Extensions of Abelian Automaton Groups

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Background

For our purposes, a **Mealy Automaton** is a tuple $\mathcal{A} = (S, \tau)$ where S is the **State Set**, and $\tau : S \times \mathbf{2} \to S \times \mathbf{2}$ is the **transition function**. Given a state $s \in S$, we can treat it as a length preserving function $\underline{s} : \mathbf{2}^* \to \mathbf{2}^*$ as follows:

$$\underline{s}(\varepsilon) = \varepsilon$$
$$\underline{s}(ax) = a'\underline{s}'(x) \quad \text{(where } (s', a') = \tau(s, a))$$

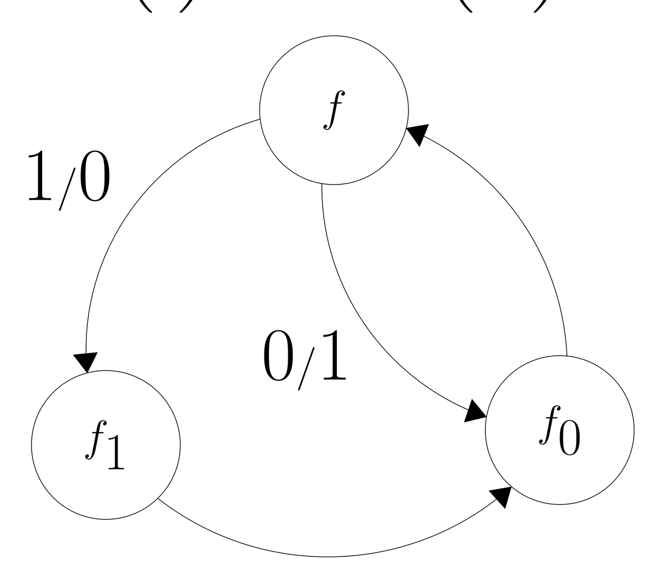
For us, each $\tau(s, -) : \mathbf{2} \to \mathbf{2}$ will be a permutation, as this will ensure inverses exist when we define $\mathcal{G}(\mathcal{A})$, the group of functions the automaton generates.
A state is called **Odd** if its permutation flips its input, and **Even** if it copies its input.

Groups

We define $\mathcal{G}(\mathcal{A})$ to be the group generated by $\{\underline{s} | s \in \mathcal{A}\}$. We will restrict ourselves to Abelian groups, and write our groups additively.
For $f \in \mathcal{G}(\mathcal{A})$, put $\partial_0 f$ as the unique function such that for all $w \in \mathbf{2}^*$, $f(0w) = (f(0))(\partial_0 f)(w)$. Define $\partial_1 f$ symmetrically.
In the abelian case, $\mathcal{G}(\mathcal{A})$ will always be $\mathbb{Z}^m$ or $(\mathbb{Z}/2\mathbb{Z})^m$. We restrict our attention to the $\mathbb{Z}^m$ case.

An Automaton: $\mathcal{A}_2^3$

Taking $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$, and $\bar{e} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ gives the following machine, called $\mathcal{A}_2^3$. Here $f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $f_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $f_1 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$:



$$f(0110) = 1f_0(110) = 11f(10) = 110f_1(0) = 1100$$

$$\partial_0 f = f_0 \text{ and } \partial_0 f_0 = \partial_1 f_0 = f$$

Main Theorems

Theorem. *Every nontrivial abelian automaton $\mathcal{A}$ can be located at $\bar{e}_1$ in some $\mathfrak{C}(\mathbf{A}, \bar{e})$*

Theorem. *If $rp = q$ in $\mathbb{Z}[x]$, then $\mathfrak{C}(\mathbf{A}, p \cdot \bar{e}_1) \hookrightarrow \mathfrak{C}(\mathbf{A}, q \cdot \bar{e}_1)$, with a canonical injection $\varphi_r : \bar{v} \mapsto r \cdot \bar{v}$. In particular, if r is a unit, then $p \cdot \mathcal{G} \cong q \cdot \mathcal{G}$. This map preserves both the group and residuation structure.*

These allow us to completely understand the residuation vector $\bar{e}$.

