Module 5: Dictionaries II

CS 240 - Data Structures and Data Management

Mark Petrick

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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Dictionary ADT: Review

A *dictionary* is a collection of *key-value pairs* (KVPs), supporting operations *search*, *insert*, and *delete*.

Realizations

- **Unordered array or linked list**: $\Theta(1)$ insert, $\Theta(n)$ search and delete
- **Ordered array**: $\Theta(\log n)$ search, $\Theta(n)$ insert and delete
- **Balanced search trees** (AVL trees):
  $\Theta(\log n)$ search, insert, and delete

Self-Organizing Search

- Unordered linked list
  search: $\Theta(n)$, *insert*: $\Theta(1)$, *delete*: $\Theta(1)$ (after a search)
- Linear search to find an item in the list
- Is there a more useful ordering?

Self-Organizing Search

- Unordered linked list
  search: $\Theta(n)$, *insert*: $\Theta(1)$, *delete*: $\Theta(1)$ (after a search)
- Linear search to find an item in the list
- Is there a more useful ordering?

- No: if items are accessed equally likely
- Yes: otherwise (we have a probability distribution for items)

- Optimal static ordering: sorting items by their probabilities of access in non-increasing order minimizes the expected cost of Search.
- Proof Idea: For any other ordering, exchanging two items that are out-of-order according to their access probabilities makes the total cost decrease.

Optimal Static Ordering

A list of elements ordered by non-increasing probability of access has minimum expected access cost

- $L = \langle\, x_1, x_2, \ldots, x_n \,\rangle$
  Expected access cost in L is
  $E(L) = \sum_{i=1}^{n} P(x_i)T(x_i) = \sum_{i=1}^{n} P(x_i) \cdot i$
  $P(x_i)$ - access probability for x_i
  $T(x_i)$ - position of x_i in L
- Example
  $P(a) = 0.3$ $P(b) = 0.5$ $P(c) = 0.2$
  $L = \langle\, a, b, c \,\rangle$
  $E(L) = 0.3 + 0.5 * 2 + 0.2 * 3 = 1.9$
  $L = \langle\, b, a, c \,\rangle$
  $E(L) = 0.5 + 0.3 * 2 + 0.2 * 3 = 1.7$

Optimal Static Ordering

A list of elements ordered by non-increasing probability of access has minimum expected access cost

Proof by Contradiction

- $L = \langle\, x_1, \ldots, x_k, x_{k+1}, \ldots, x_n \,\rangle$
  Suppose the access cost of L is optimal and there is k such that $P(x_k) < P(x_{k+1})$

$$E(L) = P(x_k) \cdot k + P(x_{k+1}) \cdot (k+1) + \sum_{i \neq k, k+1} P(x_i) \cdot i$$

- Create another list L' by swapping x_k and x_{k+1}.
  $L' = \langle\, x_1, \ldots, x_{k+1}, x_k \ldots, x_n \,\rangle$

$$E(L') = P(x_{k+1}) \cdot k + P(x_k) \cdot (k+1) + \sum_{i \neq k, k+1} P(x_i) \cdot i$$

- $E(L') - E(L) = P(x_k) - P(x_{k+1}) < 0 \Rightarrow E(L') < E(L)$
  Contradiction

Dynamic Ordering

- What if we do not know the access probabilities ahead of time?
- Move-To-Front(MTF): Upon a successful search, move the accessed item to the front of the list
- Transpose: Upon a successful search, swap the accessed item with the item immediately preceding it

Dynamic Ordering

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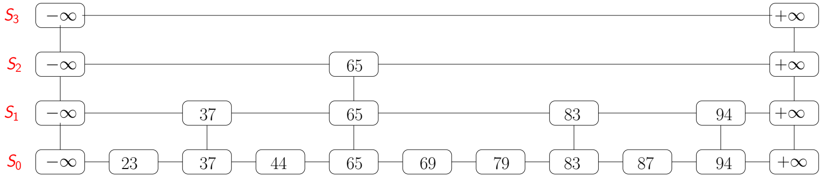
Performance of dynamic ordering:

- Both can be implemented in arrays or linked lists.
- Transpose does not adapt quickly to changing access patterns.
- MTF Works well in practice.
- Theoretically MTF is "competitive":
  No more than twice as bad as the optimal "offline" ordering.

