

# Module 3: Sorting and Randomized Algorithms

## CS 240 - Data Structures and Data Management

Mark Petrick

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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## Selection vs. Sorting

The *selection problem* is: Given an array  $A$  of  $n$  numbers, and  $0 \leq k < n$ , find the element in position  $k$  of the sorted array.

**Observation:** the  $k$ th largest element is the element at position  $n - k$ .

Best heap-based algorithm had time cost  $\Theta(n + k \log n)$ .

For median selection,  $k = \lfloor \frac{n}{2} \rfloor$ , giving cost  $\Theta(n \log n)$ .

This is the same cost as our best sorting algorithms.

**Question:** Can we do selection in linear time?

## Selection vs. Sorting

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**Question:** Can we do selection in linear time?

The *quick-select* algorithm answers this question in the affirmative.

**Observation:** Finding the element at a given position is tough, but finding the position of a given element is simple.

# Crucial Subroutines

*quick-select* and the related algorithm *quick-sort* rely on two subroutines:

- *choose-pivot*( $A$ ): Choose an index  $i$  such that  $A[i]$  will make a good pivot (hopefully near the middle of the order).
- *partition*( $A, p$ ): Using pivot  $A[p]$ , rearrange  $A$  so that all items  $\leq$  the pivot come first, followed by the pivot, followed by all items greater than the pivot.

## Selecting a pivot

Ideally, we would select a median as the pivot.

*But this is the problem we're trying to solve!*

**First idea:** Always select first element in array

```
choose-pivot1(A)  
1.  return 0
```

We will consider more sophisticated ideas later on.

# Partition Algorithm

*partition*( $A, p$ )

$A$ : array of size  $n$ ,  $p$ : integer s.t.  $0 \leq p < n$

1.  $\text{swap}(A[0], A[p])$
2.  $i \leftarrow 1, \quad j \leftarrow n - 1$
3. **loop**
4.     **while**  $i < n$  and  $A[i] \leq A[0]$  **do**
5.          $i \leftarrow i + 1$
6.     **while**  $j \geq 1$  and  $A[j] > A[0]$  **do**
7.          $j \leftarrow j - 1$
8.     **if**  $j < i$  **then break**
9.     **else**  $\text{swap}(A[i], A[j])$
10. **end loop**
11.  $\text{swap}(A[0], A[j])$
12. **return**  $j$

**Idea:** Keep swapping the outer-most wrongly-positioned pairs.

# QuickSelect Algorithm

*quick-select1*( $A, k$ )

$A$ : array of size  $n$ ,  $k$ : integer s.t.  $0 \leq k < n$

1.  $p \leftarrow \text{choose-pivot1}(A)$
2.  $i \leftarrow \text{partition}(A, p)$
3. **if**  $i = k$  **then**
4.     **return**  $A[i]$
5. **else if**  $i > k$  **then**
6.     **return** *quick-select1*( $A[0, 1, \dots, i - 1], k$ )
7. **else if**  $i < k$  **then**
8.     **return** *quick-select1*( $A[i + 1, i + 2, \dots, n - 1], k - i - 1$ )

# Analysis of quick-select1

**Worst-case analysis:** Recursive call could always have size  $n - 1$ .

Recurrence given by 
$$T(n) = \begin{cases} T(n-1) + cn, & n \geq 2 \\ d, & n = 1 \end{cases}$$

Solution:  $T(n) = cn + c(n-1) + c(n-2) + \cdots + c \cdot 2 + d \in \Theta(n^2)$



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**Best-case analysis:** First chosen pivot could be the  $k$ th element  
No recursive calls; total cost is  $\Theta(n)$ .

# Average-case analysis of quick-select1

Assume all  $n!$  permutations are equally likely.

Average cost is sum of costs for all permutations, divided by  $n!$ .

Define  $T(n, k)$  as average cost for selecting  $k$ th item from size- $n$  array:

$$T(n, k) = cn + \frac{1}{n} \left( \sum_{i=0}^{k-1} T(n-i-1, k-i-1) + \sum_{i=k+1}^{n-1} T(i, k) \right)$$

We could analyze this recurrence directly,  
or be a little lazier and still get the same asymptotic result.

For simplicity, define  $T(n) = \max_{0 \leq k < n} T(n, k)$ .

## Average-case analysis of quick-select1

The cost is determined by  $i$ , the position of the pivot  $A[0]$ .

