Complex Integration

1. State and prove Cauchy's theorem / Cauchy's integral theorem.

Statement: If $f(z)$ is analytic inside and on a simple closed curve C, then $\oint_C f(z)\,dz = 0$.

Proof: Let $f(z) = u(x,y) + iv(x,y)$ be analytic.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---(1)}$$

Since $z = x+iy$, so $dz = dx + i\,dy$

Now $\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy)$

$$= \oint_C (u\,dx + iu\,dy + iv\,dx - v\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) \quad \cdots(1)$$

By Green's theorem we have,

$$\oint_C (u\,dx - v\,dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

and $\oint_C (v\,dx + u\,dy) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$

where R is the region bounded by C.

Hence (1) becomes,

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

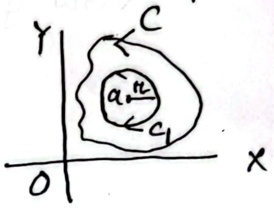
$$= 0 + i\cdot 0$$

$$= 0$$

2. State and Prove Cauchy's integral formula.

Statement: If $f(z)$ is analytic inside and on a simple closed curve C, and 'a' is any point within C, then $\oint_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

Proof: Since $f(z)$ is analytic inside and on C, $\frac{f(z)}{z-a}$ is also analytic inside and on C, except at the point $z=a$. Hence, we draw a small circle with centre at $z=a$ and radius r lying entirely inside C.



Now, $\frac{f(z)}{z-a}$ is analytic in the region enclosed between C and C_1.

Hence, by Cauchy's extended theorem,

$$\oint_C \frac{f(z)dz}{z-a} = \oint_{C_1} \frac{f(z)\,dz}{z-a} \quad \text{---} (1)$$

On C_1, any point z is given by $z = a + re^{i\theta}$

$\therefore dz = ire^{i\theta}d\theta$

where θ varies from 0 to 2π.

$$\therefore \oint_{C_1} \frac{f(z)dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta}).ire^{i\theta}d\theta}{re^{i\theta}}$$

$$= i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

As $r \to 0$, the circle tends to a point.

Taking limit $r \to 0$, we get

$$\oint_{C_1} \frac{f(z)dz}{z-a} = i\int_{\theta=0}^{2\pi} f(a)\,d\theta$$

$$= if(a)\,[\theta]_{\theta=0}^{2\pi}$$

$$= 2\pi i\, f(a)$$

So from (1), we get $\oint_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

* In general, $\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{\lfloor n} f^{(n)}(a)$

where $n = 0, 1, 2, 3, \ldots$

and $f^{(0)}(a) = f(a)$

* <u>Cauchy's extended theorem</u>: If $f(z)$ is analytic within and on the boundary of a region bounded by two closed curves C_1 and C_2, then

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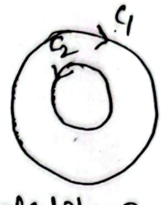
$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$



3. Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^z\,dz}{z-2}$ if C is (a) the circle $|z| = 3$, (b) the circle $|z| = 1$.

<u>Solution</u>: (a) Here $f(z) = e^z$ is analytic inside and on the circle $|z| = 3$ and $z = a = 2$ is a point inside the given circle.

Then by using the Cauchy's integral formula, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\therefore \oint_C \frac{e^z}{z-2}\,dz = 2\pi i \cdot e^2$$

$$\text{or, } \frac{1}{2\pi i} \oint_C \frac{e^z}{z-2}\,dz = e^2$$

(b) Here $f(z) = \frac{e^z}{z-2}$ is analytic inside and on the circle $|z| = 1$ and $z = 2$ is a point outside the given circle.

Then by using the Cauchy's integral theorem, $\oint_C f(z)\,dz = 0$ we get—

$$\oint_C \frac{e^z}{z-2}\,dz = 0 \text{ or, } \frac{1}{2\pi i} \oint_C \frac{e^z}{z-2}\,dz = 0$$

4. Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ if C is the circle $|z|=5$.

Solution: Here $f(z)=\sin 3z$ is analytic inside and on the circle $|z|=5$ and $z=a=-\frac{\pi}{2}$ lies inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \text{ we get}$$

$$\oint_C \frac{\sin 3z}{z-(-\pi/2)}\,dz = 2\pi i f(-\tfrac{\pi}{2})$$

$$\text{or, } \oint_C \frac{\sin 3z}{z+\pi/2} = 2\pi i$$

$\because f(-\frac{\pi}{2}) = \sin 3(-\frac{\pi}{2})$
$= -(-1)$
$= 1$

5. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ if C is (a) the circle $|z-1|=4$, (b) the ellipse $|z-2|+|z+2|=6$.

