Engineering Statistics and Complex Variables by FT

MATH 281

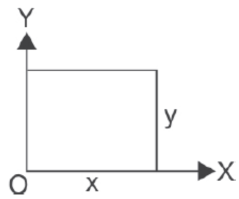
Complex Variables

Define: Complex variable, Analytic/Holomorphic/Monogenic/Regular Function

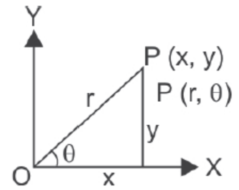
7.2 COMPLEX VARIABLE

$x + iy$ is a complex variable and it is denoted by z.

(1) $z = x + iy$ where $i = \sqrt{-1}$ (Cartesian form)



(2) $z = r(\cos\theta + i\sin\theta)$ (Polar form)



(3) $z = re^{i\theta}$ (Exponential form)

7.3 FUNCTIONS OF A COMPLEX VARIABLE

$f(z)$ is a function of a complex variable z and is denoted by w.

$$w = f(z)$$

$$w = u + iv$$

where u and v are the real and imaginary parts of $f(z)$.

7.4 NEIGHBOURHOOD OF Z_0

Let z_0 is a point in the complex plane and let z be any positive number, then the set of points z such that

$$|z - z_0| < \in$$

is called $\in$-neighbourhood of z_0.

Closed set

A set S is said to be closed if it contains all of its limits point.

Interior Point

A point z_0 is called a interior point of a point set S if there exists a neighbourhood of z_0 lying wholly in S.

Define limited continuity of a complex function f(z). Prove that the function f(z) defined Y dot.

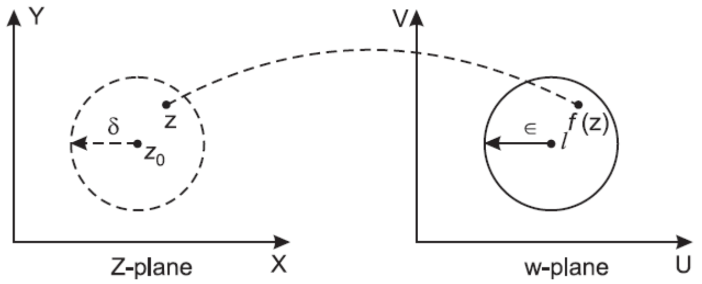
7.5 LIMIT OF A FUNCTION OF A COMPLEX VARIABLE

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0. Then $f(z)$ is said to have the limit l as z approaches z_0 along any path if given an arbitrary real number $\in > 0$, however small there exists a real number $\delta > 0$, such that

$|f(z) - l| < \in$ whenever $0 < |z - z_0| < \delta$

i.e. for every $z \neq z_0$ in δ-disc (dotted) of z-plane, $f(z)$ has a value lying in the $\in$-disc of w-plane

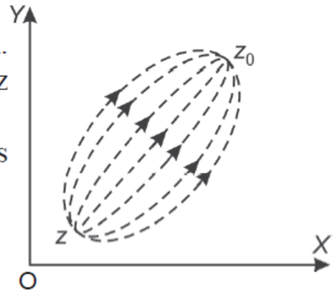
In symbolic form, $\lim\limits_{z \to z_0} f(z) = l$



Note: (I) δ usually depends upon $\in$.

(II) $z \to z_0$ implies that z approaches z_0 along any path. The limits must be independent of the manner in which z approaches z_0

If we get two different limits as $z \to z_0$ along two different paths then limits does not exist.



Example 1. *Prove that* $\lim\limits_{z \to 1-i} \dfrac{(z^2 + 4z + 3)}{z + 1} = 4 - i$

7.6 CONTINUITY

The function $f(z)$ of a complex variable z is said to be continuous at the point z_0 if for any given positive number $\in$, we can find a number δ such that $|f(z) - f(z_0)| < \in$ for all points z of the domain satisfying

$$|z - z_0| < \delta$$

$f(z)$ is said to be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

7.7 CONTINUITY IN TERMS OF REAL AND IMAGINARY PARTS

If $w = f(z) = u(x, y) + iv(x, y)$ is continuous function at $z = z_0$ then $u(x, y)$ and $v(x, y)$ are separately continuous functions of x, y at (x_0, y_0) where $z_0 = x_0 + i\,y_0$.

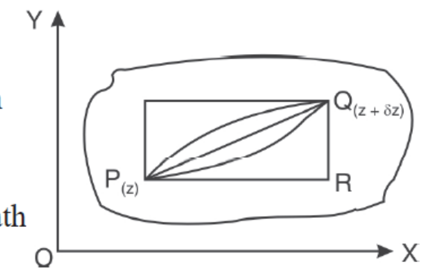
Conversely, if $u(x, y)$ and $v(x, y)$ are continuous functions of x, y at (x_0, y_0) then $f(z)$ is continuous at $z = z_0$.

