Fractional Elements in Abelian Automata Groups

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A theorem of Nekrashevych and Sidki shows the Mealy Automata structures one can place on $\mathbb{Z}^m$ are parametrized by a family of matrices (called "$\frac{1}{2}$-integral") and a choice of residuation vector $\bar{e} \in \mathbb{Z}^m$. While the impact of the chosen matrix is well understood, the impact of the residuation vector on the resulting structure was seemingly sporadic.

In this paper, we characterize the impact of the residuation vector $\bar{e}$ by recognizing an initial structure when $\bar{e}$ is the first standard basis vector. All other choices of $\bar{e}$ extend this initial structure by adding "fractional elements" in a way we make precise.

Keywords: Abelian Automata and Group Theory and Transducer and Module Theory

1. Background

Finite State Automata are combinatorial objects which encode relations between words over some alphabet. Automata provide deep connections between combinatorics, algebra, and logic, and are essential tools in contemporary computer science. One such link is in the decidability of truth in a structure whose relations are all computable by automata. One can combine these automata into more complicated automata representing logical sentences in such a way that a sentence is true if and only if a simple reachability condition holds [1]. This gives a simple proof that the theory of $\mathbb{N}$ with $+$ and $<$, for example, is decidable.

Different kinds of automata encode different kinds of information, and in this article we will be interested in **Mealy Automata** which encode functions from a set of words to itself. Indeed, the functions we consider will all be invertible (and the inverses are computable by automata as well), and thus they will generate **Automata Groups**. These groups are surprisingly complicated, and a classification of all groups generated by three state automata over the alphabet $\mathbf{2} = \{0, 1\}$ is an extremely difficult problem, though much impressive progress has been made [3]. This complexity can be useful, as automata groups have become a rich source of examples and counterexamples in group theory [12,16,5]. Most notably, automata groups provide examples of finitely generated infinite torsion groups, with application to Burnside's Problem [4,8], and automata groups have provided important examples of groups of intermediate growth, providing counterexamples to Milnor's Conjecture regarding the existence of such groups [7]. In fact, one of the simplest conceivable automata (shown below) already generates the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$, as is shown in [6].
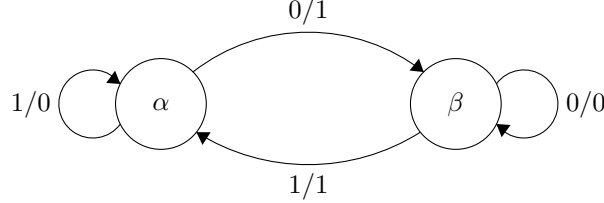
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Fig. 1: An automaton generating the lamplighter group
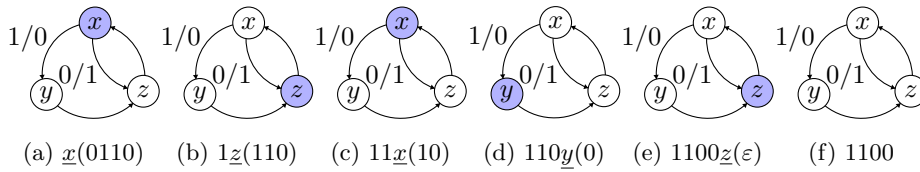
1.1. *Some Important Definitions*

Denote $\mathbf{2} = \{0, 1\}$. For our purposes, a **Mealy Automaton** is a tuple $\mathcal{A} = (S, \tau)$ where S is the **State Set**, and $\tau : S \times \mathbf{2} \to S \times \mathbf{2}$ is the **Transition Function**. We represent $\mathcal{A}$ as a (directed, multi-)graph with S as vertices and an edge from s_1 to s_2 (labeled by a/b) exactly when $\tau(s_1, a) = (s_2, b)$. Following Sutner [19], whenever we have two parallel edges labeled $0/0$ and $1/1$, we instead write one unlabeled edge to remove clutter. We write $(\partial_a s, \underline{s}(a)) = \tau(s, a)$ and call $\partial_0 s$ (resp. $\partial_1 s$) the **0-residual** (resp. **1-residual**) of s [a].

We can extend $\underline{s}$ to a length preserving function on the free monoid $\mathbf{2}^*$ as follows (here juxtaposition is concatenation, and the empty word ε is the identity):

$\underline{s} : \mathbf{2}^* \to \mathbf{2}^*$

$$\underline{s}(\varepsilon) = \varepsilon$$
$$\underline{s}(ax) = a' \underline{s'}(x) \quad (\text{where } (s', a') = \tau(s, a))$$

(a) $\underline{x}(0110)$ (b) $1\underline{z}(110)$ (c) $11\underline{x}(10)$ (d) $110\underline{y}(0)$ (e) $1100\underline{z}(\varepsilon)$ (f) 1100

Fig. 2: An example computation – $\underline{x}(0110) = 1100$.

Clearly we can treat $\underline{s}$ as a function on $\mathbf{2}^\omega$, the set of infinite words, instead. In this case, automata provide a computable way of encoding complicated continuous functions from cantor space to itself, with ties to descriptive set theory [17]. If each function $\underline{s} : \mathbf{2}^\omega \to \mathbf{2}^\omega$ for some automaton $\mathcal{A}$, we let $\mathcal{G}(\mathcal{A})$ denote the group generated by these functions. We write our group additively, and denote the identity

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[a] Residuals are also called **sections**, **restrictions**, and **states** in the literature

by I[b].

