$$

Using T, we build a matrix of pure point measures $\delta_T = (\delta_{T_{ij}})_{0 \le i,j \le 1}$, where $\delta_S := \sum_{x \in S} \delta_x$ with $\delta_\varnothing = 0$. This gives rise to an analytic matrix-valued function via

$$B(k)\,:=\,\overline{\widehat{\delta_T}(k)}\,,$$

which we call the Fourier matrix of ϱ ; see [4, 5]. Note that $B(0) = M_{\varrho}$. The Fourier matrix provides more information than M_{ϱ} as it encodes the column positions of each letter in the corresponding words that contain them, whereas the entries of M_{ϱ} only count the letters 0 and 1 in w_0 and w_1 , respectively.

The following two examples are paradigmatic for the two principal situations among aperiodic binary constant-length substitutions.

Example 2.1. Consider the Thue–Morse substitution, as given by

$$\varrho_{\mathrm{TM}}: \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10. \end{cases}$$

Here, one has $T_{\text{TM}} = \begin{pmatrix} \{0\} & \{1\} \\ \{1\} & \{0\} \end{pmatrix}$, which gives

$$\delta_{T_{\mathrm{TM}}} = \begin{pmatrix} \delta_0 & \delta_1 \\ \delta_1 & \delta_0 \end{pmatrix} \quad \text{and} \quad B_{\mathrm{TM}}(k) = \begin{pmatrix} 1 & \mathrm{e}^{2\pi\mathrm{i}k} \\ \mathrm{e}^{2\pi\mathrm{i}k} & 1 \end{pmatrix}. \qquad \diamondsuit$$

Example 2.2. On the other hand, for the period doubling substitution,

$$\varrho_{\mathrm{pd}}: \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 00, \end{cases}$$

the corresponding matrices are $T_{\rm pd}=\left(\{0\}\atop\{1\}\right.\ \varnothing\right)$ together with

$$\delta_{T_{\mathrm{pd}}} = \begin{pmatrix} \delta_0 & \delta_0 + \delta_1 \\ \delta_1 & 0 \end{pmatrix} \quad \text{and} \quad B_{\mathrm{pd}}(k) = \begin{pmatrix} 1 & 1 + \mathrm{e}^{2\pi\mathrm{i}k} \\ \mathrm{e}^{2\pi\mathrm{i}k} & 0 \end{pmatrix}.$$

3. Lyapunov exponents

Using the ergodic transformation $k \mapsto Lk \mod 1$ defined on the 1-torus \mathbb{T} , which is represented by [0,1) with addition modulo 1 and equipped with Lebesgue measure, one can use the Fourier matrix B(k) to build the matrix cocycle

$$B^{(n)}(k) := B(k)B(Lk)\cdots B(L^{n-1}k),$$

where the somewhat unusual extension to the right originates from the underlying spectral problem of binary substitutions; see Remark 3.1 below and [5, 6, 10]. Recall that the integer $L \geq 2$ is the common length of the words w_0 and w_1 from the definition of the binary substitution ϱ . We note further that the inverse cocycle $(B^{(n)}(k))^{-1}$ exists for almost every $k \in \mathbb{R}$, because $\det(B(k)) = 0$ for at most a countable subset of \mathbb{R} . Due to the ergodicity of the transformation $k \mapsto Lk$ on \mathbb{T} relative to Lebesgue measure, Oseledec's multiplicative ergodic theorem ensures the existence of the Lyapunov exponents and the corresponding subspaces in which they represent the asymptotic exponential growth of the vector norms; see [12]. More precisely, if $v \in \mathbb{C}^2$ is any (fixed) row vector, the values

