Big Data Computing

Master's Degree in Computer Science

2019-2020

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SAPIENZA
UNIVERSITÀ DI ROMA

Recap from Previous Lecture

- We want to find an effective way to measure the **trustworthiness** of a page within the Web graph

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- We want to find an effective way to measure the **trustworthiness** of a page within the Web graph
- More generally, we want to assign a score which indicates the **importance** of a node in a graph
- Derive such a score from the structural properties of the graph only (i.e., via **link analysis**)
- Exploit the fact that the Web is an example of a **scale-free network**

Computing Node Importance

Several **link analysis** approaches to compute **web page importance**

PageRank

Hubs and Authorities (HITS)

Personalized PageRank

Web Spam Detection

PageRank

One Slide PageRank

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- Assigns a numerical score to each web page with the purpose of indicating its relative importance within the whole collection

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PageRank's Intuition: Links as Votes

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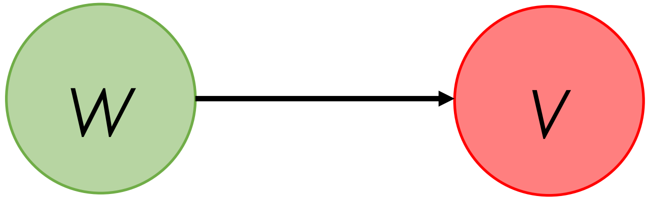

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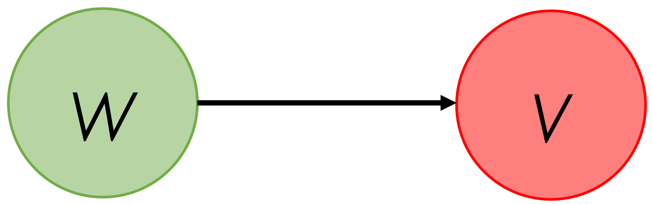

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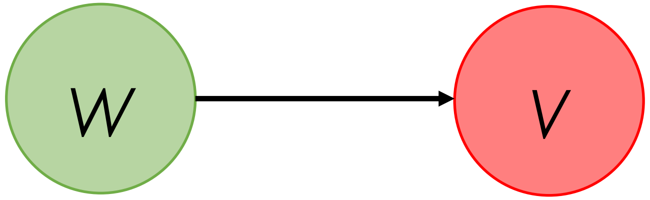

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Different web pages have different in-degree (scale-free network)

www.stanford.edu has more than 23K in-links

www.uniroma1.it/~tolomei has one or two in-links!

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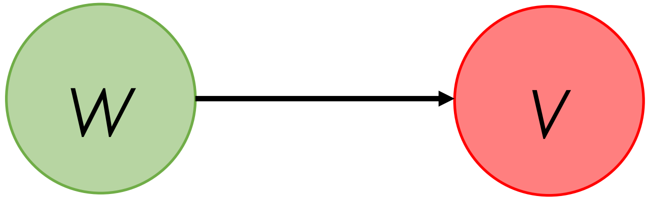

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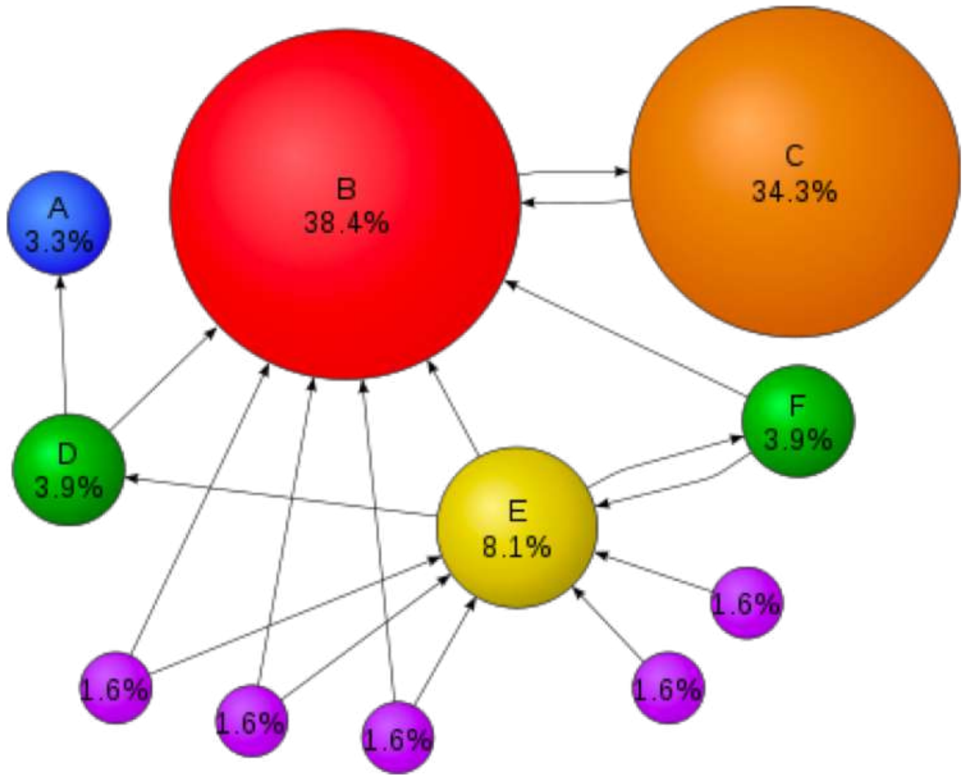
Recursive definition

PageRank Scores: Example



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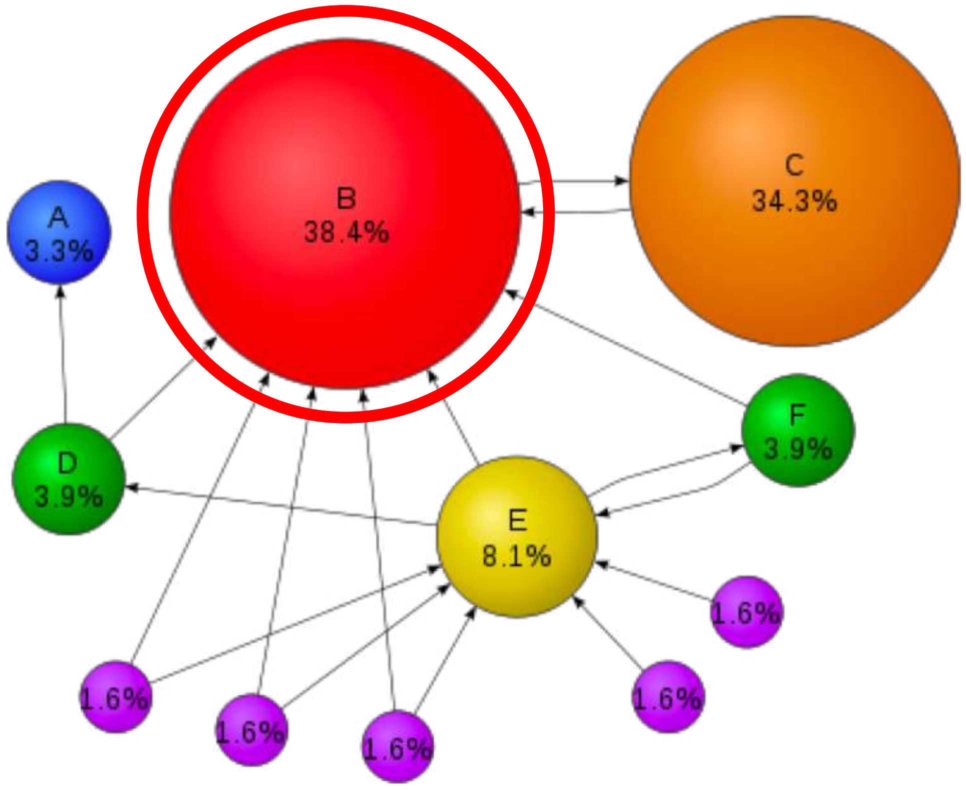

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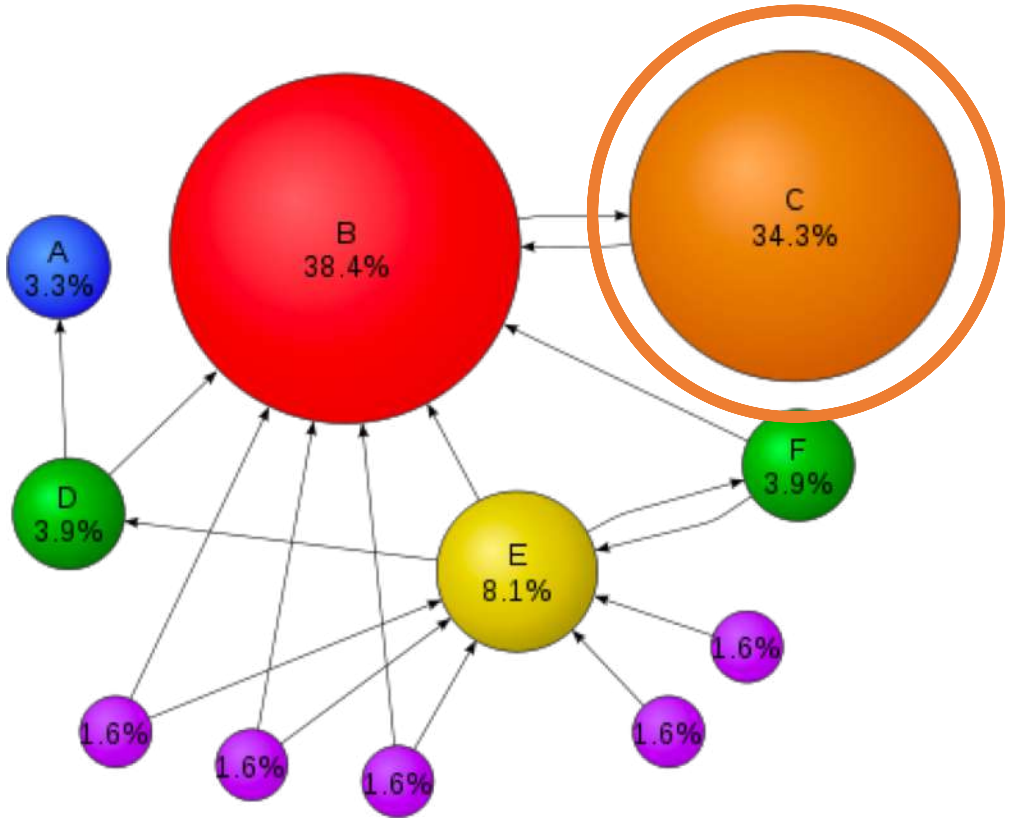

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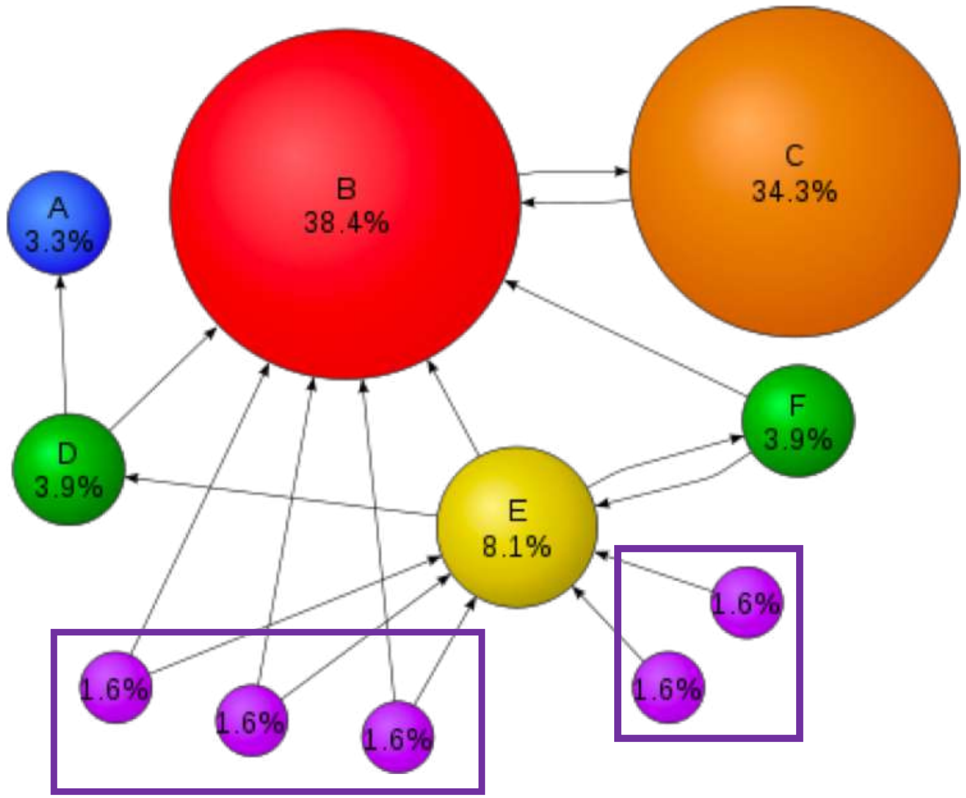

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Many other less important nodes

PageRank: Prelminaries

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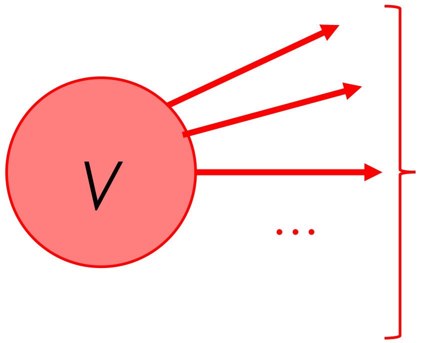
$|O_v| = o_v$ Out-degree of node *v*



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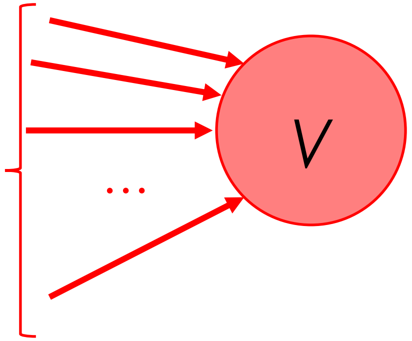

PageRank: First Simple Recursive Formulation

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If a page w has importance r_w and out-degree o_w, each out-link will get an **equal proportion** of the importance, i.e., r_w/o_w

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If a page w has importance r_w and out-degree o_w, each out-link will get an **equal proportion** of the importance, i.e., r_w/o_w

Each page v's importance can be computed just as the **sum of votes** of all its **incoming links** (i.e., in-degree)

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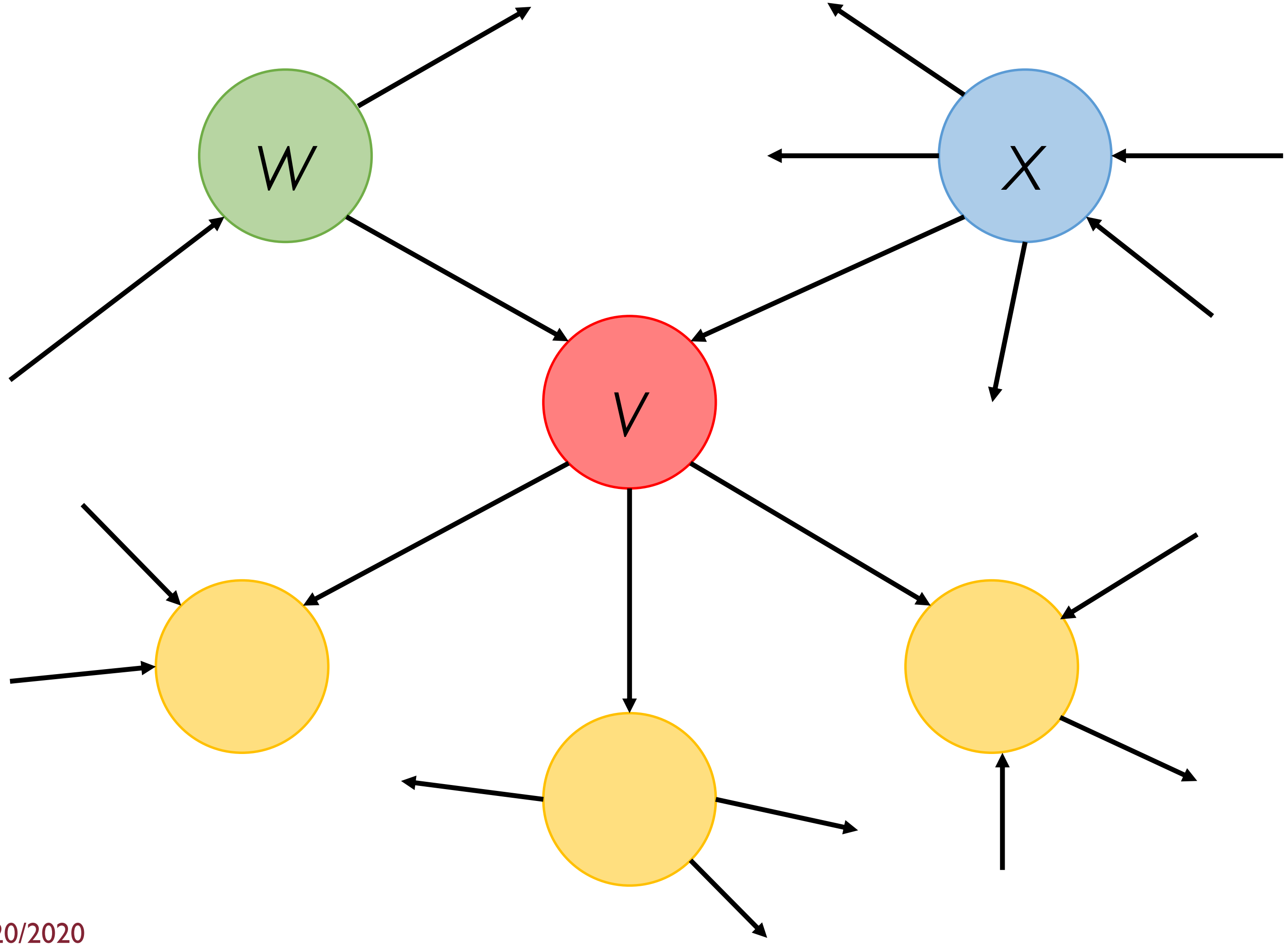

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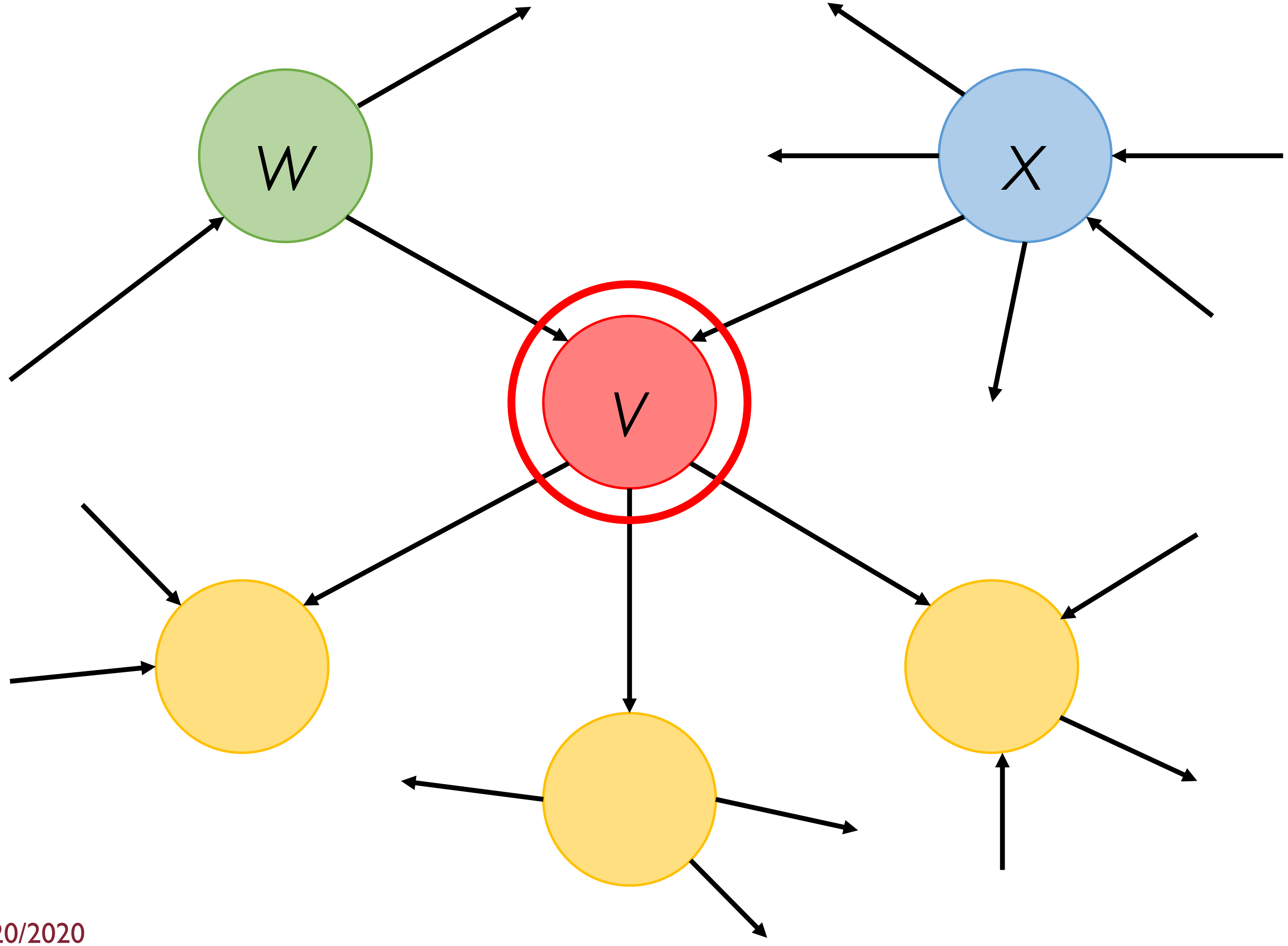

What is r_v?

