Example 19. *Define a harmonic function and conjugate harmonic function. Find the harmonic conjugate function of the function $U(x, y) = 2x(1 - y)$.* (*U.P., III Semester Dec. 2009*)

Solution. See Art. 4.15

Here, we have $U(x, y) = 2x(1 - y)$. Let V be the harmonic conjugate of U.

By total differentiation

$$\left[\begin{array}{l} U = 2x - 2xy \\ \dfrac{\partial U}{\partial x} = 2 - 2y \\ \dfrac{\partial U}{\partial y} = -2x \end{array}\right]$$

$$\begin{aligned} dV &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy \\ &= -\frac{\partial U}{\partial y}dx + \frac{\partial U}{\partial x}dy \\ &= -(-2x)\,dx + (2 - 2y)\,dy + C \\ &= 2x\,dx + (2\,dy - 2y\,dy) + C \\ V &= x^2 + 2y - y^2 + C \end{aligned}$$

Hence, the harmonic conjugate of U is $x^2 + 2y - y^2 + C$ **Ans.**

Example 24. *An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function.* (*R.G.P.V., Bhopal, III Semester, Dec. 2001*)

Solution. Let $\psi(x, y)$ be a stream function

We know that $$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = \left(-\frac{\partial \phi}{\partial y}\right)dx + \left(\frac{\partial \phi}{\partial x}\right)dy \qquad \text{[C-R equations]}$$

$$\begin{aligned} &= \{-(3x^2 - 3y^2)\}\,dx + 6xy\,dy \\ &= -3x^2\,dx + (3y^2\,dx + 6xy\,dy) \\ &= -d(x^3) + 3d(xy^2) \\ \psi &= \int -d(x^3) + 3d(xy^2) + c \\ \psi &= -x^3 + 3xy^2 + c \end{aligned}$$

ψ is the required stream function. **Ans.**

Example 23. *If* $w = \phi + i\psi$ *represents the complex potential for an electric field and*

$$\psi = x^2 - y^2 + \frac{x}{x^2 + y^2},$$

determine the function ϕ.

Solution. $w = \phi + i\psi$ and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$

$$\frac{\partial \psi}{\partial x} = 2x + \frac{(x^2 + y^2)\cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2y - \frac{x(2y)}{(x^2 + y^2)^2} = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

We know that, $d\phi = \dfrac{\partial \phi}{\partial x} dx + \dfrac{\partial \phi}{\partial y} dy = \dfrac{\partial \psi}{\partial y} dx - \dfrac{\partial \psi}{\partial x} dy$

$$= \left(-2y - \frac{2xy}{(x^2 + y^2)^2}\right) dx - \left(2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right) dy$$

$$\phi = \int \left[-2y - \frac{2xy}{(x^2 + y^2)^2}\right] dx + c$$

This is an exact differential equation.

$$\phi = -2xy + \frac{y}{x^2 + y^2} + C$$ **Ans.**

Which is the required function.

Example 34. *Show that e^x (x cos y – y sin y) is a harmonic function. Find the analytic function for which e^x (x cos y – y sin y) is imaginary part.*

(*U.P., III Semester, June 2009, R.G.P.V., Bhopal, III Semester, June 2004*)

Solution. Here $v = e^x (x \cos y - y \sin y)$

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial v}{\partial x} = e^x (x \cos y - y \sin y) + e^x \cos y = \psi_2 (x, y), \quad \text{(say)} \quad \ldots (1)$$

$$\frac{\partial v}{\partial y} = e^x (-x \sin y - y \cos y - \sin y) = \psi_1 (x, y) \quad \text{(say)} \quad \ldots (2)$$

$$\frac{\partial^2 v}{\partial x^2} = e^x (x \cos y - y \sin y) + e^x \cos y + e^x \cos y$$

$$= e^x (x \cos y - y \sin y + 2 \cos y) \quad \ldots (3)$$

and $$\frac{\partial^2 v}{\partial y^2} = e^x (-x \cos y + y \sin y - 2 \cos y) \quad \ldots (4)$$

Adding equations (3) and (4), we have

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \Rightarrow v \text{ is a harmonic function.}$$

Now putting $x = z$, $y = 0$ in (1) and (2), we get

$$\psi_2 (z, 0) = z e^z + e^z \qquad \psi_1 (z, 0) = 0$$

Hence by Milne-Thomson method, we have

$$f(z) = \int [\psi_1 (z, 0) + i \psi_2 (z, 0)] \, dz + C$$

$$= \int [0 + i(z e^z + e^z)] \, dz + C = i(z e^z - e^z + e^z) + C = i z e^z + C.$$

This is the required analytic function. **Ans.**

9.2. Show that the functions of Problem 9.1 are harmonic in the w plane under the transformation $z = w^3$.

Solution

Suppose $z = w^3$. Then $x + iy = (u + iv)^3 = u^3 - 3uv^2 + i(3u^2 v - v^3)$ and $x = u^3 - 3uv^2$, $y = 3u^2 v - v^3$.

