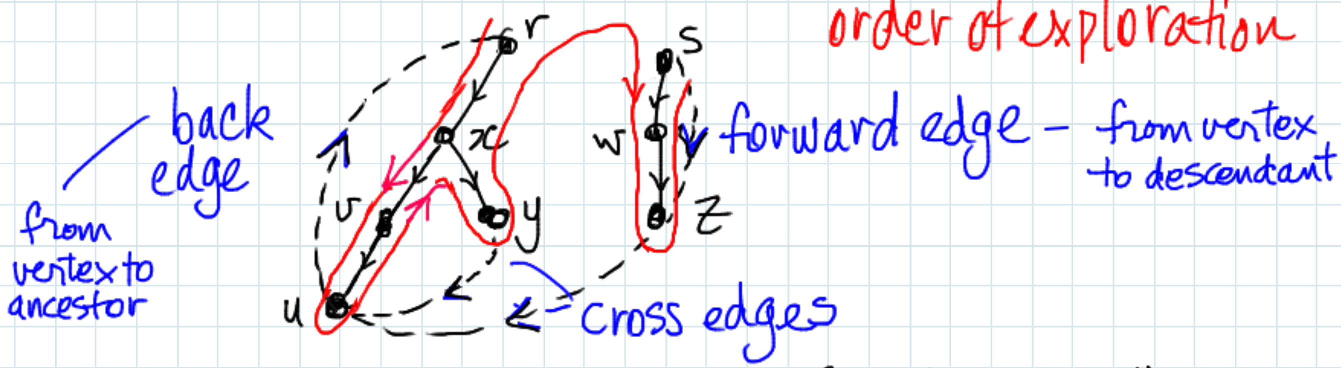
Depth First Search on Directed Graphs



$\overset{1}{d(r)}, \overset{2}{d(x)}, \overset{3}{d(v)}, \overset{4}{d(u)}, \overset{5}{f(u)}, \overset{6}{f(v)}, \overset{7}{d(y)}, \overset{8}{f(y)}, \overset{9}{f(x)}, \overset{10}{f(r)}, \overset{11}{d(s)} \ldots$

parenthesis system

DFS(v)
  mark(v) ← discovered
  d(v) ← time; time ← time+1   /* discover time
  for each neighbour u of v do
    if u is undiscovered then
      DFS(u); (v, u) is a tree edge
    else (*)
  mark(v) ← finished
  f(v) ← time; time ← time+1   /* finish time

(*) label back, forward, cross edges
  if u not finished then (v, u) is back edge
  else if d(u) > d(v) then (v, u) is forward edge
  else if d(u) < d(v) then (v, u) is cross edge

DFS takes O(n+m)

Note that result depends on vertex ordering.

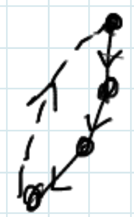
Applications of DFS

1. detecting cycles in directed graphs

<u>Lemma</u>. A directed graph has a [directed] cycle iff DFS has a back edge

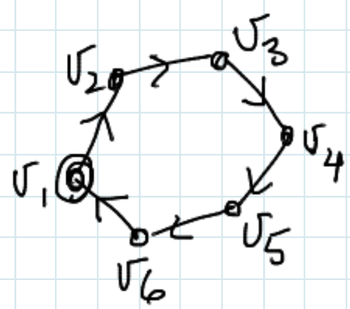
Proof

$\Leftarrow$



back edge gives directed cycle

$\Rightarrow$



Suppose there is a directed cycle. Let v_1 be first vertex discovered in DFS.

Number vertices of cycle $v_1 \cdots v_k$

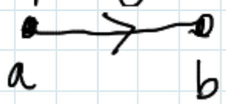
<u>Claim</u> (v_k, v_1) is a back edge

<u>Pf</u> Because we must discover & explore all v_i before we finish v_1, when we test edge (v_k, v_1) we label it a back edge.

Applications of DFS

2. Topological sort of directed acyclic graph

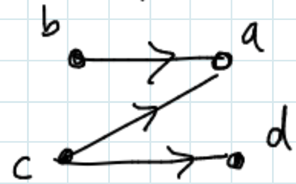
acyclic = no directed cycle



Edge (a,b) means a must come before b (e.g. job scheduling)

Find a linear order of vertices satisfying all edges. (possible iff no directed cycle).

