Module 8: Data Structures for Multi-Dimensional Data

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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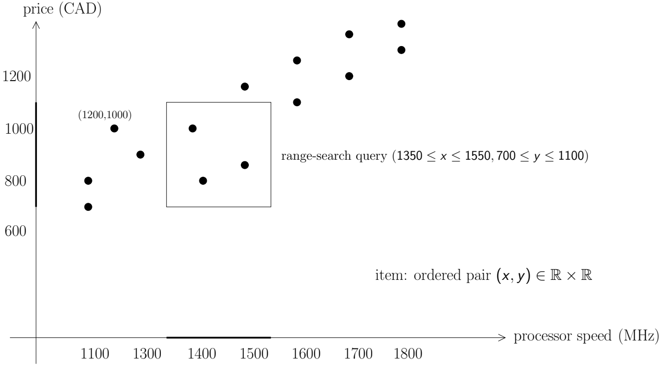
Multi-Dimensional Data

- Various applications
  - Attributes of a product (laptop: price, screen size, processor speed, RAM, hard drive, $\cdots$)
  - Attributes of an employee (name, age, salary, $\cdots$)
- Dictionary for multi-dimensional data
  A collection of d-dimensional items
  Each item has d aspects (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
  Operations: insert, delete, range-search query
- (Orthogonal) Range-search query: specify a range (interval) for certain aspects, and find all the items whose aspects fall within given ranges.
  Example: laptops with screen size between 11 and 13 inches, RAM between 8 and 16 GB, price between 1,500 and 2,000 CAD

Multi-Dimensional Data

- Each item has d aspects (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
- Aspect values (x_i) are numbers
- Each item corresponds to a point in d-dimensional space
- We concentrate on $d = 2$, i.e., points in Euclidean plane



One-Dimensional Range Search

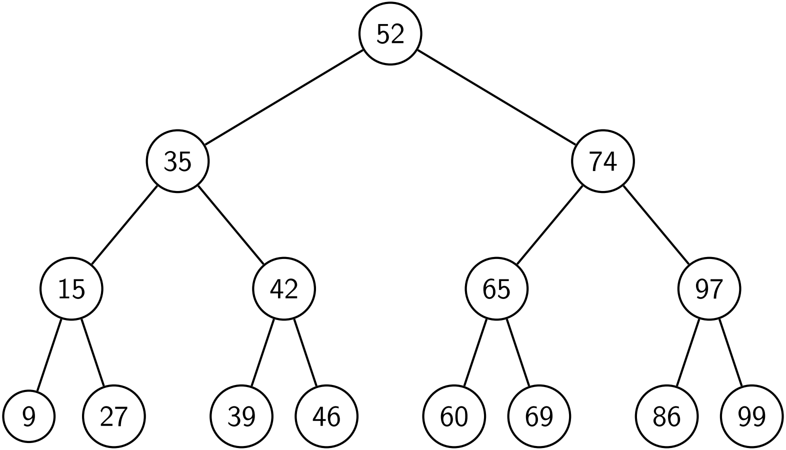
- First solution: ordered arrays
  - Running time: $O(\log n + k)$, k: number of reported items
  - Problem: does not generalize to higher dimensions
- Second solution: balanced BST (e.g., AVL tree)

BST-RangeSearch(T, k_1, k_2)
T: A balanced search tree, k_1, k_2: search keys
Report keys in T that are in range $[k_1, k_2]$

```
if T = nil then return
if key(T) < k1 then
    BST-RangeSearch(T.right, k1, k2)
if key(T) > k2 then
    BST-RangeSearch(T.left, k1, k2)
if k1 ≤ key(T) ≤ k2 then
    BST-RangeSearch(T.left, k1, k2)
    report key(T)
    BST-RangeSearch(T.right, k1, k2)
```

