Generalization of a Complex Adjoint Matrix to Construct Spherical Monogenics

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1. Background and Motivation

2. Using Reproducing Kernel and Optimizations

Clifford Algebra

Definition

Let $(e_1, e_2, \cdots e_m)$ be an orthonormal basis for $\mathbb{R}^m$. The non-commutative **Clifford algebra** $\mathbb{R}_m$ is built by the rules:

$$\begin{aligned} e_j^2 &= -1, \;\; j = 1, \cdots, m, \\ e_j e_i &= -e_i e_j, \;\; i \neq j, \;\; i, j = 1, \cdots, m. \end{aligned}$$

The basis for $\mathbb{R}_m$ is obtained by considering for any set $A = \{j_1, j_2, \cdots, j_h\} \subset \{1, \cdots m\} = M$, ordered by the element $e_A = e_{j_1} e_{j_2} \cdots e_{j_h}$.

Definition

Let $\Omega \subset \mathbb{R}^m$ be open and $f \in C^1(\Omega, \mathbb{R}_m)$. Then f is said to be left monogenic in Ω if

$$\partial_x f = 0$$

where $\partial_x = \sum_{j=1}^{m} e_j \partial_{x_j}$ is the Dirac operator.

A left monogenic homogeneous polynomial Y_k on $\mathbb{R}^m$ of degree $k \geq 0$ is called a solid inner spherical monogenic of order k.

The set of all solid inner spherical monogenic of order k on $\mathbb{R}^m$ will be denoted by $\mathcal{M}_l^+(k)$.

$$\dim \mathcal{M}_l^+(k) = \frac{(m+k-2)!}{(m-2)!k!} = \binom{m+k-2}{k} = d_{k,m}. \qquad (1)$$

Let's remember these for later:

- if $m = 2$ then $d_{k,m} = 1$,
- if $m = 3$ then $d_{k,m} = k + 1$,
- if $m = 4$, $k = 2$ then $d_{k,m} = \binom{m+k-2}{k} = 6$.

Definition

We define the operator L_c from $C^2(\bar{B}(1), \mathbb{C}_m)$ to $L^2(\bar{B}(1), \mathbb{C}_m)$ by

$$L_c f(x) = \partial_x((1 - |x|^2)\partial_x f(x)) + 4\pi^2 c^2 |x|^2 f(x)$$

where $B(1)$ is the unit ball in $\mathbb{R}^m$ and ∂_x is the Dirac Operator. We define the Eigenfunctions of the operator L_c as **Clifford Prolate Spheroidal Wave Functions (CPSWFs).**

Theorem (B.G., Hogan, Lakey)

Let $n = 2N$ be an even integer, and $\psi_{n,m}^{k,c,i}(x)$ be a CPSWF. Then, $\psi_{2N,m}^{k,c,i}(x) = P_{N,m}^{k,c}(|x|^2) Y_k^i(x)$, where $P_{N,m}^{k,c}(|x|^2)$ is a radial function.

Definition

We define truncated Fourier transformation from $L^2(\bar{B}(1), \mathbb{C}_m)$ to $L^2(\bar{B}(1), \mathbb{C}_m)$ by

$$\mathcal{G}_c f(x) = \chi_{\bar{B}(1)}(x) \int_{\bar{B}(1)} e^{2\pi i c <x,y>} f(y) dy,$$

Theorem (B.G., Hogan, Lakey)

The operators L_c and $\mathcal{G}_c$ commute.

Theorem (B.G., Hogan, Lakey)

It's possible to see that eigenfunctions of operator L_c are also the eigenfunction of $\mathcal{G}_c$, in $L^2(B(1), \mathbb{C}_m)$.

