Module 4: Dictionaries and Balanced Search Trees

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Dictionary ADT

A *dictionary* is a collection of *items*, each of which contains

- a *key*
- some *data*,

and is called a *key-value pair* (KVP). Keys can be compared and are (typically) unique.

Operations:

- *search*(k)
- *insert*(k, v)
- *delete*(k)
- optional: *join*, *isEmpty*, *size*, *etc.*

Examples: symbol table, license plate database

Elementary Implementations

Common assumptions:

- Dictionary has n KVPs
- Each KVP uses constant space
  (if not, the "value" could be a pointer)
- Comparing keys takes constant time

Unordered array or linked list

search $\Theta(n)$

insert $\Theta(1)$

delete $\Theta(n)$ (need to search)

Ordered array

search $\Theta(\log n)$

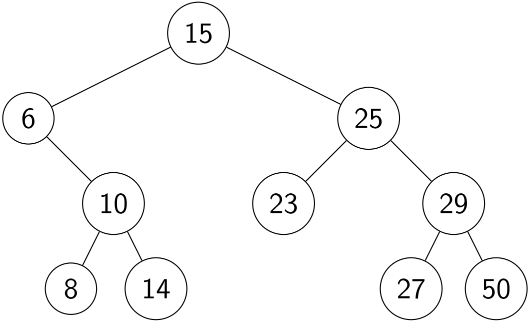
insert $\Theta(n)$

delete $\Theta(n)$

Binary Search Trees (review)

Structure A BST is either empty or contains a KVP, left child BST, and right child BST.

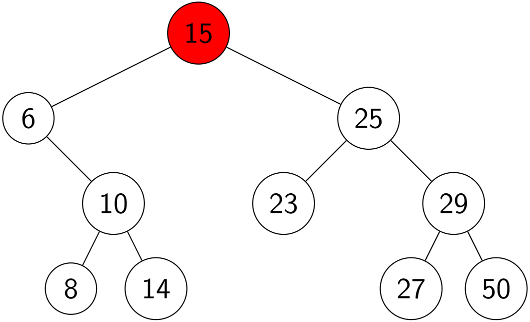
Ordering Every key k in $T.left$ is less than the root key.
Every key k in $T.right$ is greater than the root key.



BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

Example: *search*(24)



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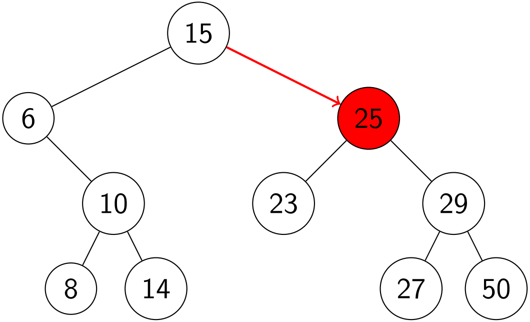

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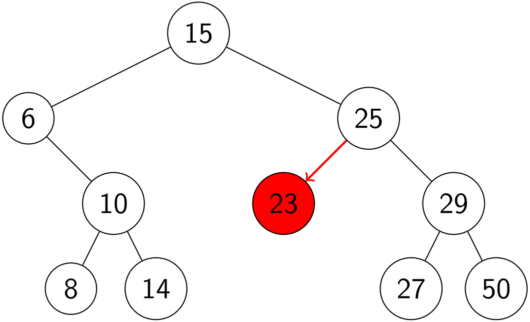

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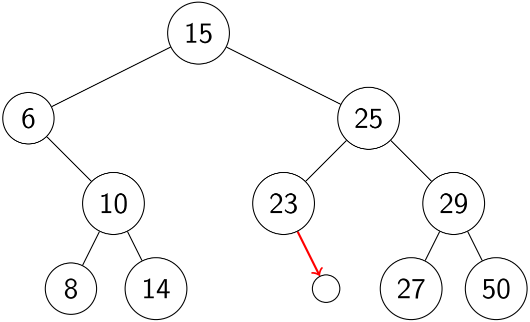
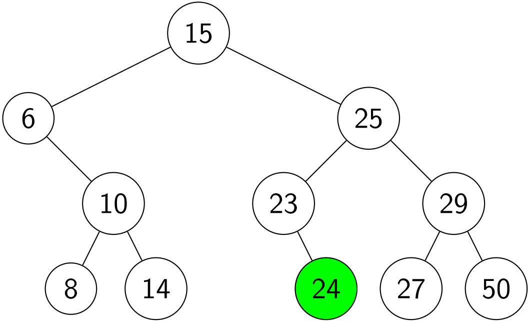


BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

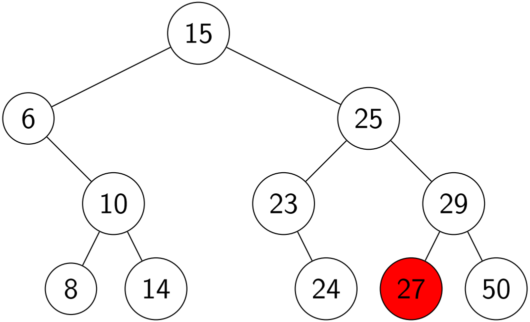
insert(k, v) Search for k, then insert (k, v) as new node

Example: *insert*$(24, \ldots)$



BST Delete

- If node is a leaf, just delete it.



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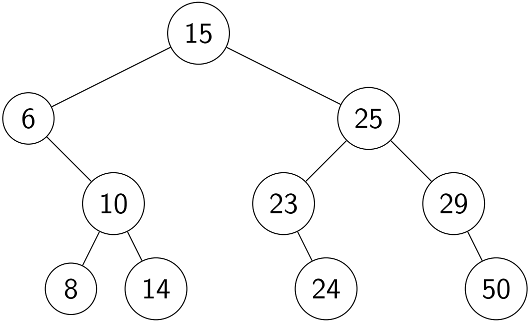
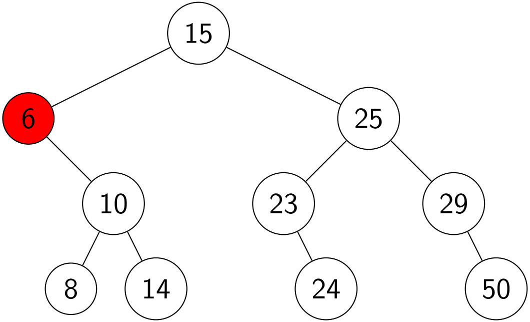


BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up



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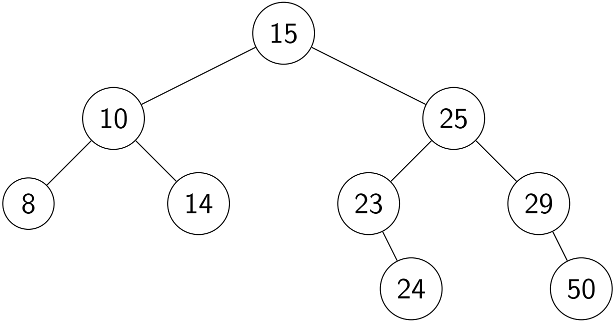

