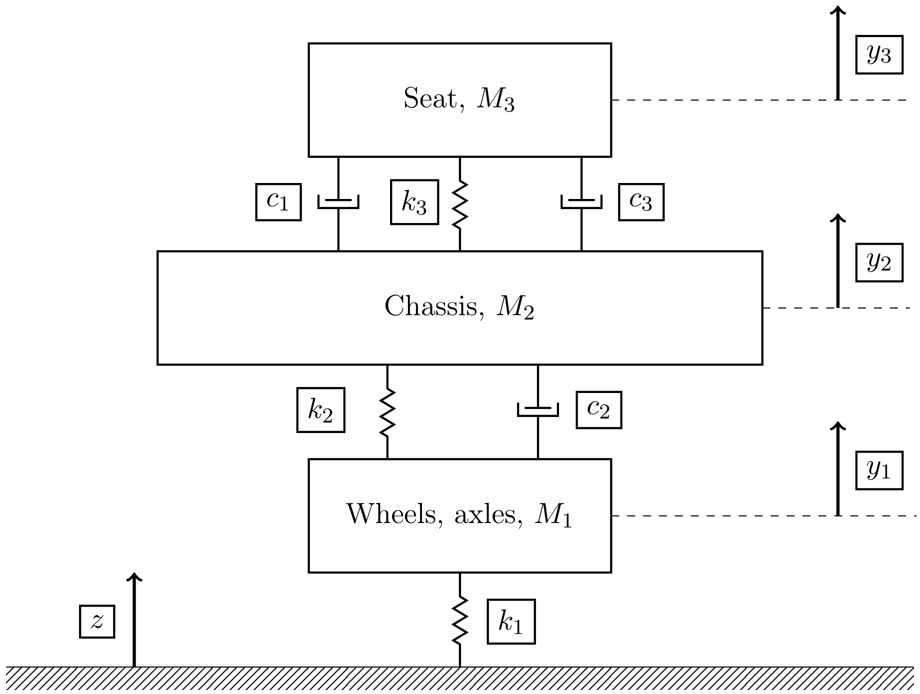
INTRODUCTION



The quarter car model is a simplified representation of a vehicle's suspension system, often used in automotive engineering to analyze the dynamic response of a vehicle to road disturbances. By focusing on a single wheel, this model provides an effective means to understand and control the vertical motion of the vehicle's body and suspension, capturing essential behaviors like ride comfort. The model being studies involved an unsprung mass (M_1), modeling the wheel, and two sprung masses, modeling the chassis and seat (M_2 and M_3). For simplicity, we assume the seat and driver masses are combined to one mass (M_3), making this a 3-DOF (Degrees-of-Freedom) model instead of a 4-DOF one. k_1 models the tyre, k_2 and c_2 model the vehicle passive suspension, k_3 and c_1 model the seat stiffness and dampening, and finally, c_3 models the seat-back friction. z denotes the road profile (or road displacement), and y_1, y_2, y_3 denote the displacement for the masses M_1, M_2, and M_3 respectively.

CHAPTER 1

System Modeling

1.1. ODE Modeling

We derive the equations describing the motion of the three masses involved in the system using Newton's second law of motion shown below.

$$\sum F = M\ddot{y} \tag{1.1}$$

We first start by applying Equation 1.1 to M_1, M_2, and M_3 to obtain the Ordinary Differential Equations describing the motion of the quarter car model system.

$$M_1\ddot{y}_1 = -(k_2 + k_3)y_1 + k_2y_2 - c_2\dot{y}_1 + c_2\dot{y}_2 + k_3z \tag{1.2}$$

$$M_2\ddot{y}_2 = k_2y_1 - (k_1 + k_2)y_2 + k_1y_3 + c_2\dot{y}_1 - (c_1 + c_2 + c_3)\dot{y}_2 + (c_1 + c_3)\dot{y}_3 \tag{1.3}$$

$$M_3\ddot{y}_3 = k_1y_2 - k_1y_3 + (c_1 + c_3)\dot{y}_2 - (c_1 + c_3)\dot{y}_3 \tag{1.4}$$

1.2. Laplace Transform

We then move on to apply the Laplace transform to the Ordinary Differential Equations we obtained as shown below. We also assume that initial conditions are zero.

$$y_1(0) = 0, \frac{dy_1}{dt}(0) = 0, y_2(0) = 0, \frac{dy_2}{dt}(0) = 0, y_3(0) = 0, \frac{dy_3}{dt}(0) = 0$$

We also note the following terms defining the Laplace domain equivalents of the input/output variables of the system.

