Sofia University Faculty of Physics

My Journey Through Theoretical Physics

IVO ILIEV

To my friends,
my familly, my teachers and collegues.
Also, to Mister T
I pitty the fool

Contents

Foreword

"It is so shocking to find out how many people do not believe that they can learn, and how many more believe learning to be difficult."

Frank Herbert, Dune

Preface

The idea behind this book is to systematically put a lot of knowledge from different fields in the same place. It is envisioned to serve a purpose of an index book, a glossary introduction to modern physics. The depth may be lacking in some respects but the core idea is to have a background knowledge of many approaches and frameworks.

Physics is a journey. In this book, I've tried to illustrate my own journey through theoretical physics in hopes that it might help others in their own studies.

The book is mainly divided into two parts. The first one is dedicated to physics. Many topics are considered there - quantum field theory, general relativity, nonlinear dynamics and a bunch more. Of course, not all topics are treated equally both in scope and depth. That does not reflect their importance, it merely reflects the fact that I am more interested in them.

The second part is about mathematical preliminaries. Again, some things are explained in detail and others are only brushed upon. However, I've tried to include everything that is absolutely *needed* if you would like to work in theoretical physics.

Again, this book is not trying to be exactly pedagogical. It is designed as an extended handbook, somewhere one can search for a definition, a formula or a general description of a phenomenon or mathematical idea.

It must be noted that this book is, and always will be, **completely free**. You are free to redistribute it, modify it and use it in any way you want, without asking for permission from the author. The LaTeX file and the documentation for it are available on `https://github.com/HallowDance/book`. The only thing I'll politely ask (but not enforce) is that you don't use parts of the book or the book as a whole for commercial purposes.

Part I

Physics

Chapter 1

General Relativity

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1.1 Anti De Sitter Space (AdS)

AdS_n is an n-dimensional solution for the theory of gravitation with Einstein-Hilbert action with negative cosmological constant Λ, i.e. the theory described by the following Lagrangian density:

$$\mathcal{L} = \frac{1}{16\pi G_{(n)}}(R - 2\Lambda), \tag{1.1}$$

where $G_{(n)}$ is the gravitational constant in n-dimensional spacetime. Therefore, it's a solution of the Einstein field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and $g_{\mu\nu}$ is the metric of the spacetime. Introducing the radius α as

$$\Lambda = \frac{-(n-1)(n-2)}{2\alpha^2} \tag{1.3}$$

this solution can be immersed in a $n+1$ dimensional spacetime with signature $(-,-,+,\cdots,+)$ by the following constraint:

$$-X_1^2 - X_2^2 + \sum_{i=3}^{n+1} X_i^2 = -\alpha^2 \tag{1.4}$$

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Chapter 2

Quantum Field Theory

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2.1 Rarita-Schwinger equation

Consider the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}_\mu \left(\epsilon^{\mu\kappa\rho\nu}\gamma_5\gamma_\kappa\partial_\rho - im\sigma^{\mu\nu}\right)\psi_\nu \tag{2.1}$$

This equation obviously controls the propagation of the wave function of a spin- object such as the gravitino. The equation of motion for this Lagrangian are known as the *Rarita-Schwinger equation*:

$$\left(\epsilon^{\mu\kappa\rho\nu}\gamma_5\gamma_\kappa\partial_\rho - im\sigma^{\mu\nu}\right)\psi_\nu \tag{2.2}$$

In the massless case, the Rarita-Schwinger equation has a fermionic gauge symmertrty, it is invariant under the gauge transformation:

$$\psi_\mu \to \psi_\mu + \partial_\mu\epsilon, \tag{2.3}$$

where $\epsilon \equiv \epsilon_\alpha$ is an arbitrary spinor field.

2.1.1 Massless case

Consider a massless Rarita-Schwinger field, described by the Lagrangian

$$\mathcal{L}_{RS} = \bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho \tag{2.4}$$

where the sum over spin indices is implicit, ψ_μ are Majorana spinors and the quantity $\gamma^{\mu\nu\rho}$ is equal to

$$\gamma^{\mu\nu\rho} \equiv \frac{1}{3!} \gamma^{[\mu} \gamma^{\nu} \gamma^{\rho]} \tag{2.5}$$

Varying the Lagrangian yealds after some calculation

$$\delta\mathcal{L}_{\mathcal{RS}} = 2\delta\bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho + \text{boundary terms} \tag{2.6}$$

Now imposing that $\mathcal{L}_{RS} = 0$ we get the equation of motion for a massless Majorana Rarita-Schwinger spinor:

$$\gamma^{\mu\nu\rho} \partial_\nu \psi_\rho = 0 \tag{2.7}$$

2.1.2 Massive case

The description of massive, higher-spin fields through the Rarita-Schwinger equation is not well defined physically. Coupling the RS Largrangian to electromagnetism leads to an equation with solutions representing wavefronts, some of which propagate faster than light. However, it was shown by Das and Freedman that local supersymmetry can circumvent this problem.

2.2 Wilson Loops and Large N

This section is dedicated to a short introduction to the methods used in nonperturbative investigations of QCD and other gauge theories. The main attention is payed to Wilson loops, both on the lattice and in the continuum, which play a central role in modern formulations of gauge theories and to the method of the large N expansion.

2.2.1 Wilson Loops

In essence, Wilson loops are phase factors in Abelian or non-Abelian gauge theories. Wilson loops are observable in quantum theory by the Aharonov-Bohm effect. Wilson loops play a central role in the lattice formulation of gauge theories. QCD can be reformulated through Wilson loops in a manifest gauge-invariant way. Ananlogues of the Wilson loops are extremely useful in solving various kinds of matrix models.

Phase factors in QED

Let us first examine the familiar setting of QED. An Abelian phase factor is defined by the formula

$$U\left[\Gamma_{yx}\right] = e^{ie\int_{\Gamma_{yx}} \mathrm{d}z^\mu A_\mu(z)} \tag{2.8}$$

Under the gauge transformation this becomes

$$A_\mu(z) \xrightarrow{\text{g.t.}} A_\mu(z) + \frac{1}{e}\partial_\mu \alpha(z) \tag{2.9}$$

and the Abelian phase factor transforms as

$$U\left[\Gamma_{yx}\right] \xrightarrow{\text{g.t.}} \mathrm{e}^{i\alpha y} U\left[\Gamma_{yx}\right] \mathrm{e}^{-i\alpha x} \tag{2.10}$$

It is easy to show that a wave function at the point x is transformed as

$$\phi(x) \xrightarrow{\text{g.t.}} \mathrm{e}^{i\alpha(x)}\psi(x), \tag{2.11}$$

therefore the phase factor is transformed as the product $\psi(y)\psi^\dagger(x)$:

$$U\left[\Gamma_{yx}\right] \overset{g.t.}{\sim} \text{"}\psi(y)\psi^\dagger(x)\text{"} \tag{2.12}$$

A wave function at the point x transforms like one at the point y after multiplication by a phase factor:

$$U\left[\Gamma_{yx}\right]\psi(x) \overset{g.t.}{\sim} \text{"}\psi(y)\text{"}, \tag{2.13}$$

and analogously

$$\psi^\dagger(y) U\left[\Gamma_{yx}\right] \overset{g.t.}{\sim} \text{"}\psi^\dagger(x)\text{"}. \tag{2.14}$$

The phase factor plays the role of a parallel transporter in an electromagnetic field, and to compare phases of a wave function at points x and y, we should first make a parallel transport along some contour Γ_{yx}. The result is of course Γ-dependent except when $A_\mu(z)$ is a pure gauge (meaning that tje field strenght $F_{\mu\nu}(z)$ is vanishing along the contour). Certain subtlelies occur for not simply connected spaces, namely the Aharonov-Bohm effect.

Propagators in external field

Let us consider a quantum particle in a classical electromagnetic field. To introduce electromagnetism field, ∂_μ is to be replaced by the covariant derivative

$$\partial_\mu \to \Delta_\mu = \partial_\mu - ieA_\mu(x). \tag{2.15}$$

For the propagator we get

$$G(x,y;A) = \frac{1}{2}\int_0^\infty \mathrm{d}\tau e^{-\frac{1}{2}\tau m^2} \int\limits_{\substack{z_\mu(0)=x_\mu \\ z_\mu(\tau)=y_\mu}} \mathcal{D}z_\mu(t) e^{-\frac{1}{2}\int_0^\tau \mathrm{d}t \dot{z}_\mu^2(t) + ie\int_0^\tau \mathrm{d}t \dot{z}^\mu(t) A_\mu(z(t))} \tag{2.16}$$

Although this may seem cumbersome, one can easily spot that the expotent is justjust the classical (Eucleadian) *action* of a particle in an external electromagnetic field. The path integral

Figure 2.1: Aharonov-Bohm effect

representation above for the propagator of a scalar particle is due to Feynman. We can alternatively rewrite it in a more compact form

$$G(x,y;A)=\sum_{\Gamma_{yx}} e^{S_{\rm free}[\Gamma_{yx}]+ie\int_{\Gamma_{yx}} {\rm d}z^\mu A_\mu(z)}, \tag{2.17}$$

where we represented the (parametric invariant) integral over dt as the contour integral along the trajectory Γ_{yx} over

$${\rm d}z^\mu = {\rm d}t\dot{z}^\mu(t). \tag{2.18}$$

The transition amptitude of a quantum particlee in a classical electromagnetic field is the sum over paths of the Abelian phase factor(2.8).

Aharonov-Bohm effect

Transverse components of the electromagnetic field describe photons. Longitudal components are related to gauging the phase of a wave function, i.e. permit one to compare its values at different space-time points when an electronelectron is placed in an external electromagnetic field.

In quantum mechanics, the wave-function phase itself is not observable. Only the phase differences are observable, e.g. via interference phenomena. The phase difference depends on the value of the phase factor for a given path Γ_{yx} along which the parallel transport is performed.

The phase factors are observable in quantum theory, in contrast to classical theory. This is seen in the Aharonov-Bohm effect, whose scheme is depicted in Fig. 2.2.1. Electrons do not pass inside the solenoid where the magnetic field is concentrated. Nevertheless, a phase difference arises between the electron beams passing through the two slits. The interference picture changes with the value of the electric current. The phase difference depends on (the real part of)

$$e^{ie\int_{\Gamma^+_{yx}} {\rm d}z^\mu A_\mu(z)} e^{-ie\int_{\Gamma^-_{yx}} {\rm d}z^\mu A_\mu(z)} = e^{ie\oint_\Gamma {\rm d}z^\mu A_\mu(z)} = e^{ie\int {\rm d}\sigma^{\mu\nu}F_{\mu\nu}} = e^{ieHS} \tag{2.19}$$

where the contour Γ is composed from Γ^+_{yx} and Γ^-_{xy}. It does not depend on the shape of the two sub-contours but depends only on HS - the magnetic flux through the solenoid.

2.2.2 Yang-Mills Theories

Modern theories of fundamental particles are gauge theories. The principle of local gauge invariance was introduced bt H.Weyl for the electromagnetic interaction in analogy with general covariance in Einstein's theory of gravitation. An extension to non-Abelian gauge groups was given by Yang and Mills in 1954.

Aa crucial role in gauge theories is played by the phase factor which is associated with parallel transport in an external gauge field. The phase factors are observable in quantum theory, in contrast to classical theory. This is analogous to the Aharonov-Bohm effect for the electromagnetic field.

Gauge invariance

The principle of local gauge invariance deals with the gauge transformation of a matter field ψ, which is given by:

$$\psi(x) \xrightarrow{g.t.} \psi'(t) = \Omega(x)\psi(x). \tag{2.20}$$

Here $\Omega(x) \in G$ with G being a semisimple Lie group which is called the gauge group ($G = SU(3)$ for QCD. (2.20) demonstrates that ψ belongs to the fundamental representation of G.

The unitary gauge group is when

$$\Omega^{-1}(x) = \Omega^{\dagger}(x), \tag{2.21}$$

while an extension to the other Lie groups is straightforward. Then we have

$$\psi^{\dagger}(x) \xrightarrow{g.t.} \psi'^{\dagger} = \psi^{\dagger}(x)\Omega^{\dagger}(x). \tag{2.22}$$

In analogy with QCD, the gauge group $G = SU(N)$ is usually associated with color and the proper index of ψ is called the color index.

The gauge transformation (2.20) of the matter field ψ can be compensated by a transformation of the non-Abelian gauge field $\mathcal{A}_\mu$ which belongs to the adjoint representation of G:

$$\mathcal{A}_\mu(x) \xrightarrow{g.t.} \mathcal{A}'_\mu(x) = \Omega(x)\mathcal{A}_\mu(x)\Omega^{\dagger}(x) + i\Omega(x)\partial_\mu\Omega^{\dagger}(x) \tag{2.23}$$

It is conveniant to introduce the Hermitian matrix

$$[\mathcal{A}_\mu(x)]^{ij} = g\sum_a A^a_\mu(x)\,[t^a]^{ij} \tag{2.24}$$

where g is the gauge coupling constant.

The matrices $[t^a]^{ij}$ are the generators of G ($a = 1, \cdots, N^2-1$ for $SU(N)$) which are normalized such that

$$\mathrm{Tr} t^a t^b = \delta^{ab}, \tag{2.25}$$

where this is a trace over the matrix indices i and j.

Quite often another normalization of the generators with an extra factor of $1/2$ in front of the delta is used for historical reasons, in particular, because $\tilde{t}^a = \sigma/2$ for the $SU(2)$ group where the sigmas are the Pauli matrices. This results in the redefinition of the coupling constant, $\tilde{g}^2 = 2g^2$.

Equation (2.24) can be inverted to give

$$A^a_\mu(x) = \frac{1}{g}\mathrm{Tr}\mathcal{A}_\mu(x)t^a. \tag{2.26}$$

Substituting

$$\Omega(x) = e^{i\alpha(x)} \tag{2.27}$$

we obtain for an infinitesimal α:

$$\delta\mathcal{A}_\mu(x) \overset{g.t.}{=} \nabla^{\mathrm{adj}}_\mu\alpha(x). \tag{2.28}$$

Here

$$\nabla_\mu^{\rm adj}\alpha \equiv \partial_\mu\alpha - i\left[\mathcal{A}_\mu, \alpha\right] \tag{2.29}$$

is the covariant derivative in the adjoint representation of G, while

$$\nabla_\mu^{\rm fun}\psi \equiv \partial_\mu\psi - i\mathcal{A}_\mu\psi \tag{2.30}$$

is that in the fundamental representation. It is evident that

$$\nabla_\mu^{\rm adj}B(x) = \left[\nabla_\mu^{\rm fun}, B(x)\right], \tag{2.31}$$

where $B(x)$ is a (matrix-valued) funtion of x.

The QCD action is given in the matrix notation as

$$S\left[\mathcal{A}, \psi, \bar{\psi}\right] = \int {\rm d}^4x \left[\bar{\psi}\gamma_\mu(\partial_\mu - i\mathcal{A}_\mu)\psi + m\bar{\psi}\psi + \frac{1}{4g^2}{\rm Tr}\mathcal{F}_{\mu\nu}^2\right], \tag{2.32}$$

where

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i\left[\mathcal{A}_\mu, \mathcal{A}_\nu\right] \tag{2.33}$$

is the (Hermitian) matrix of the non-Abelian field strength.

This action is invariant under the local gauge transformation since

$$\mathcal{F}_{\mu\nu}(x) \xrightarrow{g.t.} \Omega(x)\mathcal{F}_{\mu\nu}(x)\Omega^\dagger(x) \tag{2.34}$$

or

$$\delta\mathcal{F}_{\mu\nu}(x) \stackrel{g.t.}{=} i\left[\mathcal{F}_{\mu\nu}(x), \alpha(x)\right] \tag{2.35}$$

for the infinitesimal gauge transformation.

For the Abelian group $G = U(1)$, the formulas recover those for QED.

Non-Abelian phase factors (Wilson loops)

To compare phases of wave functions at distinct points, a non-Abelian extension of the parallel transporter is needed. The proper extension of the Abelian formula (2.8):

$$U\left[\Gamma_{yx}\right] = Pe^{i\int_{\Gamma_{yx}} {\rm d}z^\mu \mathcal{A}_\mu(z)}, \tag{2.36}$$

includes the symbol P of path-ordering.

Although the matrices $\mathcal{A}_\mu$ do not commute, the path-ordered exponential on the right hand side is defined unambiguosly. This becomes obvious if we rewrite the phase factor in an equivalent form

$$Pe^{i\int_{\Gamma_{yx}} {\rm z}^\mu \mathcal{A}_\mu(z)} = Pe^{i\int_0^\tau {\rm d}t\dot{z}^\mu(t)\mathcal{A}_\mu(z(t))}. \tag{2.37}$$

The path-ordered exponential in (2.36) can be understood as

$$U\left[\Gamma_{yx}\right] = \prod_{t=0}^{\tau}\left[1 + i{\rm d}t\dot{z}(t)\mathcal{A}_\mu(z(t))\right] \tag{2.38}$$

Imagining this as a contour integration we can rewrite the former expression as

$$U\left[\Gamma_{yx}\right] = \prod_{z\in\Gamma_{yx}} \left[1 + i\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)\right]. \tag{2.39}$$

If the contour Γ_{yx} is discretized, then the non-Abelian phase factor can be approximated to be

$$U\left[\Gamma_{yx}\right] = \lim_{M\to\infty} \prod_{i=1}^{M} \left[1 + i(z_i - z_{i-1})^{\mu}\mathcal{A}_{\mu}\left(\frac{z_i + z_{i-1}}{2}\right)\right], \tag{2.40}$$

which reproduces (2.39) in the limit $z_{i-1} \to z_i$. The non-Abelian phase factor (2.36) is an element of the gauge group G itself, while $\mathcal{A}_{\mu}$ belongs to the Lie algebra of G.

Let us recall that matrices are rearranged in inverse order under Hermitian conjugation:

$$U^{\dagger}\left[\Gamma_{yx}\right] = U\left[\Gamma_{xy}\right]. \tag{2.41}$$

The notation Γ_{yx} means the orientation of the contour from x to y, while Γ_{xy} will denote the opposite orientation. These two result in opposite orders of multiplication for the matrices in the path-ordered product. Furthermore, the phase factors obey the backtracking (*zig-zag*) condition

$$U\left[\Gamma_{yx}\right] U\left[\Gamma_{xy}\right] = 1. \tag{2.42}$$

The gauge field $\mathcal{A}_{\mu}$ in the discretized phase factor (2.40) is chosen at the *center* of the i-th interval in order to satisty Eq. (2.42) at finite discretization.

Under the gauge transformation (2.23), $U\left[\Gamma_{yx}\right]$ transforms as

$$U\left[\Gamma_{yx}\right] \xrightarrow{g.t.} \Omega(y)U\left[\Gamma_{y}x\right]\Omega^{\dagger}(x). \tag{2.43}$$

This formula stems from the fact that

$$\left[1 + i\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)\right] \xrightarrow{g.t.} \left[1 + i\mathrm{d}z^{\mu}\mathcal{A}'_{\mu}(z)\right] = \Omega(z + \mathrm{d}z)\left[1 + i\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)\right]\Omega^{\dagger}(z) \tag{2.44}$$

which can be proven by substituting (2.23), so that $\Omega^{\dagger}(z)$ and $\Omega(z)$ *cancel* in the definition (2.39) at the intermidiate point z.