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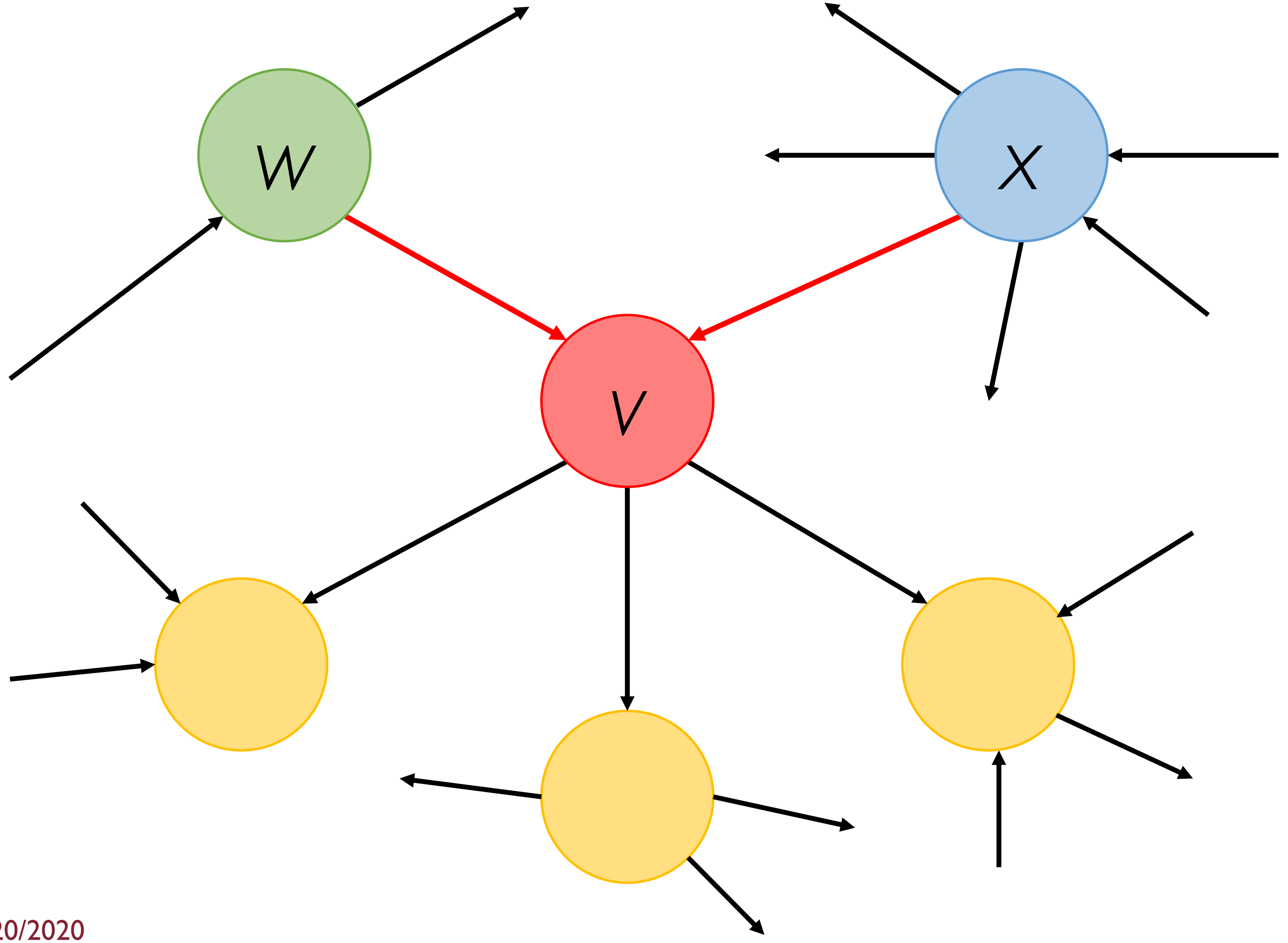

Suppose *v* has only **2** in-links coming from *w* and *x*

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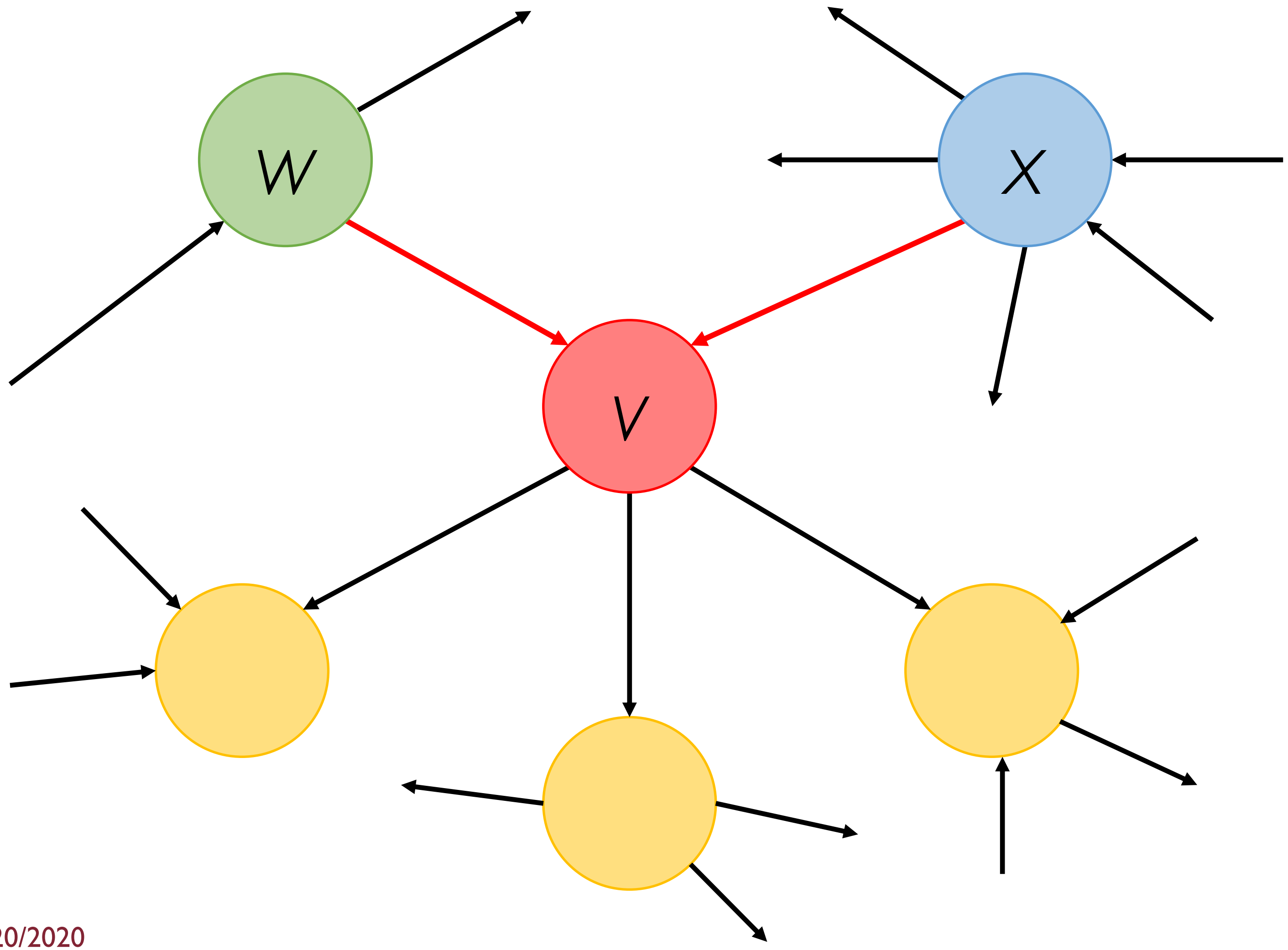
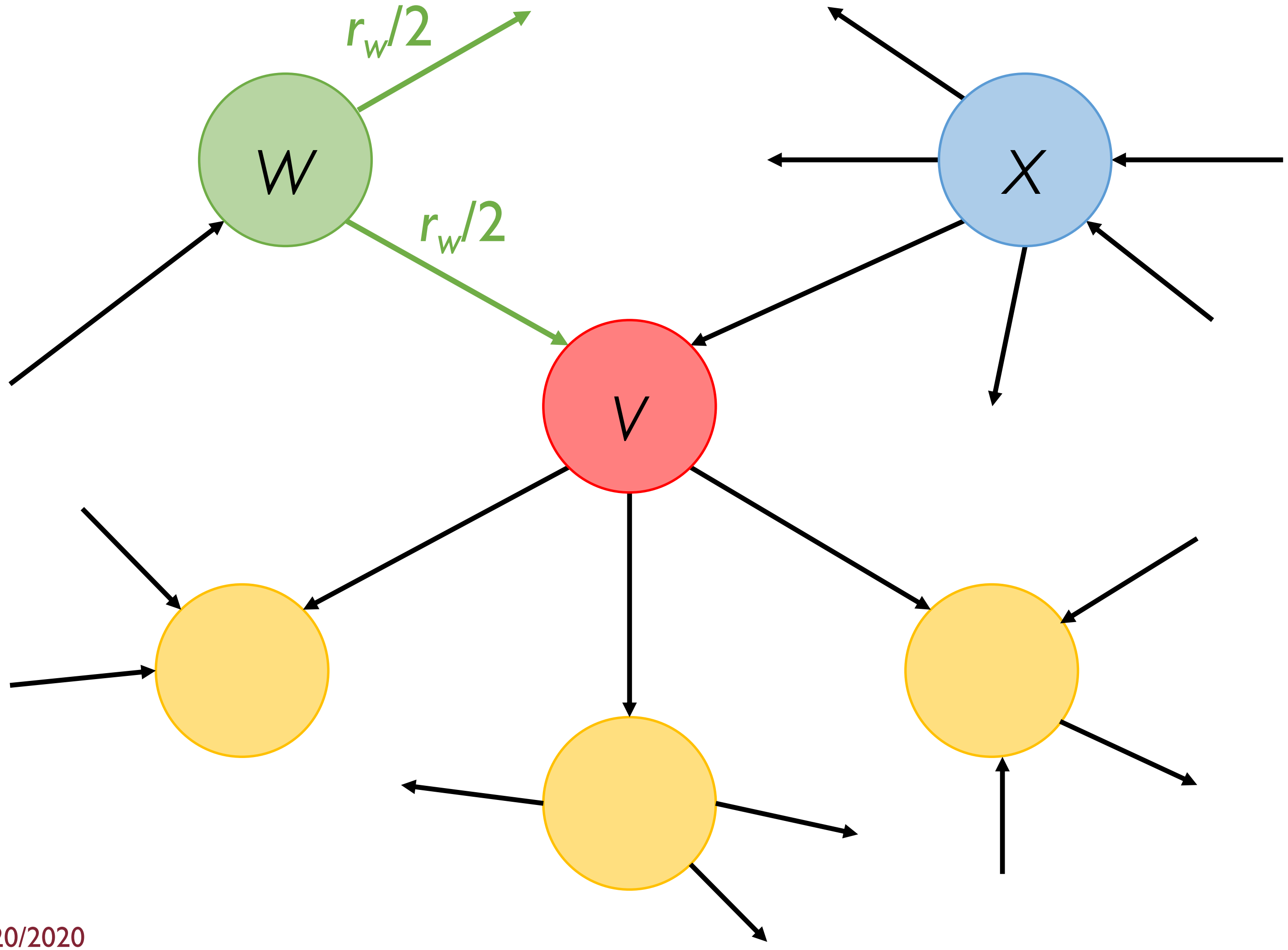


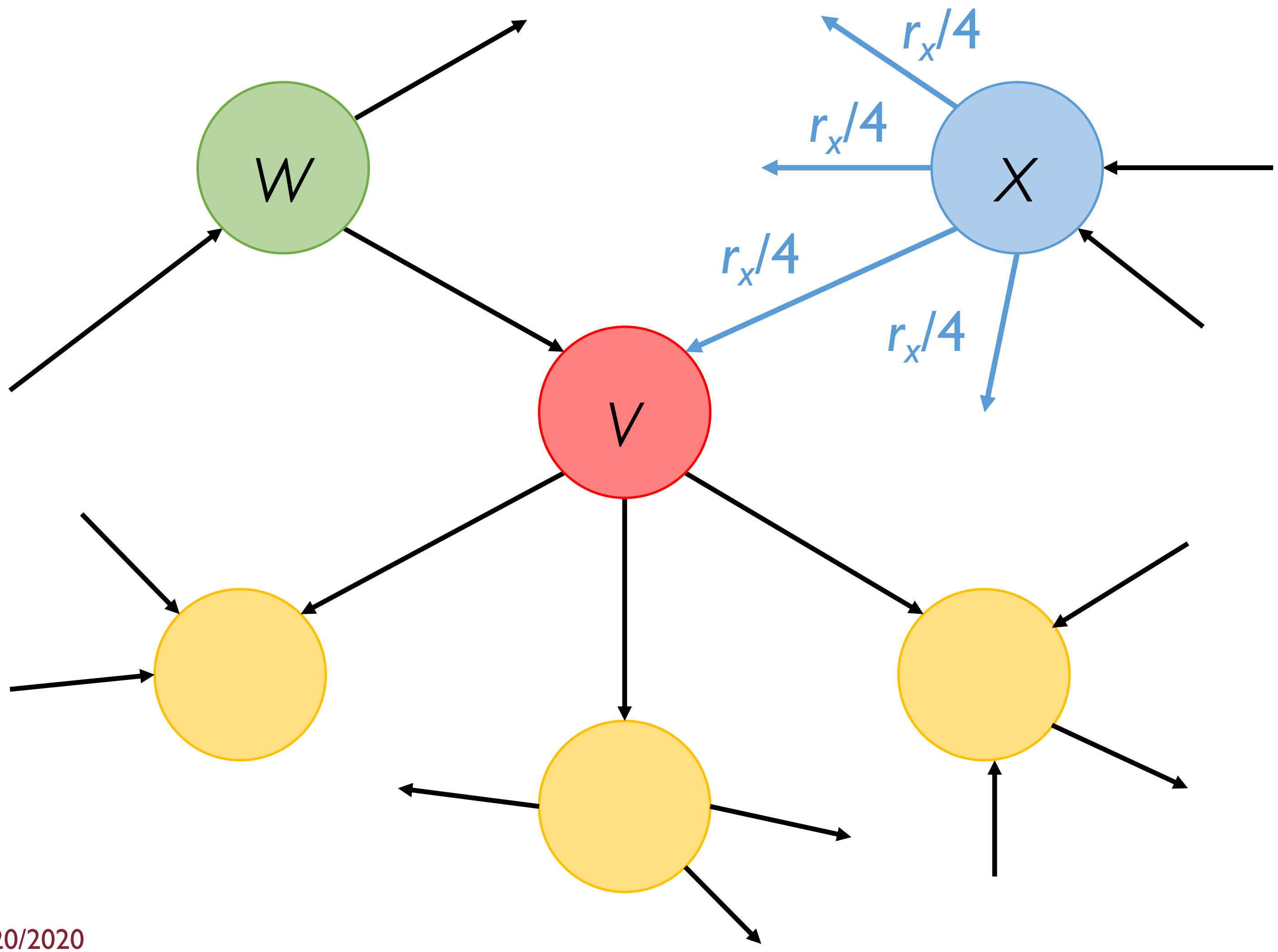
We must compute the in-link's **vote** from *w* and from *x*

PageRank: First Simple Recursive Formulation



The importance of page w (r_w) is distributed across each of its 2 outgoing links

PageRank: First Simple Recursive Formulation



The importance of page x (r_x) is distributed across each of its 4 outgoing links