$$Y_1(s) = L(y_1(t)), Y_2(s) = L(y_2(t)), Y_3(s) = L(y_3(t)), Z(s) = L(z(t))$$

- Equation 1.2 transformed into the Laplace domain.

$$s^2M_1Y_1(s) + sc_2Y_1(s) + (k_2 + k_3)Y_1(s) = sc_2Y_2(s) + k_2Y_2(s) + k_3Z(s)$$

$$(s^2M_1 + sc_2 + k_2 + k_3)Y_1(s) = (sc_2 + k_2)Y_2(s) + k_3Z(s) \tag{1.5}$$

- Equation 1.3 transformed into the Laplace domain.

$$s^2M_2Y_2(s)+s(c_1+c_2+c_3)Y_2(s)+(k_1+k_2)Y_2(s) = sc_2Y_1(s)+k_2Y_1(s)+s(c_1+c_3)Y_3(s)+k_1Y_3(s)$$

$$(s^2M_2 + s(c_1 + c_2 + c_3) + k_1 + k_2)Y_2(s) = (sc_2 + k_2)Y_1(s) + (s(c_1 + c_3) + k_1)Y_3(s) \tag{1.6}$$

- Equation 1.4 transformed into the Laplace domain.

$$s^2M_3Y_3(s) + s(c_1 + c_3)Y_3(s) + k_1Y_3(s) = s(c_1 + c_3)Y_2(s) + k_1Y_2(s)$$

$$(s^2M_3 + s(c_1 + c_3) + k_1)Y_3(s) = (s(c_1 + c_3) + k_1)Y_2(s) \tag{1.7}$$

1.3. State-space Form

In order to quickly build a simulation of the system using MATLAB, we move on to representing the system in state-space representation. We use the Ordinary Differential Equations we obtained previously (Equations 1.2, 1.3, 1.4), to build the state-space matrices. The state-space matrix is defined in Equation 1.8.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1.8}$$

We first start by defining the state variables.

$$x_1 = y_1, x_2 = y_2, x_3 = y_3$$

$$x_4 = \dot{y_1}, x_5 = \dot{y_2}, x_6 = \dot{y_3}$$

We also note that $\dot{x_1} = x_4$, $\dot{x_2} = x_5$, and $\dot{x_3} = x_6$. Hence, the state vector is as follows.

$$\begin{aligned} x &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T \\ &= \begin{bmatrix} y_1 & y_2 & y_3 & \dot{y_1} & \dot{y_2} & \dot{y_3} \end{bmatrix}^T \end{aligned}$$

The vector y shown below will denote the output vector.

$$y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$$

The input vector u will consist of a 1x1 vector (or a scalar) since the system will have one input only: the road profile z.

$$u = \begin{bmatrix} z \end{bmatrix}$$

Before defining the state-space matrices, the ODE's in Equations 1.2, 1.3, and 1.4 need to be slightly re-organized for better clarity, as shown below.

$$\begin{aligned}
\ddot{y_1} &= -\frac{k_2 + k_3}{M_1} y_1 + \frac{k_2}{M_1} y_2 - \frac{c_2}{M_1} \dot{y_1} + \frac{c_2}{M_1} \dot{y_2} + \frac{k_3}{M_1} z \\
\ddot{y_2} &= \frac{k_2}{M_2} y_1 - \frac{k_1 + k_2}{M_2} y_2 + \frac{k_1}{M_2} y_3 + \frac{c_2}{M_2} \dot{y_1} - \frac{c_1 + c_2 + c_3}{M_2} \dot{y_2} + \frac{c_1 + c_3}{M_2} \dot{y_3} \\
\ddot{y_3} &= \frac{k_1}{M_3} y_2 - \frac{k_1}{M_3} y_3 + \frac{c_1 + c_3}{M_3} \dot{y_2} - \frac{c_1 + c_3}{M_3} \dot{y_3}
\end{aligned}$$

Last but not least, we define the state-space matrices. The state matrix A is a 6×6 that will hold the coefficients for all the state variables and is shown below.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_2+k_3}{M_1} & \frac{k_2}{M_1} & 0 & -\frac{c_2}{M_1} & \frac{c_2}{M_1} & 0 \\ \frac{k_2}{M_2} & -\frac{k_1+k_2}{M_2} & \frac{k_1}{M_2} & \frac{c_2}{M_2} & -\frac{c_1+c_2+c_3}{M_2} & \frac{c_1+c_3}{M_2} \\ 0 & \frac{k_1}{M_3} & -\frac{k_1}{M_3} & 0 & \frac{c_1+c_3}{M_3} & -\frac{c_1+c_3}{M_3} \end{bmatrix}$$

The input matrix B is a 6×1 matrix (or vector) and is shown below.

$$B = \begin{bmatrix} 0 & 0 & 0 & \frac{k_3}{M_1} & 0 & 0 \end{bmatrix}^T$$

The 3×6 output matrix C is shown below. We are considering the significant outputs for this system to be y_1, y_2, and y_3.