For more than half of the  $n!$  permutations,  $\frac{n}{4} \leq i < \frac{3n}{4}$ .

In this case, the recursive call will have length at most  $\lfloor \frac{3n}{4} \rfloor$ , for any  $k$ .

The average cost is then given by:

$$T(n) \leq \begin{cases} cn + \frac{1}{2} \left( T(n) + T(\lfloor 3n/4 \rfloor) \right), & n \geq 2 \\ d, & n = 1 \end{cases}$$

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Rearranging gives:

$$\begin{aligned} T(n) &\leq 2cn + T(\lfloor 3n/4 \rfloor) \leq 2cn + 2c(3n/4) + 2c(9n/16) + \dots + d \\ &\leq d + 2cn \sum_{i=0}^{\infty} \left( \frac{3}{4} \right)^i \in O(n) \end{aligned}$$

Since  $T(n)$  *must* be  $\Omega(n)$  (why?),  $T(n) \in \Theta(n)$ .

# Randomized algorithms

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**Generating random numbers:** Computers can't generate randomness. Instead, some external source is used (e.g. clock, mouse, gamma rays, . . .)

This is expensive, so we use a *pseudo-random number generator (PRNG)*, a deterministic program that uses a true-random initial value or *seed*.

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**Goal:** To shift the probability distribution from what we can't control (the input), to what we can control (the random numbers).

There should be no more bad instances, just unlucky numbers.

## Expected running time

Define  $T(I, R)$  as the running time of the randomized algorithm for a particular input  $I$  and the sequence of random numbers  $R$ .

The *expected running time*  $T^{(exp)}(I)$  of a randomized algorithm for a particular input  $I$  is the “expected” value for  $T(I, R)$ :

$$T^{(exp)}(I) = \mathbf{E}[T(I, R)] = \sum_R T(I, R) \cdot Pr[R]$$



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$$T^{(exp)}(I) = \mathbf{E}[T(I, R)] = \sum_R T(I, R) \cdot Pr[R]$$

The *worst-case expected running time* is then

$$T^{(exp)}(n) = \max_{size(I)=n} T^{(exp)}(I).$$

For many randomized algorithms, worst-, best-, and average-case expected times are the same (why?).

# Randomized QuickSelect

*random*( $n$ ) returns an integer uniformly from  $\{0, 1, 2, \dots, n - 1\}$ .

**First idea:** Randomly permute the input first using *shuffle*:

*shuffle*( $A$ )

$A$ : array of size  $n$

1. **for**  $i \leftarrow 0$  to  $n - 2$  **do**
2.      $\text{swap}(A[i], A[i + \text{random}(n - i)])$

Expected cost becomes the same as the average cost, which is  $\Theta(n)$ .

# Randomized QuickSelect

**Second idea:** Change the pivot selection.

```
choose-pivot2(A)
```

1. **return** *random*( $n$ )

```
quick-select2(A, k)
```

1.  $p \leftarrow$  *choose-pivot2*(A)
2. ...

With probability at least  $\frac{1}{2}$ , the random pivot has position  $\frac{n}{4} \leq i < \frac{3n}{4}$ , so the analysis is just like that for the average-case.

The expected cost is again  $\Theta(n)$ .

*This is generally the fastest quick-select implementation.*

## Worst-case linear time

Blum, Floyd, Pratt, Rivest, and Tarjan invented the “medians-of-five” algorithm in 1973 for pivot selection:

*choose-pivot3*(*A*)

*A*: array of size *n*

1.  $m \leftarrow \lfloor n/5 \rfloor - 1$
2. **for**  $i \leftarrow 0$  to  $m$  **do**
3.      $j \leftarrow$  index of median of  $A[5i, \dots, 5i + 4]$
4.      $\text{swap}(A[i], A[j])$
5. **return** *quick-select3*( $A[0, \dots, m], \lfloor m/2 \rfloor$ )

*quick-select3*(*A*, *k*)

1.  $p \leftarrow \text{choose-pivot3}(A)$
2. ...

This *mutually recursive* algorithm can be shown to be  $\Theta(n)$  in the worst case, but it's a little beyond the scope of this course.

