IVO ILIEV

Essential Knowledge for Modern Theoretical Physics

To my friends,
my familly, my teachers and collegues.
Also, to Mister T
I pitty the fool

Contents

Foreword

”It is so shocking to find out how many people do not believe that they can learn, and how many more believe learning to be difficult.”

Frank Herbert, Dune

Preface

The idea behind this book is to systematically put a lot of knowledge from different fields in the same place. It is envisioned to serve a purpose of an index book, a glossary introduction to modern physics. The depth may be lacking in some respects but the core idea is to have a background knowledge of many approaches and frameworks.

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Part I

Physics

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General Relativity

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1.1 Anti De Sitter Space (AdS)

AdS_n is an n-dimensional solution for the theory of gravitation with Einstein-Hilbert action with negative cosmological constant Λ, i.e. the theory described by the following Lagrangian density:

$$\mathcal{L} = \frac{1}{16\pi G_{(n)}}(R - 2\Lambda), \tag{1.1}$$

where $G_{(n)}$ is the gravitational constant in n-dimensional spacetime. Therefore, it's a solution of the Einstein field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and $g_{\mu\nu}$ is the metric of the spacetime. Introducing the radius α as

$$\Lambda = \frac{-(n-1)(n-2)}{2\alpha^2} \tag{1.3}$$

this solution can be immersed in a $n+1$ dimensional spacetime with signature $(-, -, +, \cdots, +)$ by the following constraint:

$$-X_1^2 - X_2^2 + \sum_{i=3}^{n+1} X_i^2 = -\alpha^2 \tag{1.4}$$

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Quantum Field Theory

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2.1 Sample Section

Sample text

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Supersymmetry

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3.1 Supermultiplets

Definition 1 (Supermultiplet) *Representations of the supersymmetric algebra (superalgebra) are called supermultiplets.*

Indeed, these representations can be thought of as multiplets where we assemble together several different representations of the Lorentz algebra, since the latter is a subalgebra of the superalgebra.

3.1.1 Massless supermultiplets

If $P^2 = 0$, then we can take P_μ to a canonical form by applying boost and rotations until it reads

$$\sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu = \left(\sigma^0 + \sigma^3\right) E = \begin{bmatrix} 0 & 0 \\ 0 & 2E \end{bmatrix} \tag{3.1}$$

The supersymmetric algebra becomes

$$\begin{bmatrix} \{Q_1, \bar{Q}_{\dot{1}}\} & \{Q_1, \bar{Q}_{\dot{2}}\} \\ \{Q_2, \bar{Q}_{\dot{1}}\} & \{Q_2, \bar{Q}_{\dot{2}}\} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4E \end{bmatrix} \tag{3.2}$$

intended as acting on the states of the multiplet we are looking for. In particular,

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 0 \tag{3.3}$$

which implies that

$$||Q_1|\omega\rangle||^2 = 0 = ||\bar{Q}_{\dot{1}}|\omega\rangle||^2 \tag{3.4}$$

and thus

$$Q_1|\omega\rangle = 0 = \bar{Q}_{\dot{1}}|\omega\rangle. \tag{3.5}$$

This means that as operators Q_1 and $\bar{Q}_{\dot{1}}$ anihilate the multiplet.

The only nontrivial anticommutation relation that is left is:

$$\{Q_2, \bar{Q}_{\dot{2}}\} = 1 \tag{3.6}$$

If we call

$$\alpha = \frac{1}{2\sqrt{E}}Q_2, \quad \alpha^\dagger = \frac{1}{2E}\bar{Q}_{\dot{2}} \tag{3.7}$$

then the anticommutation relation that is left is:

$$\{\alpha, \alpha^\dagger\} = 1 \tag{3.8}$$

with $\{\alpha, \alpha\} = 0$.

We can build the representation starting from a state $|\lambda\rangle$ such that

$$\alpha|\lambda\rangle = 0 \tag{3.9}$$

Lets suppose that it has *helicity* λ:

$$M_{12}\lambda \equiv J_3|\lambda\rangle = \lambda|\lambda\rangle. \tag{3.10}$$

It is easy to compute the helicity of $\alpha^\dagger|\lambda\rangle$:

$$M_{12}\bar{Q}_{\dot{2}}|\lambda\rangle = \left(\bar{Q}_{\dot{2}}M_{12} + \frac{1}{2}\bar{Q}_{\dot{2}}\right)|\lambda\rangle = (\lambda + \frac{1}{2}\bar{Q}_{\dot{2}})|\lambda\rangle \tag{3.11}$$

3.2 General

Definition 2 (R-symmetry) *In supersymmetric theories, an R-symmetry is the symmetry transforming different supercharges into each other. In the simplest case of the $\mathcal{N} = 1$ supersymmetry, such R-symmetry is isomorphic to a global $U(1)$ group or it's discrete subgroup. For extended supersymmetry theories, the R-symmetry group becomes a global non-abelian group.*

