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# ASSIGNMENT 9

DUE: Wednesday November 22, 7 PM. DO NOT COPY. ACKNOWLEDGE YOUR SOURCES.

Please read `http://www.student.cs.uwaterloo.ca/~cs341` for general instructions and policies. In particular, note that "giving" an algorithm includes justifying correctness and run time.

1. [10 marks] **Decision vs Optimization**

   (a) [5 marks] Suppose you have a polynomial time algorithm for the following decision problem: Given a list $n$ numbers, $a_1, a_2, \ldots, a_n$, indexed by $S = \{1, \ldots, n\}$, is there a partition $S = A \cup B$ with $A \cap B = \phi$ such that $\sum_{i \in A} a_i = \sum_{i \in B} a_i$.

   Show that you can use this algorithm to find such a partition $A, B$ (if it exists) in polynomial time. If the original algorithm runs in time $O(n^p)$, give a bound on the run time of your algorithm.

   (b) [5 marks] SATISFIABILITY. In class we have seen the NP-complete problem 3-SAT, where each clause has 3 literals. Recall that a literal is a variable $x_i$ or the negation of a variable $\neg x_i$. The variant where each clause has at most 2 literals is solvable in polynomial time. However, the following problem is NP-complete:

   **MAX 2-SAT.** Input: a number $k > 0$, a set of $n$ Boolean variables, $x_1, x_2, \ldots, x_n$ and a set $C$ of $m$ clauses, where each clause has the form $(l_i \vee l_j)$ where $l_i$ and $l_j$ are literals. Question: is there an assignment of truth-values to the variables that makes at least $k$ of the clauses true?

   Suppose you have a polynomial time algorithm for the above MAX 2-SAT decision problem. Show that you can use this algorithm to find the maximum number of clauses that can be made true, and to find a truth-value assignment that satisfies that number of clauses, both in polynomial time.

2. [10 marks] **NP and co-NP**

   (a) [5 marks] Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, both with $n$ vertices, are said to be *isomorphic* if relabelling the vertices makes them identical—more precisely, if there is a mapping $\pi : V_1 \to V_2$ one-to-one and onto, such that $(u, v) \in E_1$ iff $(\pi(u), \pi(v)) \in E_2$. Show that graph isomorphism is in NP. Be clear about your certificate and about the details of your verification algorithm and its run-time.

   (b) [5 marks] A Boolean formula $F$ in variables $x_1, x_2, \ldots, x_n$ is a *tautology* if every truth-value assignment to its variables makes the formula True. Show that the question of whether a formula is a tautology is in co-NP. Equivalently: show that the question of whether a formula is NOT a tautology is in NP.

   Be clear about your certificate and about the details of your verification algorithm and its run-time.

   (c) [BONUS. 100% in the course] Show that the question of whether a formula is a tautology is in NP.