Extensions of Abelian Automaton Groups

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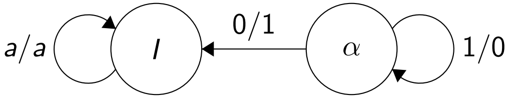
Finite State Automata

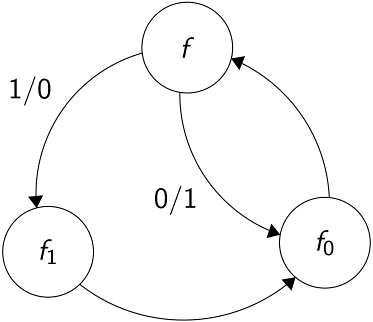
- Combinatorial Objects
- Encode length preserving functions on binary strings
  - states
  - transitions



- One function per state
- Evaluate by following edges

Evaluating Functions

- $\mathcal{A}_2^3$. This automaton will be our friend for the rest of this talk
- Defines three functions:
  - f
  - f_0
  - f_1



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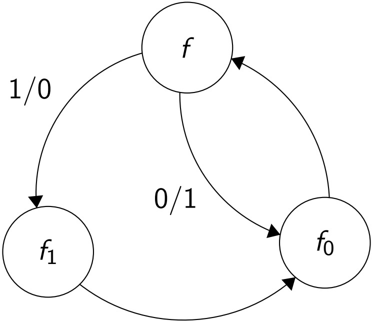
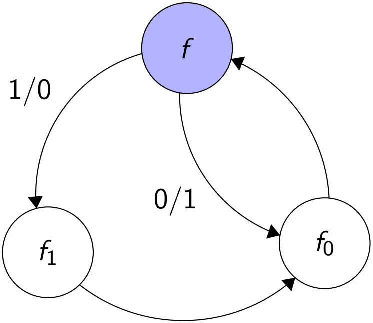


- How do we compute, say, f of a string?

Evaluating Functions



- $f(011010)$

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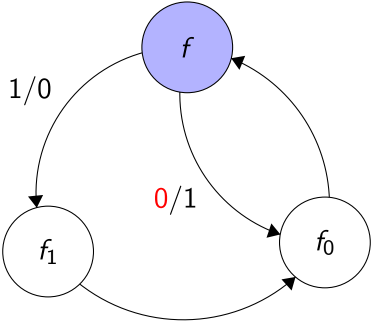

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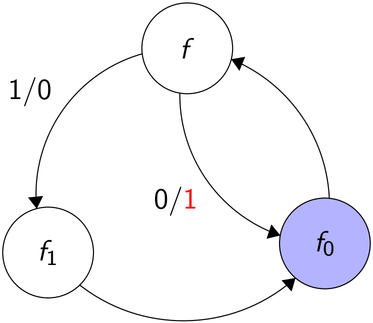

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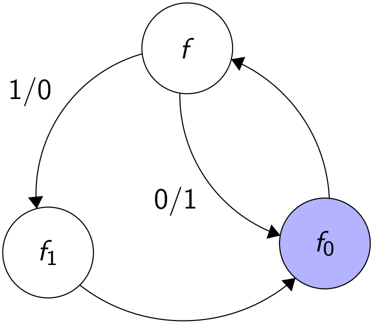

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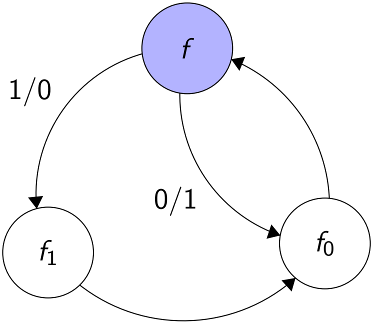

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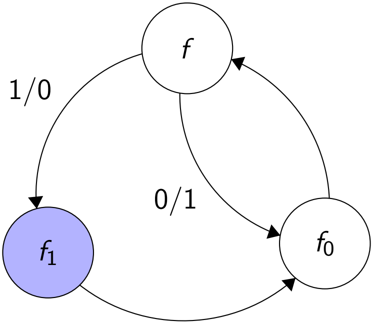

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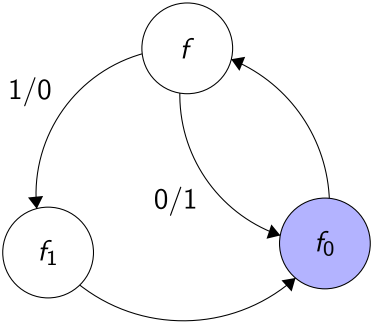

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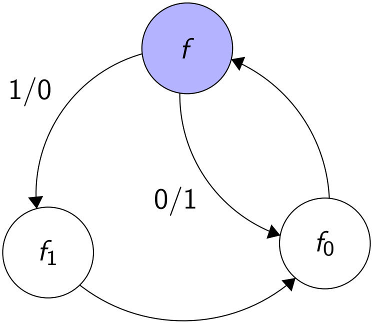