Range Search example

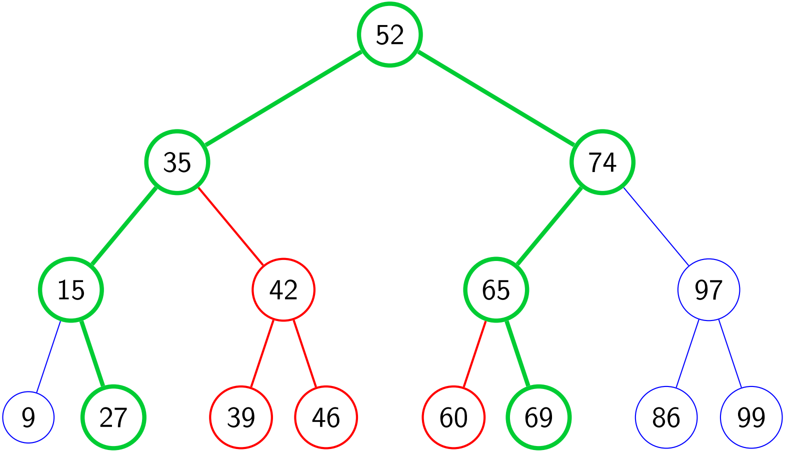
BST-RangeSearch$(T, 30, 65)$



Range Search example

BST-RangeSearch$(T, 30, 65)$

Nodes either on boundary, inside, or outside.



Range Search example

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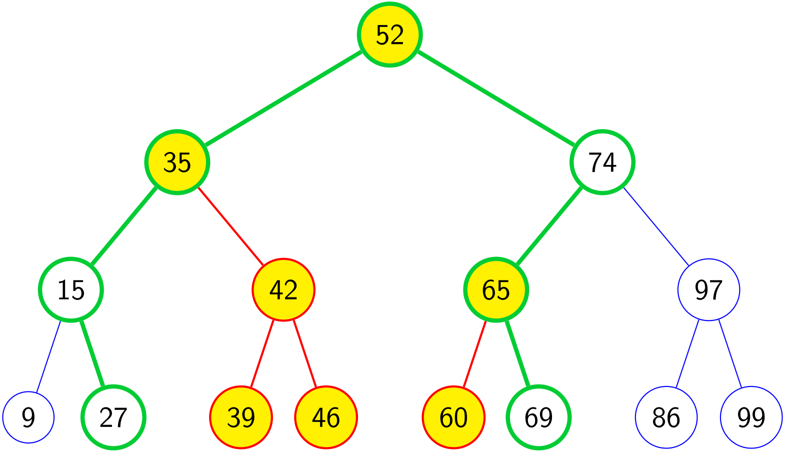


Note: Not every boundary node is returned.

One-Dimensional Range Search

- P_1: path from the root to a leaf that goes right if $k < k_1$ and left otherwise
- P_2: path from the root to a leaf that goes left if $k > k_2$ and right otherwise
- Partition nodes of T into three groups:
  1. boundary nodes: nodes in P_1 or P_2
  2. inside nodes: non-boundary nodes that belong to either (a subtree rooted at a right child of a node of P_1) or (a subtree rooted at a left child of a node of P_2)
  3. outside nodes: non-boundary nodes that belong to either (a subtree rooted at a left child of a node of P_1) or (a subtree rooted at a right child of a node of P_2)

One-Dimensional Range Search

- P_1: path from the root to a leaf that goes right if $k < k_1$ and left otherwise
- P_2: path from the root to a leaf that goes left if $k > k_2$ and right otherwise
- k: number of reported items
- Nodes visited during the search:
  - $O(\log n)$ boundary nodes
  - $O(k)$ inside nodes
  - No outside nodes
- Running time $O(\log n + k)$

2-Dimensional Range Search

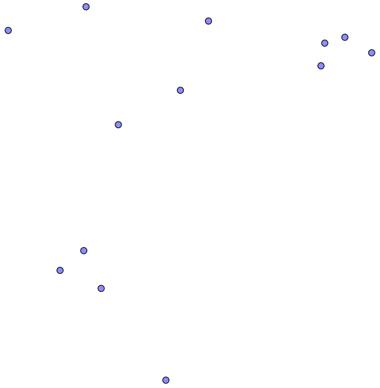
- Each item has 2 aspects (coordinates): (x_i, y_i)
- Each item corresponds to a point in Euclidean plane
- Options for implementing d-dimensional dictionaries:
  - Reduce to one-dimensional dictionary: combine the d-dimensional key into one key
    Problem: Range search on one aspect is not straightforward
  - Use several dictionaries: one for each dimension
    Problem: inefficient, wastes space
  - Partition trees
    - A tree with n leaves, each leaf corresponds to an item
    - Each internal node corresponds to a region
    - quadtrees, kd-trees
  - multi-dimensional range trees

