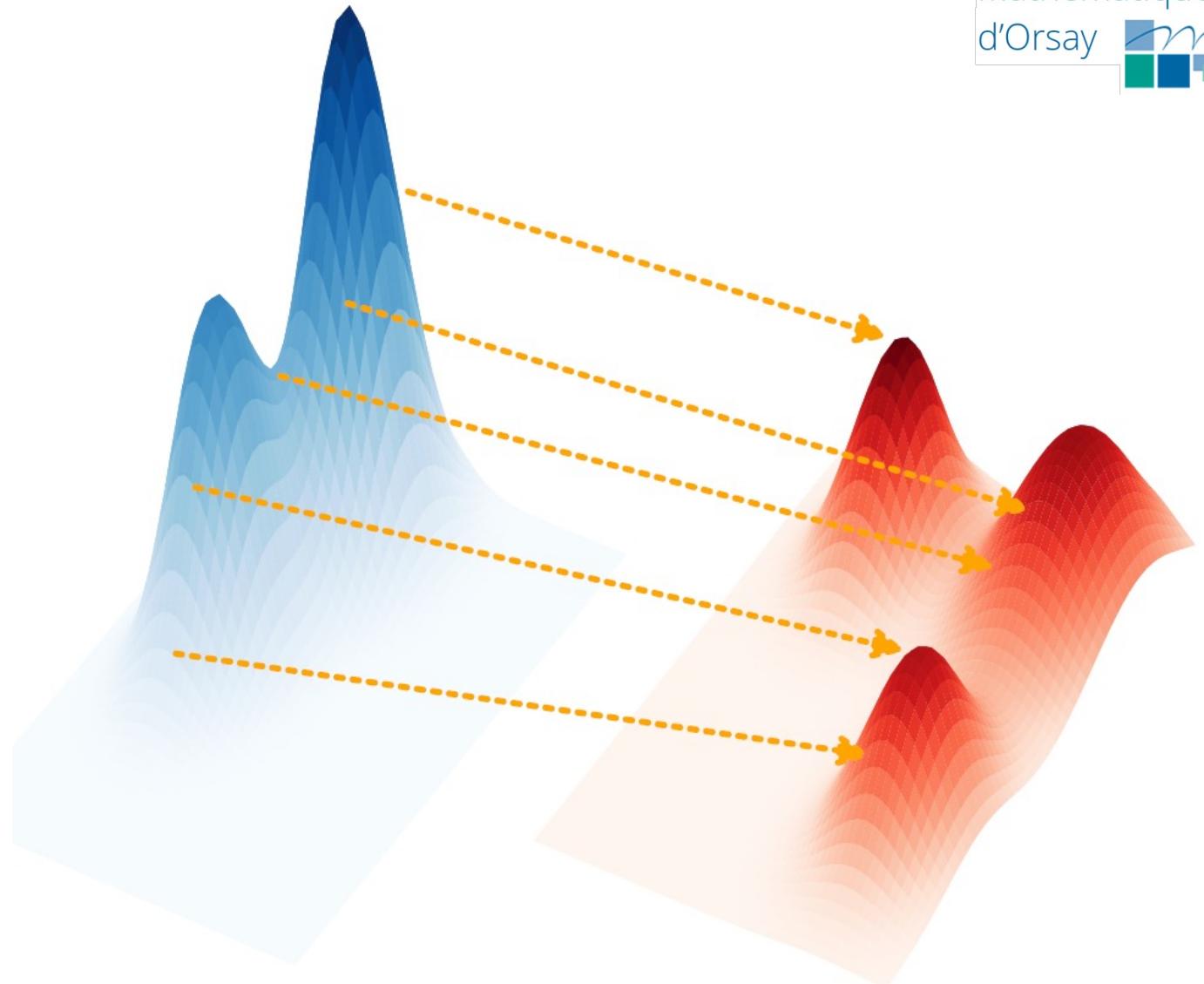


Le transport optimal numérique

BARILLER Halvard
BERTRAND Virgile
LAHMI Elona

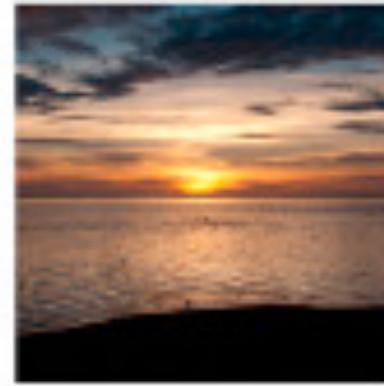


Applications diverses

Color Transfer



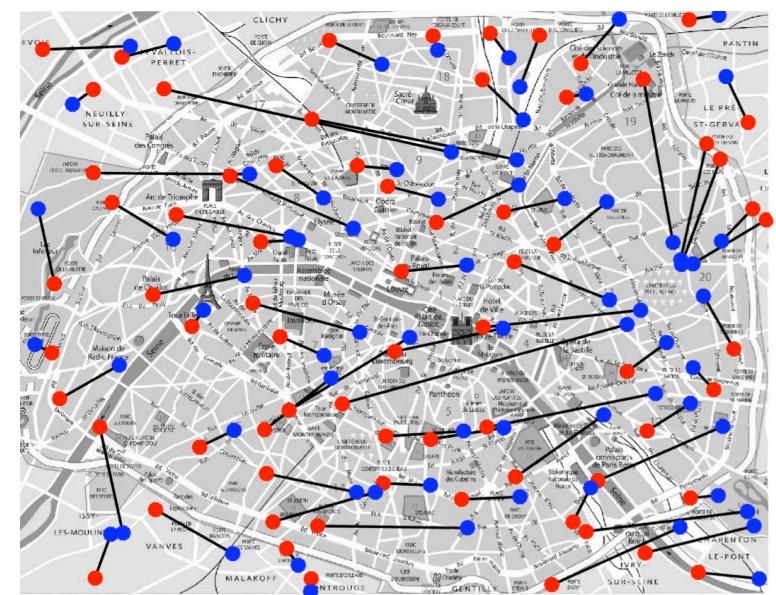
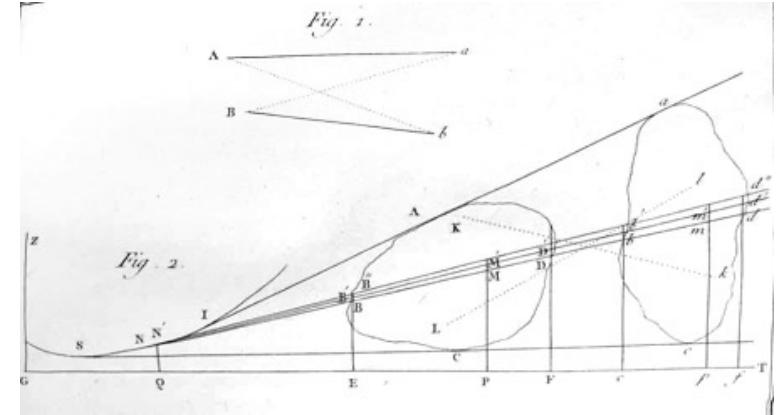
Reference



Result



Colorization



Introduction au problème discret

$$\min \left\{ \sum_{i=1}^N \sum_{j=1}^N C_{ij} \gamma_{ij} \mid \gamma \in \Pi(\mu, \nu) \right\} \quad (\mathcal{MK})$$

$$\Pi(\mu, \nu) := \{ \gamma \in \mathbb{R}_+^{N \times N} \mid \sum_{j=1}^N \gamma_{ij} = \mu_i, \forall i \in I \text{ et } \sum_{i=1}^N \gamma_{ij} = \nu_j, \forall j \in J \}.$$

Comment résoudre ce problème ? Existence/unicité d'une solution ?