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Finally, the feed-forward D matrix will be a zero-matrix since there is no direct transmission between the input and the output.

$$D = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

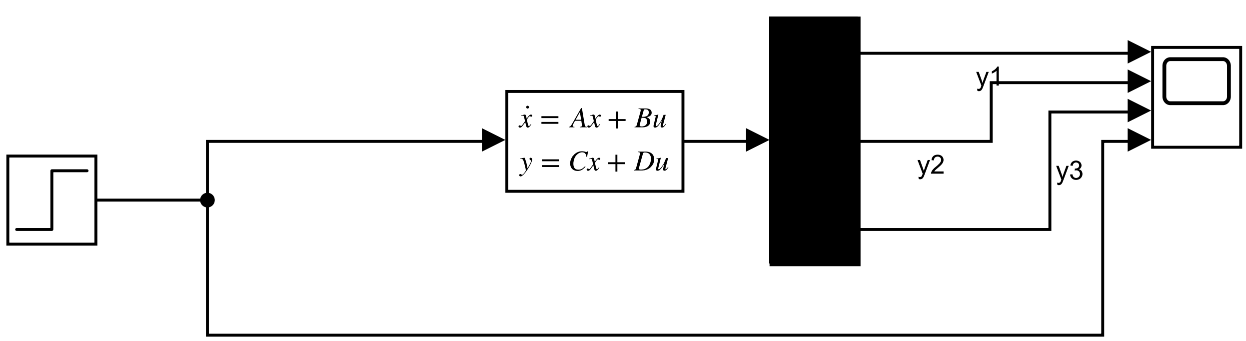


Figure 1.1: Simulink set up for statespace model in Simulink

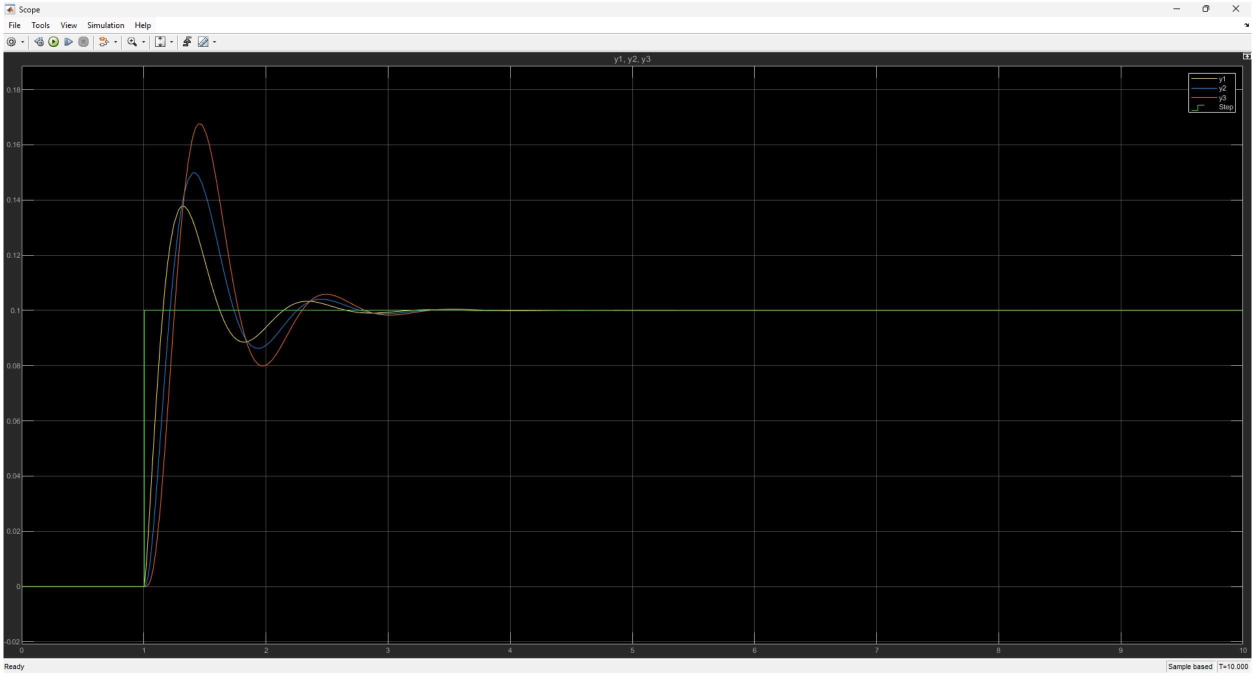


Figure 1.2: Graphical representation of input vs all displacements

CHAPTER 2

Half-car Model

2.1. Understanding Initial Design Parameters

In order to understand the forces acting on the several masses of a half car, setting the design parameters and the first governing equations is essential before formulating the ordinary differential equations for this model.

The half car shares some concepts with the quarter car suspension explained in the previous chapter, however there are a few more considerations to take in when designing this model. Unlike the previous model, this one has a rotational displacement to account for. This type of displacement is called the pitch angle of a car. The pitch angle is determined by the angular displacement of the chassis of the car with respect to the ground. The chassis experiences this motion from its C.G point, which is positioned within its mass.