(3)
$$\chi^{B}(v,k) := \lim_{n \to \infty} \frac{1}{n} \log \|vB^{(n)}(k)\|$$

exist for Lebesgue-almost every $k \in \mathbb{R}$ and are constant on a set of full measure. In fact, as a function of v, they take only finitely many values, at most two in this case. These values do not depend on the choice of the norm in (3). Moreover, in the non-degenerate case, there exists a filtration

$$\{0\} =: \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \mathcal{V}_2 := \mathbb{C}^2$$

such that χ_1^B is the corresponding exponent for all $0 \neq v \in \mathcal{V}_1$ and χ_2^B for all $v \in \mathcal{V}_2 \setminus \mathcal{V}_1$. A vector v from the Oseledec subspace $\mathcal{V}_{i+1} \setminus \mathcal{V}_i$ satisfies the property that, for almost every $k \in \mathbb{R}$, the norm $\|vB^{(n)}(k)\|$ grows like $e^{n\chi_{i+1}^B}$ as $n \to \infty$. In general, these subspaces depend on k, and the filtration simplifies in the obvious way when $\chi_1^B(k) = \chi_2^B(k)$. We refer the reader to the monographs [12] for a general overview and [42] for a more elaborate discussion of linear cocycles.

It is well known that there exist at most two distinct exponents for 2-dimensional cocycles, denoted by $\chi_{\text{max}}^B(k)$ and $\chi_{\text{min}}^B(k)$. For invertible cocycles, these exponents afford v-independent representations as

$$\chi_{\max}^B(k) := \lim_{n \to \infty} \frac{1}{n} \log ||B^{(n)}(k)||$$
 and

$$\chi_{\min}^B(k) := -\lim_{n \to \infty} \frac{1}{n} \log \| (B^{(n)}(k))^{-1} \|.$$

Moreover, the exponents are constant almost everywhere, hence effectively independent of k as well. This means that we are dealing with two *numbers*, χ_{max}^B and χ_{min}^B . It turns out that one has an unexpected connection with logarithmic Mahler measures, as we discuss below.

Remark 3.1. These exponents come up in the spectral study of the substitutions associated to these Fourier matrices, which can be derived from a renormalisation scheme involving pair correlation functions; see [5, 6, 9]. In particular, by measure-theoretic arguments, one can conclude that, if $|\chi_{\max}^B| < \log \sqrt{L}$, the diffraction measure, for an arbitrary choice of weights, never has an absolutely continuous component. We refer to the literature for further details, namely to [33] for the binary constant length case, and to [4, 10] for an appropriate extension to a family of non-Pisot substitutions, via the corresponding inflation tiling.

4. Proof of the main result

As stated in Theorem 1.1 in the Introduction, the maximal Lyapunov exponent can be written as a logarithmic Mahler measure. We prove this as Theorem 4.2 below, which is a more detailed version of Theorem 1.1; compare [33].

Lemma 4.1. Let ϱ be a substitution as specified in Eq. (2). Consider the sets

$$P_a := \left\{ m \mid \mathcal{C}_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad and \quad P_b := \left\{ m \mid \mathcal{C}_m = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\},$$

which collect bijective positions of equal type. Further, let $z = e^{2\pi ik}$ and set

$$Q(z) := \overline{\widehat{\delta_{P_a}}(k)}$$
 and $R(z) := \overline{\widehat{\delta_{P_b}}(k)}$.

Then, $\det(B(k)) = p_L(z) \cdot (Q - R)(z)$, where $p_L(z) = 1 + z + \ldots + z^{L-1}$.

Proof. Similar to the definitions of Q and R above, define the sets $P_0 := \{m \mid \mathcal{C}_m = [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}] \}$ and $P_1 := \{m \mid \mathcal{C}_m = [\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}] \}$, and let

$$S_0(z) := \overline{\widehat{\delta_{P_0}}(k)}$$
 and $S_1(z) := \overline{\widehat{\delta_{P_1}}(k)}$.

In general, the Fourier matrix of ϱ satisfies

$$B(k) = \begin{pmatrix} (S_0 + Q)(z) & (S_0 + R)(z) \\ (S_1 + R)(z) & (S_1 + Q)(z) \end{pmatrix} \text{ with } z = e^{2\pi i k}.$$

Since there are only four distinct column types, we see that

$$S_0 + S_1 + Q + R \, = \, p_L.$$

One can now verify the lemma by direct computation.

Recall that, as a consequence of Oseledec's multiplicative ergodic theorem, our Lyapunov exponents exist and are constant, for almost every $k \in \mathbb{R}$. We call them χ_{\min}^B and χ_{\max}^B .

Theorem 4.2. For any primitive, binary constant-length substitution ϱ , the extremal Lyapunov exponents are explicitly given by

$$\chi^B_{\min} = 0$$
 and $\chi^B_{\max} = \mathfrak{m}(Q - R)$,

with Q and R as in Lemma 4.1.