7.8 DIFFERENTIABILITY

Let $f(z)$ be a single valued function of the variable z, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

provided that the limit exists and is independent of the path along which $\delta z \to 0$.



Let P be a fixed point and Q be a neighbouring point. The point Q may approach P along any straight line or curved path.

7.9 ANALYTIC FUNCTION

A function $f(z)$ is said to be **analytic** at a point z_0, if f is differentiable not only at z_0 but at every point of some neighbourhood of z_0.

A function $f(z)$ is analytic in a domain if it is **analytic** at every point of the domain.

The point at which the function is not differentiable is called a **singular point** of the function.

An analytic function is also known as "holomorphic", "regular", "monogenic".

Entire Function. A function which is analytic everywhere (for all z in the complex plane) is known as an entire function.

For Example 1. Polynomials rational functions are entire.

2. $|\bar{z}|^2$ is differentiable only at $z = 0$. So it is no where analytic.

Note: (*i*) An entire is always analytic, differentiable and continuous function. But converse is not true.

(*ii*) Analytic function is always differentiable and continuous. But converse is not true.

(*iii*) A differentiable function is always continuous. But converse is not true

Q. Write down Cauchy Riemann's equations in Cartesian and Polar form.

7.10 THE NECESSARY CONDITION FOR F (Z) TO BE ANALYTIC

Theorem. *The necessary conditions for a function $f(z) = u + iv$ to be analytic at all the points in a region R are*

(*i*) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (*ii*) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ provided $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist.

7.11 SUFFICIENT CONDITION FOR F (Z) TO BE ANALYTIC

Theorem. *The sufficient condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are*

(*i*) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(*ii*) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *are continuous functions of x and y in region R.*

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

$$\boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

Proved.

Remember: **1.** If a function is analytic in a domain D, then u, v satisfy $C - R$ conditions at all points in D.

2. $C - R$ conditions are necessary but not sufficient for analytic function.

3. $C - R$ conditions are sufficient if the partial derivative are continuous.

Application in Numerical:

A function f(z) will be analytic if and only if –

i) It follows the C-R Equations

ii) It is differentiable.

✓ If we get an undefined format in general derivative then use the First Principle of derivatives or for those which are not directly differentiable.

Proof: Let $f(z)$ be an analytic function in a region R,

$$f(z) = u + iv,$$

where u and v are the functions of x and y.

Let δu and δv be the increments of u and v respectively corresponding to increments δx and δy of x and y.

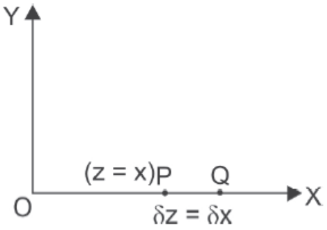
$$\therefore \quad f(z + \delta z) = (u + \delta u) + i(v + \delta v)$$

Now $$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z} = \frac{\delta u + i\delta v}{\delta z} = \frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}$$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \text{ or } f'(z) = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \quad \dots (1)$$

since δz can approach zero along any path.

(*a*) **Along real axis (*x*-axis)**



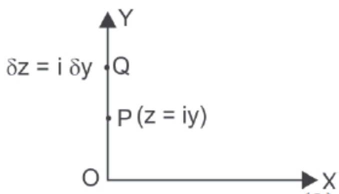
$z = x + iy$ but on x-axis, $y = 0$

$\therefore \quad z = x, \quad \delta z = \delta x, \ \delta y = 0$

Putting these values in (1), we have

$$f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \dots (2)$$

(*b*) **Along imaginary axis (*y*-axis)**



$z = x + iy$ but on y-axis, $x = 0$

$z = 0 + iy \quad \delta x = 0, \ \delta z = i\delta y.$

Putting these values in (1), we get

$$f'(z) = \lim_{\delta y \to 0}\left(\frac{\delta u}{i\delta y} + \frac{i\delta v}{i\delta y}\right) = \lim_{\delta y \to 0}\left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \dots (3)$$

If $f(z)$ is differentiable, then two values of $f'(z)$ must be the same.

Equating (2) and (3), we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

$$\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$ **are known as Cauchy Riemann equations.**

7.12 C–R EQUATIONS IN POLAR FORM

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$ *(MDU, Dec. 2010, RGPV., K.U. 2009, Bhopal, III Sem. Dec. 2007)*

Proof. We know that $x = r\cos\theta$, and u is a function of x and y.