# QuickSort

QuickSelect is based on a sorting method developed by Hoare in 1960:

*quick-sort1*(*A*)

*A*: array of size *n*

1. **if**  $n \leq 1$  **then return**
2.  $p \leftarrow \text{choose-pivot1}(A)$
3.  $i \leftarrow \text{partition}(A, p)$
4. *quick-sort1*( $A[0, 1, \dots, i - 1]$ )
5. *quick-sort1*( $A[i + 1, \dots, \text{size}(A) - 1]$ )

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**Worst case:**  $T^{(\text{worst})}(n) = T^{(\text{worst})}(n - 1) + \Theta(n)$

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**Best case:**  $T^{(\text{best})}(n) = T^{(\text{best})}(\lfloor \frac{n-1}{2} \rfloor) + T^{(\text{best})}(\lceil \frac{n-1}{2} \rceil) + \Theta(n)$

Similar to *merge-sort*;  $T^{(\text{best})}(n) \in \Theta(n \log n)$

# Average-case analysis of quick-sort1

Of all  $n!$  permutations,  $(n-1)!$  have pivot  $A[0]$  at a given position  $i$ .

Average cost over all permutations is given by:

$$T(n) = \frac{1}{n} \sum_{i=0}^{n-1} (T(i) + T(n-i-1)) + \Theta(n), \quad n \geq 2$$

It is possible to solve this recursion directly.



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It is possible to solve this recursion directly.

Instead, notice that the cost at each level of the recursion tree is  $O(n)$ . So let's consider the height (or “depth”) of the recursion, on average.

## Average depth of recursion for quick-sort1

Define  $H(n)$  as the average recursion depth for size- $n$  inputs. So

$$H(n) = \begin{cases} 1 + \frac{1}{n} \sum_{i=0}^{n-1} \max(H(i), H(n-i-1)), & n \geq 2 \\ 0, & n \leq 1 \end{cases}$$

- Let  $i$  be the position of the pivot  $A[0]$ .  
Again,  $\frac{n}{4} \leq i < \frac{3n}{4}$  for more than half of all permutations.
- Then larger recursive call has length at most  $\lfloor \frac{3n}{4} \rfloor$ .  
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- Then larger recursive call has length at most  $\lfloor \frac{3n}{4} \rfloor$ .  
This will determine the recursion depth for at least half of all inputs.
- Therefore  $H(n) \leq 1 + \frac{1}{2} (H(n) + H(\lfloor \frac{3n}{4} \rfloor))$  for  $n \geq 2$ ,  
which simplifies to  $H(n) \leq 2 + H(\lfloor \frac{3n}{4} \rfloor)$ .
- So  $H(n) \in O(\log n)$ .  
Average cost is  $O(nH(n)) \in O(n \log n)$ .  
Since best-case is  $\Theta(n \log n)$ , average must be  $\Theta(n \log n)$ .

## More notes on QuickSort

- We can randomize by using *choose-pivot2*, giving  $\Theta(n \log n)$  expected time for *quick-sort2*.
- We can use *choose-pivot3* (along with *quick-select3*) to get *quick-sort3* with  $\Theta(n \log n)$  worst-case time.
- We can use tail recursion to save space on one of the recursive calls. By making sure the other one is always smaller, the auxiliary space is  $\Theta(\log n)$  in the worst case, even for *quick-sort1*.
- QuickSort is often the most efficient algorithm in practice.

## Lower bounds for sorting

We have seen many sorting algorithms:

Sort	Running time	Analysis
Selection Sort	$\Theta(n^2)$	worst-case
Insertion Sort	$\Theta(n^2)$	worst-case
Merge Sort	$\Theta(n \log n)$	worst-case
Heap Sort	$\Theta(n \log n)$	worst-case
<i>quick-sort1</i>	$\Theta(n \log n)$	average-case
<i>quick-sort2</i>	$\Theta(n \log n)$	expected
<i>quick-sort3</i>	$\Theta(n \log n)$	worst-case

**Question:** Can one do better than  $\Theta(n \log n)$ ?

**Answer:** Yes and no! *It depends on what we allow.*

- No: Comparison-based sorting lower bound is  $\Omega(n \log n)$ .
- Yes: Non-comparison-based sorting can achieve  $O(n)$ .

# The Comparison Model

In the *comparison model* data can only be accessed in two ways:

- comparing two elements
- moving elements around (e.g. copying, swapping)

This makes very few assumptions on the kind of things we are sorting.  
We count the number of above operations.

All sorting algorithms seen so far are in the comparison model.

