Singular Points, poles, Residues

<u>Singular points</u>: All the points of the z-plane at which an analytic function does not have a unique derivative are said to be singular points.

If $f(z) = \frac{1}{(z-3)^2}$, then $z=3$ is a singularity of $f(z)$.

<u>Poles</u>: If $f(z) = \frac{\phi(z)}{(z-a)^n}$, $\phi(a) \neq 0$, where $\phi(z)$ is analytic everywhere in a region including $z=a$, and if n is a positive integer, then $f(z)$ has a singularity at $z=a$ which is called a pole of order n. If $n=1$, the pole is often called a simple pole; if $n=2$ it is called a double pole, etc.

If $f(z) = \frac{z}{(z-3)^2(z+1)}$ has two singularities; a pole of order 2 or double pole at $z=3$ and a pole of order 1 or simple pole at $z=-1$.

<u>Residues</u>: If $f(z)$ has a pole of order n at $z=a$ but is analytic at every other point inside and on a circle C with centre at a, then the Laurent's series about $z=a$ is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_n (z-a)^{-n}$$

$$\therefore f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$
$$+ \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots$$

The part $a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$ is called the analytic part, while the remainder consisting of inverse powers of $z-a$ is called the principal part.

The coefficient a_{-1}, called the residue of $f(z)$ at the pole $z=a$.

Method of finding residues:

(i) Residue of $f(z)$ at simple pole $z=a$ is

$$\lim_{x \to a} (z-a) f(z)$$

(ii) Residue of $f(z)$ at $z=a$ with pole of order n is

$$\lim_{x \to a} \frac{1}{\lfloor n-1} \frac{d^{n-1}}{dz^{n-1}} \left\{ (z-a)^n f(z) \right\}$$

Problem-1: Determine the residues of each function at its poles:

(i) $\frac{z^2}{(z-2)(z^2+1)}$

Let $f(z) = \frac{z^2}{(z-2)(z^2+1)}$

poles of $f(z)$ are given by

$$(z-2)(z^2+1) = 0$$

$\therefore \quad z = 2, \pm i$

Residue at simple pole $z = 2$ is

$$\lim_{z \to 2} (z-2) f(z)$$

$$= \lim_{z \to 2} (z-2) \cdot \frac{z^2}{(z-2)(z^2+1)}$$

simple pole $= \frac{4}{5}$

Residue at simple pole $z = i$ is

$$\lim_{z \to i} (z-i) \cdot \frac{z^2}{(z-2)(z+i)(z-i)}$$

$$= \frac{i^2}{(i-2) \cdot 2i}$$

$$= \frac{-1}{2i^2 - 4i}$$

$$= \frac{-1}{-2(1+2i)}$$

$$= \frac{1-2i}{10}$$

Residue at simple pole $z = -i$ is

$$\lim_{z \to -i} (z+i) \frac{z^2}{(z-2)(z+i)(z-i)}$$

$$= \frac{1+2i}{10}$$

(ii) $f(z) = \frac{1}{z(z+2)^3}$

poles of $f(z)$ are given by $z(z+2)^3 = 0$

$\therefore z = 0, -2$

$\therefore$ $z = 0$ is a simple pole and $z = -2$ is a pole of order 3.

Residue at simple pole $z=0$ is

$$\lim_{z\to 0} z \cdot \frac{1}{z(z+2)^3}$$

$$= \frac{1}{8}$$

Residue at $z=-2$ (pole of order 3) is

$$\lim_{z\to -2} \frac{1}{\lfloor 2} \frac{d^2}{dz^2}\left\{(z+2)^3 \cdot \frac{1}{z(z+2)^3}\right\}$$

$$= \lim_{z\to -2} \frac{1}{2} \frac{d^2}{dz^2}\left(\frac{1}{z}\right)$$

$$= \lim_{z\to -2} \frac{1}{2}\left(\frac{2}{z^3}\right)$$

$$= -\frac{1}{8}$$

Problem-2: Determine the residues of each function at its poles:

(i) $\frac{2z+3}{z^2-4}$ (ii) $\frac{z-3}{z^3+5z^2}$ (iii) $\frac{e^{zt}}{(z-2)^3}$

(iv) $\frac{z}{(z^2+1)^2}$

Cauchy's residue theorem: If $f(z)$ is analytic within and on a simple closed curve C except at a number of poles $a, b, c, \cdots$ interior to C at which the residues $a_{-1}, b_{-1}, c_{-1}, \cdots$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1}+b_{-1}+c_{-1}+\cdots)$$

$$= 2\pi i\,(\text{sum of residues})$$

Problem-3: Evaluate $\oint_C \frac{e^z dz}{(z-1)(z+3)^2}$ where C is given by (i) $|z| = \frac{3}{2}$, (ii) $|z| = 10$.

Solution: Here $f(z) = \frac{e^z}{(z-1)(z+3)^2}$

poles of $f(z)$ are given by $(z-1)(z+3)^2 = 0$

$\therefore z = 1, -3$

Residue at simple pole $z=1$ is

$$\lim_{z \to 1} \cancel{(z-1)} \cdot \frac{e^z}{\cancel{(z-1)}(z+3)^2}$$

$$= \frac{e}{16}$$

Residue at double pole $z=-3$ is

$$\lim_{z \to -3} \frac{1}{\lfloor 1} \frac{d^1}{dz^1} \left\{ (z+3)^2 \cdot \frac{e^z}{(z-1)\cancel{(z+3)^2}} \right\}$$

$$= \lim_{z \to -3} \frac{d}{dz} \left(\frac{e^z}{z-1} \right)$$

$$= \lim_{z \to -3} \frac{(z-1)e^z - e^z}{(z-1)^2}$$

$$= \frac{-5e^{-3}}{16}$$

(i) since $|z| = \frac{3}{2}$ encloses only the pole $z=1$,

the required integral $= 2\pi i \left(\frac{e}{16} \right)$

$$= \frac{\pi i e}{8}$$

(ii) since $|z| = 10$ encloses both poles $z=1$ and $z=-3$, the required integral

$$= 2\pi i \left(\frac{e}{16} - \frac{5e^{-3}}{16} \right)$$

$$= \frac{\pi i (e - 5e^{-3})}{8}$$

Problem-4: Evaluate $\oint_C \frac{z^2\,dz}{(z+1)(z+3)}$, where C is a simple closed curve enclosing all the poles.

<u>Solution</u>: Here $f(z) = \dfrac{z^2}{(z+1)(z+3)}$

poles of $f(z)$ are given by $(z+1)(z+3)=0$

$\therefore z=-1,-3$

Both poles are simple.

Residue at simple pole $z=-1$ is

$$\lim_{z\to -1} \cancel{(z+1)}\cdot\frac{z^2}{\cancel{(z+1)}(z+3)}$$

$$= \frac{1}{2}$$

Residue at simple pole $z=-3$ is

$$\lim_{z\to -3} \cancel{(z+3)}\cdot\frac{z^2}{(z+1)\cancel{(z+3)}}$$

$$= -\frac{9}{2}$$

Hence by Cauchy's residue theorem, we get

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{1}{2}-\frac{9}{2}\right)$$

$$\therefore \oint_C \frac{z^2\,dz}{(z+1)(z+2)} = -8\pi i$$

<u>OR</u> (<u>Use by Cauchy's integral formula</u>)

$$\oint_C \frac{z^2\,dz}{(z+1)(z+3)} = \oint_C \left\{1+\frac{-4z-3}{(z+1)(z+3)}\right\}dz$$

$$= \oint_C \left\{1+\frac{\frac{1}{2}}{z+1}+\frac{-9/2}{z+3}\right)dz$$

$$= \oint_C dz + \frac{1}{2}\oint_C \frac{1}{z+1}\,dz - \frac{9}{2}\oint_C \frac{1}{z+3}\,dz$$

$$= 0 + \frac{1}{2}\cdot 2\pi i\cdot 1 - \frac{9}{2}\cdot 2\pi i\cdot 1$$

$$= -8\pi i$$