13

Automorphisms of the binary tree: state-closed subgroups and dynamics of 1/2-endomorphisms

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1. Introduction

Automorphisms of regular 1-rooted trees of finite valency have been the subject of vigorous investigations in recent years as a source of remarkable groups which reflect the recursiveness of these trees (see [S1], [G2]). It is not surprising that the recursiveness could be interpreted in terms of automata. Indeed, the automorphisms of the tree have a natural interpretation as input-output automata where the states, finite or infinite in number, are themselves automorphisms of the tree. On the other hand input-output automata having the same input and output alphabets can be seen as endomorphisms of a 1-rooted tree indexed by finite sequences from this alphabet. It is to be noted that the set of automorphisms having a finite number of states and thus corresponding to finite automata, form an enumerable group called the *group of finite-state automorphisms*. The calculation of the product of two automorphisms of the tree involve calculating products between their states which are not necessarily elements of the group generated by the two automorphisms. In order to remain within the same domain of calculation we have defined a group G as *state-closed* provided the states of its elements are also elements of G [S2]. Among the outstanding examples of state-closed groups are the classes of self-reproducing (fractal-like) groups constructed in [G1, GS, BSV] which are actually generated by automorphisms with finite number of states, or equivalently, generated by finite automata. The state-closed condition has allowed the use of induction on the length function to prove detailed properties of these groups. Of course, if the group is not state-closed one may take its state-closure. In doing so, properties such as finite generation, may not be conserved. At any rate, state-closed groups which are finitely generated yet not necessarily finite-state are subgroups of another

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enumerable group called the *group of functionally recursive automorphisms* [BS1].

An important set of examples of state-closed groups of classical nature are the m-dimensional affine groups $\mathbb{Z}^m \cdot \mathrm{GL}(m, \mathbb{Z})$. It was shown in [BS2] that for every m the corresponding affine group is faithfully represented as a state-closed group of automorphisms of the 2^m-ary tree. The purpose of the present paper is to investigate state-closed groups of automorphisms of the 2-tree with emphasis on subgroups of m-dimensional affine groups.

Given an abstract group G with a subgroup H of index 2, we call a homomorphism $\rho : H \to G$ *a* $1/2$*-endomorphism* of G. The $\rho-\mathrm{core}(H)$ is the maximal subgroup K of H which is normal in G and is ρ-invariant (that is, $\rho(K) \leq K$), and ρ is called *simple* provided $\rho-\mathrm{core}(H)$ is trivial. The concept of $1/2$-endomorphism is intimately related to that state-closed group. For if G is a non-trivial state-closed group, then the stabilizer subgroup G_1 of the first level vertices of the tree is of index 2 in G and the restriction of the action of elements of G_1 to one of the two maximal subtrees provides us with such a map. On the other hand, as we will show, a group G with a $1/2$-endomorphism $\rho : H \to G$ admits representations into the automorphism group of the binary tree as a state-closed group. These representations are faithful if and only if ρ is simple. Certain classes of groups which have faithful representations as groups of automorphisms of the binary tree fail to have a faithful representation as a state-closed groups. One instance of this breakdown occurs in finitely generated free non-abelian nilpotent groups. We show that these groups cannot admit a faithful state-closed representation on the binary tree. The situation for torsion-free non-abelian polycyclic groups is mixed and the problem of describing the polycyclic state-closed groups is open. Another open problem in this context concerns the existence of non-cyclic state-closed free groups. It is to be noted that homomorphisms $\rho : H \to G$ where H is a subgroup of finite index in G (so called virtual endomorphisms of G) have been the subject of recent studies (see [GM] and [Nek]). These works as well as ours represent first explorations of a new topic in combinatorial group theory.

We call torsion-free abelian groups of finite rank m which are state-closed subgroups of the automorphism group of the binary tree *m-dimensional (binary) lattices*. A large part of our work is devoted to the classification of these lattices. We prove that the $1/2$-endomorphism associated to each such group is the restriction of an irreducible linear transformation defined on a rational vector space of dimension m. When the lattice is generated by finite-state automorphisms of the tree, we prove that its corresponding linear transformation is necessarily a contracting map, in the sense that the roots of the characteristic polynomial has absolute value less than 1. A classification of these polynomials having degree less than 6 reveal the surprising fact that the class number corresponding to each is 1. The connection between the finite-state condition for the lattice with the dynamical behavior of the associated linear transformation and the number theoretic observations about the characteristic polynomials of these transformations confirms the wide scope of interaction between these notions about tree automorphisms and other topics in mathematics. In particular, the relationship with dynamical systems as expounded in [B] is strongly enhanced.

A state-closed group G whose associated $1/2$-endomorphism is onto is called *recurrent*. We describe the topological closure of a recurrent lattice G in terms of a ring of certain infinite series which generalize the ring of dyadic integers. Here, the group of automorphisms of the tree is considered as a profinite topological group with respect to the pro-2 topology; in this setting, the topological closure of an abelian subgroup is again abelian. We also prove that the elements of the group G act on such series as *generalized adding machines*. This action may be viewed as numeration systems for abelian groups. In the case of rank 1 we get usual dyadic numeration system (or "nega-dyadic"). For rank 2 we get numeration systems similar to the numeration systems for complex numbers. For more on numeration systems one can read in [K] and [Sa].

One of the nice properties of recurrent lattices is that they are topologically determined by their $1/2$-endomorphisms. Indeed, we prove that any two recurrent lattices with the same associated $1/2$-endomorphism have equal topological closures.

One special class of m-dimensional recurrent lattices admits for every m a "large" linear group normalizer within the finite-state group of automorphisms of the binary tree. More precisely, let $\mathbb{Z}^m$ denote the m-dimensional lattice, $\varsigma : \mathrm{GL}(m,\mathbb{Z}) \to \mathrm{GL}(m,\mathbb{Z}_2)$ the natural "modulo 2" epimorphism and $B(m,\mathbb{Z})$ the pre-image of the Borel subgroup of $\mathrm{GL}(m,\mathbb{Z}_2)$. We prove that the affine group $\mathbb{Z}^m \cdot B(m,\mathbb{Z})$ admits a faithful representation as a state-closed, finite-state group of automorphisms of the binary tree. This result is optimal in the sense that $[\mathrm{GL}(m,\mathbb{Z}) : B(m,\mathbb{Z})] = (2^m-1)(2^{m-1}-1)\cdots(2^2-1)$ is the maximal odd factor of $|\mathrm{GL}(m,\mathbb{Z}_2)|$ and $B(m,\mathbb{Z})$ is a maximal subgroup of $\mathrm{GL}(m,\mathbb{Z})$ with respect to avoiding having elements of odd order. It also puts in perspective the main result in [BS2] that the affine group $\mathbb{Z}^m \cdot \mathrm{GL}(m,\mathbb{Z})$ has a faithful, finite-state, state-closed representation on a 2^m-ary tree. One needs exponentially high valency for the tree since the minimum degree of a transitive representation of $\mathrm{GL}(m,\mathbb{Z}_2)$ is 2^m-1 for $m > 4$ [KL].

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2. Tree automorphisms, automata

We present below definitions and preliminary notions about the binary tree, its automorphisms and their interpretation as automata. The one-rooted regular binary tree $\mathcal{T}_2$ may be identified with the monoid $\mathcal{M}$ freely generated by a set $Y = \{0, 1\}$ and ordered by the relation

$$v \leq u \text{ if and only if } u \text{ is a prefix of } v;$$

the identity element of $\mathcal{M}$ is the empty sequence $\emptyset$ (the root of the tree).

Let $\mathcal{A} = Aut(\mathcal{T}_2)$ be the automorphism group of the tree. The group of permutations $P(Y)$ is the cyclic group of order 2 generated by the transposition $\sigma = (0, 1)$. This permutation is extended "rigidly" to an automorphism of $\mathcal{A}$ by

$$(y \cdot u)^\sigma = y^\sigma \cdot u, \forall y \in Y, \forall u \in \mathcal{M}.$$

An automorphism $\alpha \in \mathcal{A}$ induces $\sigma^{i_\emptyset}$ where $i_\emptyset = 0, 1$, on the set $Y \subset \mathcal{M}$. Therefore the automorphism affords the representation $\alpha = \alpha' \sigma^{i_\emptyset}$, where α' fixes Y point-wise. Furthermore, α' induces for each $y \in Y$ an automorphism α'_y of the subtree whose vertices form the set $y \cdot \mathcal{M}$. On using the canonical isomorphism $yu \mapsto u$ between this subtree and the tree $\mathcal{T}$, we may consider (or, renormalize) α' as a function from Y into $\mathcal{A}$; in notational form, $\alpha' \in \mathcal{F}(Y, \mathcal{A})$. Thus, $\alpha = (\alpha_0, \alpha_1)\sigma^{i_\emptyset}$ and the group $\mathcal{A}$ is an infinitely iterated wreath product

$$\mathcal{A} = \mathcal{A} \wr \langle \sigma \rangle.$$

It is convenient to denote α by $\alpha_\emptyset$ and α'_y by α_y. In order to describe α_y we use the same procedure as in the case of α. Successive applications produce the set

$$\Sigma(\alpha) = \{\sigma^{i_u} \mid u \in \mathcal{M}\}$$

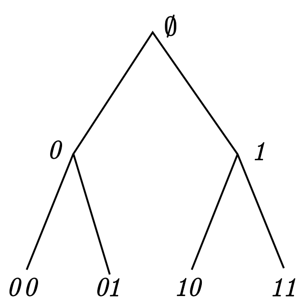

Figure 13.1 Binary tree

of permutations of Y which describes faithfully the automorphism α. Another by-product of the procedure is the set *states* of α,

$$Q(\alpha) = \{\alpha_u \mid u \in \mathcal{M}\}.$$

The definition of the product of automorphisms implies the following important properties of the function Q

$$Q(\alpha^{-1}) = Q(\alpha)^{-1},$$

$$Q(\alpha\beta) \subseteq Q(\alpha)Q(\beta), \forall \alpha, \beta \in \mathcal{A}.$$

If $Q(\alpha)$ is finite then α is said to be a *finite-state* automorphism. The set of finite-state automorphisms form the enumerable subgroup $\mathcal{F}$ of $\mathcal{A}$. The notion of finite-state automorphism is a special case of the more general functionally recursive automorphism. A finite set of automorphisms S is *functionally recursive* provided for each $\gamma \in S$, its states γ_0, γ_1 are group words in the elements of S. An automorphism α is functionally recursive provided α is an element of some functionally recursive set. The set of functionally recursive automorphisms form an enumerable group $\mathcal{R}$.

