Maximum Principles for Elliptic Operators

by

Mohammad Hamdar

May 10, 2021

Elliptic Partial Differential Equations

Let $U \subset \mathbb{R}^n$ be an open set, and $u \in C^2(U)$.
Write $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and consider the following linear differential operator of second order:

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j}(x) + \sum_{i=1}^{n} b_i(x) u_{x_i}(x) + c(x)u(x),$$

where a_{ij}, b_i and c are real valued functions in U.

Elliptic Partial Differential Equations

Let $U \subset \mathbb{R}^n$ be an open set, and $u \in C^2(U)$.
Write $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and consider the following linear differential operator of second order:

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j}(x) + \sum_{i=1}^{n} b_i(x) u_{x_i}(x) + c(x) u(x),$$

where a_{ij}, b_i and c are real valued functions in U.
L is said to be *elliptic* at a point $x \in U$ if the coefficient matrix $A = (a_{ij})_{ij}$ is a symmetric positive definite matrix.

This is equivalent to saying that if $\lambda(x), \Lambda(x)$ denote respectively the minimum and maximum eigenvalues of A, then

$$0 < \lambda(x)|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n - \{0\}$.

This is equivalent to saying that if $\lambda(x), \Lambda(x)$ denote respectively the minimum and maximum eigenvalues of A, then

$$0 < \lambda(x)|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n - \{0\}$.
Moreover, if $\Lambda(x)/\lambda(x)$ is bounded in U, we shall call L *uniformly elliptic* in U.

Example

Taking $a_{ij} = \delta_{ij}$, $b_i = 0$, and $c = 0$ we see that the Laplacian operator

$$Lu = \Delta u = \sum_{i=1}^{n} u_{x_i x_i}$$

is a uniformly elliptic operator.

Example

Taking $a_{ij} = \delta_{ij}$, $b_i = 0$, and $c = 0$ we see that the Laplacian operator

$$Lu = \Delta u = \sum_{i=1}^{n} u_{x_i x_i}$$

is a uniformly elliptic operator.

Example

In $\mathbb{R}^2$, the operator

$$u_{x_1 x_1} + x_1 u_{x_2 x_2}$$

is elliptic but not uniformly elliptic in the half plane $x_1 > 0$.

Most results concerning elliptic operators require additional conditions limiting the relative importance of the lower order terms $b_i u_{x_i}$ and cu with respect to the principal term $a_{ij} u_{x_i x_j}$. For instance, we will assume the following condition

$$\frac{|b_i(x)|}{\lambda(x)} \leq \text{constant} < \infty.$$

The Weak Maximum Principle

The Weak Maximum Principle

Theorem

Let $U \subset \mathbb{R}^n$ be a bounded domain and $u \in C^2(U) \cap C^0(\overline{U})$. If L is elliptic in U with $Lu \geq 0$ and $c = 0$ in U, then the maximum of u in $\overline{U}$ is achieved on ∂U, i.e.

$$\sup_U u = \sup_{\partial U} u.$$

The Weak Maximum Principle

Theorem

Let $U \subset \mathbb{R}^n$ be a bounded domain and $u \in C^2(U) \cap C^0(\overline{U})$. If L is elliptic in U with $Lu \geq 0$ and $c = 0$ in U, then the maximum of u in $\overline{U}$ is achieved on ∂U, i.e.

$$\sup_U u = \sup_{\partial U} u.$$

Proof.

Suppose that $Lu > 0$ in U. If u attains its maximum at $x_0 \in U$, then $Lu(x_0) = \sum_{i,j} a_{ij}(x_0) u_{x_i x_j}(x_0) \leq 0$. This is a contradiction; and so no interior maximum exists in this case. Now, there is a sufficiently large constant γ such that

$$L(u + \epsilon e^{\gamma x_1}) > 0.$$

$\square$

Notation

If U is an open set with u defined on U, we will write

$$U^+ = \{x \in U : u(x) > 0\}$$

and

$$u^+(x) = \max(u(x), 0).$$

Notation

If U is an open set with u defined on U, we will write

$$U^+ = \{x \in U : u(x) > 0\}$$

and

$$u^+(x) = \max(u(x), 0).$$

Corollary

Let U be a bounded domain in $\mathbb{R}^n$ and $u \in C^2(U) \cap C^0(\overline{U})$. If L is elliptic in U with $Lu \geq 0$ and $c \leq 0$ in U, then

$$\sup_U u \leq \sup_{\partial U} u^+.$$

Proof.

