ELSEVIER
Discrete Mathematics 152 (1996) 93–113
DISCRETE MATHEMATICS

Zero-sum problems — A survey

Yair Caro

Department of Mathematics, School of Education, University of Haifa, Oranim Tivon 36-910, Israel

Received 20 July 1993; revised 31 May 1994

Abstract

Zero-sum Ramsey theory is a newly established area in combinatorics. It brings to ramsey theory algebric tools and algebric flavour. The paradigm of zero-sum problems can be formulated as follows: Suppose the elements of a combinatorial structure are mapped into a finite group K. Does there exists a prescribed substructure the sum of the weights of its elements is 0 in K?

We survey the algebric background necessary to develop the first steps in this area and its short history dated back to a 1960 theorem of Erdős-Ginzburg and Ziv. Then a systematic survey is made to encompass most of the results published in this area until 1.1.95.

Several conjectures and open problems are cited along this manuscript with the hope to catch the eyes of the interested reader.

1. Introduction

The cornerstone of almost all recent combinatorial research on zero-sum problems is the following 30-year old theorem of Erdős–Ginzburg–ziv (originally proved in the case $m = k$):

Suppose $m \geqslant k \geqslant 2$ are integers such that $k \mid m$. Let $a_1, a_2, \ldots, a_{m+k-1}$ be a sequence of integers. Then there exists a subset I of $\{1, 2, \ldots, m + k - 1\}$, such that $|I| = m$ and $\sum_{i \in I} a_i = 0 (\text{mod } k)$.

To present the combinatorial generalizations of this beautiful result we need some definitions, which we shall use intensively later.

Suppose $m \geqslant k \geqslant 2$ are integers such that $k \mid m$. Let H be r-uniform hypergraph having $e(H) = m$ edges and $v(H) = n$ vertices:

- A k-coloring of the edges of H is a function $f: E(H) \to \{1, 2, \ldots, k\}$.
- A Z_k-coloring of the edges of H is a function $f: E(H) \to Z_k$, where Z_k is the additive group modulo-k.

Let f be a Z_k coloring of H, then we say that H is zero-sum (mod-k), relative to f, if $\sum_{e \in E(H)} f(e) \equiv 0 (\text{mod } k)$.

SSDI 0012-365X(94)00308-4

The Ramsey number $R(H, k)$ is the smallest integer t such that in any k-coloring of the edges of the complete r-uniform hypergraph on t vertices K_t^r, there exists a monochromatic copy of H.

The zero-sum Ramsey number $R(H, Z_k)$ is the smallest integer t such that in any Z_k-coloring of the edges of the complete r-uniform hypergraph on t vertices K_t^r there exists a zero-sum copy of H.

The bipartite Ramsey number $B(H, k)$ is the smallest integer t such that in any k-coloring of the edges of the complete bipartite graph $K_{t,t}$ there exists a monochromatic copy of H.

The zero-sum bipartite Ramsey number $B(H, Z_k)$ is the smallest integer t such that in any Z_k-coloring of the edges of the complete bipartite graph $K_{t,t}$ there exists a zero-sum copy of H.

The Turan number $T(n, H, k)$ is the largest integer t such that there exists a hypergraph G on n vertices and t edges and a k-coloring of $E(G)$ without a monochromatic copy of H. (Observe that if H is a graph, i.e. $r = 2$, and $k = 1$, then $T(n, H, 1)$ is reduced to the traditional Turan's number $T(n, H)$.)

The zero-sum Turan number $T(n, H, Z_k)$ is the largest integer t such that there exists a hypergraph G on n vertices and t edges and a Z_k-coloring of $E(G)$ without a zero-sum (mod-k) copy of H.

Having this, long but unavoidable, list of definitions we can describe some basic features and facts about zero-sum Ramsey/Turan numbers.

In the first place we note that the existence, in the case that $k \mid e(G)$, of the Ramsey/Turan numbers follows from the existence of the classical (traditional) Ramsey/Turan numbers, because trivially $R(G, Z_k) \leqslant R(G, k)$ and also $T(n, G, Z_k) \leqslant T(n, G, k) \leqslant kT(n, G, 1)$.

Below we list some interesting features of the zero-sum Ramsey numbers, demonstrating the main differences between them and classical Ramsey numbers.

(1) The zero-sum Ramsey/Turan numbers supply lower bounds for the classical Ramsey/Turan numbers (as noted above).

(2) A trivial property of the classical Ramsey numbers is that they share the monotonicity property in two respects:

(1) If G is a subgraph of H then $R(G, k) \leqslant R(H, k)$.

(2) If $k < m$ then $R(G, k) < R(G, m)$.

As we shall see later there are examples of graphs G and H such that $2 \mid e(G)$, $2 \mid e(H)$, G is a subgraph of H and yet $R(H, Z_2) < R(G, Z_2)$, a somewhat unexpected phenomena and, as we shall see later, far from being the only one.

(3) It is well known that $2^{n/2} < R(K_n, 2) < 4^n$ and the existence of $\lim_{n \to \infty} R(K_n, 2)^{1/n}$ is a major problem in Ramsey theory. In contrast, we show that for fixed k such that $k \mid \binom{n}{2}$, $\lim_{n \to \infty} R(K_n, Z_k)/n = 1$. In fact, a much stronger result is known.

(4) Note that for every graph G and any given integer $k \geqslant 1$ the Ramsey number $R(G, k)$ always exists while $R(G, Z_k)$ exists if and only if $k \mid e(G)$, because if $k \nmid e(G)$ then the constant coloring $f: E(K_n) \to Z_k$ given by $f(e) \equiv 1$ avoids a zero-sum copy (mod k) of G.

(5) In almost all the proofs of zero-sum theorems some algebraic tools (e.g., the Erdős–Ginzburg–Ziv theorem, Cauchy–Davenport theorem, Chevalley's theorem, Baker–Schmidt theorem, etc.) play a central role. This is mainly due to the algebraic flavour of the zero-sum problems.