Lemma

The radial part of $P_{N,m}^{k,c}(|x|^2)$ *of the CPSWF* $\bar{\psi}_{2N,m}^{k,c,i}(x)$, *satisfies*

$$4t(1-t)\frac{d^2}{dt^2}P_{N,m}^{k,c}(t) + 2(m+2k-t(2+m+2k))\frac{d}{dt}P_{N,m}^{k,c}(t) - 4\pi^2c^2tP_{N,m}^{k,c}(t) + \chi_{2N,m}^{k,c}P_{N,m}^{k,c}(t) = 0,$$

which becomes a Sturm-Liouville differential equation after multiplying by $g(t) = t^{k+\frac{m}{2}-1}$. *Therefore,* $\{P_{N,m}^{k,c}\}_{N=0}^{\infty}$ *may be normalised so that*

$$\int_0^1 P_{N,m}^{k,c}(t)P_{M,m}^{k,c}(t)t^{k+\frac{m}{2}-1}dt = \langle P_{N,m}^{k,c}, P_{M,m}^{k,c}\rangle_{g(t)} = \delta_{MN}.$$

1 Background and Motivation

2 Using Reproducing Kernel and Optimizations

Definition

A zonal function $g : S^{m-1} \to \mathbb{C}$ is of the form

$$g(x) = f(\langle x, y \rangle) \quad (y \in S^{m-1})$$

for some $f : \mathbb{R} \to \mathbb{C}$ and fixed $y \in S^{m-1}$.

Now we define a special case of zonal function.

Definition

Fix $y \in S^{m-1}$ and $c \in \mathbb{C}$. The zonal spherical harmonic Z_y is defined by

$$Z_y(x) := cRH_k(x, y),$$

where $RH_k(x, y)$ is the reproducing kernel spherical harmonics.

The collection $\{RH_k(x,\eta_i)\}_{i=1}^{2k+1}$ is a basis for $\mathcal{H}_k^3$ for almost every choice points $\eta_1, \cdots, \eta_{k+1} \in S^2$.
With the aim of obtaining an orthonormal basis for $\mathcal{H}_k^3$, we define

$$Z_t(x) := \sum_{j=1}^{2k+1} RH_k(x,\eta_j)a_{jt},$$

with $t = 1, \ldots, (2k+1)$, and, $a_{jt} \in \mathbb{C}$. Let
$G_{ij} := RH_k(\eta_i,\eta_j) = (2k+1)P_k(\langle \eta_i, \eta_j \rangle)$. If we want to have $\{Z_t(x)\}_{t=1}^{2k+1}$ be orthonormal in $L^2(S^2)$ then we can conclude that $A^*GA = I$.

Lemma

The matrix G is semidefinite.

So we compute the matrix $G := [G_{ij}]$ in dimension 3. We have $G_{ij} := (2k+1)P_k(\langle \eta_i, \eta_j \rangle)$ where η_i, and, η_j are points on the unit sphere S^2, which can be chosen randomly such that determinant of G is not zero. If $A = G^{-\frac{1}{2}}$ is a diagonal matrix then $Z_l(x)$ will be just a spherical zonal function. Otherwise, we try to generate a matrix G with diagonally dominant entries, i.e., we want to choose points η_i so that G is diagonal.

Therefore, we try to minimize the upper half of the matrix G, i.e, we try to minimize the following objective function

$$F(\eta_1, \ldots, \eta_{2k+1}) = \sum_{i<j} (2k+1)^2 P_k(\langle \eta_i, \eta_j \rangle)^2.$$

Projected Gradient Descent Method. In this method, we choose a point η_l to change in order to minimize the objective functions. The new point will be

$$new\, \eta_l(t) = \cos t\, \eta_l + \sin t\, \vec{w},$$

where $\vec{w}$ is a tangent vector to S^2 at the point η_l. We define

$$G_l(t) = F(\eta_1, \ldots, \cos t\, \eta_l + \sin t\, \vec{w}, \ldots, \eta_{2k+1}),$$

and minimize $G_l(t)$ in terms of t. The vector $\vec{w}$ should be chosen in a direction of steepest descent of G_l at the point η_l on the sphere S^2. We then repeat the same process for other points and iterate until convergence.

