Contour Integration-2

Lemma related to Cauchy's residue theorem:

(i) Cauchy's Lemma I: If γ is the arc of a small semicircle $|z-a| = r$ between the angles θ_1 and θ_2, $\theta_1 \le \theta \le \theta_2$ and if

$$\lim_{z \to a} (z-a) f(z) = A, \text{ then}$$

$$\lim_{r \to 0} \int_{\gamma} f(z)\,dz = -iA(\theta_2 - \theta_1)$$

where $f(z)$ is a continuous function.

(ii) Cauchy's Lemma II: If Γ be an arc $(\theta_1 \le \theta \le \theta_2)$ of the big semicircle $|z| = R$ and if $\lim_{z \to \infty} z f(z) = A$, then

$$\lim_{R \to \infty} \int_{\Gamma} f(z)\,dz = iA(\theta_2 - \theta_1)$$

(iii) Jordan's Lemma: If $f(z)$ is analytic except at finite numbere of singularities and if $\lim_{z \to \infty} f(z) = 0$, then

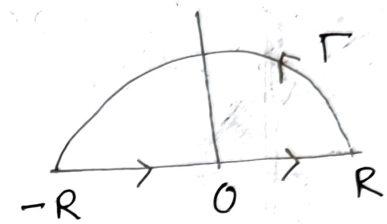
$$\lim_{R \to \infty} \int_{\Gamma} e^{imz} f(z)\,dz = 0, \quad m > 0$$

Evaluate by contour integration: (i) $\int_0^\infty \frac{dx}{x^4+a^4}$.

solution: Let us consider the integral

$$\oint_C f(z)\,dz,$$

where $f(z) = \frac{1}{z^4+a^4}$ and C is the closed contour consisting of semicircle Γ and the real axis from $-R$ to R.



Then by Cauchy's residue theorem we have,

$$\oint_C f(z)\,dz = 2\pi i\,(\text{sum of residues})$$

$$\text{or, } \int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i\,(\text{sum of residues}) \quad \text{--- (1)}$$

since $\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z}{z^4+a^4}$

$= 0$

$= A$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)\,dz = iA(\theta_2-\theta_1)$$
$$= i\cdot 0(\pi-0)$$
$$= 0$$

Poles of $f(z)$ are given by $z^4+a^4=0$

or, $z^4 = -a^4$ or, $z^4 = e^{(2n+1)\pi i} a^4$

$\therefore$ $z = a e^{(2n+1)\pi i/4}$, $n = 0, 1, 2, 3$

Only $z = ae^{\pi i/4}$ and $z = ae^{3\pi i/4}$ lie inside C.

Residue at $z = ae^{\pi i/4}$ is

$$\lim_{z \to ae^{\pi i/4}} \left\{ \left(z - ae^{\pi i/4}\right) \cdot \frac{1}{z^4 + a^4} \right\}$$

$$= \lim_{z \to ae^{\pi i/4}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3} e^{-3\pi i/4}$$

$$= -\frac{1}{4a^3} e^{\pi i/4}$$

Residue at $z = ae^{3i\pi/4}$ is

$$\lim_{z \to ae^{3\pi i/4}} \left\{ \left(z - ae^{3\pi i/4}\right) \cdot \frac{1}{z^4 + a^4} \right\}$$

$$= \lim_{z \to ae^{3\pi i/4}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3} e^{-9\pi i/4}$$

$$= \frac{1}{4a^3} e^{-\pi i/4} \quad \left[\because e^{-2\pi i} = 1\right]$$

Hence by making $R \to \infty$, relation (1) becomes

$$0 + \int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = 2\pi i \left(-\frac{1}{4a^3} e^{\pi i/4} + \frac{1}{4a^3} e^{-\pi i/4} \right)$$

or, $$2\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{-2\pi i}{4a^3}\left(e^{\pi i/4} - e^{-\pi i/4}\right)$$

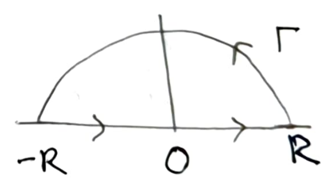
$$= \frac{-2\pi i}{4a^3} \cdot 2i \sin\frac{\pi}{4}$$

$$= \frac{\pi}{a^3} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\pi\sqrt{2}}{2a^3}$$

$$\therefore \int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$$

(ii) $$\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$$



Let us consider the integral $\oint_C f(z)\,dz$,

where $f(z) = \dfrac{1}{(z^2+1)(z^2+4)^2}$ and C is the closed contour consisting of semicircle Γ and the real axis from $-R$ to R.

