Sofia University Faculty of Physics

My Journey Through Theoretical Physics

IVO ILIEV

To my friends,
my familly, my teachers and collegues.
Also, to Mister T
I pitty the fool

Contents

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Foreword

"It is so shocking to find out how many people do not believe that they can learn, and how many more believe learning to be difficult."

Frank Herbert, Dune

Preface

The idea behind this book is to systematically put a lot of knowledge from different fields in the same place. It is envisioned to serve a purpose of an index book, a glossary introduction to modern physics. The depth may be lacking in some respects but the core idea is to have a background knowledge of many approaches and frameworks.

Physics is a journey. In this book, I've tried to illustrate my own journey through theoretical physics in hopes that it might help others in their own studies.

The book is mainly divided into two parts. The first one is dedicated to physics. Many topics are considered there - quantum field theory, general relativity, non-linear dynamics and a bunch more. Of course, not all topics are treated equally both in scope and depth. That does not reflect their importance, it merely reflects the fact that I am more interested in them.

The second part is about mathematical preliminaries. Again, some things are explained in detail and others are only brushed upon. However, I've tried to include everything that is absolutely *needed* if you would like to work in theoretical physics.

Again, this book is not trying to be exactly pedagogical. It is designed as an extended handbook, somewhere one can search for a definition, a formula or a general description of a phenomenon or mathematical idea.

It must be noted that this book is, and always will be, **completely free**. You are free to redistribute it, modify it and use it in any way you want, without asking for permission from the author. The LaTeX file and the documentation for it are available on `https://github.com/HallowDance/book`. The only thing I'll politely ask (but not enforce) is that you don't use parts of the book or the book as a whole for commercial purposes.

Part I

Physics

Chapter 1

General Relativity

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1.1 Anti De Sitter Space (AdS)

AdS_n is an n-dimensional solution for the theory of gravitation with Einstein-Hilbert action with negative cosmological constant Λ, i.e. the theory described by the following Lagrangian density:

$$\mathcal{L} = \frac{1}{16\pi G_{(n)}}(R - 2\Lambda), \tag{1.1}$$

where $G_{(n)}$ is the gravitational constant in n-dimensional spacetime. Therefore, it's a solution of the Einstein field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and $g_{\mu\nu}$ is the metric of the spacetime. Introducing the radius α as

$$\Lambda = \frac{-(n-1)(n-2)}{2\alpha^2} \tag{1.3}$$

this solution can be immersed in a $n+1$ dimensional spacetime with signature $(-,-,+,\cdots,+)$ by the following constraint:

$$-X_1^2 - X_2^2 + \sum_{i=3}^{n+1} X_i^2 = -\alpha^2 \tag{1.4}$$

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Chapter 2

Quantum Field Theory

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2.1 Rarita-Schwinger equation

Consider the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}_\mu \left(\epsilon^{\mu\kappa\rho\nu}\gamma_5\gamma_\kappa\partial_\rho - im\sigma^{\mu\nu}\right)\psi_\nu \tag{2.1}$$

This equation obviously controls the propagation of the wave function of a spin-object such as the gravitino. The equation of motion for this Lagrangian are known as the *Rarita-Schwinger equation*:

$$\left(\epsilon^{\mu\kappa\rho\nu}\gamma_5\gamma_\kappa\partial_\rho - im\sigma^{\mu\nu}\right)\psi_\nu \tag{2.2}$$

In the massless case, the Rarita-Schwinger equation has a fermionic gauge symmetry, it is invariant under the gauge transformation:

$$\psi_\mu \to \psi_\mu + \partial_\mu\epsilon, \tag{2.3}$$

where $\epsilon \equiv \epsilon_\alpha$ is an arbitrary spinor field.

2.1.1 Massless case

Consider a massless Rarita-Schwinger field, described by the Lagrangian

$$\mathcal{L}_{RS} = \bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho \tag{2.4}$$

where the sum over spin indices is implicit, ψ_μ are Majorana spinors and the quantity $\gamma^{\mu\nu\rho}$ is equal to

$$\gamma^{\mu\nu\rho} \equiv \frac{1}{3!}\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]} \tag{2.5}$$

Varying the Lagrangian yields after some calculation

$$\delta\mathcal{L}_{\mathcal{RS}} = 2\delta\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + \text{boundary terms} \tag{2.6}$$

Now imposing that $\mathcal{L}_{RS} = 0$ we get the equation of motion for a massless Majorana Rarita-Schwinger spinor:

$$\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho = 0 \tag{2.7}$$

2.1.2 Massive case

The description of massive, higher-spin fields through the Rarita-Schwinger equation is not well defined physically. Coupling the RS Largrangian to electromagnetism leads to an equation with solutions representing wavefronts, some of which propagate faster than light. However, it was shown by Das and Freedman that local supersymmetry can circumvent this problem.

2.2 Wilson Loops and Large N

This section is dedicated to a short introduction to the methods used in nonperturbative investigations of QCD and other gauge theories. The main attention is payed to Wilson loops, both on the lattice and in the continuum, which play a central role in modern formulations of gauge theories and to the method of the large N expansion.

2.2.1 Wilson Loops

In essence, Wilson loops are phase factors in Abelian or non-Abelian gauge theories. Wilson loops are observable in quantum theory by the Aharonov-Bohm effect. Wilson loops play a central role in the lattice formulation of gauge theories. QCD can be reformulated through Wilson loops in a manifest gauge-invariant way. Analogues of the Wilson loops are extremely useful in solving various kinds of matrix models.

2.2.1.1 Phase factors in QED

Let us first examine the familiar setting of QED. An Abelian phase factor is defined by the formula

$$U\left[\Gamma_{yx}\right] = e^{ie\int_{\Gamma_{yx}} \mathrm{d}z^{\mu} A_{\mu}(z)} \tag{2.8}$$

Under the gauge transformation this becomes

$$A_{\mu}(z) \xrightarrow{\text{g.t.}} A_{\mu}(z) + \frac{1}{e}\partial_{\mu}\alpha(z) \tag{2.9}$$

and the Abelian phase factor transforms as

$$U\left[\Gamma_{yx}\right] \xrightarrow{\text{g.t.}} \mathrm{e}^{i\alpha y} U\left[\Gamma_{yx}\right] \mathrm{e}^{-i\alpha x} \tag{2.10}$$

It is easy to show that a wave function at the point x is transformed as

$$\phi(x) \xrightarrow{\text{g.t.}} \mathrm{e}^{i\alpha(x)}\psi(x), \tag{2.11}$$

therefore the phase factor is transformed as the product $\psi(y)\psi^\dagger(x)$:

$$U\left[\Gamma_{yx}\right] \overset{g.t.}{\sim} \text{"}\psi(y)\psi^\dagger(x)\text{"} \tag{2.12}$$

A wave function at the point x transforms like one at the point y after multiplication by a phase factor:

$$U\left[\Gamma_{yx}\right]\psi(x) \overset{g.t.}{\sim} \text{"}\psi(y)\text{"}, \tag{2.13}$$

and analogously

$$\psi^\dagger(y)U\left[\Gamma_{yx}\right] \overset{g.t.}{\sim} \text{"}\psi^\dagger(x)\text{"}. \tag{2.14}$$

The phase factor plays the role of a parallel transporter in an electromagnetic field, and to compare phases of a wave function at points x and y, we should first make a parallel transport along some contour Γ_{yx}. The result is of course Γ-dependent except when $A_\mu(z)$ is a pure gauge (meaning that the field strength $F_{\mu\nu}(z)$ is vanishing along the contour). Certain subtleties occur for not simply connected spaces, namely the Aharonov-Bohm effect.

2.2.1.2 Propagators in external field

Let us consider a quantum particle in a classical electromagnetic field. To introduce electromagnetism field, ∂_μ is to be replaced by the covariant derivative

$$\partial_\mu \to \Delta_\mu = \partial_\mu - ieA_\mu(x). \tag{2.15}$$

For the propagator we get

$$G(x,y;A) = \frac{1}{2}\int_0^\infty \mathrm{d}\tau e^{-\frac{1}{2}\tau m^2} \int\limits_{\substack{z_\mu(0)=x_\mu \\ z_\mu(\tau)=y_\mu}} \mathcal{D}z_\mu(t) e^{-\frac{1}{2}\int_0^\tau \mathrm{d}t\dot{z}_\mu^2(t) + ie\int_0^\tau \mathrm{d}t\dot{z}^\mu(t)A_\mu(z(t))} \tag{2.16}$$

Although this may seem cumbersome, one can easily spot that the exponent is just the classical (Eucleadian) *action* of a particle in an external electromagnetic field. The path integral representation above for the propagator of a scalar particle is due to Feynman. We can alternatively rewrite it in a more compact form

$$G(x,y;A) = \sum_{\Gamma_{yx}} e^{S_{\text{free}}[\Gamma_{yx}] + ie\int_{\Gamma_{yx}} \mathrm{d}z^\mu A_\mu(z)}, \tag{2.17}$$

Figure 2.1: Aharonov-Bohm effect

where we represented the (parametric invariant) integral over $\mathrm{d}t$ as the contour integral along the trajectory Γ_{yx} over

$$\mathrm{d}z^\mu = \mathrm{d}t\dot{z}^\mu(t). \tag{2.18}$$

The transition amplitude of a quantum particle in a classical electromagnetic field is the sum over paths of the Abelian phase factor(2.8).

2.2.1.3 Aharonov-Bohm effect

Transverse components of the electromagnetic field describe photons. Longitudal components are related to gauging the phase of a wave function, i.e. permit one to compare its values at different space-time points when an electron is placed in an external electromagnetic field.

In quantum mechanics, the wave-function phase itself is not observable. Only the phase differences are observable, e.g. via interference phenomena. The phase difference depends on the value of the phase factor for a given path Γ_{yx} along which the parallel transport is performed.

The phase factors are observable in quantum theory, in contrast to classical theory. This is seen in the Aharonov-Bohm effect, whose scheme is depicted in Fig. 2.2.1.3. Electrons do not pass inside the solenoid where the magnetic field is concentrated. Nevertheless, a phase difference arises between the electron beams passing through the two slits. The interference picture changes with the value of the electric current. The phase difference depends on (the real part of)

$$e^{ie\int_{\Gamma^+_{yx}}\mathrm{d}z^\mu A_\mu(z)}e^{-ie\int_{\Gamma^-_{yx}}\mathrm{d}z^\mu A_\mu(z)} = e^{ie\oint_\Gamma \mathrm{d}z^\mu A_\mu(z)} = e^{ie\int \mathrm{d}\sigma^{\mu\nu}F_{\mu\nu}} = e^{ieHS} \tag{2.19}$$

where the contour Γ is composed from Γ^+_{yx} and Γ^-_{xy}. It does not depend on the shape of the two sub-contours but depends only on HS - the magnetic flux through the solenoid.

2.2.2 Yang-Mills Theories

Modern theories of fundamental particles are gauge theories. The principle of local gauge invariance was introduced by H.Weyl for the electromagnetic interaction in

analogy with general covariance in Einstein's theory of gravitation. An extension to non-Abelian gauge groups was given by Yang and Mills in 1954.

A crucial role in gauge theories is played by the phase factor which is associated with parallel transport in an external gauge field. The phase factors are observable in quantum theory, in contrast to classical theory. This is analogous to the Aharonov-Bohm effect for the electromagnetic field.

2.2.2.1 Gauge invariance

The principle of local gauge invariance deals with the gauge transformation of a matter field ψ, which is given by:

$$\psi(x) \xrightarrow{g.t.} \psi'(t) = \Omega(x)\psi(x). \tag{2.20}$$

Here $\Omega(x) \in G$ with G being a semisimple Lie group which is called the gauge group ($G = SU(3)$ for QCD. (2.20) demonstrates that ψ belongs to the fundamental representation of G.

The unitary gauge group is when

$$\Omega^{-1}(x) = \Omega^{\dagger}(x), \tag{2.21}$$

while an extension to the other Lie groups is straightforward. Then we have

$$\psi^{\dagger}(x) \xrightarrow{g.t.} \psi'^{\dagger} = \psi^{\dagger}(x)\Omega^{\dagger}(x). \tag{2.22}$$

In analogy with QCD, the gauge group $G = SU(N)$ is usually associated with color and the proper index of ψ is called the color index.

The gauge transformation (2.20) of the matter field ψ can be compensated by a transformation of the non-Abelian gauge field $\mathcal{A}_\mu$ which belongs to the adjoint representation of G:

$$\mathcal{A}_\mu(x) \xrightarrow{g.t.} \mathcal{A}'_\mu(x) = \Omega(x)\mathcal{A}_\mu(x)\Omega^{\dagger}(x) + i\Omega(x)\partial_\mu\Omega^{\dagger}(x) \tag{2.23}$$

It is convenient to introduce the Hermitian matrix

$$[\mathcal{A}_\mu(x)]^{ij} = g\sum_a A^a_\mu(x)\,[t^a]^{ij} \tag{2.24}$$

where g is the gauge coupling constant.

The matrices $[t^a]^{ij}$ are the generators of G ($a = 1, \cdots, N^2 - 1$ for $SU(N)$) which are normalized such that

$$\mathrm{Tr} t^a t^b = \delta^{ab}, \tag{2.25}$$

where this is a trace over the matrix indices i and j.

Quite often another normalization of the generators with an extra factor of $1/2$ in front of the delta is used for historical reasons, in particular, because $\tilde{t}^a = \sigma/2$ for the $SU(2)$ group where the sigmas are the Pauli matrices. This results in the redefinition of the coupling constant, $\tilde{g}^2 = 2g^2$.

Equation (2.24) can be inverted to give

$$A^a_\mu(x) = \frac{1}{g}\mathrm{Tr}\mathcal{A}_\mu(x)t^a. \tag{2.26}$$

Substituting

$$\Omega(x) = e^{i\alpha(x)} \tag{2.27}$$

we obtain for an infinitesimal α:

$$\delta\mathcal{A}_\mu(x) \overset{g.t.}{=} \nabla^{\mathrm{adj}}_\mu \alpha(x). \tag{2.28}$$

Here

$$\nabla^{\mathrm{adj}}_\mu \alpha \equiv \partial_\mu \alpha - i\left[\mathcal{A}_\mu, \alpha\right] \tag{2.29}$$

is the covariant derivative in the adjoint representation of G, while

$$\nabla^{\mathrm{fun}}_\mu \psi \equiv \partial_\mu \psi - i\mathcal{A}_\mu \psi \tag{2.30}$$

is that in the fundamental representation. It is evident that

$$\nabla^{\mathrm{adj}}_\mu B(x) = \left[\nabla^{\mathrm{fun}}_\mu, B(x)\right], \tag{2.31}$$

where $B(x)$ is a (matrix-valued) function of x.

The QCD action is given in the matrix notation as

$$S\left[\mathcal{A}, \psi, \bar{\psi}\right] = \int \mathrm{d}^4x \left[\bar{\psi}\gamma_\mu(\partial_\mu - i\mathcal{A}_\mu)\psi + m\bar{\psi}\psi + \frac{1}{4g^2}\mathrm{Tr}\mathcal{F}^2_{\mu\nu}\right], \tag{2.32}$$

where

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i\left[\mathcal{A}_\mu, \mathcal{A}_\nu\right] \tag{2.33}$$

is the (Hermitian) matrix of the non-Abelian field strength.

This action is invariant under the local gauge transformation since

$$\mathcal{F}_{\mu\nu}(x) \xrightarrow{g.t.} \Omega(x)\mathcal{F}_{\mu\nu}(x)\Omega^{\dagger}(x) \tag{2.34}$$

or

$$\delta\mathcal{F}_{\mu\nu}(x) \overset{g.t.}{=} i\left[\mathcal{F}_{\mu\nu}(x), \alpha(x)\right] \tag{2.35}$$

for the infinitesimal gauge transformation.

For the Abelian group $G = U(1)$, the formulas recover those for QED.

2.2.2.2 Non-Abelian phase factors (Wilson loops)

To compare phases of wave functions at distinct points, a non-Abelian extension of the parallel transporter is needed. The proper extension of the Abelian formula (2.8):

$$U\left[\Gamma_{yx}\right] = Pe^{i\int_{\Gamma_{yx}} \mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)}, \tag{2.36}$$

includes the symbol P of path-ordering.

Although the matrices $\mathcal{A}_{\mu}$ do not commute, the path-ordered exponential on the right hand side is defined unambiguously. This becomes obvious if we rewrite the phase factor in an equivalent form

$$Pe^{i\int_{\Gamma_{yx}} \mathrm{z}^{\mu}\mathcal{A}_{\mu}(z)} = Pe^{i\int_0^{\tau} \mathrm{d}t\dot{z}^{\mu}(t)\mathcal{A}_{\mu}(z(t))}. \tag{2.37}$$

The path-ordered exponential in (2.36) can be understood as

$$U\left[\Gamma_{yx}\right] = \prod_{t=0}^{\tau}\left[1 + i\mathrm{d}t\dot{z}(t)\mathcal{A}_{\mu}(z(t))\right] \tag{2.38}$$

Imagining this as a contour integration we can rewrite the former expression as

$$U\left[\Gamma_{yx}\right] = \prod_{z\in\Gamma_{yx}}\left[1 + i\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)\right]. \tag{2.39}$$

If the contour Γ_{yx} is discredited, then the non-Abelian phase factor can be approximated to be

$$U\left[\Gamma_{yx}\right] = \lim_{M\to\infty}\prod_{i=1}^{M}\left[1 + i(z_i - z_{i-1})^{\mu}\mathcal{A}_{\mu}\left(\frac{z_i + z_{i-1}}{2}\right)\right], \tag{2.40}$$

which reproduces (2.39) in the limit $z_{i-1} \to z_i$. The non-Abelian phase factor (2.36) is an element of the gauge group G itself, while $\mathcal{A}_{\mu}$ belongs to the Lie algebra of G.

Let us recall that matrices are rearranged in inverse order under Hermitian conjugation:

$$U^{\dagger}\left[\Gamma_{yx}\right] = U\left[\Gamma_{xy}\right]. \tag{2.41}$$

The notation Γ_{yx} means the orientation of the contour from x to y, while Γ_{xy} will denote the opposite orientation. These two result in opposite orders of multiplication for the matrices in the path-ordered product. Furthermore, the phase factors obey the backtracking (*zig-zag*) condition

$$U\left[\Gamma_{yx}\right] U\left[\Gamma_{xy}\right] = 1. \tag{2.42}$$

The gauge field $\mathcal{A}_{\mu}$ in the discretized phase factor (2.40) is chosen at the *center* of the i-th interval in order to satisfy Eq. (2.42) at finite discretization.

Under the gauge transformation (2.23), $U\left[\Gamma_{yx}\right]$ transforms as

$$U\left[\Gamma_{yx}\right] \xrightarrow{g.t} \Omega(y) U\left[\Gamma_{y}x\right] \Omega^{\dagger}(x). \tag{2.43}$$

This formula stems from the fact that

$$\left[1 + i\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)\right] \xrightarrow{g.t.} \left[1 + i\mathrm{d}z^{\mu}\mathcal{A}'_{\mu}(z)\right] = \Omega(z + \mathrm{d}z)\left[1 + i\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)\right]\Omega^{\dagger}(z) \tag{2.44}$$

which can be proven by substituting (2.23), so that $\Omega^{\dagger}(z)$ and $\Omega(z)$ *cancel* in the definition (2.39) at the intermediate point z.

A consequence of Eq. (2.43) is that $\psi(x)$, transported by the matrix $U\left[\Gamma_{yx}\right]$ to the point y, transforms under the gauge transformation as $\psi(y)$:

$$U\left[\Gamma_{yx}\right]\psi(x) \overset{g.t.}{\sim} \text{"}\psi(y)\text{"}. \tag{2.45}$$

Therefore, $U\left[\Gamma_{yx}\right]$ is indeed a parallel transporter. It follows from these formulas that the combination $\bar{\psi}(y)U\left[\Gamma_{yx}\right]\psi(x)$ is gauge invariant:

$$\bar{\psi}(y)U\left[\Gamma_{yx}\right]\psi(x) \xrightarrow{g.t.} \bar{\psi}(y)U\left[\Gamma_{yx}\right]\psi(x). \tag{2.46}$$

Another consequence of (2.43) is that the trace of the phase factor for a closed contour Γ is gauge invariant:

$$\mathrm{Tr}Pe^{i\oint_{\Gamma}\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)} \xrightarrow{g.t.} \mathrm{Tr}Pe^{i\oint_{\Gamma}\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)} \tag{2.47}$$

This is quite similar to the Abelian phase factor.

The sufficient and necessary condition for the phase factor to be independent on a local variation of the path is the *vanishing* of $\mathcal{F}_{\mu\nu}$. Formulas of this type are well-known in differential geometry where parallel transport around a small close contour determines the curvature. $\mathcal{F}_{\mu\nu}$ in Yang-Mills theory is the proper curvature in an internal color space while $\mathcal{A}_\mu$ is the connection.

Remark 2.1 (A brief history lesson). An analog of the phase factor was first introduced by H. Weyl in 1919, in his attempt to describe gravitational and electromagnetic interactions of electrons on equal footing. What he did is associated in modern language with the scale rather than the gauge transformation, i.e. the vector-potential was not multiplied by i as in equation (2.8). This explains the term "gauge invariance" - gauging literally means fixing a scale.

The factor of i was inserted by London in 1927 after creation of quantum mechanics and the recognition that the electromagnetic interaction corresponds to the freedom of choice of the phase of a wave function and not to a scale transformation.

2.2.3 1/N Expansion

An effective coupling constant of QCD becomes large at large distances, so fluctuations of scales of different orders of magnitude are essential and there is no small parameter. 't Hooft proposed in 1974 to use the number of colors N of the gauge group $SU(N)$ as such a parameter and to perform an expansion in $1/N$. The motivation was the $1/N$ expansion in statistical mechanics.

The $1/N$ expansion of QCD rearranges perturbation theory in a way consistent with a string picture. The accuracy of large-N QCD is of the order of ratios of meson widths to their masses (10-15%). While QCD is simplified in this limit, it is not yet solved.

2.2.3.1 Index of Ribbon Graphs

In order to describe the $1/N$-expansion of QCD, it is convenient to use the matrix-field representation

$$[A_\mu(x)]^{ij} = \sum_\alpha A^a_\mu(x)\,[t^a]^{ij}\,. \tag{2.48}$$

The matrix (2.48) is Hermitian and differs from (2.24) by a factor of g. The propagator of the matrix field $A^{ij}(x)$ reads

$$\langle A^{ij}_{\mu}(x) A^{kl}_{\nu}(y) \rangle_{\text{Gauss}} = \left(\delta^{il}\delta^{kj} - \frac{1}{N}\delta^{ij}\delta^{kl} \right) D_{\mu\nu}(x-y) \tag{2.49}$$

where we have assumed, as usual, a gauge-fixing to define the gluon propagator in perturbation theory. For instance, one has

$$D_{\mu\nu}(x-y) = \frac{1}{4\pi^2} \frac{\delta_{\mu\nu}}{(x-y)^2} \tag{2.50}$$

in the Feynman gauge. It has to be said that δ has to be substituted by a $-g_{\mu\nu}$ if we work in Minkowski space. Now, equation (2.49) can be derived immediately from the standard formula

$$\langle A^{a}_{\mu}(x) A^{b}_{\nu}(y) \rangle_{\text{Gauss}} = \delta^{ab} D_{\mu\nu}(x-y) \tag{2.51}$$

multiplying by the generators of the $SU(N)$ gauge group according to (2.48) and using the completeness condition:

$$\sum_{a=1}^{N^2-1} (t^a)^{ij} (t^a)^{kl} = \left(\delta^{il}\delta^{kj} - \frac{1}{N}\delta^{ij}\delta^{kl} \right) \quad \boxed{\text{for } SU(N)} \tag{2.52}$$

Alternatively, (2.49) can be derived directly from a path integral over the matrix fields.

We concentrate only on the structure of diagrams in the index space, i.e. the space of the indices associated with the $SU(N)$ group. We shall not consider, in most cases, space-time structures of diagrams which are prescribed by Feynman's rules.

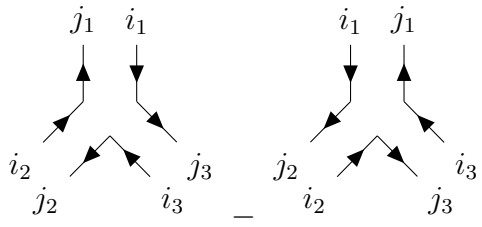
Omitting at large N the second term in parentheses on the RHS of (2.49), we depict the propagator by the *double line*

$$\langle A^{ij}_{\mu}(x) A^{kl}_{\nu}(y) \rangle_{\text{Gauss}} \propto \delta^{il}\delta^{kj} = \begin{array}{l} i \longrightarrow l \\ j \longleftarrow k \end{array} \tag{2.53}$$

Each line, often termed the index line, represents the Kronecker delta-symbol and has an orientation which is indicated by arrows. This notation is obviously consistent with the space-time structure of the propagator that describes a propagation from x to y.

Arrows are a result of the fact that the matrix A^{ij}_{μ} is Hermitian and its off-diagonal components are complex conjugate. Double lines appear generically in all models describing matrix fields in contrast to vector (in internal symmetry space) fields, whose propagators are depicted by single lines.

The three-gluon vertex is depicted in the double line notation as



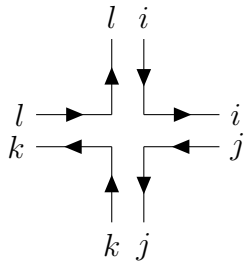
$$\propto g\left(\delta^{i_1 j_3}\delta^{i_2 j_1}\delta^{i_3 j_2} - \delta^{i_1 j_2}\delta^{i_2 i_2 j_3}\delta^{i_3 j_1}\right) \quad (2.54)$$

where the subscripts 1,2 or 3 refer to each of the three gluons. The relative minus sign arises from the commutator in the cubic-in-A term in the QCD action (2.32). The color part is antisymmetric under the interchange of gluons. The (momentum-space) space-time part

$$\gamma_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = \delta_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} + \delta_{\mu_2\mu_3}(p_2 - p_3)_{\mu_1} + \delta_{\mu_1\mu_3}(p_3 - p_1)_{\mu_2} \quad (2.55)$$

is also antisymmetric. We consider all three gluons as incoming so their momenta obey $p_1 + p_2 + p_3 = 0$. The full vertex is symmetric as is prescribed by Bose statistics.

The four-gluon vertex involves six terms - each of them is depicted by a cross - which differ by interchanging of the color indices. We depict the color structure of the four-gluon vertex for simplicity in the case when $i_1 = j_2 = i, i_2 = j_3 = j, i_3 = j_4 = k, i_4 = j_1 = l$, but i, j, k, l take on different values. Then only the following term is left:



$$\propto g^2 \quad (2.56)$$

where there are no delta-symbols since the color structure is fixed. We pick up only one color structure by equating indices pairwise.

Diagrams of perturbation theory can now be completely rewritten in the double-line notation. The simplest one describing the one-loop correction to the

gluon propagator is depicted in 2.2. The sum over the N indices is s associated with the closed index line. The contribution of this diagram is $\sim g^2 N \sim 1$. In order for the large-N limit to be nontrivial, the bare coupling constant g^2 should satisfy

$$g^2 \sim \frac{1}{N} \tag{2.57}$$

This dependence on N is also prescribed by the asymptotic-freedom formula

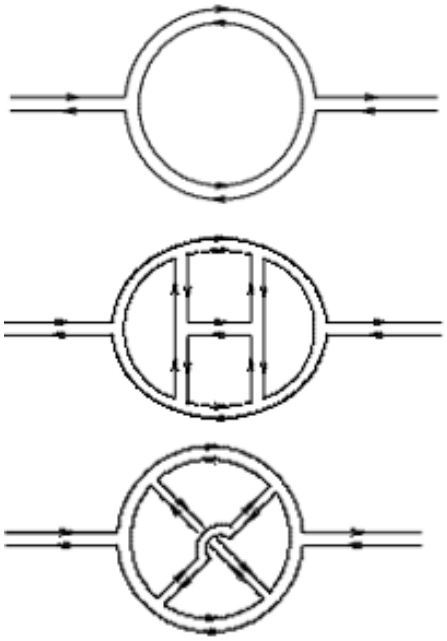


Figure 2.2: Double-line representation of a one-loop diagram for the gluon propagator, four-loop diagram and a nonplanar diagram

$$g^2 = \frac{12\pi^2}{11N\ln(\Lambda/\Lambda_{QCD})} \tag{2.58}$$

of pure $SU(N)$ gauge theory.

Thus the contribution of the first diagram in 2.2 is of order $\sim g^2 N \sim 1$ in the large-N limit. We can think of the double lines as bounding a piece of a plane. These lines represent a two-dimensional object. In mathematics these double-line graphs are often called ribbon graphs or fatgraphs. They are deeply connected with Riemann surfaces.

2.2.3.2 Remark on the U(N) gauge group

The double line representation of the diagrams holds, strictly speaking for the $U(N)$ gauge group, whose generators

$$T^A = (t^a, I/\sqrt{N}), \quad \mathrm{Tr}T^A T^B = \delta^{AB} \quad \boxed{A = 1, \cdots, N^2}. \tag{2.59}$$

obey the completeness condition

$$\sum_{A=1}^{N^2} (T^A)^{ij} (T^A)^{kl} = \delta^{il}\delta^{kj} \quad \boxed{\text{for} \quad U(N)} \tag{2.60}$$

Elements of both $SU(N)$ and $U(N)$ can be represented in the form

$$U = e^{iB} \tag{2.61}$$

where B is a general Hermitian matrix for $U(N)$ and a traceless Hermitian matrix for $SU(N)$. The large-N limit of both the $U(N)$ and $SU(N)$ groups is the same.

2.2.4 Planar and Nonplanar Graphs

The double-line representation of perturbation theory diagrams is very convenient to estimate their orders in $1/N$. Each three- or four-gluon vertex contributes a factor of g or g^2, respectively. Each closed index line contributes a factor of N, while $g^2 \sim 1/N$.

2.2.4.1 't Hooft topological expansion

Let us consider a typical diagram for the gluon propagator as depicted in 2.2 b). The sum over the N indices is associated with each of the four closed index lines, whose number is equal to the number of loops. The contribution of this diagram is then $\sim g^8 N^2 \sim 1$.

Diagrams of this type, which can be drawn on a sheet of paper without crossing any lines, are called *planar diagrams*. For such diagrams, the addition of a loop inevitably results in the addition of two three-gluon (or one four-gluon) vertices. A planar diagram with n_2 loops has n_2 closed index lines. It is of order

$$n_2-\text{loop planar diagram} \sim (g^2 N)^{n_2} \sim 1, \tag{2.62}$$

so that all planar diagrams survive in the large-N limit. Let us now consider a nonplanar diagram as the one depicted in 2.2 c). The diagram has six three-gluon vertices but only one closed index line (although it has 3 loops!). The order of this diagram is $\sim g^6 N \sim 1/N^2$.

This nonplanar diagram can be drawn without line-crossing on a surface with one handle (or hole) which in mathematics terms is called a torus of a surface with genus one. A plane is then equivalent to a surface with genus zero, which is in tern equivalent to a sphere. A general Riemanian surface with h holes has genus h.

The above evaluations of the order of the diagrams can now be described by the single formula

$$\text{genus} - h \text{ diagram} \sim \left(\frac{1}{N^2}\right)^{\text{genus}} . \tag{2.63}$$

The expansion in $1/N$ rearranges perturbation theory diagrams according to their topology as demonstrated in 1974 by 't Hooft. It is referred to as the topological expansion or the genus expansion. Only planar diagrams associated with genus zero survive in the large-N limit. The problem of summing the planar graphs is complicated but simpler than that of summing all the graphs, since the number of planar graphs with n_0 vertices grows geometrically at large n_0:

$$\sharp_p(n_0) \equiv \sharp \text{ of planar graphs} \sim \text{const}^{n_0} \tag{2.64}$$

This was shown by Tuttle and Koplic, Beveu, Nussinov, while the total number of graphs grows factorially with n_0. There is no dependence on the number of external lines of a planar graphs in (2.64), so it is assumed to be much less than n_0.

There is a big difference between the planar diagrams and the ladder diagrams which describe e^+e^- elastic scattering in QED. For the ladder with n rungs, there are $n!$ ladder diagrams, but only one of them is planar. This shows why the number of planar graphs is much smaller than the total number of graphs, most of which are non-planar. Equation (2.63) holds, strictly speaking, only for the gluon propagator, while the contribution of all planar diagrams to a connected n-point Green function is $\sim g^{n-2}$, which is its natural order in $1/N$. The three-gluon Green function is $\sim g$, the four-gluon one is $\sim g^2$ and so on. The contributions of all planar diagrams are of the same order ~ 1 in the large-N limit, independently of the number of

external lines, for the Wilson loop average

$$\Big\langle \frac{1}{N}\mathrm{Tr}P e^{ig\oint_\Gamma \mathrm{d}x^\mu A_\mu(x)}\Big\rangle = \sum_{n=0}^{\infty} i^n \oint_\Gamma \mathrm{d}x_1^{\mu_1}\int_{x_1}^{x_1}\mathrm{d}x_2^{\mu_2}\cdots\int_{x_1}^{x_{n-1}}\mathrm{d}x_n^{\mu_n}\mathrm{d}x_n^{\mu_n}G^{(n)}_{\mu_1\cdots\mu_n}(x_1,\cdots,x_n) \quad (2.65)$$

where

$$G^{(n)}_{\mu_1\cdots\mu_n}(x_1,\cdots,x_n) \equiv \frac{g^n}{N}\langle \mathrm{Tr}\left[A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\right]\rangle. \quad (2.66)$$

The factor of $1/N$, which normalizes the trace, provides the natural normalization. The ordering along a closed path implies cyclic-ordering in the index space as depicted in Fig. 2.3 where we omit the arrows for simplicity. This diagram has $n_0 = 10$ vertices, $n_1 = 12$ gluon propagators, $n_2 = 4$ closed index lines, and $B = 1$ boundaries. The color indices of the external lines are contracted by the Kronecker

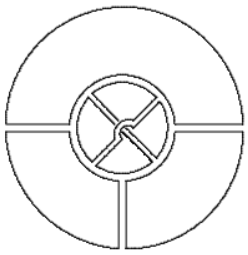


Figure 2.3: Generic double-line index diagram

delta symbols (represented by the single lines) in a cyclic order. The extra factor of $1/N$ arises from normalization. The order in $1/N$ of the diagram is $\sim 1/N^2$ in accord with (2.63). Analogously, the color indices in (2.66) are contracted in the cyclic order. The delta-symbols, which contract the color indices, are depicted by the single lines. They can be viewed as a boundary of the diagram. The actual size of the boundary is not essential - it can be shrunk to a point. Then a bounded piece of a plane will be topologically equivalent to a sphere with a puncture. We draw planar diagrams in a plane with an extended boundary (boundaries) rather than in a sphere with a puncture (punctures). The closed boundary is associated with the trace over the color indices of the multi-point Green function.

The boundary represents the Wilson loop - a trajectory of a heavy quark in the fundamental representation.

2.2.4.2 Topological expansion and quark loops

It is easy to incorporate quarks in the topological expansion. A quark field belongs to the fundamental representation of the gauge group $SU(N)$ and its propagator is represented by a single line

$$\langle \psi_i \bar{\psi}_j \rangle \propto \delta_{ij} = \ i \longrightarrow j \tag{2.67}$$

The arrow indicates, as usual, the direction of propagation of a (complex) field ψ. These arrows are often omitted for simplicity. The diagram for the gluon propagator which involves one quark loop is depicted in Fig 2.4 a). It involves one quark

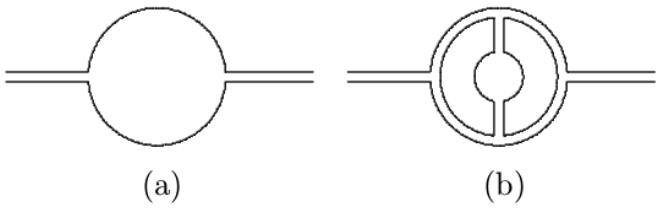


Figure 2.4: Diagrams for gluon propagator which involve quark loop

loop and has no closed index lines so that its order is $\sim g^2 \sim 1/N$. The diagram in 2.4 b) is $\sim g^6 N^2 \sim 1/N$ analogously.

It is evident from this consideration that quark loops are not accompanied by closed index lines. One should add a closed index line for each quark loop in order for a given diagram with L quark loops to have the same double-line representation as for pure gluon diagrams. Therefore, given (2.63), diagrams with L quark loops are suppressed at large N by

$$L \text{ quark loops} \sim \left(\frac{1}{N}\right)^{L+2*\text{genus}} \tag{2.68}$$

The single-line representation of the quark loops is similar to that of the Wilson loop. Such a diagram emerges in gluon corrections to the vacuum expectation value of the quark operator:

$$O = \frac{1}{N}\bar{\psi}\psi, \tag{2.69}$$

where the factor of $1/N$ is introduced to make it $\mathcal{O}(1)$ in the large-N limit. Therefore, the external boundary can be viewed as a single line associated with valence quarks.

2.2.4.3 The proof of topological expansion

To prove (2.63) and it's quark counterpart (2.68), let us consider a generic diagram in the index space which has $n_0^{(3)}$ three-point vertices (either three-gluon quark-gluon ones), $n_0^{(4)}$ four-gluon vertices, n_1 propagators (either gluon or quark ones), n_2 closed index lines, L virtual quark loops and B external boundaries. Its order in $1/N$ is

$$\frac{1}{N^B} g^{n_0^{(3)}+2n_0^{(4)}} N^{n_2} \sim N^{n_2-n_0^{(3)}/2-n_0^{(4)}-B} \tag{2.70}$$

as has already been explained. The extra factor of $1/N^2B$ arises from the extra normalization factor of $1/N$ in operators associated with external boundaries.

The number of propagators and vertices are related by

$$2n_1 = 3n_0^{(3)} + 4n_0^{(4)}, \tag{2.71}$$

since three- and four-point vertices emit three or four propagators, respectively, and each propagator connects two vertices. We then rewrite the RHS of (2.70) as

$$N^{n_2-n_0^{(3)}/2-n_0^{(4)}-B} = N^{n_2-n_1+n_0-B}, \tag{2.72}$$

where $n_0 = n_0^{(3)} + n_0^{(4)}$ is the total number of vertices. The exponent on the RHS of (2.72) can be expressed via the Euler characteristic χ of a given graph of genus h. An appropriate Riemann surface, which is associated with a given graph, is open and has $B + L$ boundaries. This surface can be closed by attaching a cap to each boundary. The single lines then become double lines together with the lines of the boundary of each cap. The number of faces for a closed Riemann surface constructed in such a manner is $n_2 + L + B$, while the number of edges and vertices are n_1 and n_0, respectively. Euler's theorem states that

$$\chi \equiv 2 - 2h = n_2 + L + B - n_1 + n_0. \tag{2.73}$$

Therefore the RHS of (2.72) can be rewritten as

$$N^{n_2-n_1+n_0-B} = N^{2-2h-L-2B}. \tag{2.74}$$

We have thus proven that the order in $1/N$ of a generic graph does not depend on its order in the coupling constant and is completely expressed via the genus h and the number of virtual quark loops L and external boundaries B by

$$\text{generic graph} \sim \left(\frac{1}{N}\right)^{2h+L+2(B-1)} \tag{2.75}$$

For $B = 1$, we recover (2.63) and (2.68).

2.2.4.4 't Hooft versus Veneziano limits

In QCD there are several species or flavours of quarks. We denote the number of flavours by N_f and associate a Greek letter α or β with a flavour index of the quark field. The quark propagator then has the Kronecker delta-symbol with respect to the flavour indices in addition to (2.67)

$$\langle \psi_i^\alpha \bar{\psi}_j^\beta \rangle \propto \delta^{\alpha\beta}\delta_{ij} \tag{2.76}$$

Their contraction results in

$$\sum_{\alpha=1}^{N_f} \delta_{\alpha\alpha} = N_f \tag{2.77}$$

Therefore, an extra factor of N_f corresponds to each closed quark loop for the N_f flavours.

The limit when N_f is fixed as $N \to \infty$ is called the 't Hooft limit. Only valence quarks are then left (the quenched approximation). In order for a meson to decay into other mesons built out of quarks, a quark-antiquark pair must be produced out of the vacuum. Consequently, the ratios of meson widths to their masses are

$$\frac{\Gamma_{\text{total}}}{M} \sim \frac{N_f}{N} \tag{2.78}$$

in the 't Hooft limit. The ratio on the LHS of (2.78) is 10-15% experimentally for the ρ-meson. The hope of solving QCD in the 't Hooft limit is the hope to describe QCD with this accuracy. An alternative large-N limit of QCD, when $N_f \sim N$ as $N \to \infty$ was proposed by Veneziano in 1976. A general diagram with L quark loops will contribute

$$L \text{ quark loops} \sim \left(\frac{N_f}{N}\right)^L \left(\frac{1}{N^2}\right)^{\text{genus}}, \tag{2.79}$$

since each quark loop results in N_f.

The quark loops are not suppressed at large N in the Veneziano limit

$$N_f \sim N \to \infty \tag{2.80}$$

if the diagram is planar.

It is the Veneziano limit that is related to the hadronic topological expansion in the dual-resonance models. In the Veneziano limit hadrons can have finite widths according to (2.78).

2.2.5 Large-N factorization

The vacuum expectation values of several colorless or white operators, which are singlets with respect to the gauge group, factorize in the large-N limit of QCD (or other matrix models) as first was noticed by A.A. Migdal and independently by E. Witten in late 1970's. The simplest gauge-invariant operators in a pure $SU(N)$ gauge theory are the closed Wilson loops

$$\Phi(C) = \frac{1}{N}\mathrm{Tr}Pe^{ig\oint_C \mathrm{d}z^\mu A_\mu(z)}. \tag{2.81}$$

They obey the factorization property

$$\langle\Phi(C_1)\cdots\Phi(C_n)\rangle = \langle\Phi(C_1)\rangle\cdots\langle\Phi(C_nC_n)\rangle + \mathcal{O}(N^{-2}) \tag{2.82}$$

The factorization implies a semiclassical nature of the large-N limit of QCD (a saddle point in the path integral for certain variables). The factorization property also holds for gauge-invariant operators constructed from quarks as in (2.69). For the case of several flavours N_f, we normalize these quark operators by

$$O_\Gamma = \frac{1}{N_f N}\bar{\Psi}\Gamma\psi \tag{2.83}$$

Here, Γ denotes one of the combination of the γ-matrices:

$$\Gamma = I, \gamma_5, \gamma_m u, i\gamma_\mu\gamma_5, \sum_{\mu\nu} = \frac{1}{2i}\left[\gamma_\mu, \gamma_\nu\right], \cdots . \tag{2.84}$$

The factorization of the gauge-invariant quark operators hold both in the 't Hooft and Veneziano limits:

$$\langle O_{\Gamma_1}\cdots O_{\Gamma_n}\rangle = \langle O_{\Gamma_1}\rangle\cdots\langle O_{\Gamma_n}\rangle + \mathcal{O}(1/(N_f N)). \tag{2.85}$$

The nonfactorized part, which is associated with connected diagrams is $\sim 1/N$ in the 't Hooft limit. This leads, in particular, to the coupling constant of meson-meson interaction of order $1/N$. The Veneziano limit it analogous to pure Yang-Mills.

The factorization can be seen (at all orders of perturbation theory) form (2.75) for the contribution of a generic connected graph of genus h with B external boundaries which are precisely associated with the Wilson loops (or the quark operators). The diagrams with gluon lines emitted and absorbed by the same

operator are products of diagrams having only one boundary. Their contribution is of order one. The diagrams with gluon lines emitted and absorbed by two different operators have two boundaries. This proves the factorization property (2.85) at all orders of perturbation theory.

The large-N factorization can also be verified beyond perturbation theory at all orders of the strong-coupling expansion of the $SU(N)$ lattice gauge theory. A non-perturbative proof of the factorization was given using quantum equations of motion (the loop equations).

2.2.6 Conclusion

We have considered in this section the basic features of the methods for non-perturbative studies of gauge theories, which were developed in the second half of 1970's- early 1980's. Their contemporary applications in high-energy physics are extremely broad: from the scattering of particles at very high energies of the attempts of constructing a unified theory of all interactions, including gravity.

2.3 $T\bar{T}$ deformation

In some sense all QFT's that we use in physics are *effective* field theories - they are valid only over some range of energy (respectively, length) scales. It has been shown that if the theory is renormalizable, this range of scales can be very large, thus making the theory more applicable and predictive. Still, even for (perturbatively) renormalizable theories, new physics enters the stage at some point, usually at some UV point (or region). Non-renormalizable theory, meaning a theory where the action involves operators with dimension greater than the spacetime dimension ($\Delta > d$) still can be applicable up to some energy scale (the so called "UV cut-off") Λ. Investigation of non-renormalizable theories has led to the idea of the "UV completion" - a way to make sense of some theories at higher energies. For example a theory might flow from a non-trivial RG fixed point, an effect called "asymptotic safety". (revize) Another possibility is that the UV limit is not a conventional UV fixed point corresponding to a local QFT but is something else entirely (eg *String Theory*). The $T\bar{T}$ deformation of 2d QFT is an example of a non-renormalizable deformation of a local QFT for which many physical quantities make sense and are finite and calculable in terms of the data of the undeformed theory. This deformation is very special in itself, it has been termed 'asymptotic fragility' - which in turn can be used as a constraint on physical theories.