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$u + iv = f(z) = f(re^{i\theta}) \qquad \text{... (1)}$$

Differentiating (1) partially w.r.t., "r", we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}).e^{i\theta} \qquad \text{... (2)}$$

Differentiating (1) w.r.t. "θ", we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\,r\,e^{i\theta}i \qquad \text{... (3)}$$

Substituting the value of $f'(re^{i\theta})e^{i\theta}$ from (2) in (3), we obtain

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)i \quad \text{or} \quad \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Equating real and imaginary parts, we get

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}} \quad \Rightarrow \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

And $$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$ **Proved.**

7.13 DERIVATIVE OF W OR F (Z) IN POLAR FORM

We know that $w = u + iv$, $\dfrac{\partial w}{\partial x} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

But $$\frac{dw}{dz} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{\sin\theta}{r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - \left(-r\frac{\partial v}{\partial r} + i\cdot r\frac{\partial u}{\partial r}\right)\frac{\sin\theta}{r} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\sin\theta \qquad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial}{\partial r}(u + iv)\sin\theta = \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial w}{\partial r}\sin\theta \qquad [\because w = u + iv]$$

$$= (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r} \qquad \text{... (1)}$$

Second form of $\frac{\partial w}{dz}$

$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial (u+iv)}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r} \qquad [w = u + iv]$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$= \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}\right)\cos\theta - \frac{\partial w}{\partial \theta}\cdot\frac{\sin\theta}{r}$$

$$= -\frac{i}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\cos\theta - \frac{\partial w}{\partial \theta}\left(\frac{\sin\theta}{r}\right)$$

$$= -\frac{i}{r}\frac{\partial}{\partial \theta}(u+iv)\cos\theta - \frac{\partial w}{\partial \theta}\left(\frac{\sin\theta}{r}\right) = -\frac{i}{r}\frac{\partial w}{\partial \theta}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r} \qquad [w = u + iv]$$

$$= -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta} \qquad \ldots (2)$$

$$\boxed{\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}} \qquad \left[-\frac{i}{r}\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial r}\right]$$

$$\boxed{\frac{dw}{dz} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}}$$

These are the two forms for $\frac{dw}{dz}$.

7.15 HARMONIC FUNCTION (*U.P., III Semester 2009-2010*)

Any function which satisfies the Laplace's equation is known as a harmonic function.

Theorem. If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic functions.

Proof. Let $f(z) = u + iv$, be an analytic function, then we have

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \qquad \ldots(1) \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \qquad \ldots(2) \end{aligned}\right] \quad C-R \text{ equations.}$$

Differentiating (1) with respect to x, we get $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y}$ ... (3)

Differentiating (2) w.r.t. 'y' we have $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial y \partial x}$ ... (4)

Adding (3) and (4) we have $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 v}{\partial x \partial y} - \dfrac{\partial^2 v}{\partial y \partial x}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \left(\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right)$$

Similarly $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Therefore both u and v are harmonic functions.

Such functions u, v are called **Conjugate harmonic functions if $u + iv$ is also analytic function.**

7.18 METHOD TO FIND THE CONJUGATE FUNCTION

Case I. Given. If $f(z) = u + iv$, and u is known.

To find. v, conjugate function.

Method. We know that $dv = \dfrac{\partial v}{\partial x} \cdot dx + \dfrac{\partial v}{\partial y} \cdot dy$ ... (1)

Replacing $\dfrac{\partial v}{\partial x}$ by $-\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$ by $\dfrac{\partial u}{\partial x}$ in (1), we get [C-R equations]

$$dv = -\frac{\partial u}{\partial y}\cdot dx + \frac{\partial u}{\partial x}\cdot dy$$

$$v = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x}\cdot dy$$

$$\Rightarrow \qquad v = \int M\, dx + \int N\, dy \qquad \text{... (2)}$$

where $\quad M = -\dfrac{\partial u}{\partial y}$ and $N = \dfrac{\partial u}{\partial x}$

so that $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(-\dfrac{\partial u}{\partial y}\right) = -\dfrac{\partial^2 u}{\partial y^2}$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}$

since u is a conjugate function, so $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$$\Rightarrow \qquad -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} \qquad \Rightarrow \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{... (3)}$$

Equation (3) satisfies the condition of an exact differential equation.
So equation (2) can be integrated and thus v is determined.

Case II. Similarly, if $v = v(x, y)$ is given
To find out u.

We know that $\quad du = \dfrac{\partial u}{\partial x} dx + i \dfrac{\partial u}{\partial y} dy \qquad$ ... (4)

On substituting the values of $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ in (4) , we get

$$du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

On integrating, we get

$$u = \int \frac{\partial v}{\partial y} dx - \int \frac{\partial v}{\partial x} dy \qquad \text{...(5)}$$

(since v is already known so $\dfrac{\partial v}{\partial y}, \dfrac{\partial v}{\partial x}$ on R.H.S. are also known)

Equation (5) is an exact differential equation. On solving (5), u can be determined.
Consequently $f(z) = u + iv$ can also be determined.