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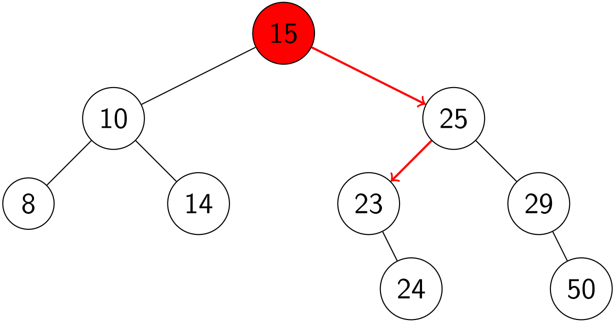

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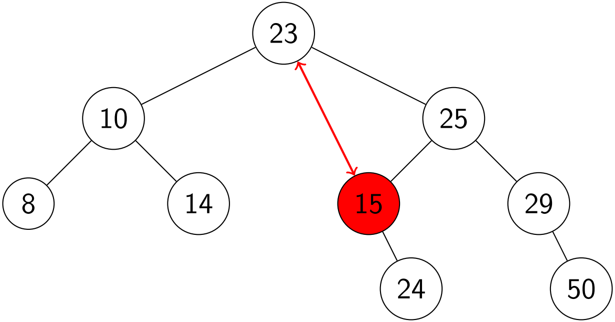

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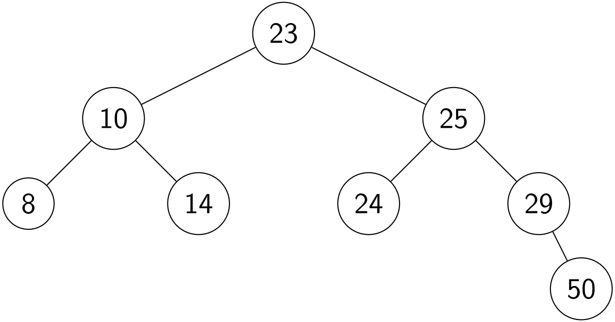

Height of a BST

search, *insert*, *delete* all have cost $\Theta(h)$, where
$h =$ height of the tree $=$ max. path length from root to leaf

If n items are *insert*ed one-at-a-time, how big is h?

- Worst-case:

Height of a BST

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- Worst-case: $n - 1 = \Theta(n)$
- Best-case:

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- Average-case:

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- Worst-case: $n - 1 = \Theta(n)$
- Best-case: $\lfloor \lg(n) \rfloor = \Theta(\log n)$
- Average-case: $\Theta(\log n)$
  (just like recursion depth in *quick-sort1*)

AVL Trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962, an *AVL Tree* is a BST with an additional structural property:
The heights of the left and right subtree differ by at most 1.

(The height of an empty tree is defined to be -1.)

At each non-empty node, we store $height(R) - height(L) \in \{-1, 0, 1\}$:

- -1 means the tree is *left-heavy*
- 0 means the tree is *balanced*
- 1 means the tree is *right-heavy*

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- -1 means the tree is *left-heavy*
- 0 means the tree is *balanced*
- 1 means the tree is *right-heavy*

- We could store the actual height, but storing balances is simpler and more convenient.

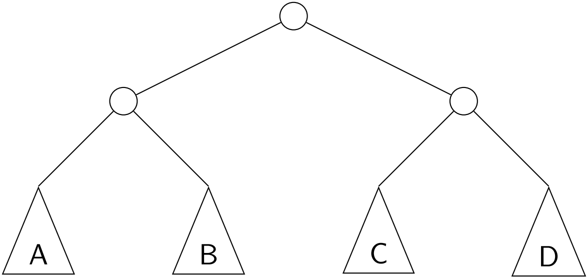
AVL insertion

To perform *insert*(T, k, v):

- First, insert (k, v) into T using usual BST insertion
- Then, move up the tree from the new leaf, updating balance factors.
- If the balance factor is -1, 0, or 1, then keep going.
- If the balance factor is ± 2, then call the *fix* algorithm to "rebalance" at that node. We are done.

How to "fix" an unbalanced AVL tree

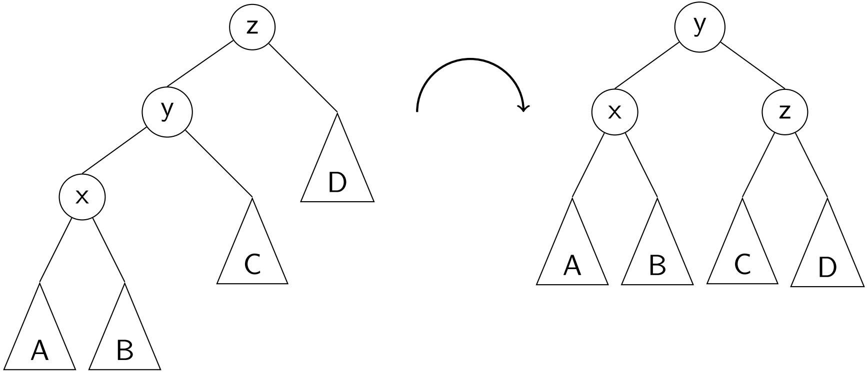
Goal: change the *structure* without changing the *order*



Notice that if heights of A, B, C, D differ by at most 1, then the tree is a proper AVL tree.

Right Rotation

This is a *right rotation* on node z:



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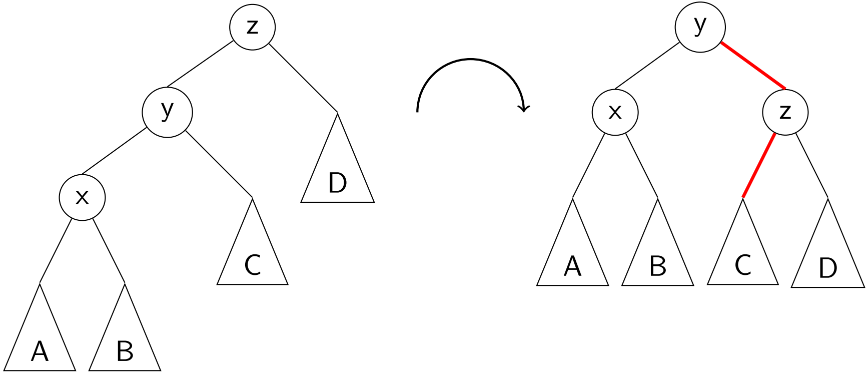
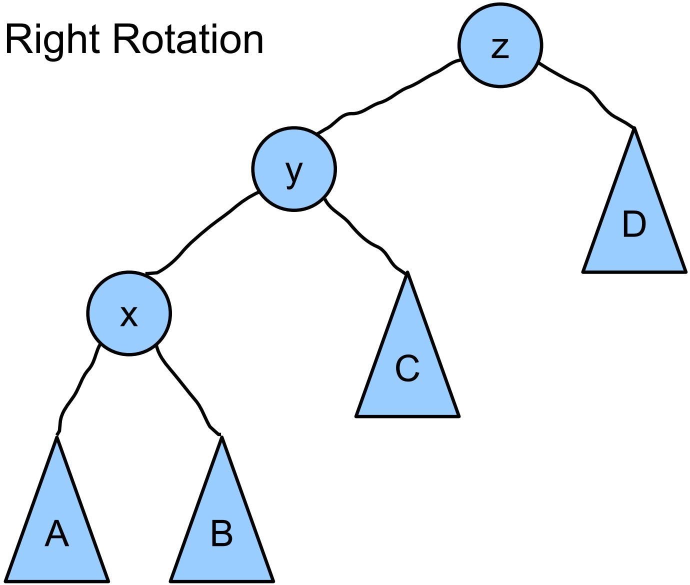


Note: Only two edges need to be moved, and two balances updated. Useful to fix left-left imbalance.