The survey is organized as follows: Section 2 provides the algebraic background. The zero-sum Ramsey numbers and Zero-sum Turan numbers are discussed in Sections 3 and 4, respectively, and finally the Zero-sum bipartite Ramsey numbers are discussed in Section 5.

Lastly, our notation is the standard following Bollobas [20], any other definition or notation will be introduced the first time we need it. I have included in the references several papers to supply complementary results from classical Ramsey/Turan Theory.

2. The algebraic background

Three well-known theorems dominate the algebraic background of the ZS-problems (where ZS denotes zero-sum). These are: Erdős–Ginzburg–Ziv theorem, Cauchy–Davenport theorem and Chevalley's theorem. I shall describe below these theorems and some of their descendants with a special emphasis on the EGZ-theorem which is the ancestor of most of the ZS-problems.

2.1. *Erdős–Ginzburg–Ziv theorem*

Theorem 2.1 (The Erdős–Ginzburg–Zin theorem (extended), Erdős et al. [45]). *Suppose $m \geqslant k \geqslant 2$ are integers such that $k \mid m$. Let $a_1, a_2, \ldots, a_{m+k-1}$ be a sequence of integers. Then there exists a subset I of $\{1, 2, \ldots, m+k-1\}$, such that $|I| = m$ and $\sum_{i \in I} a_i \equiv 0 \pmod{k}$.*

A characterization of the extremal cases in the EGZ-theorem was established in [26] (where the next result was stated in a slightly weaker form), and is later extended and simplified in [3, 48].

Theorem 2.2 (Caro [26], Alon et al. [3] and Flores and Ordaz [48]). *Let $m \geqslant k \geqslant 2$ be integers such that $k \mid m$ and let $A = \{a_1, a_2, \ldots, a_{m+k-2}\}$ be a sequence of integers that violates the EGZ-theorem. Then for every divisor d of k, the members of A belong to exactly two residue classes of Z_d say a and b, each of these residue classes contains $-1 \pmod{d}$ members of A and further $\gcd(b-a, d) = 1$.*

Bialostocki, Dierker and Lotspeich raised the following related problems (see e.g. [17, 19]): Let m and k be positive integers such that $m \geqslant k \geqslant 1$, and denote by $g(m, k)$ the least integer g such that in every Z_m-coloring of a g-element set that uses at least k distinct colors, there is a zero-sum $\pmod{m}$ subset of size m.

Conjecture 2.1. For every $m \geqslant k \geqslant 1$, $g(m, k) \leqslant 2m - k + 1$.

Conjecture 2.2. Let $k \geqslant 3$ be fixed and $m > m(k)$, then $g(m, k) = 2m - (k^2 - 5k + 12)/2 + 1$.

Many cases were solved by Bialostocki, Dierker and Lotspeich. Alon [1] announced a proof of Conjecture 2.1 for m prime and arbitrary k as well as $g(m, k) \leqslant m + 2$ if $k > m/2$. However, both conjectures are far from being solved yet.

Another conjecture of Bialostocki and Dierker states:

Conjecture 2.3 (Bialostocki [11]). Let $A = \{a_1, a_2, \ldots, a_n\}$ be a sequence of integers. Then A contains at least $\binom{\lfloor n/2 \rfloor}{m} + \binom{\lceil n/2 \rceil}{m}$ ZS-subsequences (mod m) of length m.

Recently, Kisin and Furedi and Kleitman solved this conjecture in the prime case, and gave asymptotic solution for every m (See [52, 62]. For further information see e.g. [13, 14, 19, 17, 2]).

A conjecture that extends the content of the Erdős–Ginzburg–Ziv theorem is:

Conjecture 2.4. Let $k \geqslant 2$ and m be integers. Let $a_1, a_2, \ldots, a_k$ be a sequence of integers such that $\sum_{i=1}^{k} a_i \equiv 0 \pmod m$. Let $b_1, b_2, \ldots, b_{m+k-1}$ be another sequence of integers. Then there exists a subsequence $b_{j_1}, b_{j_2}, \ldots, b_{j_k}$ such that $\sum_{i=1}^{k} a_i b_{j_i} \equiv 0 \pmod m$.

Alon [1] proved the prime case with $m = p = k$, but his method, based entirely on the Cauchy–Davenport theorem, enabled us to prove [3]:

Theorem 2.3. *Conjecture* 2.4 *holds for every prime* m, *and* $k \geqslant 2$.

Olson proved the following deep group theoretic generalization of the EGZ-theorem.

Theorem 2.4 (Olson [66]). *Let $g_1, g_2, \ldots, g_{2m-1}$ be a sequence of $2m - 1$ elements of a finite group G of order m (not necessarily abelian). Then there is a permutation of a subsequence of m terms that sum to* 0.

A related result was proved by Harborth.

Theorem 2.5 (Harborth [61]). *Let $f(m, d)$ be the minimal integer satisfying the following property: If $a_1, a_2, \ldots, a_f$ are elements of the group Z_m^d (the direct product of d copies* *of Z_m), then there is a subsequence of m elements such that $a_{i1} + a_{i2} + \cdots + a_{im} = 0$. Then*

(1) $(m-1)2^d + 1 \leqslant f(m, d) \leqslant (m-1)m^d + 1$.

(2) *the lower bound in* (1) *is attained if either $d = 1$ or if $d = 2$ and m has the form $m = 2^x 3^y$.*

Recently, Alon and Dubiner [5], using deep tools, showed $f(m, d) \leqslant c(d)m$, where $c(d)$ is a constant depends on d but not on m.

Problem 2.1. Determine $f(m, d)$ for all m and d.

2.2. Cauchy–Davenport theorem

We start with the classical theorem of Cauchy–Davenport.

Theorem 2.6 (Cauchy–Davenport [44]). *Let p be a prime and A, B be two sets of residue classes modulo p; then* $|A + B| = |\{x + y \colon x \in A, y \in B\}| \geqslant \min\{p, |A| + |B| - 1\}$.

Using induction one can derive the useful generalization.