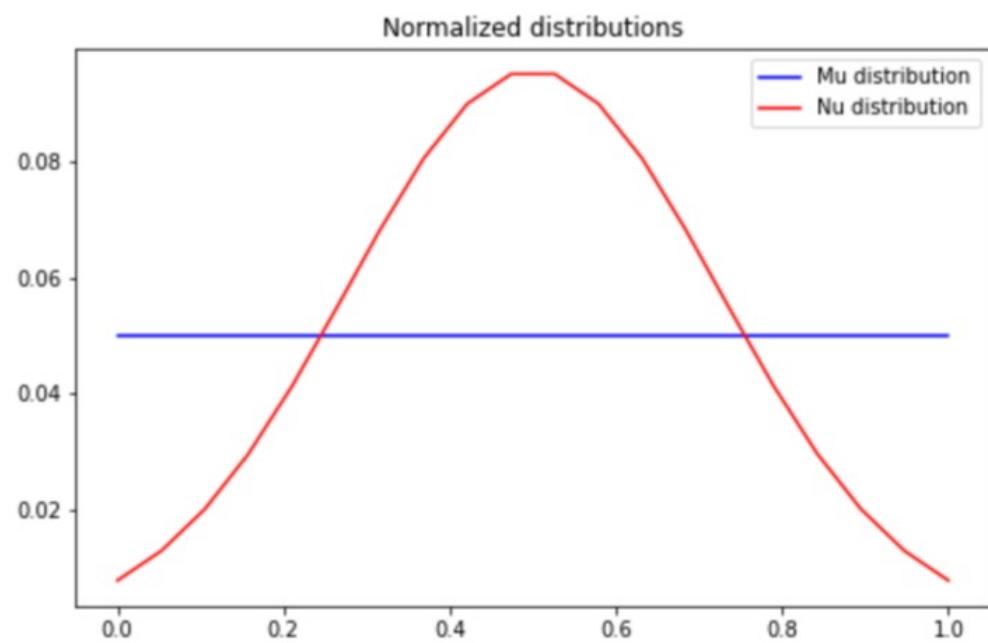
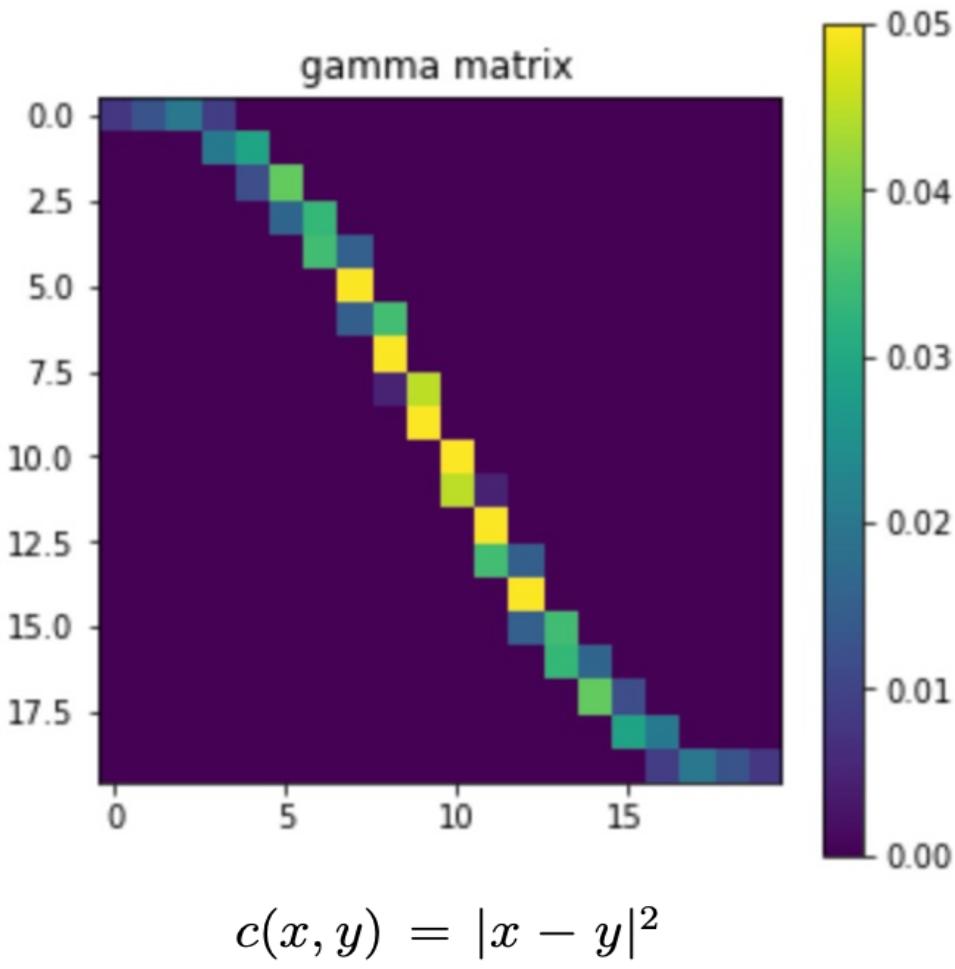
Un problème de programmation linéaire

$$\min \{ \langle \mathbf{c}, \mathbf{x} \rangle \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0 \}$$

$$Ax = b \iff \begin{pmatrix} \langle a_1, x \rangle \\ \vdots \\ \langle a_N, x \rangle \\ \langle a_{N+1}, x \rangle \\ \vdots \\ \langle a_{2N}, x \rangle \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \\ \nu_1 \\ \vdots \\ \nu_n \end{pmatrix}, \quad a_i \in \mathbb{R}^{N^2}$$

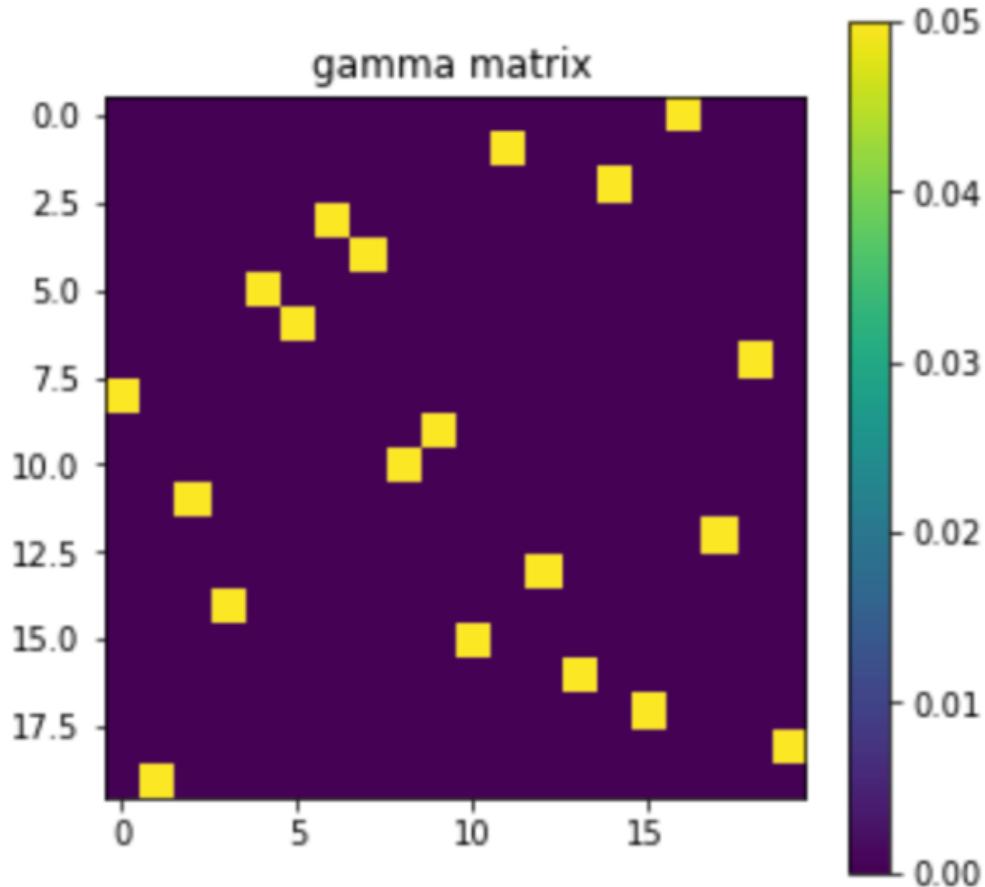
$$A = \begin{pmatrix} 1 & \dots & 1 & & 0 & \dots & 0 \\ 0 & \dots & 0 & & & & \vdots \\ \vdots & & & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & & 1 & \dots & 1 \\ & & & I_N & & \dots & I_N \end{pmatrix}$$