We can extend the definition of residuals to the whole group $\mathcal{G}(\mathcal{A})$ by defining the **0-residual** (resp. **1-residual**) of a function $f \in \mathcal{G}(\mathcal{A})$ as the unique function $\partial_0 f$ such that for all w, $f(0w) = f(0)\partial_0 f(w)$ (resp. $f(1w) = f(1)\partial_1 f(w)$). For a state $s \in S$, it is clear that $\partial_a \underline{s} = \underline{s'}$, where $(s', a') = \tau(s, a)$, so that this extends the old definition.

Thus $\mathcal{G}(\mathcal{A})$ can also be viewed as an automaton, by taking $\mathcal{G}(\mathcal{A})$ as the state set and defining $\tau(f, i) = (\partial_i f, f(i))$. Provided $\mathcal{A}$ is minimized[c], we find $\mathcal{A}$ as a natural subautomaton of $\mathcal{G}(\mathcal{A})$ by identifying $s \in \mathcal{A}$ with $\underline{s} \in \mathcal{G}(\mathcal{A})$.

We will call a function **Odd** if it flips its first bit, and **Even** otherwise, and we call an automaton **Abelian** or **Trivial** exactly when its group is. For instance, in figure 1, $\underline{\alpha}$ is odd, $\underline{\beta}$ is even, $\partial_0 \underline{\alpha} = \underline{\beta}$, and $\partial_1 \underline{\alpha} = \underline{\alpha}$. For a more in depth description of Mealy Automata and their properties, see [15,9,18]

The results of this paper stand on the shoulders of a result of Nekrashevych and Sidki that every abelian automata group is either torsion free abelian or boolean [13]. Because of this classification, much of the interesting structure of these groups comes from their residuation functions. To that end, for the duration of this paper, homomorphisms and isomorphisms are all restricted to those which preserve the residuation structure in addition to the group structure. Explicitly, we will require a homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$ to satisfy $\varphi(\partial_i f) = \partial_i \varphi f$.

It is a theorem by Sutner [19] that $\mathcal{G}(\mathcal{A})$ is abelian iff for even states $\partial_1 f - \partial_0 f = I$ and for odd states $\partial_1 f - \partial_0 f = \gamma$, where γ is independent of f. Moreover, the case $\gamma = I$ corresponds precisely to the case where $\mathcal{G}(\mathcal{A})$ is boolean. We now restrict ourselves further to the case where $\mathcal{G}(\mathcal{A})$ is free abelian, that is to say $\mathcal{G}(\mathcal{A}) \cong \mathbb{Z}^m$ for some m, and $\gamma \neq I$.[d]

1.2. *The Complete Automaton*

From the discussion above, it follows that $\mathbb{Z}^m \cong \mathcal{G}(\mathcal{A})$ carries a residuation structure, and Nekrashevych and Sidki also give a characterization of all possible such structures [13].

Without loss of generality, we can take the odd (resp. even) states to be exactly the vectors with odd (resp. even) first component. The automata structure is given by the following affine maps (which depend on a matrix $\mathbf{A}$ and an odd vector $\bar{e}$):

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[b]In general these groups need not be abelian, however we only consider abelian groups in this paper. For that reason we will use additive notation throughout.
[c]we will only consider minimized machines in this paper
[d]For historical reasons we use $\mathbb{Z}^m$ instead of $\mathbb{Z}^n$ because traditionally n is reserved for the size of the state set of an automaton.

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$$\tau_{\mathbf{A},\bar{e}}(\bar{v},0) = \begin{cases} (\mathbf{A}\bar{v},0) & \bar{v} \text{ even} \\ (\mathbf{A}(\bar{v}-\bar{e}),1) & \bar{v} \text{ odd} \end{cases} \tag{1.1}$$

$$\tau_{\mathbf{A},\bar{e}}(\bar{v},1) = \begin{cases} (\mathbf{A}\bar{v},1) & \bar{v} \text{ even} \\ (\mathbf{A}(\bar{v}+\bar{e}),0) & \bar{v} \text{ odd} \end{cases} \tag{1.2}$$

In the above definition, $\mathbf{A}$ is a "$\frac{1}{2}$–integral" matrix $\mathbf{A}$ of $\mathbb{Q}$-irreducible character. This group (with its residuation structure) is generated by a *finite* automaton exactly when $\mathbf{A}$ is a contraction (that is, all of its complex eigenvalues have norm < 1). By a $\frac{1}{2}$–integral matrix, we mean a matrix of the form

$$\begin{pmatrix} \frac{a_{11}}{2} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{2} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

where each $a_{ij} \in \mathbb{Z}$.

These matrices all have characteristic polynomial $\chi = x^n + \frac{1}{2}g(x)$, where $g \in \mathbb{Z}[x]$ and has constant term ± 1. Without loss of generality we may take $\mathbf{A}$ to be in rational canonical form, with the coefficients of χ in the first column.

