Fermions

For fermions, $P_{ij}\,\psi = -\psi$

$$\psi(x_1, x_2) = -\psi(x_2, x_1)$$

So we define

$$|x_1, x_2, \cdots\rangle_- \equiv \frac{1}{\sqrt{\#\text{Perm}}} \sum_{\text{Perm}} (-1)^P \,|\text{Perm}\{x\}\rangle$$

e.g. $|1,2\rangle_- = \frac{1}{\sqrt{2}}\left(|1,2\rangle - |2,1\rangle\right)$

Note $|\{x_i\}\rangle_- = 0$ if any $x_i = x_j$

So restrict to $x_i \neq x_j$: "Pauli exc."

Note $|x_1, x_2, x_3 \cdots\rangle_- = -|x_2, x_1, x_3 \cdots\rangle_-$

So we can restrict to representatives $x_1 < x_2 < x_3 < \cdots$

Then occupation basis is

$$|n_1 = 1, n_2 = 0, n_3 = 1\rangle = |1, 3\rangle_-$$

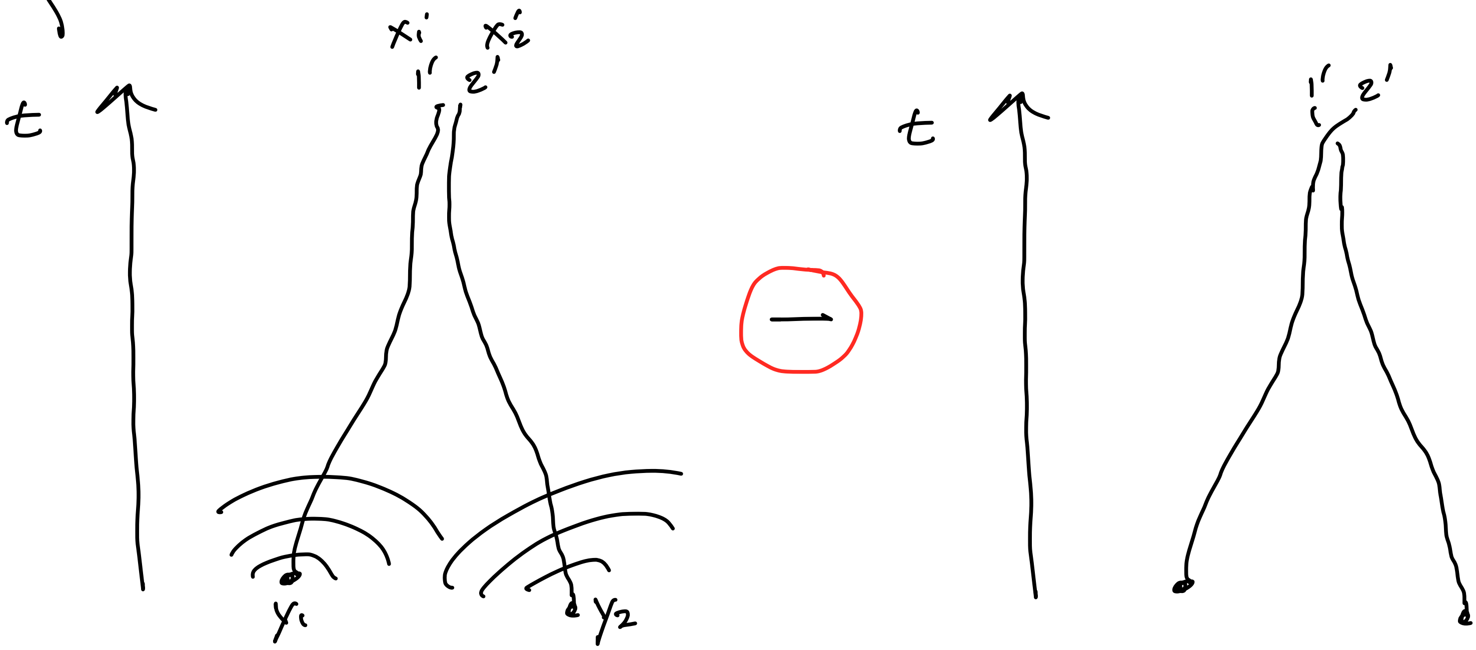
etc.