Quadtrees

- We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane
- How to build a quadtree on P:
  - Find a square R that contains all the points of P (We can compute minimum and maximum x and y values among n points)
  - Root of the quadtree corresponds to R
  - Split: Partition R into four equal subsquares (quadrants), each correspond to a child of R
  - Recursively repeat this process for any node that contains more than one point
  - Points on split lines belong to left/bottom side
  - Each leaf stores (at most) one point
  - We can delete a leaf that does not contain any point

Quadtrees

- Example: We have 13 points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{12}, y_{12})\}$ in the plane



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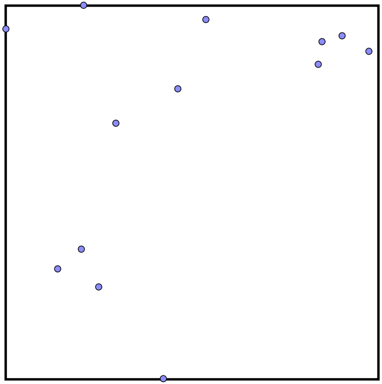

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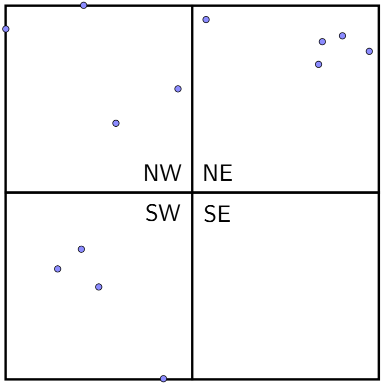

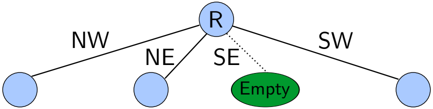

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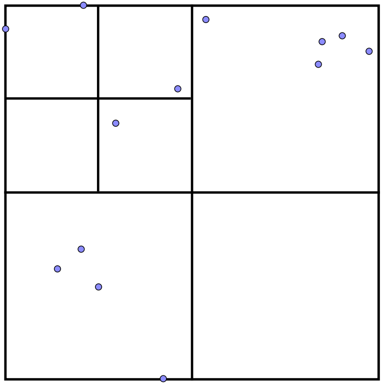

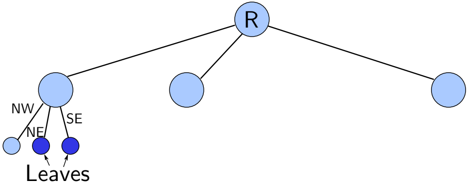

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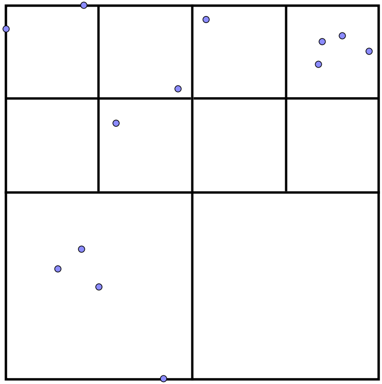

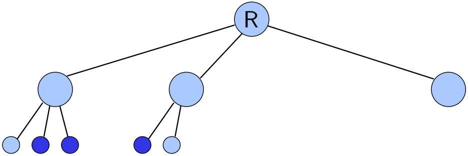

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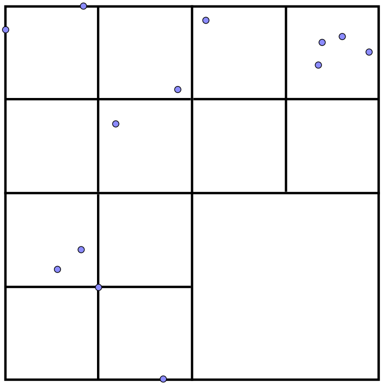

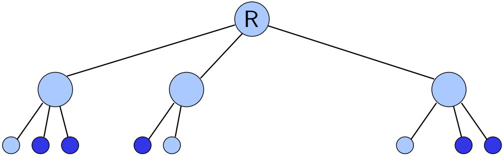

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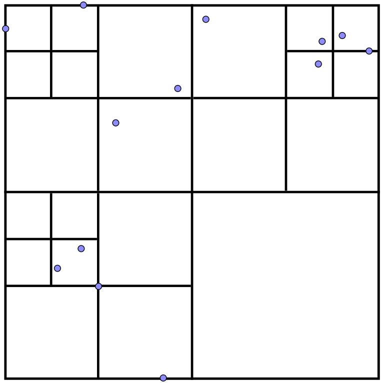