Now we see the reason for the **Residuation Vector** $\bar{e}$ in the above definition. Since $\mathbf{A} : 2\mathbb{Z} \oplus \mathbb{Z}^{m-1} \to \mathbb{Z}^m$, the transition function τ can act as simple multiplication on even vectors. However, to ensure we have an integral output, we must first make an odd vector even by adding or subtracting another odd vector $\bar{e}$. It is easy to see that this definition gives rise to the following residuation structure:

If $\bar{v}$ is even:

$$\partial_0 \bar{v} = \partial_1 \bar{v} = \mathbf{A}\bar{v}$$

If $\bar{v}$ is odd:

$$\partial_0 \bar{v} = \mathbf{A}(\bar{v} - \bar{e})$$

$$\partial_1 \bar{v} = \mathbf{A}(\bar{v} + \bar{e})$$

Following Sutner [19], for a specific matrix $\mathbf{A}$ and residuation vector $\bar{e}$ we define the **Complete Automaton** $\mathfrak{C}(\mathbf{A},\bar{e}) = (\mathbb{Z}^m, \tau_{\mathbf{A},\bar{e}})$. If $\mathcal{G}(\mathcal{A}) \cong \mathfrak{C}(\mathbf{A},\bar{e})$, we say a function $f \in \mathcal{G}(\mathcal{A})$ is **Located at** $\bar{v} \in \mathfrak{C}(\mathbf{A},\bar{e})$ iff the isomorphism between $\mathcal{G}(\mathcal{A})$ and $\mathfrak{C}(\mathbf{A},\bar{e})$ sends f to $\bar{v}$. Finally, given any state $\bar{v} \in \mathfrak{C}(\mathbf{A},\bar{e})$, closing $\{\bar{v}\}$ under residuation will result in an automaton $\mathcal{A}_{\bar{v}}$ (which will be finite whenever $\mathbf{A}$ is contracting). We say $\mathcal{A}$ is **Located at** $\bar{v} \in \mathfrak{C}(\mathbf{A},\bar{e})$ iff the isomorphism sends $\mathcal{A} \subseteq \mathcal{G}(\mathcal{A})$ to $\mathcal{A}_{\bar{v}} \subseteq \mathfrak{C}(\mathbf{A},\bar{e})$.

Keep in mind the distinction between the group of functions $\mathcal{G}(\mathcal{A})$ and a particular isomorphism between $\mathcal{G}(\mathcal{A})$ and $\mathfrak{C}(\mathbf{A},\bar{e})$. We will freely identify these objects, but the location of a particular function depends heavily on the choice of $\bar{e}$.

Nekrashevych and Sidki's theorem gives us a purely linear algebraic method for discussing these automata groups, since a restatement of their theorem says that

every torsion free abelian automata group $\mathcal{G}(\mathcal{A})$ is isomorphic to[e] $\mathfrak{C}(\mathbf{A}, \bar{e})$ for some $\mathbf{A}$ and $\bar{e}$.

Seeing this fact, it is natural to ask if, given an abelian automaton $\mathcal{A}$, we can characterize all $\mathbf{A}$ and $\bar{e}$ for which $\mathcal{A} \subseteq \mathfrak{C}(\mathbf{A}, \bar{e})$. Indeed, it is natural to ask which vector $\bar{v}$ will $\mathcal{A}$ be located at in this identification.

Nekrashevych and Sidki show that each $\mathcal{A}$ has a unique matrix $\mathbf{A}$ (up to $\mathrm{GL}(\mathbb{Q})$ similarity) which works, though their proof is nonconstructive. We call this $\mathbf{A}$ (in rational canonical form) the **Associated Matrix** of $\mathcal{A}$. Unfortunately, Nekrashevych and Sidki leave entirely open the question of which $\bar{e}$ admit $\mathcal{A}$ as a subautomaton once we have the correct $\mathbf{A}$, and moreover where $\mathcal{A}$ is located if an embedding into $\mathfrak{C}(\mathbf{A}, \bar{e})$ exists. Algorithms for determining the matrix from the automaton are given by Okano [14] and Becker [2], solving part of the problem.

In this paper we finish the job by fully characterizing the impact of $\bar{e}$ on the residuation structure of $\mathfrak{C}(\mathbf{A}, \bar{e})$. For a more detailed discussion of these linear algebraic methods and their origins, see [12,13].

1.3. *Principal Automata*

Each abelian automaton gets a unique associated matrix as above, but each matrix can be associated to infinitely many automata. It was shown by Okano [14] that there is a distinguished automaton, now called the **Principal Automaton** $\mathfrak{A}$, associated to each matrix. $\mathfrak{A}(\mathbf{A})$ is defined to be[f] $\mathfrak{A} = \mathcal{A}_{\bar{e}_1} \cup \mathcal{A}_{-\bar{e}_1} \subseteq \mathfrak{C}(\mathbf{A}, \bar{e}_1)$, though there is a longstanding conjecture (introduced in the same paper) that in most cases this is the same machine as $\mathcal{A}_{\bar{e}_1} \subseteq \mathfrak{C}(\mathbf{A}, \bar{e}_1)$. We will simply write $\mathfrak{A}$ whenever $\mathbf{A}$ is clear from context. The function located at $\bar{e}_1 \in \mathfrak{C}(\mathbf{A}, \bar{e}_1)$ will be important later on, and so we write δ for this function. Notice this means $\mathfrak{A}$ is the smallest automaton containing δ and $-\delta$.

