Optimization with Mean-Variance Portfolio Theory

1. Mean-Variance Portfolio Theory

1.1 Mean

1.1.1 Return of an asset

Multi period holding period yield refers to the total yield obtained by investors within n years of holding a certain investment product:

$$(1+R_1)\times(1+R_2)\times\ldots\times(1+R_n)-1$$

Geometric average holding period rate of return refers to the annual average rate of return actually obtained by investors within n years of holding an investment product according to the compound interest principle:

$$\sqrt{(1+R_1)\times(1+R_2)\times\ldots\times(1+R_n)}-1$$

When the earnings of each period fluctuate greatly, the arithmetic average rate of return will show an obvious upward tendency. The reason why the geometric average rate of return is better than the arithmetic average rate of return is that it introduces the formula of compound interest, that is, the compound appreciation rate of the initial investment value is measured by weighting the time, so as to overcome the upward tendency of the arithmetic average rate of return sometimes. In the economic sense, the geometric average rate of return weights the time from the perspective of compound interest. When the fluctuation of the rate of return is large, it overcomes the error caused by equal weight calculation. Since the arithmetic average rate of return is calculated with equal weight, the calculation result will be larger when the amplitude is large. Only when the returns of each period are the same throughout the investment period, the weight factor does not work, and the two average returns may be consistent.

1.1.2 Portfolio return

For the investment decision of portfolio assets, we should consider not only the income of a single asset, but also the income of the portfolio as a whole. It is necessary to determine the investment proportion of a single asset in the portfolio. The expected return of the asset portfolio is the weighted average of the expected return of all assets in the asset portfolio, in which the weight x is the ratio of each asset investment to the total investment. The formula is:

$$E(R_p)=\sum_{i=1}^{n}x_i\,E(R_i)\quad \text{i}=1,2,3\ldots n;\;\; x_1+x_2+\cdots+x_n=1$$

1.1.3 Expected return

Expected rate of return is the expected value of future rate of return:

$$E(R) = \sum_{i=1}^{n} p_i R_i$$

1.2 Variance

1.2.1 Risk of an asset

The so-called risk is the change and uncertainty of the rate of return. Generally, investment risk is defined as the deviation of actual return from expected return, which can be measured mathematically by the variance of expected return. Formula is:

$$\sigma^2 = \sum_{i=1}^{n} h_i [r_i - E(R)]$$

The greater the variance between the random variable and the mathematical expectation means the greater the risk.

1.2.2 Portfolio risk

The variance of asset portfolio is not a simple weighted average of the variance of each asset, but the square of the deviation between the return of asset portfolio and its expected return.

In the case of two asset portfolios:

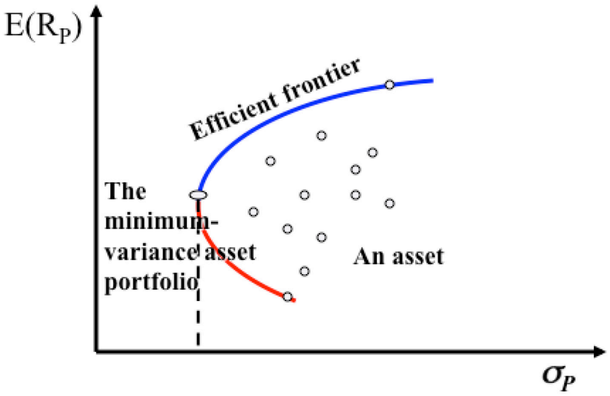
$$\sigma_p{}^2 = w_1{}^2 \sigma_1{}^2 + w_2{}^2 \sigma_2{}^2 + 2 w_1 w_2 \sigma_{1,2}$$

For the portfolio composed of N assets, the general formula of variance is：

$$\sigma_p{}^2 = \sum_{i=1}^{n} x_i{}^2 \sigma_i{}^2 - \sum_{i=1}^{n} \sum_{i=1}^{n} x_i x_j \, cov(x_i, x_j)$$

1.3 The Minimum-Variance Set and the Efficient Frontier

The portfolio composed of n basic securities has infinite combinations due to different weights. All these securities constitute a feasible set. Any given expected return has the minimum risk and any given risk level has the maximum expected return. The set of the portfolio is called Markowitz efficient set. In the feasible set, the portfolio represented by the point closest to the left is called the minimum variance portfolio. The blue line above the minimum variance portfolio represents the effective boundary or effective frontier, and the portfolio on the effective boundary is an effective portfolio.