Lemma

The direction of the tangent vector is as follows

$$\vec{w} = 2\sum_{l \neq j} (2k+1)^2 P_k(\langle \eta_l, \eta_j \rangle) P_k'(\langle \eta_l, \eta_j \rangle) \left(-\eta_j + \langle \eta_j, \eta_l \rangle \eta_l\right).$$

Lemma (De Bie., Sommen, Wutzig, 2016)

Let $P_k^i(x)$ be the basis for k, $\mathcal{M}_l^+(k)$. The reproducing kernel for the left solid inner spherical monogenics of order k, $\mathcal{M}_l^+(k)$ with dimension $d_{k,m}$ is

$$\begin{aligned} K_k(x,y) &= \sum_{i=1}^{d_{k,m}} P_k^i(x)\overline{P_k^i(y)} \\ &= \frac{2\mu+k}{2\mu}|x|^k|y|^k C_k^{\mu}(t) + (x\wedge y)|x|^{k-1}|y|^{k-1}C_{k-1}^{\mu+1}(t), \end{aligned}$$

where $\mu = \frac{m}{2}-1$ and $t = \frac{\langle x,y\rangle}{|x||y|}$, and $C_k^{\mu}(t)$ is the Gegenbauer polynomials defined on the line.

Definition

For any x, $y \in S^{d-1}$ the spherical monogenic zonal function can be defined as

$$Z(x) := K_k(x, y),$$

where $K_k(x, y)$ is the reproducing kernel.

Definition

Let $A \in M_n(\mathbb{R}_2)$, i.e., an $n \times n$ quaternionic matrix. We can write the matrix $A = A_1 + A_2 j$ where A_1, and, A_2 are in $M_{n \times n}(\mathbb{C})$. We define the complex adjoint matrix A as

$$\chi(A) := \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}.$$

Finding orthonormal basis for $\mathcal{M}_l^+(k)$: The collection of $\{K_k(x,\eta_i)\}_{i=1}^{k+1}$ are basis for $\mathcal{M}_l^+(k)$, for almost every choice points $\eta_1, \cdots, \eta_{k+1} \in S^2$. We want to obtain an orthonormal basis for $\mathcal{M}_l^+(k)$. Therefore, we define

$$Z_t(x) := \sum_{j=1}^{k+1} K_k(x,\eta_j) a_{jt},$$

where $K_k(x,y)$ is the reproducing kernel and $t = 1 \cdots (k+1)$. The coefficient matrix $A = [a_{jt}]$ defined above can be obtained as follows

$$A = G^{-\frac{1}{2}},$$

where $G := [G_{ij}] = (k+1) C_k^{\frac{1}{2}}(\langle \eta_i, \eta_j \rangle) + (\eta_i \wedge \eta_j) C_{k-1}^{\frac{3}{2}}(\langle \eta_i, \eta_j \rangle)$.

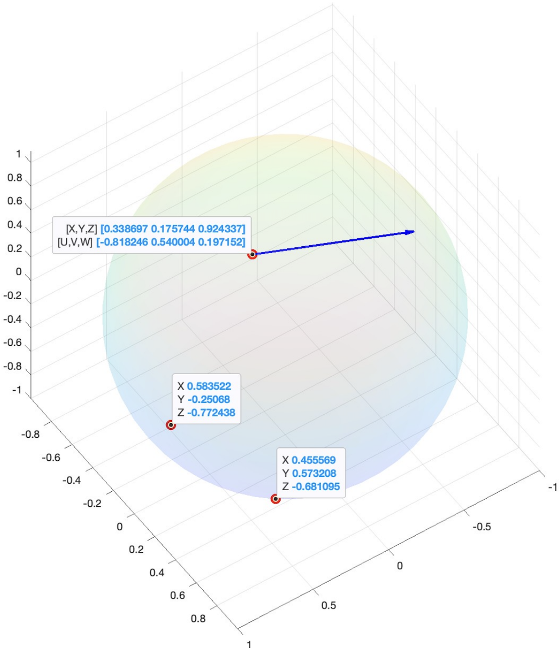
Since η_i and η_j can be chosen randomly on the sphere so we will try to obtain the matrix A such that most of the energy of the matrix are on the diagonal elements. Therefore, we will try to minimize the upper half of the matrix G, i.e. we try to minimize the following objective function