As we will see (TODO: Proposition ??), δ is located at $\bar{e} \in \mathfrak{C}(\mathbf{A}, \bar{e})$ for all $\bar{e}$. This will give us a way to compare functions in various $\mathfrak{C}(\mathbf{A}, \bar{e})$ by using δ as a kind of meterstick. Indeed, since $\mathfrak{A}$ is generated by $\pm\delta$, the next theorem will show that for every automaton $\mathcal{A}$ with associated matrix $\mathbf{A}$, $\mathcal{G}(\mathfrak{A}(\mathbf{A})) \leq \mathcal{G}(\mathcal{A})$. Thus every function in $\mathfrak{A}$ is a $\mathbb{Z}$-linear combination of functions in $\mathcal{A}$. In particular, we see $\mathfrak{A}$ is a subautomaton of $\mathcal{G}(\mathcal{A})$ for every $\mathcal{A}$ with matrix $\mathbf{A}$. While there are proofs of this claim which rely heavily on the ambient linear algebraic structure [14], we present here a construction which uses only the given automaton $\mathcal{A}$ to construct $\mathfrak{A}$. Thus every $\underline{s} \in \mathcal{G}(\mathfrak{A})$ is already in $\mathcal{G}(\mathcal{A})$, and the subgroup relation follows.

Theorem 1. *For each nontrivial abelian $\mathcal{A}$ with associated matrix $\boldsymbol{A}$, $\mathcal{G}(\mathfrak{A}(\boldsymbol{A})) \leq \mathcal{G}(\mathcal{A})$.*

Proof. It was shown in [19] that γ depends only on the matrix $\mathbf{A}$, so that for any

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[e]Recall our isomorphisms preserve the residuation structure in addition to the group structure
[f]here, as usual, $\bar{e}_1$ represents the first standard basis vector

automata $\mathcal{A}$ and $\mathcal{A}'$ with the same associated matrix $\mathbf{A}$, and for any odd states $f \in \mathcal{A}$, $f' \in \mathcal{A}'$, we have $\gamma = \partial_1 f - \partial_0 f = \partial_1 f' - \partial_0 f'$. In particular, for $\delta \in \mathfrak{A}$, we have $\gamma = \partial_1 \delta - \partial_0 \delta = \partial_1 \delta$ since $\partial_0 \delta = \mathbf{A}(\bar{e}_1 - \bar{e}_1) = \bar{0} = I$.

Since we know from the previous discussion that $\mathfrak{A}$ is generated by $\pm \delta$, we can build it by hand by leveraging the fact that $\gamma = \partial_1 \delta$ is already in $\mathcal{G}(\mathcal{A})$.

Let $\mathcal{A}$ be an abelian automaton with at least one odd state. Note that if $\mathcal{A}$ has no odd states, its group is trivial, so we may safely ignore it. Put $\gamma = \partial_1 o - \partial_0 o$ for $o \in \mathcal{A}$ odd, and construct a new automaton by closing γ under residuation. Note that this can be done using only information contained in $\mathcal{A}$, since it is easy to check that:

$$\partial_0(f + g) = \begin{cases} \partial_0 f + \partial_1 g & \text{both odd} \\ \partial_0 f + \partial_0 g & \text{otherwise} \end{cases}$$

$$\partial_1(f + g) = \begin{cases} \partial_1 f + \partial_0 g & \text{both odd} \\ \partial_1 f + \partial_1 g & \text{otherwise} \end{cases}$$

$$\partial_0(-f) = -\partial_1 f$$
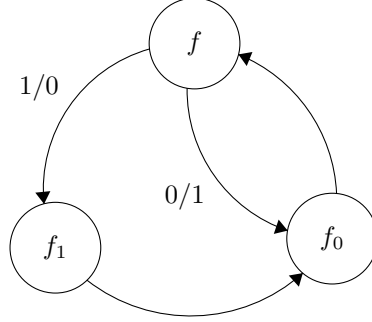
$$\partial_1(-f) = -\partial_0 f$$

Thus using the characterization by Sutner [19] that a state is odd iff it has distinct residuals, we can close γ under residuation using only information in $\mathcal{A}$. Since $\gamma \in \mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A})$ is residuation closed, this entire closure is a subset of $\mathcal{G}(\mathcal{A})$. Moreover, whenever $\mathcal{A}$ is finite, $\mathbf{A}$ is contracting and so $\mathfrak{A}$ is finite too. Thus this procedure can actually be carried out.

The above procedure has given us the smallest machine containing γ (call it $\mathfrak{B}$) but $\mathfrak{A}$ is the smallest machine containing $\pm \delta$. Thankfully, we know the desired behavior of δ, so we can manually fix this if necessary. We first add a new state with a self loop for I, and a new state δ with $\partial_0 \delta = I$ and $\partial_1 \delta = \gamma$. Call this new machine $\mathfrak{A}^+$. We can then take the inverse machine $\mathfrak{A}^-$, and we find $\mathfrak{A} = \mathfrak{A}^+ \cup \mathfrak{A}^-$ (minimizing if necessary).