Proof. Aside from the existence of the extremal Lyapunov exponents as limits, Oseledec's multiplicative ergodic theorem [12, 42] also guarantees Lyapunov regularity almost everywhere. That is, for almost every $k \in \mathbb{R}$, the sum of the exponents is given by

(4)
$$\chi_{\min}^{B}(k) + \chi_{\max}^{B}(k) = \lim_{n \to \infty} \frac{1}{n} \log \left| \det \left(B^{(n)}(k) \right) \right|,$$

where one can argue that, for the matrices above, the limit on the right-hand side converges for almost every $k \in \mathbb{R}$ to

$$\mathfrak{m}(Q-R) = \int_0^1 \log |(Q-R)(e^{2\pi it})| dt.$$

This can be seen by an application of Birkhoff's ergodic theorem, because $t\mapsto Lt$ on $\mathbb T$ is ergodic for Lebesgue measure, and $t\mapsto (p_L\cdot (Q-R))\big(\mathrm{e}^{2\pi\mathrm{i}t}\big)$ defines a function in $L^1(\mathbb T)$. The claim then follows from the multiplicative property of the determinant in conjunction with Lemma 4.1 and the fact that $\mathfrak m(p_L)=0$. This value follows via Jensen's formula because all zeros of p_L are roots of unity.

Next, we note that the row vector (1,1) is a left eigenvector of B(k), for all $k \in \mathbb{R}$, with eigenvalue $p_L(\mathrm{e}^{2\pi\mathrm{i}k})$. Hence, using this specific direction, we get one of the exponents to be $\chi_1^B = \mathfrak{m}(p_L) = 0$. From the sum in Eq. (4), and from the fact that the logarithmic Mahler measure of an integer polynomial is always non-negative, we then get that the exponent corresponding to this invariant subspace is the minimal one, $\chi_1^B = \chi_{\min}^B$, the other being $\chi_{\max}^B = \mathfrak{m}(Q - R)$.

Note that the result of this theorem is not restricted to bijective substitutions, even though only the bijective positions matter for the exponents.

Remark 4.3. In the proof of Theorem 4.2, instead of invoking Birkhoff's ergodic theorem, one can also work with the uniform distribution of $(L^n k)_{n \in \mathbb{N}}$ modulo 1 for almost every $k \in \mathbb{R}$ and Weyl's lemma. The difficulty to overcome here is that the function defined by $k \mapsto \log |\det(B(k))|$ generally has singularities. Fortunately, they are isolated (hence at most countable), and one can extend Weyl's result for locally Riemann integrable 1-periodic functions to this case via Sobol's theorem in conjunction with Diophantine approximation and discrepancy analysis; see [11] and references therein for a more comprehensive discussion.

It is also interesting to observe that one can obtain χ_{\max}^B via the (k-independent) right eigenvector $\tilde{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ of B(k), with corresponding eigenvalue $(Q - R)(e^{2\pi i k})$ as before. One then has, for almost every $k \in \mathbb{R}$, that

$$\lim_{n \to \infty} \frac{1}{n} \log ||B^{(n)}(k)\tilde{v}|| = \lim_{n \to \infty} \frac{1}{n} \Big(||\tilde{v}|| + \sum_{\ell=0}^{n-1} \log |(Q - R)(e^{2\pi i L^{\ell} k})| \Big) = \mathfrak{m}(Q - R),$$

where the last step once again relies on Birkhoff's ergodic theorem or on the remarks of the preceding paragraph.

In the general setting of Theorem 4.2, one gets a stronger result assuming periodicity. We require the following lemma, where we use the common shorthand $\varrho = (w_0, w_1)$ for the substitution from Eq. (2).

Lemma 4.4. Let ϱ be a primitive, binary constant-length substitution that defines a periodic hull. Then, one either has $\varrho = (w, w)$ with w containing at least one copy each of the letters a and b, or ϱ is bijective, and of the form $\varrho = ((ab)^m a, (ba)^m b)$ or $\varrho = ((ba)^m b, (ab)^m a)$ for some $m \in \mathbb{N}$.