Solution: (a) Here $f(z)=e^{3z}$ is analytic inside and on the given circle $|z-1|=4$, and $z=\pi i$ is a point inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \text{ we get}$$

$$\oint_C \frac{e^{3z}}{z-\pi i}\,dz = 2\pi i f(\pi i)$$
$$= 2\pi i \cdot e^{3\pi i}$$
$$= 2\pi i(\cos 3\pi + i\sin 3\pi)$$
$$= 2\pi i(-1+i\cdot 0)$$
$$= -2\pi i$$

(b) Here $f(z)=\dfrac{e^{3z}}{z-\pi i}$ is analytic inside and on the ellipse C, and $z=\pi i$ lies outside the given ellipse C.

Then by using Cauchy's integral theorem,

$$\oint_C f(z)\,dz = 0 \text{ we get}$$

$$\oint_C \frac{e^{3z}}{z-\pi i}\,dz = 0$$

** Locus of $|z-2|+|z+2|=6$ is $\frac{x^2}{3^2}+\frac{y^2}{(\sqrt{5})^2}=1$.

Its foci $(\pm ae, 0) = (\pm 3\cdot\frac{2}{3}, 0) = (\pm 2, 0)$, $e=\sqrt{1-\frac{5}{9}}$
and length of major axis $= 2\cdot 3 = 6$ $= \frac{2}{3}$

6. Evaluate $\frac{1}{2\pi i}\oint_C \frac{\cos \pi z}{z^2-1}dz$ around a rectangle with vertices at: (a) $2\pm i, -2\pm i$ (b) $-i, 2-i, 2+i, i$.

Solution: We have $\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz = \frac{1}{4\pi i}\oint_C\left[\frac{\cos\pi z}{z-1} - \frac{\cos\pi z}{z+1}\right]dz$

$$= \frac{1}{4\pi i}\left[\oint_C \frac{\cos\pi z}{z-1}dz - \oint_C \frac{\cos\pi z}{z+1}dz\right] \quad \cdots (1)$$

(a) Here $f(z) = \cos\pi z$ is analytic inside and on C, and also both points $z = \pm 1$ lie inside the rectangle $2\pm i$, $-2\pm i$.

Then by using Cauchy's integral formula, $\oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$, we get from (1)

$$\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz = \frac{1}{4\pi i}\left[2\pi i\cos\pi - 2\pi i\cos(-\pi)\right]$$

$$= \frac{1}{4\pi i}\left[-2\pi i + 2\pi i\right]$$

$$= 0$$

(b) Here only the point $z=1$ lies inside the rectangle $\pm i$, $2\pm i$.

Then by using the Cauchy's integral formula and also the Cauchy's integral theorem, we get from (1),

$$\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz = \frac{1}{4\pi i}\left[2\pi i\cos\pi - 0\right]$$

$$= -\frac{1}{2}$$

7. Show that $\frac{1}{2\pi i}\oint_C \frac{e^{zt}dz}{z^2+1} = \sin t$ if $t>0$ and C is the circle $|z|=3$

Solution: We have $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}dz = \frac{1}{2\pi i}\oint_C \frac{e^{zt}dz}{(z+i)(z-i)}$

$$= \frac{1}{2\pi i}\cdot\frac{1}{2i}\left[\oint_C \frac{e^{zt}}{z-i}dz - \oint_C \frac{e^{zt}dz}{z+i}\right] \quad \cdots (1)$$

Here $f(z) = e^{zt}$ is analytic inside and on the given circle $|z|=3$ and $z = \pm i$ are inside C.

Then by using Cauchy's integral formula, we get from (1)

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz = \frac{1}{2\pi i}\cdot\frac{1}{2i}\left[2\pi i f(i) - 2\pi i f(-i)\right]$$

$$= \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$

$$= \frac{1}{2i}\cdot 2i\sin t \quad \left[\because e^{it} - e^{-it} = 2i\sin t\right]$$

$$= \sin t$$

8. Evaluate $\oint_C \frac{e^{iz}}{z^3}\,dz$ where C is the circle $|z|=2$.

Solution: Here $f(z)=e^{iz}$ is analytic inside and on the circle $|z|=2$ and $z=a=0$ is a point inside the given circle.

Then by using Cauchy's integral formula,

$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{\lfloor n} f^{(n)}(a)$, we get

$$\oint_C \frac{f(z)}{(z-0)^3}\,dz = \frac{2\pi i}{\lfloor 2} f^{(2)}(0)$$

$$\therefore \oint_C \frac{e^{iz}}{z^3}\,dz = \frac{2\pi i}{2}\cdot(-1)$$

$$= -\pi i$$

$f(z) = e^{iz}$

$\therefore f'(z) = i e^{iz}$

$f''(z) = -e^{iz}$

$f''(0) = -1 = f^{(2)}(0)$

9. Find the value of (a) $\oint_C \frac{\sin^6 z}{z-\frac{\pi}{6}}\,dz$, (b) $\oint_C \frac{\sin^6 z\,dz}{(z-\frac{\pi}{6})^3}$ if C is the circle $|z|=1$.

