Quaternions

… define the rotation of a frame $\{\mathrm{B}\}$ wrt. a frame $\{\mathrm{I}\}$ with 4 parameters, which are derived from the axis-angle-parameterization

Rotation Matrix to Quaternions:

$$q_1 = \frac{1}{2}\cdot\sqrt{(1 + R_{11} - R_{22} - R_{33})}$$
$$q_2 = \frac{1}{4q_1}\cdot(R_{12} + R_{21})$$
$$q_3 = \frac{1}{4q_1}\cdot(R_{13} + R_{31})$$
$$q_4 = \frac{1}{4q_1}\cdot(-R_{23} + R_{32})$$

$$q_2 = \frac{1}{2}\cdot\sqrt{(1 - R_{11} + R_{22} - R_{33})}$$
$$q_1 = \frac{1}{4q_2}\cdot(R_{12} + R_{21})$$
$$q_3 = \frac{1}{4q_2}\cdot(R_{23} + R_{32})$$
$$q_4 = \frac{1}{4q_2}\cdot(R_{13} + R_{31})$$

$$q_3 = \frac{1}{2}\cdot\sqrt{(1 - R_{11} - R_{22} + R_{33})}$$
$$q_1 = \frac{1}{4q_3}\cdot(R_{13} + R_{31})$$
$$q_2 = \frac{1}{4q_3}\cdot(R_{23} + R_{32})$$
$$q_4 = \frac{1}{4q_3}\cdot(-R_{12} + R_{21})$$

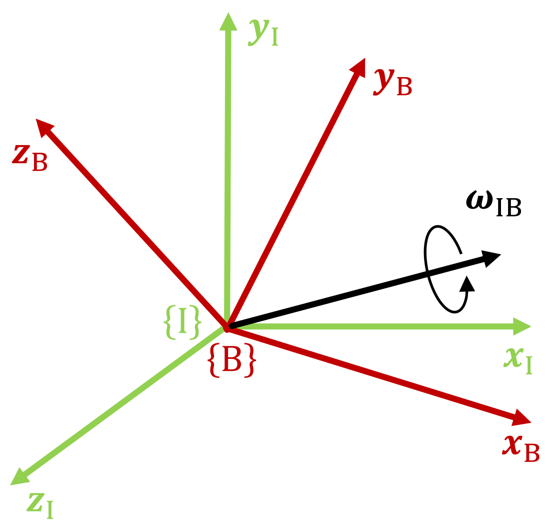
$$q_4 = \frac{1}{2}\cdot\sqrt{(1 + R_{11} + R_{22} + R_{33})}$$
$$q_1 = \frac{1}{4q_4}\cdot(-R_{23} + R_{32})$$
$$q_2 = \frac{1}{4q_4}\cdot(R_{13} - R_{31})$$
$$q_3 = \frac{1}{4q_4}\cdot(-R_{12} + R_{21})$$

➔ use the set of equations with the largest q_i to reduce numerical errors in $\frac{1}{4q_i}\cdot \ldots$

➔ 4 different quaternion vectors which represent the same rotation and arise from the 4 ambiguities of the axis-angle parameterization

Quaternion Differential Equation

… defines the change of an attitude quaternion due to the angular motion ${}^{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{IB}}$



$$ {}^{\mathrm{I}}_{\mathrm{B}}\dot{\boldsymbol{q}} = \frac{1}{2}\,{}^{\mathrm{B}}\boldsymbol{Q}_{\mathrm{IB}} \cdot {}^{\mathrm{I}}_{\mathrm{B}}\boldsymbol{q} $$

$$ {}^{\mathrm{B}}\boldsymbol{Q}_{\mathrm{IB}} = \begin{bmatrix} 0 & {}^{\mathrm{B}}\omega_{\mathrm{IB},z} & -{}^{\mathrm{B}}\omega_{\mathrm{IB},y} & {}^{\mathrm{B}}\omega_{\mathrm{IB},x} \\ -{}^{\mathrm{B}}\omega_{\mathrm{IB},z} & 0 & {}^{\mathrm{B}}\omega_{\mathrm{IB},x} & {}^{\mathrm{B}}\omega_{\mathrm{IB},y} \\ {}^{\mathrm{B}}\omega_{\mathrm{IB},y} & -{}^{\mathrm{B}}\omega_{\mathrm{IB},x} & 0 & {}^{\mathrm{B}}\omega_{\mathrm{IB},z} \\ -{}^{\mathrm{B}}\omega_{\mathrm{IB},x} & -{}^{\mathrm{B}}\omega_{\mathrm{IB},y} & -{}^{\mathrm{B}}\omega_{\mathrm{IB},z} & 0 \end{bmatrix} $$

