Analytical Mechanics Project: Rotation and Inertia of a T-Handle

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December 24, 2024

Abstract

This project explores the rotational dynamics of a T-shaped rigid body, emphasizing the derivation of its inertia tensor and Euler's equations of motion. The project includes theoretical calculations, numerical simulations, and analysis of stability under various initial conditions.

Github Repository

The full code and simulations for this project are available on GitHub: Mechanics Project - Rotation.

1 Introduction

The study of rigid body dynamics provides fundamental insights into rotational motion. This project focuses on a T-shaped structure, often referred to as a "T-handle," and investigates its inertial properties and dynamic behavior. The inertia tensor and Euler's equations are derived and applied to analyze the rotational stability under different initial perturbations. This specific choice of shape was inspired by a video taken on the international space station that can be viewed here: Dancing T-Handle

2 Inertia Tensor

The inertia tensor $\mathbf{I}$ characterizes a rigid body's resistance to angular acceleration about any axis it may be rotated around.

2.1 General Formulation

For a rigid body with mass m and a distance vector $\mathbf{r}$ from its center of mass, the inertia tensor is given by:

$$I_{ij} = \int_V \rho(\mathbf{r}) \left(\|\mathbf{r}\|^2 \delta_{ij} - r_i r_j\right) \mathrm{d}V \,,$$

where $\rho(\mathbf{r})$ is the mass density and I_{ij} is the ith and jth component of the 3×3 matrix.

2.2 Principle Axis

By definition, the inertia tensor is symmetric when in its matrix form. Thus, it admits an orthogonal diagonalization. In other words, there is a basis in which the matrix only has entries on its diagonal, such that when rotated around these 'principle axes', the angular momentum points in the same direction as angular velocity, since $L = I \cdot \omega$. Therefore, ω is an eigenvector of I.

2.3 Application to the T-Handle

The directions of these princple axes can be found manually using linear algebra techniques, or argued by logical means, such as follows. The T-handle consists of a shaft and a crossbar:

- **Shaft:** A uniform rod of length L_s and mass m_s.
- **Crossbar:** A uniform rod of length L_c and mass m_c, offset from the shaft.

  The center of mass of the system is calculated as:

$$\mathbf{r}_{\text{COM}} = \frac{m_s \mathbf{r}_s + m_c \mathbf{r}_c}{m_s + m_c}. \tag{1}$$

It is well known that a thin rod has a moment of inertia of $\frac{1}{12} m L^2$ about axes orthogonal to the rod itself.

2.3.1 Inertia Tensor for the Shaft

For the shaft centered at the origin:

$$\mathbf{I}_{\text{shaft}} = \begin{bmatrix} \frac{1}{12}m_s L_s^2 & 0 & 0 \\ 0 & \frac{1}{12}m_s L_s^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

2.3.2 Inertia Tensor for the Crossbar

For the crossbar, initially centered along its axis:

$$\mathbf{I}_{\text{crossbar}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}m_c L_c^2 & 0 \\ 0 & 0 & \frac{1}{12}m_c L_c^2 \end{bmatrix}. \tag{3}$$

Since, the default axes we used produced a diagonal matrix, we deduce that these are the principal or 'eigen' axes. These will also hold for the total inertia tensor of the system.

We can combine the moments of inertia of the crossbar and shaft using the parallel axis theorem. The parallel axis theorem states that the moment of inertia of an object about a given axis is equal to the moment of inertia about a parallel axis passing through the center of mass, plus the product of the object's mass and the square of the perpendicular distance between the two axes. Applying the parallel axis theorem:

$$\mathbf{I}_{\text{crossbar, adj}} = \mathbf{I}_{\text{crossbar}} + m_c \left(\|\mathbf{d}_c\|^2 \mathbf{I}_3 - \mathbf{d}_c \mathbf{d}_c^T \right), \tag{4}$$

where $\mathbf{d}_c$ is the displacement vector from the system's center of mass.

2.4 Total Inertia Tensor

The total inertia tensor is the sum of the adjusted tensors thus:

$$\mathbf{I}_{\text{total}} = \mathbf{I}_{\text{shaft, adj}} + \mathbf{I}_{\text{crossbar, adj}}. \tag{5}$$

$$\mathbf{I}_{\text{total}} = \mathbf{I}_{\text{shaft, adj}} + \mathbf{I}_{\text{crossbar, adj}} = \begin{pmatrix} \frac{1}{12}m_s L_s^2 + m_c d_x^2 & 0 & 0 \\ 0 & \frac{1}{12}m_s L_s^2 + \frac{1}{12}m_c L_c^2 & 0 \\ 0 & 0 & \frac{1}{12}m_c L_c^2 + m_c d_z^2 \end{pmatrix},$$

3 The Euler Equations

The general equation for torque about a point P is:

$$\vec{\tau}_{\text{about } P} = \frac{\mathrm{d}\vec{L_{aboutP}}}{\mathrm{d}t},$$

where $\vec{L}$ is the angular momentum. Assuming P is the center of mass (COM), we have that the frame can be non-inertial but cannot rotate and the equation will still hold.

For a rotating COM frame, this equation is not valid. To adapt it for use in a rotating frame, we write:

$$\vec{\tau}_P = \left.\frac{\mathrm{d}\vec{L}_P}{\mathrm{d}t}\right|_{\text{rot}} + \vec{\omega} \times \vec{L}_P,$$

where:

- $\vec{\omega}$ is the angular velocity of the rotating frame.
- The terms are expressed in the body frame aligned with the principal axes.

Body Frame Analysis

Then, since $(\vec{\omega} \times \vec{L}_P)_3 = \omega_1 L_2 - \omega_2 L_1$ and $L_2 = I_{22}\omega_2$ these trends continue, we can define an equation for torque that must be true for a rigid rotating body.