Again . . .



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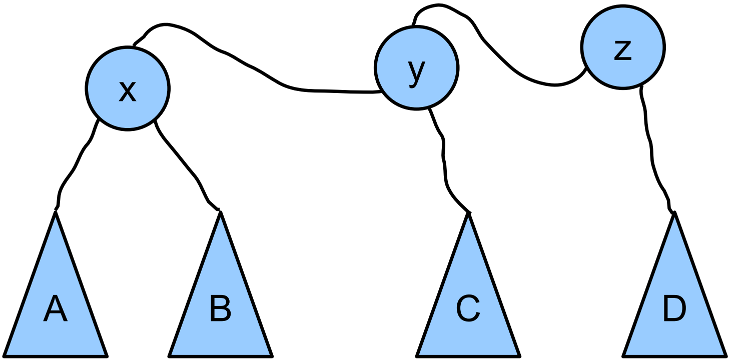

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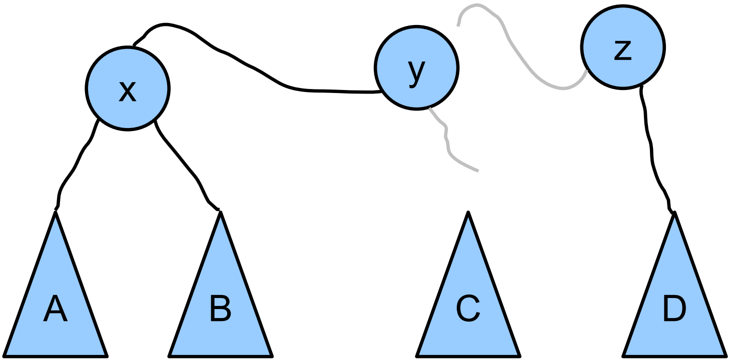

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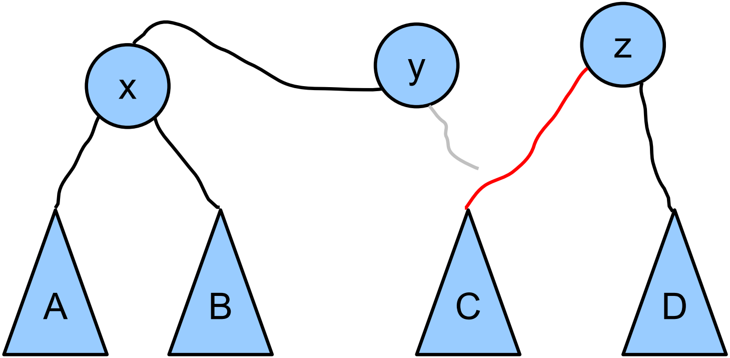

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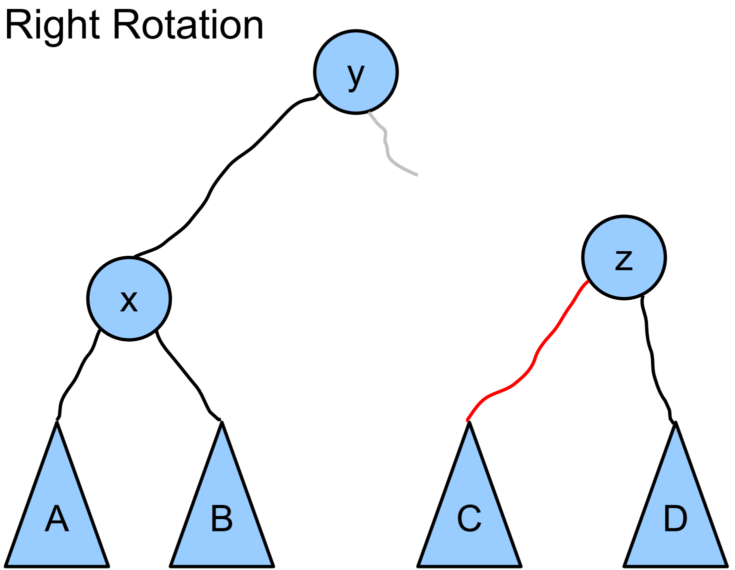

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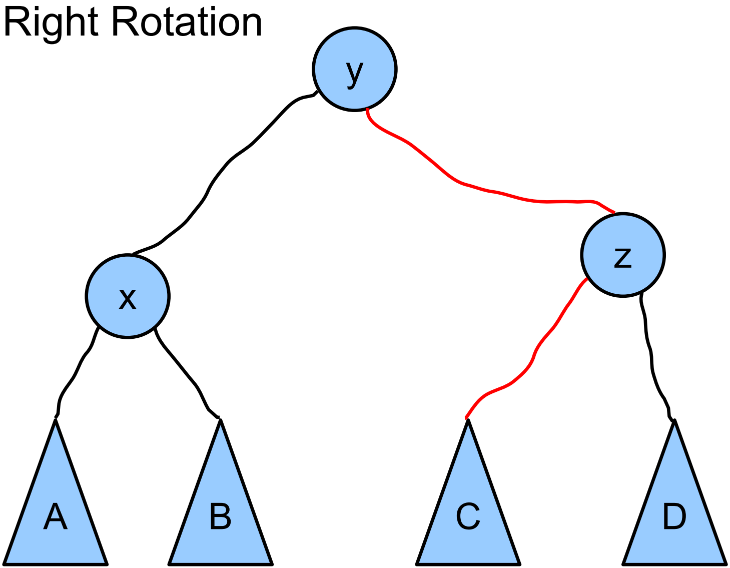
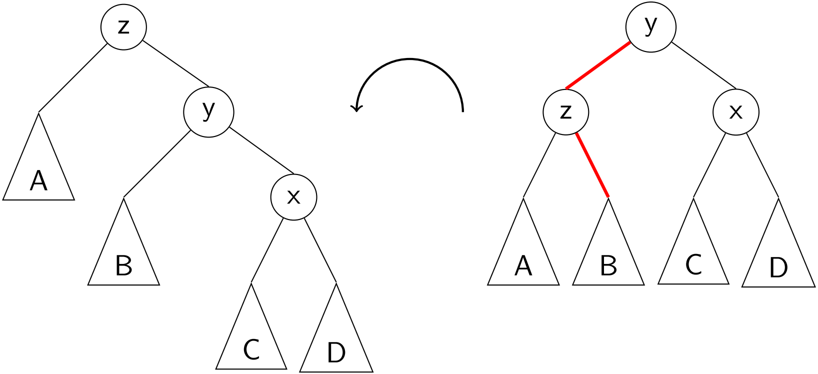