2.3.1 Overview

Consider a sequence of 2d Eucleadian field theories $\mathcal{T}$ ($t \in \mathbb{R}$) in a domain endowed with a flat Eucleadian metric η_{ij}, each with a local stress-energy tensor

$$T_{ij}^{(t)}(x) \sim \delta S^{(t)}/\delta g^{ij}(x) \tag{2.86}$$

Imagine now that the first object in the sequence, $\mathcal{T}^{(0)}$ is a conventional local QFT (massive) or a CFT (massless). We can define the deformation for our theories

$$S^{(t+\delta t)} = S^{(t)} - \delta t \int \det T^{(t)} d^2x \tag{2.87}$$

equivalently we can arrive at this by inserting $\int \det T d^2x$ into the correlation functions. We can calculate the determinant:

$$\det T = \frac{1}{2}\epsilon^{ik}\epsilon^{jl}T_{ij}T_{kl} \propto T_{zz}T_{\bar{z}\bar{z}} - T_{z\bar{z}}^2 \tag{2.88}$$

where (2.88) is written in complex (holomorphic-antiholomorhic) coordinates. In the language of conformal field theories this can be written as $T\bar{T}$. Obviously the dimension of this deformation is 4, hence one would expect that $\langle \det T \rangle \sim \Lambda^4$. The first very interesting result using these types of transformations came from Zamolodchikov (2004). By requiring that the stress-energy tensor is conserved, $\partial^i T_{ij} = 0$, we can show that

$$\frac{\partial}{\partial y_m} \epsilon^{ik} \epsilon^{jl} T_{ij}(x) T_{kl}(x+y) = \frac{\partial}{\partial x_i} \epsilon^{mk} \epsilon^{jl} T_{ij} T_{kl}(x+y) \tag{2.89}$$

Whilst at first glance this does not look very interesting, one can make the following statement:

For any transitionally invariant state the vacuum expectation value

$$\langle \epsilon^{ik} \epsilon^{jl} T_{ij}(x) T_{kl}(x) \rangle = \langle \epsilon^{ik} \epsilon^{jl} T_{ij}(x) T_{kl}(x+y) \rangle \tag{2.90}$$

The vacuum expectation value in (2.90) calculable and finite in terms of the matrix elements of T_{ij}. In some interpretation this means that the $T\bar{T}$ transformation is solvable and integrable, something that is in itself a very rare occurrence in quantum field theories. We will now follow Zamolodchikov's paper, omitting some extra detail and focusing on the $T\bar{T}$ deformations in terms of conformal field theories.

2.3.2 One-point expectation functions

The one-point expectation values $\langle \mathcal{O}_i \rangle$ of local fields control the linear reaction of the system to external forces which couple to the fields $\mathcal{O}_i$(z). Also, in view of the operation-product expansions (OPE):

$$\mathcal{O}_i(z) \mathcal{O}_j(z') = \sum)_k C_{ij}^k (z - z') \mathcal{O}_k(z') \tag{2.91}$$

the two-point correlation functions $\langle \mathcal{O}_i(z) \mathcal{O}_j(z') \rangle$ (and, by extension and repeated application of (2.91), all higher-point correlation functions) are expressed through the OPE structure functions $C_{ij}^k(z-z')$ and the one-point expectation values $\langle \mathcal{O}_k \rangle$. While this is an interesting result, it is seldom used for calculation because even though the structure functions (which describe local dynamics of the field theory) usually admit perturbative expansions, the one-point expectation values (incorporation information about the vacuum state of the theory) are typically nonpertur-

bative quantities and no general to their systematic evaluation is known.[1]. Even before Zamolodchikov it was shown that for some specific models (namely the sine-Gordon model and some minimal CFT's) the expectation value of the composite field $T\bar{T}$ is related to the trace component of the stress-energy tensor:

$$\langle T\bar{T}\rangle = -\langle\Theta\rangle^2, \tag{2.92}$$

where $\Theta = \frac{\pi}{2}T^{\mu}_{\mu}$. Zamolodchikov showed that this equations is valid for a broad spectrum of 2d quantum field theories, including some theories that are not required to be integrable. Instead of considering an infinite Eucleadian plane, let us consider a field theory on an infinite cylinder, with one of the Eucleadian axis compactified on a circle (the so-called Matsubara representation of field theories at finite temperature). It can be shown that the last equation can be generalized to:

$$\langle T\bar{T}\rangle = \langle T\rangle\langle\bar{T}\rangle - \langle\Theta\rangle\langle\Theta\rangle. \tag{2.93}$$

When the circumference of the cylinder goes to infinity (equivalently, the temperature goes to zero) the global rotational symmetry is restored, making the expectation value of the chiral components T and $\bar{T}$ vanish - this limit (2.93) reduces to (2.92). It can also be argued that the vacuum expectation values $\langle\cdots\rangle = \langle 0|\cdots|0\rangle$ are replaced by more general diagonal elements $\langle n|\cdots|n\rangle$, where $|n\rangle$ is any non-degenerate eigenstate of the energy and momentum operators (in the case of the cylinder, to make this statement precise one has to take the Hamiltonian picture in which the coordinate along the cylinder is taken as the Eucleadian time). Note that we consider quantum field theory in flat 2d space in terms of Eucleadian version of the theory. The points z of the 2d space can be labeled by the Cartesian coordinates (x, y), but we usually use complex coordinates $z = (\mathrm{z}, \mathrm{z}) = (\mathrm{x} + i\mathrm{y}, \mathrm{x} - i\mathrm{y})$. The usual normalization of the energy-momentum tensor $T_{\mu\nu}$ is assumed - for instance, in the picture where y is taken as the Eucleadian time, $-T_{yy}$ coincides with the energy density. The chiral components $T, \bar{T}, \Theta$ are normalized according to the CFT convention, namely $T = -(2\pi)T_{\mathrm{zz}},\ \bar{T} = -(2\pi)T_{\bar{\mathrm{z}}\bar{\mathrm{z}}},\ \Theta = (2\pi)T_{z\bar{z}}$.

2.3.3 Assumptions and main idea of the argument

Some of the assumptions in this section concern the local dynamics of the field theory, and others will be about the global settings. To make the proper distinction,

[1] In 2d, fairly accurate numerical estimates can be obtained for a lot of cases via a version of the Truncated Conformal Space Approach

we shall label the former with (L) and the latter with (G).

1. (L). Local translational and rotational symmetry. This implies existence of local field $T_{\mu\nu}$ (the energy-momentum tensor) which is symmetric, $T^{\mu\nu}(z) = T^{\nu\mu}(z)$, and satisfies the continuity equation $\partial_\mu T^{\mu\nu}(z) = 0$. In terms of the conventional chiral components $T = -2\pi T_{zz}, T = -2\pi T_{\bar{z}\bar{z}}$ and $\Theta = 2\pi T_{z\bar{z}} = 2\pi T_{\bar{z}z}$ the continuity equation is written as

$$\partial_{\bar{z}} T(z) = \partial_z \Theta(z), \tag{2.94}$$

$$\partial_z \bar{T}(z) = \partial_{\bar{z}\Theta(z).} \tag{2.95}$$

   This assumption is already taken into account in writing the OPE (2.91), where the structure functions C_{ij}^k are assumed to depend on the separations $z - z'$ only.

2. (G). Global translational symmetry. It is assumed that for any local field $\mathcal{O}_i(z)$ the expectation value $\langle \mathcal{O}_i(z) \rangle$ is a constant independent of z. It follows from (2.91) that the two-point correction functions depend only on the separations,

$$\langle \mathcal{O}_i(z) \mathcal{O}_j(z') \rangle = G_{ij}(z - z'). \tag{2.96}$$

3. (G). Infinite separations. We assume that at least one direction (i.e. Euclidean vector $e = (\mathrm{e}, \bar{\mathrm{e}})$) exists, such that for any $\mathcal{O}_i$ and $\mathcal{O}_j$

$$\lim_{t\to\infty} \langle \mathcal{O}_i(z + et) \mathcal{O}_j(z') \rangle = \langle \mathcal{O}_i \rangle \langle \mathcal{O}_j \rangle. \tag{2.97}$$

   The "global" assumptions 2 and 3 imply that the underlying geometry of 2D space is either an infinite plane, or an infinitely long cylinder.

4. (L) CFT limit at short distances. We will assume that the short-distance behaviour of the field theory is governed by a conformal field theory, and that certain no-resonance condition is satisfied. We will detail the content of this assumptions later. This assumption is needed in order to make the definition of the composite field $T\bar{T}$ essentially unambiguous[2]

[2]There is intrinsic ambiguity in adding certain total derivatives, which does not affect the expectation value $\langle T\bar{T} \rangle$.

The main idea of the arguments stems from a simple identity involving two-point correlation functions of the energy-momentum tensor, consequence of the assumptions 1-3 alone. Consider the following combination of two-point correlation functions

$$\mathcal{C} = \langle T(z)\bar{T}(z')\rangle - \langle \Theta(z)\Theta(z')\rangle. \tag{2.98}$$

Now, we take the derivative $\partial_{\bar{z}}$ of the above and transform it as follows. In the first term, we use (2.95) to replace the derivatives to replace the derivative $\partial_{\bar{z}}T(z)$ by $\partial_z\Theta(z)$, and then apply (2.96) to move the derivative to the second entry $\bar{T}(z')$. When the derivative $\partial_{\bar{z}\Theta(z)}$ in the second term is also moved to $\Theta(z')$, one finds

$$\langle \partial_{\bar{z}}T(z)\bar{T}(z') - \partial_{\bar{z}}\Theta(z)\Theta(z')\rangle = \langle -\Theta(z)\partial_{z'}\bar{T}(z') + \Theta(z)\partial_{\bar{z}'}\Theta(z')\rangle = 0, \tag{2.99}$$

where the equation (2.95) was used again in the last step. By similar transformations it can be shown that the derivative ∂_z also vanishes, and hence the quantity $\mathcal{C}$ is a constant, independent of the coordinates.

Note that in this derivation, only assumptions 1 and 2 are used. Adding assumption 3 allows one to relate this constant to the one-point expectation values of the fields involved. Taking the limit (2.97) of the right-hand side of (2.98), one finds:

$$\mathcal{C} = \langle T\rangle\langle\bar{T}\rangle - \langle\Theta\rangle\langle\Theta\rangle. \tag{2.100}$$

On the other hand, some reflection about equation (2.98) seems to hint to the fact that the constant $\mathcal{C}$ coincides with the expectation value of appropriately defined composite operator $T\bar{T}$. Indeed, one expects that the composite field $T\bar{T}$ can be obtained in some way form the product $T(z)\bar{T}(z')$ by bringing the points z and z' together. The main obstacle is in the presence of singular terms in the operator product expansion of $T(z)\bar{T}(z')$, which makes straightforward limits impossible. We will show in the next section that the second term in the combination $T(z)\bar{T}(z') - \Theta(z)\Theta(z')$ exactly subtracts these singular terms, so that the limit $z \to z'$ in (2.98) can be taken, leading to (2.93).

2.3.4 Operator Product Expansion

It is not difficult to repeat manipulations of the previous section, this time working not with the two-point functions (2.98), but with the combination of the operator

products $T(z)\bar{T}(z') - \Theta(z)\Theta(z')$ itself. Using (2.95) one finds

$$\partial_{\bar{z}} \left(T(z)\bar{T}(z') - \Theta(z)\Theta(z') \right) \;=\; (\partial_z \,+\, \partial_{z'})\Theta(z)\bar{T}(z') \,-\, (\partial_{\bar{z}} \,+\, \partial_{\bar{z}'})\Theta(z)\Theta(z'), \tag{2.101}$$

and

$$\partial_z(T(z)\bar{T}(z') - \Theta(z)\Theta(z')) = (\partial_z + \partial_{z'})T(z)\bar{T}(z') - (\partial_{\bar{z}} + \partial_{\bar{z}'})T(z)\Theta(z'). \tag{2.102}$$

The meaning of these expressions becomes clearer after inserting the operator product expansions

$$\Theta(z)\bar{T}(z') = \sum_i B_i(z - z')\mathcal{O}_i(z'), \tag{2.103}$$

$$(z)\Theta(z') = \sum_i A_i(z - z')\mathcal{O}_i(z'), \tag{2.104}$$

and

$$T(z)\bar{T}(z') = \sum_i D_i(z - z')\mathcal{O}_i(z'), \tag{2.105}$$

$$\Theta(z)\Theta(z') = \sum_i C_i(z - z')\mathcal{O}_i(z'), \tag{2.106}$$

where the sums involve complete sets of local fields $\{\mathcal{O}_i\}$. The equations (2.101), (2.102) then read

$$\sum_i \partial_{\bar{z}} F_i(z - z')\mathcal{O}_i(z') =$$
$$\sum_i \left(B_i(z - z')\partial_{z'}\mathcal{O}_i(z') - C_i(z - z')\partial_{\bar{z}'}\mathcal{O}_i(z') \right), \tag{2.107}$$

$$\sum_i \partial_z F_i(z - z')\mathcal{O}_i(z') =$$
$$\sum_i \left(D_i(z - z')\partial_{z'}\mathcal{O}_i(z') - A_i(z - z')\partial_{\bar{z}'}\mathcal{O}_i(z') \right), \tag{2.108}$$

where

$$F_i(z - z') = D_i(z - z') - C_i(z - z') \tag{2.109}$$

Note that the right-hand side of equations (2.107) and (2.108) involve only coordinate derivatives of local fields. It follows that any operator $\mathcal{O}_i$ appearing in the expansion

$$T(z)\bar{T}(z') - \Theta(z)\Theta(z') = \sum_i F_i(z-z')\mathcal{O}_i(z'), \tag{2.110}$$

unless itself is a coordinate derivative of another local operator, comes with a constant (i.e. coordinate-independent) coefficient F_i. In other words, the operator product expansion can be written as:

$$T(z)\bar{T}(z') - \Theta(z)\Theta(z') = \mathcal{O}_{T\bar{T}}(z') + \text{derivative terms} \tag{2.111}$$

where $\mathcal{O}_{T\bar{T}}(z)$ is some local operator. At this point it is possible to *define* the composite field $T\bar{T}$ through equation above:

$$T\bar{T}(z) := \mathcal{O}_{T\bar{T}}(z); \tag{2.112}$$

then the desired relation (2.93) follows immediately. Note that although in this way one defines $T\bar{T}$ only modulo derivative terms, in view of the assumption 2, those terms bring no contribution to the left hand side of (2.93). However, this definition may look a bit too formal to bring much insight into the meaning of (2.93). To understand the nature of the limit $z \to z'$ in (2.110), and thus to make contact with a more constructive definition of the composite field $T\bar{T}$, one needs to know more about the short-distance behaviour of the field theory. For this case, assumption 4 holds the relevant information for our construction.

2.3.5 Dimensional Analysis

As was mentioned, we assume the short-distance limit of the field theory is controlled by certain conformal field theory, which we will refer to as the CFT. More precisely, we assume that the field theory at hand is the CFT perturbed by its relevant operators. To avoid unnecessarily complex expressions, let me first assume that the perturbation is by a single operator Φ_Δ of the dimensions (Δ, Δ) with $\Delta < 1$; then the theory is described by the action

$$\mathcal{A} = A_{CFT} + \mu \int \Phi_\Delta(z) d^2z, \tag{2.113}$$

where μ is a coupling constant which has the dimension $[\text{length}]^{2\Delta-2}$. This formulation of the theory makes it possible to carry out dimensional analysis of the structure functions.

Let $\{\mathcal{O}_i\}$ be a complete set of local fields of the CFT, including primary fields as well as their descendants, and let $(\Delta_i, \bar{\Delta}_i)$ be the left and right scale dimensions of the fields $\mathcal{O}_i$. This set includes the field $T\bar{T}$ (of the dimensions) (2,2)), which in CFT is just the descendant $T\bar{T} = L_{-2}\bar{L}_{-2}I$ of the identity operator. Equivalently, this field can be defined as $T\bar{T}(z') = \lim_{z\to z'} T(z)\bar{T}(z')$, where the limit is straightforward since in CFT the above operator product has no singularity at $z = z'$.

As was explained in REF [12], the fields $\mathcal{O}_I$ of the perturbed theory are in one-to-one correspondence with the fields of the CFT (hence we use the same notations). The field $\mathcal{O}$ has the spin $s_i = \Delta_i - \bar{\Delta}_i$ and the mass dimension $d_i = \Delta_i + \bar{\Delta}_i$, and $\mathcal{O}_i$ coincides with the corresponding CFT field in the limit $\mu \to 0$. Unless certain resonance conditions are met, these properties characterize the field $\mathcal{O}_i$ uniquely. One says that the field $\mathcal{O}_i$ has n-th order resonance with the field O_j if these fields have the same spins, $s_j = s_i$, and their dimensions satisfy the equation $d_i = d_j + 2n(1-\Delta)$ (the resonance condition) with some positive integer n. When this resonance condition is fulfilled the above characterization of the field $\mathcal{O}_i$ allows for the ambiguity $\mathcal{O}_i \to \mathcal{O}_i + \text{const} * \mu^n\mathcal{O}_j$

The field $T\bar{T}$ always has the intrinsic ambiguity of the form $T\bar{T} \to T\bar{T} + \text{cosnt} * \partial_z\partial_{\bar{z}}\Theta$, where Θ is the trace component of the energy-momentum tensor tensor of the perturbed theory. Using the equation for the action, it is clear that $\Theta = (1-\Delta)\pi\mu\Phi_\Delta$, the ambiguity is due to the first-order resonance of $T\bar{T}$ with the derivative $\partial_z\partial_{\bar{z}}\Phi_\Delta$. However, this ambiguity has no effect on the expectation value of $T\bar{T}$. For the present analysis the danger is in possible resonances with non-derivative fields. Since at this time I do not know how to handle the resonance cases, we accept the following no-resonance assumption:

- (4') Dimensions Δ_i of the fields $\mathcal{O}_i$ of the CFT satisfy the condition

$$\Delta_i - 2 + n(1-\Delta) \neq 0, \quad \text{for } n = 1, 2, 3, \ldots, \tag{2.114}$$

  with the only exception of $\Delta_i = \Delta + 1$ (which is the dimension of $\partial_z\partial_{\bar{z}}\Phi_\Delta$)

According to [3], the OPE structure functions in (2.91) admit power-series expansions in μ, with the coefficients computable (in principle) through the conformal perturbation theory. Thus, the structure functions $D_i(z-z')$ in (2.106) can be written as

$$D_i(z-z') = \sum_{n=0}^{\infty}(z-z')^{\Delta_i-2+n(1-\Delta)}(\bar{z}-\bar{z}')^{\bar{\Delta}_i-2+n(1-\Delta)}D_i^{(n)}\mu^n. \tag{2.115}$$

The zero-order coefficients $D_i^{(0)}$ are taken from the unperturbed CFT, hence $D_i^{(0)} = 0$ unless $\mathcal{O}_i$ is the field $T\bar{T}$ or one of its derivatives, a nd $D_{T\bar{T}}^{(0)} = 1$. Then it follows from (2.114) that the only terms in the expansions (2.115) which carry vanishing powers of both $z - z'$ and $\bar{z} - \bar{z}'$ are the zero-order term of $D_{T\bar{T}}$, and the first-order term associated with $\mathcal{O}_i = \partial_{z'}\partial_{\bar{z}'}\Phi_\Delta$.

Similar expansion can be written down for the structure functions $C_i(z - z')$ in the OPE (2.106),

$$C_i(z - z') = \sum_{n-2}^{\infty} (z - z')^{\Delta_i - 2 + n(1-\Delta)} (\bar{z} - \bar{z}')^{\bar{\Delta}_i - 2 + n(1-\Delta)} C_i^{(n)} \mu^n. \tag{2.116}$$

Note that the sum here starts from $n = 2$, consequence of the fact that $\Theta \sim \mu\Phi_\Delta$. In this case the no-resonance condition implies that there are no terms with vanishing powers of both $z - z'$ and $\bar{z} - \bar{z}'$ at all.

Consider now the differences $F_i(z - z') = D_i(z - z') - C_i(z - z')$. It follows from some of the previous equations that, unless $\mathcal{O}_i$ is a derivative of another local field, all terms with nonzero powers of $z - z'$ or $\bar{z} - \bar{z}'$ must cancel out in this difference [3]. Therefore

$$F_i(z - z') = 0 \quad \text{unless} \quad \mathcal{O}_i = T\bar{T} \quad \text{or} \quad \mathcal{O}_i = \text{derivative}, \tag{2.117}$$

and

$$F_{T\bar{T}}(z - z') = 1. \tag{2.118}$$

One concludes that the definition of $T\bar{T}$ through the conformal perturbation theory agrees with the formal definition (2.50).

It is not difficult to generalize this analysis to the case when the CFT is perturbed by a mixture. $\sum_a \mu_a \int \Phi_{\Delta_a}(z) d^2z$ of relevant operators Φ_{Δ_a}. The dimensional analysis can be carried out in a similar straightforward way provided the no-resonance condition is modified as follows:

- 4". The dimensions Δ_i of the fields $\mathcal{O}_i$ of the CFT satisfy the conditions

$$\Delta_i - 2 + \sum_a n_a(1 - \Delta_a) \neq 0 \tag{2.119}$$

for any non-negative integers n_a such that $\sum_a n_a > 0$, with only the exeption of $\Delta_i = \Delta_a + 1$.

[3] This implies for instance $D_{T\bar{T}}^{(1)} = 0$, a statement easily verified in conformal perturbation theory

2.3.6 Further remarks

We can consider the 2D space to be a cylinder, with one of the Cartesian coordinates compactified on a circle of circumference R, $(x, y) \sim (x + R, y)$, and let $\mathbb{H}$ and $\mathbb{P}$ be the Hamiltonian and the momentum operators in the picture where the coordinate y along the cylinder is taken as the Eucleadian time. The arguments of the previous sections validate the relation (2.93) with $\langle \dots \rangle$ standing for the matrix element $\langle 0|\dots|0\rangle$, where $|0\rangle$ is the ground state of the Hamiltonian $\mathbb{H}$ (and it is assumed that the states are orthogonal, meaning $\langle 0|0\rangle = 1$). It turns out that the same relation remains valid if the vacuum expectation values there are replaced by generic diagonal matrix elements $\langle n|\dots|n\rangle$, where $|n\rangle$ is an arbitrary non-degenerate eigenstate of the energy and momentum operators,

$$\mathbb{H}\,|n\rangle = E_n\,|n\rangle\,, \quad \mathbb{P}\,|n\rangle = P_n\,|n\rangle\,, \tag{2.120}$$

and again the normalization $\langle n|n\rangle = 1$ is assumed. Indeed, of the previously listed assumptions, the local ones (1 and 4) are independent on the choice of matrix element, while assumption 2 (global translational invariance) certainly remains valid when any diagonal matrix element between energy-momentum eigenstates is taken. Hence, one can repeat the calculation at the end of the second section (which only uses assumptions 1 and 2) and again show that the combination

$$\mathcal{C}(n) = \langle n|T(z)\bar{T}(z')|n\rangle - \langle n|\Theta(z)\Theta(z')|n\rangle \tag{2.121}$$

is constant, independent of the points z and z'. In general, the asymptotic factorization (2.97) (the one in assumption 3), no longer holds, since the two-point function $\langle n|\mathcal{O}_i(x, y)\mathcal{O}_j(x', y')|n\rangle$ can pick up contributions from the intermediate states $|n'\rangle$ with $E_{n'} < E_n$ which give rise to terms growing exponentially with $|y - y'|$. However, one can write down the spectral decompositions of the two-point functions in the right-hand side of (2.121), i.e.

$$\langle n|T(z)\bar{T}(z')|n\rangle = \sum_{n'} \langle n|T(z)|n'\rangle\ \langle n'|\bar{T}(z)|n\rangle\, e^{(E_n - E_{n'})|y-y'| + i(P_n + P_{n'})(x-x')} \tag{2.122}$$

and similar decomposition of $\langle n|\Theta(z)\Theta(z')|n\rangle$, where (x, y) and (x', y') are Cartesian coordinates of the points z and z', respectively. Clearly, for the combination (2.121) to be independent of the coordinates, all terms in these decompositions with $n \neq n'$ must cancel out between the two correlators in the right-hand side of

(2.121). If $|n\rangle$ is non-degenerate, if follows that

$$\mathcal{C} = \langle n|T(z)|n\rangle \, \langle n|\bar{T}(z')|n\rangle - \langle n|\Theta(z)|n\rangle \, \langle n|\Theta(z')|n\rangle \,, \tag{2.123}$$

and by taking the limit $z \to z'$ one arrives at the desired relation

$$\langle n|T\bar{T}|n\rangle = \langle n|T|n\rangle \, \langle n|\bar{T}|n\rangle - \langle n|\Theta|n\rangle \, \langle n|\Theta|n\rangle \tag{2.124}$$

It's revealing to rewrite this relation in somewhat different form. In terms of Cartesian components of the energy-momentum tensor $T_{\mu\nu}$ it reads[4]

$$\langle n|T\bar{T}|n\rangle = -\pi^2 \left(\langle n|T_{yy}|n\rangle \, \langle n|T_{xx}|n\rangle - \langle n|T_{xy}|n\rangle \, \langle n|T_{xy}|n\rangle \right), \tag{2.125}$$

On the other hand, because we're talking about the energy-momentum tensor, we have

$$\langle n|T_{yy}|n\rangle = -\frac{1}{R}E_n(R), \quad \langle n|T_{xx}|n\rangle = -\frac{d}{dR}E_n(R), \tag{2.126}$$

and

$$\langle n|T_{xy}|n\rangle = \frac{i}{R}P_n(R). \tag{2.127}$$

where it was explicitly indicated that the energy-momentum eigenvalues depend on R. Of course, in the case of P_n, the R-dependence is fixed by the momentum quantization condition: $P_n(R) = 2\pi p_n/R$, where p_n are R-independent integers. Thus the expectation value (2.124) can be expressed in terms of the eigenvalues $E_n(R), P_n(R)$ as follows

$$\langle n|T\bar{T}|n\rangle = -\frac{\pi}{R}\left(E_n(R)\frac{d}{dR}E_n(R) + \frac{1}{R}P_n^2(R) \right). \tag{2.128}$$

Suppose that the field theory (2.113) is massive, with M_0 being the mass of it's lightest particle. Then for $R >> M_0^{-1}$ the ground-state energy $E_0(R)$ approaches its asymptotic linear form with exponential accuracy, i.e.

$$E_0(R) = F_0 R + O(e^{-M_0 R}), \tag{2.129}$$

where F_0 is the vacuum energy density in infinite space. In the same limit, the first excited state $|1\rangle$ corresponds to the one-particle state with zero momentum, hence

$$E_1(R) = F_0 R + M_0 + O(e^{-M_0 R}) \tag{2.130}$$

[4]the factor π is due to the factor 2π n the definition of the chiral components of the composite operator - T and $\bar{T}$

Then it follows from (2.124) that (up to terms $\sim e^{-}M_0R$)

$$\frac{1}{\pi^2}\langle 0|T\bar{T}|0\rangle = -F_0^2, \quad \frac{1}{\pi^2}\langle 1|T\bar{T}|1\rangle = -F_0^2 - \frac{1}{R}F_0M_0. \tag{2.131}$$

These expressions can be useful in analysis of subleading singularities in statistical systems near criticality, in the situations where the irrelevant operator $T\bar{T}$ plays a significant role. This is the case, for instance, for the Ising phase transition near the Ising tri-critical point, because the RG flow from the tri-critical fixed point (the $c = 7/10$ minimal CFT) down to the Ising fixed point (the $c = 1/2$ minimal CFT) arrives at the latter along a direction the field $T\bar{T}$ as its most significant (i.e. least irrelevant) component. [13]. Another example is the Ising field theory with pure imaginary magnetic field, taken near the Yang-Lee singularity. In such cases the final relations lead to predictions abouth the amplitudes of subleading singular terms in the expansions of the free energy and correlation length near the critical point.

2.4 $T\bar{T}$ as a TsT transformation

2.4.1 $T\bar{T}$ deformations and uniform light-cone gauge

2.4.1.1 Uniform light-cone gauge

We consider a non-linear sigma model with metric $G_{\mu\nu}(X)$, where X collectively denotes all the fields, and B-field $B_{\mu\nu}(X)$. The metric part of the action is coupled to a two-dimensional metric $\gamma^{\alpha\beta}$, which we take to have unit determinant. By construction, the theory is invariant under re-parametrizations at the classic level. For the moment, we will be interested in the classical theory, and we will not assume that the metric and B-field describe a string background. We will however assume that the metric has at least two shift isometries: one for a time-like coordinate which we denote by $t, t \to t + \delta t$, and wich yields the target-space energy E, and one for $\phi \to \phi + \delta\phi$, which yields some (angular) momentum J. In terms of the action, we have

$$S = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\tau \int_0^R \mathrm{d}\sigma \left(\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X)\right). \tag{2.132}$$

The minus sign takes into account that the world sheet metric has signature $(-,+)$. It is convinient to introduce the momenta p_μ, which are canonically conjugated to X^μ:

$$p_\mu = \frac{\delta S}{\delta \partial_\tau X^\mu} = -\gamma^{0\beta}\partial_\beta X^\nu G_{\mu\nu}(X) - \dot{X}^B_{\mu\nu}(X), \tag{2.133}$$

where we have introduced the notation $\dot{X}^\nu \equiv \partial_\sigma X^\mu$. By Noether's theorem we immediately get two conserved charges:

$$E = -\int_0^R \mathrm{d}\sigma p_t, \quad \text{and} \quad J = \int_0^R \mathrm{d}\sigma p_\phi. \tag{2.134}$$

An advantage of the first-order formalism is thath the action takes the form

$$S = \int_{-\infty}^{\infty} \mathrm{d}\tau \int_0^R \left(p_\mu \dot{X}^\mu + \frac{\gamma^{01}}{\gamma^{00}}\mathcal{C}_1 + \frac{1}{2\gamma^{00}}\mathcal{C}_2 \right), \tag{2.135}$$

where the worldsheet metric takes the form of a Lagrange multiplier and yields the two Virasoro constraints:

$$\begin{aligned} 0 &= \mathcal{C}_1 = p_\mu \dot{X}^\mu \\ 0 &= \mathcal{C}_2 = p_\mu p_\nu G^{\mu\nu} + \dot{X}^\mu \dot{X}^\nu G_{\mu\nu} + 2G^{\mu\nu}B_{\nu\rho}p_\mu \dot{X}^\rho + G^{\mu\nu}B_{\mu\rho}B_{\nu\lambda}\dot{X}^\rho \dot{X}^\lambda, \end{aligned} \tag{2.136}$$

where we suppressed the dependence of the (inverse) metric and B-field on X^μ.

Chapter 3

Supersymmetry

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Contents

3.1 Supermultiplets

Definition 3.1 (Supermultiplet)**.** *Representations of the supersymmetric algebra (superalgebra) are called supermultiplets.*

Indeed, these representations can be thought of as multiplets where we assemble together several different representations of the Lorentz algebra, since the latter is a subalgebra of the superalgebra.

3.1.1 Massless supermultiplets

If $P^2 = 0$, then we can take P_μ to a canonical form by applying boost and rotations until it reads

$$\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = \left(\sigma^0 + \sigma^3\right) E = \begin{bmatrix} 0 & 0 \\ 0 & 2E \end{bmatrix} \tag{3.1}$$

The supersymmetric algebra becomes,

$$\begin{bmatrix} \{Q_1, \bar{Q}_{\dot{1}}\} & \{Q_1, \bar{Q}_{\dot{2}}\} \\ \{Q_2, \bar{Q}_{\dot{1}}\} & \{Q_2, \bar{Q}_{\dot{2}}\} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4E \end{bmatrix} \tag{3.2}$$

intended as acting on the states of the multiplet we are looking for. In particular,

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 0 \tag{3.3}$$

which implies that

$$||Q_1|\omega\rangle||^2 = 0 = ||\bar{Q}_{\dot{1}}|\omega\rangle||^2 \tag{3.4}$$

and thus

$$Q_1|\omega\rangle = 0 = \bar{Q}_{\dot{1}}|\omega\rangle. \tag{3.5}$$

This means that as operators Q_1 and $\bar{Q}_{\dot{1}}$ annihilate the multiplet.

The only non-trivial anticommutation relation that is left is:

$$\{Q_2, \bar{Q}_{\dot{2}}\} = 1 \tag{3.6}$$

If we call

$$\alpha = \frac{1}{2\sqrt{E}} Q_2, \quad \alpha^\dagger = \frac{1}{2E} \bar{Q}_{\dot{2}} \tag{3.7}$$

then the anticommutation relation that is left is:

$$\{\alpha, \alpha^\dagger\} = 1 \tag{3.8}$$

with $\{\alpha, \alpha\} = 0$.

We can build the representation starting from a state $|\lambda\rangle$ such that

$$\alpha|\lambda\rangle = 0 \tag{3.9}$$

Lets suppose that it has *helicity* λ:

$$M_{12}\lambda \equiv J_3|\lambda\rangle = \lambda|\lambda\rangle. \tag{3.10}$$

It is easy to compute the helicity of $\alpha^\dagger|\lambda\rangle$:

$$M_{12}\bar{Q}_{\dot{2}}|\lambda\rangle = \left(\bar{Q}_{\dot{2}}M_{12} + \frac{1}{2}\bar{Q}_{\dot{2}}\right)|\lambda\rangle = (\lambda + \frac{1}{2}\bar{Q}_{\dot{2}})|\lambda\rangle \tag{3.11}$$

In the last line, we have used the fact that $\left[M_{12}, \bar{Q}_{\dot{2}}\right] = \frac{1}{2}\bar{Q}_{\dot{2}}$. Thus we found out that

$$\alpha^\dagger|\lambda\rangle = |\lambda + \frac{1}{2}\rangle \tag{3.12}$$

Since $(\alpha^\dagger)^2 = 0$, this stops here. Hence we have

$$\alpha^\dagger|\lambda + \frac{1}{2}\rangle = 0. \tag{3.13}$$

Massless multiplets are thus composed of one boson and one fermion. Since physical particles must come in CPT conjugate representation (or, there are no spin-$\frac{1}{2}$ one dimensional representations of the massless little group of the Lorentz group), one must add the CPT conjugate multiplet where helicities are flipped.

Example 1 (Examples of massless supermultiplets)**.**

- *The scalar multiplet is obtained by setting $\lambda = 0$. Then we have*

$$\alpha^\dagger\,|0\rangle = \left|\frac{1}{2}\right\rangle \tag{3.14}$$

The full multiplet is composed of two states with $\lambda = 0$ and a doublet with $\lambda = \pm\frac{1}{2}$. These are the degrees of freedom of a complex scalar and a Weyl (chiral) fermion.

- *The vector multiplet is obtained starting from a $\lambda = \frac{1}{2}$ state. We get*

$$\alpha^\dagger \left|\frac{1}{2}\right\rangle = |1\rangle\,. \tag{3.15}$$

To this we add the CPT conjugate multiplet, to obtain two pairs of states, one with $\lambda = \pm\frac{1}{2}$ and the other with $\lambda = \pm 1$. These are the degrees of freedom of a Weyl fermion and of a massless vector. The latter is usually interpreted as a gauge boson.

- *Another multiplet is obtained starting from $\lambda = \frac{3}{2}$:*

$$\alpha^\dagger \left|\frac{3}{2}\right\rangle = |2\rangle\,. \tag{3.16}$$

Adding the CPT conjugate, one has a pair of bosonic degrees of freedom with $\lambda = \pm 2$, which we interpret as the graviton, and a pair of fermionic degrees of freedom with $\lambda = \pm\frac{3}{2}$, which correspond to a massless spin-$\frac{3}{2}$ Rarita-Schwinger field, also called the gravition, since it is the SUSY partner of the graviton, as was just shown.

3.1.2 Supermultiplets of extended supersymmetry

Very briefly we will mention that having extended SUSY, the massless supermultiplets are longer. Let's take the algebra to be:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{IJ}, \tag{3.17}$$

where for simplicity we suppose that $Z^{IJ} = 0$ for these states. For massless states, $P_\mu = (E, 0, 0, E)$ and therefore as before we have that

$$\{Q^I_1, \bar{Q}^J_{\dot{1}}\} = 0, \tag{3.18}$$

which implies the (operator) equations $Q^I_1 = 0$ and $\bar{Q}^I_{\dot{1}} = 0$, for $I = 1, \cdots, \mathcal{N}$. The non-trivial relations are then:

$$\{Q^I_2, \bar{Q}^J_{\dot{2}}\} = 4E\delta^{IJ} \tag{3.19}$$

Of course we can define

$$\alpha_I = \frac{1}{2\sqrt{E}} Q^I_2 \tag{3.20}$$

and obtain the canonical anticommutation relations for $\mathcal{N}$ fermionic oscillators

$$\{\alpha_I, \alpha_J^\dagger\} = \delta_{IJ} \tag{3.21}$$

If we now start with a state $|\lambda\rangle$ with helicity λ which satisfies $\alpha_I |\lambda\rangle = 0$, we build a multiplet as follows:

$$\begin{aligned} \alpha_I^\dagger |\lambda\rangle &= \left|\lambda + \frac{1}{2}\right\rangle_I , \\ \alpha_I \dagger \alpha_J \dagger |\lambda\rangle &= |\lambda + 1\rangle_{[IJ]} , \\ &\vdots \\ \alpha_1^\dagger \cdots \alpha_{\mathcal{N}}^\dagger |\lambda\rangle &= \left|\lambda + \frac{\mathcal{N}}{2}\right\rangle \end{aligned} \tag{3.22}$$

It is very important to note that there are $\mathcal{N}$ states with helicity $\lambda + \frac{1}{2}$, $\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$ states with helicity $\lambda + 1$ and so on, until we reach a single state with helicity $\lambda + \frac{\mathcal{N}}{2}$ (it is totally antisymmetric in $\mathcal{N}$ indices I). In total, the supermultiplet is composed of $2^{\mathcal{N}}$ states, half of them bosonic and half of them fermionic.

Interestingly, in this case we can now have self-CPT conjugate multiplets. Take for example $\mathcal{N} = 4$ and start from $\lambda = -1$. Then $\lambda + \frac{\mathcal{N}}{2} = 1$ and the multiplet spans states of opposite helicities, thus filling complete representations of the Lorenz group. Indeed, it contains one pair of states with $\lambda \pm 1$ (a vector, i.e a gauge boson), 4 pairs of states with $\lambda = \pm\frac{1}{3}$ (4 Weyl Fermions) and 6 states with $\lambda = 0$ (6 real scalars, or equivalently 3 complex scalars).

Another example is $\mathcal{N} = 8$ supersymmetry. Here if we start with $\lambda = -2$ we end up with $\lambda + \frac{\mathcal{N}}{2} = 2$. Thus in this case we have the graviton in the self-CPT conjugate multiplet, corresponding to the pair of states with $\lambda \pm 2$. In addition, we have 8 massless gravitini with $\lambda \pm \frac{3}{2}$, 28 massless vectors with $\lambda = \pm 1$, 56 massless Weyl fermions with $\lambda = \pm\frac{1}{2}$ and finally 70 real scalars with $\lambda = 0$. This is the content of $\mathcal{N} = 8$ *supergravity*, which is the only multiplet of $\mathcal{N} = 8$ supersymmetry with $|\lambda| < 2$. The latter condition is necessary in order to have consistent couplings (higher spin fields cannot be coupled in a consistent way with gravity and lower spin fields).

From the theoretical standpoint, this is a very nice result, because we have a theory where *everything is determined* from symmetry alone: the complete spectrum and all the couplings. Unfortunately, this theory is also completely unphysical. To mention one problem, it has no room for fermions in complex representations of the gauge group, which are present in the Standard Model.

3.1.3 Massive supermultiplets

When $P^2 = M^2 > 0$, by boosts and rotation P_μ can be put in the following form

$$P_\mu = (M, 0, 0, 0) \tag{3.23}$$

Then we have

$$\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = M\sigma^0 = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \tag{3.24}$$

so that the superalgebra reads

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2M\delta_{\alpha\dot{\alpha}} \tag{3.25}$$

Note that $[M_{12}, Q_1] = i(\sigma_{12})_1{}^1 Q_1 = \frac{1}{2}Q_1$, thus it is Q_1 that raises the helicity, in the same way as $\bar{Q}_{\dot{2}}$. We make the redefinition

$$\alpha_1 = \frac{1}{\sqrt{2M}}\bar{Q}_{\dot{1}}, \quad \alpha_1^\dagger = \frac{1}{\sqrt{2M}}Q_1, \tag{3.26}$$

$$\alpha_2 = \frac{1}{\sqrt{2M}}Q_2, \quad \alpha_2^\dagger = \frac{1}{\sqrt{2M}}\bar{Q}_{\dot{2}}, \tag{3.27}$$

so that we have the canonical anticommutation relations of two fermionic oscillators:

$$\{\alpha_a, \alpha_b^\dagger\} = \delta_{ab}, \quad a, b = 1, 2. \tag{3.28}$$

If we start with $\alpha_a |\lambda\rangle = 0$, $M_{12} |\lambda\rangle = \lambda |\lambda\rangle$, then we build the multiplet as:

$$\alpha_1^\dagger |\lambda\rangle = \left|\lambda + \frac{1}{2}\right\rangle_1, \tag{3.29}$$

$$\alpha_2^\dagger |\lambda\rangle = \left|\lambda + \frac{1}{2}\right\rangle_2, \tag{3.30}$$

$$\alpha_1^\dagger \alpha_2^\dagger |\lambda\rangle = |\lambda + 1\rangle . \tag{3.31}$$

There are 4 states now (compared to the 2 in the massless case), two bosons and two fermions.

Example 2 (Examples of massive supermultiplets)**.**

- *In the case of the massive scalar multiplet, we start from $\lambda = -\frac{1}{2}$ and obtain two states with $\lambda = 0$ and one state with $\lambda = \frac{1}{2}$. These are the degrees of freedom of one massive complex scalar and one massive Weyl fermion.*

Note that the latter might not be familiar. Indeed, one cannot write the usual Dirac mass term for a Weyl fermion. Instead, one can write what is called a Majorana mass term:

$$\mathcal{L} \supset m\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta + h.c. \tag{3.32}$$

Note that the total degrees of freedom of a massless scalar multiplet is the same as that of a massive one.

- *For a massive vector multiplet, start from $\lambda = 0$ to obtain 2 states with $\lambda = \frac{1}{2}$ and one state with $\lambda = 1$. To this we add the CPT conjugate multiplet so that in the end we have one pair with $\lambda = \pm 1$, two pairs with $\lambda = \pm\frac{1}{2}$ and two states with $\lambda = 0$. According to the massive little group, this corresponds to 1 massive vector (with $\lambda = \pm 1, 0$), 1 real scalar and 1 massive Dirac fermion. Note however that the content in degrees of freedom is the same as that of one massless vector multiplet together with one massless scalar multiplet. This hints that the consistent way to treat massive vectors in a supersymmetric field theory will be through a SUSY version of the Brout-Englert-Higgs mechanism.*

3.2 General

Definition 3.2 (R-symmetry)**.** *In supersymmetric theories, an R-symmetry is the symmetry transforming different supercharges into each other. In the simplest case of the $\mathcal{N} = 1$ supersymmetry, such R-symmetry is isomorphic to a global $U(1)$ group or its discrete subgroup. For extended supersymmetry theories, the R-symmetry group becomes a global non-abelian group.*

Remark 3.1. In the case of the discrete subgroup $\mathbb{Z}_2$, the R-symmetry is called *R-parity*

Definition 3.3 (Extended supersymmetry)**.** *In supersymmetric theories, when $\mathcal{N} > 1$ the algebra is said to have extended supersymmetry.*

3.3 Bogomol'nyi-Prasad-Sommerfield (BPS) states

Definition 3.4 (BPS state)**.** *A massive representation of an extended supersymmetry algebra that has mass equal to the supersymmetry central charge Z is called an BPS state.*

Quantum mechanically speaking, if the supersymmetry remains unbroken, exact solutions to the modulus of Z exist. Their importance arises as the multiplets shorten for generic representations, with stability and mass formula exact.

Example 3 ($d = 4$,$\mathcal{N} = 2$)**.** *The generators for the odd part of the superalgebra have relations:*

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B \tag{3.33}$$

$$\{Q^A_\alpha, Q_{\beta B}\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z} \tag{3.34}$$

$$\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB} Z, \tag{3.35}$$

where $\alpha\dot{\beta}$ are the Lorentz group indices and A, B are the R-symmerty indices. If we take linear combinations of the above generators as follows:

$$R^A_\alpha = \xi^{-1} Q^A_\alpha + \xi\sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}B} \tag{3.36}$$

$$T^A_\alpha = \xi^{-1} Q^A_\alpha - \xi\sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}B} \tag{3.37}$$

and consider a state ψ which has momentum $(M, 0, 0, 0)$, we have:

$$\left(R^1_1 + (R^1_1)^\dagger\right)^2 \psi = 4(M + Re(Z\xi^2))\psi, \tag{3.38}$$

but because this is the square of a Hermitian operator, the right hand side coefficient must be positive for all ξ. In particular, the strongest result from this is

$$M \geq |Z| \tag{3.39}$$

3.4 Supersymmetric theories on curved manifolds

Remark: Supersymmetric theories may be defined only on backgrounds admitting solutions to certain Killing spinor equations,

$$(\nabla_\mu - iA_\mu)\,\zeta + iV_\mu\zeta + iV^\nu\sigma_{\mu\nu}\zeta = 0 \tag{3.40}$$

$$(\nabla_\mu + iA_\mu)\,\tilde{\zeta} - iV_\mu\tilde{\zeta} - iV^\nu\tilde{\sigma}_{\mu\nu}\tilde{\zeta} = 0 \tag{3.41}$$

which in four dimensions and Euclidean signature are equivalent to the requirement that the manifold is complex and the metric Hermitian.

3.5 Supersymmetric Chern-Simons-matter theories

In this section we will introduce the basic building blocks of supersymmetric Chern-Simons-matter theories. We will work in Euclidean space, and we will put the theories on the three-sphere, since we are eventually interested in computing the free energy of the gauge theory in this curved space.