Example



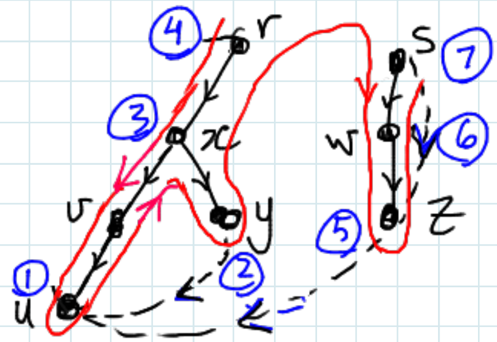
topological sort: b c a d
or c d b a or ...

One solution: find vertex v with no in-edge (= source)
Remove v and repeat.

Ex. Do this in $O(n+m)$ time.

Solution using DFS: (also $O(n+m)$)
Use reverse of finish order.

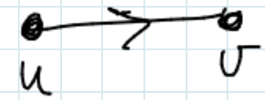
Example
(first ex. minus back edge)



finish order

reverse finish order   s, w, z, r, x, y, u
this is a topological order

Claim For every directed edge (u, v)
finish $(v) <$ finish (u)



Pf case1 u discovered before v
Then v is discovered and finished before u is finished.

case 2 v discovered before u
Because G has no directed cycle, we can't reach u in DFS(v). So v finished before u is discovered and finished.

3. Finding strongly connected components in a directed graph.

strongly connected = forall vertices u, v, there is a path $u \to v$ and a path $v \to u$.

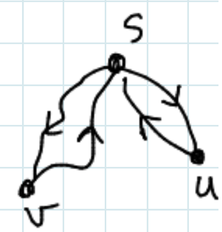
Easy to test if G is strongly connected because we don't need to test all pairs u, v.

Let s be a vertex

<u>Claim</u>. G is strongly connected iff for all vertices v, there is a path $s \to v$ and a path $v \to s$.

Pf. $\Rightarrow$ clear

$\Leftarrow$ to get from $u \to v$: $u \to s \to v$

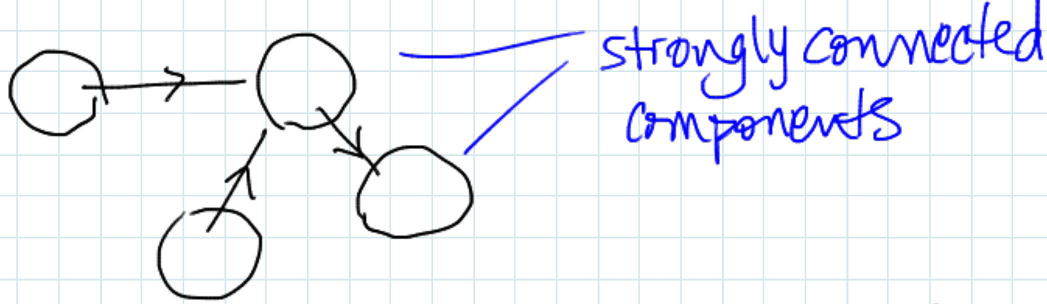


To test if there's a path $s \to v$ $\forall v$ — do DFS(s).

How can we test if there's a path $v \to s$ $\forall v$?

Reverse edge directions and do DFS(s).

More generally, structure of a digraph is



Contracting strongly conn. components gives <u>acylic</u> graph (think about the reason)

How to find strongly connected components.

[History Tarjan '72, Kosaraju '80's, Gabow '99
All linear time and all simple, but different.
We'll do Kosaraju's method.]