Proof. Clearly, any substitution $\varrho = (w, w)$ with |w| > 1 defines a periodic hull, and primitivity implies that w must contain both letters. Consequently, we can now focus on $\varrho = (w_0, w_1)$ with $w_0 \neq w_1$. Let us begin with the cases of equal prefix.

If $\varrho = (rus, rvs)$, where |u| = |v| and arbitrary $r, s \in \Sigma_2$, we may consider $\varrho' = (sru, srv)$ instead, because ϱ and ϱ' are conjugate and thus define the same hull [8, Prop. 4.6]. Since we only consider the case $w_0 \neq w_1$, we must have at least one position where they differ, and we may assume that this happens at the suffix.

For $\varrho = (aua, avb)$, the words ab and ba are both legal (as u must contain the letter b by primitivity), and an iteration of the corresponding seeds under ϱ results in the sequences

$$a|b \mapsto \dots a|a \dots \mapsto \dots a|a \dots \mapsto \dots$$

 $b|a \mapsto \dots b|a \dots \mapsto \dots b|a \dots \mapsto \dots$

that converge to two-sided fixed points. Since they are proximal (equal to the right, but not to the left) by construction, the hull of ϱ must be aperiodic [8, Cor. 4.2]. A completely analogous argument works for $\varrho = (bua, bvb)$, which is again aperiodic.

Likewise, for $\varrho = (aub, ava)$, the word ab is legal, hence also ba. Using the latter as seed, we get the iteration

$$b|a \mapsto \dots a|a \dots \mapsto \dots b|a \dots \mapsto \dots$$

that ultimately alternates between two elements that form a proximal pair, which implies aperiodicity. Analogously, for $\varrho = (bub, bva)$, we get a proximal pair (and hence aperiodicity) from an iteration that starts with the legal seed b|a, which is mapped to a|b and then alternates between b|b and a|b.

Consequently, periodic cases for $w_0 \neq w_1$ can only occur if the two words have unequal prefix and unequal suffix. When $\varrho = (aub, bva)$, the seed b|a is legal, which under iteration alternates between a|a and b|a; when $\varrho = (bua, avb)$, one has the matching situation with a|b and a|a, so both cases possess proximal pairs and are thus aperiodic.

Finally, if $\varrho = (aua, bvb)$, one gets a proximal pair if and only if aa or bb is legal, and the same statement applies to $\varrho' = (bub, ava)$. The only way this can be avoided is if w_0 and w_1 both alternate between a and b, which indeed gives periodic hulls, and these substitutions are the two other cases stated.

Theorem 4.5. If the primitive, binary constant-length substitution ϱ defines a periodic hull, the extremal Lyapunov exponents satisfy $\chi_{\min}^B = \chi_{\max}^B = 0$.

Proof. In view of Lemma 4.4, we have to check the claim for three cases. When $\varrho = (w, w)$, where w contains both letters and $L = |w| \ge 2$, we consider an arbitrary starting vector $v = (\alpha, \beta) \ne 0$. For n > 1, one then has

$$vB^{(n)}(k) = (\alpha S_0(e^{2\pi i k}) + \beta S_1(e^{2\pi i k})) \cdot (1, 1)B^{(n-1)}(Lk),$$

where (1,1) is a left eigenvector of $B^{(n-1)}(Lk)$. With $S_0 + S_1 = p_L$ in this case, this gives

$$||vB^{(n)}(k)|| = |\alpha S_0(e^{2\pi i k}) + \beta S_1(e^{2\pi i k})| ||(1,1)|| \prod_{\ell=1}^{n-1} |p_L(e^{2\pi i L^{\ell} k})|.$$

The first term on the right-hand side only vanishes at isolated (and hence countably many) values of k, which we may exclude. Then, a calculation with Birkhoff averages shows that, for almost every $k \in \mathbb{R}$, we get

$$\lim_{n \to \infty} \frac{1}{n} \log ||vB^{(n)}(k)|| = \mathfrak{m}(p_L) = 0,$$

which establishes the claim in this case.