Résolution : algorithme du simplexe



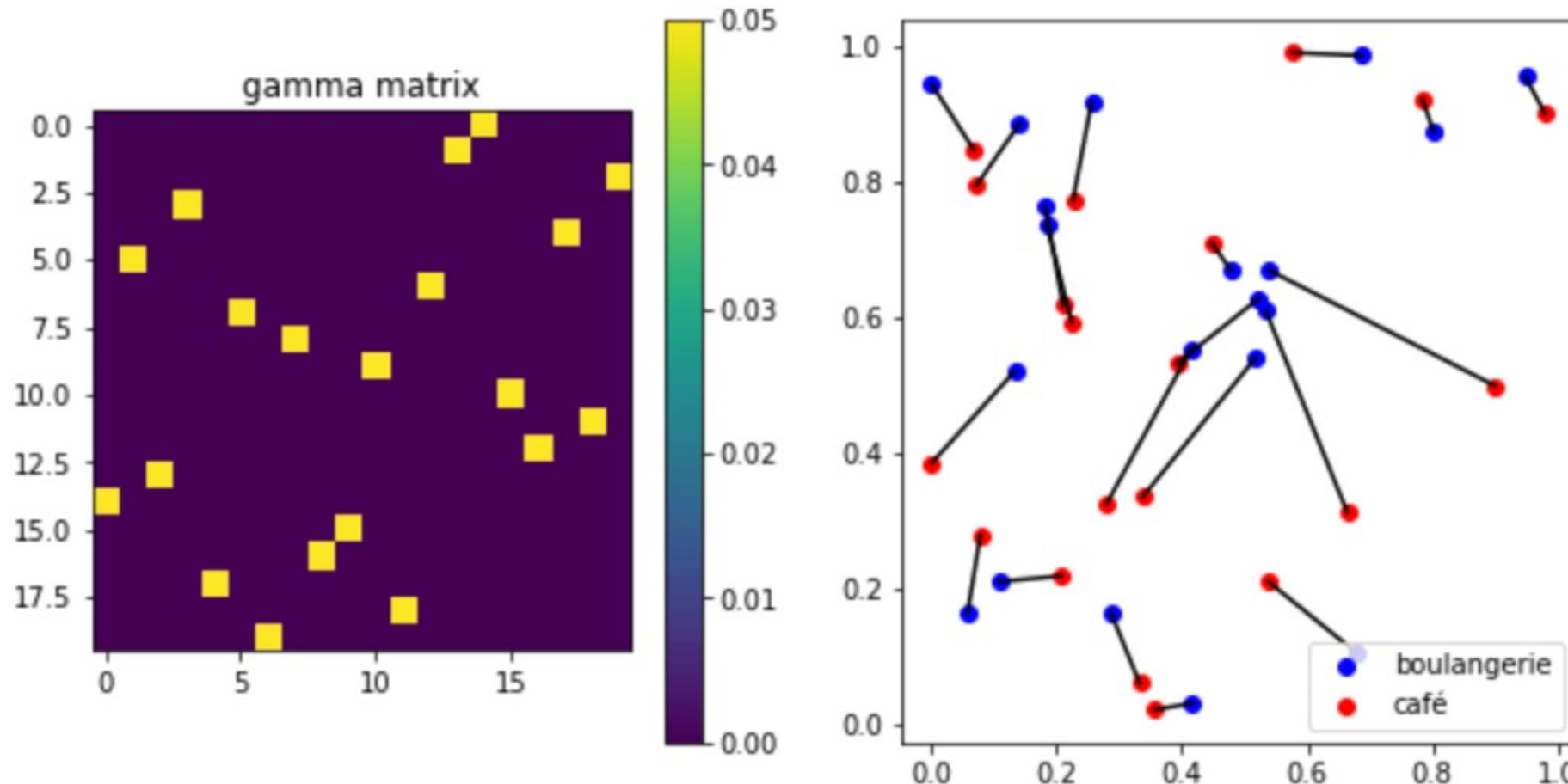
$$\mu(x) = \chi_{[0,1]}(x), \nu(y) = \exp(-10(y - 0.5)^2)$$

Permutation optimale pour distributions uniformes



$$\mu_i = \nu_j = \frac{1}{N}, \quad \forall i, j$$

Un problème d'appariement



$$C_{ij} = |x_i^1 - y_j^1|^2 + |x_i^2 - y_j^2|^2$$

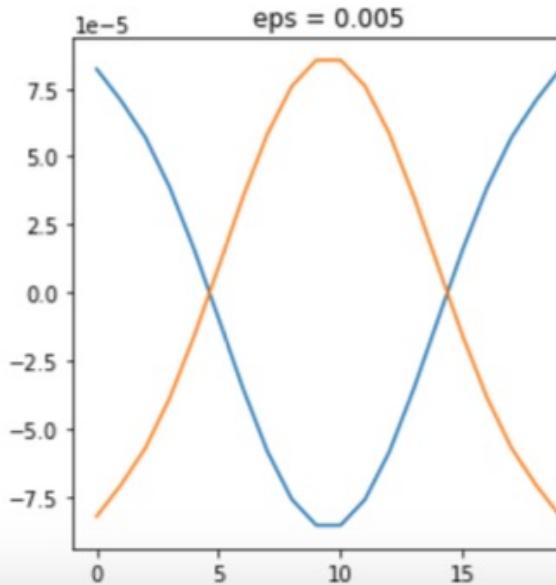
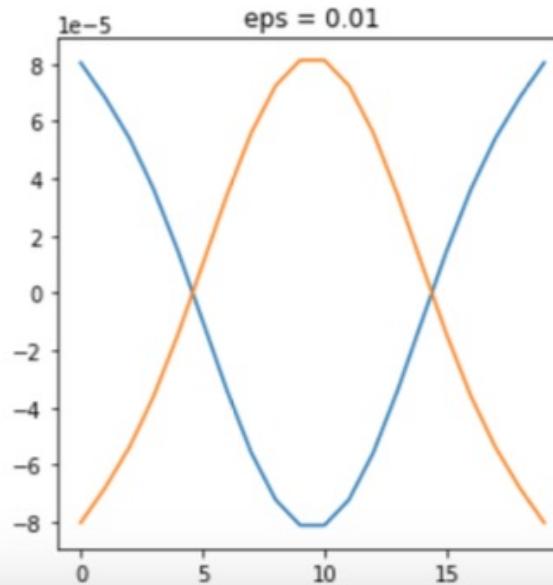
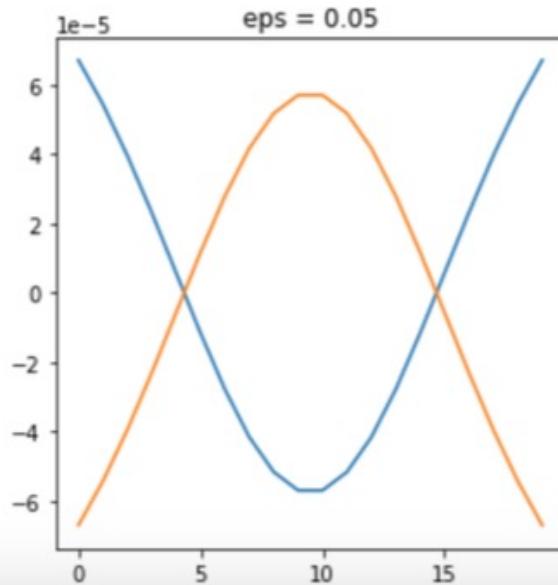
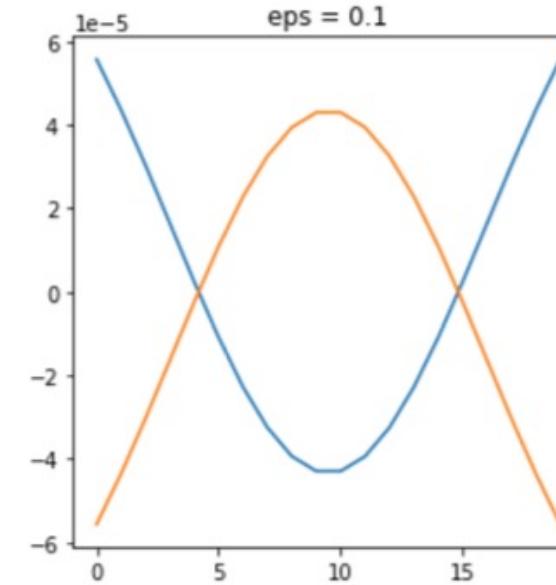
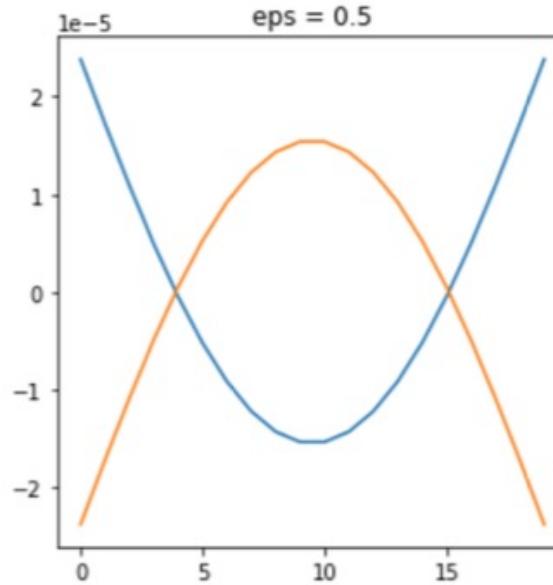
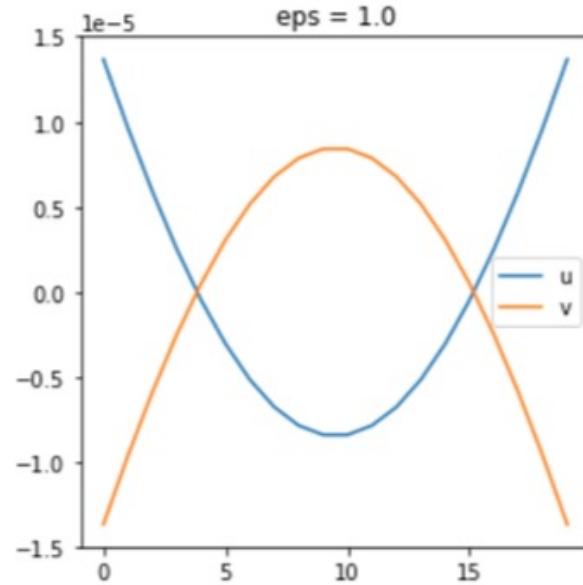
Le problème dual et la fonction log-sum-exp