The interpretation of α as an automaton proceeds as follows: the input and output alphabets are the same set $Y = \{0, 1\}$; the set of states is $Q(\alpha)$; the initial state is α; let $y \in Y$, $\alpha_u \in Q(\alpha)$ and z the image of y under σ^{i_u}, then the state-transition function is $y : \alpha_u \mapsto \alpha_{uy}$ and the output function is $\alpha_u : y \mapsto z$. Thus, a finite-state automorphism corresponds to a finite automaton. A finite automaton is usually depicted by a directed graph called *the Moore diagram*. The vertices of the diagram correspond to the states of the automaton; and the arrows correspond to the transitions. For every $y \in Y$ and every state α_u we draw an arrows from α_u to α_{uy} and label it by $(y|z)$, where $z = y^{\sigma^{i_u}}$ as above. Then the arrows correspond to the transitions while the labels show the output. The set of infinite sequences $c = (c_0, c_1, c_2, \ldots)$ with $c_i \in \{0, 1\}$ correspond to ends or boundary points of the tree. The action of an automorphism extends naturally to the boundary. Let $c = (c_0, c_1, c_2, \ldots)$. For $\alpha = (\alpha_0, \alpha_1)\sigma$, we have

$$c^\alpha = \begin{cases} (1, (c_1, c_2, \ldots)^{\alpha_0}), & \text{if } c_0 = 0, \\ (0, (c_1, c_2, \ldots)^{\alpha_1}), & \text{if } c_0 = 1 \end{cases}$$

and for $\alpha = (\alpha_0, \alpha_1)$,

$$c^\alpha = \begin{cases} (0, (c_1, c_2, \ldots)^{\alpha_0}), & \text{if } c_0 = 0, \\ (1, (c_1, c_2, \ldots)^{\alpha_1}), & \text{if } c_0 = 1. \end{cases}$$

The boundary points $c = (c_0, c_1, c_2, \ldots)$ also correspond to a dyadic integer $\xi = c_0 + c_1 2 + c_2 2^2 + \cdots + c_i 2^i + \cdots$ and the action of the tree

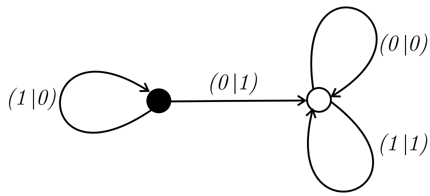

Figure 13.2 Adding machine

automorphism α can thus be translated to an action on the ring of dyadic integers. For example, consider the automorphism $\tau = (e, \tau)\sigma$. Then

$$c^{\tau} = \begin{cases} (1, c_1, c_2, \ldots), & \text{if } c_0 = 0, \\ (0, (c_1, c_2, \ldots)^{\tau}), & \text{if } c_0 = 1, \end{cases}$$

which translates to the binary addition

$$\xi^{\tau} = 1 + \xi,$$

and this fact justifies referring to τ as the *binary adding machine*. The diagram of the automaton corresponding to the binary adding machine is shown in Figure 13.2.

We have developed sufficient language to give the definition of three examples of self reproducing groups.

(i) Let $\alpha = (e, \alpha_1)$ where $\alpha_1 = (\sigma, \alpha_{11})$ and $\alpha_{11} = (\sigma, \alpha)$. The group $\langle \alpha, \alpha_1, \alpha_{11}, \sigma \rangle$ is a state-closed infinite 2-group of intermediate growth [G1].

(ii) Let $\alpha = (\alpha_0, \alpha), \alpha_0 = (\alpha_{00}, e), \alpha_{00} = (\sigma, \sigma)$. Then the group $\langle \alpha, \alpha_0, \alpha_{00}, \sigma \rangle$ is a state-closed infinite 2-group [S2].

(iii) Let $\tau = (e, \tau)\sigma, \mu = (e, \mu^{-1})\sigma$. Then $\langle \tau, \mu \rangle$ is state-closed, torsion-free, and is just-nonsolvable [BSV].

3. State-closed groups

Let G be a non-trivial state-closed group. Since G is non-trivial, some of states of its elements are active, and so the state-closed condition implies that there exists an element of G that is active; that is, G is transitive on the first level of the binary tree. The first level stabilizer G_1 is a subgroup of index 2 in the group G and $G = G_1 \cup G_1 a$ for some choice $a \in G \setminus G_1$. Then in the action of G on the tree, $a = (a_0, a_1)\sigma$ and $h = (h_0, h_1)$ for all $h \in G_1$. Since G is state-closed, $a_0, a_1 \in G$, and likewise, $h_0, h_1 \in G$ for all $h \in G_1$. Therefore the projections π_0, π_1 of G_1 on its first and second coordinates are $1/2$-endomorphisms from the subgroup G_1 into the group G. We note that in case G is recurrent, $\pi_0(G_1) = G$

and therefore G is transitive on all levels of the binary tree. Now, since $a^2 = (a_0a_1, a_1a_0) \in G_1$ we have $a_1 = a_0^{-1}\pi_0(a^2)$. Also, we have for every $h \in G_1$, $h^a = \sigma(a_0^{-1}, a_1^{-1})(h_0, h_1)(a_0, a_1)\sigma = (h_1^{a_1}, h_0^{a_0})$; thus $\pi_1(h^a) = h_0^{a_0} = \pi_0(h)^{a_0}$. Hence, the projections π_0, π_1 satisfy the following conditions

$$a_1 = a_0^{-1}\pi_0(a^2), \quad \pi_1(h) = \pi_0(h^{a^{-1}})^{a_0}.$$

In case G is abelian, the second condition simplifies to $\pi_1(h) = \pi_0(h)$ for all $h \in G_1$.

Examples. (1) Define the following sequence of elements of $Aut(\mathcal{T}_2)$, $\sigma_{\mathbf{0}} = \sigma$, $\sigma_{\mathbf{1}} = (e, \sigma_{\mathbf{0}})$ and for $i \geq 1$, $\sigma_{\mathbf{i}} = (e, \sigma_{\mathbf{i-1}})$. Define also for $0 \leq n \leq \infty$ the subgroups $P_n = \langle \sigma_{\mathbf{i}} \mid 0 \leq i \leq n \rangle$. Then P_n is isomorphic to the wreath product, iterated n times, of cyclic groups of order 2. It is easy to see from the definition of the generators $\sigma_{\mathbf{i}}$ that P_n is state-closed and moreover, P_∞ is recurrent.

(2) Let $\tau = (e, \tau)\sigma$ be the binary adding machine. Define $\tau_{\mathbf{0}} = \tau$ and $\tau_{\mathbf{i}} = (e, \tau_{\mathbf{i-1}})$ for $i \geq 1$. Define also the subgroups $\Upsilon_n = \langle \tau_i \mid 0 \leq i \leq n \rangle$. Then Υ_n is state-closed. When n is finite this group factors as $\Upsilon_n = N \cdot P_{n-1}$, where N is the normal closure of $\langle \tau_{\mathbf{n}} \rangle$ in Υ_n and is free abelian group of rank $n + 1$. We note that $\Upsilon_0, \ldots, \Upsilon_\infty$ are recurrent groups.

Proposition 3.1. *Let G be a group of automorphisms of the binary tree, generated by a finite set S. Then, G is state-closed if and only if S is functionally recursive.*

Proof. Let $S = \{\alpha, \beta, \ldots, \gamma\}$. Suppose G is state-closed. Then as $\alpha_0 \in G$, it is a word $\alpha_0 = \alpha_0(\alpha, \beta, \ldots, \gamma)$ in the elements of S, and so every state of every element in S is also a word in the elements of S. Thus, S is functionally recursive. On the other hand, if S is functionally recursive, then by definition, every state δ of every element of S is some word $\delta = \delta(\alpha, \beta, \ldots, \gamma)$ in the elements of S; therefore, $\delta \in G$. $\square$

Lemma 3.1. *Let G be a state-closed group and $\widehat{G}$ its topological closure. Then $\widehat{G}$ is also state-closed.*

Proof. An automorphism $\omega \in \mathcal{A}$ belongs to $\widehat{G}$ if and only if for every $n \in \mathbb{N}$, the action of ω on the first n levels of the tree coincides with an action of some $g \in G$ on these levels (g depends on n). If $\omega = (\omega_0, \omega_1)\sigma^i$, $i \in \{0, 1\}$ then $g = (g_0, g_1)\sigma^i$ for some $g_0, g_1 \in G$ and then the action of ω_0 on first $n - 1$ levels of the tree coincides with the action of g_0; the same is true for ω_1 and g_1 respectively. Thus the action of ω_0 and ω_1 on any finite number of levels coincides with actions of some elements of G, thus they belong to $\widehat{G}$. $\square$

Definition 1. Let $qN_{\mathcal{A}}(G) = \{\alpha \in \mathcal{A} \mid G^\alpha \leq G\}$ denote the *semi-normalizer* of G in the group $\mathcal{A}$ of automorphisms of the tree and let $C_{\mathcal{A}}(G)$ denote the centralizer of G in $\mathcal{A}$. For any element $\alpha \in \mathcal{A}$ we denote (α, α) by $\alpha^{(1)}$ and inductively, for any $n \geq 0$, $(\alpha^{(n)}, \alpha^{(n)})$ by $\alpha^{(n+1)}$.

We are able to produce information about the form of elements of the semi-normalizer and centralizer of a recurrent group.

Proposition 3.2. *Let G be a recurrent group, $\widehat{G}$ its topological closure and a an active element of G.*

(i) *Given $\alpha \in qN_{\mathcal{A}}(G)$, there exist $i \in \{0, 1\}$, $\beta \in qN_{\mathcal{A}}(G)$, $u \in G$ such that*
$$\alpha = \beta^{(1)}(e, u)a^i \text{ and } (a_0^\beta u a_0^{-1}, u^{-1}a_1^\beta a_1^{-1}) \in G.$$

(ii) *Suppose G is also abelian. Then u in the above formula satisfies*
$$u^2 = \left(a_0^{-1}a_1\right)^{-1}\left(a_0^{-1}a_1\right)^\beta.$$
In addition, $C_{\mathcal{A}}(G)$ coincides with $\widehat{G}$.

Proof. (i) Let $a = (a_0, a_1)\sigma \in G$, $h = (h_0, h_1) \in G_1$, $\alpha \in qN_{\mathcal{A}}(G)$. There exists a unique $i \in \{0, 1\}$ such that $\alpha' = \alpha a^{-i}$ is inactive; clearly, $\alpha' \in qN_{\mathcal{A}}(G)$. Thus we may assume $\alpha = (\alpha_0, \alpha_1)$. Now, $a^\alpha = (\alpha_0^{-1}a_0\alpha_1, \alpha_1^{-1}a_1\alpha_0)\sigma \in G$, $h^\alpha = (h_0^{\alpha_0}, h_1^{\alpha_1}) \in G_1$. Since G is state-closed, $\alpha_0^{-1}a_0\alpha_1 = k$, $\alpha_1^{-1}a_1\alpha_0 = k' \in G$ and also $h_0^{\alpha_0}, h_1^{\alpha_1} \in G$. Since G is recurrent, h_0 can be equal to any element of G, thus $\alpha_0 \in qN_{\mathcal{A}}(G)$. We find that $\alpha_1 = a_0^{-1}\alpha_0 k$. Thus, $\alpha = \beta^{(1)}(e, u)$ for $u = \left(a_0^{-1}\right)^{\alpha_0} k \in G$ and $\beta = \alpha_0 \in qN_{\mathcal{A}}(G)$. The second part follows from computing the commutator $[\beta^{(1)}(e, u), a^{-1}]$.

(ii) As G is abelian, then for all $h \in G_1$, $h = (h_0, h_0)$ and so, $a_0^\beta u a_0^{-1} = u^{-1}a_1^\beta a_1^{-1}$; thus, $u^2 = \left(a_0^{-1}a_1\right)^{-1}\left(a_0^{-1}a_1\right)^\beta$ follows. Again, as G is abelian then so is $\widehat{G}$; thus, $\widehat{G} \leq C_{\mathcal{A}}(G)$. Now let $\alpha \in C_{\mathcal{A}}(G)$. Then from part (i), $\alpha = \beta^{(1)}(e, u)a^i$ and clearly, $\alpha' = \beta^{(1)}(e, u) \in C_{\mathcal{A}}(G)$. On applying α' to G_1, we conclude that $\beta \in C_{\mathcal{A}}(G)$, since $\pi_0(G_1) = G$. Now $a^{\alpha'} = (\beta^{-1}a_0\beta u, u^{-1}\beta^{-1}a_1\beta)\sigma = (a_0 u, u^{-1}a_1)\sigma = a$ implies $u = e$. Hence, $\alpha = \beta^{(1)}a^i$, $\beta \in C_{\mathcal{A}}(G)$. Successive developments of α yield $\alpha \in \widehat{G}$. □

3.1. State-closed representations

Given a group G, we describe below all the state-closed representations of G on the binary tree. Consider a subgroup H of G among the subgroups of index 2 in G. This subgroup contains the subgroup G^2 generated by the squares of the elements of G and G^2 itself contains the commutator subgroup G' of G. Given such a subgroup H, we choose a $1/2$-endomorphism $\rho : H \to G$, $a \in G \setminus H$ and $a_0 \in G$. We will prove that the quadruple (H, ρ, a, a_0) defines uniquely a state-closed representation of G on the binary tree.