A consequence of Eq. (2.43) is that $\psi(x)$, transported by the matrix $U\left[\Gamma_{yx}\right]$ to the point y, transforms under the gauge transformation as $\psi(y)$:

$$U\left[\Gamma_{yx}\right]\psi(x) \overset{g.t.}{\sim} \text{"}\psi(y)\text{"}. \tag{2.45}$$

Therefore, $U\left[\Gamma_{yx}\right]$ is indeed a parallel transporter. It follows from these formulas that the combination $\bar{\psi}(y)U\left[\Gamma_{yx}\right]\psi(x)$ is gauge invariant:

$$\bar{\psi}(y)U\left[\Gamma_{yx}\right]\psi(x) \xrightarrow{g.t.} \bar{\psi}(y)U\left[\Gamma_{yx}\right]\psi(x). \tag{2.46}$$

Another consequence of (2.43) is that the trace of the phase factor for a closed controur Γ is gauge invariant:

$$\mathrm{Tr}Pe^{i\oint_{\Gamma}\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)} \xrightarrow{g.t.} \mathrm{Tr}Pe^{i\oint_{\Gamma}\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)} \tag{2.47}$$

This is quite similar to the Abelian phase factor.

The sufficient and necessary condition for the phase factor to be independent on a local variation of the path is the *vanishing* of $\mathcal{F}_{\mu\nu}$. Formulas of this type are well-known in differential geometry where parallel transport around a small close contour determines the curvature. $\mathcal{F}_{\mu\nu}$ in Yang-Mills theory is the proper curvature in an internal color space while $\mathcal{A}_\mu$ is the connection.

Remark 2.1 (A brief history lesson). An analog of the phase factor was first introduced by H. Weyl in 1919, in his attempt to describe gravitational and electromagnetic interactions of electrons on equal footing. What he did is associated in modern language with the scale rather than the gauge transformation, i.e. the vector-potential was not multiplied by i as in equation (2.8). This explains the term "gauge invariance" - gauging literaly means fixing a scale.

The factor of i was inserted by London in 1927 after creation of quantum mechanics and the recognition that the electromagnetic interaction corresponds to the freedom of choice of the phase of a wave function and not to a scale transformation.

2.2.3 1/N Expansion

An effective coupling constant of QCD becomes large at large distances, so fluctuations of scales of different orders of magnitude are essential and there is no small parameter. 't Hooft proposed in 1974 to use the number of colors N of the gauge group $SU(N)$ as such a parameter and to perform an expansion in $1/N$. The motivation was the $1/N$ expansion in statistical mechanics.

The $1/N$ expansion of QCD rearranges perturbation theory in a way consistent with a string picture. The accuracy of large-N QCD is of the order of ratios of meson widths to their masses (10-15%). While QCD is simplified in this limit, it is not yet solved.

Index of Ribbon Graphs

In order to describe the $1/N$-expansion of QCD, it is conveniant to use the matrix-field representation

$$[A_\mu(x)]^{ij} = \sum_\alpha A^a_\mu(x)\,[t^a]^{ij}\,. \tag{2.48}$$

The matrix (2.48) is Hermitian and differs from (2.24) by a factor of g. The propagator of the matrix field $A^{ij}(x)$ reads

$$\langle A^{ij}_\mu(x) A^{kl}_\nu(y)\rangle_{\text{Gauss}} = \left(\delta^{il}\delta^{kj} - \frac{1}{N}\delta^{ij}\delta^{kl}\right) D_{\mu\nu}(x-y) \tag{2.49}$$

where we have assumed, as usual, a gauge-fixing to define the gluon propagator in perturbation theory. For instance, one has

$$D_{\mu\nu}(x-y) = \frac{1}{4\pi^2}\frac{\delta_{\mu\nu}}{(x-y)^2} \tag{2.50}$$

in the Feynman gauge. It has to be said that δ has to be substituted by a $-g_{\mu\nu}$ if we work in Minkowski space. Now, equation (2.49) can be derived immediately from the standard formula

$$\langle A^a_\mu(x) A^b_\nu(y)\rangle_{\text{Gauss}} = \delta^{ab} D_{\mu\nu}(x-y) \tag{2.51}$$

multiplying by the generators of the $SU(N)$ gauge group according to (2.48) and using the completeness condition:

$$\sum_{a=1}^{N^2-1} (t^a)^{ij}(t^a)^{kl} = \left(\delta^{il}\delta^{kj} - \frac{1}{N}\delta^{ij}\delta^{kl}\right) \quad \boxed{\text{for } SU(N)} \tag{2.52}$$

Alternatively, (2.49) can be derived directly from a path integral over the matrix fields.

We concentrate only on the structure of diagrams in the index space, i.e. the space of the indices associated with the $SU(N)$ group. We shall not consider, in most cases, space-time structures of diagrams which are prescribed by Feynman's rules.

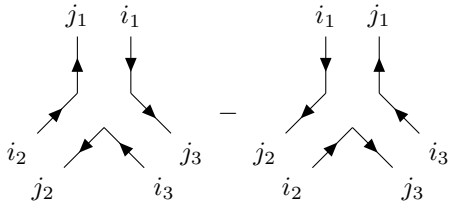
Omitting at large N the second term in parentheses on the RHS of (2.49), we depict the propagator by the *double line*

$$\langle A_\mu^{ij}(x) A_\nu^{kl}(y)\rangle_{\text{Gauss}} \propto \delta^{il}\delta^{kj} = \begin{array}{l} i \longrightarrow l \\ j \longleftarrow k \end{array} \tag{2.53}$$

Each line, often termed the index line, represents the Kronecker delta-sybol and has an orientation which is indicated by arrows. This notation is obviously consistent with the space-time structure of the propagator that describes a propagation from x to y.

Arrows are a result of the fact that the matrix A_μ^{ij} is Hermitian and its off-diagonal components are complex conjugate. Double lines appear generically in all models describing matrix fields in contrast to vector (in internal symmetry space) fields, whose propagators are depicted by single lines.

The three-gluon vertex is depicted in the double line notation as



$$\propto g\left(\delta^{i_1 j_3}\delta^{i_2 j_1}\delta^{i_3 j_2} - \delta^{i_1 j_2}\delta^{i_2 i_2 j_3}\delta^{i_3 j_1}\right) \tag{2.54}$$

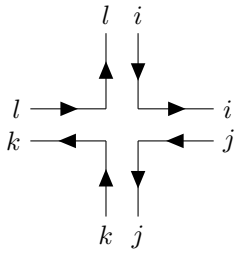
where the subscripts 1,2 or 3 refer to each of the three gluons. The relative minus sign arises from the commutator in the cubic-in-A term in the QCD action (2.32). The color part is antisymmetric under the interchange of gluons. The (momentum-space) space-time part

$$\gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = \delta_{\mu_1\mu_2}(p_1-p_2)_{\mu_3} + \delta_{\mu_2\mu_3}(p_2-p_3)_{\mu_1} + \delta_{\mu_1\mu_3}(p_3-p_1)_{\mu_2} \tag{2.55}$$

is also antisymmetric. We consider all three gluons as incoming so their momenta obey $p_1+p_2+p_3=0$. The full vertex is symmetric as is prescribed by Bose statistics.

The four-gluon vertex involves six terms - each of them is depicted by a cross - which differ by interchanging of the color indeces. We depict the color structure of the four-gluon vertex for simplicity in the case when $i_1=j_2=i, i_2=j_3=j, i_3=j_4=k, i_4=j_1=l$, but i,j,k,l take on

different values. Then only the following term is left:



$$\propto g^2 \tag{2.56}$$

where there are no delta-symbols since the color structure is fixed. We pick up only one color structure by equating indices pairwise.

Diagrams of perturbation theory can now be completely rewritten in the double-line notation. The simplest one describing the one-loop correction to the gluon propagator is depicted in 2.2. The sum over the N indices is s associated with the closed index line. The contribution of this diagram is $\sim g^2 N \sim 1$. In order for the large-N limit to be nontrivial, the bare coupling constant g^2 should satisfy

$$g^2 \sim \frac{1}{N} \tag{2.57}$$

This dependence on N is also prescribed by the asymptotic-freedom formula

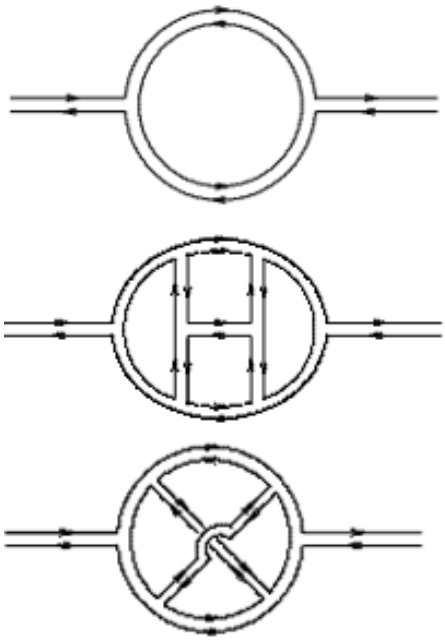


Figure 2.2: Double-line representation of a one-loop diagram for the gluon propagator, four-loop diagram and a nonplanar diagram

$$g^2 = \frac{12\pi^2}{11Nln(\Lambda/\Lambda_{QCD})} \tag{2.58}$$

of pure $SU(N)$ gauge theory.

Thus the contribution of the first diagram in 2.2 is of order $\sim g^2 N \sim 1$ in the large-N limit. We can think of the double lines as bounding a piece of a plane. These lines represent a two-dimensional object. In mathematics these double-line graphs are often called ribbon graphs or fatgraphs. They are deeply connected with Riemann surfaces.

Remark on the U(N) gauge group

The double line representation of the diagrams holds, strictly speaking for the $U(N)$ gauge group, whose generators

$$T^A = (t^a, I/\sqrt{N}), \quad \mathrm{Tr} T^A T^B = \delta^{AB} \quad \boxed{A = 1, \cdots, N^2}. \tag{2.59}$$

obey the completeness condition

$$\sum_{A=1}^{N^2} (T^A)^{ij} (T^A)^{kl} = \delta^{il}\delta^{kj} \quad \boxed{\text{for} \quad U(N)} \tag{2.60}$$

Elements of both $SU(N)$ and $U(N)$ can be represented in the form

$$U = e^{iB} \tag{2.61}$$

where B is a general Hermitian matrix for $U(N)$ and a traceless Hermitian matrix for $SU(N)$. The large-N limit of both the $U(N)$ and $SU(N)$ groups is the same.

2.2.4 Planar and Nonplanar Graphs

The double-line representation of perturbation theory diagrams is very convenient to estimate their orders in $1/N$. Each three- or four-gluon vertex contributes a factor of g or g^2, respectively. Each closed index line contributes a factor of N, while $g^2 \sim 1/N$.

't Hooft topological expansion

Let us consider a typical diagram for the gluon propagator as depicted in 2.2 b). The sum over the N indeces is associated with each of the four closed index lines, whose number is equal to the number of loops. The contribution of this diagram is then $\sim g^8 N^2 \sim 1$.

Diagrams of this type, which can be drawn on a sheet of paper without crossing any lines, are called *planar diagrams*. For such diagrams, the addition of a loop inevitably results in the addition of two three-gluon (or one four-gluon) vertices. A planar diagram with n_2 loops has n_2 closed index lines. It is of order

$$n_2-\text{loop planar diagram} \sim (g^2 N)^{n_2} \sim 1, \tag{2.62}$$

so that all planar diagrams survive in the large-N limit. Let us now consider a nonplanar diagram as the one depicted in 2.2 c). The diagram has six three-gluon vertices but only one closed index line (although it has 3 loops!). The order of this diagram is $\sim g^6 N \sim 1/N^2$.

This nonplanar diagram can be drawn without line-crossing on a surface with one handle (or hole) which in mathematics terms is called a torus of a surface with genus one. A plane is

then equivalent to a surface with genus zero, which is in tern equivalent to a sphere. A general Riemanian surface with h holes has genus h.

The above evaluations of the order of the diagrams can now be described by the single formula

$$\text{genus} - h \text{ diagram} \sim \left(\frac{1}{N^2}\right)^{\text{genus}} . \tag{2.63}$$

The expansion in $1/N$ rearranges perturbation theory diagrams according to their topology as demonstrated in 1974 by 't Hooft. It is referred to as the topological expansion or the genus expansion. Only planar diagrams associated with genus zero survive in the large-N limit. The problem of summing the planar graphs is complicated but simpler than that of summing all the graphs, since the number of planar graphs with n_0 vertices grows geometrically at large n_0:

$$_p(n_0) \equiv \text{ of planar graphs} \sim \text{const}^{n_0} \tag{2.64}$$

This was shown by Tuttle and Koplic, Beveu, Nussinov, while the total number of graphs grows factorially with n_0. There is no dependence on the number of external lines of a planar graphs in (2.64), so it is assumed to be much less than n_0.

There is a big difference between the planar diagrams and the ladder diagrams which describe e^+e^- elastic scattering in QED. For the ladder with n rungs, there are $n!$ ladder diagrams, but only one of them is planar. This shows why the number of planar graphs is much smaller than the total number of graphs, most of which are non-planar. Equation (2.63) holds, strictly speaking, only for the gluon propagator, while the contribution of all planar diagrams to a connected n-point Green function is $\sim g^{n-2}$, which is its natural order in $1/N$. The three-gluon Green function is $\sim g$, the four-gluon one is $\sim g^2$ and so on. The contributions of all planar diagrams are of the same order ~ 1 in the large-N limit, independently of the number of external lines, for the Wilson loop average

$$\Big\langle \frac{1}{N}\mathrm{Tr}Pe^{ig\oint_\Gamma \mathrm{d}x^\mu A_\mu(x)}\Big\rangle = \sum_{n=0}^{\infty} i^n \oint_\Gamma \mathrm{d}x_1^{\mu_1}\int_{x_1}^{x_1}\mathrm{d}x_2^{\mu_2}\cdots\int_{x_1}^{x_{n-1}}\mathrm{d}x_n^{\mu_n}\mathrm{d}x_n^{\mu_n}G^{(n)}_{\mu_1\cdots\mu_n}(x_1,\cdots,x_n) \tag{2.65}$$

where

$$G^{(n)}_{\mu_1\cdots\mu_n}(x_1,\cdots,x_n) \equiv \frac{g^n}{N}\langle \mathrm{Tr}\left[A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\right]\rangle. \tag{2.66}$$

The factor of $1/N$, which normalizes the trace, provides the natural normalization. The ordering along a closed path implies cyclic-ordering in the index space as depicted in Fig. 2.3 where we omit the arrows for simplicity. This diagram has $n_0 = 10$ vertices, $n_1 = 12$ gluon propagators, $n_2 = 4$ closed index lines, and $B = 1$ boundaries. The color indices of the external lines are contracted by the Kronecker delta symbols (represented by the single lines) in a cyclic order. The extra factor of $1/N$ arises from normalization. The order in $1/N$ of the diagram is $\sim 1/N^2$ in accord with (2.63). Ananlogously, the color indices in (2.66) are contracted in the cyclic order. The delta-symbols, which contract the color indices, are depicted by the single lines. They can be viewed as a boundary of the diagram. The actual size of the boundary is not essential - it can be shrunk to a point. Then a bounded piece of a plane will be topologically equivalent to a sphere

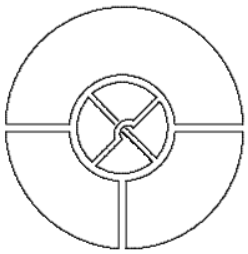


Figure 2.3: Generic double-line index diagram

with a puncture. We draw planar diagrams in a plane with an extended boundary (boundaries) rather than in a sphere awith a puncture (punctures). The closed boundary is associated with the trace over the color indeces of the multi-point Green function.

The boundary represents the Wilson loop - a trajectory of a heavy quark in the fundamental representation.

Topological expansion and quark loops

It is easy to incorporate quarks in the topological expansion. A quark field belongs to the fundamental representation of the gauge group $SU(N)$ and its propagator is represented by a single line

$$\langle \psi_i \bar{\psi}_j \rangle \propto \delta_{ij} = \ i \longrightarrow j \tag{2.67}$$

The arrow indicates, as usual, the direction of propagation of a (complex) field ψ. These arrows are often omitted for simplicity. The diagram for the gluon propagator which involves one quark loop is depicted in Fig 2.4 a). It involves one quark loop and has no closed index lines so that

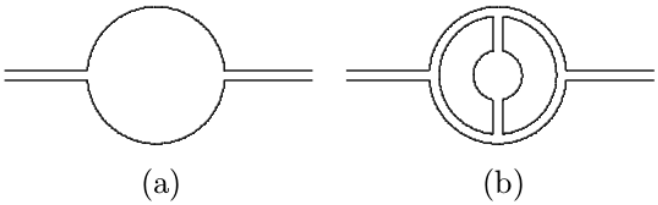


Figure 2.4: Diagrams for gluon propagator whcih invlove quark loop

its order is $\sim g^2 \sim 1/N$. The diagram in 2.4 b) is $\sim g^6 N^2 \sim 1/N$ analogously.

It is evident from this consideration that quark loops are not accompanied by closed index lines. One should add a closed index line for each quark loop in order for a given diagram with L quark loops to have the same double-line representation as for pure gluon diagrams. Therefore,

given (2.63), diagrams with L quark loops are suppressed at large N by

$$L \text{ quark loops} \sim \left(\frac{1}{N}\right)^{L+2*\text{genus}} \tag{2.68}$$

The single-line representation of the quark loops is similar to that of the Wilson loop. Such a diagram emerges in gluon corrections to the vacuum expectation value of the quark operator:

$$O = \frac{1}{N}\bar{\psi}\psi, \tag{2.69}$$

where the factor of $1/N$ is introduced to make it $\mathcal{O}(1)$ in the large-N limit. Therefore, the external boundary can be viewed as a single line associated with valence quarks.

The proof of topological expansion

To prove (2.63) and it's quark counterpart (2.68), let us consider a generic diagram in the index space which has $n_0^{(3)}$ three-point vertices (either three-gluon quark-gluon ones), $n_0^{(4)}$ four-gluon vertices, n_1 propagators (either gluon or quark ones), n_2 closed index lines, L virtual quark loops and B external boundaries. Its order in $1/N$ is

$$\frac{1}{N^B} g^{n_0^{(3)}+2n_0^{(4)}} N^{n_2} \sim N^{n_2-n_0^{(3)}/2-n_0^{(4)}-B} \tag{2.70}$$

as has already been explained. The extra factor of $1/N^2 B$ arises from the extra normalization factor of $1/N$ in operators associated with external boundaries.

The number of propagators and vertices are related by

$$2n_1 = 3n_0^{(3)} + 4n_0^{(4)}, \tag{2.71}$$

since three- and four-point vertices emit three or four propagators, respectively, and each propagator connects two vertices. We then rewrite the RHS of (2.70) as

$$N^{n_2-n_0^{(3)}/2-n_0^{(4)}-B} = N^{n_2-n_1+n_0-B}, \tag{2.72}$$

where $n_0 = n_0^{(3)} + n_0^{(4)}$ is the total number of vertices. The exponent on the RHS of (2.72) can be expressed via the Euler characteristic χ of a given graph of genus h. An appropriate Riemann surface, which is associated with a given graph, is open and has $B + L$ boundaries. This surface can be closed by attaching a cap to each boudary. The single lines then become double lines together with the lines of the boundary of each cap. The number of faces for a closed Riemann surface constructed in such a manner is $n_2 + L + B$, while the number of edges and vertices are n_1 and n_0, respectively. Euler's theorem states that

$$\chi \equiv 2 - 2h = n_2 + L + B - n_1 + n_0. \tag{2.73}$$

Therefore the RHS of (2.72) can be rewritten as

$$N^{n_2-n_1+n_0-B} = N^{2-2h-L-2B}. \tag{2.74}$$

We have thus proven that the order in $1/N$ of a generic graph does not depend on its order in the coupling constant and is completely expressed via the genus h and the number of virtual quark loops L and external boundaries B by

$$\text{generic graph} \sim \left(\frac{1}{N}\right)^{2h+L+2(B-1)} \tag{2.75}$$

For $B = 1$, we recover (2.63) and (2.68).