Lemma 3 in [19] says that $\mathcal{G}(\mathcal{A}_{\bar{v}}) = \mathcal{G}(\mathcal{A}_{\overline{w}})$ whenever $\overline{w}$ transitions into $\bar{v}$. Because of this, $\mathcal{G}(\mathfrak{A}^+) = \mathcal{G}(\mathfrak{B})$, and since $\mathfrak{A}^-$ contains exactly the negations of elements of $\mathfrak{A}^+$ (all of which are sums of elements in $\mathcal{G}(\mathcal{A})$), we find $\mathcal{G}(\mathfrak{A}) = \mathcal{G}(\mathfrak{A}^+) \leq \mathcal{G}(\mathcal{A})$. $\qquad \square$
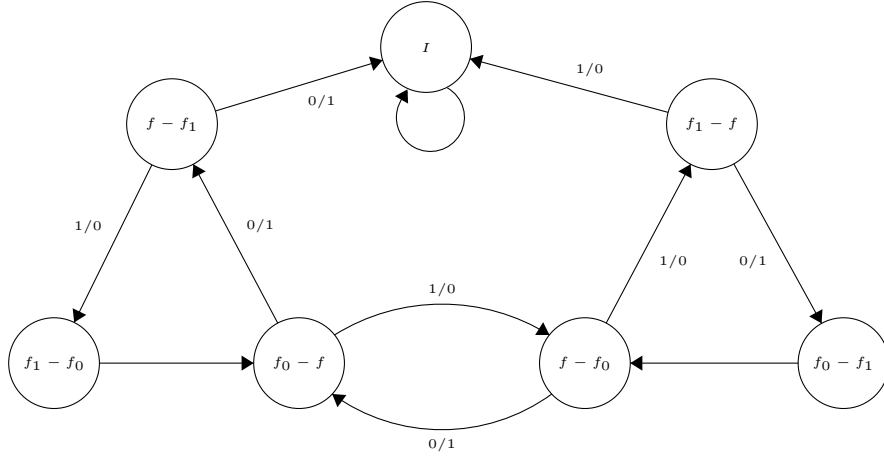
1.4. An Example

Consider the following machine, $\mathcal{A}_2^3$:

As before, the unlabeled transitions both copy the input bit, however these have been omitted for cleanliness.

Then by letting $\gamma = \partial_1 f - \partial_0 f = f_1 - f_0$, and closing under residuation using the above algorithm, we construct the following machine (γ is shown at the bottom left):

It is easy to check that this is the principal machine for $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$, where $f - f_1 = \delta$ is located at $\bar{e}_1 \in \mathfrak{C}(\mathbf{A}, \bar{e}_1)$. Moreover, one can check that $f \in \mathcal{A}_2^3$ as above is located at $\bar{e}_1 \in \mathfrak{C}\left(\mathbf{A}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right)$. Note that $\delta = f - f_1$ is indeed located at

$$\bar{e}_1 - \partial_1 \bar{e}_1 = \bar{e}_1 - \mathbf{A}\left(\bar{e}_1 + \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = \bar{e}_1 - \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \in \mathfrak{C}\left(\mathbf{A}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right).$$

When running the algorithm in this case, we do not need to separately add $\pm\delta$ or the inverse machine. Here δ is already in the closure of γ under residuation, and the machine is already closed under negation. The Strongly Connected Component Conjecture predicts that this will be the case whenever $\mathbf{A}$ has characteristic polynomial other than $x^m - \frac{1}{2}$, which corresponds to the so called sausage automata.

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Unfortunately, this conjecture is yet unproven, and so in the above proof we had to explicitly add in these extra states.

2. Fractional Extensions

Going forward, $\mathcal{G} = \mathfrak{C}(\mathbf{A}, \bar{e}_1)$ will denote $\mathcal{G}(\mathfrak{A})$ for some principal machine $\mathfrak{A}$.

Since $\mathbf{A}$ sends $2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$ to $\mathbb{Z}^m$, $\mathbf{A}^{-1}$ sends $\mathbb{Z}^m$ to $2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$, and so has only integer entries. Thus we can give $\mathcal{G}$ the structure of a $\mathbb{Z}[x]$-module where $x \cdot \bar{v} = \mathbf{A}^{-1}\bar{v}$, extended linearly. Further, since $\mathbf{A}$ has irreducible characteristic polynomial so does $\mathbf{A}^{-1}$. Thus this module is cyclic, and is generated by $\bar{e}_1 = \delta$. The cyclicity of this module tells us that we can identify the set of states $\mathbb{Z}^m$ with $\mathbb{Z}[x]/\chi^*$ where χ^* is the characteristic polynomial of $\mathbf{A}^{-1}$ (of degree m). This identifies a vector $\bar{v}$ with the polynomial $p_{\bar{v}}$ whose coefficients are the coordinates of $\bar{v}$ (the constant term is the first coefficient). Said another way, $\bar{v} = p_{\bar{v}} \cdot \bar{e}_1$.

Now for $p \in \mathbb{Z}[x]$ with odd constant term, we write $p^{-1} \cdot \mathcal{G}$ in place of $\mathcal{G}(\mathfrak{C}(\mathbf{A}, p \cdot \bar{e}_1))$. That is to say, $p^{-1} \cdot \mathcal{G}$ has as its states $\mathbb{Z}^m$ and as its odd residuations $\partial_0 \bar{v} = \mathbf{A}(\bar{v} - p \cdot \bar{e}_1)$, and $\partial_1 \bar{v} = \mathbf{A}(\bar{v} + p \cdot \bar{e}_1)$. We will only discuss polynomials p with an odd constant term, as this ensures $p \cdot \bar{e}_1$, our residuation vector, is odd.