$$(\mathcal{MK}_d) = \sup_{u,v \in \mathbb{R}^N} \left\{ \sum_{i=1}^N u_i \mu_i + \sum_{j=1}^N v_j \nu_j \mid \forall (i,j) \in I \times J, u_i + v_j \leq C_{ij} \right\}$$

$$v_j = - \max_i u_i - C_{ij} = - \lim_{\varepsilon \rightarrow 0} f_\varepsilon(u - C_{.,j}) \quad f_\varepsilon(x) = \varepsilon \log \left(\sum_{i=1}^N \exp\left(\frac{x_i}{\varepsilon}\right) \right)$$

$$(\mathcal{MK}_d) = \max_{u \in \mathbb{R}^N} \sum_{i=1}^N u_i \mu_i - \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^N \nu_j f_\varepsilon(u - C_{.,j})$$

Potentiels de Kantorovich



La régularisation entropique

$$F(\gamma) := \sum_{ij} C_{ij} \gamma_{ij} + \varepsilon \text{Ent}(\gamma)$$

$$\text{Ent}(\gamma) = \sum_{ij} e(\gamma_{ij}) \quad e(r) = \begin{cases} r(\log r - 1) & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ +\infty & \text{if } r < 0. \end{cases}$$

$$\implies (\mathcal{MK}) \quad \min \left\{ F(\gamma) \mid \sum_j \gamma_{ij} = \mu_i, \sum_i \gamma_{ij} = \nu_j \right\}$$

La régularisation entropique

$$\min \left\{ F(\gamma) \mid \sum_j \gamma_{ij} = \mu_i, \sum_i \gamma_{ij} = \nu_j \right\}$$

$$\gamma_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} \operatorname{argmin} \left\{ \operatorname{Ent}(\gamma) \mid \gamma \in \Pi(\mu, \nu), \sum_{i,j} \gamma_{ij} C_{ij} = \mathcal{MK}(\mu, \nu) \right\}$$

Formulation point selle du problème primal régularisé

$$(\mathcal{H}) = \min \left\{ F(\gamma) \mid \sum_j \gamma_{ij} = \mu_i, \sum_i \gamma_{ij} = \nu_j \right\}$$

$$L(\gamma, u, v) = F(\gamma) + \sum_{i=1}^N u_i (\mu_i - \sum_{j=1}^N \gamma_{ij}) + \sum_{j=1}^N v_j (\nu_j - \sum_{i=1}^N \gamma_{ij})$$

$$(\mathcal{H}) = \inf_{\gamma \in K} F(\gamma) = \inf_{\gamma \in \mathbb{R}^{N \times N}} \sup_{u, v \in \mathbb{R}_+^N} L(\gamma, u, v)$$

Problème dual régularisé

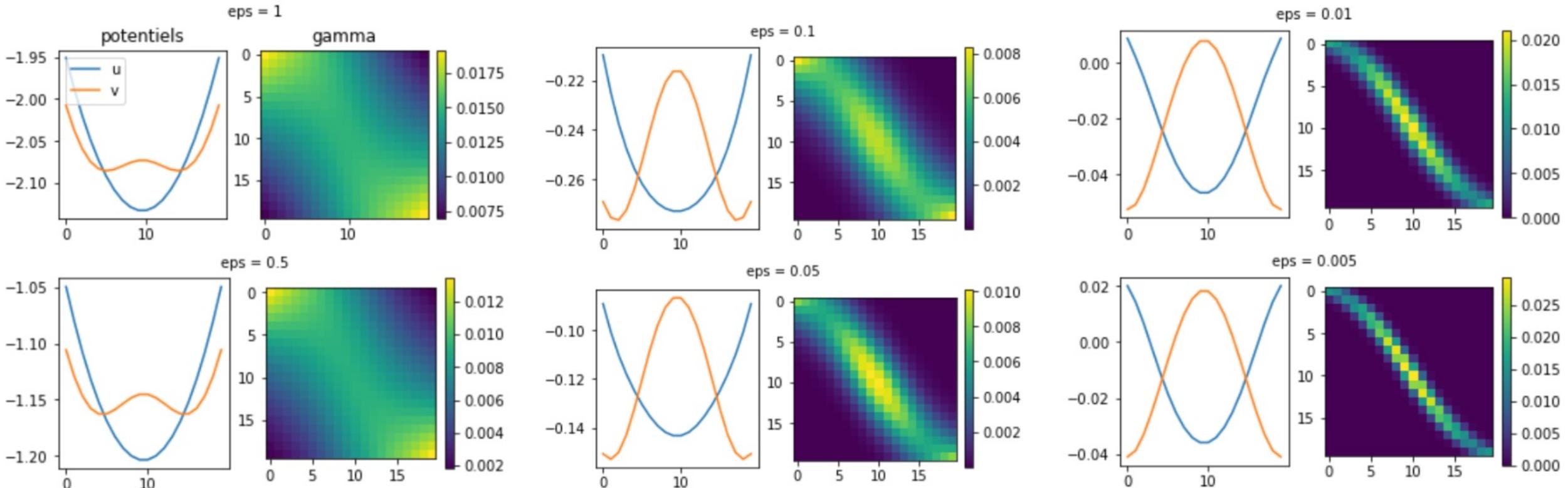
$$\gamma \in \arg \min L \implies \forall k, l \in \mathbb{R}^N, \ \gamma_{kl} = a_k b_l \bar{\gamma}_{kl}$$

$$a_k = \exp\left(\frac{u_k}{\varepsilon}\right), \ b_l = \exp\left(\frac{v_l}{\varepsilon}\right) \text{ et } \bar{\gamma}_{kl} = \exp\left(-\frac{C_{kl}}{\varepsilon}\right)$$

$$(\mathcal{H}_D) = \sup_{u, v \in \mathbb{R}_+^N} - \sum_{i, j=1}^N a_i b_j \bar{\gamma}_{ij} \varepsilon + \sum_{i=1}^N u_i \mu_i + \sum_{j=1}^N v_j \nu_j$$

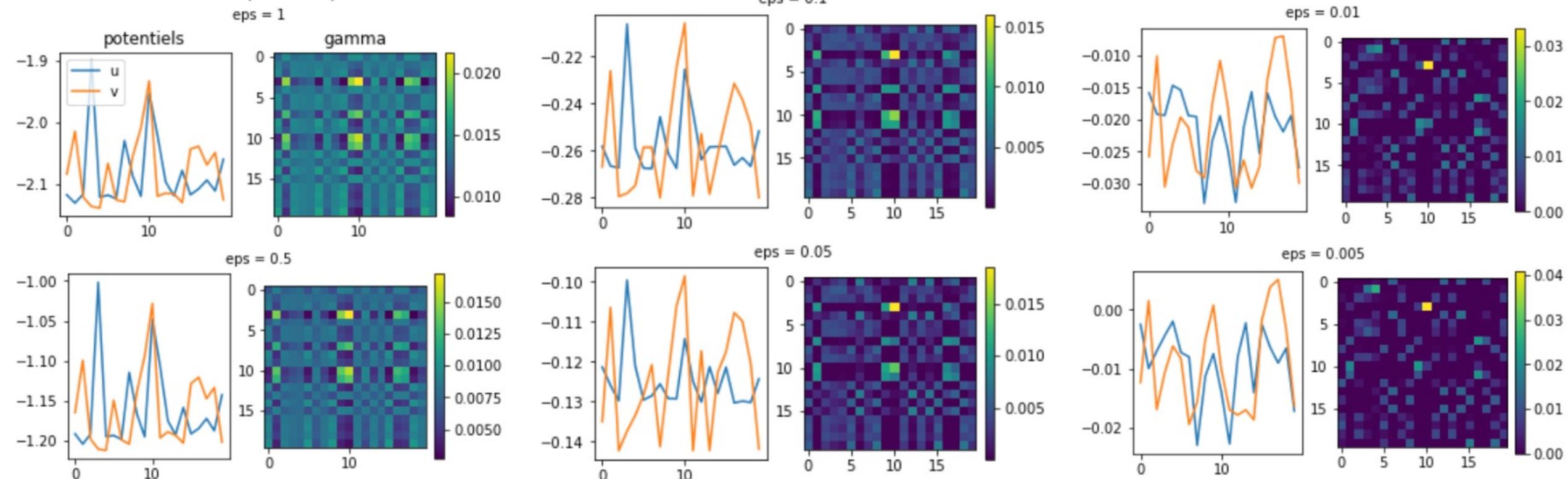
Résolution du problème dual régularisé (1)