't Hooft versus Veneziano limits

In QCD there are several species or flavours of quarks. We denote the number of flavours by N_f and associate a Greek letter α or β with a flavour index of the quark field. The quark propagatror then has the Kronecker delta-symbol with respect to the flavour indices in addition to (2.67)

$$\langle \psi_i^\alpha \bar{\psi}_j^\beta \rangle \propto \delta^{\alpha\beta}\delta_{ij} \tag{2.76}$$

Their contraction results in

$$\sum_{\alpha=1}^{N_f} \delta_{\alpha\alpha} = N_f \tag{2.77}$$

Therefore, an extra factor of N_f corresponds to each closed quark loop for the N_f flavours.

The limit when N_f is fixed as $N \to \infty$ is called the 't Hooft limit. Only valence quarks are then left (the quenched approximation). In order for a meson to decay into other mesons built out of quarks, a quark-antiquark pair must be produced out of the vacuum. Consequently, the ratios of meson widths to their masses are

$$\frac{\Gamma_{\text{total}}}{M} \sim \frac{N_f}{N} \tag{2.78}$$

in the 't Hooft limit. The ratio on the LHS of (2.78) is 10-15% experimentally for the ρ-meson. The hope of solving QCD in the 't Hooft limit is the hope to describe QCD with this accuracy. An alternative large-N limit of QCD, when $N_f \sim N$ as $N \to \infty$ was proposed by Veneziano in 1976. A general diagram with L quark loops will contribute

$$L \text{ quark loops} \sim \left(\frac{N_f}{N}\right)^L \left(\frac{1}{N^2}\right)^{\text{genus}}, \tag{2.79}$$

since each quark loop results in N_f.

The quark loops are not suppressed at large N in the Veneziano limit

$$N_f \sim N \to \infty \tag{2.80}$$

if the diagram is planar.

It is the Veneziano limit that is related to the hadronic topological expansion in the dual-resonance models. In the Veneziano limit hadrons can have finite widths according to (2.78).

2.2.5 Large-N factorization

The vacuum expectation values of several colorless or white operators, which are singlets with respect to the gauge group, factorize in the large-N limit of QCD (or other matrix models) as first was noticed by A.A. Migdal and independently by E. Witten in late 1970's. The simplest gauge-invariant operators in a pure $SU(N)$ gauge theory are the closed Wilson loops

$$\Phi(C) = \frac{1}{N}\mathrm{Tr}Pe^{ig\oint_C \mathrm{d}z^\mu A_\mu(z)}. \tag{2.81}$$

They obey the factorization property

$$\langle\Phi(C_1)\cdots\Phi(C_n)\rangle = \langle\Phi(C_1)\rangle\cdots\langle\Phi(C_nC_n)\rangle + \mathcal{O}(N^{-2}) \tag{2.82}$$

The factorization implies a semiclassical nature of the large-N limit of QCD (a saddle point in the path integral for certain variables). The factorization property also holds for gauge-invariant operators constructed from quarks as in (2.69). For the case of several flavours N_f, we normalize these quark operators by

$$O_\Gamma = \frac{1}{N_f N}\bar{\Psi}\Gamma\psi \tag{2.83}$$

Here, Γ denotes one of the combination of the γ-matrices:

$$\Gamma = I, \gamma_5, \gamma_m u, i\gamma_\mu\gamma_5, \sum_{\mu\nu} = \frac{1}{2i}\left[\gamma_\mu, \gamma_\nu\right], \cdots . \tag{2.84}$$

The factorization of the gauge-invariant quark operators hold both in the 't Hooft and Veneziano limits:

$$\langle O_{\Gamma_1}\cdots O_{\Gamma_n}\rangle = \langle O_{\Gamma_1}\rangle\cdots\langle O_{\Gamma_n}\rangle + \mathcal{O}(1/(N_f N)). \tag{2.85}$$

The nonfactorized part, which is associated with connected diagrams is $\sim 1/N$ in the 't Hooft limit. This leads, in particular, to the coupling constant of meson-meson interaction of order $1/N$. The Veneziano limit it analogous to pure Yang-Mills.

The factorization can be seen (at all orders of perturbation theory) form (2.75) for the contribution of a generic connected graph of genus h with B external boundaries which are precisely associated with the Wilson loops (or the quark operators). The diagrams with gluon lines emitted and absorbed by the same operator are products of diagrams having only one boundary. Their contribution is of order one. The diagrams with gluon lines emitted and absorbed by two different operators have two boundaries. This proves the factorization property (2.85) at all orders of perturbation theory.

The large-N factorization can also be verified beyond perturbation theory at all orders of the strong-coupling expansion of the $SU(N)$ lattice gauge theory. A non-perturbative proof of the factorization was given using quantum equations of motion (the loop equations).

2.2.6 Conclusion

We have considered in this section the basic features of the methods for nonperturbative studies of gauge theories, which were developed in the second half of 1970's- early 1980's. Their contemporary applications in high-energy physics are extremely broad: from the scattering of particles

at very high energies ot the attempts of constructing a unified theory of all interactions, including gravity.

Chapter 3

Supersymmetry

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Contents

3.1 Supermultiplets

Definition 3.1 (Supermultiplet)**.** *Representations of the supersymmetric algebra (superalgebra) are called supermultiplets.*

Indeed, these representations can be thought of as multiplets where we assemble together several different representations of the Lorentz algebra, since the latter is a subalgebra of the superalgebra.

3.1.1 Massless supermultiplets

If $P^2 = 0$, then we can take P_μ to a canonical form by applying boost and rotations until it reads

$$\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = \left(\sigma^0 + \sigma^3\right) E = \begin{bmatrix} 0 & 0 \\ 0 & 2E \end{bmatrix} \tag{3.1}$$

The supersymmetric algebra becomes,

$$\begin{bmatrix} \{Q_1, \bar{Q}_{\dot{1}}\} & \{Q_1, \bar{Q}_{\dot{2}}\} \\ \{Q_2, \bar{Q}_{\dot{1}}\} & \{Q_2, \bar{Q}_{\dot{2}}\} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4E \end{bmatrix} \tag{3.2}$$

intended as acting on the states of the multiplet we are looking for. In particular,

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 0 \tag{3.3}$$

which implies that

$$||Q_1|\omega\rangle||^2 = 0 = ||\bar{Q}_{\dot{1}}|\omega\rangle||^2 \tag{3.4}$$

and thus

$$Q_1|\omega\rangle = 0 = \bar{Q}_{\dot{1}}|\omega\rangle. \tag{3.5}$$

This means that as operators Q_1 and $\bar{Q}_{\dot{1}}$ annihilate the multiplet.

The only nontrivial anticommutation relation that is left is:

$$\{Q_2, \bar{Q}_{\dot{2}}\} = 1 \tag{3.6}$$

If we call

$$\alpha = \frac{1}{2\sqrt{E}} Q_2, \quad \alpha^\dagger = \frac{1}{2E} \bar{Q}_{\dot{2}} \tag{3.7}$$

then the anticommutation relation that is left is:

$$\{\alpha, \alpha^\dagger\} = 1 \tag{3.8}$$

with $\{\alpha, \alpha\} = 0$.

We can build the representation starting from a state $|\lambda\rangle$ such that

$$\alpha|\lambda\rangle = 0 \tag{3.9}$$

Lets suppose that it has *helicity* λ:

$$M_{12}\lambda \equiv J_3|\lambda\rangle = \lambda|\lambda\rangle. \tag{3.10}$$

It is easy to compute the helicity of $\alpha^\dagger|\lambda\rangle$:

$$M_{12}\bar{Q}_{\dot{2}}|\lambda\rangle = \left(\bar{Q}_{\dot{2}} M_{12} + \frac{1}{2}\bar{Q}_{\dot{2}}\right)|\lambda\rangle = (\lambda + \frac{1}{2}\bar{Q}_{\dot{2}})|\lambda\rangle \tag{3.11}$$

In the last line, we have used the fact that $\left[M_{12}, \bar{Q}_{\dot{2}}\right] = \frac{1}{2}\bar{Q}_{\dot{2}}$. Thus we found out that

$$\alpha^\dagger|\lambda\rangle = |\lambda + \frac{1}{2}\rangle \tag{3.12}$$

Since $(\alpha^\dagger)^2 = 0$, this stops here. Hence we have

$$\alpha^\dagger|\lambda + \frac{1}{2}\rangle = 0. \tag{3.13}$$

Massless multiplets are thus composed of one boson and one fermion. Since physical particles must come in CPT conjugate representation (or, there are no spin-$\frac{1}{2}$ one dimensional representations of the massless little group of the Lorentz group), one must add the CPT conjugate multiplet where helicities are flipped.

Example 1 (Examples of massless supermultiplets)**.**

- *The scalar multiplet is obtained by setting $\lambda = 0$. Then we have*

$$\alpha^\dagger |0\rangle = \left|\frac{1}{2}\right\rangle \tag{3.14}$$

The full multiplet is composed of two states with $\lambda = 0$ and a doublet with $\lambda = \pm\frac{1}{2}$. These are the degrees of freedom of a complex scalar and a Weyl (chiral) fermion.

- *The vector multiplet is obtained starting from a $\lambda = \frac{1}{2}$ state. We get*

$$\alpha^\dagger \left|\frac{1}{2}\right\rangle = |1\rangle\,. \tag{3.15}$$

To this we add the CPT conjugate multiplet, to obtain two pairs of states, one with $\lambda = \pm\frac{1}{2}$ and the other with $\lambda = \pm 1$. These are the degrees of freedom of a Weyl fermion and of a massless vector. The latter is usually interpreted as a gauge boson.

- *Another multiplet is obtained starting from $\lambda = \frac{3}{2}$:*

$$\alpha^\dagger \left|\frac{3}{2}\right\rangle = |2\rangle\,. \tag{3.16}$$

Adding the CPT conjugate, one has a pair of bosonic degrees of freedom with $\lambda = \pm 2$, which we interpret as the graviton, and a pair of fermionic degrees of freedom with $\lambda = \pm\frac{3}{2}$, which correspond to a massless spin-$\frac{3}{2}$ Rarita-Schwinger field, also called the gravition, since it is the SUSY partner of the graviton, as was just shown.

3.1.2 Supermultiplets of extended supersymmetry

Very briefly we will mention that having extended SUSY, the massless supermultiplets are longer. Let's take the algebra to be:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{IJ}, \tag{3.17}$$

where for simplicity we suppose that $Z^{IJ} = 0$ for these states. For massless states, $P_\mu = (E, 0, 0, E)$ and therefore as before we have that

$$\{Q^I_1, \bar{Q}^J_1\} = 0, \tag{3.18}$$

which implies the (operator) equations $Q^I_1 = 0$ and $\bar{Q}^I_1 = 0$, for $I = 1, \cdots, \mathcal{N}$. The nontrivial relations are then:

$$\{Q^I_2, \bar{Q}^J_2\} = 4E\delta^{IJ} \tag{3.19}$$

Of course we can define

$$\alpha_I = \frac{1}{2\sqrt{E}} Q^I_2 \tag{3.20}$$

and obtain the canonical anticommutation relations for $\mathcal{N}$ fermionic oscillators

$$\{\alpha_I, \alpha^\dagger_J\} = \delta_{IJ} \tag{3.21}$$

If we now start with a state $|\lambda\rangle$ with helicity λ which satisfies $\alpha_I |\lambda\rangle = 0$, we build a multiplet as follows:

$$\begin{aligned} \alpha_I^\dagger |\lambda\rangle &= \left|\lambda + \frac{1}{2}\right\rangle_I , \\ \alpha_I \dagger \alpha_J \dagger |\lambda\rangle &= |\lambda + 1\rangle_{[IJ]} , \\ &\vdots \\ \alpha_1^\dagger \cdots \alpha_{\mathcal{N}}^\dagger |\lambda\rangle &= \left|\lambda + \frac{\mathcal{N}}{2}\right\rangle \end{aligned} \tag{3.22}$$

It is very important to note that there are $\mathcal{N}$ states with helicity $\lambda + \frac{1}{2}$, $\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$ states with helicity $\lambda + 1$ and so on, until we reach a single state with helicity $\lambda + \frac{\mathcal{N}}{2}$ (it is totally antisymmetric in $\mathcal{N}$ indices I). In total, the supermultiplet is composed of $2^{\mathcal{N}}$ states, half of them bosonic and half of them fermionic.

Interestingly, in this case we can now have self-CPT conjugate multiplets. Take for example $\mathcal{N} = 4$ and start from $\lambda = -1$. Then $\lambda + \frac{\mathcal{N}}{2} = 1$ and the multiplet spans states of opposite helicities, thus filling complete representations of the Lorenz group. Indeed, it contains one pair of states with $\lambda \pm 1$ (a vector, i.e a gauge boson), 4 pairs of states with $\lambda = \pm\frac{1}{3}$ (4 Weyl Fermions) and 6 states with $\lambda = 0$ (6 real scalars, or equivalently 3 complex scalars).

Another example is $\mathcal{N} = 8$ supersymmetry. Here if we start with $\lambda = -2$ we end up with $\lambda + \frac{\mathcal{N}}{2} = 2$. Thus in this case we have the graviton in the self-CPT conjugate multiplet, corresponding to the pair of states with $\lambda \pm 2$. In addition, we have 8 massless gravitini with $\lambda \pm \frac{3}{2}$, 28 massless vectors with $\lambda = \pm 1$, 56 massless Weyl fermions with $\lambda = \pm\frac{1}{2}$ and finally 70 real scalars with $\lambda = 0$. This is the content of $\mathcal{N} = 8$ *supergravity*, which is the only multiplet of $\mathcal{N} = 8$ supersymmetry with $|\lambda| < 2$. The latter condition is necessary in order to have consistent couplings (higher spin fields cannot be coupled in a consistent way with gravity and lower spin fields).

From the theoretical standpoint, this is a very nice result, because we have a theory where *everything is determined* from symmetry alone: the complete spectrum and all the couplings. Unfortunately, this theory is also completely unphysical. To mention one problem, it has no room for fermions in complex representations of the gauge group, which are present in the Standard Model.

3.1.3 Massive supermultiplets

When $P^2 = M^2 > 0$, by boosts and rotation P_μ can be put in the following form

$$P_\mu = (M, 0, 0, 0) \tag{3.23}$$

Then we have

$$\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = M\sigma^0 = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \tag{3.24}$$

so that the superalgebra reads

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2M\delta_{\alpha\dot{\alpha}} \tag{3.25}$$

Note that $[M_{12}, Q_1] = i(\sigma_{12})_1{}^1 Q_1 = \frac{1}{2}Q_1$, thus it is Q_1 that raises the helicity, in the same way as $\bar{Q}_{\dot{2}}$. We make the redefinition

$$\alpha_1 = \frac{1}{\sqrt{2M}}\bar{Q}_{\dot{1}}, \quad \alpha_1^\dagger = \frac{1}{\sqrt{2M}}Q_1, \tag{3.26}$$

$$\alpha_2 = \frac{1}{\sqrt{2M}}Q_2, \quad \alpha_2^\dagger = \frac{1}{\sqrt{2M}}\bar{Q}_{\dot{2}}, \tag{3.27}$$

so that we have the canonical anticommutation relations of two fermionic oscillators:

$$\{\alpha_a, \alpha_b^\dagger\} = \delta_{ab}, \quad a, b = 1, 2. \tag{3.28}$$

If we start with $\alpha_a |\lambda\rangle = 0$, $M_{12}|\lambda\rangle = \lambda|\lambda\rangle$, then we build the multiplet as:

$$\alpha_1^\dagger |\lambda\rangle = \left|\lambda + \frac{1}{2}\right\rangle_1, \tag{3.29}$$

$$\alpha_2^\dagger |\lambda\rangle = \left|\lambda + \frac{1}{2}\right\rangle_2, \tag{3.30}$$

$$\alpha_1^\dagger \alpha_2^\dagger |\lambda\rangle = |\lambda + 1\rangle. \tag{3.31}$$

There are 4 states now (compared to the 2 in the massless case), two bosons and two fermions.

Example 2 (Examples of massive supermultiplets)**.**

- *In the case of the massive scalar multiplet, we start from $\lambda = -\frac{1}{2}$ and obtain two states with $\lambda = 0$ and one state with $\lambda = \frac{1}{2}$. These are the degrees of freedom of one massive complex scalar and one massive Weyl fermion. Note that the latter might not be familiar. Indeed, one cannot write the usual Dirac mass term for a Weyl fermion. Instead, one can write what is called a Majorana mass term:*

  $$\mathcal{L} \supset m\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta + h.c. \tag{3.32}$$

  Note that the total degrees of freedom of a massless scalar multiplet is the same as that of a massive one.

- *For a massive vector multiplet, start from $\lambda = 0$ to obtain 2 states with $\lambda = \frac{1}{2}$ and one state with $\lambda = 1$. To this we add the CPT conjugate multiplet so that in the end we have one pair with $\lambda = \pm 1$, two pairs with $\lambda = \pm\frac{1}{2}$ and two states with $\lambda = 0$. According to the massive little group, this corresponds to 1 massive vector (with $\lambda = \pm 1, 0$), 1 real scalar and 1 massive Dirac fermion. Note however that the content in degrees of freedom is the same as that of one massless vector multiplet together with one massless scalar multiplet. This hints that the consistent way to treat massive vectors in a supersymmetric field theory will be through a SUSY version of the Brout-Englert-Higgs mechanism.*

  What is broken!

3.2 General

Definition 3.2 (R-symmetry)**.** *In supersymmetric theories, an R-symmetry is the symmetry transforming different supercharges into each other. In the simplest case of the $\mathcal{N} = 1$ supersymmetry, such R-symmetry is isomorphic to a global $U(1)$ group or it's discrete subgroup. For extended supersymmetry theories, the R-symmetry group becomes a global non-abelian group.*

Remark 3.1. In the case of the discrete subgroup $\mathbb{Z}_2$, the R-symmetry is called *R-parity*

Definition 3.3 (Extended supersymmetry)**.** *In supersymmetric theories, when $\mathcal{N} > 1$ the algebra is said to have extended supersymmetry.*

3.3 Bogomol'nyi-Prasad-Sommerfield (BPS) states

Definition 3.4 (BPS state)**.** *A massive representation of an extended supersymmetry algebra that has mass equal to the supersymmetry central charge Z is called an BPS state.*

Quantum mechanically speaking, if the supersymmetry remains unbroken, exact solutions to the modulus of Z exist. Their importance arises as the multiplets shorten for generic representations, with stability and mass formula exact.

Example 3 ($d = 4$,$\mathcal{N} = 2$)**.** *The generators for the odd part of the superalgebra have relations:*

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B \tag{3.33}$$

$$\{Q^A_\alpha, Q_{\beta B}\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z} \tag{3.34}$$

$$\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB} Z, \tag{3.35}$$

where $\alpha\dot{\beta}$ are the Lorentz group indices and A, B are the R-symmerty indices. If we take linear combinations of the above generators as follows:

$$R^A_\alpha = \xi^{-1} Q^A_\alpha + \xi\sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}B} \tag{3.36}$$

$$T^A_\alpha = \xi^{-1} Q^A_\alpha - \xi\sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}B} \tag{3.37}$$

and consider a state ψ which has momentum $(M, 0, 0, 0)$, we have:

$$\left(R^1_1 + (R^1_1)^\dagger\right)^2 \psi = 4(M + Re(Z\xi^2))\psi, \tag{3.38}$$

but because this is the square of a Hermitian operator, the right hand side coefficient must be positive for all ξ. In particular, the strongest result from this is

$$M \geq |Z| \tag{3.39}$$

3.4 Supersymmetric theories on curved manifolds

Remark: Supersymmetric theories may be defined only on backgrounds admitting solutions to certain Killing spinor equations,

$$(\nabla_\mu - iA_\mu)\,\zeta + iV_\mu\zeta + iV^\nu\sigma_{\mu\nu}\zeta = 0 \tag{3.40}$$

$$(\nabla_\mu + iA_\mu)\,\tilde{\zeta} - iV_\mu\tilde{\zeta}0iV^\nu\tilde{\sigma}_{\mu\nu}\tilde{\zeta} = 0 \tag{3.41}$$

which in four dimensions and Euclidean signature are equivalent to the requirement that the manifold is complex and the metric Hermitian.