We call $p^{-1} \cdot \mathcal{G}$ the **Fractional Extension** of $\mathcal{G}$ by p. To first justify the use of the word "extension", notice $\mathcal{G} \hookrightarrow p^{-1} \cdot \mathcal{G}$ for all p by the homomorphism $\bar{v} \mapsto p \cdot \bar{v}$. Further, if p is not a unit in $\mathbb{Z}[x]/\chi^*$, this homomorphism is *not* surjective. That is to say $\mathcal{G}$ is a proper subgroup of $p^{-1} \cdot \mathcal{G}$. This observation is true in more generality, as shown below. Recall we work in the category whose arrows also preserve the residuation structure, and thus an isomorphism identifying two vectors will show that those vectors compute the same function on $\mathbf{2}^\omega$. However, it means we must show that our embeddings genuinely do preserve this structure.

Theorem 2. *If $rp = q$ in $\mathbb{Z}[x]/\chi^*$, then $p^{-1} \cdot \mathcal{G} \hookrightarrow q^{-1} \cdot \mathcal{G}$, with a canonical injection $\varphi_r : \bar{v} \mapsto r \cdot \bar{v}$. In particular, if r is a unit, then $p^{-1} \cdot \mathcal{G} \cong q^{-1} \cdot \mathcal{G}$.*

Proof. It is clear that φ_r preserves the $\mathbb{Z}[x]$-module structure, so it remains to show that it preserves the residuation structure.

Let $rp = q$, $f \in p^{-1} \cdot \mathcal{G}$ located at $\bar{v}$. Consider $f' \in q^{-1} \cdot \mathcal{G}$ located at $r \cdot \bar{v}$. First note f and f' have the same parity, since r has odd constant term, and so $\bar{v}$ and $r \cdot \bar{v}$ have the same parity. Now, consider the residuals of f and f'.

If f is even, then

$$\partial_0 f' = \mathbf{A}(r \cdot \bar{v}) = r \cdot \mathbf{A}\bar{v} = r \cdot \partial_0 f$$

If f is odd, then

$$\partial_0 f' = \mathbf{A}(r \cdot \bar{v} - q \cdot \bar{e}_1) = r \cdot \mathbf{A}(\bar{v} - p \cdot \bar{e}_1) = r \cdot \partial_0 f$$

A similar argument shows $\partial_1 f' = r \cdot \partial_1 f$

If r is a unit, then r^{-1} also has odd constant term (since $rr^{-1} = 1$ has odd constant term) and so φ_r is an isomorphism with inverse $\varphi_{r^{-1}}$. $\qquad\square$

The previous proof has justified the use of the word "extension", but it is still not clear why this extension should be "fractional". As the previous proof shows, $p \cdot \bar{v} \in p^{-1} \cdot \mathcal{G}$, computes exactly the same function as $\bar{v} \in \mathcal{G}$. However, most vectors cannot be written as a multiple of p. What do they do as functions? We call such vectors (and their corresponding functions) **Fractional**, due to the following analogy:

Say we are only allowed to compute with $\mathbb{Z}$, but we want the ability to work with fractions. We can approximate $\mathbb{Q}$ by allowing fractions with fixed denominator. If we write $\frac{1}{n}\mathbb{Z} = \{\frac{k}{n} \mid k \in \mathbb{Z}\}$, then we clearly see $\mathbb{Z} \cong \frac{1}{n}\mathbb{Z} \leq \mathbb{Q}$ for every n. So if we want to be able to talk about fractions like $\frac{1}{3}$, we might work in the "extension" $\frac{1}{3}\mathbb{Z}$, where $\mathbb{Z} \hookrightarrow \frac{1}{3}\mathbb{Z}$ by the embedding $k \mapsto 3k$. The new elements, then, are "fractional" in the obvious sense. Once we have made this identification, we can (computationally) forget about the fact that we're working in an extension at all. The equation $4 + 6 = 10$ remains true, we simply reinterpret this as being $\frac{4}{3} + \frac{6}{3} = \frac{10}{3}$.

Scaling $\mathcal{G} \cong \mathbb{Z}^m$ by some polynomial p^{-1} is *exactly* the same operation. However, in this new higher dimensional setting, we have more degrees of freedom, and must therefore index by polynomials p instead of mere constants. Note this explains all of the ambiguity regarding the location of a function f in an extension $\mathfrak{C}(\mathbf{A}, \bar{e})$. Since $\bar{e}$ corresponds to our choice of p^{-1}, the change in location of f is entirely analogous to the change in position of $\frac{1}{2} \in \mathbb{Q}$ in various $\frac{1}{n}\mathbb{Z}$. In $n = 10$, $\frac{1}{2}$ shows up at 5. In $n = 6$, $\frac{1}{2}$ shows up at 3. But $\frac{1}{2}$ doesn't appear when $n = 3$... The seemingly sporadic $\bar{e}$ in which a function f can be found is explained by exactly the same phenomenon!

Of course, there is a minimal n for which $\frac{1}{2} \in \frac{1}{n}\mathbb{Z}$. Moreover, once we know it, we can characterize exactly where $\frac{1}{2}$ will be: in all extensions $\frac{1}{m}\mathbb{Z}$ where $n \mid m$. In fact, we can do the same thing for automata.

3. Characterizing Automata

Since each (minimal, abelian) automaton $\mathcal{A}$ is a subautomaton of some $\mathfrak{C}(\mathbf{A}, \bar{e})$, equivalently some $p^{-1} \cdot \mathcal{G}$, there should be a minimal $\bar{e}$ (up to multiplication by units) which still has $\mathcal{A}$ as a subautomaton.