Theorem 2.7. *Let p be prime and $A_1, A_2, \ldots, A_k$ be sets of residue classes modulo p; then*

$$\begin{aligned} |A_1 + A_2 + \cdots + A_k| &= |\{x_1 + x_2 + \cdots + x_k \colon x_i \in A_i\}| \\ &\geqslant \min\{p, |A_1| + |A_2| + \cdots + |A_k| - k + 1\}. \end{aligned}$$

Some further results concerning the Cauchy–Davenport theorem have been obtained by Chowla, Hamidoune and Danilov (see e.g. [51, 58, 43]). Certainly one of the most challenging related problems here was:

Conjecture 2.5 (Erdős–Heilbronn [46]). If $a_1, a_2, \ldots, a_k$ are distinct residues modulo p, then the pair sums $a_i + a_j$, $i \neq j$, represent at least $\min\{p, 2k-3\}$ residue classes modulo p.

Recently, Hamidoune and das-Silva [59] announced a complete solution of the Erdős–Heilbronn conjecture. Alon, Nathanson and Rusza devised an ingenious algebraic method that among other things solves far reaching generalizations of the Erdős–Heilbronn conjecture [9]. Kneser proved a far reaching generalization of Cauchy–Davenport theorem. Here I state a special case of his theorem.

Theorem 2.8 (Kneser [63]). *Let A and B be subsets of Z_m, then there exists a subgroup H of Z_m such that*

(1) $A + B + H = A + B$,

(2) $|A + B| \geqslant |A + H| + |B + H| - |H|$.

2.3. Chevalley's Theorem and its generalizations

The last algebraic tool is the celebrated theorem of Chevalley [21]. I give here only one of the many forms of this useful result. There are many generalizations, conjectures and striking applications of this theorem, most of them given in a survey article of Alon [2]; see also [6, 7].

Theorem 2.9 (Chevalley [21]). *For $j = 1, 2, \ldots, n$, let $f_j(x_1, x_2, \ldots, x_m)$ be a polynomial in m variables with integers coefficients and with no constant term. Let p be a prime and consider the system of equations $f_j(x_1, x_2, \ldots, x_m) \equiv 0 (\mathrm{mod}\, p)$ for $j = 1, 2, \ldots, n$. If $m > \sum_{j=1}^{n} \deg f_j$, then there exists a non-trivial solution to the system.*

An important generalization of Chevalley's theorem is given by:

Theorem 2.10 (Baker–Schmidt [10]). *Let q be a prime-power. If $t \geqslant d(q-1)+1$ and $h_1(x_1, x_2, \ldots, x_t)$, $h_2(x_1, x_2, \ldots, x_t)$, $\ldots$, $h_n(x_1, x_2, \ldots, x_t) \in Z[x_1, x_2, \ldots, x_t]$ satisfy $h_1(0) = \cdots = h_n(0) = 0$ and also $\sum_{i=1}^{n} \deg h_i \leqslant d$, then there exists an $0 \neq \alpha \in \{0, 1\}^t$ such that $h_1(\alpha) \equiv \cdots h_n(\alpha) \equiv 0 (\mathrm{mod}\, q)$.*

A related result for non-prime power moduli was given in 1969 by:

Theorem 2.11 (Van Emde Boas–Kruyswijk [73]). *Let Z_k^r denote the sum of r copies of the group Z_k (i.e., the abelian group of all vectors of length r over Z_k). Let $v_1, v_2, \ldots, v_p$ be a sequence of p (not necessarily distinct) members of Z_k^r. If $p > r(k-1)\log_2 k$ then there is a non-empty subset I of $\{1, 2, \ldots, p\}$ such that $\sum_{i \in I} v_i = 0$, in Z_k^r.*

3. ZS-Ramsey numbers for graphs and hypergraphs

Recall the definitions of the classical, respectively, zero-sum Ramsey numbers.

The Ramsey number $R(H, k)$ is the smallest integer t such that in any k-coloring of the edges of the complete r-uniform hypergraph on t vertices K_t^r, there exists a monochromatic copy of H.

The zero-sum Ramsey number $R(H, Z_k)$ is the smallest integer t such that in any Z_k-coloring of the edges of the complete r-uniform hypergraph on t vertices K_t^r there exists a zero-sum copy of H.

Bialostocki and Dierker [14] were the first to introduce the concept of the Ramsey numbers $R(G, Z_m)$ where $m = e(G)$. This notion was later extended in [26] to the more general concept of $R(G, Z_k)$ where $k \mid e(G)$, and as we shall see later this extension is particularly suitable for the algebraic approach.

3.1. Stars, matchings and forests

The existence of the ZS-Ramsey numbers follows from the following simple result.

Theorem 3.1 (Bialostocki and Dierker [14], and Caro [26]). (1) *Suppose* $k \mid e(G)$ *then* $R(G, Z_k) \leqslant R(G, k)$.

(2) *If* $k = e(G)$ *then also* $R(G, 2) \leqslant R(G, Z_k)$.

Theorem 3.2 (see [14] for the case $k = n$ and [26] for the case $k \mid n$). *Let* $K_{1,n}$ *be the star on* n *edges and suppose* $k \mid n$. *Then*

$$R(K_{1,n}, Z_k) = \begin{cases} n + k - 1 & \textit{if } n \equiv k \equiv 0 (\mathrm{mod}\ 2) \\ n + k & \textit{otherwise} \end{cases}$$

The following related theorem, which was proved in a weaker form in [26], determines the directed ZS-Ramsey number for stars. The directed zero-sum Ramsey number $R*(K_{1,n}, Z_k)$ is the smallest integer t such that in any Z_k-coloring of the edges of the complete graph K_t and with any orientation of the edges of K_t there exists a zero-sum (mod k) copy of the directed star $K_{1,n}$ in which all the edges are directed out from the center.

Theorem 3.3 (Caro [26]). *Suppose* $k \mid n$, *then* $R*(K_{1,n}, Z_k) = 2(n + k - 1)$.

Theorem 3.4 (see [14] for the case $k = t$ and Caro [25] for the case $k \mid t$). *Let* tK_2 *be the matching consisting of* t *disjoint edges and suppose* $k \mid t$. *Then* $R(tK_2, Z_k) = 2t + k - 1$.