Chapter 4

Entaglement Entropy

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4.1 Introduction to Entaglement entropy

Entaglement entropy is an important way to quantify quantum correlations in a system. For gauge theories, the definition of entaglement entropy turns out to be subtle. Physicall, this is because there are non-local excitations in the system - for example, loops of electric or magnetic flux created by Wilson or 't Hooft loop operators, which can cut across the boundary of the region of interest. More precisely, it is because the Hilbert space of gauge invariant states, $\mathcal{H}_{g\text{inv}}$, does not admit a tensor product decomposition between the region of interest and its complement.

A definition of the entaglement entropy has been given for a general gauge theory based on an extended Hilbert space (EHS) construction. For a non-Abelian theory without matter, on a spatial latice, this takes the rather compact form

$$S_{\text{EE}} = -\sum_i p_i \log p_i + \sum_i \log\big(d_a^i\big) - \sum_i p_i \text{Tr}_{\mathcal{H}_i} \bar{\rho}_i \log(\bar{\rho}_i). \tag{4.1}$$

The index i in the summation denotes various sectors which are determined by the value of the normal component of the electric field at all the boundary points. The probability to be in the sector i is given by p_i ,and d_a^i, with index a denoting the particular boundary point, is the dimension of the representation specifying the outgoing electric flux at that boundary point in

the i^{th} sector. The last term is a weighted sum with $\bar{\rho}_i$ being the reduced density matrix in the i^{th} sector and the trace being taken over the Hilbert space $\mathcal{H}_i$ in this sector. It has been argued that this EHS definition agrees with the replica trick method of calculating the entaglement entropy. In the discussion below, we will sometimes refer to the first two terms in (4.1) as the classical terms and the last term as the quantum term.

It should be obvious that the middle term in (4.1), the one depending on d_a^i is absent in the Abelian theory. One can argue that the lack of a tensor product decomposition of the Hilbert space is related to the presence of a non-trivial centre. While the Hilbert of gauge-invariant states does not admit a tensor product decomposition, it could be written as a sum of tensor product terms,

$$\mathcal{H}_{g\text{inv}} = \bigoplus_i \mathcal{H}_i \otimes \mathcal{H}'_i, \tag{4.2}$$

where each factor $\mathcal{H}_i \otimes \mathcal{H}'_i$ is the Hilbert space in a sector corresponding to a particular value for the centre. Different choices for the algebra of gauge-invariant operators lead to different choices of centres and to a different value of the entaglement entropy in general. In particular, keeping all gauge-invariant operators in the region of interest in the algebra gives rise to the electric centre which is specified by the normal component of the electric field along the boundary. It is this definition which agrees with the EHS definition for the Abelian case. Other choices of centres can also be made. In particular, removing all the tangental components of the electric field at the boundary from the algebra gives rise to a magnetic centre.

Here, we are interested in studying the behaviour of the first two terms of (4.1), which correspond to the non-extractable contributions to the entanglement entropy, in the continuum limit. These terms depend on the distribution p_i which determines the probability for being in the various superselection sectors. For the electric centre case[1], by studying the correlation functions of the electric field on the boundary, we will find, both for the Abelian and non-Abelian cases, that the distribution is typically determined by very high momentum modes localized close to the boundary. As a result, we will argue that the contribution form these terms drops out of the mutual information of disjoint regions or the relative entropy between two states which only have finite energy excitations about the vacuum.

4.2 Classical Term in the Continuum Limit

4.2.1 $U(1)$ Abelian Theory

Let us begin by considering a 3+1-dimensional free $U(1)$ theory. Since the theory is free, the classical term is determined by the two-point function of the normal component of the electric field,

$$G_{rr} = \langle E_r E_r \rangle, \tag{4.3}$$

[1]In the non-Abelian case, the electric centre definition is sometimes taken to be different from the EHS definition with the middle term in (4.1) being absent. We will not be very careful about such distinctions. Our considerations, studying the probability distribution p_i will apply to both of these cases.

with the two electric fields E_r being inserted at two different points on the boundary which is an S^2 of radius R. The classical term is then given by

$$-\sum p_i \log(p_i) = \alpha_1 \frac{A}{\epsilon^2} - \frac{1}{2}\log\big(\det G_{rr}^{-1}\big), \tag{4.4}$$

with ϵ being a short distance cut-off. Thus, up to a non-universal area lawa divergence, G_{rr} determines the classical term. Let us note in passing that the first term on the RHS of (4.4) depends on the measure for the sum over over all electric field configurations. Starting from the lattice and passing to the continuum gives a well defined measure.

Since the two electric fields in G_{rr} are inserted at points on the boundary S^2 it is a function of the angular separation of the two points. In fact, it turns out to be divergent and some care needs to be exercised in regulating this divergence and defining it. After introducing a short distance cut-off Δ for the radial momentum, the resulting answer was shown to be

$$G_{rr}^{lm} = \frac{1}{\pi R^4}\left(\log\frac{R^2}{\Delta^2}\right) l(l+1). \tag{4.5}$$

Here we have carried out a Fourier transform to go from the angular separation variables to the angular momentum variables labelled by the integers (l, m), as per standard conventions. Note that the result diverges as $\Delta \to 0$, due to the contribution of modes with very high values of the radial momentum, which dominate the result in this limit. The factor of $l(l+1)$ means that the Green function is proportional to the two-dimenstional Laplacian for a free scalar on the boundary.

From G_{rr}, we can calculate the classical piece (4.4) which turns out to be,

$$-\sum_i p_i \log(p_i) = \alpha_2 \frac{A}{\epsilon^2} - \frac{1}{6}\log\left(\frac{A}{\epsilon^2}\right) + \cdots, \tag{4.6}$$

where the ellipsis refer to non-universal finite pieces. The logarithmic term arises from the two-dimenstional scalar Laplacian mentioned above, while the pre-factor $\log\big(R^2/\Delta^2\big)/\pi R^4$ contributes to the non-universal area law divergent pieces above.

It is important to emphasise that we have introduced two cut-offs above, Δ and ϵ, and worked in the limit where Δ goes to zero first. This was true in (**??**), for example, since the angular momentum l along the S^2 was being kept fixed while $\Delta \to 0$. It is in this limit that G_{rr} depends on the two-dimenstional Laplacian for a free scalar with a logarithmic dependence on the radial cut-off Δ. The behaviour of modes whose momentum along S^2 is comparable to the radial cutoff is more complicated and less universal. In the discussion below, when we analyse the contribution of the classical term to the mutual information and relative entropy, we will comment on such modes as well and show that their contribution too drops out from these quantities.

Having understood the case of a spherical region, we can now turn to the more general situation. Let us begin by first considering the half space $z < 0$, with the remaining spatial coordinates, (x, y) taking values in $(-\infty, \infty)$. The boundary of this region is the two-plane $z = 0$. The normal component of the electric field is along the z direction. It is easy to see by standard quantisation of the electromagnetic field that

$$\langle E_z(\mathbf{x_1}) E_z(\mathbf{x_2})\rangle = \int \frac{d^3k}{(2\pi)^2}\frac{\omega_k}{2}\left(1 - \frac{k_z^2}{k^2}\right) e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \tag{4.7}$$

(The two-point function is computed at equal time.) Here $\omega_k = k = \sqrt{k_z^2 + k_x^2 + k_y^2}$. Setting the two points to be at the boundary, $z = 0$, we get

$$\langle E_z(\mathbf{k}_\parallel)_z(\mathbf{k}'_\parallel)\rangle = (2\pi)^2\delta^{(2)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel)k_\parallel^2 \int \frac{dk_z}{4\pi}\frac{1}{k}. \tag{4.8}$$

We see that the integral on the RHS is logarithmically divergent since $k \sim k_z$ for large k_z. Thus, as noted above, the result needs to be regulated by introducing a cut-off for momentum along the z direction, normal to the boundary. One efficient way to do this is to take the two points located at slightly different values in the z direction, with $\Delta z = \delta$, instead of both of them being exactly at the boundary. Repeating the calculation above now gives,

$$\langle E_z(\mathbf{k}_\parallel)_z(\mathbf{k}'_\parallel)\rangle = (2\pi)^2\delta^{(2)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel)k_\parallel^2 \int \frac{dk_z}{4\pi}\frac{e^{ik_z\Delta}}{k}. \tag{4.9}$$

We see that the divergence in the integral at large k_z is regulated resulting in

$$\langle E_z(\mathbf{k}_\parallel)_z(\mathbf{k}'_\parallel)\rangle \sim (2\pi)^2\delta^{(2)}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel)k_\parallel^2 \log\big(k_\parallel\Delta\big). \tag{4.10}$$

We can see that the result above is analogous to what was obtained for the spherical region case. In particular, the result diverges e logarithmically as $\Delta \to 0$ and is proportional to $k_\parallel^2$ which is the eigenvalue of the two-dimenstional scalar Laplacian on the boundary. One difference is that in the spherical case, the logarithmic divergence due to the modes with high value of the normal component is cut-off by the radius R, whereas in the infinite plane case, where this cut-off is not availiable, it is cut-off by $k_\parallel$.

From this example of the half plane and the spherical region, it is now clear that we expect the two-point function in any compact region a result analogous to the sphere case, namely going like $\log(R/\Delta)$, with Δ being the cut-off fir the normal component of momentum, and R being an IR scale provided by the size of the region of interest, and also being proportional to the two-dimenstional Laplacian along the boundary.

4.3 Entanglement and Dualities

4.4 Conclusions

Chapter 5

Introduction to Conformal Field Theory

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5.1 String theory as a CFT

String theories in the conformal gauge are two-dimensional conformal field theories. Thus, instead of the operatorial analysis, one can give an equivalent description by using the language of conformal field theory in which one works with the OPE rather then commutators or anticommutators and that contributes to simplify many calculations.

5.1.1 Variables and coordinates in the CFT formulation

In the case of a closed string it is convenient to introduce the variables z and $\bar{z}$ that are related to the world sheet variables τ and σ through a conformal transformation:

$$z = e^{2i(\tau-\sigma)} \quad ; \quad \bar{z} = e^{2i(\tau+\sigma)} \tag{5.1}$$

In the case of an euclidean world sheet ($\tau \to -i\tau$), z and $\bar{z}$ are complex conjugates of each other. In terms of them we can write the bosonic coordinate X^μ as follows:

$$X^\mu = (z, \bar{z}) = \frac{1}{2}\left[X^\mu(z) + \tilde{X}^\mu(\bar{z})\right] \tag{5.2}$$

where

$$X^{\mu}(z) = \hat{q}^{\mu} - i\sqrt{2\alpha'}\log z\alpha_0^{\mu} + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^{\mu}}{n}z^{-n} \tag{5.3}$$

and

$$\tilde{X}^{\mu}(\bar{z}) = \hat{q}^{\mu} - i\sqrt{2\alpha'}\log \bar{z}\tilde{\alpha}_0^{\mu} + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^{\mu}}{n}\bar{z}^{-n} \tag{5.4}$$

with $\alpha_0^{\mu} = \tilde{\alpha}_0^{\mu} = \sqrt{\frac{\alpha'}{2}}\hat{p}^{\mu}$

In the case of **an open string theory** one can introduce the variables:

$$z = e^{i(\tau-\sigma)}; \quad \bar{z} = e^{i(\tau+\sigma)} \tag{5.5}$$

and the string coordinate can be written as

$$X^{\mu}(z,\bar{z}) = \frac{1}{2}\left[X^{\mu}(z) + X^{\mu}(\bar{z})\right] \tag{5.6}$$

where X^{μ} is given in (5.3) and $\alpha_0^{\mu} = \sqrt{2\alpha'}\hat{p}^{\mu}$. In superstring theory we must also introduce a conformal field with conformal dimension equal to $1/2$ corresponding to the fermionic coordinate.

In the closed string case we have two in-dependent fields for the holomorphic and anti-holomorphic sectors which are obtained through the Wick rotation ($\tau \to -i\tau$) and the conformal transformation $(\tau,\sigma) \to (z,\bar{z})$

$$\Psi^{\mu}(z) \sim \sum_t \psi_t z^{-t-1/2}; \quad \tilde{\Psi}^{\mu}(\bar{z}) \sim \sum_t \tilde{\psi}_t \bar{z}^{-t-1/2} \tag{5.7}$$

In the open string case, applying the same operations we get again eqs(5.7), but this time with the same oscillators.

In what follows we will explicitly consider only the **holomorphic sector for the closed string**. In the case of an open string it is sufficient to consider the string coordinate at the string endpoint $\sigma = 0$. In both cases it is convenient to introduce a *bosonic dimensionless variable*:

$$x^{\mu}(z) \equiv \frac{X^{\mu}(z)}{\left(\sqrt{2\alpha'}\right)} = \tilde{q}^{\mu} - i\alpha_0^{\mu}\log z + i\sum_{n\neq 0}\frac{\alpha_n}{n}z^{-n} \tag{5.8}$$

where $\tilde{q} = \hat{q}/\sqrt{2\alpha'}$ and a *fermionic* one:

$$\psi^{\mu}(z) = -i\sum_t \psi_t z^{-t-1/2} \tag{5.9}$$

5.1.2 Operator Product Expansion

The theory can be quantized by imposing the following OPEs

$$x^{\mu}(z)\,x^{\nu}(\omega) = -\eta^{\mu\nu}\log(z-\omega) + \cdots; \quad \psi^{\mu}(z)\,\psi^{\nu}(\omega) = -\frac{\eta^{\mu\nu}}{z-\omega} + \cdots, \tag{5.10}$$

where the dots denote finite terms for $z \to \omega$. In terms of the previous conformal fields we can define the generators of superconformal transformations:

$$G(z) = -\frac{1}{2}\psi.\partial x \quad ; T(z) = T^{x}(z) + T^{\psi}(z) = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}\partial\psi.\psi \tag{5.11}$$

These conformal fields satisfy the following OPEs:

$$T(z)\,T(\omega) = \frac{\frac{d}{d\omega}T(\omega)}{z-\omega} + 2\frac{T(\omega)}{(z-\omega)^2} + \frac{c/2}{(z-\omega)^4} + \cdots \tag{5.12}$$

$$T(z)\,G(\omega) = \frac{\partial/\partial\omega G(\omega)}{z-\omega} + \frac{3}{2}\frac{G(\omega)}{(z-\omega)^2} + \cdots \tag{5.13}$$

$$G(z)\,G(\omega) = \frac{2T(z)}{z-\omega} + \frac{d}{(z-\omega)^3} + \cdots \tag{5.14}$$

We have to translate the L_0 operator in the R sector by a constant:

$$L_0 \to L_0^{conf} \equiv L_0 + \frac{d}{16} = \sum_{n=1}^{\infty}(\alpha_{-n}.\alpha_n + n\psi_{-n}.\psi_n) + \alpha' p^2 + \frac{d}{16} \tag{5.15}$$

Therefore in the R sector we have two L_0 operators that are related by eq.(5.15). L_0 determines the spectrum of superstring while L_0^{conf} encodes the correct conformal properties of the R sector.

5.1.3 Primary fields & Highest weight states

In conformal field theory one introduces the concept of conformal or *primary field*

Definition 5.1 (Primary field)**.** *A conformal field $\Phi(z)$ is called a primary field with dimension h if it satisfies the following OPE with the energy-momentum tensor:*

$$T(z)\,\Phi(\omega) = \frac{\partial_\omega \Phi(\omega)}{z-\omega} + h\frac{\Phi(\omega)}{(z-\omega)^2} \tag{5.16}$$

From it one can compute the corresponding *highest weight state* $|\Phi\rangle$ by means of the following limiting procedure

$$|\Phi\rangle = \lim_{z\to 0}\Phi(z)|0\rangle \quad , \quad \langle\Phi| = \lim_{z\to 0}\langle 0|\Phi^\dagger(z) \sim \lim_{z\to\infty}\langle 0|\left(z^2\right)^h\Phi(z) \tag{5.17}$$

The hermitian conjugate field $\Phi^\dagger$ in the previous expression has been defined as the field transformed under the conformal transformation $z \to 1/z$. It is easy to show that:

$$L_0|\Phi\rangle = h|\Phi\rangle \quad ; \quad L_n|\Phi\rangle = 0 \tag{5.18}$$

5.1.4 BRST charge & Vertex operators

With the introduction of ghosts, the string action in the conformal gauge becomes invariant under the BRST transformations and the physical states are characterized by the fact that they are annihilated by the BRST charge that in the bosonic case is given by

$$Q \equiv \oint \frac{dz}{2\pi i} c(z)\, J_{BRST} \equiv \oint \frac{dz}{2\pi i} c(z)\left[T^x(z)\frac{1}{2}T^{bc}(z)\right] \tag{5.19}$$

where $T^x(z) = -1/2\,(\partial x)^2$ and $T^{bc}(z) = \sum_n L_n z^{-n-2}$. The physical states are annihilated by the BRST charge

$$Q|\phi_{phys}\rangle = 0 \tag{5.20}$$

By using the OPE it can be shown that in the bosonic string the most general BRST invariant *vertex operator* has the following form

$$\mathcal{W} = c\,(z)\,\mathcal{V}_{\alpha}^{x}\,(z) \tag{5.21}$$

where $\mathcal{V}_{\alpha}^{x}\,(z)$ is a conformal field with dimension equal to 1 that depends only on the string coordinate x^{μ}.

Chapter 6

Strings in Curved Backgrounds

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Chapter 7

Introduction to Density Functional Theory

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Density functional theory (DFT) is a computational quantum mechanical modelling method used in physics, chemistry and materials science to investigate the electronic structure (principally the ground state) of many-body systems, in particular atoms, molecules, and the condensed phases. Let us consider a N-electron system (atom, molecule or solid) in the Born-Oppenheimer and non-relativistic approximations:

$$\begin{aligned} \psi_{total} &= \psi_{electronic} \otimes \psi_{nuclear} \\ v &<< c \end{aligned} \tag{7.1}$$

7.1 The many-body problem

We can write the electronic Hamiltonian of such a system in the position representation and atomic units as:

$$\mathcal{H}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n) = -\frac{1}{2}\sum_{i=1}^{N}\nabla_{r_i}^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=0 \\ j\neq i}}^{N}\frac{1}{|\vec{r}_i - \vec{r}_j|} + \sum_{i=1}^{N} v_{\mathrm{ne}}(\vec{r}_i), \tag{7.2}$$

where $v_{\text{ne}} = -\sum_\alpha Z_\alpha / \left|\vec{r}_i - \vec{R}_\alpha\right|$ is the nuclei-electron interaction ($\vec{R}_\alpha$ and Z_α are the position and charges of the nuclei, respectively). From now on we will omit the vector symbols for ease of notation. The stationary states are determined by the time-independent Schrodinger equation

$$\mathcal{H}(r_1, r_2, \cdots, r_N)\Psi(x_1, x_2, \cdots, x_N) = E\Psi(x_1, x_2, \cdots, x_N), \tag{7.3}$$

where $\Psi(x_1, x_2, \cdots, x_N)$ is a wave function written with space spin coordinates $x_i = (r_i, \sigma_i)$(with $r_i \in \mathbb{R}^3$ and $\sigma_i =\uparrow$ or $\downarrow$) which is antisymmetric with respect to the change of two coordinates, and E is the associated energy.