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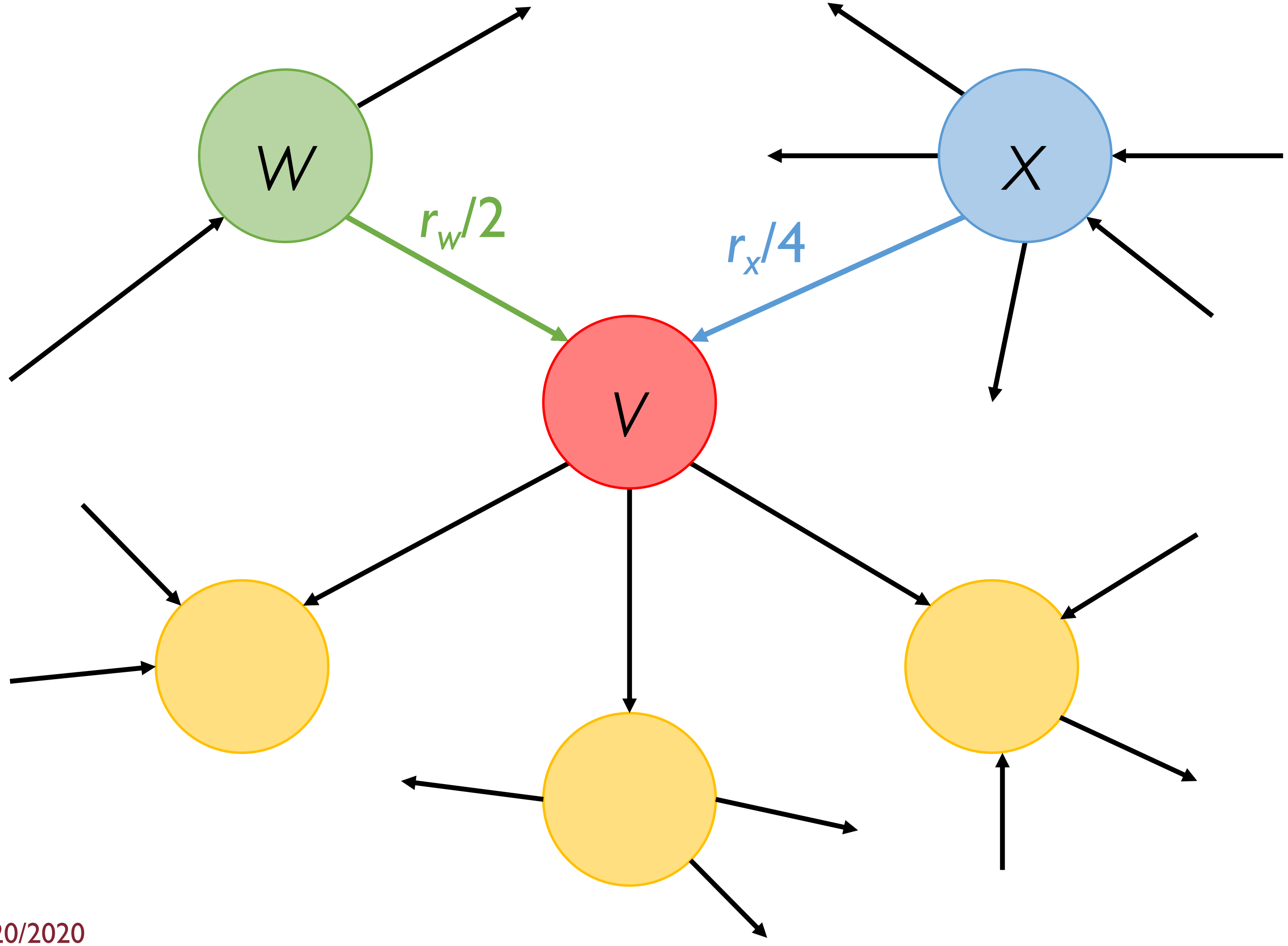
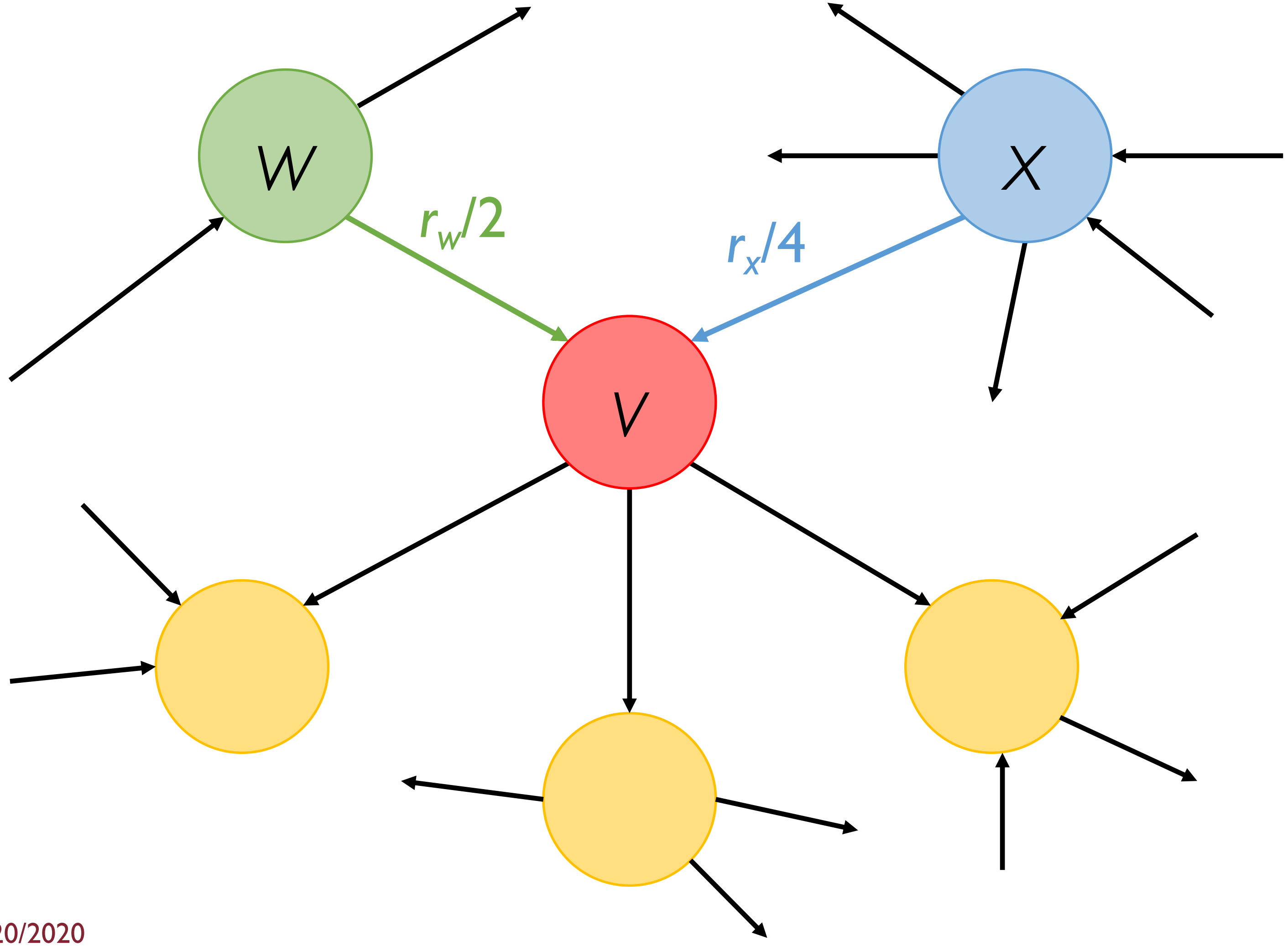


The importance of page v (r_v) is just the **sum** of its incoming links' votes

PageRank: First Simple Recursive Formulation



The importance of page v (r_v) is just the **sum** of its incoming links' votes

$$\boxed{r_v = r_w/2 + r_x/4}$$

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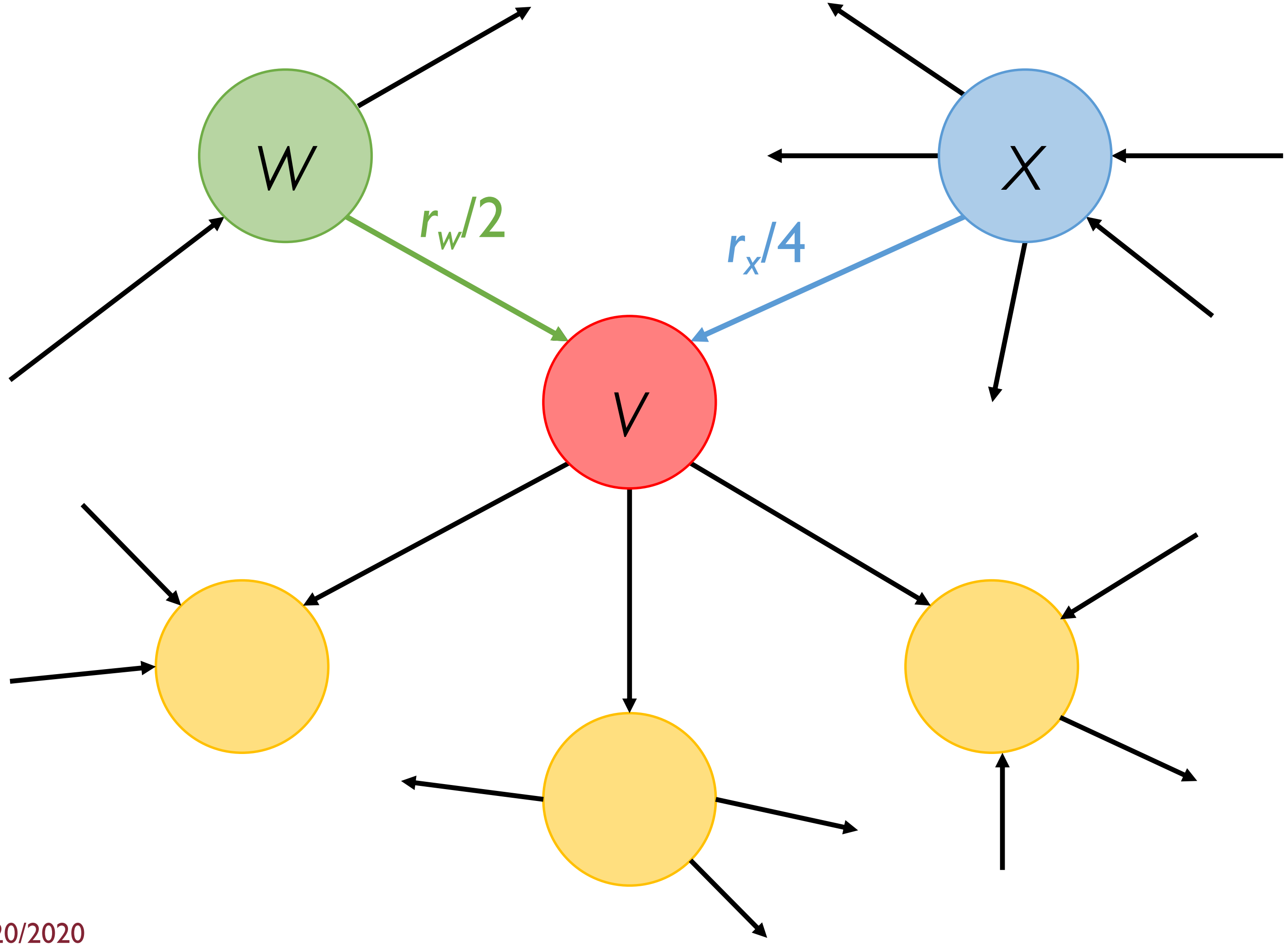

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$$\boxed{r_v = \sum_{u \in I_v} \frac{r_u}{o_u}}$$

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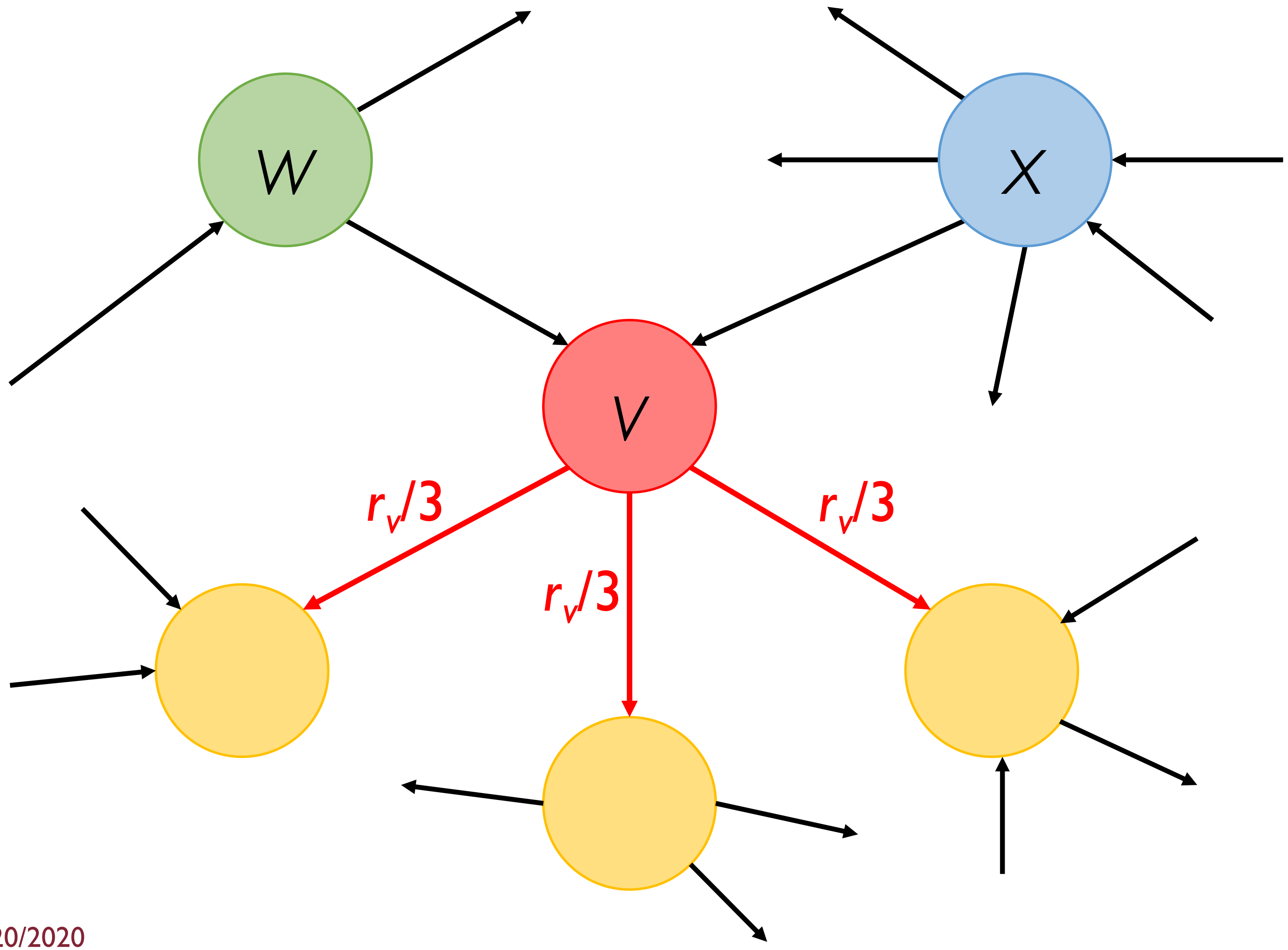


Similarly, page v **uniformly** distributes its importance r_v to its outgoing links

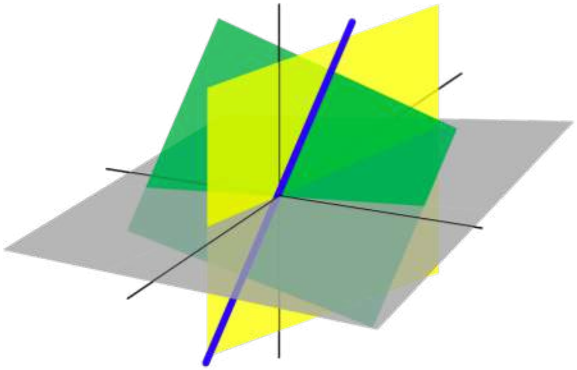
PageRank's Interpretations

$\boxed{\textbf{2} \text{ main perspectives}}$

PageRank's Interpretations

2 main perspectives

Linear Algebra



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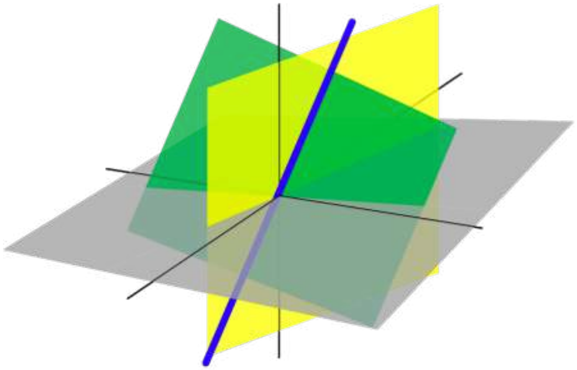

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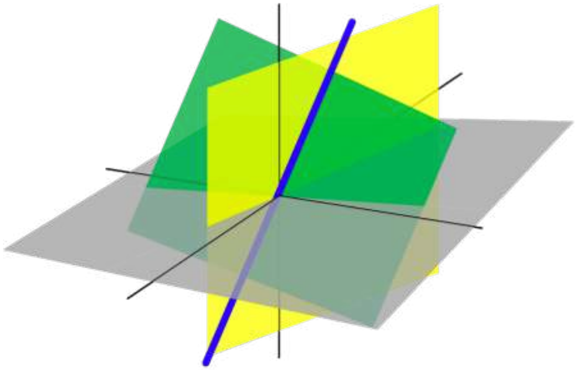
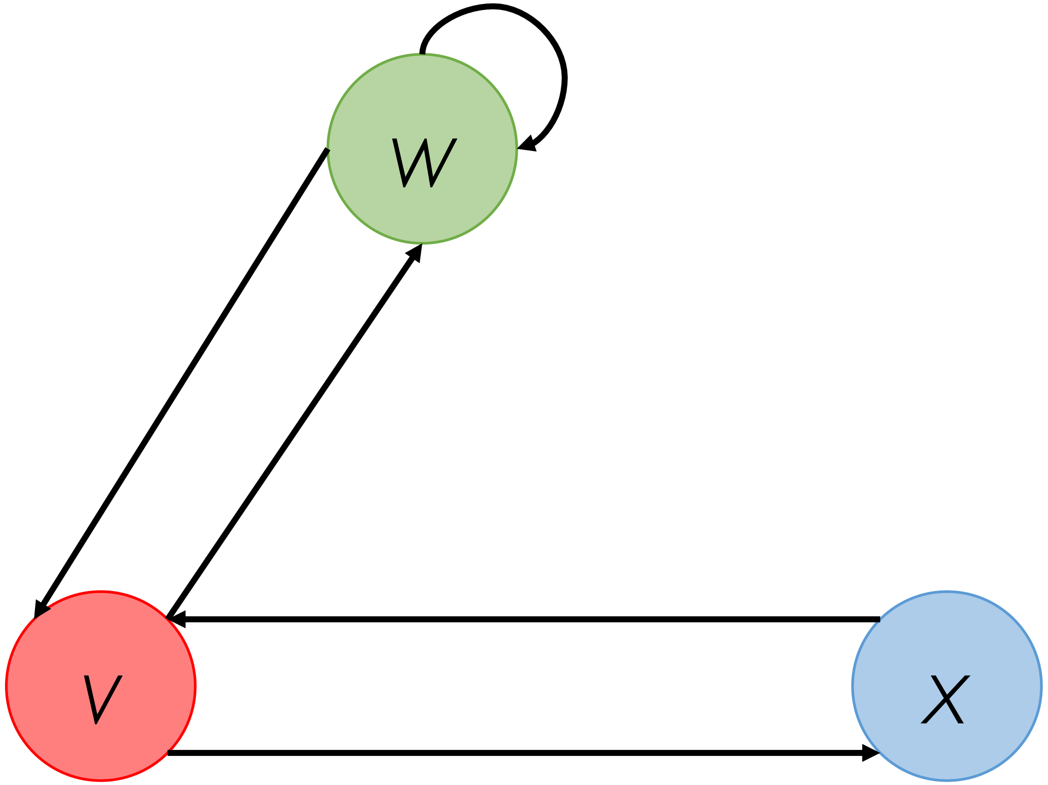


Probabilistic

PageRank: The "Flow" Model



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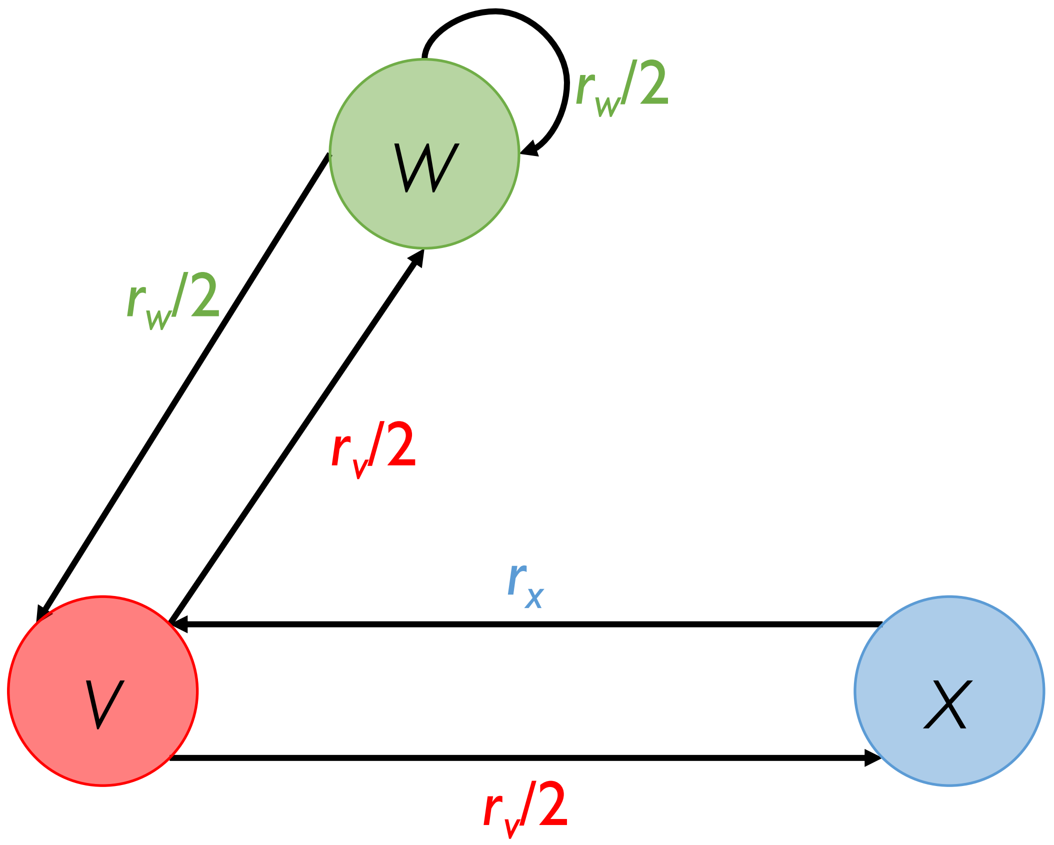

