Example Proof Remark

IVO ILIEV

Essential Knowledge for Modern Theoretical Physics

To my friends,
my familly, my teachers and collegues.
Also, to Mister T
I pitty the fool

Contents

Foreword

"It is so shocking to find out how many people do not believe that they can learn, and how many more believe learning to be difficult."

Frank Herbert, Dune

Preface

The idea behind this book is to systematically put a lot of knowledge from different fields in the same place. It is envisioned to serve a purpose of an index book, a glossary introduction to modern physics. The depth may be lacking in some respects but the core idea is to have a background knowledge of many approaches and frameworks.

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Part I

Physics

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General Relativity

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1.1 Anti De Sitter Space (AdS)

AdS_n is an n-dimensional solution for the theory of gravitation with Einstein-Hilbert action with negative cosmological constant Λ, i.e. the theory described by the following Lagrangian density:

$$\mathcal{L} = \frac{1}{16\pi G_{(n)}}(R - 2\Lambda), \tag{1.1}$$

where $G_{(n)}$ is the gravitational constant in n-dimensional spacetime. Therefore, it's a solution of the Einstein field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and $g_{\mu\nu}$ is the metric of the spacetime. Introducing the radius α as

$$\Lambda = \frac{-(n-1)(n-2)}{2\alpha^2} \tag{1.3}$$

this solution can be immersed in a $n+1$ dimensional spacetime with signature $(-,-,+,\cdots,+)$ by the following constraint:

$$-X_1^2 - X_2^2 + \sum_{i=3}^{n+1} X_i^2 = -\alpha^2 \tag{1.4}$$

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Quantum Field Theory

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2.1 Rarita-Schwinger equation

Consider the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}_\mu \left(\epsilon^{\mu\kappa\rho\nu}\gamma_5\gamma_\kappa\partial_\rho - im\sigma^{\mu\nu}\right)\psi_\nu \tag{2.1}$$

This equation obviously controls the propagation of the wave function of a spin- object such as the gravitino. The equation of motion for this Lagrangian are known as the *Rarita-Schwinger equation*:

$$\left(\epsilon^{\mu\kappa\rho\nu}\gamma_5\gamma_\kappa\partial_\rho - im\sigma^{\mu\nu}\right)\psi_\nu \tag{2.2}$$

In the massless case, the Rarita-Schwinger equation has a fermionic gauge symmertrty, it is invariant under the gauge transformation:

$$\psi_\mu \to \psi_\mu + \partial_\mu\epsilon, \tag{2.3}$$

where $\epsilon \equiv \epsilon_\alpha$ is an arbitrary spinor field.

2.1.1 Massless case

Consider a massless Rarita-Schwinger field, described by the Lagrangian

$$\mathcal{L}_{RS} = \bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho \tag{2.4}$$

where the sum over spin indices is implicit, ψ_μ are Majorana spinors and the quantity $\gamma^{\mu\nu\rho}$ is equal to

$$\gamma^{\mu\nu\rho} \equiv \frac{1}{3!}\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]} \tag{2.5}$$

Varying the Lagrangian yealds after some calculation

$$\delta\mathcal{L}_{\mathcal{RS}} = 2\delta\bar{\psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + \text{boundary terms} \tag{2.6}$$

Now imposing that $\mathcal{L}_{RS} = 0$ we get the equation of motion for a massless Majorana Rarita-Schwinger spinor:

$$\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho = 0 \tag{2.7}$$

2.1.2 Massive case

The description of massive, higher-spin fields through the Rarita-Schwinger equation is not well defined physically. Coupling the RS Largrangian to electromagnetism leads to an equation with solutions representing wavefronts, some of which propagate faster than light. However, it was shown by Das and Freedman that local supersymmetry can circumvent this problem.

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Supersymmetry

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3.1 Supermultiplets

Definition 3.1 (Supermultiplet) *Representations of the supersymmetric algebra (superalgebra) are called supermultiplets.*

Indeed, these representations can be thought of as multiplets where we assemble together several different representations of the Lorentz algebra, since the latter is a subalgebra of the superalgebra.

3.1.1 Massless supermultiplets

If $P^2 = 0$, then we can take P_μ to a canonical form by applying boost and rotations until it reads

$$\sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu = \left(\sigma^0 + \sigma^3\right) E = \begin{bmatrix} 0 & 0 \\ 0 & 2E \end{bmatrix} \tag{3.1}$$

The supersymmetric algebra becomes,

$$\begin{bmatrix} \{Q_1, \bar{Q}_{\dot{1}}\} & \{Q_1, \bar{Q}_{\dot{2}}\} \\ \{Q_2, \bar{Q}_{\dot{1}}\} & \{Q_2, \bar{Q}_{\dot{2}}\} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4E \end{bmatrix} \tag{3.2}$$

intended as acting on the states of the multiplet we are looking for. In particular,

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 0 \tag{3.3}$$

which implies that

$$||Q_1|\omega\rangle||^2 = 0 = ||\bar{Q}_{\dot{1}}|\omega\rangle||^2 \tag{3.4}$$

and thus

$$Q_1|\omega\rangle = 0 = \bar{Q}_{\dot{1}}|\omega\rangle. \tag{3.5}$$

This means that as operators Q_1 and $\bar{Q}_{\dot{1}}$ annihilate the multiplet.

The only nontrivial anticommutation relation that is left is:

$$\{Q_2, \bar{Q}_{\dot{2}}\} = 1 \tag{3.6}$$

If we call

$$\alpha = \frac{1}{2\sqrt{E}} Q_2, \quad \alpha^\dagger = \frac{1}{2E} \bar{Q}_{\dot{2}} \tag{3.7}$$

then the anticommutation relation that is left is:

$$\{\alpha, \alpha^\dagger\} = 1 \tag{3.8}$$

with $\{\alpha, \alpha\} = 0$.

