Laurent's series

SOS-91 / P-166 Expand $f(z) = \frac{1}{z-3}$ in a Laurent series valid for (a) $|z| < 3$, (b) $|z| > 3$.

Solution: We have, $f(z) = \frac{1}{z-3}$ ... (1)

(a) Since $|z| < 3$, so from (1) we get

$$f(z) = -\frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1}$$
$$= -\frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \cdots\cdots\right)$$
$$= -\frac{1}{3} - \frac{z}{9} - \frac{z^2}{27} - \frac{z^3}{81} - \cdots\cdots$$

(b) Since $|z| > 3$, so from (1) we get

$$f(z) = \frac{1}{z}\left(1 - \frac{3}{z}\right)^{-1}$$
$$= \frac{1}{z}\left(1 + \frac{3}{z} + \frac{9}{z^2} + \frac{27}{z^3} + \cdots\cdots\right)$$
$$= \frac{1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \frac{27}{z^4} + \cdots\cdots$$

SOS-92 / P-166 Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent series valid for:

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z-1| > 1$

(e) $0 < |z-2| < 1$

Solution: We have, $f(z) = \frac{z}{(z-1)(2-z)}$

$$= \frac{1}{z-1} + \frac{2}{2-z} \quad \cdots (1)$$

(a) Since $|z| < 1$, so from (1) we get

$$f(z) = -\frac{1}{1-z} + \frac{1}{1-z/2}$$
$$= -(1-z)^{-1} + (1 - z/2)^{-1}$$
$$= -(1 + z + z^2 + z^3 + \cdots\cdots) + \left(1 + z/2 + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$$
$$= -z/2 - \frac{3z^2}{4} - \frac{7z^3}{8} - \cdots\cdots$$

(b) Since $1 < |z| < 2$, so from (1) we get

$$f(z) = \frac{1}{z(1-\frac{1}{z})} + \frac{1}{1 - z/2}$$
$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} + (1 - z/2)^{-1}$$
$$= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) + \left(1 + z/2 + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$$

$\therefore f(z) = \cdots\cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z/2 + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\cdots$

(c) Since $|z|>2$, so from (1) we get—

$$f(z) = \frac{1}{z(1-\frac{1}{z})} - \frac{2}{z(1-2/z)}$$

$$= \frac{1}{z}(1-\frac{1}{z})^{-1} - \frac{2}{z}(1-\frac{2}{z})^{-1}$$

$$= \frac{1}{z}(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots) - \frac{2}{z}(1+\frac{2}{z}+\frac{4}{z^2}+\cdots\cdots)$$

$$= \frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\cdots - \frac{2}{z} - \frac{4}{z^2} - \frac{8}{z^3} - \cdots$$

$$= -\frac{1}{z} - \frac{3}{z^2} - \frac{7}{z^3} - \cdots\cdots$$

(d) Since $|z-1|>1$, so from (1) we get—

$$f(z) = \frac{1}{z-1} - \frac{2}{z-2}$$

$$= \frac{1}{z-1} - \frac{2}{(z-1)-1}$$

$$= \frac{1}{z-1} - \frac{2}{(z-1)(1-\frac{1}{z-1})}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}(1-\frac{1}{z-1})^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left\{1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\cdots\cdots\right\}$$

$$= \frac{1}{z-1} - \frac{2}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots\cdots$$

$$= -\frac{1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots\cdots$$

(e) Since $0<|z-2|<1$, so from (1) we get

$$f(z) = \frac{1}{(z-2)+1} - \frac{2}{z-2}$$

$$= \{1+(z-2)\}^{-1} - \frac{2}{z-2}$$

$$= 1-(z-2)+(z-2)^2-(z-2)^3+\cdots\cdots - \frac{2}{z-2}$$

$$= 1 - \frac{2}{z-2} - (z-2) + (z-2)^2 - (z-2)^3 + \cdots\cdots$$

SOS-93 / P-166 Expand $f(z)=\frac{1}{z(z-2)}$ in a Laurent series valid for (a) $0<|z|<2$, (b) $|z|>2$.

Solution: We have,
$$f(z)=\frac{1}{z(z-2)}$$
$$=\frac{-\frac{1}{2}}{z}+\frac{\frac{1}{2}}{z-2}$$
$$=\frac{1}{2}\left[\frac{1}{z-2}-\frac{1}{z}\right]\cdots(1)$$

(a) Since $0<|z|<2$, so from (1) we get
$$f(z)=\frac{1}{2}\left[\frac{1}{-2(1-z/2)}-\frac{1}{z}\right]$$
$$=\frac{1}{2}\left[-\frac{1}{2}(1-z/2)^{-1}-\frac{1}{z}\right]$$
$$=\frac{1}{2}\left[-\frac{1}{2}\left(1+z/2+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)-\frac{1}{z}\right]$$
$$=-\frac{1}{2z}-\frac{1}{4}-\frac{z}{8}-\frac{z^2}{16}-\frac{z^3}{32}-\cdots\cdots$$

(b) Since $|z|>2$, so from (1) we get
$$f(z)=\frac{1}{2}\left[\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\right]$$
$$=\frac{1}{2}\left[\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right)-\frac{1}{z}\right]$$
$$=\frac{1}{z^2}+\frac{2}{z^3}+\frac{4}{z^4}+\cdots\cdots$$

SOS-94 / P-166 Find an expression of $f(z)=\frac{z}{z^2+1}$ valid for $|z-3|>2$.

Solution: Since $|z-3|>2$, so from $f(z)=\frac{z}{z^2+1}$, we get
$$f(z)=\frac{z}{z^2\left(1+\frac{1}{z^2}\right)}$$
$$=\frac{1}{z}\left(1+\frac{1}{z^2}\right)^{-1}$$
$$=\frac{1}{z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\frac{1}{z^8}-\cdots\right)$$
$$=\frac{1}{z}-\frac{1}{z^3}+\frac{1}{z^5}-\frac{1}{z^7}+\frac{1}{z^9}-\cdots\cdots$$

SOS-95
P-166

Expand $f(z) = \dfrac{1}{(z-2)^2}$ in a Laurent series valid for (a) $|z| < 2$, (b) $|z| > 2$.

Solution: we have, $f(z) = \dfrac{1}{(z-2)^2}$ ... (1)

(a) Since $|z| < 2$, so from (1) we get

$$f(z) = \frac{1}{\{-2(1-z/2)\}^2}$$

$$= \frac{1}{4(1-z/2)^2}$$

$$= \frac{1}{4}(1-z/2)^{-2}$$

$$= \frac{1}{4}\left\{1 + 2\cdot\left(\frac{z}{2}\right) + 3\cdot(z/2)^2 + 4(z/2)^3 + \cdots \infty\right\}$$

$$= \frac{1}{4} + \frac{z}{4} + \frac{3z^2}{16} + \frac{z^3}{8} + \cdots\cdots,$$

(b) since $|z| > 2$, so from (1) we get

$$f(z) = \frac{1}{\{z(1-\frac{2}{z})\}^2}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-2}$$

$$= \frac{1}{z}\left\{1 + 2\left(\frac{2}{z}\right) + 3\left(\frac{2}{z}\right)^2 + 4\left(\frac{2}{z}\right)^3 + \cdots \infty\right\}$$

$$= \frac{1}{z} + \frac{4}{z^2} + \frac{12}{z^3} + \frac{32}{z^4} + \cdots,$$