Notice that if we locate $\mathcal{A}$ at $\bar{e}_1 \in p^{-1} \cdot \mathcal{G}$, then there can be no smaller polynomial q (in the division ordering) which also places $\mathcal{A}$ at an integral position. The following theorem shows this is always possible[g].

Theorem 3. *Every strongly connected, minimal, nontrivial abelian automaton $\mathcal{A}$ can be located at $\bar{e}_1$ in $p^{-1} \cdot \mathcal{G}$ for some p.*

Proof. Let f be an odd state in $\mathcal{A}$. Then at least one of $\partial_0 f$ and $\partial_1 f$ is not equal to f. So there is some nontrivial cycle from f to itself, which we can represent

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[g]In [19], it is shown that every abelian automaton has a terminal strongly connected component that contains all the interesting data of the original automaton. Since we only consider minimal machines, this theorem classifies all abelian automata of interest

by a matrix equation relating $\bar{v}_f$, and $\bar{e}$. (Here $\bar{v}_f$ is where f will be located, and $\bar{e}$ will be the residuation vector). We can then rearrange this equation to obtain $p_1(\mathbf{A})\bar{v}_f = p_2(\mathbf{A})\bar{e}$.

Now $p_1, p_2 \in \mathbb{Z}[x]$, and $\mathbf{A}$ has irreducible character over $\mathbb{Z}$. Then the eigenvalues of $p(\mathbf{A})$ are precisely $p(\lambda)$ where λ is an eigenvalue of $\mathbf{A}$, so $\mathbf{A}$'s invertibility implies the invertibility of both $p_1(\mathbf{A})$ and $p_2(\mathbf{A})$. Thus

$$\bar{e} = p_2(\mathbf{A})^{-1} p_1(\mathbf{A}) \bar{v}_f$$

Choosing $\bar{v}_f = \bar{e}_1$ gives a value for the residuation vector $\bar{e}$, and (since $\mathcal{G}$ is cyclic as a $\mathbb{Z}[x]$-module) a value $\bar{e}$ induces a polynomial $p_{\bar{e}}$ such that $p_{\bar{e}} \cdot \bar{e}_1 = \bar{e}$. Then, by construction, $\mathcal{A}$ is a subautomaton of $p_{\bar{e}}^{-1} \cdot \mathcal{G}$, and is anchored with f at $\bar{e}_1$. As desired. □

Corollary. *The function δ is always located at $\bar{e}$ inside $\mathfrak{C}(\mathbf{A}, \bar{e})$.*

For any automaton $\mathcal{A}$, we can now completely characterize in which $\mathfrak{C}(\mathbf{A}, \bar{e})$ it can be located, and at what vectors. First locate $\mathcal{A}$ at $\bar{e}_1 \in p^{-1} \cdot \mathcal{G}$, and then to locate it at any odd vector $\bar{v}$, scale both sides by $p_{\bar{v}}$ to see $\mathcal{A}$ located at $\bar{v} \in (p_{\bar{v}}p)^{-1} \cdot \mathcal{G}$. In the above proof, the choice of $\bar{v}_f = \bar{e}_1$ was arbitrary, and we can directly locate $\mathcal{A}$ at a different odd vector $\bar{v}'$ by setting $\bar{v}_f = \bar{v}'$. This will give the same result as locating it at $\bar{e}_1$ and then multiplying by $p_{\bar{v}'}$, again, by cyclicity. The same observation shows that, given some polynomial q (equivalently some vector $q \cdot \bar{e}_1$) $\mathcal{A}$ is located somewhere in $q^{-1} \cdot \mathcal{G} = \mathfrak{C}(\mathbf{A}, q \cdot \bar{e}_1)$ if and only if $p \mid q$. Further, it will be located at exactly $p^{-1}q \cdot \bar{e}_1$.

3.1. *An Example*

Recall the abelian automaton $\mathcal{A}_2^3$ from earlier in the paper:
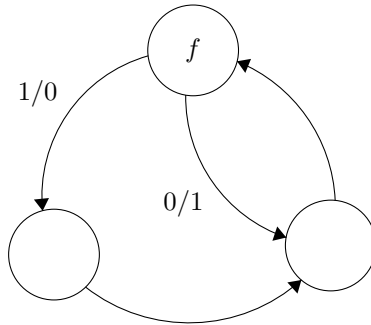
Say we want to find $\bar{v}$ and $\bar{e}$ such that $\mathcal{A}_2^3$ is located at $\bar{v} \in \mathfrak{C}(\mathbf{A}, \bar{e})$.

Using the algorithm described by Becker [2], we find $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$.

Then notice $\partial_0 \partial_0 f = f$. So $\mathbf{A}^2(\bar{v}_f - \bar{e}) = \bar{v}_f$, and $\mathbf{A}^2 \bar{v}_f - \bar{v}_f = \mathbf{A}^2 \bar{e}$. Thus

$$\bar{e} = \mathbf{A}^{-2}(\mathbf{A}^2 - I)\bar{v}_f$$

Choosing $\bar{v}_f = \bar{e}_1$ gives $\bar{e} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Then $f = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in (3 + 2x)^{-1} \cdot \mathcal{G}$

3.2. *Limiting Object*

Just as we can recover $\mathbb{Q}$ as a limit of the fractional groups $\frac{1}{n}\mathbb{Z}$, we can define a group $\widetilde{\mathcal{G}}$ as the limit of our $p^{-1} \cdot \mathcal{G}$. Indeed, just as $\mathbb{Q}$ eliminates the parameter n in $\frac{1}{n}\mathbb{Z}$, $\widetilde{\mathcal{G}}$ contains every automaton $\mathcal{A}$ at exactly one position, while removing the need for the parameter p (and thus, the parameter $\bar{e}$ in $\mathfrak{C}(\mathbf{A}, \bar{e})$). Moreover, it is still effective to work with $\widetilde{\mathcal{G}}$, so we do not lose any of the computability benefits of working with automata groups.