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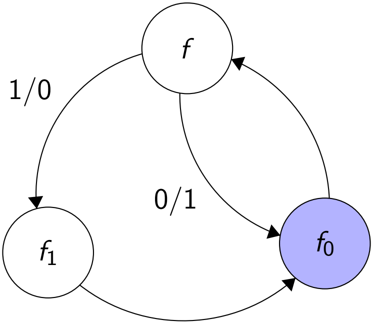

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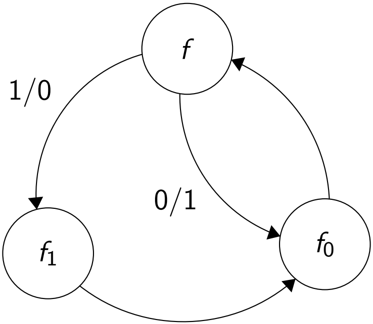

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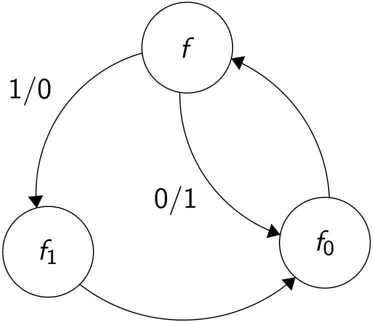

- $\partial_0 f = f_0$ and $\partial_1 f = f_1$
- f is odd, f_0 and f_1 are even

Definition

For f and g in an automaton $\mathcal{A}$, write $f + g$ for the function $(f + g)(x) = f(g(x))$

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- We are interested in the Abelian case.
- For all of our machines, $f+g=g+f$
- Given a machine $\mathcal{A}$, this condition is checkable in polynomial time

Definition

Recall a Group is a set $\mathcal{G}$ *equipped with*

- $0 \in \mathcal{G}$
- $+ : \mathcal{G} \to \mathcal{G} \to \mathcal{G}$ *(associative)*
- $- : \mathcal{G} \to \mathcal{G}$
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- Each function is invertible iff each state is invertible in one step

The Inverse Automaton

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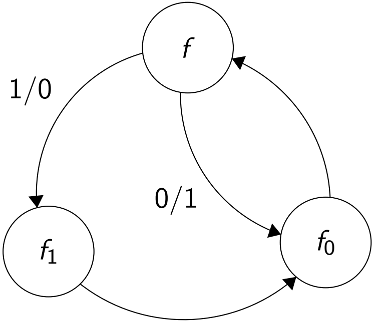

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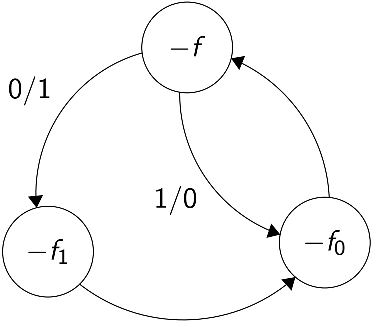

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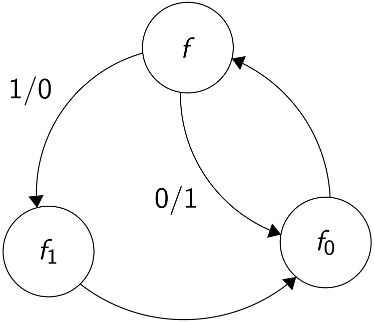

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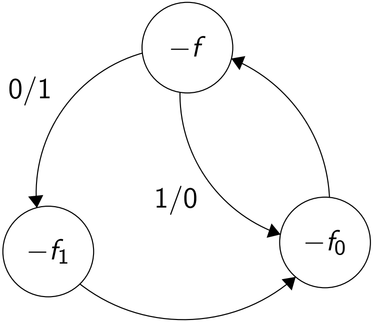

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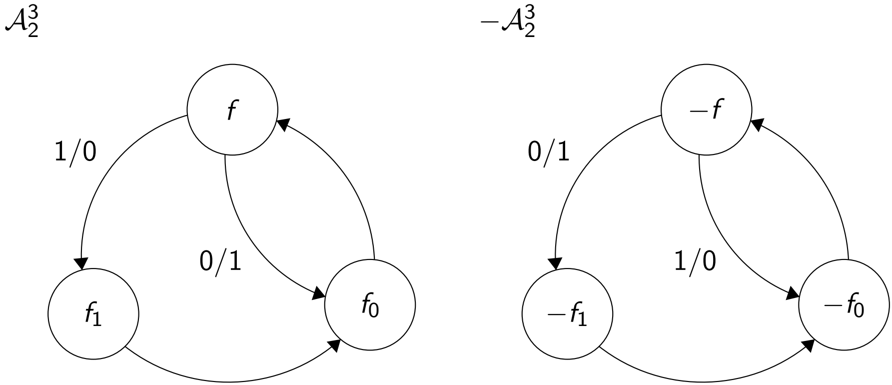

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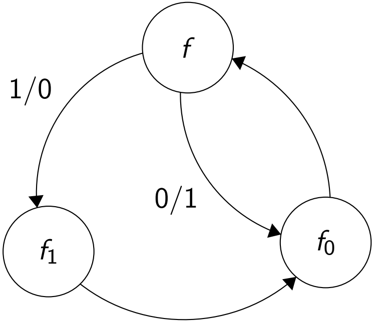

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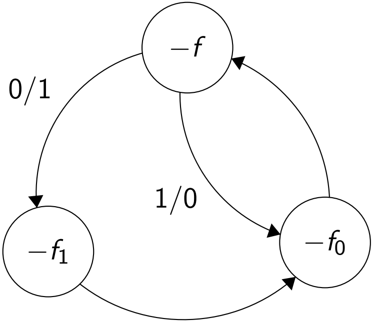

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- But: $(f + (-f))(01) = f((-f)(01)) = f(11) = 01$
- An easy induction shows these are actually inverses.

