Contoure Integration-1

Evaluate by contour integration:

(i) $\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}$ (ii) $\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2}$, $a^2<1$

(iii) $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, $|a|<1$

(iv) $\int_0^{\pi} \frac{\sin^2\theta \, d\theta}{5-4\cos\theta}$ (v) $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$

Solution: (i) let $I = \int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}$

$$= \text{Real part of } \int_0^{2\pi} \frac{e^{i3\theta} \, d\theta}{5-4\cos\theta}$$

$$= \text{R P of } \int_0^{2\pi} \frac{e^{i3\theta} \, d\theta}{5-4\cdot\frac{1}{2}(e^{i\theta}+e^{-i\theta})}$$

Put $e^{i\theta} = z$

$\therefore e^{i\theta} \cdot i\, d\theta = dz$

or, $d\theta = \frac{dz}{iz}$

$\therefore I = \text{R.P. of } \oint_C \frac{z^3 \cdot \frac{dz}{iz}}{5-2(z+\frac{1}{z})}$, where C is the unit circle, $|z|=1$.

$$= \text{RP of } \frac{1}{i}\oint_C \frac{z^3 \, dz}{5z-2z^2-2}$$

$$= \text{RP of } \frac{-1}{i}\oint_C \frac{z^3 \, dz}{2z^2-5z+2}$$

$$= \text{RP of } \frac{-1}{i}\oint_C f(z)\,dz \quad \ldots (1)$$

where $f(z) = \frac{z^3}{2z^2-5z+2}$

poles of $f(z)$ are given by

$2z^2-5z+2=0$

or, $(z-2)(2z-1)=0$

$\therefore z = 2, \frac{1}{2}$

Only simple pole $z=\frac{1}{2}$ within C.

$1-a^2$

$= 0$

Residue at $z=\frac{1}{2}$ is

$$\lim_{z\to\frac{1}{2}} (z-\tfrac{1}{2})\cdot\frac{z^3}{2z^2-5z+2}$$

$$=\lim_{z\to\frac{1}{2}} (z-\tfrac{1}{2})\cdot\frac{z^3}{(z-2)(2z-1)}$$

$$=\lim_{z\to\frac{1}{2}} \cancel{(z-\tfrac{1}{2})}\cdot\frac{z^3}{2(z-2)\cancel{(z-\tfrac{1}{2})}}$$

$$=\frac{\frac{1}{8}}{2(\frac{1}{2}-2)}$$

$$=-\frac{1}{24}$$

By Cauchy's residue theorem we have,

$$\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{24}\right)$$

$$=-\frac{\pi i}{12}$$

$$\therefore\ I = \text{R.P. of } \frac{-1}{i}\oint_C f(z)\,dz$$

$$=\text{R.P. of } \frac{-1}{i}\cdot\left(-\frac{\pi i}{12}\right)$$

$$=\frac{\pi}{12}$$

(ii) let $I=\displaystyle\int_0^{\pi}\frac{\cos 2\theta\,d\theta}{1-2a\cos\theta+a^2}$

$$=\frac{1}{2}\int_0^{2\pi}\frac{\cos 2\theta\,d\theta}{1-2a\cos\theta+a^2}\quad\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x)=f(x)\right]$$

$$=\tfrac{1}{2}\text{R.P of }\frac{1}{2}\int_0^{2\pi}\frac{e^{i2\theta}\,d\theta}{1-2a\cos\theta+a^2}$$

$$=\text{RP of }\frac{1}{2}\int_0^{2\pi}\frac{e^{i2\theta}\,d\theta}{1-2a\cdot\frac{1}{2}(e^{i\theta}+e^{-i\theta})+a^2}$$

Put $e^{i\theta}=z \quad \therefore\ d\theta=\frac{dz}{iz}$

$$\therefore I = RP \text{ of } \frac{1}{2}\oint_C \frac{z^2 \cdot \frac{dz}{iz}}{1-a\left(z+\frac{1}{z}\right)+a^2}$$

$$= RP \text{ of } \frac{1}{2i}\oint_C \frac{z^2 dz}{z-az^2-a+a^2z}$$

$$= R.P \text{ of } \frac{-1}{2i}\oint_C \frac{z^2 dz}{az^2-a^2z-z+a}$$

$$= R.P. \text{ of } \frac{-1}{2i}\oint_C f(z)\,dz$$

where $f(z) = \dfrac{z^2}{az^2-a^2z-z+a}$

poles of $f(z)$ are given by

$$az^2-a^2z-z+a=0$$

$$\text{or, } (z-a)(az-1)=0$$

$$\therefore z = a, \frac{1}{a}$$

Only simple pole $z=a$ lies inside C.