7.52 SINGULAR POINT

A point at which a function $f(z)$ is not analytic is known as a singular point or **singularity** of the function.

For example, the function $\dfrac{1}{z-2}$ has a singular point at $z - 2 = 0$ or $z = 2$.

Isolated singular point. If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$; otherwise it is called non-isolated.

For example, the function $\dfrac{1}{(z-1)(z-3)}$ has two isolated singlar points, namely $z = 1$ and $z = 3$. $\quad$ [Put $(z-1)(z-3) = 0 \quad \Rightarrow \quad z = 1, 3$].

Example of non-isolated singularity. Function $\dfrac{1}{\sin\dfrac{\neq}{z}}$ is not analytic at the points where $\sin\dfrac{\neq}{z} = 0$, *i.e.*, at the points $\dfrac{\pi}{z} = n\pi$ *i.e.*, the points $z = \dfrac{1}{n}\ (n = 1, 2, 3, \ldots)$. Thus $z = 1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots, z = 0$ are the points of singularity. $z = 0$ is the **non-isolated singularity** of the function $\dfrac{1}{\sin\dfrac{\neq}{z}}$ because in the neighbourhood of $z = 0$, there are infinite number of other singularities $z = \dfrac{1}{n}$, where n is very large.

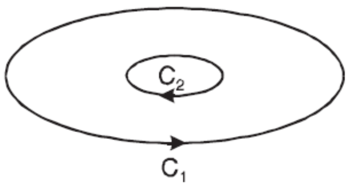
Show that an analytic function of constant modulus is constant.

Harmonic function, Pole, fixed point, Mobius Transformation, By linear transformation

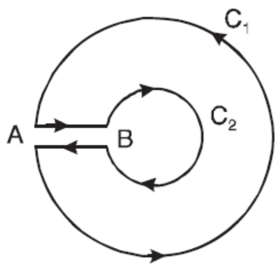
(*i*) **Simply connected Region.** A connected region is said to be a simply connected if all the interior points of a closed curve C drawn in the region D are the points of the region D.

(*ii*) **Multi-Connected Region.** Multi-connected region is bounded by more than one curve. We can convert a multi-connected region into a simply connected one, by giving it one or more cuts.

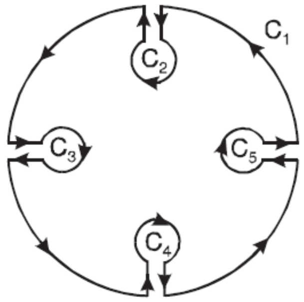
Note. A function $f(z)$ is said to be **meromorphic** in a region R if it is analytic in the region R except at a finite number of poles.



Multi-Connected Region



Simply Connected Region



Simply Connected Region

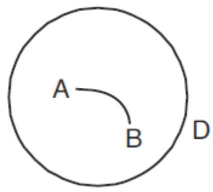
(*iii*) **Single-valued and Multi-valued function**

If a function has only one value for a given value of z, then it is a single valued function.

For example $\quad f(z) = z^2$

If a function has more than one value, it is known as multi-valued function,

For example $\quad f(z) = z^{\frac{1}{2}}$



(*vi*) **Multiple point.** If an equation is satisfied by more than one value of the variable in the given range, then the point is called a multiple point of the arc.

(*vii*) **Jordan arc.** A continuous arc without multiple points is called a Jordan arc.

Regular arc. If the derivatives of the given function are also continuous in the given range, then the arc is called a regular arc.

(*viii*) **Contour.** A contour is a Jordan curve consisting of continuous chain of a finite number of regular arcs.

The contour is said to be closed if the starting point A of the arc coincides with the end point B of the last arc.

(*ix*) **Zeros of an Analytic function.**

The value of z for which the analytic function $f(z)$ becomes zero is said to be the zero of $f(z)$. **For example,** (1) Zeros of $z^2 - 3z + 2$ are $z = 1$ and $z = 2$.

(2) Zeros of $\cos z$ is $\pm\ (2n-1)\dfrac{\pi}{2}$, where $n = 1, 2, 3$..........

4.4 Properties of Integrals

Suppose $f(z)$ and $g(z)$ are integrable along C. Then the following hold:

(a) $\displaystyle\int_C f(z) + g(z)\} \, dz = \int_C f(z)\, dz + \int_C g(z)\, dz$

(b) $\displaystyle\int_C Af(z)\, dz = A\int_C f(z)\, dz$ where $A =$ any constant

(c) $\displaystyle\int_a^b f(z)\, dz = -\int_b^a f(z)\, dz$

(d) $\displaystyle\int_a^b f(z)\, dz = \int_a^m f(z)\, dz + \int_m^b f(z)\, dz$ where points a, b, m are on C

(e) $\displaystyle\left|\int_C f(z)\, dz\right| \le ML$

where $|f(z)| \le M$, i.e., M is an *upper bound* of $|f(z)|$ on C, and L is the *length* of C.