Theorem 3.1. *Let G be a group, (H, ρ, a, a_0) a quadruple as defined above and $a_1 = a_0^{-1}\rho(a^2)$. Also, let σ be the rigid extension of the transposition $(0, 1)$ to an automorphism of the binary tree $\mathcal{T}_2$. Then the map $\varphi : G \to Aut(\mathcal{T}_2)$ defined recursively by the rules:*

$$(ha)^{\varphi} = (\rho(h)^{\varphi} a_0^{\varphi}, \rho\left(h^{a^{-1}}\right)^{a_0\varphi} a_1^{\varphi})\sigma$$
$$h^{\varphi} = (\rho(h)^{\varphi}, \rho\left(h^{a^{-1}}\right)^{a_0\varphi})$$

is a homomorphism such that the first level stabilizer of G^{φ} coincides with H^{φ}. The kernel of φ is equal to $\rho-\mathrm{core}(H)$.

Proof. It follows from the definition of the map φ that $(ha)^{\varphi} = h^{\varphi}a^{\varphi}$ for every $h \in H$. Thus, in order to prove that φ is a homomorphism it is sufficient to check the following equalities:

$$\left(a^2\right)^{\varphi} = (a^{\varphi})^2,$$
$$(h_1h_2)^{\varphi} = h_1^{\varphi}h_2^{\varphi},$$
$$\left(h^a\right)^{\varphi} = (h^{\varphi})^{a^{\varphi}}.$$

We prove them by induction on the tree level.

$$1)\ \left(a^2\right)^{\varphi}(a^{\varphi})^{-2} = \left(\rho\left(a^2\right)^{\varphi}, \rho\left(\left(a^2\right)^{a^{-1}}\right)^{a_0\varphi}\right)\left(\left(a_0^{\varphi}, a_1^{\varphi}\right)\sigma\right)^{-2}$$
$$= ((a_0a_1)^{\varphi}, (a_1a_0)^{\varphi})\left(a_0^{\varphi}a_1^{\varphi}, a_1^{\varphi}a_0^{\varphi}\right)^{-1}$$
$$= \left((a_0a_1)^{\varphi}\left(a_0^{\varphi}a_1^{\varphi}\right)^{-1}, (a_1a_0)^{\varphi}\left(a_1^{\varphi}a_0^{\varphi}\right)^{-1}\right).$$

The verification of the homomorphism condition clearly reduces to the next level.

$$2)\ (h_1h_2)^{\varphi}\left(h_1^{\varphi}h_2^{\varphi}\right)^{-1} = \left(\rho(h_1h_2)^{\varphi}, \rho\left(h_1^{a^{-1}}h_2^{a^{-1}}\right)^{a_0\varphi}\right)\cdot$$
$$\left(\rho(h_1)^{\varphi}\rho(h_2)^{\varphi}, \rho\left(h_1^{a^{-1}}\right)^{a_0\varphi}\rho\left(h_2^{a^{-1}}\right)^{a_0\varphi}\right)^{-1}$$
$$= \Bigg((\rho(h_1)\rho(h_2))^{\varphi}(\rho(h_1)^{\varphi}\rho(h_2)^{\varphi})^{-1},$$
$$\left(\rho\left(h_1^{a^{-1}}\right)^{a_0}\rho\left(h_2^{a^{-1}}\right)^{a_0}\right)^{\varphi}\left(\rho\left(h_1^{a^{-1}}\right)^{a_0\varphi}\rho\left(h_2^{a^{-1}}\right)^{a_0\varphi}\right)^{-1}\Bigg).$$

Again, the reduction in this case is clear.

$$3)\ \left(h^{a}\right)^{\varphi}\left(\left(h^{\varphi}\right)^{a^{\varphi}}\right)^{-1}=\left(\rho\left(h^{a}\right)^{\varphi},\rho\left(h\right)^{a_0\varphi}\right)\cdot$$

$$\left(\sigma\left(\left(a_0^{\varphi}\right)^{-1},\left(a_1^{\varphi}\right)^{-1}\right)\left(\rho\left(h\right)^{\varphi},\rho\left(h^{a^{-1}}\right)^{a_0\varphi}\right)\left(a_0^{\varphi},a_1^{\varphi}\right)\sigma\right)^{-1}$$

$$=\left(\rho\left(h^{a}\right)^{\varphi}\left(\rho\left(h^{a^{-1}}\right)^{a_0\varphi a_1^{\varphi}}\right)^{-1},\rho\left(h\right)^{a_0\varphi}\left(\rho\left(h\right)^{\varphi a_0^{\varphi}}\right)^{-1}\right)$$

$$=\left(\rho\left(h^{a}\right)^{\varphi}\left(\rho\left(h^{a}\right)^{a_1^{-1}\varphi a_1^{\varphi}}\right)^{-1},\rho\left(h\right)^{a_0\varphi}\left(\rho\left(h\right)^{\varphi a_0^{\varphi}}\right)^{-1}\right);$$

the last equality follows from

$$\rho\left(h^{a^{-1}}\right)=\rho\left(h^{a}\right)^{\rho\left(a^{-2}\right)}=\rho\left(h^{a}\right)^{a_1^{-1}a_0^{-1}}.$$

Now we consider whether the first coordinate of $(h^{a})^{\varphi}\left((h^{\varphi})^{a^{\varphi}}\right)^{-1}$ is trivial. The following sequence of equivalent statements lead to the desired reduction

$$\rho\left(h^{a}\right)^{\varphi}\left(\rho\left(h^{a}\right)^{a_1^{-1}\varphi a_1^{\varphi}}\right)^{-1}=e,$$

$$\rho\left(h^{a}\right)^{a_1^{-1}\varphi a_1^{\varphi}}=\rho\left(h^{a}\right)^{\varphi},$$

$$a_1^{\varphi}\rho\left(h^{a}\right)^{\varphi}\left(a_1^{\varphi}\right)^{-1}=\left(\rho\left(h^{a}\right)^{a_1^{-1}}\right)^{\varphi}.$$

The question of triviality of the second coordinate of $(h^{a})^{\varphi}\left((h^{\varphi})^{a^{\varphi}}\right)^{-1}$ reduces more simply to the next level. Now let N be the kernel of the homomorphism φ. Then for every $h \in N$, h^{φ} is inactive, and consequently, $h \in H$. Thus $h^{\varphi} = e = (e, e) = (\rho(h)^{\varphi}, \rho(h^{a^{-1}})^{a_0\varphi})$. Hence $\rho(h) \in N$ and N is ρ-invariant. On the other hand, if M is a subgroup of H which is normal in G and is also ρ-invariant then for every $h \in M$ the elements $\rho(h)$ and $\rho(h^{a^{-1}})^{a_0}$ belong to M. It follows inductively from the representation $h^{\varphi} = (\rho(h)^{\varphi}, \rho(h^{a^{-1}})^{a_0\varphi})$ that h^{φ} is trivial. Hence, $M \leq N$ and kernel of φ is $\rho-\mathrm{core}(H)$. □

Remarks. We maintain the above notation.

(1) The relationship between the $\ker(\rho)$ and $\rho-\mathrm{core}(H)$ is not totally clear. Yet, it is obvious that $D = \ker(\rho) \cap \ker(\rho)^{a}$ is contained in $\rho-\mathrm{core}(H)$. Thus if ρ is simple then $D = \{e\} = [\ker(\rho), \ker(\rho)^{a}]$. We conclude that in case the abelian subgroups are cyclic (for instance, when G is a free group), then the condition ρ is simple implies ρ is a monomorphism. In the other direction, if G

is abelian then $\ker(\rho) = \ker(\rho)^a = D$, and all simple $1/2$-endomorphisms are also monomorphisms.

(2) Suppose that G is a normal subgroup of some group F and suppose that ρ is a restriction of an endomorphism $\widehat{\rho}$ of F. Suppose in addition that G is of finite index k in F, an let F^k be the subgroup of F generated by the k-th powers of its elements. Then $L = \left(F^k\right)^2 \leq H$ and so, $L^\rho = L^{\widehat{\rho}} \leq L$; thus, $L \leq \rho-\mathrm{core}\,(H)$. It follows then that if $F = G$ and ρ is simple then G is an elementary abelian 2-group.

3.2. Extensions and restrictions of $1/2$-endomorphisms

The next results concern manners of producing simple $1/2$-endomorphisms.

Lemma 3.2. *Let G be a group and H a subgroup of index 2. Suppose that $\rho : H \to G$ is a simple $1/2$-endomorphism of G. Then the restriction of ρ to $H \cap H^\rho$ is a simple $1/2$-endomorphism of H^ρ.*

Proof. Since H is not ρ-invariant, there exists an element b in $\rho(H)$ outside H. Therefore, $G = H\langle b\rangle$. Let K be a subgroup of $H \cap H^\rho$, which is normal in H^ρ and is ρ-invariant. Then K is contained in $K^{\rho^{-1}}$ which is a normal subgroup of H. Therefore, $K^{\rho^{-1}b}$ is normal in H and $K = K^b \leq K^{\rho^{-1}b}$. Now let $M = K^{\rho^{-1}} \cap K^{\rho^{-1}b}$. Then M is a normal subgroup of G and $K \leq M$. Also, we have

$$K \leq M \leq K^{\rho^{-1}}, M^\rho \leq K$$

and thus M is ρ-invariant. Since ρ is simple, it follows that both M and K are trivial. □

Proposition 3.3. *Let G be a group and H a subgroup of index 2. Suppose that $\rho : H \to G$ is a simple homomorphism. Let $\widetilde{G} = G \times G$ be the direct product of G with itself, $\widetilde{H} = H \times G$, and $\widetilde{\rho} : H \times G \to G \times G$ the map defined by $\widetilde{\rho} : (h, x) \to (x, h^\rho)$ for all $h \in H, x \in G$. Then $\widetilde{H}$ is a subgroup of index 2 in $\widetilde{G}$ and $\widetilde{\rho}$ is a simple homomorphism.*

Proof. Let $\widetilde{K}$ be a subgroup of $\widetilde{H}$, normal in $\widetilde{G}$ and $\widetilde{\rho}$-invariant. Then the projection of $\widetilde{K}$ on its first coordinate produces a subgroup K of H, normal in G. Furthermore, if $y = (h, x) \in \widetilde{K}$ then we have $h \in K$, $y^{\widetilde{\rho}}(x, h^\rho) \in \widetilde{K}$, $x \in H$, $y^{\widetilde{\rho}^2} = (h^\rho, x^\rho) \in \widetilde{K}$ and so $h^\rho \in K$; that is, K is a ρ-invariant. We conclude that K is trivial and thus $y = (e, x)$, $y^{\widetilde{\rho}} = (x, e) = (e, e)$, $y = e$; hence $\widetilde{K}$ is trivial. □

A direct application of this proposition is

Corollary 3.1. *Let $k \geq 0$, $m = 2^k$. Consider the free-abelian group $G = \mathbb{Z}^m$ of rank m, its subgroup $H = 2\mathbb{Z} \times \mathbb{Z}^{m-1}$ and the rational vector space $V = \mathbb{Q}^m$. Define the following linear transformations of V represented by the matrices*

$$\mathbf{A}_0 = \frac{1}{2},\ \mathbf{A}_1 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}, \ldots, \mathbf{A}_k = \begin{pmatrix} 0 & \mathbf{I}_{2^{k-1}} \\ \mathbf{A}_{k-1} & 0 \end{pmatrix}$$

with respect to the canonical basis. Then $\mathbf{A}_k$ defines a simple homomorphism from H into G, for all $k \geq 0$.