Thus $\oint_C f(z)\,dz = 2\pi i$ (sum of residues)

or, $$\int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \text{(sum of residues)} \quad \cdots (1)$$

$\because \lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z}{(z^2+1)(z^2+4)^2}$

$= 0$

$= A$

$\therefore \lim_{R\to\infty} \int_\Gamma f(z)dz = iA(\theta_2 - \theta_1)$

$= i \cdot 0 (\pi - 0)$

$= 0$

Hence by making $R \to \infty$, relation (1) becomes,

$$0 + \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = 2\pi i (\text{sum of residues}) \quad \text{--- (2)}$$

Poles of $f(z)$ are given by

$(z^2+1)(z^2+4)^2 = 0$

$\therefore z = \pm i, \pm 2i$

But only two poles $z = i$, $z = 2i$ (pole of 2nd order) lie inside C.

Residue at $z = i$ is $\lim_{z\to i} (z-i) \cdot \frac{1}{(z^2+1)(z^2+4)^2}$

$= \lim_{z\to i} \frac{1}{(z+i)(z^2+4)^2}$

$= \frac{1}{2i(-1+4)^2}$

$= \frac{1}{18i}$

Residue at double pole $z = 2i$ is

$$\lim_{z \to 2i} \frac{1}{\lfloor 2-1} \frac{d}{dz}\left\{(z-2i)^2 \cdot \frac{1}{(z^2+1)(z^2+4)^2}\right\}$$

$$= \lim_{z \to 2i} \frac{d}{dz}\left\{\frac{1}{(z^2+1)(z+2i)^2}\right\}$$

$$= \lim_{z \to 2i} \frac{0 - [2z(z+2i)^2 + (z^2+1)\cdot 2(z+2i)\cdot 1]}{(z^2+1)^2 (z+2i)^4}$$

$$= \frac{-[(4i)(4i)^2 + (-4+1)\cdot 2 \cdot 4i]}{(-4+1)^2 (4i)^4}$$

$$= \frac{64i + 24i}{9 \times 16 \times 16}$$

$$= \frac{88i}{9 \times 16 \times 16}$$

$$= \frac{11i}{9 \times 2 \times 16}$$

$$= \frac{11i}{18 \times 16}$$

So from (2) we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = 2\pi i \left(\frac{1}{18i} + \frac{11i}{18 \times 16}\right)$$

$$\text{or, } 2\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = 2\pi i \left(\frac{16-11}{18i \times 16}\right)$$

$$\therefore \int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{5\pi}{288}$$

(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x-1)(x^2+1)}$

Let us consider the integral $\oint_C f(z)\,dz$, where $f(z) = \dfrac{1}{(z-1)(z^2+1)}$ and C is the closed contour consisting of a big semicircle Γ and the real axis $-R$ to R indented at $z=1$. The small semicircle is denoted by γ, its centre is at $z=1$ and radius r.