1.4 Solution to the Markowitz Model

In order to find a solution, we can use Lagrange multipliers λ and μ. We form the Lagrangian:

$$L = \frac{1}{2}\sum_{i,j=0}^{n} w_i w_j \sigma_{i,j} - \lambda\left(\sum_{i=1}^{n} w_i r_i - r\right) - \mu\left(\sum_{i=1}^{n} w_i - 1\right)$$

We differentiate the Lagrangian with respect to each variable and set this derivative to zero.

For the two-variable case:

$$L = \frac{1}{2}\left({w_1}^2{\sigma_1}^2 + {w_2}^2{\sigma_2}^2 + w_1 w_2 \sigma_{1,2} + w_2 w_1 \sigma_{2,1}\right)$$
$$-\lambda(r_1 w_1 - r_2 w_2 - r) - \mu(w_1 + w_2 - 1)$$

Hence, we can obtain:

$${\sigma_1}^2 w_1 + w_2\sigma_{1,2} - \lambda r_1 - \mu = 0$$
$${\sigma_2}^2 w_2 + w_1\sigma_{2,1} - \lambda r_2 - \mu = 0$$

This gives us two equations. In this way, we get general form for n variables now can be written by obvious generalization.

$$\sum_{j=1}^{n} \sigma_{ij}\, w_j - \lambda r_i - \mu = 0$$

$$\sum_{i=1}^{n} r_i\, w_i = r$$

$$\sum_{i=1}^{n} w_i = 1$$

Notice that all n+2 equations are linear, so they can be solved with linear algebra methods.

2. The two-fund theorem

Points in the minimum-variance set should satisfy the system of n+2 linear equations:

$$\sum_{j=1}^{n} \sigma_{ij} - \lambda \overline{r_i} - \mu = 0 \quad \text{for i=1, 2, ..., n} \tag{a}$$

$$\sum_{i=1}^{n} \omega_i \, \overline{r_i} = \bar{r} \tag{b}$$

$$\sum_{i=1}^{n} \omega_i = 1 \tag{c}$$

Suppose the above equation has two known solutions, $\omega^1 = (\omega_1^1, \omega_2^1, \ldots, \omega_n^1)$, λ^1, μ^1 and $\omega^2 = (\omega_1^2, \omega_2^2, \ldots, \omega_n^2)$, λ^2, μ^2, corresponding to expected rate of return $\bar{r}^1$ and $\bar{r}^2$. We multiply the first by α and the second by $1 - \alpha$ to form a combination. The new combination could also satisfy the above n + 2 equations, corresponding to expected rate of return $\alpha\bar{r}^1$ and $(1 - \alpha)\bar{r}^2$.

We could check it: $\alpha\bar{r}^1 + (1 - \alpha)\bar{r}^2$ is a legitimate portfolio with a sum of weights of 1, and thus (c) is satisfied. The expected return is $\alpha\bar{r}^1 + (1 - \alpha)\bar{r}^2$ and thus (b) is satisfied for the value. What is more, both solutions make the left side of (a) equal to zero, and their combination does also. Thus (a) is satisfied. So We could safely draw a conclusion that the combination portfolio $\alpha\overline{\omega}^1 + (1 - \alpha)\overline{\omega}^2$ is also a solution, which represents a point in the minimum-variance set.

From the above we can conclude that: suppose ω^1 and ω^2 are two different portfolios in the minimum-variance set. When the value of α changes over $\left(-\infty, \ +\infty\right)$, the portfolios defined by $\alpha\overline{\omega}^1 + (1 - \alpha)\overline{\omega}^2$ is the entire minimum-variance set. We could easily find the two original solutions (on the upper portion of the minimum-variance set), which will generate all other efficient points (as well as all other points in the minimum-variance set).

This result is often stated in a form that has operational significance for investors:

The two-fund theorem: Two efficient funds (portfolios) can be established so that any efficient portfolio can be duplicated, in terms of mean and variance, as a combination of these two. In other words, all investors seeking efficient portfolios need only invest in combinations of these two funds.