Using Dirac notation, the Schrodinger equation can be rewritten in a representation-independent formalism

$$\hat{\mathcal{H}} \left|\Psi\right\rangle = E \left|\Psi\right\rangle, \tag{7.4}$$

where the Hamiltonian is formally written as:

$$\hat{\mathcal{H}} = \hat{T} + \hat{W}_{ee} + \hat{V}_{ne}, \tag{7.5}$$

where $\hat{T}$ is the kinetic-energy operator, $\hat{W}_{ee}$ is the electron-electron interaction operator, and $\hat{V}_{ne}$ is the nuclei-electron interaction operator. These operators can be expressed in convinient ways in second quantization because of the fact that they will be independent of the number of electrons (i.e. we work in Fock space). The kinetic-energy operator becomes:

$$\hat{T} = -\frac{1}{2}\sum_{\sigma=\uparrow,\downarrow}\int \hat{\psi}^\dagger_\sigma(r)\nabla^2\hat{\psi}_\sigma(r)dr \tag{7.6}$$

$\hat{W}_{ee}$, the electron-electron operator, becomes:

$$\hat{W}_{ee} = \frac{1}{2}\sum_{\sigma_1=\uparrow,\downarrow}\sum_{\sigma_2=\uparrow,\downarrow}\int\int \hat{\psi}^\dagger_{\sigma_2}(r_2)\hat{\psi}^\dagger_{\sigma_1}(r_1)w_{ee}(r_1, r_2)\hat{\psi}^\dagger_{\sigma_1}(r_1)\hat{\psi}^\dagger_{\sigma_2}(r_2)dr_1dr_2, \tag{7.7}$$

with $w_{ee}(r_1, r_2) = \frac{1}{|r_1 - r_2|}$, and $\hat{V}_{ne}$ is the nuclei-electron operator:

$$\hat{V}_{ne} = \sum_{\sigma=\uparrow,\downarrow}\int \hat{\psi}^\dagger_\sigma(r)v_{ne}(r)\hat{\psi}_\sigma(r)dr. \tag{7.8}$$

In these expressions, $\hat{\psi}^\dagger_\sigma(r)$ and $\hat{\psi}^\dagger_\sigma(r)$ are the creation and annihilation field operators, repsectively, which obey fermionic anticommutation rules:

$$\{\hat{\psi}^\dagger_\sigma(r), \hat{\psi}^\dagger_{\sigma'}(r')\} = 0 \tag{7.9}$$

$$\{\hat{\psi}_\sigma(r), \hat{\psi}_{\sigma'}(r')\} = 0 \tag{7.10}$$

$$\left\{\hat{\psi}^\dagger_\sigma(r), \hat{\psi}^\dagger_{\sigma'}(r')\right\} = \delta(r - r')\delta_{\sigma\sigma'} \tag{7.11}$$

It is also convinient to define the density operator

$$\hat{n}(r) = \sum_{\sigma=\uparrow,\downarrow}\hat{\psi}^\dagger_\sigma(r)\hat{\psi}_\sigma(r), \tag{7.12}$$

the one-particle density-matrix operator

$$\hat{n}_1(r,r') = \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^\dagger_\sigma(r')\hat{\psi}_\sigma(r) \tag{7.13}$$

and the pair-density operator

$$\begin{aligned}\hat{n}_2(r_1,r_2) &= \sum_{\sigma_1=\uparrow,\downarrow}\sum_{\sigma_2=\uparrow,\downarrow} \hat{\psi}^\dagger_{\sigma_2}(r_2)\hat{\psi}^\dagger_{\sigma_1}(r_1)\hat{\psi}_{\sigma_1}(r_1)\hat{\psi}_{\sigma_2}(r_2) \\ &= \hat{n}(r_2)\hat{n}(r_1) - \hat{r_1}\delta(r_1-r_2) \\ &= \hat{n}(r_1)\hat{n}(r_2) - \hat{n}(r_1)\delta(r_1-r_2)\end{aligned} \tag{7.14}$$

Now we can write a compact form for the kinetic and interaction operators in thethe Hamiltonian:

$$\hat{T} = -\frac{1}{2}\int \left[\nabla^2_r \hat{n}_1(r,r')\right]_{r'=r} \mathrm{d}r, \tag{7.15}$$

$$\hat{W}_{ee} = \frac{1}{2}\int\int \omega_{ee}(r_1,r_2)\hat{n}_2(r_1,r_2)\mathrm{d}r_1\mathrm{d}r_2, \tag{7.16}$$

$$\hat{V}_{ne} = \int v_{ne}(r)\hat{n}(r)\mathrm{d}r. \tag{7.17}$$

We can also use the second-quantization formalism in an orthonormal spin-orbital basis $\{\psi_p(x)\}$, where $x = (r,\sigma)$. For this, we expand the field operators as

$$\hat{\psi}^\dagger_\sigma(r) = \sum_p \psi^*_p(x)\hat{a}^\dagger_p \tag{7.18}$$

and

$$\hat{\psi}_\sigma(r) = \sum_p \psi_p(x)\hat{a}_p \tag{7.19}$$

where $\hat{a}^\dagger_p$ and $\hat{a}_p$ are the creation and annihilation operators in this basis, which still obey anticommutation rules: $\left\{\hat{a}^\dagger_p, \hat{a}^\dagger_q\right\} = \{\hat{a}_p, \hat{a}_q\} = 0$ and $\left\{\hat{a}^\dagger_p, \hat{a}_q\right\} = \delta_{pq}$. The expressions of the operators are then

$$\hat{T} = \sum_{pq} t_{pq}\hat{a}^\dagger_p\hat{a}_q \tag{7.20}$$

$$\hat{W}_{ee} = \frac{1}{2}\sum_{pqrs} \langle\psi_p\psi_q|\psi_r\psi_s\rangle\, \hat{a}^\dagger_p\hat{a}^\dagger_q\hat{a}_s\hat{a}'_r \tag{7.21}$$

$$\hat{V}_{ne} = \sum_{pq} v_{ne,pq}\hat{a}^\dagger_p\hat{a}_q \tag{7.22}$$

where t_{pq} and $v_{\mathrm{ne},pq}$ are the one-electron kinetic and nuclei-electron integrals, respectively, and $\langle\psi_p\psi_q|\psi_r\psi_s\rangle$ are the two-electron integrals.

The quantity of primary interest is the ground-state energy E_0. The variational theorem establishes that E_0 can be obtained by the following minimization

$$E_0 = \min_\Psi \langle\Psi|\hat{H}|\Psi\rangle\,, \tag{7.23}$$

where the search is over all N-electron antisymmetric wave functions Ψ, normalized to unity $\langle\Psi|\Psi\rangle = 1$. DFT is based on a reformulation of the variational theorem in terms of the one-electron density defined as[1]

$$n(r) = N \int \cdots \int |\Psi(x, x_2, \cdots, x_N|^2 \mathrm{d}\sigma \mathrm{d}x_2 \cdots \mathrm{d}x_N), \tag{7.24}$$

which is normalized to the electron number, $\int n(r)\mathrm{d}r = N$.

7.2 The universal density functional

7.2.1 The Hohenberg-Kohn theorem

Consider an electronic system with an arbitrary external local potential $v(r)$ in place of $v_{\mathrm{ne}}(r)$. A corresponding ground-state wave function Ψ (there can be several of them if the ground staate is degenerate) can be obtained by solving the Schrodinger equation, from which the associated ground-state density $n(r)$ can be deduced. Therefore, one has a mapping from the potential $v(r)$ to the considered ground-state density $n(r)$

$$v(r) \to n(r). \tag{7.25}$$

In 1964, Hohenberg and Kohn showed that this mapping can be inverted, i.e. the ground-state density $n(r)$ determines the potential $v(r)$ up to an arbitrary additive constant

$$\boxed{n(r) \xrightarrow{Hohenberg-Kohn} v(r) + \text{const.}} \tag{7.26}$$

Proof. The two-step proof by contradiction proceeds as follows. We consider two locallocal potentials $v_1(r)$ and $v_2(r)$ differing by more than an additive constant, $v_1(r) \neq v_2(r) + \text{const}$, and we denote by E_1 and E_2 the ground-state energies of the Hamiltonians $\hat{H}_1 = \hat{T} + \hat{W}_{ee} + \hat{V}_1$ and $\hat{H}_2 = \hat{T} + \hat{W}_{ee} + \hat{V}_2$, respectively. Now, assume that $\hat{H}_1$ and $\hat{H}_2$ have the same ground-state wave functions Ψ, i.e. $\hat{H}_1 |\Psi\rangle = E_1 |\Psi\rangle$ and $\hat{H}_2 |\Psi\rangle = E_2 |\Psi\rangle$. Then, substacting these two equations gives

$$(\hat{V}_1 - \hat{V}_2) |\Psi\rangle = (E_1 - E_2) |\Psi\rangle , \tag{7.27}$$

or, in position representation

$$\sum_{i=1}^{N} [v_1(r_i) - v_2(r_i)] \Psi(x_1, x_2, \cdots, x_N) = (E_1 - E_2) \Psi(x_1, x_2, \cdots, x_N) \tag{7.28}$$

□

which implies $v_1(r) - v_2(r) = \text{const}$, in contradiction with the initial hypothesis. Note that, to eliminate Ψ it was assumed that $\Psi(x_1, x_2, \cdots, x_N) \neq 0$ for all spatial coordinates $(r_1, r_2, \cdots, r_N)$ and at least one fixed set of spin coordinates $(\sigma_1, \sigma_2, \cdots, \sigma_N)$. This is in fact true "almost

[1] The integration over the spin coordinate σ in the following equation just means a sum over the two values $\sigma = \uparrow$ and $\sigma = \downarrow$

everywhere" for "reasonably well behaved" potentials. In this case, we thus conclude that two local potentials differing by more than an additive constant cannot share the same ground-state wave function.

Let then Ψ_1 and Ψ_2 be (necessarily different) ground-state wave functions of $\hat{H}_1$ and $\hat{H}_2$, respectively, and assume that Ψ_1 and Ψ_2 have the same ground-state density $n(r)$. The variational theorem leads to the following inequality.

$$E_1 = \langle\Psi_1|\hat{H}_1|\Psi_1\rangle < \langle\Psi_2|\hat{H}_1|\Psi_2\rangle = \langle\Psi_2|\hat{H}_2 + \hat{V}_1 - \hat{V}_2|\Psi_2\rangle = E_2 + \int [v_1(r) - v_2(r)]\, n(r)\mathrm{d}r, \tag{7.29}$$

where the strict inequality comes from the fact that Ψ_2 cannot be a ground-state wave function of $\hat{H}_1$, as shown in the first step of the proof. Symmetrically, by exchanging the role of system 1 and 2, we have the stict inequality

$$E_2 < E_1 + \int [v_2(r) - v_1(r)]\, n(r)\mathrm{d}r. \tag{7.30}$$

Adding the two previous equations, we get

$$E_1 + E_2 < E_1 + E_2, \tag{7.31}$$

which finally leads to the conclusion that there cannot exist two local potentials differing by more than an additive constant which have the same ground-state density. Note that this proof does not assume non-degenerate ground states (contrary to the original Hohenberg-Kohn proof) ■

So, the ground-state density $n(r)$ determinese the potential $v(r)$, which in turn determines the Hamiltonian, and thus everything about the many-body problem. In other words, the potential v is a unique (up to an additive constant) functional of the ground-state density n, and all other properties as well. The ground-state wave function Ψ for the potential $v(r)$ is itself a functional of n, denoted by $\Psi[n]$, which was exploited by Hohenberg-Kohn to define the universal (i.e., independent from the external potential) density functional

$$F[n] = \langle\Psi[n]|\, \hat{T} + \hat{W}_{ee}\, |\Psi[n]\rangle\,, \tag{7.32}$$

which can be used to define the total electronic energy functional

$$E[n] = F[n] + \int v_{ne}(r)n(r)\mathrm{d}r, \tag{7.33}$$

for the specific potential $v_{ne}(r)$ of the system considered. Note that, forfor degenerate ground states, $\Psi[n]$ is not unique but stands for any degenerate ground-state wave function. However, all $\Psi[n]$ give te same $F[n]$, which is thus a unique functional of n.

Hohenberg and Kohn further showed that the density functional $E[n]$ satisfies a variational property: the ground-state energy E_0 of the system considered is obtained by minimizing this functional with respecst to N-electr-electron densities n that are ground-state densities associated with some local potential (referred to as v-representable densities).

$$E_0 = \min_n \left\{ F[n] + \int v_{ne}(r)n(r)\mathrm{d}r \right\}, \tag{7.34}$$

the minimum being reached for a ground-state density $n_0(r)$ corresponding to the potential $v_{ne}(r)$. The existance of a mapping from a ground-state density to a local potential, the existance of the universal density functional, and the variational property with respect to the density constitutes the *Hohenberg-Kohn theorem.*

Part II

Mathematics

Chapter 8

Group Theory

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Contents

"The Universe is an enormous direct product of resentations of symmetry groups."

Hermann Weyl

8.1 Basic Definitions

Definition 8.1 ((Group) Homomorphism)**.** *Let $(G, *)$ and $(H, \cdot)$ be two groups. A (group) homomorphism from G to H is a function $h : G \to H$ such that for all x, y in G it holds that $h(x * y) = h(u) \cdot h(v)$*

DICKS

Definition 8.2 (Coset)**.** *Let G be a group and H is a subgroup of G. Consider an element $g \in G$. Then, $gH = \{gh : h \in H\}$ is the left coset of H in G with respect to g, and $Hg = \{hg : h \in H\}$ is the right coset of H in G with respect to g*

Remark 8.1. In general the left and right cosets are not groups.

Definition 8.3 (Normal Subgroup)**.** *A subgroup H of G is called normal if and only if the left and right sets of cosets coincide, that is if $gH = Hg$ for all $g \in G$*

Definition 8.4 (Representation)**.** *A representation of a group G on a vector space V over a filed K is a group homomorphism from G to $GL(V)$, the general linear group on V. That is, a representation is a map*

$$\rho : G \to GL(V) \tag{8.1}$$

such that

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \quad \forall g_1, g_2 \in G. \tag{8.2}$$

V is often called the *representation space* and the dimension of V is called the *dimension* of the representation. It is common practice to refer to V itself as the representation when the homomorphism is clear from the context.

Example 4. *Consider the complex number $u = e^{2\pi/3}$ which has the property $u^3 = 1$. The cyclic group $C_3 = \{1, u, u^2\}$ has a representation ρ on $\mathbb{C}^2$ given by:*

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u) = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}, \quad \rho(u^2) = \begin{bmatrix} 1 & 0 \\ 0 & u^2 \end{bmatrix}, \quad . \tag{8.3}$$

Another representation for C_3 on $\mathbb{C}^2$, isomorphic to the previous one, is

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(u^2) = \begin{bmatrix} u^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad . \tag{8.4}$$

Definition 8.5 (Subrepresentation)**.** *A subspace W of V that is invariant under the group action is called a subrepresentation.*

Definition 8.6 ((Ir)reducible representation)**.** *If V has exactly two representations, namely the zero-dimensional subspace and V itself, then the representation is said to be irreducible; if it has a proper representation of nonzero dimension, the representation is said to be reducible. The representation of dimension zero is considered to be neither reducible nor irreducible.*

Definition 8.7 (Quotent Group)**.** *Let N be a normal subgroup of a group G. We define the set G/N to be the set of all left cosets of N in G, i.e., $G/N = \{aN : a \in G\}$. Define an operation on G/N as follows. For each aN and bN in G/N, the product of aN and bN is $(aN)(bN)$. This defines an operation of G/N if we impose $(aN)(bN) = (ab)N$, because $(ab)N$ does not depend on the choice of the representatives a and b: if $xN = aN$ and $yN = bN$ for some $x, y \in G$, then:*

$(ab)N = a(bN) = a(yN) = a(Ny) = (aN)y = (xN)y = x(Ny) = x(yN) = (xy)N$

Here it was used in an important way that N is a normal subgroup. It can be shown that this operation on G/N is associative, has identity element N and the inverse of an element $aN \in G/N$ is $a^{-1}N$. Therefore, the set G/N together with the defined operation forms a group; this is known as the quotient group or factor group of G by N

8.2 Lie groups and representations

Every symmetry group that we work with in field theory is a Lie group. We will focus on $SU(N)$ as a working example, but everything generalizes for all other Lie groups. $SU(N)$ consists of (Hermitian and unit determinant) matrices of size $N \times N$ that act on $\mathbb{C}^n$. $SU(N)$ has $dim(SU(N))$ basis elements and each of these basis elements is an $N \times N$ matrix. Note that $dim(SU(N)) \neq N$.

For every Lie group there is an associated Lie algebra (sometimes called the tangent space in the literature). The Lie algebra of $SU(N)$ is denoted by $\mathfrak{su}(N)$.$\mathfrak{su}(N)$ has the same dimension as $SU(N)$ has the same dimension as $SU(N)$. We denote a basis of $\mathfrak{su}(N)$ by $T^a,\ a = 1, \cdots, dim(SU(N))$. Note that T^a are also $N \times N$ matrices. Every Lie algebra has an additional structure called the *Lie Bracket*, which classifies the algebra, denoted by

Definition 8.8 (Lie Bracket)**.**

$$\left[T^a, T^b\right] = f_c^{ab} T^c \tag{8.5}$$

f_c^{ab} are called the *structure constants* of the Lie algebra. They encode the structure of the group. We can gen an element of the Lie group by exponentiating an element of the Lie algebra:

$$\begin{gathered} \text{Lie algebra} \xrightarrow{exp} \text{Lie group} \\ exp(i\omega_a T^a) \in SU(N) \end{gathered} \tag{8.6}$$

We have chosen T^a to be Hermitian, resulting in a factor of i in the exponent. If we choose T^a to be anti-Hermitian, then there would be no factor of i. However, we always work with Hermitian operators, so there is always a factor of i. Since an element of the Lie group can be found by exponentating some element of the Lie algebra and we can expand any element of the Lie algebra in terms of the basis of the Lie algebra (for example, T^a), then this basis generates any element of the Lie algebra and by exponentating, generates any element of the Lie group (that is connected to the origin)[1] For this reason, the basis of the Lie algebra are called the *generators*.

However, $SU(N)$ only acts on $\mathbb{C}^N$. What if we want a group with the same structure (constants) as $SU(N)$[2] but that act on different spaces such as $\mathbb{C}^m$, for $m \neq N$? Representations of $SU(N)$ allow us to do this. They allow us to construct groups with the same structure (constants) as $SU(N)$ but which act on different spaces.

8.2.1 Representations

A representation of a group $\mathcal{G}$ is a homomorphism from $\mathcal{G}$ to the space of linear maps acting on a representation space, $\mathcal{V}_{rep}$:

$$\rho : \mathcal{G} \to GL(\mathcal{V}_{rep}) \tag{8.7}$$

[1] A group can be split up into different components. The Lorentz group, for example, consists of 4 disconnected components. One of this components, called the proper orthochronous Lorentz group $SO(1,3)^+$ is connected to the origin. Any relativistic theory will be invariant under $SO(1,3)^+$ (or $SO(3,1)$, depending on convention) but not neccesarily under the other components. Other groups, such as $SU(N)$ have only one component, which is connected to the origin. As such, exponenting the Lie algebra generates all of $SU(N)$.

[2] It should be noted that $SU(N)$ have different structure constants for different N

Every linear map can be thought of as a matrix and so an element of $\rho(\mathcal{G})$, say R, can be thought of as a matrix. $\rho(\mathcal{G})$ is a group of matrices and is called a *representation* of the group. Each R acts on $\mathcal{V}_{rep}$:

$$R : \mathcal{V}_{rep} \to V_{rep} \tag{8.8}$$

Since R is a matrix that acts on $\mathcal{V}$, it has to be a matrix of size $dim(\mathcal{V}_{rep}) \times dim(\mathcal{V}_{rep})$. The degree of the representation is the dimension of the representation space:

$$deg(\rho) = dim(\mathcal{V}_{rep}) \tag{8.9}$$

So the representation $\rho(\mathcal{G})$ has dimension equal to $dim(\mathcal{G})$ and each element of $\rho(\mathcal{G})$ is a square matrix of size $dim(\mathcal{V}_{rep})$. We have to be careful when talking about the dimensions of $\mathcal{G}$ and $\mathcal{V}_{rep}$ as it is easy to get confused when considering both. Examples will help clarify these concept. Different representations can act on different spaces. Representations are useful for describing how different states transform under the action of a particular symmetry.