$$ {}^{\mathrm{B}}\boldsymbol{Q}_{\mathrm{IB}} = \begin{bmatrix} -{}^{\mathrm{B}}\boldsymbol{\Omega}_{\mathrm{IB}} & {}^{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{IB}} \\ -{}^{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{IB}}^{\mathrm{T}} & 0 \end{bmatrix} $$

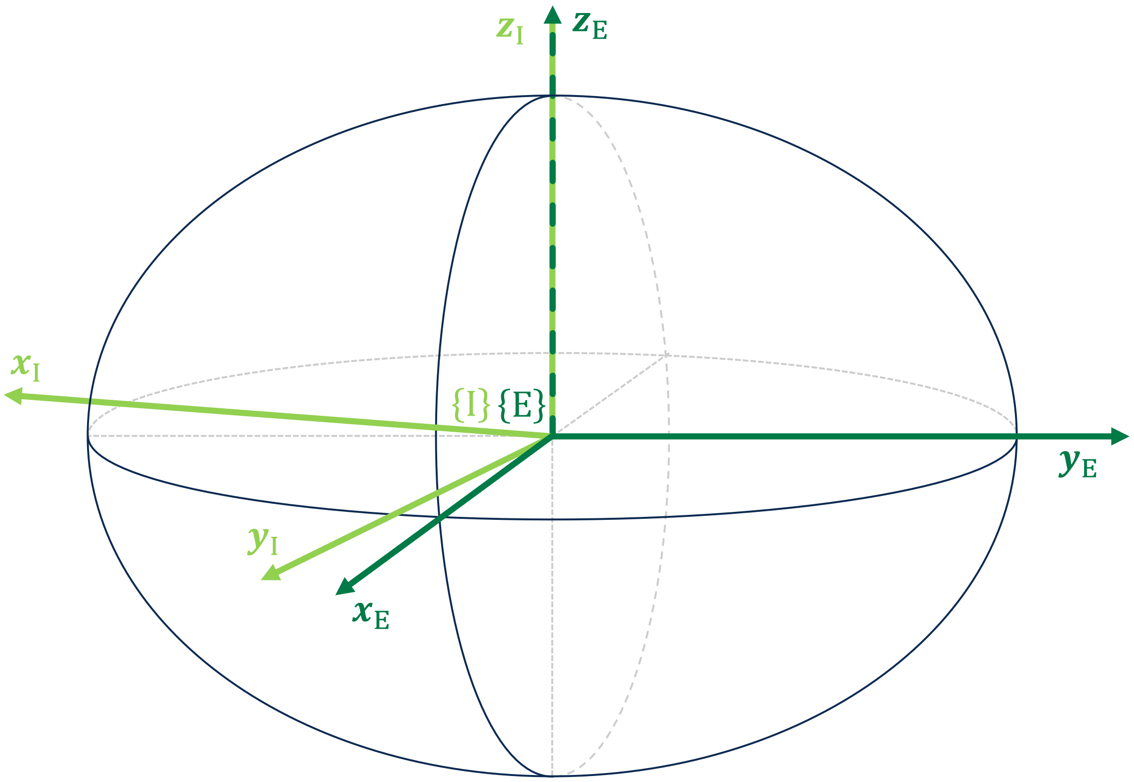
➔ contains no trigonometric functions!!

➔ contains no singularities

➔ less parameters than rotation matrix differential eq.

