Analytic function-2

The complex potential function

The analytic function $W = \phi(x,y) + i\psi(x,y)$ is referred to as the complex potential function. Its real part $\phi(x,y)$ represents the velocity potential function and the imaginary part $\psi(x,y)$ represents the stream function.

Problem-1(a). If $W = \phi + i\psi$ represents the complex potential for an electric field and $\psi = 3x^2y - y^3$, find the potential function ϕ.

Solution: Given $\psi = 3x^2y - y^3$

$$\frac{\partial \psi}{\partial y} = 3x^2 - 3y^2$$
$$= \psi_1(x,y), \text{ say}$$

and $\frac{\partial \psi}{\partial x} = 6xy$
$$= \psi_2(x,y), \text{ say}$$

By Milne's method we have,

$$W'(z) = \psi_1(z,0) + i\psi_2(z,0)$$
$$= 3z^2 + i \cdot 0$$
$$= 3z^2$$

integrating w.r.t. z, we get

$$W(z) = z^3 + C$$

$$\text{or } \phi + i\psi = (x+iy)^3 + c_1 + ic_2$$

$$\text{or, } \phi + i\psi = x^3 + i3x^2y - 3xy^2 - iy^3 + c_1 + ic_2$$

$\therefore \phi = x^3 - 3xy^2 + c_1$ is the required potential function.

Problem-1(b). An incompressible fluid flowing over the xy-plane has the velocity potential

$$\varphi = x^2 - y^2 + \frac{x}{x^2+y^2}$$

Examine if this is possible and find a stream function ψ.

Solution: Given, $\varphi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$

$$\therefore \frac{\partial \varphi}{\partial x} = 2x + \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2} = \varphi_1(x,y), \text{ say}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 2 + \frac{(x^2+y^2)^2.(-2x) - (y^2-x^2).2(x^2+y^2).2x}{(x^2+y^2)^4}$$

$$= 2 + \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial \varphi}{\partial y} = -2y + \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2} = \varphi_2(x,y), \text{ say}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = -2 + \frac{(x^2+y^2)^2(-2x) - (-2xy).2(x^2+y^2).2y}{(x^2+y^2)^4}$$

$$= -2 + \frac{-2x^3 + 6xy^2}{(x^2+y^2)^3}$$

$\therefore \dfrac{\partial^2 \varphi}{\partial x^2} + \dfrac{\partial^2 \varphi}{\partial y^2} = 0$ i.e., φ is harmonic.

Hence it can be a possible form of the velocity potential function.

By Milne's method, we have

$$f'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$$

$$= 2z - \frac{1}{z^2} - i.0$$

integrating it, we get $f(z) = z^2 + \dfrac{1}{z} + C$

$$\text{or, } \varphi + i\psi = (x+iy)^2 + \frac{1}{x+iy} + c_1 + ic_2$$

$$= x^2 - y^2 + i2xy + \frac{x-iy}{x^2+y^2} + c_1 + ic_2$$

$\therefore \psi = 2xy - \dfrac{y}{x^2+y^2} + c_2$ is the required.

Problem-1(c): If $W = \varphi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, find φ. Ans. $\varphi = -2xy + \dfrac{y}{x^2+y^2} + c_1$

Problem-1(d). Prove that the real and imaginary parts of an analytic function $f(z) = u(x,y) + iv(x,y)$ satisfies the Laplace's equation.

Solution: Since $f(z) = u(x,y) + iv(x,y)$ is an analytic function, so we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{---(1)} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---(2)}$$

Differentiate (1) partially wrt x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{---(3)}$$

Differentiate (2) partially wrt y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \text{---(4)}$$

Adding (3) and (4) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or,} \quad \nabla^2 u = 0$$

similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ or $\nabla^2 v = 0$

i.e. u and v satisfy their Laplace's equations.

[Both u and v are harmonic functions]

Problem-2. Show that an analytic function with constant real part is constant.

Solution: Let $f(z) = u + iv$ be an analytic function.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{---(1)} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---(2)}$$

Given that $u = \text{constant} = c_1$, say

$$\therefore \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0$$

so from (1) and (2) we get, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$

i.e. v is independent of x and y

$\Rightarrow v = \text{Constant} = c_2$, say

$\therefore f(z) = u + iv = c_1 + ic_2$ is a constant

Problem-3(a). Show than an analytic function with Constant imaginary part is constant.