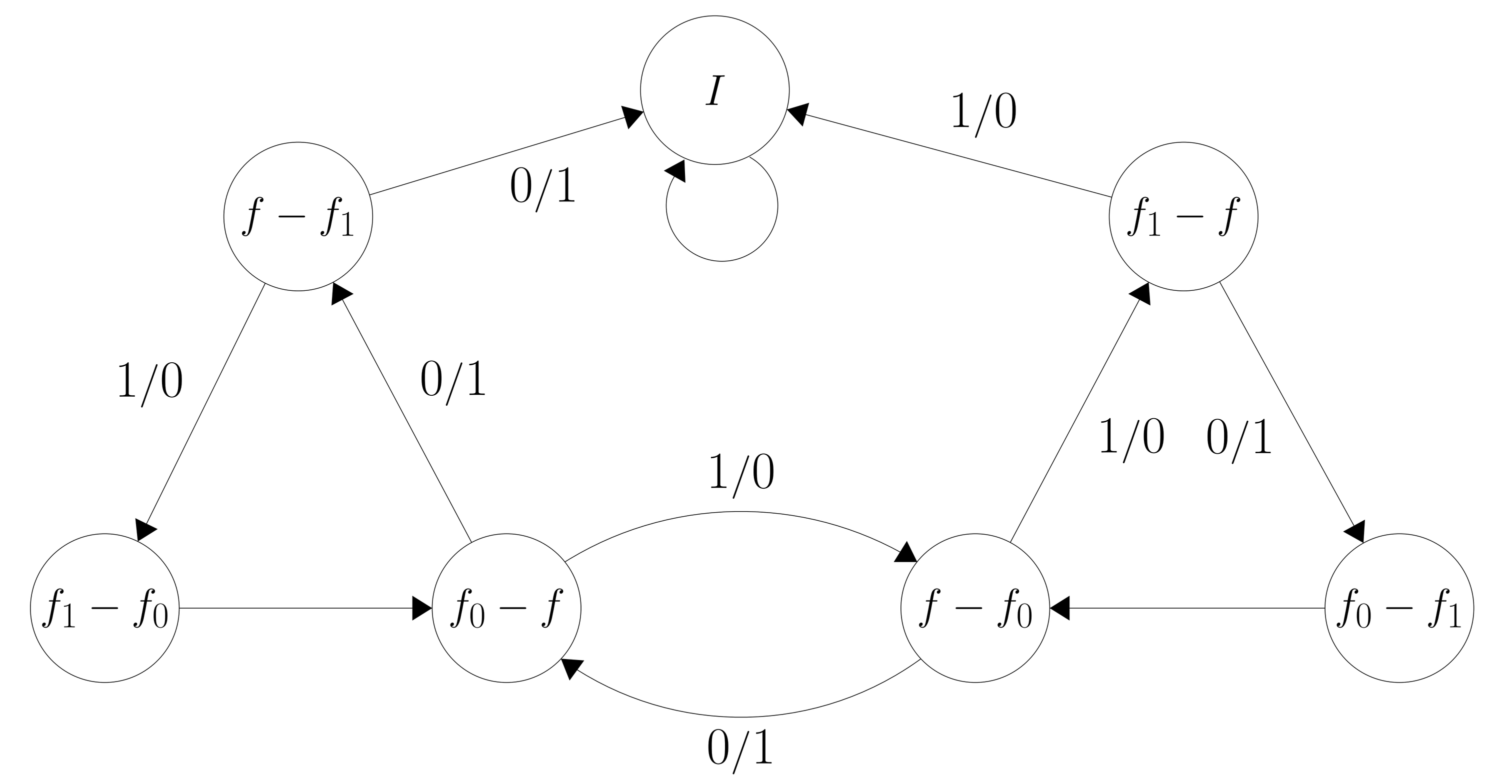
- First find $\bar{r}$ such that $\mathcal{A}$ has a state at $\bar{e}_1$.
- $\mathcal{A}$ is a subautomaton of $\mathfrak{C}(\mathbf{A}, \bar{e})$ if and only if $p_{\bar{r}}$ divides $p_{\bar{e}}$
- Also, if $\mathcal{A}$ *is* a subautomaton, then $qp_{\bar{r}} = p_{\bar{e}}$, and $\mathcal{A}$ is located at $q \cdot \bar{e}_1$