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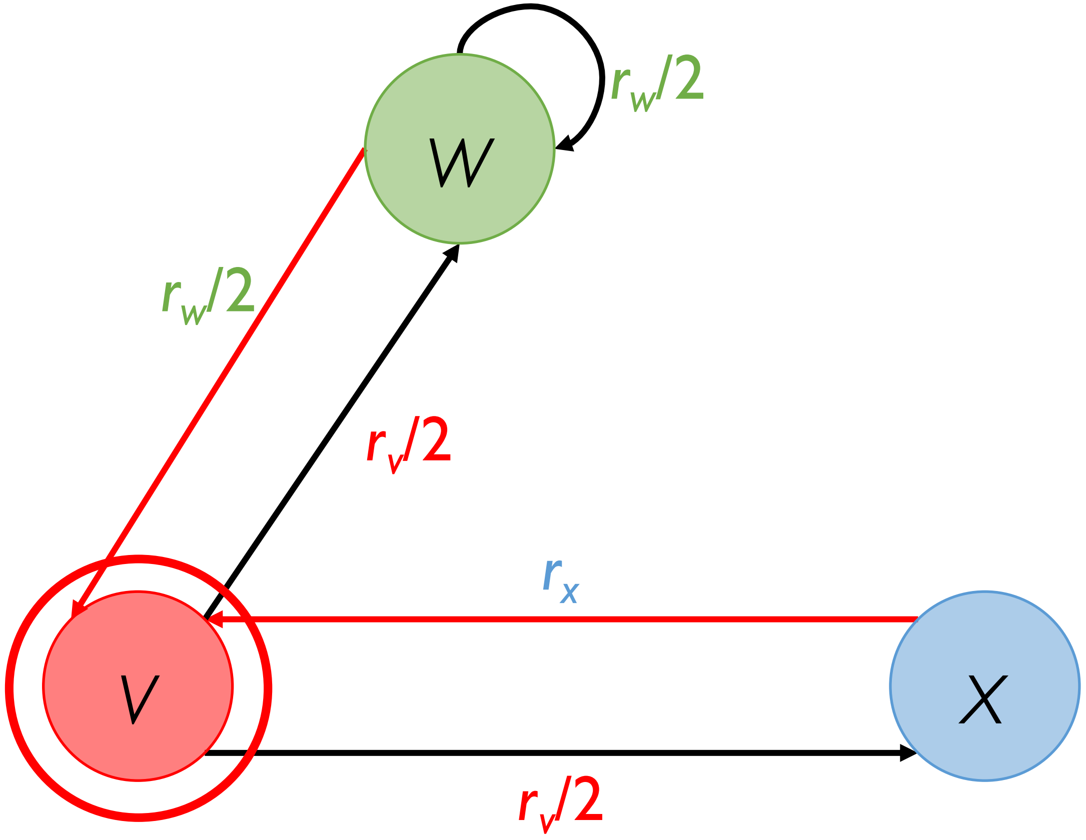

$$\begin{cases} \boxed{r_v = r_w/2 + r_x} \\ \\ \end{cases}$$

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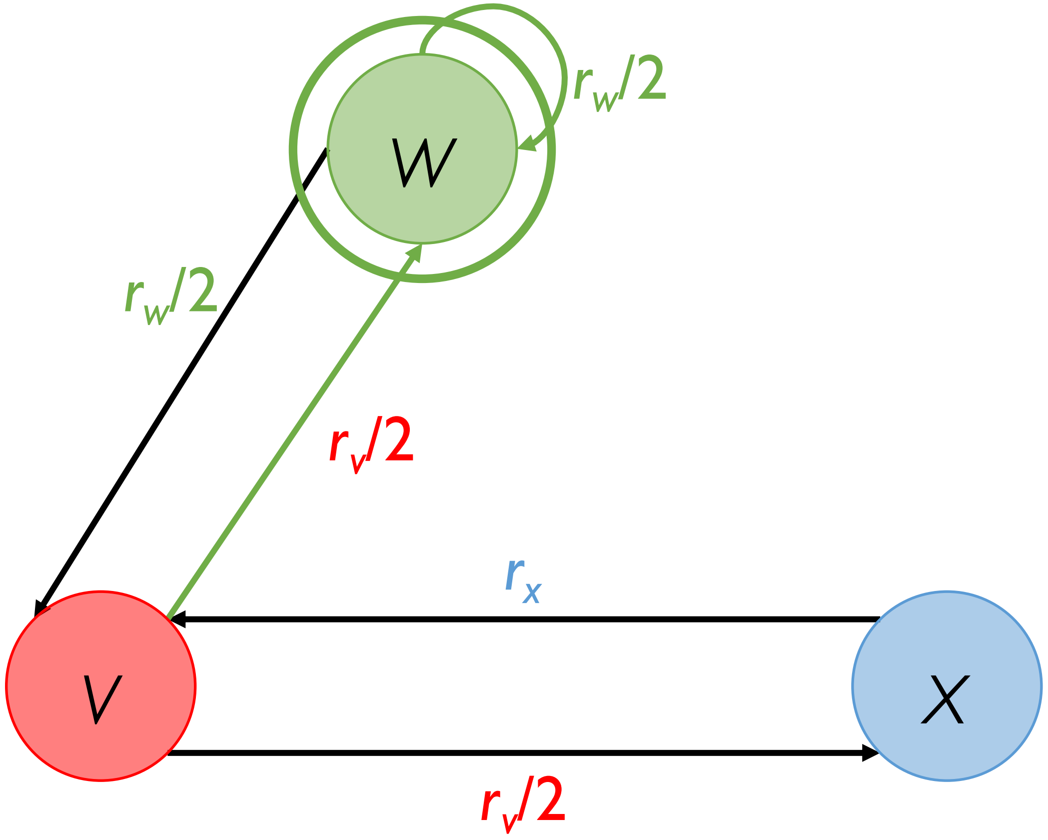

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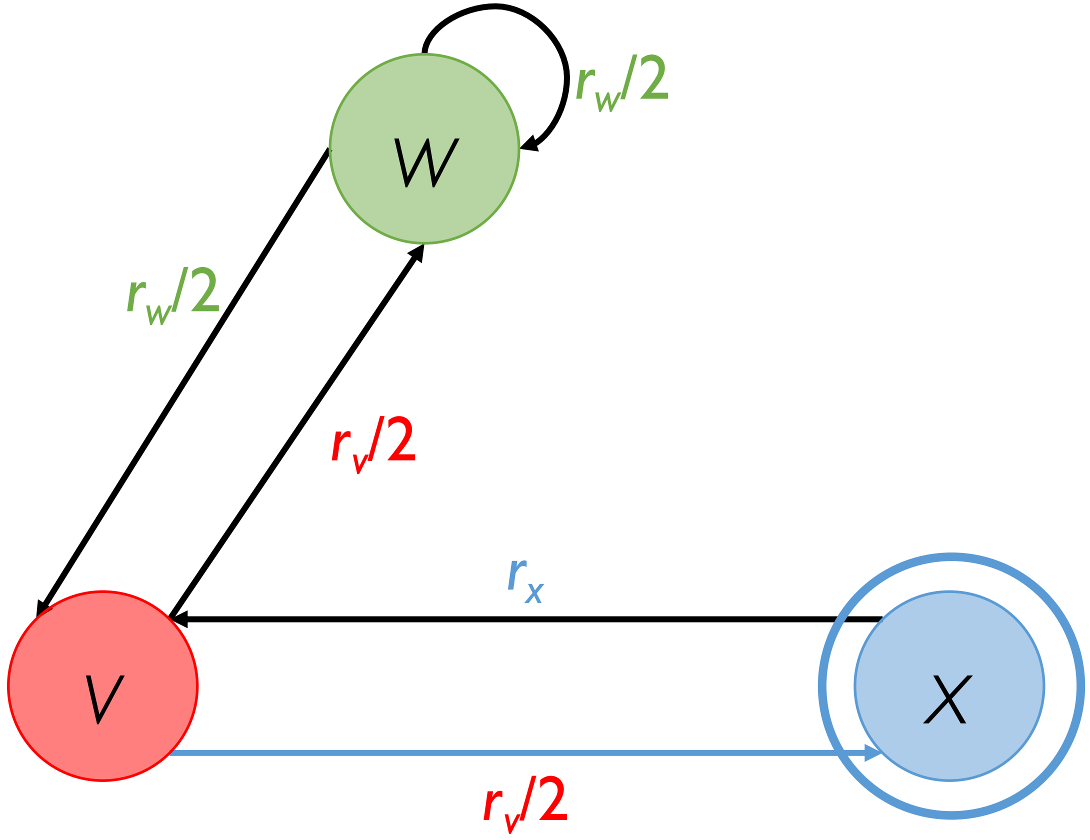

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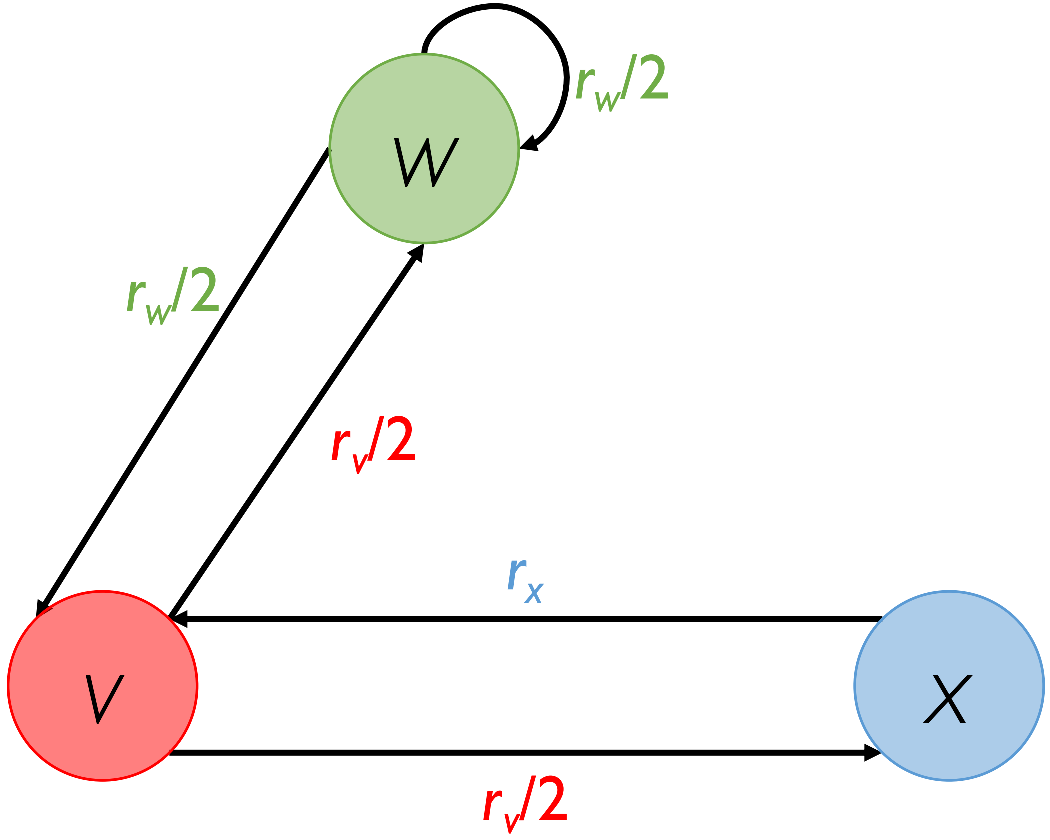

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"Flow" Equations

Solving the System of "Flow" Equations

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3 equations with 3 unknowns: r_v, r_w, and r_x

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No unique solution!

Infinitely many apart from a constant scale factor

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$$\begin{cases} r_v = r_w/2 + r_x \\ r_w = r_v/2 + r_w/2 \\ r_x = r_v/2 \\ \boxed{r_v + r_w + r_x = 1} \end{cases}$$

Additional constraint (equation) enforces the uniqueness of the solution

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$$\boxed{r_v = r_w = \frac{2}{5} \quad r_x = \frac{1}{5}}$$

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This may work for very small systems of linear equations (e.g., using Gaussian elimination)

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In the case of web pages we might have **100s of billions** of equations!

We need a new formulation

PageRank: The Matrix Formulation

Represent the Web graph of documents $G=(V, E)$ s.t. $|V|=N$
as a **column stochastic matrix** **M** of size NxN

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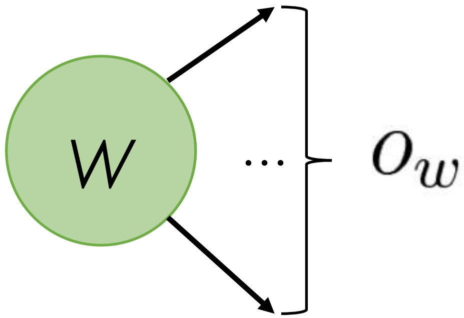

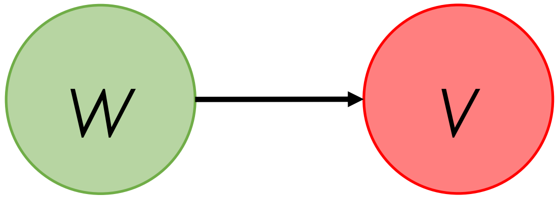

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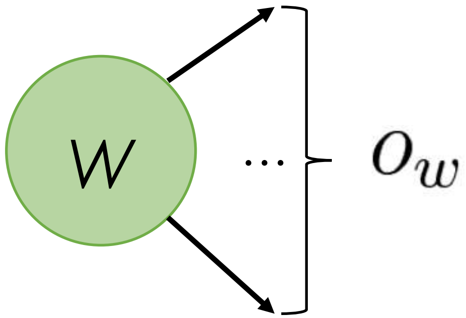

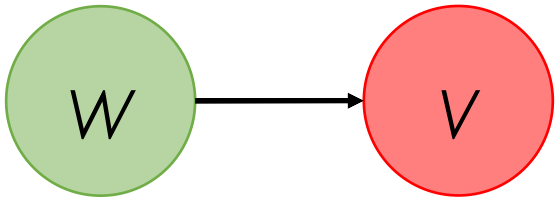

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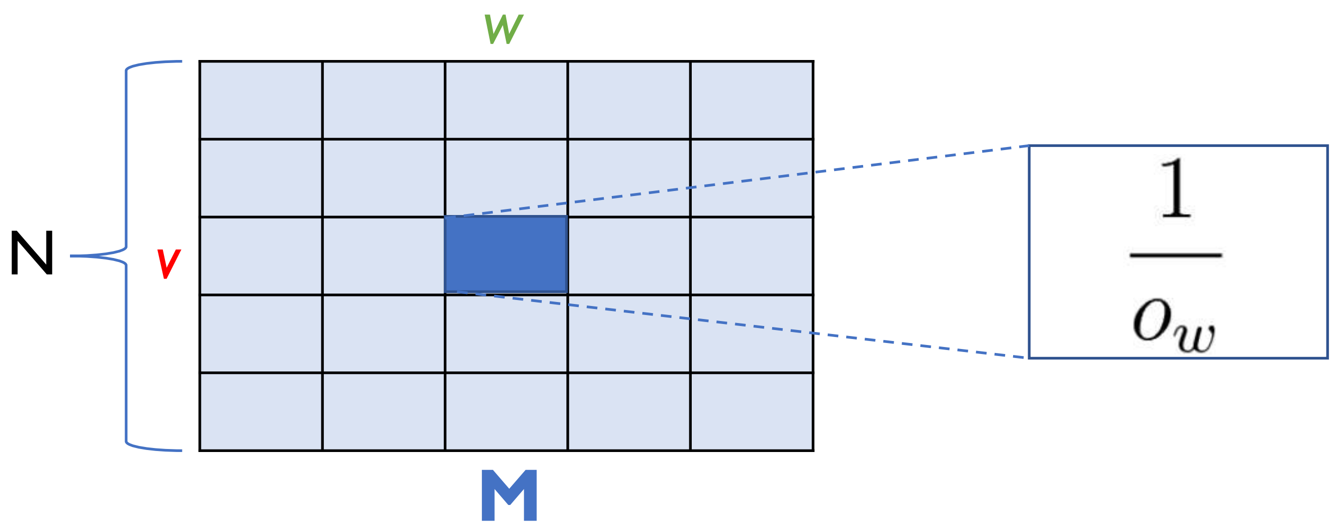

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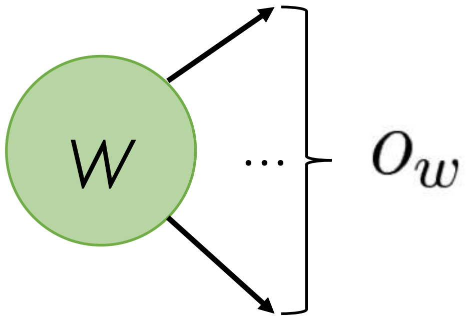
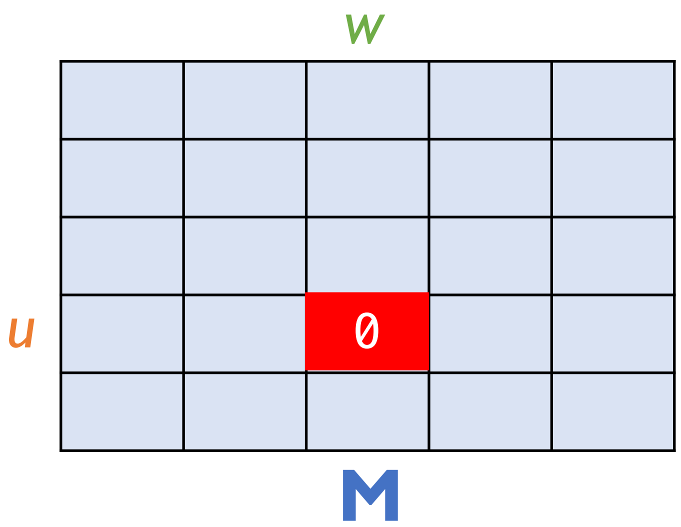


For any other page *u* which *w* is not pointing to **M**[*u*, *w*] = 0



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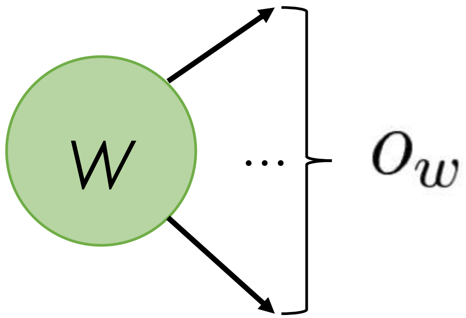

M is **column stochastic** because, by design, each of its **column sums up to 1**

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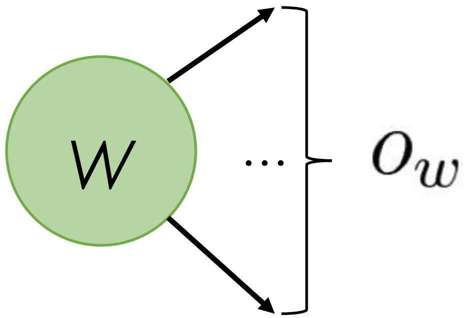

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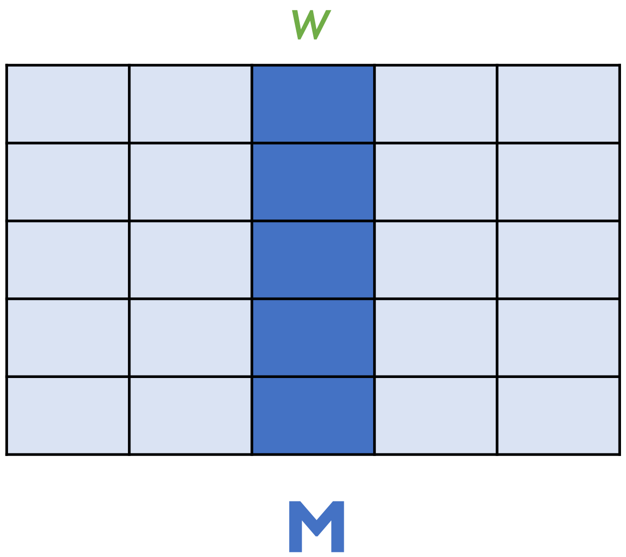

The *w*-th column will contain o_w <= N non-zero entries, each evaluating to $1/o_w$

$$\sum_{v=1}^{N} m_{v,w} = o_w \times \frac{1}{o_w} = 1$$

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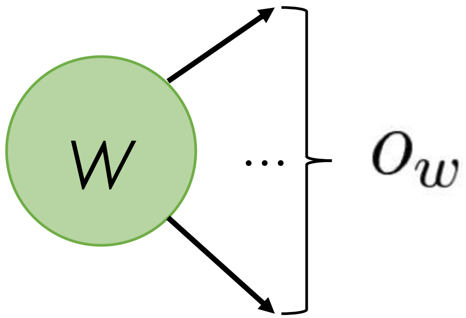