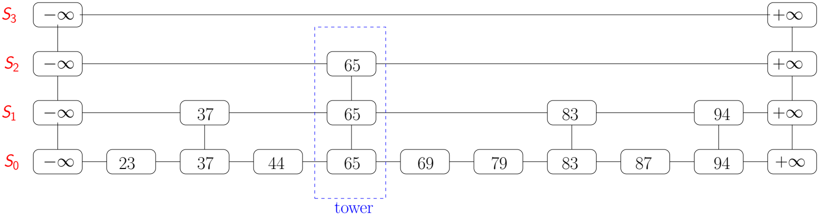
Skip Lists

- Randomized data structure for dictionary ADT
- A hierarchy of ordered linked lists
- A skip list for a set S of items is a series of lists $S_0, S_1, \cdots, S_h$ such that:
  - Each list S_i contains the special keys $-\infty$ and $+\infty$
  - List S_0 contains the keys of S in non-decreasing order
  - Each list is a subsequence of the previous one, i.e., $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$
  - List S_h contains only the two special keys



Skip Lists

- A skip list for a set S of items is a series of lists $S_0, S_1, \cdots, S_h$
- A two-dimensional collection of positions: levels and towers
- Traversing the skip list: after(p), below(p)



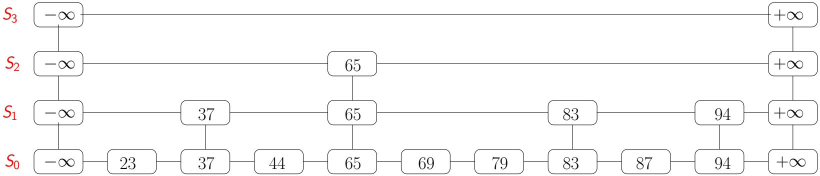
Search in Skip Lists

skip-search(L, k)
L: A skip list, k: a key
    $p \leftarrow$ topmost left position of L
    $S \leftarrow$ stack of positions, initially containing p
    while $\mathit{below}(p) \neq \mathit{null}$ **do**
        $p \leftarrow \mathit{below}(p)$
        while $\mathit{key}(\mathit{after}(p)) < k$ **do**
            $p \leftarrow \mathit{after}(p)$
        push p onto S
    return S

- S contains positions of the largest key **less than** k at each level.
- $\mathit{after}(\mathit{top}(S))$ will have key k, iff k is in L.
- drop down: $p \leftarrow \mathit{below}(p)$
- scan forward: $p \leftarrow \mathit{after}(p)$

Search in Skip Lists

Example: Skip-Search$(S, 87)$



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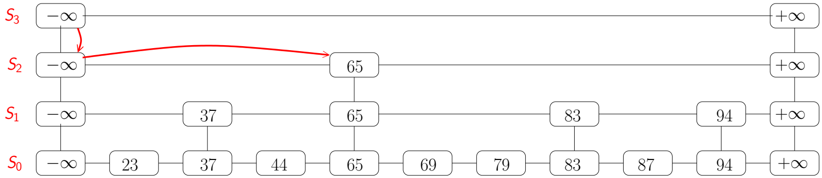

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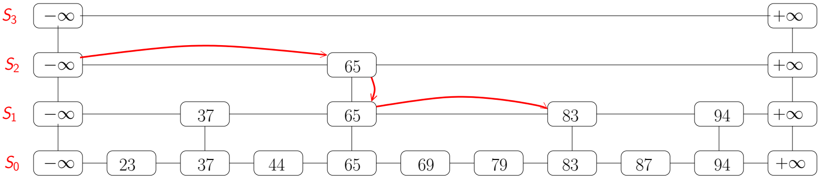