According to the two-fund theorem, two mutual funds could provide a complete investment service for anyone. No one need to purchase individual stocks separately; they only need to purchase shares in the mutual funds. This conclusion is based on the assumption that everyone only cares about mean and variance; that everyone has the same assessment of the means, variances, and covariances; and that a single-period framework is appropriate. All the assumptions are tenuous. However, as an investor, if you don't have time and interest to do detailed evaluation, you might choose two funds managed by people whose assessments you trust, and invest in the two funds.

In order to solve (a-c) for all values of $\bar{r}$, we just need to find the two original solutions and then form combinations of them. A particularly simple way to specify two solutions is to specify values of λ and μ. Convenient choices are: (a) $\lambda = 0, \mu = 1$ and (b) $\lambda = 1, \mu = 0$. The solution obtained may not meet the constraint $\sum_{i=1}^{\pi} \omega_i = 1$, but this can be remedied by normalizing all ω_i's by a common scale factor. The solution obtained from (a) ignores the constraint of the expected rate of return, so it is the minimum-variance point.

3. The one-fund theorem

If risk-free lending and borrowing are allowed, the efficient set consists of a single straight line, that is, the top of the triangular feasible region. This line is tangent to the original feasible set of risky assets. There will be a point F in the original feasible set that is on the line segment defining the overall efficient set (Obviously, F is the tangent point between the effective set line and the initial feasible set). lt is clear that any efficient point (any point on the line) can be expressed as a combination of the asset and the risk-free asset. By changing the weights of the two assets, we can get different effective points (the weight of risk-free assets can be negative, that is, it can buy risky assets through borrowing funds, resulting in leverage.). The portfolio at the cut-off point can be considered as a fund composed of multiple assets and sold as an asset unit.

The role of this fund is summarized by the following statement:

The one-fund theorem: There is a single fund F of risky assets such that any efficient portfolio can be constructed as a combination of the fund F and the risk-free asset.

That is, as long as the risk-free assets are determined, the risk portfolio is unique and determined. According to the one-fund theorem, all investors will buy a single fund composed of risky assets, and they may borrow or lend funds at a risk-free interest rate.

4. Optimizing: Maximizing the Sharpe Ratio

4.1 Sharpe Ratio

The Sharpe ratio of a given portfolio $x = [x_1, x_2, \dots, x_n]^T$ is the ratio of its expected excess return (more than risk-free return) to its volatility (standard deviation) of return:

$$Sharpo\ ratio = \frac{\mu^T x - r_f}{\sqrt{x^T V x}} \qquad (a)$$

μ : the vector of mean returns
V : the covariance matrix of return
x : the vector of asset allocation
r_f : the risk-free return

4.2 The Programming Model

Indices & Sets

$i \in I$: names of stocks

Data

μ_i : the average return of stock i
σ_{ij} : the covariance of stock i and stock j
r_f : the risk-free return
V : the covariance matrix of return

$$V = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_n^2 \end{bmatrix}$$

μ : the vector of mean returns

$$\mu = [\mu_1, \mu_2, \dots, \mu_n]^T$$

Decision variables

x_i : the proportion of stock i
x : the vector of asset allocation

$$x = [x_1, x_2, \dots, x_n]^T$$

Formulation

$$maximize\ \frac{\mu^T x - r_f}{\sqrt{x^T V x}} \qquad (b)$$

$s.t.$

$$\sum_i x_i = 1$$

$$0 \le x$$

Assumption: There exists a vector x satisfying:

$$\sum_i x_i = 1$$

$$0 \leq x$$

Such that

$$\mu^T x - r_f > 0.$$

This assumption is easy to satisfy, it simply says that we can find a portfolio of stocks that exceeds the risk-free return.

We can easily rewrite the formulation

$$f(x) \ = \frac{\mu^T x - r_f}{\sqrt{x^T V x}} = \frac{\mu^T x - r_f \sum_i x_i}{\sqrt{x^T V x}} \ = \frac{\hat{\mu}^T x}{\sqrt{x^T V x}},$$

where for each index i, we define $\hat{\mu}_i = \mu_j - r_f$.