We can build the representation starting from a state $|\lambda\rangle$ such that

$$\alpha|\lambda\rangle = 0 \tag{3.9}$$

Lets suppose that it has *helicity* λ:

$$M_{12}\lambda \equiv J_3|\lambda\rangle = \lambda|\lambda\rangle. \tag{3.10}$$

It is easy to compute the helicity of $\alpha^\dagger|\lambda\rangle$:

$$M_{12}\bar{Q}_{\dot{2}}|\lambda\rangle = \left(\bar{Q}_{\dot{2}} M_{12} + \frac{1}{2}\bar{Q}_{\dot{2}}\right)|\lambda\rangle = (\lambda + \frac{1}{2}\bar{Q}_{\dot{2}})|\lambda\rangle \tag{3.11}$$

In the last line, we have used the fact that $\left[M_{12}, \bar{Q}_{\dot{2}}\right] = \frac{1}{2}\bar{Q}_{\dot{2}}$. Thus we found out that

$$\alpha^\dagger|\lambda\rangle = |\lambda + \frac{1}{2}\rangle \tag{3.12}$$

Since $(\alpha^\dagger)^2 = 0$, this stops here. Hence we have

$$\alpha^\dagger|\lambda + \frac{1}{2}\rangle = 0. \tag{3.13}$$

Massless multiplets are thus composed of one boson and one fermion. Since physical particles must come in CPT conjugate representation (or, there are no spin-$\frac{1}{2}$ one dimensional representations of the massless little group of the Lorentz group), one must add the CPT conjugate multiplet where helicities are flipped. [Examples of massless supermultiplets]

- The *scalar* multiplet is obtained by setting $\lambda = 0$. Then we have

$$\alpha^\dagger |0\rangle = \left|\frac{1}{2}\right\rangle \tag{3.14}$$

The full multiplet is composed of two states with $\lambda = 0$ and a doublet with $\lambda = \pm\frac{1}{2}$. These are the degrees of freedom of a complex scalar and a Weyl (chiral) fermion.

- The *vector* multiplet is obtained starting from a $\lambda = \frac{1}{2}$ state. We get

$$\alpha^\dagger \left|\frac{1}{2}\right\rangle = |1\rangle\,. \tag{3.15}$$

To this we add the CPT conjugate multiplet, to obtain two pairs of states, one with $\lambda = \pm\frac{1}{2}$ and the other with $\lambda = \pm 1$. These are the degrees of freedom of a Weyl fermion and of a massless vector. The latter is usually interpreted as a gauge boson.

- Another multiplet is obtained starting from $\lambda = \frac{3}{2}$:

$$\alpha^\dagger \left|\frac{3}{2}\right\rangle = |2\rangle\,. \tag{3.16}$$

Adding the CPT conjugate, one has a pair of bosonic degrees of freedom with $\lambda = \pm 2$, which we interpret as the *graviton*, and a pair of fermionic degrees of freedom with $\lambda = \pm\frac{3}{2}$, which correspond to a massless spin-$\frac{3}{2}$ Rarita-Schwinger field, also called the *gravition*, since it is the SUSY partner of the graviton, as was just shown.

3.1.2 Supermultiplets of extended supersymmetry

Very briefly we will mention that having extended SUSY, the massless supermultiplets are longer. Let's take the algebra to be:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^{IJ}, \tag{3.17}$$

where for simplicity we suppose that $Z^{IJ} = 0$ for these states. For massless states, $P_\mu = (E, 0, 0, E)$ and therefore as before we have that

$$\{Q^I_1, \bar{Q}^J_{\dot 1}\} = 0, \tag{3.18}$$

which implies the (operator) equations $Q^I_1 = 0$ and $\bar{Q}^I_{\dot 1} = 0$, for $I = 1, \cdots ,$. The nontrivial relations are then:

$$\{Q^I_2, \bar{Q}^J_{\dot 2}\} = 4E\delta^{IJ} \tag{3.19}$$

Of course we can define

$$\alpha_I = \frac{1}{2\sqrt{E}} Q_2^I \tag{3.20}$$

and obtain the canonical anticommutation relations for fermionic oscillators

$$\{\alpha_I, \alpha_J^\dagger\} = \delta_{IJ} \tag{3.21}$$

If we now start with a state $|\lambda\rangle$ with helicity λ which satisfies $\alpha_I |\lambda\rangle = 0$, we build a multiplet as follows:

$$\begin{aligned} \alpha_I^\dagger |\lambda\rangle &= \left|\lambda + \frac{1}{2}\right\rangle_I , \\ \alpha_I \dagger \alpha_J \dagger |\lambda\rangle &= |\lambda + 1\rangle_{[IJ]} , \\ &\vdots \\ \alpha_1^\dagger \cdots \alpha^\dagger |\lambda\rangle &= \left|\lambda + \frac{}{2}\right\rangle \end{aligned} \tag{3.22}$$

It is very important to note that there are states with helicity $\lambda + \frac{1}{2}$, $\frac{1}{2}(-1)$ states with helicity $\lambda + 1$ and so on, until we reach a single state with helicity $\lambda + \frac{}{2}$ (it is totally antisymmetric in indices I). In total, the supermultiplet is composed of 2 states, half of them bosonic and half of them fermionic.

Interestingly, in this case we can now have self-CPT conjugate multiplets. Take for example $= 4$ and start from $\lambda = -1$. Then $\lambda + \frac{}{2} = 1$ and the multiplet spans states of opposite helicities, thus filling complete representations of the Lorenz group. Indeed, it contains one pair of states with $\lambda \pm 1$ (a vector, i.e a gauge boson), 4 pairs of states with $\lambda = \pm\frac{1}{3}$ (4 Weyl Fermions) and 6 states with $\lambda = 0$ (6 real scalars, or equivalently 3 complex scalars).

Another example is $= 8$ supersymmetry. Here if we start with $\lambda = -2$ we end up with $\lambda + \frac{}{2} = 2$. Thus in this case we have the graviton in the self-CPT conjugate multiplet, corresponding to the pair of states with $\lambda \pm 2$. In addition, we have 8 massless gravitini with $\lambda \pm \frac{3}{2}$, 28 massless vectors with $\lambda = \pm 1$, 56 massless Weyl fermions with $\lambda = \pm\frac{1}{2}$ and finally 70 real scalars with $\lambda = 0$. This is the content of $= 8$ *supergravity*, which is the only multiplet of $= 8$ supersymmetry with $|\lambda| < 2$. The latter condition is necessary in order to have consistent couplings (higher spin fields cannot be coupled in a consistent way with gravity and lower spin fields).