A compact car is taken as an example to set the initial design parameters. The car's length is taken to be 4 meters. The distances between the driver seat, front and back axle of the car from the chassis's C.G point is shown in Figure 2.1. The distances shown in the figure are in meter units.

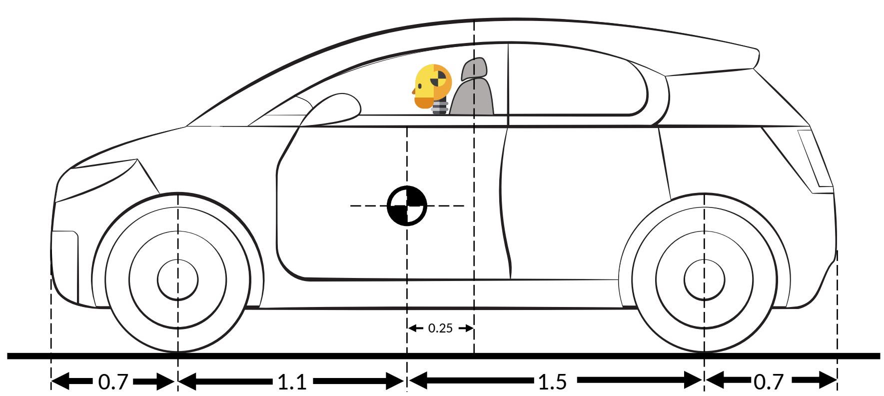


Figure 2.1: Initial design parameters of the compact car showcasing the positions of the front axle, back axle, and driver seat from the car chassis's C.G point

2.2. Force Analysis for the Half Car Model

First lets explain the displacement and displacement components in the half car model as seen in Figure 2.2 -

- Y_s - Vertical Displacement of the Driver Seat M_s
- θ - Resultant Angular Displacement of the Chassis M_c inducing Moment of Inertia I_c
- Y_c - Resultant Vertical Displacement of the Chassis M_c
- Y_5 - Vertical Displacement of the Chassis M_c on the Driver spring damper system
- Y_4 - Vertical Displacement of the Chassis M_c due to Rear Axle Suspension System
- Y_3 - Vertical Displacement of the Chassis M_c due to Front Axle Suspension System
- Y_2 - Vertical Displacement of the Rear Wheel, Axle M_f
- Y_1 - Vertical Displacement of the Front Wheel, Axle M_b
- d - Distance from the driver seat to the C.G point of the chassis
- a - Distance from the front axle to the C.G point of the chassis
- b - Distance from the rear axle to the C.G point of the chassis