We can prove that for any scalar $\lambda > 0$, $f(\lambda x) = f(x)$.

$$f(\lambda x) \ = \frac{\hat{\mu}^T \lambda x}{\sqrt{\lambda x^T V \lambda x}} = \frac{\hat{\mu}^T x}{\sqrt{x^T V x}} \ = f(x)$$

We consider the problem:

$$maximize \ \frac{1}{\sqrt{y^T V y}} \qquad (c)$$

$s.t.$

$$\hat{\mu}^T y \ = \ 1$$
$$0 \leq y$$

Suppose the optimal solution of (c) is $\bar{y}$. Because $\hat{\mu}^T \bar{y} \ = \ 1$, we know that $\bar{y}$ is not identically zero, and by $0 \leq \bar{y}$, $\sum_i y_i > 0$.

Define the vector

$$\bar{x} = \frac{\bar{y}}{\sum_i y_i}.$$

with the limitation that

$$\sum_i x_i \ = 1.$$

Therefore, $\bar{x}$ is the feasible solution for (b).

In addition, we can calculate that

$$f(\bar{x}) = f(\bar{y}) \ = \frac{1}{\sqrt{\bar{y}^T V \bar{y}}}.$$

Therefore, we prove that the optimal value of (b) is at least as good as that of (c). The converse is proved in a similar way. We conclude that (b) and (c) are equivalent.

We can solve the problem (b) by just solving problem (c). And we can rewrite problem (c) as a standard convex quadratic programming

$$minimize\ \sqrt{y^T V y} \qquad (d)$$

$$s.t.$$

$$\hat{\mu}^T y = 1$$
$$0 \leq y$$

In the python program, we use Sequential Least Squares Programming optimization algorithm to get the optimal solution.

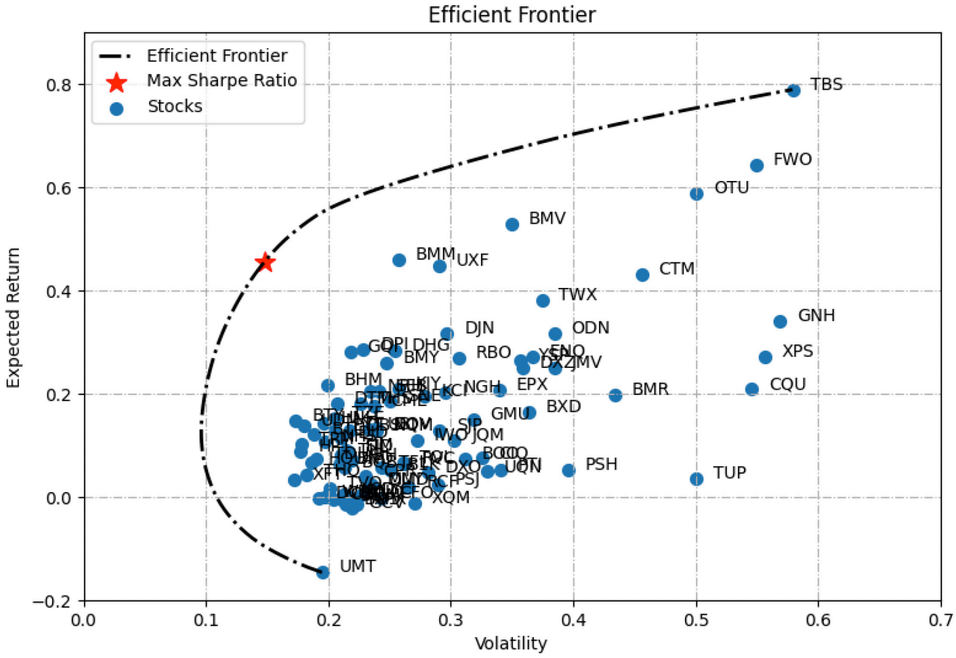
5. Portfolio Realization Process and Final Income

5.1 Stock pool filtering

From the project assignment, we should manage $100,000 starting from January 1, 2018, until December 31, 2019. One alternative is that you can invest them in stocks or government bonds. The bond has a 3% annual return, and it matures every six months. We just buy the stocks and hold them until 2020-1-1 because the volatility of the stocks are greatly increasing from 2018 to 2020.

Firstly, we would abandon the stocks which went on the market later than 2015.1.1. Because of lack of sufficient data to get the accurate prediction in the future, it is wise to abandon these stocks (Including B3N / BQU / JFM / T43 / XUD)

Secondly, according to the Mean-Variance Model, we choose the stocks which are the point close to the efficient frontier. These are the efficient portfolios, in the sense that they provide the best mean-variance combinations for most investors. We limit our investigation to this frontier.



