Random Variable and Probability Distribution

Def^n: A random variable is a function that associates a real number with each element in the sample space. We shall use a capital letter, say X, to denote a random variable and its corresponding small letter, x in this case, for one of its values.

1st coin

2nd coin	H	T
H	HH	HT
T	TH	TT

→ Sample space when two coins are tossed.

1st and 2nd coin

3rd coin	H H	H T	T H	T T
H	H H H	H T H	T H H	T T H
T	T H H	H T T	T H T	T T T

Now

$X \rightarrow$ represents the numbers of Head then

X :	0	1	2	3
$P(X=x)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

→ random variable

↓

Probability distribution.

Discreate Sample Space

If a sample space contains a finite number of possibilities or an unending sequence with as many element as there are whole numbers, it is called a discreate sample space.

Continuous Sample space:

If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a continuous space.

Probability Function:

The set of ordered pairs $(x, f(x))$ is a probability function of the discreate random variable X if, for each possible out come x,

1. $f(x) \geq 0$
2. $\sum_{x} f(x) = 1$
3. $P(X=x) = f(x)$

Cumulative distribution function:

The c.d.f $F(x)$ of a discreate random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t),$$

for $-\infty < x < \infty$.

Ex: 3.8
P-104

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers find the probability distribution.

Sol^n: Let $X \to$ be a random variable whose values x are the possible numbers of defective computers purchased by the school. Then x can be any numbers 0, 1 and 2. Now

we have $f(x) = P(X=x)$

$$f(0) = P(X=0) = \frac{\binom{3}{0}\binom{5}{2}}{\binom{8}{2}} = \frac{10}{28}$$

$$f(1) = P(X=1) = \frac{\binom{3}{1}\binom{5}{1}}{\binom{8}{2}} = \frac{15}{28}$$

$$f(2) = P(X=2) = \frac{\binom{3}{2}\binom{5}{0}}{\binom{8}{2}} = \frac{3}{28}$$

Thus the Probability distribution of X is

x	0	1	2
$f(x) = P(X=x)$	$\frac{10}{28}$	$\frac{15}{28}$	$3/28$

Here $F(1) = P(X \le 1) = f(0) + f(1)$

$$= \frac{10}{28} + \frac{15}{28}$$

$$= \frac{25}{28}$$

Ex: Find cdf of the random variable X.

x	0	1	2	3	4
$P(X=x)$	$1/16$	$1/4$	$3/8$	$1/4$	$1/16$

$F(0) = f(0) = \frac{1}{16}$

$F(1) = f(0) + f(1) = \frac{5}{16}$

$F(2) = f(0) + f(1) + f(2) = \frac{11}{16}$

$F(3) = F(2) + f(3) = \frac{15}{16}$

$F(4) = F(3) + f(4) = \frac{15}{16} + \frac{1}{16} = 1$

$$F(x) = \begin{cases} 0 \; ; & x < 0. \\ \frac{1}{16} \; ; & 0 \le x < 1 \\ \frac{5}{16} \; ; & 1 \le x < 2 \\ \frac{11}{16} \; ; & 2 \le x < 3 \\ \frac{15}{16} \; ; & 3 \le x < 4 \\ 1 \; ; & x \geq 4 \end{cases}$$

$f(2) = F(2) - F(1)$

$= \frac{11}{16} - \frac{5}{16}$

$= \frac{3}{8}$

3.13 The probability distribution X, the numb[er of] imperfections per 10 meters of a synthetic fabri[c] in continuous rolls of uniform width, is giv[en] by

x	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

Construct the cdf of X.

$F(0) = f(0) = 0.41$

$F(1) = f(0) + f(1) = 0.41 + 0.37 = 0.78$

$\vdots$

**6. Two tetrahedral dice have the number 1, 2, 3, 4 on their faces. The dice are thrown together. Let S = the sum of their two scores and let D = the difference between their two scores.

a) Show that $P(S=6) = \frac{3}{16}$

b) Find the probability distribution for the random variable S and D.

c) Find $P(S \le 7)$

d) Show that $P(D=1) = 3/8$

e) Find $P(D \geq 2)$.