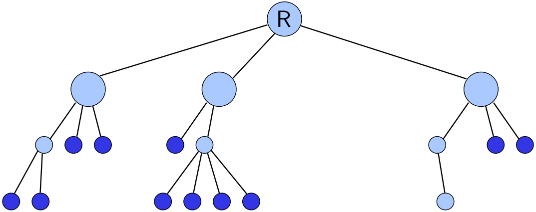

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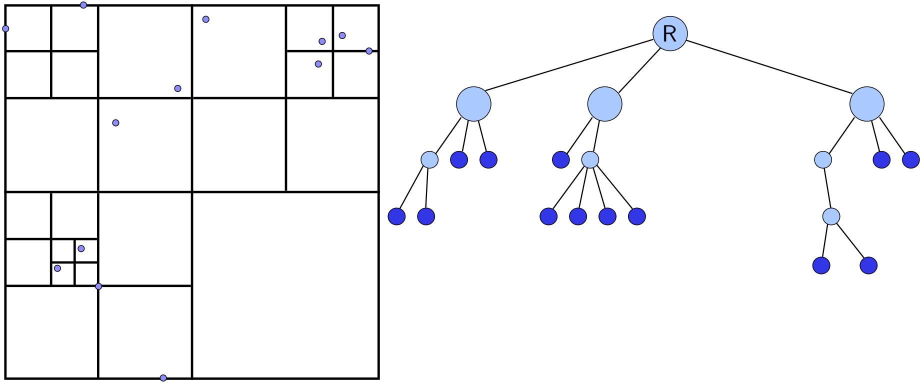


Quadtree Operations

- Search: Analogous to binary search trees
- Insert:
  - Search for the point
  - Split the leaf if there are two points
- Delete:
  - Search for the point
  - Remove the point
  - If its parent has only one child left, delete that child and continue the process toward the root.

Quadtree: Range Search

```
QTree-RangeSearch(T, R)
T: A quadtree node, R: Query rectangle
    if (T is a leaf) then
        if (T.point ∈ R) then
            report T.point
    for each child C of T do
        if C.region ∩ R ≠ ∅ then
            QTree-RangeSearch(C, R)
```

- spread factor of points P: $\beta(P) = d_{max}/d_{min}$
- $d_{max}(d_{min})$: maximum (minimum) distance between two points in P
- height of quadtree: $h \in \Theta(\log_2 \frac{d_{max}}{d_{min}})$
- Complexity to build initial tree: $\Theta(nh)$
- Complexity of range search: $\Theta(nh)$ even if the answer is $\emptyset$

Quadtree Conclusion

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (usually the boundary box is padded to get a power of two).
- Space wasteful
- Major drawback: can have very large height for certain nonuniform distributions of points
- Easily generates to higher dimensions (octrees, *etc.*).

kd-trees

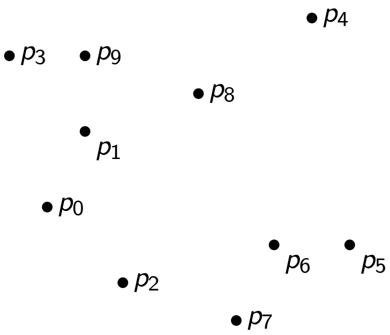
- We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane
- Quadtrees split square into quadrants regardless of where points actually lie
- kd-tree idea: Split the points into two (roughly) equal subsets
- How to build a kd-tree on P:
  - Split P into two equal subsets using a vertical line
  - Split each of the two subsets into two equal pieces using horizontal lines
  - Continue splitting, alternating vertical and horizontal lines, until every point is in a separate region
- Complexity: $\Theta(n \log n)$, height of the tree: $\Theta(\log n)$

kd-trees

- We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane
- Quadtrees split square into quadrants regardless of where points actually lie
- kd-tree idea: Split the points into two (roughly) equal subsets
- More details:
  - Initially, we sort the n points according to their x-coordinates.
  - The root of the tree is the point with median x coordinate (index $\lfloor n/2 \rfloor$ in the sorted list)
  - All other points with x coordinate less than or equal to this go into the left subtree; points with larger x-coordinate go in the right subtree.
  - At alternating levels, we sort and split according to y-coordinates instead.
- Complexity: $\Theta(n \log n)$, height of the tree: $\Theta(\log n)$

kd-trees

- kd-tree idea: Split the points into two (roughly) equal subsets
- A balanced binary tree