If ϱ is bijective, we have L=2m+1 for the two remaining cases by Lemma 4.4. In line with our previous reasoning, the Fourier matrix is of the form

$$B(k) = \begin{pmatrix} Q(z) & R(z) \\ R(z) & Q(z) \end{pmatrix}$$
 with $z = e^{2\pi i k}$,

where the polynomials Q and R satisfy $Q(z)+R(z)=p_L(z)=1+z+\cdots+z^{2m}$, which is cyclotomic, so that $\mathfrak{m}(Q+R)=0$. Also, due to the alternating structure of $\varrho(a)$ and $\varrho(b)$, one has $(Q-R)(z)=\pm(Q+R)(-z)$. This means that Q-R is cyclotomic as well, and $\mathfrak{m}(Q-R)=0$. Now, one sees that

$$(1,\pm 1)B^{(n)}(k) = (1,\pm 1)\prod_{\ell=0}^{L-1} (Q \pm R) (e^{2\pi i L^{\ell} k})$$

which, for almost every $k \in \mathbb{R}$, gives the two exponents as $\mathfrak{m}(Q+R)=0$ and $\mathfrak{m}(Q-R)=0$ by a simple calculation as in Remark 4.3. This implies our claim for these two cases.

The converse of Theorem 4.5 does not hold. For example, both the Thue–Morse and the period doubling substitutions, given in Examples 2.1 and 2.2, have $\chi_{\min}^B = \chi_{\max}^B = 0$; this means that the norm of the resulting vector after applying their respective cocycles to any starting vector v does not grow exponentially. However, one must be careful here, as zero Lyapunov exponents do not exclude sub-exponential growth behaviour.

5. Examples: From Polynomials to substitutions

Theorem 1.1, and the more specific Theorem 4.2, allow one, given a binary constant-length substitution, to write down a polynomial whose logarithmic Mahler measure determines the maximal Lyapunov exponent related to the substitution. But the nature of our results allows one to go the other way as well, as we now do. We explain how, given a specific polynomial, one can build a substitution associated to the maximal Lyapunov exponent for the cocyle $B^{(n)}(k)$. We also comment on the essential uniqueness of these substitutions and the properties of their Fourier matrices. We focus on specific classes of height-1 polynomials that have been important in the study of the logarithmic Mahler measure in the context of Lehmer's problem.

Example 5.1 (Littlewood polynomials). Recall that a polynomial $q(z) = \sum_{m=0}^{n-1} c_m z^m$ of degree n-1 with coefficients $c_m \in \{-1,1\}$ is called a *Littlewood polynomial of order* n-1; see [18, 20, 36]. As before, let \mathcal{C}_m be the m^{th} column of ϱ . Starting with the polynomial, we choose \mathcal{C}_m to be

$$C_m = \begin{cases} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \text{if } c_m = 1, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \text{if } c_m = -1, \end{cases}$$

and we build the substituted words for 0 and 1 by looking at the concatenation $C_0 C_1 \cdots C_{L-1}$. Since there are only two possible column types present, we see immediately that the sets P_a and P_b from Lemma 4.1 satisfy

$$P_a \cup P_b = \{0, 1, \dots, L-1\},\$$

and also that the resulting substitution ϱ must be bijective. By construction, we have

(5)
$$\chi_{\max}^B = \mathfrak{m}(Q - R) = \mathfrak{m}(q)$$

for the cocycle defined by the Fourier matrix associated to ϱ , which explicitly reads

$$B(k) = \begin{pmatrix} Q(z) & R(z) \\ R(z) & Q(z) \end{pmatrix}$$
 where $z = e^{2\pi i k}$.

Note that, in this case, χ_{\max}^B can also be calculated by observing that (1, -1) is a k-independent left eigenvector of B(k) with eigenvalue $(Q - R)(e^{2\pi i k})$, thus also giving (5).