Remark 1 *In the case of the discrete subgroup $\mathbb{Z}_2$, the R-symmetry is called R-parity*

Definition 3 (Extended supersymmetry) *In supersymmetric theories, when $\mathcal{N} > 1$ the algebra is said to have extended supersymmetry.*

3.3 Bogomol'nyi-Prasad-Sommerfield (BPS) states

Definition 4 (BPS state) *A massive representation of an extended supersymmetry algebra that has mass equal to the supersymmetry central charge Z is called an BPS state.*

Quantum mechanically speaking, if the supersymmetry remains unbroken, exact solutions to the modulus of Z exist. Their importance arises as the multiplets shorten for generic representations, with stability and mass formula exact.

Example 1 ($d = 4$,$\mathcal{N} = 2$) *The generators for the odd part of the superalgebra have relations:*

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B \tag{3.12}$$

$$\{Q^A_\alpha, Q_{\beta B}\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z} \tag{3.13}$$

$$\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB} Z, \tag{3.14}$$

where $\alpha\dot{\beta}$ are the Lorentz group indeces and A, B are the R-symmerty indeces. If we take linear combinations of the above generators as follows:

$$R^A_\alpha = \xi^{-1} Q^A_\alpha + \xi\sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}B} \tag{3.15}$$

$$T^A_\alpha = \xi^{-1} Q^A_\alpha - \xi\sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}B} \tag{3.16}$$

and consider a state ψ which has momentum $(M, 0, 0, 0)$, we have:

$$\left(R^1_1 + (R^1_1)^\dagger\right)^2 \psi = 4(M + Re(Z\xi^2))\psi, \tag{3.17}$$

but because this is the square of a Hermitian operator, the right hand side coefficient must be positive for all ξ. In particular, the strongest result from this is

$$M \geq |Z| \tag{3.18}$$

3.4 Supersymmetric theories on curved manifolds

Remark: Supersymmetric theories may be defined only on backgrounds admitting solutions to certain Killing spinor equations,

$$(\nabla_\mu - iA_\mu)\,\zeta + iV_\mu\zeta + iV^\nu\sigma_{\mu\nu}\zeta = 0 \tag{3.19}$$

$$(\nabla_\mu + iA_\mu)\,\tilde{\zeta} - iV_\mu\tilde{\zeta}0iV^\nu\tilde{\sigma}_{\mu\nu}\tilde{\zeta} = 0 \tag{3.20}$$

which in four dimensions and Euclidean signature are equivalent to the requirement that the manifold is complex and the metric Hermitian.

Part II

Mathematics

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Group Theory

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4.1 Basic Definitions

Definition 5 ((Group) Homomorphism) *Let $(G, *)$ and $(H, \cdot)$ be two groups. A (group) homomorphism from G to H is a function $h : G \to H$ such that for all x, y in G it holds that $h(x * y) = h(u) \cdot h(v)$*

Definition 6 (Coset) *Let G be a group and H is a subgroup of G. Consider an element $g \in G$. Then, $gH = \{gh : h \in H\}$ is the left coset of H in G with respect to g, and $Hg = \{hg : h \in H\}$ is the right coset of H in G with respect to g*

Remark 2 *In general the left and right cosets are not groups.*

Definition 7 (Normal Subgroup) *A subgroup H of G is called normal if and only if the left and right sets of cosets coincide, that is if $gH = Hg$ for all $g \in G$*

Definition 8 (Representation) *A representation of a group G on a vector space V over a filed K is a group homomorphism from G to $GL(V)$, the general linear group on V. That is, a representation is a map*

$$\rho : G \to GL(V) \tag{4.1}$$

such that

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \quad \forall g_1, g_2 \in G. \tag{4.2}$$

V is often called the *representation space* and the dimension of V is called the *dimension* of the representation. It is common practice to refer to V itself as the representation when the homomorphism is clear from the context.