But now $n_x = 0$ or 1 because of of exclusion.

$$\mathcal{H} = \text{span}\left(|110010\cdots\rangle, |11110\rangle \cdots\right)$$

e ○ e

Exclusion can be understood as destructive interference phenomena:



Path 1 — Path 2

$w(x) =$

[See: R. Shankar's QM]

$$\lim_{1' \to 2'} \left(e^{i S_1/\hbar} - e^{i S_2/\hbar} \right) \to 0$$

$$\lim_{1' \to 2'} S_1 = S_2 \qquad S = \int \mathcal{L}(x)\, dt$$

Amplitude vanishes for fermions to end up at same place!

$$\psi(x_1, x_2, t=0) = \frac{1}{\sqrt{2}}\left(w(x_1 - y_1)\, w(x_2 - y_2) - \leftrightarrow \right)$$

$\downarrow$ Schrodinger.

$$\psi(x_1, x_2, t)$$

$$P_{12 \to 1'2'} \quad |\psi(x_1', x_2', t)|^2$$

For non-interacting fermions, we thus have $H = \sum_\alpha \epsilon_\alpha \hat{n}_\alpha \qquad \hat{n}_\alpha = 0/1$

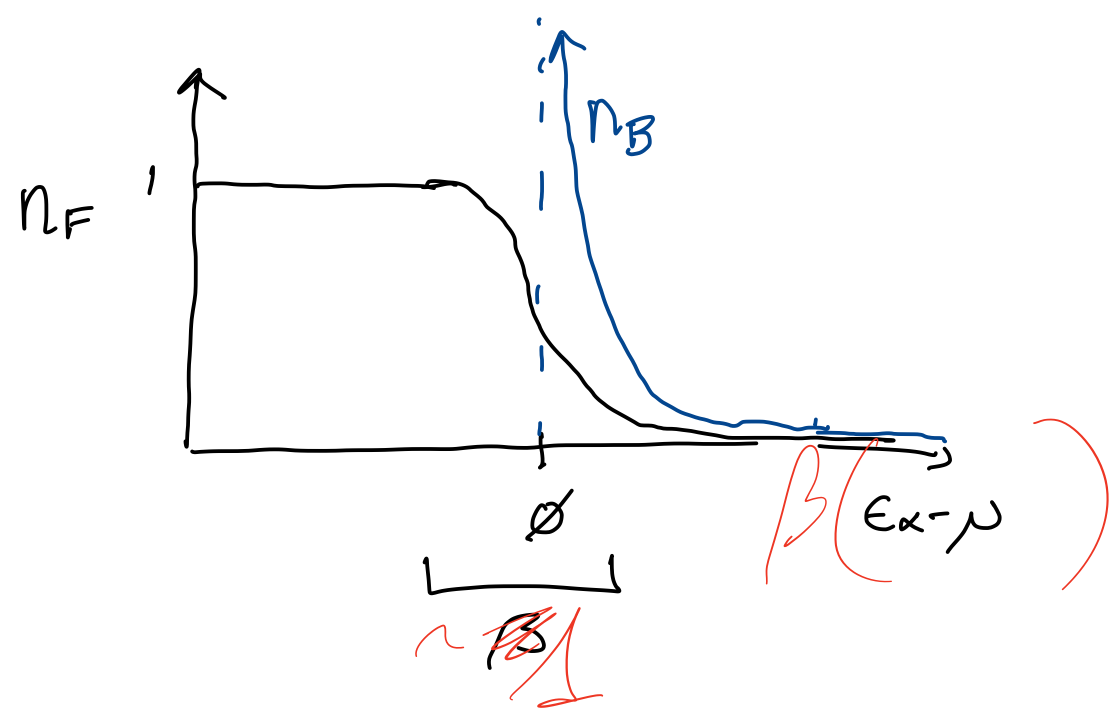
$$Z = \sum_{\{n_\alpha\}} e^{-\beta \sum_\alpha (\epsilon_\alpha - \mu) n_\alpha}$$

$$= \prod_\alpha \left(\sum_{n=0}^{1} e^{-\beta(\epsilon_\alpha - \mu) n} \right)$$

$$= \prod_\alpha \left(1 + e^{-\beta(\epsilon_\alpha - \mu)} \right)$$

$$\langle n_\alpha \rangle = \frac{e^{-\beta(\epsilon_\alpha - \mu)}}{1 + e^{-\beta(\epsilon_\alpha - \mu)}} = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} + 1}$$

This motivates $\langle n \rangle_{B/F} = \dfrac{1}{e^{\beta(\epsilon_\alpha - \mu)} \mp 1}$



Fermionic raising/lowering operators

Similar to bosons, we define ops
$c^\dagger_\alpha$, c_α, $n_\alpha = c^\dagger_\alpha c_\alpha$.