Both in [14, 25], it was shown that Theorem 3.4 holds for matchings in hypergraphs as well.

Denote by $\mathrm{Tr}(m)$ the family of all the trees on m edges (hence $m + 1$ vertices).

Theorem 3.5 (see Bialostocki and Dierker [13] for m a prime, Furedi and Kleitman [51] and Schrijver and Seymour [68, 69] for arbitrary m). $R(\mathrm{Tr}(m), Z_m) = m + 1$.

3.2. Small graphs

The interest in small graphs is motivated by at least two reasons:

1. We hope to discover either a new proof technique or at least some phenomena that extend to general results.
2. For small graphs the Ramsey numbers are more tractable, usually far below the limit of computation, and it is interesting to compare the exact results with the general theoretic bounds.

An early result of Chung and Graham can be reformulated as a zero-sum theorem.

Theorem 3.6 (Chung and Graham [42]).

$$R(K_3, 2) = 6 < R(K_3, Z_3) = 11 < R(K_3, 3) = 17.$$

For small graphs, having at most 4 edges the ZS-Ramsey numbers were calculated in [15]. I also give the zero-sum (mod 2) Ramsey numbers, based on a recent results of [4, 34, 35] (see Theorem 3.15).

The limited data given in Tables 1–3 and the contrast between Theorems 3.2–3.5 and 3.6 (in the first four cases the Ramsey numbers and the ZS-Ramsey numbers

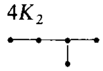
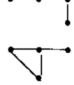
Table 1
2 edges

G	$R(G, Z_2)$	$R(G, 2)$
P_3	3	3
$2K_2$	5	5

Table 2
3 edges

G	$R(G, Z_3)$	$R(G, 2)$
K_3	11	6
P_4	5	5
$K_{1,3}$	6	6
$P_3 \cup K_2$	6	6
$3K_2$	8	8

Table 3
4 edges

	G	$R(G, Z_4)$	$R(G, 2)$	$R(G, Z_2)$
	C_4	6	6	4
	$K_{1,4}$	7	7	5
	P_5	6	6	5
(i)	$K_3 \cup K_2$	8	7	6
	$2P_3$	7	7	6
	$P_4 \cup K_2$	8	8	6
(ii)	$K_{1,3} \cup K_2$	8	7	7
	$P_3 \cup 2K_2$	9	9	7
	$4K_2$	11	11	9
$(Q_5) =$		6	6	5
		7	7	4

coincide), suggest some questions on the relations between the Ramsey numbers and their ZS-analogues. A conjecture along this line was raised in [15].

Conjecture 3.1. Let G range over all forests having m edges. Then $\lim_{m\to\infty} \max_G\{R(G, z_m)/R(G, 2)\} = 1$.

As we shall see later Conjecture 3.1 becomes false if e.g. instead of forests we take disjoint copies of K_3. From Table 3 we observe that (i) represents a case when for two graphs H $(=K_3)$, $G(=K_3 \cup K_2)$, H is a subgraph of G and yet $R(H, Z_{e(H)}) > R(G, Z_{e(G)})$. This is in sharp contrast to the monotone property of the Ramsey numbers. There are many examples of this kind, e.g. $R(C_{4n}, Z_2) = 4n$ while $R(2nK_2, Z_2) = 4n + 1$. Observe that (ii) represents a case when even for a forest G it is possible that $R(G, Z_{e(G)}) > R(G, 2)$, but this example does not contradict Conjecture 3.1.

Problem 3.1. (1) Let H be a graph and suppose $n < m$ are integers which both divide $e(H)$. Is it true that $R(H, Z_n) < R(H, Z_m)$?

(2) Does there exist a tree T on m edges such that $R(T, Z_m) > R(T, 2)$?

It is well known [22] that for $n > 3$, $R(K_{1,n-1} \cup K_2, 2) = 2n - 1$.
In [15] we find:

Theorem 3.7 (Bialostocki and Dierker [15]). *Let p be a prime, then $2p - 1 \leqslant R(K_{1,p-1} \cup K_2, Z_p) \leqslant 2p + 2$.*

Problem 3.2. For $k \mid n$ determine $R(K_{1,n-1} \cup K_2, Z_k)$.

3.3. Multiple copies of graphs and hypergraphs

Motivated by the exact determination of $R(tK_2, Z_t)$, in [16, 24], the paper [39] begins the investigation of the ZS-Ramsey numbers for multiple copies of a graph. Denote by tG the vertex-disjoint union of t copies of G. The basic result is:

Theorem 3.8 (Bialostocki and Dierker [16], Caro [24] and Caro and Roditty [39]). *Let G be a graph on n vertices, m edges, and independence number β:*

(1) *if $k \mid t$, then $R(tG, Z_k) \leqslant (t + k - 1)n$,*

(2) *if $k \mid t$ and $q \mid m$ then $R(tG, Z_{qk}) \leqslant R(G, Z_q) + (t + k - 2)n$,*

(3) $t(2n - \beta) \leqslant R(tG, 2) \leqslant R(tG, Z_{mt}) \leqslant t[R(G, Z_m) + 1] - 1$.

In [16, 24] it is shown that Theorem 3.8 remains true for hypergraphs. Using Theorem 3.8 we can determine several ZS-Ramsey numbers of multiple copies of a given graph G. The list below is taken from [16].

Theorem 3.9 ([16])

(1) $R(tP_4, 2) = R(tP_4, Z_{3t}) = 6t - 1$,
(2) $R(t(P_3 \cup K_2), 2) = R(T(P_3 \cup K_2), Z_{3t}) = 7t - 1$,
(3) $R(tP_5, 2) = R(tP_5, Z_{4t}) = 7t - 1$,
(4) $R(t(2P_3), 2) = R(t(2P_3), Z_{4t}) = 8t - 1$,
(5) $R(t(P_4 \cup K_2), 2) = R(t(P_4 \cup K_2), Z_{4t}) = 9t - 1$,
(6) $R(t(P_3 \cup 2K_2), 2) = R(t(P_3 \cup 2K_2), Z_{4t}) = 10t - 1$,
(7) $R(tQ_5, 2) = R(tQ_5, Z_{4t}) = 7t - 1$ *(see Q_5 in Table* 3),
(8) $5t \leqslant R(tK_3, Z_{3t}) \leqslant 6t + 5$, *for* $t > 1$ [16, 24]).