➔ quaternions are the attitude parameterization of choice

Earth Frame



$\{\mathrm{I}\}$ … Inertial Frame — Earth Centered Inertial Frame (ECI)

$\{\mathrm{E}\}$ … Earth Frame — Earth Centered Earth Fixed (ECEF)

Earth Frame {E}: Earth Centered Earth Fixed Frame (ECEF)

- fixed to the earth with its origin at the earth's center
- its *z*-axis points towards the north pole
- its x-axis points towards the intersection of equator and prime meridian (which passes Greenwich)
- its y-axis completes the right-handed frame

Position Vector of {E} in {I} expressed in {I}

$$\boxed{{}^{\mathrm{I}}\boldsymbol{r}_{\mathrm{IE}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}}$$

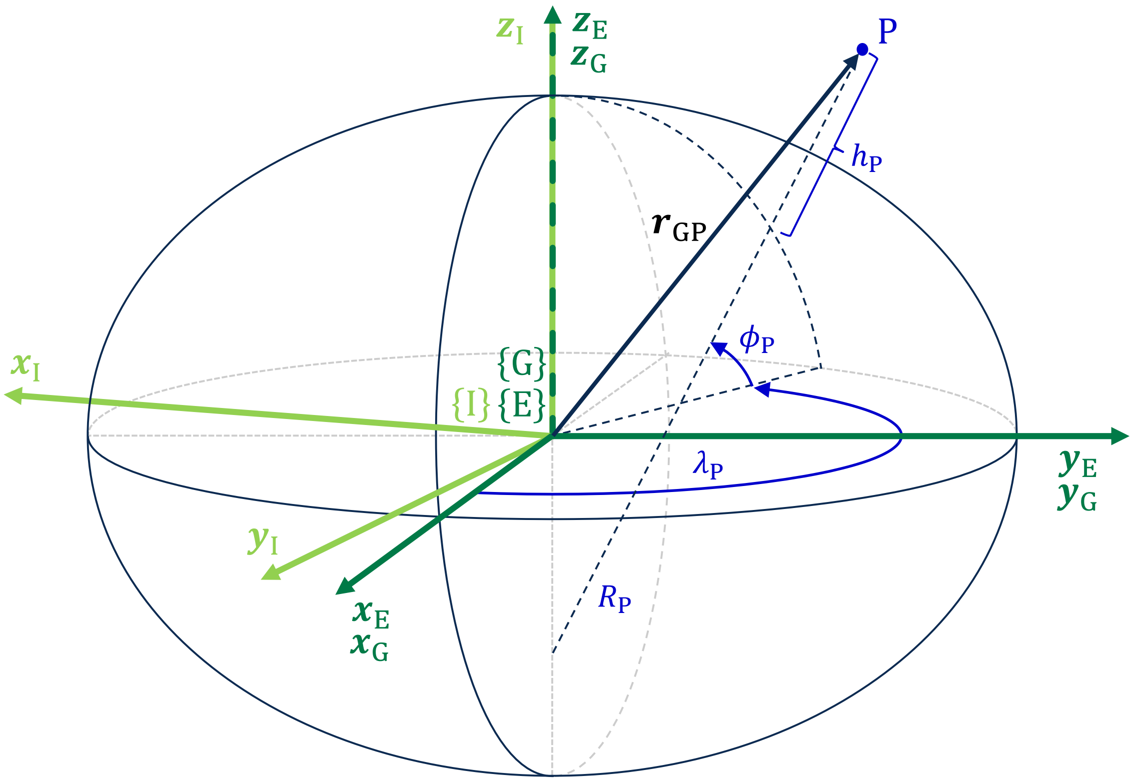
Rotation Matrix of {E} wrt. {I}

$$\boxed{{}^{\mathrm{I}}_{\mathrm{E}}\boldsymbol{R} = \begin{bmatrix} \cos(\omega_e \cdot t) & -\sin(\omega_e \cdot t) & 0 \\ \sin(\omega_e \cdot t) & \cos(\omega_e \cdot t) & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\boxed{{}^{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{IE}} = \begin{bmatrix} 0 \\ 0 \\ \omega_e \end{bmatrix}}$$

$$\omega_e = \frac{2\pi}{24\mathrm{h}} = \frac{2\pi}{24 \cdot 60 \cdot 60 s} = 7.2921 \cdot 10^{-5}\,\frac{\mathrm{rad}}{\mathrm{s}}$$