The first governing equations are the following -

$$y_3 = y_c + a\theta \tag{2.1}$$

$$y_4 = y_c - b\theta \tag{2.2}$$

$$y_5 = y_c - d\theta \tag{2.3}$$

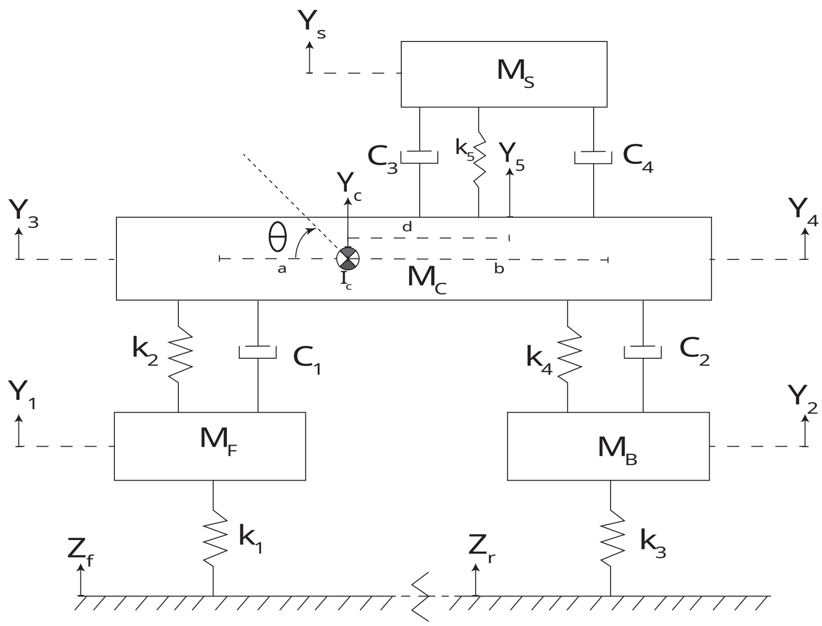


Figure 2.2: A Half Car Model

For this model lets assume a positive angle displacement in the mass M_c pitching away from the mass M_f. Equations 2.1, 2.2, 2.3 define y_3, y_4, y_5 as displacements with respect to the chassis displacement y_c. The chassis M_c is connected to three masses M_s, M_f, M_b via suspension elements, at different fixed positions d, a, b from the C.G, exerting their own forces at different periods in time as seen in Figure 2.2. This induces an angular disturbance in the chassis position relative to the ground. Hence we need to take the distance times the angular displacement θ in the three above equations.

By applying equation 1.1 from the Chapter 1, the Ordinary Differential Equations describing the motion of the Half Car Model is formed.

$$M_F\ddot{y_1} = -(k1 + k2)y_1 + k_2y_c + ak_2\theta - c_1\dot{y_1} + c_1\dot{y_c} + ac_1\dot{\theta} + k_1z_f \tag{2.4}$$

$$M_B\ddot{y_2} = -(k3 + k4)y_2 + k_4y_c - bk_4\theta - c_2\dot{y_2} + c_2\dot{y_c} - bc_2\dot{\theta} + k_3z_r \tag{2.5}$$

$$M_C\ddot{y}_c = k_2y_1+k_4y_2-(k_2+k_4+k_5)y_c+c_2y_c+(-ak_2+bk_4+dk_5)\theta+k_5y_s+c_1\dot{y}_1+c_2\dot{y}_2-(c_1+c_2+c_3+c_4)\dot{y}_c$$

$$+(-ac_1+bc_2+dc_3+dc_4)\dot{\theta}+(c_3+c_4)\dot{y}_1 \tag{2.6}$$

$$M_s\ddot{y}_c = k_5y_c+dk_5\theta-k5y_s+(c_3+c_4)\dot{y}_c+(dc_3+dc_4)\dot{\theta}-(c_3+c_4)\dot{y}_s \tag{2.7}$$

The torque observed on the chassis is derived using the governing Equation 1.1

$$\sum\tau = I\ddot{\theta} = r*F*sin\theta \tag{2.8}$$

The pitch angles observed would be minimal, and negligible enough to ignore the sine term in the torque equation. Applying the Equation 2.8 we obtain the angular momentum of the chassis mass.

$$I_c\ddot{\theta} = ak_2y_1-bk_4y_2+(-ak_2+bk_4+dk_5)y_c-(a^2k_2+b^2k_4-d^2k_5)\theta-dk_5y_s+(ac_1)\dot{y}_1-bc_2\dot{y}_2$$

$$+(-ac_1+bc_2+dc_3+dc_4)\dot{y}_c-(a^2c_1+b^2c_2-d^2c_3-d^2c_4)\dot{\theta}-(dc_3+dc_4)\dot{y}_s \tag{2.9}$$

2.3. Generating the State Space Matrices

Matlab Simulink is going to be used for generating the road profile responses for the Half Car Model as well. To build the simulation, the state variables need to be defined and the state matrices need to be generated.

$$x_1 = y_1, x_2 = y_2, x_3 = y_c, x_4 = \theta, x_5 = y_s$$

$$x_6 = \dot{y_1}, x_7 = \dot{y_2}, x_8 = \dot{y_c}, x_9 = \dot{\theta}, x_{10} = \dot{y_s}$$

The state vector is as follows.

$$\begin{aligned} x &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{bmatrix}^T \\ &= \begin{bmatrix} y_1 & y_2 & y_c & \theta & y_s & \dot{y_1} & \dot{y_2} & \dot{y_c} & \dot{\theta} & \dot{y_s} \end{bmatrix}^T \end{aligned}$$

The vector y shown below will denote the output vector.

$$y = \begin{bmatrix} y_c & \theta & y_s \end{bmatrix}^T$$

The input vector u will consist of a 2x1 vector, since the system will now have two inputs, the road profile affecting the front and back suspension at different time periods.

$$u = \begin{bmatrix} z_f & z_r \end{bmatrix}^T$$

The input matrix B is a 10×2 matrix (or vector) and is shown below.