Residue at $z=a$ is,

$$\lim_{z\to a} \cancel{(z-a)} \frac{z^2}{\cancel{(z-a)}(az-1)}$$

$$= \frac{a^2}{a^2-1}$$

Hence by Cauchy's residue theorem we get

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{a^2}{a^2-1}\right)$$

$$\therefore I = R.P. \text{ of } \frac{-1}{2i}\oint_C f(z)\,dz$$

$$= R.P. \text{ of } \frac{-1}{2i}\cdot 2\pi i\left(\frac{a^2}{a^2-1}\right)$$

$$= \frac{\pi a^2}{1-a^2}$$

(iii) Let $I = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}$

$$= \int_0^{2\pi} \frac{d\theta}{1 - a(e^{i\theta} + e^{-i\theta}) + a^2}$$

$$= \oint_C \frac{dz/iz}{1 - a(z + 1/z) + a^2}$$

Put $e^{i\theta} = z$

$\therefore d\theta = \frac{dz}{iz}$

Where C is the unit circle, $|z| = 1$.

$$= \frac{1}{i} \oint_C \frac{dz}{z - az^2 - a + a^2 z}$$

$$= \frac{-1}{i} \oint_C \frac{dz}{az^2 - a^2 z - z + a}$$

$$= \frac{-1}{i} \oint_C f(z)\,dz$$

where $f(z) = \frac{1}{az^2 - a^2 z - z + a}$

Poles of $f(z)$ are given by

$$az^2 - a^2 z - z + a = 0$$

or, $(z - a)(az - 1) = 0$

$\therefore z = a, 1/a$

Only simple pole $z = a$ lies inside C.

Residue at $z = a$ is $\lim_{z \to a} (z - a) \cdot \frac{1}{(z - a)(az - 1)}$

$$= \frac{1}{a^2 - 1}$$

Hence by Cauchy's residue theorem we get

$$\oint_C f(z)\,dz = 2\pi i \left(\frac{1}{a^2 - 1} \right)$$

$$\therefore I = -\frac{1}{i} \oint_C f(z)\,dz$$

$$= -\frac{1}{i} \cdot 2\pi i \left(\frac{1}{a^2 - 1} \right)$$

$$= \frac{2\pi}{1 - a^2}$$

(iv) Let $I = \int_0^{\pi} \frac{\sin^2\theta \, d\theta}{5-4\cos\theta}$

$= \frac{1}{2}\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{5-4\cos\theta}$ $\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x)=f(x)\right]$

$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{2\pi} \frac{(1-\cos 2\theta)\, d\theta}{5-4\cos\theta}$

$=$ R.P of $\frac{1}{4}\int_0^{2\pi} \frac{(1-e^{i2\theta})\, d\theta}{5-4\cos\theta}$

$=$ R.P of $\frac{1}{4}\int_0^{2\pi} \frac{(1-e^{i2\theta})\, d\theta}{5-2(e^{i\theta}+e^{-i\theta})}$

$=$ RP of $\frac{1}{4}\oint_C \frac{(1-z^2)\cdot\frac{dz}{iz}}{5-2(z+\frac{1}{z})}$, Put $e^{i\theta}=z$ $\therefore d\theta = \frac{dz}{iz}$

$=$ R.p of $\frac{1}{4i}\oint_C \frac{(1-z^2)dz}{5z-2z^2-2}$ where C is the unit circle, $|z|=1$.

$=$ R.P. of $\frac{-1}{4i}\oint_C \frac{(1-z^2)\, dz}{2z^2-5z+2}$

$=$ R.P. of $\frac{-1}{4i}\oint_C f(z)\, dz$

where $f(z) = \frac{1-z^2}{2z^2-5z+2}$

Poles of $f(z)$ are given by $2z^2-5z+2=0$
or, $(z-2)(2z-1)=0$
$\therefore z = 2, \frac{1}{2}$

Only simple pole $z=\frac{1}{2}$ lies inside C.