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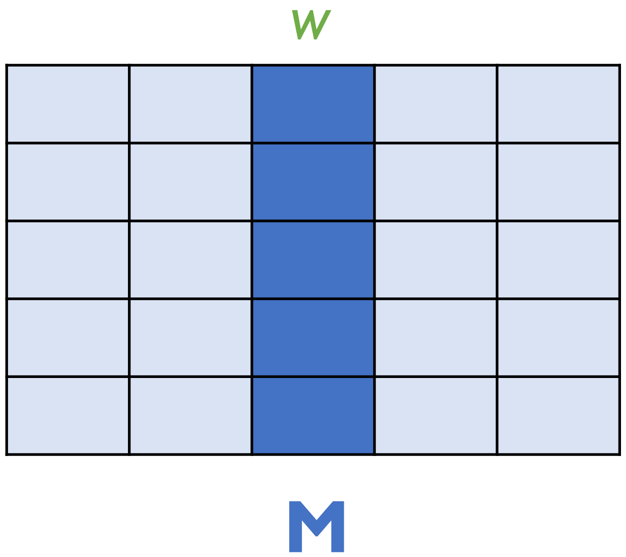

<u>Note:</u>
We are implicitly assuming there exists **at least one** outgoing link from each node

A Formal View of the Matrix **M**

$$\mathbf{A}_{N\times N} \quad a_{v,w} = \begin{cases} 1 & \text{if } w \in O_v \\ 0 & \text{otherwise} \end{cases}$$

Traditional adjacency matrix

A Formal View of the Matrix **M**

$$\mathbf{A}_{N\times N} \quad a_{v,w} = \begin{cases} 1 & \text{if } w \in O_v \\ 0 & \text{otherwise} \end{cases}$$

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$$\mathbf{M}_{N\times N} \quad m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$$

Column stochastic matrix

$$\boxed{\mathbf{M} = (\mathbf{L}^{-1}\mathbf{A})^T}$$

PageRank: The Matrix Formulation

$\mathbf{r}$ Nx1 **rank vector** with an entry for each page

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All the rank scores must sum up to 1

PageRank: The Matrix Formulation

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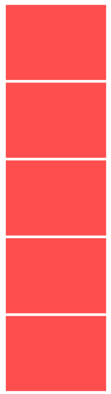
r_v  Rank score of page *v*

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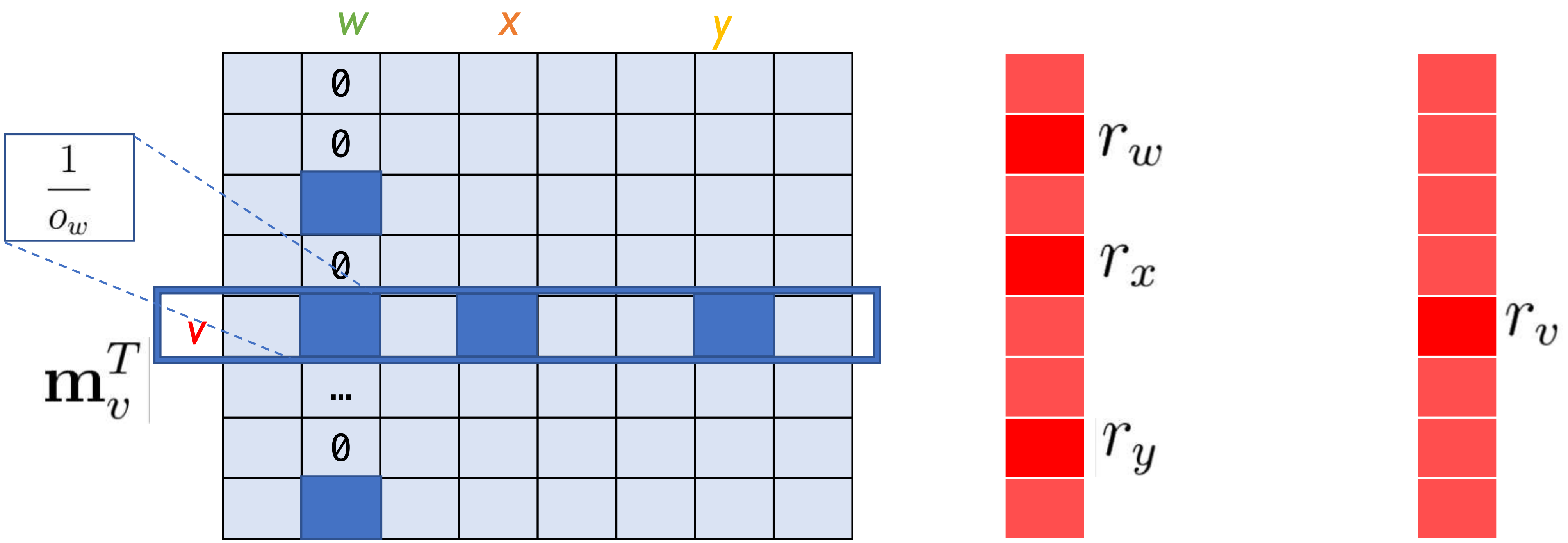
All the rank scores must sum up to 1

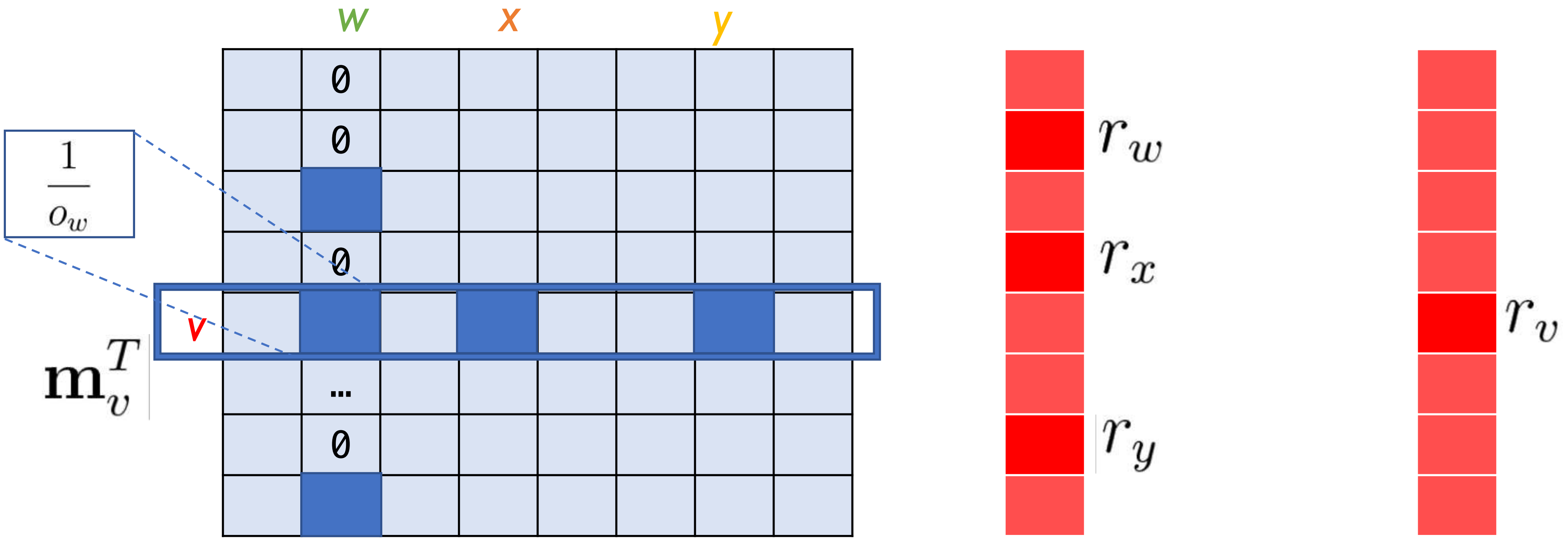
$$r_v = \sum_{w \in I_v} \frac{r_w}{o_w} \quad \Rightarrow \quad \mathbf{r} = \mathbf{M}\mathbf{r}$$

Flow equations in matrix form

PageRank: The Matrix Formulation

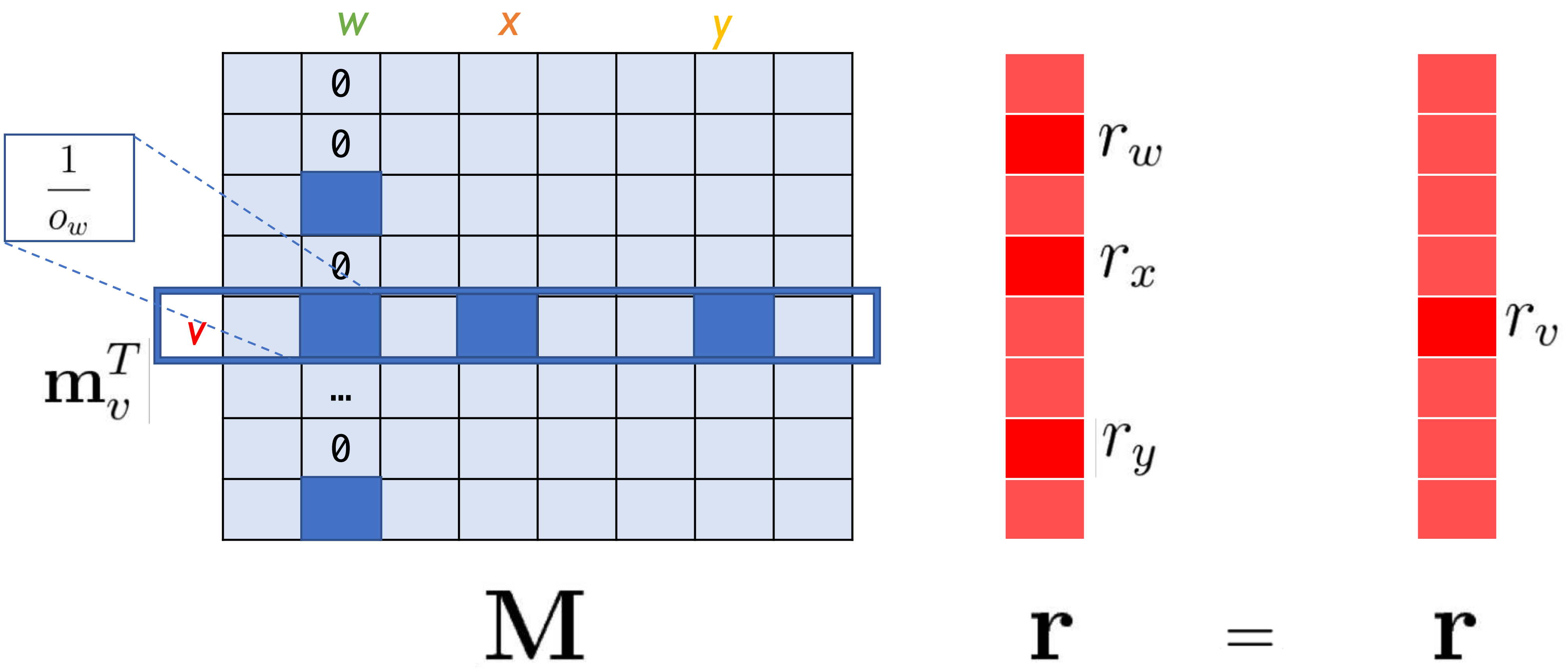


PageRank: The Matrix Formulation



$$r_v = \mathbf{m}_v^T \cdot \mathbf{r} = \sum_{w=1}^{N} m_{v,w} \times r_w = \sum_{w=1}^{N} \frac{1}{o_w} \times r_w = \sum_{w=1}^{N} \frac{r_w}{o_w} = \sum_{w \in I_v} \frac{r_w}{o_w}$$

PageRank: The Matrix Formulation



PageRank: The Eigenvector Formulation

$$\boxed{\mathbf{M}\mathbf{r} = \mathbf{r}}$$

Doesn't it look familiar?

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x is an eigenvector

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So, the rank vector **r** is an **eigenvector** of the matrix **M**

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So, the rank vector **r** is an **eigenvector** of the matrix **M**

In fact, **r** is the eigenvector corresponding to the **eigenvalue** λ = 1

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For a fixed eigenvalue, eigenvectors are just scalar multiples of each other

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Since PageRank should reflect only the relative importance of the nodes, choose $\mathbf{r} = \mathbf{r}^*$ as the eigenvector whose entries sum up to 1

This may be referred to as the **probabilistic eigenvector** corresponding to the eigenvalue $\lambda = 1$

PageRank: The Eigenvector Formulation

$$\boxed{\mathbf{Mr} = \mathbf{r}}$$

We know from linear algebra theory that for any **stochastic** matrix **M** its **largest eigenvalue** is $\lambda = 1$

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Note:

So far, we have assumed that **M** is (column) stochastic yet this may not be the case for the general Web graph…

PageRank: Quick Recap

We start from "flow" equations

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We reformulate the system of linear equations using linear algebra
(i.e., stochastic matrix $\mathbf{M}$ and rank vector $\mathbf{r}$)

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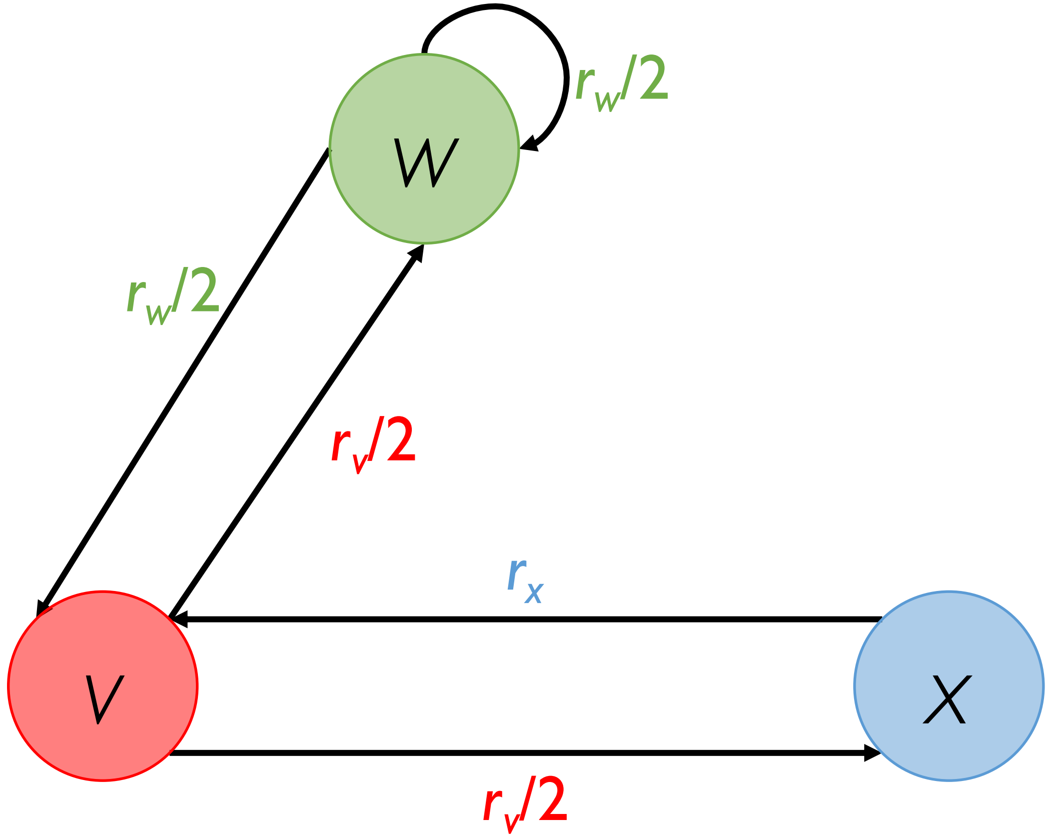
⇩

We know how to solve this efficiently using **power iteration** method

PageRank: The "Flow" Model



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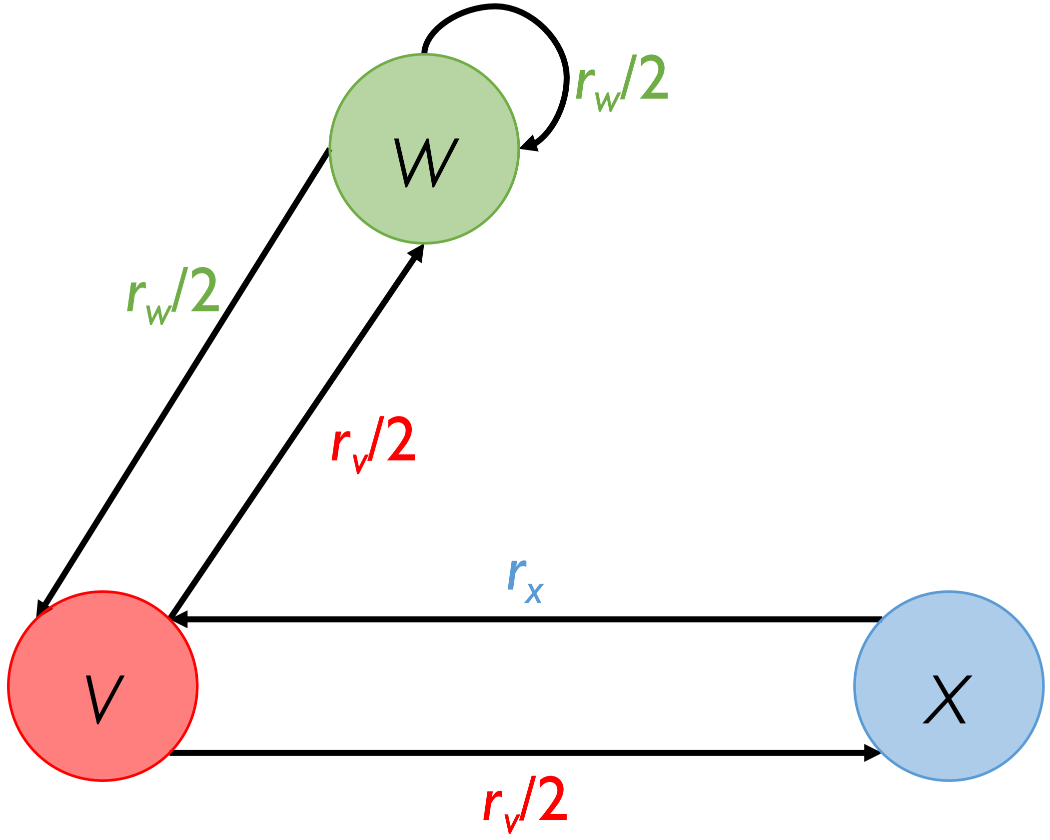

$$
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r_w = r_v/2 + r_w/2 \\
r_x = r_v/2
\end{cases}
$$

0	1/2	1
1/2	1/2	0
1/2	0	0

$$\mathbf{r} = \mathbf{M} \quad \mathbf{r}$$

PageRank: Power Iteration Method

At the beginning, we assume all pages have the same rank score, **uniformly distributed** across the N pages

$$\textbf{init:}\ \ t = 0; \mathbf{r}(t) = (1/N, 1/N, \ldots, 1/N)^T$$

PageRank: Power Iteration Method

Keep updating the rank vector **r** **until convergence**

init: $t = 0; \mathbf{r}(t) = (1/N, 1/N, \ldots, 1/N)^T$

repeat:

$$\mathbf{r}(t+1) = \mathbf{M}\mathbf{r}(t)$$

until $\delta(\mathbf{r}(t+1), \mathbf{r}(t)) < \epsilon$

$\epsilon > 0$

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$$\left\{ \begin{array}{c} \delta(\mathbf{r}(t+1), \mathbf{r}(t)) = |\mathbf{r}(t+1) - \mathbf{r}(t)| \\ \text{or} \\ \delta(\mathbf{r}(t+1), \mathbf{r}(t)) = ||\mathbf{r}(t+1) - \mathbf{r}(t)|| \end{array} \right.$$

Power Iteration Method: Example

$$\begin{cases} r_v = r_w/2 + r_x \\ r_w = r_v/2 + r_w/2 \\ r_x = r_v/2 \end{cases}$$

$$\mathbf{r} = \mathbf{M}\,\mathbf{r}, \qquad \mathbf{M} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$

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$$\mathbf{r}(0) = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

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$$\underset{\mathbf{r}(0)}{\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}} \qquad \underset{\mathbf{r}(1) = \mathbf{M}\mathbf{r}(0)}{\begin{bmatrix} 3/6 \\ 1/3 \\ 1/6 \end{bmatrix}}$$