EntDual sur l'exemple de la question 1.3



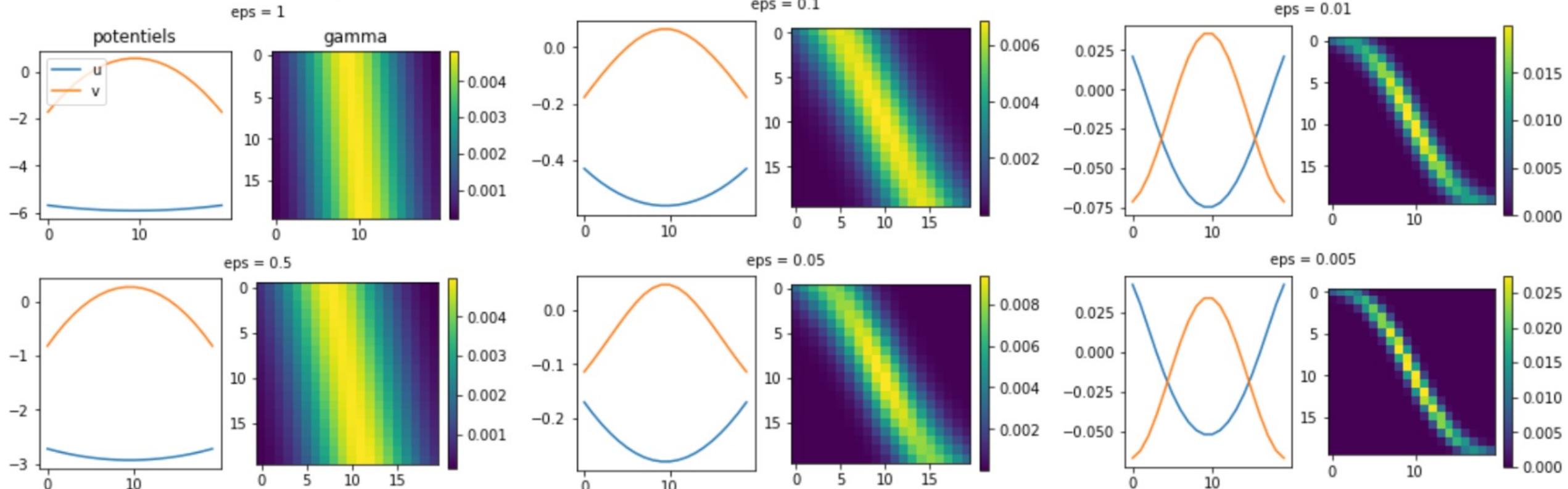
Résolution du problème dual régularisé (2)

EntDual sur l'exemple de la question 1.5



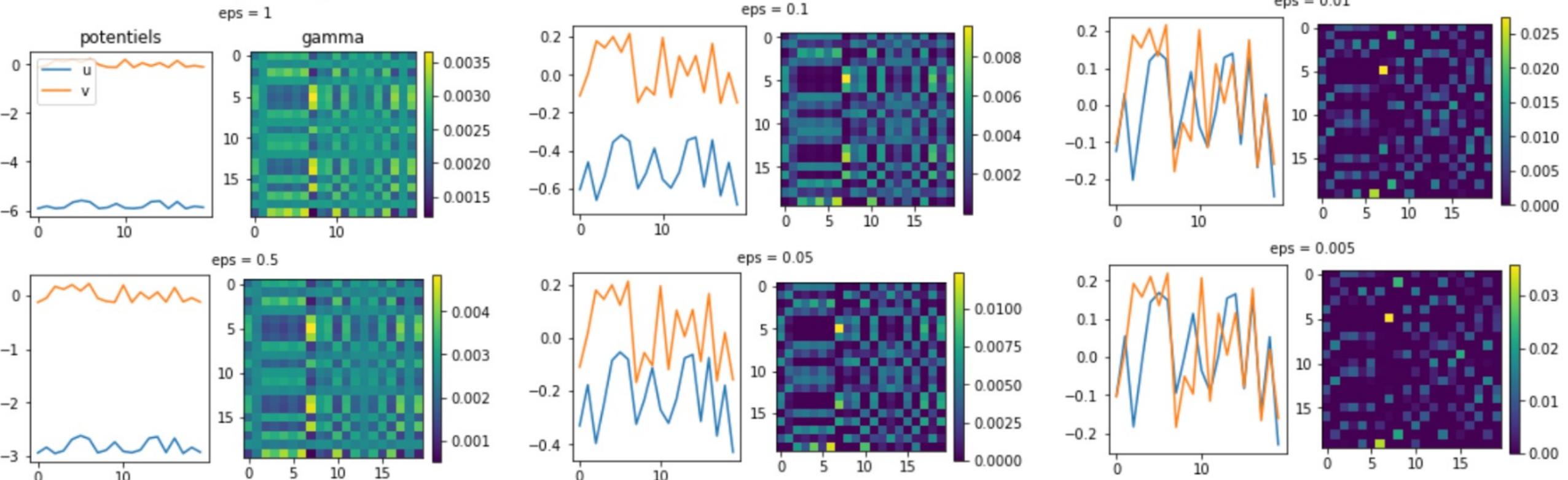
Algorithme de Sinkhorn (1)

Sinkhorn sur l'exemple de la question 1.3



Algorithme de Sinkhorn (2)

Sinkhorn sur l'exemple de la question 1.5



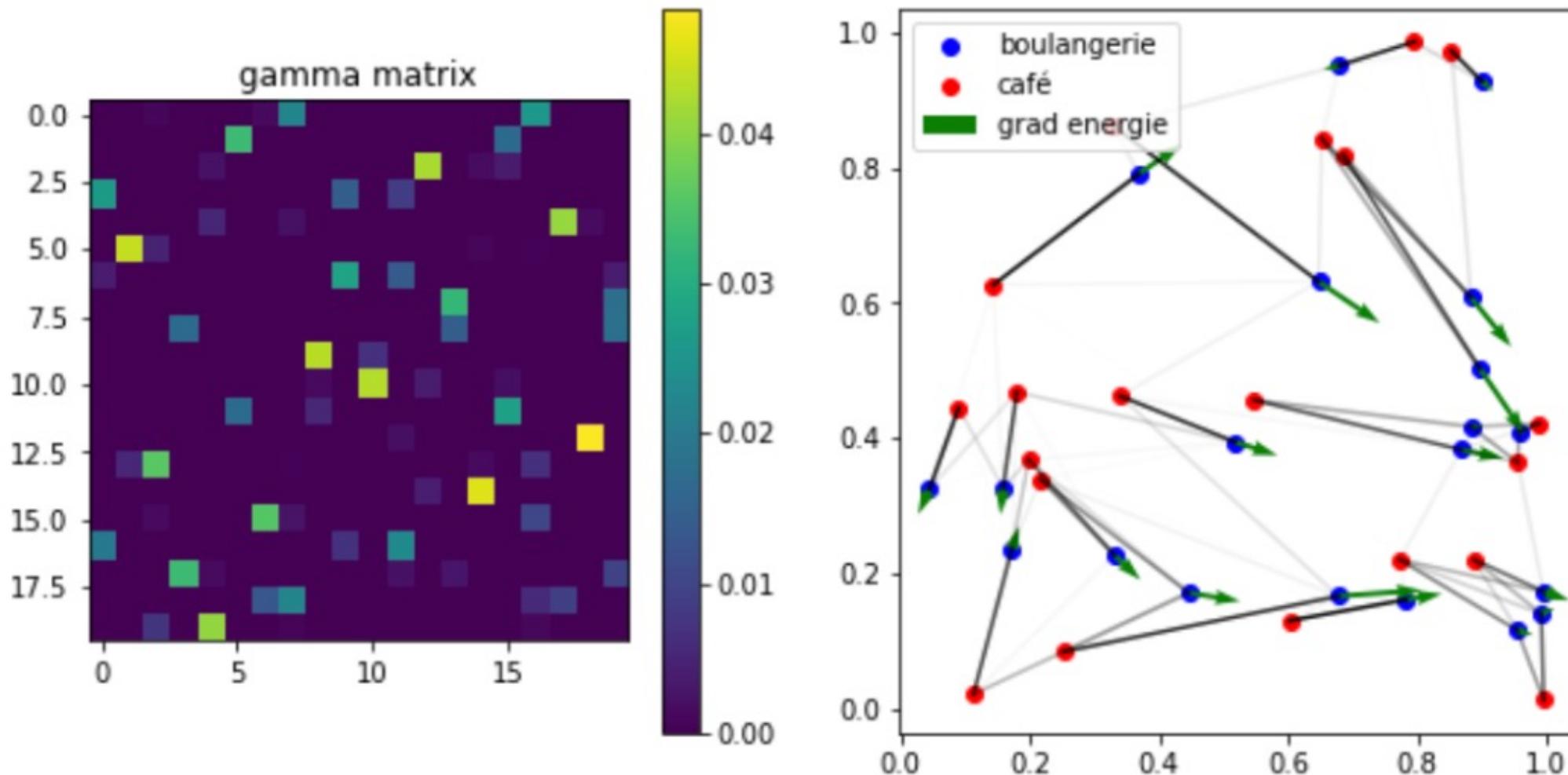
Wasserstein float

$$\mathcal{E}(z) = \mathcal{W}_\varepsilon(\mu, \nu) = \sum_{ij}^N C_{ij} \gamma_{ij} + \varepsilon \sum_{ij}^N \gamma_{ij} \left(\log \left(\frac{\gamma_{ij}}{\mu_i \nu_j} \right) - 1 \right)$$

