1 Problem statement and boundary integral formulation

Assume that a single bubble occupies D, which is a circle of radius R_b and center at the origin. Let $\mathcal{C} = \cup_{n\in\mathbb{Z}^2}(D+n)$ be the periodic bubble crystal.

Consider now a perturbed crystal, where D is replaced by a defect circle D_d of radius $R_d < R_b$. Let $\mathcal{C}_d = D_d \cup \left(\cup_{n\in\mathbb{Z}^2\setminus\{0,0\}} D+n\right)$ be the perturbed crystal. We consider the following problem

$$\begin{cases} \nabla\cdot\dfrac{1}{\rho}\nabla u + \dfrac{\omega^2}{\kappa}u = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\mathcal{C}_d, \\ \nabla\cdot\dfrac{1}{\rho_b}\nabla u + \dfrac{\omega^2}{\kappa_b}u = 0 \quad \text{in} \quad \mathcal{C}_d, \\ u_+ - u_- = 0 \quad \text{on} \quad \partial\mathcal{C}_d, \\ \dfrac{1}{\rho}\dfrac{\partial u}{\partial\nu}\Big|_+ - \dfrac{1}{\rho_b}\dfrac{\partial u}{\partial\nu}\Big|_- = 0 \quad \text{on} \quad \partial\mathcal{C}_d \end{cases} \tag{1}$$

Here, $\partial/\partial\nu$ denotes the outward normal derivative and $|_\pm$ denote the limits from outside and inside D.

Let

$$v = \sqrt{\frac{\kappa}{\rho}}, \quad v_b = \sqrt{\frac{\kappa_b}{\rho_b}}, \quad k = \frac{\omega}{v} \quad \text{and} \quad k_b = \frac{\omega}{v_b}$$

be respectively the speed of sound outside and inside the bubbles, and the wavenumber outside and inside the bubbles. We also introduce two dimensionless contrast parameters

$$\delta = \frac{\rho_b}{\rho} \quad \text{and} \quad \tau = \frac{k_b}{k} = \frac{v}{v_b} = \sqrt{\frac{\rho_b\kappa}{\rho\kappa_b}}.$$

Let $G(x,y)$ be the Green's function corresponding to the periodic crystal, i.e. G satisfies

$$\Delta G + (k^2 + (k_b^2 - k^2)\chi(\mathcal{C}))G = \delta(x-y)$$

Let $\mathcal{S}_D^k$ be the free-space single layer potential defined by

$$\mathcal{S}_D^k[\phi](x) = \int_{\partial D}\Gamma(x,y)\phi(y)\,\mathrm{d}\sigma(y), \quad x\in\mathbb{R}^2,$$

and let $\mathcal{S}_D^\#$ be the single layer potential associated to the Green's function G, i.e.

$$\mathcal{S}_D^\#[\phi](x) = \int_{\partial D}G(x,y)\phi(y)\,\mathrm{d}\sigma(y), \quad x\in\mathbb{R}^2.$$

We seek a solution $u(x)$ of the form

$$u(x) = \begin{cases} \mathcal{S}_{D_d}^{k_b}[\phi_1](x) & x\in D_d \\ \mathcal{S}_{D_d}^{k}[\phi_2](x) + S_D^k[\phi_3](x) & x\in D\setminus D_d \\ \mathcal{S}_D^\#[\phi_4](x) & x\in\mathbb{R}^2\setminus D. \end{cases}$$

A solution of this form satisfies the differential equation in (1). The boundary conditions in equation (1) implies that the layer densities $\phi_i,\ i=1,2,3,4$ satisfies the system of boundary integral equations $\mathcal{A}(\omega,\delta)\Phi = 0$, where

$$\mathcal{A}(\omega,\delta) = \begin{pmatrix} \mathcal{S}_D^{k_b} & -\mathcal{S}_D^k & -\mathcal{S}_{D_d,D}^k & 0 \\ 0 & \mathcal{S}_{D,D_d}^k & \mathcal{S}_{D_d}^k & -\mathcal{S}_D^\# \\ -\frac{1}{2}I + \mathcal{K}_{D_d}^{k_b,*} & -\delta\left(\frac{1}{2}I + (\mathcal{K}_D^k)^*\right) & -\delta\frac{\partial\mathcal{S}_{D_d,D}^k}{\partial\nu} & 0 \\ 0 & \frac{\partial\mathcal{S}_{D,D_d}^k}{\partial\nu} & -\frac{1}{2}I + (\mathcal{K}_D^k)^* & -\left(\frac{1}{2}I + (\mathcal{K}_D^\#)^*\right) \end{pmatrix}, \text{ and } \Phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}.$$

Here the operator $\mathcal{S}_{D_d,D}^k = \mathcal{S}_D^k|_{x\in\partial D_d}$ is the restriction of $\mathcal{S}_D^k$ onto ∂D_d.