8.2.2 Some noteworthy points

- The number of basis elements in the representation is the same as the number of basis elements of the group, $\mathcal{G}$. Another way of saying this is $dim(\rho(\mathcal{G})) = dim(\mathcal{G})$.
- A matrix in the representation, say R, acts on a representation space, $\mathcal{V}_{rep}$, by matrix multiplication. An element of $\mathcal{V}_{rep}$ can be considered as a column vector of $dim(\mathcal{V}_{rep})$. So R has to be a matrix of size $dim(\mathcal{V}_{rep}) \times dim(\mathcal{V}_{rep})$. This is why we have to seperate out the dimension of ρ from the degree. The examples below will clarify this idea.
- Recap: The representation of the symmetry group can be described using the generators. The generators are a basis for the representation on the Lie algebra. The number of generators is the number of basis elements of the symmetry group $= dim(\mathcal{G})$. Each generator has size $dim(\mathcal{V}_{rep}) \times dim(\mathcal{V}_{rep})$. Each representation has the same structure constants.

8.2.3 Example

If we consider the group $\mathcal{G} = SU(2)$ when thinking about spin angular momentum, this can give an insight into how representations work. For $SU(N)$, we normally work with irreducible Representation spaces. Consider each irreducible representation space labelled by j seperately, i.e. $\mathcal{V}_{rep} = \mathcal{V}_j$. Call the $3 = dim(SU(2))$ generators of the representation $T^a = J^a$, where we normally call $J^1 = J^x, J^2 = J^y, J^3 = J^z$ angular momentum. Examine the following table in detail to understand how the generators can be different sized matrices.

It is important to notice that there are always **only** $dim(SU(2)) = 3$ generators T^a. However, these generators can be different sized matrices depending on which representation space $\mathcal{V}_j$ the generators act on. Each representation has the same structure constants. As such, each representation has the same structure of the original group but allows us to act on different spaces.

j	basis for $\mathcal{V}_j$	$dim(\mathcal{V}_j)$ = degree of rep	$T^a = J^a$
$j = 0$	$\|0, 0\rangle$	1	$T^a = 0$, so that it doesn't transform under SU(2)
$j = \frac{1}{2}$	$\|\frac{1}{2}, m\rangle$	2	T^a are 2x2 matrices, equal to $\sigma^a/2$, where σ are the Pauli matrices
$j = 1$	$\|1, m\rangle$	3	T^a are 3x3 matrices, equal to the generalization of the Pauli matrices

Table 8.1: Some representations for SU(2) angular momentum

8.2.4 More points

- In general, consider $\chi_1 \in \mathcal{V}_{rep}$ and let T^a be a generator of a representation of the group, then it acts on the representation space as:

$$T^a \chi_1 = \chi_2 \in \mathcal{V}_{rep} \tag{8.10}$$

- One way to know what representation you are in is to see what size matrices the generators T^a are. Some examples are given later. If the generators are $m \times m$ matrices, then an element of the representation of the group, g, can be written as $g = exp(\lambda^a T^a)$. So g is also an $m \times m$ matrix, i.e, every representation of the Lie algebra gives a representation of the group by exponentating. Since the generators must act on elements of the representation space, and the generators are matrices of size $m \times m$, the elements of the representation space must be m component column vectors.

- A gauge theoroy has the Lagrangian $\mathcal{L} = \bar{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi$, where the field Ψ is in a representation space of the gauge group and D_μ is the covariant derivative. D_μ is defined to transform under an element of the representation, U, of the gauge group as

$$D_\mu \Psi \to U D_\mu \Psi = (U D_\mu U^{-1})(U\Psi) \tag{8.11}$$

  and so $D_\mu \to U D_\mu U^{-1}$. This transformation property is called the adjoint action of the group on the Lie algebra. Because of this transform property, the covariant derivative is said to transform in th adjoint action, or just adjointly. $D_\mu = \partial_\mu - igA_\mu(x)$, where $A_\mu(x)$ is an element of the representation of the Lie algebra. We can expand $A_\mu(x)$ in a basis of the representation so that $A_\mu(x) = A^a + \mu(x)T^a$, where $A^a_\mu(x)$ are the gauge field coefficients with corresponding (matrix) generators T^a. The gauge field A^a_μ are said to transform under the adjoint action in order to make the covariant derivative transform correctly. The transformation is the standard transformation:

$$A_\mu(x) \to U(x)A_\mu U(x)^\dagger - \frac{1}{e}(\partial_\mu U(x))U(x)^\dagger \tag{8.12}$$

  This is precisely the right transformation to make the covariant derivative transform as required

$$D_\mu \chi(x) \to U(x) D_\mu \Psi(x) \tag{8.13}$$

The generators **do not** have to be in the adjoint representation of the group and in general they are not. They just have to transform adjointly.

- We want to consider infinitesmal transformations. Let $U = exp(\lambda^a T^a) = 1 + \lambda^a T^a +$ higher order terms. Call $\lambda = \lambda_a T^a$. We know

$$\begin{aligned} A_\mu(x) &\to U A_\mu U^{-1} - \frac{1}{e}(\partial_\mu U)U^{-1}(x) && (8.14)\\ &= A_\mu - \frac{1}{e}\left\{\partial_\mu \lambda + e\left[A_\mu, \lambda\right]\right\} && (8.15)\\ &= A_\mu - \frac{1}{e} D_\mu \lambda && (8.16) \end{aligned}$$

Because of laziness, we don't want to write out $[A_\mu, \lambda]$ all the time. We give it a special name: the adjoint action of the Lie algebra on itself. Write it $A_\mu \cdot \lambda = [A_\mu, \lambda]$ or $(A_\mu \cdot \lambda)^{ad} = [A_\mu, \lambda]$. This comes from the adjoint action of the group on the Lie algebra in infinitimesal form. But we can expand in a basis, $A_\mu - A_\mu^a T^a, \lambda = \lambda^a T^a$. The A_μ^a, λ^a are field coefficents when expanded in the (matrix) basis T^a. They are fields which commute. But T^a are matrices which in general don't commute. So

$$\begin{aligned} D_\mu \lambda &= \partial_\mu \lambda + (A_\mu \cdot \lambda)^{ad} \\ &= \partial_\mu \lambda + e\left[A_\mu, \lambda\right] \\ &= \partial_\mu \lambda^a T^a + e A_\mu^a \lambda^b \left[T^a, T^b\right] \\ &= \partial_\mu \lambda^a T^a + e A_\mu^a \lambda^b f_c^{ab} T^c \quad \text{using } \left[T^a, T^b\right] = f_c^{ab} T^c \end{aligned} \tag{8.17}$$

In component form, the covariant derivative acts on an element of the Lie algebra as

$$\begin{aligned} (D_\mu \lambda)_a &= \partial_\mu \lambda_a + (A_\mu \cdot \lambda)_a^{ad} \\ &= \partial_\mu \lambda_a + e\left[A_\mu, \lambda\right]_a \\ &= \partial_\mu \lambda_a + e A_\mu^b \lambda^c f_a^{bc} \end{aligned} \tag{8.18}$$

- The main point to focus on here is the adjoint action. Do not be confused between the adjoint ($(A_\mu \cdot \lambda)^{ad} = [A_\mu, \lambda]$) and the adjoint representation. They are two different things. The adjoint action is true for all representations, i.e, for all $m \times m$ matrices T^a. But the adjoint re[representation is a specific representation of the group. In the adjoint representation, the matrices have a specific size, where $m = dim(G)$. This is standard (sometimes confusing) terminology.

8.2.5 Certain Types of Representations

When thinking about representations of $SU(N)$ as $m \times m$ matrices, there are three natural numbers to consideer for m. One is $m = 1$, i.e, a scalar. This is called the trivial representation. Another is $m = n$, where the $SU(n)$ matrices themselves are the representation. This is called the fundamental representation. The last is when $m = dim(SU(n))$. This is called the adjoint representation.

The **trivial** representation is whre $\rho(g) = 0$ so that all generators of the representation have $T^a = 0$. This is the $j = 0$ representation of $SU(2)$. $|j = 0, 0\rangle$ is said to be a singlet or scalar as it doesn't transform under $SU(2)$.

The **fundamental** representation is the representation which is equal to the group. This is the homomorphism $\rho(g) = g$. In the fundamental representation, when considering $SU(n)$, the representation $\rho(SU(n)) = SU(n)$, i.e. the $SU(n)$ matrices are the representation matrices. The representation is classified by $n \times n$ matrices. $SU(n)$ must act on a column vector of size n and hence the dimension of the representation space must be n. This is the $j = 1/2$ representation of $SU(2)$.

In the **adjoint** representation, it can be shown that the generators have the component form $(T^a_{ad})_{bc} = f^a_{bc}$, where f^a_{bc} are the anti-symmetric structure constants. If we know the structure constants, then we can construct the adjoint representation. E.g, $f_{abc} = \varepsilon_{abv}$ for $SU(2)$.

8.2.6 Usefulness of Lie group representations

Representation spaces are used to clarify spin and isospin using representations of $SU(2)$. Hadrons are described by representation spaces of $SU(3)_{\text{flavour}}$, for example the eight fold way. Representation spaces of the Poincaré group are classifeid in terms of a particles spin and momentum.

Historically, weak decays were seen to involve flavour changing decays and break parity in experiment. We want to construct a Lagrangian to describe weak decays and to make theoretical predictions. Parity sends left handed fields to right handed fields and so we put the left handed components of each generation of matter into a two component column vector. We call this L. As an example, for the electron and neutrino we have $L = (\nu_e, e_L)^T$. We require that the terms involving L in the Lagrangian be invariant under a $SU(2)$ gauge symmetry in order to introduce gauge fields which mediate a force between the components. We call this $SU(2)_{\text{Left}}$. The kinetic term for L contains a covariant derivative in the form given above. It contains generators for the representation of $SU(2)_{\text{L}}$. These generators have to act on L. L is two component and so we see that the generators, T^a must be 2×2 matrices. This is justthe fundamental representation of $SU(2)_{\text{L}}$. So L is said to be in the fundamental representation.

Now, in order to maximally break parity, we do not want the right handed parts of the matter fields to feel force. So we do not put the right handed parts of matter fields into a doublet. The right handed part of the matter fields do not transform under $SU(2)_{\text{L}}$. They are in the trivial or singlet representation of $SU(2)_{\text{L}}$. This is the $j = 0$ representation of $SU(2)$ discussed above.

In QCD, we introduce a new symmetry called $SU(3)_{\text{colour}}$. We say that each quark field Ψ now comes with a colour: either red, blue or green. Put the different coloured quark fields into a column vector $\chi = (\Psi_{\text{red}}, \Psi_{\text{blue}}\Psi_{\text{green}})^T$. For example, using the up-type quark $\chi\,(u_{\text{red}}, u_{\text{blue}}, u_{\text{green}})^T$. Make a Lagrangian that will be invariant udner a $SU(3)_{\text{colour}}$ gauge symmetry:

$$\mathcal{L} = \bar{\chi}\,(i\gamma^\mu D_\mu - m)\,\chi. \tag{8.19}$$

Since the χ is three component, the generators for the representation of the $SU(3)_{\text{colour}}$ symmetry will have to act on χ and so these generators will have to be 3×3 matrices. This is the fundamental representation of the group $SO(3)_{\text{colour}}$. The quark fields are said to be in the fundamental representation of $SO(3)_{\text{colour}}$. Leptons do not feel the strong force eand so transform trivially

under $SO(3)_{\text{colour}}$. This is the representation where the generators for $SO(3)_{\text{colour}}$ are $T^a = 0$. It is similar to the right handed fields in the $SU(2)_{\text{Left}}$ representation.

When working in the fundamental representation, it is ok to think of the group representation transformations just as the group transformations and the representation of the Lie algebra as just the Lie algebra.

Chapter 9

Differential Geometry

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Contents

9.1 General

Definition 9.1 (Einstein Manifold)**.** *A Riemannian manifold M is called an Einstein manifold if its Ricci tensor is proportional to the metric, i.e.*

$$Ric_g = \lambda g \tag{9.1}$$

Remark 9.1. An Einsteinian manifold, where $\lambda = 0$ is called a *Ricci-flat* manifold.

9.2 Differential Forms

"Hamiltonian mechanics cannot be understood without differential forms"

V. I. Arnold

In this section we will give a very brief introduction to differential forms and the general rules for calculus with differential forms.

9.2.1 Exterior forms

We begin with the more general notion of a *exterior form*, which is generally a poly-linear map from a vector space to an algebraic filed.

Definitions

Let $\mathbb{L}^n$ be an n-dimensional real vector space.

Definition 9.2 (Exterior algebraic form of degree k)**.** *An exterior algebraic form of degree k, also known as a k-form, is a function of k vectors which is k-linear and anti-symmetric. Namely:*

$$\omega(\lambda_1\xi_1' + \lambda_2\xi_1'', \xi_2, \cdots, \xi_k) = \lambda_1\omega(\xi_1', \xi_2, \cdots, \xi_k) + \lambda_2\omega(\xi_1'', \xi_2, \cdots, \xi_k), \tag{9.2}$$

and

$$\omega(\xi_{i_1}, \cdots, \xi_{i_k}) = (-1)^\nu\omega(\xi_1, \cdots, \xi_k) \tag{9.3}$$

with $(-1)^\nu = 1$ if the permutation $(i_1, \cdots, i_k)$ is even and $(-1)^\nu = -1$ if the same permutation is odd. Here, $\xi_i \in \mathbb{L}^n$ and $\lambda_i \in \mathbb{R}$.

The set of all k-forms in $\mathbb{L}^n$ forms a real vector space. Indeed, one has that:

$$(\omega_1 + \omega_2)(\xi) = \omega_1(\xi) + \omega_2(\xi), \quad \xi = \{\xi_1, \cdots, \xi_k\} \tag{9.4}$$

and

$$(\lambda\omega)(\xi) = \lambda\omega(\xi). \tag{9.5}$$

Since $\mathbb{L}^n$ is a vector space, we can always suppose that it is equipped with a coordinate system. Let us denote these coordinates by $x_1, \cdots, x_n$. Now, we can think of these coordinates as 1-forms so that $x_i(\xi) = \xi_i$ and that is the i-th coordinate of the vector ξ. These coordinates form a basis of 1-forms in the 1-form vector space (which is also called the *dual space* $(\mathbb{L}^n)^*$).

Any 1-form can then be written as a linear combination of basis 1-forms:

$$\omega_1 = a_1x_1 + \cdots + a_nx_n. \tag{9.6}$$

Exterior Products

Definition 9.3 (Exterior Product)**.** *An exterior product of k 1-forms $\omega_1, \omega_2, \cdots, \omega_k$ is a k-form defined by*

$$(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k)(\xi_1, \xi_2, \cdots, \xi_k) = det\omega_i(\xi_j). \tag{9.7}$$

Remark 9.2. Exterior products of basis 1-forms $x_{i_1} \wedge \cdots \wedge x_{i_k}$ with $i_1 < \cdots < i_k$ for a basis in the space of k-forms. The dimension the latter space is obviously C_n^k.

A general k-form can be written as

$$\omega^k = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \cdots i_k} x_{i_1} \wedge \cdots \wedge x_{i_k} \tag{9.8}$$

where $a_{i_1 \cdots i_k}$ are real numbers.

Definition 9.4 (Exterior product of an k-form and an l-form)**.** *The exterior product $\omega^k \wedge \omega^l$ of a k-form with an l-form on $\mathbb{L}^n$ is the $k+l$-form on $\mathbb{L}^n$ defined as:*

$$(\omega^k \wedge \omega^l)(\xi_1, \cdots, \xi_{k+l}) = \sum (-1)^\nu \omega^k(\xi_{i_1}, \cdots, \xi_{i_k}) \wedge \omega^l(\xi_{j_1}, \cdots, \xi_{j_l}) \tag{9.9}$$

where $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$ and the sum is taken over all permutations $(i_1, \cdots, i_k, j_1, \cdots, j_l)$ with $(-1)^\nu$ being +1 for even permutations and -1 for odd permutation.

One can check that this definition is consistent with the definition for exterior product of 1-forms. Furthermore it can be shown that the exterior product is distributive, associative and *skew-commutative*. The latter means that $\omega^k \wedge \omega^l = (-1)^{kl} \omega^l \wedge \omega^k$. If ω is a 1-form (or a form of any odd degree) one can easily show that $\omega \wedge \omega = 0$. We will present a brief version of the proof here:

Proof. Let ω be an exterior k-form, where k is an odd integer. Then the exterior product takes the form:

$$\omega \wedge \omega = (-1)^{k^2} \omega \wedge \omega \tag{9.10}$$

But k is odd, therefore k^2 is odd. From that we see that

$$\omega \wedge \omega = (-1)^{k^2} \omega \wedge \omega = -\omega \wedge \omega \tag{9.11}$$

From this we conclude that $\omega \wedge \omega = 0$. □

Behaviour under mappings

Let $f : \mathbb{L}^m \to \mathbb{L}^n$ be a linear map and ω^k an exterior k-form on $\mathbb{L}^n$. We can define a k-form $(f^*\omega^k)$ on $\mathbb{L}^m$ by

$$(f^*\omega^k)(\xi_1, \cdots, \xi_k) = \omega^k(f\xi_1, \cdots, f\xi_k) \tag{9.12}$$

Notice that the obtained mapping of forms f^* acts in *opposite* direction to f. Namely, $f^* : \Omega_k(\mathbb{L}^n) \to \Omega_k(\mathbb{L}^b)$, where $\Omega_k(\mathbb{L}^n)$ is the vector space of k-forms on $\mathbb{L}^n$.

9.2.2 Differential Forms

After the general discussion about exterior forms it is time to restrict ourselves to a more practical object, namely the differential form. Understanding differential forms or at least having working knowledge about the topic is paramount for the study of differential geometry and physics on curved manifolds. Although this may seem abstract at first, we urge the reader to push through the mathematical definitions and grasp the essence of the idea, that is how do we perform calculus on curved manifolds without the notion of coordinates.

Definition 9.5 (Differential Form)**.** *A differential k-form $\omega^k|_x$ at a point x of a manifold M is an exterior k-form on the tangent space TM_x to M at x, i.e., a k-linear skew-symmetric function of k vectors $\xi_1, \cdots, \xi_k$ tangent to M at x. If such a form is given at every point x of M and if it is differentiable, we say that we are given a k-form ω^k on the manifold M.*

We intoduce the coordinate basis 1-forms dx_i with $i = 1, \cdots, n$ ($dimM = n$). The notation dx_i is used for the exterior forms emphasizes that these basis forms act on TM_x at a given point x of M. In the neighborhood of x one can always write the general differential k-form as

$$\omega^k = \sum_{i_1 < \cdots < i_k} a_{i_1, \cdots, i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \tag{9.13}$$

where $a_{i_1 \cdots i_k}$ are smooth functions of x. Let's give a simple example.