Geodetic Frame



φ_P … (geodetic) latitude (ger: "Breitengrad") of P
λ_P … longitude (ger: "Längengrad") of P
h_P … altitude of P

Geodetic Frame {G}: World Geodetic System Frame (WGS84)

- often also referred to as Longitude Latitude Height (LLH) Coordinates
- same coordinate frame as ECEF frame
- coordinate description based on reference ellipsoid
- many different reference ellipsoid standardizations exist
- WGS84 is a reference ellipsoid which is widely used (e.g. for GPS applications)

World Geodetic System (1984)

$a = 6\,378\,137.0$ m

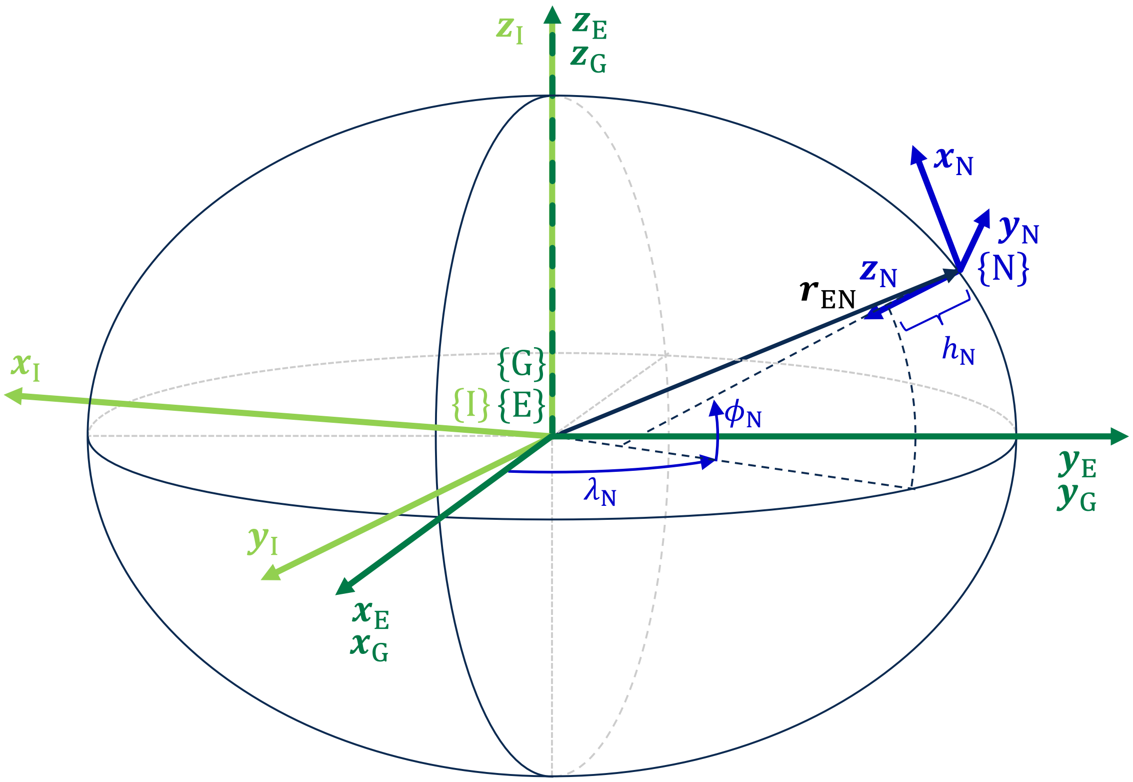
$e = 0.08181919$

Position Vector of {P} in {E} expressed in {E} (based on Geodetic Coordinates)