The substitution corresponding to q=Q-R is essentially unique, up to the obvious freedom that emerges from the relation $\mathfrak{m}(-q)=\mathfrak{m}(q)$, that is, from changing all signs. This is the case precisely because a given sequence of signs uniquely specifies the columns of ϱ . For example, let us consider the polynomial $q(z)=-1-z+z^2-z^3+z^4$, whence we get the substitutions

$$\varrho_q: \begin{cases} 0 \mapsto 11010 \\ 1 \mapsto 00101 \end{cases} \quad \text{and} \quad \varrho_{-q}: \begin{cases} 0 \mapsto 00101 \\ 1 \mapsto 11010 \end{cases}$$

with associated Fourier matrices

$$B_q(k) = \begin{pmatrix} e^{4\pi i k} + e^{8\pi i k} & 1 + e^{2\pi i k} + e^{6\pi i k} \\ 1 + e^{2\pi i k} + e^{6\pi i k} & e^{4\pi i k} + e^{8\pi i k} \end{pmatrix} \text{ and } B_{-q}(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_q(k).$$

Both induce a cocycle whose maximum Lyapunov exponent is

$$\chi_{\text{max}}^B = \mathfrak{m}(q) \approx 0.656256.$$

The Fourier matrices associated to bijective substitutions enjoy further properties such as simultaneous diagonalisibility and a k-independent expression for the Oseledec splitting. In these cases, one always has $\mathcal{V}_1 = \mathbb{C}(1,1)$. This means that, for all vectors $v \neq 0$ in this subspace, the asymptotic exponential growth rate is 0, for almost every $k \in \mathbb{R}$. One also sees that every column sum and every row sum of B(k) is the cyclotomic polynomial p_L ; for rows it is due to the bijectivity of the substitution, for columns it follows from the constant-length property.

Example 5.2 (Newman polynomials). For the class of $\{0,1\}$ -polynomials, also known as Newman polynomials [36], one has R=0, so the associated Fourier matrix is

$$B(k) = \begin{pmatrix} (S_0 + Q)(z) & S_0(z) \\ S_1(z) & (S_1 + Q)(z) \end{pmatrix}$$
 with $z = e^{2\pi i k}$,

which leads to $\chi_{\text{max}}^B = \mathfrak{m}(Q)$ by Theorem 4.2. If either S_0 or S_1 is zero, $M_{\varrho} = B(0)$ is a triangular matrix, and cannot be primitive. This can be avoided if there are at least two columns with a coincidence. If there is only one such column, one can still construct a primitive substitution by recalling that $\mathfrak{m}(-q) = \mathfrak{m}(q)$, so one only needs to exchange the two bijective column types.

As a concrete example, consider $q(z) = 1 + z^2$. The two standard choices

$$\varrho_q: \begin{cases} 0 \mapsto 000 \\ 1 \mapsto 101 \end{cases} \quad \text{and} \quad \varrho_{q'}: \begin{cases} 0 \mapsto 010 \\ 1 \mapsto 111 \end{cases}$$

both give non-primitive substitutions; in fact, their substitution matrices are not even irreducible. However,

$$\varrho_{-q}: \begin{cases}
0 \mapsto 101 \\
1 \mapsto 000
\end{cases} \quad \text{and} \quad \varrho_{-q'}: \begin{cases}
0 \mapsto 111 \\
1 \mapsto 010
\end{cases}$$

are both primitive and aperiodic, and have $\chi_{\max}^B = \mathfrak{m}(q)$. One can see in this example that replacing q by -q really means an exchange of w_0 and w_1 in the definition of ϱ .

As another example, consider the reciprocal Newman polynomial

$$q(z) \, = \, 1 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9 + z^{10} + z^{11} + z^{14}$$

 \Diamond

taken from [21, p. 1375]. One choice for a substitution (with L=15) is

which means $S_1(z) = z$ and $S_0(z) = z^2 + z^{12} + z^{13}$, together with Q = q. Here, one has $\mathfrak{m}(q) \approx \log(1.265122)$. This value is strictly smaller than $\log(\lambda_p)$, where $\lambda_p \approx 1.324718$ is the plastic number as described in the Introduction. Recall that $\log(\lambda_p)$ is the sharp lower bound for $\mathfrak{m}(p)$ over all non-reciprocal integer polynomials p that are not a product of a monomial with cyclotomic polynomials.

Note that, when associating a polynomial to a binary constant-length substitution ϱ , it is only the bijective columns of ϱ that are determined by the non-zero coefficients. Thus, we can extend the construction to generic height-1 polynomials, even when the constant term vanishes. However, when interested in the logarithmic Mahler measure, one can assume that the constant coefficient does not vanish.