Solution: Let $f(z) = u+iv$ be an analytic function.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (1)}$$

Given that $v =$ Constant $= c_1$, say

$$\therefore \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial x} = 0 \quad [\text{by } (1)]$$

$\Rightarrow u$ is independent of x and y

$\Rightarrow u =$ Constant $= c_2$, say

$$\therefore f(z) = u + iv = c_2 + ic_1 = \text{Constant}$$

Problem-3(b). Determine the analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Solution: Given that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$= u_1(x,y), \text{ say}$$

$$\frac{\partial u}{\partial y} = -6xy - 6y$$

$$= u_2(x,y), \text{ say}$$

By Milne's method, we have

$$f'(z) = u_1(z,0) - iu_2(z,0)$$

$$= 3z^2 + 6z - i \cdot 0$$

Integrating it, we get $f(z) = 3 \cdot \frac{z^3}{3} + 6 \cdot \frac{z^2}{2} + c$

$= z^3 + 3z^2 + c$, where c is the complex constant.

Problem-4: Show that an analytic function with constant absolute value/modulus is constant.

Solution: Let an analytic function be $f(z) = u + iv$

$$\therefore \ |f(z)| = \sqrt{u^2+v^2}$$

But we are given, $|f(z)| = \text{Constant} = K$, say

$$\therefore \ u^2+v^2 = K^2$$

By differentiation, $uu_x + vv_x = 0$, $uu_y + vv_y = 0$

Now we use $v_x = -u_y$ in the first equation and $v_y = u_x$ in the second, we get

$$uu_x - vu_y = 0 \quad \text{--} \ (1)$$

$$uu_y + vu_x = 0 \quad \text{--} \ (2)$$

Multiplying (1) by u and (2) by v, then adding and also multiplying (1) by $-v$ and (2) by u, then adding we get

$$(u^2+v^2)u_x = 0, \quad (u^2+v^2)u_y = 0$$

If $K^2 = u^2+v^2 = 0$, then $u = 0 = v$, hence $f = 0$.

If $K \neq 0$, then $u_x = u_y = 0$, hence by Cauchy-Riemann equations, also $v_x = v_y = 0$.
Together, $u = \text{Constant}$ and $v = \text{Constant}$, hence $f = \text{Constant}$.

Problem-5: Test whether the function $f(z) = z^3 + z$ is analytic or not.

Solution: We have, $f(z) = z^3 + z$

$$= (x+iy)^3 + (x+iy)$$

$$= x^3 + i3x^2y - 3xy^2 - iy^3 + x + iy$$

or, $u + iv = (x^3 - 3xy^2 + x) + i(3x^2y - y^3 + y)$

Equating the real and imaginary parts, we get

$$u = x^3 - 3xy^2 + x$$

$$v = 3x^2y - y^3 + y$$

$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1, \quad \frac{\partial u}{\partial y} = -6xy$

$\frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 1$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow$ C-R equations are satisfied.

$\therefore f(z) = z^3 + z$ is analytic.

Problem-6: Find the constants a, b and c if $f(z) = x + ay + i(bx + cy)$ is analytic.

Solution: Given, $f(z) = x + ay + i(bx + cy)$

or, $u + iv = x + ay + i(bx + cy)$

$\therefore u = x + ay, \quad v = bx + cy$

$\therefore \frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$

$\frac{\partial v}{\partial x} = b, \quad \frac{\partial v}{\partial y} = c$

Since $f(z)$ is analytic, so Cauchy-Riemann (C-R) equations are satisfied.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$\therefore 1=c, \quad a=-b$$

$$\Rightarrow c=1, \; a=-b, \; b \text{ may be any value.}$$

Problem-7: Determine b such that $u=e^{bx}\cos 5y$ is harmonic.