Hence by Cauchy's residue theorem we have,

$$\oint_C f(z)\,dz = 2\pi i\,(\text{sum of residues})$$

$$\text{or, } \int_\Gamma f(z)\,dz + \int_{-R}^{1-r} f(x)\,dx + \int_\gamma f(z)\,dz + \int_{1+r}^{R} f(x)\,dx$$

$$= 2\pi i\,(\text{sum of residues}) \quad (1)$$

$$\therefore \lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z}{(z-1)(z^2+1)}$$

$$= 0$$

$$= A$$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)\,dz = iA\,(\theta_2-\theta_1)$$

$$= i\cdot 0\,(\pi - 0)$$

$$= 0$$

Also $\lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \cancel{(z-1)} \cdot \frac{1}{\cancel{(z-1)}(z^2+1)}$

$$= \frac{1}{2}$$

$$= A$$

$\therefore \lim_{r \to 0} \int_{\gamma} f(z) dz = -iA(\theta_2 - \theta_1)$

$$= -i \cdot \frac{1}{2} (\pi - 0)$$

$$= -\frac{\pi i}{2}$$

Poles of $f(z)$ are given by

$$(z-1)(z^2+1) = 0$$

$$\therefore z = 1, \pm i$$

Both $z = 1, -i$ lie outside the contour but $z = i$ lies in C.

Residue at $z = i$ is $\lim_{z \to i} (z-i) \cdot \frac{1}{(z-1)(z^2+1)}$

$$= \lim_{z \to i} \frac{1}{(z-1)(z+i)}$$

$$= \frac{1}{(i-1) \cdot 2i}$$

$$= -\frac{1-i}{4}$$

Hence by making $R \to \infty$, $r \to 0$, relation (1) becomes,

$$0 + \int_{-\infty}^{1} f(x) dx - \frac{\pi i}{2} + \int_{1}^{\infty} f(x) dx = 2\pi i \left(-\frac{1-i}{4}\right)$$

or, $\int_{-\infty}^{\infty} f(x) dx = \pi i \left(-\frac{1-i}{2}\right) + \frac{\pi i}{2}$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x-1)(x^2+1)} = -\frac{\pi}{2}$$

(iv) $\int_0^\infty \frac{x \sin x \, dx}{x^2 + a^2}$

let us consider the integral $\oint_C f(z)\,dz$, where $f(z) = \frac{z e^{iz}}{z^2+a^2}$ and C is the closed contour consisting of semicircle Γ and the real axis −R to R.

Hence by Cauchy's residue theorem we have, $\oint_C f(z)\,dz = 2\pi i$ (sum of residues)

or, $\int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i$ (sum of residues)

$\because \lim_{z\to\infty} \phi(z) = \lim_{z\to\infty} \frac{z}{z^2+a^2}$ $\left[\because f(z) = \frac{z e^{iz}}{z^2+a^2} = e^{iz}\phi(z), \text{ where } \phi(z) = \frac{z}{z^2+a^2}\right.$

$= \lim_{z\to\infty} \frac{1}{2z}$ [using L Hospital's rule

$= 0$

Hence by Jordan's Lemma we get

$\lim_{R\to\infty} \int_\Gamma f(z)\,dz = \lim_{R\to\infty} \int_\Gamma e^{iz}\phi(z)\,dz$

$= 0$

poles of $f(z)$ are given by $z^2 + a^2 = 0$

$\therefore z = \pm ia$

Only simple pole $z = ia$ lies inside C.

Residue at $z = ai$ is $\lim_{z \to ai} (z-ai) \cdot \frac{z e^{iz}}{z^2+a^2}$

$$= \lim_{z \to ai} \frac{z e^{iz}}{z+ai}$$

$$= \frac{iae^{-a}}{2ai}$$

$$= \frac{e^{-a}}{2}$$

Hence by making $R \to \infty$, relation (1) becomes

$$0 + \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \left(\frac{e^{-a}}{2} \right)$$

$$\text{or, } \int_{-\infty}^{\infty} \frac{x e^{ix}\,dx}{x^2+a^2} = \pi i e^{-a}.$$

$$\text{or, } \int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x)\,dx}{x^2+a^2} = \pi i e^{-a}$$

Equating imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{x \sin x\,dx}{x^2+a^2} = \pi e^{-a}$$

$$\text{or, } 2\int_{0}^{\infty} \frac{x \sin x\,dx}{x^2+a^2} = \pi e^{-a}$$

$$\therefore \int_{0}^{\infty} \frac{x \sin x\,dx}{x^2+a^2} = \frac{\pi}{2} e^{-a}.$$