$$ {}^{\mathrm{E}}\boldsymbol{r}_{\mathrm{EP}} = \underbrace{\begin{bmatrix} {}^{\mathrm{E}}r_{\mathrm{EP},x} & {}^{\mathrm{E}}r_{\mathrm{EP},y} & {}^{\mathrm{E}}r_{\mathrm{EP},z} \end{bmatrix}^{\mathrm{T}}}_{\substack{\text{cartesian}\\ \text{coordinates}\\ \text{(ECEF)}}} = {}^{\mathrm{E}}\boldsymbol{r}_{\mathrm{GP}} = \mathbf{r}\left({}^{\mathrm{G}}\boldsymbol{r}_{\mathrm{GP}}\right) = \mathbf{r}\,\underbrace{\left(\phi_{\mathrm{P}}, \lambda_{\mathrm{P}}, h_{\mathrm{P}}\right)}_{\substack{\text{geodetic}\\ \text{coordinates}\\ \text{(WGS84)}}} = \begin{bmatrix} (R_{\mathrm{P}} + h_{\mathrm{P}}) \cdot \mathrm{c}\,\phi_{\mathrm{P}}\,\mathrm{c}\,\lambda_{\mathrm{P}} \\ (R_{\mathrm{P}} + h_{\mathrm{P}}) \cdot \mathrm{c}\,\phi_{\mathrm{P}}\,\mathrm{s}\,\lambda_{\mathrm{P}} \\ (R_{\mathrm{P}}(1 - e^2) + h_{\mathrm{P}}) \cdot \mathrm{s}\,\phi_{\mathrm{P}} \end{bmatrix} $$

→ closed form solution for the inverse operation ${}^{\mathrm{E}}\boldsymbol{r}_{\mathrm{EP}} \rightarrow {}^{\mathrm{G}}\boldsymbol{r}_{\mathrm{GP}}$ exists, but is mathematically complex and is therefore omitted here.

Navigation Frame



$\{I\}$ … Inertial Frame	Earth Centered Inertial Frame (ECI)
$\{E\}$ … Earth Frame	Earth Centered Earth Fixed (ECEF)
$\{G\}$ … Geodetic Frame	World Geodetic System Frame (WGS84)
$\{N\}$ … Navigation Frame	North East Down (NED) / East North Up (ENU)

Navigation Frame {N}: North East Down (NED) / East North Up (ENU)

- reference frame for relative navigation
- fixed on earth (often at initial robot position or a reference position on a local map)
- its *z*-axis points downwards towards the direction of gravity (not necessarily to the earth's center)
- its x-axis points north
- its y-axis points east to complete the right-handed frame

Position Vector of {N} in {E} expressed in {E} (based on Geodetic Coordinates)