# Lower bound for sorting in the comparison model

**Theorem.** Any correct **comparison-based** sorting algorithm requires at least  $\Omega(n \log n)$  comparison operations.

**Proof.**

- A correct algorithm takes different **actions** (moves, swaps, etc.) for each of the  $n!$  possible permutations.
- The choice of actions is determined only by comparisons.

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- The algorithm can be viewed as a **decision tree**.  
Each internal node is a comparison, each leaf is a set of actions.
- Each permutation must correspond to a leaf.



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- The choice of actions is determined only by comparisons.
- The algorithm can be viewed as a **decision tree**.  
Each internal node is a comparison, each leaf is a set of actions.
- Each permutation must correspond to a leaf.
- The worst-case number of comparisons is the longest path to a leaf.
- Since the tree has at least  $n!$  leaves, the height is at least  $\lg n!$ .
- Therefore worst-case number of comparisons is  $\Omega(n \log n)$ . □

# Non-comparison-based sorting

- Assume keys are numbers in base  $R$  ( $R$ : radix)
  - ▶  $R = 2, 10, 128, 256$  are the most common.
- Assume all keys have the same number  $m$  of digits.
  - ▶ Can achieve after padding with leading 0s.
- Can sort based on individual digits.

*MSD-Radix-sort*( $A, l, r, d$ )

$A$ : array of size  $n$ , contains  $m$ -digit radix- $R$  numbers

$l, r, d$ : integers,  $0 \leq l, r \leq n - 1$ ,  $1 \leq d \leq m$

**if**  $l < r$

1. partition  $A[l..r]$  into bins according to  $d$ th digit
2.     **if**  $d < m$
3.         **for**  $i \leftarrow 0$  to  $R - 1$  **do**
4.             let  $l_i$  and  $r_i$  be boundaries of  $i$ th bin
5.             MSD-Radix-sort( $A, l_i, r_i, d + 1$ )

- To partition: Sort by counting.

# Count Sort

## *key-indexed-count-sort*(A)

A: array of size  $n$  containing numbers in  $\{0, \dots, R - 1\}$

// count how many of each kind there are

1.  $C \leftarrow$  array of size  $R$ , filled with zeros
2. **for**  $i \leftarrow 0$  to  $n - 1$  **do**
3.     increment  $C[A[i]]$

// find left boundary for each kind

4.  $I \leftarrow$  array of size  $R$ ,  $I[0] = 0$
5. **for**  $i \leftarrow 1$  to  $R - 1$  **do**
6.      $I[i] \leftarrow I[i - 1] + C[i - 1]$

// copy, then move back in sorted order

7.  $B \leftarrow \text{copy}(A)$
8. **for**  $i \leftarrow 0$  to  $n - 1$  **do**
9.      $A[I[B[i]]] \leftarrow B[i]$
10.    increment  $I[d]$

- **Time:**  $\Theta(n + R)$ .
- **Optional:** re-use  $C$  rather than introducing  $I$ .

# LSD-Radix-Sort

- Drawback of MSD-Radix-Sort: many recursions

*LSD-radix-sort*( $A$ )

$A$ : array of size  $n$ , contains  $m$ -digit radix- $R$  numbers

1.     **for**  $d \leftarrow m$  down to 1 **do**
2.         Sort  $A$  by  $d$ th digit

- Sort-routine must be *stable*: equal items stay in original order.
  - ▶ CountSort, InsertionSort, MergeSort are (usually) stable.
  - ▶ HeapSort, QuickSort *not* stable as implemented.
- Loop-invariant:  $A$  is sorted w.r.t. digits  $d, \dots, m$  of each entry.
- **Time cost**:  $\Theta(m(n + R))$  if we use CountSort
- **Auxiliary space**:  $\Theta(n + R)$

# Summary of sorting

- Randomized algorithms can eliminate “bad cases”
- Best-case, worst-case, average-case, expected-case can all differ
- Sorting is an important and *very* well-studied problem
- Can be done in  $\Theta(n \log n)$  time; faster is not possible for general input
- HeapSort is the only fast algorithm we have seen with  $O(1)$  auxiliary space.
- QuickSort is often the fastest in practice
- MergeSort is also  $\Theta(n \log n)$ , selection & insertion sorts are  $\Theta(n^2)$ .
- CountSort, RadixSort can achieve  $o(n \log n)$  if the input is special