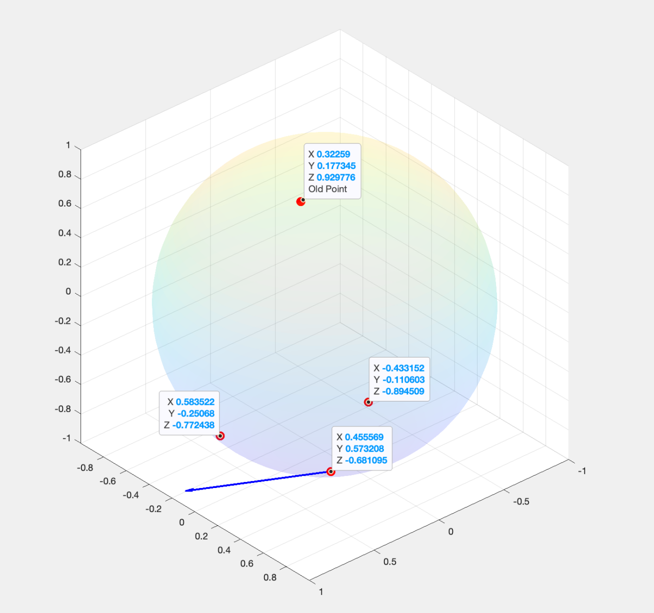
$$F = \sum_{i<j} [(k+1)^2 |C_k^{\frac{1}{2}}(\langle \eta_i, \eta_j \rangle)|^2 + |(\eta_i \wedge \eta_j)|^2 |C_{k-1}^{\frac{3}{2}}(\langle \eta_i, \eta_j \rangle)|^2].$$

Lemma (B.G., Hogan, Lakey)

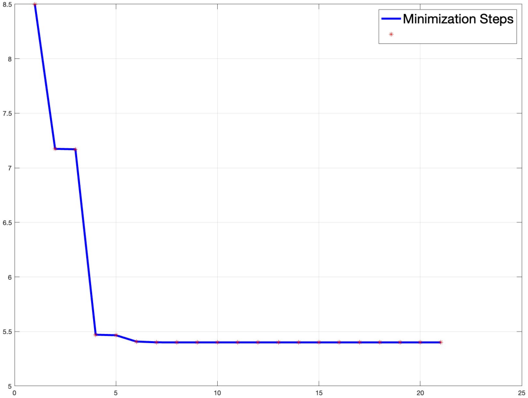
The direction of the tangent vector is as follows

$$w = -\sum_{l \neq j} \Big[(k+1)^2 C_k^{\frac{1}{2}} \langle \eta_l, \eta_j \rangle (C_k^{\frac{1}{2}})' \langle \eta_l, \eta_j \rangle \eta_j \\ - \langle \eta_l, \eta_j \rangle \eta_j (C_k^{\frac{3}{2}} \langle \eta_l, \eta_j \rangle)^2 \\ + (1 - (\langle \eta_l, \eta_j \rangle)^2) C_k^{\frac{3}{2}} \langle \eta_l, \eta_j \rangle (C_k^{\frac{3}{2}})' (\langle \eta_l, \eta_j \rangle) \eta_j \Big].$$


[X,Y,Z] [0.338697 0.175744 0.924337]
[U,V,W] [-0.818246 0.540004 0.197152]
X 0.583522
Y -0.25068
Z -0.772438
X 0.455569
Y 0.573208
Z -0.681095
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
-1
-0.5
0
0.5


X 0.32259
Y 0.177345
Z 0.929776
Old Point
X -0.433152
Y -0.110603
Z -0.894509
X 0.583522
Y -0.25068
Z -0.772438
X 0.455569
Y 0.573208
Z -0.681095
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
-1
-0.5
0
0.5

The amount of Objective Function is $F = 5.399999$.