Formally, we define $\widetilde{\mathcal{G}}$ to be $\varinjlim p^{-1} \cdot \mathcal{G}$ where the colimit (in the category of abelian groups) is taken over the poset of polynomials (with odd constant term) under the divisibility ordering. This colimit can then be given residuation structure in a unique way which is compatible with the residuation structure on all the $p^{-1} \cdot \mathcal{G}$.

Explicitly, we look at the set

$$\widetilde{\mathcal{G}} = \left\{ \frac{\bar{v}}{p} \;\middle|\; \bar{v} \in \mathbb{Z}^m, p \in \mathbb{Z}[x], p_0 \text{ odd} \right\} / \sim$$

where we quotient by $\frac{\bar{v}}{p} \sim \frac{q \cdot \bar{v}}{pq}$ for every q with odd coefficient. We first endow this with a group structure by declaring $[\frac{\bar{v}}{p}] + [\frac{\overline{w}}{q}] = [\frac{q \cdot \bar{v} + p \cdot \overline{w}}{pq}]$. Finally, we give $\widetilde{\mathcal{G}}$ residuation structure too, by declaring $\partial_0[\frac{\bar{v}}{p}] = [\frac{\partial_0 \bar{v}}{p}]$ where the residuation on the right hand side takes place in $p^{-1} \cdot \mathcal{G}$. We define $\partial_1[\frac{\bar{v}}{p}]$ analogously, and these operations are quickly seen to be well defined. This structure remains computable, because any finite computation we wish to do will have a least common denominator, and we can simply work in a good enough approximation.

4. Conclusion

We have shown that the residuation vector $\bar{e}$ corresponds to how fine an approximation of $\widetilde{\mathcal{G}}$ one wants. This is because each $\mathfrak{C}(\mathbf{A}, \bar{e})$ corresponds to $p_{\bar{e}}^{-1} \cdot \mathcal{G}$, with progressively larger $\bar{e}$ corresponding to progressively more complicated fractional elements, which approximate $\widetilde{\mathcal{G}}$. This allows us to characterize which automata show up in which $\mathfrak{C}(\mathbf{A}, \bar{e})$ (and, moreover, where they show up) by finding a minimal (in the division ordering) $p_{\bar{e}}$ in which a given automaton is found.

Further, the existence of the universal object $\widetilde{\mathcal{G}}$ sheds new light on the connection between affine tiles [10,11] and abelian automata noted by Sutner [19]. Indeed it is easy to see that in $\widetilde{\mathcal{G}}$ every strongly connected component (and thus every

subautomaton of interest) has each vector in the attractor of the iterated function system given by the residuation functions $\{\bar{v} \mapsto \mathbf{A}\bar{v}, \bar{v} \mapsto \mathbf{A}(\bar{v} \pm \bar{e}_1)\}$. Thus, in particular, the size of the principal machine is bounded by the number of integral points in this attractor. Even in $\mathbb{Z}^2$, however, there are examples where this bound is not tight.

The relation between automata and polynomials discussed in this paper also provides a new take on a proof technique for the longstanding Strongly Connected Component Conjecture. This conjecture asserts that principal machines $\mathfrak{A}$ have only one strongly connected component (plus the self looping identity state) whenever their matrix has a characteristic polynomial that is *not* of the form $x^n + \frac{1}{2}$. The new way of looking at residuation vectors allows us to rewrite the residual functions as $\partial_i \bar{v} = \mathbf{A}(\bar{v} - (-1)^i \delta)$ for $\bar{v}$ odd. It is easy to see, then, that the following polynomials correspond to paths ending in δ, since they undo residuation:

$$P_\epsilon(x) = 1$$
$$P_{w0}(x) = xP_w(x) + 0$$
$$P_{w1}(x) = xP_w(x) + 1$$
$$P_{w\bar{1}}(x) = xP_w(x) - 1$$

Sutner made a similar observation, and described Path Polynomials [19] which allow us to reason about the existence of directed paths between states in an automaton by purely algebraic means. However, the traditional path polynomials are clunky and not always defined, since they correspond to paths *starting* at δ, and so $P'_{w0} \cdot \delta$ is only well defined if $P'_w \cdot \delta$ is even (and P'_{w1} and $P'_{w\bar{1}}$ are only well defined if $P_w \cdot \delta$ is odd). Since the polynomials defined above move *backwards* along transitions instead of forwards, they are always well defined.

The existence of a path polynomial p which is congruent to $-1 \bmod \chi^*$ then shows the existence of a path from $-\delta$ to δ. Then to prove the SCC conjecture, it suffices to prove that whenever $\mathbf{A}$ does not have characteristic $x^n + \frac{1}{2}$ there is a polynomial $p \in \{-1, 0, 1\}[x]$ which is congruent to $-1 \bmod \chi^*$.

Acknowledgements

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