(a) $\Phi = x^2 - y^2 + 2y = (u^3 - 3uv^2)^2 - (3u^2 v - v^3)^2 + 2(3u^2 v - v^3)$

$= u^6 - 15u^4 v^2 + 15u^2 v^4 - v^6 + 6u^2 v - 2v^3$

Then $\partial^2 \Phi / \partial u^2 = 30u^4 - 180u^2 v^2 + 30v^4 + 12v$, $\partial^2 \Phi / \partial v^2 = -30u^4 + 180u^2 v^2 - 30v^4 - 12v$ and $(\partial^2 \Phi / \partial u^2) + (\partial^2 \Phi / \partial v^2) = 0$ as required.

(b) We must show that $\Phi = \sin(u^3 - 3uv^2) \cosh(3u^2 v - v^3)$ satisfies $(\partial^2 \Phi / \partial u^2) + (\partial^2 \Phi / \partial v^2) = 0$. This can readily be established by straightforward but tedious differentiation.

This problem illustrates a general result proved in Problem 9.4.

Example 5.1

A patient is given a drip feed containing a particular chemical and its concentration in his blood is measured, in suitable units, at one hour intervals for the next five hours. The doctors believe the figures to be subject to random errors, arising both from the sampling procedure and the subsequent chemical analysis, but that a linear model is appropriate.

Table 5.3

Time, x (hours)	0	1	2	3	4	5
Concentration, y	2.4	4.3	5.2	6.8	9.1	11.8

Note

You can see from the intervals that x is a controlled variable.

(i) Find the equation of the regression line of y on x.
(ii) Illustrate the data and your regression line on a scatter diagram.
(iii) Estimate the concentration of the chemical in the patient's blood at
  (a) 3½ hours,
  (b) 10 hours after treatment started.

Comment on the likely accuracy of your predictions.

(iv) Calculate the residuals for each data pair. Check that the sum of the residuals is zero and find the sum of the squares of the residuals.

Solution

(i) $n = 6$

Table 5.4

x	y	x^2	xy
0	2.4	0	0.0
1	4.3	1	4.3
2	5.2	4	10.4
3	6.8	9	20.4
4	9.1	16	36.4
5	11.8	25	59.0
15	39.6	55	130.5

$$\bar{x} = \frac{\Sigma x}{n} = \frac{15}{6} = 2.5 \quad \bar{y} = \frac{\Sigma y}{n} = \frac{39.6}{6} = 6.6$$

$$S_{xx} = \Sigma x^2 - n\bar{x}^2 = 55 - 6 \times 2.5^2 = 17.5$$

$$S_{xy} = \Sigma xy - n\overline{xy} = 130.5 - 6 \times 2.5 \times 6.6 = 31.5$$

$$b = \frac{S_{xy}}{S_{xx}} = \frac{31.5}{17.5} = 1.8$$

Hence the least squares regression line is given by

$$y - \bar{y} = b(x - \bar{x})$$

$$y - 6.6 = 1.8(x - 2.5)$$

$$y = 2.1 + 1.8x$$

(ii)

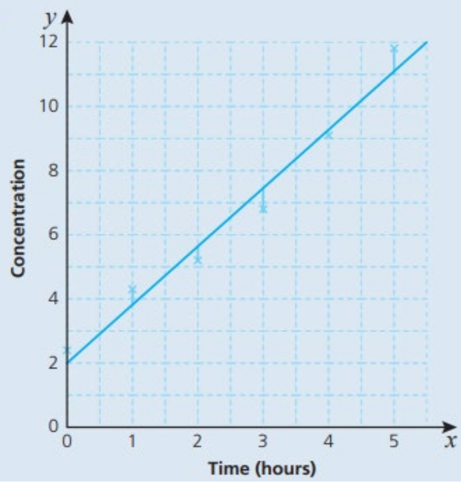

Figure 5.3

(iii) When $x = 3.5$,

$y = 2.1 + 1.8 \times 3.5$
$= 8.4$.

When $x = 10$,

$y = 2.1 + 1.8 \times 10$
$= 20.1$.

The concentration of 8.4 lies between the measured values of 6.8 at time 3 hours and 9.1 at time 4 hours, so the prediction seems quite reasonable. The time ten hours is a long way outside the set of data times; there is no indication that the linear relationship can be extrapolated to such a time, even though there seems to be a good fit, so the prediction is probably unreliable.

> **Note**
>
> In this case, the value of x from which you want to predict the value of y lies within the range of data values.
>
> This is an example of *interpolation* (meaning literally within the points).