From the theoretical standpoint, this is a very nice result, because we have a theory where *everything is determined* from symmetry alone: the complete spectrum and all the couplings. Unfortunately, this theory is also completely unphysical. To mention one problem, it has no room for fermions in complex representations of the gauge group, which are present in the Standard Model.

3.1.3 Massive supermultiplets

When $P^2 = M^2 > 0$, by boosts and rotation P_μ can be put in the following form

$$P_\mu = (M, 0, 0, 0) \tag{3.23}$$

Then we have

$$\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = M\sigma^0 = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \tag{3.24}$$

so that the superalgebra reads

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2M\delta_{\alpha\dot{\alpha}} \tag{3.25}$$

Note that $[M_{12}, Q_1] = i(\sigma_{12})_1{}^1 Q_1 = \frac{1}{2}Q_1$, thus it is Q_1 that raises the helicity, in the same way as $\bar{Q}_{\dot{2}}$. We make the redefinition

$$\alpha_1 = \frac{1}{\sqrt{2M}}\bar{Q}_{\dot{1}}, \quad \alpha_1^\dagger = \frac{1}{\sqrt{2M}}Q_1, \tag{3.26}$$

$$\alpha_2 = \frac{1}{\sqrt{2M}}Q_2, \quad \alpha_2^\dagger = \frac{1}{\sqrt{2M}}\bar{Q}_{\dot{2}}, \tag{3.27}$$

so that we have the canonical anticommutation relations of two fermionic oscillators:

$$\{\alpha_a, \alpha_b^\dagger\} = \delta_{ab}, \quad a, b = 1, 2. \tag{3.28}$$

If we start with $\alpha_a |\lambda\rangle = 0$, $M_{12}|\lambda\rangle = \lambda|\lambda\rangle$, then we build the multiplet as:

$$\alpha_1^\dagger |\lambda\rangle = \left|\lambda + \frac{1}{2}\right\rangle_1, \tag{3.29}$$

$$\alpha_2^\dagger |\lambda\rangle = \left|\lambda + \frac{1}{2}\right\rangle_2, \tag{3.30}$$

$$\alpha_1^\dagger \alpha_2^\dagger |\lambda\rangle = |\lambda + 1\rangle. \tag{3.31}$$

There are 4 states now (compared to the 2 in the massless case), two bosons and two fermions. [Examples of massive supermultiplets]

- In the case of the *massive scalar multiplet*, we start from $\lambda = -\frac{1}{2}$ and obtain two states with $\lambda = 0$ and one state with $\lambda = \frac{1}{2}$. These are the degrees of freedom of one massive complex scalar and one massive Weyl fermion. Note that the latter might not be familiar. Indeed, one cannot write the usual Dirac mass term for a Weyl fermion. Instead, one can write what is called a Majorana mass term:

$$\mathcal{L} \supset m\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta + h.c. \tag{3.32}$$

  Note that the total degrees of freedom of a massless scalar multiplet is the same as that of a massive one.

- For a *massive vector multiplet*, start from $\lambda = 0$ to obtain 2 states with $\lambda = \frac{1}{2}$ and one state with $\lambda = 1$. To this we add the CPT conjugate multiplet so that in the end we have one pair with $\lambda = \pm 1$, two pairs with $\lambda = \pm\frac{1}{2}$ and two states with $\lambda = 0$. According to the massive little group, this corresponds to 1 massive vector (with $\lambda = \pm 1, 0$), 1 real scalar and 1 massive Dirac fermion. Note however that the content in degrees of freedom is the same as that of one massless vector multiplet together with one massless scalar multiplet. This hints that the consistent way to treat massive vectors in a supersymmetric field theory will be through a SUSY version of the Brout-Englert-Higgs mechanism.

What is broken!

3.2 General

Definition 3.2 (R-symmetry) *In supersymmetric theories, an R-symmetry is the symmetry transforming different supercharges into each other. In the simplest case of the* $= 1$ *supersymmetry, such R-symmetry is isomorphic to a global* $U(1)$ *group or it's discrete subgroup. For extended supersymmetry theories, the R-symmetry group becomes a global non-abelian group.*

In the case of the discrete subgroup $\mathbb{Z}_2$, the R-symmetry is called *R-parity*

Definition 3.3 (Extended supersymmetry) *In supersymmetric theories, when* > 1 *the algebra is said to have extended supersymmetry.*

3.3 Bogomol'nyi-Prasad-Sommerfield (BPS) states

Definition 3.4 (BPS state) *A massive representation of an extended supersymmetry algebra that has mass equal to the supersymmetry central charge* Z *is called an BPS state.*

Quantum mechanically speaking, if the supersymmetry remains unbroken, exact solutions to the modulus of Z exist. Their importance arises as the multiplets shorten for generic representations, with stability and mass formula exact.

[$d = 4, = 2$] The generators for the odd part of the superalgebra have relations:

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B \tag{3.33}$$

$$\{Q^A_\alpha, Q_{\beta B}\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z} \tag{3.34}$$

$$\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB} Z, \tag{3.35}$$

where $\alpha\dot{\beta}$ are the Lorentz group indices and A, B are the R-symmerty indices. If we take linear combinations of the above generators as follows:

$$R^A_\alpha = \xi^{-1} Q^A_\alpha + \xi\sigma^0_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}B} \tag{3.36}$$

$$T^A_\alpha = \xi^{-1} Q^A_\alpha - \xi\sigma^0_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}B} \tag{3.37}$$

and consider a state ψ which has momentum $(M, 0, 0, 0)$, we have:

$$\left(R^1_1 + (R^1_1)^\dagger\right)^2 \psi = 4(M + Re(Z\xi^2))\psi, \tag{3.38}$$

but because this is the square of a Hermitian operator, the right hand side coefficient must be positive for all ξ. In particular, the strongest result from this is

$$M \geq |Z| \tag{3.39}$$

3.4 Supersymmetric theories on curved manifolds

Remark: Supersymmetric theories may be defined only on backgrounds admitting solutions to certain Killing spinor equations,

$$(\nabla_\mu - iA_\mu)\,\zeta + iV_\mu\zeta + iV^\nu\sigma_{\mu\nu}\zeta = 0 \tag{3.40}$$

$$(\nabla_\mu + iA_\mu)\,\tilde{\zeta} - iV_\mu\tilde{\zeta}0iV^\nu\tilde{\sigma}_{\mu\nu}\tilde{\zeta} = 0 \tag{3.41}$$

which in four dimensions and Euclidean signature are equivalent to the requirement that the manifold is complex and the metric Hermitian.