Considering Theorem 3.9 (8) in [16] the authors raised the following problem and conjecture.

Problem 3.3. (1) Does $R(tK_3, Z_{3t})/R(tK_3, 2)$ tend to a limit when t tends to infinity?
(2) Does there exist a graph G such that limsup $R(tG, Z_{te(G)})/R(tG, 2) > 1$?

Conjecture 3.2. Let G be a forest on m edges; then for $t > t(m)$, $R(tG, Z_{mt}) = R(tG, 2)$.

Clearly, when Conjecture 3.2 is true then Problem 3.3(2) has a negative answer if G is a forest. Unfortunately, we do not know the truth status of Conjecture 3.2, while the next result shows that Problem 3.3(2) has in fact a positive answer with $G = K_3$.

Theorem 3.10 (Caro [30]). (1) *For t odd* $6t + 3 \leqslant R(tK_3, Z_{3t}) \leqslant 6t + 5$,
(2) limsup $R(tK_3, Z_{3t})/R(tK_3, 2) = 1.2$.

Another natural problem raised by Roditty [67] and solved by him for some simple cases is:

Problem 3.4. Suppose $k \mid tn$. Determine $R(tK_{1,n}, Z_k)$.

3.4. The defect $\mathrm{def}_t(H)$ *and its applications*

Below is described the 'defect method' that generalizes the proof technique presented in Theorem 3.4 from matchings to arbitrary hypergraphs.

Denote by K_n^r the complete r-uniform hypergraph on n-vertex set.

Definition. Let H be r-uniform hypergraph on n vertices and let $f: E(H) \to Z_t$. Set $f(H) = \sum_{e \in E(H)} f(e)$. The defect of H with respect to t, written as $\mathrm{def}_t(H)$, is the least non-negative integer d such that: for every Z_t-coloring $f: E(K_{2n}^r) \to Z_t$ either

(1) for every copy H_1 of H in K_{2n}^r there exists a vertex disjoint copy of H, say H_2, such that $f(H_1) \equiv f(H_2) \pmod t$, or

(2) there exists K_{n+d}^r, containing two copies of H, say H_1 and H_2, such that $f(H_1) \not\equiv f(H_2) \pmod t$.

The concept of defect introduced in [25] and proved to be a useful tool in attacking ZS-problems for multiple copies, as we shall see in Theorem 3.12.

Theorem 3.11 (Caro [25]). *Let $t \geqslant 2$ be an integer and H be r-uniform hypergraph; then* $\mathrm{def}_t(H) \in \{0, 1\}$.

The main result in this direction, which improves Theorem 3.8, is given by:

Theorem 3.12 (Caro [25]). *Let H be r-uniform hypergraph on n vertices.*

(1) *Let t be a prime, and suppose* $\mathrm{def}_t(H) = d$.

Then $tn \leqslant R(tH, Z_t) \leqslant t(n + d) - d$. In particular, if $d = 0$ then $R(tH, Z_t) = tn$.

(2) *Let m and t be integers such that $t \mid m$ and t is a prime. Suppose further that* $\mathrm{def}_t(H) = 0$. *Then $R(mH, Z_t) = mn$.*

(3) *Let m and k be integers such that $k \mid m$. Then $mn \leqslant R(mH, Z_k) \leqslant mn + k - 1$.*

Example. It is easy to see that for $t \geqslant 2$, $\mathrm{def}_t(P_n) = 0$, hence if t is a prime and $t \mid m$ then $R(mP_n, Z_t) = mn$.

Based upon many special cases the following conjectures are formulated in [25].

Conjecture 3.3. Let H be r-uniform hypergraph and t be an integer. Then $\mathrm{def}_t(H) = 1$ iff H is a vertex-disjoint union of complete r-uniform hypergraphs.

Conjecture 3.4. Let H be r-uniform hypergraph on n vertices and let $t \geqslant 2$ be an integer. Suppose $\mathrm{def}_t(H) = d$, then $R(tH, Z_t) = t(n + d) - d$.

Recall Theorem 3.8 (3) and Theorem 3.12; the following generalization was proved in [25].

Theorem 3.13 (Caro [26]). *Let H be r-uniform hypergraph and suppose $k \mid t$ and $m \mid e(H)$; then $R(tH, Z_{mk}) \leqslant tR(H, Z_m) + k - 1$.*

Observed that if $k = t$ and $m = e(H)$, then Theorem 3.13 reduced to Theorem 3.8(3), and if $m = 1$ it is reduced to Theorem 3.12(3).

Problem 3.5. Find non-trivial lower bounds for $R(tH, Z_k)$ when $k \mid te(H)$.

3.5. The order of magnitude of $R(G, Z_k)$

Recall Theorem 3.1 which states that $R(G, Z_k) \leqslant R(G, k)$ and if $k = e(G)$ then also $R(G, 2) \leqslant R(G, Z_k)$. Which one is closer to the truth? Representatives of this problem are given by:

Problem 3.6. (i) Let k be fixed and suppose $k \mid e(G)$. What can be said about $R(G, Z_k)$ as G gets larger?

(ii) Determine the order of magnitude of $R(K_n, Z_{\binom{n}{2}})$.

The main results in this direction are:

Theorem 3.14 (Caro [28]). *Suppose $k \mid \binom{n}{2}$ and $k \geqslant c\binom{n}{2}$, with c a positive constant. Then*

$$R(K_n, Z_k) \geqslant e^{c(n-1)/8}.$$

Moreover, $R(K_n, Z_k)$ remains sup polynomial even if $k > n^{1.5+\varepsilon}$.

For $k = 2$ we can answer Problem 3.6(i) completely.