One Example with $k = 2$

Now we are able to calculate the matrix G and as a result A. From the definition of G in the entries of G is in the even subalgebra $\mathbb{R}_3^0 \oplus \mathbb{R}_3^2$ of $\mathbb{R}_3$. Before proceeding further and computing the matrix A, we need to introduce another isomorphism. Let $x = x_0 + x_{12}e_{12} + x_{13}e_{13} + x_{23}e_{23} \in \mathbb{R}_3^0 \oplus \mathbb{R}_3^2$ in which $x_0, x_{12}, x_{13}, x_{23} \in \mathbb{R}$. We define the mapping $\tau : \mathbb{R}_3^0 \oplus \mathbb{R}_3^2 \to \mathbb{H}$ by $\tau(x) = y$ where $y = x_0 + x_{12}k + x_{13}j + x_{23}i$ where i, j, k are the imaginary units in the quaternions. Then τ is $*-$isomorphism. Since G is positive semidefinite $\tau(G)$ is also semidefinite.

Theorem (B.G., Hogan, Lakey)

The coefficient matrix $A = [a_{jl}]$ *can be computed as follows*

$$A = \tau^{-1}\left(\chi^{-1}\left(\left(\chi(\tau G)\right)^{-\frac{1}{2}}\right)\right),$$

where χ *is the complex adjoint matrix.*

One Example with $k = 2$

$$Z_1(x) = K_2(x, \eta_1)a_{11} + K_2(x, \eta_2)a_{21} + K_2(x, \eta_3)a_{31}$$

Where

$a_{11} = (0.5281) + (0.0347)e_{12} + (0.0251)e_{13} + (-0.0140)e_{23},$
$a_{21} = (0.0332) + (0.0354)e_{12} + (0.1147)e_{13} + (0.0204)e_{23},$
$a_{31} = (0.0449) + (0.0795)e_{12} + (0.0410)e_{13} + (-0.0280)e_{23}.$

Respectively we can obtain $Z_2(x)$ and $Z_3(x)$.

Now in dimnesion $m \geq 4$. We see that

$$G_{ij} = K_k(\eta_i, \eta_j) = \frac{(k+m-2)}{(m-2)} C_k^{\mu}(\langle \eta_i, \eta_j \rangle) + (\eta_i \wedge \eta_j) C_{k-1}^{\mu+1}(\langle \eta_i, \eta_j \rangle).$$

From the definition, the entries of G is in the even subalgebra $\mathbb{R}_m^+$ of $\mathbb{R}_m$.

To compute the matrix coefficient A, we require

- A homomorphism from $\mathbb{R}_{m-1}$ to $\mathbb{R}_m^+$.
- A monomorphism from $M_n(\mathbb{R}_m)$ to $M_{2n}(\mathbb{R}_m^+)$.

In [3] page 64 the isomorphism $\psi : \mathbb{R}_{m-1} \to \mathbb{R}_m^+$ is introduced. Let $u \in \mathbb{R}_{m-1}$ as $u = u_e + u_o$. The isomorphism is given by $\psi(u) = \psi(u_e + u_o) = u_e + e_m u_o$.
So $\psi^{-1} : \mathbb{R}_m^+ \to \mathbb{R}_{m-1}$, let $v \in \mathbb{R}_m^+$ be written as $v = v' + e_m v''$, where $v' \in \mathbb{R}_m^+$ includes all components of v except e_m, and, $v'' \in \mathbb{R}_m^-$ includes all components of v except e_m. The inverse isomorphism $\psi^{-1} : \mathbb{R}_m^+ \to \mathbb{R}_{m-1}$ is defined as

$$\psi^{-1}(v) = v' + v''.$$

Theorem (3)

Let $G \in M_n(\mathbb{R}_m^+)$. We can define $\tilde{\tau} : M_n(\mathbb{R}_m^+) \to M_n(\mathbb{R}_{m-1})$ by $\tilde{\tau}(G) := [\tilde{\tau}(G)]_{ij} = \psi^{-1}(G_{ij})$.

The construction of τ can be as follows

$$\tau_3(e_0) = e_0$$

$$\tau_3(e_{12}) = e_1 \quad \tau_3(e_{13}) = e_2$$
$$\tau_3(e_{23}) = e_{12}$$

which is the way of construction implying $\tau_3 : \mathbb{R}_3^+ \to \mathbb{R}_2$ is an isomorphism.