3.5.1 Conventions

In Euclidean space, the fermions ψ and $\bar{\psi}$ are independent and they transform and they transform in the same representation of the Lorentz group. Their index structure is:

$$\psi^\alpha, \quad \bar{\psi}^\alpha. \tag{3.42}$$

We will take γ_μ to be the Pauli matrices, which are hermitian, and

$$\gamma_{\mu\nu} = \frac{1}{2}\left[\gamma_\mu, \gamma_n u\right] = i\epsilon_{\mu\nu\rho}\gamma^\rho \tag{3.43}$$

We introduce the usual symplectic product through the antisymmetric matrix

$$C_{\alpha\beta} = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}. \tag{3.44}$$

We set $C = -1$ and denote the matrix by $\epsilon_{\alpha\beta}$. The product is

$$\bar{\epsilon}\lambda = \bar{\epsilon}^{\alpha} C_{\alpha\beta} \lambda^{\beta}. \tag{3.45}$$

Notice that

$$\bar{\epsilon}\gamma^{\mu}\lambda = \bar{\epsilon}^{\beta} C_{\beta\gamma} (\gamma^{\mu})^{\gamma}_{\alpha} \lambda^{\alpha}. \tag{3.46}$$

It is easy to check that

$$\bar{\epsilon}\lambda = \lambda\bar{\epsilon}, \quad \bar{\epsilon}\gamma^{\mu}\lambda = -\lambda\gamma^{\mu}\bar{\epsilon}, \tag{3.47}$$

and in particular

$$(\gamma^{\mu}\bar{\epsilon})\lambda = -\bar{\epsilon}\gamma^{\mu}\lambda. \tag{3.48}$$

We also have the following Fierz identities

$$\bar{\epsilon}(\epsilon\psi) + \epsilon(\bar{\epsilon}\psi) + (\bar{\epsilon}\epsilon)\psi = 0 \tag{3.49}$$

and

$$\epsilon(\bar{\epsilon}\psi) + 2(\bar{\epsilon}\epsilon)\psi + (\bar{\epsilon}\gamma_{\mu}\psi)\gamma^{\mu}\epsilon = 0. \tag{3.50}$$

3.5.2 Vector multiplet and supersymmetric Chern-Simons theory

We first start with theories based on vector multiplets. The three dimensional Euclidean $\mathcal{N} = 2$ vector superfield V has the following content

$$V: \quad A_{\mu}, \sigma, \lambda, \bar{\lambda}, D, \tag{3.51}$$

where A_{μ} is a gauge field, σ is an auxiliary scalar field, $\lambda, \bar{\lambda}$ twi-component complex Dirac spinors, and D is an auxiliary scalar. This is just the dimensional reduction of the $\mathcal{N} = 1$ vector multiplet in 4 dimensions, and σ is the reduction of the fourth component of A_{μ}. All fields are valued in the Lie algebra $\mathfrak{g}$ of the gauge group G. For $G = U(N)$ our convention is that $\mathfrak{g}$ are Hermitian matrices. It follows that the gauge covariant derivative is given by

$$\partial_{\mu} + i\,[A_{\mu}, .\,] \tag{3.52}$$

while the gauge field strength is

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + i\,[A_{\mu}, A_{\nu}]\,. \tag{3.53}$$

The transformations of the fields are generated by two independent complex structures ϵ, $\bar{\epsilon}$. They are given by,

$$\begin{aligned}
\delta A_\mu &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon),\\
\delta\sigma &= \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon),\\
\delta\lambda &= -\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma + \frac{2i}{3}\sigma\gamma^\mu D_\mu\epsilon,\\
\delta\bar{\lambda} &= -\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} + D\bar{\epsilon} - i\gamma^\mu\bar{\epsilon}D_\mu\sigma - \frac{2i}{3}\sigma\gamma^\mu D_\mu\bar{\epsilon},\\
\delta D &= -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon + \frac{i}{2}\left[\bar{\epsilon}\lambda,\sigma\right] + \frac{i}{2}\left[\bar{\lambda}\epsilon,\sigma\right] - \frac{i}{6}(D_\mu\bar{\epsilon}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu D_\mu\epsilon),
\end{aligned} \tag{3.54}$$

and we split naturally

$$\delta = \delta_\epsilon + \delta_{\bar{\epsilon}}. \tag{3.55}$$

Here we follow the conventions of [26], be we change the sign of the gauge connection: $A_\mu \to -A_\mu$. The derivative D_μ is covariant with respect to both the gauge field and the spin connection. On all the fields, except D, the commutator $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ becomes a sum of translation, gauge transformation, Lorentz rotation, dilation and R-rotation:

$$\begin{aligned}
\left[\delta_\epsilon,\delta_{\bar{\epsilon}}\right] A_\mu &= iv^\nu\partial_\nu A_\mu + i\partial_\mu\nu^\nu A_\nu - D_\mu\Lambda,\\
\left[\delta_\epsilon,\delta_{\bar{\epsilon}}\right]\sigma &= i\nu^\mu\partial_\mu\sigma + i\left[\Lambda,\sigma\right] + \rho\sigma,\\
\left[\delta_\epsilon,\delta_{\bar{\epsilon}}\right]\lambda &= i\nu^\mu\partial_\mu\lambda + \frac{i}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\lambda + i\left[\Lambda,\lambda\right] + \frac{3}{2}\rho\lambda + \alpha\lambda,\\
\left[\delta_\epsilon,\delta_{\bar{\epsilon}}\right]\bar{\lambda} &= i\nu^\mu\partial_\mu\bar{\lambda} + \frac{i}{4}\Theta_{\mu\nu}\bar{\lambda} + i\left[\Lambda,\bar{\lambda}\right] + \frac{3}{2}\rho\bar{\lambda} - \alpha\bar{\lambda},\\
\left[\delta_\epsilon,\delta_{\bar{\epsilon}}\right] D &= i\nu^\mu\partial_\mu D + i\left[\Lambda,D\right] + 2\rho D + \frac{1}{3}\sigma(\bar{\epsilon}\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon - \epsilon\gamma^\mu\gamma^\nu D_\mu D_\nu\bar{\epsilon}),
\end{aligned} \tag{3.56}$$

where

$$\begin{aligned}
\nu^\mu &= \bar{\epsilon}\gamma^\mu\epsilon,\\
\Theta^{\mu\nu} &= D^{[\mu}v^{\nu]} + \nu^\lambda\omega^{\mu\nu}{}_\lambda,\\
\Lambda &= \nu^\mu iA_\mu + \sigma\bar{\epsilon}\epsilon\\
\rho &= \frac{i}{3}(\bar{\epsilon}\gamma^\mu D_\mu\epsilon + D_\mu\bar{\epsilon}\gamma^\mu\epsilon),\\
\alpha &= \frac{i}{3}(D_\mu\bar{\epsilon}\gamma^\mu\epsilon - \bar{\epsilon}\gamma^\mu D_\mu\epsilon).
\end{aligned} \tag{3.57}$$

Here, $\omega_\lambda^{\mu\nu}$ is the spin connection. As a check, let us calculate the commutator acting on σ. We have,

$$\begin{aligned}
[\delta_\epsilon, \delta_{\bar\epsilon}]\,\sigma &= \delta_\epsilon\left(\frac{1}{2}\bar\epsilon\lambda\right) - \delta_{\bar\epsilon}\left(-\frac{1}{2}\bar\lambda\epsilon\right) \\
&= \frac{1}{2}\bar\epsilon\left(-\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma\right) + \frac{i}{3}\bar\epsilon\gamma^\mu(D_\mu\epsilon)\sigma \\
&+ \frac{1}{2}(-\frac{1}{2}\gamma^{\mu\nu}\bar\epsilon F_{\mu\nu} + D\epsilon - i\gamma^\mu\bar\epsilon D_\mu\sigma)\epsilon - \frac{i}{3}\gamma^\mu(D_\mu\bar\epsilon)\epsilon\sigma \\
&= i\bar\epsilon\gamma^\mu\epsilon D_\mu\sigma + \rho\sigma. \qquad (3.58)
\end{aligned}$$

In order for the supersymmetry algebra to close, the last term in the RHS of $[\delta_\epsilon, \delta_{\bar\epsilon}]\,D$ must vanish. This is the case if the Killing spinors satisfy:

$$\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon = h\epsilon, \quad \gamma^\mu\gamma^\nu D_\mu D_\nu\bar\epsilon = h\bar\epsilon \qquad (3.59)$$

for some scalar function h. A sufficient condition for this is to simply have

$$D_\mu\epsilon = \frac{i}{2r}\gamma_\mu\epsilon, \quad D_\mu\bar\epsilon = \frac{i}{2r}\gamma y_\mu\bar\epsilon \qquad (3.60)$$

and

$$h = -\frac{9}{4r^2} \qquad (3.61)$$

where r is the radius of the three-sphere. This condition is satisfied by one of the Killing spinors on the three-sphere (the one which is constant in the left-invariant frame). Notice that, with this choice ρ vanishes.

The Euclidean SUSY Chern-Simons (CS) action, in flat space, is given by

$$S_{SCS} = -\int d^3x \mathrm{Tr}\left(A\wedge dA + \frac{2i}{3}A^3 - \bar\lambda\lambda + 2D\sigma\right) \qquad (3.62)$$

$$= -\int d^3x \mathrm{Tr}\left(\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\right) - \bar\lambda\lambda + 2D\sigma\right). \qquad (3.63)$$

Here Tr denotes the trace in the fundamental representation. The part of the action involving the gauge connection A is the standard, bosonic CS action in three dimensions. This action was first considered from the point of view of QFT, in [19], where the total action for a non-abelian gauge field was the sum of the standard Yang-Mills action and the CS action. In [57], the CS action was considered by itself and shown to lead to a topological gauge theory.

We can check that the SUSY CS action is invariant under the supersymmetry generated by δ_ϵ (the proof for $\delta_{\bar{\epsilon}}$ is similar). The SUSY variation of the integrand of the action is

$$\begin{aligned}(2\delta A_\mu \partial_\nu A_\rho + 2i\delta A_\mu A_\nu A_\rho)\epsilon^{\mu\nu\rho} - \bar{\lambda}\delta\lambda + 2(\delta D)\sigma + 2D\delta\sigma = \\ - i\bar{\lambda}\gamma_\mu\epsilon\partial_\nu A_\rho\epsilon^{\mu\nu\rho} + \bar{\lambda}\gamma_\mu\epsilon A_\nu A_\rho\epsilon^{\mu\nu\rho} - \bar{\lambda}(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D + i\gamma^\mu D_\mu\sigma)\epsilon - \frac{2i}{3}\bar{\lambda}\gamma^\mu D_\mu\epsilon\sigma \\ - i(D_\mu\bar{\lambda})\gamma^\mu\sigma\epsilon + i\left[\bar{\lambda}\epsilon, \sigma\right]\sigma - \frac{i}{3}\bar{\lambda}\gamma^\mu D_\mu\epsilon\sigma - \bar{\lambda}\epsilon D. \quad (3.64)\end{aligned}$$

It is obvious that the terms involving D cancel. Let us consider the terms involving the gauge field. Using the gamma matrix comutator, we find

$$\frac{1}{2}\bar{\lambda}\gamma^{\mu\nu}F_{\mu\nu}\epsilon = i\bar{\lambda}\gamma_\rho\epsilon\epsilon^{\mu\nu\rho}\partial_\mu A_\nu - \bar{\lambda}\gamma_\rho\epsilon\epsilon^{\mu\nu\rho}A_\mu A_\nu \quad (3.65)$$

which cancels the first two terms in the variation. Let us now look at the remaining terms. The covariant derivative of $\bar{\lambda}$ is

$$D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} + \frac{i}{2r}\gamma_\mu\bar{\lambda} + i\left[A_\mu, \bar{\lambda}\right]. \quad (3.66)$$

If we integrate by parts the term involving the derivative of λ we find in total

$$\begin{aligned}i\bar{\lambda}\gamma^\mu\epsilon\partial_\mu\sigma + i\bar{\lambda}\gamma^\mu\partial_\mu\epsilon\sigma + \frac{1}{2r}(\gamma^\mu\bar{\lambda})\gamma_\mu\epsilon + \left[A_\mu, \bar{\lambda}\right]\gamma^\mu\epsilon\sigma = \\ = i\bar{\lambda}\gamma^\mu\epsilon\partial_\mu\sigma + i\bar{\lambda}\gamma^\mu D_\mu\epsilon\sigma + \left[A_\mu, \bar{\lambda}\right]\gamma^\mu\epsilon\sigma \quad (3.67)\end{aligned}$$

The derivative of σ cancels against the corresponding term in the covariant derivative of σ. Putting all together, we find

$$i\bar{\lambda}\gamma^\mu(D_\mu\epsilon)\sigma - i\bar{\lambda}\gamma^\mu(D_\mu\epsilon)\sigma + \left[A_\mu, \bar{\lambda}\right]\gamma^\mu\epsilon\sigma + \bar{\lambda}\gamma^\mu\epsilon\left[A_\mu, \sigma\right] + i\left[\bar{\lambda}\epsilon, \sigma\right]\sigma. \quad (3.68)$$

The last three terms cancel due to the cyclic property of the trace. This proves the invariance of the SUSY CS theory.

In the path integral, the SUSY CS action enters the form

$$\exp\left(\frac{ik}{4\pi}S_{\text{SCS}}\right) \quad (3.69)$$

where k plays the role of the inverse coupling constant and it is referred to as the level of the CS theory. In a consistend quantum theory, k must be an integer. This is due to the fact that the Chern-Simons action for the connection A is not

invariant under large gauge transformations, but changes by an integer times $8\pi^2$. The quantization of k guarantees that (3.69) remains invariant.

Of course, there is another Lagrangian for vector multiplets, namely the Yang-Mills Lagrangian,

$$\mathcal{L}_{\rm YM} = {\rm Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}\left(D + \frac{\sigma}{r}\right)^2 + \frac{i}{2}\bar\lambda\gamma^\mu D_\mu\lambda + \frac{i}{2}\bar\lambda\left[\sigma,\lambda\right] - \frac{1}{4r}\bar\lambda\lambda\right]. \tag{3.70}$$

In the flat space limit $r \to \infty$, this becomes the standard (Euclidean) super Yang-Mills theory in three dimensions. The Lagrangian (3.70) is not only invariant under the SUSY transformations, but it can also be written as a superderivative,

$$\bar\epsilon\epsilon\mathcal{L}_{\rm YM} = \delta_{\bar\epsilon}\delta_\epsilon{\rm Tr}\left(\frac{1}{2}\bar\lambda\lambda - 2D\sigma\right). \tag{3.71}$$

This will be important later on.

3.5.3 Supersymmetric matter multiplets

We will now add supersymmetric matter, i.e. a chiral multiplet Φ in a representation R of the gauge group. Its components are

$$\Phi: \quad \phi, \bar\phi, \psi, \bar\psi, F, \bar F. \tag{3.72}$$

The supersymmetry transformations are

$$\begin{aligned}
\delta\phi &= \bar\epsilon\psi,\\
\delta\bar\psi &= \epsilon\bar\psi,\\
\delta\psi &= i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi + \frac{2\Delta i}{3}\gamma^\mu D_\mu\epsilon\phi + \bar\epsilon F,\\
\delta\bar\psi &= i\gamma^\mu\bar\epsilon D_\mu\bar\psi + i\bar\phi\sigma\bar\epsilon + \frac{2\Delta i}{3}\bar\phi\gamma^\mu D_\mu\bar\epsilon + \bar F\epsilon,\\
\delta F &= \epsilon(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi) + \frac{i}{3}(2\Delta - 1)D_\mu\epsilon\gamma^\mu\psi,\\
\delta\bar F &= \bar\epsilon(i\gamma^\mu D_\mu\bar\psi - i\bar\psi\sigma + i\bar\psi\bar\lambda) + \frac{i}{3}(2\Delta - 1)D_\mu\bar\epsilon\gamma^\mu\bar\psi
\end{aligned} \tag{3.73}$$

where Δ is the possible anomalous dimension of ϕ. For theories with $\mathcal{N} \geq 3$ supersymmetry, the field has the canonical dimension

$$\Delta = \frac{1}{2}, \tag{3.74}$$

but in general this is not the case.

The commutators of these transformations are given by

$$[\delta_\epsilon, \delta_{\bar\epsilon}]\,\phi = i\nu^\mu\partial_\mu\phi + i\Lambda\phi + \Delta\rho\phi - \Delta\alpha\phi \tag{3.75}$$

$$[\delta_\epsilon, \delta_{\bar\epsilon}]\,\bar\phi = i\nu^\mu\partial_\mu\bar\phi - i\Lambda\bar\phi + \Delta\rho\bar\phi + \Delta\alpha\bar\phi \tag{3.76}$$

$$[\delta_\epsilon, \delta_{\bar\epsilon}]\,\psi = i\nu^\mu\partial_\mu\psi + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\psi + i\Lambda\psi + \left(\Delta + \frac{1}{2}\right)\rho\psi + (1-\Delta)\alpha\psi \tag{3.77}$$

$$[\delta_\epsilon, \delta_{\bar\epsilon}]\,\bar\psi = i\nu^\mu\partial_\mu\bar\psi + \frac{1}{4}\Theta_{\mu\nu}\psi^{\mu\nu}\bar\psi - i\bar\psi\Lambda + \left(\Delta + \frac{1}{2}\right)\rho\bar\psi + (\Delta-1)\,\alpha\bar\psi, \tag{3.78}$$

$$[\delta_\epsilon, \delta_{\bar\epsilon}]\,F = i\nu^\mu\partial_\mu F + i\Lambda F + (\Delta+1)\,\rho F + (2-\Delta)\,\alpha F, \tag{3.79}$$

$$[\delta_\epsilon, \delta_{\bar\epsilon}]\,\bar F = i\nu^\mu\partial_\mu\bar F - i\bar F\Lambda + (\Delta+1)\,\rho\bar F + (\Delta-2)\,\alpha\bar F. \tag{3.80}$$

The lowest components of the superfields are assigned the dimension Δ and R-charge $\mp\Delta$. The supersymmetry algebra closes off-shell when the Killing spinors $\epsilon, \bar\epsilon$ satisfy (3.59) and h is given by (3.61). As a check, we compute

$$[\delta_\epsilon, \delta_{\bar\epsilon}]\,\phi = \tag{3.81}$$

$$= eps\bar{i}lon\left(i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi + \frac{2i\Delta}{3}\gamma^\mu(D_\mu\epsilon)\phi\right) = i\nu^\mu D_\mu\phi + i\sigma\bar\epsilon\epsilon + \frac{2i\Delta}{3}\left(\bar\epsilon\gamma^\mu D_\mu\epsilon\right), \tag{3.82}$$

which is the wished-for result.

Let us now consider supersymmetric Lagrangians for the matter hypermultiplet. If the fields have their canonical dimensions, the Lagrangian:

$$\mathcal{L} = D_\mu\bar\phi D^\mu\phi - i\bar\psi\gamma^\mu D_\mu\psi + \frac{3}{4r^2}\bar\phi\phi + i\bar\psi\sigma\psi + i\bar\psi\lambda\phi - i\bar\phi\bar\lambda\psi + i\bar\psi D\phi + \bar\phi\sigma^2\phi + \bar F F \tag{3.83}$$

is invariant under supersymmetry if the Killing spinors $\epsilon, \bar\epsilon$ satisfy (3.59), with h given as in (3.61). The quadratic part of the Lagrangian for ϕ gives indeed the standard conformal coupling for a scalar field. We recall that the action for a massless scalar field in a curved space of n dimensions contains a coupling to the curvature R given by

$$S = \int \mathrm{d}^n x\sqrt{g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi R\phi^2), \tag{3.84}$$

where ξ is a constant. This action is conformally invariant when

$$\xi = \frac{1}{4}\frac{n-2}{n-1}. \tag{3.85}$$

If the spacetime is an n-sphere of radius r, the curvature is

$$R = \frac{n(n-1)}{r^2} \tag{3.86}$$

and the conformal coupling of the scalar leads to an effective mass term of the form

$$\frac{n(n-2)}{4r^2}\phi^2 \tag{3.87}$$

which in $n = 3$ dimensions gives the quadratic term for ϕ in (3.83).

If the fields have non-canonical dimensions, the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{mat}} = D_\mu\bar{\phi}D^\mu\phi + \bar{\phi}\sigma^2\phi + \frac{i(2\Delta-1)}{r}\bar{\phi}\sigma\phi + \frac{\Delta(2-\Delta)}{r^2}\bar{\phi}\phi + i\bar{\phi}D\phi + \bar{F}F \\ - i\bar{\psi}\gamma^\mu D_\mu\psi + i\bar{\psi}\sigma\psi - \frac{2\Delta-1}{2r}\bar{\psi}\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi\end{aligned} \tag{3.88}$$

is supersymmetry, provided the parameters $\epsilon, \bar{\epsilon}$ satisfy the Killing spinor conditions (3.59). The Lagrangian in (3.88) is not only invariant under the supersymmetries $\delta_{\epsilon,\bar{\epsilon}}$ but it can be written as a total superderivative,

$$\bar{\epsilon}\epsilon\mathcal{L}_{\text{mat}} = \delta_{\bar{\epsilon}}\delta_\epsilon\left(\bar{\psi}\psi - 2i\bar{\phi}\sigma\phi + \frac{2(\Delta-1)}{r}\bar{\phi}\phi\right). \tag{3.89}$$

3.5.4 ABJM theory

The theory proposed by Aharony, Bergman, Jafferis and Maldacena in [4, 3] to describe N M2 branes is a praticular example of a supersymmetric Chern-Simons theory. It consists of two copies of Chern-Simons theory with gauge groups $U(N_1)$, $U(N_2)$, and opposite levels $k, -k$. In addition, we have four matter supermultiplets Φ_i, $i = 1, \ldots, 4$, in the bifundamental representation of the gauge group $U(N_1) \times U(N_2)$. This theory can be represented as a quiver[1], with two nodes representing the Chern-Simons theories, and four edges between the nodes representing the matter supermultiplets, see the figure. In addition, there is a superpotential invloving the matter fields, which after integrating out the auxiliary fields in the Chern-Simons-matter system, reads (on $\mathbb{R}^3$.

$$W = \frac{4\pi}{k}\text{Tr}\left(\Phi_1\Phi_2^\dagger\Phi_3\Phi_4^\dagger - \Phi_1\Phi_2^\dagger\Phi_3\Phi_4^\dagger\right), \tag{3.90}$$

[1]A quiver is a directed graph where loops and multiple arrows between two vertices are allowed, i.e. a multidigraph. They are commonly used in representation theory: a representation V of a quive assigns a vector space $V(x)$ to each vertex x of the quiver and a linear map $V(a)$ to each arrow a.

where we have used the standard superspace notation for $\mathcal{N} = 1$ supermultiplets.

3.6 A brief review of Chern-Simons theory

Since one crucial ingredient in the theories we are considering is Chern-Simons theory, we review here some results concerning the perturbative of structure of this theory on general three-manifolds. Chern-Simons theory on three-manifolds is an important subject in itself, hence we present a rather formal treatment with at least some pedagogical value.

3.6.1 Perturbative approach

In this section, we will denote the bosonic Chern-Simons action by

$$S = -\frac{1}{4\pi}\int_M \mathrm{Tr}(A\wedge \mathrm{d}A + \frac{2i}{3}A\wedge A\wedge A) \tag{3.91}$$

where we use the conventions for Hermitian connections, and we included the factor $1/4\pi$ in the action for notational convenience. The group of gauge transformations $\mathcal{G}$ acts on the gauge connections as follows,

$$A \to A^U = UAU^{-1} - iU\mathrm{d}U^{-1}, \quad U \in \mathcal{G}. \tag{3.92}$$

We will assume that the theory is defined on a compact three-manifold M. The partition function is defined as

$$Z(M) = \frac{1}{\mathrm{vol}(\mathcal{G})}\int [\mathcal{D}A]\, e^{ikS} \tag{3.93}$$

where we recall that $k \in \mathbb{Z}$.

There are many different approaches to the calculation of (3.93), but the obvious strategy is to use perturbation theory. Notice that, since the theory is defined on a compact manifold, there are no IR divergences and we just have to deal with UV divergences, as in standard QFT. Once these are treated appropriately, the partition function (3.93) is a well-defined observable. In perturbation theory we evaluate (3.93) by expanding around saddle points. These are flat connections, which are in one-to-one correspondance with group homomorphisms

$$\pi_1(M) \to G \tag{3.94}$$

modulo conjugation. For example, if $M = \mathbb{S}^3/\mathbb{Z}_p$ is the lens space $L(p,1)$, one has $\pi_1(L(p,1)) = \mathbb{Z}_p$, and flat connections are labelled by homomorphisms $\mathbb{Z}_p \to G$. Let us assume that thesea rare a discrete set of points(this happens, for example. if M] is a rotational homology sphere, since in that case $\pi_1(M)$ is a finite group). We weill label the falt connnections with an index c, and a flat connection will be denoted by $A^{(c)}$. Each flat connection leads to a covariant derivative:

$$\mathrm{d}_{A^{(c)}} = \mathrm{d} + i[A^(c), \cdot], \tag{3.95}$$

and flatness implies that

$$\mathrm{d}^2_{A^{(c)}} = iF_{A^{(c)}} = 0. \tag{3.96}$$

Therefore the covariant derivative leads to a complex

$$0 \to \Omega^0(M, \mathbf{g}) \xrightarrow{\mathrm{d}_{A^{(c)}}} \Omega^1(M, \mathbf{g}) \xrightarrow{\mathrm{d}_{A^{(c)}}} \Omega^3(M, \mathbf{g}). \tag{3.97}$$

The first two terms in this complex have a natural interpretation in the context of gauge theories: $\Omega^0(M, \mathbf{g})$ is the Lie algebraalgebra of the group of gauge transformations, and we can write a gauge transformation as

$$U = e^{i\phi} \quad \phi \in \Omega^0(M, \mathbf{g}) \tag{3.98}$$

The elements of $\Omega^0(M, \mathbf{g})$ generate infinitesimal gauge transformations,

$$\delta A = -\mathrm{d}_A \phi \tag{3.99}$$

The second term, $\Omega^1(M, \mathbf{g})$, can be identifeid with the tangent space to the space of gauge connections. The first map in the complex (3.97) is interpreted as (minus) an infinitesimal gauge transformation in the background of $A^{(c)}$.

One can recall that the space of $\mathbf{g}$ valued forms on M has a natural inner product given by

$$\langle a, b \rangle = \int_M \mathrm{Tr}(a \wedge *b), \tag{3.100}$$

where $*$ is the Hodge operator. With respect to this product, we can define an adjoint operator on $\mathbf{g}$-valued p-forms in the same way that is done for the usual de Rham operator,

$$\mathrm{d}^\dagger_{A^{(c)}} = (-1)^{3(1+p)+1}(*\mathrm{d}_{A^{(c)}}) * . \tag{3.101}$$

We then have the orthogonal decompositions

$$\begin{aligned}\Omega^0(M,\mathbf{g}) &= \mathrm{Ker}\,\mathrm{d}_{A^{(c)}} \oplus \mathrm{Im}\,\mathrm{d}^\dagger_{A^{(c)}} \\ \Omega^1(M,\mathbf{g}) &= \mathrm{Ker}\,\mathrm{d}^\dagger_{A^{(c)}} \oplus \mathrm{Im}\,\mathrm{d}_{A^{(c)}}\end{aligned} \tag{3.102}$$

These decompositions are easily proved. For the first one, for example, we just have to note that

$$a \in \mathrm{Ker}\,\mathrm{d}_{A^{(c)}} \Rightarrow \langle \mathrm{d}_{A^{(c)}} a, \phi\rangle = \langle a, \mathrm{d}^\dagger_{A^{(c)}}\phi\rangle = 0, \quad \forall\phi \tag{3.103}$$

therefore

$$(\mathrm{Ker}\,\mathrm{d}_{A^{(c)}})^\perp = \mathrm{Im}\,\mathrm{d}^\dagger_{A^{(c)}}. \tag{3.104}$$

One also has the analogue of the Laplace-Beltrami operator acting on p-forms

$$\Delta^p_{A^{(c)}} = \mathrm{d}^\dagger_{A^{(c)}}\mathrm{d}_{A^{(c)}} + \mathrm{d}_{A^{(c)}}\mathrm{d}^\dagger_{A^{(c)}}. \tag{3.105}$$

In the following we will assume that

$$H^1(M,\mathbf{g}) = 0. \tag{3.106}$$

This means that the conection $A^{(c)}$ is *isolated*. However we will consider the possibility that $A^{(c)}$ has a non-trivial isotropy group $\mathcal{H}_c$. We recall that the isotropy group of a connection $A^{(c)}$ is the subgroup of gauge transformations which leave $A^{(c)}$ invariant,

$$\mathcal{H}_c = \left\{\phi \in \mathcal{G} \,\middle|\, \phi(A^{(c)}) = A^{(c)}\right\}. \tag{3.107}$$

The Lie algebra of this group is given by zero-forms annihilated by the covariant derivative (3.95),

$$\mathrm{Lie}(\mathcal{H}_c) = H^0(M,\mathbf{g}) = \mathrm{Ker}\,\mathrm{d}_{A^{(c)}}, \tag{3.108}$$

which is in general non-trivial. A connection is *irreducible* if its isotopy group is equal to the center of the group. In particular, if $A^{(c)}$ is irreducible one has

$$H^0(M,\mathbf{g}) = 0 \tag{3.109}$$

It can be shown that the isotropy group $\mathcal{H}_c$ consists of constant gauge translations that leave $A^{(c)}$ invariant,

$$\phi A^{(c)}\phi^{-1} = A^{(c)}. \tag{3.110}$$

They are in one-to-one correspondence with a subgroup of G which we will denote by H_c.

In the semiclassical approximation, $Z(M)$ is written as a sum of terms associated to saddle-points:

$$Z(M) = \sum_c Z^{(c)}(M), \tag{3.111}$$

where c labels the different flat connections $A^{(c)}$ on M. Each of the $Z^{(c)}(M)$ will be a perturbative series in $1/k$ of the form

$$Z^{(c)}(M) = Z^{(c)}_{1-\text{loop}}(M) \exp\left\{\sum_{l=1}^{\infty} S_l^{(c)} k^{-l}\right\}, \tag{3.112}$$

where $S_l^{(c)}$ is the $(l+1)$-loop contribution around the flat connection $A^{(c)}$. In order to derive this expansion, we split the connection into a "background", which is the flat connection $A^{(c)}$, plus a "fluctuation" B:

$$A = A^{(c)} + B \tag{3.113}$$

Expanding around this, we find

$$S(A) = S(A^{(c)}) + S(B), \tag{3.114}$$

where

$$S(B) = -\frac{1}{4}\int_M \text{Tr}(B \wedge \text{d}_{A^{(c)}} B + \frac{2i}{3} B^3). \tag{3.115}$$

The first term in (3.114) is the classical Chern-Simons invariant of the connection $A^{(c)}$. Since Chern-Simonstheory is a gauge theory, in order to proceed we have to fix the gauge. We will follow the detailed analysis of [2]. Our gauge choice will be the standard, covariant gauge,

$$g_{A^{(c)}}(B) = \text{d}^\dagger_{A^{(c)}} B = 0 \tag{3.116}$$

where $g_{A^{(c)}}$ is the gauge fixing function. We recall that in the standard Fadeev-Popov (FP) gauge fixing one first defines

$$\Delta^{-1}_{A^{(c)}}(B) = \int \mathcal{D}U \delta(g_{A^{(c)}}(B^U)), \tag{3.117}$$

and then inserts into the path integral

$$1 = \left[\int \mathcal{D}U \delta(g_{A^{(c)}}(B^U))\right] \Delta_{A^{(c)}}(B). \tag{3.118}$$

The key new ingredient here is the presence of a non-trivial isotropy group $\mathcal{H}_c$ for the flat connection $A^{(c)}$. When there is a non-trivial isotropy group, the gauge-fixing condition does not fix the gauge completely, since

$$g_{A^{(c)}}(B^\phi) = \phi g_{A^{(c)}}(B)\phi^{-1}, \quad \phi \in \mathcal{H}_c, \tag{3.119}$$

i.e. the basic assumption that $g(A) = 0$ only cuts the gauge orbit once is not true, and there is a residual symmetry by the isotropy group. Another way to see this is that the standard FP determinant vanishes due to zero modes. In fact, the standard calculation of (3.117) (which is valid if the standard isotropy group of $A^{(c)}$ is trivial) gives

$$\Delta^{-1}_{A^{(c)}}(B) = \left|\det \frac{\delta g_{A^{(c)}}(B^U)}{\delta U}\right|^{-1} = \left|\det \mathrm{d}^\dagger_{A^{(c)}}\mathrm{d}_A\right|^{-1}. \tag{3.120}$$

Note that when $\mathcal{H}_c \neq 0$, the operator $\mathrm{d}_{A^{(c)}}$ has zero modes due to the nonvanishing of (3.108), and the FP procedure is ill-defined. The correct way to proceed in the calculation of (3.117) is to split the integration over the gauge group into two pieces. The first piece is the integration over the isotropy group. Due to (3.119), the integrand does not depend on it, and we obtain a factor of $\mathrm{Vol}(\mathcal{H}_c)$. The second piece gives an integration over the remaining part of the gauge transformations, which has as its Lie algebra

$$(\mathrm{Ker}\ \mathrm{d}_{A^{(c)}})^\perp . \tag{3.121}$$

The integration over this piece leads to the standard FP determinant (3.120) but with the zero modes removed. We then find,

$$\Delta^{-1}_{A^{(c)}}(B) = \mathrm{Vol}(\mathcal{H}_\rfloor)\left|\det \mathrm{d}^\dagger_{A^{(c)}}\mathrm{d}_A\right|^{-1}_{(\mathrm{Ker}\ \mathrm{d}_{A^{(c)}})^\perp} \tag{3.122}$$

This phenomenon was first observed by Rozansky in [47], and developed in this language in [2]. As usual, the determinant appearing here can be written as a path integral over ghost fields, with action

$$S_{\mathrm{ghosts}}(C, \bar{C}, B) = \langle \bar{C}, \mathrm{d}^\dagger_{A^{(c)}}\mathrm{d}_A C\rangle, \tag{3.123}$$

where $C, \bar{C}$ are Grassmannian fields taking values in

$$(\mathrm{Kerd}_{A^{(c)}})^\perp . \tag{3.124}$$

The action for the ghosts can be divided into a kinetic term plus an interaction term between the ghost fields and the fluctuation B:

$$S_{\text{ghosts}}(C,\bar{C},B) = \langle \bar{C}, \Delta^0_{A^{(c)}} C\rangle + i\langle \bar{C}, \mathrm{d}^\dagger_{A^{(c)}}\,[B,C]\rangle. \tag{3.125}$$

The modified FP gauge-fixing leads then to the path integral

$$\begin{aligned} Z^{(c)}(M) &= e^{ikS(A^{(c)})}\int_{\Omega^1(M,\mathbf{g})} \mathcal{D}B e^{ikS(B)}\Delta_{A^{(c)}}(B)\delta\left(\mathrm{d}^\dagger_{A^{(c)}}B\right) = \\ &= \frac{e^{ikS(A^{(c)})}}{\mathrm{Vol}(\mathcal{H}_c)}\int_{\Omega^1(M,\mathbf{g})}\mathcal{D}B\delta(\mathrm{d}^\dagger_{A^{(c)}}B)\int_{(\mathrm{Ker}\ \mathrm{d}_{A^{(c)}})^\perp}\mathcal{D}C\mathcal{D}\bar{C}e^{ikS(B)-S_{\text{ghosts}}(C,\bar{C},B)}. \end{aligned} \tag{3.126}$$

Finally, we analyze the delta constraint on B. Due to the decomposition of $\Omega^1(M,\mathbf{g})$ in (3.102), we can write

$$B = \mathrm{d}_{A^{(c)}}\phi + B', \tag{3.127}$$

where

$$\phi \in (\mathrm{Ker}\ \mathrm{d}_{A^{(c)}})^\perp, \quad B' \in \mathrm{Ker}\ \mathrm{d}^\dagger_{A^{(c)}}. \tag{3.128}$$

The presence of the operator $\mathrm{d}_{A^{(c)}}$ in the change of variables (3.127) leads to a non-trivial Jacobian. Indeed, we have

$$||B||^2 = \langle \phi, \Delta^0_{A^{(c)}}\phi\rangle + ||B'||^2 \tag{3.129}$$

and the measure in the functional integral becomes

$$\mathcal{D}B = \left(\det{}'\Delta^0_{A^{(c)}}\right)^{\frac{1}{2}}\mathcal{D}\phi\mathcal{D}B', \tag{3.130}$$

where the $'$ indicates, as usual, that we are removing the zero nodes. Notice that the operator in the last equation is positive-definite, so the square root of its determinant is well-defined. We also have that

$$\delta\left(\mathrm{d}^\dagger_{A^{(c)}}B\right) = \delta\left(\Delta^0_{A^{(c)}}\phi\right) = \left(\det{}'{\Delta^0_{A^{(c)}}}^{-1}\right)\delta(\phi), \tag{3.131}$$

which is a straighforward generalization of the standard formula

$$\delta(ax) = \frac{1}{|a|}\delta(x). \tag{3.132}$$

We conclude that the delta function, together with the Jacobian in (3.130), lead to the following factor in the path integral:

$$(\det{}'\Delta^0_{A^{(c)}})^{-\frac{1}{2}}. \tag{3.133}$$

In addition, the delta function sets $\phi = 0$. The only thing that remains is the integration over B' which we relabel $B' \to B$. The final result for the gauge-fixed path integral is then

$$Z^{(c)}(M) = \frac{e^{ikS(A^{(c)})}}{\mathrm{Vol}(\mathcal{H}_c)} \left(\det{}' \Delta^0_{A^{(c)}}\right)^{-\frac{1}{2}} \int_{(\mathrm{Ker}\, \mathrm{d}^\dagger_{A^{(c)}})^\perp} \mathcal{D}C\mathcal{D}\bar{C} e^{ikS(B) - S_{\mathrm{ghosts}}(C,\bar{C},B)} \quad (3.134)$$

This is the starting point to perform gauge-fixed perturbation theory in: Chern-Simons theory.

3.6.2 The one-loop contribution

We now consider the one-loop contribution of a saddle-point to the path integral. This has been studied in many papers [41,58,87,88]. We will follow the detailed presentation in [2]. Before proceeding, we should specify what is the regularization mathod that we will use to define the functional determinants appearing in our calculation. A natural and useful regularization for quantum field theories in curved space is zeta-functional regularization. We recall that the zeta function of a self-adjoint operator T with eigenvalues $\lambda_n > 0$ is defined as

$$\zeta_T(s) = \sum_n \lambda_n^{-s}. \quad (3.135)$$

Under appropriate conditions, this defines a meromorphic[2] function on the complex $s-$plane which is regular at $s = 0$. Since

$$= \zeta'_T(0) = \sum_n \log \lambda_n \quad (3.136)$$

[2]A meromorphic function on an open subset D of the complex plane (or a higher dimensional complex disk) is a function that is holomorhic on all of D except for a discrete set of isoleted poles.

Chapter 4

Entaglement Entropy

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Contents

4.1 Introduction to Entaglement entropy

Entaglement entropy is an important way to quantify quantum correlations in a system. For gauge theories, the definition of entaglement entropy turns out to be subtle. Physicall, this is because there are non-local excitations in the system - for example, loops of electric or magnetic flux created by Wilson or 't Hooft loop operators, which can cut across the boundary of the region of interest. More precisely, it is because the Hilbert space of gauge invariant states, $\mathcal{H}_{g\text{inv}}$, does

not admit a tensor product decomposition between the region of interest and its complement.

A definition of the entaglement entropy has been given for a general gauge theory based on an extended Hilbert space (EHS) construction. For a non-Abelian theory without matter, on a spatial latice, this takes the rather compact form

$$S_{\rm EE} = -\sum_i p_i \log p_i + \sum_i \log\big(d_a^i\big) - \sum_i p_i {\rm Tr}_{\mathcal{H}_i} \bar{\rho}_i \log(\bar{\rho}_i). \tag{4.1}$$

The index i in the summation denotes various sectors which are determined by the value of the normal component of the electric field at all the boundary points. The probability to be in the sector i is given by p_i ,and d_a^i, with index a denoting the particular boundary point, is the dimension of the representation specifying the outgoing electric flux at that boundary point in the $i^{\rm th}$ sector. The last term is a weighted sum with $\bar{\rho}_i$ being the reduced density matrix in the $i^{\rm th}$ sector and the trace being taken over the Hilbert space $\mathcal{H}_i$ in this sector. It has been argued that this EHS definition agrees with the replica trick method of calculating the entaglement entropy. In the discussion below, we will sometimes refer to the first two terms in (4.1) as the classical terms and the last term as the quantum term.

It should be obvious that the middle term in (4.1), the one depending on d_a^i is absent in the Abelian theory. One can argue that the lack of a tensor product decomposition of the Hilbert space is related to the presence of a non-trivial centre. While the Hilbert of gauge-invariant states does not admit a tensor product decomposition, it could be written as a sum of tensor product terms,

$$\mathcal{H}_{g{\rm inv}} = \bigoplus_i \mathcal{H}_i \otimes \mathcal{H}_i', \tag{4.2}$$

where each factor $\mathcal{H}_i \otimes \mathcal{H}_i'$ is the Hilbert space in a sector corresponding to a particular value for the centre. Different choices for the algebra of gauge-invariant operators lead to different choices of centres and to a different value of the entaglement entropy in general. In particular, keeping all gauge-invariant operators in the region of interest in the algebra gives rise to the electric centre which is specified by the normal component of the electric field along the boundary. It is this definition which agrees with the EHS definition for the Abelian case. Other choices of centres can also be made. In particular, removing all the tangental components of the electric field at the boundary from the algebra gives rise to a magnetic centre.

Here, we are interested in studying the behaviour of the first two terms of (4.1), which correspond to the non-extractable contributions to the entanglement

entropy, in the continuum limit. These terms depend on the distribution p_i which determines the probability for being in the various superselection sectors. For the electric centre case[1], by studying the correlation functions of the electric field on the boundary, we will find, both for the Abelian and non-Abelian cases, that the distribution is typically determined by very high momentum modes localized close to the boundary. As a result, we will argue that the contribution form these terms drops out of the mutual information of disjoint regions or the relative entropy between two states which only have finite energy excitations about the vacuum.

4.2 Classical Term in the Continuum Limit

4.2.1 $U(1)$ Abelian Theory

Let us begin by considering a 3+1-dimensional free $U(1)$ theory. Since the theory is free, the classical term is determined by the two-point function of the normal component of the electric field,

$$G_{rr} = \langle E_r E_r \rangle, \tag{4.3}$$

with the two electric fields E_r being inserted at two different points on the boundary which is an S^2 of radius R. The classical term is then given by

$$-\sum p_i \log(p_i) = \alpha_1 \frac{A}{\epsilon^2} - \frac{1}{2}\log\big(\det G_{rr}^{-1}\big), \tag{4.4}$$

with ϵ being a short distance cut-off. Thus, up to a non-universal area lawa divergence, G_{rr} determines the classical term. Let us note in passing that the first term on the RHS of (4.4) depends on the measure for the sum over over all electric field configurations. Starting from the lattice and passing to the continuum gives a well defined measure.

Since the two electric fields in G_{rr} are inserted at points on the boundary S^2 it is a function of the angular separation of the two points. In fact, it turns out to be divergent and some care needs to be exercised in regulating this divergence and

[1]In the non-Abelian case, the electric centre definition is sometimes taken to be different from the EHS definition with the middle term in (4.1) being absent. We will not be very careful about such distinctions. Our considerations, studying the probability distribution p_i will apply to both of these cases.

defining it. After introducing a short distance cut-off Δ for the radial momentum, the resulting answer was shown to be

$$G^{lm}_{rr} = \frac{1}{\pi R^4}\left(\log\frac{R^2}{\Delta^2}\right) l(l+1). \tag{4.5}$$

Here we have carried out a Fourier transform to go from the angular separation variables to the angular momentum variables labelled by the integers (l, m), as per standard conventions. Note that the result diverges as $\Delta \to 0$, due to the contribution of modes with very high values of the radial momentum, which dominate the result in this limit. The factor of $l(l+1)$ means that the Green function is proportional to the two-dimenstional Laplacian for a free scalar on the boundary.

From G_{rr}, we can calculate the classical piece (4.4) which turns out to be,

$$-\sum_i p_i \log(p_i) = \alpha_2 \frac{A}{\epsilon^2} - \frac{1}{6}\log\left(\frac{A}{\epsilon^2}\right) + \cdots, \tag{4.6}$$

where the ellipsis refer to non-universal finite pieces. The logarithmic term arises from the two-dimenstional scalar Laplacian mentioned above, while the pre-factor $\log(R^2/\Delta^2)/\pi R^4$ contributes to the non-universal area law divergent pieces above.

It is important to emphasise that we have introduced two cut-offs above, Δ and ϵ, and worked in the limit where Δ goes to zero first. This was true in (4.6), for example, since the angular momentum l along the S^2 was being kept fixed while $\Delta \to 0$. It is in this limit that G_{rr} depends on the two-dimenstional Laplacian for a free scalar with a logarithmic dependence on the radial cut-off Δ. The behaviour of modes whose momentum along S^2 is comparable to the radial cutoff is more complicated and less universal. In the discussion below, when we analyse the contribution of the classical term to the mutual information and relative entropy, we will comment on such modes as well and show that their contribution too drops out from these quantities.