Part II

Mathematics

4

Group Theory

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”The Universe is an enormous direct product of resentations of symmetry groups.”

Hermann Weyl

4.1 Basic Definitions

Definition 4.1 ((Group) Homomorphism) *Let $(G, *)$ and $(H, \cdot)$ be two groups. A (group) homomorphism from G to H is a function $h : G \to H$ such that for all x, y in G it holds that $h(x * y) = h(u) \cdot h(v)$*

Definition 4.2 (Coset) *Let G be a group and H is a subgroup of G. Consider an element $g \in G$. Then, $gH = \{gh : h \in H\}$ is the left coset of H in G with respect to g, and $Hg = \{hg : h \in H\}$ is the right coset of H in G with respect to g*

In general the left and right cosets are not groups.

Definition 4.3 (Normal Subgroup) *A subgroup H of G is called normal if and only if the left and right sets of cosets coincide, that is if $gH = Hg$ for all $g \in G$*

Definition 4.4 (Representation) *A representation of a group G on a vector space V over a filed K is a group homomorphism from G to $GL(V)$, the general linear group on V. That is, a representation is a map*

$$\rho : G \to GL(V) \tag{4.1}$$

such that

$$\rho(g_1g_2) = \rho(g_1)\rho(g_2), \quad \forall g_1, g_2 \in G. \tag{4.2}$$

V is often called the *representation space* and the dimension of V is called the *dimension* of the representation. It is common practice to refer to V itself as the representation when the homomorphism is clear from the context.

Consider the complex number $u = e^{2\pi/3}$ which has the property $u^3 = 1$. The cyclic group $C_3 = \{1, u, u^2\}$ has a representation ρ on $\mathbb{C}^2$ given by:

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u) = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}, \quad \rho(u^2) = \begin{bmatrix} 1 & 0 \\ 0 & u^2 \end{bmatrix}, \quad . \tag{4.3}$$

Another representation for C_3 on $\mathbb{C}^2$, isomorphic to the previous one, is

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u^2) = \begin{bmatrix} u^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad . \tag{4.4}$$

Definition 4.5 (Subrepresentation) *A subspace W of V that is invariant under the group action is called a subrepresentation.*

Definition 4.6 ((Ir)reducible representation) *If V has exactly two representations, namely the zero-dimensional subspace and V itself, then the representation is said to be irreducible; if it has a proper representation of nonzero dimension, the representation is said to be reducible. The representation of dimension zero is considered to be neither reducible nor irreducible.*

Definition 4.7 (Quotent Group) *Let N be a normal subgroup of a group G. We define the set G/N to be the set of all left cosets of N in G, i.e., $G/N = \{aN : a \in G\}$. Define an operation on G/N as follows. For each aN and bN in G/N, the product of aN and bN is $(aN)(bN)$. This defines an operation of G/N if we impose $(aN)(bN) = (ab)N$, because $(ab)N$ does not depend on the choice of the representatives a and b: if $xN = aN$ and $yN = bN$ for some $x, y \in G$, then:*

$(ab)N = a(bN) = a(yN) = a(Ny) = (aN)y = (xN)y = x(Ny) = x(yN) = (xy)N$

Here it was used in an important way that N is a normal subgroup. It can be shown that this operation on G/N is associative, has identity element N and the inverse of an element $aN \in G/N$ is $a^{-1}N$. Therefore, the set G/N together with the defined operation forms a group; this is known as the quotient group or factor group of G by N

5

Differential Geometry

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5.1 General

Definition 5.1 (Einstein Manifold) *A Riemannian manifold M is called an Einstein manifold if its Ricci tensor is proportional to the metric, i.e.*

$$Ric_g = \lambda g \tag{5.1}$$

An Einsteinian manifold, where $\lambda = 0$ is called a *Ricci-flat* manifold.

5.2 Differential Forms

> "Hamiltonian mechanics cannot be understood without differential forms"
>
> V. I. Arnold

In this section we will give a very brief introduction to differential forms and the general rules for calculus with differential forms.

5.2.1 Exterior forms

We begin with the more general notion of a *exterior form*, which is generally a poly-linear map from a vector space to an algebraic filed.

5.2.1.1 Definitions

Let $\mathbb{L}^n$ be an n-dimensional real vector space.

Definition 5.2 (Exterior algebraic form of degree k) *An exterior algebraic form of degree k, also known as a k-form, is a function of k vectors which is k-linear and anti-symmetric. Namely:*

$$\omega(\lambda_1\xi_1' + \lambda_2\xi_1'', \xi_2, \cdots, \xi_k) = \lambda_1\omega(\xi_1', \xi_2, \cdots, \xi_k) + \lambda_2\omega(\xi_1'', \xi_2, \cdots, \xi_k), \quad (5.2)$$

and

$$\omega(\xi_{i_1}, \cdots, \xi_{i_k}) = (-1)^\nu\omega(\xi_1, \cdots, \xi_k) \quad (5.3)$$

with $(-1)^\nu = 1$ if the permutation $(i_1, \cdots, i_k)$ is even and $(-1)^\nu = -1$ if the same permutation is odd. Here, $\xi_i \in \mathbb{L}^n$ and $\lambda_i \in \mathbb{R}$.

The set of all k-forms in $\mathbb{L}^n$ forms a real vector space. Indeed, one has that:

$$(\omega_1 + \omega_2)(\xi) = \omega_1(\xi) + \omega_2(\xi), \quad \xi = \{\xi_1, \cdots, \xi_k\} \quad (5.4)$$

and

$$(\lambda\omega)(\xi) = \lambda\omega(\xi). \quad (5.5)$$

Since $\mathbb{L}^n$ is a vector space, we can always suppose that it is equipped with a coordinate system. Let us denote these coordinates by $x_1, \cdots, x_n$. Now, we can think of these coordinates as 1-forms so that $x_i(\xi) = \xi_i$ and that is the i-th coordinate of the vector ξ. These coordinates form a basis of 1-forms in the 1-form vector space (which is also called the *dual space* $(\mathbb{L}^n)^*$).