Theorem 3.15 (Caro [34]). *Let G be a graph on n vertices and an even number of edges. Then*

$$R(G, Z_2) = \begin{cases} n+2 & \text{if } G = K_n, n \equiv 0, 1 \pmod 4, \\ n+1 & \text{if } G = K_p \cup K_q, \binom{p}{2} + \binom{q}{2} \equiv 0 \pmod 2, \\ n+1 & \text{if all the degrees in } G \text{ are odd}, \\ n & \text{otherwise}. \end{cases}$$

Theorem 3.16 (Caro [30]). *Let k be fixed and suppose $k \mid \binom{n}{2}$, then there exists a constant $c(k)$ such that $R(K_n, Z_k) \leqslant n + c(k)$ (if k is odd then $c(k) \leqslant R(K_{2k-1}, k)$ while if k is even then $c(k) \leqslant R(K_{3k-1}, k)$).*

Taking care of the constant $c(k)$ in Theorem 3.16 it is shown:

Theorem 3.17 (Alon and Caro [4]). (1) *Under the assumptions of Theorem* 3.16 *and with k an odd prime power, if $n \geqslant n(k)$ then $R(K_n, Z_k) \leqslant n + 2k - 2$, also the equality holds if k is a prime and $k \mid n$.*

(2) *Under the assumptions of Theorem* 3.16 *and for any k, if $n \geqslant n(k)$ then $R(K_n, Z_k) \leqslant n + k(k+1)(k+2)\log_2 k$.*

Extensions of Theorems 3.16 and 3.17 to hypergraphs are also known (see e.g. [4, 36]). Due to lack of monotonicity and in view of Theorems 3.15–3.17 the following problem seems inevitable.

Problem 3.7. Let $k \geqslant 3$ be fixed integer. Does there exist a minimal integer $n(k)$ such that if G has $n \geqslant n(k)$ vertices and $k \mid e(G)$ then $R(G, Z_k) \leqslant n + 2k - 2$.

Concerning the lower bound for ZS-Ramsey numbers we have:

Theorem 3.18 (Caro [30]). *Let G be a connected graph such that* $e(G) \equiv 1 \pmod 2$ *and every cut of G contains an even number of edges. Then* $R(G, Z_{e(G)}) \geqslant 2R(G, 2) - 1$.

Remark. The set of graphs satisfying the conditions of Theorem 3.18 is exactly the set of the connected eulerian graphs having an odd number of edges.

An immediate consequence is:

Theorem 3.19 (Caro [30]). (1) *Suppose* $n \equiv 3 \pmod 4$; *then*

$$R(K_n, Z_{\binom{n}{2}}) \geqslant 2R(K_n, 2) - 1.$$

(2) *Let* $n > 4$ *be odd; then* $R(C_n, Z_n) \geqslant 2R(C_n, 2) - 1 = 4n - 3$.

In view of Theorem 3.19 the next conjecture seems plausible.

Conjecture 3.5. $R(C_n, Z_n) = 4n - 3$, for n odd $\geqslant 5$.

The next conjecture and result are probably the first step to show that in general the Ramsey numbers are much larger than the zero-sum Ramsey numbers. More precisely:

Conjecture 3.6. If $k \mid e(G)$ then $R(G, Z_k) \ll R(G, k)$.

Theorem 3.20 (Caro [32]). *Let H be r-uniform hypergraph on m edges, and let a, b be two integers such that* $ab \mid m$ *and* $\gcd(a, b) = 1$; *then*

$$R(H, Z_{ab}) \leqslant \min\{R(K_{R(H, Z_a)}, b), R(K_{R(H, Z_b)}, a)\}.$$

Consider, for example, $n \equiv 5 \pmod 8$. Let $a = 2$, $b = \binom{n}{2}/2$; then $\gcd(a, b) = 1$ and by Theorems 3.20 and 3.15: $R(K_n, Z_{\binom{n}{2}}) \leqslant R(K_{n+2}, \binom{n}{2}/2) \ll R(K_n, \binom{n}{2})$.

I think the following problem was never considered before and, if true, might have some applications when combined with Theorem 3.20 in solving Conjecture 3.6.

Problem 3.8. It is true that for $n > 3$, $R(K_{n+1}, k) < R(K_n, k + 1)$.

In closing we offer two more problems to consider.

Problem 3.9. Does there exist an absolute constant c such that for every graph G, $R(G, Z_{e(G)}) \leqslant cR(G, 2)$?

Problem 3.10. Determine $R(K_4, Z_3)$ and $R(K_4, Z_6)$.

3.6. Some sporadic results

In this section we describe some results that do not fit the former sections.

Theorem 3.21 (Caro [30]). (1) *If* $k \mid m$ *or* $k \mid n$ *then* $R(K_{m,n}, Z_k) \leqslant m + n + k - 1$.
(2) $R(K_{m,n}, Z_{mn}) \leqslant (2n-2)\binom{2m-1}{m} + 2m$.

Denote by $\mathrm{Tr}(r, m)$ the family of all the r-uniform hypertrees on m edges (hence $m(r-1)+1$ vertices). We have the following extension of Theorem 3.5.

Theorem 3.22 ([Bialostocki and Dierker [16], and Schrijver and Seymour [68, 69]).

$$R(\mathrm{Tr}(r, m), Z_m) = R(\mathrm{Tr}(r, m), 2) = m(r-1) + 1.$$

Definition. Let $F = \{e_1, e_2, \ldots, e_t\}$ be t r-element sets. Suppose there exists a set Q, with cardinality $0 \leqslant q < r$, such that $e_i \cap e_j = Q$ for $1 \leqslant i < j \leqslant t$. Then F is called a delta system of type $S(r, q, t)$. Observe that $S(2, 0, t) = tK_2$ while $S(2, 1, t) = K_{1,t}$.

We have the following extension of Theorems 3.2 and 3.4.

Theorem 3.23 (Caro [25]). *Suppose* $k \mid t$; *then* $(r-q)t + q - 1 \leqslant R(S(r, q, t), Z_k) \leqslant (r-q)t + k + q - 1$.

Remark. The proof of the lower bound $(r-q)t + k - 1$ given in [25] is incorrect, but the lower bound $(r-q)t + q - 1$ is trivial.