Having understood the case of a spherical region, we can now turn to the more general situation. Let us begin by first considering the half space $z < 0$, with the remaining spatial coordinates, (x, y) taking values in $(-\infty, \infty)$. The boundary of this region is the two-plane $z = 0$. The normal component of the electric field is along the z direction. It is easy to see by standard quantisation of the electromagnetic field that

$$\langle E_z(\mathbf{x_1}) E_z(\mathbf{x_2})\rangle = \int \frac{d^3k}{(2\pi)^2}\frac{\omega_k}{2}\left(1 - \frac{k_z^2}{k^2}\right) e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \tag{4.7}$$

(The two-point function is computed at equal time.) Here $\omega_k = k = \sqrt{k_z^2 + k_x^2 + k_y^2}$. Setting the two points to be at the boundary, $z = 0$, we get

$$\langle E_z(\mathbf{k}_\parallel) E_z(\mathbf{k}'_\parallel)\rangle = (2\pi)^2 \delta^{(2)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) k_\parallel^2 \int \frac{dk_z}{4\pi} \frac{1}{k}. \tag{4.8}$$

We see that the integral on the RHS is logarithmically divergent since $k \sim k_z$ for large k_z. Thus, as noted above, the result needs to be regulated by introducing a cut-off for momentum along the z direction, normal to the boundary. One efficient way to do this is to take the two points located at slightlA different values in the z direction, with $\Delta z = \delta$, instead of both of them being exactly at the boundary. Repeating the calculation above now gives,

$$\langle E_z(\mathbf{k}_\parallel) E_z(\mathbf{k}'_\parallel)\rangle = (2\pi)^2 \delta^{(2)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) k_\parallel^2 \int \frac{dk_z}{4\pi} \frac{e^{ik_z\Delta}}{k}. \tag{4.9}$$

We see that the divergence in the integral at large k_z is regulated resulting in

$$\langle E_z(\mathbf{k}_\parallel) E_z(\mathbf{k}'_\parallel)\rangle \sim (2\pi)^2 \delta^{(2)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) k_\parallel^2 \log\big(k_\parallel \Delta\big). \tag{4.10}$$

We can see that the result above is analogous to what was obtained for the spherical region case. In particular, the result diverges e logarithmically as $\Delta \to 0$ and is proportional to $k_\parallel^2$ which is the eigenvalue of the two-dimenstional scalar Laplacian on the boundary. One difference is that in the spherical case, the logarithmic divergence due to the modes with high value of the normal component is cut-off by the radius R, whereas in the infinite plane case, where this cut-off is not availiable, it is cut-off by $k_\parallel$.

From this example of the half plane and the spherical region, it is now clear that we expect the two-point function in any compact region a result analogous to the sphere case, namely going like $\log(R/\Delta)$, with Δ being the cut-off fir the normal component of momentum, and R being an IR scale provided by the size of the region of interest, and also being proportional to the two-dimenstional Laplacian along the boundary.

We are now ready to consider the mutual information and relative entropy in this theory. Consider the mutual information in the vacuum state first. Suppose there are two compact disjoint spatial regions A, B. The mutual information is given by

$$I(A, B) = S_A + S_B - S_{AB} \tag{4.11}$$

where S_{AB} is the entaglement between the region $A \cup B$ abd the rest. We are interested in the classical term's contribution to $I(A, B)$. Since the $U(1)$ theory under consideration is free, this is determined by the two-point function, as discussed above, with $p(E_n)$ the probability to be in the sector where the normal component of the electric field takes value E_n being given by

$$p(E_n(\mathbf{x})) = N \exp\left(-\frac{1}{2}\int d^2x d^2y E_n(\mathbf{x}) G^{-1}_{nn}(\mathbf{x}-\mathbf{y}) E_n(\mathbf{y})\right). \tag{4.12}$$

Here N is a normalization, E_n denotes the normal component of the electric field, $\mathbf{xy}$ are two points on the boundary and G_{nn} is the Green function for the normal component E_n.

When we are dealing with the two disjoint regions A and B, we have two boundaries which are also disjoint. Thus the two point function which appears on the RHS is evaluated when the two points $\mathbf{x}, \mathbf{y}$ are both on the boundary of A or on the boundary of B, or when one point is on the boundary of A and the other on the boundary of B. We saw on general grounds in the previous subsection that when the two points are on the same boundary the two-point function diverges and, after a cut-off is introduced, is proportional to $\log(R/\Delta)$, where R is an IR scale set by the size of the region A or B. There is no such divergence when the two points are located on the two different boundaries of A and B respectively, since in this case the two points cannot come close to each other. Hence, the two-point function which appears in the exponent in the equation above will be dominated by the contribution when the two points are on the same boundary; with the contribution from when they are on separate boundaries being suppresed parametrically by a factor of $[\log(R/\Delta)]^{-1}$. As a result, the probability, $p(E^A_n, E^B_n)$, for the E_n to take the value E^A_n, E^B_n on the two boundaries $\partial A, \partial B$ will simply be the product

$$p(E^A_n, E^B_n) = p(E^A_n) p(E^B_n), \tag{4.13}$$

where $p(E^A_n)$, for example, is the probability that E_n takes value E^A_n on ∂A regardless of any value E_n takes on ∂B. (4.13) is true up to subleading terms which vanish in the continuum limit when $\Delta/R \to 0$. It then follows that the classical term will cancel out and not contribute to the mutual information in the continuum limit.

Similarly, we can consider the relative entropy between two states. Here we will be interested in states which only carry a finite energy above the ground state. Let the corresponding density matrices for some spatial region be ρ^1 and ρ^2 in these

two cases; the relative entropy is given by

$$S(\rho^1 \mid \rho^2) = \mathrm{Tr}\left(\rho^1 \log \rho^1\right) - \mathrm{Tr}\left(\rho^1 \log \rho^2\right) \tag{4.14}$$

It is easy to see that this becomes

$$S(\rho^1 \mid \rho^2) = \sum_i p_i^1 \log\left(\frac{p_i^1}{p_i^2}\right) + \sum_i p_i^1 \mathrm{Tr}_{\mathcal{H}_i} \log\left(\frac{\bar{\rho}_i^1}{\bar{\rho}_i^2}\right), \tag{4.15}$$

where the bared rhos are the normalised density matrices in the i^{th} sector in state 1 or 2, respectively. We can see, by an argument analogous to the one above for mutual information that the first term, which is due to the classical contribution to the entanglement, again vanishes in the continuum limit. The argument goes as follows. The probability $p_i^{1,2}$ is determined by the correlation functions of E_n in state $1, 2$. The two-point function in the two states to the leading order will be the same, and in turn, equal to that in the vacuum (4.12), since it's dominated by very high momentum modes whose behaviour will be the same as in the vacuum for states that which only carry a finity energy above the vacuum. For a region of size R, this two-point function goes like $\log(R/\Delta)$ and therefore diverges when $\Delta \to 0$. Connected higher point correlations can arise is states which are not the vacuum, but these will be finite and thus subdominant compared to the two-point function. Therefore, p_i^1, p_i^2 will be the same up to corrections which vanish as $[\log(R/\Delta)]^{-1}$, and as a result,

$$\log\left(\frac{p_i^1}{p_i^2}\right) \sim \frac{1}{\log\left(\frac{R}{\Delta}\right)}. \tag{4.16}$$

Since p_i^1 is normalised so that $\sum_i p_i^1 = 1$, we see that the first term will be of order $[\log(R/\Delta)]^{-1}$ and thus will vanish.

Before we proceed, some comments are worth to be made. We mentioned that in general, the two-point function would depend on the boundary Laplacian. For a spherical region, the zero mode of the Laplacian needs to be excluded when computing the determinant, due to the Gauss law constraint. More generally, all zero modes need to be handled with care, but such rigor goes beyond the scope of this chapter.

The divergent behaviour of the two-point function going like $\log(R/\Delta)$ is true for modes which carry momentum along the boundary that is much smaller than Δ^{-1}, (i.e., $k_\parallel \ll \Delta^{-1}$). We can also consider modes which have $k_\parallel \sim \Delta^{-1}$. In this case, the two-point function is again dominated by the contribution where the two

points lie on the same boundary. Consider, without loss of generality, two regions corresponding to $z < 0$ and $z > L$. When the two points are on the two boundaries, $z = 0, z = L$, the two-point function becomes

$$\langle E_z(\mathbf{k}_\parallel) E_z(\mathbf{k}'_\parallel)\rangle = (2\pi)^2 \delta^{(2)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) k_\parallel^2 \int \frac{dk_z}{4\pi} \frac{e^{ik_z L}}{\sqrt{k_\parallel^2 + k_z^2}}. \tag{4.17}$$

Integrating, we get

$$\langle E_z(\mathbf{k}_\parallel) E_z(\mathbf{k}'_\parallel)\rangle = (2\pi)^2 \delta^{(2)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) k_\parallel^2 K_0(k_\parallel L), \tag{4.18}$$

where K_0 is called the modifed Bessel function of the second kind. For $k_\parallel \sim \Delta^{-1} \gg L^{-1}$, we get

$$\langle E_z(\mathbf{k}_\parallel) E_z(\mathbf{k}'_\parallel)\rangle = (2\pi)^2 \delta^{(2)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) \left(\frac{1}{\Delta^3 L}\right)^{1/2} e^{-L/\Delta}, \tag{4.19}$$

so that the two-point function (where the two points lie on the two boundaries for modes with $k_\parallel \sim \Delta^{-1}$) is exponentially suppressed, compared to the case when the two points lie on the same boundary. Thus, the contribution of these modes will also drop out in the classical part of the mutual information. Similarly, if we are considering two states whose behaviour at the cut-off scale Δ is the same as that of the vacuum, then the contribution of modes with $k_\parallel \sim \Delta^{-1}$ will also drop out in the relative entropy of these two states.

Let us also briefly consider the case of the magnetic centre. In this case, the different superselection sectors are specified by the normal component of the magnetic field, $\mathbf{B}$, and the probability of being in a superselection sector is specified by the two point function of the normal component of $\mathbf{B}$. For the planar boundary considered above at $z = 0$, it is easy to see that the two-point function is given by,

$$\langle B_z(\mathbf{k}_\parallel) B_z(\mathbf{k}'_\parallel)\rangle = (2\pi)^2 \delta^{(2)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) \int \frac{dk_z}{4\pi} \frac{e^{ik_z \Delta}}{k}. \tag{4.20}$$

This is obviously the same as the two-point function for the electric field (4.9). It is also straightforward to generalize this discussion for a gauge field to other dimensions. In $d+1$ dimensions, (4.7) is replaced by

$$\langle E_z(\mathbf{x_1}) E_z(\mathbf{x_2})\rangle = \int \frac{d^d k}{(2\pi)^d} \frac{\omega_k}{2} \left(1 - \frac{k_z^2}{k^2}\right) e^{i\mathbf{k}\cdot(\mathbf{x_1} - \mathbf{x_2})}, \tag{4.21}$$

where we are still considering the region $z < 0$ with a boundary which now has extent in $d-1$ spatial dimensions.

It then follows that

$$\begin{aligned}\langle E_z(\mathbf{k}_\parallel)E_z(\mathbf{k}'_\parallel)\rangle &= (2\pi)^{d-1}\delta^{d-1}(\mathbf{k}_\parallel+\mathbf{k}'_\parallel)k_\parallel^2\int\frac{dk_z}{4\pi}\frac{e^{ik_z\Delta}}{k}\\ &\sim (2\pi)^{d-1}\delta^{d-1}(\mathbf{k}_\parallel+\mathbf{k}'_\parallel)k_\parallel^2\log\big(k_\parallel\Delta\big),\end{aligned}\tag{4.22}$$

showing that the logarithmic divergence, and boundary Laplacian are universal features in all dimensions $d \geq 2$.[2]The logarithmic divergence then implies that our arguments go through for the $U(1)$ theory in any dimension greater than two. It also follows that the mutual information and relative entropy are independent of the classical term. Our next consideration is the logical extension of the free $U(1)$ theory - we want to study theories with charged matter. On the lattice, charged matter adds degrees of freedom that live on the spatial lattice sites in the standard manner. The discussion above can be readily extended to such a case. The Gauss law constraint still results in a non-trivial centre, and the full Hilbert space admits a decomposition of the form given in (4.2). The label i denotes sectors where the centre takes a fixed value. Also, the extended Hilbert space in this case is obtained by taking the tensor product of the Hilbert spaces of the gauge degrees of freedom (living on the links) and the matter degrees of freedom (living on the sites). Passing to the continuum in the electric centre case, or equivalently, the extended Hilbert space case, the different sectors are specified by the different values taken by the normal component of the electric field on the boundary of the region of interest. The probability p_i is a functional of these boundary values, $p_i \equiv p_i\,[E_n]$. The theory is no longer free, therefore there are non-zero correlation functions of higher order, besides the two-point function.

As long as the interactions become weak in the UV sector, at the scale of the lattice, one would expect that the theory is perturbative at said scale and that the divergence seen in the free field case would dominate over the perturbative corrections due to the interactions. Thus, the leading correlation would again be the two-point function, which has the same behaviour as in (4.10) and would diverge in the continuum limit. This should be the case in $d = 2$, since in that case the gauge coupling is superrenormalisable. In $d = 3$, the gauge coupling grows

[2]In $d = 2$, despite the fact that the compact theory is confined on large length-scales, it is not on small length-scales, and thus we expect this result to continue to hold as well.

logarithmically and becomes strong in the UV; it also becomes strong in the UV in $d > 3$, where the gauge coupling is non-renormalisable. Therefore, it is unclear what happens in the continuum limit for those cases, or even if such a limit exists for them. One interesting possibility is that the theory flows to a non-trivial fixed point in the UV. In this case, the behaviour of the two-point function of the electric field would be determined by its anomalous dimension at said UV fixed point. The short distance contribution would take the form

$$\langle E_z(\mathbf{k}_\parallel) E_z(\mathbf{k}'_\parallel)\rangle \sim \delta^{(d-1)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) k_\parallel^2 \int \frac{dk_z}{k^{(d-2(\delta-1))}} e^{ik_z\Delta} \tag{4.23}$$

where δ is the anomalous dimension. If $\delta > (d+1)/2$, so that the anomalous dimension exceeds the engineering dimension, then the correlations will be even more strongly dominated by the short distance modes and one expects the arguments for the independence of the mutual information and relative entropy from the classical term to apply. For $\delta < (d+1)/2$, the situation is less clear. If the correlations of E_n are finite and non-divergent, then it could be that the classical piece continues to contribute in the continuum limit to the mutual information and the relative entropy.

Suppose that the theory is strong coupled in the UV and flows to a weakly coupled one, which is close to the free non-interacting theory, in the IR at an energy scacle of order $E \sim \Lambda$. In such cases, one would still expect thath the contribution to the classical term from modes with energies $E \leq \Lambda$ is approximately governed by thr two-point function and thus, the contribution of these modes should drop out in the mutual information or the relative entropy. It will be worth trying to make this intuitive argument more precise.

4.2.2 Non-Abelian Theory

Let us now turn our attention to the non-Abelian case. In this case, in the continuum, the different sectors in the electric centre or the EHS definition are specified by the value of $\mathrm{Tr}(E_n^2)$ and other Casismirs (i.e., appropriate local gauge invariant operators) on the boundary, where E_n is the normal component of the electric field. The probability to be in a particular sector is a functional of $\mathrm{Tr}(E_n^2)$ and the other Casismirs and we denote it for ease of notation as $p\,[E_n^2]$.

Let us start by considering the free $SU(N)$ Yang-Mills theory, say, in 3+1 dimensions, and consider the region of space $z < 0$. The boundary lies at $z = 0$.

It is easy to show that in this case the two-point function is given by

$$\langle E_z(\mathbf{k}_\parallel)^2 E_z(\mathbf{k}'_\parallel)^2\rangle \sim (N^2-1)\delta^{(2)}(\mathbf{k}_\parallel+\mathbf{k}'_\parallel)\int d^2\bar{k}_\parallel \bar{k}_\parallel^2 (\mathbf{k}_\parallel-\bar{\mathbf{k}}_\parallel)^2 \tag{4.24}$$

$$\int \frac{dk_z}{\sqrt{k_z^2+\bar{k}_\parallel^2}} e^{ik_z\Delta}\int \frac{dk_z}{\sqrt{k_z^{2\prime}+(\mathbf{k}_\parallel-\bar{\mathbf{k}}_\parallel)^2}} e^{ik'_x\Delta} \tag{4.25}$$

where we have separated the two points in the z direction by a value $\Delta z=\Delta$. We see that now the integral over $\bar{k}_\parallel$, the momentum along the boundary, is also divergent, since E_z^2 is a more singular operator than E_z. Introducing a cut-off along the boundary directions and not distinguishing it form Δ, the cut-off along the normal direction, we get

$$\langle E_z(\mathbf{k}_\parallel)^2 E_z(\mathbf{k}'_\parallel)^2\rangle \sim (N^2-1)\delta^{(2)}(\mathbf{k}_\parallel+\mathbf{k}'_\parallel)\frac{1}{\Delta^6}, \tag{4.26}$$

up to logarithmic corrections, which we have not kept carefully. This is much more singular than the two-point function of E_z in the Abelian case.

It is also easy to calculate the higher point correlators for E_z^2 in the free case. The contributions to the n-point function can be classified in terms of the number of disconnected components. The dominant contribution comes from the maximally disconnected component going like the product $\langle E_z^2(1)E_z^2(2)\rangle\langle E_z^2(3)E_z^2(4)\rangle\cdots\langle E_z^2(n-1)E_z^2(n)\rangle$ for n even, which diverges as $\Delta^{-2(n+1)}$. In contrast, the fully connected contribution goes like Δ^{-6}, as before, and is therefore subdominant. As a result, the two-point correlation function dominates the correlation functions and hence the probability distribution and therefore the classical term is determined by the two-point function (4.26). This two-point function is dominated by the UV modes at the cut-off. As for the higher-order Casismirs, the two-point function is even more singular and therefore dominated even more strongly.

On turning on the gauge coupling, these correlations will change. However, Yang-Mills theory is known to be asymptotically free in 3+1 dimensions and superrenormalisable in 2+1 dimensions. Therefore, in these cases, we expect the behaviour at the scale of the lattice to continue to be that of the free theory to good aproximation, and the two-point and higher point correlations to diverge, as described above. These divergences of course occur because points on the boundary can come close together. When we consider two disconnected boundaries, this means that the correlations will be dominated by their value when the points lie

on the same boundary. As a result, in the continuum limit, the joint probability should satisfy the condition

$$p((E_n^2)^A, (E_n^2)^B) = p((E_n^2)^A)p((E_n^2)^B), \tag{4.27}$$

analogous to (4.13) in the Abelian case, and the contribution of the classical piece to the mutual information should therefore vanish. Note that this conclusion is equally true in the electric centre definition, where the classical piece is given by the first term on the RHS of (4.1), and in the EHS definition, where it is given by the sum of the first two terms on the RHS of (4.1). Both these terms vanish if (4.27) is true. Similarly, one can argue that the relative entropy for states which only differ from the vacuum with finite energy excitations, will also not recieve a contribution from the classical piece.

In higher dimensions $d > 3$, the theory is non-renormalisable and the situation is more interesting. A continuum limut might exist with the theory flowing to a non-trivial fixed point in the UV, and as in the discussion above for the Abelian case, the anomalous dimension of $\mathrm{Tr}(E^2)$ would then determine the short distance nature of the two-point and higher point correlators. Similarly, adding matter can change the bahaviour of the theory. An important example of this type is if the resulting theory becomes conformally invariant. Once again, the anomalous dimension of $\mathrm{Tr}(E^2)$ plays an important role in determining the nature of the short distance correlators. If the correlators are smooth at short distances, and do not diverge as $\Delta \to 0$, then the classical piece could potentially contribute to both the mutual information and the relative entropy.

4.2.3 p-Form Abelian Gauge Theory

Our discussion of the Abelian $U(1)$ gauge theory can be generalised to the case of a general p-form Abelian theory in $d+1$ dimensions. The action is given by

$$S = -\frac{1}{2(p+1)!}\int d^{d+1}x H_{\mu_1\mu_2\cdots\mu_{p+1}}H^{\mu_1\mu_2\cdots\mu_{p+1}} \tag{4.28}$$

where

$$H_{\mu_1\mu_2\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1}B_{\mu_2\mu_3\cdots\mu_{p+1}]}, \tag{4.29}$$

and the square brackets indicate complete anti-symmetrisation.

The EHS is obtained by working the the following gauge

$$B_{0\mu_1\cdots\mu_{p-1}} = 0. \tag{4.30}$$

The resulting constrains, analogous to the Gauss law, are

$$\partial^i H_{0i\mu_1\cdots\mu_{p-1}} = 0. \tag{4.31}$$

The different superselection sectors, for the electric centre choice are therefore specified by the value for the normal component of the electric field, $H_{0n\mu_1\cdots\mu_p} \equiv H_{0i\mu_1\cdots\mu_p} n^i$.

The resulting classical term is then determined by the two-point function of the normal component. For a planar boundary, which we continue to denote as $z = 0$, we have

$$\langle H_{0zi_1i_2\cdots i_{p-1}}(\mathbf{k}_\parallel) H_{0zj_1j_2\cdots j_{p-1}}(\mathbf{k}'_\parallel)\rangle \sim \delta^{(d-1)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) k_\parallel^2 (\delta^T_{i_1[j_1}\delta^T_{i_2j_2}\cdots\delta^T_{i_{p-1}j_{p-1}]}) \times \int \frac{dk_z}{\sqrt{k_z^2 + k_\parallel^2}} e^{ik_z\Delta} \tag{4.32}$$

Here

$$\delta^T_{ij} \equiv \delta_{ij} - \frac{(k_\parallel)_i (k_\parallel)_j}{k_\parallel^2} \tag{4.33}$$

is the delta function transverse to the spatial momentum along the boundary $\mathbf{k}_\parallel$, and the square brackets in the product of the delta functions indicates complete anti-symmetrisation with respect to the second indices $j_1, j_2, \cdots, j_{p-1}$.

We see that the two-point function generally involves a Laplacian for a $(p-1)$-form living on the boundary which is $(d-1)$-dimensional. The contribution is once again dominated by short distance modes and logarithmically dependent on the cut-off Δ. As a result, we see that the contribution of the classical piece drops out of the mutual information or relative entropy.[3]

Let us also note that when we work with a region which is not the half space and the boundary is no longer planar, the logarithmic divergence will be cut off by the size of the region of interest. Also, in those more general cases, one should be more careful about the zero modes and their contributions.

4.3 Entanglement and Dualities

In this section, we will study some aspects of the duality between a theory with a p-form gauge potential in $d+1$ dimensions and the dual theory, which is equipped

[3] As in the gauge field case, the logarithmic dependence on Δ is true for modes with $k_\parallel \ll \Delta^{-1}$. However, it also follows that these modes have zero contribution to the mutual information and the relative entropy in the classical piece.

with a $(d-p-1)$-form potential. This is a broad generalisation of the electric-magnetic duality in 3+1 dimensions for a gauge field, and is also closely related to the Kramers-Wannier duality on the lattice. Our interest will be to study how the different choices for the centre of the algebra of observables transform under this duality and the accompanying changes in the entaglement entropy and related quantities. In particular, we will consider two choices of centers for the p-form theory, called the electric and magnetic centers and study how they map under duality. We will find that the magnetic centre choice for a region R in the p-form case maps to an algebra of observables in the dual theory which is closesly related but not identical to the electric centre of the dual $(d-p-1)$-form theory in a suitable region $\tilde{R}$ of the dual lattice. In the continuum limit, these differences become uninmortant for ultraviolet insensitive quantities, like the relative entropy and mutual information, as we mentioned above, and the two dual theories therefore agree.

4.4 Conclusions

In this section :

Chapter 5

Strings and Conformal Field Theory

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Contents

5.1 String theory as a CFT

String theories in the conformal gauge are two-dimensional conformal field theories. Thus, instead of the operatorial analysis, one can give an equivalent description by using the language of conformal field theory in which one works with the OPE rather then commutators or anticommutators and that contributes to simplify many calculations.

5.1.1 Variables and coordinates in the CFT formulation

In the case of a closed string it is convenient to introduce the variables z and $\bar{z}$ that are related to the world sheet variables τ and σ through a conformal transformation:

$$z = e^{2i(\tau-\sigma)} \quad ; \quad \bar{z} = e^{2i(\tau+\sigma)} \tag{5.1}$$

In the case of an euclidean world sheet ($\tau \to -i\tau$), z and $\bar{z}$ are complex conjugates of each other. In terms of them we can write the bosonic coordinate X^μ as follows:

$$X^\mu = (z, \bar{z}) = \frac{1}{2}\left[X^\mu(z) + \tilde{X}^\mu(\bar{z})\right] \tag{5.2}$$

where

$$X^\mu(z) = \hat{q}^\mu - i\sqrt{2\alpha'}\log z\alpha_0^\mu + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}z^{-n} \tag{5.3}$$

and

$$\tilde{X}^\mu(\bar{z}) = \hat{q}^\mu - i\sqrt{2\alpha'}\log \bar{z}\tilde{\alpha}_0^\mu + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}\bar{z}^{-n} \tag{5.4}$$

with $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\hat{p}^\mu$

In the case of **an open string theory** one can introduce the variables:

$$z = e^{i(\tau-\sigma)}; \quad \bar{z} = e^{i(\tau+\sigma)} \tag{5.5}$$

and the string coordinate can be written as

$$X^{\mu}(z,\bar{z}) = \frac{1}{2}\left[X^{\mu}(z) + X^{\mu}(\bar{z})\right] \tag{5.6}$$

where X^{μ} is given in (5.3) and $\alpha_0^{\mu} = \sqrt{2\alpha'}\hat{p}^{\mu}$. In superstring theory we must also introduce a conformal field with conformal dimension equal to 1/2 corresponding to the fermionic coordinate.

In the closed string case we have two in-dependent fields for the holomorphic and anti-holomorphic sectors which are obtained through the Wick rotation ($\tau \to -i\tau$) and the conformal transformation $(\tau,\sigma) \to (z,\bar{z})$

$$\Psi^{\mu}(z) \sim \sum_t \psi_t z^{-t-1/2}; \quad \tilde{\Psi}^{\mu}(\bar{z}) \sim \sum_t \tilde{\psi}_t \bar{z}^{-t-1/2} \tag{5.7}$$

In the open string case, applying the same operations we get again eqs(5.7), but this time with the same oscillators.

In what follows we will explicitly consider only the **holomorphic sector for the closed string**. In the case of an open string it is sufficient to consider the string coordinate at the string endpoint $\sigma = 0$. In both cases it is convenient to introduce a *bosonic dimensionless variable*:

$$x^{\mu}(z) \equiv \frac{X^{\mu}(z)}{\left(\sqrt{2\alpha'}\right)} = \tilde{q}^{\mu} - i\alpha_0^{\mu}\log z + i\sum_{n\neq 0}\frac{\alpha_n}{n}z^{-n} \tag{5.8}$$

where $\tilde{q} = \hat{q}/\sqrt{2\alpha'}$ and a *fermionic* one:

$$\psi^{\mu}(z) = -i\sum_t \psi_t z^{-t-1/2} \tag{5.9}$$

5.1.2 Operator Product Expansion

The theory can be quantized by imposing the following OPEs

$$x^{\mu}(z)\,x^{\nu}(\omega) = -\eta^{\mu\nu}\log(z-\omega) + \cdots; \quad \psi^{\mu}(z)\,\psi^{\nu}(\omega) = -\frac{\eta^{\mu\nu}}{z-\omega} + \cdots, \tag{5.10}$$

where the dots denote finite terms for $z \to \omega$. In terms of the previous conformal fields we can define the generators of superconformal transformations:

$$G(z) = -\frac{1}{2}\psi.\partial x \quad ; T(z) = T^x(z) + T^\psi(z) = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}\partial\psi.\psi \tag{5.11}$$

These conformal fields satisfy the following OPEs:

$$T(z)\,T(\omega) = \frac{\frac{d}{d\omega}T(\omega)}{z-\omega} + 2\frac{T(\omega)}{(z-\omega)^2} + \frac{c/2}{(z-\omega)^4} + \cdots \tag{5.12}$$

$$T(z)\,G(\omega) = \frac{\partial/\partial\omega G(\omega)}{z-\omega} + \frac{3}{2}\frac{G(\omega)}{(z-\omega)^2} + \cdots \tag{5.13}$$

$$G(z)\,G(\omega) = \frac{2T(z)}{z-\omega} + \frac{d}{(z-\omega)^3} + \cdots \tag{5.14}$$

We have to translate the L_0 operator in the R sector by a constant:

$$L_0 \to L_0^{conf} \equiv L_0 + \frac{d}{16} = \sum_{n=1}^{\infty}(\alpha_{-n}.\alpha_n + n\psi_{-n}.\psi_n) + \alpha' p^2 + \frac{d}{16} \tag{5.15}$$

Therefore in the R sector we have two L_0 operators that are related by eq.(5.15). L_0 determines the spectrum of superstring while L_0^{conf} encodes the correct conformal properties of the R sector.

5.1.3 Primary fields & Highest weight states

In conformal field theory one introduces the concept of conformal or *primary field*

Definition 5.1 (Primary field)**.** *A conformal field $\Phi(z)$ is called a primary field with dimension h if it satisfies the following OPE with the energy-momentum tensor:*

$$T(z)\,\Phi(\omega) = \frac{\partial_\omega\Phi(\omega)}{z-\omega} + h\frac{\Phi(\omega)}{(z-\omega)^2} \tag{5.16}$$

From it one can compute the corresponding *highest weight state* $|\Phi\rangle$ by means of the following limiting procedure

$$|\Phi\rangle = \lim_{z\to 0}\Phi(z)|0\rangle \quad , \quad \langle\Phi| = \lim_{z\to 0}\langle 0|\Phi^\dagger(z) \sim \lim_{z\to\infty}\langle 0|\left(z^2\right)^h\Phi(z) \tag{5.17}$$

The hermitian conjugate field $\Phi^\dagger$ in the previous expression has been defined as the field transformed under the conformal transformation $z \to 1/z$. It is easy to show that:

$$L_0|\Phi\rangle = h|\Phi\rangle \quad ; \quad L_n|\Phi\rangle = 0 \tag{5.18}$$

5.1.4 BRST charge & Vertex operators

With the introduction of ghosts, the string action in the conformal gauge becomes invariant under the BRST transformations and the physical states are characterized by the fact that they are annihilated by the BRST charge that in the bosonic case is given by

$$Q \equiv \oint \frac{dz}{2\pi i} c\left(z\right) J_{BRST} \equiv \oint \frac{dz}{2\pi i} c\left(z\right) \left[T^{x}\left(z\right) \frac{1}{2} T^{bc}\left(z\right) \right] \tag{5.19}$$

where $T^{x}\left(z\right) = -1/2\left(\partial x\right)^{2}$ and $T^{bc}\left(z\right) = \sum_{n} L_{n} z^{-n-2}$. The physical states are annihilated by the BRST charge

$$Q\left|\phi_{phys}\right\rangle = 0 \tag{5.20}$$

By using the OPE it can be shown that in the bosonic string the most general BRST invariant *vertex operator* has the following form

$$\mathcal{W} = c\left(z\right) \mathcal{V}_{\alpha}^{x}\left(z\right) \tag{5.21}$$

where $\mathcal{V}_{\alpha}^{x}\left(z\right)$ is a conformal field with dimension equal to 1 that depends only on the string coordinate x^{μ}.

5.2 T Duality

The compactification of a dimension in string theory is characterized by the appearance of new interesting phenomena with respect to those already present in field theory. In fact, in the case of a closed string, together with the Kaluza-Klein (K-K) excitations, a new kind of states called winding states appear in the spectrum. It turns out that the bosonic closed string theory is invariant under the exchange of the winding modes with the K-K modes according to a transformation that is called T-duality. In the supersymmetric case, instead, this transformation is in general not a symmetry anymore but brings from a certain string theory to another string theory. For instance T-duality along a certain direction acts interchanging the IIA with the IIB theory. In the case of an open string, instead, this analysis naturally leads to the existence of other objects called Dp-branes.

5.2.1 Equations of motion & momentum

The most general solution of the eqs. of motion for the bosonic closed string can be written as:

$$X^{\mu}(\tau,\sigma) = q^{\mu} + \sqrt{2\alpha'}\,(\alpha_0^{\mu} + \tilde{\alpha}_0^{\mu})\,\tau - \sqrt{2\alpha'}\,(\alpha_0^{\mu} - \tilde{\alpha}_0^{\mu})\,\sigma +$$

$$+ i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left(\frac{\alpha_n^{\mu}}{n} e^{-2in(\tau-\sigma)} + \frac{\tilde{\alpha}_n^{\mu}}{n} e^{-2in(\tau+\sigma)} \right) \quad (5.22)$$

where the momentum of the string is given by

$$p^{\mu} = \frac{1}{\sqrt{2\alpha'}}\,(\alpha_0^{\mu} + \tilde{a}_o^{\mu}) \quad (5.23)$$

In the uncompactified case the two zero modes must be identified because the string coordinate must be invariant under $\sigma \to \sigma + \pi$ and the expression for the momentum becomes:

$$p^{\mu} = \sqrt{\frac{2}{\alpha'}}\alpha_0^{\mu} \quad (5.24)$$

5.2.2 Compactification of a space dimension

Let us compactify one of the space dimensions along a circle with radius R. For the coordinate X, corresponding to that dimension we have:

$$X \sim X + 2\pi R \quad (5.25)$$

The conjugate momentum corresponding to the compactified direction must be quantized as:

$$p = \frac{n}{R} \quad ; \quad n \in Z \quad (5.26)$$

Also, we have

$$\pi\sqrt{2\alpha'}\,(a_0 - \tilde{a_0}) = 2\pi\omega R \quad ; \quad \omega \in Z \quad (5.27)$$

where ω corresponds to the number of times the close string winds around the compact direction. Equations (5.24),(5.26) and (5.27) imply the zero modes for the compact direction must have the following expression

$$\alpha_0 = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} + \frac{\omega R}{\alpha'} \right) \quad and \quad \tilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} - \frac{\omega R}{\alpha'} \right) \quad (5.28)$$

For the Virassoro operator L_0, including the contribution from the uncompactified directions, we get:

$$L_0 = \frac{\alpha'}{4}\hat{p}^2 + \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}.\alpha_n = \frac{\alpha'}{4}\hat{p}^2 + \frac{\alpha'}{4}\left(\frac{n}{R} + \frac{\omega R}{\alpha'}\right)^2 + \sum_{n=1}^{\infty} \alpha_{-n}.\alpha_n \quad (5.29)$$

and

$$\tilde{L_0} = \frac{\alpha'}{4}\hat{p}^2 + \frac{\alpha'}{4}\left(\frac{n}{R} - \frac{\omega R}{\alpha'}\right)^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}.\tilde{\alpha_n} \quad (5.30)$$

5.2.3 Mass operator & winding modes

The mass operator becomes

$$M^2 = \frac{2}{\alpha'}\left[\sum_{n=1}^{\infty} \alpha_{-n}.\alpha_n + \tilde{\alpha}_{-n}.\tilde{\alpha_n} - 2\right] + \left(\frac{n}{R}\right)^2 + \left(\frac{\omega R}{\alpha'}\right)^2 \quad (5.31)$$

The last term corresponds to a new kind of excitations that are called *winding modes* because they can be thought of as generated by the winding of the closed string around the compact direction.

From (5.31) we see that the spectrum of the theory is invariant under the exchange of KK modes with winding modes together with an inversion of the radius of compactification:

$$\omega \leftrightarrow n \quad ; \quad R \leftrightarrow \hat{R} \equiv \frac{\alpha'}{R} \quad (5.32)$$

This is called a *T-Duality* transformation and $\hat{R}$ is the compactification radius of the T-dual theory. **Correlators and the partition function are invariant under T-duality**. This means that T-duality is a symmetry of the bosonic closed string theory. As a consequence of this invariance, whenever we have to consider compactified theories, we can limit ourselves to the case $R \geq \sqrt{\alpha'}$. That is the reason why $\sqrt{\alpha'}$ is often called the *minimal length* of the string theory.

5.2.4 Action of T-duality on the string coordinate

Writing:

$$X = \frac{1}{2}\left(X_- + X_+\right) \quad (5.33)$$

where

$$X_- = q + 2\sqrt{2\alpha'}\,(\tau - \sigma)\,\alpha_0 + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-2in(\tau-\sigma)} \tag{5.34}$$

and

$$X_- = q + 2\sqrt{2\alpha'}\,(\tau + \sigma)\,\tilde{\alpha}_0 + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\tilde{\alpha}_n}{n}e^{-2in(\tau+\sigma)} \tag{5.35}$$

it is evident that the T-dual coordinate must satisfy the conditions:

$$\partial_\tau X \to \partial\tau\hat{X} = -\partial_\sigma X \quad ; \quad \partial_\sigma X \to \partial_\sigma\hat{X} = -\partial_\tau X \tag{5.36}$$

They are satisfied if the T-dual coordinate is equal to

$$\hat{X} = \frac{1}{2}\,(X_- - X_+) \tag{5.37}$$

Therefore the T-duality transformation acts on the right sector as a parity operator changing sign of the right moving coordinate and leaving unchanged the left moving one.

5.2.5 T-duality in open string theory

In an open string theory the string coordinate does not satisfy any periodicity requirement on σ. This implies that in its compactified version there are only K-K modes, while the winding modes are absent. This could suggest that T-duality is not a symmetry of the open string theory. Such a conclusion, however, leads to some problem when we remember that theories with open strings also contain closed strings. Let us consider a theory with open and closed strings with $d-p-1$ directions compactified on circles with radii R^l and take the limit

$$R^l \to 0 \quad \forall \quad \text{compact direction} \tag{5.38}$$

In this limit the open string theory loses effectively $d-p-1$ directions. Therefore in this limit the open string will appear to only be living in a $p+1$-dimensional subspace of the entire d-dimensional target space. We can conclude that But in we would end up with a theory in which open strings live in a p+1-dimensional subspace of the entire space-time, while closed strings live in the entire d-dimensional target space. This mismatch can be solved by requiring that, in the T-dual picture, open string still can oscillate in d dimensions, while their endpoints are fixed on a $p+1$-dimensional hyperplane that we call Dp-brane. Open strings with their

endpoints fixed on these hyperplanes satisfy Dirichlet boundary conditions in the $d-p-1$ transverse directions. We can extend the definition of a T-dual coordinate to the open string case. We obtain:

$$\hat{X}^l = \frac{1}{2}\left[X_-^l - X_+^l\right] \tag{5.39}$$

where now the left and right movers contain the same set of oscillators:

$$X_-^l = q^l + c^l + \sqrt{2\alpha'}\,(\tau-\sigma)\,\alpha_-^l + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^l}{n}e^{-in(\tau-\sigma)} \tag{5.40}$$

and

$$X_+^l = q^l + c^l + \sqrt{2\alpha'}\,(\tau+\sigma)\,\alpha_-^l + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^l}{n}e^{-in(\tau+\sigma)} \tag{5.41}$$

One can immediately see that T-duality has transformed a string coordinate satisfying Neumann boundary conditions into a T-dual one satisfying Dirichlet boundary conditions.

5.2.6 Dp-branes

The fact that open strings satisfy Dirichlet boundary conditions implies the existence in the theory of objects, called the *Dp*-branes, that are characterized by the fact that open strings have their endpoints attached to them. It can be easily shown that:

$$\hat{X}^l(\pi) \sim \hat{X}^l(0) \tag{5.42}$$

That means that in the T-dual theory the two endpoints of the string are attached to the same D-brane. Up to now we have treated a *Dp*-brane as a pure geometrical hyperplane to which open strings are attached and we have completely disregarded the excitations of the attached open strings. But we will see that, as soon as we let them come into play, they provide dynamical degrees of freedom to the *Dp*-brane. Among all possible excitations of an open string the massless ones have the peculiarity of not changing the energy of the *Dp*-brane to which the open string is attached. Therefore from the brane point of view they can be interpreted as collective coordinates of the brane. For the treatment of compactification in the presence of Chan-Paton factors, see [53].

The fundamental observation made by Polchinski has been to identify the Dp-branes required by T-duality with the p-branes obtained as classical solutions of

the low-energy string effective action. Therefore, on the one hand the p-branes are new non-perturbative states of string theory and on the other hand have the important property that open string have their endpoints attached on them. The latter property will allow one to compute their interactions and more in general to study their properties by computing open string one-loop diagrams. On the other hand we should not be worried that the Dirichlet boundary conditions break Poincaré invariance because this happens in presence of any kind of solitonic state. In the next chapter we will introduce the boundary state and we will show that it provides a stringy description of these new states.

5.3 Bosonic Boundary State

For now, we treat Dp-branes as static and rigid objects. The open string with the endpoint at $\sigma = 0$ attached to a Dp-brane satisfies the usual Neumann boundary conditions along the directions longitudinal to the world volume of the brane

$$\partial_\sigma X^\alpha|_{\sigma=0} = 0 \qquad \alpha = 0, 1, \cdots, p \tag{5.43}$$

and Dirichlet boundary conditions along the directions transverse to the brane

$$X^i|_{\sigma=0} = y^i \qquad i = p+1, \cdots, d-1 \tag{5.44}$$

where y^i are the coordinates of the brane and d is the dimension of the Minkowski spacetime, that in the case of the bosonic string is equal to $d = 26$.

The interaction between two Dp-branes is given by the vacuum fluctuation of an open string that is stretching between them. This means that their interaction is simply given by the one-loop open string "free-energy" which is usually represented by the annulus or equivalently by the cylinder diagram. From either of those two diagrams it is easy to see that by exchanging the variables σ and τ the one-loop open string amplitude can also be viewed as a tree diagram of a closed string created from the vacuum, propagating for a while and then annihilating again into the vacuum. These two equivalent descriptions of the same diagram are called respectively the 'open-channel' and the 'closed-channel'. We want to stress that the physical content of the two descriptions is a priori completely different. In the first case we describe the interaction between two Dp-branes as a one-loop amplitude of open strings, which is the amplitude of a quantum theory of open strings, while in the second case we describe the same interaction as a tree-level amplitude of closed

strings, which is instead a classical amplitude in a theory of closed strings. The fact that these two descriptions are equivalent is a consequence of the conformal symmetry of string theory that allows one to connect the two a priori different descriptions.

5.3.1 Transformation from the open channel to the closed channel

To show that, let us consider a one-loop diagram with an open string circulating in it and stretching between two parallel Dp-branes with coordinates respectively given by $(y^{p+1}, \cdots, y^{d-1})$ and $(w^{p+1}, \cdots, w^{d-1})$. The open string satisfies the boundary conditions in (5.43) both at $\sigma = 0$ and $\sigma = \pi$ along the world-volume directions of the brane, while along the transverse directions satisfies the following equations

$$X^i|_{\sigma=0} = y^i \quad X^i|_{\sigma=\pi} = \omega^i \quad i = p+1, \cdots, d-1 \quad , \tag{5.45}$$

where we take σ and τ in the two intervals $\sigma \in [0, \pi]$ and $\tau \in [0, T]$.

The goal is to find a conformal transformation acting on the open string boundary conditions in order to transform them into the boundary conditions for a closed string propagating between two Dp-branes. In terms of the complex coordinate $\zeta \equiv \sigma + i\tau$ a conformal transformation simply transforms $\zeta \to f(\zeta)$, where $f(\zeta)$ is an arbitrary holomorphic function of ζ. Lets consider the following transformation:

$$\zeta \to -i\zeta \tag{5.46}$$

which in terms of (σ, τ) has the form:

$$(\sigma, \tau) \to (\tau, \sigma) \tag{5.47}$$

In order to have the closed string variables σ and τ vary in the intervals $\sigma \in [0, \pi]$ and $\tau \in [0, \hat{T}]$, corresponding to a closed string propagating between the two D branes, one must also perform the following conformal rescaling:

$$\sigma \to \frac{\pi}{T}\sigma \quad \tau \to \frac{\pi}{T}\tau \tag{5.48}$$

where $\hat{T} = \pi^2/T$.

5.3.2 Boundary states

Thus we have constructed a conformal transformation that brings us from the open string channel to the closed string channel. In the closed string channel we need to construct the two **boundary states**, $|B_x\rangle$, that describe the Dp-branes at $\tau = 0$ and $\tau = \hat{T}$ respectively. The equations that characterize these states are obtained by applying the conformal transformation previously constructed to the boundary conditions for the open string. At $\tau = 0$ we get the following conditions:

$$\partial_\tau X^\alpha|_{\tau=0} |B_X\rangle = 0 \quad \alpha = 0, \cdots, p \tag{5.49}$$

$$X^i|_{\tau=0} |B_X\rangle = y^i \quad i = p+1, \cdots, d-1 \tag{5.50}$$

Analogous equations can be obtained for the D_p-brane at $\tau = \hat{T}$.