The next result will be used in the final section of the paper in order to extend certain simple $1/2$-endomorphisms associated to lattices to their affine groups.

Lemma 3.3. *Let G be a group which admits a factorization $G = MH$ where H is a subgroup of index 2 and M a normal subgroup such that $C_G(M) \leq M$. Furthermore, let $\rho : H \to G$ be a homomorphism, $N = H \cap M$ and η the restriction of ρ to N. Suppose $\eta(N) \leq M$ and η simple. Then ρ is also simple.*

Proof. Suppose η is simple and define $K = \rho-\mathrm{core}(H)$, $D = K \cap M$. Then D is a normal subgroup of G contained in N and

$$D^\eta = (K \cap N)^\eta \leq K^\rho \cap N^\eta \leq K \cap M = D,$$

is a subgroup of the $\eta-\mathrm{core}(M)$. Thus D is trivial and as both K and M are normal subgroups of G, K centralizes M. We conclude that $K \leq M$, $K \leq \eta-\mathrm{core}(M)$ and $K = \{e\}$. □

3.3. State-closed solvable groups

We present in this section a number of results, some positive and others negative, for state-closed representations of finitely generated solvable groups.

Proposition 3.4. *Let G be an abelian state-closed group of automorphisms of the binary tree. Then G is either torsion-free or an elementary abelian 2-group. If G is an elementary abelian 2-group then it is a subgroup of the topological closure of $\langle \sigma, \sigma^{(1)}, \sigma^{(2)}, \ldots \rangle$.*

Proof. Suppose the group is not torsion-free. Then the set of involutions $\Omega_2(G)$ is not contained in the stabilizer subgroup G_1, for otherwise this set would be ρ-invariant and therefore trivial. Let a be an involution such that $G = G_1 \oplus \langle a \rangle$. But then, $G_1^2 = G^2$ and thus $G^2 = \{e\}$. As a is active, $a = (a', a')\sigma$, and the elements of G_1 have the form $h = (x, x)$, with $a', x \in G$. It becomes clear on developing the elements of G that these belong to the topological closure of $\langle \sigma, \sigma^{(1)}, \sigma^{(2)}, \ldots \rangle$. □

Theorem 3.2. *Let G be a finitely generated free nilpotent group of class k, H a subgroup of G of finite index containing the commutator subgroup G' and $\rho : H \to G$ a homomorphism. Then G' is ρ-invariant. Therefore, G is a state closed group acting on the binary tree if and only if $k = 1$.*

Proof. We proceed by induction on the nilpotency class of k of G and may assume $k \geq 2$.

There exists a free generating set $S = \{a_1, a_2, \ldots, a_m\}$ of G modulo G' and positive integers $n_1, n_2, \ldots, n_m$ such that $U = \{a_1^{n_1}, a_2^{n_2}, \ldots, a_m^{n_m}\}$ is a free generating set for H modulo G'.

First will show that $\gamma_k(G)$ is ρ-invariant. Note that

$$\gamma_k(H) \leq \gamma_k(G) \leq H,$$
$$\gamma_k(H)^\rho = \gamma_k(H^\rho) \leq \gamma_k(G).$$

It is well-known that $\gamma_k(G)$ is generated by $a = [a_{i_1}, a_{i_2}, \ldots, a_{i_k}]$ where $a_{i_s} \in S$. Likewise, $\gamma_k(H)$ is generated by $b = [a_{i_1}^{n_{i_1}}, a_{i_2}^{n_{i_2}}, \ldots, a_{i_k}^{n_{i_k}}]$. Note that $b = [a_{i_1}, a_{i_2}, \ldots, a_{i_k}]^n = a^n$ where $n = n_{i_1}, n_{i_2} \ldots n_{i_k}$. Now, $b^\rho = [a_{i_1}^{n_{i_1}\rho}, a_{i_2}^{n_{i_2}\rho}, \ldots, a_{i_k}^{n_{i_k}\rho}] = (a^\rho)^n$. Since $G/\gamma_k(G)$ is torsion-free it follows that $a^\rho \in \gamma_k(G)$ and $\gamma_k(G)$ is ρ-invariant. Therefore ρ induces a homomorphism $\overline{\rho} : H/\gamma_k(G) \to G/\gamma_k(G)$ and the proof of the first statement follows by induction. The second statement is an immediate conclusion. □

Remark. In the above proposition, the hypothesis that the subgroup H contain G' is necessary. For let G be the free nilpotent class group of nilpotency class 2, freely generated by a, b, and let $z = [a, b]$. Consider the subgroup $H = \langle a^k, b\rangle$ of G where $k \geq 2$. Then $[G : H] = k^2$ and H is not a normal subgroup of G. The map $\rho : a^k \to b, b \to a^{-1}$ extends to an epimorphism $\rho : H \to G$ and $\rho(z^k) = \rho[a^k, b] = [b, a^{-1}] = z$. If K is a nontrivial subgroup of H and is normal in G, then K contains z^{ik} for some $i \geq 1$; choose ik to be minimal. Then on applying ρ to K we produce z^i which shows that K cannot be ρ-invariant and therefore it follows that ρ is simple.

Theorem 3.3. *Let G be a finitely generated nilpotent group and suppose G is a state-closed group of automorphisms of the binary tree. Then G is torsion-free or a finite 2-group.*

Proof. It is well-known that the set $T(G)$ of torsion elements of G is indeed a finite subgroup. We proceed by induction on $|T(G)|$. Suppose $T(G)$ is nontrivial. Choose an involution a in G outside G_1; therefore, $G = G_1\langle a\rangle$. Recall that π_0 is the projection map of the stabilizer subgroup G_1 on its first coordinate. The group G is described by the quadruple (G_1, π_0, a, a_0). We may obtain other faithful state-closed representations of G by different choices of a_0. Let $a_0 = e$. Then we have $a = (e, a_1)\sigma$, and since $o(a) = 2$, we find that $a_1 = e$ and $a = \sigma$. By Lemma 3.5, π_0 is a simple 1/2-endomorphism of $(G_1)^{\pi_0}$. As $|T(G_1)| < |T(G)|$, we conclude by induction that $(G_1)^{\pi_0}$ is either torsion-free or a finite 2-group. Since $a = \sigma$, we have that $G_1 \leq (G_1)^{\pi_0} \times (G_1)^{\pi_0}$ and so, G_1 is either torsion-free or a finite 2-group. We have to discuss the first alternative only. In this case, $T(G) = \langle\sigma\rangle$ and is central. We conclude that $G^2 = G_1^2$ and thus G is an elementary abelian 2-group. □

The question as to which finitely generated solvable groups admit faithful state-closed representations is open. We give below some examples of non-abelian groups with such representations.

Examples. We have shown in Lemma 3.3 of [BS2] that if ξ is a dyadic unit then $\lambda = \lambda^{(1)}(e, \tau^{(\xi-1)/2})$ conjugates $\tau = (e, \tau)\sigma$ to τ^{ξ}. The affine group of the dyadic integers is a metabelian group and is state-closed.

If $\xi = -1$ then $\lambda = \lambda^{(1)}(e, \tau^{-1})$ inverts τ and the group $G = \langle\tau, \lambda\rangle$ is state-closed and polycyclic. If $\xi = 3$, then $\lambda = \lambda^{(1)}(e, \tau)$ conjugates $\tau = (e, \tau)\sigma$ to τ^3 and $G = \langle\tau, \lambda\rangle$ is a metabelian torsion-free recurrent group but is not polycyclic.

More polycyclic examples will be constructed in the last section of this paper.

4. Lattices of finite rank

4.1. Generating pairs

Let G be an m-dimensional lattice. We recall the formulas in Section 3,

$$a_1 = a_0^{-1}\pi_0(a^2),$$

$$\pi_1(h) = \pi_0(h^{a^{-1}})^{a_0} = \pi_0(h).$$

Therefore the elements of G have the following developments:

$$a = (a_0, a_0^{-1}\pi_0(a^2))\sigma,$$

$$h = (\pi_0(h), \pi_0(h)), \quad h \in H.$$

Any subgroup of G of finite index also has rank m, and thus so does the first level stabilizer G_1. We choose a free generating set $\{v_\mathbf{1}, v_\mathbf{2}, \ldots, v_\mathbf{m}\}$ of G such that G_1 is freely generated by $\{2v_\mathbf{1}, v_\mathbf{2}, \ldots, v_\mathbf{m}\}$ and let $a = v_\mathbf{1}$. Consider the vector space $V = \mathbb{Q} \otimes G$ and denote the extension of π_0 to V by A. Then A is an invertible linear transformation of V and the elements of G are represented in additive notation by

$$v_{\mathbf{1}} = (a_0, -a_0 + 2A(v_{\mathbf{1}}))\sigma,$$
$$h = (A(h), A(h)) \text{ for all } h \in G_1.$$

On choosing $r = -a_0 + A(v_{\mathbf{1}})$ in V, $v_{\mathbf{1}}$ may be re-written as

$$v_{\mathbf{1}} = (A(v_{\mathbf{1}}) - r, A(v_{\mathbf{1}}) + r)\sigma.$$

In this form the development of the elements of G simplify to

$$\begin{cases} u = (A(u) - r, A(u) + r)\sigma & \text{for all } u \in G \setminus G_1, \\ h = (A(h), A(h)) & \text{for all } h \in G_1. \end{cases}$$

Note that $r \in G + A(G) \setminus G$. The matrix representation of A with respect to the basis $\{v_{\mathbf{1}}, v_{\mathbf{2}}, \ldots, v_{\mathbf{m}}\}$ has the form

$$\mathbf{A} = \begin{pmatrix} \frac{a_{11}}{2} & a_{12} & \cdots & a_{1m} \\ \frac{a_{21}}{2} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{m1}}{2} & a_{m2} & \cdots & a_{mm} \end{pmatrix}, \tag{1}$$

where all the a_{ij}'s are integers; this is so since A maps the group $\langle 2v_{\mathbf{1}}, v_{\mathbf{2}}, \ldots, v_{\mathbf{m}}\rangle$ into the group $\langle v_{\mathbf{1}}, v_{\mathbf{2}}, \ldots, v_{\mathbf{m}}\rangle$. Considering that G_1 is not A-invariant, the first column $\mathbf{A}(v_{\mathbf{1}})$ is not an integral vector, yet $2\mathbf{A}(v_{\mathbf{1}})$ is integral. The same observation holds for $r = -a_0 + \mathbf{A}(v_{\mathbf{1}})$. The characteristic polynomial $f(x)$ of A has the form $f(x) = x^m + \frac{1}{2}g(x)$ where $g(x)$ is an integral polynomial of degree $m - 1$.

Definition 2. Let $\mathbf{A}$ be an invertible $m \times m$ matrix with rational coefficients. Then $W = \mathbf{A}^{-1}(\mathbb{Z}^m) \cap \mathbb{Z}^m$ is called the $\mathbb{Z}$-*domain* of $\mathbf{A}$. If W has index n in $\mathbb{Z}^m$ then $\mathbf{A}$ is called $1/n$-*integral*. If the restriction of $\mathbf{A}$ to W defines a simple homomorphism then $\mathbf{A}$ is said to be *simple*.

For an arbitrary basis of the group G, the matrix $\mathbf{A}$ associated to the $1/2$-endomorphism π_0 from the stabilizer subgroup G_1 into the group G must be $1/2$-integral. It is clear that a matrix is $1/2$-integral if and only if the sum of any two (not necessary distinct) of its non-integral columns is integral. It follows from the arguments above that any $1/2$-integral matrix is conjugate to a matrix of the type (1). The following proposition, which is an analogue of [BJ, Prop. 10.1], gives a criterion for a $1/n$-integral matrix $\mathbf{A}$ to be simple, when considered as a homomorphism from its $\mathbb{Z}$-domain into $\mathbb{Z}^m$.