Any 1-form can then be written as a linear combination of basis 1-forms:

$$\omega_1 = a_1 x_1 + \cdots + a_n x_n. \tag{5.6}$$

5.2.1.2 Exterior Products

Definition 5.3 (Exterior Product) *An exterior product of k 1-forms $\omega_1, \omega_2, \cdots, \omega_k$ is a k-form defined by*

$$(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k)(\xi_1, \xi_2, \cdots, \xi_k) = det\omega_i(\xi_j). \tag{5.7}$$

Exterior products of basis 1-forms $x_{i_1} \wedge \cdots \wedge x_{i_k}$ with $i_1 < \cdots < i_k$ for a basis in the space of k-forms. The dimension the latter space is obviously C_n^k. A general k-form can be written as

$$\omega^k = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \cdots i_k} x_{i_1} \wedge \cdots \wedge x_{i_k} \tag{5.8}$$

where $a_{i_1 \cdots i_k}$ are real numbers.

Definition 5.4 (Exterior product of an k-form and an l-form) *The exterior product $\omega^k \wedge \omega^l$ of a k-form with an l-form on $\mathbb{L}^n$ is the $k+l$-form on $\mathbb{L}^n$ defined as:*

$$(\omega^k \wedge \omega^l)(\xi_1, \cdots, \xi_{k+l}) = \sum (-1)^\nu \omega^k(\xi_{i_1}, \cdots, \xi_{i_k}) \wedge \omega^l(\xi_{j_1}, , \xi_{j_l}) \tag{5.9}$$

where $i_1 < \cdots < i_k$ and $j_1, < \cdots < j_l$ and the sum is taken over all permutations $(i_1, \cdots, i_k, j_1, \cdots, j_l)$ with $(-1)^\nu$ being +1 for even permutations and -1 for odd permutation.

One can check that this definition is consistent with the definition for exterior product of 1-forms. Furthermore it can be shown that the exterior product is distributive, associative and *skew-commutative*. The latter means that $\omega^k \wedge \omega^l = (-1)^{kl} \omega^l \wedge \omega^k$. If ω is a 1-form (or a form of any odd degree) one can easily show that $\omega \wedge \omega = 0$. We will present a brief version of the proof here: Let ω be an exterior k-form, where k is an odd integer. Then the exterior product takes the form:

$$\omega \wedge \omega = (-1)^{k^2} \omega \wedge \omega \tag{5.10}$$

But k is odd, therefore k^2 is odd. From that we see that

$$\omega \wedge \omega = (-1)^{k2} \omega \wedge \omega = -\omega \wedge \omega \tag{5.11}$$

From this we conclude that $\omega \wedge \omega = 0$.

5.2.1.3 Behaviour under mappings

Let $f : \mathbb{L}^m \to \mathbb{L}^n$ be a linear map and ω^k an exterior k-form on $\mathbb{L}^n$. We can define a k-form $(f^*\omega^k)$ on $\mathbb{L}^m$ by

$$(f^*\omega^k)(\xi_1, \cdots, \xi_k) = \omega^k(f\xi_1, \cdots, f\xi_k) \tag{5.12}$$

Notice that the obtained mapping of forms f^* acts in *opposite* direction to f. Namely, $f^* : \Omega_k(\mathbb{L}^n) \to \Omega_k(\mathbb{L}^b)$, where $\Omega_k(\mathbb{L}^n)$ is the vector space of k-forms on $\mathbb{L}^n$.

5.2.2 Differential Forms

After the general discussion about exterior forms it is time to restrict ourselves to a more practical object, namely the differential form. Understanding differential forms or at least having working knowledge about the topic is paramount for the study of differential geometry and physics on curved manifolds. Although this may seem abstract at first, we urge the reader to push through the mathematical definitions and grasp the essence of the idea, that is how do we perform calculus on curved manifolds without the notion of coordinates.

Definition 5.5 (Differential Form) *A differential k-form $\omega^k|_x$ at a point x of a manifold M is an exterior k-form on the tangent space TM_x to M at x, i.e., a k-linear skew-symmetric function of k vectors $\xi_1, \cdots, \xi_k$ tangent to M at x. If such a form is given at every point x of M and if it is differentiable, we say that we are given a k-form ω^k on the manifold M.*

We intoduce the coordinate basis 1-forms dx_i with $i = 1, \cdots, n$ ($dimM = n$). The notation dx_i is used for the exterior forms emphasizes that these basis forms act on TM_x at a given point x of M. In the neighborhood of x one can always write the general differential k-form as

$$\omega^k = \sum_{i_1<\cdots<i_k} a_{i_1,\cdots,i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \tag{5.13}$$

where $a_{i_1\cdots i_k}$ are smooth functions of x. Let's give a simple example. The differential of some scalar function $f(x)$ defined on a manifold M is a differential 1-form. We introduce

$$df = \sum_{k=1}^{n} \left.\frac{\partial f}{\partial x_k}\right|_x dx_k, \tag{5.14}$$

where dx_k are basis differential 1-forms. The value of this 1-form on a vector $\xi \in TM_x$ is given by

$$df(\xi) = \sum_{k=1}^{n} \left.\frac{\partial f}{\partial x_k}\right|_x \xi_k. \tag{5.15}$$

5.2.2.1 Behaviour under mappings

Let $f : M \to N$ be a differentiable map of a smooth manifold M to a smooth manifold N, and let ω be a differential k-form o N. The mapping f induces the mapping $f_* : TM_x \to TM_{f(x)}$ of tangent spaces. The latter mapping f_* is called the *differential* of the map f. The mapping f_* is a mapping of linear spaces and gives rise to the mapping of forms defined on corresponding tangent spaces. As a result a well-defined differential k-form $(f^*\omega)$ exists on M:

$$(f^*\omega)(\xi_1, \cdots, \xi_k) = \omega(f_*\xi_1, \cdots, f_*\xi_k). \tag{5.16}$$

5.2.3 Integration of Differential Forms over Chains

5.2.3.1 Integration of k-form over k-dimensional cell

Let D be a bounded convex polyhedron in $\mathbb{L}^k$ and $x_1, \cdots, x_k$ an oriented coordinate system on $\mathbb{L}^k$. Any differential k-form on $\mathbb{L}^k$ can be written as $\omega^k = \phi(x)dx_1 \wedge \cdots \wedge dx_k$, where $\phi(x)$ is a differentiable function on $\mathbb{R}^k$. We define the integral of the form ω^k over D as the integral of the function $\phi(x)$:

$$\int_D \omega_k = \int_D \phi(x)dx_1dx_2\cdots dx_k. \tag{5.17}$$

Definition 5.6 (k-dimensional cell) *A k-dimensional cell σ of an n-dimensional manifold M is a polyhedron D in $\mathbb{L}^n$ with a differentiable map $f : D \to M$.*

One can think about σ as a "curvilinear polyhedron" - the image of D on M. If ω is a differentiable k-form on M, we define the integral of a form over the cell σ as

$$\int_\sigma \omega = \int_D f^*\omega, \tag{5.18}$$

where f^* is a mapping of k-forms induced by f.