4.9 Green's Theorem in the Plane

Let $P(x, y)$ and $Q(x, y)$ be continuous and have continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C. *Green's theorem* states that

$$\oint_C P\, dx + Q\, dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy \tag{4.7}$$

The theorem is valid for both simply- and multiply-connected regions.

7.24 CAUCHY'S INTEGRAL THEOREM

(*AMIETE, Dec. 2009, U.P. III Semester, 2009-2010, R.G.P.V., Bhopal, III Semester, Dec. 2002*)

If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve c, then $\int_c f(z)dz = 0$.

Proof. Let the region enclosed by the curve c be R and let

$$f(z) = u + iv, \quad z = x + iy, \; dz = dx + idy$$

$$\int_c f(z)\, dz = \int_c (u+iv)(dx+idy) = \int_c (u\, dx - v\, dy) + i\int_c (v\, dx + u\, dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy + i\iint_c \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy \qquad \text{(By Green's theorem)}$$

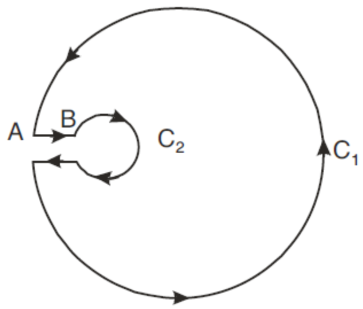
Replacing $-\dfrac{\partial v}{\partial x}$ by $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$ by $\dfrac{\partial u}{\partial x}$ we get

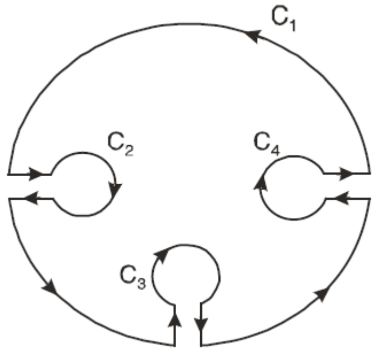
$$\int_c f(z)\, dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\, dy + i\iint_c \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\, dy = 0 + i0 = 0$$

7.25 EXTENSION OF CAUCHY'S THEOREM TO MULTIPLE CONNECTED REGION

If $f(z)$ is analytic in the region R between two simple closed curves c_1 and c_2 then

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$





Proof. $\int f(z)\,dz = 0$

where the path of integration is along AB, and curves c_2 in clockwise direction and along BA and along c_1 in anticlockwise direction.

$$\int_{AB} f(z)\,dz - \int_{c_2} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{c_1} f(z)dz = 0$$

$$\Rightarrow \qquad -\int_{c_2} f(z)\,dz + \int_{c_1} f(z)\,dz = 0 \qquad \text{as } \int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz$$

$$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$$

Proved.

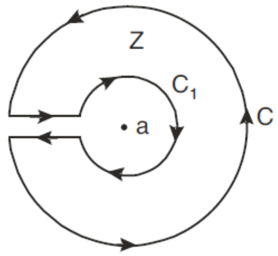
Corollary. $\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz + \int_{c_4} f(z)\,dz$

7.26 CAUCHY INTEGRAL FORMULA

If $f(z)$ is analytic within and on a closed curve C, and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_z \frac{f(z)}{z-a}\,dz$$

(*AMIETE June 2010, U.P., III Semester Dec. 2009 R.G.P.V., Bhopal, III Semester, June 2008*)



Proof. Consider the function $\frac{f(z)}{z-a}$, which is analytic at all points within C, except $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1; hence by Cauchy's Integral Theorem for multiple connected region, we have

$$\int_c \frac{f(z).dz}{z-a} = \int_{c_1} \frac{f(z)}{z-a}\,dz = \int_{c_1} \frac{f(z)-f(a)+f(a)}{z-a}\,.\,dz$$

$$= \int_{c_1} \frac{f(z)-f(a)}{z-a}\,dz + f(a)\int_{c_1} \frac{dz}{z-a} \qquad \text{... (1)}$$

For any point on C_1

$$\text{Now, } \int_{c_1} \frac{f(z)-f(a)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})-f(a)}{re^{i\theta}}\, ire^{i\theta}d\theta \qquad [\, z-a = re^{i\theta} \text{ and } dz = ire^{i\theta}d\theta \,]$$

$$= \int_0^{2\pi} [\, f(a+re^{i\theta}) - f(a)]\; id\theta = 0 \qquad \text{(where } r \text{ tends to zero).}$$

$$\int_{c_1} \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = \int_0^{2\pi} id\theta = i[\theta]_0^{2\pi} = 2\pi i$$

Putting the values of the integrals in R.H.S. of (1), we have

$$\int_c \frac{f(z)\,dz}{z-a} = 0 + f(a)\,(2\pi i) \;\Rightarrow\; f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$$

Proved.