Let us start with single orbital $\alpha = 1$:

⚠ Pre-factors!

$$c^\dagger |n\rangle = |n+1\rangle \qquad c^\dagger|1\rangle = 0$$
$$c\,|n\rangle = |n-1\rangle \qquad c|0\rangle = 0$$

But because $n_\alpha = 0, 1$

$$c_\alpha^2 = (c^\dagger_\alpha)^2 = 0, \qquad c^\dagger_\alpha c_\alpha = n_\alpha$$

For a single orbital,

$$c = \begin{array}{cc} \begin{array}{cc} |0\rangle & |1\rangle \end{array} & \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \end{array} \qquad c^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

But this implies $c\,c^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$c^\dagger c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = n$$

So $$c^\dagger c + c\,c^\dagger = \mathbb{1} = \{c^\dagger, c\}$$

$$\{A, B\} \equiv AB + BA$$

"anti-commutation relation"

$$c^{\dagger 2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \ ✓$$

Ladder op's operating on different orbitals, c_1 and c_2

~~But what about c_1 + c_2?~~

Do we have $\{c_1^+, c_2^+\} = 0$ or

$\underbrace{[c_1^+, c_2^+] = 0}_{\text{wrong!}}$??

~~Now we need to be careful~~

~~because $|x_1, x_2\rangle_- = -|x_2, x_1\rangle_-$!!~~

Let us define

$$c_x^+ |x_1, \cdots, x_N\rangle_- = \overbrace{|x_1, \cdots, x_N, x_{N+1} = x\rangle_-}^{=0 \text{ if } x_i = x \text{ for any } i \in \{1, N\} \; \triangle!}$$

Now

$$c_y^+ c_x^+ |x_1, \cdots, x_N\rangle = c_y^+ |x_1, \cdots, x_N, x_{N+1} = x\rangle_-$$

$$= |x_1, \cdots, x_N, x_{N+1} = x, x_{N+2} = y\rangle_-$$

But similarly

$$c_x^+ c_y^+ |x_1, \cdots, x_N\rangle = |x_1, \cdots, x_N, x_{N+1} = y, x_{N+2} = x\rangle_-$$

This implies $c_x^+ c_y^+ = - c_y^+ c_x^+$

$$\hookrightarrow \boxed{\begin{aligned} \{c_x^+, c_y^+\} &= 0 \\ \{c_x, c_y\} &= 0 \\ \{c_x^+, c_y\} &= \delta_{x,y} \end{aligned}}$$

$\triangle!$ Bosons → commutators || Fermions → anticomm's

Because $|\{n_\alpha\}\rangle \equiv |x_1, x_2, \cdots\rangle_-$ for $x_1 < x_2 < \cdots$,

$$|x_1, x_2, \cdots\rangle_- = c^+_{x_N} \cdots c^+_{x_2} c^+_{x_1} |\rangle_-$$

<u>if</u> $x_1 < x_2, \cdots$.

But this leads to "$-$" signs if you create in a different order! In particular

$$c^+_\alpha |n_1, \cdots, n_\alpha, \cdots\rangle = \eta |n_1, \cdots, n_\alpha + 1, \cdots\rangle$$

$$\text{where} \quad \eta = \underbrace{(-1)^{\sum_{\beta>\alpha} n_\beta}}_{\text{"Jordan Wigner String"}}$$

So even though the Hilbert space of fermions <u>looks</u> like $S = 1/2$ spins,

$$|\{n_\alpha\}\rangle = |1001101\cdots\rangle \sim |\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow\cdots\rangle$$

fermion $\hat{H}$ are built from c^+_α, c_α which behave different from S^+_α, S^-_α!

Pressure of a quantum gas

For $\hat{H} = \sum \frac{p_i^2}{2m}$ in a $V = L^3$ box, the classical pressure is $\beta P = \frac{N}{V}$. Here we derive QM correction in the Grand ensemble (see Kardar 7.2 for Canonical)

The eigenstates are $|\vec{k}\rangle$ for

$$\vec{k} = \frac{2\pi}{L}(i, j, k), \quad \epsilon_k = \frac{\hbar^2 k^2}{2m}$$

(non-relatavistic)