Example 2 *Consider the complex number* $u = e^{2\pi/3}$ *which has the property* $u^3 = 1$*. The cyclic group* $C_3 = \{1, u, u^2\}$ *has a representation* ρ *on* $\mathbb{C}^2$ *given by:*

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u) = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}, \quad \rho(u^2) = \begin{bmatrix} 1 & 0 \\ 0 & u^2 \end{bmatrix}, \quad . \tag{4.3}$$

Another representation for C_3 *on* $\mathbb{C}^2$*, isomorphic to the previous one, is*

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u^2) = \begin{bmatrix} u^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad . \tag{4.4}$$

Definition 9 (Subrepresentation) *A subspace* W *of* V *that is invariant under the group action is called a subrepresentation.*

Definition 10 ((Ir)reducible representation) *If* V *has exactly two representations, namely the zero-dimensional subspace and* V *itself, then the representation is said to be irreducible; if it has a proper representation of nonzero dimension, the representation is said to be reducible. The representation of dimension zero is considered to be neither reducible nor irreducible.*

Definition 11 (Quotent Group) *Let* N *be a normal subgroup of a group* G*. We define the set* G/N *to be the set of all left cosets of* N *in* G*, i.e.,* $G/N = \{aN : a \in G\}$*. Define an operation on* G/N *as follows. For each* aN *and* bN *in* G/N*, the product of* aN *and* bN *is* $(aN)(bN)$*. This defines an operation of* G/N *if we impose* $(aN)(bN) = (ab)N$*, because* $(ab)N$ *does not depend on the choice of the representatives* a *and* b*: if* $xN = aN$ *and* $yN = bN$ *for some* $x, y \in G$*, then:*

$(ab)N = a(bN) = a(yN) = a(Ny) = (aN)y = (xN)y = x(Ny) = x(yN) = (xy)N$

Here it was used in an important way that N *is a normal subgroup. It can be shown that this operation on* G/N *is associative, has identity element* N *and the inverse of an element* $aN \in G/N$ *is* $a^{-1}N$*. Therefore, the set* G/N *together with the defined operation forms a group; this is known as the quotient group or factor group of* G *by* N

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Differential Geometry

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5.1 General

Definition 12 (Einstein Manifold) *A Riemannian manifold M is called an Einstein manifold if its Ricci tensor is proportional to the metric, i.e.*

$$Ric_g = \lambda g \tag{5.1}$$

Remark 3 *An Einsteinian manifold, where $\lambda = 0$ is called a Ricci-flat manifold.*

5.2 Contact Manifolds

Definition 13 (Riemannian Cone) *Given a Riemannian manifold (M, g), its Riemannian cone is a product*

$$(M \times \mathbb{R}^{>0}) \tag{5.2}$$

of M with the half-line $\mathbb{R}^{>0}$ equipped with the cone metric

$$t^2 g + dt^2, \tag{5.3}$$

where t is a parameter in $\mathbb{R}^{>0}$

Definition 14 (Contact Manifold) *A manifold M, equipped with a 1-form θ is contact if and only if the 2-form*

$$t^2 d\theta + 2tdt \cdot \theta \tag{5.4}$$

on its cone is symplectic.

Definition 15 (Sasakian Manifold) *A contact Riemannian manifold is called a Sasakian manifold, if its Riemannian cone with the cone metric is a Kähler manifold with Kähler form*

$$t^2 d\theta + 2tdt \cdot \theta. \tag{5.5}$$

Example 3 *Consider the manifold $\mathbb{R}^{2n+1}$ with coordinates $(\vec{x}, \vec{y}, z)$, endowed with contact form*

$$\theta = \frac{1}{2}dz + \Sigma_i y_i dx_i \tag{5.6}$$

and Riemannian metric

$$g = \Sigma_i (dx_i)^2 + (dy_i)^2 + \theta^2 \tag{5.7}$$

Definition 16 (Sasaki-Einstein Manifold) *A Sasaki-Einstein manifold is a Riemanian manifold (S, g) that is both Sasakian and Einstein*

Example 4 *The odd dimensional sphere S^{2n-1}, equipped with its standard Einstein metric is a Sasaki-Einstein manifold. In this case, the Kähler cone is $\mathbb{C}^2\backslash\{0\}$, equipped with its flat metric.*

5.3 Symplectic Geometry

Theorem 1 (Duistermaat-Heckman Formula) *For a compact symplectic manifold M of dimension $2n$ with symplectic form ω and with a Hamiltonian $U(1)$ action whose moment map is denoted by μ, the following formula holds:*

$$\int_M \frac{\omega^n}{n!} e^{-\mu} = \sum_i \frac{e^{-\mu(x_i)}}{e(x_i)} \tag{5.8}$$

Here, x_i are the fixed points of the $U(1)$ action and they are assumed to be isolated, and $\mathrm{e}(x_i)$ is the product of the weights of the $U(1)$ action on the tangent space at x_i.

5.4 Complex Manifolds

Definition 17 (Hermitian Metric) *If a Riemannian metric g of a complex manifold M satisfies*

$$g_p(J_pX, J_pY) = g_p(X, Y) \tag{5.9}$$

at each point $p \in M$ and for any $X, Y \in T_pM$, g is said to be a Hermitian metric. Here, J_p denotes the almost complex structure on M.