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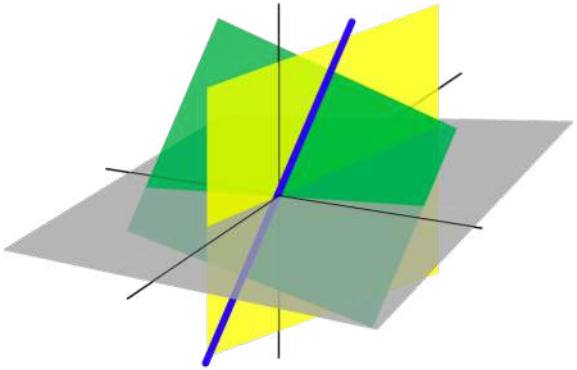
We came up with the same set of solutions for r_v, r_w, and r_x without explicitly solving the system of equations

PageRank's Interpretations

2 main perspectives

Linear Algebra



Probabilistic

Random Walk Interpretation of Page Rank

Imagine a **random surfer** navigating through the pages of the Web graph

Random Walk Interpretation of Page Rank

Initially, at time $t=0$ the surfer can be on **any** web page

www.donuts.com

www.krustyburger.com

www.duffbeer.com

…

www.moes.com

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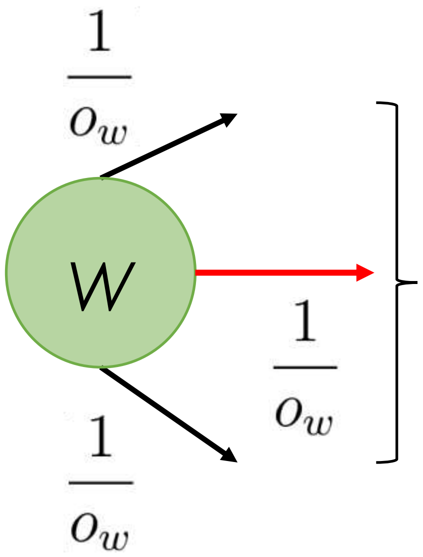
Each web page has **equal probability** 1/N to be chosen as starting point

Random Walk Interpretation of Page Rank

At any given time t, the surfer is on some web page w

Random Walk Interpretation of Page Rank

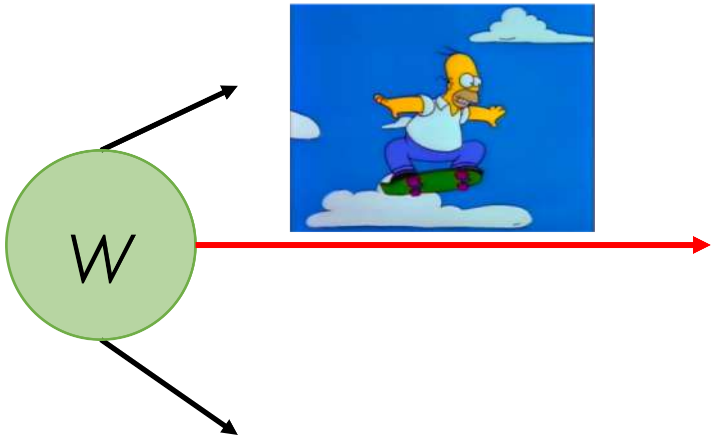
At time *t+1*, the surfer follows one of the outgoing links from web page *w*, chosen **uniformly at random**



o_w out-degree of node *w*

Random Walk Interpretation of Page Rank

The surfer ends up into some other web page *v* pointed by *w*



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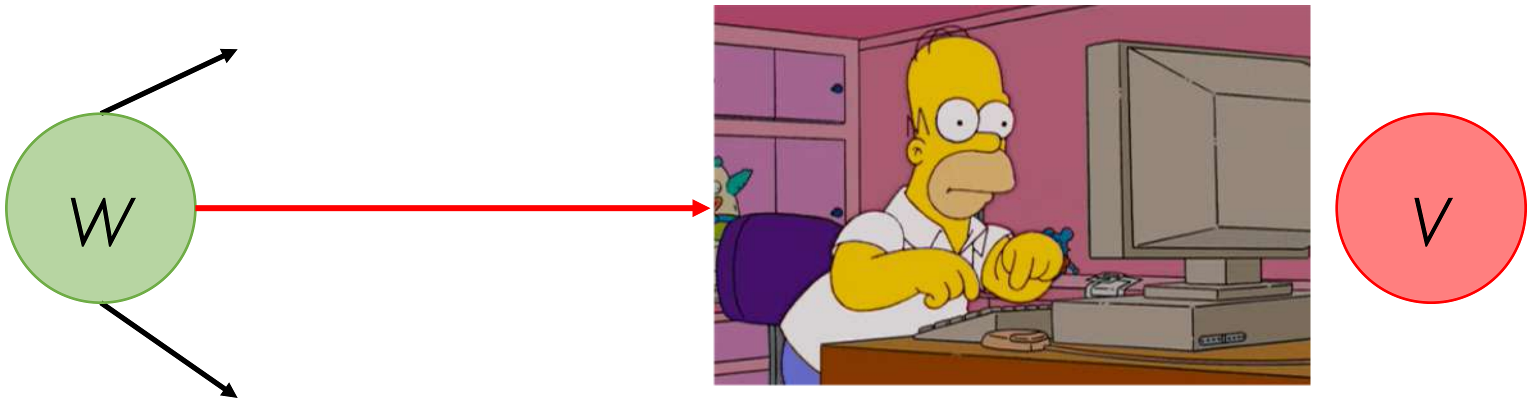

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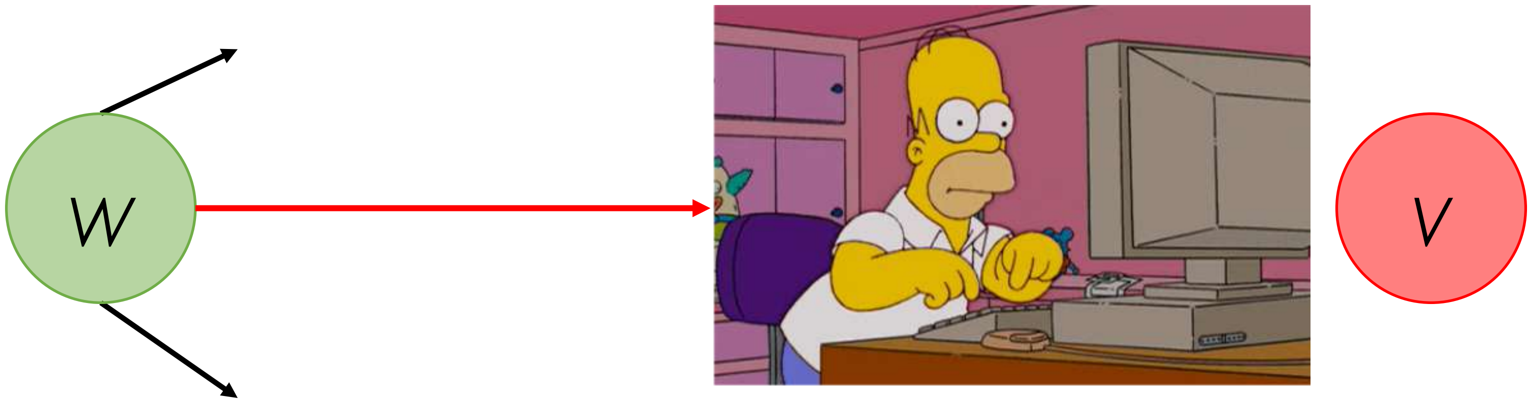

This process repeats indefinitely and is known as **random walk**

Transition Matrix **M**

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The *v*-th, *w*-th entry of **M** indicates the probability of a random surfer moving from page *w* to page *v*

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Such a matrix describes a **Markov chain** over the finite state space V of nodes (i.e., pages) of the Web graph

Random Walk Interpretation of Page Rank

X Discrete-Valued Random Variable taking on |V| = N possible values

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Probability distribution over web pages at time *t*

Random Walks as Markov Chains

Random Walks are also known as **stochastic processes** with **Markov property** (i.e., **Markov chains**)

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The **transition probability** of moving to the next state **depends** only on the **present state** and not on the previous states

$$P(X_{t+1} = v | X_1 = x_1, X_2 = x_2, \ldots, X_t = x_t) = P(X_{t+1} = v | X_t = x_t)$$

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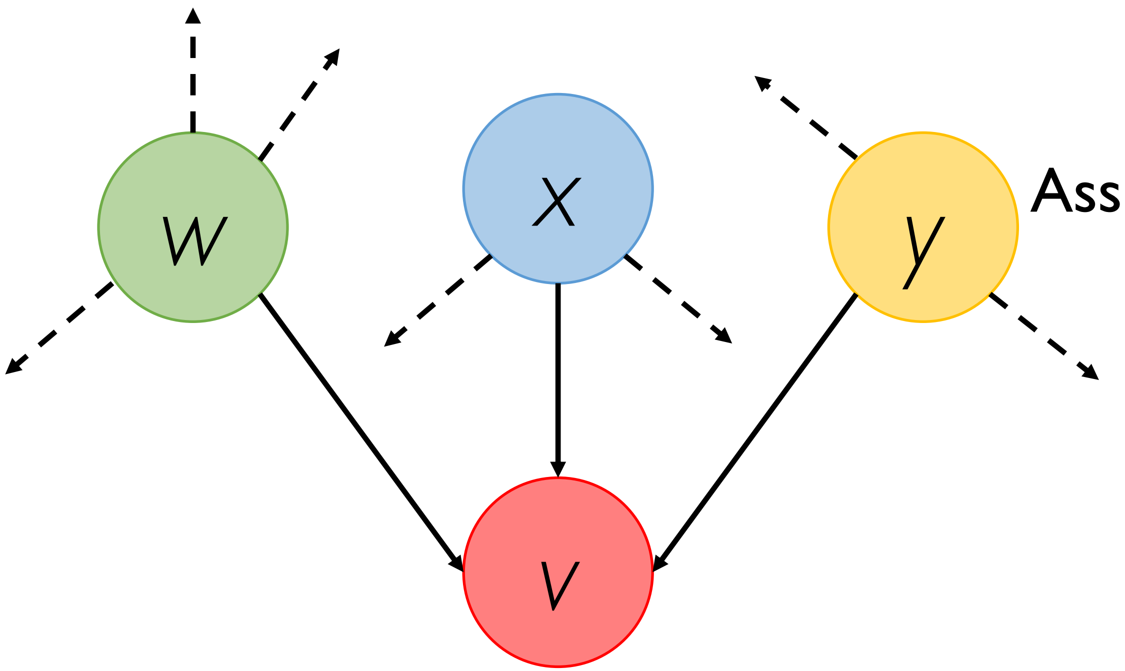
The probability that the random surfer will be on page *v* at time *t+1* depends only on where the surfer was at time *t*

Random Walk Interpretation of Page Rank

Where is the random surfer at time $t+1$ knowing where he was at time t?
Suppose we want to estimate $P(X_{t+1} = v)$



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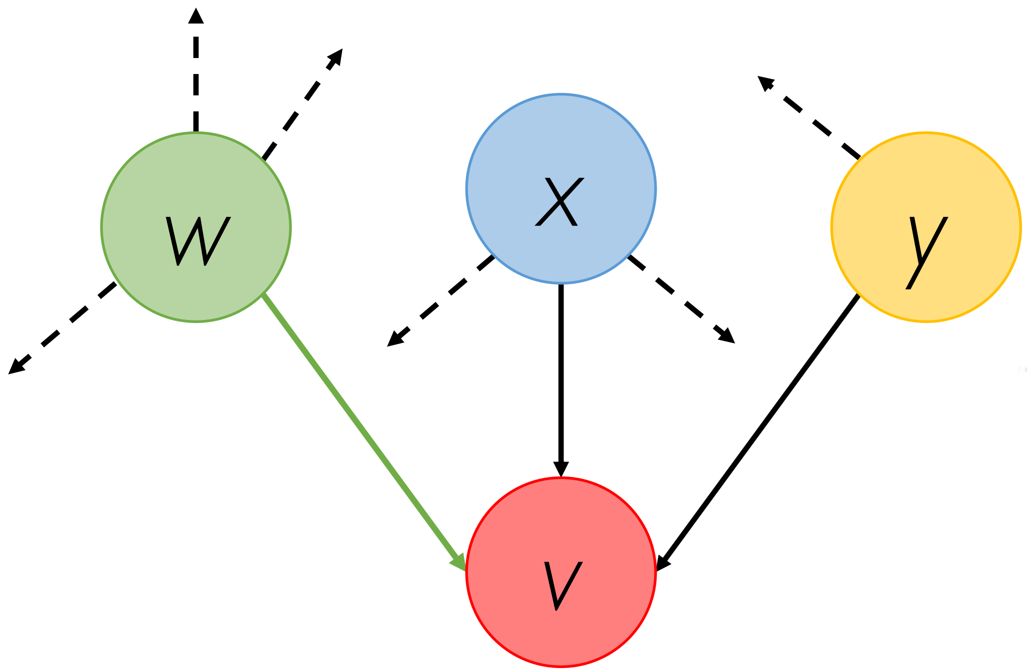

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$$P(X_{t+1} = v) = \boxed{P(X_t = w, Z_w = v) +}$$

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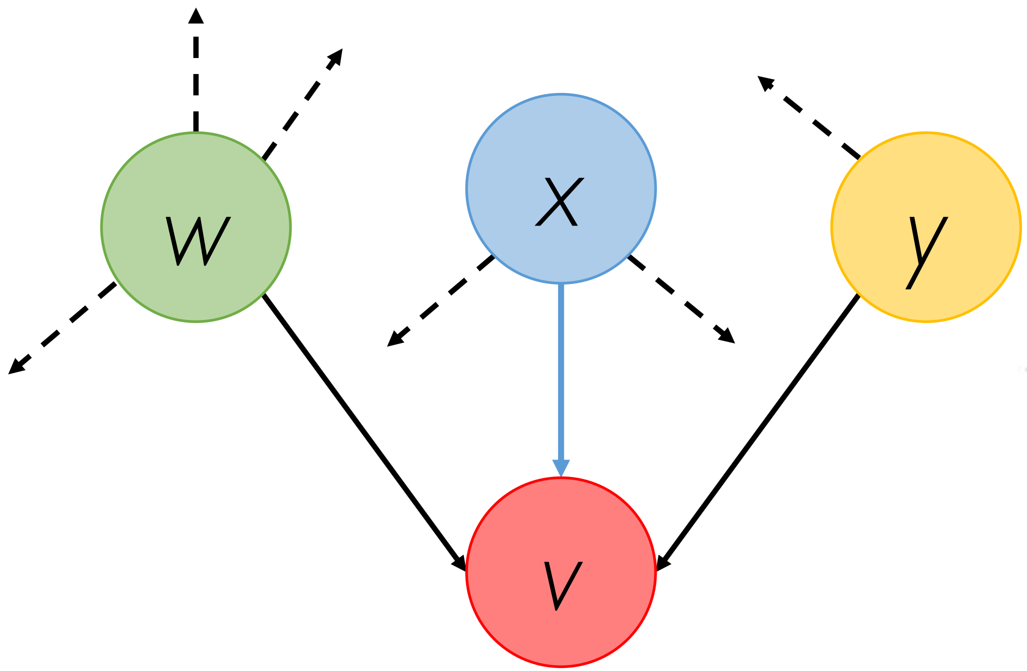

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$$\boxed{P(X_t = x, Z_x = v) +}$$

Random Walk Interpretation of Page Rank

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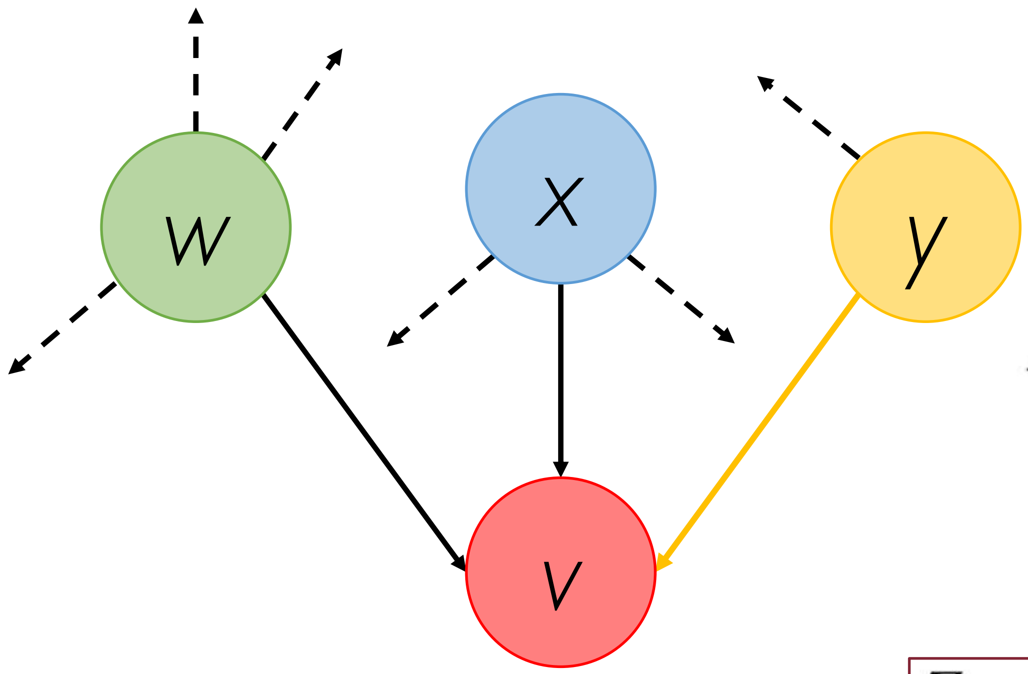
Suppose we want to estimate $P(X_{t+1} = v)$



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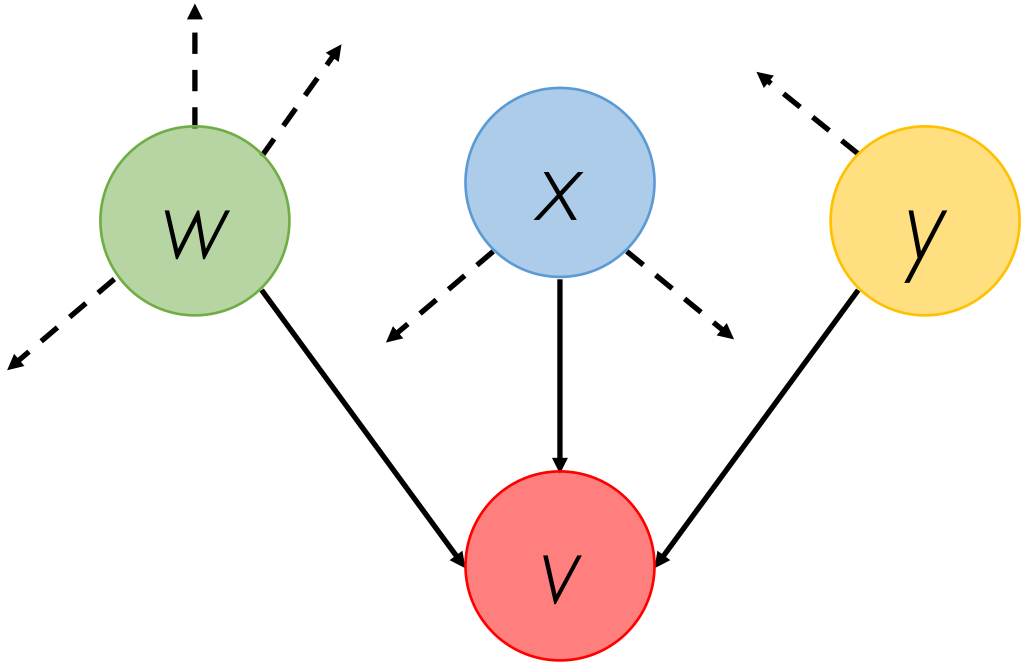
$$P(X_{t+1} = v) = P(X_t = w, Z_w = v) +$$
$$P(X_t = x, Z_x = v) +$$
$$\boxed{P(X_t = y, Z_y = v)}$$