7.27 CAUCHY INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION

(R.G.P.V., Bhopal, III Semester, Dec. 2007)

If a function $f(z)$ is analytic in a region R, then its derivative at any point $z = a$ of R is also analytic in R, and is given by

$$f'(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^2} dz$$

where c is any closed curve in R surrounding the point $z = a$.

Proof. We know Cauchy's Integral formula

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)} dz \qquad \ldots (1)$$

Differentiating (1) w.r.t. 'a', we get

$$f'(a) = \frac{1}{2\pi i} \int_c f(z) \frac{\partial}{\partial a}\left(\frac{1}{z-a}\right) dz$$

$$\boxed{f'(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^2} dz}$$

Similarly, $\boxed{f''(a) = \frac{2!}{2\pi i} \int_c \frac{f(z)\,dz}{(z-a)^3}} \Rightarrow \boxed{f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)\,dz}{(z-a)^{n+1}}}$

7.50 ZERO OF ANALYTIC FUNCTION

A zero of analytic function $f(x)$ is the value of z for which $f(z) = 0$.

Example 92. *Find out the zeros and discuss the nature of the singularities of*

$$f(z) = \frac{(z-2)}{z^2} \sin\left(\frac{1}{z-1}\right) \qquad \textit{(R.G.P.V. Bhopal, III Semester, Dec. 2004)}$$

Solution. Poles of $f(z)$ are given by equating to zero the denominator of $f(z)$ i.e. $z = 0$ is a pole of order two.

zeros of $f(z)$ are given by equating to zero the numerator of $f(z)$ *i.e.*, $(z-2)\sin\left(\frac{1}{z-1}\right) = 0$

$$\Rightarrow \qquad \text{Either } z - 2 = 0 \quad \text{or} \quad \sin\left(\frac{1}{z-1}\right) = 0$$

$$\Rightarrow \qquad z = 2 \qquad \text{and} \qquad \frac{1}{z-1} = n\pi$$

$$\Rightarrow \qquad z = 2, \qquad z = \frac{1}{n\pi} + 1, \quad n = \pm 1, \pm 2, \ldots\ldots\ldots$$

Thus, $z = 2$ is a simple zero. The limit point of the zeros are given by

$$z = \frac{1}{n\pi} + 1 \quad (n = \pm 1, \pm 2, \ldots\ldots\ldots\ldots) \quad \text{is } z = 1.$$

Hence $z = 1$ is an isolated essential singularity. **Ans.**

7.56 DEFINITION OF THE RESIDUE AT A POLE

Let $z = a$ be a pole of order m of a function $f(z)$ and C_1 circle of radius r with centre at $z = a$ which does not contain any other singularities except at $z = a$ then $f(z)$ is analytic within the annulus $r < |z - a| < R$ can be expanded within the annulus. Laurent's series:

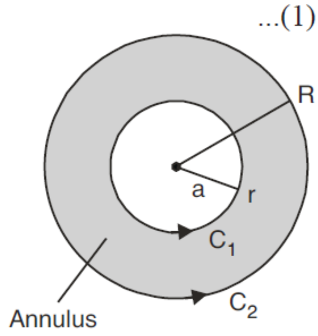
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \qquad \ldots(1)$$

where
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad \ldots(2)$$

and
$$b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-a)^{-n+1}}\,dz \qquad \ldots(3)$$

$|z - a| = r$ being the circle C_1.

Particularly,
$$b_1 = \frac{1}{2\pi i}\int_{C_1} f(z)\,dz$$



The coefficient b_1 is called residue of $f(z)$ at the pole $z = a$. It is denoted by symbol Res. $(z = a) = b_1$.

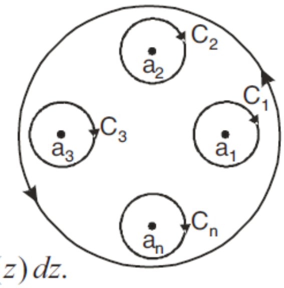
7.57 RESIDUE AT INFINITY

Residue of $f(z)$ at $z = \infty$ is defined as $-\frac{1}{2\pi i}\int_C f(z)\,dz$ where the integration is taken round C in anti-clockwise direction.

where C is a large circle containing all finite singularities of $f(z)$.