The construction of τ can be as follows

$$\tau_4(e_0) = e_0$$

$$\tau_4(e_{12}) = e_1 \quad \tau_4(e_{13}) = e_2 \quad \tau_4(e_{14}) = e_3$$
$$\tau_4(e_{23}) = e_{12} \quad \tau_4(e_{24}) = e_{13}$$
$$\tau_4(e_{34}) = e_{23}$$

$$\tau_4(e_{1234}) = e_{123}.$$

which is the way of construction implying $\tau_4 : \mathbb{R}_4^+ \to \mathbb{R}_3$ is an isomorphism.

The construction of τ can be as follows

$$\tau_m(e_0) = e_0$$

$$\tau_m(e_{12}) = e_1 \quad \tau_m(e_{13}) = e_2 \quad \ldots \tau_m(e_{1\,(m-1)}) = e_{m-2} \quad \tau_m(e_{1m}) = e_{m-1}$$

$$\tau_m(e_{23}) = e_{12} \quad \tau_m(e_{24}) = e_{13} \quad \ldots \quad \tau_m(e_{2m}) = e_{1(m-1)}$$

$$\tau_m(e_{34}) = e_{23} \quad \ldots \quad \tau_m(e_{3m}) = e_{2(m-1)}$$

$$\vdots$$

$$\tau_m(e_{(m-1)m}) = e_{(m-2)(m-1)}.$$

$$\tau_m(e_{1234}) = e_{123} \ldots$$

$$\tau_m(e_{(m-3)\,(m-2)(m-1)m}) = \tau_m(e_{1(m-3)})\tau_m(e_{1(m-2)})\tau_m(e_{1(m-1)})\tau_m(e_{1m})$$

$$\vdots$$

Lemma (B.G., Hogan, Lakey)

For any $b \in \mathbb{R}_m^+$, *we define* $\varphi : \mathbb{R}_m^+ \to \mathbb{R}_m^+$ *by* $\varphi(b) := \overline{e_{23\ldots m}}\, b\, e_{23\ldots m}$. *Then*

(i) φ *is an isomorphism from* $\mathbb{R}_m^+$ *to itself,*

(ii) $\varphi(b)e_1 = e_1 b$, *and,* $e_1\varphi(b) = e_1 b$, *for all* $b \in \mathbb{R}_m^+$,

(iii) $\varphi^2(b) = b$, *for all* $b \in \mathbb{R}_m^+$.

For the construction, we can show that φ satisfies

$$\varphi(e_{k_1 k_2 \ldots k_{2l}}) = \begin{cases} e_{k_1 k_2 \ldots k_{2l}} & \text{if} \quad k_1 \neq 1, \\ -e_{k_1 k_2 \ldots k_{2l}} & \text{if} \quad k_1 = 1, \end{cases} \qquad (2)$$

with $k_1 < k_2 < \cdots < k_{2l}$.

By the isomorphism defined in the previous slide, we can define the matrix version of φ, i.e., $\tilde{\varphi} : M_n(\mathbb{R}_m^+) \to M_n(\mathbb{R}_m^+)$ as $\tilde{\varphi}(A^+) := [\varphi(a^+)]_{ij}$ where a^+s are the elements of A^+.

Theorem (B.G., Hogan, Lakey)

There is a monomorphism, $\chi_m(A) := \begin{bmatrix} A^+ & \tilde{A}^+ \\ -\tilde{\varphi}(\tilde{A}^+) & \tilde{\varphi}(A^+) \end{bmatrix}$*, from* $M_n(\mathbb{R}_m)$ *into* $M_{2n}(\mathbb{R}_m^+)$*.*

BHL Algorithm for dimension m for calculating coefficient matrix A

1 Choose $\{\eta_i\}_{i=1}^{d_k^m}$ points on the sphere S^{m-1}.

2 We then move these points using the method of projected gradient descent on the sphere to minimize

$$F(\eta_1, \ldots, \eta_{d_k^m}) = \sum_{i<j} \left[\left(\frac{(k+m-2)}{(m-2)} \right)^2 C_k^{\mu}(\langle \eta_i, \eta_j \rangle)^2 + (1 - (\langle \eta_i, \eta_j \rangle)^2) C_{k-1}^{\mu+1}(\langle \eta_i, \eta_j \rangle)^2 \right]. \tag{3}$$

3 Calculate the matrix $G \in M_n(\mathbb{R}_m^+)$ defined in which the elements are defined in (3).

4 Apply the operator $\tilde{\tau} : M_n(\mathbb{R}_m^+) \to M_n(\mathbb{R}_{m-1})$ on G. If $H := \tilde{\tau}(G)$ is a complex matrix then go to step (6) otherwise go to step (5).