Proposition 4.1. *Let* $\mathbf{A}$ *be an* $m \times m$ *matrix over* $\mathbb{Q}$*. Suppose* $\mathbf{A}$ *is an* $1/n$*-integral matrix. Then* $\mathbf{A}$ *is simple if and only if its characteristic polynomial is not divisible by a monic polynomial with integral coefficients. Furthermore, if* n *is a prime number, then* $\mathbf{A}$ *is simple if and only if its characteristic polynomial of* $\mathbf{A}$ *is irreducible.*

Proof. Let $W = \mathbf{A}^{-1}(\mathbb{Z}^m) \cap \mathbb{Z}^m$ be the $\mathbb{Z}$-domain of $\mathbf{A}$.

(i) Suppose $\mathbf{A}$ is not simple and let $\{0\} \neq U \leq W$ be such that $\mathbf{A}(U) \leq U$ and $\mathbf{C}$ be the restriction of $\mathbf{A}$ to U. Then the characteristic polynomial of $\mathbf{C}$ is a monic polynomial with integral coefficients and is a factor of the characteristic polynomial of $\mathbf{A}$. In the other direction, suppose $f(x) = x^k + a_1 x^{k-1} + \cdots + a_k \in \mathbb{Z}[x]$ is an irreducible factor of the characteristic polynomial of $\mathbf{A}$. Let $\widehat{U} \leq \mathbb{Q}^m$ be the kernel of the operator $f(\mathbf{A})$. Then for arbitrary nonzero element $v \in \widehat{U}$ the vectors v, $\mathbf{A}(v)$, $\mathbf{A}^2(v), \ldots \mathbf{A}^{k-1}(v)$ form a basis of the space $\widehat{U}$ and the matrix of the operator $\mathbf{A}|_{\widehat{U}}$ in this basis is obviously integral. Therefore there exists a nonzero integer q such that all the vectors $qv, q\mathbf{A}(v), q\mathbf{A}^2(v), \ldots q\mathbf{A}^{k-1}(v)$ are integral and form a basis of the space $\widehat{U}$ and the matrix of $\mathbf{A}|_{\widehat{U}}$ is integral. Thus $U = \widehat{U} \cap \mathbb{Z}^m$ is a nontrivial invariant group, and $\mathbf{A}$ is not simple.

(ii) Suppose n is a prime number. Then any $1/n$-integral matrix is similar to a matrix of the type

$$\mathbf{A} = \begin{pmatrix} \frac{a_{11}}{n} & a_{12} & \cdots & a_{1m} \\ \frac{a_{21}}{n} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{m1}}{n} & a_{m2} & \cdots & a_{mm} \end{pmatrix}.$$

Let $f(x)$ be the characteristic polynomial of $\mathbf{A}$, then the polynomial $nf(x)$ has integral coefficients. It is clear that in any nontrivial decomposition of $nf(x)$ into a product of two polynomials with integral coefficients one of the factors will have a leading coefficient equal to 1. Thus the irreducibility of $f(x)$ is equivalent to the simplicity of $\mathbf{A}$. □

Corollary 4.1. *Let* G *be an* m*-dimensional lattice and* K *a sub-lattice of* G*. Then* K *is also* m*-dimensional.*

Proof. Suppose K is a proper non-trivial subgroup of G, which is state-closed. Since $\mathbb{Q} \otimes K_1 = \mathbb{Q} \otimes K$ and is A-invariant and as A is irreducible, we get that $\mathbb{Q} \otimes K = \mathbb{Q} \otimes G$ and therefore K and G have equal ranks. □

Example. We cannot conclude in the above corollary that $K = G$. For, let G be the 2-dimensional lattice generated by $\alpha = \left(e, \alpha\beta^2\right)\sigma$, $\beta = (\alpha, \alpha)$. Then $K = \langle \alpha, \beta^2 \rangle$ is a proper sub-lattice of G.

We have obtained the following variant of Theorem 3.1 for torsion-free abelian groups.

Proposition 4.2. *Let $G = \mathbb{Z}^m$ and let $\mathbf{A}$ be an $m \times m$ invertible rational matrix which is $1/2$-integral and simple. Let $H = \mathbf{A}^{-1}(G) \cap G$ and $r \in G + \mathbf{A}(G) \setminus G$. Then the pair $(\mathbf{A}, r)$ determines uniquely a representation φ of the group G as an m-dimensional lattice, by the rules*

$$v^{\varphi} = \begin{cases} \left((\mathbf{A}(v) - r)^{\varphi}, (\mathbf{A}(v) + r)^{\varphi}\right)\sigma & \text{if } v \in G \setminus H \\ (\mathbf{A}(v)^{\varphi}, \mathbf{A}(v)^{\varphi}) & \text{if } v \in H. \end{cases}$$

Let us start again with an abstract torsion-free abelian group G of rank m and a faithful state-closed representation φ of G described by the quadruple (H, ρ, a, a_0). The following steps lead to the description of the representation φ by the pair of parameters $(\mathbf{A}, r)$.

(i) Identify G with the additive group $\mathbb{Z}^m$, fix the canonical basis $\{e_1, e_2, \ldots, e_m\}$. Identify a with e_1 and identify the basis of H with $\{2e_1, e_2, \ldots, e_m\}$.
(ii) Define the vector space $V = \mathbb{Q}^m$ and extend ρ to a linear transformation A of V which is represented in the canonical basis as an irreducible $1/2$-integral matrix $\mathbf{A}$.
(iii) Define $r = -a_0 + \mathbf{A}(e_1)$. Then $H = \mathbf{A}^{-1}(G) \cap G$, $r \in G + \mathbf{A}(G) \setminus G$. Hence the representation φ is described simply by the *generating pair* $(\mathbf{A}, r)$.

If $\mathbf{A}$ is an invertible integral matrix, then for $\mathbf{A}' = \mathbf{TAT}^{-1}$ and $r' = \mathbf{T}r$, the pair $(\mathbf{A}', r')$ defines a state-closed representation φ' of G and $\varphi = T\varphi'$; in particular, $G^{\varphi} = G^{\varphi'}$. Therefore, in classifying the generating pairs $(\mathbf{A}, r)$, we may restrict $\mathbf{A}$ to representatives of the similarity classes of the $1/2$-integral irreducible matrices under conjugation by $\mathbf{T} \in \mathrm{GL}(m, \mathbb{Z})$.

4.2. 1-dimensional lattices

We give below a description of the 1-dimensional lattices.

Theorem 4.1. *Let $G = \langle g \rangle$ be a cyclic group of infinite order, let $\varphi : G \to Aut(\mathcal{T}_2)$ be a faithful state-closed representation of G and denote g^{φ} by α. Furthermore, let $\tau = (e, \tau)\sigma$, $\mu = (e, \mu^{-1})\sigma$. Then there exist integers c, d such that $\alpha = \left(\alpha^c, \alpha^d\right)\sigma$ and $c + d$ odd. If $\alpha = \left(\alpha^c, \alpha^{1-c}\right)\sigma$, then $\tau = \alpha^{-2c+1}$. If $\alpha = \left(\alpha^c, \alpha^{-1-c}\right)\sigma$, then $\mu = \alpha^{2c+1}$. In addition, any $\alpha = \left(\alpha^c, \alpha^d\right)\sigma$ with $c + d$ odd is conjugate to $\tau = (e, \tau)\sigma$ by a functionally recursive automorphism of the tree.*

Proof. Let $(\mathbf{A}, r)$ be the generating pair of φ. Obviously, $H = \langle g^2 \rangle$. On denoting g^φ by α we have $\alpha = (\alpha^c, \alpha^d)\sigma$, for some integers c, d. In additive notation, $\rho : 2\alpha \mapsto (c+d)\alpha$ and so $\mathbf{A} = (\frac{c+d}{2})$ and $r = \frac{d-c}{2}$. Since $\mathbf{A}$ is simple, $f = c + d$ is odd. Let $c + d = \pm 1$. When $\alpha = \left(\alpha^c, \alpha^{1-c}\right)\sigma$, we have

$$\alpha^2 = (\alpha, \alpha), \alpha^{-2c} = \left(\alpha^{-c}, \alpha^{-c}\right), \alpha^{-2c+1} = \left(e, \alpha^{1-2c}\right)\sigma$$

and thus $\tau = \alpha^{-2c+1}$. Similarly, if $\alpha = \left(\alpha^c, \alpha^{-1-c}\right)\sigma$, we have $\mu = \alpha^{2c+1}$. Let $w = (c + d - 1)/2$. Define the automorphisms of the tree, $\lambda = (\lambda, \lambda\alpha^{(-c+d)w})$ and $\gamma = (\lambda^{-1}\gamma, \alpha^{-c}\lambda^{-1}\gamma)$. Then $\{\alpha, \lambda, \gamma\}$ is a functionally recursive set and it can be verified that $\lambda^{-1}\alpha\lambda = \alpha^{c+d}$ and $\gamma^{-1}\alpha\gamma = \tau$. $\square$

5. Finite-state lattices

5.1. Contracting maps

If A is a linear transformation of a vector space over a subfield of the complex numbers, then the *spectral radius* of A, denoted by $\kappa(A)$, is the largest absolute value of the eigenvalues of A. If $\kappa(A) < 1$ then A is a *contracting map*.

Theorem 5.1. *Suppose G is an m-dimensional lattice defined by (A, r). Then G is finite-state if and only if A is a contracting map.*

Proof. Since G is a lattice, the characteristic polynomial $f(x)$ of A is irreducible. Therefore the eigenvalues λ_i of A are all distinct and thus A is diagonalizable over the complex numbers. We will fix a basis $\{e_\mathbf{1}, e_\mathbf{2}, \ldots e_\mathbf{m}\}$ for G, which will be identified with the Euclidean basis for $\mathbb{R}^m$. Let $\{\varepsilon_\mathbf{1}, \varepsilon_\mathbf{2}, \ldots, \varepsilon_\mathbf{m}\}$ be the basis of $\mathbb{C} \otimes G$, formed by the eigenvectors of A such that $A(\varepsilon_\mathbf{i}) = \lambda_i \varepsilon_\mathbf{i}$. Let $(\xi_1(v), \xi_2(v), \ldots, \xi_m(v))$ be the coordinate vector of $v \in G$ with respect to this basis.

(i) Suppose by contradiction that G is finite state and $\kappa(A) \geq 1$. Let $\lambda = \lambda_1$, $|\lambda| \geq 1$. Since the group G has rank m there exists a vector $v \in G$ with a nonzero first coordinate $\xi_1(v)$. Let us fix this v. We are going to find a sequence $\{v_\mathbf{n}\}$ of states of v such that the sequence $\{|\xi_1(v_\mathbf{n})|\}$ is nondecreasing. Using this we will prove that v has infinitely many states. The sequence will be defined inductively. Set $v_\mathbf{0} = v$. For every $n \geq 0$, $v_\mathbf{n+1}$ is a state of $v_\mathbf{n}$ and is defined by the rules below.

If $v_\mathbf{n}$ is inactive then $v_\mathbf{n+1} = A(v_\mathbf{n})$. Then $v_\mathbf{n+1}$ is a state of $v_\mathbf{n}$ and $\xi_1(v_\mathbf{n+1}) = \lambda\xi_1(v_\mathbf{n})$ with $|\ \xi_1(v_\mathbf{n+1})\ | = |\ \lambda\ | \cdot |\ \xi_1(v_\mathbf{n})\ | \geq |\ \xi_1(v_\mathbf{n})\ |$.