The cell σ inherits an orientation from the orientation of $\mathbb{L}^k$. The k-dimensional cell which differs from σ only by the choice of orientation is called the *negative* of σ and is denoted by $-\sigma$ or by $(-1)\sigma$. One can show that under a change of orientation the integral changes sign:

$$\int_{-\sigma} \omega = -\int_\sigma \omega. \tag{5.19}$$

5.2.3.2 Chains and the Boundary Operator

It is convenient to generalize our definition of the integral of a form over *cell* to the integral over a *chain*.

Definition 5.7 (Chain of a Manifold) *A chain of dimension k on a manifold M consists of a finite collection of k-dimensional oriented cells $\sigma_1, \cdots, \sigma_r$ in M and integers $m_1, \cdots, m_r$ called multiplicities. A chains is denoted by*

$$c_k = m_1\sigma_1 + \cdots + m_r\sigma_r. \tag{5.20}$$

One can introduce the structure of a commutative group on a set of k-chains on M with natural definitions of addition of chains $c_k + b_k$.

Definition 5.8 (Boundary Operator) *The boundary of a convex oriented k-polyhedron D on $\mathbb{L}^k$ is the $(k-1)$-chain ∂D on $\mathbb{L}^k$ defined as*

$$\partial D = \sum_i \sigma_i \tag{5.21}$$

where the cells σ_i are the $(k-1)$-dimensional faces of D with orientations inherited from the orientation of $\mathbb{L}^k$.

One can easily extend this definition to the definition of the *boundary of a cell $\partial\sigma$* on M and then to the boundary of a chain. Indeed, defining:

Definition 5.9 (Boudary of a Chain)

$$\partial c_k = m_1\partial\sigma_1 + \cdots + m_r\partial\sigma_r \tag{5.22}$$

We can see that ∂c_k is a $(k-1)$-chain on M. Additionally, we define a 0-chain as a collection of points with multiplicities. Furthermore we define the boundary of an oriented interval $\vec{AB}$ as $B - A$. The boundary of a point is empty.

It is straightforward to show that the boundary of the boundary of a cell is zero. Therefore

$$\partial(\partial c_k) = 0 \tag{5.23}$$

for any k-chain c_k. We denote this property as

$$\partial\partial = \partial^2 = 0 \tag{5.24}$$

5.2.3.3 Integration of a k-form over a k-chain

An integral of a k-form over a k-chain is then defined as

$$\int_{c_k} \omega^k = \sum m_i \int_{\sigma_i} \omega^k. \tag{5.25}$$

5.2.4 Exterior Differentiation and Stokes Formula

Definition 5.10 (Exterior Derivative of a Form) *An exterior derivative of a differential k-form ω is a $(k+1)$-form $\Omega = d\omega^k$. Given a set of coordinates $\{dx_{i_j}\}$, we have:*

$$\Omega = d\omega^k = \sum da_{i_1 \cdots i_k} \wedge dx_{i_1} \wedge \wedge dx_{i_k} \tag{5.26}$$

implying that we have defined ω^k as in (5.13)*. Here, da is a 1-form, the differential of the function $a(x)$.*

One can show that the definition does not actually depend on the choice of coordinates. We can think of the 1-form given by the differential of a scalar function as of an external derivative of a 0-form. It is easy to show that $d(df) = 0$, if f belongs to the set of 0-forms. Indeed If f belongs to the set of 0-forms, then by definition

$$df = \sum \frac{\partial a}{\partial x^i} dx^i, \tag{5.27}$$

which is a 1-form. Taking an exterior derivative again yields

$$\begin{aligned} d(df) &= d\left(\sum \frac{\partial a}{\partial x^i} dx^i\right) = \\ &= \sum \frac{\partial^2}{\partial x^i \partial x^j} dx^i \wedge x^j = \\ &= \frac{\partial^2 g}{dx^i dx^j} dx^i \wedge dx^j + \frac{\partial^2 g}{dx^j dx^i} dx^j \wedge dx^i \\ &= 0 \end{aligned} \tag{5.28}$$

Here we have used that $\frac{\partial g^2}{dx^i dx^j} = \frac{\partial g^2}{dx^j dx^i}$ and $dx^i \wedge dx^j = -dx^j \wedge dx^i$. Using this result, one can show that it holds for forms of any degree.

Another useful formula is that for differentiating an exterior product of form:

$$d(\omega^k \wedge \omega^l) = d\omega^k \wedge d\omega^l + (-1)^k \omega^k \wedge d\omega^l \tag{5.29}$$

Lastly, if $f : M \to N$ is a smooth map and ω is a k-form on N, we have

$$f^*(d\omega) = d(f^*\omega). \tag{5.30}$$

5.2.4.1 Stokes' Formula

One of the most-important formulae in differential geometry is the *Newton-Leibniz-Gauss-Green-Ostrogradski-Stokes-Poincaré* formula:

$$\boxed{\int_{\partial c} \omega = \int_c d\omega} \tag{5.31}$$

where c is any $(k+1)$-chain on a manifold M and ω is any k-form on M. In the case when the boundary $\partial c = 0$, we have $\int_c d\omega = 0$, which corresponds to integration of a complete derivative over a closed surface.