5 Apply the operator $\chi_{m-1} : M_n(\mathbb{R}_{m-1}) \to M_{2n}(\mathbb{R}_{m-1}^+)$ on H. Then go to step (3).

6 Calculate the matrix

$$A = \tilde{\tau}^{-1} \circ \chi_{m-1}^{-1} \circ \tilde{\tau}^{-1} \circ \cdots \circ \chi_2^{-1} \circ \tilde{\tau}^{-1}\left(B^{-\frac{1}{2}}\right), \quad (4)$$

where $B = H$ is the complex matrix from step 4.

For $k = 2$ and dimension $m = 4$ we obtain the 6 monogenic homogeneous functions. By any optimization methods we see that the best optimal points on the sphere are

$$\begin{aligned}
\eta_1 &= (-0.3055, -0.3675, -0.1685, -0.8621),\\
\eta_2 &= (-0.3916, 0.0379, -0.5478, 0.7383),\\
\eta_3 &= (-0.1693, -0.5860, -0.7911, -0.0449),\\
\eta_4 &= (0.6053, -0.3942, -0.4204, 0.5491),\\
\eta_5 &= (-0.6560, 0.3052, 0.4443, 0.5283),\\
\eta_6 &= (0.9383, 0.3275, 0.1098, 0.0166).
\end{aligned}$$

The objective function of those points takes the value $F(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6) = 40.4376$. Therefore,

Therefore, by $Z_j(x) = \sum_{i=1}^{6} K_2(x, \eta_i) a_{ij}$ we can see for instance, a_{i1}'s

$$a_{11} = (0.5)\, e_0 + (-0.0001751)\, e_{1234},$$

$$\begin{aligned} a_{21} = {} & (0.00456)\, e_0 + (0.01032)\, e_{12} + (-0.01597)\, e_{13} \\ & + (0.04519)\, e_{14} + (-0.02264)\, e_{23} + (0.02916)\, e_{24} + (0.05645)\, e_{34} \\ & + (-0.001705)\, e_{1234}, \end{aligned}$$

$$\begin{aligned} a_{31} = {} & (0.005577)\, e_0 + (0.0119)\, e_{12} + (0.02261)\, e_{13} + (-0.0159)\, e_{14} \\ & + (0.01897)\, e_{23} + (-0.04864)\, e_{24} + (-0.06711)\, e_{34} \\ & + (-0.0001817)\, e_{1234}, \end{aligned}$$

$$
\begin{aligned}
a_{41} = (0.001499)\, e_0 + (-0.02649)\, e_{12} + (-0.02269)\, e_{13} \\
+ (-0.02975)\, e_{14} + (-0.005987)\, e_{23} + (0.0501)\, e_{24} + (0.04863)\, e_{34} \\
+ (-0.0009728)\, e_{1234},
\end{aligned}
$$

$$
\begin{aligned}
a_{51} = (0.003591)\, e_0 + (0.02954)\, e_{12} + (0.01926)\, e_{13} + (0.07882)\, e_{14} \\
+ (0.007936)\, e_{23} + (-0.002199)\, e_{24} + (-0.02181)\, e_{34} \\
+ (0.0009455)\, e_{1234},
\end{aligned}
$$

$$
\begin{aligned}
a_{61} = (0.005036)\, e_0 + (-0.02784)\, e_{12} + (-0.01157)\, e_{13} \\
+ (-0.08013)\, e_{14} + (-0.003863)\, e_{23} + (-0.01768)\, e_{24} \\
+ (0.002964)\, e_{34} + (0.001643)\, e_{1234}.
\end{aligned}
$$

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