7.64 CAUCHY'S RESIDUE THEOREM *(MDU, DEC. 2008)*

If $f(z)$ is analytic in a closed curve C, except at a finite number of poles within C, then $\int_c f(z)\,dz = 2\pi i$ **(sum of residues at the poles within C).**



Proof. Let $C_1, C_2, C_3 ...., C_n$ be the non-intersecting circles with centres at $a_1, a_2, a_3, ..., a_n$ respectively, and radii so small that they lie entirely within the closed curve C. Then $f(z)$ is analytic in the multiple connected region lying between the curves C and $C_1, C_2, ..., C_n$.

Applying Cauchy's theorem

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz + ... + \int_{c_n} f(z)\,dz.$$

$$= 2\pi i\,[\text{Res } f(a_1) + \text{Res } f(a_2) + \text{Res } f(a_3) + ... + \text{Res } f(a_n)]$$ **Proved.**

State Taylor's theorem in complex domain.

6.7 Taylor's Theorem

Let $f(z)$ be analytic inside and on a simple closed curve C. Let a and $a+h$ be two points inside C. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots \tag{6.3}$$

or writing $z = a + h$, $h = z - a$,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots \tag{6.4}$$

This is called *Taylor's theorem* and the series (6.3) or (6.4) is called a *Taylor series* or *expansion* for $f(a+h)$ or $f(z)$.

The region of convergence of the series (6.4) is given by $|z-a| < R$, where the radius of convergence R is the distance from a to the nearest singularity of the function $f(z)$. On $|z-a| = R$, the series may or may not converge. For $|z-a| > R$, the series diverges.

If the nearest singularity of $f(z)$ is at infinity, the radius of convergence is infinite, i.e., the series converges for all z.

If $a = 0$ in (6.3) or (6.4), the resulting series is often called a *Maclaurin series*.

6.9 Laurent's Theorem

Let C_1 and C_2 be concentric circles of radii R_1 and R_2, respectively, and center at a [Fig. 6-1]. Suppose that $f(z)$ is single-valued and analytic on C_1 and C_2 and, in the ring-shaped region $\mathcal{R}$ [also called the *annulus* or *annular region*] between C_1 and C_2, is shown shaded in Fig. 6-1. Let $a+h$ be any point in $\mathcal{R}$. Then we have

$$f(a+h) = a_0 + a_1 h + a_2 h^2 + \cdots + \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \frac{a_{-3}}{h^3} + \cdots \quad (6.5)$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad n = 0, 1, 2, \ldots \\ a_{-n} &= \frac{1}{2\pi i} \oint_{C_1} (z-a)^{n-1} f(z)\, dz \quad n = 1, 2, 3, \ldots \end{aligned} \quad (6.6)$$

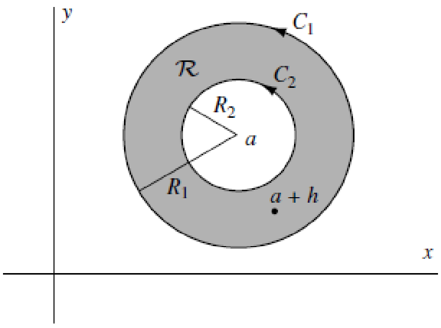


Fig. 6-1

C_1 and C_2 being traversed in the positive direction with respect to their interiors.

In the above integrations, we can replace C_1 and C_2 by any concentric circle C between C_1 and C_2 [see Problem 6.100]. Then, the coefficients (6.6) can be written in a single formula,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad n = 0, \pm 1, \pm 2, \ldots \quad (6.7)$$

With an appropriate change of notation, we can write the above as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \quad (6.8)$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta \qquad n = 0, \pm 1, \pm 2, \ldots \quad (6.9)$$

This is called *Laurent's theorem* and (6.5) or (6.8) with coefficients (6.6), (6.7), or (6.9) is called a *Laurent series* or *expansion*.

The part $a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$ is called the *analytic part* of the Laurent series, while the remainder of the series, which consists of inverse powers of $z-a$, is called the *principal part*. If the principal part is zero, the Laurent series reduces to a Taylor series.

3.13 Curves

Suppose $\phi(t)$ and $\psi(t)$ are real functions of the real variable t assumed continuous in $t_1 \le t \le t_2$. Then the parametric equations

$$z = x + iy = \phi(t) + i\psi(t) = z(t), \quad t_1 \le t \le t_2 \tag{3.16}$$

define a *continuous curve* or *arc* in the z plane joining points $a = z(t_1)$ and $b = z(t_2)$ [see Fig. 3-5].

If $t_1 \ne t_2$ while $z(t_1) = z(t_2)$, i.e., $a = b$, the endpoints coincide and the curve is said to be *closed*. A closed curve that does not intersect itself anywhere is called a *simple closed curve*. For example, the curve of Fig. 3-6 is a simple closed curve while that of Fig. 3-7 is not.