Solution: Given, $u=e^{bx}\cos 5y$

$$\therefore \frac{\partial u}{\partial x}=be^{bx}\cos 5y$$

$$\frac{\partial^2 u}{\partial x^2}=b^2e^{bx}\cos 5y$$

$$\frac{\partial u}{\partial y}=e^{bx}(-5\sin 5y)$$

$$\frac{\partial^2 u}{\partial y^2}=-5e^{bx}\cdot 5\cos 5y$$

$$=-25e^{bx}\cos 5y$$

$\because$ u is harmonic function, so

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$\text{or, } b^2e^{bx}\cos 5y-25e^{bx}\cos 5y=0$$

$$\text{or, } e^{bx}\cos 5y\,(b^2-25)=0$$

$$\text{or, } b^2-25=0 \quad [\because e^{bx}\cos 5y\neq 0]$$

$$\therefore b=\pm 5$$

Problem-8: (a) Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic. (b) Find v such that $f(z) = u + iv$ is analytic.

Solution: Given, $u = e^{-x}(x\sin y - y\cos y)$

$$\therefore \frac{\partial u}{\partial x} = e^{-x} \cdot \sin y + (-e^{-x})(x\sin y - y\cos y)$$

$$= e^{-x}(\sin y - x\sin y + y\cos y)$$

$$= u_1(x,y), \text{ say}$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x}(\sin y - x\sin y + y\cos y) + e^{-x}(-\sin y)$$

$$= -e^{-x}(2\sin y - x\sin y + y\cos y) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y - 1\cdot\cos y + y\sin y)$$

$$= u_2(x,y), \text{ say}$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-x\sin y + \sin y + 1\cdot\sin y + y\cos y)$$

$$= e^{-x}(-x\sin y + 2\sin y + y\cos y) \quad \text{--- (2)}$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since u satisfies Laplace's equation, so u is harmonic function.

<u>2nd part</u>: (b) By Milne's method we have,

$$f'(z) = u_1(z,0) - iu_2(z,0)$$

$$= 0 - i(ze^{-z} - e^{-z})$$

Integrating, $f(z) = -iz\cdot(-e^{-z}) + i\int 1\cdot(-e^{-z})\,dz + i\int e^{-z}\,dz + c$

$$= ize^{-z} + c$$

or, $u+iv = i(x+iy)\,e^{-(x+iy)} + c$

$= i(x+iy)\,e^{-x}.\,e^{-iy} + c$

$= i(x+iy).\,e^{-x}(\cos y - i\sin y) + c$

$= ixe^{-x}\cos y + xe^{-x}\sin y - ye^{-x}\cos y + iye^{-x}\sin y + c_1 + ic_2$

$= (xe^{-x}\sin y - ye^{-x}\cos y + c_1) + i(xe^{-x}\cos y + ye^{-x}\sin y + c_2)$

Equating imaginary parts, we get

$$v = e^{-x}(x\cos y + y\sin y) + c_2$$

Problem-9. In a two dimensional flow of a fluid, the velocity potential $\varphi = x^2 - y^2$. Find the stream function ψ.

Solution: Given that $\varphi = x^2 - y^2$

$\therefore \frac{\partial \varphi}{\partial x} = 2x$

$= \varphi_1(x, y)$, say

and $\frac{\partial \varphi}{\partial y} = -2y$

$= \varphi_2(x, y)$, say

By Milne's method we have,

$W'(z) = \varphi_1(z, 0) - i\varphi_2(z, 0)$

$= 2z - i.0$

$= 2z$

Integrating it, we get $W(z) = \cancel{2}\frac{z^2}{\cancel{2}} + c$

or, $W(z) = z^2 + C$

or, $\phi + i\psi = (x+iy)^2 + c_1 + ic_2$

$= x^2 - y^2 + i2xy + c_1 + ic_2$

Equating imaginary parts, we get

$\psi = 2xy + c_2$ is the required stream function.

Problem-10. Show that xy^2 cannot be real part of an analytic function.

<u>Solution:</u> Given $u = xy^2$

$\therefore \frac{\partial u}{\partial x} = y^2, \quad \frac{\partial^2 u}{\partial x^2} = 0$ --- (1)

$\frac{\partial u}{\partial y} = 2xy, \quad \frac{\partial^2 u}{\partial y^2} = 2x$ --- (2)

Adding (1) and (2) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2x$$
$$\neq 0$$

$\therefore u$ is not harmonic function.

i.e. u cannot be a real part of an analytic function.