2 Numerical implementation

We seek a spatial discretization of the boundary integral formulation. The factors $\mathcal{S}_D^{\#}$ and $(\mathcal{K}_D^{\#})^*$ require the Green's function G for the crystal, so the equation (1) has to be solved numerically. We will apply the method found in [1] Applying the Floquet transform we can decompose G into the α-quasiperiodic Green's function G_α which satisfies

$$\Delta G_\alpha + (k^2 + (k_b^2 - k^2)\chi(\mathcal{C}))G_\alpha = \sum_{n\in\mathbb{Z}} \delta(x-y-n)e^{in\cdot\alpha}$$

Let $Y = [-1/2, 1/2]^2$. For a fixed $y \in \mathbb{R}^2$, the function $u(x) = G_\alpha(x,y)$ is a solution to the problem

$$\begin{cases} \nabla \cdot \frac{1}{\rho}\nabla u + \frac{\omega^2}{\kappa} u = \delta(x-y) \quad \text{in} \quad Y\backslash D, \\ \nabla \cdot \frac{1}{\rho_b}\nabla u + \frac{\omega^2}{\kappa_b} u = \delta(x-y) \quad \text{in} \quad D, \\ u_+ - u_- = 0 \quad \text{on} \quad \partial D, \\ \frac{1}{\rho}\frac{\partial u}{\partial \nu}\Big|_+ - \frac{1}{\rho_b}\frac{\partial u}{\partial \nu}\Big|_- = 0 \quad \text{on} \quad \partial D \\ e^{-i\alpha\cdot x}u \text{ is periodic.} \end{cases} \tag{2}$$

Because of the reciprocity relation $G_\alpha(x,y) = G_\alpha(y,x)$, also $e^{-i\alpha\cdot y}G_\alpha(x,y)$ is periodic in y, so we can restrict to the case $y \in Y$. Then G_α can be written

$$G_\alpha(x,y) = \begin{cases} \Gamma_\alpha^{k_b}(x,y) + S_D^{k_b}[\psi_b](x) & x \in D \\ \Gamma_\alpha^{k}(x,y) + S_D^{k}[\psi](x) & x \in Y\backslash\bar{D} \end{cases}$$

Using the jump relations for the single layer potentials, we find that

$$\mathcal{B}(\omega,\delta)[\Psi] = F$$

where

$$\mathcal{B}(\omega,\delta) = \begin{pmatrix} \mathcal{S}_D^{k_b} & -\mathcal{S}_D^{\alpha,k} \\ -\frac{1}{2} + \mathcal{K}_D^{k_b,*} & -\delta(\frac{1}{2} + (\mathcal{K}_D^{-\alpha,k})^*) \end{pmatrix}, \ \Psi = \begin{pmatrix} \psi_b \\ \psi \end{pmatrix}, \ F = \begin{pmatrix} \Gamma_\alpha^k - \Gamma_\alpha^{k_b} \\ \delta\frac{\partial \Gamma_\alpha^k}{\partial\nu} - \frac{\partial \Gamma_\alpha^{k_b}}{\partial\nu} \end{pmatrix}$$

Recall that the quasi-periodic Green's function Γ_α^k, defined as the solution to the equation

$$\Delta\Gamma_\alpha^k(x,y) + k^2\Gamma_\alpha\alpha^k(x,y) = \sum_{n\in\mathbb{Z}}\delta(x-y-n)e^{in\cdot\alpha}$$

can be expanded as

$$\Gamma_\alpha^k(x,y) = -\frac{i}{4}\sum_{m\in\mathbb{Z}^2} H_0^{(1)}(k|x-y-m|)e^{im\cdot\alpha} \tag{3}$$

We need to expand the function $\Gamma_\alpha^k(x,y)$ in terms of the polar coordinates (r,θ) of x. We will use the following versions of Graf's addition theorem.

$$H_l^{(1)}(kr_2)e^{il\theta_2} = \begin{cases} \sum_{n=-\infty}^{\infty} H_{l-n}^{(1)}(kb)e^{i(l-n)\beta}J_n(kr_1)e^{in\theta_1} & \text{if } r_1 < b \\ \sum_{n=-\infty}^{\infty} H_{l-n}^{(1)}(kr_1)e^{i(l-n)\theta_1}J_n(kb)e^{in\beta} & \text{if } r_1 > b \end{cases}$$

In these equations we have $x_1 = r_1e^{i\theta_1}, x_2 = r_2e^{i\theta_2}$ and $x_2 = x_1 + be^{i\theta}$.