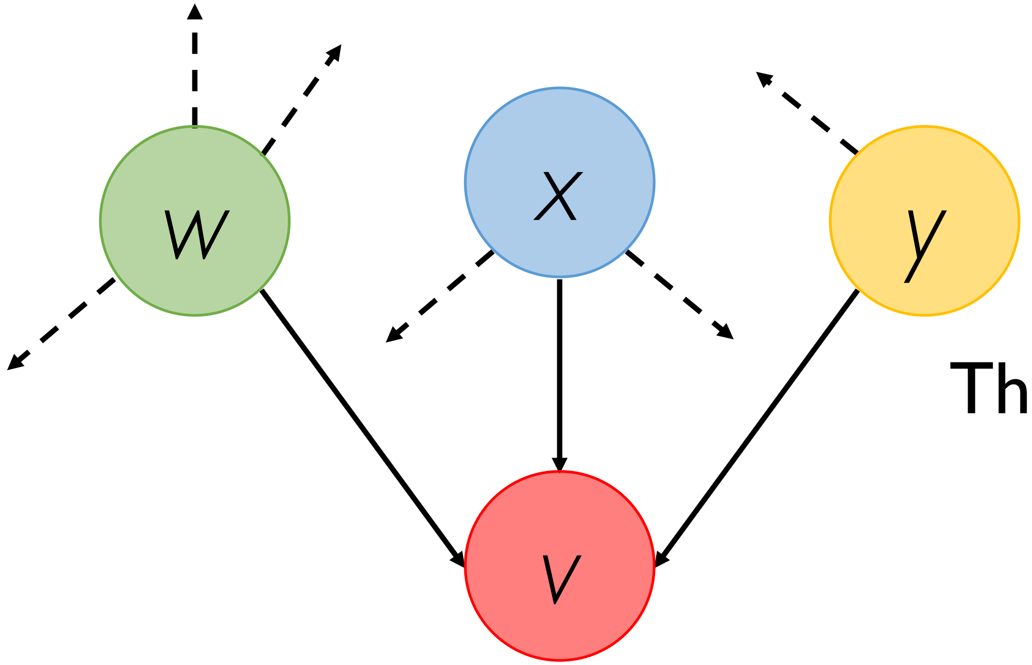
$$\boxed{Z_u \sim \text{Uniform}(1, o_u)}$$

Random Walk Interpretation of Page Rank



$$\boxed{\mathbf{p}(t+1) = \mathbf{M}\mathbf{p}(t)}$$

Random Walk Interpretation of Page Rank

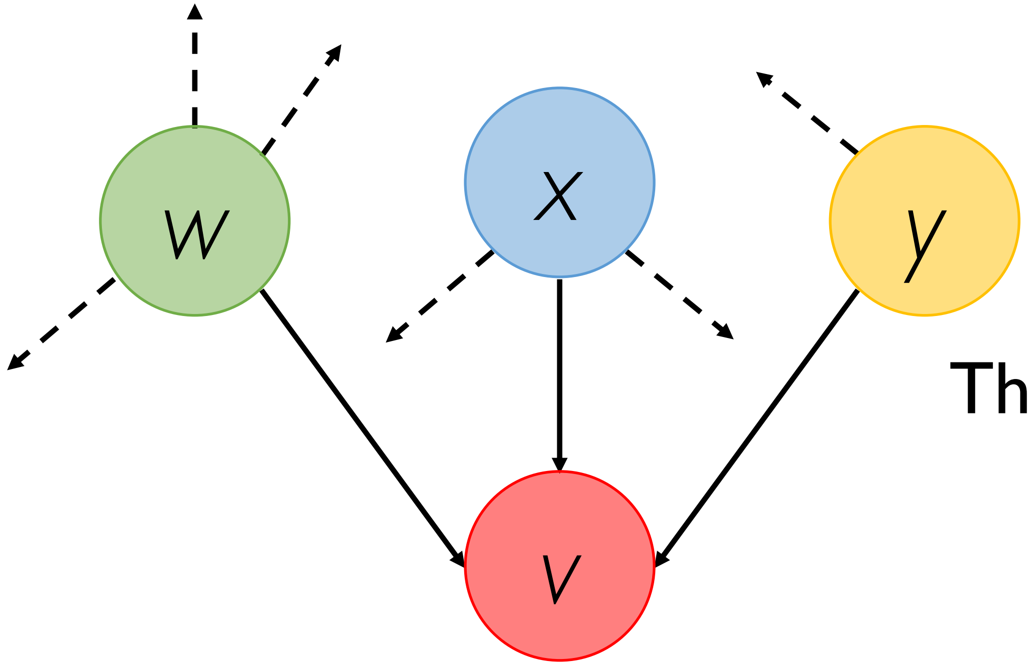


$$\boxed{\mathbf{p}(t+1) = \mathbf{M}\mathbf{p}(t)}$$

This resembles our PageRank equation

$$\boxed{\mathbf{r}(t+1) = \mathbf{M}\mathbf{r}(t)}$$

Random Walk Interpretation of Page Rank



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This resembles our PageRank equation

$$\boxed{\mathbf{r}(t+1) = \mathbf{M}\mathbf{r}(t)}$$

Solving the former is equivalent to solving the latter!

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Random Walk Interpretation of Page Rank

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The probability that page *i* will be visited after one step corresponds to the *i*-th entry of **p**(1), obtained as follows:

$$\mathbf{p}(1) = \mathbf{M}\mathbf{p}(0)$$

Random Walk Interpretation of Page Rank

Initially, the stochastic vector **p**(0) is a **uniform probability distribution**

The probability that page *i* will be visited after one step corresponds to the *i*-th entry of **p**(1), obtained as follows:

$$\mathbf{p}(1) = \mathbf{M}\mathbf{p}(0)$$

More generally, the probability of visiting *any* web page after *t* steps is:

$$\boxed{\mathbf{p}(t) = \mathbf{M}^t\mathbf{p}(0)}$$

Random Walk Interpretation of Page Rank

$$\mathbf{p}(0) = (\underbrace{1/N}_{P(X_0=1)}, \ldots, \underbrace{1/N}_{P(X_0=w)}, \ldots, \underbrace{1/N}_{P(X_0=N)})^T$$

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$$\vdots$$

$$\mathbf{p}(k) = \mathbf{M}\mathbf{p}(k-1) = \underbrace{\mathbf{M} \times \mathbf{M} \times \ldots \times \mathbf{M}}_{\mathbf{M}^k}\mathbf{p}(0)$$

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Random Walk Interpretation of Page Rank

$\{\mathbf{p}(t)\}_{t=0,1,\ldots,T}$

Discrete Stochastic Process

Markov chain

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Suppose that our random surfer reaches a so-called **steady state**

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$\mathbf{p}^*$ is the **stationary distribution** of the random walk

Equivalence between Formulations

Linear Algebra

Probabilistic

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System of linear "flow" equations

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So the PageRank vector $\mathbf{r}^*$ corresponds to the **stationary distribution** $\mathbf{p}^*$ for the random walk on the graph encoded by $\mathbf{M}$!

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So the PageRank vector $\mathbf{r}^*$ corresponds to the **stationary distribution** $\mathbf{p}^*$ for the random walk on the graph encoded by **M**!

Intuitively, the PageRank vector indicates for each web page the probability that a random surfer will eventually get to that page

Hang on a second…

Linear Algebra

Probabilistic

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How do we know that the power iteration method always converge to $\mathbf{r}^*$?

existence

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How do we know that a Markov chain always converge to a steady-state $\mathbf{p}^*$?

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existence and **uniqueness** of $\mathbf{r}^*$ ($\mathbf{p}^*$) are guaranteed under certain conditions on the matrix $\mathbf{M}$

Existence and Uniqueness of PageRank

If **M** is a **column stochastic** matrix with **all positive entries**:

- $\lambda = 1$ is an eigenvalue of **M** with multiplicity one
- $\lambda = 1$ is the largest eigenvalue of **M**
- There exists a unique (right) eigenvector $\mathbf{r}^*$ associated with the eigenvalue $\lambda = 1$ with the sum of its entries equal to 1

Perron-Frobenius theorem (circa 1910)

Existence and Uniqueness of PageRank

If **M** is a **column stochastic** matrix with **all positive entries**, then **M has** a **unique** steady-state vector $\mathbf{p}^*$ such that for any $\mathbf{p}(0)$

$\mathbf{p}(t) = \mathbf{M}^t\mathbf{p}(0)$ converges to $\mathbf{p}^*$ as $t \to \infty$

Perron-Frobenius theorem (circa 1910)

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We know that the largest eigenvalue of a stochastic matrix is $\lambda = 1$ (though we haven't proved it)

The steady-state vector is the unique eigenvector associated with the largest eigenvalue $\lambda = 1$

Are We Done, Then?

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$$\mathbf{M}_1 = \begin{bmatrix} 0.6 & 0.5 & 0 \\ 0.4 & 0.3 & 1 \\ 0 & 0.2 & 0 \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} 0.6 & 0.5 & 0.1 \\ 0.2 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{bmatrix}$$

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Both $\mathbf{M}_1$ and $\mathbf{M}_2$ are column stochastic, but only $\mathbf{M}_2$ is positive

So? Should We Give Up?

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We show how they fixed the issues with the original definition of **M** to accommodate for the heterogeneity of the Web graph

By doing so, we know that a solution to our PageRank problem **exists** and is **unique**!

Google's PageRank

Problems with Original PageRank Formulation

We cannot directly apply the Perron-Frobenius theorem to the original Web graph matrix $\mathbf{M}$

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We show why this causes the problem of **existence** and **convergence** of PageRank when applied to the original matrix **M**

Then we discuss how Brin and Page fixed this in their seminal paper which sets up the rising of Google

Problems with Original PageRank Formulation

2 main **issues** to solve:

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Dead End

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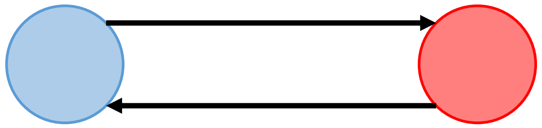
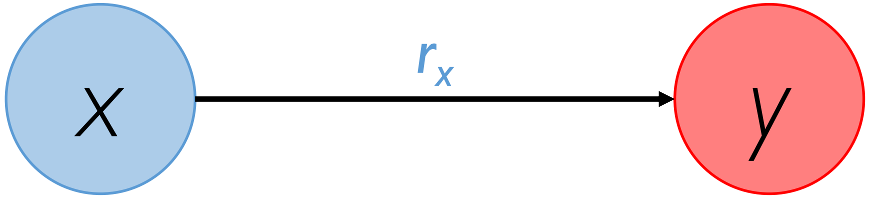
Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by small group of pages



The "Dead End" Problem (Dangling Nodes)

Example:



$$r_y = r_x$$

$\mathbf{M}$

0	0
1	0

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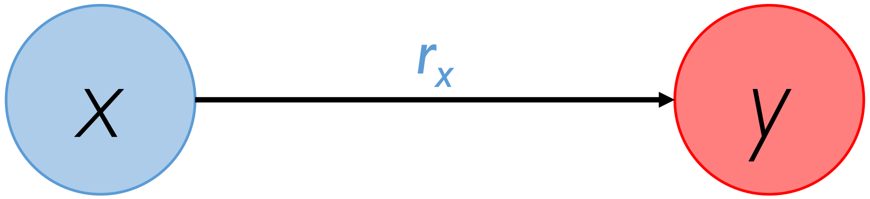

$$r_y = r_x$$

$$\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

When a web page has no outgoing links (**dangling node**) the resulting column vector in the matrix **M** is **not stochastic** anymore!

*Previously, we assumed each web page has at least one outgoing link, and therefore **M** was stochastic*

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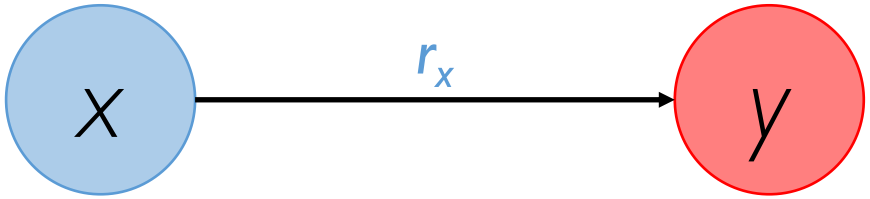

$$r_y = r_x$$

$\mathbf{M}$

0	0
1	0

Assume the following initialization for **r**:

$$\mathbf{r}(0) = \begin{bmatrix} r_x^{(0)} \\ r_y^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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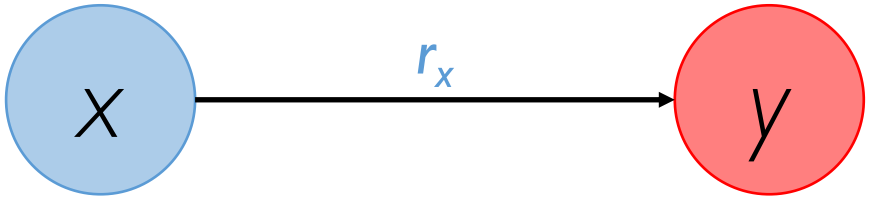

$$r_y = r_x$$

$$\begin{array}{c}\mathbf{M}\\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\end{array}$$

$$\mathbf{r}(1) = \mathbf{M} \quad \mathbf{r}(0)$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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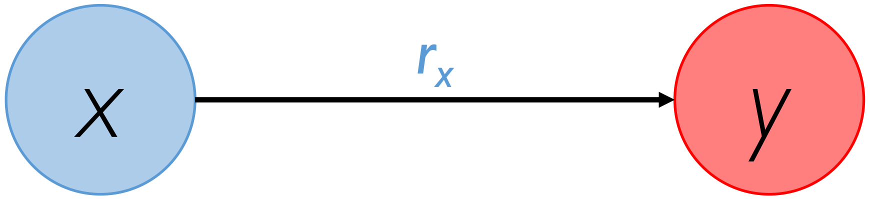

$$r_y = r_x$$

$$\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{r}(2) = \mathbf{M} \quad \mathbf{r}(1)$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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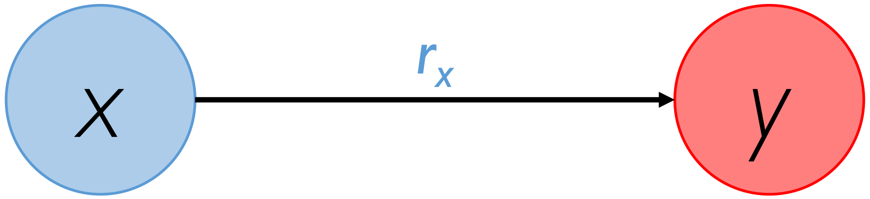


$$r_y = r_x$$

M

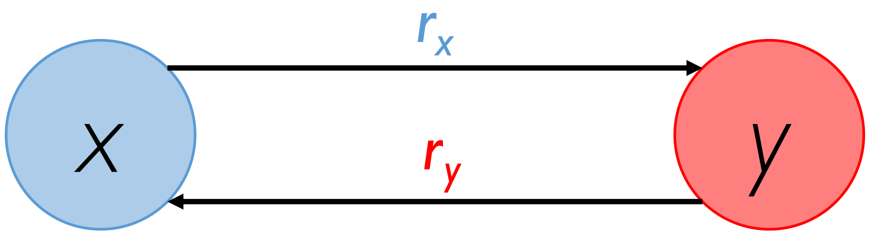
0	0
1	0

$$\mathbf{r}(0) = \begin{bmatrix}1\\0\end{bmatrix} \quad \mathbf{r}(1) = \begin{bmatrix}0\\1\end{bmatrix} \quad \mathbf{r}(2) = \begin{bmatrix}0\\0\end{bmatrix} \quad \cdots \quad \mathbf{r}(t-1) = \begin{bmatrix}0\\0\end{bmatrix} \quad \mathbf{r}(t) = \begin{bmatrix}0\\0\end{bmatrix}$$

The PageRank vector vanishes to **0**!

The "Spider Trap" Problem

Example:



$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

M

0	1
1	0

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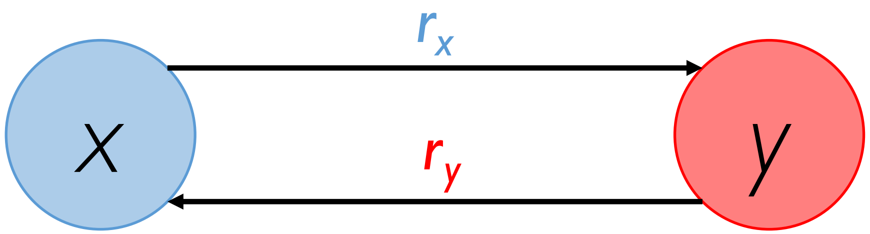

$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

$$\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

M is column stochastic non-negative (but **not strictly positive**)

Does PageRank converge regardless of the initialization of **r**?

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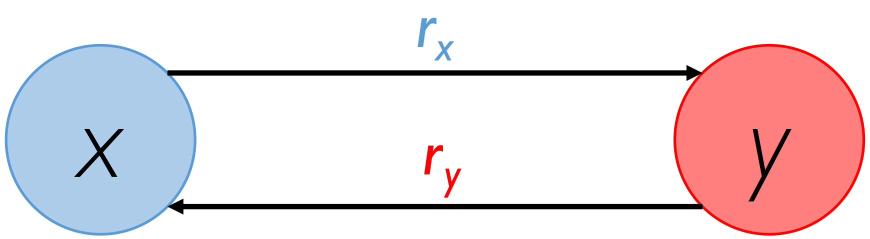

$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

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Assume the same initialization as before for **r**:

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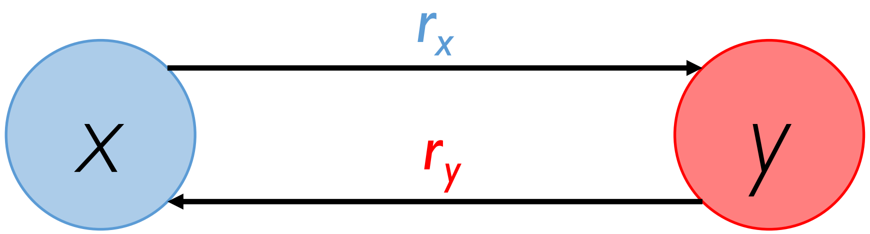

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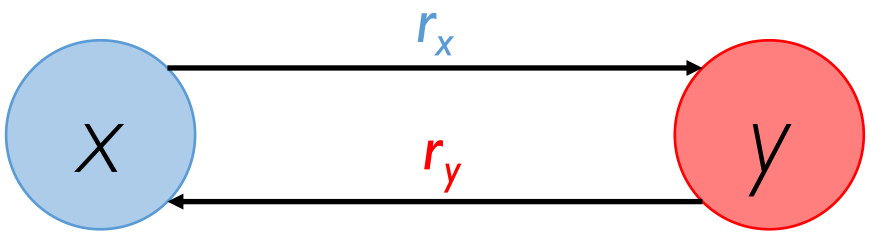

$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

M

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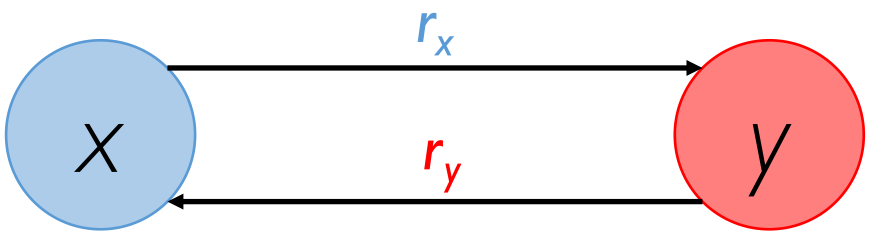

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$$\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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The PageRank vector keeps alternating its components and **never** converges!