5.2.5 Homologies and Cohomologies

5.2.5.1 Closed and Exact Forms

Definition 5.11 (Closed Form) *A differential form ω on a manifold M is said to be closed if $d\omega = 0$.*

In particular, on a 3D Riemannian manifold, we have $d\omega^2_{\vec{A}} = \left(\nabla \cdot \vec{A}\right)\omega^3 = 0$, which is equivalent to $\left(\nabla \cdot \vec{A}\right) = 0$, i.e the corresponding vector field is divergenless. We can apply Stokes' formula for a closed form, getting

$$\int_{c_{k+1}} \omega^k = 0 \quad \text{if } dw^k = 0. \tag{5.32}$$

Definition 5.12 (Exact Form) *A differential form ω on a manifold M is said to be exact if there exists such a differential form μ that $\omega = d\mu$.*

Since $d(d\omega) = 0$ as we've proven all exact forms are closed. However, there are some closed forms which are not exact. Let's give a short example. Consider the circle S^1, parametrized by an angle $\phi \in [0, 2\pi]$. One can introduce the 1-form ω^1, defined by $\omega^1(\partial_t\gamma) = \partial_t\gamma$, where $\partial_t\gamma$ is the "velocity" along the path $\phi = \gamma(t)$. Obviously, this "velocity", belongs to the tangent space of S^1 at the point with coordinate ϕ. The form is closed - $d\omega$ is a 2-form, and we can't have 2-forms on a one-dimensional manifold. However:

$$\int_{S^1} \omega^1 = \int_0^T dt\partial_t\gamma = 2\pi, \tag{5.33}$$

the length of the circle. Although the boundary ∂S^1 is zero, the integral is not zero, and therefore ω^1 is not exact.

One can notice that locally the introduced 1-form can be written as $\omega^1 = d\phi$, which can look like a contradiction. It is easy to see where the problem lies - writing $\omega^1 = d\phi$ is not valid for $\phi = 0$. We've come across an example of a general result, namely

Theorem 5.1 (Poincaré's Lemma) *Any closed form is locally exact.*

The existence of locally but not globally exact closed forms is related to some topological properties of the underlying manifold M.

5.2.5.2 Cycles and Boundaries

Definition 5.13 (Cycle on a Manifold) *A cycle on a manifold M is a chain whose boundary is equal to zero.*

Using Stokes theorem, we have

$$\int_{c_{k+1}} d\omega^k = 0 \quad \text{if } \partial c_{k+1} = 0. \tag{5.34}$$

Chains that can be considered boundaries of some other chains are called *boundaries*. Since $\partial = 0$, all boundaries are cycles. However, not all cycles are boundaries. The existence of cycles that are not boundaries is again related to some topological properties of the manifold. A fairly simple example is found on the 2-torus. The 2-torus is the direct product of 2 circles - $T^2 = S^1 \times S^1$. Each of the S^1 is a cycle, but none of them are boundaries.

5.2.5.3 Homologies and Cohomologies

The set of all k-forms on M is a vector space, the set of all *closed* k-forms is a subspace of that space and the set of differentials of $(k-1)$-forms (the exact k-forms) are a subspace of the subspace of closed forms. We can now define:

Definition 5.14 *The quotient space:*

$$\frac{(closed\ forms)}{(exact\ forms)} = H^k(M, \mathbb{R}) \tag{5.35}$$

is called the k-th cohomology group of the manifold M. An element of this group is a class of closed forms, differing from each other only by an exact form.

For the circle S^1, we have $H^1(S^1, \mathbb{R}) = \mathbb{R}$.

Definition 5.15 (Betti Number) *The dimension of H^k is called the k-th Betti number of M.*

Obviously, the first Betti number of S^1 is 1.

The cohomology groups of M are important *topological* properties of M.

Definition 5.16 (Homologous Cycles) *Let us now consider two k-cyclces a and b, such that their difference is a boundary of a $(k+1)$-chain, i.e. $a - b = \partial c_{k+1}$. Such cycles are called homologous.*

Let us have two k-cycles, a and b, homologous to each other and a closed form ω^k. From (5.32) *we can see that*

$$\int_a \omega^k = \int_b \omega^k. \tag{5.36}$$

In other words, homologous cycles can be replaced with one another for integration paths.

Definition 5.17 (Homology Group) *The quotient group*

$$\frac{(cycles)}{(boundaries)} = H_k(M) \tag{5.37}$$

is called the k-th homology group of M. An element of this group is a class of cycles homologous to each other. The rank of this group is also equal to the k-th Betti number of M.

5.2.6 Homologies and Homotopies

There are important relations between homology and homotopy groups of a topological space M.

Let us suppose that $\pi_1(M)$ and $H_1(M)$ are the fundamental and the first homology group of M, respectively. Then $H_1(M) = \pi_1(M)/[\pi_1, \pi_1]$, where $[\pi_1, \pi_1]$ is the commutator in the fundamental group. In particular, if $\pi_1(M)$ is Abelian, then $\pi_1(M) = H_1(M)$. For the higher homotopy groups, there is another result, known as the Gurevich theorem.

Theorem 5.2 (Gurevich theorem) *If $\pi_k(M) = 0$ for all $k < n$, then*

$$\pi_n(M) = H_n(M). \tag{5.38}$$

As a general rule of thumb, homology (and cohomology) groups are usually easier to calculate than homotopy groups. FUCK

5.3 Contact Manifolds

Definition 5.18 (Riemannian Cone) *Given a Riemannian manifold (M, g), its Riemannian cone is a product*

$$(M \times \mathbb{R}^{>0}) \tag{5.39}$$

of M with the half-line $\mathbb{R}^{>0}$ equipped with the cone metric

$$t^2 g + dt^2, \tag{5.40}$$

where t is a parameter in $\mathbb{R}^{>0}$

Definition 5.19 (Contact Manifold) *A manifold M, equipped with a 1-form θ is contact if and only if the 2-form*

$$t^2 d\theta + 2tdt \cdot \theta \tag{5.41}$$

on its cone is symplectic.

Definition 5.20 (Sasakian Manifold) *A contact Riemannian manifold is called a Sasakian manifold, if its Riemannian cone with the cone metric is a Kähler manifold with Kähler form*

$$t^2 d\theta + 2tdt \cdot \theta. \tag{5.42}$$

Consider the manifold $\mathbb{R}^{2n+1}$ with coordinates $(\vec{x}, \vec{y}, z)$, endowed with contact form

$$\theta = \frac{1}{2}dz + \Sigma_i y_i dx_i \tag{5.43}$$

and Riemannian metric

$$g = \Sigma_i (dx_i)^2 + (dy_i)^2 + \theta^2 \tag{5.44}$$

Definition 5.21 (Sasaki-Einstein Manifold) *A Sasaki-Einstein manifold is a Riemanian manifold (S, g) that is both Sasakian and Einstein*

The odd dimensional sphere S^{2n-1}, equipped with its standard Einstein metric is a Sasaki-Einstein manifold. In this case, the Kähler cone is $\mathbb{C}^2 \backslash \{0\}$, equipped with its flat metric.