The previous equations can be expressed using the closed string oscillators, obtaining:

$$\left(\alpha_n^\alpha + \tilde{\alpha}_{-n}^\alpha\right) |B_X\rangle = 0 \quad ; \quad \left(\alpha_n^i - \tilde{\alpha}_{-n}^i\right) |B_X\rangle = 0 \quad \forall n \neq 0 \tag{5.51}$$

$$\hat{p}^\alpha |B_X\rangle = 0 \quad \left(\hat{q}^i - y^i\right) |B_X\rangle = 0. \tag{5.52}$$

Introducing the matrix

$$S^{\mu\nu} = \left(\eta^{\alpha\beta}, -\delta^{ij}\right) \tag{5.53}$$

the equations for the non-zero modes can be rewritten as

$$\left(\alpha_n^\mu + S_\nu^\mu \tilde{\alpha}_{-n}^\nu\right) |B_X\rangle = 0 \quad \forall n \neq 0 \tag{5.54}$$

The state satisfying the previous condition can be determined to be:

$$|B_X\rangle = N_p \delta^{d-p-1} \left(\hat{q}^i - y^i\right) \left(\prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_{-n} S . \tilde{\alpha}_{-n}}\right) |0\rangle_\alpha |0\rangle_{\tilde{\alpha}} |p=0\rangle , \tag{5.55}$$

where N_p is an yet-to-be fixed normalization constant. We can find this constant by computing the interaction between the two parallel Dp-branes both in the open and the close string channel and comparing the results. We find that

$$N_p = \frac{T_p}{2}, \quad T_p = \frac{\sqrt{\pi}}{2^{\frac{d-10}{4}}} \left(2\pi\sqrt{\alpha'}\right)^{\frac{d}{2}-2-p} \tag{5.56}$$

5.3.3 Ghost contributions to the boundary state

A BRST invariant boundary state is in fact the **product** of the boundary state $|B_X\rangle$ for the bosonic coordinate, that was constructed in the previous section, and of $|B_{gh}\rangle$, the ghost contribution:

$$|B\rangle = |B_X\rangle\,|B_{gh}\rangle \tag{5.57}$$

BRST invariance requires that the total boundary state satisfies the equation

$$\left(Q + \tilde{Q}\right)|B\rangle = 0, \tag{5.58}$$

where the BRST charge is equal to

$$Q = \sum_n c_n L_n^X + \sum_{n=-1}^{\infty} c_{-n} L_n^{gh} + \sum_{n=2}^{\infty} L_{-n}^{gh} c_n \tag{5.59}$$

The overlap conditions (5.51) and (5.52) imply that the boundary state for the bosonic coordinate satisfies the following eqs:

$$\left(L_m^X - \tilde{L}_{-m}^X\right)|B_X\rangle = 0 \tag{5.60}$$

Using the expressions for Q and $\tilde{Q}$ and using (5.60) we can see that (5.58) implies the following overlap conditions for the ghost boundary state

$$(c_n + \tilde{c}_{-n})\,|B_{gh}\rangle = 0 \quad ; \quad \left(b_n - \tilde{b}_{-n}\right)|B_{gh}\rangle = 0. \tag{5.61}$$

Also

$$\left(L_m^{gh} - \tilde{L}_{-m}^{gh}\right)|B_{gh}\rangle = 0 \tag{5.62}$$

Equations (5.61) are satisfied by the state

$$|B_{gh}\rangle = e^{\sum_{n=1}^{\infty}\left(c_{-n}\tilde{b}_{-n} - b_{-n}\tilde{c}_{-n}\right)}\left(\frac{c_0 + \tilde{c}_0}{2}\right)|q=1\rangle\,|\tilde{q=1}\rangle \tag{5.63}$$

where $|q=1\rangle$ is the state, annihilated by the following oscillators

$$c_n\,|q=1\rangle = 0 \quad \forall n \geq 1 \quad ; \quad b_m\,|q=1\rangle = 0 \quad \forall m \geq 0 \tag{5.64}$$

5.4 Dynamic D Branes

Until now, we've considered Dp-branes as static and rigid objects to which open strings are attached. We have not discussed the fact that they can be boosted, rotated and the excitations of the attached open strings provide dynamical degrees of freedom to them.[54] In particular the massless excitations that have the property of not changing the energy of the brane, can be interpreted as collective coordinates of the Dp-branes. In this sections, boosted and rotated boundary states are constructed. Also we will construct a boundary state that contains a constant abelian gauge field, living in the world volume of the brane. We will also show that some of those states are related to T-duality.

5.4.1 Boosted and Rotated Boundary State

In the previous chapters we have considered the boundary state, corresponding to a static Dp-brane. We want to extend this notion to a boosted and a rotated boundary state. It is easy to see that a boost along a longitudinal direction of a brane does not modify the boundary state from the previous chapter as expected from Poincaré invariance of the classical solution along the longitudinal directions of the brane. Therefore, it is sufficient to concentrate on a boost along one the transverse directions. Let us call that direction k. The way to construct the boundary state is to once again start from the boundary conditions for an open string attached to such a Dp-brane and then translate them into the language of the closed string channel via a conformal transformation and a conformal rescaling.

5.4.2 Boosted Boundary State

The boundary conditions for an open string attached to a Dp-brane boosted with velocity v in the direction of k are[10]

$$\partial_\sigma X^\alpha|_{\sigma=0} = 0 \qquad \alpha = 1, \cdots, p \tag{5.65}$$

$$\partial_\sigma \left(X^0 + vX^k\right)|_{\sigma=0} = 0 \tag{5.66}$$

$$X^i|_{\sigma=0} = y^i, \qquad i = p+1, \cdots, D-1, \quad and \quad i \neq k \tag{5.67}$$

$$\left(\frac{X^k + vX^0}{\sqrt{1-v^2}}\right)|_{\sigma=0} = \frac{y^k}{\sqrt{1-v^2}} \tag{5.68}$$

where $\vec{y}$ is a vector belonging to the space transverse to the Dp-brane and therefore has zero component along the time and the other world volume directions of the Dp-brane. Translating to the **closed channel**, these conditions define the equations, characterizing the boosted boundary state $|B, v, y\rangle$ in the bosonic string:

$$\partial_\tau X^\alpha|_{\tau=0}\,|B,v,y\rangle = 0 \qquad \alpha = 1,\cdots,p \tag{5.69}$$

$$\partial_\tau \left(X^0 + vX^k\right)|_{\tau=0}\,|B,v,y\rangle = 0 \tag{5.70}$$

$$\left(X^i - y^i\right)|_{\sigma=0}\,|B,v,y\rangle = 0, \qquad i = p+1,\cdots,D-1, \quad \textit{and} \quad i \neq k \tag{5.71}$$

$$\left(\frac{\left(X^k - y^k\right) + vX^0}{\sqrt{1-v^2}}\right)|_{\sigma=0}\,|B,v,y\rangle = 0 \tag{5.72}$$

the last equation can be exchanged for a less restrictive one

$$\partial_\sigma \left(X^k + vX^0\right)|_{\tau=0}\,|B,v\rangle = 0, \tag{5.73}$$

which corresponds to the case of *a brane, delocalized in the k direction.*

The only overlap conditions that differ from those of the static case are those in the directions of the boost, namely the time and the k directions and they are equal to

$$\left(\hat{p}^0 + v\hat{p}^k\right)|B,v,y\rangle = 0 \tag{5.74}$$

$$\left[\left(\alpha_n^0 + \tilde{\alpha}_{-n}^0\right) + v\left(\alpha_n^k + \tilde{\alpha}_{-n}^k\right)\right]|B,v,y\rangle = 0 \quad \forall n \neq 0 \tag{5.75}$$

$$\frac{\hat{q}^k + vq^0}{\sqrt{1-v^2}}\,|B,v,y\rangle = \frac{y^k}{\sqrt{1-v^2}}\,|B,v,y\rangle \tag{5.76}$$

$$\left[\left(\alpha_n^k\tilde{\alpha}_{-n}^k\right) + v\left(\alpha_n^0 + \tilde{\alpha}_{-n}^0\right)\right]|B,v,y\rangle = 0 \quad \forall n \neq 0 \tag{5.77}$$

From these conditions we can now determine the explicit expression for the boosted boundary state $|B, v, y\rangle$. For the zero mode part, (5.76) tells us that the boundary state must contain a delta function of the type

$$\delta\left(\frac{q^k + vq^0 - y^k}{\sqrt{1-v^2}}\right) = \sqrt{1-v^2}\,\delta\left(q^k + vq^0 - y^k\right) \tag{5.78}$$

Because the operator that acts on the boundary state in (5.74) commutes with the δ-function in (5.78), in order to satisfy (5.74) it is sufficient to write the zero mode part as follows:

$$\sqrt{1-v^2}\delta\left(q^k + vq^0 - y^k\right)|p=0\rangle \tag{5.79}$$

It is straightforward to check that it satisfies both zero mode eqs. (5.74) and (5.76). Now we must consider the non-zero modes part of the overlap conditions. It can be seen that in order to satisfy both (5.75) and (5.77), the non-zero mode part of the boundary state must have the following structure

$$\prod_{n=1}^{\infty}\left(e^{-\frac{1}{n}\alpha_{-n}\cdot M(v)\cdot\tilde{\alpha}_{-n}}\right)|0\rangle_{\alpha}|0\rangle_{\tilde{\alpha}} \tag{5.80}$$

where the matrix M is obtained from the matrix S in (5.53) by substituting its elements $(S_{00}, S_{0k}, S_{k0}, S_{kk})$ with the correspondent ones

$$M_{00} = M_{kk} = -\frac{1+v^2}{1-v^2} \quad ; \quad M_{0k} = M_{k0} = -\frac{2v}{1-v^2} \tag{5.81}$$

Putting eqs. (5.79) and (5.79) together we get the final expression for the boosted boundary state:

$$\boxed{|B,v,y\rangle = \frac{T_p}{2}\prod_{i=p+1,i\neq j}^{d-1}\left[\delta\left(\hat{q}^i - y^i\right)\right]\sqrt{1-v^2}\delta\left(q^k + vq^0 - y^k\right)e^{-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\cdot M\cdot\tilde{\alpha}_{-n}}|0\rangle_{\alpha}|0\rangle_{\tilde{\alpha}}} \tag{5.82}$$

Although we have fixed the normalization factor to be $T_p/2$ as in the static case, the overlap conditions do not allow us to fix it uniquely. In general, the boundary state (5.82) could include an arbitrary function $N(v)$ of the physical velocity of the brane, that can only be determined by requiring agreement between the calculation of the interaction between two D-branes in the closed and open string channels. In this case, however, we have an independent way of uniquely fixing its normalization by applying to the static boundary state $|B_X\rangle$ in (5.55) an operator that performs a boost along the direction k transverses to the world volume of the D-brane

$$|B,y,\omega\rangle = e^{i\omega^k J^{0}k}\left|B,^{(\omega)}\, y\right\rangle, \tag{5.83}$$

where ω is related to the physical velocity via the relation

$$v = \tanh\omega \tag{5.84}$$

and ${}^{(\omega)}y = y^k \cosh\omega$ is the boosted position of the D-brane and the generator of the Lorentz transformation is equal to

$$J^{\mu\nu} = q^\mu p^\nu - q^\nu p^\nu - i\sum_{n=1}^{\infty}\left(a_n^{\dagger\mu}a_n^\nu - a_n^{\dagger\nu}a_n^\mu + \tilde{a}_n^{\dagger\mu}\tilde{a}_n^\nu - \tilde{a}_n^{\dagger\nu}\tilde{a}_n^\mu\right) \tag{5.85}$$

with $a_n\sqrt{n} = \alpha_n$ and $a_n^\dagger\sqrt{n} = \alpha_{-n}$ with $n > 0$. After some algebra it can be seen that the boosted boundary state can in equation (5.83) can be written in the following form

$$|B, y, \omega(v)\rangle = \frac{T_p}{2}\prod_{i=p+1, i\neq k}^{d-1}[\delta(\hat{q}^i - y^i)]\frac{1}{\cosh\omega}\delta(q^k + \tanh\omega q^0 - y^k)\times$$
$$\times\, e^{-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\cdot M(\omega)\cdot\tilde{\alpha}_{-n}}\left|0\right\rangle_\alpha\left|0\right\rangle_{\tilde{\alpha}}\left|p=0\right\rangle, \tag{5.86}$$

which exactly coincides with the previous expression, meaning that the overall normalization that we chose for the boosted state is correct.

5.4.2.1 Rotated Boundary State

The previous construction can easily be generalized for the case of a rotated D-brane. Obviously, the configuration of a Dp-brane embedded in a d-dimensional space-time is invariant under rotations in the longitudinal space as well as in the transverse space. This implies that the boundary state is invariant under rotations in the planes (α, β) or $(i, j), \forall\alpha, \beta \in 0, \cdots, p$ and $\forall i, j \in p+1, \cdots, d-1$. This means that in order to get a new boundary state, we must consider a Dp-brane which is rotated with an angle ω in one of the planes specified by the directions (α, k).

The open string attached to this brane at $\sigma = 0$ satisfies the boundary conditions

$$\partial_\sigma X^\beta|_{\sigma=0} = 0 \quad \forall\beta \in \{0, \cdots, p\}, \beta \neq \alpha \tag{5.87}$$

$$\partial_\sigma(X^\alpha\cos\omega + X^k\sin\omega)|_{\sigma=0} = 0 \tag{5.88}$$

$$X^i|_{\sigma=0} = y^i i = p+1, \cdots, D-1, i \neq k \tag{5.89}$$

$$(X^k\cos\omega - X^\alpha\sin\omega - y^k\cos\omega)|_{\sigma=0} = 0 \tag{5.90}$$

The the rotated boundary state in the directions different from (α, k) must satisfy the same overlap conditions as the unrotated case. On the other hand, in the directions (α, k) we must impose the following conditions

$$\partial_\tau \left(X^\alpha \cos\omega + X^k \sin\omega \right) |_{\tau=0} \, |B, \omega, y\rangle - 0 \tag{5.91}$$

and

$$\left(X^k \cos\omega - X^\alpha \sin\omega - y^k \cos\omega \right) |_{\tau=0} \, |B, \omega, y\rangle = 0 \tag{5.92}$$

In terms of oscillators, the overlap conditions become:

$$\left(\hat{p}^\alpha \cos\omega + \hat{p}^k \sin\omega \right) |B, \omega, y\rangle = 0 \tag{5.93}$$

$$\left[\left(\alpha_n^\alpha + \tilde{\alpha}_{-n}^\alpha \right) + \tan\omega \left(\alpha_n^k + \tilde{\alpha}_{-n}^k \right) \right] |B, \omega, y\rangle = 0 \quad \forall n \neq 0 \tag{5.94}$$

$$\left(\hat{q}^k \cos\omega - \hat{q}^\alpha \sin\omega \right) |B, \omega, y\rangle = y^k \cos\omega \, |B, \omega, y\rangle \tag{5.95}$$

$$\left[\left(\alpha_n^k - \tilde{\alpha}^k{-n} \right) - \tan\omega \left(\alpha_n^\omega - \tilde{\alpha}_{-n}^\omega \right) \right] |B, \omega, y\rangle = 0 \quad \forall n \neq 0 \tag{5.96}$$

From here, it can be shown that the rotated boundary state has the form

$$|B, y, \omega(v)\rangle = \frac{T_p}{2} \prod_{i=p+1, i\neq k}^{d-1} [\delta(\hat{q}^i - y^i)] \delta(q^k + \sin\omega q^\alpha - y^k \cos\omega) \times \\ \times \, e^{-\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot M(\omega) \cdot \tilde{\alpha}_{-n}} \, |0\rangle_\alpha \, |0\rangle_{\tilde{\alpha}} \, |p = 0\rangle \,, \tag{5.97}$$

where in this case the matrix M can be obtained from the matrix S, substituting its elements in the following way:

$$M_{\alpha\alpha} = -M_{kk} = \cos 2\omega \quad M_{0k} = M_{k0} = \sin 2\omega \tag{5.98}$$

Once again, the boundary state for a rotated D-brane can be obtained by acting on the static boundary state in (5.55) with the rotation operator

$$|B, y, \omega\rangle = e^{i\omega^k J^\alpha k} \left| B, ^{(\omega)} y \right\rangle , \tag{5.99}$$

where $J^{\mu\nu}$ is the previously defined quantity and ${}^{(\omega)}y^k = y^k \cos\omega$ is the rotated position of the D-brane. The previous considerations can be easily extended to the fermionic coordinate obtaining the boosted boundary state in the case of superstring.

5.4.2.2 Compactified boundary state

In this section we explore the case of the boundary state describing a Dp-brane when all directions are compactified on circles. From it, by decompactifying some direction we can obtain the boundary state in which only some directions are compactified. For simplicity, we take all radii to be equal to R, but from the formulas that we will get, it will be trivial to extend the results to arbitrary radii. Firstly, it is convenient to introduce notation, such that we can distinguish between position and momentum operators for the momentum and winding degrees of freedom. We will require them to satisfy the following commutation relations:

$$[q_\omega^\mu, p_\omega^\nu] = i\eta^{\mu\nu} \quad ; \quad [q_n^\mu, p_n^\nu] = i\eta^{\mu\nu} \tag{5.100}$$

Here the subscripts n and ω omega correspond to the momentum and winding degrees of freedom respectively and all other commutators vanish. By denoting with $|n^\mu, \omega^\nu\rangle$ an eigenstate of the two "momentum" operators

$$p_n^\rho |n, \omega\rangle = \frac{n^\rho}{R} |n, \omega\rangle \quad ; \quad p_\omega^\rho |n, \omega\rangle = \frac{\omega^\rho R}{\alpha'} |n, \omega\rangle \tag{5.101}$$

it is easy to convince oneself that the previous state can also be written as follows

$$|n, \omega\rangle = e^{iq_n \cdot n/R} e^{iq_w \cdot wR/\alpha'} |0, 0\rangle \tag{5.102}$$

where $|0, 0\rangle$ is the state with zero momentum and winding number. The state in eq.(5.102) can be normalized as:

$$\langle n, \omega | n', \omega' \rangle = \Phi \delta_{nn'} \delta_{\omega\omega'} \tag{5.103}$$

where Φ is the "self dual" volume that has the following properties:

$$\Phi = 2\pi R \;\; if \;\; R \to \infty \quad ; \quad \Phi = \frac{2\pi\alpha'}{R} \;\; if \;\; R \to 0 \tag{5.104}$$

Let us use this formalism to write the boundary state in the compactifed case. In this case the part corresponding to the non-zero modes is unchanged, while the one corresponding to the zero modes of the bosonic coordinate becomes [24]

$$|\Omega\rangle = \mathcal{N}_p \prod_{\alpha=0}^{p} \left[\sum_{\omega^\alpha} e^{i(q_\omega^\alpha - y^\alpha)\omega_\alpha R/\alpha'} \right] \prod_{i=P+1}^{d-1} \left[\sum_{n^i} e^{i(q_n^i - y^i)n_i/R} \right] |n = 0, \omega = 0\rangle \tag{5.105}$$

where the parameters y^α y^i correspond respectively to Wilson lines[1] turned on along the world volume of the brane to the position of the brane in the transverse directions.

The previous boundary state satisfies the overlap conditions:

$$\left(e^{i(R/\alpha')q_\omega^\alpha} - e^{i(R/\alpha')y^\alpha}\right)|\Omega\rangle = p_n^\alpha |\Omega\rangle = 0, \;\; \alpha = 0, \cdots, p \tag{5.106}$$

and

$$\left(e^{iq_n^i/R} - e^{iy^i/R}\right)|\Omega\rangle = p_\omega^i |\Omega\rangle = 0, \;\; i = p+1, \cdots, 9-p \tag{5.107}$$

The overall normalization can be determined by comparing the calculation of the brane interaction done in the closed and open string channels. In this way one gets the following relation [24]:

$$\mathcal{N}_p^2 \frac{\alpha'}{4}\Phi^d = \frac{VC_1}{2\pi} \tag{5.108}$$

where

$$V = (2\pi R)^{p+1}\left(\frac{2\pi\alpha'}{R}\right)^{d-p-1} \quad ; \quad C_1 = (2\pi)^{-d}(2\alpha')^{-d/2} \tag{5.109}$$

From eq.(5.108) we get

$$\mathcal{N}_p = \sqrt{\frac{2VC_1}{\pi\alpha'(2\pi R)^d}}(2\pi R)^{d-p-1}\left[\left(\frac{2\pi R}{\Phi}\right)^{d/2}(2\pi R)^{p+1-d}\right] \tag{5.110}$$

After some calculation it can be seen that the part outside of the square brackets is just the normalization of the boundary state in the uncompactifed case, therefore we get:

$$\mathcal{N}_p = \frac{T_p}{2}\left[\left(\frac{2\pi R}{\Phi}\right)^{d/2}(2\pi R)^{p+1-d}\right] \tag{5.111}$$

where T_p was defined back in (5.56). Now we can show that the factor $\mathcal{N}_p$ reduces to $T_p/2$ in the decompactified limit. In the decompactified limit, meaning $R \to \infty$ it can easily be checked that the following relations hold:

$$e^{i(q_\omega^\alpha - y^\alpha)\omega_\alpha(R/\alpha')}|0,0\rangle \to |0,0\rangle \tag{5.112}$$

[1]In gauge theory, a Wilson line, or Wilson loop is a gauge-invariant observable obtained from the holonomy of the gauge connection around a given loop. We will not be investigating this object any further as it is outside of the scope of the thesis.

and

$$e^{i(q_n^i - y^i)n_i/R} \left|0,0\right\rangle \to R \int dk e^{i(q_n - y)k} \left|0,0\right\rangle = \tag{5.113}$$

$$= (2\pi R) \int \frac{dk}{2\pi} e^{i(q_n - y)k} \left|0,0\right\rangle = 2\pi R \delta(q-y) \left|0,0\right\rangle . \tag{5.114}$$

In the first relation we have taken into account that in the decompactification limit $R \to 0$ only the term with $\omega = 0$ survives and in the second relation we have substituted the summation with an integral by introducing $k = n/R$. Using the two previous relations and the first equation in eq. 5.104 we can see that the normalization factor indeed reduces to $T_p/2$ in the decompactified limit, which is the correct result. An interesting case arises when we decompactify only time and the transverse directions. The zero mode then becomes

$$\frac{T_p}{2} \left(\frac{2\pi R}{\Phi}\right)^{p/2} \prod_{\alpha=1} p \left[\sum_{\omega_\alpha} e^{i\theta^\alpha \omega^\alpha} \left|n^\alpha = 0, \omega^\alpha\right\rangle\right] \left|k^0 = 0\right\rangle \prod_{i=p+1}^{d-1} \left[\delta(q^y - y^i) \left|k^i = 0\right\rangle\right] \tag{5.115}$$

where $\theta = -yR/\alpha'$.

5.4.3 Interaction Between Branes

In this section, the interaction between a Dp brane located at y_1 and a Dp' brane located at y_2, is studied. We consider the case with $NN \equiv min\left\{p, p'; \right\} + 1$ directions common to the brane world volumes,
$DD \equiv min\left\{d-p-1, d-p'-1; \right\}$ directions transverse to both, and $\nu = (d - NN - DD)$ directions of mixed type. We will not consider instantonic D-branes, hence also $NN \geq 1$. The two D-branes simply interact via tree-level exchange of closed strings whose propagator is

$$D = \frac{\alpha'}{4\pi} \int \frac{d^2 z}{|z|^2} z^{L_0} \bar{z}^{\widetilde{L_0}} \tag{5.116}$$

so that indicating with $|B_1\rangle$ and $|B_2\rangle$ the boundary states describing the two D-branes the static amplitude is given by

$$A = \langle B_1 | D | B_2 \rangle = \frac{T_p T_{p'}}{4} \frac{\alpha'}{4\pi} \int_{|z|\leq 1} \frac{d^z}{|z|^2} \mathcal{A}\mathcal{A}^{(\prime)}, \tag{5.117}$$

where we have indicated with $\mathcal{A}$ and $\mathcal{A}^{(\prime)}$ respectively the non zero mode and the zero mode contribution in which the previous amplitude can be factorized. we do not have any intercept as we had for the bosonic string because we assume that both L_0 and $\tilde{L}_0$ contain the ghost degrees of freedom.

Chapter 6

More on T-duality and the TsT transformation

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Contents

In this chapter we will briefly delve into bosonic and fermionic T duality and the associated TsT transformation in the context of a string σ model.

6.1 The notion of Bosonic and Fermionic T-duality

Lets consider a generic (classical[1]) string σ model of the form

$$S \propto \int \mathrm{d}^2\sigma \partial_+ Z^M \mathcal{E}_{MN}(Z) \partial_- Z^N \equiv \int \mathrm{d}^2\sigma \mathcal{L}, \quad M, N = 1, \ldots, D, \tag{6.1}$$

[1]A dilaton ϕ enters the string action at a higher order in the coupling α'. At the classical level the dilaton has to be introduced in the corresponding supergravity (e.g. the RR-forms appear always as $e^{\phi} F_{\mu_1,\ldots,\mu_p}$). As we will not do explicit field redifinitions, we neglect it and its behaviour under T duality from the start. Working at the classical level we also disregard any prefactors of the action and are only interested in its schematical form.

where we work in conformal gauge for the sake of convenience, and understand Z^M as

$$Z^M = (X^\mu(\sigma), \theta^\Delta(\sigma)) \tag{6.2}$$

with some bosonic fields X^μ and some fermionic Grassmann-valued fields θ^Δ. We refer to the parity of the coordinate Z^M as $s(M)$. $\mathcal{E}_{MN}(Z)$ is the background field describing the coupling between the fields[2] with parity $s(\mathcal{E}_{MN}) = s(M) + s(N)$, so that $s(\mathcal{L} = 0)$.

Now we assume the model has a manifest isometry and choose the associated coordinate to be Z^1, meaning the symmetry is realised as a shift of Z^1. We write $Z^M = (Z^1, Z^{\underline{M}})$ with $\underline{M} = 2, \ldots, D$, so that $\mathcal{E}_{MN} \equiv \mathcal{E}_{MN}(Z^{\underline{M}})$. Z^1 can be either bosonic or fermionic[3]. This allows us to rewrite the Lagrangian by introducing gauge fields $A_\pm$:

$$\partial_\pm Z^1 \propto A_\pm, \quad \mathcal{L} \to \mathcal{L} - \bar{Z}^1(\partial_+ A_- \partial_- A_+), \tag{6.3}$$

where the Lagrange multiplier $\bar{Z}^1$ ensures $A_\pm = \partial_\pm Z^1$ by its equations of motion. Integrating out $A_\pm$ instead of $\bar{Z}^1$ yields the action of the dual model

$$\bar{S} \propto \int \mathrm{d}^2\sigma \partial_+ \bar{X}^M \bar{\mathcal{E}}_{MN} \partial_- \bar{X}^N, \tag{6.4}$$

[2] $\mathcal{E}_{MN}$ could be decomposed into its graded symmetric (metric like) and graded skewsymmetric (B-field like) part: $\mathcal{E}_{MN} = \mathcal{G}_{MN} + \mathcal{B}_{MN}$. But only the order θ^0 terms in $\mathcal{G}_{\mu\nu}$ and respectively $B_{\mu\nu}$ have direct physical interpretation as the components of the metric and the B field. We shall stick to the quite abstract "background" $\mathcal{E}_{MN}$ as it is practical and sufficient for our further considerations.

[3] In the fermionic case the generator Q dual to the isometry coordinate has to anticommute with itself in order to correspond to a shift isometry. In other words, fermionic T-duality requires a nullpotent supercharge $Q, Q^2 = 0$.

Chapter 7

Schwarzian Quantum Mechanics

Contents

7.1 Overview

In recent times it has been recognized that holographic conformal field theories at finite temperature exhibit characteristics of many-body quantum chaos. The Sachdev-Ye-Kitaev model[49] is a soluble many-body quantum system with a well-controlled large N limit that exhibits maximal chaos and other characteristics that indicate it has a holographic dual given by a 2D gravity theory on AdS_2.[36, 45] The Schwarzian theory describes the quantum dynamics of a single 1D degree of freedom $f(\tau)$ and forms the theoretical gateway between the microscopic SYK model and the 2D dilaton gravity theory.

The main object of study is the finite temperature correlation functions in the 1D QM theory described by the action[38]

$$S[f] = -C \int_0^\beta d\tau \left(\{f, \tau\} + \frac{2\pi^2}{\beta^2} f'^2 \right) \tag{7.1}$$

$$= -C \int_0^\beta d\tau \{F, \tau\}, \quad F \equiv \tan\left(\frac{\pi f(\tau)}{\beta} \right), \tag{7.2}$$

where C is the coupling constant of the zero-temperature theory. We will set $C = 1/2$ from here on out, unless explicitly stated. Here $f(\tau + \beta) = f(\tau) + \beta$ runs over the space Diff(S^1) of diffeomorphisms on the thermal circle and most importantly

$$\{f, \tau\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{7.3}$$

denotes the Schwarzian derivative.

It is easy to show that this action is invariant under $SL(2,\mathbb{R})$ Möbius transformations that act of F via

$$F \to \frac{aF + b}{cF + d} \tag{7.4}$$

This model possesses a corresponding set of conserved charges l_a that generate the $\mathfrak{sl}(2,\mathbb{R})$ algebra $[l_a, l_b] = i\epsilon_{abc}l_c$ and commute with the Hamiltonian H. It just happens to be that the Hamiltonian is equal to the $SL(2,\mathbb{R})$ Casimir operator, $H = \frac{1}{2}l_a l_a$. This means that the energy spectrum and dynamics are uniquely determined by the $SL(2,\mathbb{R})$ symmetry. The Schwarzian theory is integrable and expected to be exactly soluble at any value of the inverse temperature β. In the following, we will label the energy eigenvalues E in terms of the $SL(2,\mathbb{R})$ spin $j = -\frac{1}{2} + ik$ via

$$E(k) = -j(j+1) = \frac{1}{4} + k^2 \tag{7.5}$$

We will drop the constant $\frac{1}{4}$, that can be achieved by choosing appropriate normal ordering in the quantum theory. If we mod out by the overall $SL(2,\mathbb{R})$ symmetry, the partition sum

$$Z(\beta) = \int_{\mathcal{M}} \mathcal{D}f e^{-S[f]} \tag{7.6}$$

reduces to an integral over the infinite dimensional quotient space

$$\mathcal{M} = Diff(S^1)/SL(2,\mathbb{R}) \tag{7.7}$$

dadsadsadasasdad This space M equals the coadjoint orbit of the identity element $\mathbb{1} \in$ Diff(S^1) which is known to be a symplectic manifold that upon quantization gives rise to the identity representation of the Virasoro group Diff(S^1), i.e. the identity module of the Virasoro algebra. We chose the functional measure $d\mu(f)$ to be the one derived form the symplectic form on $\mathcal{M}$ which is shown in [5][11] to

take the form $\mathcal{D}f = \prod_\tau df/f'$. The fact that the space $\mathcal{M}$ is symplectic is exploited in [51] to show that the partition function Z is one-loop exact and given by

$$Z(\beta) = \left(\frac{\pi}{\beta}\right)^{3/2} e^{\pi^2/\beta} = \int_0^\infty d\mu(k) e^{-\beta E(k)} \tag{7.8}$$

with $E(k)$ as in (7.5) and where the integration measure is given in terms of k by

$$d\mu(k) = dk^2 \sinh(2\pi k). \tag{7.9}$$

This exact result for the spectral density

$$\rho(E) = \sinh\Big(2\pi\sqrt{E}\Big) \tag{7.10}$$

if further indication that the Schwarzian theory is completely soluble.

For our analysis we will make use of the more detailed property that the space $\mathcal{M}$ in (7.7) is not just any phase space, but forms the quantizable coadjoint orbit space that gives rise to the identity module of the Virasoro algebra. This observation implies that the correlation functions of the Schwarzian theory,

$$\langle \mathcal{O}_1, \cdots, \mathcal{O}_n \rangle = \frac{1}{Z} \int_{\mathcal{M}} \mathcal{D}f e^{-S[f]} \mathcal{O}_1 \cdots \mathcal{O}_n = \frac{1}{Z} \operatorname{Tr}\big(e^{-\beta H} \mathcal{O}_1 \cdots \mathcal{O}_n\big), \tag{7.11}$$

can be obtained by taking a suitable large c limit of well-studied correlation functions of an exactly soluble 2D CFT with Virasoro symmetry.

7.2 Schrödinger formulation

In this section, we outline the Hamiltonian formulation of the Schwarzian theory, and how it is relate to other 1D systems with $SL(2,\mathbb{R})$ symmetry. We temporary set $\beta = 2\pi$.

7.2.1 Zero Temperature

We first consider the Schwarzian theory at zero temperature. In this limit, the f'^2-term is dropped in the action (7.2), reducing it to the pure Schwarzian action $S = \int d\tau \{f, \tau\}$. To transit to a Hamiltonian description, it is useful to rewrite the Lagrangian into a first order form as

$$L = \pi_\phi \phi' + \pi_f f' - (\pi_\phi^2 + \pi_f e^\phi). \tag{7.12}$$

This first order form makes clear that the Schwarzian theory has a four dimensional phase space, labeled by two pairs of canonical variables (ϕ, π_ϕ) and (f, π_f). Alternatively, we may view the quantity π_f as a Lagrange multiplier, enforcing the constrained $f' = e^\phi$. Setting $\phi = \log f'$ and integrating out π_ϕ, it is readily seen that the above first-order Lagrangian indeed reduces to the Schwarzian theory. Upon quantization, the variables satisfy canonical commutation relations $[f, \pi_f] = i$ and $[\phi, \pi_\phi] = i$.

The invariance of the Schwarzian action under Möbius transformations

$$f \to \frac{af + b}{cf + d} \tag{7.13}$$

implies the presence of a set of conserved charges

$$l_{-1} = \pi_f, \quad l_0 = f\pi_f + \pi_\phi, \quad l_1 = f^2\pi_f + 2f\pi_\phi + e^\phi \tag{7.14}$$

that satisfy an $\mathfrak{sl}(2, \mathbb{R})$ algebra. The Hamiltonian H is equal to the quadratic Casimir

$$H = \pi_\phi^2 + \pi_f e^\phi = l_0^2 - \frac{1}{2}\{l_{-1}, l_1\} \tag{7.15}$$

and thus manifestly commutes with the $SL(2, \mathbb{R})$ symmetry generators. In particular, we can define a mutual eigenbasis of H and $\pi_f = l_{-1}$

$$\pi_f \left|\lambda, k\right\rangle = \lambda \left|\lambda, k\right\rangle, \quad H \left|\lambda, k\right\rangle = E(k) \left|\lambda, k\right\rangle, \quad E(k) \equiv \frac{1}{4} + k^2, \tag{7.16}$$

which spans the complete Hilbert space of the theory.

The 1D Schwarzian theory is closely related to the free particle on the 3D Euclidian AdS space H_3^+ with coordinates $\phi, f, \bar{f}$ and metric $ds^2 = d\phi^2 + 2e^{-\phi} df d\bar{f}$, and to 1D Liouville theory.

7.2.2 Finite temperature

Putting the theory at finite temperature, we reintroduce the f'^2 term in the action (7.2). The effect pf this term in the first order formulation is taken into account by changing the Hamiltonian

$$H = \pi_\phi^2 + \pi_f e^\phi + e^{2\phi} \tag{7.17}$$

We now solve the constraint $f' = e^\phi$, reducing the added term to $e^{2\phi} = f'^2$. The new Hamiltonian still has $SL(2,\mathbb{R})$ symmetry, generated by the charges:

$$l_{-1} = \cos^2(f)\pi_f - \sin(2f)\pi_\phi + \cos(2f)e^\phi \tag{7.18}$$

$$l_0 = \frac{1}{2}\sin(2f)\pi_f + \cos(2f)\pi_\phi + \sin(2f)e^\phi \tag{7.19}$$

$$l_1 = \sin^2(f)\pi_f + \sin(2f)\pi_\phi - \cos(2f)e^\phi \tag{7.20}$$

This charges satisfy $[l_0, l_{\pm 1}] = \mp l_{\pm 1}$ and $[l_1, l_{-1}] = 2l_0$ and they all commute with the Hamiltonian, which can again be identified as the quadratic Casimir $H = l_0^2 - \frac{1}{2}\{l_1, l_{-1}\}$. The $SL(2,\mathbb{R})$ symmetry generated by these charges acts via broken linear transformations on the uniformizing variable F

$$F \to \frac{aF + b}{cF + d}, \quad F = \tan(f/2), \quad ad - bc = 1. \tag{7.21}$$

Since $\pi_f = l_1 + l_{-1}$ commutes with H, we can again define a mutual eigenbasis (7.16) that spans the entire Hilbert space of the model. The Schrödinger wavefunctions of the eigenstates take the form

$$\Psi_{\lambda,k}(f,\phi) = e^{i\lambda f}\psi_{\lambda,k}(\phi), \tag{7.22}$$

where $\psi_{\lambda,k}(\phi)$, solves the Schrödinger equation

$$(-\partial_\phi^2 + \lambda e^\phi + e^{2\phi})\psi_{\lambda,k}(\phi) = k^2\psi_{\lambda,k}(\phi) \tag{7.23}$$

given by a 1D particle in the so-called Morse potential $V(\phi) = \lambda e^\phi + e^{2\phi}$. The solutions of this equation are given in terms of Whittaker W-functions[1]. The full eignemode functions normalized in the flat measure $df d\phi$ are given by

$$\Psi_{\lambda,k}(f,\phi) = \sqrt{\frac{k\sinh(2\pi k)}{4\pi^3}}|\Gamma(ik + \lambda/2 + 1/2)|e^{i\lambda f}e^{-\phi/2}W_{-\lambda/2,ik}(2e^\phi). \tag{7.24}$$

[1]For a small resume of Whittaker functions, check the appendices

7.3 Particle in a magnetic field

There exists an interesting and useful connection between the Schwarzian model and a particle of the hyperbolic plane H_2^+ in a constant magnetic field[31]. The Landau problem on H_2^+ was first analyzed by A. Comtet and P.J. Houston in [16, 17]. On of the main results in this papers, which we will use is an explicit formula for the spectral density of states.

Writing the H_2^+ metric as $ds^2 = d\phi^2 + e^{-2\phi}df^2$, the Lagranigan of the particle is given by

$$S = \int dt \left(\frac{1}{4}\phi'^2 + \frac{1}{4}e^{-2\phi}f'^2 + Bf'e^{-\phi} \right) \tag{7.25}$$

which identifies the magnetic vector potential as $qA_f = Be^{-\phi}$, where q is the charge of the particle. For a fixed constant B, we get the following Hamiltonian for the system

$$H_B = p_\phi^2 + (p_f e^\phi - B)^2, \tag{7.26}$$

where with p_ϕ and p_f we have denoted the canonical conjugate variables. The model is again invariant under Möbius transformations and possesses a corresponding set of $SL(2,\mathbb{R})$ generators:

$$l_{-1} = p_f, \quad l_0 = fp_f + p_\phi, \quad l_1 = f_f^p + 2fp_\phi - p_f e^{2\phi} + 2Be^\phi \tag{7.27}$$

Once again, the Hamiltonian is equal to the quadratic Casimir. The normalized simultaneous eigenmodes of p_f (with eigenvalue ν and H_B (with eigenvalue $E(k) = \frac{1}{4} + k^2 + B^2$ take the form [16]

$$\Psi_{\nu,k}(f,\phi) = \sqrt{\frac{k\sinh(2\pi k)}{4\pi^3|\nu|}|\Gamma(ik - B + 1/2|}e^{i\nu f}e^{-\phi/2}W_{B,ik}(2|\nu|e^\phi), \tag{7.28}$$

which should be compared with formula (7.24) for the eigenmodes of the Schwarzian model.

Using the above formula for the eigenmodes, it is straightforward to compute the density of states for the Landau problem on H_2^+. The result for the spectral measure reads:

$$d\mu_B(k) = \rho_B(k)dk = dk^2 \frac{\sinh(2\pi k)}{\cosh(2\pi k) = \cos(2\pi B)} \tag{7.29}$$

We can use this result to compute the spectral measure of the Schwarzian theory via the following observation[31]. Upon shifting $\phi \to \phi - \log(-2B)$ with $B \to i\infty$, the Hamiltonian H_B reduces to

$$H_B = p_\phi^2 + p_f e^\phi + B^2, \tag{7.30}$$

which, up to irrelevant constant B^2 contribution, coincides with the Hamiltonian (7.15) for the Schwarzian model at zero temperature. We can use this correspondence to derive the exact formula for the spectral measure (7.10), that we gave in the introduction to the Schwarzian model. Starting from the result (7.29) and observing that $\cos(2\pi B)$ diverges as $B \to i\infty$, we deduce that (up to an irrelevant overall normalization)

$$d\mu(k) = dk^2 \sinh(2\pi k), \tag{7.31}$$

which is equivalent to (7.10).

Chapter 8

Partition function in two dimensions

In this section we will study the path integral formulation of the Schwarzian theory at finite temperature. In particular, we will use its relationship to the group $\text{Diff}(S^1)$ to reformulate 1D Schwarzian QM as a suitable large c limit of 2D Virasoro CFT, that we have analyzed in the first chapter. Similar ideas are analyzed in [37].

The partition function of the Schwarzian theory (7.2) is defined as the integral

$$Z(\beta) = \int \frac{\mathcal{D}f}{SL(2,\mathbb{R})} e^{-S[f]} \tag{8.1}$$

over invertible functions f, satisfying the periodicity and monotonicity constraints $f(\tau+\beta) = f(\tau) + \beta$ and $f'(\tau) > 0$. The space of functions with these properties specifies the group of diffeomorphisms of the circle, also known as the Virassoro group.

The $SL(2,\mathbb{R})$ quotient in (8.1) indicates that the functional integral runs over the infinite dimensional quotent space

$$\mathcal{M} = \text{Diff}(S^1)/SL(2,\mathbb{R}) \tag{8.2}$$

of diffeomorphisms modulo the group of Möbius transformations acting on $F = \tan\left(\frac{\pi f}{\beta}\right)$.This space $\mathcal{M}$ is called the coadjoint orbit of the identity element $\mathbb{1} \in \text{Diff}(S^1)$, which is known to be a symplectic manifold[30]. It symplectic form takes

the following form

$$\omega = \int_0^{2\pi} dx \left[\frac{df' \wedge df''}{f'^2 - df \wedge df'} \right] \tag{8.3}$$

This observation was used by Stanford and Witten [51] to calculate the functional integral with the help of the Duistermaat-Heckman (DH) formula[21].

The DH formula applies to any integral over a symplectic manifold of the schematic form

$$I = \int dpdqe^{-H(p,q)} \tag{8.4}$$

where $H(p,q)$ generates, via the Poisson bracket $\{q,p\} = 1$, a $U(1)$ symmetry of the manifold. The derivation using the DH formula goes beyond the scope of this work. In place of that, we will give the result for the spectral density using the knowledge of ZZ branes, that we have defined in the previous chapter.

8.1 Spectral Density from ZZ branes

Firstly, we must recall that the *identity character* of a $c > 1$ CFT takes the form

$$\mathrm{Tr}(q^{L_0}) \equiv \chi_0(q) = \frac{q^{\frac{1-c}{24}}(1-q)}{\eta(\tau)} \tag{8.5}$$

where $\eta(\tau)$ denotes the Dedekind eta function:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \tag{8.6}$$

with $q = e^{2\pi i \tau}$. The factor $(1-q)$ in the above formula for the identity character accounts for the presence of the null state $L_{-1}|0\rangle = 0$. The identity character represents the chiral genus one partition function of the identity sector of the Virasoro CFT. Alternatively, we can identify $\chi_0(q)$ with the partition function of the Virasoro CFT on the annulus. This annulus partition function is equal to a trace over an open string sector of the Virasoro CFT, or by using the duality between the open and closed string channels, as the transition amplitude between two ZZ boundary states[23, 15].

$$\chi_0(q) = \langle ZZ|\tilde{q}|ZZ\rangle \tag{8.7}$$

The ZZ boundary state is given as an integral over Ishibashi boundary states (up to irrelevant constant factors) [23]:

$$|ZZ\rangle = \int_0^\infty dP \psi_{ZZ}(P) \, |P\rangle\rangle, \tag{8.8}$$

where

$$\Psi_{ZZ}(P) = \frac{2\pi i P}{\Gamma(1-2ibP)\Gamma(1+\frac{2iP}{b})} \tag{8.9}$$

In the limit we are considering the boundary states are associated to a circle with a radius that goes to zero and this allows us to approximate $|P\rangle\rangle \to |P\rangle$. This is the main feature that will allow computation of correlation functions since it can be used to turn a correlation function between ZZ-branes into an integral of a correlation function on the sphere. Using this and taking $\tilde{q} = e^{-\frac{\beta}{b^2}}$, where β is the temperature of the Schwarzian theory, the partition function becomes

$$Z = \int_0^\infty dP |\Psi_{ZZ}(P)|^2 e^{-\beta \frac{P^2}{b^2}}, \quad |\Psi_{ZZ}(P)|^2 = \sinh(2\pi b P) \sinh\left(\frac{2\pi P}{b}\right) \tag{8.10}$$

For small b this integral is dominated by states with P of order b. Therefore we define $P = kb$ and take the $b \Rightarrow 0$ limit; we can reproduce the result of Witten and Stanford. There is a third way, using a modular bootstrap and taking the large c limit, which is discussed in detail in [38]

8.2 Schwarzian correlators from ZZ branes

In this short section we will exploit the relationship between the Schwarzian theory and Virasoro CFT to compute finite temperature correlation functions of $SL(2,\mathbb{R})$ invariant operators in the Schwarzian theory. The simplest such operator is the Schwarzian itself. It's correlation functions are completely fixed by symmetries. Let's introduce

$$\mathrm{Sch}(f,\tau) \equiv \{\tan \frac{\pi f(\tau)}{\beta}, \tau\} \tag{8.11}$$

From the 2D perspective, we can obtain these correlation functions via the dictionary

$$T(\omega) \leftrightarrow \frac{1}{12}\mathrm{Sch}(f,\tau) \tag{8.12}$$

These correlation functions are fixed by Virasoro-ward identities. The case of a single insertion corresponds to taking the derivative of the partition function in

the open channel with respect to the modulus q:

$$< T(z) >= \frac{1}{Z}\frac{\partial Z}{\partial \log q}. \tag{8.13}$$

It is instructive to evaluate both sides via the representation of the partition function in terms of the ZZ boundary states

$$\langle ZZ|T|ZZ\rangle = \frac{1}{Z}\int dk^2 e^{-\beta k^2}\sinh(2\pi k)\,\langle k|T|k\rangle\,. \tag{8.14}$$

The stress-tensor one-point function is constant on the cylinder, upon mapping the plane to the cylinder via $z = e^{-\frac{\varpi}{b}}$, using the standard anomalous transformation law for the stress tensor and that $h_K = \frac{Q^2}{4} + b^2k^2$, one can find that

$$< \mathrm{Sch}(f,\tau) >= \frac{2\pi^2}{\beta^2} + \frac{3}{\beta} \tag{8.15}$$

Correlation functions of more insertions can be deduced in the same way, but we haven't calculated them in this work.