Problems with Original PageRank Formulation

2 main **issues** to solve:

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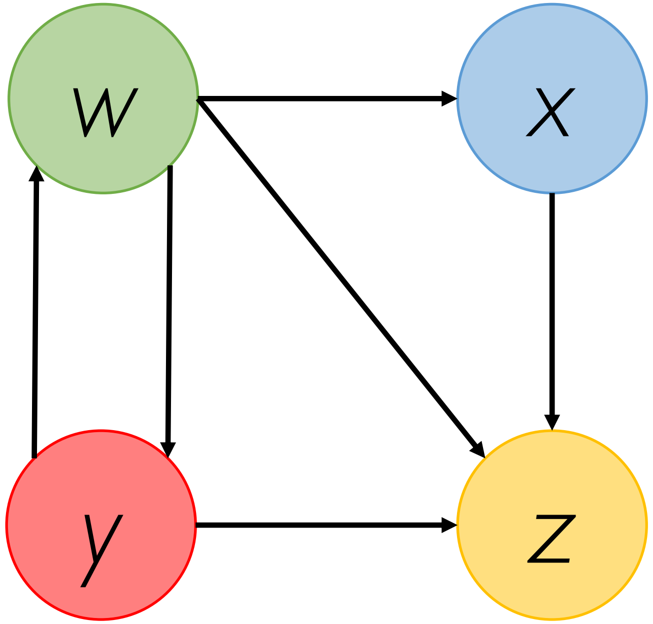
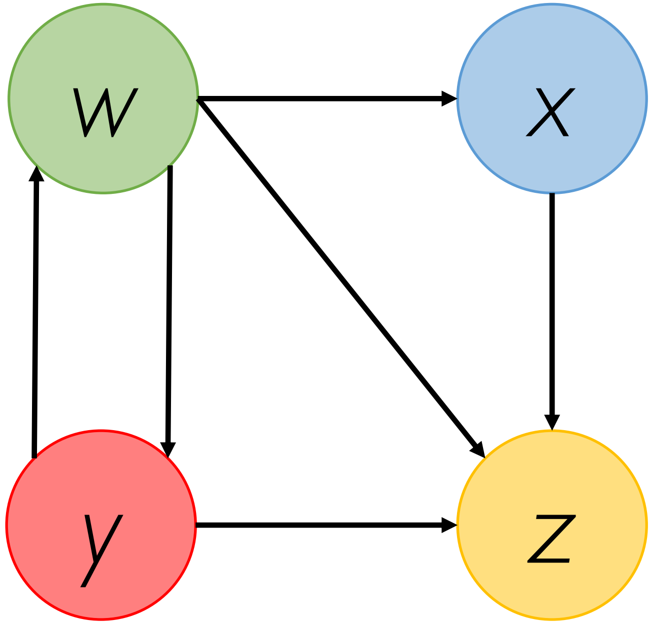
M is stochastic but **not** strictly positive

Deal with Dangling Nodes



$$\mathbf{M} = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 1/2 & 0 \end{bmatrix} \end{array}$$

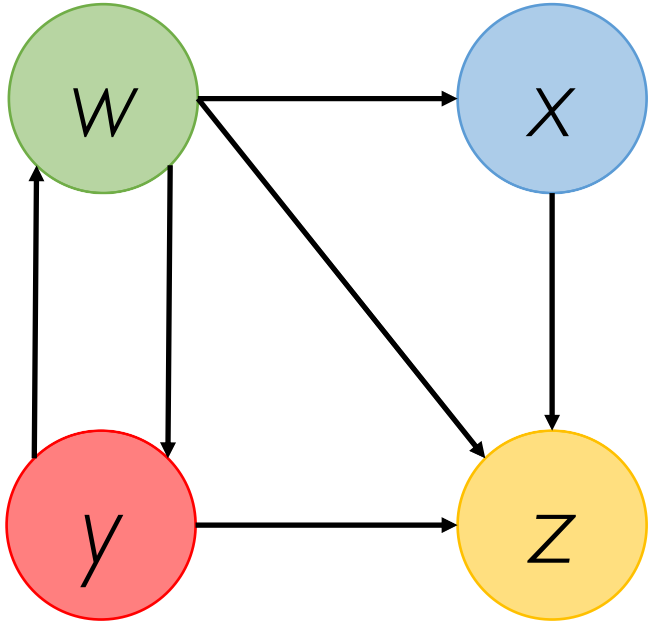
Deal with Dangling Nodes



z is a dangling node

$$\mathbf{M} = \begin{array}{c c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 0 & 1/2 & \boxed{0} \\ 1/3 & 0 & 0 & \boxed{0} \\ 1/3 & 0 & 0 & \boxed{0} \\ 1/3 & 1 & 1/2 & \boxed{0} \end{bmatrix} \end{array}$$

Deal with Dangling Nodes



z is a dangling node

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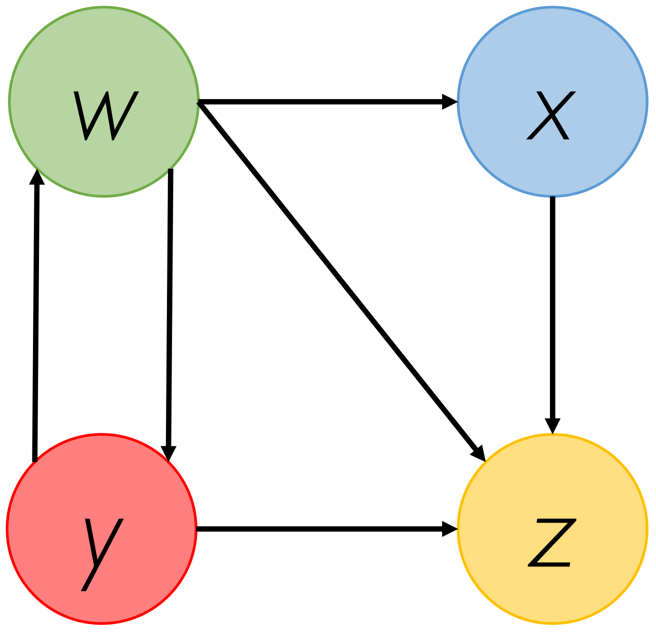
M is **not**
(column) stochastic

Deal with Dangling Nodes



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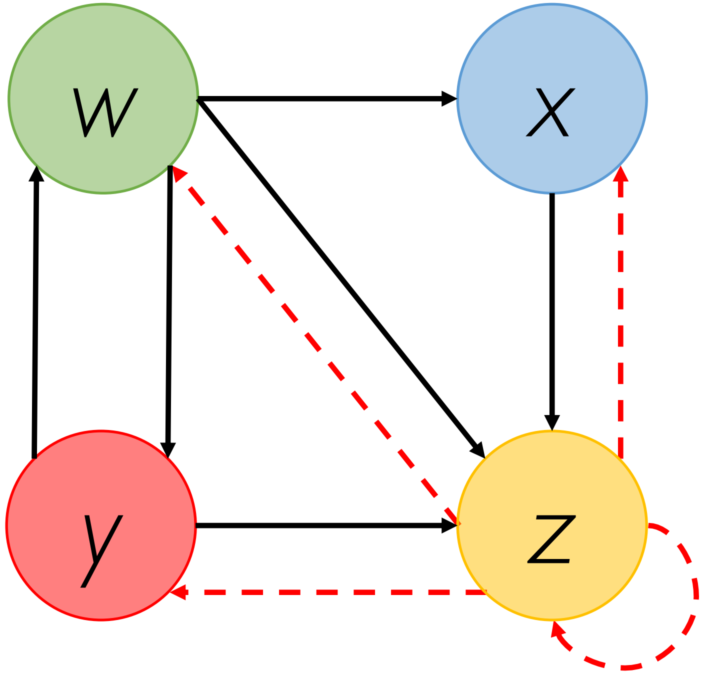
If we apply simplified PageRank to **M** the rank vector **r** will eventually vanish to **0**

Deal with Dangling Nodes



$$
\begin{array}{c c}
 & \begin{array}{cccc} w & x & y & z \end{array} \\
\mathbf{M}' = \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 0 & 1/2 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 1 & 1/2 & 1/4 \end{bmatrix}
\end{array}
$$

Solution: Teleporting

Create **artificial links** from any dangling node to any other node

Deal with Dangling Nodes: Teleporting

This adjustment is justified by modeling the behaviour of a web surfer

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After reading a page with no out-going link, jump to a page picked **uniformly at random** amongst the N

Deal with Dangling Nodes: Teleporting

Initially, we set $\mathbf{M}_{N\times N}$ $m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$

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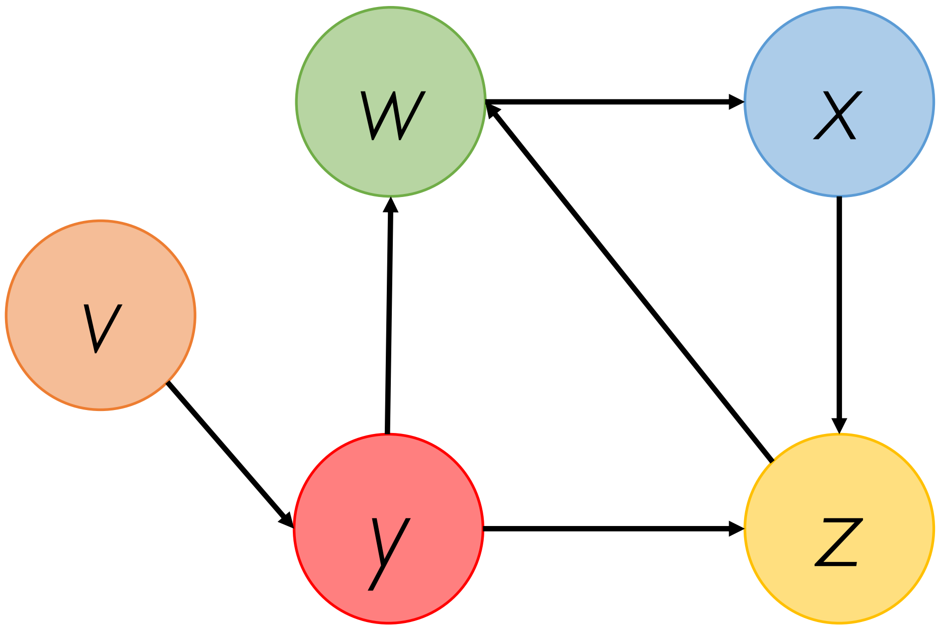
$$\boxed{\mathbf{M} \rightsquigarrow \mathbf{M}'}$$

This transformation allows **M'** to be **column stochastic**

Deal with Spider Traps



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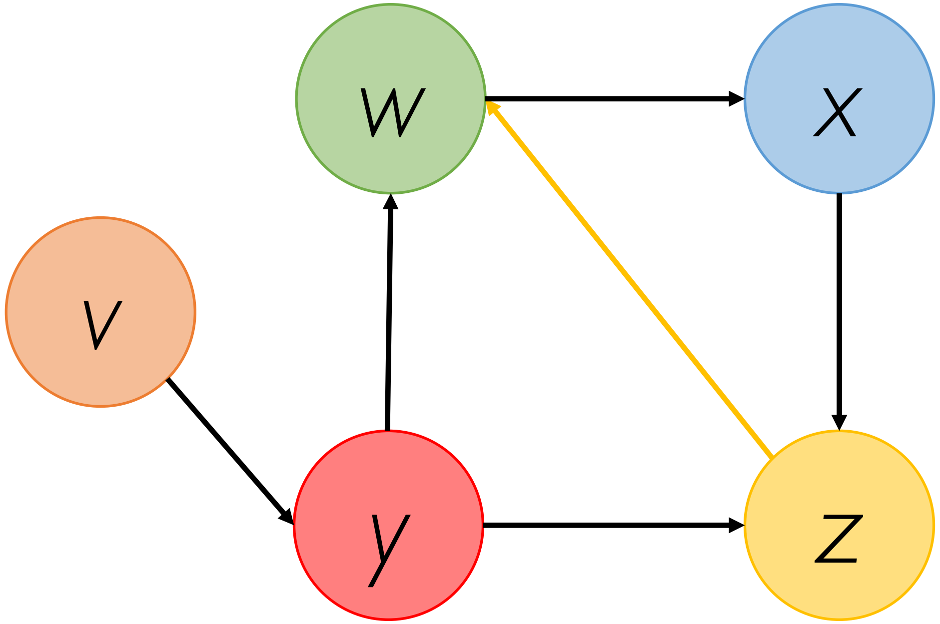

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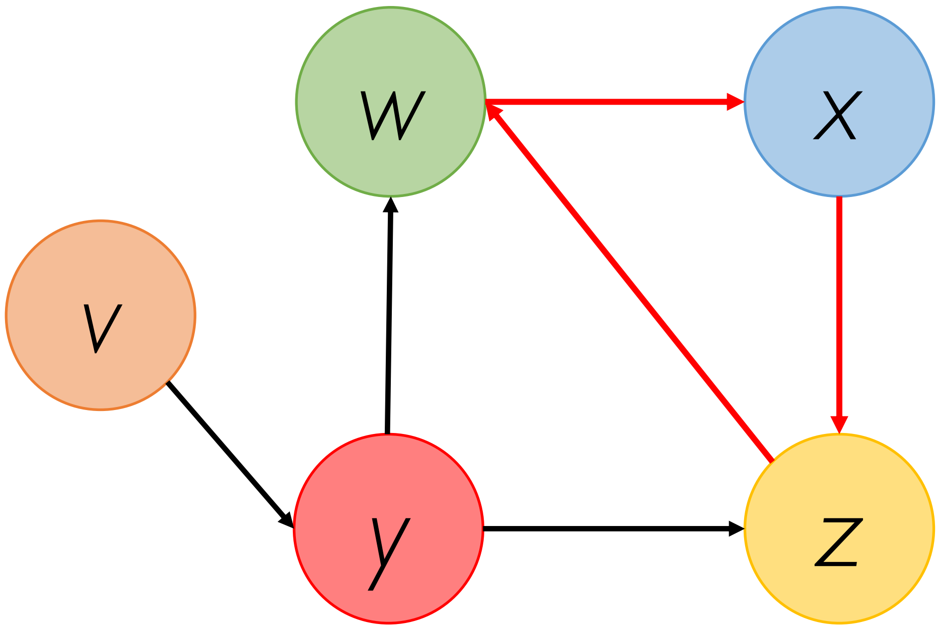
z is not a dangling node anymore

M is (column) stochastic

Deal with Spider Traps



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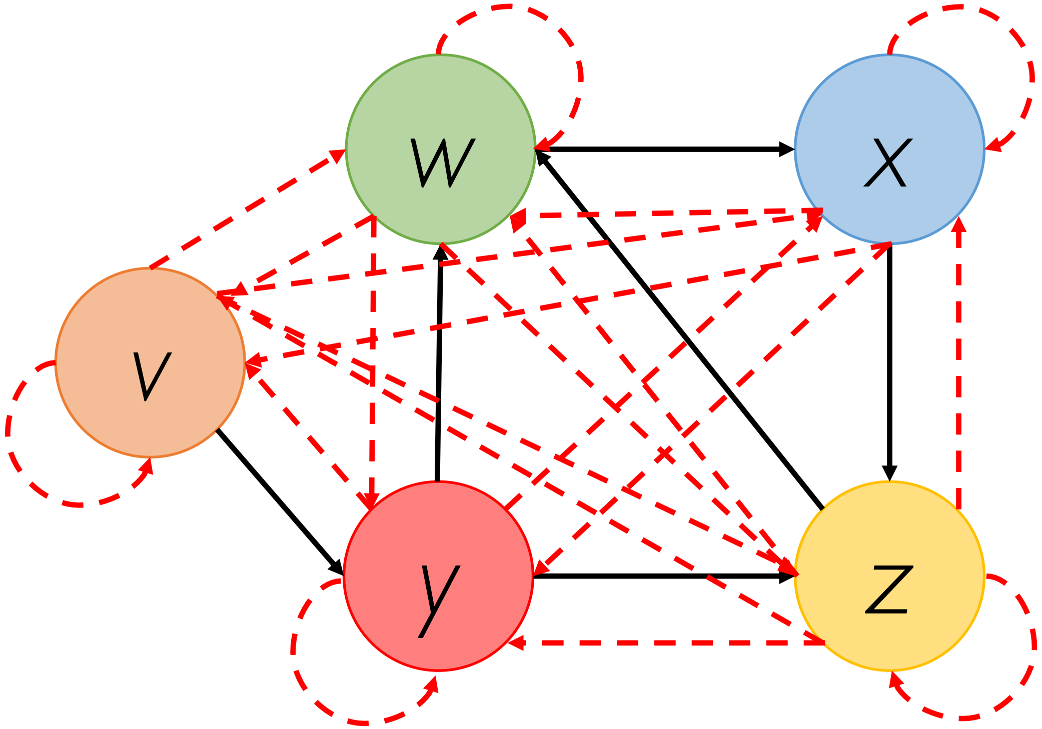
If we apply simplified PageRank to **M** some entries of the rank vector **r** will eventually drop to 0, as we get stuck in *w*, *x*, *z*

Deal with Spider Traps: Teleporting (Again!)



$$\mathbf{M}' = \begin{array}{c} \\ v \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} v & w & x & y & z \end{array} \\ \begin{bmatrix} 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.455 & 0.88 \\ 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.88 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.88 & 0.455 & 0.03 \end{bmatrix} \end{array}$$

Solution: Probabilistic Teleporting

Create **artificial links** from each node to every other node and follow each of it with probability (1-*d*)/N

Deal with Spider Traps: Probabilistic Teleporting

To avoid the surfer to get stuck in a spider trap

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On each page *w* the surfer will either follow one of its outgoing links with probability *d* or jump to another page with probability (1-*d*)

Deal with Spider Traps: Probabilistic Teleporting

To avoid the surfer to get stuck in a spider trap

On each page *w* the surfer will either follow one of its outgoing links with probability *d* or jump to another page with probability (1-*d*)

d is called **damping factor**

d = **0.85** in the original Google formulation

The Google's PageRank Formulation

$$\mathbf{M}_{N\times N} \quad m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$$

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$$\mathbf{G} = d\mathbf{M}' + \frac{1-d}{N} \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}}_{\mathbf{1}_{N\times N}}$$

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$\boxed{\mathbf{M}' \rightsquigarrow \mathbf{G}}$

Ensure the matrix is **strictly positive**

Why Does Teleporting Solve Our Problem?

$$\mathbf{G} = d\mathbf{M}' + \frac{1-d}{N} \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}}_{\mathbf{1}_{N \times N}}$$

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The Perron-Frobenius theorem now applies to **G** and guarantees the existence (convergence) and uniqueness of the steady-state eigenvector $\mathbf{r}^*$

$$\mathbf{r}(t) = \mathbf{G}^t\mathbf{r}(0)$$

$$\mathbf{r} \rightsquigarrow \mathbf{r}^* \text{ as } t \rightarrow \infty$$

How Do We Actually Compute PageRank?

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Problem:

G represents a **fully-connected** graph with a huge number of nodes (web pages)

G is a dense matrix

How Do We Actually Compute PageRank?

Assuming the number of web pages in the graph is $N=10^9$

G will have N^2 entries = 10^{18}

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Note: The Web contains far more than $N=10^9$ pages!

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$$\mathbf{r} = \mathbf{G}\mathbf{r}$$

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Approximately 10 links per web page reduces the amount of memory required to store **M'** by a factor of 8 w.r.t. **G** (10^{10} vs. 10^{18} entries)

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Approximately 10 links per web page reduces the amount of memory required to store **M'** by a factor of 8 w.r.t. **G** (10^{10} vs. 10^{18} entries)

We can work with **M'** rather than **G**

Re-Arrange the Equation

$$\boxed{\mathbf{r} = d\mathbf{M}'\mathbf{r} + \left[\frac{1-d}{N}\right]_{N\times 1}}$$

At each iteration we can compute PageRank vector as follows:

1. $\mathbf{r}(t+1) = d\mathbf{M}'\mathbf{r}(t)$
2. $\mathbf{r}(t+1) = \mathbf{r}(t+1) + \left[\frac{1-d}{N}\right]_{N\times 1}$

Add the constant (1-d)/N to each component of **r**(t+1)

PageRank: Pseudocode

Algorithm: PageRank

Input : A directed Web graph $G = (V, E)$, where $|V| = N$ and its associated matrix $\mathbf{M}_{N \times N}$ defined as follows: $\mathbf{M}_{v,w} = \frac{1}{o_w}$ if w points to v, 0 otherwise ($o_w = |O_w|$ where $O_w = \{x \in V : (w, x) \in E\}$);
A *damping factor* $d \in (0, 1)$;
A *tolerance* $\epsilon > 0$.

Output: The PageRank vector $\mathbf{r}^*_{N \times 1}$

Init : $t \leftarrow 0; \mathbf{r}(t) \leftarrow \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$;

repeat
    $t \leftarrow t + 1$;
    /* Compute the temporary PageRank score of every page v */
    for $i \leftarrow 1$ **to** N **do**
        $r_v^{\text{tmp}}(t) \leftarrow \sum_{w \in I_v} \frac{r_w(t-1)}{o_w}$; /* $r_v^{\text{tmp}}(t) = 0$ if v has no in-links */
    end
    /* Adjust the PageRank score of each page v with *teleporting* */
    for $i \leftarrow 1$ **to** N **do**
        $r_v(t) \leftarrow d \times r_v^{\text{tmp}}(t) + \frac{1-d}{N}$;
    end
until $|\mathbf{r}(t) - \mathbf{r}(t-1)| < \epsilon$
return $\mathbf{r}^* = \mathbf{r}(t)$;

Take-Home Message of Today

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Take-Home Message of Today

- We present an example of **link analysis** algorithm: **PageRank**
- **Goal:** Find an **importance score** associated with each web page
- Represent the Web graph as a matrix **M** where a link between page *v* and *w* is a **vote** from *v* to *w*
- **2** different yet equivalent approaches:
  - **Linear Algebra** → Matrix eigenvector
  - **Probabilistic** → Stationary distribution of Markov chain (**random walk**)

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- Google solution: **probabilistic teleport links**
- Still efficiently computable from the original, sparse matrix **M**