Solution: Here $f(z)=\sin^6 z$ is analytic inside and on the circle $|z|=1$ and $z=a=\frac{\pi}{6}$ is a point inside the given circle.

(a) Then by using Cauchy's integral formula,

$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$ we get

$$\oint_C \frac{\sin^6 z}{z-\frac{\pi}{6}}\,dz = 2\pi i\left(\sin\frac{\pi}{6}\right)^6$$

$$= 2\pi i\cdot\left(\frac{1}{2}\right)^6$$

$$= 2\pi i\cdot\frac{1}{64}$$

$$= \frac{\pi i}{32}$$

(b) Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{\lfloor n} f^{(n)}(a) \text{ we get}$$

$$\oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}dz = \frac{2\pi i}{\lfloor 2} f^{(2)}\left(\frac{\pi}{6}\right) \quad \cdots (1)$$

We have $f(z) = \sin^6 z$

$\therefore f'(z) = 6\sin^5 z \cdot \cos z$

$f''(z) = 30\sin^4 z \cos^2 z + 6\sin^5 z(-\sin z)$

$$\therefore f''\left(\frac{\pi}{6}\right) = f^{(2)}\left(\frac{\pi}{6}\right) = 30 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{\sqrt{3}}{2}\right)^2 - 6 \cdot \left(\frac{1}{2}\right)^6$$

$$= 30 \cdot \frac{1}{16} \cdot \frac{3}{4} - 6 \cdot \frac{1}{64}$$

$$= \frac{90-6}{64}$$

$$= \frac{84}{64}$$

$$= \frac{21}{16}$$

So from (1) we get

$$\oint_C \frac{\sin^6 z \, dz}{(z-\frac{\pi}{6})^3} = \frac{2\pi i}{2} \cdot \frac{21}{16}$$

$$= \frac{21\pi i}{16}$$

10. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$ if $t>0$ and C is the circle $|z|=3$.

Solution: We have, $$\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}$$

$$= \frac{1}{4iz}\left[\frac{1}{(z-i)^2} - \frac{1}{(z+i)^2}\right]$$

$$\therefore \oint_C \frac{e^{zt}\,dz}{(z^2+1)^2} = \frac{1}{4i}\left[\oint_C \frac{\frac{e^{zt}}{z}\,dz}{(z-i)^2} - \oint_C \frac{\frac{e^{zt}}{z}\,dz}{(z+i)^2}\right]$$

Here $f(z) = \frac{e^{zt}}{z}$ is analytic inside on the given circle $|z|=3$ and $z = \pm i$ are inside C.

Then by using Cauchy's integral theorem,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{\lfloor n} f^{(n)}(a) \text{ we get}$$

$$\oint_C \frac{e^{zt}dz}{(z^2+1)^2} = \frac{1}{4i}\left[\frac{2\pi i}{\lfloor 1} f'(i) - \frac{2\pi i}{\lfloor 1} f'(-i)\right] \cdots (1)$$

we have, $f(z) = \dfrac{e^{zt}}{z}$

$$\therefore f'(z) = \frac{z.e^{zt}.t - 1.e^{zt}}{z^2}$$

$$\therefore f'(i) = \frac{ite^{it} - e^{it}}{i^2}$$

$$= e^{it} - ite^{it} \quad [\because i^2 = -1]$$

Also, $f'(-i) = \dfrac{-ite^{-it} - e^{-it}}{(-i)^2}$

$$= ite^{-it} + e^{-it}$$

So from (1) we get

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}dz}{(z^2+1)^2} = \frac{1}{4i}\left[e^{it} - ite^{it} - ite^{-it} - e^{-it}\right]$$

$$= \frac{1}{4i}\left[(e^{it} - e^{-it}) - it(e^{it} + e^{-it})\right]$$

$$= \frac{1}{4i}\left[2i\sin t - it.2\cos t\right]$$

$$= \frac{1}{4i}.2i\left[\sin t - t\cos t\right]$$

$$= \frac{1}{2}(\sin t - t\cos t) \quad \left[\begin{array}{l}\because e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \\ \Rightarrow 2\cos\theta = e^{i\theta} + e^{-i\theta} \\ 2i\sin\theta = e^{i\theta} - e^{-i\theta}\end{array}\right]$$

11. Evaluate $\oint_C \dfrac{e^z dz}{z(1-z)^3}$ if (i) 0 lies inside C and 1 lies outside C, (ii) 1 lies inside C and 0 lies outside C, (iii) 0 and 1 lie inside C.

Solution: (i) Since 0 lies inside C and 1 lies outside C.