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{k_1}{M_f} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{k_3}{M_b} & 0 & 0 & 0 \end{bmatrix}^T$$

The system matrix A is now a 10×10 matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{M_f} & 0 & \frac{k_2}{M_f} & \frac{ak_2}{M_f} & 0 & -\frac{c_1}{M_f} & 0 & \frac{c_1}{M_f} & \frac{ac_1}{M_f} & 0 \\ 0 & -\frac{k_3+k_4}{M_b} & \frac{k_4}{M_b} & -\frac{bk_4}{M_b} & 0 & 0 & -\frac{c_2}{M_b} & \frac{c_2}{M_b} & -\frac{bc_2}{M_b} & 0 \\ \frac{k_2}{M_c} & \frac{k_4}{M_c} & -\frac{k_2+k_4+k_5}{M_c} & \frac{-ak_2+bk_4+dk_5}{M_c} & \frac{k_5}{M_c} & \frac{c_1}{M_c} & \frac{c_2}{M_c} & -\frac{c_1+c_2+c_3+c_4}{M_c} & -\frac{-ac_1+bc_2+dc_3+dc_4}{M_c} & \frac{c_3+c_4}{M_c} \\ \frac{ak_2}{I_c} & -\frac{bk_4}{I_c} & \frac{-ak_2+bk_4+dk_5}{I_c} & -\frac{a^2k_2+b^2k_4-d^2k_5}{I_c} & -\frac{dk_5}{I_c} & \frac{ac_1}{I_c} & -\frac{bc_2}{I_c} & \frac{-ac_1+bc_2+dc_3+dc_4}{I_c} & -\frac{a^2c_1+b^2c_2-d^2c_3-d^2c_4}{I_c} & -\frac{dc_3+dc_4}{I_c} \\ 0 & 0 & -\frac{k_5}{M_s} & \frac{dk_5}{M_s} & -\frac{k_5}{M_s} & 0 & 0 & \frac{c_3+c_4}{M_s} & \frac{dc_3+dc_4}{M_s} & -\frac{c_3+c_4}{I_c} \end{bmatrix}$$

The 3×10 output matrix C is shown below. We are considering the significant outputs for this system to be y_c, θ, and y_s.

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, the feed-forward D matrix will also be a zero-matrix just like the quarter car since there is no direct transmission between the input and the output.

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T$$

Now that the state space matrices have been generated, these can be fed into the Simulink model to generate the road profile responses.

2.4. Designing a Simulink Model for a Half Car

The state space matrices are fed into the 'Half-car-state-space' block through the workspace.

The road profile model circuit generates two speed bump signals at a certain time difference. This is to simulate the half car to first dampen the speed bump with its front suspension and then after a few seconds it should be able to do the same with its rear suspension. Gaussian noise has also been added to the signals to simulate a noise road profile input. Other blocks in the circuit such as the saturation, step signal and the product blocks have a role in processing the signal to produce a single sample of positive peak signal.

After receiving the road profile input the state space block generates the vertical and angular responses (in degrees) as seen in Figure 2.3.