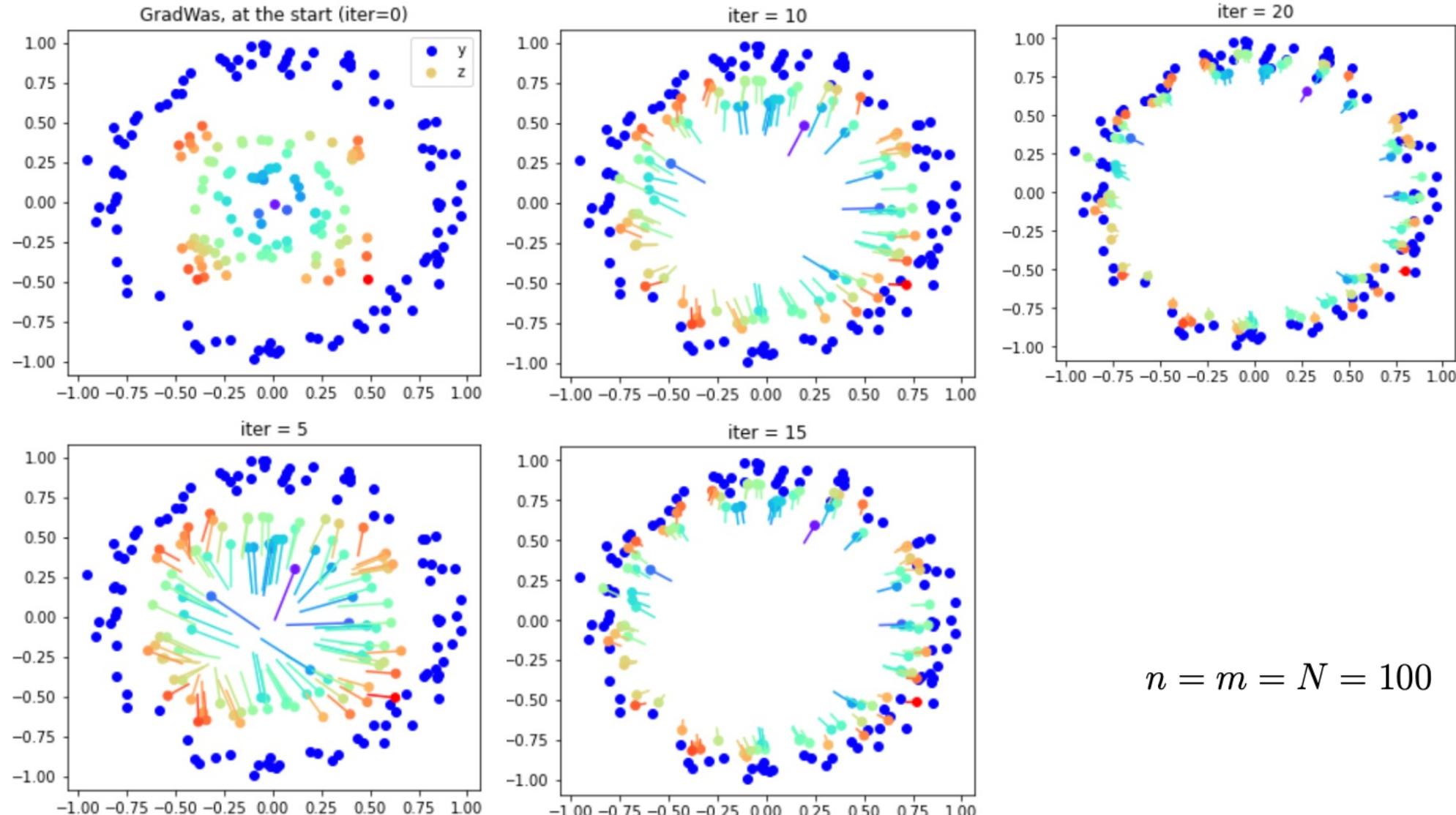
$$\min_{z_i} \mathcal{E}(z) := \mathcal{W}_\varepsilon\left(\frac{1}{n} \sum_i \delta_{z_i}, \frac{1}{m} \sum_i \delta_{y_i}\right).$$