The exact determination of $R(S(r, q, t), Z_k)$ seems difficult!

Definition. Let $t = \chi(G, q)$ denote the smallest integer t such that the vertex set V of G can be partitioned into t classes $V(G) = \bigcup_{i=1}^{t} V_i$, such that the number of edges in each induced subgraph $\langle V_i \rangle$, $1 \leqslant i \leqslant t$, is divisible by q. This parameter was coined as the 'chromatic number module q'.

Theorem 3.24 (Caro [37]). *Let G be a graph and $q \geqslant 2$ be fixed.*
(1) $\chi(G, q) \leqslant 2q - 1$.
(2) *There exists an algorithm that computes* $\chi(G, q)$ *in a polynomial time,* (0 $(n^{8(q-1)^2+1})$ *if q is a prime power*).
(3) *For almost all graphs* $\chi(G, q) \leqslant 2$ [1].

4. ZS-Turan numbers for graphs

Recall the definitions of the classical, respectively, zero-sum Turan numbers.

The Turan number $T(n, H, k)$ is the largest integer t such that there exists a hypergraph G on n vertices and t edges and a k-coloring of $E(G)$ without a monochromatic copy of H. (Observe that if H is a graph, i.e. $r = 2$, and $k = 1$, then $T(n, H, 1)$ reduce to the traditional Turan's number $T(n, H)$.)

The zero-sum Turan number $T(n, H, Z_k)$ is the largest integer t such that there exists a hypergraph G on n vertices and t edges and a Z_k-coloring of $E(G)$ without a zero-sum (mod-k) copy of H.

ZS-Turan numbers were introduced in [12, 28]. The basic existence theorem is given by:

Theorem 4.1 (Bialostocki et al. [12] and Caro [28]). (1) *If* $k \mid e(G)$ *then* $T(n, G, Z_k) \leqslant T(n, G, k) \leqslant kT(n, G, 1)$.

(2) *If* $k = e(G)$ *then* $T(n, G, 2) \leqslant T(n, G, Z_k)$.

The importance of Theorem 4.1(1) is that is shows that if G is bipartite then $T(n, G, Z_k)$ is of the same order of magnitude as $T(n, G, 1) = o(n^2)$.

4.1. Complete graphs and non-bipartite graphs

For complete graphs we have the following important relation.

Theorem 4.2 (see [12] for $k = \binom{m}{2}$, [28] for $k \mid \binom{m}{2}$). *Suppose* $k \mid \binom{m}{2}$; *then* $T(n, K_m, Z_k) = T(n, K_{R(K_m, Z_k)}, 1)$.

Using the known Turan numbers for complete graphs and the few known zero-sum Ramsey numbers we obtain:

Theorem 4.3 ([12, 28, 32]). (1) $T(n, K_3, Z_3) = T(n, K_{11}, 1)$,

(2) *for* $m \equiv 0, 1 \pmod 4$, $T(n, K_m, Z_2) = T(n, K_{m+2}, 1)$,

(3) *if* k *is an odd prime and* $m \geqslant m(k)$ *and* $k \mid \binom{m}{2}$, *then* $T(n, K_m, Z_k) = T(n, K_{m+2k-2}, 1)$

Theorem 4.4 ([32]). *Let* G *be a connected graph such that* $e(G) \equiv 1 \pmod 2$, *but every cut of* G *contains an even number of edges. Let* $\chi(G) = k$ *be the chromatic number of* G. *Then* $T(n, G, Z_{e(G)}) \geqslant T(n, G, 2) + T(n, K_3, 1) \geqslant (1 + o(1))(1 - 1/2(k-1)^2)n^2$.

Problem 4.1. Let G be a graph on m edges and suppose $\chi(G) = k \geqslant 3$. Derive an asymptotic for $T(n, G, Z_m)$.

4.2. Trees and matchings

For stars we have an almost complete results.

Theorem 4.5 (see [12] for $k = n$, [28] for $k \mid m$). (1) *If* $m \equiv 0 (\mathrm{mod}\, 2)$ *and* $n \equiv 1$ *(mod 2) then* $T(n, K_{1,m}, Z_m) = (m-1)n - 1$; *otherwise,* $T(n, K_{1,m}, Z_m) = (m-1)n$.

(2) *Suppose* $k \mid m$ *and* $n > 2(m-1)(k-1)$, *then if* $n - 1 \equiv k \equiv m \equiv 0 (\mathrm{mod}\, 2)$, $T(n, K_{1,m}, Z_k) = (m+k-2)n/2 - 1$; *otherwise* $T(n, K_{1,m}, Z_k) = [(m+k-2)n/2]$.

For $\mathrm{Tr}(m)$ the set of all trees on m edges we have:

Theorem 4.6 ([12]).

(1) *Let* $m \mid n$ *and* $n > m^2$; *then* $T(n, \mathrm{Tr}(m), Z_m) = (m-1)n$.

(2) $\lim_{n \to \infty} T(n, \mathrm{Tr}(m), Z_m)/n = m - 1$.

For Tr(3) we have the following complete result.

Theorem 4.7 ([12]). *Suppose* $n \geqslant 9$; *then*

$$T(n, \mathrm{Tr}(3), Z_3) = \begin{cases} 6k & \text{if } n = 3k, \\ 6k & \text{if } n = 3k+1, \\ 6k+2 & \text{if } n = 3k+2. \end{cases}$$

Table 4 shows the asymptotics of all ZS-Turan numbers for graphs having three edges. The data is taken from [12].

Observe the bizarre behavior of $T(n, P_4, Z_3)$ relative to the other forests whose zero-sum Turan numbers are bounded above by twice the traditional Turan numbers. This phenomena extends to larger paths, namely:

Theorem 4.8 (Caro and Roditty [40]). *Suppose* $m \equiv 0$ (mod 4). *Then*:

(1) $T(n, P_m, Z_{m-1}) \geqslant (n - m + 1)(m - 1)$,

(2) *For* $n > m^2$, $T(n, P_m, Z_{m-1}) > 2T(n, P_m, 1) \approx n(m-2)$.