Example 5. *The differential of some scalar function $f(x)$ defined on a manifold M is a differential 1-form. We introduce*

$$df = \sum_{k=1}^{n} \left.\frac{\partial f}{\partial x_k}\right|_x dx_k, \tag{9.14}$$

where dx_k are basis differential 1-forms. The value of this 1-form on a vector $\xi \in TM_x$ is given by

$$df(\xi) = \sum_{k=1}^{n} \left.\frac{\partial f}{\partial x_k}\right|_x \xi_k. \tag{9.15}$$

Behaviour under mappings

Let $f : M \to N$ be a differentiable map of a smooth manifold M to a smooth manifold N, and let ω be a differential k-form o N. The mapping f induces the mapping $f_* : TM_x \to TM_{f(x)}$ of tangent spaces. The latter mapping f_* is called the *differential* of the map f. The mapping f_* is a mapping of linear spaces and gives rise to the mapping of forms defined on corresponding tangent spaces. As a result a well-defined differential k-form $(f^*\omega)$ exists on M:

$$(f^*\omega)(\xi_1, \cdots, \xi_k) = \omega(f_*\xi_1, \cdots, f_*\xi_k). \tag{9.16}$$

9.2.3 Integration of Differential Forms over Chains

Integration of k-form over k-dimensional cell

Let D be a bounded convex polyhedron in $\mathbb{L}^k$ and $x_1, \cdots, x_k$ an oriented coordinate system on $\mathbb{L}^k$. Any differential k-form on $\mathbb{L}^k$ can be written as $\omega^k = \phi(x) dx_1 \wedge \cdots \wedge dx_k$, where $\phi(x)$ is a

differentiable function on $\mathbb{R}^k$. We define the integral of the form ω^k over D as the integral of the function $\phi(x)$:

$$\int_D \omega_k = \int_D \phi(x) dx_1 dx_2 \cdots dx_k. \tag{9.17}$$

Definition 9.6 (k-dimensional cell)**.** *A k-dimensional cell σ of an n-dimensional manifold M is a polyhedron D in $\mathbb{L}^n$ with a differentiable map $f : D \to M$.*

One can think about σ as a "curvilinear polyhedron" - the image of D on M. If ω is a differentiable k-form on M, we define the integral of a form over the cell σ as

$$\int_\sigma \omega = \int_D f^* \omega, \tag{9.18}$$

where f^* is a mapping of k-forms induced by f.

The cell σ inherits an orientation from the orientation of $\mathbb{L}^k$. The k-dimensional cell which differs from σ only by the choice of orientation is called the *negative* of σ and is denoted by $-\sigma$ or by $(-1)\sigma$. One can show that under a change of orientation the integral changes sign:

$$\int_{-\sigma} \omega = -\int_\sigma \omega. \tag{9.19}$$

Chains and the Boundary Operator

It is convenient to generalize our definition of the integral of a form over *cell* to the integral over a *chain*.

Definition 9.7 (Chain of a Manifold)**.** *A chain of dimension k on a manifold M consists of a finite collection of k-dimensional oriented cells $\sigma_1, \cdots, \sigma_r$ in M and integers $m_1, \cdots, m_r$ called multiplicities. A chains is denoted by*

$$c_k = m_1\sigma_1 + \cdots + m_r\sigma_r. \tag{9.20}$$

One can introduce the structure of a commutative group on a set of k-chains on M with natural definitions of addition of chains $c_k + b_k$.

Definition 9.8 (Boundary Operator)**.** *The boundary of a convex oriented k-polyhedron D on $\mathbb{L}^k$ is the $(k-1)$-chain ∂D on $\mathbb{L}^k$ defined as*

$$\partial D = \sum_i \sigma_i \tag{9.21}$$

where the cells σ_i are the $(k-1)$-dimensional faces of D with orientations inherited from the orientation of $\mathbb{L}^k$.

One can easily extend this definition to the definition of the *boundary of a cell* $\partial\sigma$ on M and then to the boundary of a chain. Indeed, defining:

Definition 9.9 (Boudary of a Chain)**.**

$$\partial c_k = m_1\partial\sigma_1 + \cdots + m_r\partial\sigma_r \tag{9.22}$$

We can see that ∂c_k is a $(k-1)$-chain on M. Additionally, we define a 0-chain as a collection of points with multiplicities. Furthermore we define the boundary of an oriented interval $\vec{AB}$ as $B - A$. The boundary of a point is empty.

It is straightforward to show that the boundary of the boundary of a cell is zero. Therefore

$$\partial(\partial c_k) = 0 \tag{9.23}$$

for any k-chain c_k. We denote this property as

$$\partial\partial = \partial^2 = 0 \tag{9.24}$$

Integration of a k-form over a k-chain

An integral of a k-form over a k-chain is then defined as

$$\int_{c_k} \omega^k = \sum m_i \int_{\sigma_i} \omega^k. \tag{9.25}$$

9.2.4 Exterior Differentiation and Stokes Formula

Definition 9.10 (Exterior Derivative of a Form)**.** *An exterior derivative of a differential k-form ω is a $(k+1)$-form $\Omega = d\omega^k$. Given a set of coordinates $\{dx_{i_j}\}$, we have:*

$$\Omega = d\omega^k = \sum da_{i_1 \cdots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{9.26}$$

implying that we have defined ω^k as in (9.13)*. Here, da is a 1-form, the differential of the function $a(x)$.*

One can show that the definition does not actually depend on the choice of coordinates. We can think of the 1-form given by the differential of a scalar function as of an external derivative of a 0-form. It is easy to show that $d(df) = 0$, if f belongs to the set of 0-forms. Indeed

Proof. If f belongs to the set of 0-forms, then by definition

$$df = \sum \frac{\partial a}{\partial x^i} dx^i, \tag{9.27}$$

which is a 1-form. Taking an exterior derivative again yields

$$\begin{aligned} d(df) &= d\left(\sum \frac{\partial a}{\partial x^i} dx^i\right) = \\ &= \sum \frac{\partial^2}{\partial x^i \partial x^j} dx^i \wedge x^j = \\ &= \frac{\partial^2 g}{dx^i dx^j} dx^i \wedge dx^j + \frac{\partial^2 g}{dx^j dx^i} dx^j \wedge dx^i \\ &= 0 \end{aligned} \tag{9.28}$$

Here we have used that $\frac{\partial g^2}{dx^i dx^j} = \frac{\partial g^2}{dx^j dx^i}$ and $dx^i \wedge dx^j = -dx^j \wedge dx^i$. □

Using this result, one can show that it holds for forms of any degree.
Another useful formula is that for differentiating an exterior product of form:

$$d(\omega^k \wedge \omega^l) = d\omega^k \wedge d\omega^l + (-1)^k \omega^k \wedge d\omega^l \tag{9.29}$$

Lastly, if $f : M \to N$ is a smooth map and ω is a k-form on N, we have

$$f^*(d\omega) = d(f^*\omega). \tag{9.30}$$

Stokes' Formula

One of the most-important formulae in differential geometry is the *Newton-Leibniz-Gauss-Green-Ostrogradski-Stokes-Poincaré* formula:

$$\boxed{\int_{\partial c} \omega = \int_c d\omega} \tag{9.31}$$

where c is any $(k+1)$-chain on a manifold M and ω is any k-form on M. In the case when the boundary $\partial c = 0$, we have $\int_c d\omega = 0$, which corresponds to integration of a complete derivative over a closed surface.

9.2.5 Homologies and Cohomologies

Closed and Exact Forms

Definition 9.11 (Closed Form)**.** *A differential form ω on a manifold M is said to be closed if $d\omega = 0$.*

In particular, on a 3D Riemannian manifold, we have $d\omega^2_{\vec{A}} = \left(\nabla \cdot \vec{A}\right) \omega^3 = 0$, which is equivalent to $\left(\nabla \cdot \vec{A}\right) = 0$, i.e the corresponding vector field is divergenless. We can apply Stokes' formula for a closed form, getting

$$\int_{\partial c_{k+1}} \omega^k = 0 \quad \text{if } dw^k = 0. \tag{9.32}$$

Definition 9.12 (Exact Form)**.** *A differential form ω on a manifold M is said to be exact if there exists such a differential form μ that $\omega = d\mu$.*

Since $d(d\omega) = 0$ as we've proven all exact forms are closed. However, there are some closed forms which are not exact. Let's give a short example.

Example 6. *Consider the circle S^1, parametrized by an angle $\phi \in [0, 2\pi]$. One can introduce the 1-form ω^1, defined by $\omega^1(\partial_t \gamma) = \partial_t \gamma$, where $\partial_t \gamma$ is the "velocity" along the path $\phi = \gamma(t)$. Obviously, this "velocity", belongs to the tangent space of S^1 at the point with coordinate ϕ. The form is closed - $d\omega$ is a 2-form, and we can't have 2-forms on a one-dimensional manifold. However:*

$$\int_{S^1} \omega^1 = \int_0^T dt \partial_t \gamma = 2\pi, \tag{9.33}$$

the length of the circle. Although the boundary ∂S^1 is zero, the integral is not zero, and therefore ω^1 is not exact.

One can notice that locally the introduced 1-form can be written as $\omega^1 = d\phi$, which can look like a contradiction. It is easy to see where the problem lies - writing $\omega^1 = d\phi$ is not valid for $\phi = 0$. We've come across an example of a general result, namely

Theorem 9.1 (Poincaré's Lemma)**.** *Any closed form is locally exact.*

The existence of locally but not globally exact closed forms is related to some topological properties of the underlying manifold M.

Cycles and Boundaries

Definition 9.13 (Cycle on a Manifold)**.** *A cycle on a manifold M is a chain whose boundary is equal to zero.*

Using Stokes theorem, we have

$$\int_{c_{k+1}} d\omega^k = 0 \quad \text{if } \partial c_{k+1} = 0. \tag{9.34}$$

Chains that can be considered boundaries of some other chains are called *boundaries*. Since $\partial\partial = 0$, all boundaries are cycles. However, not all cycles are boundaries. The existence of cycles that are not boundaries is again related to some topological properties of the manifold. A fairly simple example is found on the 2-torus. The 2-torus is the direct product of 2 circles - $T^2 = S^1 \times S^1$. Each of the S^1 is a cycle, but none of them are boundaries.

Homologies and Cohomologies

The set of all k-forms on M is a vector space, the set of all *closed* k-forms is a subspace of that space and the set of differentials of $(k-1)$-forms (the exact k-forms) are a subspace of the subspace of closed forms. We can now define:

Definition 9.14. *The quotient space:*

$$\frac{(\textit{closed forms})}{(\textit{exact forms})} = H^k(M, \mathbb{R}) \tag{9.35}$$

is called the k-th cohomology group of the manifold M. An element of this group is a class of closed forms, differing from each other only by an exact form.

For the circle S^1, we have $H^1(S^1, \mathbb{R}) = \mathbb{R}$.

Definition 9.15 (Betti Number)**.** *The dimension of H^k is called the k-th Betti number of M.*

Obviously, the first Betti number of S^1 is 1. The cohomology groups of M are important *topological* properties of M.

Definition 9.16 (Homologous Cycles)**.** *Let us now consider two k-cyclces a and b, such that their difference is a boundary of a $(k+1)$-chain, i.e. $a - b = \partial c_{k+1}$. Such cycles are called homologous.*

Let us have two k-cycles, a and b, homologous to each other and a closed form ω^k. From (9.32) we can see that

$$\int_a \omega^k = \int_b \omega^k. \tag{9.36}$$

In other words, homologous cycles can be replaced with one another for integration paths.

Definition 9.17 (Homology Group)**.** *The quotient group*

$$\frac{(cycles)}{(boundaries)} = H_k(M) \tag{9.37}$$

is called the k-th homology group of M. An element of this group is a class of cycles homologous to each other. The rank of this group is also equal to the k-th Betti number of M.

9.2.6 Homologies and Homotopies

There are important relations between homology and homotopy groups of a topological space M.

Let us suppose that $\pi_1(M)$ and $H_1(M)$ are the fundamental and the first homology group of M, respectively. Then $H_1(M) = \pi_1(M)/[\pi_1, \pi_1]$, where $[\pi_1, \pi_1]$ is the commutator in the fundamental group. In particular, if $\pi_1(M)$ is Abelian, then $\pi_1(M) = H_1(M)$. For the higher homotopy groups, there is another result, known as the Gurevich theorem.

Theorem 9.2 (Gurevich theorem)**.** *If $\pi_k(M) = 0$ for all $k < n$, then*

$$\pi_n(M) = H_n(M). \tag{9.38}$$

As a general rule of thumb, homology (and cohomology) groups are usually easier to calculate than homotopy groups.

9.3 Contact Manifolds

Definition 9.18 (Riemannian Cone)**.** *Given a Riemannian manifold (M, g), its Riemannian cone is a product*

$$(M \times \mathbb{R}^{>0}) \tag{9.39}$$

of M with the half-line $\mathbb{R}^{>0}$ equipped with the cone metric

$$t^2 g + dt^2, \tag{9.40}$$

where t is a parameter in $\mathbb{R}^{>0}$

Definition 9.19 (Contact Manifold)**.** *A manifold M, equipped with a 1-form θ is contact if and only if the 2-form*

$$t^2 d\theta + 2tdt \cdot \theta \tag{9.41}$$

on its cone is symplectic.

Definition 9.20 (Sasakian Manifold)**.** *A contact Riemannian manifold is called a Sasakian manifold, if its Riemannian cone with the cone metric is a Kähler manifold with Kähler form*

$$t^2 d\theta + 2tdt \cdot \theta. \tag{9.42}$$

Example 7. *Consider the manifold $\mathbb{R}^{2n+1}$ with coordinates $(\vec{x}, \vec{y}, z)$, endowed with contact form*

$$\theta = \frac{1}{2} dz + \Sigma_i y_i dx_i \tag{9.43}$$

and Riemannian metric

$$g = \Sigma_i (dx_i)^2 + (dy_i)^2 + \theta^2 \tag{9.44}$$

Definition 9.21 (Sasaki-Einstein Manifold)**.** *A Sasaki-Einstein manifold is a Riemanian manifold (S, g) that is both Sasakian and Einstein*

Example 8. *The odd dimensional sphere S^{2n-1}, equipped with its standard Einstein metric is a Sasaki-Einstein manifold. In this case, the Kähler cone is $\mathbb{C}^2 \backslash \{0\}$, equipped with its flat metric.*

9.4 Symplectic Geometry

Theorem 9.3 (Duistermaat-Heckman Formula)**.** *For a compact symplectic manifold M of dimension $2n$ with symplectic form ω and with a Hamiltonian $U(1)$ action whose moment map is denoted by μ, the following formula holds:*

$$\int_M \frac{\omega^n}{n!} e^{-\mu} = \sum_i \frac{e^{-\mu(x_i)}}{e(x_i)} \tag{9.45}$$

Here, x_i are the fixed points of the $U(1)$ action and they are assumed to be isolated, and $e(x_i)$ is the product of the weights of the $U(1)$ action on the tangent space at x_i.

9.5 Complex Manifolds

Definition 9.22 (Hermitian Metric)**.** *If a Riemannian metric g of a complex manifold M satisfies*

$$g_p(J_p X, J_p Y) = g_p(X, Y) \tag{9.46}$$

at each point $p \in M$ and for any $X, Y \in T_p M$, g is said to be a Hermitian metric. Here, J_p denotes the almost complex structure on M.

Definition 9.23 (Hermitian Manifold)**.** *The pair (M, g) is called a Hermitian manifold*

Theorem 9.4. *A complex manifold always admits a Hermitian metric.*

Proof. Let g be any Riemannian metric of a complex manifold M. Define a new metric $\hat{g}$ by

$$\hat{g}_p(X, Y) \equiv \frac{1}{2} \left[g_p(X, Y) + g_p(J_p X, J_p Y) \right]. \tag{9.47}$$

Clearly $\hat{g}_p(J_p X, J_p Y) = \hat{g}_p(X, Y)$. Moreover, $\hat{g}$ is positive definite provided that g is. Hence, $\hat{g}$ is a Hermitian metric on M □

Definition 9.24 (Kähler Form)**.** *Let (M, g) be a Hermitian manifold. Define a tensor field Ω whose action on $X, Y \in T_pM$ is*

$$\Omega_p(X, Y) = g_p(J_pX, Y) \tag{9.48}$$

Note that Ω is anti-symmetric, $\Omega(X, Y) = g(JX, Y) = g(J^2X, JY) = -g(JY, X) = -\Omega(Y, X)$. Hence, Ω defines a two-form, called the Khäler form of a Hermitian metric g.

Definition 9.25 (Kähler Manifold)**.** *A Kähler manifold is a Hermitian manifold (M, g) whose Kähler form Ω is closed, $d\Omega = 0$. The metric g is called the Kähler metric of M.*

Remark 9.3. Not all complex manifolds admit Kähler metrics

Theorem 9.5. *A Hermitian manifold (M, g) is a Kähler manifold if and only if the almost complex structure J satisfies*

$$\nabla_\mu J = 0 \tag{9.49}$$

where ∇_μ is the Levi-Cevita connection associated with g.

Proof. We first note that for any r-form ω, $\mathrm{d}\omega$ is written as

$$d\omega = \nabla\omega \equiv \frac{1}{r!}\nabla_\mu\omega_{\nu_1\cdots\nu_r}dx^\mu \wedge dx^{\nu_1} \wedge \cdots \wedge dx^\nu \tag{9.50}$$

Now we prove that $\nabla_\mu J = 0$ if and only if $\nabla_\mu\Omega = 0$. We verify the following equalities:

$$\begin{aligned}
(\nabla_Z\Omega)(X, Y) &= \nabla_Z\left[\Omega(X, Y)\right] - \Omega(\nabla_ZX, Y) - \Omega(X, \nabla_zY) && (9.51)\\
&= \nabla_Z\left[g(JX, Y)\right] - g(J\nabla_ZX, Y) - g(JX, \nabla_ZY) && (9.52)\\
&= (\nabla_Zg)(JX, Y) + g(\nabla_ZJX, Y) - g(J\nabla_ZX, Y) && (9.53)\\
&= g(\nabla_ZJX - J\nabla_ZX, Y) = g((\nabla_ZJ)X, y) && (9.54)
\end{aligned}$$

where $\nabla_Zg = 0$ has been used. Since this is true for any X, Y, Z, if follows that $\nabla_Z\Omega = 0$ if and only if $\nabla_ZJ = 0$. □

The last theorem shows that the Riemann structure is compatable with the Hermitian structure in the Kähler manifold.

We can also characterize Kähler manifolds as Hermitian manifolds for which the Cristoffel symbols of the Levi-Chevita connection are pure. In other words, Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$ may be non zero, but all "mixed" symbols like $\Gamma^{\bar{i}}_{jk}$, for example, vanish. This means that (anti-)holomorphic vectors get parallel transported to (anti-)holomorphic vectors.

Kähler manifolds are manifolds on which we can always find a *holomorphic* change of coordinates which, at some given point, sets the metric to its cannonical form, and its first derivatives to zero.

Equivalently, an n-dimensional Kähler manifold are precisely $2n$-dimensional Riemannian manifolds with holonomy group contained in $U(1)$

Definition 9.26 (Hopf Surface)**.** *Let $\mathbb{Z}$ act on $\mathbb{C}^n\backslash\{0\}$ by $(z_1, \cdots, z_n) \to \left(\lambda^k z_1, \cdots, \lambda^k z_n\right)$ for $k \in \mathbb{Z}$. For $0 < \lambda < 1$ the action is free and discrete. The quotient complex manifold*

$X = (\mathbb{C}^n \backslash \{0\})/\mathbb{Z}$ is diffeomorphic to $S^1 \times S^{2n-1}$. For $n = 1$ this manifold is isomorphic to a complex torus $\mathbb{C}/\Gamma$. The lattice Γ can be determined explicitly.
In other words, a Hopf manifold is obtained as a quotient of the complex vector space (with zero deleted) $\mathbb{C}^n \backslash \{0\}$ by a free action of the group $\Gamma \cong \mathbb{Z}$ of integers, with the generator γ of Γ acting by holomorphic contractions.

9.6 Gauss-Bonet Theorem

The Gauss-Bonet theorem bridges the gap between topology and differential geometry. Its importance lies in relating geometrical information of a surface to a purely topological characteristic, which has resulted in varied and powerful applications. Gauss first proved this theorem in 1827, for the case of a hyperbolic triangle. It established a remarkable invariant relating curvature to the notion of angle withing the surface. However, with the developments in topology in the 19th and 20th centuries, this theorem has become an invaluable piece of modern mathematics.