For S:

2nd Dice \ 1st Dice	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

For D:

2nd dice \ 1st dice	1	2	3	4
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

a) $P(S=6) = \frac{3}{16}$ ✓

b)

s	2	3	4	5	6	7	8
$P(S=s)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

d	0	1	2	3
$P(D=d)$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{2}{16}$

c) $P(S \le 7)$

$= 1 - \frac{1}{16}$

$= \frac{15}{16}$

d) $P(D \geq 2)$

$= \frac{4}{16} + \frac{2}{16}$

$= \frac{6}{16}$

10. The discreate random variable X has c.d.f $F(x)$ defined by

$$F(x) = \frac{2+x}{7} \; ; \; x = 1, 2, 3, 4 \; \& \; 5$$

a) Find $P(X \leq 3)$

b) Show that $P(X=4) = 1/7$

c) Find the Probability distribution for X.

a) Given $F(x) = \frac{2+x}{7}$

$$F(1) = \frac{3}{7} = f(1)$$

$$\Rightarrow F(2) = f(1) + f(2)$$

$$\Rightarrow \frac{4}{7} - 3/7 = f(2)$$

$$\Rightarrow f(2) = 1/7$$

and $f(3) = F(3) - F(2) = \frac{5}{7} - \frac{4}{7} = 1/7$

a) $P(X \leq 3) = F(3) = \frac{2+3}{7} = \frac{5}{7}$

b) $P(X=4) = f(4) = F(4) - F(3)$

$$= \frac{2+4}{7} - \frac{2+3}{7}$$

$$= 1/7$$

c)

x	1	2	3	4	5
$P(X=x)$	$\frac{3}{7}$	$1/7$	$1/7$	$1/7$	$1/7$

12) The discreate random variable X has c.d.f. $F(x)$ defined by

$$F(x) = \frac{(x+k)^2}{16}; \quad x = 1, 2, 3$$

a) Find the value of k.

b) Find the probability distribution for k.

we have

$$F(x_0) = \sum_{x \le x_0} P(X=x)$$

$$\Rightarrow F(3) = \frac{(3+k)^2}{16}$$

$$\Rightarrow 1 = \frac{(3+k)^2}{16}$$

$$\Rightarrow k+3 = \pm 4$$

$$\underline{k=1}$$

b)

$$F(1) = \frac{(1+1)^2}{16} = \frac{4}{16} = f(1)$$

$$F(2) = \frac{3^2}{16} = \frac{9}{16}$$

$$f(2) = F(2) - F(1) = \frac{5}{16}$$

$$f(3) = F(3) - F(2) = 1 - \frac{9}{16} = 7/16$$

x	1	2	3
$P(X=x$	$\frac{4}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

9. The random variable X has the following distribution

x:	1	2	3
$P(X=x)$:	a	b	a

write down $E(X)$.

$$1 \quad 2 \quad 3$$
$$\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4}$$
$$\frac{1}{4} + 1 + \frac{3}{4} = 2$$

a) $E(X) = a + 2b + 3a$ (2)

$= 4a + 2b$

b) Given that $Var(X) = 0.75$, find the values of a & b.

we have,

$$Var(X) = (a + 4b + 9a) - 2^2 = 0.75$$

$$\Rightarrow 10a + 4b = 4.75 \longrightarrow (1)$$

Again $\sum_{\forall x} x\, P(X=x) = 1$

$\Rightarrow a + b + a = 1$

$\Rightarrow 2a + b = 1$

$\Rightarrow 10a + 5b = 5$

~~$4a = 2b = 2$~~

$10a = 5 - 5b$

a =

(1) $\Rightarrow$ $5 - 5b + 4b = 4.75$

$\Rightarrow -b = 4.75 - 5$

$+b = +0.25$

$\Rightarrow b = 1/4$

$2a + b = 1$

$\Rightarrow 2a = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore a = 3/8$

* Expected value of x

$$\mu = E(x) = \sum_{\forall x} x\, P(X=x)$$

$$E(2x-1) = 2E(x) - 1$$

* Variance of X: (σ^2)

$$\cancel{Var(x) = \sum_{\forall x} (x-\mu)^2}$$

$$\sigma^2 = E(x^2) - \mu^2$$

* $E(x^2) \neq [E(x)]^2$

Ex:

x:	1	2	3	Var: x^2	1	4	9
$P(X=x)$:	$1/3$	$1/2$	$1/6$				

$$\mu = E(x) = \sum_{\forall} x\, P(X=x)$$

$$= 1 \times 1/3 + 2 \times 1/2 + 3 \times 1/6$$

$$= 11/6$$

$$Var(x) = 1 \times 1/3 + 4 \times 1/2 + 9 \times 1/6$$

$$= \frac{17}{36}$$

(7)