Search in Skip Lists

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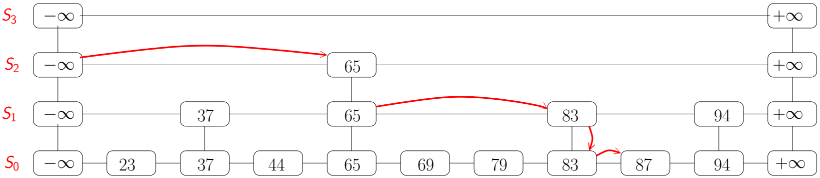

Search in Skip Lists

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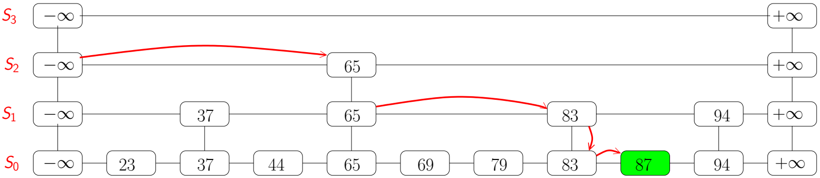
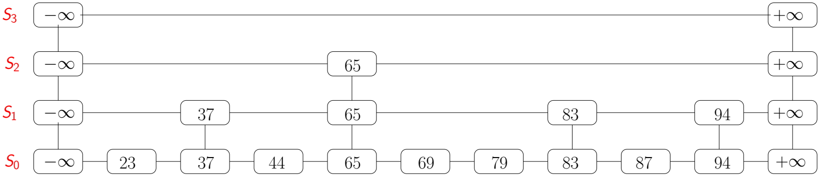


Insert in Skip Lists

- *Skip-Insert*(S, k, v)
  - Randomly compute the height of new item: repeatedly toss a coin until you get tails, let i the number of times the coin came up heads
  - Search for k in the skip list and find the positions $p_0, p_1, \cdots, p_i$ of the items with largest key less than k in each list $S_0, S_1, \cdots, S_i$ (by performing *Skip-Search*(S, k))
  - Insert item (k, v) into list S_j after position p_j for $0 \leq j \leq i$ (a tower of height i)

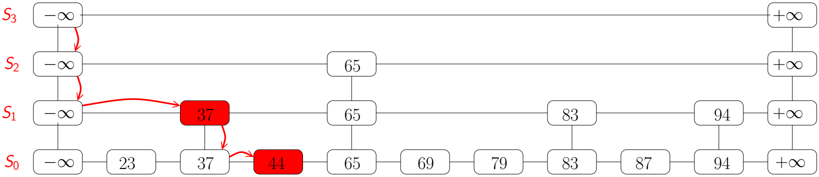
Insert in Skip Lists

Example: Skip-Insert$(S, 52, v)$
Coin tosses: H,T $\Rightarrow i = 1$



Insert in Skip Lists

Example: Skip-Insert(S, 52, v)
Coin tosses: H,T $\Rightarrow i = 1$
Skip-Search(S, 52)



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Example: Skip-Insert$(S, 52, v)$
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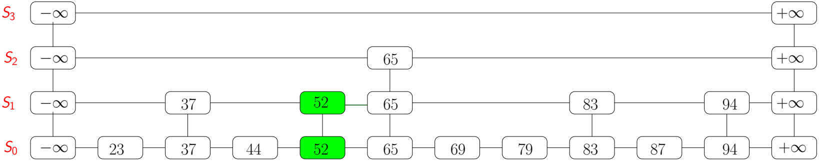
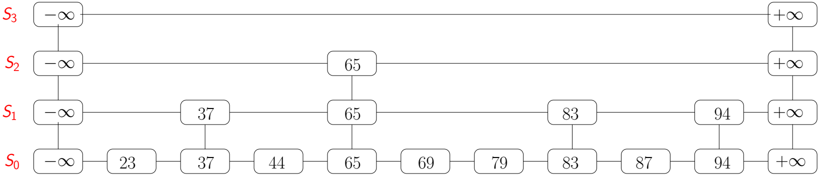