Idea: Vertices $1 \ldots n$

Run DFS (vertex ordering resolves what vertex comes next)

Let finish order be $f_1\ f_2 \cdots f_n$

G^R = G with all edges reversed

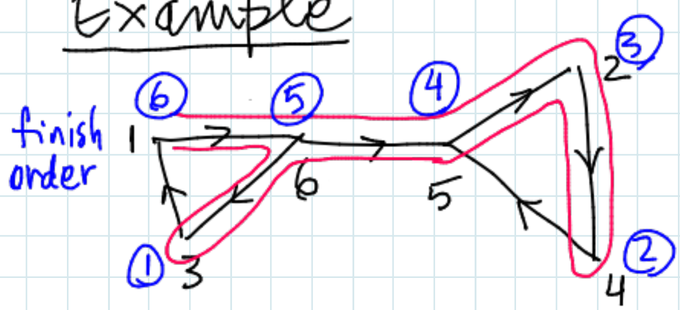
run DFS on G^R with vertex order $f_n\ f_{n-1} \cdots f_1$

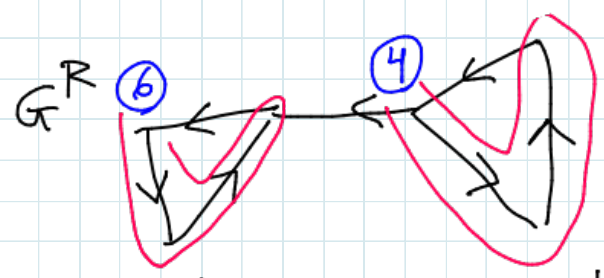
Lemma Trees in 2nd DFS are exactly the strongly connected components.

For pseudo code, see CLRS.

Run time is $O(n+m)$

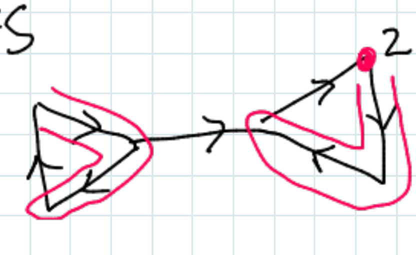
Example





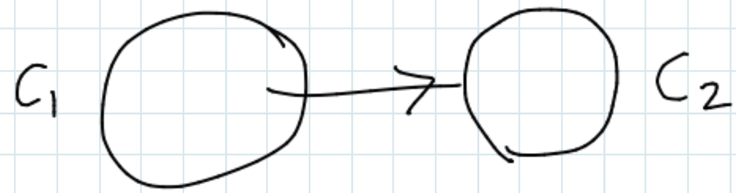
2 DFS trees, 2 components.

If first DFS started at 2



last finished vertex still on left. So same 2 trees.

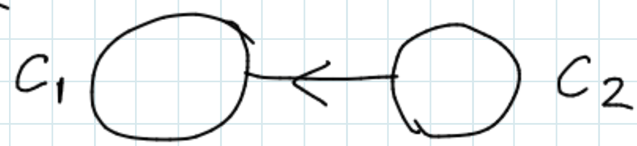
Idea of why this works:



If DFS of G starts at v in C_1 then it reaches $C_1 \cup C_2$ and v is last finished.

If DFS of G starts at v in C_2 then it reaches all of C_2 and new DFS tree starts at C_1. So last finished vertex is in C_1

in G^R



DFS starts in C_1 — so it does not reach C_2 — need a new tree for C_2.

Formal Proof of Lemma

i.e. $\exists$ path $u \to v$ and $v \to u$

Must prove: vertices u, v are strongly connected iff they are in the same tree of DFS of G^R

$\Rightarrow$ Suppose without loss of generality u discovered first in DFS of G^R. Then, since there is a $u \to v$ path in in G^R, v is discovered before u is finished.

$\Leftarrow$ suppose u, v in same tree of DFS G^R. Let r = root.

<u>Claim</u> r and u are strongly connected.

Then, by same argument, r and v are strongly connected. Therefore u and v are too.

<u>Pf. of claim</u> r is root of tree containing u.

So $\exists$ path $r \to u$ in G^R, i.e. a path $u \to r$ in G.

We must show $\exists$ path $r \to u$ in G.

When we started the tree rooted at r, u was undiscovered too. Why did we pick r? It had a higher finish time in DFS of G,

$u \rightsquigarrow r$ finished later

In DFS of G:

If u discovered before r, then r is discovered & finished before u is finished. Contra.

So r is discovered before u but finished later.

Then u is in r's tree so $\exists$ path $r \to u$ $\square$