If $v_\mathbf{n}$ is active then the vectors $A(v_\mathbf{n}) + r$ and $A(v_\mathbf{n}) - r$ are states of $v_\mathbf{n}$. If $\xi_1(r) \neq 0$ then either $|\xi_1\left(A\left(v_\mathbf{n}\right) + r\right)| > |\xi_1\left(A\left(v_\mathbf{n}\right)\right)| \geq |\xi_1(v_\mathbf{n})|$ or $|\xi_1\left(A\left(v_\mathbf{n}\right) - r\right)| > |\xi_1\left(A\left(v_\mathbf{n}\right)\right)| \geq |\xi_1(v_\mathbf{n})|$. Then we choose $v_\mathbf{n+1}$ to be equal to one of the vectors $A(v_{\mathbf{n}}) + r$, $A(v_{\mathbf{n}}) - r$ so that $|\xi_1(v_{\mathbf{n+1}})| > |\xi_1(v_{\mathbf{n}})|$. If $\xi_1(r) = 0$ then we choose $v_{\mathbf{n+1}} = A\,(v_{\mathbf{n}}) + r$. Then $|\xi_1(v_{\mathbf{n+1}})| = |\lambda| \cdot |\xi_1\,(v_{\mathbf{n}})| \geq |\xi_1\,(v_{\mathbf{n}})|$.

An infinite number of elements from the sequence $\{v_{\mathbf{n}}\}$ are active, otherwise all the vectors $A^n(v)$ are eventually integral which would contradict the simplicity of the matrix $\mathbf{A}$. Thus in the case where $\xi_1(r) \neq 0$, the sequence $\{|\xi_1(v_{\mathbf{n}})|\}$ is nondecreasing and has infinitely many different elements. If $\xi_1(r) = 0$ then $\xi_1(v_{\mathbf{n}}) = \lambda^n \xi_1(v)$. Therefore the sequence $\{\xi_1(v_{\mathbf{n}})\}$ may contain a finite number of different elements only when λ is a root of unity. But by Proposition 4.1 this contradicts the simplicity of $\mathbf{A}$. Hence in all cases, v has an infinite number of different states.

(ii) Suppose $\kappa = \kappa(A) < 1$. We will prove that every element $v \in G$ has finite number of states. Define the max-norm of a vector by $\|u\| = \max\{|\xi_1(u)|, \ldots, |\xi_m(u)|\}$. Then $\|A^n(u)\| \leq \kappa^n \|u\|$ for all $n \in \mathbb{N}$ and $u \in G$. Any state of v, seen as a tree automorphism, is equal to v or to a vector of the type $A^n(v) \pm A^{n_1}(r) \pm A^{n_2}(r) \pm \cdots \pm A^{n_l}(r)$, where $n > n_1 > n_2 > \ldots > n_l > 0$ is a decreasing sequence of positive integers. The norms of the latter states can be estimated as follows:

$$
\begin{aligned}
&\|A^n(v) \pm A^{n_1}(r) \pm A^{n_2}(r) \pm \cdots \pm A^{n_l}(r)\| \leq \\
&\qquad \|A^n(v)\| + \|A^{n_1}(r)\| + \|A^{n_2}(r)\| + \cdots + \|A^{n_l}(r)\| \leq \\
&\quad \|A^n(v)\| + \|A^{n-1}(r)\| + \|A^{n-2}(r)\| + \cdots + \|A(r)\| + \|r\| \leq \\
&\qquad\qquad \kappa^n \|v\| + \left(\kappa^{n-1} + \kappa^{n-2} + \cdots + \kappa + 1\right) \|r\| \leq \\
&\qquad\qquad\qquad \kappa^n \|v\| + \sum_{s=0}^{\infty} \kappa^s \|r\| \leq \|v\| + (1-\kappa)^{-1} \|r\|.
\end{aligned}
$$

Therefore all the states of the tree automorphism v lie inside some finite ball with respect to the max-norm which itself is contained in a finite Euclidean ball. As the states are integral vectors, there exists only a finite number of them. $\square$

Example. Let $\mathbf{A}$ be the $m \times m$ matrix

$$
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & 1 \\
1/2 & 0 & \ldots & \ldots & 0
\end{pmatrix}.
$$

Then the characteristic polynomial of $\mathbf{A}$ is $f(x) = x^m - 1/2$ and $\mathbf{A}$ is an irreducible 1/2-integral matrix with spectral radius $\sqrt[m]{1/2}$. If we choose $r = \frac{1}{2} e_m$ then the generating pair $(\mathbf{A}, r)$ defines a group G with the following set of free

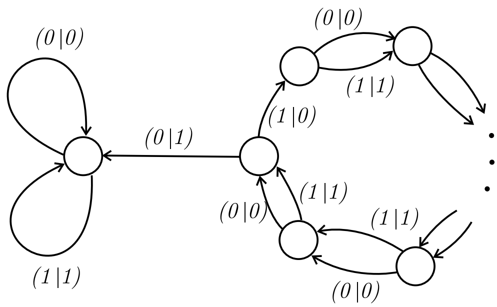

Figure 13.3 Automaton generating $\mathbb{Z}^n$

generators

$$\alpha = (e, \alpha^{(m-1)})\sigma, \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m-1)}.$$

Thus, G is state-closed and finite-state. Indeed G is recurrent, as the first level stabilizer G_1 is freely generated by $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m-1)}, \alpha^2 = (\alpha^{(m-1)}, \alpha^{(m-1)}) = \alpha^{(m)}$ whose projection on the first coordinate produces G. The automata corresponding to the generators of G are obtained by choosing different initial states in the Moore diagram shown on Figure 13.3.

The group G is a minimal lattice. To prove this, it is sufficient to show that the state-closure of any non-trivial cyclic subgroup generated by $\beta = \alpha^{i_0}\alpha^{(1)i_1}\alpha^{(2)i_2}\ldots\alpha^{(m-1)i_{m-1}}$ is the whole group G. We define a norm of β by

$$\mid \beta \mid = \mid i_0 \mid + \mid i_1 \mid + \cdots + \mid i_{m-1} \mid .$$

Consider a minimal counterexample; that is, β has minimal non-zero norm such that the set of states $Q(\beta)$ does not generate G. We may choose β such that $i_0 \geq 0$. If $i_0 = 2i'$ then $\beta = (\beta_0, \beta_0) \in G_1$ and $\beta_0 = \alpha^{(m-1)i'}\alpha^{i_1}\alpha^{(1)i_2}\ldots\alpha^{(m-2)i_{m-1}}$. If $i_0 = 2i' + 1$ then $\beta = (\beta_0, \beta_1)\sigma$ where again $\beta_0 = \alpha^{(m-1)i'}\alpha^{i_1}\alpha^{(1)i_2}\ldots\alpha^{(m-2)i_{m-1}}$ which has norm $\mid i' \mid + \mid i_1 \mid + \cdots + \mid i_{m-1} \mid$. Thus by the minimality of β, we have $i_0 = 0$. A repetition of this argument leads to a contradiction.

Theorem 5.2. *Let G be finite-state lattice of rank m defined by the pair (A, r). Then G is a recurrent lattice and the characteristic polynomial of A^{-1} is an integral monic irreducible polynomial. Furthermore, for a fixed m, there exist only finitely many* $\mathrm{GL}(m, \mathbb{Z})$*-similarity classes of linear transformations A which are* $1/2$*-integral, irreducible and contracting.*

Proof. Let $f(x) = x^m + \frac{1}{2}(a_{m-1}x^{m-1} + \ldots + a_0)$ be the characteristic polynomial of A. Then as the spectral radius of A is less than 1, and since the coefficients

$\frac{1}{2}a_i$ of $f(x)$ are symmetric polynomials in the roots, we get that $|a_i| < 2\binom{m}{i}$. In particular, $1 \le |a_0| < 2$ and $|a_0| = 1$. Since the a_i's are integers, the number of possible characteristic polynomials $f(x)$ is finite. As the matrix $\mathbf{A}$ is conjugate to a matrix of the type (1) and has determinant $\pm\frac{1}{2}$, the image of G_1 under A is G; that is, A is recurrent. Now, the characteristic polynomial of A^{-1} is the integral irreducible polynomial $h(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + 2$, where $|a_0| = 1$. By a theorem of Latimer and MacDuffee [N] the number of GL$(m, \mathbb{Z})$-similarity classes of linear transformations with characteristic polynomial $h(x)$ is equal to the number of ideal classes in the order $\mathbb{Z}[\alpha]$, where α is a root of $h(x)$, and therefore is finite. We conclude that there is a finite number of similarity classes of A^{-1} and therefore of A as well. $\square$

Remarks. (i) Although a finite-state lattice is recurrent, the converse is not necessarily true. For example, let G be determined by $(\mathbf{A}, \mathbf{r})$ where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1/2 & 1 \end{pmatrix}$. Then, since $\det(\mathbf{A}) = -1/2$, the group G is recurrent. However, it is not finite-state since $\kappa(\mathbf{A}) > 1$.

(ii) If a lattice G is finite-state we cannot conclude that it is a minimal lattice. For, let $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 5/2 & -2 \end{pmatrix}$ and let G be the group generated by $\alpha = (e, \alpha^2\beta^5)\sigma$, $\beta = (\alpha^{-1}\beta^{-2}, \alpha^{-1}\beta^{-2})$. We verify directly that the states of α are contained in $K = \langle \alpha, \beta^5 \rangle$ and therefore G is not a minimal lattice.

5.2. Finite-state lattices of small rank

Theorem 5.3. *Consider the automorphisms of the binary tree $\tau = (e, \tau)\sigma$, $\mu = (e, \mu^{-1})\sigma$. A subgroup G of automorphisms of the binary tree is a 1-dimensional finite-state lattice if and only if it is generated by an l-th root of τ or μ for some odd integer l.*

Proof. Let $G = \langle \alpha \rangle$ be a 1-dimensional finite-state lattice with generating pair (A, r). By Proposition 5.1, $A = \pm 1/2$. When $(A, r) = (1/2, 1/2)$, we get a cyclic group generated by the adding machine $\tau = (e, \tau)\sigma$ and when $(A, r) = (-1/2, -1/2)$ we get a cyclic group generated by $\mu = (e, \mu^{-1})\sigma$. By Theorem 4.1, α is an l-th root of τ or μ for some odd integer l. On the other hand it can be checked directly that a group generated by an l-th root of τ or μ for an odd integer l is state-closed and finite-state. $\square$

Theorem 5.4. *Let $\mathbf{A}$ be a 2×2 rational $1/2$-integral matrix. Then $(\mathbf{A}, r)$ is a generating pair for a 2-dimensional lattice if and only if $\mathbf{A}$ is GL$(2, \mathbb{Z})$-similar to* *the companion matrix of one of the six polynomials* $x^2 \pm \frac{1}{2}$, $x^2 \pm \frac{1}{2}x + \frac{1}{2}$, $x^2 \pm x + \frac{1}{2}$.

Proof. The matrix $\mathbf{A}$ is equivalent to $\begin{pmatrix} \frac{1}{2}a_{11} & a_{12} \\ \frac{1}{2}a_{21} & a_{22} \end{pmatrix}$ where the a_{ij}'s are integers. By Theorem 5.2 its characteristic polynomial is $f(x) = x^2 + \frac{1}{2}a_1 x + \frac{1}{2}a_0$ with $|a_0| < 2$, $|a_1| < 4$. Thus, $|a_0| = 1$, $|a_1| = 0, 1, 2, 3$. The roots of $f(x)$ are $z_1 = \frac{1}{4}(-a_1 + \sqrt{a_1^2 - 8a_0})$, $z_2 = \frac{1}{4}(-a_1 - \sqrt{a_1^2 - 8a_0})$. If $\Delta = a_1^2 - 8a_0 \leq 0$ then $a_0 = 1$, $|a_1| = 0, 1, 2$ follow. On the other hand, suppose $\Delta = a_1^2 - 8a_0 > 0$; that is, $a_1^2 \geq 8a_0$. If $a_0 = 1$ then $a_1 = \pm 3$, $\Delta = 1$, and so $|z_1|$ or $|z_2| = 1$. Thus, $a_0 = -1$, $\Delta = a_1^2 + 8$. On substituting the possible values of a_1 in z_i we find that the only possibility for a_1 is 0. Hence the pair (a_0, a_1) varies over the set $\{(1,0), (1,\pm 1), (1,\pm 2), (-1,0)\}$.