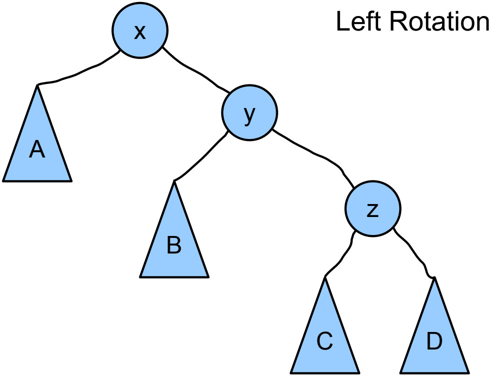
Left Rotation

This is a *left rotation* on node z:



Again, only two edges need to be moved and two balances updated.
Useful to fix right-right imbalance.

Again ...



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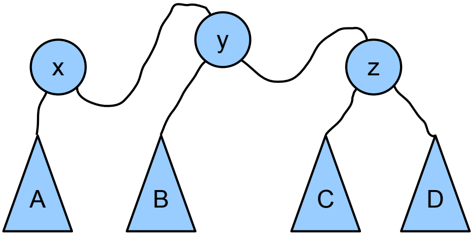
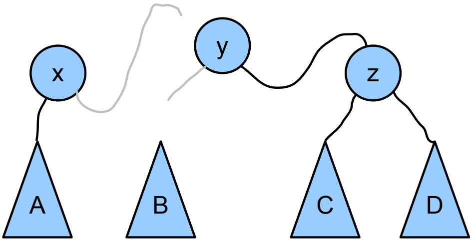


Again ...

Left Rotation



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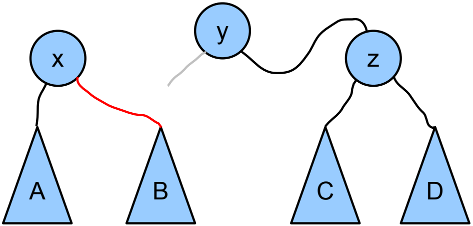

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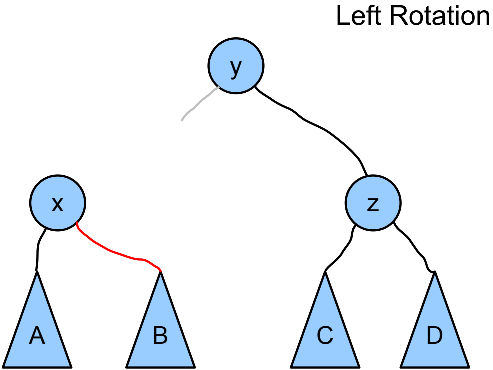

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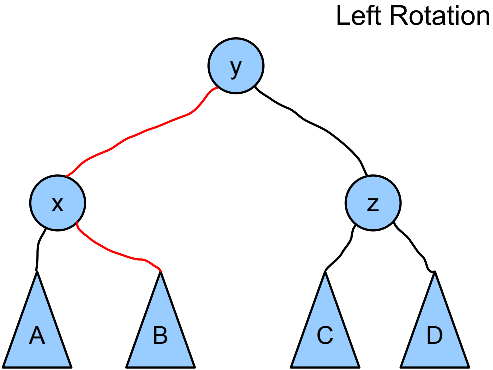


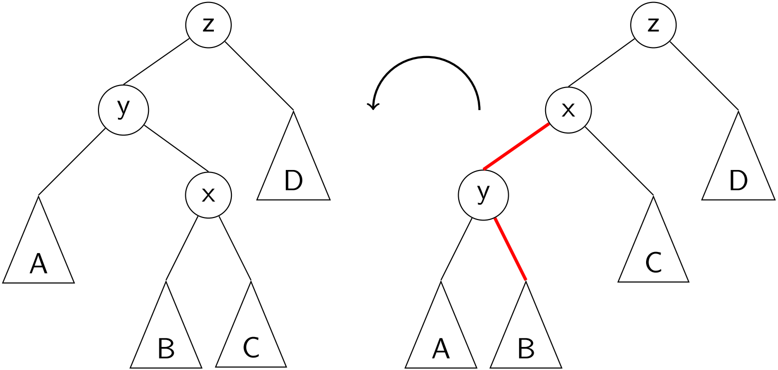
Pseudocode for rotations

rotate-right(T)
T: AVL tree
returns rotated AVL tree
1. $newroot \leftarrow T.left$
2. $T.left \leftarrow newroot.right$
3. $newroot.right \leftarrow T$
4. **return** $newroot$

rotate-left(T)
T: AVL tree
returns rotated AVL tree
1. $newroot \leftarrow T.right$
2. $T.right \leftarrow newroot.left$
3. $newroot.left \leftarrow T$
4. **return** $newroot$

Double Right Rotation

This is a *double right rotation* on node z:



First, a left rotation on the left subtree (y). Second, a right rotation on the whole tree (z).

Useful for left-right imbalance.

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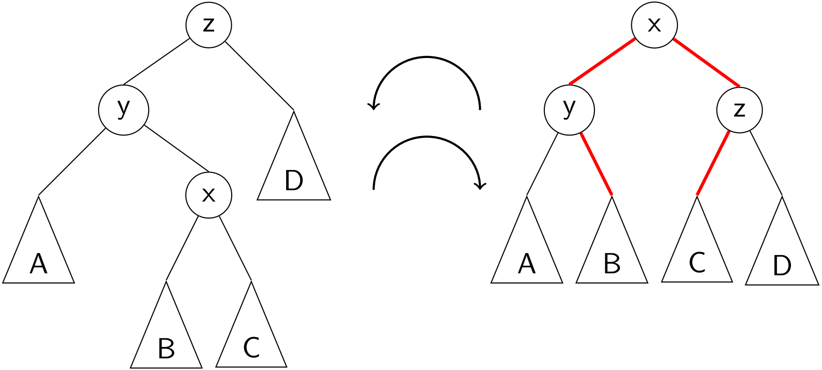

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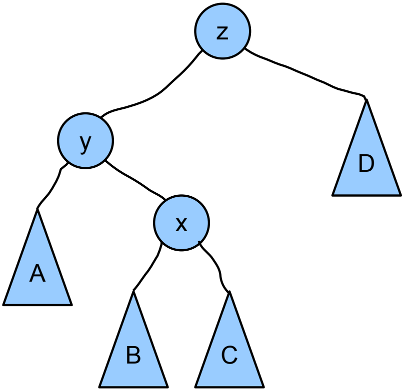