Example 5.3 (Borwein polynomials). When considering Borwein polynomials, one can choose the columns of an associated substitution just as in Example 5.1, but with the additional freedom to vary the choice for a zero coefficient. Similar to Example 5.1, starting with the polynomial, we choose \mathcal{C}_m to be

$$C_m = \begin{cases} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \text{if } c_m = 1, \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \text{if } c_m = -1, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & \text{if } c_m = 0, \end{cases}$$

and we build the substituted words for 0 and 1 by looking at the concatenation $C_0 C_1 \cdots C_{L-1}$. But now, there are two choices for every zero coefficient of the polynomials, so that, if p is a polynomial with n zero coefficients, there are 2^n distinct binary constant-length substitutions whose maximal Lyapunov exponents are all given by $\mathfrak{m}(p)$. On top of this freedom, we can also still work both with p or with -p, as we saw earlier.

As a concrete example, we consider Lehmer's polynomial $\ell_{\rm L}$ from (3), where $c_2=c_8=0$. Recall from the Introduction that this polynomial is irreducible, and has precisely one real root outside the unit circle, which is Salem. Recall further that $\ell_{\rm L}$ is the polynomial with the smallest known positive logarithmic Mahler measure, $\mathfrak{m}(\ell_{\rm L})\approx\log(1.176281)$. Here,

$$\varrho_{\ell_{\mathbf{L}}}: \begin{cases} 0 \mapsto 001111111000 \\ 1 \mapsto 11100000011 \end{cases}$$

is one of the eight substitutions that correspond to the polynomial $\ell_{\rm L}$.

With this connection and representation, we obtain the following equivalent reformulation of the strong version of Lehmer's problem for Borwein polynomials.

Question 5.4. Does there exist a primitive, binary constant-length substitution ϱ with maximal Lyapunov exponent $0 < \chi_{\max}^B < \mathfrak{m}(\ell_L) \approx \log(1.176280818)$?

6. Extensions and outlook

Lyapunov exponents are neither restricted to contant-length substitutions nor to binary alphabets. In fact, there are many extensions possible; see [4, 6] and references therein for more. In general, however, the Lyapunov exponents are no longer logarithmic Mahler measures themselves, though they often still satisfy interesting estimates in such a setting.

Moreover, there is actually also a generalisation to higher dimensions, as briefly stated in [34]. Here, one considers stone inflation rules of finite local complexity [8] and selects a suitable marker (or reference point) in each prototile (such that the tiling and the point set are mutually locally derivable [8, Sec. 5.2] from each other). One particular class emerges from *block substitutions*, such as those discussed in [7, 29].

Example 6.1. A simple bijective example is given by

$$a \mapsto \begin{array}{c} b & a \\ a & b \end{array}, \quad b \mapsto \begin{array}{c} a & b \\ b & a \end{array}$$

which is primitive and aperiodic. Here, one can represent the two letters by unit squares with a reference point at their lower left corners. Then, with $k = (k_1, k_2) \in \mathbb{R}^2$, one finds the Fourier matrix

$$B(k) = \begin{pmatrix} 1 + xy & x + y \\ x + y & 1 + xy \end{pmatrix}$$
 where $(x, y) = (e^{2\pi i k_1}, e^{2\pi i k_2}),$

which satisfies $(1, \pm 1)B(k) = ((1+xy)\pm (x+y))(1, \pm 1)$. Since 1+x+y+xy = (1+x)(1+y) and 1-x-y+xy = (1-x)(1-y), all factors are cyclotomic. Consequently, for almost every $k \in \mathbb{R}^2$, one gets the Lyapunov exponents as

$$\chi_{\min}^{B} = \mathfrak{m}(1 + x + y + xy) = 0$$
 and $\chi_{\max}^{B} = \mathfrak{m}(1 - x - y - xy) = 0$,

which resembles Example 2.1 in many ways.

As another example, consider the bijective block substitution

$$a \mapsto \begin{pmatrix} b & a & b \\ a & a & a \\ b & a & b \end{pmatrix}, \quad b \mapsto \begin{pmatrix} a & b & a \\ b & b & b \\ a & b & a \end{pmatrix}$$

that emerges from the squiral tiling [7]. Here, one has $\chi_{\min}^B = \mathfrak{m}((1+x+x^2)(1+y+y^2)) = 0$ as before, while

$$\chi_{\text{max}}^B = \mathfrak{m}(x + y(1 + x + x^2) + xy^2 - (1 + x^2)(1 + y^2)) \approx 0.723909$$

 \Diamond

is strictly positive.