Now we try to determine the possible reductions of the matrix $\mathbf{A}$ modulo conjugations by invertible integral matrices $\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$. On conjugating $\mathbf{A}$ by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, we may assume $a_{21} \geq 0$. We have trace$(\mathbf{A}) = \frac{1}{2}a_{11} + a_{22} = -\frac{1}{2}a_1$, $\det(\mathbf{A}) = \frac{1}{2}a_{11}a_{22} - \frac{1}{2}a_{21}a_{12} = \frac{1}{2}a_0$. Therefore, $a_{22} = -\frac{1}{2}a_1 - \frac{1}{2}a_{11}$, $a_{21}a_{12} = -(\frac{1}{2}a_{11}^2 + \frac{1}{2}a_1 a_{11} + a_0)$. On conjugating $\mathbf{A}$ by $\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ we transform $\frac{1}{2}a_{11}$ into $\frac{1}{2}a_{11} + s a_{12}$, and on conjugating by $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, we transform $\frac{1}{2}a_{11}$ into $\frac{1}{2}(a_{11} - s a_{21})$. Thus, we can reduce the absolute value of $\frac{1}{2}a_{11}$ unless possibly when $|a_{11}| \leq |a_{12}|, \frac{1}{2}|a_{21}|$. Hence we may assume, $2a_{11}^2 \leq \left|\frac{1}{2}a_{11}^2 + \frac{1}{2}a_1 a_{11} + a_0\right|$. The possible solutions (a_0, a_1, a_{11}) are contained in the table

a_0	1	1	1	1	1	-1
a_1	0	-1	1	2	-2	0
a_{11}	0	$-1, 0$	$0, 1$	$0, 1$	$-1, 0$	0

On using the fact that $a_{22} = -\frac{1}{2}(a_1 + a_{11})$ is an integer, the possibilities for a_{11} are reduced further and on using the formula $a_{12}a_{21} = -(\frac{1}{2}a_{11}^2 + \frac{1}{2}a_1 a_{11} + a_0)$, we obtain the following table

a_0	1	1	1	1	1	-1
a_1	0	-1	1	2	-2	0
a_{11}	0	-1	1	0	0	0
a_{22}	0	1	-1	-1	1	0
$a_{12}a_{21}$	-1	-2	-2	-1	-1	1

The different columns of the table correspond to different characteristic polynomials. As we may always choose $a_{21} > 0$, there are at most 8 classes. Further equivalences may occur only within the same column of the table. Suppose a_{21} is an even integer $2a'_{21}$, as may occur in the second and third columns. Then $\begin{pmatrix} a_{22} & -2a_{12} \\ -a'_{21} & a_{11} \end{pmatrix}$ conjugates $\mathbf{A}$ into $\begin{pmatrix} \frac{1}{2}a_{11} & 2a_{12} \\ \frac{1}{2}a'_{21} & a_{22} \end{pmatrix}$. Thus, in the third and fourth columns a_{21} may be chosen to be equal to 1. Considering that $|a_{11}| = 1$ in these columns, we may conjugate the corresponding matrix into one where $a_{11} = 0$. Hence, the equivalence classes of $\mathbf{A}$ are represented uniquely by the companion matrix of the characteristic polynomial of $\mathbf{A}$. $\square$

Remark. The polynomials $f(x) = x^m + \frac{1}{2}(a_{m-1}x^{m-1} + \ldots + a_0)$ such that the absolute value of each of their roots is less than 1 present some remarkable features. As we showed in Theorem 5.2, there is a finite number of such polynomials for each fixed degree m. The proof of the above theorem shows that the number for $m = 2$ is 6, and that the class number corresponding to each polynomial is 1. We note that Pavel Guerzhoy has investigated with the use of the Number Theoretic package PARI-GP ([PG] polynomials of higher degree. He reproduced our 6 polynomials of degree 2, produced a complete list of 14 polynomials of degree 3, 36 polynomials of degree 4 and 58 of degree 5. In addition, it turns out that the class number in all these cases is 1. We note that though the class number eventually increases for higher degrees, still 1 seems to be quite predominant.

6. Recurrent lattices

6.1. Adding machines

As we commented earlier, the automorphism $\tau = (e, \tau)\sigma$ which generates a lattice of rank 1 represents binary addition in its action on the dyadic integers and was thus called an adding machine. We will show below that the notion of an adding machine can be generalized to lattices of arbitrary rank. Let G be an m-dimensional recurrent lattice defined by the generating pair (A, r). Also let $A^{-k}(G) = \{v \in G \mid A^k(v) \in G\}$ and recall that G_k is the pointwise stabilizer of the k-th level of the tree.

Lemma 6.1. *The group $A^{-n}(G)$ coincides with G_n for every n.*

Proof. We proceed by induction on n. We have $A^0(G) = G = G_0$. Assume $n \geq 1$. The group $A^{-n}(G)$ stabilizes the first level. For every $g \in A^{-n}(G)$, we have $g = (A(g), A(g))$ and $A(g) \in A^{-n+1}(G)$ which by the inductive hypothesis is the stabilizer of the $(n-1)$-st level of the tree. Thus $g \in G_n$ and $A^{-n}(G) \leq G_n$. On the other hand, we have $A(G_1) = G$, $A(G_2) \leq G_1, \ldots, A(G_n) \leq G_{n-1}$, and thus $G_n \leq A^{-n}(G)$. Hence $G_n = A^{-n}(G)$. □

Since $A(G_1) = G$, it follows that there exists $d \in G \setminus G_1$ such that $r = A(d)$; we fix such a d. Therefore, $A^{-1}(d) \in G_1 \setminus G_2$ and for all $n \geq 0$, $A^{-n}(d) \in G_n \setminus G_{n+1}$. We define an A-adic number as the series $c_0 d + c_1 A^{-1}(d) + \cdots + c_n A^{-n}(d) + \cdots$ where the c_n's are 0 or 1. These A-dic numbers belong to the closure of $\widehat{G}$ of the group G in the automoprhism group of the tree with respect to the 2-adic topology. We will show that the group G acts as an adding machine on these numbers.

Lemma 6.2. *The set of all A-adic numbers coincides with the topological closure $\widehat{G}$ of the group G. The map Ψ from the boundary of the binary tree to $\widehat{G}$, defined by*

$$\Psi(c_0, c_1, \ldots, c_n, \ldots) = c_0 d + c_1 A^{-1}(d) + \cdots + c_n A^{-n}(d) + \cdots$$

is well-defined and bijective.

Proof. We have to prove that the series $c_0 d + c_1 A^{-1}(d) + c_2 A^{-2}(d) + \cdots + c_n(A^{-n} d) + \cdots$ is convergent; equivalently, the sequence of its partial sums is a Cauchy sequence. This follows directly from the fact that $G_n = A^{-n}(G)$. Thus the map Ψ is well defined. In order to prove that every element of $\widehat{G}$ can be uniquely expanded in such a way, it is sufficient to prove that for every n and every $g \in G$ there exists a unique element of the form $g_n = c_0 d + c_1 A^{-1}(d) + \cdots + c_n A^{-n}(d)$ where $c_i \in \{0, 1\}$ such that $g - g_n \in G_n$. Let us prove this by induction on n. If $g \in G_1$ then $g - d \notin G_1$. On the other hand, if $g \notin G_1$ then $g - d \in G_1$. Thus the assertion is true for $n = 1$. Suppose it is true for $n = k-1$, $k \geq 2$. Since $g - g_{k-1} \in G_{k-1} = A^{-k+1}(G)$ and A is injective, there exists unique $h \in G$ such that $g - g_{k-1} = A^{-k+1}(h)$. If $h \in G_1 = A^{-1}(G)$ then $g - g_{k-1} \in A^{-k}(G) = G_k$ and we put $g_k = g_{k-1}$. Then $g - g_k \in G_k$ but $g - \left(g_{k-1} + A^{-k+1}(d)\right) = A^{-k+1}(h) - A^{-k+1}(d) = A^{-k+1}(h-d) \notin G_k$. If $h \notin G_1$ then $h - d \in G_1 = A^{-1}(G)$ and if we put $g_k = g_{k-1} + A^{-k+1}(d)$ then $g - g_k = A^{-k+1}(h-d) \in A^{-k}(G) = G_k$ but $g - g_k = A^{-k+1}(h) \notin A^{-k}(G) = G_k$. Thus, in any case, g_k can be chosen uniquely. □

The following proposition shows that the above identification Ψ is compatible with the action of G both on $\widehat{G}$ and on the boundary of the tree.

Proposition 6.1. *Let Ψ be the bijection from the boundary of the tree to $\widehat{G}$. Then for every element w of the boundary and $g \in G$ we have*

$$w^g = \Psi^{-1}\left(\Psi(w) + g\right).$$

Proof. Let $g \in G$, $w = (c_0, c_1, \dots)$ be arbitrary infinite path of the binary tree ($c_i \in \{0, 1\}$) and $w' = (c_1, c_2, \dots)$. Denote by $\psi(g)$ the tree automorphism defined by the rule

$$w^{\psi(g)} = \Psi^{-1}\left(\Psi\left(w\right) + g\right).$$

We have to prove that $\psi(g) = g$. If $g \in G_1$ then

$$\begin{aligned} g + \Psi(w) = g + c_0 d + c_1 A^{-1}(d) + \cdots = c_0 d \\ + A^{-1}\left(A\left(g\right) + c_1 d + c_2 A^{-1}\left(d\right) + \cdots\right). \end{aligned}$$

Therefore $w^{\psi(g)} = \Psi^{-1}\left(\Psi\left(w\right) + g\right)$ is the sequence $\left(c_0, \left(w'\right)^{\psi(g_0)}\right)$ where, $g_0 = A(g)$. Thus, in this case, $\psi(g) = \left(\psi\left(A\left(g\right)\right), \psi\left(A\left(g\right)\right)\right)$. If $g \notin G_1$ and $c_0 = 0$ then

$$\begin{aligned} g + \Psi(w) &= g + c_1 A^{-1}(d) + \cdots = d + (g - d) + c_1 A^{-1}(d) + \cdots \\ &= 1 \cdot d + A^{-1}(A\left(g - d\right) + c_1 d + + c_2 A^{-1}\left(d\right) + \cdots). \end{aligned}$$

If $g \notin G_1$ and $c_0 = 1$ then

$$\begin{aligned} g + \Psi(w) = g + d + c_1 A^{-1}(d) + c_2 A^{-2}\left(d\right) \cdots = 0 \cdot d \\ + A^{-1}(A\left(g + d\right) + c_1 + c_2 A^{-1}\left(d\right) + c_3 A^{-2}\left(d\right) + \cdots). \end{aligned}$$

Therefore in this case

$$\begin{aligned} \psi(g) &= \left(\psi\left(A\left(g - d\right)\right), \psi\left(A\left(g + d\right)\right)\right)\sigma \\ &= \left(\psi\left(A\left(g\right) - r\right), \psi\left(A\left(g\right) + r\right)\right)\sigma. \end{aligned}$$

We conclude that the map ψ satisfies the same recurrence as that which defines the group G and therefore $\psi(g) = g$ for every $g \in G$. $\square$

Now it follows from Proposition 6.1 that the natural action of $\widehat{G}$ on itself is identified, by using the bijection Ψ, with the action of $\widehat{G}$ on the tree.