* At a fair a roll-a-Penny stall can be played with 1p & 2p Coins. If the coin lands inside a square the player receives the coin plus 2 other coins of the same value, otherwise the coin is lost. The probability of winning the prize with a 1p coin is $\frac{19}{40}$ and the probability for a 2p coin is $\frac{11}{40}$.

a) Find the expected winnings for each coin
b) Would you play this game and why?
c) The stall was eventually closed down by the management. Give a possible reason for this.

a)

p	1	2
$P(X=p)$	$\frac{19}{40}$	$\frac{11}{40}$

Distribution for 1 p

x	-1	2
$P(X=x)$	$\frac{21}{40}$	$\frac{19}{40}$

$$E(x) = -\frac{21}{40} + \frac{38}{40}$$

$$= \frac{17}{40}$$

Distribution for 2p

x	-2	4
$P(X=x)$	$\frac{29}{40}$	$\frac{11}{40}$

$$E(X=2) = \cancel{58} - \frac{58}{40} + \frac{44}{40}$$

$$= -\frac{14}{40}$$

$$= \frac{-7}{20}$$

b) Yes I ~~will~~ shall play with 1p since expected value is +ve.

c) It was making a loss because people saw that 1p was better.

Mathematical Expectation

Let x be a random variable with probability distribution $f(x)$. The mean or expected value of x is

$$\mu = E(x) = \sum_{x} x f(x) \quad ; \text{ if } x \text{ is discrete.}$$

and
$$\mu = E(x) = \int_{-\infty}^{\infty} x f(x) dx \; ; \text{ if } x \text{ is Continuous}$$

$$E(ax) = a\,E(x)$$
$$E(ax+b) = a\,E(x)+b$$
$$Var(ax) = a^2\,Var(x)$$
$$Var(ax+b) = a^2\,Var(x)$$

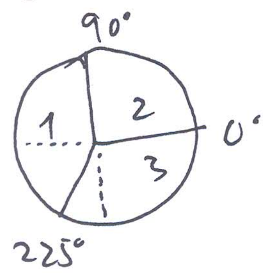
Ex:

x	-2	0	2	4
$P(X=x)$	0.3	0.4	0.2	0.1

a) Find $E(\frac{1}{2}x+1)$ and $Var(\frac{1}{2}x+1)$

b) $E(3-2x)$

4. A ~~Spspine~~ Spinner is made from the disc in the diagram and the random variable X represent the number it lands on after being spun.



Find distribution, $E(x)$ and $Var(X)$

x	1	2	3
$P(X=x)$	$3/8$	$1/4$	$3/8$

$E(x) = 2$, $\quad Var(x) = 3/4$

Ann / Bill

Find $E(2x-1)$
$= 2E(x)-1$
$= 4-1$
$= 3$

$E(3x-3)$
$= 3E(x)-3$
$= 3\times 2-3 = 3$

Find Var for them

3, 27/4

Probability Density Function:

The function $f(x)$ is a probability density function for the continuous random variable X, defined over the set of real numbers R, if

1. $f(x) \geq 0 \quad \forall\, x \in \mathbb{R}$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a < X < b) = \int_a^b f(x)\,dx.$

Ex: 3.11

$$f(x) = \begin{cases} x^2/3 ; & -1 < x < 2 \\ 0, & \text{other wise} \end{cases}$$

a)

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$$

$$= 0 + \int_{-1}^{2} x^2/3\,dx + 0$$

$$= \frac{1}{3}\left[x^3/3\right]_{-1}^{2}$$

$$= \frac{1}{9}\left[8 - (-1)\right] = 1.$$

b)

$$P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3}\,dx$$

$$= \left[\frac{x^3}{9}\right]_0^1$$

$$= \left(\frac{1}{9} - 0\right)$$

$$= \frac{1}{9}$$