Table 4

G	$\approx T(n, G, 1)$	$\approx T(n, G, 2)$	$\approx T(n, G, Z_3)$
K_3	$0.25n^2$	$0.4n^2$	$0.45n^2$
P_4	n	$2n$	$3n$
$K_{1,3}$	n	$2n$	$2n$
$3K_2$	$2n$	$4n$	$4n$
$P_3 \cup K_2$	n	$2n$	$2n$

Problem 4.2. Determine $T(n, P_m, Z_k)$ for each pair k, m such that $k \mid m-1$.

The following conjecture will have many consequences if true:

Conjecture 4.1. Let $m \geqslant k \geqslant 2$ be positive integers such that $k \mid m$. Let G be a graph with minimal degree $\delta(G) \geqslant m+k-1$. Then in any Z_k-coloring of $E(G)$ and for any tree T_m on m edges there exists in G a zero-sum (mod k) copy of T_m.

This conjecture is known to be true for $k=2$ and $m \equiv 0 \pmod 2$ and in some other special cases [40]. If true it will imply that $T(n, T_m, Z_k) \leqslant (m+k-2)n$ which is far better than the bound given in Theorem 4.1.

Theorem 4.9 (Caro [24]). *Suppose $k \mid t$; then $T(n, tG, Z_k) \leqslant T(n, (t+k-1)G, 1)$.*

Using Theorem 4.9 we solved in [39] the following.

Theorem 4.10 (Caro and Roditty [39]). *Suppose $k \mid t$; then*

$$T(n, tK_2, Z_k) = T(n, (t+k-1)K_2, 1)$$
$$= \binom{t+k-2}{2} + (t+k-2)(n-t-k+2).$$

4.3. Topological graphs

Definition. A graph H is a topological extension of a graph G if H is obtained from G upon replacing edges of G by paths. The sets of all topological extensions of G is denoted by TG. Some results were obtained concerning topological graphs.

Theorem 4.11 ([29]). *Let F_t be the set of all cycles of length at least t; then*
(1) $T(n, F_3, Z_2) = [3(n-1)/2]$,
(2) $T(n, F_t, Z_2) \leqslant c(t)n$, *when $c(t)$ is a positive constant depending on t only.*

This was further extended to answer a conjecture of Bialostocki.

Theorem 4.12 ([39]). *Let G be a non-empty graph and let $k > 1$ be fixed. Then there exists a positive constant $c(G, k)$ such that $T(n, TG, Z_k) \leqslant c(G, k)n$.*

5. Zero-sum bipartite Ramsey numbers

The zero-sum bipartite Ramsey number $B(H, Z_k)$ is the smallest integer t such that in any Z_k-coloring of the edges of the complete bipartite graph $K_{t,t}$ there exists a zero-sum copy of H.

For most of the results in this section consult [31]. For classical bipartite Ramsey theory consult [60].

Definition. For a bipartite graph G define $m(G) = \min\{|A|, V(G) = A \cup B, |A| \geqslant |B|\}$ where the minimum ranges over all the representations of G as a bipartite graph with classes A and B (e.g. $m(K_{6,8}) = 8$ while $m(K_{3,7} \cup K_{4,5}) = 11$).

Theorem 5.1. *Let G be a bipartite graph such that $2 \mid e(G)$; then*

(1) *if $m(G) \equiv 1 \pmod 2$ then $B(G, Z_2) = m(G)$,*

(2) *if $m(G) \equiv 0 \pmod 2$ then $B(G, Z_2) \leqslant m(G) + 1$,*

(3) *if $m(G) \equiv 0 \pmod 2$ and A realizes $m(G)$, $|A| > |B|$, and for every vertex $x \in A$ $\deg x \equiv 0 \pmod 2$, then $B(G, Z_2) = m(G)$.*

Some results concerning $B(G, Z_k)$ are known, namely:

Theorem 5.2. *Let $n \geqslant m \geqslant 1$; then*

(1) $$B(K_{n,m}, Z_k) \leqslant \begin{cases} m+k-1 & \text{if } k \mid m \text{ and } m \leqslant n \leqslant m+k-2, \\ n & \text{if } k \mid m \text{ and } m+k-1 \leqslant n, \\ n+k-1 & \text{if } k \mid n \text{ but } k \nmid m. \end{cases}$$

(2) Set
$$f(k) = \begin{cases} k-1 & \text{if } k \text{ is a prime,} \\ [(k-1)^{0.5}] & \text{otherwise.} \end{cases}$$

Then $B(K_{n,m}, Z_k) \geqslant \max\{m + f(k), n\}$.

(3) *In particular, if k is a prime such that $k \mid m$ and $m \leqslant n \leqslant m+k-2$. Then $B(K_{n,m}, Z_k) = m + k - 1$.*

Theorem 5.3. *Suppose $k \mid n^2$ and further $n^2/k = t$ where t is a fixed integer. Then $B(K_{n,n}, Z_k) \geqslant n\mathrm{e}^{n/4t^2}/2\mathrm{e}$*

Theorem 5.4.
$$B(K_{m,n}, Z_{mn}) \leqslant \min\left\{(2n-2)\binom{2m-1}{m}, (2m-2)\binom{2n-1}{n}\right\}.$$

Combining Theorem 5.3 (a sharper form exists for $k = n^2$) and Theorem 5.4 it can be shown that

$$n2^{n/2}/2e \leqslant B(K_{n,n}, Z_{n^2}) \leqslant n4^n \lll n^n/3n < B(K_{n,n}, n^2).$$

Theorem 5.5. *Let $n \geqslant k \geqslant 2$ be integers such that $k \mid n$. Then*

$$B(nK_2, Z_k) = B(K_{1,n}, Z_k) = n + k - 1.$$

There are some interesting problems on zero-sum bipartite Ramsey numbers of which I have chosen the following.

Problem 5.1. Determine $B(G, Z_2)$ for every bipartite graph G such that $2 \mid e(G)$, or at least for all trees.

Problem 5.2 (Bialostocki [11]). Prove that for $n > 1$, $B(K_{2,n}, Z_{2n}) \leqslant 4n - 3$.

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