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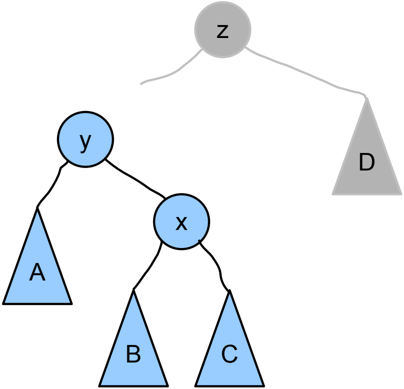

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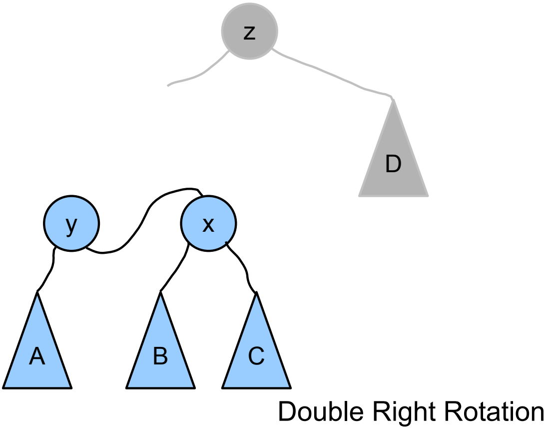

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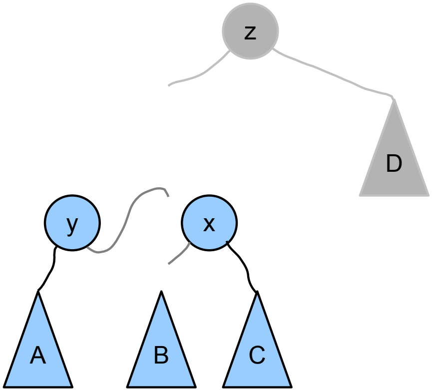

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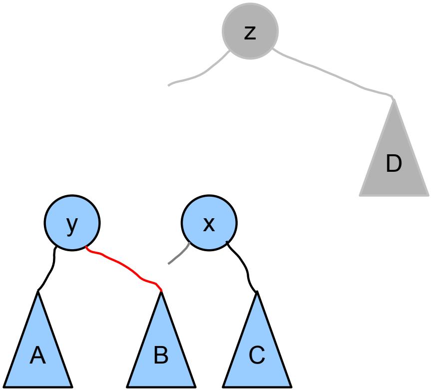

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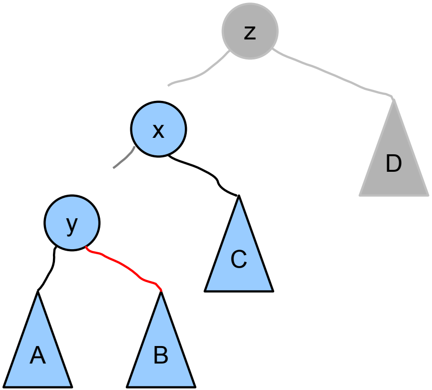

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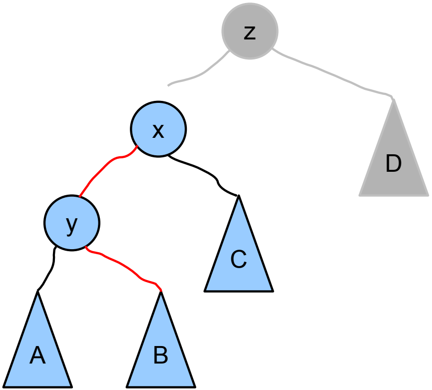

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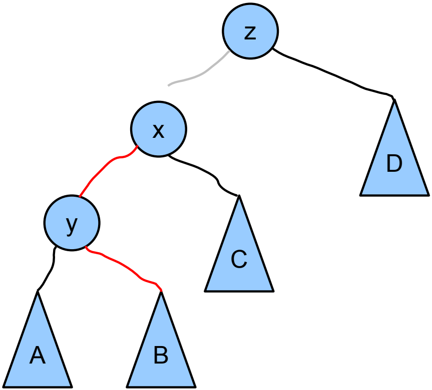

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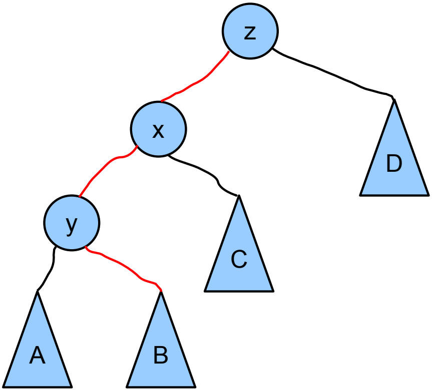

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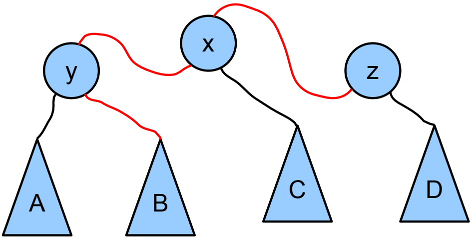

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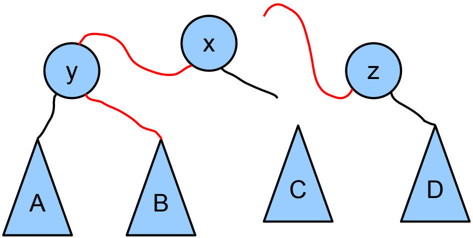

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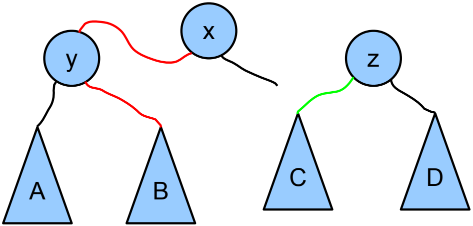

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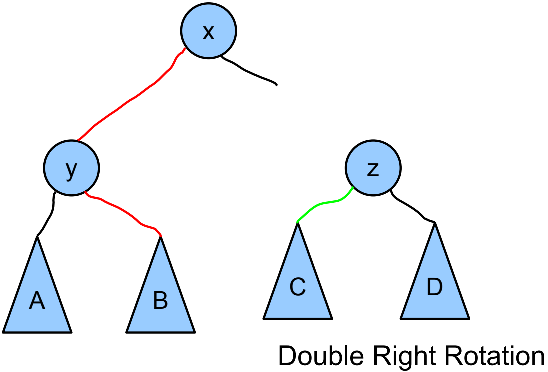

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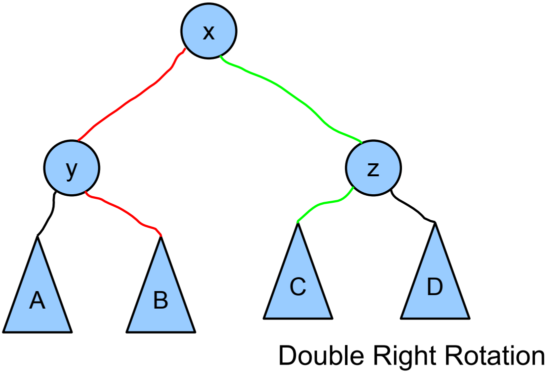