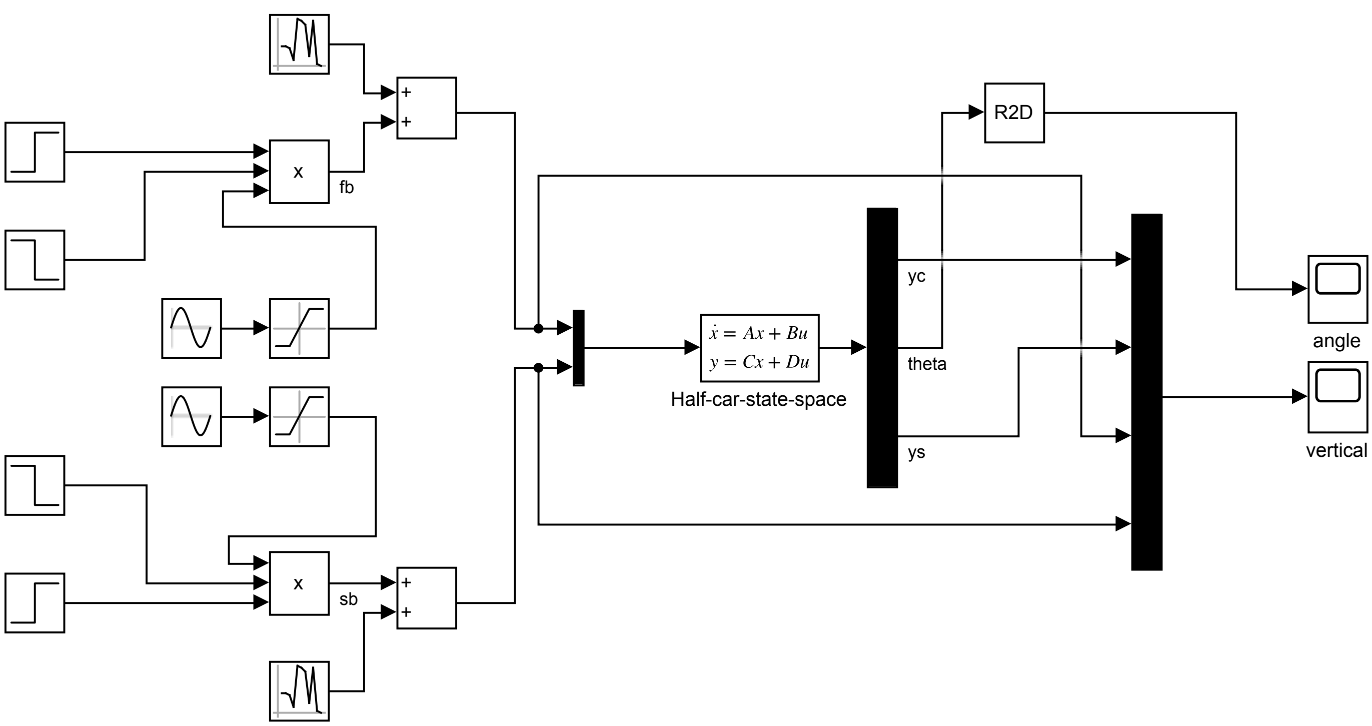


Figure 2.3: Simulink Model for a Half Car

The plots for the vertical and angular displacement are shown in Figures 2.4 and 2.5. y_c and y_s are two vertical displacements recorded in the first plot along with the road inputs.The speed bump signals are sent to the model 2 seconds apart.

The speed bumps have an amplitude of 10 cm, whereas the y_s maximum displacement shows just under 8 cm. The model dampens out the noisy input well. The dampening oscillations are noticed to be a little erratic due to the position of the driver seat with respect to the chassis. This causes irregular motion in the system. However the model is able to dampen the oscillations with minimal variation.

The angular displacement plots have damped oscillations, but the deviations seen in the angle plot are minimal, the maximum deviation being just under ± 4 degrees. The angle plot is also plotted over a longer duration than the vertical displacement. This is to show the low variability in the angle disturbance.