Example

So $p \cdot \bar{v} \in \mathfrak{C}(\mathbf{A}, p \cdot \bar{e}_1)$, computes exactly the same function as $\bar{v} \in \mathfrak{C}(\mathbf{A}, \bar{e})$. However, most vectors cannot be written as $p \cdot \bar{v}$. What do they do as functions? We call such vectors (and their corresponding functions) **Fractional**, due to the following observation and theorem: Let δ be the function computed by $\bar{e}_1 \in \mathfrak{C}(\mathbf{A}, \bar{e})$. Then $3\bar{e}_1 \in \mathfrak{C}(\mathbf{A}, 3 \cdot \bar{e}_1)$ should compute the same function. So we should expect the function $\bar{e}_1 \in \mathfrak{C}(\mathbf{A}, 3 \cdot \bar{e}_1)$ to behave like "$\frac{1}{3}\delta$", and in fact it does.

Initial Object: The Principal Automaton

So $\mathfrak{C}(\mathbf{A}, \bar{e}_1)$ is a subautomaton of every $\mathfrak{C}(\mathbf{A}, p \cdot \bar{e})$, and it is reasonable to ask if there is a corresponding finite automaton whose group embeds into the group generated by every other automaton with the same matrix.
Such a machine was found by Okano, and is called the **Principal Automaton** of a matrix. It is located at $\bar{e}_1$ in $\mathfrak{C}(\mathbf{A}, \bar{e}_1)$ for any matrix $\mathbf{A}$, and can be directly constructed by taking differences of the states in a given automaton $\mathcal{A}$.
Below is the result of this construction for $\mathcal{A}_2^3$:



Complete Automaton

Theorem (Nekrashevych and Sidki). *If $\mathcal{G}$ is an automaton group and $\varphi : \mathcal{G} \to \mathbb{Z}^m$ is a group isomorphism, then there is a matrix $\mathbf{A}$ of $\mathbb{Q}$-irreducible character and an odd vector $\bar{e}$ such that if $\mathbb{Z}^m$ is equipped with the following residuation structure, then φ preserves residuation:*
If $\bar{v}$ is even:

$$\partial_0 \bar{v} = \partial_1 \bar{v} = \mathbf{A}\bar{v}$$

If $\bar{v}$ is odd:

$$\partial_0 \bar{v} = \mathbf{A}(\bar{v} - \bar{e})$$
$$\partial_1 \bar{v} = \mathbf{A}(\bar{v} + \bar{e})$$

Further, $\mathbf{A}$ is unique up to conjugation, and can always be taken to be "$\frac{a}{2} - integral$", meaning $\mathbf{A}$ is of the form

$$\begin{pmatrix} \frac{a_{11}}{2} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{2} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

where each $a_{ij} \in \mathbb{Z}$.
These matrices all have characteristic polynomial
$\chi = x^n + \frac{1}{2}g(x)$, where $g \in \mathbb{Z}[x]$ and has constant term ± 1.

Definition. This construction is called $\mathfrak{C}(\mathbf{A}, \bar{e})$

Theorem. *Every abelian automaton $\mathcal{A}$ can be embedded in $\mathfrak{C}(\mathbf{A}, \bar{e})$ for some $\mathbf{A}$ and $\bar{e}$.*

It is natural to wonder what vectors $\bar{e}$ admit a given automaton $\mathcal{A}$ as a subautomaton of $\mathfrak{C}(\mathbf{A}, \bar{e})$. We answer this question here.

$\mathbb{Z}[x]$-modules

$\mathcal{G}(\mathcal{A})$ naturally allows scalars in $\mathbb{Z}$ by putting

$$n \cdot f = \underbrace{f + f + \ldots + f}_{n \text{ times}}$$

We allow scalars in $\mathbb{Z}[x]$, polynomials with integer coefficients, by embedding $\mathcal{G}(\mathcal{A})$ in $\mathfrak{C}(\mathbf{A}, \bar{e})$, and putting $p \cdot \bar{v} = p(\mathbf{A}^{-1})\bar{v}$.

Theorem. *Because $\mathbf{A}$ has irreducible character, every vector $\bar{v}$ is equal to $p_{\bar{v}} \cdot \bar{e}_1$ for some $p_{\bar{v}}$. Here $\bar{e}_1$ is the unit vector $(1, 0, \ldots, 0)$.*

Acknowledgements

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