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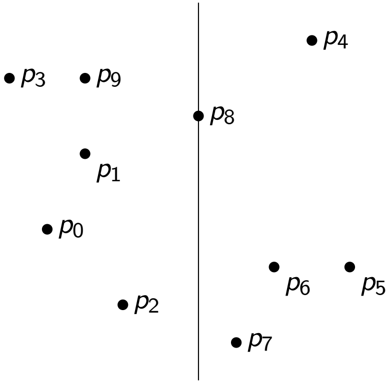

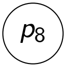

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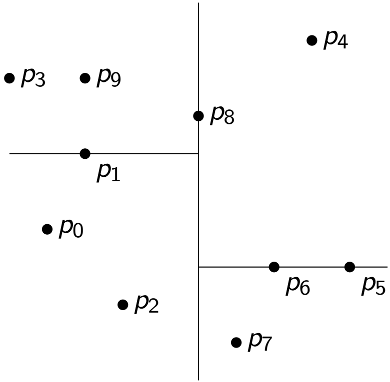

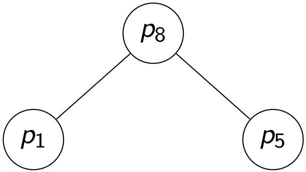

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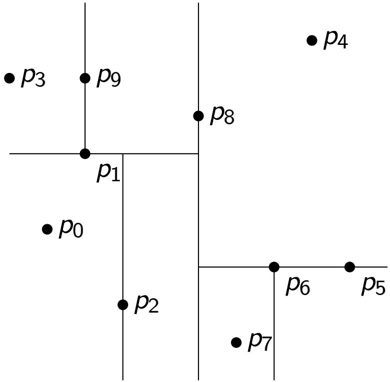

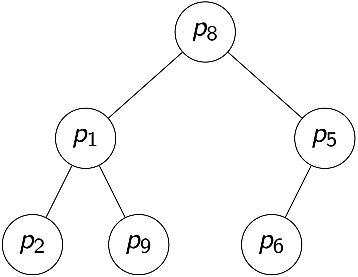

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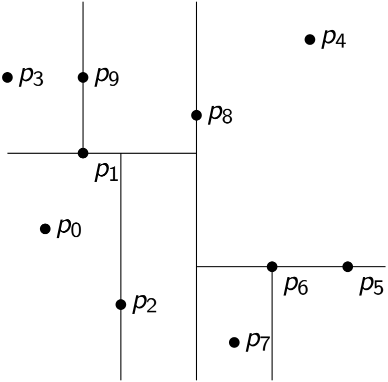
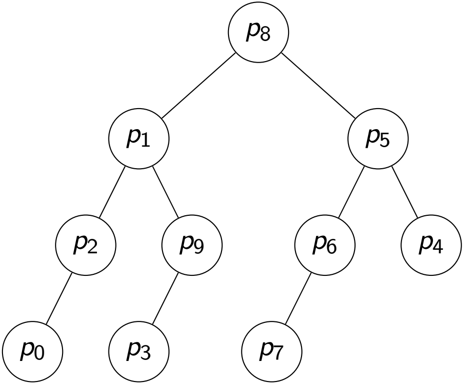




kd-tree: Range Search

kd-rangeSearch(T, R)
T: A kd-tree node, R: Query rectangle

```
if T is empty then return
if T.point ∈ R then
    report T.point
for each child C of T do
    if C.region ∩ R ≠ ∅ then
        kd-rangeSearch(C, R)
```

kd-tree: Range Search

```
kd-rangeSearch(T, R, split[← 'x'])
T: A kd-tree node, R: Query rectangle
    if T is empty then return
    if T.point ∈ R then
        report T.point
    if split = 'x' then
        if T.point.x ≥ R.leftSide then
            kd-rangeSearch(T.left, R, 'y')
        if T.point.x < R.rightSide then
            kd-rangeSearch(T.right, R, 'y')
    if split = 'y' then
        if T.point.y ≥ R.bottomSide then
            kd-rangeSearch(T.left, R, 'x')
        if T.point.y < R.topSide then
            kd-rangeSearch(T.right, R, 'x')
```

kd-tree: Range Search Complexity

- The complexity is $O(k + U)$ where k is the number of keys reported and U is the number of regions we go to but unsuccessfully
- U corresponds to the number of regions which intersect but are not fully in R
- Those regions have to intersect one of the four sides of R
- $Q(n)$: Maximum number of regions in a kd-tree with n points that intersect a vertical (horizontal) line
- $Q(n)$ satisfies the following recurrence relation:

$$Q(n) = 2Q(n/4) + O(1)$$

- It solves to $Q(n) = O(\sqrt{n})$
- Therefore, the complexity of range search in kd-trees is $O(k + \sqrt{n})$

kd-tree: Higher Dimensions

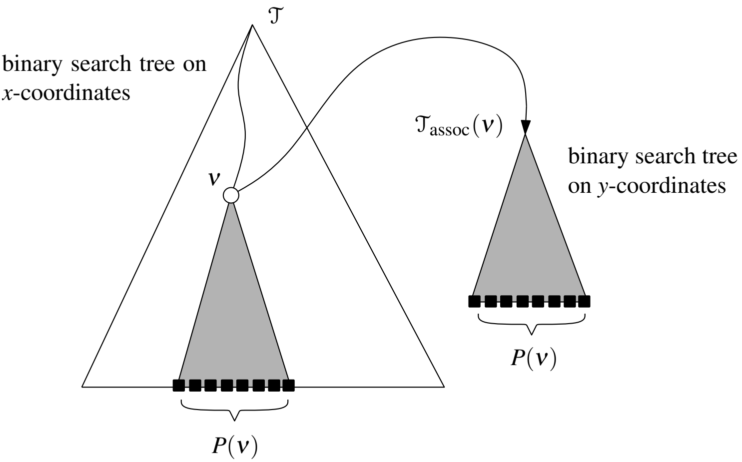
- kd-trees for d-dimensional space
  - At the root the point set is partitioned based on the first coordinate
  - At the children of the root the partition is based on the second coordinate
  - At depth $d-1$ the partition is based on the last coordinate
  - At depth d we start all over again, partitioning on first coordinate
- Storage: $O(n)$
- Construction time: $O(n \log n)$
- Range query time: $O(n^{1-1/d} + k)$

(Note: d is considered to be a constant.)

Range Trees

- We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane
- A range tree is a tree of trees (a *multi-level* data structure)
- How to build a range tree on P:
  - Build a balanced binary search tree τ determined by the x-coordinates of the n points
  - For every node $v \in \tau$, build a balanced binary search tree $\tau_{assoc}(v)$ (associated structure of τ) determined by the y-coordinates of the nodes in the subtree of τ with root node v

Range Tree Structure



Range Trees: Operations

- Search: trivially as in a binary search tree
- Insert: insert point in τ by x-coordinate
- From inserted leaf, walk back up to the root and insert the point in all associated trees $\tau_{assoc}(v)$ of nodes v on path to the root
- Delete: analogous to insertion
- Note: re-balancing is a problem!

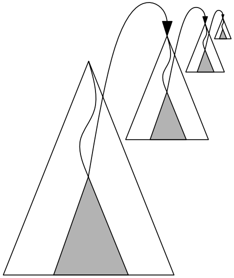
Range Trees: Range Search

- A two stage process
- To perform a range search query $R = [x_1, x_2] \times [y_1, y_2]$:
  - Perform a range search (on the x-coordinates) for the interval $[x_1, x_2]$ in τ (*BST-RangeSearch*(τ, x_1, x_2))
  - For every outside node, do nothing.
  - For every "top" inside node v, perform a range search (on the y-coordinates) for the interval $[y_1, y_2]$ in $\tau_{assoc}(v)$. During the range search of $\tau_{assoc}(v)$, do not check any x-coordinates (they are all within range).
  - For every boundary node, test to see if the corresponding point is within the region R.
- Running time: $O(k + \log^2 n)$
- Range tree space usage: $O(n \log n)$

Range Trees: Higher Dimensions

- Range trees for d-dimensional space
  - Storage: $O(n\,(\log n)^{d-1})$
  - Construction time: $O(n\,(\log n)^{d-1})$
  - Range query time: $O((\log n)^{d} + k)$

(Note: d is considered to be a constant.)



Range Trees: Higher Dimensions

- Space/time trade-off
  - Storage: $O(n\,(\log n)^{d-1})$ kd-trees: $O(n)$
  - Construction time: $O(n\,(\log n)^{d-1})$ kd-trees: $O(n \log n)$
  - Range query time: $O((\log n)^d + k)$ kd-trees: $O(n^{1-1/d} + k)$

(Note: d is considered to be a constant.)