Example. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}$, $r = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$. Then $\mathbf{A}^{-1} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and the multiplicative group G is freely generated by $\alpha = (e, \alpha^{(1)})\sigma, \alpha^{(1)}$. In additive notation, G has the basis $\alpha = v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha^{(1)} = v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We compute

$$d = \mathbf{A}^{-1} r = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_1, \mathbf{A}^{-1} d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_2, \dots ,$$

$$\mathbf{A}^{-2i} d = 2^i v_1, \mathbf{A}^{-(2i+1)} d = 2^i v_2, \dots$$

An element $w \in \widehat{G}$, in additive notation, has the form

$$w = \xi_1 v_1 + \xi_2 v_2,$$

where $\xi_1 = \sum c_i 2^i$, $\xi_2 = \sum d_i 2^i$ are dyadic integers. The element of the boundary which corresponds to w is $\Psi^{-1}(w) = (f_0, f_1, \dots, f_{2i}, f_{2i+1}, \dots)$ where $f_{2i} = c_i$, $f_{2i+1} = d_i$. Now, it can be checked that applying α to $f = \sum f_i 2^i$ corresponds to calculating $\Psi^{-1}(w + v_1)$.

6.2. Topological closure of recurrent lattices

Let G be a recurrent lattice and (A, r) its generating pair. We will prove in this section that A determines the topological closure $\widehat{G}$ of G, independently of r. Recall from Proposition 3.2 that the centralizer of a recurrent abelian group G is equal to $\widehat{G}$. As was mentioned earlier, there exists $d \in G \setminus G_1$ such that $r = A(d)$. Then the closure $\widehat{G}$, by Proposition 6.1, can be naturally identified with the group of formal series of the form $c_0 d + c_1 A^{-1}(d) + c_2 A^{-2}(d) + \cdots$, $c_i \in \{0, 1\}$.

Lemma 6.3. *For any formal power series $f(x) = 1 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots \in \mathbb{Z}[[x]]$ the operator $f(A^{-1}) = I + b_1 A^{-1} + b_2 A^{-2} + \cdots + b_n A^{-n} + \cdots$ is a well-defined continuous automorphism of the group $\widehat{G}$.*

Proof. For any $g \in \widehat{G}$, the sequence of partial sums of the series $g + b_1 A^{-1}(g) + b_2 A^{-2}(g) + \cdots + b_n A^{-n}(g) + \cdots$ is a Cauchy sequence and thus the series is convergent to an element of the group $\widehat{G}$. Thus $f(A^{-1})$ is an endomorphism of the group $\widehat{G}$ which leaves invariant $G_n = A^{-n}(G)$ for all $n \geq 0$. Hence, $f(A^{-1})$ is a continuous function. Now, since every series $f(x) = 1 + b_1 x + \cdots + b_n x^n + \cdots \in \mathbb{Z}[[x]]$ is a unit in this ring, the endomorphism $f(A^{-1})$ is an automorphism of the group $\widehat{G}$. □

Theorem 6.1. *Let L and M be two recurrent m-dimensional lattices with the respective generating pairs $(A, A(d_1))$ and $(A, A(d_2))$. Then their closures $\widehat{L}$ and $\widehat{M}$ are equal.*

Proof. Since d_2 is an integral vector, there exists a sequence $(c_0, c_1, \dots, c_n, \dots)$ with $c_n \in \{0, 1\}$ such that $d_2 = c_0 d_1 + c_1 A^{-1}(d_1) + \cdots + c_n A^{-n}(d_1) + \cdots$. Since $A(d_2)$ is not integral, $c_0 = 1$. By Lemma 6.3, the sum $B = 1 + c_1 A^{-1} + c_2 A^{-2} + \cdots + c_n A^{-n} + \cdots$ defines an automorphism of the group $\widehat{L}$. Obviously, A commutes with B and thus the generating pair of the group $B^{-1}(L)$ is $\left(BAB^{-1}, BA(d_1)\right) = (A, A(d_2))$. We conclude that $B^{-1}(L) = M$ and $\widehat{M} = \widehat{L}$. □

We conclude that recurrent lattices defined by matrices with the same characteristic polynomial have equal completions, irrespective of the vector r.

7. Finite-state representations of affine groups

Let V be the free abelian group of rank m generated by the canonical basis of column vectors $\{v_{\mathbf{1}}, v_{\mathbf{2}}, \dots, v_{\mathbf{m}}\}$ and let W be the subgroup of V generated by $\{2v_{\mathbf{1}}, v_{\mathbf{2}}, \dots, v_{\mathbf{m}}\}$. Consider the matrix defined on the vector space $\mathbb{Q} \otimes V$,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1 \\ 1/2 & 0 & \dots & \dots & 0 \end{pmatrix}$$

Conjugation of elementary transformations $E_{i,j}(t)$ by $\mathbf{A}^{-1}$ has the following effect:

$$\mathbf{A}E_{i,j}(t)\mathbf{A}^{-1} = E_{i-1,j-1}(\delta t),$$

where the indices are written modulo m and

$$\delta = \begin{cases} 1 & \text{if } i = j, \text{ or } 1 < i, j, \\ 2 & \text{if } 1 = j < i, \\ 1/2 & \text{if } 1 = i < j. \end{cases}$$

Let Γ be the subgroup of $\mathrm{GL}(m, \mathbb{Z})$ consisting of matrices $\mathbf{B} = (b_{ij})$ where b_{ij} is even for $i < j$. Then Γ is a maximal subgroup of $\mathrm{GL}(m, \mathbb{Z})$ with respect to the property of not containing non-trivial elements of odd order. The group Γ is generated by the set of elementary matrices

$$S = \left\{E_{i,j}(1) \text{ for } i > j,\ E_{i,j}(2) \text{ for } i < j,\ E_{i,i}(-1) \text{ for all } i\right\}.$$

We verify the following conditions:

$$W \text{ is } \Gamma\text{-invariant, } \mathbf{A}(W) = V, \mathbf{A}\Gamma\mathbf{A}^{-1} = \Gamma.$$

Let G be the semi-direct product of V by Γ. The elements of G are written as $(v, \mathbf{B})$, where $v \in V$, $\mathbf{B} \in \Gamma$ and the product is defined by $(v, \mathbf{B})(v', \mathbf{B}') = (v + \mathbf{B}(v'), \mathbf{B} \cdot \mathbf{B}')$. We simplify the notation $(v, \mathbf{B})$ as $v \cdot \mathbf{B}$. Then $H = W \cdot \Gamma$ is a subgroup of index 2 in G. Also, $G = VH$ and, clearly, V is self-centralizing in G. Define the $1/2$-endomorphism $\rho : H \to G$ by

$$\rho : w \cdot \mathbf{B} \mapsto \mathbf{A}(w) \cdot \mathbf{ABA}^{-1}.$$

Then ρ is an isomorphism from H onto G and so by using it we may define a recurrent representation φ of G on the binary tree. Since ρ restricted to W is simple we conclude from Lemma 3.3 that ρ is also simple and so, the representation φ is faithful.

7.1. Polycyclic examples

Using the affine group, we can prove the existence of subgroups of G which are torsion-free polycyclic and non-abelian. To this effect let $\mathbf{C} = \mathbf{A^{-1}} - \mathbf{I}$. Then $\mathbf{C} \in \Gamma$; this follows from the fact that $x^n - 2 = (x-1)(x^{n-1} + \cdots + x + 1) - 1$. Since $\mathbf{A^{-1}}$ is irreducible on $\mathbb{Q} \otimes V$ then so is $\mathbf{C}$. Let $M = V\langle \mathbf{C} \rangle$. Then by Lemma 3.3, the map ρ induces a simple $1/2$-endomorphism on M. One may generalize this construction by considering the group $\mathbf{U}$ of units in the ring of algebraic integers of $\mathbb{Q}[\mathbf{C}]$. Then $\mathbf{U}$ is a subgroup of Γ. Dirichlet's Unit Theorem [BSh] provides us with a torsion-free abelian subgroup F of D of higher rank for polynomials of higher degrees m. Then $M = VF$ is a state-closed metabelian polycyclic group.

7.2. Finite-state state-closed representation of affine groups

We proceed to construct a concrete representation $\varphi : G\,(= V\Gamma) \to Aut(\mathcal{T}_2)$ and show that the image is generated by finite-state automorphisms of the binary tree.

Choose $\left(v_1^{\varphi}\right)_0 = e$. Then,

$$\begin{aligned} v_1^{\varphi} &= (e, 2\mathbf{A}(v_1)^{\varphi})\sigma = (e, v_m^{\varphi})\sigma, \\ v_2^{\varphi} &= (v_1^{\varphi}, v_1^{\varphi}), \ldots, \\ v_i^{\varphi} &= (v_{i-1}^{\varphi}, v_{i-1}^{\varphi}), \ldots, \\ v_m^{\varphi} &= (v_{m-1}^{\varphi}, v_{m-1}^{\varphi}). \end{aligned}$$

Drop φ from the notation; so,

$$\begin{aligned} v_1 &= (e, v_m)\sigma, \\ v_i &= (v_{i-1}, v_{i-1}) \text{ for } 2 \le i \le m-1, \\ v_m &= (v_{m-1}, v_{m-1}). \end{aligned}$$

The representation of $\mathbf{B} \in \Gamma$ is given by

$$\mathbf{B}^{\varphi} = (\rho(\mathbf{B})^{\varphi}, \rho(\mathbf{B}^{v_1^{-1}})^{\varphi}) = (\rho(\mathbf{B})^{\varphi}, (A(I - \mathbf{B})v_1)) \cdot \rho(\mathbf{B})^{\varphi});$$

and so, in particular, for the generators of Γ in S,

$$E_{i,j}(t)^{\varphi} = (E_{i-1,j-1}(\delta t)^{\varphi}, \big(\mathbf{A}(I - E_{i,j}(t))v_1\big) \cdot E_{i-1,j-1}(\delta t)^{\varphi}).$$

Note that

$$\mathbf{A}(I - E_{i,j}(t))v_1 = \begin{cases} 0 & \text{if } j \neq 1, \\ -tv_{i-1} & \text{if } i \neq 1,\ j = 1, \\ \frac{1}{2}\,(1 - t)\,v_m & \text{if } i = 1,\ j = 1. \end{cases}$$

Hence, on removing the φ from the notation, we have

$$E_{i,j}(t) = \begin{cases} (E_{i-1,j-1}(\delta t), E_{i-1,j-1}(\delta t)) & \text{if } j \neq 1, \\ (E_{i-1,m}(\delta t), (-tv_{i-1}) \cdot E_{i-1,m}(\delta t)) & \text{if } i \neq 1,\ j = 1, \\ (E_{m,m}(\delta t), \big(\frac{1-t}{2} v_m\big) \cdot E_{m,m}(\delta t)) & \text{if } i = 1,\ j = 1. \end{cases}$$

We have arrived finally at the form the generators take in this representation:

$$\begin{aligned} E_{1,1}(-1) &= (E_{m,m}(-1), v_m \cdot E_{m,m}(-1)), & \\ E_{j,j}(-1) &= (E_{j-1,j-1}(-1), E_{j-1,j-1}(-1)) & \text{if } 1 < j, \\ E_{1,j}(2) &= (E_{m,j-1}(1), E_{m,j-1}(1)) & \text{if } 1 < j, \\ E_{i,j}(2) &= (E_{i-1,j-1}(2), E_{i-1,j-1}(2)) & \text{if } 1 < i < j, \\ E_{i,1}(1) &= (E_{i-1,m}(\delta), (-v_{i-1}) \cdot E_{i-1,m}(\delta)) & \text{if } 1 < i, \\ E_{i,j}(1) &= (E_{i-1,j-1}(1), E_{i-1,j-1}(1)) & \text{if } 1 < j < i, \end{aligned}$$

The representation of G is finite-state. This is so because the set of states of each v_j is equal to $\{e, v_i \mid 1 \leq i \leq m\}$ and the states of the given generators $E_{i,j}(t)$ of Γ are contained in the product of sets $Y \cdot S$ where $Y = \{e, \pm v_i \mid 1 \leq i \leq m\}$.

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