> **Note**
>
> In this second case, the value of x from which you want to predict the value of y lies outside the range of data values.
>
> This is an example of *extrapolation* (meaning literally outside the points).

(iv) For each pair of data, (x, y), the corresponding point on the regression line is denoted by $(x, \hat{y})$. So the predicted value is $\hat{y} = 2.1 + 1.8x$. Corresponding values of y and $\hat{y}$ are tabulated below, together with the residuals and their squares.

> **Note**
>
> So the actual data point (x,y) is in the same vertical line as $(x,\hat{y})$ on scatter diagram.

Table 5.5

x	y	$\hat{y}$	$y - \hat{y}$	$(y - \hat{y})$
0	2.4	2.1	0.3	0.09
1	4.3	3.9	0.4	0.16
2	5.2	5.7	-0.5	0.25
3	6.8	7.5	-0.7	0.49
4	9.1	9.3	-0.2	0.04
5	11.8	11.1	0.7	0.49
15	39.6	39.6	0.0	1.52

You can see that the sum of the residuals, $\Sigma(y - \hat{y})$ is zero, and that the sum of the squares of the residuals, $\Sigma(y - \hat{y})^2$, is 1.52.

The number of runs scored by two batsmen A and B in different innings are as follows.

A	12	115	6	73	7	19	119	36	84	29
B	47	12	76	42	4	51	37	48	13	0

(i) Who is better run scorer?

(ii) Who is more consistant?

Let x_1 and x_2 be the runs of batsmen A and B.

x_1	x_1^2	x_2	x_2^2
12	144	47	2209
115	13225	12	144
6	36	76	144
73	5329	42	5776
7	49	4	16
19	361	51	2601
119	14161	37	1369
6	1296	48	2304
84	7059	13	169
29	841	0	0
500	42498	330	14588

$$n_1 = n_2 = 10,\ \bar{X}_1 = \frac{500}{10} = 50;\ \bar{X}_2 = \frac{330}{10} = 33.$$

(i) Here $\bar{X}_1$(batsman A) > $\bar{X}_2$ (batsman B)

So batsman A is the better run scorer.

(ii) $$\sigma_1 = \sqrt{\frac{\Sigma X_1^2}{n} - \left(\frac{\Sigma X_1}{n}\right)^2}$$

$$= \sqrt{\frac{42498}{10} - \left(\frac{500}{10}\right)^2} = 41.8306$$

$$\sigma_2 = \sqrt{\frac{\Sigma X_2^2}{n} - \left(\frac{\Sigma X_2}{n}\right)^2}$$

$$= \sqrt{\frac{14588}{10} - \left(\frac{330}{10}\right)^2} = 19.2302$$

coefficient of variation $CV_1 = \frac{\sigma_1}{\bar{X}_1} \times 100$

$$= \frac{41.8306}{50} \times 100 = 83.66\%$$

$$CV_2 = \frac{\sigma_2}{\bar{X}_2} \times 100 = \frac{19.2302}{33} \times 100$$

$$= 58.27\%$$

Here CV_2 (batsman B) < CV_1, (batsman A)

∴ Batsman B is more consistant.

Example:

The following data give the hardness (X) and tensile strength (Y) of 7 samples of metal in certain units. Find the linear regression equation of Y on X.

X:	146	152	158	164	170	176	182
Y:	65	78	77	89	82	85	86

Solution:

Regression equation of Y on X is given by

$$Y = a+bX$$

The normal equations are:

$$\sum Y = Na + b\sum X \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\sum XY = a\sum X + b\sum X^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Calculation of regression equations

X	Y	X^2	Y^2	XY
146	75	21316	5625	10950
152	78	23104	6084	11856
158	77	24964	5929	12166
164	89	26896	7921	14596
170	82	28900	6724	13940
176	85	30976	7225	14960
182	86	33124	7396	15652
1148 $(=\sum X)$	572 $(=\sum Y)$	189280 $(=\sum X^2)$	46904 $(=\sum Y^2)$	94120 $(=\sum XY)$

Here, N = 7

Substituting the values in equations (1) and (2), we get

572 =7a +1148 b ………………………… (5)

94120 = 1148 a +189280 b ……………… (4)

Multiplying the equation (3) by 164, we get

93808 =1148 a +188272 b ………..(5)

Subtracting this equation from (4), we get

b = 0.31

Putting this value of b in equation (3), we have

$$572 = 7a + 1148 \times 0 \cdot 31$$
$$\Rightarrow 572 = 7a + 355 \cdot 88$$
$$\Rightarrow 7a = 572 - 355 \cdot 88$$
$$\Rightarrow 7a = 216 \cdot 12$$
$$\Rightarrow a = \frac{216 \cdot 12}{7} = 30 \cdot 87$$

$\therefore$ **The linear regression equation of Y on X is Y = 30.87 + 0.31X**