$\frac{\langle k|p^2|k\rangle}{2m}$

$$Q = \prod_{\vec{k}} \begin{cases} 1 + e^{-\beta(\epsilon_k - \mu)} & \text{Fer.} \\ \left(1 - e^{-\beta(\epsilon_k - \mu)}\right)^{-1} & \text{Bos.} \end{cases}$$

$$\Omega = -\frac{1}{\beta} \ln Q = -\frac{1}{\beta} \sum_k \left(\mp \ln\left(1 \mp e^{-\beta(\epsilon_k - \mu)}\right)\right)$$

low-T limit

Suppose gas is dilute, $\beta(\epsilon_k - \mu) \gg 1$:

$$\ln(1 + x) = -\sum_{m=1}^{\infty} (-1)^m \frac{x^m}{m}$$

$$\Omega_k = \mp\frac{1}{\beta} \sum_{m=1}^{\infty} \frac{(\pm)^m}{m} e^{-\beta m(\epsilon_k - \mu)}$$

Note $\sum_k \simeq \left(\frac{L}{2\pi}\right)^D \int d^D k = V \cdot \int \frac{d^D k}{(2\pi)^D}$

So $\Omega = V \cdot \int \frac{d^D k}{(2\pi)^D} \Omega_k$

$$\Omega = \frac{V}{\beta}\int \frac{d^D k}{(2\pi)^D}\left(\mp\sum_{m=1}^{\infty}\frac{(\pm)^m}{m}e^{-\beta m(\epsilon_k-\mu)}\right)$$

$$= \sum_{m=1}^{\infty}\Omega_m = \mp\frac{V}{\beta}\sum_m \frac{(\pm)^m}{m}e^{m\beta\mu}\left(\frac{1}{\lambda\sqrt{m}}\right)^D$$

$$= -\frac{V}{\beta\lambda^D}\sum_m \frac{(\pm)^{m-1}}{m}\frac{e^{\beta m\mu}}{\sqrt{m}^D} \qquad \lambda = \sqrt{\frac{\hbar^2 \pi}{k_B T\, m\, 2}}$$

We then obtain P, N

$$P = -\frac{\partial\Omega}{\partial V} = \sum_m P_m = \frac{1}{\beta\lambda^D}\sum_m \frac{(\pm)^{m-1}}{m}\frac{e^{\beta m\mu}}{\sqrt{m}^D}$$

$$\frac{N}{V} = -\frac{1}{V}\frac{\partial\Omega}{\partial\mu} = \frac{1}{\lambda^D}\sum_m \frac{(\pm)^{m-1}}{\sqrt{m}^D}e^{\beta m\mu} \qquad \text{(c.f. Kardar 7.36)}$$

Now recall virial expansion

$$\beta P = \frac{N}{V}\left(1 + B_2(T)\frac{N}{V} + B_3(T)\left(\frac{N}{V}\right)^2 + \cdots\right)$$

Keeping to 2nd order in $e^{\beta\mu}$,

$$\left(e^{\beta\mu} \pm \frac{1}{2}\frac{e^{2\beta\mu}}{\sqrt{2}^D} + \cdots\right) = \left(e^{\beta\mu} \pm \frac{e^{2\beta\mu}}{\sqrt{2}^D} + \cdots\right)\times$$

$$\left(1 + B_2(T)\frac{e^{\beta\mu}}{\lambda^D} + \cdots\right)$$

2nd Order:

$$\pm\frac{1}{2}\frac{e^{2\beta\mu}}{\sqrt{2}^D} = \pm\frac{e^{2\beta\mu}}{\sqrt{2}^D} + B_2(T)\, e^{2\beta\mu}/\lambda^D$$

$$\pm \frac{1}{2} \frac{e^{2\beta\mu}}{\sqrt{2}^D} = \pm \frac{e^{2\beta\mu}}{\sqrt{2}^D} + B_2(T)\, e^{2\beta\mu}/\lambda^D$$

$$\boxed{B_2(T) = \mp \frac{\lambda^D}{2^{(D+2)/2}}}$$

2nd Virial Coefficient of quantum gas

For Bosons, negative correction to pressure; for fermions, positive.

Classically, for pairwise interactions

$$B_2 = -\frac{1}{2} \int d^3q \left(e^{-\beta V(q)} - 1 \right)$$

In Kardar 7.2, this is shown to be consistent with quantum result if we take

$$\beta V(r) = \mp e^{-2\pi r^2/\lambda^2}$$

So bosons/fermions behave as if there is short-range ($r \sim \lambda$) attractive/repulsive interaction. Of course really, there is no interaction: it is quantum statistics!

Important when $\lambda^D \cdot \frac{N}{V} \approx 1$

"Quantum Degeneracy"

We'll study the quantum degenerate limits next.