A more interesting class of correlation functions are those involving the bi-local operators:

$$\mathcal{O}(\tau_1,\tau_2) \equiv \left(\frac{\sqrt{f'(\tau_1)f'(\tau_2)}}{\frac{\beta}{\pi}\sin\frac{\pi}{\beta}[f(\tau_1)-f(\tau_2)]}\right)^{2l}. \tag{8.16}$$

These operators naturally live on the 2D space κ parametrized by pairs of points (τ_1,τ_2) on the thermal circle. The space κ is called *kinematic space*, since it plays an analogous geometrical role as the kinematic space associated with 2D holographic CTFs.

To exhibit the geometry of kinematic space κ, ket us - motivated by the form (8.16) of the bi-local operators - associate to any point $(u,v) \in \kappa$ a classical field $\varphi_{cl}(u,v)$ via

$$e^{\varphi_{cl}(u,v)} = \frac{\sqrt{f'(u)f'(v)}}{\frac{\beta}{\pi}\sin\frac{\pi}{\beta}[f(u)-f(v)]} \tag{8.17}$$

This field satisfies the Liouville equation

$$\partial_u\partial_v\varphi_{cl}(u,v) = e^{2\varphi_{cl}(u,v)} \tag{8.18}$$

Hence kinematic space κ naturally comes with a constant curvature metric $ds^2 = e^{2\phi(u,v)}dudv$ and looks like a hyperbolic cylinder with an asymptotic boundary located at $u = v$. Note, however, that the metric on kinematic space is now a dynamical quantity that depends on the dynamical diffeomorphism $f(\tau)$.

Chapter 9

ZZ-branes

Contents

9.1 Overview

Two-dimensional Euclidean AdS_2 (the pseudosphere) was quantized using Liouville quantum field theory by Zamolodchikov and Zamolodchikov. In [58] they generalized the quantization of Liouville theory on the disk [23] to the non-compact geometry of the pseudosphere. The main difference between the quantization of the disk and the pseudosphere is the assumption with regard to the pseudosphere that the two-point correlation function factorizes when the geodesic distance separating the two operators diverges[6].

Using conformal bootstrap methods the Zamolodchikovs found a number of conformal invariant boundary conditions, that may be imposed at "infinity" of the pseudosphere, and that are consistent with the above assumption. These boundary conditions were labeled by two positive integers $(\hat{m}, \hat{n}$, where the "basic" $(1, 1)$ boundary condition played a role quite similar to the $(1, 1)$ Cardy boundary state in the minimal conformal field theories[1]. In the context of (p, q) minimal non-critical string theory the boundary conditions of the Zamolodchikovs were given an interpretation as branes, the so-called **ZZ branes**[50].

[1]for more information, check the apendices

9.2 Properties of ZZ branes

In this subsection, we will review the properties of the FZZT and most importantly, the ZZ branes. This will be important later, when we come back to the Partition functions of Schwarzian quantum mechanics. In order to get a ZZ brane, we must take the tensor product of a FZZT boundary state form Liouville theory and a Cardy state from the matter. This gives the boundary state[23]

$$|\sigma; k, l\rangle = \sum_{k',l'} \int_0^\infty dP \cos(2\pi P\sigma) \frac{\Psi^*(P) S(k,l;k',l')}{\sqrt{S(1,1;k',l')}} |P\rangle\rangle_L |k',l'\rangle\rangle_M \tag{9.1}$$

Here (k,l) labels the matter Cardy state associated to the minimal model primary $\mathcal{O}_{k,l}$ and $|P\rangle\rangle_L$ and $|k',l'\rangle\rangle_M$ are Liouville and matter Ishibashi states, respectively. The Liouvlille alnd matter wavefunctions are $\cos 2\pi P\sigma \Psi(P)$ and $S(k,l;k',l')$, where

$$\Psi(P) = \mu^{-\frac{iP}{b}} \frac{\Gamma(1+\frac{2iP}{b})\Gamma(1+2iPb)}{i\pi P} \tag{9.2}$$

$$S(k,l;k',l') = (-1)^{kl'+k'l} \sin\left(\frac{\pi p l l'}{q}\right) \sin\left(\frac{\pi q k k'}{p}\right) \tag{9.3}$$

What can be seen from these equations is that the matter wavefunction is essentially the modular S-matrix of the minimal model. Finally, the parameter σ is related to the boundary cosmological constant μ_B via (up to a rescaling for convenience)

$$\frac{\mu_B}{\sqrt{\mu}} = \cosh \pi b\sigma \tag{9.4}$$

From here, we the one-point functions of physical operators on the disk with the FZZT boundary condition can be calculated, as it is in [50].

The previous calculation are evidence that in the full string theory, where the boundary states are representatives of the BRST cohomology, the following is true

$$|\sigma; k, l\rangle = \sum_{m',n'} \left| \sigma + \frac{i(m'q+n'p)}{\sqrt{pq}}; 1, 1 \right\rangle \tag{9.5}$$

modulo BRST exact states. The result (9.5), which relates branes with different matter states is a fully quantum result. This relation is difficult to understand

semiclassically, where branes with different matter states appear distinct. But there is no contradiction, because (9.5) involves a shift of σ by an imaginary quantity, which amounts to analytic continuation of μ_B from the semiclassical region where it is real and positive.

According to (9.5), the FZZT branes with $(k, l) = (1, 1)$ form a *complete basis* of all the FZZT branes of the theory. The branes with other matter states should be thought of as multi-brane states formed out of these elementary FZZT branes. This allows us to simplify our discussion henceforth by restricting our attention, without loss of generality, to the elementary FZZT (and ZZ) branes with $(k, l) = (1, 1)$. We will also simplify the notation by dropping the label $(1, 1)$ from the boundary states; this label will be implicit for the rest of the section.

A second interesting property of the one-point functions is that they are clearly invariant under the transformations

$$\sigma \Rightarrow -\sigma, \quad \sigma \pm 2i\sqrt{pq} \tag{9.6}$$

Again, this is evidence that the states labeled by σ should be identified under the transformations (9.6) modulo BRST exact states. Therefore it makes more sense to define

$$z = \cosh \frac{\pi\sigma}{\sqrt{pq}} \tag{9.7}$$

and to label the states by z

$$|\sigma\rangle \Rightarrow |z\rangle \tag{9.8}$$

such that two states $ketz$ and $ketz'$ are equal if and only if $z = z'$.

Now we turn our attention to the ZZ boundary states. As was for the case of the FZZT boundary states, these are formed by tensoring a Liouville ZZ boundary state and a matter Cardy state (in this case, the $(1, 1)$ matter state). However, here there are subtleties arising from the fact that b^2 is rational: the Liouville ZZ states are in one-to-one correspondence with the degenerate representations of Liouville theory, which have rather different properties at generic b and at b^2 rational. In either case, the prescription given in [58] for constructing the ZZ boundary states is to take the formula for the irreducible character for a rational b^2:

$$\hat{chi}_{t,m,n}(q) = \frac{1}{\eta(q)} \sum_{j=0}^{t} \left(q^{-N(t-2j,m,n)^2/4pq} - q^{-N(t-2j,m,-n)^2/4pq} \right) \tag{9.9}$$

and replace each term $\frac{1}{\eta(q)} q^{(} - N^2/4pq)$ with an FZZT boundary state with $\sigma = iN$. This (9.9)

$$\begin{aligned} |t,m,n\rangle &= \sum_{j=0}^{t} \left(\left| z = \cos \frac{\pi N(t-2j,m,n)}{pq} \right\rangle - \left| z = \cos \frac{\pi N(t-2j,m,-n)}{pq} \right\rangle \right) \\ &= (t+1) \left(\left| z = (-1)^t \cos \frac{\pi(mq+np)}{pq} \right\rangle - \left| z = (-1)^t \cos \frac{\pi(mq-np)}{pq} \right\rangle \right) \end{aligned} \tag{9.10}$$

In this second equation we can observe that we lose dependence of j. We recognize the quantity in parentheses to be a ZZ state with $t = 0$; thus we conclude that

$$|t,m,n\rangle = \begin{cases} +(t+1)\,|t=0,m,n\rangle & \text{t even} \\ -(t+1)\,|t=0,m,q-n\rangle & \text{t odd} \end{cases} \tag{9.11}$$

It is also straightforward to show using (9.5) that

$$|t,m,n\rangle = |t,p-m,q-n\rangle \tag{9.12}$$

and that

$$|t,m,n\rangle = 0 \quad \text{when} \quad m = p \text{ or } n = q \tag{9.13}$$

One should keep in mind that the above three equations are meant to be true modulo BRST null states.

One can identify the states with $t = 0$ appearing in (9.11) are identical to the ZZ boundary states for generic b, which can be written as differences of just two FZZT states

$$\begin{aligned} |t=0,m,n\rangle &= \left| z = \cos \frac{\pi\sigma(m,n)}{\sqrt{pq}} \right\rangle - \left| z = \cos \frac{\pi\sigma(m,-n)}{\sqrt{pq}} \right\rangle \\ &= 2 \sum_{k',l'} \int_0^\infty dP \sinh\left(\frac{2\pi mP}{b}\right) \sinh(2\pi nPb) \Psi^*(P) \sqrt{S(1,1;k',l')}\, |P\rangle\rangle_L\, |k',l'\rangle\rangle_M \end{aligned} \tag{9.14}$$

with

$$\sigma(m,n) = i(\frac{m}{b} + nb) \tag{9.15}$$

The boundary cosmological constant corresponding to $\sigma(m,n)$ is

$$\mu_B(m,n) = \sqrt{\mu}(-1)^m \cos\left(\pi n b^2\right) \tag{9.16}$$

The two subtracted FZZT states in (9.14) have the same boundary cosmological constant.

Using the above identifications we can reduce any ZZ brane down to a linear combination of $(t=0,m,n)$ branes with $1 \leq m \leq p-1, 1 \leq n \leq q-1$ and $mq - np > 0$. We call these $(p-1)(q-1)/2$ branes the *principal ZZ branes*. It is easy to see from (9.14) that the one-point functions of physical operators are sufficient to distinguish the principal ZZ branes from one another. Thus the principal ZZ branes form a complete and linearly independent basis of physical states with ZZ-type boundary conditions.

We will conclude this section by discussing an interesting feature of the principal ZZ branes. The ground ring one-point functions in the principal ZZ brane states can be normalized so that (we will drop the label $t=0$ from these states from this point onwards so the notation won't become bloated)

$$\langle \hat{\mathcal{O}}_{r,s} \left| m,n \right\rangle = U_{s-1}\left((-1)^m \cos\frac{\pi np}{q}\right) U_{r-1}\left((-1)^n \cos\frac{\pi mq}{p}\right) \langle 0 \left| m,n \right\rangle, \tag{9.17}$$

where $\langle 0 \left| m,n \right\rangle$ denotes the ZZ partition function (i.e. the one-point function of the identity operator). This is consistent with the multiplication rule

$$\hat{\mathcal{O}}_{r,s} = U_{s-1}(\hat{x}) U_{r-1}(\hat{y}) \tag{9.18}$$

assuming that the principal ZZ branes are eigenstates of the ring generators:

$$\hat{x} \left| m,n \right\rangle = x_{mn} \left| m,n \right\rangle \tag{9.19}$$

$$\hat{y} \left| m,n \right\rangle = y_{mn} \left| m,n \right\rangle \tag{9.20}$$

with eigenvalues

$$x_{mn} = (-1)^m \cos\frac{\pi np}{q}, \quad y_{mn} = (-1)^n \cos\frac{\pi mq}{p} \tag{9.21}$$

Let's make a few more comments about the result (9.17)

- It is clear that the general FZZT boundary states labeled by σ will not be an eigenstate of the ring generators: this property is special to the ZZ boundary states

- Once we have normalized the ring elements to bring their one-point functions to the form (9.17), the ZZ branes with other matter labels (k, l) will not be eigenstates of the ring elements. Of course, we could have normalized the ring elements with respect to a different (k, l); then (9.17) would have applied to this matter label. Still, it is natural to assume that the branes with matter label $(1, 1)$ are eigenstates of the ring.

- Finally, one can notice that x_{mn} is essentially the value of the boundary cosmological constant associated with the (m, n) ZZ brane.

Chapter 10

Statistical Physics

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Contents

10.1 The Ising Model

10.1.1 Estimation of the Critical Temperature in 2d and 3d

In 1936, Peierls with the help of H. Bethe, used a geometrical argument which provides us with a quick method to obtain an estimation for the critical temperature, as approached from below. The method followed here is an adaptation of Peierls'. He counts how many spins can be enclosed with known information from the partition function $\mathcal{Z}$, to obtain

$$e^{-E/T_c} \sim \frac{1}{3} \tag{10.1}$$

Instead, we shall trace a random path in the dual lattice and use energy arguments. As detailed later, the dual to the square net of lattice points (here our spins)

is also a square net (here considered to represent spin-spin interaction). It is to be understood that the pictures are equivalent due to the square being its own dual. For our estimation, we start by assuming the lowest energy state. In this picture, all spin states are aligned, which would correspond to a trivial interaction picture. Then, if a pocket of anti-aligned spins is created, it requires energy ΔE, proportional to the length L of the domain wall in the dual picture

$$\Delta E = 2JL \tag{10.2}$$

The unique dependence on L (i.e., the independence of the number of spin enclosed), grants freedom for the shape of the pocket and its connectedness (cf. figures 10.1 and 10.2). As illustrated below, we see that a boundary of length L

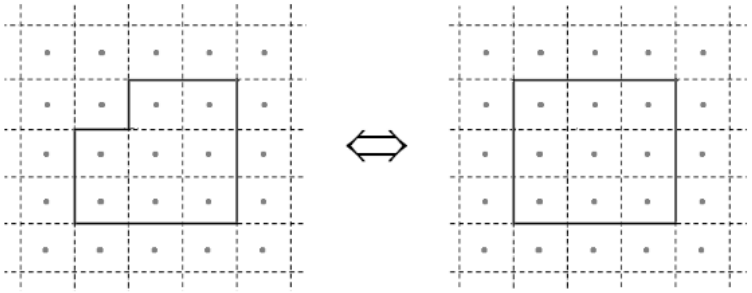

Figure 10.1: The lattice of dots represents the spin lattice, and the dashed lines are the interaction lattice. $L = 12$ in both graphs, therefore both require the same energy, yet do not have the same number of spins flipped.

can be thought of as an L steps closed path along the array's links. At each site, a choice of two directions is sufficient to guarantee a closed path[1].

There are then 2^L ways to realise a boundary given all L, all topologies included, thus according to statistical mechanics

$$\Delta S = \log\left(2^L\right) = L\log(2) \tag{10.3}$$

Combined with equation (10.2) we can express the variation in freefree energy:

$$\Delta F = \Delta E - T\Delta S = L(2J - T\log(2)) \tag{10.4}$$

[1]In the most general case, this number is four. However, when we exclude the possibility for back-tracking, we are left with three. Furthermore, as we are looking for an upper bound, two is sufficient

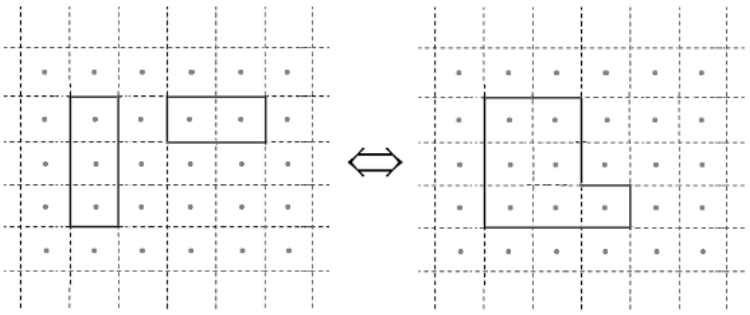

Figure 10.2: Same convention as above. Here $L = 14$ in both cases, therefore both graphs are energetically equivalent, despite the different topologies.

At low temperature, where we can observe ferromagnetic behaviour, we can assume stability of the creation of such boundaries, i.e $\Delta F \geq 0$. We can obtain a first approximation to the critical temperature T_c, as approached from below by plugging (10.4) into the inequality

$$T_c \approx \frac{2}{\log(2)} J \leq T \tag{10.5}$$

which gives

$$\boxed{T_C \approx 2.885J} \tag{10.6}$$

We can expand this argument for three dimensions if we suppose equation (10.2) is still holds. On the other hand, we should only consider 5^L ways to close a path in the cubic array. The same calculation lead to an evaluation of the critical temperature of the 3d Ising model

$$T_{C_{\text{Ising}}} = \frac{2}{\log(5)} J \approx 1.243J \tag{10.7}$$

This appears to be a lower temperature than in 2d. One is tempted to conclude that thermal fluctuations are stronger than spin-spin interactions, as dimensions increase. This approach, although naïve, has the power to estimate a value of T_C despite not being able to solve the model exactly.

10.1.2 Kramers-Wannier duality

The Kramers-Wannier duality is an extended symmetry in statistical physics. It relates the free energy of a two-dimensional square-lattice Ising model at low temperature to that of another Ising model at high temperature. Kramers and Wannier

used this duality trick to find the exact location of the critical point for the Ising model on the square lattice.

Let us investigate the behaviour of an $M \times M$ links square lattice[2], in the absence of an external field. To do so, we shall use the high, then low temperature series expansion. We will mostly use geometrical arguments to consolidate the expansion.

10.1.2.1 High-Temperature Expansion

Let us suppose the two coupling constants for the vertical and horizontal link-link interaction to be identical $J_V = J_H = J$. If that is the case, we can write the partition function

$$Z_M = \sum_s e^{J\sum_{\langle i,j\rangle} s_i s_j + J\sum_{\langle i,k\rangle s_i s_k}} \tag{10.8}$$

Using $K = \beta J$ and writing the exponent as a sum of hyperbolic functions, we can rewrite Z_M as follows,

$$Z_M = (\cosh K)^{2M} \sum_s \left(\prod_{\langle i,j\rangle} (1 + s_i s_j \tanh K) \right) \tag{10.9}$$

At high temperature, the system is in the disordered phase. Because of the unique dependence of Z_M on $\tanh K$, let us observe the limit $\lim_{T\to\infty} \tanh K = 0$. It is now natural to expand om the high temperature limit. In fact, expanding the product will produce $2^2 M$ terms that can be thought of graphically. Returning to figures 10.1 and 10.2, we can now rigorously construct them, and they turn out to be highter order terms. The process is to represent each of the three possible terms than can be produced: to every $s_i s_j$ product there is an associated vertical or horizontal link whereas to each factor 1 produced, nothing is drawn. Repeating the process for all $2^2 M$ terms becomes reminiscent of the random path created in the previous section. In effect, it corresponds to the geometrical quantity we implicitly used with Peierls' argument, which outputs valid polygons given a length L:

$$\Phi = \sum_P (\tanh K)^{p+q} \tag{10.10}$$

[2]In the thermodynamical limit, when $M \to \infty$, the number of links and lattice points N are indistinguishable. The calculation is made in the interaction picture, and remains valid in the spin picture.

where $p + q = L$ are the horizontal and vertical links drawn. Thus we have a pre-expansion partition function written as

$$Z_M = 2^N (\tanh K) M \Phi \tag{10.11}$$

Now remains to sum over all polygons. Taking into account $s \in \mathbb{Z}_2$, we can expect the conditions - listed in the previous section - for the polygons to be valid to be naturally satisfied: in effect, the sum is null unless it outputs an even L per polygon.

The first term of the sum is clearly 1, corresponding to no domain wall. The first non-trivial term is the smallest domain wall: a square of length $L = 4$, which can be placed anywhere on the lattice, hence occur N times.

10.1.2.2 Low-Temperature Expansion

Not immediately obvious is that we can draw a correspondence between the high-temperature limit in the interaction picture to the low temperature limit in the spin picture. One can see that in the first case, the first term corresponds to no domain walls, indeed at $T = 0$ all spins are aligned. One spin flipped is bounded by the next term in the polygon expansion: the $L = 4$ square.

Even without much details, this could be anticipated from the asymptotic bound behaviour and odd parity of $\tanh K$. However, if we define a new variable K^*

$$\tanh K = e^{-2K^*}, \tag{10.12}$$

we can rewrite the expansion of the partition function and arrive at the following relation

$$\frac{Z(K^*)}{(e^{K^*})^N} = \frac{Z(K)}{(2\cosh^2 K)^N} \tag{10.13}$$

By construction, we associated K with the high temperature and K^* with the low, such that they are inversely proportional to each other. Rearranging both equations, we can make the duality finally obvious

Kramers-Wannier duality

$$\frac{Z(K^*)}{\sinh^{N/2}(2K^*)} = \frac{Z(K)}{\sinh^{N/2}(2K)} \tag{10.14}$$

The final argument is to suppose there are strictly two phases, hence one critical temperature. If this is the case, then $K = K^*$, or

$$\sinh(2K_C) = 1 \tag{10.15}$$

which gives the following result for the critical temperature of the 2d Ising model

$$T_C^{2d} = 2.269J \tag{10.16}$$

10.1.2.3 Exact Critical Temperature in 2d

Now, with the low-high temperature self-duality made clear, we have acquired the power to switch from one end of the temperature scale to the other, by simply switching from the lattice to the interaction picture. This is particularly useful when one calculation is hard or even impossible in one limit, but feasible in the other limit.

10.1.3 Note on Correspondence to String Theory

Based purely on the dimensionsional arguments , one can draw parallels between flipped spins and their domain walls, with particles and their world line.

Lattice Dimension	**Domain Wall Dimension**	**String Theory Correspondant**
2d	1d	kink particle and its (0+1)-d world line
3d	2d	string and its (1+1)-d world line
4d	3d	2-brane and its (2+1)-d world line

Table 10.1: Lattice points and domain walls correspondence to String theory

The high-low temperature Kramers-Wannier (self)-duality of the two-dimensional Ising model is a particular example of the more general weak-strong $S-duality$. Here the gauge group (and the dual group) is $\mathbb{Z}_2$, as we saw earlier.

10.1.4 Conclusion

We have derived two exact solutions to the Ising model. In doing so, we have discovered that at low dimensions the mean-field approximation leads to erraneous

results. As we increase in dimensions, and reach 4d, mean-field theory is not only accurate, but the only way to provide insight into our model.

Ironicall, the only thing the Ising model has failed to do is fulfil its original intent and accurately reproduce ferromagnets. On the other hand, it allowes us to explore exactly and analytically critical behaviours that are insensitive to microscopic scales, hence the usefulness of universality classes. It is indeed sufficient to study the Ising model as a representative, to characterise the entire class. We had already noted the appearance of the divergencies and the importance of the correlation length, as hinting the need to introduce a quantum field language. Indeed, using the correlation length as a variable, instead of the lattice length, in other words coarse graining in the context of a renormalisation group transformation, could be grounding for future work. Further investigation into the conformality at the critical point, of the Ising model and all models whose Hamiltonian is $\mathbb{Z}_2$ invariant at the same dimensions, would provide elegant solutions.

Chapter 11

Lattices and Structure Definitions

Contents

11.1 Dual Lattice

In this section we define the notian of the *dual* of a lattice and see some of its applications.

Definition 11.1. *(Dual Lattice) For a full-rank lattice* Λ *we define its dual lattice (sometimes known as the reciprocal lattice):*

$$\Lambda^* = \{y \in \mathbb{R}^n | \forall x \in \Lambda, \langle x, y \rangle \in \mathbb{Z}\}. \tag{11.1}$$

In general, we define:

$$\Lambda^* = \{y \in \text{span}(\Lambda) | \forall x \in \Lambda, \langle x, y \rangle \in \Lambda\}. \tag{11.2}$$

In other words, the dual of Λ is the set of all points (in the span of Λ) whose inner product with any of the points in Λ is an integer. As we will show later, Λ^* is indeed a lattice, as the name suggests.

Example 4. *The lattice of integer points satisfies $(\mathbb{Z}^n)^* = \mathbb{Z}$ (and hence can be called self-dual). Similarly, $(2\mathbb{Z}^n)^* = \frac{1}{2}\mathbb{Z}^n$, and this gives some justification to the name reciprocal lattice.*

From the above definition, we have the following geometrical interpretation of the dual lattice. For any vector x, the set of all points whose inner product with x is integer forms a set of hyperplanes perpendicular to x and separated by distance $1/||x||$. Hence, any vector x in a lattice Λ imposes the constraint that all points in Λ^* lie in one of the hyperplanes defined by x. The next figure tries to illustrate that notion.

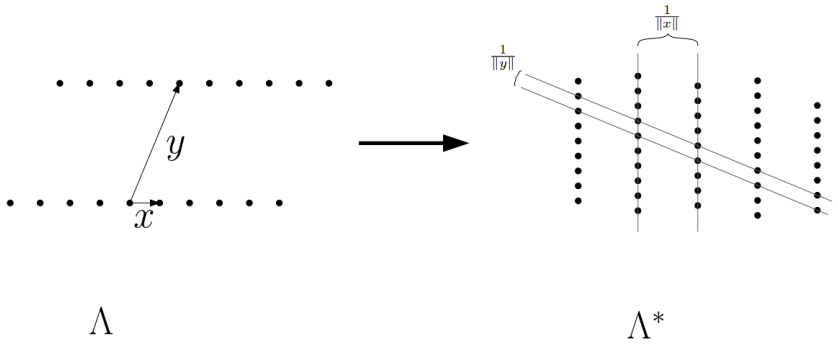

Figure 11.1: A lattice and its dual

Definition 11.2. *(Dual Basis) For a basis $B = (b_1, \ldots, b_n) \in \mathbb{R}^{m \times n}$, define the dual basis $D = (d_1, \ldots, d_n) \in \mathbb{R}^{m \times n}$ as the unique basis that satisfies*

- $\text{span}(D) = \text{span}(B)$
- $B^T D = \mathbb{I}$

The second condition can be interpreted as saying that $\langle b_i, d_j \rangle = \delta_{ij}$ where δ_{ij} is the usual Kroneker delta. It is not hard to check that D is indeed unique. In fact, for the case of a full-rank lattice, D is given by $(B^T)^{-1}$; in general, we get $D = B(B^T B)^{-1}$ (and we could use this as our definition of a dual basis).

In the next claim, we show that if B is a basis of a lattice Λ, then the dual basis of B is a basis of Λ^*. In particular, this shows that Λ^* is indeed a lattice.

Proposition 11.1. *If D is the dual basis of B then $(\mathcal{L}(B))^* = \mathcal{L}(D)$.*

Proof. We first show that $\mathcal{L}(D)$ is contained in $(\mathcal{L}(B))^*$. Any $x \in \mathcal{L}(B)$ can be written as $\sum a_i b_i$ for some $a_i \in \mathbb{Z}$. Therefore, for any j we have

$$\langle x, d_j \rangle = \sum_i a_i \langle b_i, d_j \rangle = a_i \in \mathbb{Z} \tag{11.3}$$

and we get $D \subseteq (\mathcal{L}(B))^*$. It is easy to check that $(\mathcal{L}(B))^*$ is closed under addition, hence, $\mathcal{L}(D) \subseteq (\mathcal{L}(B))^*$. To complete the proof, we show that $(\mathcal{L}(B))^*$ is contained in $\mathcal{L}(D)$. Take any $y \in (\mathcal{L}(B))^*$. Since $y \in \text{span}(B) = \text{span}(D)$, we can write $y = \sum a_i d_i$ for some $a_i \in \mathbb{R}$. Now for all $j, \mathbb{Z} \ni \langle y, b_j \rangle = \sum a_i \langle d_i, b_j \rangle = a_j$. Hence, $y \in \mathcal{L}(D)$ and the proof is complete. □

Proposition 11.2. *For any lattice Λ, $(\Lambda^*)^* = \Lambda$.*

Proof. Let B be a basis of Λ. Then $B(B^T B)^{-1}$ is a basis of Λ^* and

$$(B(B^T B)^{-1}) \cdot ((B(B^T B)^{-1})^T \cdot B(B^T B)^{-1})^{-1} = B \tag{11.4}$$

is a basis of $(\Lambda^*)^*$. □

The next claim says that the volume of the basic parallelepiped of Λ^* is the reciprocal of that of Λ. For example, it implies that the volume of the basic parallelepiped of a self-dual lattice must be 1 (as in the case with $\mathbb{Z}^n$).

Proposition 11.3. *For any lattice Λ, $\det(\Lambda^*) = 1/\det(\Lambda)$.*

Proof. For full-rank lattices,

$$\det(\Lambda^*) = \left|\det\left((B^T)^{-1}\right)\right| = \left|\frac{1}{\det(B^T)}\right| = \left|\frac{1}{\det(B)}\right| = \frac{1}{\det(\Lambda)}. \tag{11.5}$$

In general,

$$\begin{aligned}\det(\Lambda^*) &= \sqrt{\det(D^T D)} \\ &= \sqrt{\det(((B^T B)^{-1})^T B^T \cdot B(B^T B)^{-1})} \\ &= \sqrt{(\det(B^T B)^{-1})} \\ &= \frac{1}{\sqrt{\det(B^T B)}} = \frac{1}{\det(\Lambda)}. \end{aligned} \tag{11.6}$$

□

The following two claims give some relations between properties of a latticlattice and that of its dual. Such properties are known as transference theorems.

Proposition 11.4. *For any rank n lattice Λ, $\lambda_1(\Lambda) \cdot \lambda_1(\Lambda^*) \leq n$.*

Proof. By Minkowski's bound,

$$\lambda_1(\Lambda) \leq \sqrt{n} \cdot (\det(\Lambda))^{\frac{1}{n}} \tag{11.7}$$

and

$$\lambda_1(\Lambda^*) \leq \sqrt{n} \cdot (\det(\Lambda^*))^{\frac{1}{n}} = \frac{\sqrt{n}}{(\det(\Lambda))^{\frac{1}{n}}}. \tag{11.8}$$

□

Proposition 11.5. *For any rank lattice Λ, $\lambda_1(\Lambda) \cdot \lambda_n(\Lambda^*) \geq 1$.*

Proof. Let $v \in \Lambda$ be such that $||v|| = \lambda_1(\Lambda)$. Take any set $x_1, \ldots, x_n$ of n linearly independent vectors in Λ^*. Not all of them are orthogonal to v. Hence, there exists an i such that $\langle x_i, v\rangle \neq 0$. By the definition of the dual lattice, we have $\langle x_i, v\rangle \in \mathbb{Z}$ and hence $||x_i|| \geq \frac{1}{||v||}$. □

For a basis $b_1, \ldots, b_n$, let π_i denote the projection on the space $\text{span}(b_1, \ldots, b_{i-1})^\perp$. In particular, $\pi_1(b_1), \ldots, \pi_n(b_n)$ is the Gram-Schmidt orthogonalization of $b_1, \ldots, b_n$.

Proposition 11.6. *Let B, D be dual bases. Then, for all i, $B' = (\pi_i(b_i), \ldots, \pi_i(b_n))$ and $D' = (d_i, \ldots, d_n)$ are also dual bases.*

Proof. First, notice that $\text{span}(B') = \text{span}(b_1, \ldots, b_{i-1})^\perp)$. Moreover, since $d_i, \ldots, d_n$ are orthogonal to $b_1, \ldots, b_{i-1}$ and linearly independent, $\text{span}(D') = \text{span}(b_1, \ldots, b_{i-1})^\perp$. Hence, $\text{span}(B') = \text{span}(D')$. Finally, we have that for any $j, k \geq i$,

$$\langle d_j, \pi_i(b_k) \rangle = \langle d_j, b_k \rangle = \delta_{jk} \tag{11.9}$$

where the first equality holds since $d_j \in \text{span}(b_1, \ldots, b_{i-1})^\perp$. □

Proposition 11.7. *Let $b_1, \ldots, b_n$ be some basis and let $\tilde{b}_1, \ldots, \tilde{b}_n$ be its Gram-Schmidt orthogonalization. Let $d_n, \ldots, d_1$ be the dual basis of $b_1, \ldots, b_n$ in reverse order and let $\tilde{d}_n, \ldots, \tilde{d}_1$ be its Gram-Schmidt orthogonalization (using this order). Then, for all i,*

$$\tilde{d}_i = \frac{\tilde{b}_i}{||\tilde{b}_i||^2} \tag{11.10}$$

Proof. The proof is by induction on n. Assume the claim holds for lattices of rank $n-1$ and let us prove it for lattices of rank n. First, notice that $\tilde{b}_1 = b_1$ and $\tilde{d}_1$ is the projection of d_1 on $\text{span}(d_2, \ldots, d_n)^\perp = \text{span}(b_1)$. Hence, $\tilde{d}_1 \in \text{span}(b_1)$ and $\langle \tilde{d}_1, b_1 \rangle = \langle d_1, b_1 \rangle = 1$. This implies that

$$\tilde{d}_1 = \frac{b_1}{||b_1||^2} = \frac{\tilde{b}_1}{||\tilde{b}_1||^2} \tag{11.11}$$

We can now complete the proof by applying the inductive hypothesis to the bases $(\pi_2(b_2), \ldots, \pi_2(b_n))$ and $d_2, \ldots, d_n$. Indeed,11.6 says that these are dual bases, and moreover, the Gram-Schmidt orthogonalization of the former is $\tilde{b}_2, \ldots, \tilde{b}_n$. □

11.2 Korkine-Zolotarev bases

In this section we define the notion of a Korkine-Zolotarev (KZ) basis. This gives one way to formalize the idea of a 'shortest possible' basis.

Definition 11.3 (Korkine-Zolotarev basis)**.** *For a rank n lattice Λ, we define its Korkine-Zolotarev (KZ) basis $b_1, \ldots, b_n$ recursively as follows. We let b_1 be the shortest vector in Λ. We then let Λ' be the lattice given by the projection of Λ on the subspace of* $\text{span}(\Lambda)$ *orthogonal to b_1. Let $c_2, \ldots, c_n$ be the KZ basis of Λ'. Define $b_i = c_i + \alpha_i b_1$ where $\alpha_i \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ is the unique number such that $b_i \in \Lambda$.*

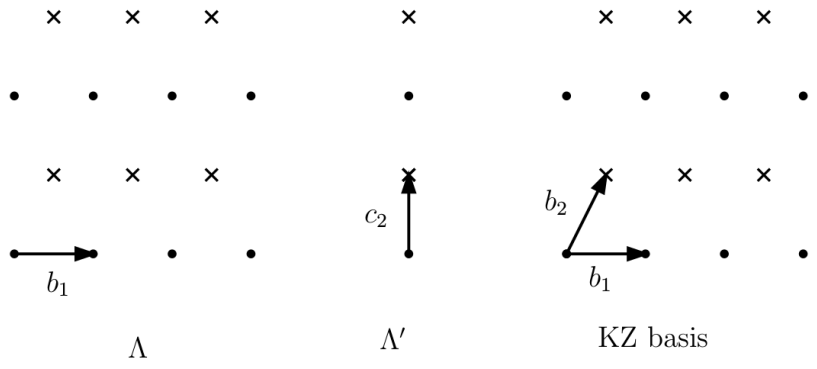

Figure 11.2: A lattice and its KZ basis

It is not too difficult to verify that Λ' is indeed a lattice. Moreover, b_1 is a primitive vector in Λ (since it is a shortest vector) and hence the vectors $b_1, \ldots, b_n$ defined above indeed former a basis of Λ. The deffinition is illustrated in 11.2.

As a first application of KZ bases, we prove the following lemma by Lagarias, Lenstra and Schnorr. Recall that for any basis, $b_1, \ldots, b_n$, we have that $\min(||\tilde{b}_1||, \ldots, ||\tilde{b}_n||) \leq \lambda_1(\Lambda)$. The lemma says that any lattice has a basis where this lower bound is not far from being tight.

Lemma 11.1 (Lagarias, Lenstra, Schnorr)**.** *For any lattice Λ, there exists a basis $b_1, \ldots, b_n$ such that*

$$\text{span}(||\tilde{b}_1||, \ldots, ||\tilde{b}_n||) \geq \frac{1}{n} \cdot \lambda_1(\Lambda) \tag{11.12}$$

Proof. Let $d_1, \ldots, d_n$ be a KZ basis of Λ^* and let $b_n \ldots, b_1$ be its dual basis in reverse order. We claim that $b_n, \ldots, b_1$ satisfies the lemma. By 11.7, we know that $\tilde{b}_i = \frac{\tilde{d}_i}{||\tilde{d}_i||^2}$. Hence, its enough to show that $\max(||\tilde{d}_1||, \ldots, ||\tilde{d}_n||) \leq \frac{n}{\lambda_1(\Lambda)}$. First, $\tilde{d}_1$ is the shortest vector in Λ^*. By 11.4, we have $||\tilde{d}_1|| \leq \frac{n}{\lambda_1(\Lambda)}$. Next, $\tilde{d}_2 = \pi_2(d_2)$ is the shortest vector in $\mathcal{L}(\pi_2(d_2), \ldots, \pi_2(d_n))$. By 11.6, the dual of this lattice is $\mathcal{L}(b_2, \ldots, b_n)$. But $\lambda_1(\mathcal{L}(b_2, \ldots, b_n)) \geq \lambda_1(\Lambda)$(since $\mathcal{L}(b_2, \ldots, b_n)$ is a sublattice of Λ) and hence

$$||\tilde{d}_2|| \leq \frac{n-1}{\lambda_1(\mathcal{L}(b_1, \ldots, b_n))} \leq \frac{n}{\lambda_1(\Lambda)}. \tag{11.13}$$

We continue similarly for all i. □

We complete this chapter with the following somewhat surprising result by Lenstra and Schnorr. Recall that Minkowski's bound says that for any lattice Λ,

$\lambda_1(\Lambda) \leq \sqrt{n}(\det(\Lambda))^{\frac{1}{n}}$. However, it is easy to see that in many cases Minkowski's bound is far from being tight. Nevertheless, the following lemma implies that being able to find vectors of lenght at most $\sqrt{n}(\det(\Lambda))^{\frac{1}{n}}$ is enough to imply an n-approximation to SVP (shortest vector problem).

Lemma 11.2. *Assume there exists an algorithm A that given a basis B, finds a non-zero vector $v \in \mathcal{L}(B)$ such that*

$$||v|| \leq f(n) \cdot (\det(\mathcal{L}(B)))^{\frac{1}{n}} \tag{11.14}$$

for some non-decreasing function $f(n)$. Then, we can approximate SVP to within $(f(n))^2$.

Proof. By applying A to $\mathcal{L}(B)$ and $\mathcal{L}(B)^*$ we obtain $u \in \mathcal{L}(B), v \in \mathcal{L}(B)^*$ such that $||u|| \leq f(n) \cdot (\det(\mathcal{L}(B)))^{\frac{1}{n}}, ||v|| \leq f(n) \cdot (\det(\mathcal{L}(B)))^{-\frac{1}{n}}$. In particular, $||u|| \cdot ||v|| \leq (f(n))^2$. So the result follows from the following lemma. □

Lemma 11.3. *Assume there exists an algorithm A that given a basis B, finds non-zero vectors $u \in \mathcal{L}(B), v \in \mathcal{L}(B)^*$ such taht $||u|| \cdot ||v|| \leq g(n)$ for some non-decreasing function $g(n)$. Then, we can approximate SVP to within $g(n)$.*

Proof. First, we describe a recursive procedure that given a lattice, outputs a set of vectors $u_1, \ldots, u_n$ in Λ and a basis $v_1, \ldots, v_n$ of Λ^*. The procedure is recursive. First, apply A to obtain a pair u_1, v_1. Without loss of generality, we can assume that v_1 is primitive (indeed, write $v_1 = \sum a_i b_i$ and replace it by $v_1/\gcd(a_1, \ldots, a_n)$). Let Λ' be the projection of Λ^* on the subspace of $\text{span}(\Lambda^*)$ orthogonal to v_1. Then apply the procedure recursively to Λ' and let $u_2, \ldots, u_n, v_2', \ldots, v_n'$ be the result. Define $v_i = v_i' + \alpha_i v_1$ for the unique $\alpha_i \in (-\frac{1}{2}, \frac{1}{2}]$ for which $v_i \in \Lambda^*$. This completes the description of the procedure.

It can be checked that the output of the procedure satisfies that $v_1, \ldots, v_n$ is a basis of Λ^* and that for all $i, ||u_i|| \cdot ||v_i|| \leq g(n - i + 1) \leq g(n)$. Let $b_n, \ldots, b_1$ be the reversed dual basis of $v_1, \ldots, v_n$. Then,

$$\min ||\tilde{b}_i|| = \min \frac{1}{||\tilde{v}_i||} \leq \frac{1}{g(n)} \min ||u_i||. \tag{11.15}$$

Hence,

$$\min ||u_i|| \leq g(n) \cdot \min ||\tilde{b}_i|| \leq g(n) \cdot \lambda_1(\Lambda) \tag{11.16}$$

where we used that $b_n, \ldots, b_1$ is a basis of Λ. Therefore, by outputting the shortest vector among $u_1, \ldots, u_n$ we obtain a $g(n)$ application to SVP. □

Chapter 12

Introduction to Density Functional Theory

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Contents

Density functional theory (DFT) is a computational quantum mechanical modelling method used in physics, chemistry and materials science to investigate the electronic structure (principally the ground state) of many-body systems, in particular atoms, molecules, and the condensed phases. Let us consider a N-electron system (atom, molecule or solid) in the Born-Oppenheimer and non-relativistic approximations:

$$\begin{aligned} \psi_{total} &= \psi_{electronic} \otimes \psi_{nuclear} \\ v &<< c \end{aligned} \tag{12.1}$$

12.1 The many-body problem

We can write the electronic Hamiltonian of such a system in the position representation and atomic units as:

$$\mathcal{H}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n) = -\frac{1}{2}\sum_{i=1}^{N} \nabla_{r_i}^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=0 \\ j\neq i}}^{N} \frac{1}{|\vec{r}_i - \vec{r}_j|} + \sum_{i=1}^{N} v_{\text{ne}}(\vec{r}_i), \tag{12.2}$$

where $v_{\text{ne}} = -\sum_\alpha Z_\alpha / \left|\vec{r}_i - \vec{R}_\alpha\right|$ is the nuclei-electron interaction ($\vec{R}_\alpha$ and Z_α are the position and charges of the nuclei, respectively). From now on we will omit the vector symbols for ease of notation. The stationary states are determined by the time-independent Schrodinger equation

$$\mathcal{H}(r_1, r_2, \cdots, r_N)\Psi(x_1, x_2, \cdots, x_N) = E\Psi(x_1, x_2, \cdots, x_N), \tag{12.3}$$

where $\Psi(x_1, x_2, \cdots, x_N)$ is a wave function written with space spin coordinates $x_i = (r_i, \sigma_i)$(with $r_i \in \mathbb{R}^3$ and $\sigma_i = \uparrow$ or $\downarrow$) which is antisymmetric with respect to the change of two coordinates, and E is the associated energy.