In the following, pick x on the boundary ∂D, i.e. $x = R_be^{i\theta}$. Furthermore, pick $y = r'e^{i\theta'}$ inside Y. Using the addition formulas, we have

$$H_0^{(1)}(k|x-y-m|) = \begin{cases} \sum_{n=-\infty}^{\infty}(-1)^nH_{-n}^{(1)}(k|y+m|)e^{-in\theta'_m}J_n(kR_b)e^{in\theta} & \text{if } R_b < |y+m| \\ \sum_{n=-\infty}^{\infty}(-1)^nH_{-n}^{(1)}(kR_b)e^{-in\theta}J_n(k|y+m|)e^{in\theta'_m} & \text{if } R_b > |y+m| \end{cases}$$

For $m \neq 0$ we have $R_b < |y+m|$ and

$$H_{-n}^{(1)}(k|y+m|)e^{-in\theta'_m} = \sum_{l=-\infty}^{\infty} H_{-n-l}^{(1)}(k|m|)e^{i(-n-l)\theta_m}J_l(kr')e^{il\theta'}$$

Plugging in above expressions into equation 3, we find

$$\Gamma_\alpha^k(x,y) = -\frac{i}{4}\sum_{n=-\infty}^{\infty}\left[M_n e^{in\theta} + \sum_{l=-\infty}^{\infty}\left[\sum_{m\in\mathbb{Z}^2, m\neq 0} H_{-n-l}(k|m|)e^{i(-n-l)\theta_m}e^{im\cdot\alpha}\right](-1)^n J_l(kr')e^{il\theta'}J_n(kR_b)e^{in\theta}\right],$$

where the terms M_n, corresponding to $m = 0$, are given by

$$M_n = \begin{cases}(-1)^n H_{-n}(kr')e^{-in\theta'}J_n(kR_b) & \text{if } r' > R_b \\ (-1)^n H_n(kR_b)J_{-n}(kr')e^{-in\theta'} & \text{if } r' < R_b.\end{cases}$$

The two different cases correspond to the source y being inside or outside the bubble. Define the lattice sum Q_n as

$$Q_n = \sum_{m\in\mathbb{Z}^2, m\neq 0} H_n(k|m|)e^{in\theta_m}e^{im\cdot\alpha}.$$

Then the equation for Γ_α^k is

$$\Gamma_\alpha^k(x,y) = -\frac{i}{4}\sum_{n=-\infty}^{\infty}\left[M_n + \sum_{l=-\infty}^{\infty} Q_{-n-l}(-1)^n J_l(kr')e^{il\theta'}J_n(kR_b)\right]e^{in\theta}. \tag{4}$$

This can be viewed as a Fourier series expansion of Γ_α^k as a function of $x \in S^1$. The n:th Fourier coefficient is

$$-\frac{i}{4}\left[M_n + \sum_{l=-\infty}^{\infty} Q_{-n-l}(-1)^n J_l(kr')e^{il\theta'}J_n(kR_b)\right]$$

For $x \in \partial D$ we have

$$\frac{\partial \Gamma_\alpha^k}{\partial \nu(x)} = \frac{\partial \Gamma_\alpha^k}{\partial r}$$

Differentiating equation 4 we find

$$\frac{\partial \Gamma_\alpha^k}{\partial \nu(x)} = -\frac{i}{4}\sum_{n=-\infty}^{\infty}\left[M_n' + \sum_{l=-\infty}^{\infty} Q_{-n-l}(-1)^n J_l(kr')e^{il\theta'}kJ_n'(kR_b)\right]e^{in\theta},$$

where

$$M_n' = \begin{cases}(-1)^n kH_{-n}(kr')e^{-in\theta'}J_n'(kR_b) & \text{if } r' > R_b \\ (-1)^n kH_n'(kR_b)J_{-n}(kr')e^{-in\theta'} & \text{if } r' < R_b.\end{cases}$$

References

[1] H. Ammari, B. Fitzpatrick, H. Lee, S. Yu, and H. Zhang. Subwavelength phononic bandgap opening in bubbly media. *ArXiv e-prints*, February 2017.