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- We will focus on the $\mathbb{Z}^m$ case here

Theorem

$\mathbb{Z}^m$ equipped with a matrix $\mathbf{A}$ and $\bar{e}$ forms a (infinite state) abelian automaton (called $\mathfrak{C}(\mathbf{A}, \bar{e})$) with residuation as shown below. Further, for every abelian automaton $\mathcal{A}$ whose group is $\mathbb{Z}^m$, there exists an $\mathbf{A}$ and $\bar{e}$ such that $\mathcal{A}$ is a finite subautomaton. The odd states are exactly the states with odd first component.

$$\mathbf{A} = \begin{pmatrix} \frac{a_1}{2} & 1 & 0 & \cdots & 0 \\ \frac{a_2}{2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{m-1}}{2} & 0 & 0 & \cdots & 1 \\ \frac{a_m}{2} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

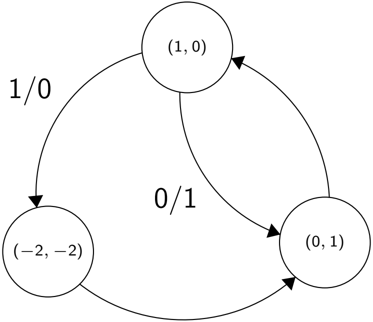
$$\partial_0 \bar{v} = \begin{cases} A(\bar{v}) & \bar{v} \text{ is even} \\ A(\bar{v} - \bar{e}) & \bar{v} \text{ is odd} \end{cases}$$

$$\partial_1 \bar{v} = \begin{cases} A(\bar{v}) & \bar{v} \text{ is even} \\ A(\bar{v} + \bar{e}) & \bar{v} \text{ is odd} \end{cases}$$

- $a_i \in \mathbb{Z}$
- $\mathbf{A}$ has irreducible characteristic polynomial
- $\bar{e}$ (the Residuation Vector) is odd

Example:

Take $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$, and $\bar{e} = (3, 2)$. Then:



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- It can also be shown that for any $\bar{e}$, if $\mathcal{A}$ is a subautomaton, its location in the structure is unique
- There are infinitely many choices of $\bar{e}$ though, and the goal is to understand them.

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Definition

For $p \in \mathbb{Z}[x]$ and $\bar{v} \in \mathfrak{C}(\mathbf{A}, \bar{e})$, put $p \cdot \bar{v} = (p(\mathbf{A}^{-1}))\bar{v}$

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- So different residuation vectors give groups which *extend* the group $\mathfrak{C}(\mathbf{A}, \bar{e}_1)$
- Also, if $p_{\bar{e}}$ divides $p_{\bar{r}}$, then $\mathfrak{C}(\mathbf{A}, \bar{r})$ extends $\mathfrak{C}(\mathbf{A}, \bar{e})$.

Theorem

If $\mathcal{A}$ is an automaton whose group is $\mathbb{Z}^m$, then for each odd state in $\mathcal{A}$, there is exactly one $\bar{e}$ which locates that state at $\bar{e}_1$ in $\mathfrak{C}(\mathbf{A}, \bar{e})$. Further, if $\bar{e}$ and $\bar{r}$ are two such residuation vectors, they differ by a unit. This procedure is effective.

Theorem

If $\mathcal{A}$ has a state located at $\bar{v} \in \mathfrak{C}(\mathbf{A}, \bar{e})$, then $\bar{v}$ is located at $p \cdot \bar{v} \in \mathfrak{C}(\mathbf{A}, p \cdot \bar{e})$

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- Incredibly, we now understand residuation vectors!
- First find $\bar{r}$ such that $\mathcal{A}$ has a state at $\bar{e}_1$.
- $\mathcal{A}$ is a subautomaton of $\mathfrak{C}(\mathbf{A}, \bar{e})$ if and only if $p_{\bar{r}}$ divides $p_{\bar{e}}$
- Also, if $\mathcal{A}$ *is* a subautomaton, then $qp_{\bar{r}} = p_{\bar{e}}$, and $\mathcal{A}$ is located at $q \cdot \bar{e}_1$

- It turns out we can "scale by an infinite polynomial" to get a universal structure which contains *every* automaton (with the correct matrix $\mathbf{A}$) at exactly one location.
- The construction is a bit involved, so we don't have time to discuss it, but it is computable, and removes the need for the extra parameter $\bar{e}$.

Questions?