# What to do with NP-hard optimization problems

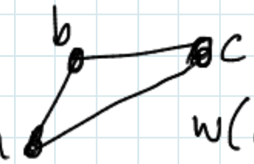
- efficient exhaustive search (backtracking, branch & bound) - exponential time.
- heuristics
  - local search - start with some solution and try to improve it via small "local" changes.
    "simulated annealing" overcomes local optima.
- approximation algs. - today's topic

Example

TSP for points in the plane w/ Euclidean distances

Complete graph

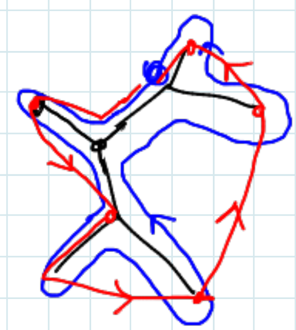
triangle inequality




$w(a,c) \le w(a,b) + w(b,c)$

Approx. Alg.

- compute MST (in black)
- take a tour by walking around it. (in blue)
  (we visit every vertex but maybe more than once)
- take shortcuts to avoid revisiting (in red)
  Note: $\Delta$ inequality $\Rightarrow$ short cuts shorter



We can do this in poly. time.

let $\ell$ be length of resulting tour
$\ell_{TSP}$ = " of min TSP tour

<u>Claim</u> $\ell \leq 2\ell_{TSP}$        [Note $\ell_{TSP} \leq \ell$]

So in poly. time we find a tour within 2× optimum.
We call this <u>approximation factor 2</u>

<u>Pf of Claim</u> $\ell_{MST}$ = length of MST

$\ell_{MST} \leq \ell_{TSP}$ because deleting one edge of TSP gives <u>a</u> spanning tree

$\ell \leq 2\ell_{MST}$ because we use every MST edge twice, then take short cuts. (use $\triangle$ ineq.)

Putting these together:

$\ell \leq 2\ell_{TSP}$

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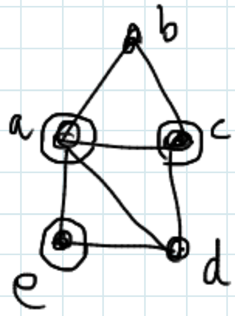
We say this alg. has approximation factor 2.

X

## Example Vertex Cover




$G = (V, E)$

want set $C \subseteq V$ s.t.

$\forall (u,v) \in E$, $u \in C$ or $v \in C$ (or both)

minimize $|C|$

Greedy approximation alg.

$C \leftarrow \emptyset$   $F \leftarrow E$   // F is uncovered edges

while $F \neq \emptyset$

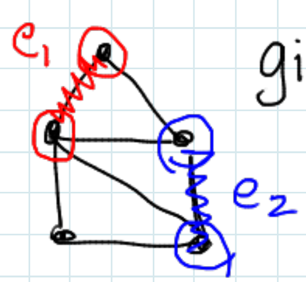
  pick $e = (u,v)$ from F

  add u and v to C

  remove edges incident to u from F

  - - - - -       v - -.

end




gives $|C| = 4$ (min. is 3)

Note that the alg. takes poly. time.

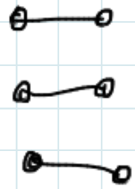
Let $C$ = vertex cover found by alg.

$C_{OPT}$ = a min vertex cover.

Claim $|C| \leq 2 \cdot |C_{OPT}|$

Pf. The set of edges you pick forms a matching $M$

(no 2 edges are incident)



$$|C| = 2|M|$$

Any vertex cover must have at least one vertex from each edge in a matching.

$$|M| \le |C_{OPT}|$$

Thus $|C| \le 2|C_{OPT}|$

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This alg. has approx. factor 2.

General TSP cannot be approximated to within constant factor in poly. time (unless P = NP).

Suppose we have a poly. time alg. for TSP that guarantees a tour of length $\leq k \cdot l_{TSP}$

<u>Claim</u> Then we can make a poly. time alg. for Hamiltonian cycle.

And hence P = NP.

Alg. for Ham. cycle:

Input: $G = (V, E)$ $|V| = n$

Construct $G' = (V, V \times V)$ — complete graph

$$w(e) = \begin{cases} 1 & e \in E \\ k \cdot n & \text{otherwise} \end{cases}$$

Run approx TSP alg. on $G'$ to get tour, length $l$

if $l \leq k \cdot n$ output YES (G has Ham. cycle)
else output NO

This is a poly. time alg.

Correctness:

In $G'$, a tour that only uses edges of G has length n.

a tour that uses at least one edge not in G has length $\geq (n-1) + k \cdot n > k \cdot n$ (assuming $n > 1$)

Claim. $l \leq k \cdot n$ iff G has Ham. cycle.

Pf. $\Rightarrow$ $l \leq k \cdot n \Rightarrow l = n$ so G has Ham. cycle

$\Leftarrow$ G has Ham. cycle $\Rightarrow$ $G'$ has a tour of length n $\Rightarrow$ k-approx has length $\leq k \cdot n$.