Double Left Rotation

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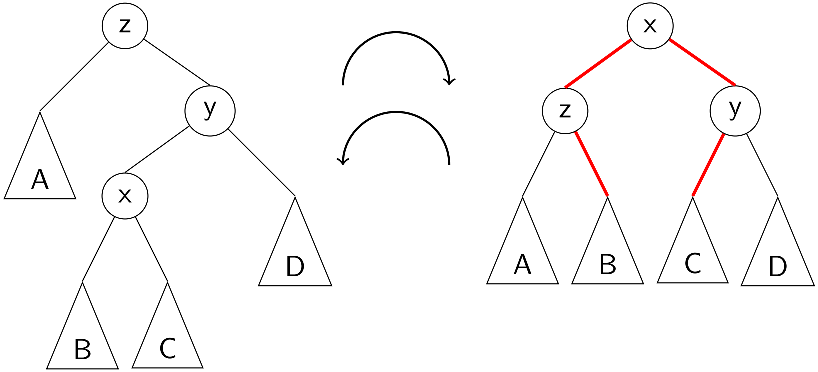


Right rotation on right subtree (y), followed by left rotation on the whole tree (z).
Useful for right-left imbalance.

Fixing a slightly-unbalanced AVL tree

Idea: Identify one of the previous 4 situations, apply rotations

```
fix(T)
T: AVL tree with T.balance = ±2
returns a balanced AVL tree
    if T.balance = −2 then
        if T.left.balance = 1 then
            T.left ← rotate-left(T.left)
        return rotate-right(T)
    else if T.balance = 2 then
        if T.right.balance = −1 then
            T.right ← rotate-right(T.right)
        return rotate-left(T)
```

AVL Tree Operations

search: Just like in BSTs, costs $\Theta(height)$

insert: Shown already, total cost $\Theta(height)$

- *fix* restores the height of the tree it fixes to what it was,
- so *fix* will be called *at most once*.

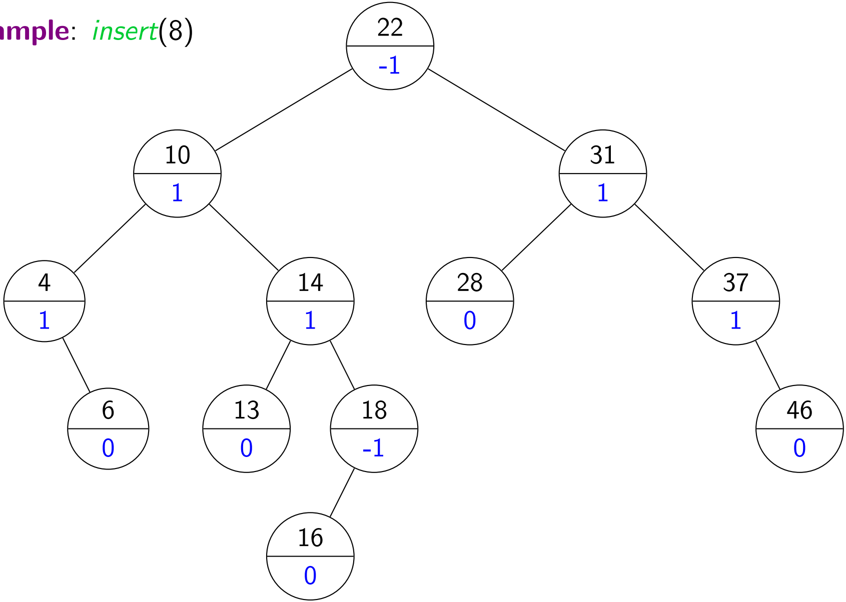
delete: First search, then swap with successor (as with BSTs), then move up the tree and apply *fix* (as with *insert*).

- *fix* may be called $\Theta(height)$ times.

Total cost is $\Theta(height)$.

AVL tree examples

Example: *insert*(8)



AVL tree examples

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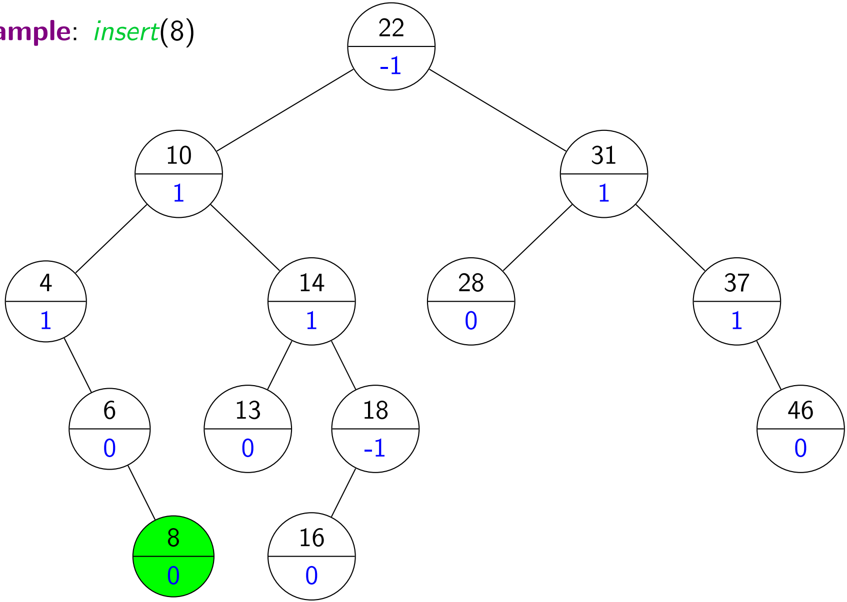

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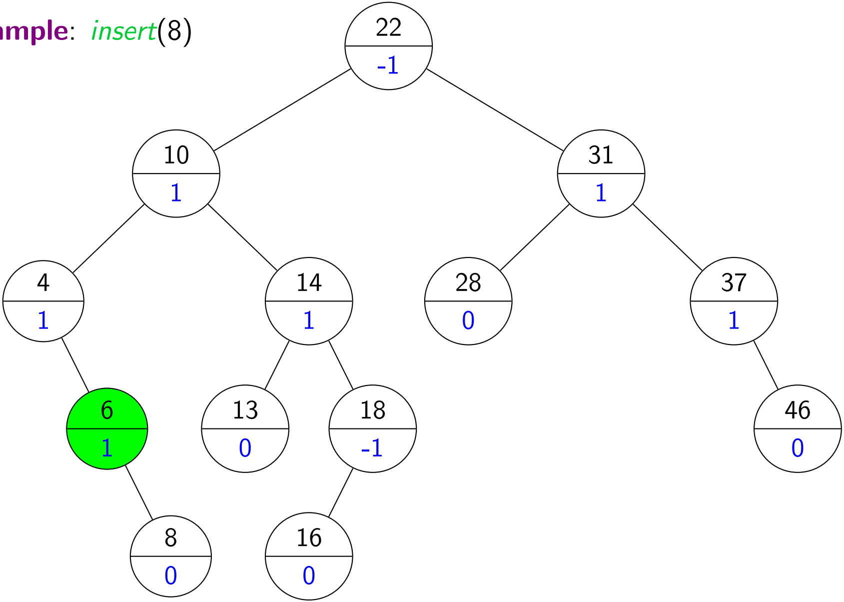