5.4 Symplectic Geometry

Theorem 5.3 (Duistermaat-Heckman Formula) *For a compact symplectic manifold M of dimension $2n$ with symplectic form ω and with a Hamiltonian $U(1)$ action whose moment map is denoted by μ, the following formula holds:*

$$\int_M \frac{\omega^n}{n!} e^{-\mu} = \sum_i \frac{e^{-\mu(x_i)}}{e(x_i)} \tag{5.45}$$

Here, x_i are the fixed points of the $U(1)$ action and they are assumed to be isolated, and $e(x_i)$ is the product of the weights of the $U(1)$ action on the tangent space at x_i.

5.5 Complex Manifolds

Definition 5.22 (Hermitian Metric) *If a Riemannian metric g of a complex manifold M satisfies*

$$g_p(J_p X, J_p Y) = g_p(X, Y) \tag{5.46}$$

at each point $p \in M$ and for any $X, Y \in T_pM$, g is said to be a Hermitian metric. Here, J_p denotes the almost complex structure on M.

Definition 5.23 (Hermitian Manifold) *The pair (M, g) is called a Hermitian manifold*

Theorem 5.4 *A complex manifold always admits a Hermitian metric.*

Let g be any Riemannian metric of a complex manifold M. Define a new metric $\hat{g}$ by

$$\hat{g}_p(X, Y) \equiv \frac{1}{2}\left[g_p(X, Y) + g_p(J_pX, J_pY)\right]. \tag{5.47}$$

Clearly $\hat{g}_p(J_pX, J_pY) = \hat{g}_p(X, Y)$. Moreover, $\hat{g}$ is positive definite provided that g is. Hence, $\hat{g}$ is a Hermitian metric on M

Definition 5.24 (Kähler Form) *Let (M, g) be a Hermitian manifold. Define a tensor field Ω whose action on $X, Y \in T_pM$ is*

$$\Omega_p(X, Y) = g_p(J_pX, Y) \tag{5.48}$$

Note that Ω is anti-symmetric, $\Omega(X, Y) = g(JX, Y) = g(J^2X, JY) = -g(JY, X) = -\Omega(Y, X)$. Hence, Ω defines a two-form, called the Khäler form of a Hermitian metric g.

Definition 5.25 (Kähler Manifold) *A Kähler manifold is a Hermitian manifold (M, g) whose Kähler form Ω is closed, $d\Omega = 0$. The metric g is called the Kähler metric of M.*

Not all complex manifolds admit Kähler metrics

Theorem 5.5 *A Hermitian manifold (M, g) is a Kähler manifold if and only if the almost complex structure J satisfies*

$$\nabla_\mu J = 0 \tag{5.49}$$

where ∇_μ is the Levi-Cevita connection associated with g.

We first note that for any r-form ω, $d\omega$ is written as

$$d\omega = \nabla\omega \equiv \frac{1}{r!}\nabla_\mu\omega_{\nu_1\cdots\nu_r}dx^\mu \wedge dx^{\nu_1} \wedge \cdots \wedge dx^\nu \tag{5.50}$$

Now we prove that $\nabla_\mu J = 0$ if and only if $\nabla_\mu\Omega = 0$. We verify the following equalities:

$$\begin{aligned}
(\nabla_Z\Omega)(X, Y) &= \nabla_Z\left[\Omega(X, Y)\right] - \Omega(\nabla_ZX, Y) - \Omega(X, \nabla_zY) && (5.51)\\
&= \nabla_Z\left[g(JX, Y)\right] - g(J\nabla_ZX, Y) - g(JX, \nabla_ZY) && (5.52)\\
&= (\nabla_Zg)(JX, Y) + g(\nabla_ZJX, Y) - g(J\nabla_ZX, Y) && (5.53)\\
&= g(\nabla_ZJX - J\nabla_ZX, Y) = g((\nabla_ZJ)X, y) && (5.54)
\end{aligned}$$

where $\nabla_Z g = 0$ has been used. Since this is true for any X, Y, Z, if follows that $\nabla_Z \Omega = 0$ if and only if $\nabla_Z J = 0$. The last theorem shows that the Riemann structure is compatable with the Hermitian structure in the Kähler manifold.

We can also characterize Kähler manifolds as Hermitian manifolds for which the Cristoffel symbols of the Levi-Chevita connection are pure. In other words, Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$ may be non zero, but all "mixed" symbols like $\Gamma^{\bar{i}}_{jk}$, for example, vanish. This means that (anti-)holomorphic vectors get parallel transported to (anti-)holomorphic vectors.

Kähler manifolds are manifolds on which we can always find a holomorphic change of coordinates which, at some given point, sets the metric to its cannonical form, and its first derivatives to zero.

Equivalently, an n-dimensional Kähler manifold are precisely $2n$-dimensional Riemannian manifolds with holonomy group contained in $U(1)$

Definition 5.26 (Hopf Surface) *Let $\mathbb{Z}$ act on $\mathbb{C}^n \backslash \{0\}$ by $(z_1, \cdots, z_n) \to (\lambda^k z_1, \cdots, \lambda^k z_n)$ for $k \in \mathbb{Z}$. For $0 < \lambda < 1$ the action is free and discrete. The quotient complex manifold $X = (\mathbb{C}^n \backslash \{0\}) / \mathbb{Z}$ is diffeomorphic to $S^1 \times S^{2n-1}$. For $n = 1$ this manifold is isomorphic to a complex torus $\mathbb{C}/\Gamma$. The lattice Γ can be determined explicitly.*
In other words, a Hopf manifold is obtained as a quotient of the complex vector space (with zero deleted) $\mathbb{C}^n \backslash \{0\}$ by a free action of the group $\Gamma \cong \mathbb{Z}$ of integers, with the generator γ of Γ acting by holomorphic contractions.