Residue at simple pole $z=\frac{1}{2}$ is

$\lim_{z\to\frac{1}{2}} (z-\frac{1}{2})\frac{1-z^2}{(z-2)(2z-1)}$

$= \lim_{z\to\frac{1}{2}} \frac{1-z^2}{2(z-2)}$

$= \lim_{z\to\frac{1}{2}} \frac{1-\frac{1}{4}}{1-4}$

$= -\frac{1}{4}$

By Cauchy's residue theorem we get

$\oint_C f(z)\, dz = 2\pi i\left(-\frac{1}{4}\right)$

$\therefore\ I = R.P.$ of $\frac{-1}{4i} \cdot 2\pi i\left(-\frac{1}{4}\right)$

$= \frac{\pi}{8}$

(v) Let $I = \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$

$= \int_0^{2\pi} \frac{d\theta}{5+3\cdot\frac{1}{2i}(e^{i\theta}-e^{-i\theta})}$

$= \oint_C \frac{dz/iz}{5+\frac{3}{2i}\left(z-\frac{1}{z}\right)}$ Put $e^{i\theta} = z$ $\therefore d\theta = \frac{dz}{iz}$

$= 2\oint_C \frac{dz}{10iz+3z^2-3}$

$= 2\oint_C f(z)\,dz$

where $f(z) = \frac{1}{3z^2+10iz-3}$

Poles of $f(z)$ are given by $3z^2+10iz-3=0$

or, $3z^2+10iz+i^2 3=0$

or, $3z^2+9iz+iz+i^2 3=0$

or, $3z(z+3i)+i(z+3i)=0$

or, $(z+3i)(3z+i)=0$

$\therefore z = -3i, -i/3$

since $|-i/3| = \frac{1}{3}$, so only simple pole $z=-i/3$ lies inside C.

Residue at simple pole $z = -\frac{i}{3}$ is

$\lim_{z\to -i/3} (z+i/3) \cdot \frac{1}{(z+3i)(3z+i)}$

$= \lim_{z\to -i/3} \frac{1}{3(z+3i)}$

$= \frac{1}{3(-i/3+3i)}$

$= \frac{1}{8i}$

Hence by Cauchy's residue theorem, we get

$\oint_C f(z)dz = 2\pi i\left(\frac{1}{8i}\right)$

$\therefore I = 2\cdot 2\pi i\left(\frac{1}{8i}\right)$

$= \frac{\pi}{2}$

Evaluate by contour integration: $\int_0^{2\pi} \frac{\sin 2\theta \, d\theta}{1-2a\cos\theta + a^2}, a^2<1$

Solution: Let $I = \int_0^{2\pi} \frac{\sin 2\theta \, d\theta}{1-2a\cos\theta+a^2}$

$= \text{imaginary part of } \int_0^{2\pi} \frac{e^{i2\theta} d\theta}{1-2a\cos\theta+a^2}$

$= \text{I.P. of } \int_0^{2\pi} \frac{e^{i2\theta} d\theta}{1-2a\cdot\frac{1}{2}(e^{i\theta}+e^{-i\theta})+a^2}$

Let us put $e^{i\theta} = z \quad \therefore e^{i\theta} i d\theta = dz$

or, $d\theta = \frac{dz}{iz}$

$\therefore I = \text{I.P. of } \oint_C \frac{z^2 \cdot \frac{dz}{iz}}{1-a(z+\frac{1}{z})+a^2}$, where C is the unit circle $|z|=1$.

$= \text{I.P. of } \frac{1}{i}\oint_C \frac{z^2 dz}{z-az^2-a+a^2z}$

$= \text{I.P. of } \frac{-1}{i}\oint_C \frac{z^2 dz}{az^2-z-a^2z+a}$

$= \text{I.P. of } \frac{-1}{i}\oint_C f(z)\,dz \quad \text{---} (1)$

where $f(z) = \frac{z^2}{az^2-z-a^2z+a}$

poles of $f(z)$ are given by

$az^2-z-a^2z+a=0$

or, $z(az-1)-a(az-1)=0$

or, $(az-1)(z-a)=0$

$\therefore z = \frac{1}{a}, a$

since $a^2<1$, so $z=a$ lies inside C.

Residue at simple pole $z=a$ is

$\lim_{z\to a} \cancel{(z-a)} \cdot \frac{z^2}{(az-1)\cancel{(z-a)}}$

$= \frac{a^2}{a^2-1}$

Hence by Cauchy's residue theorem, we get

$\oint_C f(z)\,dz = 2\pi i\left(\frac{a^2}{a^2-1}\right)$

So from (1) we get

$I = \text{I.P. of } \frac{-1}{i} \cdot 2\pi i\left(\frac{a^2}{a^2-1}\right)$

$= \text{I.P. of } \left(\frac{2\pi a^2}{1-a^2} + i\cdot 0\right)$

$= 0$