5.2 Trading strategy

5.2.1 Determine the proportions (ω) of each stock

Sharpe Ratio is the factor we choose, it represents the additional amount of return that an investor receives per unit of increase in risk. By seeking for the largest Sharpe Ratio, we find the proportions of each stock in the investment portfolio.

$$\text{Sharpe Ratio} = \frac{\mathrm{ER}_{\mathrm{P}} - ER_f}{\sigma_p}$$

The result of ω in investment portfolios purchased on 2018.1.2:

Stocks	Weights	Stocks	Weights
BMM	0.2039	GQI	0.1383
BMV	0.1477	OTU	0.0047
BMY	0.0158	RBO	0.0080
CTM	0.0084	SNE	0.0349
DHG	0.0061	TBS	0.0976
DJN	0.0685	TWX	0.0194
DPI	0.0126	UXF	0.1572
FWO	0.0635	YSP	0.0132

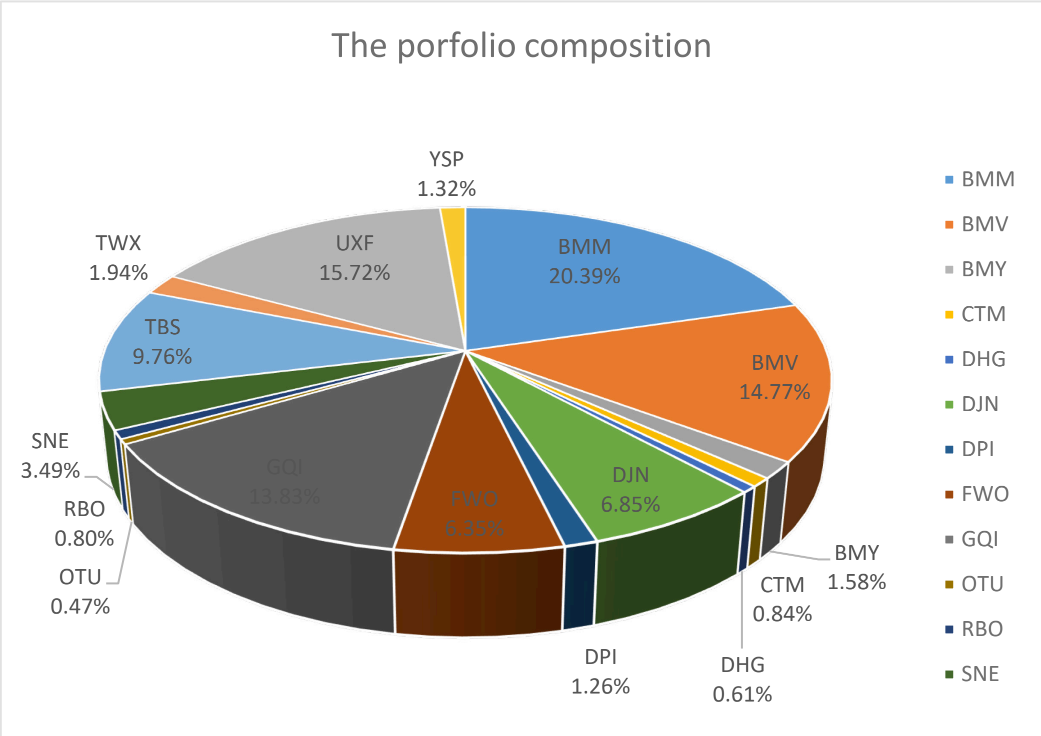


Chart 1 the proportions (ω)of investment portfolios

5.2.2 Trading process

To avoid the greatly increasing volatility from 2018 to 2020， we prefer to only purchase the stocks and bond on 2018-1-2 and hold them until 2020. 2018.1.2-2020.1.2.

Stock: We decide to purchase an investment portfolio (the proportions (ω)of investment portfolios in Table 1) on 2018-1-2.

Bond: We purchase the bond with the rest money on the bond purchase date.

Cash: Considering the low interested rate， we would not hold any cash.

In all, the value of the portfolio is $167458.30.

Reference:

[1] Cornuejols G, Tütüncü R. Optimization methods in finance[M]. Cambridge University Press, 2006.
[2] Luenberger D G. Investment science[M]. Oxford university press, 1998.