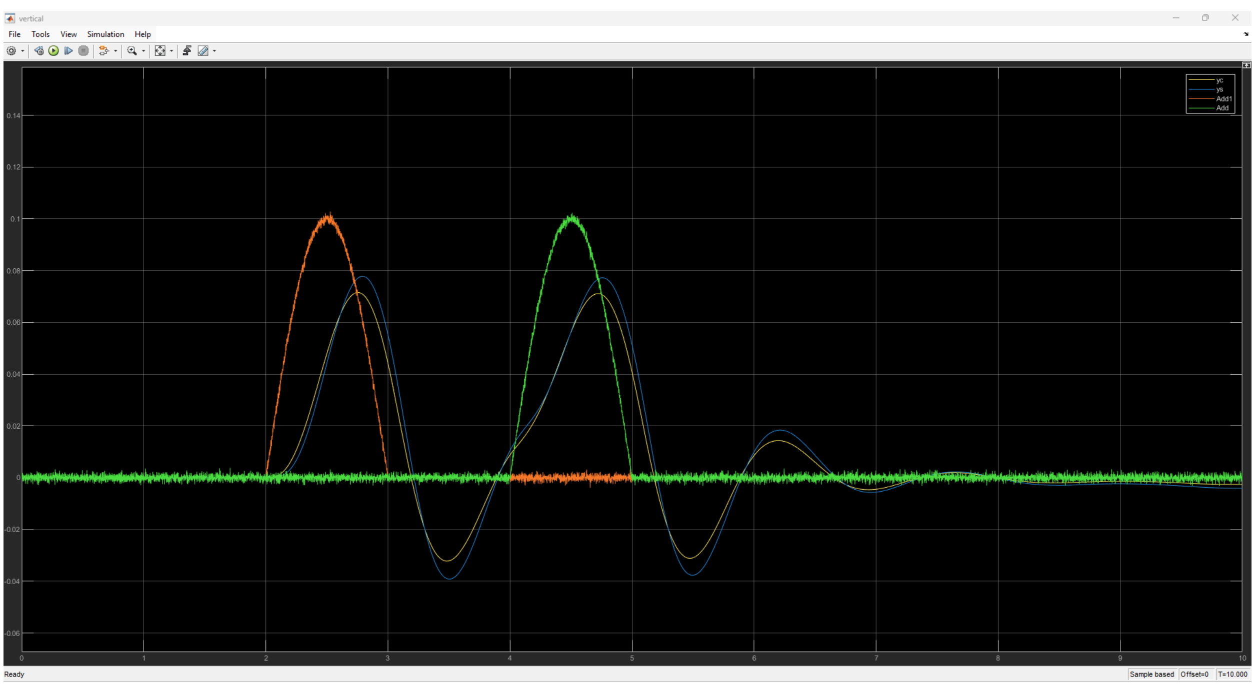


Figure 2.4: Speed bump response showcasing the displacement and noise damping capability of the model

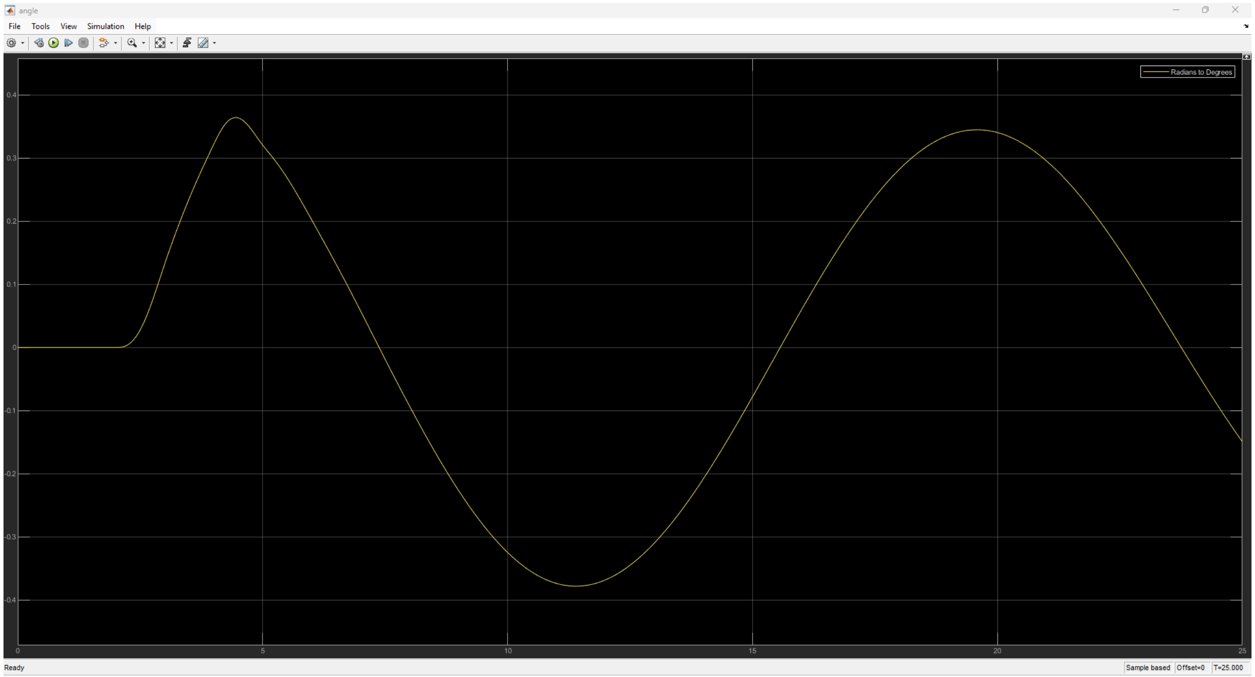


Figure 2.5: Speed bump response showcasing the minimal angular displacement

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