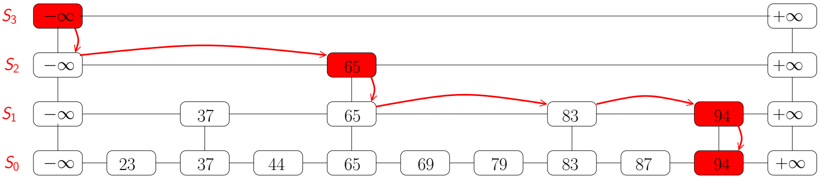
Insert in Skip Lists

Example: Skip-Insert(S, 100, v)
Coin tosses: H,H,H,T $\Rightarrow i = 3$



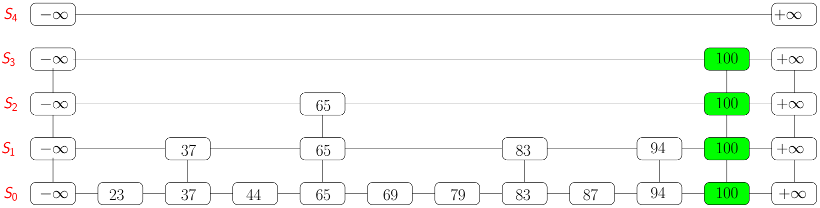
Insert in Skip Lists

Example: Skip-Insert$(S, 100, v)$
Coin tosses: H,H,H,T $\Rightarrow i = 3$
Skip-Search$(S, 100)$



Insert in Skip Lists

Example: Skip-Insert$(S, 100, v)$
Coin tosses: H,H,H,T $\Rightarrow i = 3$
Height increase

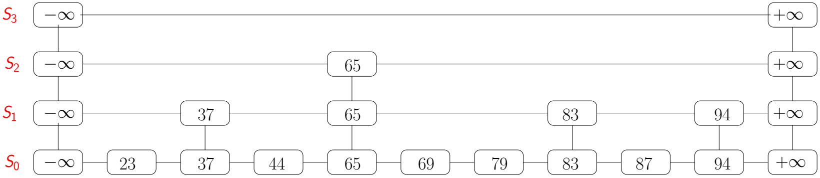


Delete in Skip Lists

- Skip-Delete(S, k)
  - Search for k in the skip list and find all the positions $p_0, p_1, \ldots, p_i$ of the items with the largest key smaller than k, where p_j is in list S_j. (this is the same as Skip-Search)
  - For each i, if $key(after(p_i)) == k$, then remove $after(p_i)$ from list S_i
  - Remove all but one of the lists S_i that contain only the two special keys

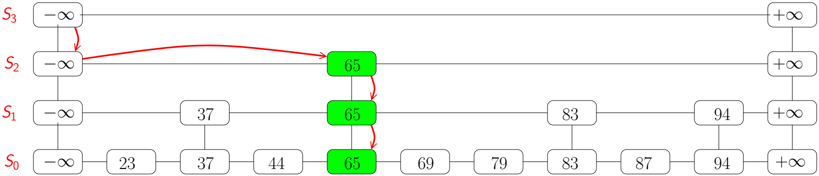
Delete in Skip Lists

Example: Skip-Delete$(S, 65)$



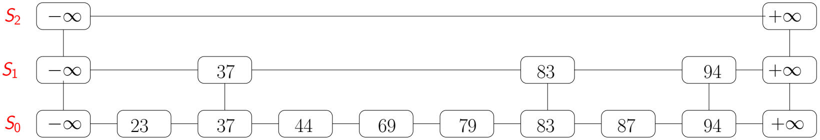
Delete in Skip Lists

Example: Skip-Delete(S, 65)
Skip-Search(S, 65)



Delete in Skip Lists

Example: Skip-Delete(S, 65)



Summary of Skip Lists

- Expected space usage: $O(n)$
- Expected height: $O(\log n)$
  A skip list with n items has height at most $3 \log n$ with probability at least $1 - 1/n^2$
- *Skip-Search*: $O(\log n)$ expected time
- *Skip-Insert*: $O(\log n)$ expected time
- *Skip-Delete*: $O(\log n)$ expected time
- Skip lists are fast and simple to implement in practice