It is clear that one can now repeat a lot of our previous analysis for the class of bijective block substitutions, in any dimension. As is implicit in [34], the blocks need not be cubes, as long as they have length at least 2 in each direction. We leave further experimentation along

 \Diamond

these lines to the interested reader. Outside the bijective class, interesting new phenomena are possible as follows.

Example 6.2. Consider the binary block substitution

$$a \mapsto \begin{array}{c} b & a \\ b & b \end{array}, \quad b \mapsto \begin{array}{c} a & a \\ a & a \end{array}$$

which is clearly primitive. It has a coincidence, so that the higher-dimensional analogue of Dekking's result, see [7, 13, 29], implies the pure point spectral nature of the corresponding dynamical system (under the action of the \mathbb{Z}^2 -shift).

Here, the Fourier matrix reads

$$B(k) = \begin{pmatrix} xy & (1+x)(1+y) \\ 1+x+y & 0 \end{pmatrix}, \text{ where } (x,y) = (e^{2\pi i k_1}, e^{2\pi i k_2})$$

and $\det(B(k)) = -(1+x)(1+y)(1+x+y)$. As before, we get $\chi_{\min}^B = \mathfrak{m}((1+x)(1+y)) = 0$, while the sum then satisfies

$$\begin{split} \chi_{\min}^B + \chi_{\max}^B &= \chi_{\max}^B = \mathfrak{m} \left((1+x)(1+y)(1+x+y) \right) = \mathfrak{m} (1+x+y) \\ &= \frac{3\sqrt{3}}{4\pi} L(2,\chi_{-3}) = L'(-1,\chi_{-3}) = 2 \int_0^{1/3} \log \left(2\cos(\pi t) \right) \mathrm{d}t \, \approx \, 0.323066, \end{split}$$

with $L(s, \chi_{-3})$ denoting the Dirichlet *L*-series of the character $\chi_{-3}(n) = \left(\frac{-3}{n}\right)$, written in terms of the Legendre symbol; compare [23]. This special value of a Mahler measure also appears in [40, 41], and various other relations of this kind are known [17], such as

$$\mathfrak{m}(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3).$$

The latter can clearly be realised by a block substitution in three dimensions.

Remark 6.3. It seems hardly known that the Mahler measure from Example 6.2, together with its integral representation, first appeared in Wannier's calculation of the groundstate entropy of the antiferromagnetic Ising model on the triangular lattice [43]. This might be due to the fact that the numerical value originally given by him was incorrect, though corrected in an erratum 23 years later.

To be more precise, Wannier gets the entropy s as a double integral from which we obtain

$$s = \frac{1}{2} \int_0^1 \int_0^1 \log(1 + 4\cos(2\pi v)^2 - 4\cos(2\pi u)\cos(2\pi v)) du dv$$

$$= \frac{1}{2} \int_0^1 \int_0^1 \log\left|(1 + y^2 - xy)(x + xy^2 - y)\right|_{x = e^{2\pi i u}, y = e^{2\pi i v}} du dv$$

$$= \frac{1}{2} \left(\mathfrak{m}(1 + y^2 - xy) + \mathfrak{m}(1 + y^2 - x^{-1}y)\right) = \mathfrak{m}(1 + y^2 - xy).$$

Now, one clearly has

$$\mathfrak{m}(1+y^2-xy) = \mathfrak{m}(1+y^2+xy) = \mathfrak{m}(1+x+y),$$

via a change of variables $u = u' + \frac{1}{2}$ for the first identity and the invariance of the Mahler measure under an invertible linear map with integer coefficients, as in [28, Exc. 3.1], which is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ in this case.

It is well known that the connection between the Mahler measure and special values of L-series has deep roots; in particular, see [25], and [17] for a survey with several examples and references. Also, a connection between Mahler measures and Lyapunov exponents is known from [26]. On the other hand, our observation shows that these quantities also occur in the spectral theory of dynamical systems, in a rather elementary way, and it seems an interesting problem to analyse this connection further.

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