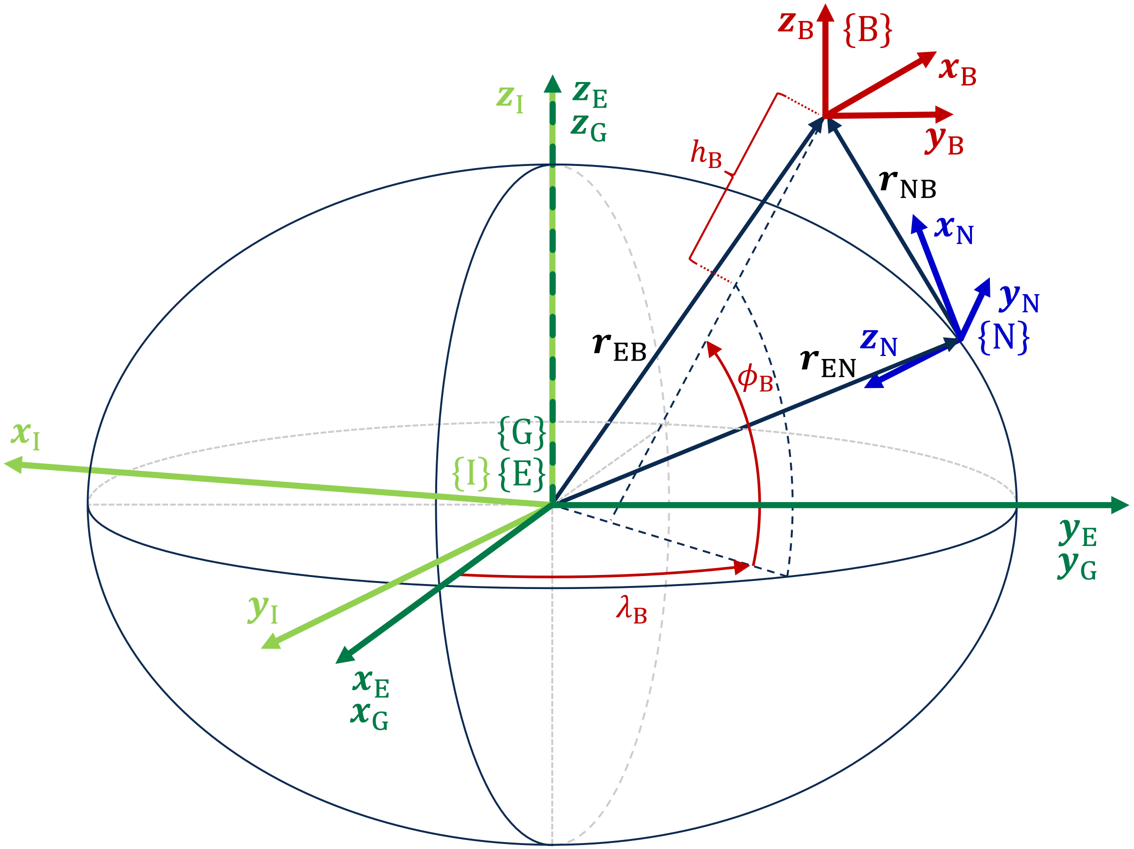
$$ {}^{\mathrm{E}}\boldsymbol{r}_{\mathrm{EN}} = \left({}^{\mathrm{E}}\boldsymbol{r}_{\mathrm{GN}}\right) = \mathbf{r}\left({}^{\mathrm{G}}\boldsymbol{r}_{\mathrm{EN}}\right) = \mathbf{r}(\phi_{\mathrm{N}}, \lambda_{\mathrm{N}}, h_{\mathrm{N}}) = \begin{bmatrix} (R_{\mathrm{N}} + h_{\mathrm{N}}) \cdot \mathrm{c}\,\phi_{\mathrm{N}}\,\mathrm{c}\,\lambda_{\mathrm{N}} \\ (R_{\mathrm{N}} + h_{\mathrm{N}}) \cdot \mathrm{c}\,\phi_{\mathrm{N}}\,\mathrm{s}\,\lambda_{\mathrm{N}} \\ (R_{\mathrm{N}}(1-e^2) + h_{\mathrm{N}}) \cdot \mathrm{s}\,\phi_{\mathrm{N}} \end{bmatrix} = \text{const.} $$

→ no closed form solution for the inverse operation ${}^{\mathrm{E}}\boldsymbol{r}_{\mathrm{EN}} \rightarrow {}^{\mathrm{G}}\boldsymbol{r}_{\mathrm{GN}}$!!!

Rotation Matrix of {N} wrt. {E} based on Geodetic Coordinates

$$ {}^{\mathrm{E}}_{\mathrm{N}}\boldsymbol{R} = \mathbf{R}\left({}^{\mathrm{G}}\boldsymbol{r}_{\mathrm{EN}}\right) = \mathbf{R}(\phi_{\mathrm{N}}, \lambda_{\mathrm{N}}) = \begin{bmatrix} -\mathrm{s}\,\phi_{\mathrm{N}}\,\mathrm{c}\,\lambda_{\mathrm{N}} & -\mathrm{s}\,\lambda_{\mathrm{N}} & -\mathrm{c}\,\phi_{\mathrm{N}}\,\mathrm{c}\,\lambda_{\mathrm{N}} \\ -\mathrm{s}\,\phi_{\mathrm{N}}\,\mathrm{s}\,\lambda_{\mathrm{N}} & \mathrm{c}\,\lambda_{\mathrm{N}} & -\mathrm{c}\,\phi_{\mathrm{N}}\,\mathrm{s}\,\lambda_{\mathrm{N}} \\ \mathrm{c}\,\phi_{\mathrm{N}} & 0 & -\mathrm{s}\,\phi_{\mathrm{N}} \end{bmatrix} = \text{const.} $$

Body Frame



{I} … Inertial Frame — Earth Centered Inertial Frame (ECI)

{E} … Earth Frame — Earth Centered Earth Fixed (ECEF)

{G} … Geodetic Frame — World Geodetic System Frame (WGS84)