The Euler equations for rotational dynamics are thus:

$$\begin{aligned}
\tau_1 &= I_{11}\dot{\omega}_1 + \omega_2\omega_3(I_{33} - I_{22}), \\
\tau_2 &= I_{22}\dot{\omega}_2 + \omega_3\omega_1(I_{11} - I_{33}), \\
\tau_3 &= I_{33}\dot{\omega}_3 + \omega_1\omega_2(I_{22} - I_{11}).
\end{aligned}$$

4 Tennis Racket Theorem (The Dzhanibekov effect)

For a rotating body, if $I_{11} \neq I_{22} \neq I_{33}$, for the moments of inertia in the principle, 'eigen' basis then dynamics are different when rotated around each axis. Essentially, for the non extreme, middle value (say I_{22}) that corresponds to a certain principle axis, then rotating around this principle axis will be unstable, in the sense that rotations around it with small perturbations in the direction of other axes will cause these small perturbations to grow and oscillate, becoming unstable in nature.

For the T-handle, we consider rotational motion about all three principal axes under different initial conditions.

5 Switching Back to Ground Frame

To visualize the T-handle from an inertial perspective, we must convert its body-frame solution into the ground frame at each time step. The simulation provides the angular velocity $\boldsymbol{\omega}(t)$ in the body frame, which changes the T-handle's orientation over time. At each discrete time step Δt, we approximate the incremental rotation by exponentiating the skew-symmetric matrix associated with $\boldsymbol{\omega}$, i.e.,

$$R_{\Delta t} \approx \exp\big([\boldsymbol{\omega}]\,\Delta t\big),$$

where $[\boldsymbol{\omega}]$ is the skew-symmetric form of $\boldsymbol{\omega}$. Multiplying the previous global orientation matrix R by $R_{\Delta t}$ yields the updated orientation:

$$R \leftarrow R\,R_{\Delta t}.$$

By applying R to the T-handle's coordinates in the body frame, we obtain the corresponding coordinates in the ground frame. Consequently, the animation shows the T-handle's tumbling motion in an inertial reference frame, illustrating how the angular velocity and angular momentum vectors evolve in real time.

6 Numerical Simulations

The dynamics were simulated numerically using Python. Key parameters include:

- Time span: $t \in [0, 20]$ seconds.
- Initial angular velocities: `omega0` values such as [10, 0.1, 0.1] and [1, 0.1, 0.1].

- The respective masses and lengths of the T-handle shaft and crossbar.

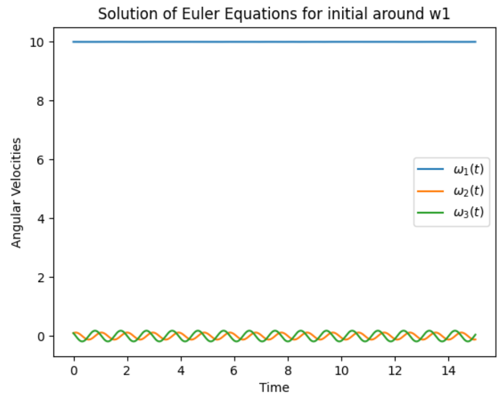
The following cases were studied:

1. Stable rotation about ω_3.
2. Unstable perturbation along ω_2.
3. Small perturbation along ω_1.

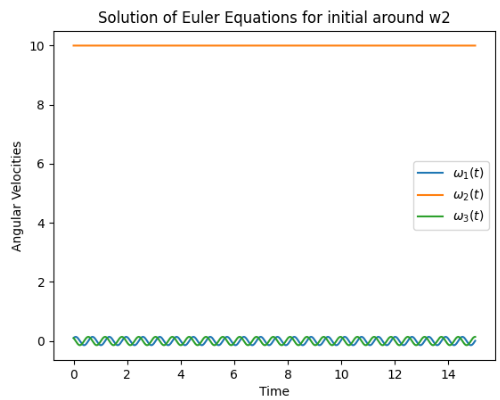
To solve the system and evolve it forward over time, the Runge-Kutta 4-5th-order method was used in Python's SciPy library. The data was then plotted and animated over the time frame using MatPlotLib. Please refer to the Github repository for animations and code.

7 Results and Discussion

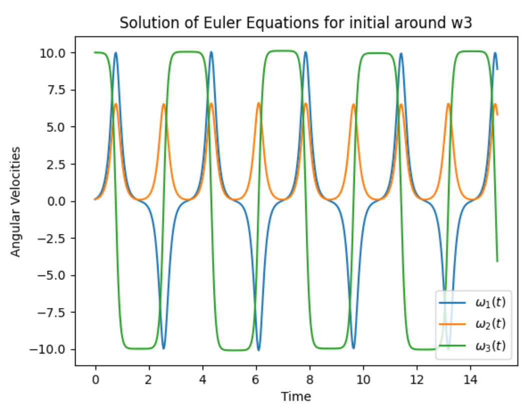
Figures and tables summarize the rotational behavior of the T-handle under each condition. The angular velocity over time in the body frame is displayed in the plots. Two are stable and the third is not:



(a) `[10, 0.1, 0.1]`



(b) `[0.1, 10, 0.1]`



(c) `[0.1, 0.1, 10]`

Figure 1: Simulated angular velocity components for different initial conditions.

8 Conclusion

This study demonstrates the interplay of inertial properties and rotational dynamics in a T-shaped rigid body. The results in the plots and animations verify the expectations laid out by the Tennis Racket Theorem.

Future work could explore external torques, different geometries, and real-world applications.

9 References

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