Define the elliptic operator

$$L_0(u) = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j}(x) + \sum_{i=1}^{n} b_i(x) u_{x_i}(x).$$

Since $Lu \geq 0$ in U, we have that

$$L_0(u) = Lu - c(x)u(x) \geq -c(x)u(x),$$

so that $L_0(u) \geq 0$ in U^+. Hence, we can use the weak maximum principle for $L_0 u$, and the result then follows. $\square$

Theorem (Uniqueness of Solutions to the Dirichlet Problem)

Let $U \subset \mathbb{R}^n$ and L an elliptic operator in U with $c \leq 0$ in U. Then there exists at most one solution $u \in C^2(U) \cap C^0(\overline{U})$ to the Dirichlet Problem

$$\begin{aligned} Lu &= f \text{ in } U \\ u &= g \text{ on } \partial U \end{aligned}$$

where f is defined in U and $g \in C^0(\partial U)$.

Theorem (Uniqueness of Solutions to the Dirichlet Problem)

Let $U \subset \mathbb{R}^n$ and L an elliptic operator in U with $c \leq 0$ in U. Then there exists at most one solution $u \in C^2(U) \cap C^0(\overline{U})$ to the Dirichlet Problem

$$
\begin{aligned}
Lu &= f \text{ in } U \\
u &= g \text{ on } \partial U
\end{aligned}
$$

where f is defined in U and $g \in C^0(\partial U)$.

Theorem (Comparison Principle)

Let $U \subset \mathbb{R}^n$ and L an elliptic operator in U with $c \leq 0$ in U. If $u, v \in C^2(U) \cap C^0(\overline{U})$ such that $Lu \geq Lv$ in U and $u \leq v$ on ∂U, then $u \leq v$ in U.

The Strong Maximum Principle

The Strong Maximum Principle

Lemma (Hopf's Boundary Lemma)

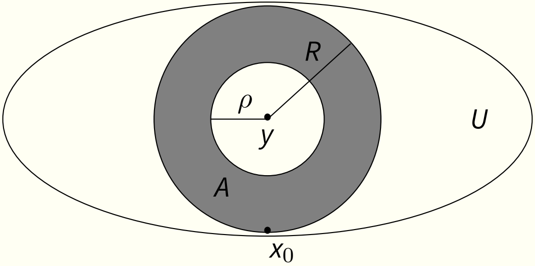
Let $U \subset \mathbb{R}^n$ be a domain and $u \in C^2(U)$. Let L be uniformly elliptic in U and $Lu \geq 0$ throughout U. Further, let x_0 be a point on ∂U such that:

- u is continuous at x_0
- $u(x) < u(x_0)$ for all $x \in U$
- U satisfies the interior sphere condition at x_0;

then:

1) If $c(x) = 0$ in U, then $\frac{\partial u}{\partial \nu}(x_0) > 0$.
2) If $c(x) \leq 0$ and $c(x)/\lambda(x)$ is bounded in U, then the same conclusion holds provided $u(x_0) \geq 0$.
3) If $u(x_0) = 0$, then the same conclusion holds irrespective of the sign of c.

Proof.



We introduce an auxiliary function v by defining

$$v(x) = e^{-\alpha r^2} - e^{-\alpha R^2},$$

where α is chosen so that $Lv \geq 0$ in A. □

we have that, in A,

$$L(u - u(x_0) + \epsilon v) \geq 0,$$

and $u - u(x_0) + \epsilon v \leq 0$ on ∂A. Consequently, by the weak maximum principle,

$$u - u(x_0) + \epsilon v \leq 0 \text{ in } A. \tag{1}$$

We can then take normal derivatives at x_0.

Theorem (Strong Maximum Principle)

Let $U \subset \mathbb{R}^n$ be a domain and $u \in C^2(U)$. Let L be uniformly elliptic in U with $Lu \geq 0$ in U. Then the following are true:

- If $c = 0$ in U and u achieves its maximum in the interior of U, then u is constant.
- If $c \leq 0$ in U, c/λ bounded in U, and u achieves a non-negative maximum in the interior U, then u is constant.
- If $u = 0$ at an interior maximum of U, then $u \equiv 0$ irrespective of the sign of c.

Theorem (Uniqueness of solutions to the Neumann Problem)

Let $U \subset \mathbb{R}^n$ be a bounded domain and $u \in C^2(U) \cap C^0(\bar{U})$. Let L be uniformly elliptic in U with $c \leq 0$ and c/λ is bounded in U. Assume that U satisfies the interior sphere condition at each point of ∂U, and the normal derivative of u is defined everywhere on ∂U. If

$$\begin{aligned} Lu &= 0 \text{ in } U \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial U \end{aligned}$$

then u is constant in U. If, also, $c < 0$ at some point in U, then $u \equiv 0$.

Thank You!