9.6.1 Topological preliminaries

Definition 9.27 (A surface)**.** *S is a two dimensional sub-manifold of Eucledian space. The word **regular** ensures differentiability in a neighborhood of each point of the surface; however, we note that the full definition of this term includes other criteria which will be inconsequential for us here*

Definition 9.28 (A triangulation)**.** *of a regular surface R is a finite collection J of triangles $\{T_j\}\, j = 1^n$ such that*

$$\bigcup_{j=1}^{n} T_j = R \tag{9.55}$$

and the only possible intersection of T_i and T_j with $i \neq j$ is a common edge or a common vertex.

Proposition 9.1. *Every regular surface S admits a triangulation.*

Definition 9.29 (Orientable surface)**.** *A surface is orientable if all of the triangles in a given triangulation have compatable orientations (i.e, they preserve the notion of clockwise-anticlockwise as a path moves between triangles).*

Definition 9.30 (Euler characteristic)**.** *$\chi(s)$ of a surface S is defined as*

$$\chi(S) = V - E + F, \tag{9.56}$$

where V, E, F are the number of vertices, edges and faces of a given triangulation of S.

Proposition 9.2. *The Euler characteristic is a topological invariant.*

Remark 9.4. All surfaces (in $N = 2$ dimensions) are classified up to homeomorphism by their genus, which is closely related to the Euler characteristic. Essentially, the number of holes in a surface, classifies it topologically.

Definition 9.31 (Fundamental group (topology)). *Let X be a space and x_0 be a point in X. A path in X that begins and ends at x_0 is a **loop** based at x_0. The **fundamental group** $\pi_1(X, x_0)$ is the seth of path homotopy classes of loops based at x_0, with some operation, namely concatenation.*

In order to probe the so-called Jordan Curve Theorem, we list several lemmas without proof.

Proposition 9.3 (Jordan Separation Theorem). *Let C be a simple closed curve in S^2. Then $S^2 - C$ is not connected.*

Proposition 9.4. *Lex X be the union of open sets U and V. Suppose*

$$U \cup V = A \cup A' \cup B \tag{9.57}$$

that is, it's a union of three disjoint open sets. Given

$$a \in A, a' \in A', b \in B \tag{9.58}$$

path connected in U and V, then $\pi_1(X, a)$ is not infinite cyclic (i.e., not isomorphic to $\mathbb{Z}, +, \nvdash$).

Proposition 9.5. *(A nonseparation theorem) Let A be an arc in S^2. Then $S^2 - A$ is connected.*

Proposition 9.6 (The Jordan Curve Theorem). *Let C be a simple closed curve in S^2. Then, $S^2 - C$ has two and only two components W_1 and W_2, of which C is the common boundary.*

Proof. First, we show that $S^2 - C$ has exactly two components. Using the proposition above, we write C as the union of two arcs C_1 and C_2 which intersect in exactly two points p and q. Let X be $S^2 - p - q$. Let

$$U \equiv S^2 - C_1 \tag{9.59}$$

and

$$V \equiv S^2 - C_2 \tag{9.60}$$

We note that

$$X = U \cup V \tag{9.61}$$

and

$$S^2 - C = U \cap V, \tag{9.62}$$

which we know has at least two components by proposition 9.4. Suppose, in order to find a contradiction, that $S^2 - C$ has more than two components. Let A and A' be two components, and call the union of the rest of the components B. Each of these sets is open because $S^2 - C$ is (locally) connected. Given,

$$a \in A, a' \in A', b \in B, \tag{9.63}$$

by the nonseparation theorem above, U and V are path-connected, because no arc separates S^2. Thus $\pi_1(X, a)$ is not infinite cyclic. However, we recall that $X = S^2 - p - q$ is homeomorphic to the punctured plane, which has infinite cyclic fundamental group, which is a contradiction.

Next, we show that C is the common boundary of W_1 and W_2, which are the regions that we wish to show are separated by the curve. We note that W_1 and W_2 must be open because

neither can contain a limit point of the other (because S^2 is locally connected). Therefore, since S^2 is the disjoint union of W_1,W_2 and C, it follows that

$$\bar{W_1} - W_1 \quad \text{aswellas} \quad \bar{W_2} - W_2, \tag{9.64}$$

must be contained in C. Now, we show the converse, namely that given a point $x \in C$ we know that $x \in \bar{W_1} - W_1$.

Thus, let U be a neighborhood of x. Then, using the fact that C is homeomorphic to S^1, we break it up into two curves C_1 and C_2 such that C_1 is entirely contained within U. We let a and b to be points of W_1 and W_2, respectively. Again, using the nonseparation theorem, we know that we can find a path in $S^2 - C$, call it α, that connects a and b, because C_2 does not separate S^2. Therefore, $\alpha(I)$ must contain a point y such that

$$y \in \bar{W_1} - W_1, \quad y \in C. \tag{9.65}$$

We know that such y must exist because otherwise $\alpha(I)$, which we know is a connected set, would lie in the disjoint union of open sets W_1 and $S^2 - \bar{W_1}$ but also intersect both of them. Because we know that

$$C \supset \bar{W_1} - W_1, \tag{9.66}$$

if follows that $y \in C$.. Now, as we know that α does not intersect C_2, the point y must be on C_1 and thusly

$$y \in U \tag{9.67}$$

Therefore, as we wished to show, U intersects $\bar{W_1} - W_1$ in the point y, and thusly we have shown that C is the common boundary of W_1 and W_2.

■

□

Definition 9.32 (Homotopic maps)**.** *We say that f and g, continuous maps from X to Y are homotopic if there exists a map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. This can be though of as a continuous deformation of one map into the other.*

Definition 9.33 (Convex maps)**.** *A convex map is a map such that for any line connecting two points of the map, every point on that line is contained in the map.*

Definition 9.34 (Rotation index)**.** *If $\alpha : [a, b] \to \mathbb{R}^2$ is a convex map, then the rotation index of α is*

$$\int_a^b \kappa(s)ds = \theta(b) - \theta(a) = 2\pi I(9.68)$$

where $\kappa(s)$ is the Gaussian curvature.

Remark 9.5. We note that the rotation index is homotopy invariant over α.

Definition 9.35 (Secant Map)**.** *$\psi : T \to S^1$ is the secant map from the triangular region*

$$T = \{(t_1, t_2) \in [0, 1] \times [0, l] \, ; 0 \leq t_1 \leq t_2 \leq l\}, \tag{9.69}$$

onto the 1-spehre:

$$\Psi(x) = \begin{cases} \frac{\alpha(t_2)-\alpha(t_1)}{|\alpha(t_2)-\alpha(t_1)|} & t_1 \neq t_2, (t_1, t_2) \in T \setminus \{0, l\} \\ \frac{\alpha'(t)}{|\alpha'(t)|} & t_1 = t_2 \\ \frac{-\alpha'(0)}{|\alpha(0)|} & (t_1, t_2) = (0, l) \end{cases} \tag{9.70}$$

We now present a proof of the Turning Tangents theorem. Roughly, this theorem states that a loop on a surfaces turns 2π radians. This allows us to index the rotation of the boundary of a region of a surface, which is a key piece of the proof of the local Gauss-Bonet theorem.

Theorem 9.6 (Turning Tangents)**.** *If $\beta : [0, l] \to \mathbb{R}^2$ is a simple, closed, regular, plane curve, then the rotation index of β is ± 1.*

Proof. We assume the Jordan Curve Theorem as proven. Now, we take some line b and position it such that it is tangent to β at the point p. We note that due to this choice of b, the curve lies entirely on one side of the line. Then we choose a parametrization $\alpha \in [0, l]$ such that

$$\alpha(0) = p. \tag{9.71}$$

We observe that the secant map Ψ is a continuous function, which is trivially verified. Now, we let A,B,C denote the vertices of T at $(0, 0), (0, l), (l, l)$, respectively. We notice that the side AC is simply the map of tangents to α into S^1, the degree of which is the rotation of α. Next, we recognize that the tangent map must be homotopic to the restriction of ψ to the other two sides of the triangle AB and BC. Thus, it suffices to show that the rotation index of Ψ on

$$AB \cup BC = \pm 1. \tag{9.72}$$

To do so, we assume that we have an orientation such that the angle from $\alpha'(0)$ to $-\alpha'(0)$ is π. This implies that ψ on AB, i.e.,

$$[0, 0] \to [0, l] \tag{9.73}$$

covers half of S^1 in the positive direction, and Ψ on BC covers the other half. Since the rotation index is clearly 1 for S^1, we're done. We note we can reverse the assumed orientation to obtain the opposite index. Therefore the rotation index of β is ± 1. □

9.6.2 Local Gauss-Bonnet Theorem

Remark 9.6. We will now prove the local case of Gauss-Bonnet. However, in order to do this, we first need to define some key resources. These notions of curvature tell us roughly what a surface looks like both locally and globally. The theorem tells us that there is a remarkable invariance on surfaces that is balanced by the total curvature and geodesic curvature. Gauss proved this by considering the hyperbolic triangle.

Definition 9.36 (Simple region)**.** *A simple region R of a surface S is a region such that R is homeomorphic to the disk.*

Definition 9.37. *Given a parametrization of a surface, $\boldsymbol{x} : U \to S$, we have the following quantities*

$$E = \langle x_u, x_u \rangle, F = \langle x_u, x_v \rangle, G = \langle x_v, x_v \rangle \tag{9.74}$$

These are the coefficients of the first fundamental form of a surface.

Definition 9.38 (Orthogonal parametrization)**.** *Given a parametrization of a surface,*

$$\boldsymbol{x} : U \to S, \tag{9.75}$$

we call that parametrization orthogonal if $F = 0$*.*

Definition 9.39 (Geodesic Curvature)**.** *The geodesic curvature* k_g *of a curve is a measure of the amount of deviance of the curve from the shortest arc between two points on a surface.*

Definition 9.40 (Gaussian Curvature)**.** *The Gaussian curvature* κ *of a surface is an intrinsic measure of the curvature of a surface at a point. It is calculated by considering the maximal and minimal curvatures on the surface at a point. Formally, these values are multiplied to give* κ*.*

Now we can state and prove the local variant of the Gauss-Bonnet theorem.

Theorem 9.7 (**Gauss-Bonnet, Local**)**.** *Let* $R \in \boldsymbol{x}(U)$ *be a simple region of* S *with orthogonal parametrization, and choose* $\alpha : I \to S$ *such that* $\alpha(I) = \partial R$*. Assume that* α *is a positively oriented and parametrized piecewise by arc-length* s_i*. Let* $\{\theta_i\}_{i=0}^{k}$ *be the external angles of* α *at the vertices* $\{\alpha(s_i)\}_{i=0}^{k}$*, then*

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_{g(s)} ds + \int\int_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi \tag{9.76}$$

Proof. We first let $u = u(s)$ and $v = v(s)$ be the expression of α in the parametrization $\mathbf{x}$. We recall that

$$k_{g(s)} = \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) + \frac{d\varphi}{ds}, \tag{9.77}$$

where we denote the differentiable function that measures the positive angle from $\mathbf{x}$ to $\alpha'(s)$ in $[s_i, s_{i+1}]$ as $\psi(s_i)$. We now integrate the above expression, adding up the values for each $[s_i, s_{i+1}]$:

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_{g(s)} = \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) ds + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds, \tag{9.78}$$

now using the Gauss-Green theorem in the uv-plane on the right hand side of the above equation, we obtain the expression:

$$\iint_{\mathbf{x}^{-1}(R)} \frac{E_v}{2\sqrt{EG_v}} + \frac{G_u}{2\sqrt{EG_u}} du dv + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds. \tag{9.79}$$

We note that by the Gauss Formula,

$$-\iint_{\mathbf{x}^{-1}(R)} \kappa \sqrt{EG} du dv = -\iint_R \kappa d\sigma, \tag{9.80}$$

also, recalling the Theorem of Turning Tangents, we know that,

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds = \sum_{i=0} k \int_{s_i}^{s_{i+1}} \varphi_i(s_i + 1) - \varphi_i(s_i) = \pm 2\pi - \sum_{i=0}^{k} \theta_i, \tag{9.81}$$

which we get because the theorem does not account for the discontinuities along the curve at the theta values. As we have assumed a positive orientation, we have

$$\sum_{i=0}^{k} \int_{s_i}^{s_{s+1}} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi, \tag{9.82}$$

we note that we can obtain the opposite sign by assuming the opposite orientation, and thus we have proven the local case of the Gauss-Bonnet theorem. □

9.6.3 Global Gauss-Bonnet Theorem

Remark 9.7. We have proven the local case of this theorem and the global theorem tells us similar information. We prove this generalization by using the local theorem in each triangular region of our triangulation of the given surface. This theorem leads to a series of very deep corollaries.

Theorem 9.8. *Let $R \in S$ be a regular regin of an oriented surface. Let ∂R be made up by closed, piecewise, simple, regular curves*

$$C_1, \cdots C_n, \tag{9.83}$$

then

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi\chi(R). \tag{9.84}$$

Proof. Let J denote a triangulation of R such that each triangle T_j is contained in a neighborhood of orthogonal parametrization compatible with the orientation of S. We note that such a triangulation exists by the proposition proven above. Now, we simply apply the local Gauss-Bonnet theorem to each T_j of the above triangulation, and we have:

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{j,k=1}^{F,3} \theta_{jk} = 2\pi F, \tag{9.85}$$

where the indexing of each theta accounts for each angle of the triangles in J. We note that F here is the number of faces in our triangulation. We denote the interior angles of the triangles by $\varphi_{jk} = \pi - \theta_{jk}$. We calculate, in general, that,

$$\sum_{j,k} \theta_{jk} = 3\pi F - \sum_{j,k=1} \varphi_{jk}. \tag{9.86}$$

Now, we introduce notation to assist in counting the vertices and edges of our triangulation, so the number of external/internal edge and vertices are V_e, E_e and V_i, E_i respectively. Since the C_i are closed, however, we know $V_e = E_e$, and thus inductively, $3F = 2E_i + E_e$. This implies,

$$\sum_{j,k} \theta_{jk} = 2\pi E_i + \pi E_e - \sum_{j,k=1} \varphi_{jk}. \tag{9.87}$$

We not that the vertices must belong to either some T_j or a C, so $V_e = V_{et} + V_{ec}$, and then since the sum of the angles around each internal vertex is 2π,

$$\sum_{j,k=1} \theta_{jk} = 2\pi E - 2\pi V + \sum_i \theta_i. \tag{9.88}$$

Now, we collect the terms to find:

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi(F - E + V). \tag{9.89}$$

But, by definition of a triangulation $F - E + V = \chi(R)$, thus

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi\chi(R). \tag{9.90}$$

□

9.6.4 Applications

Remark 9.8. If R is a simple region, then

$$\chi(R) = 1.$$

Proof. We note that R is homeomorphic to the disk and thus homeomorphic to a single triangle. Thus, by the definition of the Euler characteristic,

$$\chi(R) = V - E + F = 3 - 3 + 1 = 1. \tag{9.91}$$

□

Corollary 9.1. *If R is a simple region of a surface S, then,*

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_C \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi. \tag{9.92}$$

The next corollary is often the presented form of the Gauss-Bonnet theorem. We simply take into account that the boundary of a compact surface is empty, and therefore it can be thought of as a region with empty boundary. Thusly, the terms that depend on the boundary drop out of the equation and we are left with the following powerful statement.

Corollary 9.2. *If S is an orientable compact surface, then:*

$$\iint_S \kappa d\sigma = 2\pi\chi(S). \tag{9.93}$$

Remark 9.9. The Euler characteristic $\chi(S)$ can be written as

$$\chi(S) = 2 - 2g \tag{9.94}$$

where g is the genus of a surface, i.e., the number of holes in the surface.

Remark 9.10. This elegant formulation of the theorem introduces the notion that the total curvature depends exclusively on the topological characteristic - the genus. This implies that all possible embeddings of a surface of genus g have the same total curvature, which is a hightly non-intuitive result.

Definition 9.41 (Index at a singular point)**.** *The index I of v at the singular point p is defined as follows. Let $\boldsymbol{x}$ be an orthogonal parametrization such that $\boldsymbol{x}(0,0) = p$ and the orientation is compatable with that of the surface S. Let $\alpha : [0, l] \to S$ be a closed simple regular parametrized curve such that it is the boundary of a simple region $R \subset S$, where the only singular point in R is p. Now, we have a function $\varphi(t)$ with $t \in [0, l]$, such that it measures the angle from $\boldsymbol{x_u}$ to the restriction of v to α, then:*

$$2\pi I = \varphi(l) - \varphi(0) = \int_0^l \frac{d\varphi}{dt} dt. \tag{9.95}$$

Proposition 9.7. *The index is independent of the choice of parametrization.*

Proposition 9.8. *The index is independent of the choice of α.*

Theorem 9.9 (Poincaré-Hopf Index Theorem)**.** *The sum of the indices of a differentiable vector field v with isolated singular points on a compact surface S is equal to the Euler characteristic of S*

Proof. Let $S \subset \mathbb{R}^3$ be a compact surface and v a differentiable vector field with exclusively isolated singular points. We notice that, due to compactness, these singular points must be finite in number otherwise there would exist a non-isolated singular point as a limit point of the others. We let $\{\mathbf{x_a}\}$ be a family of parametrizations such taht each is compatable with tje orientation of S. Now, we let J be a triangulation of S with the condtitions that each T in J is contained in a coordinate neighborhood of $\{\mathbf{x_a}\}$, each triangle T contains at most one singular point, and each triangle is positively oriented with no singular points on its boundary. Now we apply the local Gauss-Bonnet theorem to each triangle and sum up the result. However, we recall that each triangle appears twice in this formulation, in opposite orientation, so we have:

$$\iint_S \kappa d\sigma - 2\pi \sum_{i=1}^{k} I_i = 0 \tag{9.96}$$

to which we apply the most general form of the Gauss-Bonnet to obtain,

$$\sum_{i=1}^{k} I_i = \frac{1}{2\pi} \iint_S \kappa d\sigma = \chi(S). \tag{9.97}$$

□

Remark 9.11. This result guarantees that a vector field on any surface homeomorphic to a sphere must have at least two isolated singular points, because the Euler characteristic of the 2-spehre is two. The solution is particularly remarkable because it shows that the sum of the indices of a vector field does not depend on v but rather on the topology of S which is a non-intuitive idea. More tangibly, the Poincaré-Hopf index theorem implies the previos theorem.

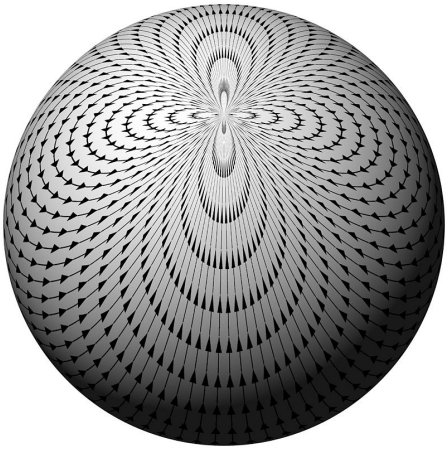

Figure 9.1: An isolated singular point guaranteed by the Poincaré-Hopf theorem[1]

Theorem 9.10 (The Hairy Ball Theorem)**.** *You can't comb a hairy ball without a bald spot. Or, in more precise terms, let f be a continuous function that assigns a vector in $\mathbb{R}^3$ to each point on S^2, such that $f(p)$ is tangent to S^2 at p. Then, there is at least one p for such that $f(p) = 0$.*

Proof. Assume, in order to find a contradiction, that $f(p) \neq 0$ for all p on the 2-sphere. Then, there exists a vector field v on S^2 such that v has no singular points. However, given the above theorem, this is a contradiction, as we know that there must exist two distinct isolated singular points on this vector field, thusly there can be no such vector field on the sphere or on any manifold topologically homeomorphic to the sphere. □

asdsadd

[1]Image Courtesy of Wikipedia: RokerHRO - Own work, CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=8257798`

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