Sinkhorn et champ de gradient

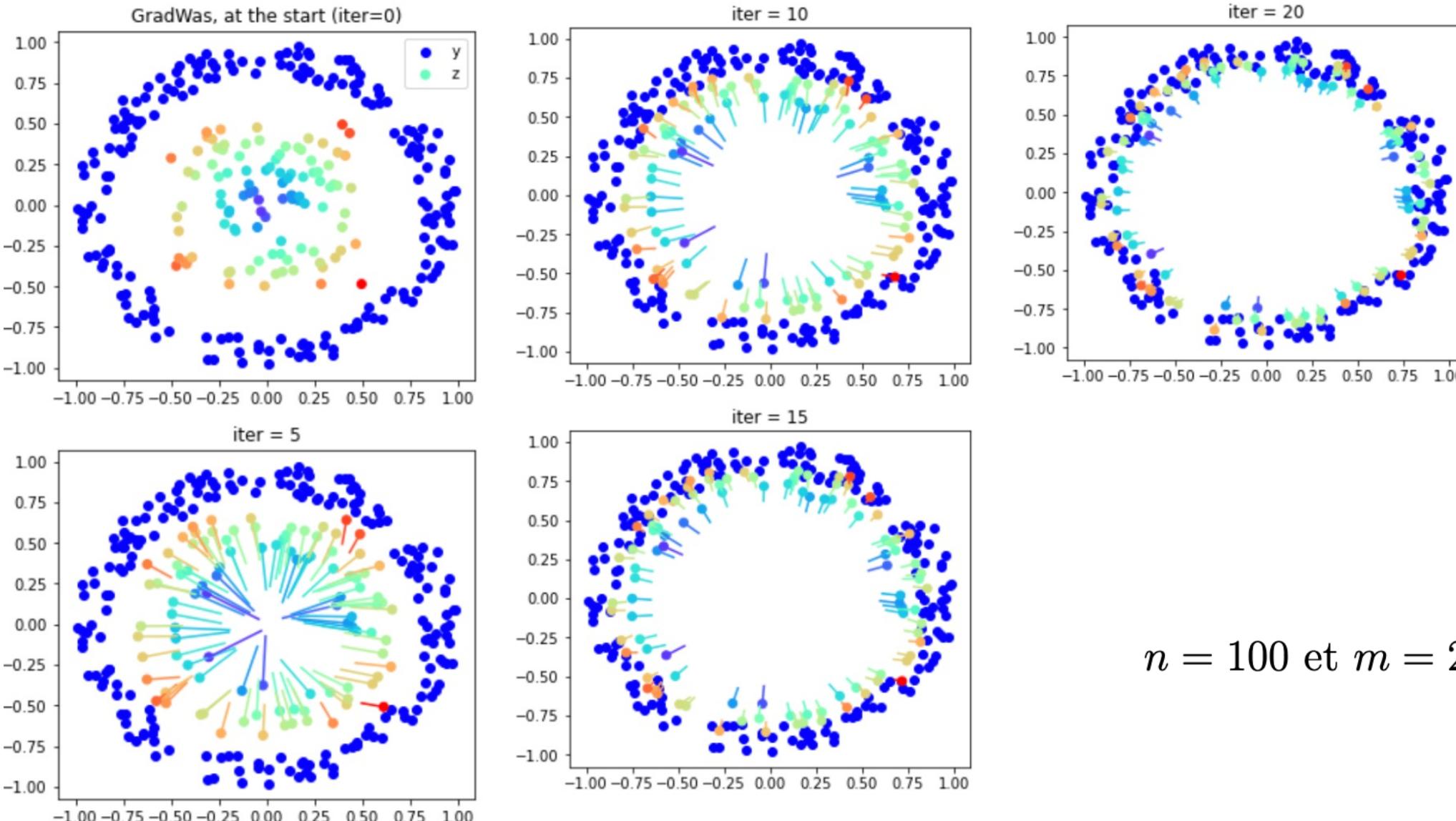
Sinkhorn 2d avec l'exemple de Q1.5



Descente de gradient pour la distance de Wasserstein

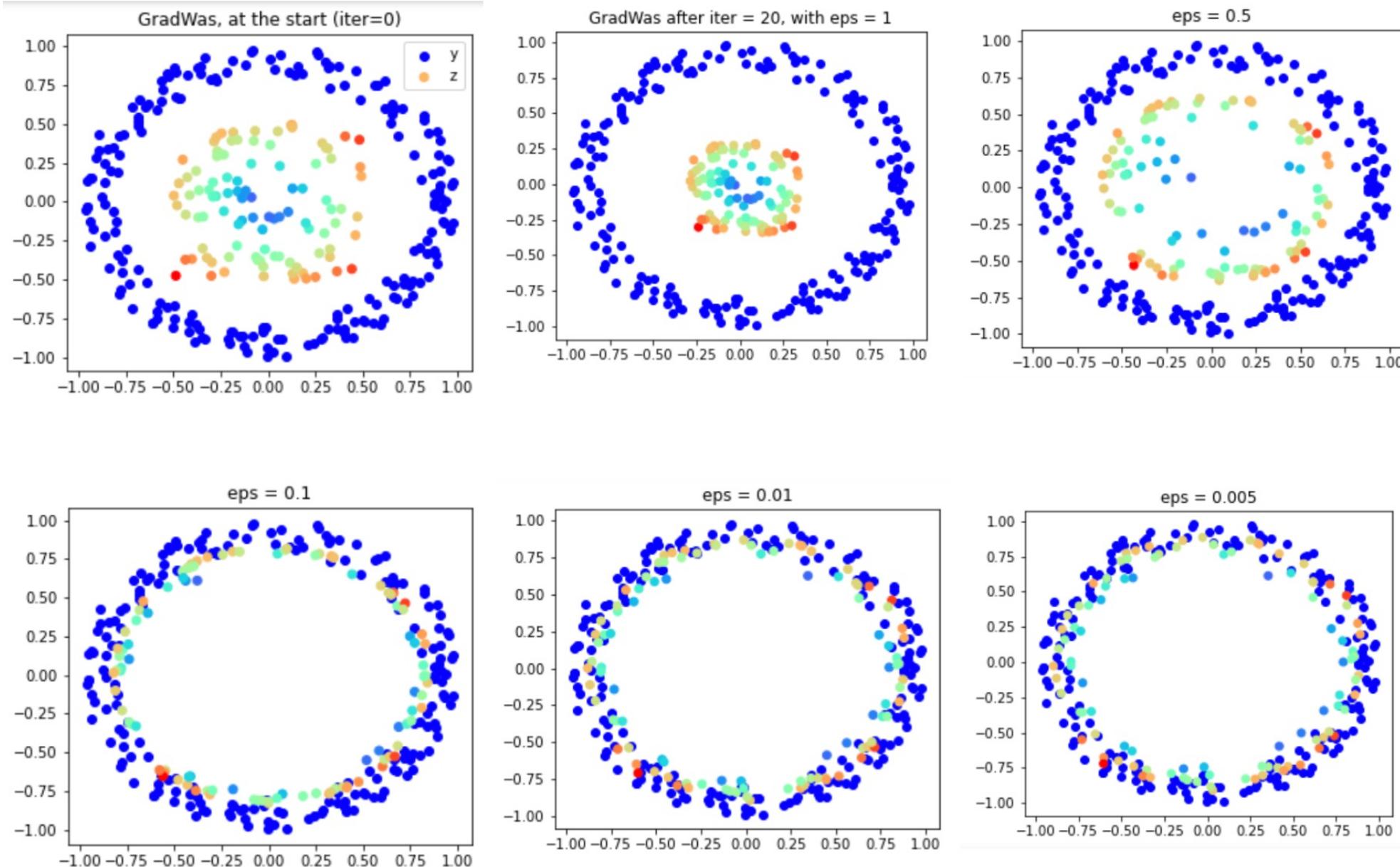


Descente de gradient pour la distance de Wasserstein



$n = 100$ et $m = 200$

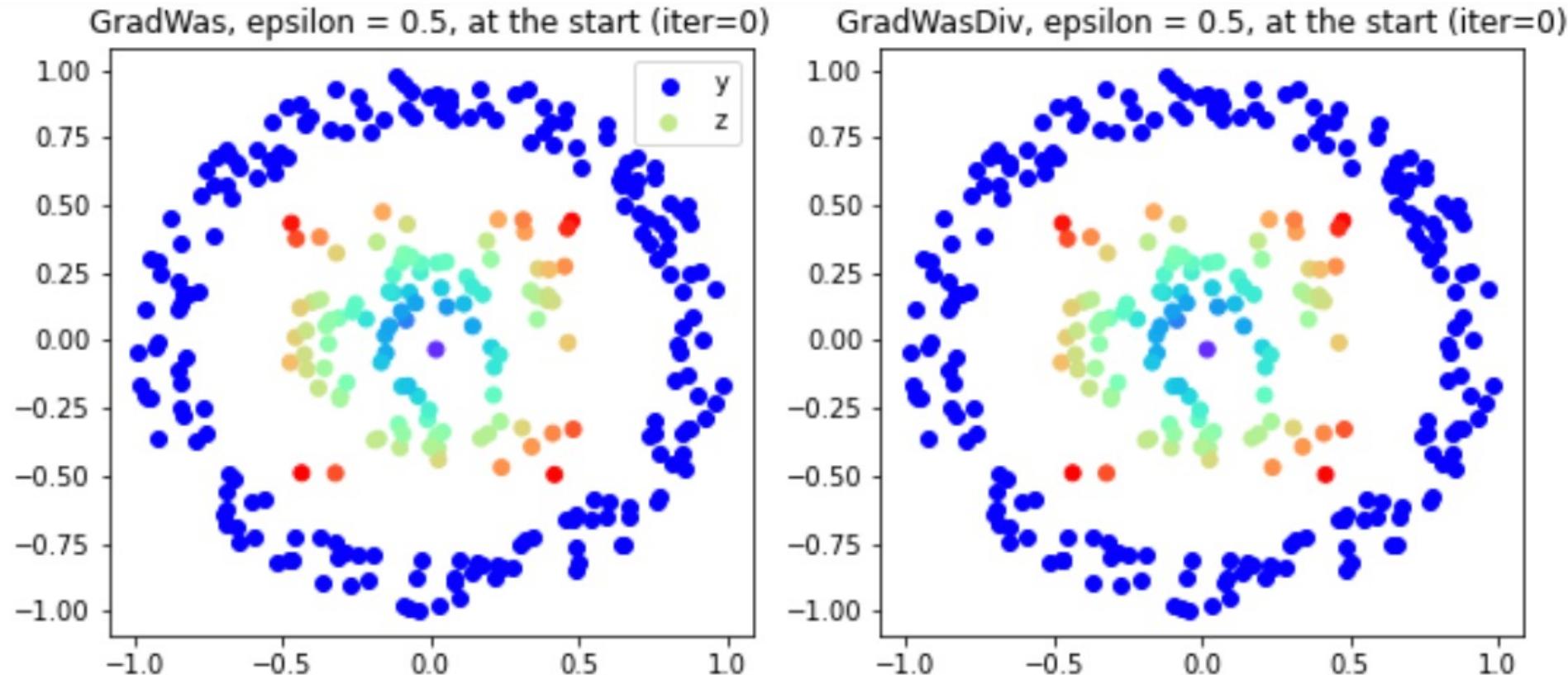
Vitesses de convergence



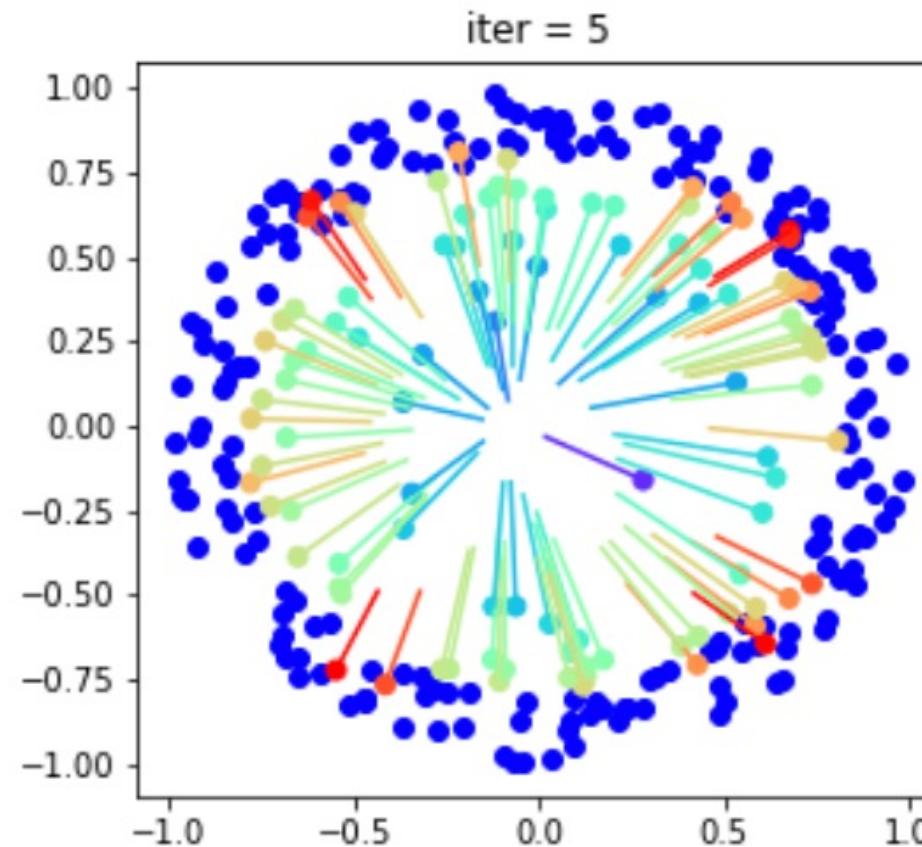
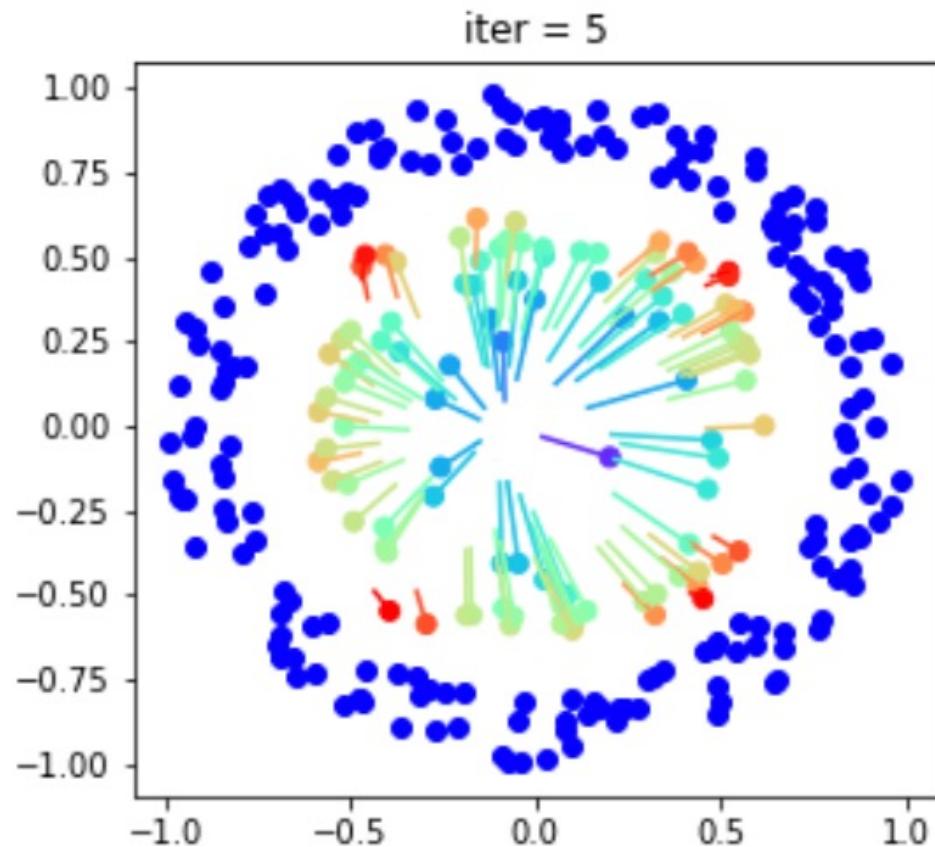