Definition 18 (Hermitian Manifold) *The pair (M, g) is called a Hermitian manifold*

Theorem 2 *A complex manifold always admits a Hermitian metric.*

Proof 1 *Let g be any Riemannian metric of a complex manifold M. Define a new metric $\hat{g}$ by*

$$\hat{g}_p(X, Y) \equiv \frac{1}{2}\left[g_p(X, Y) + g_p(J_pX, J_pY)\right]. \tag{5.10}$$

Clearly $\hat{g}_p(J_pX, J_pY) = \hat{g}_p(X, Y)$. Moreover, $\hat{g}$ is positive definite provided that g is. Hence, $\hat{g}$ is a Hermitian metric on M

Definition 19 (Kähler Form) *Let (M, g) be a Hermitian manifold. Define a tensor field Ω whose action on $X, Y \in T_pM$ is*

$$\Omega_p(X, Y) = g_p(J_pX, Y) \tag{5.11}$$

Note that Ω is anti-symmetric, $\Omega(X, Y) = g(JX, Y) = g(J^2X, JY) = -g(JY, X) = -\Omega(Y, X)$. Hence, Ω defines a two-form, called the Khäler form of a Hermitian metric g.

Definition 20 (Kähler Manifold) *A Kähler manifold is a Hermitian manifold (M, g) whose Kähler form Ω is closed, $d\Omega = 0$. The metric g is called the Kähler metric of M.*

Remark 4 *Not all complex manifolds admit Kähler metrics*

Theorem 3 *A Hermitian manifold (M, g) is a Kähler manifold if and only if the almost complex structure J satisfies*

$$\nabla_\mu J = 0 \tag{5.12}$$

where ∇_μ is the Levi-Cevita connection associated with g.

Proof 2 *We first note that for any r-form ω, $d\omega$ is written as*

$$d\omega = \nabla\omega \equiv \frac{1}{r!}\nabla_\mu \omega_{\nu_1\cdots\nu_r} dx^\mu \wedge dx^{\nu_1} \wedge \cdots \wedge dx^\nu \tag{5.13}$$

Now we prove that $\nabla_\mu J = 0$ if and only if $\nabla_\mu \Omega = 0$. We verify the following equalities:

$$\begin{aligned}
(\nabla_Z\Omega)(X,Y) &= \nabla_Z\left[\Omega(X,Y)\right] - \Omega(\nabla_Z X, Y) - \Omega(X, \nabla_z Y) &(5.14)\\
&= \nabla_Z\left[g(JX,Y)\right] - g(J\nabla_Z X, Y) - g(JX, \nabla_Z Y) &(5.15)\\
&= (\nabla_Z g)(JX,Y) + g(\nabla_Z JX, Y) - g(J\nabla_Z X, Y) &(5.16)\\
&= g(\nabla_Z JX - J\nabla_Z X, Y) = g((\nabla_Z J)X, y) &(5.17)
\end{aligned}$$

where $\nabla_Z g = 0$ has been used. Since this is true for any X, Y, Z, if follows that $\nabla_Z\Omega = 0$ if and only if $\nabla_Z J = 0$.

The last theorem shows that the Riemann structure is compatable with the Hermitian structure in the Kähler manifold.

We can also characterize Kähler manifolds as Hermitian manifolds for which the Cristoffel symbols of the Levi-Chevita connection are pure. In other words, Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$ may be non zero, but all "mixed" symbols like $\Gamma^{\bar{i}}_{jk}$, for example, vanish. This means that (anti-)holomorphic vectors get parallel transported to (anti-)holomorphic vectors.

Kähler manifolds are manifolds on which we can always find a *holomorphic* change of coordinates which, at some given point, sets the metric to its cannonical form, and its first derivatives to zero.

Equivalently, an n-dimensional Kähler manifold are precisely $2n$-dimensional Riemannian manifolds with holonomy group contained in $U(1)$

Definition 21 (Hopf Surface) *Let $\mathbb{Z}$ act on $\mathbb{C}^n\backslash\{0\}$ by $(z_1, \cdots, z_n) \to \left(\lambda^k z_1, \cdots, \lambda^k z_n\right)$ for $k \in \mathbb{Z}$. For $0 < \lambda < 1$ the action is free and discrete. The quotient complex manifold $X = \left(\mathbb{C}^n\backslash\{0\}\right)/\mathbb{Z}$ is diffeomorphic to $S^1 \times S^{2n-1}$. For $n = 1$ this manifold is isomorphic to a complex torus $\mathbb{C}/\Gamma$. The lattice Γ can be determined explicitly.*
In other words, a Hopf manifold is obtained as a quotient of the complex vector space (with zero deleted) $\mathbb{C}^n\backslash\{0\}$ by a free action of the group $\Gamma \cong \mathbb{Z}$ of integers, with the generator γ of Γ acting by holomorphic contractions.