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{f(z)}{z-0}dz \text{ where } f(z) = \frac{e^z}{(1-z)^3}$$

$$= 2\pi i f(0)$$

$$= 2\pi i \cdot 1$$

$$= 2\pi i$$

(ii) Since 1 lies inside C and 0 lies outside C.

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{f(z)\,dz}{(z-1)^3}$$

$$= \frac{2\pi i}{\lfloor 2} f''(1)$$

Where, $f(z) = -\frac{e^z}{z}$

$$\therefore f'(z) = \frac{e^z}{z^2} - \frac{e^z}{z}$$

$$f''(z) = -\frac{2e^z}{z^3} + \frac{e^z}{z^2} + \frac{e^z}{z^2} - \frac{e^z}{z}$$

$$\therefore f''(1) = -2e^1 + 2e^1 - e^1$$

$$= -e$$

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \frac{2\pi i}{2} \cdot (-e)$$

$$= -\pi i e$$

(iii) Since 0 and 1 lie inside C, so we express $\frac{1}{z(1-z)^3}$ in partial fractions.

Let $\frac{1}{z(1-z)^3} = \frac{1}{z} + \frac{1}{1-z} + \frac{A}{(1-z)^2} + \frac{B}{(1-z)^3}$ ...(1)

$\Rightarrow 1 = (1-z)^3 + z(1-z)^2 + Az(1-z) + Bz$ ...(2)

Putting $z=1$ in (2), we get $1 = B$

Equating the coefficients of z^2 from the both sides of (2), we get

$$0 = 3 - 2 - A \quad \text{or, } A = 1$$

So from (1) we get

$$\frac{1}{z(1-z)^3} = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{(1-z)^2} + \frac{1}{(1-z)^3}$$

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{e^z dz}{z} + \oint_C \frac{e^z dz}{1-z} + \oint_C \frac{e^z dz}{(1-z)^2} + \oint_C \frac{e^z dz}{(1-z)^3}$$

$$= 2\pi i (e^0) - 2\pi i \cdot (e^1) + \frac{2\pi i}{\lfloor 1} f'(1) - \frac{2\pi i}{\lfloor 2} f''(1)$$

$$= 2\pi i - 2\pi i e + 2\pi i e - \frac{2\pi i}{2} \cdot e$$

$$= \pi i (2 - e)$$

12. What is the value of $\oint_C \frac{z^2+1}{z^2-1}\,dz$ if C is a circle of unit radius with centre at (i) $z=1$ and (ii) $z=-1$.

Solution: (i) If C is a circle of unit radius with centre at $z=1$, then

$$\oint_C \frac{(z^2+1)dz}{z^2-1} = \oint_C \frac{\frac{z^2+1}{z+1}}{z-1}\,dz$$

$$= 2\pi i f(1) \quad \text{where } f(z)=\frac{z^2+1}{z+1}$$

$$= 2\pi i \cdot 1$$

$$= 2\pi i$$

(ii) If C is a circle of unit radius with centre $z=-1$, then

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = \oint_C \frac{\frac{z^2+1}{z-1}}{z+1}\,dz$$

$$= 2\pi i f(-1) \quad \text{where } f(z)=\frac{z^2+1}{z^2-1}$$

$$= 2\pi i(-1)$$

$$= -2\pi i$$

13. Using Cauchy's integral formula, evaluate $\oint_C \frac{z\,dz}{(z-1)(z-2)}$ where C is the circle $|z-2|=\frac{1}{2}$

Solution: Since $z=2$ is the only point lies inside the circle $|z-2|=\frac{1}{2}$,

$$\therefore \oint_C \frac{z\,dz}{(z-1)(z-2)} = \oint_C \frac{\left(\frac{z}{z-1}\right)dz}{z-2}$$

$$= 2\pi i f(2) \quad \text{where } f(z)=\frac{z}{z-1}$$

$$= 2\pi i \cdot 2$$

$$= 4\pi i$$

14. Evaluate $\oint_C \frac{dz}{(z^2+4)^2}$, where C is the circle $|z-i|=2$

Solution: Let $F(z)=\frac{1}{(z^2+4)^2}$

$\therefore$ Singular points of $f(z)$ are $z=\pm 2i$. Among this only $z=2i$ lies inside the circle $|z-i|=2$.

$$\therefore \oint_C \frac{dz}{(z^2+4)^2} = \oint_C \frac{\frac{1}{(z+2i)^2}dz}{(z-2i)^2}$$

$$= \frac{2\pi i}{\lfloor 1}f'(2i)$$

$$= 2\pi i \cdot \frac{1}{32i}$$

$$= \frac{\pi}{16}$$

where $f(z)=\frac{1}{(z+2i)^2}$

$$\therefore f'(z) = -\frac{2}{(z+2i)^3}$$

$$\therefore f(2i) = -\frac{2}{(4i)^3}$$

$$= \frac{1}{32i}$$