Using Dirac notation, the Schrodinger equation can be rewritten in a representation-independent formalism

$$\hat{\mathcal{H}} \langle \Psi | = E \langle \Psi |, \tag{12.4}$$

where the Hamiltonian is formally written as:

$$\hat{\mathcal{H}} = \hat{T} + \hat{W}_{ee} + \hat{V}_{ne}, \tag{12.5}$$

where $\hat{T}$ is the kinetic-energy operator, $\hat{W}_{ee}$ is the electron-electron interaction operator, and $\hat{V}_{ne}$ is the nuclei-electron interaction operator. These operators can be expressed in convinient ways in second quantization because of the fact that they will be independent of the number of electrons (i.e. we work in Fock space). The kinetic-energy operator becomes:

$$\hat{T} = -\frac{1}{2} \sum_{\sigma=\uparrow,\downarrow} \int \hat{\psi}^\dagger_\sigma(r) \nabla^2 \hat{\psi}_\sigma(r) dr \tag{12.6}$$

$\hat{W}_{ee}$, the electron-electron operator, becomes:

$$\hat{W}_{ee} = \frac{1}{2} \sum_{\sigma_1=\uparrow,\downarrow} \sum_{\sigma_2=\uparrow,\downarrow} \int \int \hat{\psi}^\dagger_{\sigma_2}(r_2) \hat{\psi}^\dagger_{\sigma_1}(r_1) w_{ee}(r_1, r_2) \hat{\psi}^\dagger_{\sigma_1}(r_1) \hat{\psi}^\dagger_{\sigma_2}(r_2) dr_1 dr_2, \tag{12.7}$$

with $w_{ee}(r_1, r_2) = \frac{1}{|r_1 - r_2|}$, and $\hat{V}_{ne}$ is the nuclei-electron operator:

$$\hat{V}_{ne} = \sum_{\sigma=\uparrow,\downarrow} \int \hat{\psi}^\dagger_\sigma(r) v_{ne}(r) \hat{\psi}_\sigma(r) dr. \tag{12.8}$$

In these expressions, $\hat{\psi}^\dagger_\sigma(r)$ and $\hat{\psi}^\dagger_\sigma(r)$ are the creation and annihilation field operators, repsectively, which obey fermionic anticommutation rules:

$$\{\hat{\psi}^\dagger_\sigma(r), \hat{\psi}^\dagger_{\sigma'}(r')\} = 0 \tag{12.9}$$

$$\{\hat{\psi}_\sigma(r), \hat{\psi}_{\sigma'}(r')\} = 0 \tag{12.10}$$

$$\left\{\hat{\psi}^\dagger_\sigma(r), \hat{\psi}^\dagger_{\sigma'}(r')\right\} = \delta(r - r')\delta_{\sigma\sigma'} \tag{12.11}$$

It is also convinient to define the density operator

$$\hat{n}(r) = \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^\dagger_\sigma(r) \hat{\psi}_\sigma(r), \tag{12.12}$$

the one-particle density-matrix operator

$$\hat{n}_1(r, r') = \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^\dagger_\sigma(r') \hat{\psi}_\sigma(r) \tag{12.13}$$

and the pair-density operator

$$\begin{aligned}\hat{n}_2(r_1, r_2) &= \sum_{\sigma_1=\uparrow,\downarrow} \sum_{\sigma_2=\uparrow,\downarrow} \hat{\psi}^\dagger_{\sigma_2}(r_2)\hat{\psi}^\dagger_{\sigma_1}(r_1)\hat{\psi}_{\sigma_1}(r_1)\hat{\psi}_{\sigma_2}(r_2) \\ &= \hat{n}(r_2)\hat{n}(r_1) - \hat{r_1}\delta(r_1 - r_2) \\ &= \hat{n}(r_1)\hat{n}(r_2) - \hat{n}(r_1)\delta(r_1 - r_2) \end{aligned} \tag{12.14}$$

Now we can write a compact form for the kinetic and interaction operators in thethe Hamiltonian:

$$\hat{T} = -\frac{1}{2}\int \left[\nabla^2_r \hat{n}_1(r, r')\right]_{r'=r} \mathrm{d}r, \tag{12.15}$$

$$\hat{W}_{ee} = \frac{1}{2}\int\int \omega_{ee}(r_1, r_2)\hat{n}_2(r_1, r_2)\mathrm{d}r_1\mathrm{d}r_2, \tag{12.16}$$

$$\hat{V}_{ne} = \int v_{ne}(r)\hat{n}(r)\mathrm{d}r. \tag{12.17}$$

We can also use the second-quantization formalism in an orthonormal spin-orbital basis $\{\psi_p(x)\}$, where $x = (r, \sigma)$. For this, we expand the field operators as

$$\hat{\psi}^\dagger_\sigma(r) = \sum_p \psi^*_p(x)\hat{a}^\dagger_p \tag{12.18}$$

and

$$\hat{\psi}_\sigma(r) = \sum_p \psi_p(x)\hat{a}_p \tag{12.19}$$

where $\hat{a}^\dagger_p$ and $\hat{a}_p$ are the creation and annihilation operators in this basis, which still obey anticommutation rules: $\left\{\hat{a}^\dagger_p, \hat{a}^\dagger_q\right\} = \{\hat{a}_p, \hat{a}_q\} = 0$ and $\left\{\hat{a}^\dagger_p, \hat{a}_q\right\} = \delta_{pq}$. The expressions of the operators are then

$$\hat{T} = \sum_{pq} t_{pq}\hat{a}^\dagger_p\hat{a}_q \tag{12.20}$$

$$\hat{W}_{ee} = \frac{1}{2}\sum_{pqrs} \langle\psi_p\psi_q|\psi_r\psi_s\rangle\, \hat{a}^\dagger_p\hat{a}^\dagger_q\hat{a}_s\hat{a}'_r \tag{12.21}$$

$$\hat{V}_{ne} = \sum_{pq} v_{ne,pq}\hat{a}^\dagger_p\hat{a}_q \tag{12.22}$$

where t_{pq} and $v_{\mathrm{ne},pq}$ are the one-electron kinetic and nuclei-electron integrals, respectively, and $\langle\psi_p\psi_q|\psi_r\psi_s\rangle$ are the two-electron integrals.

The quantity of primary interest is the ground-state energy E_0. The variational theorem establishes that E_0 can be obtained by the following minimization

$$E_0 = \min_\Psi \langle \Psi | \hat{H} | \Psi \rangle \,, \tag{12.23}$$

where the search is over all N-electron antisymmetric wave functions Ψ, normalized to unity $\langle \Psi | \Psi \rangle = 1$. DFT is based on a reformulation of the variational theorem in terms of the one-electron density defined as[1]

$$n(r) = N \int \cdots \int |\Psi(x, x_2, \cdots, x_N|^2 \mathrm{d}\sigma \mathrm{d}x_2 \cdots \mathrm{d}x_N), \tag{12.24}$$

which is normalized to the electron number, $\int n(r)\mathrm{d}r = N$.

12.2 The universal density functional

12.2.1 The Hohenberg-Kohn theorem

Consider an electronic system with an arbitrary external local potential $v(r)$ in place of $v_{\mathrm{ne}}(r)$. A corresponding ground-state wave function Ψ (there can be several of them if the ground staate is degenerate) can be obtained by solving the Schrodinger equation, from which the associated ground-state density $n(r)$ can be deduced. Therefore, one has a mapping from the potential $v(r)$ to the considered ground-state density $n(r)$

$$v(r) \to n(r). \tag{12.25}$$

In 1964, Hohenberg and Kohn showed that this mapping can be inverted, i.e. the ground-state density $n(r)$ determines the potential $v(r)$ up to an arbitrary additive constant

$$\boxed{n(r) \xrightarrow{Hohenberg-Kohn} v(r) + \text{const.}} \tag{12.26}$$

Proof. The two-step proof by contradiction proceeds as follows. We consider two locallocal potentials $v_1(r)$ and $v_2(r)$ differing by more than an additive constant, $v_1(r) \neq v_2(r) + \text{const}$, and we denote by E_1 and E_2 the ground-state energies of the Hamiltonians $\hat{H}_1 = \hat{T} + \hat{W}_{ee} + \hat{V}_1$ and $\hat{H}_2 = \hat{T} + \hat{W}_{ee} + \hat{V}_2$, respectively. Now, assume

[1] The integration over the spin coordinate σ in the following equation just means a sum over the two values $\sigma = \uparrow$ and $\sigma = \downarrow$

that $\hat{H}_1$ and $\hat{H}_2$ have the same ground-state wave functions Ψ, i.e. $\hat{H}_1 |\Psi\rangle = E_1 |\Psi\rangle$ and $\hat{H}_2 |\Psi\rangle = E_2 |\Psi\rangle$. Then, substacting these two equations gives

$$(\hat{V}_1 - \hat{V}_2) |\Psi\rangle = (E_1 - E_2) |\Psi\rangle , \tag{12.27}$$

or, in position representation

$$\sum_{i=1}^{N} [v_1(r_i) - v_2(r_i)] \Psi(x_1, x_2, \cdots, x_N) = (E_1 - E_2) \Psi(x_1, x_2, \cdots, x_N) \tag{12.28}$$

□

which implies $v_1(r) - v_2(r) = \text{const}$, in contradiction with the initial hypothesis. Note that, to eliminate Ψ it was assumed that $\Psi(x_1, x_2, \cdots, x_N) \neq 0$ for all spatial coordinates $(r_1, r_2, \cdots, r_N)$ and at least one fixed set of spin coordinates $(\sigma_1, \sigma_2, \cdots, \sigma_N)$. This is in fact true "almost everywhere" for "reasonably well behaved" potentials. In this case, we thus conclude that two local potentials differing by more than an additive constant cannot share the same ground-state wave function.

Let then Ψ_1 and Ψ_2 be (necessarily different) ground-state wave functions of $\hat{H}_1$ and $\hat{H}_2$, respectively, and assume that Ψ_1 and Ψ_2 have the same ground-state density $n(r)$. The variational theorem leads to the following inequality.

$$E_1 = \langle \Psi_1 | \hat{H}_1 | \Psi_1 \rangle < \langle \Psi_2 | \hat{H}_1 | \Psi_2 \rangle = \langle \Psi_2 | \hat{H}_2 + \hat{V}_1 - \hat{V}_2 | \Psi_2 \rangle = E_2 + \int [v_1(r) - v_2(r)] \, n(r) \mathrm{d}r, \tag{12.29}$$

where the strict inequality comes from the fact that Ψ_2 cannot be a ground-state wave function of $\hat{H}_1$, as shown in the first step of the proof. Symmetrically, by exchanging the role of system 1 and 2, we have the stict inequality

$$E_2 < E_1 + \int [v_2(r) - v_1(r)] \, n(r) \mathrm{d}r. \tag{12.30}$$

Adding the two previous equations, we get

$$E_1 + E_2 < E_1 + E_2, \tag{12.31}$$

which finally leads to the conclusion that there cannot exist two local potentials differing by more than an additive constant which have the same ground-state density. Note that this proof does not assume non-degenerate ground states (contrary to the original Hohenberg-Kohn proof) ■

So, the ground-state density $n(r)$ determinese the potential $v(r)$, which in turn determines the Hamiltonian, and thus everything about the many-body problem. In other words, the potential v is a unique (up to an additive constant) functional of the ground-state density n, and all other properties as well. The ground-state wave function Ψ for the potential $v(r)$ is itself a functional of n, denoted by $\Psi[n]$, which was exploited by Hohenberg-Kohn to define the universal (i.e., independent from the external potential) density functional

$$F[n] = \langle \Psi[n] | \, \hat{T} + \hat{W}_{ee} \, | \Psi[n] \rangle \, , \tag{12.32}$$

which can be used to define the total electronic energy functional

$$E[n] = F[n] + \int v_{ne}(r) n(r) \mathrm{d}r, \tag{12.33}$$

for the specific potential $v_{ne}(r)$ of the system considered. Note that, forfor degenerate ground states, $\Psi[n]$ is not unique but stands for any degenerate ground-state wave function. However, all $\Psi[n]$ give te same $F[n]$, which is thus a unique functional of n.

Hohenberg and Kohn further showed that the density functional $E[n]$ satisfies a variational property: the ground-state energy E_0 of the system considered is obtained by minimizing this functional with respecst to N-electr-electron densities n that are ground-state densities associated with some local potential (referred to as v-representable densities).

$$E_0 = \min_n \left\{ F[n] + \int v_{ne}(r) n(r) \mathrm{d}r \right\}, \tag{12.34}$$

the minimum being reached for a ground-state density $n_0(r)$ corresponding to the potential $v_{ne}(r)$. The existance of a mapping from a ground-state density to a local potential, the existance of the universal density functional, and the variational property with respect to the density constitutes the *Hohenberg-Kohn theorem*.

Part II

Mathematics

Chapter 13

Group Theory

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"The Universe is an enormous direct product of resentations of symmetry groups."

Hermann Weyl

13.1 Basic Definitions

Definition 13.1 ((Group) Homomorphism)**.** *Let $(G, *)$ and $(H, \cdot)$ be two groups. A (group) homomorphism from G to H is a function $h : G \to H$ such that for all x, y in G it holds that $h(x * y) = h(u) \cdot h(v)$*

Definition 13.2 (Coset)**.** *Let G be a group and H is a subgroup of G. Consider an element $g \in G$. Then, $gH = \{gh : h \in H\}$ is the left coset of H in G with respect to g, and $Hg = \{hg : h \in H\}$ is the right coset of H in G with respect to g*

Remark 13.1. In general the left and right cosets are not groups.

Definition 13.3 (Normal Subgroup)**.** *A subgroup H of G is called normal if and only if the left and right sets of cosets coincide, that is if $gH = Hg$ for all $g \in G$*

Definition 13.4 (Representation)**.** *A representation of a group G on a vector space V over a filed K is a group homomorphism from G to $GL(V)$, the general linear group on V. That is, a representation is a map*

$$\rho : G \to GL(V) \tag{13.1}$$

such that

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \quad \forall g_1, g_2 \in G. \tag{13.2}$$

V is often called the *representation space* and the dimension of V is called the *dimension* of the representation. It is common practice to refer to V itself as the representation when the homomorphism is clear from the context.

Example 5. *Consider the complex number $u = e^{2\pi/3}$ which has the property $u^3 = 1$. The cyclic group $C_3 = \{1, u, u^2\}$ has a representation ρ on $\mathbb{C}^2$ given by:*

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u) = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}, \quad \rho(u^2) = \begin{bmatrix} 1 & 0 \\ 0 & u^2 \end{bmatrix}, \quad . \tag{13.3}$$

Another representation for C_3 on $\mathbb{C}^2$, isomorphic to the previous one, is

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u^2) = \begin{bmatrix} u^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad . \tag{13.4}$$

Definition 13.5 (Subrepresentation)**.** *A subspace W of V that is invariant under the group action is called a subrepresentation.*

Definition 13.6 ((Ir)reducible representation)**.** *If V has exactly two representations, namely the zero-dimensional subspace and V itself, then the representation is said to be irreducible; if it has a proper representation of nonzero dimension, the representation is said to be reducible. The representation of dimension zero is considered to be neither reducible nor irreducible.*

Definition 13.7 (Quotent Group)**.** *Let N be a normal subgroup of a group G. We define the set G/N to be the set of all left cosets of N in G, i.e., $G/N = \{aN : a \in G\}$. Define an operation on G/N as follows. For each aN and bN in G/N, the product of aN and bN is $(aN)(bN)$. This defines an operation of G/N if we impose $(aN)(bN) = (ab)N$, because $(ab)N$ does not depend on the choice of the representatives a and b: if $xN = aN$ and $yN = bN$ for some $x, y \in G$, then:*

(ab)N = a(bN) = a(yN) = a(Ny) = (aN)y = (xN)y = x(Ny) = x(yN) = (xy)N

Here it was used in an important way that N is a normal subgroup. It can be shown that this operation on G/N is associative, has identity element N and the inverse of an element $aN \in G/N$ is $a^{-1}N$. Therefore, the set G/N together with the defined operation forms a group; this is known as the quotient group or factor group of G by N

13.2 Lie groups and representations

Every symmetry group that we work with in field theory is a Lie group. We will focus on $SU(N)$ as a working example, but everything generalizes for all other Lie groups. $SU(N)$ consists of (Hermitian and unit determinant) matrices of size $N \times N$ that act on $\mathbb{C}^n$. $SU(N)$ has $dim(SU(N))$ basis elements and each of these basis elements is an $N \times N$ matrix. Note that $dim(SU(N)) \neq N$.

For every Lie group there is an associated Lie algebra (sometimes called the tangent space in the literature). The Lie algebra of $SU(N)$ is denoted by $\mathfrak{su}(N)$.$\mathfrak{su}(N)$ has the same dimension as $SU(N)$ has the same dimension as $SU(N)$. We denote a basis of $\mathfrak{su}(N)$ by $T^a,\ a = 1, \cdots, dim(SU(N))$. Note that T^a are also $N \times N$ matrices. Every Lie algebra has an additional structure called the *Lie Bracket*, which classifies the algebra, denoted by

Definition 13.8 (Lie Bracket)**.**

$$\left[T^a, T^b\right] = f_c^{ab} T^c \tag{13.5}$$

f_c^{ab} are called the *structure constants* of the Lie algebra. They encode the structure of the group. We can gen an element of the Lie group by exponentiating an element of the Lie algebra:

$$\begin{gathered} \text{Lie algebra} \xrightarrow{exp} \text{Lie group} \\ exp(i\omega_a T^a) \in SU(N) \end{gathered} \tag{13.6}$$

We have chosen T^a to be Hermitian, resulting in a factor of i in the exponent. If we choose T^a to be anti-Hermitian, then there would be no factor of i. However, we always work with Hermitian operators, so there is always a factor of i. Since an element of the Lie group can be found by exponentating some element of the Lie algebra and we can expand any element of the Lie algebra in terms of the basis

of the Lie algebra (for example, T^a), then this basis generates any element of the Lie algebra and by exponentating, generates any element of the Lie group (that is connected to the origin)[1] For this reason, the basis of the Lie algebra are called the *generators*.

However, $SU(N)$ only acts on $\mathbb{C}^N$. What if we want a group with the same structure (constants) as $SU(N)$[2] but that act on different spaces such as $\mathbb{C}^m$, for $m \neq N$? Representations of $SU(N)$ allow us to do this. They allow us to construct groups with the same structure (constants) as $SU(N)$ but which act on different spaces.

13.2.1 Representations

A representation of a group $\mathcal{G}$ is a homomorphism from $\mathcal{G}$ to the space of linear maps acting on a representation space, $\mathcal{V}_{rep}$:

$$\rho : \mathcal{G} \to GL(\mathcal{V}_{rep}) \tag{13.7}$$

Every linear map can be thought of as a matrix and so an element of $\rho(\mathcal{G})$, say R, can be thought of as a matrix. $\rho(\mathcal{G})$ is a group of matrices and is called a *representation* of the group. Each R acts on $\mathcal{V}_{rep}$:

$$R : \mathcal{V}_{rep} \to V_{rep} \tag{13.8}$$

Since R is a matrix that acts on $\mathcal{V}$, it has to be a matrix of size $dim(\mathcal{V}_{rep}) \times dim(\mathcal{V}_{rep})$. The degree of the representation is the dimension of the representation space:

$$deg(\rho) = dim(\mathcal{V}_{rep}) \tag{13.9}$$

So the representation $\rho(\mathcal{G})$ has dimension equal to $dim(\mathcal{G})$ and each element of $\rho(\mathcal{G})$ is a square matrix of size $dim(\mathcal{V}_{rep})$. We have to be careful when talking about the dimensions of $\mathcal{G}$ and $\mathcal{V}_{rep}$ as it is easy to get confused when considering both. Examples will help clarify these concept. Different representations can act

[1] A group can be split up into different components. The Lorentz group, for example, consists of 4 disconnected components. One of this components, called the proper orthochronous Lorentz group $SO(1,3)^+$ is connected to the origin. Any relativistic theory will be invariant under $SO(1,3)^+$ (or $SO(3,1)$, depending on convention) but not neccesarily under the other components. Other groups, such as $SU(N)$ have only one component, which is connected to the origin. As such, exponenting the Lie algebra generates all of $SU(N)$.

[2] It should be noted that $SU(N)$ have different structure constants for different N

on different spaces. Representations are useful for describing how different states transform under the action of a particular symmetry.

13.2.2 Some noteworthy points

- The number of basis elements in the representation is the same as the number of basis elements of the group, $\mathcal{G}$. Another way of saying this is $dim(\rho(\mathcal{G})) = dim(\mathcal{G})$.
- A matrix in the representation, say R, acts on a representation space, $\mathcal{V}_{rep}$, by matrix multiplication. An element of $\mathcal{V}_{rep}$ can be considered as a column vector of $dim(\mathcal{V}_{rep})$. So R has to be a matrix of size $dim(\mathcal{V}_{rep}) \times dim(\mathcal{V}_{rep})$. This is why we have to seperate out the dimension of ρ from the degree. The examples below will clarify this idea.
- Recap: The representation of the symmetry group can be described using the generators. The generators are a basis for the representation on the Lie algebra. The number of generators is the number of basis elements of the symmetry group $= dim(\mathcal{G})$. Each generator has size $dim(\mathcal{V}_{rep}) \times dim(\mathcal{V}_{rep})$. Each representation has the same structure constants.

13.2.3 Example

If we consider the group $\mathcal{G} = SU(2)$ when thinking about spin angular momentum, this can give an insight into how representations work. For $SU(N)$, we normally work with irreducible Representation spaces. Consider each irreducible representation space labelled by j seperately, i.e. $\mathcal{V}_{rep} = \mathcal{V}_j$. Call the $3 = dim(SU(2))$ generators of the representation $T^a = J^a$, where we normally call $J^1 = J^x, J^2 = J^y, J^3 = J^z$ angular momentum. Examine the following table in detail to understand how the generators can be different sized matrices.

It is important to notice that there are always **only** $dim(SU(2)) = 3$ generators T^a. However, these generators can be different sized matrices depending on which representation space $\mathcal{V}_j$ the generators act on. Each representation has the same structure constants. As such, each representation has the same structure of the original group but allows us to act on different spaces.

j	basis for $\mathcal{V}_j$	$dim(\mathcal{V}_j)$ = degree of rep	$T^a = J^a$
$j = 0$	$\|0, 0\rangle$	1	$T^a = 0$, so that it doesn't transform under SU(2)
$j = \frac{1}{2}$	$\left\|\frac{1}{2}, m\right\rangle$	2	T^a are 2x2 matrices, equal to $\sigma^a/2$, where σ are the Pauli matrices
$j = 1$	$\|1, m\rangle$	3	T^a are 3x3 matrices, equal to the generalization of the Pauli matrices

Table 13.1: Some representations for SU(2) angular momentum

13.2.4 More points

- In general, consider $\chi_1 \in \mathcal{V}_{rep}$ and let T^a be a generator of a representation of the group, then it acts on the representation space as:

$$T^a \chi_1 = \chi_2 \in \mathcal{V}_{rep} \tag{13.10}$$

- One way to know what representation you are in is to see what size matrices the generators T^a are. Some examples are given later. If the generators are $m \times m$ matrices, then an element of the representation of the group, g, can be written as $g = exp(\lambda^a T^a)$. So g is also an $m \times m$ matrix, i.e, every representation of the Lie algebra gives a representation of the group by exponentating. Since the generators must act on elements of the representation space, and the generators are matrices of size $m \times m$, the elements of the representation space must be m component column vectors.

- A gauge theoroy has the Lagrangian $\mathcal{L} = \bar{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi$, where the field Ψ is in a representation space of the gauge group and D_μ is the covariant derivative. D_μ is defined to transform under an element of the representation, U, of the gauge group as

$$D_\mu \Psi \to U D_\mu \Psi = (U D_\mu U^{-1})(U\Psi) \tag{13.11}$$

  and so $D_\mu \to U D_\mu U^{-1}$. This transformation property is called the adjoint action of the group on the Lie algebra. Because of this transform property,

the covariant derivative is said to transform in th adjoint action, or just adjointly. $D_\mu = \partial_\mu - igA_\mu(x)$, where $A_\mu(x)$ is an element of the representation of the Lie algebra. We can expand $A_\mu(x)$ in a basis of the representation so that $A_\mu(x) = A^a + \mu(x)T^a$, where $A^a_\mu(x)$ are the gauge field coefficients with corresponding (matrix) generators T^a. The gauge field A^a_μ are said to transform under the adjoint action in order to make the covariant derivative transform correctly. The transformation is the standard transformation:

$$A_\mu(x) \to U(x)A_\mu U(x)^\dagger - \frac{1}{e}(\partial_\mu U(x))U(x)^\dagger \tag{13.12}$$

This is precisely the right transformation to make the covariant derivative transform as required

$$D_\mu \chi(x) \to U(x) D_\mu \Psi(x) \tag{13.13}$$

The generators **do not** have to be in the adjoint representation of the group and in general they are not. They just have to transform adjointly.

- We want to consider infinitesmal transformations. Let $U = exp(\lambda^a T^a) = 1 + \lambda^a T^a +$ higher order terms. Call $\lambda = \lambda_a T^a$. We know

$$A_\mu(x) \to U A_\mu U^{-1} - \frac{1}{e}(\partial_\mu U)U^{-1}(x) \tag{13.14}$$

$$= A_\mu - \frac{1}{e}\left\{\partial_\mu \lambda + e\left[A_\mu, \lambda\right]\right\} \tag{13.15}$$

$$= A_\mu - \frac{1}{e} D_\mu \lambda \tag{13.16}$$

Because of laziness, we don't want to write out $[A_\mu, \lambda]$ all the time. We give it a special name: the adjoint action of the Lie algebra on itself. Write it $A_\mu \cdot \lambda = [A_\mu, \lambda]$ or $(A_\mu \cdot \lambda)^{ad} = [A_\mu, \lambda]$. This comes from the adjoint action of the group on the Lie algebra in infinitimesal form. But we can expand in a basis, $A_\mu - A^a_\mu T^a, \lambda = \lambda^a T^a$. The A^a_μ, λ^a are field coefficents when expanded in the (matrix) basis T^a. They are fields which commute. But T^a are matrices which in general don't commute. So

$$\begin{aligned} D_\mu \lambda &= \partial_\mu \lambda + (A_\mu \cdot \lambda)^{ad} \\ &= \partial_\mu \lambda + e\left[A_\mu, \lambda\right] \\ &= \partial_\mu \lambda^a T^a + e A^a_\mu \lambda^b \left[T^a, T^b\right] \\ &= \partial_\mu \lambda^a T^a + e A^a_\mu \lambda^b f^{ab}_c T^c \quad \text{using } \left[T^a, T^b\right] = f^{ab}_c T^c \end{aligned} \tag{13.17}$$

In component form, the covariant derivative acts on an element of the Lie algebra as

$$\begin{aligned}(D_\mu\lambda)_a &= \partial_\mu\lambda_a + (A_\mu\cdot\lambda)^{ad}_a \\ &= \partial_\mu\lambda_a + e\left[A_\mu,\lambda\right]_a \\ &= \partial_\mu\lambda_a + eA^b_\mu\lambda^c f^{bc}_a \end{aligned} \tag{13.18}$$

- The main point to focus on here is the adjoint action. Do not be confused between the adjoint ($(A_\mu\cdot\lambda)^{ad} = [A_\mu,\lambda]$) and the adjoint representation. They are two different things. The adjoint action is true for all representations, i.e, for all $m\times m$ matrices T^a. But the adjoint re[representation is a specific representation of the group. In the adjoint representation, the matrices have a specific size, where $m = dim(G)$. This is standard (sometimes confusing) terminology.

13.2.5 Certain Types of Representations

When thinking about representations of $SU(N)$ as $m\times m$ matrices, there are three natural numbers to consideer for m. One is $m = 1$, i.e, a scalar. This is called the trivial representation. Another is $m = n$, where the $SU(n)$ matrices themselves are the representation. This is called the fundamental representation. The last is when $m = dim(SU(n))$. This is called the adjoint representation.

The **trivial** representation is whre $\rho(g) = 0$ so that all generators of the representation have $T^a = 0$. This is the $j = 0$ representation of $SU(2)$. $|j = 0, 0\rangle$ is said to be a singlet or scalar as it doesn't transform under $SU(2)$.

The **fundamental** representation is the representation which is equal to the group. This is the homomorphism $\rho(g) = g$. In the fundamental representation, when considering $SU(n)$, the representation $\rho(SU(n)) = SU(n)$, i.e. the $SU(n)$ matrices are the representation matrices. The representation is classified by $n\times n$ matrices. $SU(n)$ must act on a column vector of size n and hence the dimension of the representation space must be n. This is the $j = 1/2$ representation of $SU(2)$.

In the **adjoint** representation, it can be shown that the generators have the component form $(T^a_{ad})_{bc} = f^a_{bc}$, where f^a_{bc} are the anti-symmetric structure constants. If we know the structure constants, then we can construct the adjoint representation. E.g, $f_{abc} = \varepsilon_{abv}$ for $SU(2)$.

13.2.6 Usefulness of Lie group representations

Representation spaces are used to clarify spin and isospin using representations of $SU(2)$. Hadrons are described by representation spaces of $SU(3)_{\text{flavour}}$, for example the eight fold way. Representation spaces of the Poincaré group are classifeid in terms of a particles spin and momentum.

Historically, weak decays were seen to involve flavour changing decays and break parity in experiment. We want to construct a Lagrangian to describe weak decays and to make theoretical predictions. Parity sends left handed fields to right handed fields and so we put the left handed components of each generation of matter into a two component column vector. We call this L. As an example, for the electron and neutrino we have $L = (\nu_e, e_L)^T$. We require that the terms involving L in the Lagrangian be invariant under a $SU(2)$ gauge symmetry in order to introduce gauge fields which mediate a force between the components. We call this $SU(2)_{\text{Left}}$. The kinetic term for L contains a covariant derivative in the form given above. It contains generators for the representation of $SU(2)_{\text{L}}$. These generators have to act on L. L is two component and so we see that the generators, T^a must be 2×2 matrices. This is justthe fundamental representation of $SU(2)_{\text{L}}$. So L is said to be in the fundamental representation.

Now, in order to maximally break parity, we do not want the right handed parts of the matter fields to feel force. So we do not put the right handed parts of matter fields into a doublet. The right handed part of the matter fields do not transform under $SU(2)_{\text{L}}$. They are in the trivial or singlet representation of $SU(2)_{\text{L}}$. This is the $j = 0$ representation of $SU(2)$ discussed above.

In QCD, we introduce a new symmetry called $SU(3)_{\text{colour}}$. We say that each quark field Ψ now comes with a colour: either red, blue or green. Put the different coloured quark fields into a column vector $\chi = (\Psi_{\text{red}}, \Psi_{\text{blue}}\Psi_{\text{green}})^T$. For example, using the up-type quark $\chi\,(u_{\text{red}}, u_{\text{blue}}, u_{\text{green}})^T$. Make a Lagrangian that will be invariant udner a $SU(3)_{\text{colour}}$ gauge symmetry:

$$\mathcal{L} = \bar{\chi}\,(i\gamma^\mu D_\mu - m)\,\chi. \tag{13.19}$$

Since the χ is three component, the generators for the representation of the $SU(3)_{\text{colour}}$ symmetry will have to act on χ and so these generators will have to be 3×3 matrices. This is the fundamental representation of the group $SO(3)_{\text{colour}}$. The quark fields are said to be in the fundamental representation of $SO(3)_{\text{colour}}$. Leptons do not feel the strong force eand so transform trivially under $SO(3)_{\text{colour}}$.

This is the representation where the generators for $SO(3)_{\text{colour}}$ are $T^a = 0$. It is similar to the right handed fields in the $SU(2)_{\text{Left}}$ representation.

When working in the fundamental representation, it is ok to think of the group representation transformations just as the group transformations and the representation of the Lie algebra as just the Lie algebra.

13.3 Automorphic Forms

In this section we provide a pedestrian introduction to automorphic forms and theta series, emphasizing on examples important for string theory and M-theory. Automorphic forms play an important role in physics, especially in the context of string theory and M-theory dualities. Notably, U-dualities, first discovered as symmetries of classical toroidal compactifications of 11-dimensional supergravity by Cremmer and Julia [1] and later on elevated to quantum postulates by Hull and Townsend [2], motivate the study of automorphic forms for exceptional arithmetic groups $E_n(\mathbb{Z})(n = 6, 7, 8$ or their A_n and D_n analogues for $1 \leq n \leq 5)$ - see *e.g* [3] for a review of U-duality. These chapter is aimed to provide a pedestrian introduction to these seemingly abstract mathematical objects, designed to offer a concrete footing for physicists. The basic concepts are introduced via the simple $Sl(2)$ Eisenstein (not to be confused with Einstein!) and theta series. The general construction of continuous representations and of their accompanying Eisenstein series is detailed for $Sl(3)$. Thereafter, unipotent representations and their theta series for arbitrary simply-laced groups are presented.

13.3.1 Eisenstein and Jacobi Theta series

The general mechanism underlying automorphic forms is best illustrated b taking a representation-theoretic tour of two familiar $Sl(2,\mathbb{Z})$ examples:

13.3.1.1 $Sl(2,\mathbb{Z})$ Eisenstein Series

Our first example is the non-holomorphic Eisenstein series

$$\mathcal{E}_s^{Sl(2)}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \left(\frac{\tau_2}{|m+n\tau|^2}\right)^s \tag{13.20}$$

which, for $s = 3/2$, appears in string theory as the description of the complete, non-perturbative, four-graviton scattering amplitude at low energies [6]. It is a function of the complex modulus τ, taking values on the Poincaré upper half plane, or equivalently points in the symmetric space $\mathcal{M} = K\backslash G = SO(2)\backslash Sl(2,\mathbb{R})$ with coset representatie

$$e = \frac{1}{\sqrt{\tau_2}} \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} \in Sl(2,\mathbb{R}) \tag{13.21}$$

The Eisenstein series (13.20) is invariant under the modular transformation

$$\tau \to (a\tau + b)/(c\tau + d) \tag{13.22}$$

which is the right action of $g \in Sl(2,\mathbb{Z})$ on $\mathcal{M}$. Invariance follows simply from that of the lattice $\mathbb{Z} \times \mathbb{Z}$. This setup may be formalized by introducing:

(i) The linear representation ρ of $Sl(2,\mathbb{R})$ in the space $\mathcal{H}$ of functions of two variables $f(x,y)$,

$$[\rho(g)\cdot f](x,y) = f(ax+by, cx+dy), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad-bc = 1. \tag{13.23}$$

(ii) An $Sl(2,\mathbb{Z})$-invariant distribution

$$\delta_{\mathbb{Z}}(x,y) = \sum_{(m,n)\in\mathbb{Z}\backslash(0,0)} \delta(x-m)\delta(y-n) \tag{13.24}$$

in the dual space $\mathcal{H}^*$.

(iii) A vector

$$f_K(x,y) = (x^2+y^2)^{-s} \tag{13.25}$$

invariant udner the maxiamal compact subgroup $K = SO(2) \in G = Sl(2,\mathbb{R})$.

The Eisenstein series may now be recast in a general notation for automorphic forms

$$\mathcal{E}_s^{Sl(2)}(e) = \langle \delta_{\mathbb{Z}}, \rho(e)\cdot f_K\rangle, \quad e \in G. \tag{13.26}$$

The modular invariance of $\mathcal{E}_s^{Sl(2)}$ is now manifest: under the right action $e \to eg$ of $g \in Sl(2,\mathbb{Z})$, the vector $\rho(e)\cdot f_k$ transforms by $\rho(g)$, which in turn hits the $Sl(2,\mathbb{Z})$ invariant distribution $\delta_{\mathbb{Z}}$. Furhtermore, (13.26) is ensured to be a function of the

coset $K\backslash G$ by invariance of the vector f_K under the maximal compact K. Such a distinguished vector is known as *spherical*. All the automorphic forms we shall encounter can be written in terms of a triplet $(\rho, \delta_{\mathbb{Z}}, f_K)$.

Clearly any other function of the $SO(2)$ invariant norm $|x,y|_\infty \equiv \sqrt{x^2+y^2}$ would be as good a candidate for f_K. This reflects the reducibility of the representation ρ in (13.21). However, its restriction to homogeneous, even functions of degree $2s$,

$$f(x,y) = \lambda^{2s} f(\lambda x, \lambda y) = y^{-2s} f(\frac{x}{y}, 1), \tag{13.27}$$

is irreducible. The restriction of the representation ρ acts on the space of functions of a single variable $z = x/y$ by weight $2s$ conformal transformations $z \to (az+b)/(cz+d)$ and admits $f_K(z) = (1+z^2)^{-s}$ as its unique spherical vector. In these variables, the distribution $\delta_{\mathbb{Z}}$ is rather singular as its support is on all rational values $z \in \mathbb{Q}$. A related problem is that the behaviour of $\mathcal{E}_s^{Sl(2)}(\tau)$ at the cusp $\tau \to i\infty$ is difficult to assess - yet of considerable interest to physicists being the limit relevant to non-perturbative instantons.

These two problems may be evaded by performing a Poisson resummation on the integer $m \to \tilde{m}$ in the sum of the $Sl(2,\mathbb{Z})$ invariant distribution (13.24), after first separating out terms with $n = 0$. The results may be rewritten as a sum over the single variable $N = \tilde{m}n$, except for two degenerate - or "perturbative" - contributions:

$$\begin{aligned}\mathcal{E}_s^{Sl(2)} = 2\zeta(2s)\tau_2^s + \frac{2\sqrt{\pi}\tau_2^{1-s}\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)} \\ + \frac{2\pi^s\sqrt{\tau_2}}{\Gamma(s)} \sum_{N\in\mathbb{Z}\setminus\{0\}} \mu_s(N) N^{s-1/2} K_{s-1/2}(2\pi\tau_2 N) e^{2\pi i \tau_1 N}.\end{aligned} \tag{13.28}$$

In this expression, the summation measure

$$\mu_s(N) = \sum_{n|N} n^{-2s+1} \tag{13.29}$$

is of prime physical interest, as it is connected to quantum fluctuations in an instanton background.

First focus on the non-degenerate terms in the second line. Analyzing the transformation properties under the Borel and Cartan $Sl(2)$ generators $\rho\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$:

$\tau_1 \to \tau_1 + t$ and $\rho \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} : \tau_2 \to t^2\tau_2$, we readily see that they fit into the framework (13.26), upon identifying

$$f_K(z) = z^{s-1/2}K_{s-1/2}(z), \quad \delta_{\mathbb{Z}}(z) = \sum_{N\in\mathbb{Z}\setminus\{0\}} \mu_s(N)\delta(z-N), \tag{13.30}$$

and the representation ρ as

$$E_+ = iz, \quad E_- = i(z\partial_z + 2 - 2s)\partial_z, \quad H = 2z\partial_z + 2 - 2s. \tag{13.31}$$

This is of course equivalent to the representation on homegeneous functions, upon Fourier transform in the variable z. The power-like degenerate terms may be viewed as regulating the singular value of the distribution δ at $z = 0$. They may, in principle, be recovered by performing a Weyl reflection on the regular part. It is also easy to check that the spherical vector condition, $K\cdot f_K(z) \equiv (E_+ - E_-)\cdot f_K(z) = 0$, is the modified Bessel equation whose unique decaying solution at $z \to \infty$ is the spherical vector in (13.30).

While the representation ρ and its spherical vector f_K are easily understood, the distribution $\delta_{\mathbb{Z}}$ requires additional technology. Remarkably, the summation measure (13.29) can be written as an infinite product

$$\mu_s(z) = \prod_{p\in \text{ primes}} f_p(z), \quad f_p(z) = \frac{1 - p^{-2s+1}|z|_p^{2s-1}}{1 - p^{-2s+1}}\gamma_p(z). \tag{13.32}$$

(A simple trial computation of $\mu_s(2.3^2)$ will easily convince the reader of this equality.) Here $|z|_p$ is the p-adic[3] norm of z, i.e $|z|_p = p^{-k}$ with k the largest integer such that p^k divides z. The function $\gamma_p(z)$ is unity if z is a p-adic integer ($|z|_p \leq 1$) and vanishes otherwise. Therefore $\mu(z)$ vanishes unless z is an integer N. Equation (13.26) can therefore be expressed as

$$\mathcal{E}_s^{Sl(2)}(e) = \sum_{z\in\mathbb{Q}} \prod_{p\in\text{primes},\infty} f_p(z)\rho(e)\cdot f_K(z), \tag{13.33}$$

The key observation now is that f_p is in fact the spherical vector for the representation of $Sl(2,\mathbb{Q}_p)$, just as $f_\infty := f_K$ is the spherical vector of $Sl(2,\mathbb{R})$. The reader

[3]A useful physics introduction to p-adic and adelic fields in [10]. It is worth noting that a special function theory analogous to that over the complex numbers exists for the p-adics.

can be convinced of this important fact, by evaluating the p-adic Fourier transform of $f_p(y)$ on y, thereby reverting to the $Sl(2)$ representation on homogeneous functions: the result

$$\tilde{f}_p(x) = \int_{\mathbb{Q}_p} \mathrm{d}z f_p(z) e^{ixz} = |1, x|_p^{-2s} \equiv \max(1, |x|_p)^{-2s}, \tag{13.34}$$

is precisely the p-adic counterpart of the real spherical vector $f_K(x) = (1+x^2)^{-s} \equiv |1, x|_\infty^{-2}$. The analogue of the decay condition is that f_p should have support over the p-adic integers only, which holds by virtue of the factor $\gamma_p(y)$. It is easy to check that the formula (13.33) in this representation reproduces the (13.20).

Thus, the $Sl(2,\mathbb{Z})-$invariant distribution $\delta_{\mathbb{Z}}$ can be straightforwardly obtained by computing the spherical vector over all p-adic fields $\mathbb{Q}_p$. More conceptually, the Eisenstein series may be written *adelically* (or *globally*) as

$$\mathcal{E}_s^{SL(2)}(e) = \sum_{z\in\mathbb{Q}} \rho(e) \cdot f_{\mathbb{A}}(z), \quad f_{\mathbb{A}} = \prod_{p=\text{prime},\infty} f_p(z), \tag{13.35}$$

where the sum $z \in \mathbb{Q}$ is over principle adeles[4], and $f_{\mathbb{A}}$ is the spherical vector of $Sl(2,\mathbb{A})$, invariant under the maximal compact subgroup $K(\mathbb{A}) = \prod_d Sl(2,\mathbb{Z}_p) \times U(1)$ of $Sl(2,\mathbb{A})$. This relation between functions on $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$ and functions on $G(\mathbb{Q})\backslash G(\mathbb{A})/K(\mathbb{A})$ is known as the Strong Approximation Theorem, and is a powerful tool in the study of automorphic forms.

13.3.1.2 Jacobi Theta Series

Our next example, the Jacobi theta series, demonstates the key role played by Fourier invariant Gaussian characters - "*the Fourier transform of the Gaussian is the Gaussian*". Our later generalization will involve cubic type characters invariant under Fourier transform.

In contrast to the Eisenstein series, the Jacobi theta series

$$\theta(\tau) = \sum_{m\in\mathbb{Z}} e^{i\pi\tau m^2}, \tag{13.36}$$

is a modular form for a congruence subgroup $\Gamma_0(2)$ of $Sl(2,\mathbb{Z})$ with modular weight $1/2$ and a non-trivial multiplier system. It may, nevertheless, be cast in the framework (13.26), with a minor caveat. The representation ρ now acts on functions of

[4]Adeles are infinite sequences $(z_p)_{p=\text{prime},\infty}$ where all but a finite set of z_p are p-adic integers. Principle adeles are constant sequences $z_p = z \in \mathbb{Q}$, isomorphic to $\mathbb{Q}$ itself.

a single variable x as

$$E_+ = i\pi x^2, \quad H = \frac{1}{2}(x\partial_x + \partial_x x), \quad E_- = \frac{i}{4\pi}\partial_x^2 \tag{13.37}$$

Here, the action of E_+ and H may be read off form the usual Borel and Cartan actions of $Sl(2)$ ot τ while the generator E_- follows by noting that the Weyl reflection $S : \tau \to -1/\tau$ can be compensated by Fourier transform on the integer m. The invariance of the "comb" distribution $\delta_{\mathbb{Z}} = \sum_{m\in\mathbb{Z}} \delta(x-m)$ under Fourier transform is just the Poisson resummation formula.

Finally (the caveat), the compact generator $K = E_+ - E_-$ is exactly the Hamiltonian of the harmonic oscillator, which notoriously does not admit a normalizable zero energy eigenstate, but rather the Fourier-invariant ground state $f_\infty(x) = e^{-\pi x^2}$ of eigenvalue $i/2$. This relaxation of the spherical vector condition is responsible for the non-trivial modular weight and multiplier system. Correspondingly, ρ does not represent the group $Sl(2,\mathbb{R})$, but rather its double cover, the metaplectic group.

Just as for the Eisenstein series, an adelic formula for the summation measure exists: note that the p-adic spherical vector must be invariant under the compact generator S which acts by Fourier transform. Remarkably, the function $f_p(x) = \gamma_p(x)$, imposing support on the integers only is Fourier-invariant - it is the p-adic Gaussian! One therefore recovers the "comb" distribution with uniform measure. Note that the $Sl(2) = Sp(1)$ theta series generalizes to higher symplectic groups under the title of Siegel theta series, relying in the same way on Gaussian Poisson resummation.

13.3.2 Continuous representations and Eisenstein series

The two $Sl(2)$ examples demonstrate that the essential ingredients for automorphic forms with respect to an arithmetic group $G(\mathbb{Z})$ are (i) an irreducible representation of ρ of G and (ii) corresponding spherical vectors over $\mathbb{R}$ and $\mathbb{Q}_p$. We now explain how to construct these representations by quantizing coadjoint orbits.aaa

Chapter 14

Differential Geometry

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14.1 General

Definition 14.1 (Einstein Manifold)**.** *A Riemannian manifold M is called an Einstein manifold if its Ricci tensor is proportional to the metric, i.e.*

$$Ric_g = \lambda g \tag{14.1}$$

Remark 14.1. An Einsteinian manifold, where $\lambda = 0$ is called a *Ricci-flat* manifold.

Definition 14.2 (Covering space)**.** *Let X be a topological space. A covering space of X is a topological space C together with a continuous surjective map*

$$p : C \to X \tag{14.2}$$

such that for every $x \in X$, there exists an open neighborhood U of x, such that $p^{-1}(U)$ is a union of disjoint open sets in C, each of which is mapped homeomorphically onto U by p.

The map p is called the covering map, the space X is often called the base space of the covering, and the space C is called the total space of the covering. For any point x in the base the inverse image of x in C is necessarily a discrete space called the fiber over x. The special open neighbourhoods U of x given in the definition are called *evenly covered neighbourhoods*. The evenly covered neighbourhoods form an open cover of the space X. The homeomorphic copies of C of an evenly covered neighborhood U are called the *sheets* over U.

14.2 Differential Forms

"Hamiltonian mechanics cannot be understood without differential forms"

V. I. Arnold

In this section we will give a very brief introduction to differential forms and the general rules for calculus with differential forms.

14.2.1 Exterior forms

We begin with the more general notion of a *exterior form*, which is generally a poly-linear map from a vector space to an algebraic filed.

14.2.1.1 Definitions

Let $\mathbb{L}^n$ be an n-dimensional real vector space.

Definition 14.3 (Exterior algebraic form of degree k)**.** *An exterior algebraic form of degree k, also known as a k-form, is a function of k vectors which is k-linear and anti-symmetric. Namely:*

$$\omega(\lambda_1\xi_1' + \lambda_2\xi_1'', \xi_2, \cdots, \xi_k) = \lambda_1\omega(\xi_1', \xi_2, \cdots, \xi_k) + \lambda_2\omega(\xi_1'', \xi_2, \cdots, \xi_k), \tag{14.3}$$

and

$$\omega(\xi_{i_1}, \cdots, \xi_{i_k}) = (-1)^\nu\omega(\xi_1, \cdots, \xi_k) \tag{14.4}$$

with $(-1)^\nu = 1$ if the permutation $(i_1, \cdots, i_k)$ is even and $(-1)^\nu = -1$ if the same permutation is odd. Here, $\xi_i \in \mathbb{L}^n$ and $\lambda_i \in \mathbb{R}$.