{N} … Navigation Frame — North East Down (NED) / East North Up (ENU)

{B} … Body Frame — Body Fixed Frame

Body Frame {B}: Body Fixed Frame

- fixed on body
- different sensors usually have different positions and attitudes within the body frame
  → transformations assumed to be known (extrinsic calibration)
  → sensors assumed to have the same frame as the body (same position/attitude)

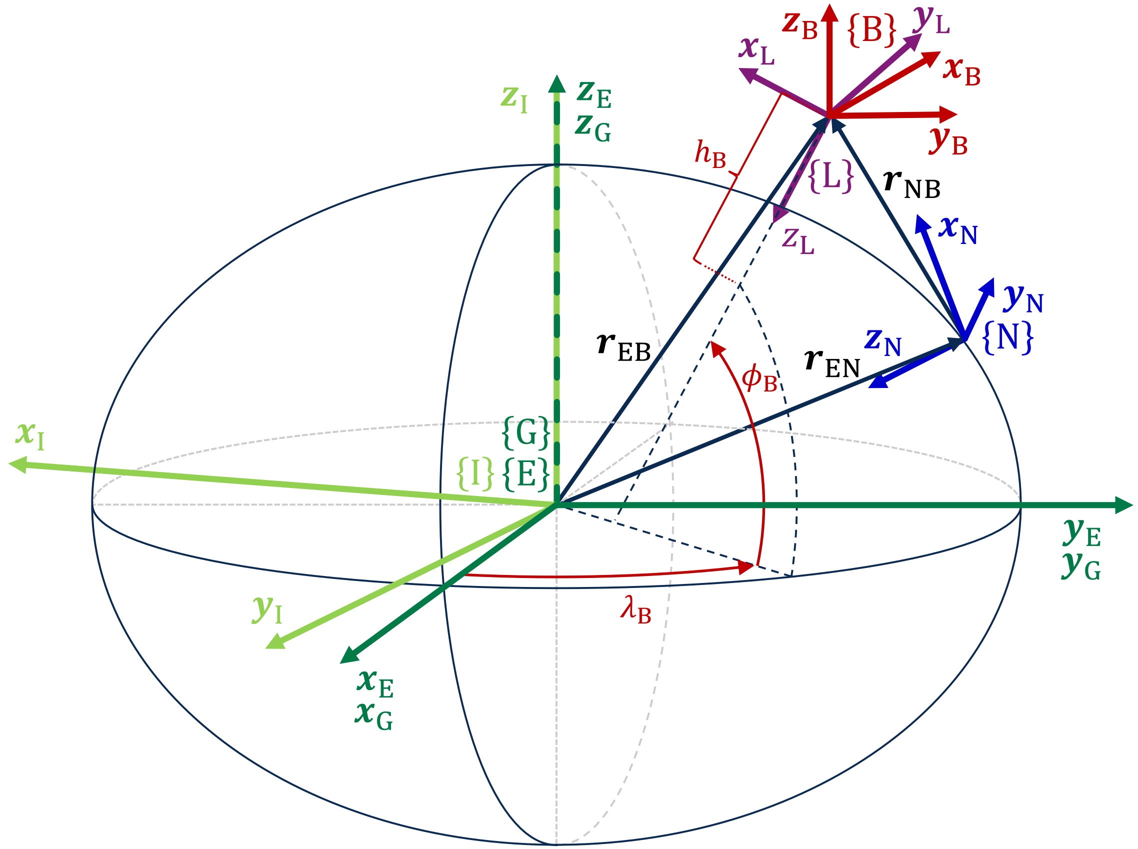
Position Vector of {B} in {N} expressed in {N}

$$ {}^{\mathrm{N}}\boldsymbol{r}_{\mathrm{NB}} = \begin{bmatrix} {}^{\mathrm{N}}r_{\mathrm{NB},x} \\ {}^{\mathrm{N}}r_{\mathrm{NB},y} \\ {}^{\mathrm{N}}r_{\mathrm{NB},z} \end{bmatrix} $$