If $\phi(t)$ and $\psi(t)$ [and thus $z(t)$] have continuous derivatives in $t_1 \le t \le t_2$, the curve is often called a *smooth curve* or *arc*. A curve, which is composed of a finite number of smooth arcs, is called a *piecewise* or *sectionally smooth* curve or sometimes a *contour*. For example, the boundary of a square is a piecewise smooth curve or contour.

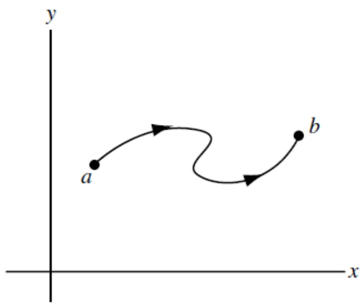


Fig. 3-5

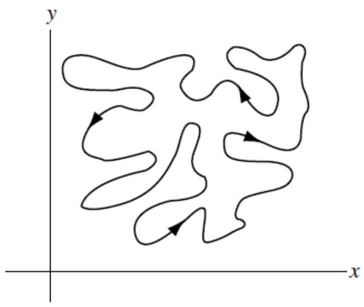


Fig. 3-6

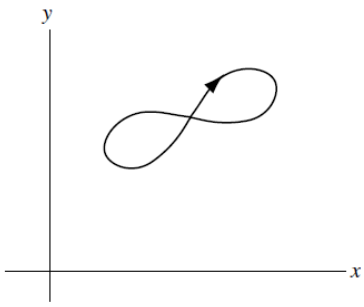


Fig. 3-7

7.65 EVALUATION OF REAL DEFINITE INTEGRALS BY CONTOUR INTEGRATION

A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated by using Cauchy's theorem of residues. For finding the integrals we take a closed curve C, find the poles of the function $f(z)$ and calculate residues at those poles only which lie within the curve C.

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of the residues of } f(z) \text{ at the poles within } C)$$

We call the curve, a contour and the process of integration along a contour is called contour integration.

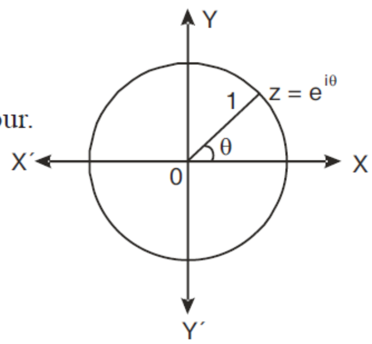
7.66 INTEGRATION ROUND UNIT CIRCLE OF THE TYPE

$$\int_0^{2\pi} f(\cos\theta,\ \sin\theta)\,d\theta$$

where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

Convert $\sin\theta$, $\cos\theta$ into z.

Consider a circle of unit radius with centre at origin, as contour.



$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left[z - \frac{1}{z}\right], \qquad z = re^{i\theta} = 1.\, e^{i\theta} = e^{i\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left[z + \frac{1}{z}\right]$$

As we know

$$z = e^{i\theta} \Rightarrow dz = e^{i\theta}\, i\, d\theta = z\, i\, d\theta \text{ or } d\theta = \frac{dz}{iz}$$

The integrand is converted into a function of z.

Then apply Cauchy's residue theorem to evaluate the integral.

Transformation:

Conformal Mapping, Bilinear transformation.

Statistics

Define: Measures of Dispersion, Random Variable with classification, Normal Distribution, Binomial Distribution, Poisson Distribution, Moment, Kurtosis, Skewness of Data,

Define: Conditional probability, Estimator with properties of good estimator, Population with classification.

Probability

Probability: Probability is the concept which numerically measures the degree of uncertainty and therefore certainty of the occurrence of the event.

Define: Probability Mass Function, Probability Density Function, Mutually exclusive and exhausting events

State conditional probability of two events.

Show that Poisson distribution is the limiting form of the binomial distribution.

Write down the probability mass function of Binomial distribution. Show that the main and variance of poison distribution are same.

Statistical Hypothesis: A statement framed in terms of restrictions on the statistical model.

There is a reason to doubt the Hypothesis if the outcomes we observe have a low probability when the hypothesis is time.

Ex: Coin Tossed- Claim p(tail) = 0.8 when tossed.

Tossed 10 times and lands tails up 4 times.

Binomial: 10C4 $(0.8)^4(0.2)^6 = 0.005$

When the claim is true, we can thus take the outcomes of the experiment is evidence against the claim.

On the other hand, seven tails in the 10 tosses 10C7 $(0.8)^7(0.2)^3 = 0.201$

When the claim is time, we have little reason to doubt the claim.

Define hypothesis testing with the application.

Define: Null and Alternative Hypothesis Write down the procedures consisting of a statistical hypothesis.