The set of all k-forms in $\mathbb{L}^n$ forms a real vector space. Indeed, one has that:

$$(\omega_1 + \omega_2)(\xi) = \omega_1(\xi) + \omega_2(\xi), \quad \xi = \{\xi_1, \cdots, \xi_k\} \tag{14.5}$$

and

$$(\lambda\omega)(\xi) = \lambda\omega(\xi). \tag{14.6}$$

Since $\mathbb{L}^n$ is a vector space, we can always suppose that it is equipped with a coordinate system. Let us denote these coordinates by $x_1, \cdots, x_n$. Now, we can think

of these coordinates as 1-forms so that $x_i(\xi) = \xi_i$ and that is the i-th coordinate of the vector ξ. These coordinates form a basis of 1-forms in the 1-form vector space (which is also called the *dual space* $(\mathbb{L}^n)^*$).

Any 1-form can then be written as a linear combination of basis 1-forms:

$$\omega_1 = a_1 x_1 + \cdots + a_n x_n. \tag{14.7}$$

14.2.1.2 Exterior Products

Definition 14.4 (Exterior Product)**.** *An exterior product of k 1-forms $\omega_1, \omega_2, \cdots, \omega_k$ is a k-form defined by*

$$(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k)(\xi_1, \xi_2, \cdots, \xi_k) = det\omega_i(\xi_j). \tag{14.8}$$

Remark 14.2. Exterior products of basis 1-forms $x_{i_1} \wedge \cdots \wedge x_{i_k}$ with $i_1 < \cdots < i_k$ for a basis in the space of k-forms. The dimension the latter space is obviously C_n^k.

A general k-form can be written as

$$\omega^k = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \cdots i_k} x_{i_1} \wedge \cdots \wedge x_{i_k} \tag{14.9}$$

where $a_{i_1 \cdots i_k}$ are real numbers.

Definition 14.5 (Exterior product of an k-form and an l-form)**.** *The exterior product $\omega^k \wedge \omega^l$ of a k-form with an l-form on $\mathbb{L}^n$ is the $k+l$-form on $\mathbb{L}^n$ defined as:*

$$(\omega^k \wedge \omega^l)(\xi_1, \cdots, \xi_{k+l}) = \sum (-1)^\nu \omega^k(\xi_{i_1}, \cdots, \xi_{i_k}) \wedge \omega^l(\xi_{j_1}, \cdots, \xi_{j_l}) \tag{14.10}$$

where $i_1 < \cdots < i_k$ and $j_1, < \cdots < j_l$ and the sum is taken over all permutations $(i_1, \cdots, i_k, j_1, \cdots, j_l)$ with $(-1)^\nu$ being +1 for even permutations and -1 for odd permutation.

One can check that this definition is consistent with the definition for exterior product of 1-forms. Furthermore it can be shown that the exterior product is distributive, associative and *skew-commutative*. The latter means that $\omega^k \wedge \omega^l = (-1)^{kl} \omega^l \wedge \omega^k$. If ω is a 1-form (or a form of any odd degree) one can easily show that $\omega \wedge \omega = 0$. We will present a brief version of the proof here:

Proof. Let ω be an exterior k-form, where k is an odd integer. Then the exterior product takes the form:

$$\omega \wedge \omega = (-1)^{k^2} \omega \wedge \omega \tag{14.11}$$

But k is odd, therefore k^2 is odd. From that we see that

$$\omega \wedge \omega = (-1)^{k^2} \omega \wedge \omega = -\omega \wedge \omega \tag{14.12}$$

From this we conclude that $\omega \wedge \omega = 0$. □

14.2.1.3 Behaviour under mappings

Let $f : \mathbb{L}^m \to \mathbb{L}^n$ be a linear map and ω^k an exterior k-form on $\mathbb{L}^n$. We can define a k-form $(f^*\omega^k)$ on $\mathbb{L}^m$ by

$$(f^*\omega^k)(\xi_1, \cdots, \xi_k) = \omega^k(f\xi_1, \cdots, f\xi_k) \tag{14.13}$$

Notice that the obtained mapping of forms f^* acts in *opposite* direction to f. Namely, $f^* : \Omega_k(\mathbb{L}^n) \to \Omega_k(\mathbb{L}^b)$, where $\Omega_k(\mathbb{L}^n)$ is the vector space of k-forms on $\mathbb{L}^n$.

14.2.2 Differential Forms

After the general discussion about exterior forms it is time to restrict ourselves to a more practical object, namely the differential form. Understanding differential forms or at least having working knowledge about the topic is paramount for the study of differential geometry and physics on curved manifolds. Although this may seem abstract at first, we urge the reader to push through the mathematical definitions and grasp the essence of the idea, that is how do we perform calculus on curved manifolds without the notion of coordinates.

Definition 14.6 (Differential Form)**.** *A differential k-form $\omega^k|_x$ at a point x of a manifold M is an exterior k-form on the tangent space TM_x to M at x, i.e., a k-linear skew-symmetric function of k vectors $\xi_1, \cdots, \xi_k$ tangent to M at x. If such a form is given at every point x of M and if it is differentiable, we say that we are given a k-form ω^k on the manifold M.*

We intoduce the coordinate basis 1-forms dx_i with $i = 1, \cdots, n$ ($dimM = n$). The notation dx_i is used for the exterior forms emphasizes that these basis forms

act on TM_x at a given point x of M. In the neighborhood of x one can always write the general differential k-form as

$$\omega^k = \sum_{i_1<\cdots<i_k} a_{i_1,\cdots,i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \tag{14.14}$$

where $a_{i_1\cdots i_k}$ are smooth functions of x. Let's give a simple example.

Example 6. *The differential of some scalar function $f(x)$ defined on a manifold M is a differential 1-form. We introduce*

$$df = \sum_{k=1}^{n} \left.\frac{\partial f}{\partial x_k}\right|_x dx_k, \tag{14.15}$$

where dx_k are basis differential 1-forms. The value of this 1-form on a vector $\xi \in TM_x$ is given by

$$df(\xi) = \sum_{k=1}^{n} \left.\frac{\partial f}{\partial x_k}\right|_x \xi_k. \tag{14.16}$$

14.2.2.1 Behaviour under mappings

Let $f : M \to N$ be a differentiable map of a smooth manifold M to a smooth manifold N, and let ω be a differential k-form o N. The mapping f induces the mapping $f_* : TM_x \to TM_{f(x)}$ of tangent spaces. The latter mapping f_* is called the *differential* of the map f. The mapping f_* is a mapping of linear spaces and gives rise to the mapping of forms defined on corresponding tangent spaces. As a result a well-defined differential k-form $(f^*\omega)$ exists on M:

$$(f^*\omega)(\xi_1, \cdots, \xi_k) = \omega(f_*\xi_1, \cdots, f_*\xi_k). \tag{14.17}$$

14.2.3 Integration of Differential Forms over Chains

14.2.3.1 Integration of k-form over k-dimensional cell

Let D be a bounded convex polyhedron in $\mathbb{L}^k$ and $x_1, \cdots, x_k$ an oriented coordinate system on $\mathbb{L}^k$. Any differential k-form on $\mathbb{L}^k$ can be written as $\omega^k = \phi(x) dx_1 \wedge \cdots \wedge dx_k$, where $\phi(x)$ is a differentiable function on $\mathbb{R}^k$. We define the integral of the form ω^k over D as the integral of the function $\phi(x)$:

$$\int_D \omega_k = \int_D \phi(x) dx_1 dx_2 \cdots dx_k. \tag{14.18}$$

Definition 14.7 (k-dimensional cell)**.** *A k-dimensional cell σ of an n-dimensional manifold M is a polyhedron D in $\mathbb{L}^n$ with a differentiable map $f : D \to M$.*

One can think about σ as a "curvilinear polyhedron" - the image of D on M. If ω is a differentiable k-form on M, we define the integral of a form over the cell σ as

$$\int_\sigma \omega = \int_D f^* \omega, \tag{14.19}$$

where f^* is a mapping of k-forms induced by f.

The cell σ inherits an orientation from the orientation of $\mathbb{L}^k$. The k-dimensional cell which differs from σ only by the choice of orientation is called the *negative* of σ and is denoted by $-\sigma$ or by $(-1)\sigma$. One can show that under a change of orientation the integral changes sign:

$$\int_{-\sigma} \omega = -\int_\sigma \omega. \tag{14.20}$$

14.2.3.2 Chains and the Boundary Operator

It is convenient to generalize our definition of the integral of a form over *cell* to the integral over a *chain*.

Definition 14.8 (Chain of a Manifold)**.** *A chain of dimension k on a manifold M consists of a finite collection of k-dimensional oriented cells $\sigma_1, \cdots, \sigma_r$ in M and integers $m_1, \cdots, m_r$ called multiplicities. A chains is denoted by*

$$c_k = m_1\sigma_1 + \cdots + m_r\sigma_r. \tag{14.21}$$

One can introduce the structure of a commutative group on a set of k-chains on M with natural definitions of addition of chains $c_k + b_k$.

Definition 14.9 (Boundary Operator)**.** *The boundary of a convex oriented k-polyhedron D on $\mathbb{L}^k$ is the $(k-1)$-chain ∂D on $\mathbb{L}^k$ defined as*

$$\partial D = \sum_i \sigma_i \tag{14.22}$$

where the cells σ_i are the $(k-1)$-dimensional faces of D with orientations inherited from the orientation of $\mathbb{L}^k$.

One can easily extend this definition to the definition of the *boundary of a cell* $\partial\sigma$ on M and then to the boundary of a chain. Indeed, defining:

Definition 14.10 (Boudary of a Chain)**.**

$$\partial c_k = m_1 \partial \sigma_1 + \cdots + m_r \partial \sigma_r \tag{14.23}$$

We can see that ∂c_k is a $(k-1)$-chain on M. Additionally, we define a 0-chain as a collection of points with multiplicities. Furthermore we define the boundary of an oriented interval $\vec{AB}$ as $B - A$. The boundary of a point is empty.

It is straightforward to show that the boundary of the boundary of a cell is zero. Therefore

$$\partial(\partial c_k) = 0 \tag{14.24}$$

for any k-chain c_k. We denote this property as

$$\partial\partial = \partial^2 = 0 \tag{14.25}$$

14.2.3.3 Integration of a k-form over a k-chain

An integral of a k-form over a k-chain is then defined as

$$\int_{c_k} \omega^k = \sum m_i \int_{\sigma_i} \omega^k. \tag{14.26}$$

14.2.4 Exterior Differentiation and Stokes Formula

Definition 14.11 (Exterior Derivative of a Form)**.** *An exterior derivative of a differential k-form ω is a $(k+1)$-form $\Omega = d\omega^k$. Given a set of coordinates $\{dx_{i_j}\}$, we have:*

$$\Omega = d\omega^k = \sum da_{i_1 \cdots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{14.27}$$

implying that we have defined ω^k as in (14.14)*. Here, da is a 1-form, the differential of the function $a(x)$.*

One can show that the definition does not actually depend on the choice of coordinates. We can think of the 1-form given by the differential of a scalar function as of an external derivative of a 0-form. It is easy to show that $d(df) = 0$, if f belongs to the set of 0-forms. Indeed

Proof. If f belongs to the set of 0-forms, then by definition

$$df = \sum \frac{\partial a}{\partial x^i} dx^i, \tag{14.28}$$

which is a 1-form. Taking an exterior derivative again yields

$$\begin{aligned} d(df) &= d\left(\sum \frac{\partial a}{\partial x^i} dx^i\right) = \\ &= \sum \frac{\partial^2}{\partial x^i \partial x^j} dx^i \wedge x^j = \\ &= \frac{\partial^2 g}{dx^i dx^j} dx^i \wedge dx^j + \frac{\partial^2 g}{dx^j dx^i} dx^j \wedge dx^i \\ &= 0 \end{aligned} \tag{14.29}$$

Here we have used that $\frac{\partial g^2}{dx^i dx^j} = \frac{\partial g^2}{dx^j dx^i}$ and $dx^i \wedge dx^j = -dx^j \wedge dx^i$. □

Using this result, one can show that it holds for forms of any degree.

Another useful formula is that for differentiating an exterior product of form:

$$d(\omega^k \wedge \omega^l) = d\omega^k \wedge d\omega^l + (-1)^k \omega^k \wedge d\omega^l \tag{14.30}$$

Lastly, if $f : M \to N$ is a smooth map and ω is a k-form on N, we have

$$f^*(d\omega) = d(f^*\omega). \tag{14.31}$$

14.2.4.1 Stokes' Formula

One of the most-important formulae in differential geometry is the *Newton-Leibniz-Gauss-Green-Ostrogradski-Stokes-Poincaré* formula:

$$\boxed{\int_{\partial c} \omega = \int_c d\omega} \tag{14.32}$$

where c is any $(k+1)$-chain on a manifold M and ω is any k-form on M. In the case when the boundary $\partial c = 0$, we have $\int_c d\omega = 0$, which corresponds to integration of a complete derivative over a closed surface.

14.2.5 Homologies and Cohomologies

14.2.5.1 Closed and Exact Forms

Definition 14.12 (Closed Form)**.** *A differential form ω on a manifold M is said to be closed if $d\omega = 0$.*

In particular, on a 3D Riemannian manifold, we have $d\omega^2_{\vec{A}} = \left(\nabla \cdot \vec{A}\right) \omega^3 = 0$, which is equivalent to $\left(\nabla \cdot \vec{A}\right) = 0$, i.e the corresponding vector field is divergenless. We can apply Stokes' formula for a closed form, getting

$$\int_{\partial c_{k+1}} \omega^k = 0 \quad \text{if } dw^k = 0. \tag{14.33}$$

Definition 14.13 (Exact Form)**.** *A differential form ω on a manifold M is said to be exact if there exists such a differential form μ that $\omega = d\mu$.*

Since $d(d\omega) = 0$ as we've proven all exact forms are closed. However, there are some closed forms which are not exact. Let's give a short example.

Example 7. *Consider the circle S^1, parametrized by an angle $\phi \in [0, 2\pi]$. One can introduce the 1-form ω^1, defined by $\omega^1(\partial_t \gamma) = \partial_t \gamma$, where $\partial_t \gamma$ is the "velocity" along the path $\phi = \gamma(t)$. Obviously, this "velocity", belongs to the tangent space of S^1 at the point with coordinate ϕ. The form is closed - $d\omega$ is a 2-form, and we can't have 2-forms on a one-dimensional manifold. However:*

$$\int_{S^1} \omega^1 = \int_0^T dt \partial_t \gamma = 2\pi, \tag{14.34}$$

the length of the circle. Although the boundary ∂S^1 is zero, the integral is not zero, and therefore ω^1 is not exact.

One can notice that locally the introduced 1-form can be written as $\omega^1 = d\phi$, which can look like a contradiction. It is easy to see where the problem lies - writing $\omega^1 = d\phi$ is not valid for $\phi = 0$. We've come across an example of a general result, namely

Theorem 14.1 (Poincaré's Lemma)**.** *Any closed form is locally exact.*

The existence of locally but not globally exact closed forms is related to some topological properties of the underlying manifold M.

14.2.5.2 Cycles and Boundaries

Definition 14.14 (Cycle on a Manifold)**.** *A cycle on a manifold M is a chain whose boundary is equal to zero.*

Using Stokes theorem, we have

$$\int_{c_{k+1}} d\omega^k = 0 \quad \text{if } \partial c_{k+1} = 0. \tag{14.35}$$

Chains that can be considered boundaries of some other chains are called *boundaries*. Since $\partial\partial = 0$, all boundaries are cycles. However, not all cycles are boundaries. The existence of cycles that are not boundaries is again related to some topological properties of the manifold. A fairly simple example is found on the 2-torus. The 2-torus is the direct product of 2 circles - $T^2 = S^1 \times S^1$. Each of the S^1 is a cycle, but none of them are boundaries.

14.2.5.3 Homologies and Cohomologies

The set of all k-forms on M is a vector space, the set of all *closed* k-forms is a subspace of that space and the set of differentials of $(k-1)$-forms (the exact k-forms) are a subspace of the subspace of closed forms. We can now define:

Definition 14.15. *The quotient space:*

$$\frac{(\textit{closed forms})}{(\textit{exact forms})} = H^k(M, \mathbb{R}) \tag{14.36}$$

is called the k-th cohomology group of the manifold M. An element of this group is a class of closed forms, differing from each other only by an exact form.

For the circle S^1, we have $H^1(S^1, \mathbb{R}) = \mathbb{R}$.

Definition 14.16 (Betti Number)**.** *The dimension of H^k is called the k-th Betti number of M.*

Obviously, the first Betti number of S^1 is 1. The cohomology groups of M are important *topological* properties of M.

Definition 14.17 (Homologous Cycles)**.** *Let us now consider two k-cyclces a and b, such that their difference is a boundary of a $(k+1)$-chain, i.e. $a - b = \partial c_{k+1}$. Such cycles are called homologous.*

Let us have two k-cycles, a and b, homologous to each other and a closed form ω^k. From (14.33) we can see that

$$\int_a \omega^k = \int_b \omega^k. \tag{14.37}$$

In other words, homologous cycles can be replaced with one another for integration paths.

Definition 14.18 (Homology Group)**.** *The quotient group*

$$\frac{(cycles)}{(boundaries)} = H_k(M) \tag{14.38}$$

is called the k-th homology group of M. An element of this group is a class of cycles homologous to each other. The rank of this group is also equal to the k-th Betti number of M.

14.2.6 Homologies and Homotopies

There are important relations between homology and homotopy groups of a topological space M.

Let us suppose that $\pi_1(M)$ and $H_1(M)$ are the fundamental and the first homology group of M, respectively. Then $H_1(M) = \pi_1(M)/[\pi_1, \pi_1]$, where $[\pi_1, \pi_1]$ is the commutator in the fundamental group. In particular, if $\pi_1(M)$ is Abelian, then $\pi_1(M) = H_1(M)$. For the higher homotopy groups, there is another result, known as the Gurevich theorem.

Theorem 14.2 (Gurevich theorem)**.** *If $\pi_k(M) = 0$ for all $k < n$, then*

$$\pi_n(M) = H_n(M). \tag{14.39}$$

As a general rule of thumb, homology (and cohomology) groups are usually easier to calculate than homotopy groups.

14.3 Contact Manifolds

Definition 14.19 (Riemannian Cone)**.** *Given a Riemannian manifold (M, g), its Riemannian cone is a product*

$$(M \times \mathbb{R}^{>0}) \tag{14.40}$$

of M with the half-line $\mathbb{R}^{>0}$ equipped with the cone metric

$$t^2 g + dt^2, \tag{14.41}$$

where t is a parameter in $\mathbb{R}^{>0}$

Definition 14.20 (Contact Manifold)**.** *A manifold M, equipped with a 1-form θ is contact if and only if the 2-form*

$$t^2 d\theta + 2tdt \cdot \theta \tag{14.42}$$

on its cone is symplectic.

Definition 14.21 (Sasakian Manifold)**.** *A contact Riemannian manifold is called a Sasakian manifold, if its Riemannian cone with the cone metric is a Kähler manifold with Kähler form*

$$t^2 d\theta + 2tdt \cdot \theta. \tag{14.43}$$

Example 8. *Consider the manifold $\mathbb{R}^{2n+1}$ with coordinates $(\vec{x}, \vec{y}, z)$, endowed with contact form*

$$\theta = \frac{1}{2} dz + \Sigma_i y_i dx_i \tag{14.44}$$

and Riemannian metric

$$g = \Sigma_i (dx_i)^2 + (dy_i)^2 + \theta^2 \tag{14.45}$$

Definition 14.22 (Sasaki-Einstein Manifold)**.** *A Sasaki-Einstein manifold is a Riemanian manifold (S, g) that is both Sasakian and Einstein*

Example 9. *The odd dimensional sphere S^{2n-1}, equipped with its standard Einstein metric is a Sasaki-Einstein manifold. In this case, the Kähler cone is $\mathbb{C}^2 \backslash \{0\}$, equipped with its flat metric.*

14.4 Symplectic Geometry

Theorem 14.3 (Duistermaat-Heckman Formula)**.** *For a compact symplectic manifold M of dimension $2n$ with symplectic form ω and with a Hamiltonian $U(1)$ action whose moment map is denoted by μ, the following formula holds:*

$$\int_M \frac{\omega^n}{n!} e^{-\mu} = \sum_i \frac{e^{-\mu(x_i)}}{e(x_i)} \tag{14.46}$$

Here, x_i are the fixed points of the $U(1)$ action and they are assumed to be isolated, and $e(x_i)$ is the product of the weights of the $U(1)$ action on the tangent space at x_i.

14.5 Complex Manifolds

Definition 14.23 (Hermitian Metric)**.** *If a Riemannian metric g of a complex manifold M satisfies*

$$g_p(J_pX, J_pY) = g_p(X, Y) \tag{14.47}$$

at each point $p \in M$ and for any $X, Y \in T_pM$, g is said to be a Hermitian metric. Here, J_p denotes the almost complex structure on M.

Definition 14.24 (Hermitian Manifold)**.** *The pair (M, g) is called a Hermitian manifold*

Theorem 14.4. *A complex manifold always admits a Hermitian metric.*

Proof. Let g be any Riemannian metric of a complex manifold M. Define a new metric $\hat{g}$ by

$$\hat{g}_p(X, Y) \equiv \frac{1}{2}\left[g_p(X, Y) + g_p(J_pX, J_pY)\right]. \tag{14.48}$$

Clearly $\hat{g}_p(J_pX, J_pY) = \hat{g}_p(X, Y)$. Moreover, $\hat{g}$ is positive definite provided that g is. Hence, $\hat{g}$ is a Hermitian metric on M □

Definition 14.25 (Kähler Form)**.** *Let (M, g) be a Hermitian manifold. Define a tensor field Ω whose action on $X, Y \in T_pM$ is*

$$\Omega_p(X, Y) = g_p(J_pX, Y) \tag{14.49}$$

Note that Ω is anti-symmetric, $\Omega(X, Y) = g(JX, Y) = g(J^2X, JY) = -g(JY, X) = -\Omega(Y, X)$. Hence, Ω defines a two-form, called the Khäler form of a Hermitian metric g.

Definition 14.26 (Kähler Manifold)**.** *A Kähler manifold is a Hermitian manifold (M, g) whose Kähler form Ω is closed, $d\Omega = 0$. The metric g is called the Kähler metric of M.*

Remark 14.3. Not all complex manifolds admit Kähler metrics

Theorem 14.5. *A Hermitian manifold (M, g) is a Kähler manifold if and only if the almost complex structure J satisfies*

$$\nabla_\mu J = 0 \tag{14.50}$$

where ∇_μ is the Levi-Cevita connection associated with g.

Proof. We first note that for any r-form ω, $\mathrm{d}\omega$ is written as

$$d\omega = \nabla\omega \equiv \frac{1}{r!}\nabla_\mu \omega_{\nu_1\cdots\nu_r} dx^\mu \wedge dx^{\nu_1} \wedge \cdots \wedge dx^\nu \tag{14.51}$$

Now we prove that $\nabla_\mu J = 0$ if and only if $\nabla_\mu \Omega = 0$. We verify the following equalities:

$$\begin{aligned}
(\nabla_Z \Omega)(X,Y) &= \nabla_Z \left[\Omega(X,Y)\right] - \Omega(\nabla_Z X, Y) - \Omega(X, \nabla_z Y) && (14.52)\\
&= \nabla_Z \left[g(JX,Y)\right] - g(J\nabla_Z X, Y) - g(JX, \nabla_Z Y) && (14.53)\\
&= (\nabla_Z g)(JX,Y) + g(\nabla_Z JX, Y) - g(J\nabla_Z X, Y) && (14.54)\\
&= g(\nabla_Z JX - J\nabla_Z X, Y) = g((\nabla_Z J)X, y) && (14.55)
\end{aligned}$$

where $\nabla_Z g = 0$ has been used. Since this is true for any X, Y, Z, if follows that $\nabla_Z \Omega = 0$ if and only if $\nabla_Z J = 0$. □

The last theorem shows that the Riemann structure is compatable with the Hermitian structure in the Kähler manifold.

We can also characterize Kähler manifolds as Hermitian manifolds for which the Cristoffel symbols of the Levi-Chevita connection are pure. In other words, Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$ may be non zero, but all "mixed" symbols like $\Gamma^{\bar{i}}_{jk}$, for example, vanish. This means that (anti-)holomorphic vectors get parallel transported to (anti-)holomorphic vectors.

Kähler manifolds are manifolds on which we can always find a *holomorphic* change of coordinates which, at some given point, sets the metric to its cannonical form, and its first derivatives to zero.

Equivalently, an n-dimensional Kähler manifold are precisely $2n$-dimensional Riemannian manifolds with holonomy group contained in $U(1)$

Definition 14.27 (Hopf Surface)**.** *Let $\mathbb{Z}$ act on $\mathbb{C}^n\backslash\{0\}$ by $(z_1, \cdots, z_n) \to \left(\lambda^k z_1, \cdots, \lambda^k z_n\right)$ for $k \in \mathbb{Z}$. For $0 < \lambda < 1$ the action is free and discrete. The quotient complex manifold $X = \left(\mathbb{C}^n\backslash\{0\}\right)/\mathbb{Z}$ is diffeomorphic to $S^1 \times S^{2n-1}$. For $n = 1$ this manifold is isomorphic to a complex torus $\mathbb{C}/\Gamma$. The lattice Γ can be determined explicitly.*
In other words, a Hopf manifold is obtained as a quotient of the complex vector space (with zero deleted) $\mathbb{C}^n\backslash\{0\}$ by a free action of the group $\Gamma \cong \mathbb{Z}$ of integers, with the generator γ of Γ acting by holomorphic contractions.

14.6 Gauss-Bonet Theorem

The Gauss-Bonet theorem bridges the gap between topology and differential geometry. Its importance lies in relating geometrical information of a surface to a purely topological characteristic, which has resulted in varied and powerful applications. Gauss first proved this theorem in 1827, for the case of a hyperbolic triangle. It established a remarkable invariant relating curvature to the notion of angle withing the surface. However, with the developments in topology in the 19th and 20th centuries, this theorem has become an invaluable piece of modern mathematics.

14.6.1 Topological preliminaries

Definition 14.28 (A surface)**.** *S is a two dimensional sub-manifold of Eucledian space. The word **regular** ensures differentiability in a neighborhood of each point of the surface; however, we note that the full definition of this term includes other criteria which will be inconsequential for us here*

Definition 14.29 (A triangulation)**.** *of a regular surface R is a finite collection J of triangles $\{T_j\}\, j = 1^n$ such that*

$$\bigcup_{j=1}^{n} T_j = R \tag{14.56}$$

and the only possible intersection of T_i and T_j with $i \neq j$ is a common edge or a common vertex.

Proposition 14.1. *Every regular surface S admits a triangulation.*

Definition 14.30 (Orientable surface)**.** *A surface is orientable if all of the triangles in a given triangulation have compatable orientations (i.e, they preserve the notion of clockwise-anticlockwise as a path moves between triangles).*

Definition 14.31 (Euler characteristic)**.** *$\chi(s)$ of a surface S is defined as*

$$\chi(S) = V - E + F, \tag{14.57}$$

where V, E, F are the number of vertices, edges and faces of a given triangulation of S.

Proposition 14.2. *The Euler characteristic is a topological invariant.*

Remark 14.4. All surfaces (in $N = 2$ dimensions) are classified up to homeomorphism by their genus, which is closely related to the Euler characteristic. Essentially, the number of holes in a surface, classifies it topologically.

Definition 14.32 (Fundamental group (topology))**.** *Let X be a space and x_0 be a point in X. A path in X that begins and ends at x_0 is a **loop** based at x_0. The **fundamental group** $\pi_1(X, x_0)$ is the seth of path homotopy classes of loops based at x_0, with some operation, namely concatenation.*

In order to probe the so-called Jordan Curve Theorem, we list several lemmas without proof.

Proposition 14.3 (Jordan Separation Theorem)**.** *Let C be a simple closed curve in S^2. Then $S^2 - C$ is not connected.*

Proposition 14.4. *Lex X be the union of open sets U and V. Suppose*

$$U \cup V = A \cup A' \cup B \tag{14.58}$$

that is, it's a union of three disjoint open sets. Given

$$a \in A, a' \in A', b \in B \tag{14.59}$$

path connected in U and V, then $\pi_1(X, a)$ is not infinite cyclic (i.e., not isomorphic to $\mathbb{Z}, +, \mathbb{0}$).

Proposition 14.5. *(A nonseparation theorem) Let A be an arc in S^2. Then $S^2 - A$ is connected.*

Proposition 14.6 (The Jordan Curve Theorem)**.** *Let C be a simple closed curve in S^2. Then, $S^2 - C$ has two and only two components W_1 and W_2, of which C is the common boundary.*

Proof. First, we show that $S^2 - C$ has exactly two components. Using the proposition above, we write C as the union of two arcs C_1 and C_2 which intersect in exactly two points p and q. Let X be $S^2 - p - q$. Let

$$U \equiv S^2 - C_1 \tag{14.60}$$

and

$$V \equiv S^2 - C_2 \tag{14.61}$$

We note that

$$X = U \cup V \tag{14.62}$$

and

$$S^2 - C = U \cap V, \tag{14.63}$$

which we know has at least two components by proposition 14.4. Suppose, in order to find a contradiction, that $S^2 - C$ has more than two components. Let A and A' be two components, and call the union of the rest of the components B. Each of these sets is open because $S^2 - C$ is (locally) connected. Given,

$$a \in A, a' \in A', b \in B, \tag{14.64}$$

by the nonseparation theorem above, U and V are path-connected, because no arc separates S^2. Thus $\pi_1(X, a)$ is not infinite cyclic. However, we recall that $X = S^2 - p - q$ is homeomorphic to the punctured plane, which has infinite cyclic fundamental group, which is a contradiction.

Next, we show that C is the common boundary of W_1 and W_2, which are the regions that we wish to show are separated by the curve. We note that W_1 and W_2 must be open because neither can contain a limit point of the other (because S^2 is locally connected). Therefore, since S^2 is the disjoint union of W_1,W_2 and C, it follows that

$$\bar{W_1} - W_1 \quad \text{aswellas} \quad \bar{W_2} - W_2, \tag{14.65}$$

must be contained in C. Now, we show the converse, namely that given a point $x \in C$ we know that $x \in \bar{W_1} - W_1$.

Thus, let U be a neighborhood of x. Then, using the fact that C is homeomorphic to S^1, we break it up into two curves C_1 and C_2 such that C_1 is entirely contained within U. We let a and b to be points of W_1 and W_2, respectively. Again, using the nonseparation theorem, we know that we can find a path in $S^2 - C$, call it α, that connects a and b, because C_2 does not separate S^2. Therefore, $\alpha(I)$ must contain a point y such that

$$y \in \bar{W_1} - W_1, \quad y \in C. \tag{14.66}$$

We know that such y must exist because otherwise $\alpha(I)$, which we know is a connected set, would lie in the disjoint union of open sets W_1 and $S^2 - \bar{W_1}$ but also intersect both of them. Because we know that

$$C \supset \bar{W_1} - W_1, \tag{14.67}$$

if follows that $y \in C$.. Now, as we know that α does not intersect C_2, the point y must be on C_1 and thusly

$$y \in U \tag{14.68}$$

Therefore, as we wished to show, U intersects $\bar{W}_1 - W_1$ in the point y, and thusly we have shown that C is the common boundary of W_1 and W_2.

■

□

Definition 14.33 (Homotopic maps)**.** *We say that f and g, continuous maps from X to Y are homotopic if there exists a map $F : X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. This can be though of as a continuous deformation of one map into the other.*

Definition 14.34 (Convex maps)**.** *A convex map is a map such that for any line connecting two points of the map, every point on that line is contained in the map.*

Definition 14.35 (Rotation index)**.** *If $\alpha : [a,b] \to \mathbb{R}^2$ is a convex map, then the rotation index of α is*

$$\int_a^b \kappa(s)ds = \theta(b) - \theta(a) = 2\pi I \tag{14.69}$$

where $\kappa(s)$ is the Gaussian curvature.

Remark 14.5. We note that the rotation index is homotopy invariant over α.

Definition 14.36 (Secant Map)**.** *$\psi : T \to S^1$ is the secant map from the triangular region*

$$T = \{(t_1, t_2) \in [0,1] \times [0,l]\,; 0 \le t_1 \le t_2 \le l\}\,, \tag{14.70}$$

onto the 1-spehre:

$$\Psi(x) = \begin{cases} \frac{\alpha(t_2)-\alpha(t_1)}{|\alpha(t_2)-\alpha(t_1)|} & t_1 \neq t_2, (t_1,t_2) \in T \setminus \{0,l\} \\ \frac{\alpha'(t)}{|\alpha'(t)|} & t_1 = t_2 \\ \frac{-\alpha'(0)}{|\alpha(0)|} & (t_1,t_2) = (0,l) \end{cases} \tag{14.71}$$

We now present a proof of the Turning Tangents theorem. Roughly, this theorem states that a loop on a surfaces turns 2π radians. This allows us to index the rotation of the boundary of a region of a surface, which is a key piece of the proof of the local Gauss-Bonet theorem.

Theorem 14.6 (Turning Tangents)**.** *If $\beta : [0, l] \to \mathbb{R}^2$ is a simple, closed, regular, plane curve, then the rotation index of β is ± 1.*

Proof. We assume the Jordan Curve Theorem as proven. Now, we take some line b and position it such that it is tangent to β at the point p. We note that due to this choice of b, the curve lies entirely on one side of the line. Then we choose a parametrization $\alpha \in [0, l]$ such that

$$\alpha(0) = p. \tag{14.72}$$

We observe that the secant map Ψ is a continuous function, which is trivially verified. Now, we let A,B,C denote the vertices of T at $(0, 0), (0, l), (l, l)$, respectively. We notice that the side AC is simply the map of tangents to α into S^1, the degree of which is the rotation of α. Next, we recognize that the tangent map must be homotopic to the restriction of ψ to the other two sides of the triangle AB and BC. Thus, it suffices to show that the rotation index of Ψ on

$$AB \cup BC = \pm 1. \tag{14.73}$$

To do so, we assume that we have an orientation such that the angle from $\alpha'(0)$ to $-\alpha'(0)$ is π. This implies that ψ on AB, i.e.,

$$[0, 0] \to [0, l] \tag{14.74}$$

covers half of S^1 in the positive direction, and Ψ on BC covers the other half. Since the rotation index is clearly 1 for S^1, we're done. We note we can reverse the assumed orientation to obtain the opposite index. Therefore the rotation index of β is ± 1. □

14.6.2 Local Gauss-Bonnet Theorem

Remark 14.6. We will now prove the local case of Gauss-Bonnet. However, in order to do this, we first need to define some key resources. These notions of curvature tell us roughly what a surface looks like both locally and globally. The theorem tells us that there is a remarkable invariance on surfaces that is balanced by the total curvature and geodesic curvature. Gauss proved this by considering the hyperbolic triangle.

Definition 14.37 (Simple region)**.** *A simple region R of a surface S is a region such that R is homeomorphic to the disk.*

Definition 14.38. *Given a parametrization of a surface, $\boldsymbol{x} : U \to S$, we have the following quantities*

$$E = \langle x_u, x_u \rangle, F = \langle x_u, x_v \rangle, G = \langle x_v, x_v \rangle \tag{14.75}$$

These are the coefficients of the first fundamental form of a surface.

Definition 14.39 (Orthogonal parametrization)**.** *Given a parametrization of a surface,*

$$\boldsymbol{x} : U \to S, \tag{14.76}$$

we call that parametrization orthogonal if $F = 0$.

Definition 14.40 (Geodesic Curvature)**.** *The geodesic curvature k_g of a curve is a measure of the amount of deviance of the curve from the shortest arc between two points on a surface.*

Definition 14.41 (Gaussian Curvature)**.** *The Gaussian curvature κ of a surface is an intrinsic measure of the curvature of a surface at a point. It is calculated by considering the maximal and minimal curvatures on the surface at a point. Formally, these values are multiplied to give κ.*

Now we can state and prove the local variant of the Gauss-Bonnet theorem.

Theorem 14.7 (**Gauss-Bonnet, Local**)**.** *Let $R \in \boldsymbol{x}(U)$ be a simple region of S with orthogonal parametrization, and choose $\alpha : I \to S$ such that $\alpha(I) = \partial R$. Assume that α is a positively oriented and parametrized piecewise by arc-length s_i. Let $\{\theta_i\}_{i=0}^k$ be the external angles of α at the vertices $\{\alpha(s_i)\}_{i=0}^k$, then*

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_{g(s)} ds + \int\int_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi \tag{14.77}$$

Proof. We first let $u = u(s)$ and $v = v(s)$ be the expression of α in the parametrization $\mathbf{x}$. We recall that

$$k_{g(s)} = \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) + \frac{d\varphi}{ds}, \tag{14.78}$$

where we denote the differentiable function that measures the positive angle from $\mathbf{x}$ to $\alpha'(s)$ in $[s_i, s_{i+1}]$ as $\psi(s_i)$. We now integrate the above expression, adding up the values for each $[s_i, s_{i+1}]$:

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_{g(s)} = \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) ds + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds, \tag{14.79}$$

now using the Gauss-Green theorem in the uv-plane on the right hand side of the above equation, we obtain the expression:

$$\iint_{\mathbf{x}^{-1}(R)} \frac{E_v}{2\sqrt{EG_v}} + \frac{G_u}{2\sqrt{EG_u}} du dv + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds. \tag{14.80}$$

We note that by the Gauss Formula,

$$- \iint_{\mathbf{x}^{-1}(R)} \kappa \sqrt{EG} du dv = - \iint_R \kappa d\sigma, \tag{14.81}$$

also, recalling the Theorem of Turning Tangents, we know that,

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds = \sum_{i=0} k \int_{s_i}^{s_{i+1}} \varphi_i(s_i + 1) - \varphi_i(s_i) = \pm 2\pi - \sum_{i=0}^{k} \theta_i, \tag{14.82}$$

which we get because the theorem does not account for the discontinuities along the curve at the theta values. As we have assumed a positive orientation, we have

$$\sum_{i=0}^{k} \int_{s_i}^{s_{s+1}} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi, \tag{14.83}$$

we note that we can obtain the opposite sign by assuming the opposite orientation, and thus we have proven the local case of the Gauss-Bonnet theorem. □

14.6.3 Global Gauss-Bonnet Theorem

Remark 14.7. We have proven the local case of this theorem and the global theorem tells us similar information. We prove this generalization by using the local theorem in each triangular region of our triangulation of the given surface. This theorem leads to a series of very deep corollaries.

Theorem 14.8. *Let $R \in S$ be a regular regin of an oriented surface. Let ∂R be made up by closed, piecewise, simple, regular curves*

$$C_1, \cdots C_n, \tag{14.84}$$

then

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi\chi(R). \tag{14.85}$$

Proof. Let J denote a triangulation of R such that each triangle T_j is contained in a neighborhood of orthogonal parametrization compatible with the orientation of S. We note that such a triangulation exists by the proposition proven above. Now, we simply apply the local Gauss-Bonnet theorem to each T_j of the above triangulation, and we have:

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{j,k=1}^{F,3} \theta_{jk} = 2\pi F, \tag{14.86}$$

where the indexing of each theta accounts for each angle of the triangles in J. We note that F here is the number of faces in our triangulation. We denote the interior angles of the triangles by $\varphi_{jk} = \pi - \theta_{jk}$. We calculate, in general, that,

$$\sum_{j,k} \theta_{jk} = 3\pi F - \sum_{j,k=1} \varphi_{jk}. \tag{14.87}$$

Now, we introduce notation to assist in counting the vertices and edges of our triangulation, so the number of external/internal edge and vertices are V_e, E_e and V_i, E_i respectively. Since the C_i are closed, however, we know $V_e = E_e$, and thus inductively, $3F = 2E_i + E_e$. This implies,

$$\sum_{j,k} \theta_{jk} = 2\pi E_i + \pi E_e - \sum_{j,k=1} \varphi_{jk}. \tag{14.88}$$

We not that the vertices must belong to either some T_j or a C, so $V_e = V_{et} + V_{ec}$, and then since the sum of the angles around each internal vertex is 2π,

$$\sum_{j,k=1} \theta_{jk} = 2\pi E - 2\pi V + \sum_{i} \theta_i. \tag{14.89}$$

Now, we collect the terms to find:

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi(F - E + V). \tag{14.90}$$

But, by definition of a triangulation $F - E + V = \chi(R)$, thus

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi\chi(R). \tag{14.91}$$

□

14.6.4 Applications

Remark 14.8. If R is a simple region, then

$$\chi(R) = 1.$$

Proof. We note that R is homeomorphic to the disk and thus homeomorphic to a single triangle. Thus, by the definition of the Euler characteristic,

$$\chi(R) = V - E + F = 3 - 3 + 1 = 1. \tag{14.92}$$

□

Corollary 14.1. *If R is a simple region of a surface S, then,*

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_C \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi. \tag{14.93}$$

The next corollary is often the presented form of the Gauss-Bonnet theorem. We simply take into account that the boundary of a compact surface is empty, and therefore it can be thought of as a region with empty boundary. Thusly, the terms that depend on the boundary drop out of the equation and we are left with the following powerful statement.

Corollary 14.2. *If S is an orientable compact surface, then:*

$$\iint_S \kappa d\sigma = 2\pi\chi(S). \tag{14.94}$$

Remark 14.9. The Euler characteristic $\chi(S)$ can be written as

$$\chi(S) = 2 - 2g \tag{14.95}$$

where g is the genus of a surface, i.e., the number of holes in the surface.

Remark 14.10. This elegant formulation of the theorem introduces the notion that the total curvature depends exclusively on the topological characteristic - the genus. This implies that all possible embeddings of a surface of genus g have the same total curvature, which is a hightly non-intuitive result.

Definition 14.42 (Index at a singular point)**.** *The index I of v at the singular point p is defined as follows. Let $\boldsymbol{x}$ be an orthogonal parametrization such that $\boldsymbol{x}(0,0) = p$ and the orientation is compatable with that of the surface S. Let $\alpha : [0,l] \to S$ be a closed simple regular parametrized curve such that it is the boundary of a simple region $R \subset S$, where the only singular point in R is p. Now, we have a function $\varphi(t)$ with $t \in [0,l]$, such that it measures the angle from $\boldsymbol{x_u}$ to the restriction of v to α, then:*

$$2\pi I = \varphi(l) - \varphi(0) = \int_0^l \frac{d\varphi}{dt} dt. \tag{14.96}$$

Proposition 14.7. *The index is independent of the choice of parametrization.*

Proposition 14.8. *The index is independent of the choice of α.*

Theorem 14.9 (Poincaré-Hopf Index Theorem)**.** *The sum of the indices of a differentiable vector field v with isolated singular points on a compact surface S is equal to the Euler characteristic of S*

Proof. Let $S \subset \mathbb{R}^3$ be a compact surface and v a differentiable vector field with exclusively isolated singular points. We notice that, due to compactness, these singular points must be finite in number otherwise there would exist a non-isolated singular point as a limit point of the others. We let $\{\mathbf{x_a}\}$ be a family of parametrizations such taht each is compatable with tje orientation of S. Now, we let J be a triangulation of S with the condtitions that each T in J is contained in a coordinate neighborhood of $\{\mathbf{x_a}\}$, each triangle T contains at most one singular point, and each triangle is positively oriented with no singular points on its boundary. Now we apply the local Gauss-Bonnet theorem to each triangle and sum up the result. However, we recall that each triangle appears twice in this formulation, in opposite orientation, so we have:

$$\iint_S \kappa d\sigma - 2\pi \sum_{i=1}^{k} I_i = 0 \tag{14.97}$$

to which we apply the most general form of the Gauss-Bonnet to obtain,

$$\sum_{i=1}^{k} I_i = \frac{1}{2\pi} \iint_S \kappa d\sigma = \chi(S). \tag{14.98}$$

□

Remark 14.11. This result guarantees that a vector field on any surface homeomorphic to a sphere must have at least two isolated singular points, because the Euler characteristic of the 2-spehre is two. The solution is particularly remarkable because it shows that the sum of the indices of a vector field does not depend on v but rather on the topology of S which is a non-intuitive idea. More tangibly, the Poincaré-Hopf index theorem implies the previos theorem.

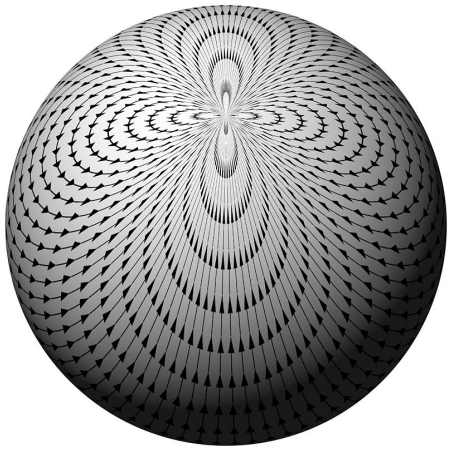

Figure 14.1: An isolated singular point guaranteed by the Poincaré-Hopf theorem[2]

Theorem 14.10 (The Hairy Ball Theorem)**.** *You can't comb a hairy ball without a bald spot. Or, in more precise terms, let f be a continuous function that assigns a vector in $\mathbb{R}^3$ to each point on S^2, such that $f(p)$ is tangent to S^2 at p. Then, there is at least one p for such that $f(p) = 0$.*

Proof. Assume, in order to find a contradiction, that $f(p) \neq 0$ for all p on the 2-sphere. Then, there exists a vector field v on S^2 such that v has no singular points. However, given the above theorem, this is a contradiction, as we know that

[2]Image Courtesy of Wikipedia: RokerHRO - Own work, CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=8257798`

there must exist two distinct isolated singular points on this vector field, thusly there can be no such vector field on the sphere or on any manifold topologically homeomorphic to the sphere. □

asdsadd

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