Divergence de Wasserstein

$$\tilde{\mathcal{W}}_\varepsilon(\mu, \nu) = \mathcal{W}_\varepsilon(\mu, \nu) - \frac{1}{2}\mathcal{W}_\varepsilon(\mu, \mu) - \frac{1}{2}\mathcal{W}_\varepsilon(\nu, \nu)$$

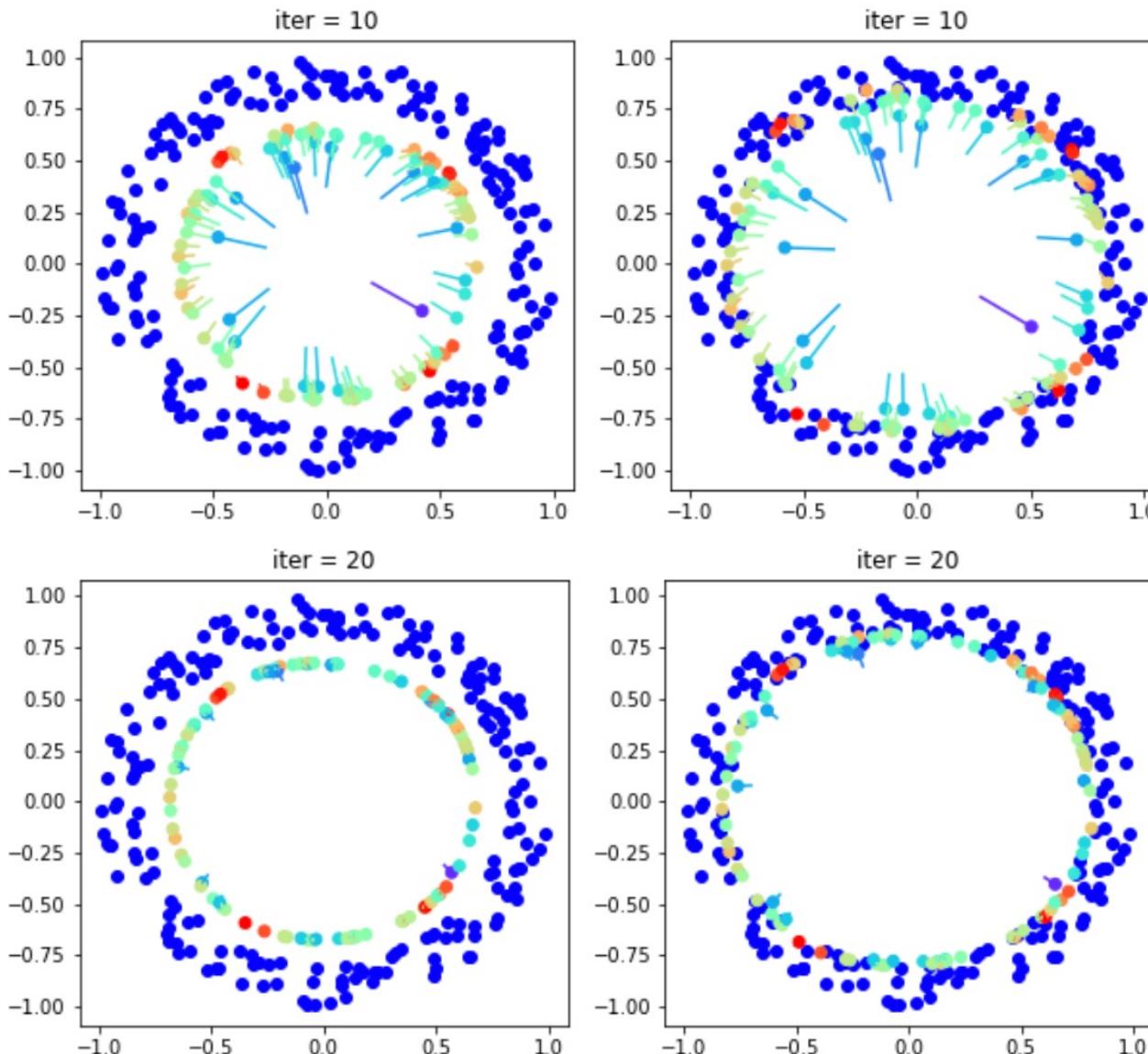
Comparaison gradient et divergence de Wasserstein



Comparaison gradient et divergence de Wasserstein



Comparaison gradient et divergence de Wasserstein



Comparaison gradient et divergence de Wasserstein