Rotation Matrix of {B} wrt. {N} based on Geodetic Coordinates

$$ {}^{\mathrm{N}}_{\mathrm{B}}\boldsymbol{R} = \mathbf{R}\left({}^{\mathrm{N}}_{\mathrm{B}}\boldsymbol{q}\right) = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2\cdot(q_1 q_2 - q_3 q_4) & 2\cdot(q_1 q_3 + q_2 q_4) \\ 2\cdot(q_1 q_2 + q_3 q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2\cdot(q_2 q_3 - q_1 q_4) \\ 2\cdot(q_1 q_3 - q_2 q_4) & 2\cdot(q_2 q_3 + q_1 q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix} $$

Local Frame



{I} … Inertial Frame	Earth Centered Inertial Frame (ECI)
{E} … Earth Frame	Earth Centered Earth Fixed (ECEF)
{G} … Geodetic Frame	World Geodetic System Frame (WGS84)
{N} … Navigation Frame	North East Down (NED) / East North Up (ENU)
{B} … Body Frame	Body Fixed Frame
{L} … Local Frame	Local Body Fixed North East Down Frame

Local Frame {L}: Local Body Fixed North East Down Frame

- local north east down frame
- fixed on body → attitude depends on the geodetic position of the body
- its *z*-axis points downwards towards the direction of gravity (not necessarily to the earth's center)
- its x-axis points north
- its y-axis points east to complete the right-handed frame

Position Vector of {L} in {E} expressed in {E} (based on Geodetic Coordinates)

$$ {}^{\mathrm{E}}\boldsymbol{r}_{\mathrm{EL}} = {}^{\mathrm{E}}\boldsymbol{r}_{\mathrm{EB}} = \left({}^{\mathrm{E}}\boldsymbol{r}_{\mathrm{GB}}\right) = \mathbf{r}\left({}^{\mathrm{G}}\boldsymbol{r}_{\mathrm{EB}}\right) = \mathbf{r}(\phi_{\mathrm{B}}, \lambda_{\mathrm{B}}, h_{\mathrm{B}}) = \begin{bmatrix} (R_{\mathrm{B}} + h_{\mathrm{B}}) \cdot \mathrm{c}\,\phi_{\mathrm{B}}\,\mathrm{c}\,\lambda_{\mathrm{B}} \\ (R_{\mathrm{B}} + h_{\mathrm{B}}) \cdot \mathrm{c}\,\phi_{\mathrm{B}}\,\mathrm{s}\,\lambda_{\mathrm{B}} \\ (R_{\mathrm{B}}(1-e^2) + h_{\mathrm{B}}) \cdot \mathrm{s}\,\phi_{\mathrm{B}} \end{bmatrix} \neq \text{const.} $$

→ no closed form solution for the inverse operation ${}^{\mathrm{E}}\boldsymbol{r}_{\mathrm{EL}} \rightarrow {}^{\mathrm{G}}\boldsymbol{r}_{\mathrm{GB}}$!!!

Rotation Matrix of {L} wrt. {E} based on Geodetic Coordinates

$$ {}^{\mathrm{E}}_{\mathrm{L}}\boldsymbol{R} = \mathbf{R}\left({}^{\mathrm{G}}\boldsymbol{r}_{\mathrm{GB}}\right) = \mathbf{R}(\phi_{\mathrm{B}}, \lambda_{\mathrm{B}}) = \begin{bmatrix} -\mathrm{s}\,\phi_{\mathrm{B}}\,\mathrm{c}\,\lambda_{\mathrm{B}} & -\mathrm{s}\,\lambda_{\mathrm{B}} & -\mathrm{c}\,\phi_{\mathrm{B}}\,\mathrm{c}\,\lambda_{\mathrm{B}} \\ -\mathrm{s}\,\phi_{\mathrm{B}}\,\mathrm{s}\,\lambda_{\mathrm{B}} & \mathrm{c}\,\lambda_{\mathrm{B}} & -\mathrm{c}\,\phi_{\mathrm{B}}\,\mathrm{s}\,\lambda_{\mathrm{B}} \\ \mathrm{c}\,\phi_{\mathrm{B}} & 0 & -\mathrm{s}\,\phi_{\mathrm{B}} \end{bmatrix} \neq \text{const.} $$