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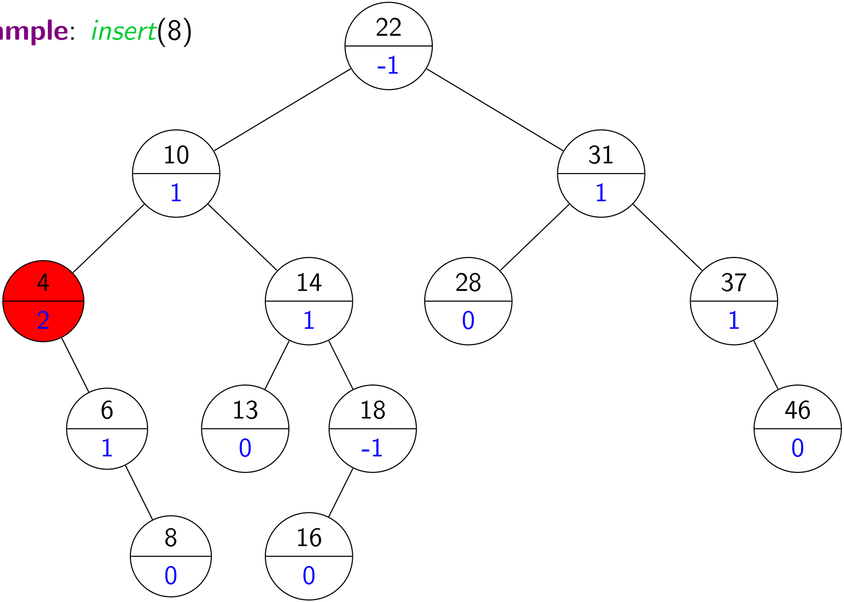

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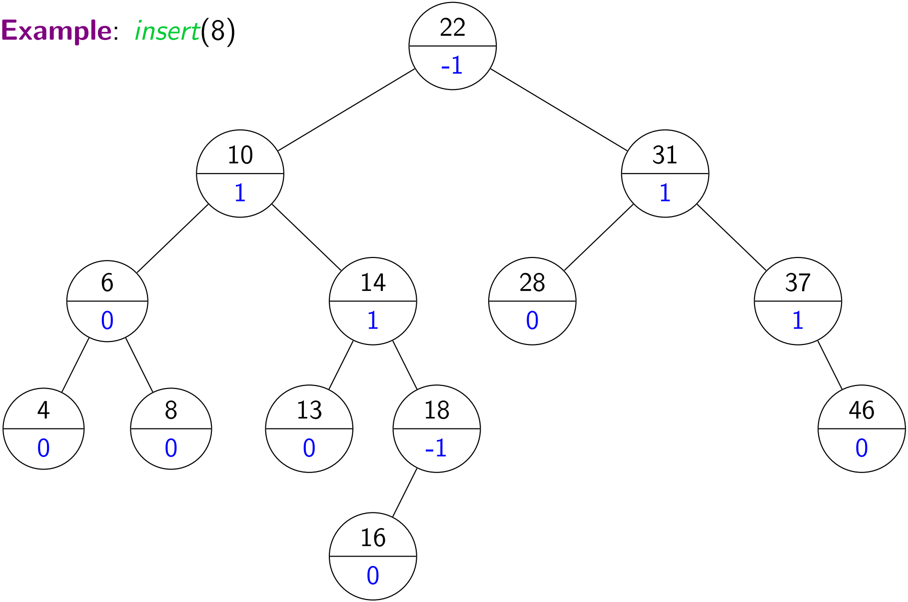
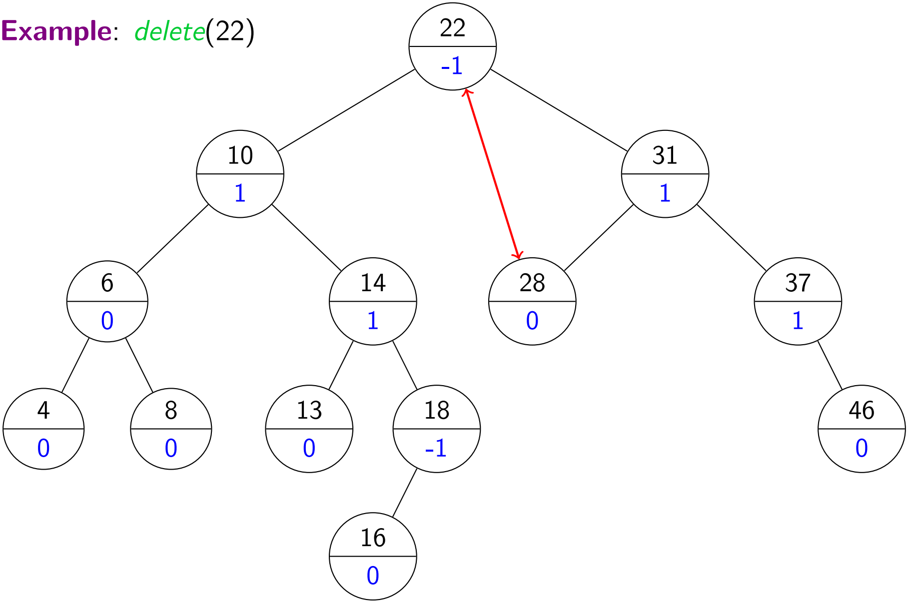


AVL tree examples

Example: *delete*(22)



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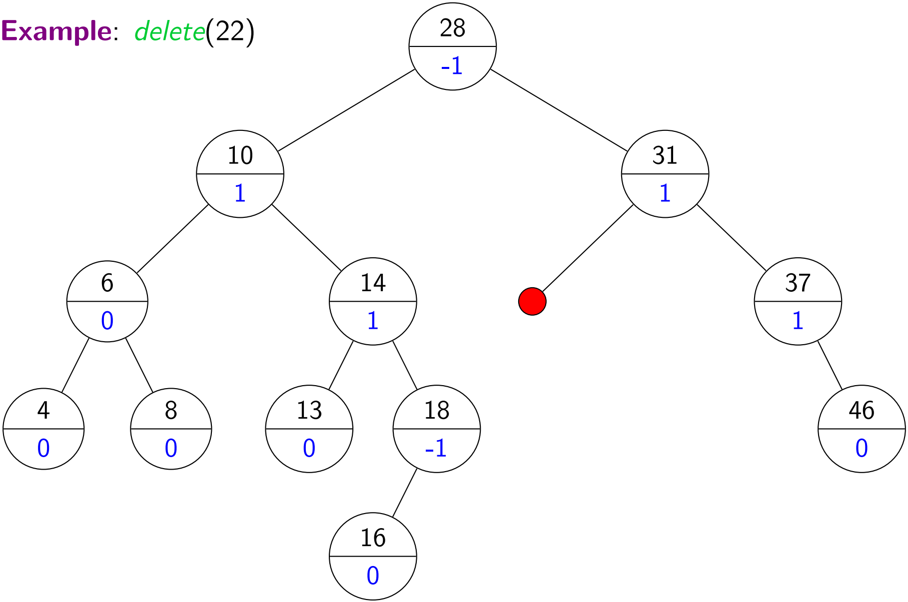

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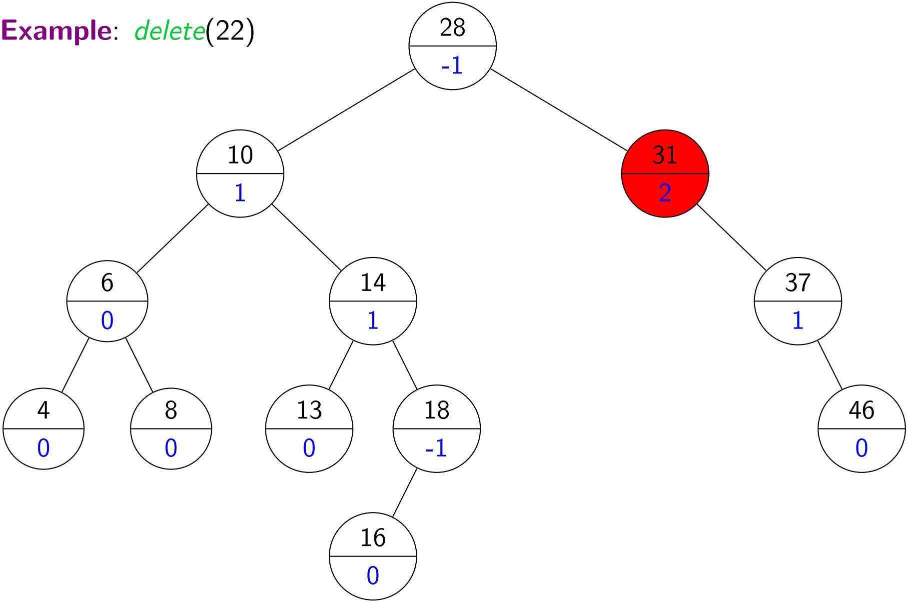

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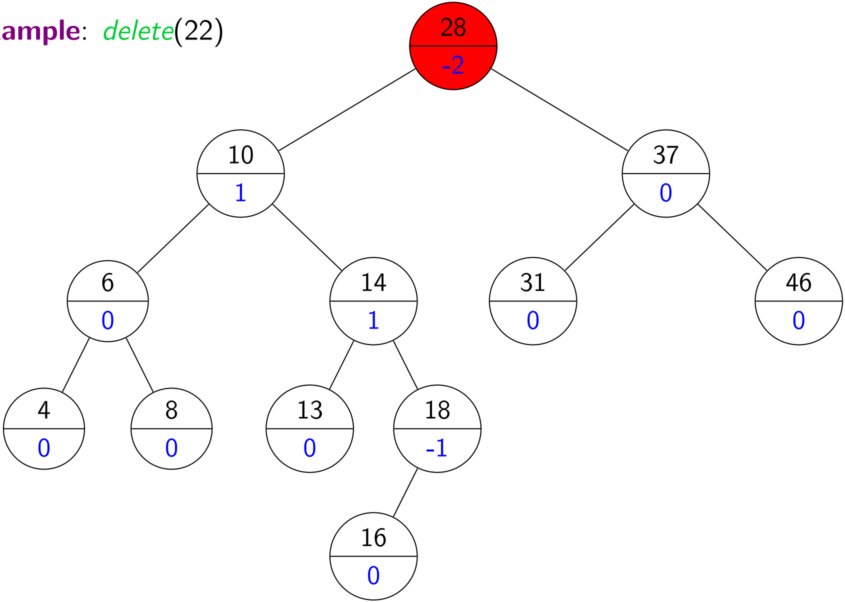

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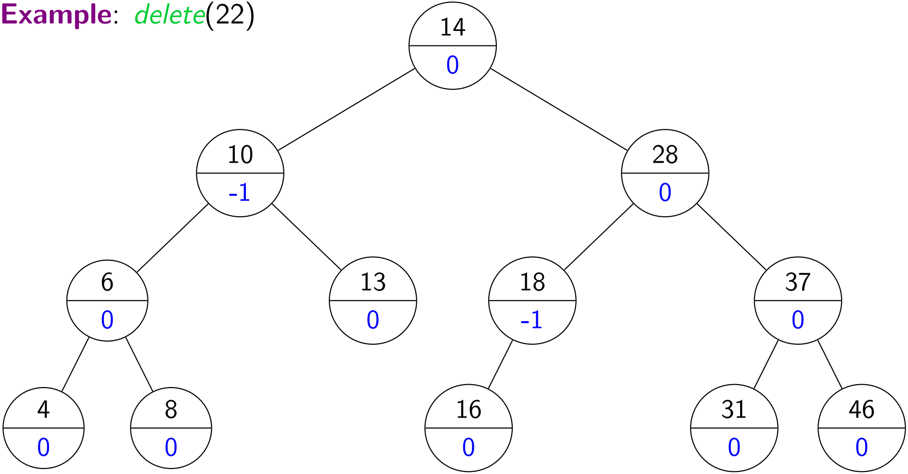

Height of an AVL tree

Define $N(h)$ to be the *least* number of nodes in a height-h AVL tree.

One subtree must have height at least $h-1$, the other at least $h-2$:

$$N(h) = \begin{cases} 1 + N(h-1) + N(h-2), & h \geq 1 \\ 1, & h = 0 \\ 0, & h = -1 \end{cases}$$

What sequence does this look like?

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What sequence does this look like? The Fibonacci sequence!

$$N(h) = F_{h+3} - 1 = \left\lceil \frac{\varphi^{h+3}}{\sqrt{5}} \right\rfloor - 1, \text{ where } \varphi = \frac{1+\sqrt{5}}{2}$$

AVL Tree Analysis

Easier lower bound on $N(h)$:

$$N(h) > 2N(h-2) > 4N(h-4) > 8N(h-6) > \cdots > 2^i N(h-2i) \geq 2^{\lfloor h/2 \rfloor}$$

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Since $n > 2^{\lfloor h/2 \rfloor}$, $h \leq 2 \lg n$,
and thus an AVL tree with n nodes has height $O(\log n)$.
Also, $n \leq 2^{h+1} - 1$, so the height is $\Theta(\log n)$.

$\Rightarrow$ *search*, *insert*, *delete* all cost $\Theta(\log n)$.